SCIENTIA MAGNA

An international journal

Edited by School of Mathematics
Northwest University, P.R.China

Vol. 13, No. 1, 2018 ISSN 1556-6706

SCIENTIA MAGNA

An international journal

Edited by

School of Mathematics

Northwest University

Xi'an, Shaanxi, China

Information for Authors

Scientia Magna is a peer-reviewed, open access journal that publishes original research articles in all areas of mathematics and mathematical sciences. However, papers related to Smarandache's problems will be highly preferred.

The submitted manuscripts may be in the format of remarks, conjectures, solved/unsolved or open new proposed problems, notes, articles, miscellaneous, etc. Submission of a manuscript implies that the work described has not been published before, that it is not under consideration for publication elsewhere, and that it will not be submitted elsewhere unless it has been rejected by the editors of Scientia Magna.

Manuscripts should be submitted electronically, preferably by sending a PDF file to ScientiaMagna@hotmail.com.

On acceptance of the paper, the authors will also be asked to transmit the TeX source file. PDF proofs will be e-mailed to the corresponding author.

Contents

Scientia Magna
Vol. 13 (2018), No. 1, 1-11

A multiplier transformation and conditions for parabolic starlike and uniformly convex functions

Richa Brar[1] and Sukhwinder Singh Billing[2]

[1] Department of Mathematics, Sri Guru Granth Sahib World University,
Fatehgarh Sahib-140 407, Punjab, India
E-mail: richabrar4@gmail.com
[2]Department of Mathematics, Sri Guru Granth Sahib World University,
Fatehgarh Sahib-140 407, Punjab, India
E-mail: ssbilling@gmail.com

Abstract In the present paper, we establish a subordination theorem involving a multiplier transformation. As special cases of our main result, we obtain certain sufficient conditions for parabolic starlikeness, starlikeness, uniform convexity and convexity of analytic functions.

Keywords Analytic function, parabolic starlike function, uniformly convex function, differential subordination, multiplier transformation.
2010 Mathematics Subject Classification 30C80, 30C45.

§1. Introduction and preliminaries

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N}; z \in \mathbb{E})$$

which are analytic and p-valent in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. Obviously, $\mathcal{A}_1 = \mathcal{A}$, the class of all analytic functions f, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g in $\mathbb{E}$ (written as $f \prec g$), if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1, \phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)), \ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1),

is said to be the best dominant of (1). The best dominant is unique up to a rotation of $\mathbb{E}$.
A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order $\alpha(0 \leq \alpha < p)$ in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \mathbb{E}.$$

Let $\mathcal{S}_p^*(\alpha)$ denote the class of p-valent starlike functions of order α. Write $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$, the class of p-valent starlike functions.
A function $f \in \mathcal{A}_p$ is said to be p-valent convex of order $\alpha(0 \leq \alpha < p)$ in $\mathbb{E}$ if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{E}.$$

Let the class of such functions be denoted by $\mathcal{K}_p(\alpha)$. Let $\mathcal{K}_p(0) = \mathcal{K}_p$, the class of p-valent convex functions.
A function $f \in \mathcal{A}_p$ is said to be p-valent parabolic starlike in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - p\right|, z \in \mathbb{E}.$$

Let $\mathcal{S}_{\mathcal{P}}^p$ denote the class of p-valent parabolic starlike functions. Write $\mathcal{S}_{\mathcal{P}}^1 = \mathcal{S}_{\mathcal{P}}$, the class of parabolic starlike functions.
A function $f \in \mathcal{A}_p$ is said to be uniformly p-valent convex in $\mathbb{E}$ if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)} - (p-1)\right|, z \in \mathbb{E},$$

and is denoted by UCV_p, the class of uniformly p-valent convex functions and Let $UCV_1 = UCV$, the class of uniformly convex functions.
For $f \in \mathcal{A}_p$, we define the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n, \lambda)[f](z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \text{where } \lambda \geq 0, n \in \mathbb{Z}.$$

Recently, Billing [2–6] and Singh et al. [15, 16] investigated the operator $I_p(n, \lambda)$ and obtained certain sufficient conditions for starlike and convex functions. Earlier, this operator was studied by Aghalary et al. [1]. In 2003, Cho and Srivastava [8] and Cho and Kim [7] investigated the operator $I_1(n, \lambda)$, whereas Uralegaddi and Somanatha [17] studied the operator $I_1(n, 1)$. The operator $I_1(n, 0)$ is the well-known Sălăgean [14] derivative operator $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$.
Let $\mathcal{S}_n(\alpha)$ denote the class of functions $f \in \mathcal{A}$ for which

$$\Re\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) > \alpha, z \in \mathbb{E}, 0 \leq \alpha < 1.$$

In 1989, Owa, Shen and Obradovič [12] studied this class and proved the following result.

Theorem 1.1. *For $n \in \mathbb{N}_0$. if $f \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{n+1}[f](z)}{D^n[f](z)} - 1\right|^{1-\beta} \left|\frac{D^{n+2}[f](z)}{D^{n+1}[f](z)} - 1\right|^{\beta} < (1-\alpha)^{1-2\beta}\left(1 - \frac{3}{2}\alpha + \alpha^2\right)^{\beta}, z \in \mathbb{E},$$

for some $\alpha(0 \leq \alpha \leq 1/2)$ *and* $\beta(0 \leq \beta \leq 1)$ *then* $\Re\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) > \alpha$ *i.e.* $f \in \mathcal{S}_n(\alpha)$. Later on, Li and Owa [9] extended this result by proving the following result.

Theorem 1.2. *For* $n \in \mathbb{N}_0$, *if* $f \in \mathcal{A}$ *satisfies*

$$\left|\frac{D^{n+1}[f](z)}{D^n[f](z)} - 1\right|^\gamma \left|\frac{D^{n+2}[f](z)}{D^{n+1}[f](z)} - 1\right|^\beta < \begin{cases} (1-\alpha)^\gamma(\frac{3}{2}-\alpha)^\beta, & 0 \leq \alpha \leq \frac{1}{2}, \\ 2^\beta(1-\alpha)^{\beta+\gamma}, & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

for some $\alpha(0 \leq \alpha < 1), \beta \geq 0$ *and* $\gamma \geq 0$ *with* $\beta + \gamma > 0$, *then* $f \in S_n(\alpha)$, $n \in \mathbb{N}_0$. Let $\mathcal{S}_n(p, \lambda, \alpha)$ denote the class of functions $f \in \mathcal{A}_p$ for which

$$\Re\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right) > \frac{\alpha}{p}, z \in \mathbb{E}, 0 \leq \alpha < p.$$

In 2008, Billing et al. [15] investigated the above class and proved the following sufficient condition for a multivalent function to be a member of this class.

Theorem 1.3. *Let* $f \in \mathcal{A}_p$ *satisfy*

$$\left|\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} - 1\right|^\gamma \left|\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)} - 1\right|^\beta < M(p,\lambda,\alpha,\beta,\gamma), z \in \mathbb{E},$$

for some real numbers α, β *and* γ *such that* $0 \leq \alpha < p, \beta \geq 0, \gamma \geq 0, \beta + \gamma > 0$, *then* $f \in S_n(p, \lambda, \alpha)$, *where* $n \in \mathbb{N}_0$ *and*

$$M(p,\lambda,\alpha,\beta,\gamma) = \begin{cases} \left(1-\frac{\alpha}{p}\right)^\gamma \left(1-\frac{\alpha}{p}+\frac{1}{2(p+\lambda)}\right)^\beta, & 0 \leq \alpha \leq \frac{p}{2}, \\ \left(1-\frac{\alpha}{p}\right)^{\gamma+\beta} \left(1+\frac{1}{(p+\lambda)}\right)^\beta, & \frac{p}{2} \leq \alpha < p. \end{cases}$$

For $f \in \mathcal{A}_p$, we define a class $\mathcal{S}_n(p, \lambda)$ consisting of functions which satisfy

$$\Re\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right) > \left|\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} - p\right|, z \in \mathbb{E}, \tag{2}$$

where $\lambda \geq 0, n \in \mathbb{N}_0$. Note that $\mathcal{S}_0(p, 0) = \mathcal{S}_\mathcal{P}^p$ and $\mathcal{S}_1(p, 0) = UCV_p$. Define the parabolic domain Ω as under

$$\Omega = \{u + iv : u > \sqrt{(u-p)^2 + v^2}\}.$$

Clearly the function

$$q(z) = p + \frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

maps the unit disk $\mathbb{E}$ onto the domain Ω. Hence the condition (2) is equivalent to

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec q(z)$$

where $q(z)$ is given above.

Ronning [13] and Ma and Minda [10] studied the domain Ω and the above function $q(z)$ in a special case where $p = 1$. In the present paper, we obtain sufficient conditions for a function $f \in \mathcal{A}_p$ to be a member of class $\mathcal{S}_n(p, \lambda)$. As consequences of our main result, we obtain sufficient conditions for parabolic starlikeness, starlikeness, uniform convexity and convexity of multivalent/univalent analytic functions.

To prove our main results, we shall use the following lemma of Miller and Mocanu ([11], p.132).

Lemma 1.1. *Let q be a univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q(z)$ and suppose that either*

(i) h is convex, or

(ii) Q is starlike.

In addition, assume that

(iii) $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for all z in $\mathbb{E}$.

If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)], z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

§2. Main Result

In what follows, all the powers taken are principal ones.

Theorem 2.1. *Let $q(z)$ be a univalent function in $\mathbb{E}$ such that*

(i) $\Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)}\right] > 0$

(ii) $\Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} + \left(1 + \frac{\gamma}{\beta}\right)(p+\lambda)q(z)\right] > 0.$

If $f \in \mathcal{A}_p$ satisfies

$$\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)^{\gamma}\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}\right)^{\beta} \prec (q(z))^{\gamma}\left(q(z) + \frac{zq'(z)}{(p+\lambda)q(z)}\right)^{\beta}, \tag{3}$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec q(z), z \in \mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$ and β, γ are complex numbers such that $\beta \neq 0$.

Proof. On writing $\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} = u(z)$, in (3), we obtain:

$$(u(z))^{\gamma}\left(u(z) + \frac{zu'(z)}{(p+\lambda)u(z)}\right)^{\beta} \prec (q(z))^{\gamma}\left(q(z) + \frac{zq'(z)}{(p+\lambda)q(z)}\right)^{\beta},$$

or

$$(u(z))^{\frac{\gamma}{\beta}+1} + \frac{u(z)^{\frac{\gamma}{\beta}-1}zu'(z)}{(p+\lambda)} \prec (q(z))^{\frac{\gamma}{\beta}+1} + \frac{q(z)^{\frac{\gamma}{\beta}-1}zq'(z)}{(p+\lambda)}.$$

Let us define the function θ and ϕ as follows:

$$\theta(w) = w^{\frac{\gamma}{\beta}+1}$$

and

$$\phi(w) = \frac{w^{\frac{\gamma}{\beta}-1}}{(p+\lambda)}.$$

Therefore,

$$Q(z) = \phi(q(z))zq'(z) = \frac{q(z)^{\frac{\gamma}{\beta}-1}zq'(z)}{(p+\lambda)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = (q(z))^{\frac{\gamma}{\beta}+1} + \frac{q(z)^{\frac{\gamma}{\beta}-1}zq'(z)}{(p+\lambda)}.$$

On differentiating, we obtain $\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)}$ and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} + \left(1 + \frac{\gamma}{\beta}\right)(p+\lambda)q(z).$$

In view of the given conditions, we see that Q is starlike and $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$.
Therefore, the proof, now follows from Lemma 1.1. □

Remark. *Setting $\lambda = 0$ and $p = 1$ in Theorem 2.1, we have the following result.*
Theorem 2.2. *If $f \in \mathcal{A}$ satisfies*

$$\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right)^{\gamma}\left(\frac{D^{n+2}[f](z)}{D^{n+1}[f](z)}\right)^{\beta} \prec (q(z))^{\gamma}\left(q(z) + \frac{zq'(z)}{q(z)}\right)^{\beta}, z \in \mathbb{E},$$

then

$$\frac{D^{n+1}[f](z)}{D^n[f](z)} \prec q(z), z \in \mathbb{E},$$

where $n \in \mathbb{N}_0$ and β, γ are complex numbers such that $\beta \neq 0$.

§3. Applications

Remark 3.1. *When we select the dominant $q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ in Theorem 2.1, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}$$

$$+ \left(\frac{\gamma}{\beta} - 1\right)\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

$$1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} + \left(1 + \frac{\gamma}{\beta}\right)(p+\lambda)q(z)$$

$$= \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

$$+\left(1+\frac{\gamma}{\beta}\right)(p+\lambda)\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right].$$

Thus for $\beta, \gamma \in \mathbb{R}$ such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$, we notice that q(z) satisfies the condition (i) and (ii) of Theorem 2.1. Therefore, we immediately arrive at the following result.

Theorem 3.1. *Let γ and β be real numbers such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)^{\gamma}\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}\right)^{\beta} \prec \left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$

$$\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{1}{(p+\lambda)}\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\beta}, z \in \mathbb{E},$$

then $f \in \mathcal{S}_n(p,\lambda)$, where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.1, we get the following result.

Corollary 3.1. *Let γ and β be real numbers such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left(\frac{zf'(z)}{pf(z)}\right)^{\gamma}\left\{\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}^{\beta} \prec \left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$

$$\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{p+\frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\beta}, z \in \mathbb{E},$$

then $f \in \mathcal{S}^p_{\mathcal{P}}$.

Setting $p = 1$ in above corollary, we get:

Corollary 3.2. *Let γ and β be real numbers such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$. If $f \in \mathcal{A}$ satisfies*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta} \prec \left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$

$$\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\beta}, z \in \mathbb{E},$$

then $f \in \mathcal{S}_{\mathcal{P}}$.

Setting $\lambda = 0, n = 1$ in Theorem 3.1, we obtain the following result.

Corollary 3.3. *Let γ and β be real numbers such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left\{\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}^{\gamma}\left\{\frac{1}{p}\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right)\right\}^{\beta} \prec$$

$$\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$

$$\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{p+\frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\beta}, z\in\mathbb{E},$$

then $f \in UCV_p$.

Setting $p = 1$ in above corollary, we get:

Corollary 3.4. *Let* γ *and* β *be real numbers such that* $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right)^{\beta} \prec \left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$

$$\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\beta}, z\in\mathbb{E},$$

then $f \in UCV$.

Remark 3.2. *When we select the dominant* $q(z) = \dfrac{1+z}{1-z}$ *in Theorem 2.1, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}=\frac{1+z}{1-z}+\left(\frac{\gamma}{\beta}-1\right)\frac{2z}{1-z^2}$$

$$1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}+\left(1+\frac{\gamma}{\beta}\right)(p+\lambda)q(z)=\frac{1+z}{1-z}+\left(\frac{\gamma}{\beta}-1\right)\frac{2z}{1-z^2}$$

$$+\left(1+\frac{\gamma}{\beta}\right)(p+\lambda)\frac{1+z}{1-z}.$$

Clearly, q(z) satisfies condition (i) and (ii) of Theorem 2.1 for $\gamma = \beta = 1$ *or* $\gamma = 0$. For $\gamma = \beta = 1$, Theorem 2.1 yields:

Theorem 3.2. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\frac{I_p(n+2,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \left(\frac{1+z}{1-z}\right)^2+\frac{2z}{(p+\lambda)(1-z)^2}, z\in\mathbb{E},$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \frac{1+z}{1-z}, z\in\mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.2, we get the following result.

Corollary 3.5. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec p^2\left(\frac{1+z}{1-z}\right)^2+\frac{2pz}{(1-z)^2}, z\in\mathbb{E},$$

then $f \in \mathcal{S}_p^*$.

Setting $p = 1$ in above corollary, we get:

Corollary 3.6. *If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^2 + \frac{2z}{(1-z)^2}, z \in \mathbb{E},$$

then $f \in \mathcal{S}^*$.

Setting $\lambda = 0, n = 1$ in Theorem 3.2, we obtain the following result.

Corollary 3.7. *If $f \in \mathcal{A}_p$ satisfies*

$$1+\frac{3zf''(z)}{f'(z)}+\frac{z^2f'''(z)}{f'(z)} \prec p^2\left(\frac{1+z}{1-z}\right)^2 + \frac{2pz}{(1-z)^2}, z \in \mathbb{E},$$

then $f \in \mathcal{K}_p$.

Setting $p = 1$ in above corollary, we get the following condition for convexity.

Corollary 3.8. *If $f \in \mathcal{A}$ satisfies*

$$1+\frac{3zf''(z)}{f'(z)}+\frac{z^2f'''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^2 + \frac{2z}{(1-z)^2}, z \in \mathbb{E},$$

then $f \in \mathcal{K}$.

For $\gamma = 0$, Theorem 2.1 reduces to:

Theorem 3.3. *If $f \in \mathcal{A}_p$ satisfies*

$$\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)} \prec \frac{1+z}{1-z} + \frac{2z}{(p+\lambda)(1-z^2)}, z \in \mathbb{E},$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \frac{1+z}{1-z}, z \in \mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.3, we get the following result.

Corollary 3.9. *If $f \in \mathcal{A}_p$ satisfies*

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{p(1+z)}{1-z} + \frac{2z}{(1-z^2)}, z \in \mathbb{E},$$

then $f \in \mathcal{S}_p^*$.

Setting $p = 1$ in above corollary, we obtain the following result.

Corollary 3.10. *If $f \in \mathcal{A}$ satisfies*

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} + \frac{2z}{(1-z^2)}, z \in \mathbb{E},$$

then $f \in \mathcal{S}^*$.

Setting $\lambda = 0, n = 1$ in Theorem 3.3, we get:

Corollary 3.11. *If $f \in \mathcal{A}_p$ satisfies*

$$1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)} \prec \frac{p(1+z)}{1-z} + \frac{2z}{(1-z^2)}, z \in \mathbb{E},$$

then $f \in \mathcal{K}_p$.

Setting $p = 1$ in above corollary, we have the following result.

Corollary 3.12. *If $f \in \mathcal{A}$ satisfies*

$$1 + \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} \prec \frac{1+z}{1-z} + \frac{2z}{1-z^2}, z \in \mathbb{E},$$

then $f \in \mathcal{K}$.

Remark 3.3. *When we select the dominant $q(z) = e^z$ in Theorem 2.1, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} = 1 + \frac{\gamma z}{\beta}$$

$$1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} + \left(1 + \frac{\gamma}{\beta}\right)(p+\lambda)q(z) = 1 + \frac{\gamma z}{\beta} + \left(1 + \frac{\gamma}{\beta}\right)(p+\lambda)e^z.$$

Thus for $\beta, \gamma \in \mathbb{R}$ such that $0 \leq \frac{\gamma}{\beta} < 1$, we notice that q(z) satisfies the condition (i) and (ii) of Theorem 2.1. Therefore, we immediately arrive at the following result.

Theorem 3.4. *Let γ and β be real numbers such that $0 \leq \frac{\gamma}{\beta} < 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)^{\gamma}\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}\right)^{\beta} \prec e^{\gamma z}\left(e^z + \frac{z}{p+\lambda}\right)^{\beta}, z \in \mathbb{E},$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec e^z, z \in \mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.4, we get the following result.

Corollary 3.13. *Let γ and β be real numbers such that $0 \leq \frac{\gamma}{\beta} < 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\left(\frac{zf'(z)}{pf(z)}\right)^{\gamma}\left\{\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}^{\beta} \prec e^{\gamma z}\left(e^z + \frac{z}{p}\right)^{\beta}, z \in \mathbb{E},$$

then $f \in \mathcal{S}_p^*$.

Setting $p = 1$ in above corollary, we obtain the following condition for starlikeness.

Corollary 3.14. *Let γ and β be real numbers such that $0 \leq \frac{\gamma}{\beta} < 1$. If $f \in \mathcal{A}$ satisfies*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} \prec e^{\gamma z}\left(e^z + z\right)^{\beta}, z \in \mathbb{E},$$

then $f \in \mathcal{S}^*$.

Setting $\lambda = 0, n = 1$ in Theorem 3.4, we get:

Corollary 3.15. *Let γ and β be real numbers such that $0 \leq \frac{\gamma}{\beta} < 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\left\{\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}^{\gamma}\left\{\frac{1}{p}\left(1 + \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right)\right\}^{\beta} \prec e^{\gamma z}\left(e^z + \frac{z}{p}\right)^{\beta}, z \in \mathbb{E},$$

then $f \in \mathcal{K}_p$.

Setting $p = 1$ in above corollary, we get:

Corollary 3.16. *Let γ and β be real numbers such that $0 \leq \frac{\gamma}{\beta} < 1$. If $f \in \mathcal{A}$ satisfies*

$$\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma} \left(1 + \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right)^{\beta} \prec e^{\gamma z} \left(e^z + z\right)^{\beta}, z \in \mathbb{E},$$

then $f \in \mathcal{K}$.

Remark 3.4. *When we select the dominant $q(z) = 1 + az; 0 \leq a < 1$ in Theorem 2.1, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right) \frac{zq'(z)}{q(z)} = 1 + \left(\frac{\gamma}{\beta} - 1\right) \frac{az}{1 + az}$$

$$1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right) \frac{zq'(z)}{q(z)} + \left(1 + \frac{\gamma}{\beta}\right)(p + \lambda)q(z) = 1 + \left(\frac{\gamma}{\beta} - 1\right) \frac{az}{1 + az} + \left(1 + \frac{\gamma}{\beta}\right)(p + \lambda)(1 + az).$$

Thus for $\beta, \gamma \in \mathbb{R}$ such that $\gamma = 0$, we notice that q(z) satisfies the condition (i) and (ii) of Theorem 2.1. Therefore, we immediately get the following result.

Theorem 3.5. *If $f \in \mathcal{A}_p$ satisfies*

$$\frac{I_p(n+2, \lambda)[f](z)}{I_p(n+1, \lambda)[f](z)} \prec 1 + az + \frac{az}{(p + \lambda)(1 + az)}, z \in \mathbb{E},$$

then

$$\frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \prec 1 + az, z \in \mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.5, we get the following result.

Corollary 3.17. *If $f \in \mathcal{A}_p$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec p(1 + az) + \frac{az}{(1 + az)}, z \in \mathbb{E},$$

then $f \in \mathcal{S}_p^*$.

Setting $p = 1$ in above corollary, we obtain the following result.

Corollary 3.18. *If $f \in \mathcal{A}$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + az + \frac{az}{1 + az}, z \in \mathbb{E},$$

then $f \in \mathcal{S}^*$.

Setting $\lambda = 0, n = 1$ in Theorem 3.5, we get:

Corollary 3.19. *If $f \in \mathcal{A}_p$ satisfies*

$$1 + \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} \prec p(1 + az) + \frac{az}{(1 + az)}, z \in \mathbb{E},$$

then $f \in \mathcal{K}_p$.

Setting $p = 1$ in above corollary, we obtain the following result for convex functions.

Corollary 3.20. *If $f \in \mathcal{A}$ satisfies*

$$1 + \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} \prec 1 + az + \frac{az}{1 + az}, z \in \mathbb{E},$$

then $f \in \mathcal{K}$.

References

[1] R. Aghalary, R. M. Ali and S. B. Joshi, V. Ravichandran. Inequalities for analytic functions defined by certain linear operators. International J. Math. Sci. 4 (2005), no. 2, 267-274 .

[2] S. S. Billing. Cerain differential subordination involving a multiplier transformation. Scientia Magna 8 (2012), no. 1, 87-93.

[3] S. S. Billing. Differential inequalities implying starlikeness and convexity. Acta Universitatis Matthiae. Belii, series Mathematics 20 (2012), 3-8.

[4] S. S. Billing. A subordination theorem involving a multiplier transformation. Int. J. Ann and Appl. 1 (2013), no. 2, 100-105.

[5] S. S. Billing. A multiplier transformation and conditions for starlike and convex functions. Math. Sci. Res. J. 17 (2013), no. 9, 239-244.

[6] S. S. Billing. Differential inequalities and criteria for starlike and convex functions. Stud. Univ. Babes-Bolyai Math. Sci. 59 (2014), no. 2, 191-198.

[7] N. E. Cho and T. H. Kim. Multiplier transformations and stronlgy close-to-convex functions. Bull. Korean Math. Soc. 40 (2003), 399-410.

[8] N. E. Cho and H. M. Srivastava. Argument estimates of certain analytic functions defined by a class of multiplier transformations. Math. Comput. Modelling 37 (2003), 39-49.

[9] J. Li and S. Owa. Properties of the sălăgean operator. Georgian Math. J. 5 (1998), no. 4, 361-366.

[10] W. C. Ma and D. Minda. Uniformly convex functions. Ann. Polon. Math. 57 (1992), no.2, 165-175.

[11] S. S. Miller and P. T. Mocanu. Differential subordinations: Theory and applications. Marcel Dekker, New York and Basel, 2000.

[12] S. Owa, C. Y. Shen and M. Obradovič. Certain subcalsses of analytic functions. Tamkang J. Math. 20 (1989), 105-115.

[13] F. Ronning. Uniformly convex functions and a corresponding class of starlike functions. Proc. Amer. Math. Soc. 118 (1993), no. 1, 189-196.

[14] G. S. Sălăgean. Subclasses of univalent functions. Lecture Notes in Math., Springer-Verlag, Heidelberg, 1013 (1983) 362-372.

[15] S. Singh, S. Gupta and S. Singh. On a class of multivalent functions defined by a multiplier transformation. Matematicki Vesnik 60 (2008), 87-94.

[16] S. Singh, S. Gupta and S. Singh. On starlikeness and convexity of analytic functions satisfying a differential inequality. J. Inequal. Pure and Appl. Math. 9 (2008), no.3, Art. 81, 1-7.

[17] B. A. Uralegaddi and C. Somanatha, Cerain classes of univalent functions. Current Topics in Analytic Function Theory, H. M. Srivastava and S. Owa (ed.), World Scientific, Singapore (1992) 371-374.

Scientia Magna
Vol. 13 (2018), No. 1, 12-18

On *pgb* - Connectedness and *pgb* - Compactness in Topological Spaces

S. Sekar[1] and R. Brindha[2]

[1]Department of Mathematics, Government Arts and Science College
Thittamalai – 638 458, Erode, Tamil Nadu, India
E-mail: sekar_nitt@rediffmail.com
[2]Department of Mathematics, King College of Technology
Namakkal – 637 020, Tamil Nadu, India.
E-mail: brindhaaramasamy@gmail.com

Abstract In this paper, the authors introduce a new type of connected spaces called pre generalized b -connected spaces (briefly pgb-connected spaces) in topological spaces. The notion of pre generalized b -compact spaces is also introduced (briefly pgb-compact spaces) in topological spaces. Some characterizations and several properties concerning pgb-connected spaces and pgb-compact spaces are obtained.

Keywords pgb-closed sets, pgb-closed map, pgb-continuous map, contra pgb-continuity.
2010 Mathematics Subject Classification 54C05, 54C08, 54C10.

§1. Introduction

Topological spaces are mathematical structures that allow the formal definitions of concepts such as connectedness, compactness, interior and closure. In 1974, Das [4] defined the concept of semi-connectedness in topology and investigated its properties. Compactness is one of the most important, useful and fundamental concepts in topology. In 1981, Dorsett [6] introduced and studied the concept of semi-compact spaces. In 1990, Ganster [7] defined and investigated semi-Lindelof spaces. Since then, Hanna and Dorsett [10], Ganster and Mohammad S. Sarsak [8] investigated the properties of semi-compact spaces. The notion of connectedness and compactness are useful and fundamental notions of not only general topology but also of other advanced branches of mathematics.

Ganster and Steiner [9] introduced and studied the properties of gb-closed sets in topological spaces. Benchalli et al [2] introduced gb - compactness and gb - connectedness in topological spaces. Dontchev and Ganster [5] analyzed sg - compact space. Later, Shibani [14] introduced and analyzed rg - compactnes and rg - connectedness. Crossely et al [3] introduced semi - closure. Vadivel et al [15] studied $rg\alpha$ - interior and $rg\alpha$ - closure sets in topological spaces. The aim of this paper is to introduce the concept of pgb-connected and pgb-compactness in topological spaces.

§2. Preliminary Notes

Definition 2.1. *A subset A of a topological space (X,τ), is called pre open [11], if $A \subseteq int(cl(A))$. The complement of pre open set is said to be pre closed set . The family of all pre open sets (respectively pre closed sets) of (X,τ) is denoted by $PO(X,\tau)$[respectively $PCL(X,\tau)$].*

Definition 2.2. *A subset A of a topological space (X,τ), is called pre generalized b-closed set [12] (briefly pgb-closed set) if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is pre open in X. The complement of pgb-closed set is called pgb-open. The family of all pgb-open [respectively pgb-closed] sets of (X,τ) is denoted by $pgb-O(X,\tau)$ [respectively $pgb-CL(X,\tau)$].*

Definition 2.3. *A subset A of a topological space (X,τ) is called b-open set [1] if $A \subseteq cl(int(A)) \cup int(cl(A))$. The complement of b-open set is b-closed sets. The family of all b-open sets (respectively b-closed sets) of (X,τ) is denoted by $bO(X,\tau)$ (respectively $bCL(X,\tau)$)*

Definition 2.4. *The pgb-closure of a set A, denoted by $pgb-Cl(A)$ [13] is the intersection of all pgb-closed sets containing A.*

Definition 2.5. *The pgb-interior of a set A, denoted by $pgb-int(A)$ [13] is the union of all pgb-open sets containing A.*

§3. *pgb*-Connectedness

In this section, we introduce pre generalized b - connected space and investigate some of their properties.

Definition 3.1. *A topological space X is said to be pgb-connected if X cannot be expressed as a disjoint of two non - empty pgb-open sets in X. A subset of X is pgb-connected if it is pgb-connected as a subspace.*

Example 3.2. *Let $X = \{a,b,c\}$ and let $\tau = \{X,\varphi,\{a\}\}$. It is pgb-connected.*

Theorem 3.3. *For a topological space X, the following are equivalent.*
(i) X is pgb-connected.
(ii) X and φ are the only subsets of X which are both pgb-open and pgb-closed.
(iii) Each pgb-continuous map of X into a discrete space Y with at least two points is constant map.

Proof. (i) $\Rightarrow$ (ii) : Suppose X is *pgb* - connected. Let S be a proper subset which is both *pgb* - open and *pgb* - closed in X. Its complement $X - S$ is also *pgb* - open and *pgb* - closed. $X = S \cup (X - S)$, a disjoint union of two non empty *pgb* - open sets which is contradicts (i). Therefore $S = \varphi$ or X.
(ii) $\Rightarrow$ (i) : Suppose that $X = A \cup B$ where A and B are disjoint non empty *pgb* - open subsets of X. Then A is both *pgb* - open and *pgb* - closed. By assumption $A = \varphi$ or X. Therefore X is *pgb* - connected.
(ii) $\Rightarrow$ (iii) : Let $f : X \to Y$ be a *pgb* - continuous map. X is covered by *pgb* - open and *pgb* - closed covering $\{f^{-1}(y) : y \in Y\}$. By assumption $f^{-1}(y) = \varphi$ or X for each $y \in Y$. If $f^{-1}(y) = \varphi$ for all $y \in (Y)$, then f fails to be a map. Then there exists only one point $y \in Y$ such that $f^{-1}(y) \neq \varphi$ and hence $f^{-1}(y) = X$. This shows that f is a constant map.

(iii) $\Rightarrow$ (ii) : Let S be both *pgb* - open and *pgb* - closed in X. Suppose $S \neq \varphi$. Let $f : X \to Y$ be a *pgb* - continuous function defined by $f(S) = \{y\}$ and $f(X - S) = \{w\}$ for some distinct points y and w in Y. By (iii) f is a constant function. Therefore $S = X$. □

Theorem 3.4. *Every pgb - connected space is connected.*

Proof. Let X be *pgb* - connected. Suppose X is not connected. Then there exists a proper non empty subset B of X which is both open and closed in X. Since every closed set is *pgb* - closed, B is a proper non empty subset of X which is both *pgb* - open and *pgb* - closed in X. Using by Theorem 3.3, X is not *pgb* - connected. This proves the theorem. □

The converse of the above theorem need not be true as shown in the following example.

Example 3.5. *Let $X = \{a, b, c\}$ and let $\tau = \{X, \varphi, \{a\}, \{c\}, \{a, c\}\}$. X is connected but not pgb - connected. Since $\{a\}, \{b, c\}$ are disjoint pgb - open sets and $X = \{a\} \cup \{b, c\}$.*

Theorem 3.6. *If $f : X \to Y$ is a pgb - continuous onto and X is pgb - connected, then Y is connected.*

Proof. Suppose that Y is not connected. Let $Y = A \cup B$ where A and B are disjoint non - empty open set in Y. Since f is *pgb* - continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non - empty *pgb* - open sets in X. This contradicts the fact that X is *pgb* - connected. Hence Y is connected. □

Theorem 3.7. *If $f : X \to Y$ is a pgb - irresolut and X is pgb - connected, then Y is pgb - connected.*

Proof. Suppose that Y is not *pgb* connected. Let $Y = A \cup B$ where A and B are disjoint non - empty *pgb* open set in Y. Since f is *pgb* - irresolut and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non - empty *pgb* - open sets in X. This contradicts the fact that X is *pgb* - connected. Hence Y is *pgb* - connected. □

Definition 3.8. *A topological space X is said to be T_{pgb} - space if every pgb - closed set of X is closed subset of X.*

Definition 3.9. *Suppose that X is T_{pgb} - space then X is connected if and only if it is pgb - connected.*

Proof. Suppose that X is connected. Then X cannot be expressed as disjoint union of two non - empty proper subsets of X. Suppose X is not a *pgb* - connected space. Let A and B be any two *pgb* - open subsets of X such that $X = A \cup B$, where $A \cap B = \varphi$ and $A \subset X, B \subset X$. Since X is T_{pgb} - space and A, B are *pgb* - open. A, B are open subsets of X, which contradicts that X is connected. Therefore X is *pgb* - connected.
Conversely, every open set is *pgb* - open. Therefore every *pgb* - connected space is connected. □

Theorem 3.10. *If the pgb - open sets C and D form a separation of X and if Y is pgb - connected subspace of X, then Y lies entirely within C or D.*

Proof. Since C and D are both *pgb* - open in X, the sets $C \cap Y$ and $D \cap Y$ are *pgb* - open in Y. These two sets are disjoint and their union is Y. If they were both non - empty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely C or D. □

Theorem 3.11. *Let A be a pgb - connected subspace of X. If $A \subset B \subset pgb - cl(A)$ then B is also pgb - connected.*

Proof. Let A be *pgb* - connected and let $A \subset B \subset pgb - cl(A)$. Suppose that $B = C\ cupD$ is a separation of B by *pgb* - open sets. By using Theorem 3.10, A must lie entirely in C or D. Suppose that $A \subset C$, then $pgb - cl(A) \subset pgb - cl(B)$. Since $pgb - cl(C)$ and D are disjoint, B cannot intersect D. This contradicts the fact that C is non empty subset of B. So $D = \varphi$ which implies B is *pgb* - connected. □

Theorem 3.12. *A contra pgb - continuous image of an pgb - connected space is connected.*

Proof. Let $f : X \to Y$ is a contra *pgb* - continuous function from *pgb* - connected space X on to a space Y. Assume that Y is disconnected. Then $Y = A \cup B$, where A and B are non empty clopen sets in Y with $A \cap B = \varphi$. Since f is contra *pgb* - continous, we have $f^{-1}(A)$ and $f^{-1}(B)$ are non empty *pgb* - open sets in X with $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\varphi) = \varphi$. This shows that X is not *pgb* - connected, which is a contradiction. This proves the theorem. □

§4. *pgb* - Compactness

Definition 4.1. *A collection $\{A_\alpha : \alpha \in \Lambda\}$ of pgb -open sets in a topological space X is called a pgb - open cover of a subset B of X if $B \subset \bigcup \{A_\alpha : \alpha \in \Lambda\}$ holds.*

Definition 4.2. *A topological space X is pgb - compact if every pgb - open cover of X has a finite sub - cover.*

Definition 4.3. *A subset B of a topological space X is said to be pgb - compact relative to X, if for every collection $\{A_\alpha : \alpha \in \Lambda\}$ of pgb - open subsets of X such that $B \subset \bigcup \{A_\alpha : \alpha \in \Lambda\}$ there exists a finite subset Λ_0 of Λ such that $B \subset \bigcup \{A_\alpha : \alpha \in \Lambda_0\}$.*

Definition 4.4. *A subset B of a topological space X is said to be pgb - compact if B is pgb - compact as a subspace of X.*

Theorem 4.5. *Every pgb - closed subset of pgb - compact space is pgb - compact relative to X.*

Proof. Let A be *pgb* - closed subset of a *pgb* - compact space X. Then A^c is *pgb* - open in X. Let $M = \{G_\alpha : \alpha \in \Lambda\}$ be a cover of A by *pgb* - open sets in X. Then $M^* = M \cup A^c$ is a *pgb* - open cover of X. Since X is *pgb* - compact, M^* is reducible to a finite sub cover of X, say $X = G_{\alpha 1} \cup G_{\alpha 2} \cup G_{\alpha 3} \cup \ldots \cup G_{\alpha m} \cup A^c$, $G_{\alpha k} \in M$. But A and A^c are disjoint. Hence $A \subset G_{\alpha 1} \cup G_{\alpha 2} \cup G_{\alpha 3} \cup \ldots \cup G_{\alpha m} G_{\alpha k} \in M$, this implies that any *pgb* open cover M of A contains a finite sub - cover. Therefore A is *gb* - compact relative to X. That is, every *pgb* - closed subset of a *pgb* - compact space X is *pgb* - compact. □

Definition 4.6. *A function $f : X \to Y$ is said to be pgb - continuous if $f^{-1}(V)$ is pgb - closed in X for every closed set V of Y.*

Theorem 4.7. *A pgb - continuous image of a pgb - compact space is compact.*

Proof. Let $f : X \to Y$ be a *pgb* - continuous map from a *pgb* - compact space X onto a topological space Y. Let $\{A_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in \Lambda\}$ is a *pgb* - open cover of X. Since X is *pgb* - compact, it has a finite sub - cover say $\{f^{-1}(A_1), f^{-1} : i \in \Lambda(A_2), \ldots, f^{-1}(A_n)\}$. Since f is onto $\{A_1, A_2, \ldots, A_n\}$ is a cover of Y, which is finite. Therefore Y is compact. □

Definition 4.8. *A function $f : X \to Y$ is said to be pgb - irresolute if $f^{-1}(V)$ is pgb - closed in X for every pgb - closed set V of Y.*

Theorem 4.9. *If a map $f : X \to Y$ is pgb - irresolute and a subset B of X is pgb - compact relative to X, then the image $f(B)$ is pgb - compact relative to Y.*

Proof. Let$\{A_\alpha : \alpha \in \Lambda\}$ be any collection of *pgb* - open subsets of Y such that $f(B) \subset \bigcup \{A_\alpha : \alpha \in \Lambda\} \subset$. Then $B \subset \bigcup \{f^{-1}(A_\alpha) : \alpha \in \Lambda\}$. Since by hypothesis B is *pgb* - compact relative to X, there exists a finite subset $\Lambda_0 \in \Lambda$ such that $B \subset \bigcup \{f^{-1}(A_\alpha) : \alpha \in \Lambda_0\}$. Therefore we have $f(B) \bigcup \subset \{(A_\alpha) : \alpha \in \Lambda_0\}$, it shows that $f(B)$ is *pgb* - compact relative to Y. □

Theorem 4.10. *A space X is pgb - compact if and only if each family of pgb - closed subsets of X with the finite intersection property has a non - empty intersection.*

Proof. Given a collection A of subsets of X, let $C = \{X - A : A \in A\}$ be the collection of their complements. Then the following statements hold.
(a) A is a collection of *pgb* - open sets if and only if C is a collection of *pgb* - closed sets.
(b) The collection A covers X if and only if the intersection $\bigcap_{c \in C} C$ of all the elements of C is empty.
(c) The finite sub collection $\{A_1, A_2, \ldots A_n\}$ of A covers X if and only if the intersection of the corresponding elements $C_i = X - A_i$ of C is empty.The statement (a) is trivial, while the (b) and (c) follow from De Morgan's law.

$X - (\bigcup_{\alpha \in J} A_\alpha) = \bigcap_{\alpha \in J}(X - A_\alpha)$. The proof of the theorem now proceeds in two steps, taking contra positive of the theorem and then the complement.

The statement X is *pgb* - compact is equivalent to : Given any collection A of *pgb* - open subsets of X, if A covers X, then some finite sub collection of A covers X. This statement is equivalent to its contra positive, which is the following.

Given any collection A of *pgb* - open sets, if no finite sub - collection of A of covers X, then A does not cover X. Let C be as earlier, the collection equivalent to the following:

Given any collection C of *pgb* - closed sets, if every finite intersection of elements of C is not - empty, then the intersection of all the elements of C is non - empty. This is just the condition of our theorem. □

Definition 4.11. *A space X is said to be pgb - Lindelof space if every cover of X by pgb - open sets contains a countable sub cover.*

Theorem 4.12. *Let $f : X \to Y$ be a pgb - continuous surjection and X be pgb - Lindelof, then Y is Lindelof Space.*

Proof. Let $f : X \to Y$ be a *pgb* - continuous surjection and X be *pgb* - Lindelof. Let $\{V_\alpha\}$ be an open cover for Y. Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by *pgb* - open sets. Since X is *pgb* - Lindelof, $\{f^{-1}(V_\alpha)\}$ contains a countable sub cover, namely $\{f^{-1}(V_{\alpha n})\}$. Then $\{V_{\alpha n}\}$ is a countable subcover for Y. Thus Y is Lindelof space. □

Theorem 4.13. *Let $f : X \to Y$ be a pgb - irresolute surjection and X be pgb - Lindelof, then Y is pgb - Lindelof Space.*

Proof. Let $f : X \to Y$ be a *pgb* - irresolute surjection and X be *pgb* - Lindelof. Let $\{V_\alpha\}$ be an open cover for Y. Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by *pgb* - open sets. Since X is *pgb* - Lindelof, $\{f^{-1}(V_\alpha)\}$ contains a countable sub cover, namely $\{f^{-1}(V_{\alpha n})\}$. Then $\{V_{\alpha n}\}$ is a countable subcover for Y. Thus Y is *pgb* - Lindelof space. □

Theorem 4.14. *If $f : X \to Y$ is a pgb - open function and Y is pgb -Lindelof space, then X is Lindelof space.*

Proof. Let $\{V_\alpha\}$ be an open cover for X. Then $\{f(V_\alpha)\}$ is a cover of Y by *pgb* - open sets. Since Y is *pgb* Lindelof, $\{f(V_\alpha)\}$ contains a countable sub cover, namely $\{f(V_{\alpha n})\}$. Then $\{V_{\alpha n}\}$ is a countable sub cover for X. Thus X is Lindelof space. □

Acknowledgements

The authors gratefully acknowledge the Dr. G. Balaji, Professor of Mathematics and Head, Department of Science and Humanities, Al-Ameen Engineering College, Erode - 638 104, for encouragement and support. The authors also heartfelt thank to Dr. M. Vijayarakavan, Associate Professor, Department of Mathematics, VMKV Engineering College, Salem - 636 308, Tamil Nadu, India, for his kind help and suggestions.

References

[1] D. Andrijevic. b-open sets, Mat.Vesink, 48 (1996), 59 - 64.

[2] S. S. Benchalli and P. M. Bansali. *gb* - compactness and *gb* - connectedness in topological spaces, Int. J. Contemp. Math. Sciences, 6 (2011), no. 10, 465 - 475.

[3] S. G. Crossley and S. K. Hildebrand. Semi-topological properties, Fund. Math., 74 (1972), 233 - 254.

[4] P. Das. A note on semi connectedness, Indian J. of mechanics and mathematics, 12 (1974), 31 - 34.

[5] J. Dontchev and M. Ganster. On δ - generalized set $T_{3/4}$ spaces, Mem. Fac. Sci. Kochi Uni. Ser.A. Math., 17 (1996), 15 - 31.

[6] C. Dorsett. Semi compactness, semi separation axioms and product spaces, Bull. Malay. Math., 4 (1981), no. 1, 21 - 28.

[7] M. Ganster, D. S. Jankonc and I. L. Reilly. On compactness with respect to semi-open sets, Comment. Math. Uni. Carolinae, 31 (1990), no. 1, 37 - 39.

[8] M. Ganster and Mohammad S. Sarsak. On semi compact sets and associated properties, International Journal of Mathematics and Mathematica Successes, 2004 (2009), 8.

[9] M. Ganster and M. Steiner. On some questions about b - open sets, Questions Answers Gen. Topology, 25 (2007), no. 1, 45 - 52.

[10] F. Hanna and C. Dorsett, Semi compactness, $Q\&A$ in General Topology, 2 (1984), 38 - 47.

[11] A. S. Mashor Abd.El, M. E. Monsef and S. N. Ei - Deeb. On Pre continuous and weak pre - continuous mapping, Proc. Math. Phys. Soc. Egypt. 53 (1982), 47 - 53.

[12] S. Sekar and R. Brindha. On pre generalized b-closed set in Topological Spaces, International Journal of Pure and Applied Mathematics, 111 (2016), no. 4, 577-586.

[13] S. Sekar and R. Brindha. ON pgb-interior and pgb-closure in Topological Spaces, International Journal of Advanced Science and Engineering Research, (Accepted for publication) (2017).

[14] A. M. Shibani. rg - compact spaces and rg - connected spaces, Mathematica Pannonica, 17 (2006), 61 - 68.

[15] A. Vadivel and Vairamanickam. $rg\alpha$ - interior and $rg\alpha$ closure in topological spaces, Int. Journal of Math. Analysis, 4 (2010), no. 9, 435 - 444.

Scientia Magna
Vol. 13 (2018), No. 1, 19-31

On Partial Sum of the Hurwitz Zeta-function and its Application

Qianli Yang[1] and Hailong Li[2]

[1]Department of Mathematics School of Mathematics and Physics
Weinan Normal University, Weinan City, Shaanxi Province.
E-mail: yangql@wnu.edu.cn
[2]Department of Mathematics School of Mathematics and Physics
Weinan Normal University, Weinan City, Shaanxi Province.
E-mail: lihailong@wnu.edu.cn

Abstract In this paper, we state integral representation of the partial sum $L_u(x,a) = \sum_{0\le u\le x}(n+a)^u$ of the Hurwitz Zeta function $\zeta(-u,a)$ and proof. Use the partial sum formula of integral representation to get three results, improve and generalize results of Srivastava and Choi on the asymptotic formula for the sum of the $\upsilon-$th derivative of ψ: $\sum_{k\le x}\psi^{(\upsilon)}(k)$. Second give an asymptotic expansion for $\zeta'''(-m,a)$. Third give the formula of double gamma function of Barnes.

Keywords Partial Summation, Integaral Repesentation, Hurwitz Zeta-function, logarithmic derivative.

2010 Mathematics Subject Classification: Primary 26A48, 33B15;
Secondary 26A51, 26D07, 26D10

§1.Introduction and Proposition

In paper [1], many authors give many integral formulas of some Series by Zeta Partial Summation ion, In paper [2], Professor S.Kanemitsu exhibit the importance and usefulness of the Partial Summation formula by applying it to the sum.

$$L_u(x,a) = \sum_{0\le u\le x}(n+a)^u$$

In a Similar setting which appeared in the pursuit of the divisor problem [2] . In paper [3], professor Li give the formula $\frac{\mathbf{d}}{\mathbf{d}u}L_u(x,a)$ and many beautiful formulas. In book [6] we gave the formula $\frac{\mathbf{d}^2}{\mathbf{d}u^2}L_u(x,a)$. In this paper, we state integral representation of the partial sum $L_u(x,a) = \sum_{0\le u\le x}(n+a)^u$of the Hurwitz Zeta function $\zeta(-u,a)$, by use $L_u(x,a) = \sum_{0\le u\le x}(n+a)^u$, we give three importent results.

We use the following notation.

Notation $s = \sigma + \mathbf{i}t$ − the complex variable.

$\Gamma(s) = \int_0^{+\infty}\mathbf{e}^{-t}t^{s-1}\mathbf{d}t$ –the gamma function ($\sigma > 0$),

$\psi(s) = \frac{\Gamma'}{\Gamma}(s) = (\log\Gamma(s))'$– the digamma function,

Both of which are meromorphically continued to the whole complex plane with simple poles at non-positive integers;

$\zeta(s,a) = \sum\limits_{n=0}^{+\infty} \frac{1}{(n+a)^s}$- Hurwitz Zeta function, $\sigma > 1$, $a > 0$ the power taking the principal value.

$\zeta(s) = \zeta(s,1)$ –the Riemann Zeta function, both of which are continued meromorphically over the complex plane with a simple pole at $s = 1$.

$B_r^{(\alpha)}(x)$ the generalized Bernoulli polynomial of degree r in x, defined through the generating function

$\left(\frac{z}{\mathbf{e}^z-1}\right)^\alpha \mathbf{e}^{zx} = \sum\limits_{r=0}^{+\infty} \frac{1}{r!} B_r^{(\alpha)}(x) z^n \quad (|z| < 2\pi)$ satisfying the addition formula

$$B_r^{(\alpha+\beta)}(x+y) = \sum_{r=0}^{+\infty} \begin{pmatrix} r \\ k \end{pmatrix} B_r^{(\alpha)}(x) B_{r-k}^{(\beta)}(y) \tag{1}$$

([3], Formula (24),p.61) with the properties $B_r^{(\alpha)} = B_r^{(\alpha)}(0)$, $B_r^{(1)}(x) = B_r(x)$,$B_r^{(1)} = B_r$, where $B_r(x)$ and $B_r = B_r(0)$ are the $r-$ th Bernoulli polynomial and the $r-$ th Bernonlli number defined by (1) with $\alpha = 1$.

$\overline{B_r}(x) = B_r(\{x\})$ the $r-$th periodic Bernonulli polynomial.

Propsition1. Let $L_u(x,a) = \sum\limits_{0\le u\le x} (n+a)^u$.

Then, for any $l \in \mathbb{N}$ with $l > \operatorname{Re} u + 1$,$\operatorname{Re} a > 0$ we have

$$\begin{aligned} L_u(x,a) &= \sum_{r=1}^{l} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \frac{(-1)^r}{r!} \overline{B_r}(x)(x+a)^{u-r+1} \\ &+ \frac{(-1)^r}{r!} \frac{\Gamma(u+1)}{\Gamma(u+1-r)} \int_x^{+\infty} \overline{B_l}(t)(t+a)^{u-l} \mathbf{d}t \\ &+ \begin{cases} \frac{1}{u+1}(x+a)^{u+1} + \zeta(-u,a), & u \ne -1 \\ \log(x+a) - \psi(a), & u = -1 \end{cases} \end{aligned} \tag{2}$$

Also the asymptotic formula

$$\begin{aligned} L_u(x,a) &= \sum_{r=1}^{l} \frac{(-1)^r}{r!} \begin{pmatrix} u \\ r-1 \end{pmatrix} \overline{B_r}(x)(x+a)^{u-r+1} + \mathrm{O}\left(x^{\operatorname{Re} u - l}\right) \\ &+ \begin{cases} \frac{1}{u+1}(x+a)^{u+1} + \zeta(-u,a), & u \ne -1 \\ \log(x+a) - \psi(a), & u = -1 \end{cases} \end{aligned} \tag{3}$$

hods true as $x \to +\infty$. On the other hand, formula (2) with $x = 0$ yields the integral representation

$$\zeta(-u,a) = a^u - \frac{1}{u+1}a^{u+1} - \sum_{r=1}^{l}\frac{(-1)^2}{r}\begin{pmatrix} u \\ r-1 \end{pmatrix} B_r a^{u-r+1}$$
$$+ (-1)^{l+1}\begin{pmatrix} u \\ l \end{pmatrix}\int_0^{+\infty}\overline{B_l}(t)(t+a)^{u-1}\mathbf{d}t \tag{4}$$

Which is true for all $u \neq 1$,and l can be any natural number satisfying $l > \mathrm{Re}\,u + 1$; the integral being absolutely convergent in the region $\mathrm{Re}\,u < l-1$, where it is analytic except at $u = -1$.

Corollary 1. For any $l \in \mathbb{N}$ with $l < \mathrm{Re}\,u + 1$, $a > 0$ we have

$$\frac{d^3}{du^3}L_u(x,a) = \sum_{0\leqslant n\leqslant x}(n+a)^u \log^3(x+a)$$
$$= \sum_{r=1}^{l}\frac{(-1)^r}{r!}\overline{B_r}(x)(x+a)^{u-r+1}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\left\{[f(u,r)+\log(x+a)]^3\right.$$
$$+ \qquad 2f'(u,r)\log a + 3f(u,r)f'(u,r) + f''(u,r)\}$$
$$+ \frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\int_x^{+\infty}\overline{B_l}(t)(x+a)^{u-1}\left\{[f(u,l)+\log(x+a)]^3 + 3f(u,l)f'(u,l)\right.$$
$$+ f'(u,l)\log(x+a) + f''(u,l)\}\,\mathbf{d}t$$
$$+\begin{cases} \dfrac{(x+a)^{u+1}}{u+1}\log^3(x+a) - \dfrac{3(x+a)^{u+1}}{(u+1)^2}\log^2(x+a) + \dfrac{6(x+a)^{u+1}}{(u+1)^3}\log(x+a) & \\ -\dfrac{6(x+a)^{u+1}}{(u+1)^4} - \zeta'''(-u,a) & u \neq 1 \\ \dfrac{1}{4}\log^4(x+a) + \gamma_3(a) & u = 1 \end{cases} \tag{5}$$

(2)$f(u,r) = \psi(u+1) - \psi(u+2-r)$,$f(u,l) = \psi(u+1) - \psi(u+1-l)$,

$$\gamma_3(a) = \frac{1}{2a}\log^3 a - \frac{1}{4}\log^4 a - \int_0^{+\infty}\frac{\overline{B_1}(t)}{(t+a)^2}\left(\log^3(t+a) - 3\log^4(t+a)\right)\mathbf{d}t$$

For proof of propsition1 and corollary1, we give some lemmas

Lemma 1[4]**.** Suppose that f is of the Class C' in the closed interval $[a,b]$ $(a<b)$.then

$$\sum_{a\leqslant n\leqslant b} f(n) = \int_a^b f(t)\,\mathbf{d}t + \sum_{r=1}^{l}\frac{(-1)^r}{r!}\left(\overline{B_r}(b)f^{r-1}(b) - \overline{B_r}(a)f^{r-1}(a)\right)$$
$$+ \frac{(-1)^{l+1}}{l!}\int_a^b \overline{B_l}(t)f^{(l)}\mathbf{d}t \tag{6}$$

Lemma 2[5]**.** For any complex u and $a > 0$

$$\begin{aligned}\frac{\mathbf{d}^2}{\mathbf{d}u^2}Lu\left(x,a\right) &= \sum_{0\leqslant n\leqslant x}\left(n+a\right)^u\log^2\left(n+a\right)\\ &= \sum_{r=1}^{l}\frac{\Gamma\left(u+1\right)}{\Gamma\left(u+2-r\right)}\frac{\left(-1\right)^r}{r!}\overline{B_r}\left(x\right)\left(x+a\right)^{u-r+1}\\ &\left\{\left[\psi\left(u+1\right)-\psi\left(u+2-l\right)+\log\left(x+a\right)\right]^2\right.\\ &\left.+\ \psi'\left(u+1\right)-\psi'\left(u+2-l\right)\right\}\\ &+\frac{\left(-1\right)^l}{l!}\frac{\Gamma\left(u+1\right)}{\Gamma\left(u+1-r\right)}\int_x^{+\infty}\overline{B_l}\left(t\right)\left(t+a\right)^{u-l}\\ &\left\{\left[\psi\left(u+1\right)-\psi\left(u+1-l\right)+\log\left(x+a\right)\right]^2\right.\\ &\left.+\ \psi'\left(u+1\right)-\psi'\left(u+1-l\right)\right\}\mathbf{d}t\\ &+\begin{cases}\dfrac{\left(x+a\right)^{u+1}}{u+1}\log^2\left(x+a\right)-\dfrac{2\left(x+a\right)^{u+1}}{\left(u+1\right)^2}\log^2\left(x+a\right) & \\ +\dfrac{2\left(x+a\right)^{u+1}}{\left(u+1\right)^3}+\zeta''\left(-u,a\right) & u\neq-1\\ \dfrac{1}{3}\log^3\left(x+a\right)+\gamma_2\left(a\right) & u=-1\end{cases}\end{aligned}\tag{7}$$

Now we prove the propsition1

Since the $r-$th derivative of $f\left(t\right)=\left(t+a\right)^u$ is

$$f^r\left(t\right)=\begin{pmatrix}u\\ r\end{pmatrix}\left(r-1\right)!\left(t+a\right)^{u-r}=\frac{\Gamma\left(u+1\right)}{\Gamma\left(u+1-r\right)}\left(t+a\right)^{u-r}\tag{8}$$

We derive from (6) that

$$\begin{aligned}L_u\left(x,a\right) &= \left(-1\right)^{l+1}\begin{pmatrix}u\\ l\end{pmatrix}\int_x^{+\infty}\overline{B_l}\left(t\right)\left(t+a\right)^{u-l}\mathbf{d}t-\sum_{r=1}^{l}\frac{\Gamma\left(u+1\right)}{\Gamma\left(u+2-r\right)}\frac{\left(-1\right)^r}{r!}B_r\left(x\right)a^{u-r+1}\\ &+\sum_{r=1}^{l}\frac{\Gamma\left(u+1\right)}{\Gamma\left(u+2-r\right)}\frac{\left(-1\right)^r}{r!}\overline{B_l}\left(x\right)\left(x+a\right)^{u-r+1}+\frac{\left(-1\right)^l}{l!}\frac{\Gamma\left(u+1\right)}{\Gamma\left(u+1-l\right)}\int_x^{+\infty}\overline{B_l}\left(t\right)\left(t+a\right)^{u-l}\mathbf{d}t\\ &+a^u+\begin{cases}\dfrac{1}{u+1}\left(x+a\right)^{u+1}-\dfrac{1}{u+1}a^{u+1}, & u\neq-1\\ \log\left(x+a\right)-\log a, & u=-1\end{cases}\end{aligned}\tag{9}$$

For any natural number $l > \Re(u)+1$.

Now we note that the last integral is in (9) clearly $\mathrm{O}\left(x^{\Re(u)-l}\right)$ by the mean value theorem for partial integration. Thus, taking the limit as $x\to+\infty$ the case when $\Re\left(u\right) < -l$, we conclude that the constant term on the right-hand side of (9) must coincide with the left-hand side.

$$\lim_{x\to\infty} L_u(x,a) = \zeta(-u,a)$$

That is, (4) follows.

Then, by analytic continuation; (4) can be show n to hold true for all $u \in C\setminus\{-1\}$.

At the same time, this gives a generic definition of $\zeta(-u,a)$;

$$\zeta(-u,a) = \lim_{N\to\infty}\left(\sum_{a=0}^{N}(n+a)^u - \frac{1}{u+1}(N+a)^{u+1} - \sum_{r=1}^{l}\frac{\Gamma(u+1)}{\Gamma(u+1-r)}\frac{(-1)^r}{r!}(N+a)^{u-r+1}\right) \quad (10)$$

Where $l \in \mathbb{N}$ satisfies $l > \Re(u)+1$.

On the other hand, for $u=-1$. Formula (9) implies that the constant term is

$$\log a + \frac{1}{2a} - \sum_{r=2}^{l}\frac{B_r}{r!}a^{-r} + \int_0^\infty \overline{B_l}(t)(t+a)^{-l-i}\mathbf{d}t$$

Which must be equal to

$$\lim_{N\to\infty}\left(\sum_{n=0}^{N}(n+a)^{-1} - \log(N+a)\right),$$

Which we denote by $\gamma(a) = \gamma_0(a)$.

Upon replacing the constant term by $\zeta(-u,a)$ in (9) gives (2), which then entails (3) on replacing the integral by the above estimate $\mathrm{O}\left(x^{\Re(u)-l}\right)$. This completes the proof of Propsition1.

Now we give the proof Corollary 1.

From lemma2 (7)

When $u \neq -1$. Let

$$M_1(u) = \sum_{r=1}^{l}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\frac{(-1)^r}{r!}\overline{B_r}(x)(x+a)^{u-r+1}$$
$$\left\{[\psi(u+1)-\psi(u+2-l)+\log(x+a)]^2 + \psi'(u+1) - \psi'(u+2-l)\right\}$$

$$M_2(u) = \frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)}$$
$$\int_x^{+\infty}\overline{B_l}(t)(t+a)^{u-l}\left\{[\psi(u+1)-\psi(u+2-l)+\log(x+a)]^2 + \psi'(u+1) - \psi'(u+1-l)\right\}\mathbf{d}t$$

$$M_3(u) = \frac{(x+a)^{u+1}}{u+1}\{\log(x+a)\}^2 - \frac{2(x+a)^{u+1}}{(u+1)^2}\{\log(x+a)\}^2 + \frac{2(x+a)^{u+1}}{(u+1)^3} + \zeta''(-u,a)$$

So $\frac{\mathbf{d}^2}{\mathbf{d}u^2}Lu(x,a) = M_1(u) + M_2(u) + M_3(u)$.

We give derivation

$$\frac{\mathbf{d}^3}{\mathbf{d}u^3}Lu(x,a) = M_1'(u) + M_2'(u) + M_3'(u)$$

When $u \neq -1$.

$$\frac{\mathbf{d}}{\mathbf{d}u}\left(\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\right)=\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\{\psi(u+1)-\psi(u+2-r)\}$$

$$\frac{\mathbf{d}}{\mathbf{d}u}\left(\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\right)=\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\{\psi(u+1)-\psi(u+1-l)\}$$

Let $f(u,r)=\psi(u+1)-\psi(u+2-r)$,$f(u,l)=\psi(u+1)-\psi(u+1-l)$

$$M_1'(u)=\sum_{r=1}^{l}\frac{(-1)^r}{r!}\overline{B_r}(x)(x+a)^{u-r+1}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}$$

$$\left\{(f(u,r)+\log(x+a))^3+2f'(u,r)\log a+3f(u,r)f'(u,r)+f''(u,r)\right\}$$

$$=\sum_{r=1}^{l}\frac{(-1)^r}{r!}\overline{B_r}(x)(x+a)^{u-r+1}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\left\{[f(u,r)+\log(x+a)]^3+2f'(u,r)\log a\right.$$

$$\left.+3f(u,r)f'(u,r)+f''(u,r)\right\}.$$

$$M'_2(u)=\frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)}$$

$$\int_x^{+\infty}\overline{B_l}(t)(t+a)^{u-l}\left\{(f(u,l)+\log(x+a))^3+3f(u,l)f'(u,l)+f'(u,l)\log(x+a)+f''(u,l)\right\}\mathbf{d}t$$

$$M'_3(u)=\frac{(x+a)^{u+1}}{u+1}\log^3(x+a)-\frac{3(x+a)^{u+1}}{(u+1)^2}\log^2(x+a)+\frac{6(x+a)^{u+1}}{(u+1)^3}\log(x+a)$$

$$-\frac{6(x+a)^{u+1}}{(u+1)^4}-\zeta'''(-u,a)$$

We now complete the proof Corollary 1.

§2. Application of Proposition 1

In [3 p.13-24], many professors gave special values of $\psi(z)$ and the polygammma $\psi^{(n)}(z)$, the formula (66) of [3 p.24] gave $\sum_{k=1}^{n}\psi'(z)$, in this paper, by proposition 1, we give $\sum_{k\le x}\psi^{(v)})(k)$.

Theorem1. For $n\in\mathrm{N}$, v, $x\in R$, we have

$$\sum_{k\leqslant x}\psi^{(v)}(k)=(-1)^v v!\begin{cases}\dfrac{1}{v(v-1)}x^{1-v}-\zeta(v)+\dfrac{1}{v}\left(\overline{B_1}(x)+\dfrac{1}{2}\right)x^{-v}, & v\neq 0,1\\ x\log x-x-\zeta(0)-\left(\overline{B_1}(x)+\dfrac{1}{2}\right)\log x, & v=0\\ -\log x-\gamma-1+\left(\overline{B_1}(x)+\dfrac{1}{2}\right)x^{-1}, & v=-1\end{cases}$$

$$+(-1)^{v+r-2}(v+r-2)!\sum_{r=2}^{l}\frac{(-1)^r}{r(r-1)}\binom{-v-1}{r-2}B_r^{(2)}(\{x\}+1)x^{-v-r+1}+\mathrm{O}(x^{\mathrm{Re}(-v)-l}). \quad (11)$$

Where we recall from notion, the definition of $B_r^{(2)}(x)$:

$$B_r^{(2)}(x+1)=\sum_{k=0}^{r}\binom{r}{k}B_{r-k}(1)\overline{B}_k$$

For prove theorem 1, we give lemma3.

Lamma 3[4]**.** we have

$$\left(\overline{B}_1(x)+\frac{1}{2}\right)\overline{B}_r(x)=\frac{1}{r+1}\sum_{k=0}^{r}(-1)^{r-k+1}\binom{r+1}{k}B_{r-k+1}\overline{B}_k(x)+\overline{B}_{r+1}(x) \tag{12}$$

Now we prove theorem1.

In formula (54) of [3 p.22]

$$\psi^{(\upsilon)}(z+m)-\psi^{(\upsilon)}(z)=(-1)^{\upsilon}\upsilon!\sum_{k=1}^{m}\frac{1}{(z+k-1)^{\upsilon+1}} \tag{13}$$

whence that

$$\sum_{k\leqslant x}\psi^{(\upsilon)}(k)=(-1)^{\upsilon}\upsilon!\sum_{n=1}^{k-1}\frac{1}{n^{\upsilon+1}}+\psi(1)\,. \tag{14}$$

summing (14) over $k\leqslant x$, we obtain

$$\begin{aligned}\sum_{k\leqslant x}\psi^{(\upsilon)}(k)&=(-1)^{\upsilon}\upsilon!\sum_{k\leqslant x}\left(\sum_{n=1}^{k}\frac{1}{n^{\upsilon+1}}-\frac{1}{k^{\upsilon+1}}\right)+\psi^{\upsilon}(1)[x]\\&=(-1)^{\upsilon}\upsilon!\left(\sum_{\upsilon\leqslant x}\frac{1}{n^{\upsilon+1}}\sum_{n\leqslant k\leqslant x}1-L_{-\upsilon-1}(x)\right)+\psi^{\upsilon}(1)[x]\end{aligned}$$

After changing the order of summation. Since . $\sum\limits_{n\leqslant k\leqslant x}1=[x]-n+1$.,we see that the firist term reduces to $(-1)^{\upsilon}\upsilon!\left([x]L_{-\upsilon-1}(x)-L_{-\upsilon}(x)\right)$, Taking into account that [3 p.22]

$$\psi^{\upsilon}(1)=\begin{cases}(-1)^{\upsilon+1}\upsilon!\zeta(\upsilon+1)\,,\ \upsilon\neq 0\\ -\gamma,\ \upsilon=0\end{cases}$$

We can get

$$\sum_{k\leqslant x}\psi^{(\upsilon)}(k)=(-1)^{\upsilon}\upsilon!\left([x]L_{-\upsilon-1}(x)-L_{-\upsilon}(x)-\begin{cases}\zeta(\upsilon+1)[x],\ \upsilon\neq 0\\ \gamma[x],\ \upsilon=0\end{cases}\right) \tag{15}$$

Use $[x]=x-\left(\overline{B}_1(x)+\frac{1}{2}\right)$, we apply (8) to obtain

$$\begin{aligned}\sum_{k\leqslant x}\psi^{(\upsilon)}(k)&=(-1)^{\upsilon}\upsilon!\sum_{r=1}^{l}\frac{(-1)^r}{r}\binom{-\upsilon-1}{r-1}\overline{B}_r(x)\,x^{-\upsilon-r+1}\\&+(-1)^{\upsilon+1}\upsilon!\sum_{r=1}^{l}\frac{(-1)^r}{r}\binom{-\upsilon}{r-1}\overline{B}_r(x)\,x^{-\upsilon-r+1}\\&(-1)^{\upsilon+1}\upsilon!\sum_{r=1}^{l}\frac{(-1)^r}{r}\binom{-\upsilon-1}{r-1}\left(\overline{B}_1(x)+\frac{1}{2}\right)\overline{B}_r(x)\,x^{-\upsilon-r}\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{\upsilon}\upsilon!\begin{cases}\frac{1}{-\upsilon}x^{1-\upsilon}+\zeta\left(1+\upsilon\right)x+\frac{1}{\upsilon-1}x^{-\upsilon+1},\ \upsilon\neq 0,1\\ x\left(\log x+\gamma\right)-x-\zeta(0),\ \upsilon=0\\ x\left(-x^{-1}+\zeta(2)-\log x-\gamma\right),\ \upsilon=1\end{cases}\\
&+(-1)^{\upsilon+1}\upsilon!\begin{cases}\left(\overline{B}_1(x)+\frac{1}{2}\right)\left(-\frac{1}{\upsilon}+\zeta(1+\upsilon)\right),\ \upsilon\neq 0\\ \left(\overline{B}_1(x)+\frac{1}{2}\right)\left(\log x+\gamma\right),\ \upsilon=0\end{cases}\\
&-(-1)^{\upsilon}\upsilon!\begin{cases}\zeta(1+\upsilon)x+\left(\overline{B}_1(x)+\frac{1}{2}\right),\ \upsilon\neq 0\\ \gamma x+\left(\overline{B}_1(x)+\frac{1}{2}\right)\gamma,\ \upsilon=0\end{cases}\\
&+O\left(x^{\mathrm{Re}(-\upsilon)-l}\right)
\end{aligned} \tag{16}
$$

Where the first and the second terms combine to yield

$$
-(-1)^{\upsilon}\upsilon!\sum_{r=2}^{l}\frac{(-1)^r}{r}\binom{-\upsilon-1}{r-2}\overline{B}_r\left(x\right)x^{-\upsilon-r+1},
$$

While the third term may be written as

$$
(-1)^{\upsilon}\upsilon!\sum_{r=2}^{l}\frac{(-1)^r}{r}\binom{-\upsilon-1}{r-2}\left(\overline{B}_1\left(x\right)+\frac{1}{2}\right)\overline{B}_{r-1}\left(x\right)x^{-\upsilon-r+1}
$$

Hence we transform (16) into

$$
\begin{aligned}
\sum_{k\leqslant x}\psi^{(\upsilon)}\left(k\right)=&(-1)^{\upsilon+1}\upsilon!\begin{cases}\frac{1}{\upsilon(\upsilon-1)}x^{1-\upsilon}+\zeta\left(\upsilon\right)x-\frac{1}{\upsilon}\left(\overline{B}_1(x)+\frac{1}{2}\right)x^{-\upsilon},\ \upsilon\neq 0,1\\ -x\log x+\gamma+\zeta(0)+\left(\overline{B}_1(x)+\frac{1}{2}\right)\log x,\ \upsilon=0\\ +\log x+\gamma+1-\left(\overline{B}_1(x)+\frac{1}{2}\right)x^{-1},\ \upsilon=1\end{cases}\\
&+(-1)^{\upsilon+1}\upsilon!\sum_{r=1}^{l}\frac{(-1)^r}{r}\binom{-\upsilon-1}{r-2}\left(\frac{1}{r}\overline{B}_r\left(x\right)-\frac{1}{r-1}\left(\overline{B}_r\left(x\right)+\frac{1}{2}\right)\overline{B}_{r-1}\left(x\right)\right)x^{-\upsilon-r+1}\\
&+O\left(x^{\mathrm{Re}(-\upsilon)-l}\right)\\
&(-1)^{\upsilon+1}\upsilon!\sum_{r=1}^{l}\frac{(-1)^r}{r}\binom{-\upsilon-1}{r-1}\left(\overline{B}_1\left(x\right)+\frac{1}{2}\right)\overline{B}_r\left(x\right)x^{-\upsilon-r}\\
&+(-1)^{\upsilon}\upsilon!\begin{cases}\frac{1}{-\upsilon}x^{1-\upsilon}+\zeta\left(1+\upsilon\right)x+\frac{1}{\upsilon-1}x^{-\upsilon+1},\ \upsilon\neq 0,1\\ x\left(\log x+\gamma\right)-x-\zeta(0),\ \upsilon=0\\ x\left(-x^{-1}+\zeta(2)-\log x-\gamma\right),\ \upsilon=1\end{cases}\\
&+(-1)^{\upsilon+1}\upsilon!\begin{cases}\left(\overline{B}_1(x)+\frac{1}{2}\right)\left(-\frac{1}{\upsilon}+\zeta(1+\upsilon)\right),\ \upsilon\neq 0\\ \left(\overline{B}_1(x)+\frac{1}{2}\right)\left(\log x+\gamma\right),\ \upsilon=0\end{cases}
\end{aligned}
$$

$$- (-1)^{\upsilon}\upsilon! \begin{cases} \zeta(1+\upsilon)x + \left(\overline{B}_1(x) + \frac{1}{2}\right), \ \upsilon \neq 0 \\ \gamma x + \left(\overline{B}_1(x) + \frac{1}{2}\right)\gamma, \ \upsilon = 0 \end{cases}$$
$$+ O\left(x^{\mathrm{Re}(-\upsilon)-l}\right) \tag{17}$$

Applying Lemma 3, we may transform the penultimate term further. Since the third factor of $x^{-\upsilon-r+1}$ is

$$-\frac{1}{r\,(r+1)}\left(\overline{B}_r(x) + \sum_{k=0}^{r-1}(-1)^{r-k}\binom{r}{k}B_{r-k}\overline{B}_k(x)\right) = -\frac{1}{r\,(r+1)}\sum_{k=0}^{r}\binom{r}{k}B_{r-k}(1)\overline{B}_k(x),$$

By (1) $B_r^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{r}\binom{r}{k}B_k^{\alpha}(x)B_{r-k}^{\beta}(y)$, equal to $-\frac{1}{r(r+1)}B_r^2(\{x\}+1)$,

we conclude that the penultimate term in (17) is

$$(-1)^{\upsilon}\upsilon!\sum_{r=2}^{l-1}\frac{(-1)^r}{r}\binom{-\upsilon-1}{r-2}B_r^2(\{x\}+1)\,x^{-\upsilon-r+1}$$

Substiting this in (3 #) completes the proof.

§3. Application of Corollary 1

In [2].[6].[7]and [8] gave many fomulas related hurwitz zeta function and $\zeta'(-m,a)$,$\zeta''(-m,a)$, we use this formulas give $\zeta'''(-m,a)$.

Theorem 2. For $m \in \mathbb{N}\cup\{0\}$, $\mathrm{Re}\,a > 0$ and $m+2 \leqslant l \in \mathbb{N}$. We have

$$\begin{aligned}
&\zeta'''(-m,a)\\
&= -a^3\log^3 a + \frac{a^{m+1}}{m+1}\log^3 a - \frac{3a^{m+1}}{(m+1)^2}\log^2 a - \frac{6a^{m+1}}{(m+1)^3}\log a - \frac{6a^{m+1}}{(m+1)^4}\\
&+ \sum_{r=1}^{l}\frac{(-1)^r}{r!}B_r\frac{m!}{(m+1-r)!}a^{m-r+1}\left(\left(\sum_{k=1}^{r-1}\frac{1}{m+k-l+1} + \log a\right)^3 - 2\sum_{k=1}^{r-1}\frac{\log a}{(m+k-r+1)^2}\right.\\
&\left.-3\sum_{k_1=1}^{r-1}\sum_{k_2=1}^{r-1}\frac{1}{(m+k_2-r+1)(m+k_1-r+1)^2} + \sum_{k=1}^{r-1}\frac{1}{(m+k-r+1)^3}\right)\\
&+ \frac{(-1)^l}{l!}\frac{m!}{(m-l)!}a^{m-1}\int_0^{+\infty}\overline{B_l}(t)\left(\left(\sum_{k=1}^{l}\frac{1}{m+k-l} + \log a\right)^3 - 3\sum_{k_1=1}^{l}\sum_{k_2=1}^{l}\frac{1}{(m+k_2-r+1)(m+k_1-r+1)^2}\right.\\
&\left.-\sum_{k=1}^{l}\frac{1}{(m+k-l)^2}\log a + \sum_{k=1}^{l}\frac{1}{(m+k-l)^3}\right)\mathbf{d}t.
\end{aligned}$$

In for proof of theorem 2 we give lemma 4.

Lemma4. For $m \in \mathbb{N} \cup \{0\}$, $\operatorname{Re} a > 0$ and $m+2 \leq l \in \mathbb{N}$. We have

$$\begin{aligned}\zeta'(-m,a) &= \frac{1}{m+1}a^{m+1}\log a - \frac{1}{(m+1)^2}a^{m+1} - \frac{1}{2}a^m \log a + \frac{1}{12}a^{m-1}\log a \\ &+ \sum_{r=4}^{r+1} \frac{B_r}{r}\left(\sum_{j=0}^{r-2}(-1)^j \binom{m}{j}\frac{1}{r-1-j} + \binom{m}{r-1}\log a\right)a^{m-r+1} \\ &+ \frac{1}{m+1}\sum_{r=m+2}^{l} B_r \left(\sum_{j=0}^{r-1}(-1)^j \binom{r-m-2}{j}\frac{1}{r-j}\right)a^{m-r+1} \\ &+ (-1)^{l+1}\int_0^{+\infty}\left(\sum_{j=0}^{l-1}(-1)^j \binom{l-m-1}{j}\frac{1}{l-j}B_l(t)(t+a)^{m-l}\right)\mathbf{d}t \qquad (18)\end{aligned}$$

Now we give proof of theorem 2.

From [3]

$$\psi(z+1) - \psi(z) = \frac{1}{z} \quad (z \neq 0) \qquad (19)$$

We any easily deduce that

$$\psi^{(2\nu)}(z+m) - \psi^{(\nu)}(z) = (-1)^{(\nu)}\nu! \sum_{k=1}^{m} \frac{1}{(z+k-1)^{\nu+1}} \qquad (20)$$

So

$$\begin{aligned}f(u,r) &= \psi(u+1) - \psi(u+2-r) = \sum_{k=1}^{r-1}\frac{1}{u+k-r+1} \\ f(u,l) &= \psi(u+1) - \psi(u+1-l) = \sum_{k=1}^{r-1}\frac{1}{u+k-l} \\ f'(u,r) &= -\sum_{k=1}^{r-1}\frac{1}{(u+k-r+1)^2} \\ f'(u,l) &= -\sum_{k=1}^{r-1}\frac{1}{(u+k-l)^2} \\ f''(u,r) &= 2\sum_{k=1}^{r-1}\frac{1}{(u+k-r+1)^2}, f''(u,l) = 2\sum_{k=1}^{r-1}\frac{1}{(u+k-l)^2}\end{aligned}$$

Let $x = 0$, $\overline{B_r}(x) = B_r$, $u \neq -1$

$$\sum_{0\le n\le x} (n+a)^u \log (x+a) = a^u \log^3 a$$

$$= \sum_{r=}^{l} \frac{(-1)^r}{r!} B_r \frac{\Gamma(u+1)}{\Gamma(u+2-r)} a^{u-r+1} \left\{ \left(\sum_{k=1}^{r-1} \frac{1}{u+k-r+1} + \log a \right)^3 - 2\sum_{k=1}^{r-1} \frac{\log a}{(u+k-r+1)^2} \right.$$

$$\left. -3 \sum_{k_1=1}^{r-1} \sum_{k_2=1}^{r-1} \frac{1}{(u+k_2-r+1)(u+k_1-r+1)^2} + \sum_{k=1}^{r-1} \frac{1}{(u+k-r+1)^3} \right\}$$

$$+ \frac{(-1)^l}{l!} \frac{\Gamma(u+1)}{\Gamma(u+1-l)} a^{u-1} \int_0^{+\infty} \overline{B}_l(t) \left\{ \left(\sum_{k=1}^{l} \frac{1}{u+k-l} + \log a \right)^3 - 3\sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \frac{1}{(u+k_2-l)(u+k_1-l)^2} \right.$$

$$\left. - \sum_{k=1}^{l} \frac{\log a}{(u+k-l)^2} + \sum_{k=1}^{l} \frac{1}{(u+k-l)^3} \right\} \mathbf{d}t$$

$$+ \frac{a^{u+1}}{u+1} \log^3 a - \frac{3a^{u+1}}{(u+1)^2} \log^2 a + \frac{6a^{u+1}}{(u+1)^3} \log a - \frac{6a^{u+1}}{(u+1)^4} - \zeta'''(-u,a)$$

When $u \to m$

$$\frac{\Gamma(u+1)}{\Gamma(u+2-r)} = \frac{m!}{(m+1-r)!}, \quad \frac{\Gamma(u+1)}{\Gamma(u+1-l)} = \frac{m!}{(m-l)!}$$

Then

$$\zeta'''(-m,a)$$

$$= -a^3 \log^3 a + \frac{a^{m+1}}{m+1} \log^3 a - \frac{3a^{m+1}}{(m+1)^2} \log^2 a - \frac{6a^{m+1}}{(m+1)^3} \log a - \frac{6a^{m+1}}{(m+1)^4}$$

$$+ \sum_{r=1}^{l} \frac{(-1)^r}{r!} B_r \frac{m!}{(m+1-r)!} a^{m-r+1} \left(\left(\sum_{k=1}^{r-1} \frac{1}{m+k-l+1} + \log a \right)^3 - 2\sum_{k=1}^{r-1} \frac{\log a}{(m+k-r+1)^2} \right.$$

$$\left. -3 \sum_{k_1=1}^{r-1} \sum_{k_2=1}^{r-1} \frac{1}{(m+k_2-r+1)(m+k_1-r+1)^2} + \sum_{k=1}^{r-1} \frac{1}{(m+k-r+1)^3} \right)$$

$$+ \frac{(-1)^l}{l!} \frac{m!}{(m-l)!} a^{m-1} \int_0^{+\infty} \overline{B_l}(t) \left(\left(\sum_{k=1}^{l} \frac{1}{m+k-l} + \log a \right)^3 - 3\sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \frac{1}{(m+k_2-r+1)(m+k_1-r+1)^2} \right.$$

$$\left. - \sum_{k=1}^{l} \frac{1}{(m+k-l)^2} \log a + \sum_{k=1}^{l} \frac{1}{(m+k-l)^3} \right) \mathbf{d}t.$$

We now complete the proof of theorem 2.

In [3, p.94-96] some proofers gave some formulas of double Gamma function, in this term we give noe formula theorem 3.

Theorem3. With $\Gamma_2(a)(= G(a)^{-1})$ denoting the double gamma (or the $G-$) function of Barnes, we have for $a, b > 0$

$$\int_0^{\infty} \overline{B}_1(t) \log \frac{t+a}{t+b} \mathbf{d}t = -\log \frac{\Gamma_2(a)}{\Gamma_2(b)} + (1-a)\log\Gamma(a) + (b-1)\log\Gamma(b)$$

$$+ \frac{1}{2}\left(a^2 \log a - b^2 \log b\right) - \frac{1}{4}\left(a^2 - b^2\right) - \frac{1}{2}\left(a \log a - b \log b\right)$$

Proof of theorem: in [9 (1.3)] for $\mathrm{Re}\,u < 0,\ u \neq -1$, we have

$$\zeta'\left(-u,a\right)=\frac{1}{u+1}a^{u+1}\log a-\frac{1}{\left(u+1\right)^{2}}a^{m+1}-\frac{1}{2}a^{u}\log a$$
$$-\int_0^{\infty}\left(1+u\log(t+a)\right)\overline{B_l}\left(t\right)(t+a)^{u-1}\mathbf{d}t \tag{21}$$

By Abels continuity theorem for infinite integrals, take the limit as $u \to 1$ of the difference $\zeta'(-u,a)-\zeta'(-u,b)$, we can get

$$\zeta'(-1,a)-\zeta'(-1,b)=\frac{1}{2}\left(a^2\log a-b^2\log b\right)-\frac{1}{4}\left(a^2-b^2\right)$$
$$-\frac{1}{2}\left(a\log a-b\log b\right)-\int_0^{\infty}\overline{B}_1(t)\log\frac{t+a}{t+b}\mathbf{d}t \tag{22}$$

formula (33) 0f [10 p.94]

$$\Gamma_2(a)=A\{\Gamma(a)\}^{1-a}\exp\left[-\frac{1}{12}+\zeta'(-1,a)\right]\ (a>0)$$

A is indicates the Glaisher-Kinkelin constant [3 p94] formula [38]

Form (3.2)we get

$$\zeta'(-1,a)-\frac{1}{12}=\Gamma_2(a)\log A+(a-1)\log\Gamma(a)$$

So

$$\zeta'(-1,a)-\zeta'(-1,b)=\log\frac{\Gamma_2(a)}{\Gamma_2(b)}+(a-1)\log\Gamma(a)-(b-1)\log\Gamma(b) \tag{23}$$

From (22) and (23), we can get Theorem 3.

References

[1] Pan Chengdong and Peng Chengbiao, Analytic theory of number 1999.

[2] Shigeru Kanemitsu and Haruo Tsukada, Vistas of Special Functions, World Scientific Publishing Co.Pte.Lid.2007

[3] H.M.Srivastava and J.Choi, Seies Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston, and London 2001.

[4] Ishibashi, M. and Kanemitsu, S., Fractional part sums and divisor functions III. Kyoyo Ronshu (Kinki Univ. Intern. Rep.) 4 (1990), 1 C 40.

[5] Yang Qian-Li, Wang Yong-Xing and Lihai-Long,On Derivate of a Number Theory and Its Application Mathem Atics in Practice and Theory. Vol.35,No.8 (2005)194-199.

[6] Hailong Li,Number Theory and Special Functions, Science Press Beijing 2011.

[7] Ishibashi,M. and Kanemitsu, S., Fractional Part Sums and Divisor Function . In: Number Theory and Combinatorics . By J.A Kiyama et al, World Sci. Pub 1985,pp.119-183.

[8] Kanemitsu S, Kumagai H. and Srivastava HM. Some Integral and Asymptotic Formulas Associate with the Hurwitz Zate Function [J] Appl. Mathcomput,2004(1)31-37.

[9] Kanemitsu, S., Tanigawa, Y., Tsukada, H. and Yoshimoto, M., Contributions to the theory of the Hurwitz zeta-function. [J] Hardy-Ramanujan Journal Vol.30(2007)31-35.

[10] Kalyan Chakraborty , Shigeru Kanemitsu and Li Hai Long Quadratic reciprocity and Riemannsnon-differentiable function. [J] ResearchinNumberTheory (2015)

Scientia Magna
Vol. 13 (2018), No. 1, 32-41

Proper Bases for a class of Dirichlet series analytic in half plane

Niraj Kumar[1] and Lakshika Chutani[2]

[1]Department of Mathematics, Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110078, India
E-mail: nirajkumar2001@hotmail.com
[2]Department of Mathematics, Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110078, India
E-mail: lakshika91.chutani@gmail.com

Abstract The present paper deals with the space χ of Dirichlet series in two variables s_1, s_2 analytic in the half plane. It is shown that χ becomes a Frechet space. We have characterized the form of continuous linear functionals and continuous linear operator. Further conditions have been proved in which a base becomes a proper base for χ.

Keywords Dirichlet series, Banach algebra.
2010 Mathematics Subject Classification 30B50, 30D10.

§1. Introduction and preliminaries

Let

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n}\, e^{(\lambda_m s_1 + \mu_n s_2)} \;, \qquad (s_j = \sigma_j + it_j,\ j = 1, 2) \tag{1}$$

be a Dirichlet series of two complex variables s_1 and s_2 where $a_{m,n}{}'s \in \mathbb{C}$ and sequence of exponents $\lambda_m{}'s,\ \mu_n{}'s \in \mathbb{R}$ satisfy

$$\begin{aligned} & 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_m \to \infty \text{ as } m \to \infty \\ \text{and} \quad & 0 < \mu_1 < \mu_2 < \ldots < \mu_n \to \infty \text{ as } n \to \infty. \end{aligned}$$

along with

$$\limsup_{m,n\to\infty} \frac{\log(m+n)}{\lambda_m + \mu_n} = 0 \tag{2}$$

and

$$\limsup_{m,n\to\infty} \frac{\log^+ |a_{m,n}| + \lambda_m A_1 + \mu_n A_2}{\lambda_m + \mu_n} = 0 \tag{3}$$

where $A_1 > 0$, $A_2 > 0$ denote maximal abscissa of convergence. Here we take $\log^+ x = 0$ if $x \leq 1$ and $\log^+ x = \log x$ if $x > 1$. The series (1) converges in the domain

$$d = \{(\sigma_1 + it_1\, , \sigma_2 + it_2) \in \mathbb{C}^2 : \sigma_1 < A_1, \sigma_2 < A_2, -\infty < t_1, t_2 < \infty\}.$$

If D_1 and D_2 are two positive numbers such that

$$\frac{\log|a_{m,n}|}{\lambda_m + \mu_n} \to -D_1 \text{ as } m \to \infty. \tag{4}$$

and

$$\frac{\log|a_{m,n}|}{\lambda_m + \mu_n} \to -D_2 \text{ as } n \to \infty. \tag{5}$$

Then take $D = \min(D_1, D_2)$

$$\limsup_{m,n\to\infty} \frac{\log|a_{m,n}|}{\lambda_m + \mu_n} = -D \tag{6}$$

Suppose the series given by (1) converges absolutely in the left half plane $\sigma_1, \sigma_2 < D$ then the series is called Dirichlet series analytic in half plane. Observe that χ_D includes all functions f represented by (1) satisfying (2), (3), and (6). In the given paper we will write χ instead of χ_D.

If

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n}\, e^{(\lambda_m s_1 + \mu_n s_2)}$$

$$\text{and} \quad g(s_1, s_2) = \sum_{m,n=1}^{\infty} b_{m,n}\, e^{(\lambda_m s_1 + \mu_n s_2)}$$

then the binary operations in χ are defined as follows

$$f(s_1, s_2) + g(s_1, s_2) = \sum_{m,n=1}^{\infty} (a_{m,n} + b_{m,n})\, e^{(\lambda_m s_1 + \mu_n s_2)}$$

$$\xi.f(s_1, s_2) = \sum_{m,n=1}^{\infty} (\xi.a_{m,n})\, e^{(\lambda_m s_1 + \mu_n s_2)}$$

It is also clear that χ defines a linear space over $\mathbb{C}$.

The norm in χ is defined as

$$\|f\| = \sum_{m,n=1}^{\infty} |a_{m,n}|\, e^{(\sigma_1 \lambda_m + \sigma_2 \mu_n)} \text{ for every } \sigma_1, \sigma_2 < D. \tag{7}$$

Let χ be the topology generated by the family of norms $\{\|f\| \; ; \; \sigma_1, \sigma_2 < D\}$. Then from [1] χ is complete, metrizable and locally convex and this topology is equivalent to the topology generated by invariant metric e, where

$$e(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f - g; \sigma_{1,k}\,,\, \sigma_{2,k}\|}{1 + \|f - g; \sigma_{1,k}\,,\, \sigma_{2,k}\|} \tag{8}$$

where

$$0 < \sigma_{j,1} < \sigma_{j,2} < \ldots < \sigma_{j,k} \to D_j \text{ as } k \to \infty, \; j = 1, 2.$$

It can be easily verified that the topology induced by e on χ is the same as induced by the sequence $\{\|\ldots, \sigma_{j,k}\|, j = 1, 2\,;\, k = 1, 2 \ldots\}$.

Daoud in his papers [1] worked on entire Dirichlet series of two variables having finite order point. Kamthan in [3] considered different classes of entire functions represented by Dirichlet series in several variables and proved results on continuous linear functionals.

In [2] bornological properties for the space of entire Dirichlet series in several complex variables have been discussed. So far many researchers have worked on Dirichlet series which can be seen in [4] - [5]. The aim of the present work is to project various new aspects for the Dirichlet series in two variables.

§2. Definitions

Following definitions are required to prove the main results. For the definitions of terms used refer [6] - [7].

A sequence $\{d_{m,n}\} \subset X$ will be linearly independent if $\sum_{m,n=1}^{\infty} a_{\{m,n\}} d_{\{m,n\}} = 0$ implies $a_{\{m,n\}} = 0$ for all $m, n \geq 1$, that is for all sequences $\{a_{m,n}\}$ of complex numbers for which $\sum_{m,n=1}^{\infty} a_{\{m,n\}} d_{\{m,n\}}$ converges in χ.

The sequence $\{d_{m,n}\} \subset \chi$ spans a subspace χ_0 of χ, if χ_0 consists of all linear combinations $\sum_{m,n=1}^{\infty} a_{\{m,n\}} d_{\{m,n\}}$, such that $\sum_{m,n=1}^{\infty} a_{\{m,n\}} d_{\{m,n\}}$ converges in χ.
A sequence $\{d_{m,n}\} \subset \chi$ which is linearly independent and spans a closed subspace χ_0 of χ, will be a base in χ_0. If $\{e_{m,n}\} \subset \chi$ such that $e_{m,n}(s) = e^{(\sigma_1 \lambda_m + \sigma_2 \mu_n)}$ then clearly $\{e_{m,n}\}$ is a base in χ.

A sequence $\{d_{m,n}\} \subset \chi$ will be a proper base if it is a base and it satisfies the condition that for all sequences $\{a_{m,n}\}$ of complex numbers $\sum_{m,n=1}^{\infty} a_{m,n} d_{m,n}$ converges in χ if and only if $\sum_{m,n=1}^{\infty} a_{m,n} e_{m,n}$ converges in χ.

§3. Main Results

Theorem 2.1. *With respect to the usual addition and multiplication defined χ becomes a FK-space.*

Proof. In order to prove this theorem we need to show that χ is complete under the norm defined in (7). Let $\{f_p\}$ be any cauchy sequence in χ where

$$f_p(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n}^{(p)} \, e^{(\lambda_m s_1 + \mu_n s_2)}$$

Then for a given $\epsilon > 0$ we can find a constant M such that

$$\|f_p - f_q\| < \epsilon \quad \forall \quad p, q \geq M$$

that is

$$\sum_{m,n=1}^{\infty} |a_{m,n}^{(p)} - a_{m,n}^{(q)}|\, e^{(\sigma_1\lambda_m + \sigma_2\mu_n)} < \epsilon \ \ \forall \ \ p, q \geq M.$$

This shows that $\{a_{m,n}^{(p)}\}$ forms a cauchy sequence for all values of $m, n \geq 1$. Hence

$$\lim_{p\to\infty} a_{m,n}^{(p)} = a_{m,n} \ \forall \ m, n \geq 1.$$

Letting $q \to \infty$,

$$\sum_{m,n=1}^{\infty} |a_{m,n}^{(p)} - a_{m,n}|\, e^{(\sigma_1\lambda_m + \sigma_2\mu_n)} < \epsilon \ \forall \ p \geq M.$$

Thus $f_p \to f$ as $p \to \infty$.

Now we need to show that $f = \sum_{m,n=1}^{\infty} a_{m,n}\, e_{m,n} \ \in \ \chi$.

We can choose σ_i such that $D \ < \ \sigma_i \ + \ \epsilon \,, \ i \ = \ 1, 2$. Keeping p fixed we have

$$|a_{m,n}^{(p)}| \ < \ e^{(-D+\epsilon)(\lambda_m+\mu_n)} \ \forall \ p \ \geq \ M_1.$$

Then

$$|a_{m,n}| \leq |a_{m,n}^{(p)} - a_{m,n}| \ + \ |a_{m,n}^{(p)}|$$

$$\Rightarrow |a_{m,n}| \leq \epsilon\, e^{-(\sigma_1\lambda_m + \sigma_2\mu_n)} \ + \ e^{(-D+\epsilon)(\lambda_m+\mu_n)}.$$

Hence for all $m, n \ \geq \ M_2 \ = \max(M, M_1)$ we have

$$\limsup_{m,n\to\infty} \frac{\log |a_{m,n}|}{\lambda_m + \mu_n} \leq -D.$$

Thus $f = \sum_{m,n=1}^{\infty} a_{m,n} e_{m,n} \ \in \ \chi$ and this completes the proof. □

Theorem 2.2. *(i) A continuous linear functional ψ on $\chi(\sigma_1, \sigma_2)$ is of the form $\psi(f) = \sum_{m,n=1}^{\infty} d_{m,n}\, a_{m,n}$, $f \ = \ \sum_{m,n=1}^{\infty} a_{m,n}\, e_{m,n}$ if and only if $\left\{ \frac{|d_{m,n}|}{e^{(\sigma_1\lambda_m + \sigma_2\mu_n)}} \right\}$ is bounded for all $m, n \ \geq \ 1$.*

(ii) A continuous linear functional ψ on $\chi(\sigma_1, \sigma_2)$ is of the form $\psi(f) = \sum_{m,n=1}^{\infty} d_{m,n}\, a_{m,n}$, $f = \sum_{m,n=1}^{\infty} a_{m,n} e_{m,n}$ if and only if $\left\{ \frac{|d_{m,n}|}{e^{(\sigma_1\lambda_m + \sigma_2\mu_n)}} \right\}$ is bounded for some $\sigma_1, \sigma_2 \ < \ D$.

Proof. Let ψ be continuous linear functional on $\chi_{(\sigma_1,\sigma_2)}$. Then $\psi(f) = \sum_{m,n=1}^{\infty} d_{m,n}\, a_{m,n}$, $f \ = \sum_{m,n=1}^{\infty} a_{m,n}\, e_{m,n}$ where $d_{m,n} = \psi(e_{m,n})$. Hence there exists a constant S such that

$$|\psi(f)| \ \leq \ S\|f\|_{(\sigma_1,\sigma_2)} \ \text{ for all } \ f \ \in \chi.$$

Take $f = e_{m,n} = e^{(\sigma_1\lambda_m + \sigma_2\mu_n)} \in \chi$, this implies that

$$|d_{m,n}| \leq S\, e^{(\sigma_1\lambda_m + \sigma_2\mu_n)} \text{ for } m, n \geq 1.$$

Conversely, let f be defined as before and consider $\psi(f) = \sum_{m,n=1}^{\infty} d_{m,n}\, a_{m,n}$, where $\left\{ \frac{|d_{m,n}|}{e^{(\sigma_1\lambda_m + \sigma_2\mu_n)}} \right\}$ is bounded and hence $\psi(f)$ does exist. Since

$$\begin{aligned} |\sum_{m,n=1}^{\infty} a_{m,n}\, d_{m,n}| &\leq \sum_{m,n=1}^{\infty} |a_{m,n}\, d_{m,n}| \\ &\leq S \sum_{m,n=1}^{\infty} |a_{m,n}|\, e^{(\sigma_1\lambda_m + \sigma_2\mu_n)} \\ &< +\infty. \end{aligned}$$

Thus ψ is a continuous linear functional on $\chi_{(\sigma_1,\sigma_2)}$. This proves the first part of theorem and the proof of second part follows from the part (i). □

Theorem 2.3. *A necessary and sufficient condition that there exists a continuous linear transformation* $\psi : \chi \to \chi$ *with* $\psi(e_{m,n}) = \alpha_{m,n}$, $m, n = 1, 2, \ldots$ *is that for each* $\sigma_1, \sigma_2 < D$

$$\limsup_{m,n\to\infty} \frac{\log \|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} < D.$$

Proof. Let there exist a continuous linear transformation ψ from χ into χ with

$$\psi(e_{m,n}) = \alpha_{m,n} \; , \; m, n = 1, 2, \ldots$$

Then for a given σ_1 there exists a $\sigma_1^{'}$ such that $(\sigma_1, \sigma_1^{'} < D)$ and $\sigma_2^{'}$ corresponding to σ_2 such that $(\sigma_2, \sigma_2^{'} < D)$

$$\|\psi(e_{m,n})\|_{(\sigma_1,\sigma_2)} < P\|e_{m,n}\|_{(\sigma_1',\sigma_2')} = P\, e^{(\sigma_1\lambda_m + \sigma_2\mu_n)}$$

This implies

$$\frac{\log \|\psi(e_{m,n})\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} \leq \frac{\log P}{\lambda_m + \mu_n} + \frac{(\sigma_1\lambda_m + \sigma_2\mu_n)}{\lambda_m + \mu_n}$$

$$\limsup_{m,n\to\infty} \frac{\log \|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} \leq \sigma_1 < D$$

or

$$\limsup_{m,n\to\infty} \frac{\log \|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} \leq \sigma_2 < D.$$

Conversely let the given condition hold.

Let $f = \sum_{m,n=1}^{\infty} a_{m,n}\, e_{m,n} \in \chi$. Then there exists a $\epsilon > 0$ such that

$$\frac{\log \|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} \leq D - \epsilon \text{ for all } m \geq M_1(\epsilon), n \geq N_1(\epsilon)$$

$$\|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)} \leq e^{(D-\epsilon)(\lambda_m+\mu_n)} \text{ for all } m \geq M_1(\epsilon)\,, n \geq N_1(\epsilon).$$

Choose $\delta > 0$ such that $\delta < \epsilon$, then

$$|a_{m,n}|_{(\sigma_1,\sigma_2)} \leq e^{(-D+\delta)(\lambda_m+\mu_n)} \text{ for all } m \geq M_2(\delta)\,, n \geq N_2(\delta).$$

Hence one gets

$$|a_{m,n}|\|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)} \leq e^{(\delta-\epsilon)(\lambda_m+\mu_n)} \text{ for all } m \geq max\{M_1(\epsilon), M_2(\delta)\}\ , n \geq max\{N_1(\epsilon), N_2(\delta)\}.$$

Thus $\sum\limits_{m,n=1}^{\infty} |a_{m,n}|\|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)}$ is convergent as σ_1, σ_2 is arbitarily less than D.

We find that $\sum\limits_{m,n=1}^{\infty} a_{m,n}\,\alpha_{m,n}$ is convergent in χ. Hence there exist a transformation $\psi : \chi \to \chi$ such that $\psi(f) = \sum\limits_{m,n=1}^{\infty} a_{m,n}\,\alpha_{m,n}$ for each $f \in \chi$. Then ψ is linear and $\psi(e_{m,n}) = \alpha_{m,n}$ for $m, n = 1, 2, \ldots$ Now for given $\sigma_1, \sigma_2 < D$ there exists $\delta > 0$ such that

$$\frac{\log \|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} \leq D - \delta \text{ for all } m \geq M\,,\ n \geq N.$$

This implies

$$\|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)} \leq e^{(D-\delta)(\lambda_m+\mu_n)} \text{ for all } m \geq M\,,\ n \geq N$$

Further we have

$$\|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)} \leq e^{(D-\delta)(\lambda_m+\mu_n)} \text{ for all } m, n \geq 1.$$

Hence

$$\|\psi(f)\| \leq P \sum_{m,n=1}^{\infty} a_{m,n}\, e^{(D-\delta)(\lambda_m+\mu_n)} = P\|f\|_{D-\delta}.$$

Thus ψ is continuous. □

§4. Proper Bases and their Characterizations

To prove the main result on proper bases we first need to prove two lemmas.

Lemma 2.4. *Let $\{\alpha_{m,n}\} \subset \chi$. Then the following three properties are equivalent*

(A) $\limsup\limits_{m,n\to\infty} \dfrac{\log \|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} < D$ *for all* $\sigma_1, \sigma_2 < D$.

(B) For all sequences $\{a_{m,n}\}$ of complex numbers the convergence of $\sum\limits_{m,n=1}^{\infty} a_{m,n}\, e_{m,n}$ implies the convergence of $\sum\limits_{m,n=1}^{\infty} a_{m,n}\,\alpha_{m,n}$ in χ.

(C) For all sequences $\{a_{m,n}\}$ of complex numbers the convergence of $\sum\limits_{m,n=1}^{\infty} a_{m,n}\, e_{m,n}$ implies that $\{a_{m,n}\,\alpha_{m,n}\}$ tends to zero in χ.

Proof. Let us assume (A) holds then from the above proof it is clear that there exists a continuous linear transformation $\psi : \chi \to \chi$ with $\psi(e_{m,n}) = \alpha_{m,n}$, $m, n = 1, 2, \ldots$.By continuity of ψ,

$$\begin{aligned}\psi(\sum_{m,n=1}^{\infty} a_{m,n}\, e_{m,n}) &= \psi\left(\lim_{m,n\to\infty} \sum_{w_1=1}^{m} \sum_{w_2=1}^{n} a_{w_1,w_2}\, e_{w_1,w_2}\right) \\ &= \lim_{m,n\to\infty} \left\{ \sum_{w_1=1}^{m} \sum_{w_2=1}^{n} a_{w_1,w_2} \psi(e_{w_1,w_2}) \right\} \\ &= \sum_{m,n=1}^{\infty} a_{m,n}\, \alpha_{m,n}.\end{aligned}$$

Hence $(A) \Rightarrow (B)$. Clearly $(B) \Rightarrow (C)$ from the proof of sufficiency part of theorem **??**. We now need to prove $(C) \Rightarrow (A)$.

Assume that (C) is true and (A) is false. Then for some $\sigma_1^{'}$, $\sigma_2^{'} < D$,

$$\limsup_{m,n\,\to\,\infty} \frac{\log \|\alpha_{m,n}\|_{(\sigma_1',\sigma_2')}}{\lambda_m + \mu_n} \geq D$$

Hence there exists a sequence $\{m_{k_1}\}$ and $\{n_{k_2}\}$ of positive integers, such that

$$\limsup_{k_1,k_2\,\to\,\infty} \frac{\log \|\alpha_{m_{k_1},n_{k_2}}\|_{(\sigma_1',\sigma_2')}}{\lambda_{m_{k_1}} + \mu_{n_{k_2}}} \geq D - \frac{1}{k_1} - \frac{1}{k_2} \text{ for all } k_1\,,\, k_2 = 1, 2, \ldots$$

Define $\{a_{m,n}\}$ as

$$a_{m,n} = \left\{ \begin{array}{c} e^{-\left(D - \left(\frac{1}{k_1} + \frac{1}{k_2}\right)\right)(\lambda_{m_{k_1}} + \mu_{n_{k_2}})} \text{ for } k_1\,,\, k_2 = 1, 2, \ldots \\ 0 \text{ for } m \neq m_{k_1}\,,\, n \neq n_{k_2} \end{array} \right\}$$

So we have

$$|a_{m_{k_1},n_{k_2}}| e^{(\sigma_1 \lambda_{m_{k_1}} + \sigma_2 \mu_{n_{k_2}})} = e^{-\left(D - \left(\frac{1}{k_1} + \frac{1}{k_2} + \sigma_1\right)\right)\lambda_{m_{k_1}}} . \, e^{-\left(D - \left(\frac{1}{k_1} + \frac{1}{k_2} + \sigma_2\right)\right)\mu_{n_{k_2}}}$$

There exists k_1 , k_2 large such that $D - \left(\frac{1}{k_1} + \frac{1}{k_2} + \sigma_1\right) > 0$ and $D - \left(\frac{1}{k_1} + \frac{1}{k_2} + \sigma_2\right) > 0$.

This implies $\sum_{k_1,k_2=1}^{\infty} |a_{m_{k_1},n_{k_2}}| e^{(\sigma_1 \lambda_{m_{k_1}} + \sigma_2 \mu_{n_{k_2}})}$ converges in χ for all σ_1 , $\sigma_2 < D$. But

$$\begin{aligned}|a_{m_{k_1},n_{k_2}}| \|\alpha_{m_{k_1},n_{k_2}}\|_{(\sigma_1,\sigma_2)} &\geq e^{-\left(D - \left(\frac{1}{k_1} + \frac{1}{k_2}\right)\right)(\lambda_{m_{k_1}} + \mu_{n_{k_2}})} . \, e^{\left(D - \left(\frac{1}{k_1} + \frac{1}{k_2}\right)\right)(\lambda_{m_{k_1}} + \mu_{n_{k_2}})} \\ &= 1\end{aligned}$$

Thus $\{a_{m_{k_1},n_{k_2}} \alpha_{m_{k_1},n_{k_2}}\}$ does not tend to zero in χ which is a contradiction. Hence $(C) \Rightarrow (A)$. □

Lemma 2.4. *The following three conditions are equivalent for any sequence* $\{\alpha_{m,n}\} \subset \chi$.

(a) $\lim_{\sigma_1,\sigma_2\,\to\,D} \left\{ \liminf_{m,n\,\to\,\infty} \frac{\log \|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} \right\} \geq D.$

(b) For all sequences $\{a_{m,n}\}$ *of complex numbers the convergence of* $\sum_{m,n=1}^{\infty} a_{m,n}\,\alpha_{m,n}$ *in* χ *implies the convergence of* $\sum_{m,n=1}^{\infty} a_{m,n}\,e_{m,n}$ *in* χ.

(c) For all sequences $\{a_{m,n}\}$ *of complex numbers* $\{a_{m,n}\alpha_{m,n}\}$ *tends to zero in* χ *implies the convergence of* $\sum_{m,n=1}^{\infty} a_{m,n}\,e_{m,n}$ *in* χ.

Proof. It is clear that $(c) \Rightarrow (b)$. We shall prove that $(b) \Rightarrow (a)$ and $(a) \Rightarrow (c)$.
Firstly we suppose that (b) holds and (a) does not hold. Therefore

$$\lim_{\sigma_1,\sigma_2 \to D} \left\{ \liminf_{m,n \to \infty} \frac{\log \|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} \right\} < D.$$

Since $\|...\|_{(\sigma_1,\sigma_2)}$ increases as (σ_1,σ_2) increases this implies for each $(\sigma_1,\sigma_2) < D$,

$$\liminf_{m,n \to \infty} \frac{\log \|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} < D \quad \text{for all } (\sigma_1,\sigma_2) < D.$$

If η and γ be two positive number then there exists an increasing sequence $\{m_{r_1}\}$, $\{n_{r_2}\}$ such that

$$\frac{\log \|\alpha_{m_{r_1},n_{r_2}}\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} \le D - \eta - \gamma$$

Then

$$\|\alpha_{m_{r_1},n_{r_2}}\|_{(\sigma_1,\sigma_2)} \le e^{(D-\eta-\gamma)(\lambda_{m_{r_1}} + \mu_{n_{r_2}})}.$$

Let $\delta_1 < \eta$, $\delta_2 < \gamma$ then define $\{a_{m,n}\}$ as

$$a_{m,n} = \left\{ \begin{array}{c} e^{-(D-\delta_1-\delta_2)(\lambda_{m_{r_1}} + \mu_{n_{r_2}})} \text{ for } r_1, r_2 = 1,2,\ldots \\ 0 \text{ for } m \ne m_{r_1},\, n \ne n_{r_2} \end{array} \right\}$$

Then for every $(\sigma_1,\sigma_2) < D$,

$$\begin{aligned}
\sum_{m,n=1}^{\infty} |a_{m,n}|\|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)} &= \sum_{r_1,r_2=1}^{\infty} |a_{m_{r_1},n_{r_2}}|\|\alpha_{m_{r_1},n_{r_2}}\|_{(\sigma_1,\sigma_2)} \\
&\le \sum_{r_1,r_2=1}^{\infty} e^{(D-\eta-\gamma)(\lambda_{m_{r_1}} + \mu_{n_{r_2}})}.e^{-(D-\delta_1-\delta_2)(\lambda_{m_{r_1}} + \mu_{n_{r_2}})} \\
&= \sum_{r_1,r_2=1}^{\infty} e^{(\delta_1+\delta_2-(\eta+\gamma))(\lambda_{m_{r_1}} + \mu_{n_{r_2}})}
\end{aligned}$$

and the last series is convergent since $\delta_1 + \delta_2 < \eta + \gamma$. Hence for the sequence $\{a_{m,n}\}$, $\sum_{m,n=1}^{\infty} a_{m,n}\alpha_{m,n}$ converges in $\chi_{(\sigma_1,\sigma_2)}$ for each $(\sigma_1,\sigma_2) < D$ thus converges in χ.

But we have

$$\sum_{m,n=1}^{\infty} |a_{m,n}|\, e^{(\sigma_1\lambda_m + \sigma_2\mu_n)} = \sum_{r_1,r_2=1}^{\infty} |a_{m_{r_1},n_{r_2}}| e^{(\sigma_1\lambda_{m_{r_1}} + \sigma_2\mu_{n_{r_2}})}$$

$$= \sum_{r_1,r_2=1}^{\infty} e^{-(D-\delta_1-\delta_2)(\lambda_{m_{r_1}}+\mu_{n_{r_2}})}.e^{(\sigma_1\lambda_{m_{r_1}}+\sigma_2\mu_{n_{r_2}})}$$
$$= \sum_{r_1,r_2=1}^{\infty} e^{(\sigma_1+\delta_1-D+\delta_2)\lambda_{m_{r_1}}}.e^{(\sigma_2+\delta_2-D+\delta_1)\mu_{n_{r_2}}}.$$

Given δ_1, δ_2 choose $\sigma_1, \sigma_2 < D$ such that $(\sigma_1 + \delta_1 + \delta_2) > D$ and $(\sigma_2 + \delta_2 + \delta_1) > D$ thus the last series is divergent for σ_1, σ_2. Hence $\sum_{m,n=1}^{\infty} a_{m,n}e_{m,n}$ does not converge in χ which contradicts (b). Hence $(b) \Rightarrow (a)$.

To prove $(a) \Rightarrow (c)$ we assume that (a) is true but (c) is not true. Hence there exists a sequence $\{a_{m,n}\}$ of complex numbers for which $a_{m,n}\alpha_{m,n}$ tends to zero in χ but $\sum_{m,n=1}^{\infty} a_{m,n}e_{m,n}$ does not converge in χ. This implies that

$$\limsup_{m,n\to\infty} \frac{\log|a_{m,n}|}{\lambda_m + \mu_n} \geq -D + \delta_1 + \delta_2.$$

Hence for $\delta_1, \delta_2 > 0$ there exists a sequence $\{m_{k_1}\}$ and $\{n_{k_2}\}$ of integers such that

$$|a_{m_{k_1},n_{k_2}}| \geq e^{(-D+\delta_1+\delta_2)(\lambda_{m_{k_1}}+\mu_{n_{k_2}})}.$$

Now we choose a positive number σ_1, σ_2 such that. We can also find $\sigma_1 = \sigma_1(\eta)$ and $\sigma_2 = \sigma_2(\gamma)$ such that

$$\liminf_{m,n\to\infty} \frac{\log\|\alpha_{m,n}\|}{\lambda_m + \mu_n} \geq D - \eta - \gamma.$$

Hence there exists $M = M(\gamma)$ and $N = N(\eta)$ such that

$$\frac{\log\|\alpha_{m,n}\|_{(\sigma_1,\sigma_2)}}{\lambda_m + \mu_n} \geq D - 2\eta - 2\gamma \text{ for all } m \geq M, n \geq N.$$

Therefore

$$\begin{aligned}|a_{m_{k_1},n_{k_2}}|\|\alpha_{m_{k_1},n_{k_2}}\|_{(\sigma_1,\sigma_2)} &\geq e^{(-D+\delta_1+\delta_2)(\lambda_{m_{k_1}}+\mu_{n_{k_2}})}.e^{(D-2\eta-2\gamma)(\lambda_{m_{k_1}}+\mu_{n_{k_2}})}\\ &= e^{(\delta_1-2\eta)(\lambda_{m_{k_1}}+\mu_{n_{k_2}})}.e^{(\delta_2-2\gamma)(\lambda_{m_{k_1}}+\mu_{n_{k_2}})}\end{aligned}$$

which tends to ∞ as $k_1, k_2 \to \infty$, since $\delta_1 > 2\eta$ and $\delta_2 > 2\gamma$.

This shows $\{a_{m,n}\,\alpha_{m,n}\}$ does not tend to zero in χ and this is a contradiction. Therefore we conclude that $(a) \Rightarrow (c)$. □

The main result that follows from above lemma is stated as

Theorem 2.6. *A base in a closed subspace χ_o of χ is proper if and only if condition (a) and (A) are satisfied.*

References

[1] Daoud S., On a class of entire Dirichlet functions of several complex variables having finite order point. Portugaliae Mathematica , 1986, 43, 417-427.

[2] M.S.A. Hussein , G.S. Srivastava , A study on Bornological properties of the space of entire functions of several complex variables. Tamkang Jour. of Math,2002, 33(4), 289-301.

[3] P.K.Kamthan , A study on the space of entire functions of several complex variables. Yokohama Math. J. 1973, 21, 11-20.

[4] N. Kumar , G. Manocha , On a class of entire functions represented by Dirichlet series. J. Egypt. Math. Soc. 2013, 21, 21-24.

[5] N. Kumar , L. Chutani , G. Manocha , Certain results on a class of Entire Dirichlet series in two variables. Scientia Magna, 2015,(Accepted).

[6] Larsen R., Banach Algebras - An Introduction. Marcel Dekker Inc.,1973, New York.

[7] Larsen R., Functional Analysis - An Introduction. Marcel Dekker Inc.,1973, New York.

Scientia Magna
Vol. 13 (2018), No. 1, 42-47

A short note on pairwise fuzzy Volterra spaces

G. Thangaraj†, V. Chandiran♯

†Thiruvalluvar University, Serkkadu, Vellore - 632 115, Tamil Nadu, India.
♯Research Scholar, Thiruvalluvar University, Serkkadu, Vellore - 632 115, Tamil Nadu, India.
E-mails: g.thangaraj@rediffmail.com , profvcmaths@gmail.com

Abstract In this paper we study the conditions under which a fuzzy bitopological space becomes a pairwise fuzzy Volterra space.

Keywords Pairwise fuzzy Baire space, pairwise fuzzy submaximal space, pairwise fuzzy P-space, pairwise fuzzy hyperconnected space and pairwise fuzzy Volterra space.
2010 Mathematics Subject Classification: 54A40, 03E72.

§1. Introduction

In order to deal with uncertainties, the idea of fuzzy sets and fuzzy set operations was introduced by L. A. Zadeh in his classical paper [16] in the year 1965. This inspired mathematicians to fuzzify Mathematical Structures. The first notion of fuzzy topological space had been defined by C. L. Chang [3] in 1968. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. In 1989, A. Kandil [9] introduced the concept of fuzzy bitopological space as an extension of fuzzy topological space and as a generalization of bitopological space. The concepts of Volterra spaces have been studied extensively in classical topology in [4], [5], [6], [7] and [8]. The concept of pairwise fuzzy Volterra spaces and pairwise fuzzy weakly Volterra spaces in fuzzy setting was introduced and studied by the authors in [14]. In this paper we study under what conditions a fuzzy bitopological space becomes a pairwise fuzzy Volterra space and pairwise fuzzy Baire space, pairwise fuzzy submaximal space, pairwise fuzzy P-space and pairwise fuzzy hyperconnected space are considered for this work.

§2. Preliminaries

Now we introduce some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). In this work by (X, T_1, T_2) or simply by X, we will denote a fuzzy bitopological space due to A. Kandil.

Definition 2.1. Let λ and μ be any two fuzzy sets in a fuzzy topological space (X, T). Then we define :

(i) $\lambda \vee \mu$: X $\to$ [0,1] as follows : $(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}$.

(ii) $\lambda \wedge \mu$: X $\to [0, 1]$ as follows : $(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}$.

(iii) $\mu = \lambda^c \Leftrightarrow \mu(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined respectively as $\psi(x) = sup_i\{\lambda_i(x), x \in X\}$ and $\delta(x) = inf_i\{\lambda_i(x), x \in X\}$.

Definition 2.2.[1] Let (X, T) be a fuzzy topological space. For a fuzzy set λ of X, the interior and the closure of λ are defined respectively as $int(\lambda) = \vee\{\mu / \mu \leq \lambda, \mu \in T\}$ and $cl(\lambda) = \wedge\{\mu / \lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.1.[1] Let λ be any fuzzy set in a fuzzy topological space (X, T). Then $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$.

Definition 2.3.[2] Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called a *fuzzy G_δ-set* if $\lambda = \wedge_{i=1}^{\infty} \lambda_i$ for each $\lambda_i \in T$.

Definition 2.4.[2] Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called a *fuzzy F_σ-set* if $\lambda = \vee_{i=1}^{\infty} \lambda_i$ for each $1 - \lambda_i \in T$.

Definition 2.5.[14] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy open set* if $\lambda \in T_i$, $(i = 1, 2)$. The complement of a pairwise fuzzy open set in (X, T_1, T_2) is called a *pairwise fuzzy closed set* in (X, T_1, T_2).

Definition 2.6.[14] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy G_δ-set* if $\lambda = \wedge_{i=1}^{\infty} \lambda_i$, where (λ_i)'s are pairwise fuzzy open sets in (X, T_1, T_2).

Definition 2.7.[14] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy F_σ-set* if $\lambda = \vee_{i=1}^{\infty} \lambda_i$, where (λ_i)'s are pairwise fuzzy closed sets in (X, T_1, T_2).

Definition 2.8.[10] A fuzzy set λ in a fuzzy topological space (X, T) is called *fuzzy dense* if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$. That is., $cl(\lambda) = 1$.

Definition 2.9.[11] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy dense set* if $cl_{T_1} cl_{T_2}(\lambda) = 1 = cl_{T_2} cl_{T_1}(\lambda)$.

Definition 2.10.[10] A fuzzy set λ in a fuzzy topological space (X, T) is called *fuzzy nowhere dense* if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is., $intcl(\lambda) = 0$.

Definition 2.11.[12] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy nowhere dense set* if $int_{T_1} cl_{T_2}(\lambda) = 0 = int_{T_2} cl_{T_1}(\lambda)$.

Definition 2.12.[12] Let (X, T_1, T_2) be a fuzzy bitopological space. A fuzzy set λ in (X, T_1, T_2) is called a *pairwise fuzzy first category set* if $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). A fuzzy set in (X, T_1, T_2) which is not pairwise fuzzy first category is said to be a *pairwise fuzzy second category set* in (X, T_1, T_2).

Definition 2.13.[12] If λ is a pairwise fuzzy first category set in a fuzzy bitopological space (X, T_1, T_2), then the fuzzy set $1 - \lambda$ is called a *pairwise fuzzy residual set* in (X, T_1, T_2).

§3. Pairwise fuzzy Volterra spaces

Definition 3.1.[14] A fuzzy bitopological space (X, T_1, T_2) is said to be a *pairwise fuzzy Volterra space* if $cl_{T_i}\left(\wedge_{k=1}^{N} (\lambda_k)\right) = 1$, $(i = 1, 2)$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2).

Proposition 3.1. If λ is a pairwise fuzzy open set in a fuzzy bitopological space (X, T_1, T_2) such that $cl_{T_i}(\lambda) = 1$, $(i = 1, 2)$, then $1 - \lambda$ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2).

Proof. Let λ be a pairwise fuzzy open set in a fuzzy bitopological space (X, T_1, T_2) such that $cl_{T_i}(\lambda) = 1,\ (i = 1, 2)$. Then, we have $int_{T_1}(\lambda) = \lambda,\ int_{T_2}(\lambda) = \lambda$ and $cl_{T_i}(\lambda) = 1,\ (i = 1, 2)$. Now $int_{T_1}cl_{T_2}(1 - \lambda) = 1 - cl_{T_1}int_{T_2}(\lambda) = 1 - cl_{T_1}(\lambda) = 1 - 1 = 0$. Also, $int_{T_2}cl_{T_1}(1 - \lambda) = 1 - cl_{T_2}int_{T_1}(\lambda) = 1 - cl_{T_2}(\lambda) = 1 - 1 = 0$. Hence $int_{T_1}cl_{T_2}(1 - \lambda) = 0 = int_{T_2}cl_{T_1}(1 - \lambda)$. Therefore $1 - \lambda$ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2).

Theorem 3.1.[15] If λ is a pairwise fuzzy G_δ-set such that $cl_{T_i}(\lambda) = 1,\ (i = 1, 2)$, in a fuzzy bitopological space (X, T_1, T_2), then $1 - \lambda$ is a pairwise fuzzy first category set in (X, T_1, T_2).

Definition 3.2.[12] A fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy Baire space* if $int_{T_i}(\vee_{k=1}^{\infty}(\lambda_k)) = 0,\ (i = 1, 2)$, where (λ_k)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2).

Proposition 3.2. If the pairwise fuzzy first category sets are pairwise fuzzy closed sets in a pairwise fuzzy Baire space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Proof. Let $\lambda_k,\ (k = 1\ to\ \infty)$ be pairwise fuzzy G_δ-sets such that $cl_{T_i}(\lambda_k) = 1,\ (i = 1, 2)$ in a pairwise fuzzy Baire space (X, T_1, T_2). Then, by theorem 3.1, $(1 - \lambda_k)$'s are pairwise fuzzy first category sets in (X, T_1, T_2). Then, by hypothesis, $(1 - \lambda_k)$'s are pairwise fuzzy closed sets in (X, T_1, T_2). Hence (λ_k)'s are pairwise fuzzy open sets in (X, T_1, T_2). Now (λ_k)'s are pairwise fuzzy open sets in (X, T_1, T_2) such that $cl_{T_i}(\lambda_k) = 1,\ (i = 1, 2)$. Hence, by proposition 3.1, $(1-\lambda_k)$'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Since (X, T_1, T_2) is a fuzzy Baire bitopological space, $int_{T_i}\Big(\vee_{k=1}^{\infty}(1-\lambda_k)\Big) = 0$. Then $int_{T_i}\Big(1-\wedge_{k=1}^{\infty}(\lambda_k)\Big) = 0$. This implies that $1-cl_{T_i}\Big(\wedge_{k=1}^{\infty}(\lambda_k)\Big) = 0$. Hence we have $cl_{T_i}\Big(\wedge_{k=1}^{\infty}(\lambda_k)\Big) = 1$. Now $cl_{T_i}\Big(\wedge_{k=1}^{\infty}(\lambda_k)\Big) \leq cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big)$ implies that $1 \leq cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big)$. That is., $cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big) = 1$. Since $cl_{T_i}(\lambda_k) = 1$, $(i = 1, 2)$ $cl_{T_1}cl_{T_2}(\lambda_k) = cl_{T_1}(1) = 1$ and $cl_{T_2}cl_{T_1}(\lambda_k) = cl_{T_2}(1) = 1$. Hence (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2). Therefore, we have $cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big) = 1,\ (i = 1, 2)$ where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Hence (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Proposition 3.3. If the pairwise fuzzy residual sets are pairwise fuzzy open sets in a pairwise fuzzy Baire space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Proof. Let the pairwise fuzzy residual sets (λ_k)'s $(k = 1\ to\ \infty)$ be pairwise fuzzy open sets in a pairwise fuzzy Baire space (X, T_1, T_2). Then $(1 - \lambda_k)$'s are pairwise fuzzy first category sets in (X, T_1, T_2) such that $(1 - \lambda_k)$'s are pairwise fuzzy closed sets. Hence, by proposition 3.2, (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Theorem 3.2.[12] If λ is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2), then $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2).

Proposition 3.4. If pairwise fuzzy first category sets are pairwise fuzzy nowhere dense sets in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Proof. Let (λ_k)'s $(k = 1\ to\ \infty)$ be pairwise fuzzy G_δ-sets such that $cl_{T_i}(\lambda_k) = 1,\ (i = 1, 2)$ in a fuzzy bitopogical space(X, T_1, T_2). Then, by theorem 3.1, $(1 - \lambda_k)$'s are pairwise fuzzy first category sets in (X, T_1, T_2). By hypothesis, $(1 - \lambda_k)$'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Hence $\vee_{k=1}^{\infty}(1 - \lambda_k)$ is a pairwise fuzzy first category set in (X, T_1, T_2). Again, by hypothesis, $\vee_{k=1}^{\infty}(1 - \lambda_k)$ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2). Then, we have $int_{T_1}cl_{T_2}\big(\vee_{k=1}^{\infty}(1 - \lambda_k)\big) = 0$ and $int_{T_2}cl_{T_1}\big(\vee_{k=1}^{\infty}(1 - \lambda_k)\big) = 0$. Now $int_{T_1}\big(\vee_{k=1}^{\infty}(1 - \lambda_k)\big) \leq$

$int_{T_1}\Big(cl_{T_2}\big(\vee_{k=1}^{\infty}(1-\lambda_k)\big)\Big)$ implies that $int_{T_1}\big(\vee_{k=1}^{\infty}(1-\lambda_k)\big) \leq 0$. That is., $int_{T_1}\big(\vee_{k=1}^{\infty}(1-\lambda_k)\big) = 0$. Hence $int_{T_1}\big(1 - \wedge_{k=1}^{\infty}(\lambda_k)\big) = 0$. This implies that $1 - cl_{T_1}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 0$. That is., $cl_{T_1}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1$. Similarly, we will prove that $cl_{T_2}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1$. Hence $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1,\ (i = 1, 2)$. Now $cl_{T_i}(\lambda_k) = 1$ implies that $cl_{T_1}cl_{T_2}(\lambda_k) = 1$ and $cl_{T_2}cl_{T_1}(\lambda_k) = 1$. Hence (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2). Now $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) \leq cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big)$ implies that $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1$. Hence $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1,\ (i = 1, 2)$ where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Definition 3.3.[15] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy σ-nowhere dense set* if λ is a pairwise fuzzy F_σ-set in (X, T_1, T_2) such that $int_{T_1}int_{T_2}(\lambda) = int_{T_2}int_{T_1}(\lambda) = 0$

Proposition 3.5. If $int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = 0$, where the fuzzy sets (λ_k)'s $(k = 1\ to\ N)$ are pairwise fuzzy σ-nowhere dense sets in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Proof. Suppose that $int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = 0$, where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Then, we have $1 - int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = 1$. This implies that $cl_{T_i}\big(1 - \vee_{k=1}^{\infty}(\lambda_k)\big) = 1$ and hence we have $cl_{T_i}\big(\wedge_{k=1}^{\infty}(1 - \lambda_k)\big) = 1 \longrightarrow (A)$. Since (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2), (λ_k)'s are pairwise fuzzy F_σ-sets such that $int_{T_1}int_{T_2}(\lambda_k) = 0$ and $int_{T_2}int_{T_1}(\lambda_k) = 0$. Then $(1 - \lambda_k)$'s are pairwise fuzzy G_δ-sets and $1 - int_{T_1}int_{T_2}(\lambda_k) = 1$ and $1 - int_{T_2}int_{T_1}(\lambda_k) = 1$. Hence $(1 - \lambda_k)$'s are pairwise fuzzy G_δ-sets and $cl_{T_1}cl_{T_2}(1 - \lambda_k) = 1$ and $cl_{T_2}cl_{T_1}(1 - \lambda_k) = 1$. Hence, from (A), we have $cl_{T_i}\big(\wedge_{k=1}^{N}(1 - \lambda_k)\big) = 1,\ (i = 1, 2)$ where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Definition 3.4.[14] A fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy P-space* if countable intersection of pairwise fuzzy open sets in (X, T_1, T_2) is pairwise fuzzy open. That is., every non-zero pairwise fuzzy G_δ-set in (X, T_1, T_2) is pairwise fuzzy open in (X, T_1, T_2).

Proposition 3.6. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Baire and pairwise fuzzy P-space, then (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Proof. Let (λ_k)'s $(k = 1\ to\ N)$ be pairwise fuzzy G_δ-sets in a fuzzy bitopological s-pace (X, T_1, T_2) such that $cl_{T_i}(\lambda_k) = 1,\ (i = 1, 2)$. Then $cl_{T_1}cl_{T_2}(\lambda_k) = cl_{T_1}(1) = 1$ and $cl_{T_2}cl_{T_1}(\lambda_k) = cl_{T_2}(1) = 1$. Since (X, T_1, T_2) is a pairwise fuzzy P-space, the pairwise fuzzy G_δ-sets (λ_k)'s $(k = 1\ to\ N)$ are pairwise fuzzy open sets in (X, T_1, T_2). Then, (λ_k)'s $(k = 1\ to\ N)$ are pairwise fuzzy open sets in (X, T_1, T_2) such that $cl_{T_i}(\lambda_k) = 1,\ (i = 1, 2)$. By proposition 3.1, $(1 - \lambda_k)$'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy Baire space, $int_{T_i}\big(\vee_{\alpha=1}^{\infty}(\mu_\alpha)\big) = 0$, where (μ_α)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Let us take the first $N(\mu_\alpha)$'s as $(1 - \lambda_k)$ in (X, T_1, T_2). Then, $int_{T_i}\big(\vee_{k=1}^{N}(1 - \lambda_k)\big) \leq int_{T_i}\big(\vee_{\alpha=1}^{\infty}(\mu_\alpha)\big)$ implies that $int_{T_i}\big(\vee_{k=1}^{N}(1 - \lambda_k)\big) = 0$. Hence $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1,\ (i = 1, 2)$ where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Definition 3.5.[13] A fuzzy bitopological space (X, T_1, T_2) is said to be a *pairwise fuzzy submaximal space* if for each fuzzy set λ in (X, T_1, T_2) such that $cl_{T_i}(\lambda) = 1,\ (i = 1, 2)$, then λ is a pairwise fuzzy open set in (X, T_1, T_2).

Proposition 3.7. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Baire space and pairwise fuzzy submaximal space, then (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Proof. Let (λ_k)'s $(k = 1 \text{ to } N)$ be pairwise fuzzy G_δ-sets in a fuzzy bitopological space (X, T_1, T_2) such that $cl_{T_i}(\lambda_k) = 1,\ (i = 1, 2)$. Since (X, T_1, T_2) is a pairwise fuzzy submaximal space, the pairwise fuzzy dense sets (λ_k)'s $(k = 1 \text{ to } N)$ are pairwise fuzzy open sets in (X, T_1, T_2). Then, (λ_k)'s $(k = 1 \text{ to } N)$ are pairwise fuzzy open sets (X, T_1, T_2) such that $cl_{T_i}(\lambda_k) = 1,\ (i = 1, 2)$. By proposition 3.1, $(1 - \lambda_k)$'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy Baire space, $int_{T_i}\big(\vee_{\alpha=1}^{\infty}(\mu_\alpha)\big) = 0,\ (i = 1, 2)$, where (μ_α)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Let us take the first $N(\mu_\alpha)$'s as $(1 - \lambda_k)$ in (X, T_1, T_2). But $int_{T_i}\big(\vee_{k=1}^{N}(1 - \lambda_k)\big) \leq int_{T_i}\big(\vee_{\alpha=1}^{\infty}(\mu_\alpha)\big)$. Now $int_{T_i}\big(\vee_{k=1}^{N}(1 - \lambda_k)\big) = 0$ implies that $int_{T_i}\big(1 - \wedge_{k=1}^{N}(\lambda_k)\big) = 0$. Then $1 - cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 0$. That is., $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1,\ (i = 1, 2)$ where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Definition 3.6.[15] A fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy hyperconnected space* if λ is a pairwise fuzzy open set, then $cl_{T_i}(\lambda) = 1,\ (i = 1, 2)$.

Proposition 3.8. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy submaximal space and pairwise fuzzy hyperconnected space, then (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Proof. Let (λ_k)'s $(k = 1 \text{ to } N)$ be pairwise fuzzy G_δ-sets in a fuzzy bitopological space (X, T_1, T_2) such that $cl_{T_i}(\lambda_k) = 1,\ (i = 1, 2)$. Now $cl_{T_i}(\lambda_k) = 1$ implies that $cl_{T_1}cl_{T_2}(\lambda_k) = 1 = cl_{T_2}cl_{T_1}(\lambda_k)$. Since (X, T_1, T_2) is a pairwise fuzzy submaximal space, $cl_{T_i}(\lambda_k) = 1,\ (i = 1, 2)$ implies that (λ_k)'s are pairwise fuzzy open sets in (X, T_1, T_2). That is., $\lambda_k \in T_i$. Now $\lambda_k \in T_i$ implies that $\wedge_{k=1}^{N}(\lambda_k) \in T_i,\ (i = 1, 2)$. Also since (X, T_1, T_2) is a pairwise fuzzy hyperconnected space, $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1,\ (i = 1, 2)$ where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy Volterra space.

References

[1] K. K. Azad, On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl., **82**(1981), 14-32.

[2] G. Balasubramanian, Maximal fuzzy topologies, Kybernetika, **31(5)**(1995), 459-464.

[3] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl., **24**(1968), 182-190.

[4] David H Foster, Fuzzy topological groups, J. Math. Anal. Appl., **67**, (1979), 549-564.

[5] David Gauld and Zbigniew Piotrowski, On Volterra spaces, Far East J. Math. Sci., **1**, (1993), 209-214.

[6] David Gauld, Sina Greenwood and Zbigniew Piotrowski, On Volterra spaces - II, Papers on General Topology and its Applications, Ann. New York Acad. Sci., **806**, (1996), 169-173.

[7] David Gauld, Sina Greenwood, Zbigniew Piotrowski, On Volterra spaces - III, Topological Operations, Topology Proc., **23**, (1998), 167-182.

[8] Jiling Cao and David Gauld, Volterra spaces revisited, J. Aust. Math. Soc., **79**, (2005), 61-76.

[9] A. Kandil, Biproximities and fuzzy bitopological spaces, Simen Stevin, **63** (1989), 45-66.

[10] G. Thangaraj and G. Balasubramanian, On somewhat fuzzy continuous functions, J. Fuzzy Math., **11(2)**, (2003), 725-736.

[11] G. Thangaraj, On pairwise fuzzy resolvable and fuzzy irresolvable spaces, Bull. Cal. Math. Soc., **101** (2010), 59-68.

[12] G. Thangaraj and S. Sethuraman, On pairwise fuzzy Baire bitopological spaces, Gen. Math. Notes, **20(2)**, (2014), 12-21.

[13] G. Thangaraj and S. Sethuraman, A note on pairwise fuzzy Baire spaces, (To appear in Annals of Fuzzy Mathematics and Informatics).

[14] G. Thangaraj and V. Chandiran, On pairwise fuzzy Volterra spaces, Annals of Fuzzy Mathemarics and Informatics, **7(6)** (2014), 1005-1012.

[15] G. Thangaraj and V. Chandiran, A note on pairwise fuzzy Volterra spaces, (To appear in Annals of Fuzzy Mathematics and Informatics).

[16] L. A. Zadeh, Fuzzy sets, Information and Control, **8**, (1965), 338-353.

Scientia Magna
Vol. 13 (2018), No. 1, 48-62

Improved McClelland and Koolen-Moulton bounds for the energy of graphs

G. Sridhara[1], M. R. Rajesh Kanna[2], R. Jagadeesh[3] and I. N. Cangul[4]

[1] Maharani's Science College for Women,
J. L. B. Road, Mysore - 570 005, India.

[2] Sri D Devaraja Urs Government First Grade College,
Hunsur-571105, Mysore District, India.

[3]Research and Development Centre,
Bharathiar University, Coimbatore - 641 046, India.

[4]Faculty of Arts and Science,
Department of Mathematics,
Uludag University, Gorukle - 16059, Bursa,Turkey
(corresponding author).

e-mails: srsrig@gmail.com, mr.rajeshkanna@gmail.com, jagadeeshr1978@gmail.com,
ncangul@gmail.com

Abstract Given a graph G with n vertices and m edges, the term energy of graph $\mathcal{E}(G)$ was introduced by Gutman in chemistry, due to its relevance to the total $\pi-$electron energy of carbon compounds. In 1971 McClelland obtained both lower and upper bounds for $\pi-$electron energy. An improved upper bound was obtained by Koolen-Moulton in 2001. The lower and upper bounds for $\mathcal{E}(G)$ obtained in this paper are better than McClelland and Koolen-Moulton bounds. Also we obtained an upper bound for graph energy in terms of n as $\mathcal{E}(G) \leq \frac{n}{2}\left[1+\sqrt{\frac{n-2}{2}}\right]$.

Keywords Adjacency matrix, graph spectrum, energy of a graph.
2010 **Mathematics Subject Classification** 05C50, 05C69.

§1. Introduction and preliminaries

Let $G=(V,E)$ be a simple undirected graph with n vertices and m edges. For any $v_i \epsilon V$, the degree of the vertex v_i, denoted by d_i or $d(v_i)$, is defined as the number of edges that are incident to v_i. A graph G is said to be r-regular if each vertex of G have same degree r. A graph G is bipartite of degree r_1 and r_2, if the vertex set, V is partitioned into disjoint subsets X and Y such that no two vertices of $X(\text{or}Y)$ are adjacent. A regular bipartite graph is a bipartite graph if each vertex has same degree. A bipartite graph G is semi-regular bipartite if it is bipartite and each vertex in the same partition has same degree. Clearly regular bipartite graph is semi regular bipartite graph ($r_1 = r_2$).

Given a graph G, the energy of G defined by

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of G which are obtained from its adjacency matrix. The studies on graph energy can be found in many papers, see e.g. [4–6]. For detailed survey on applications on graph energy, see papers [1–3].

Koolen and Moulton obtained upper bounds, [10], for graph energy in terms of m and n as

$$\mathcal{E}(G) \leq \frac{2m}{n} + \sqrt{(n-1)\Big(2m - \Big(\frac{2m}{n}\Big)^2\Big)} \text{ for } 2m \geq n \tag{1.1}$$

and obtained an upper bound for bipartite graph [11] as

$$\mathcal{E}(G) \leq \frac{4m}{n} + \sqrt{(n-2)\Big(2m - 2\Big(\frac{2m}{n}\Big)^2\Big)} \text{ for } 2m \geq n. \tag{1.2}$$

Also they proved that for a graph G with n vertices

$$\mathcal{E}(G) \leq \frac{n}{2}(1 + \sqrt{n}).$$

Here in this paper we have shown that the upper bound (1.1) can be modified to a better bound for all classes of graphs with $n^2 \geq 4m$. Further results on upper bounds can also be seen in [7].

McClelland gave the following bounds for the energy of a graph

$$\sqrt{2m + n(n-1)|det(A)|^{\frac{2}{n}}} \leq \mathcal{E}(G) \leq \sqrt{2mn}, \tag{1.3}$$

[12]. The lower bound obtained in this paper is better than the McClelland bounds. Kinkar Ch. Das et. al. also obtained an improvised lower bound for non-singular graphs, [8]. J. H. Koolen et. al. have improvised McClelland bounds in the paper [9].

§2. Discussion and Main results

The following is a very useful tool on the eigenvalues:

Lemma 2.1. *Let G be a graph with $n \geq 3$ vertices and m edges. For respectively largest and smallest eigenvalues λ_1 and λ_n of G, we have*

$$\lambda_1 + \lambda_n \leq 2\sqrt{\frac{m(n-2)}{n}}.$$

Proof. For n eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of G, it is well known that

$$\sum_{i=1}^{n} \lambda_i = 0 \ \text{ and } \ \sum_{i=1}^{n} \lambda_i^2 = 2m.$$

Applying the Cauchy-Schwarz inequality for $(\lambda_2, \lambda_3, \cdots, \lambda_{n-1})$ and $\underbrace{(1, 1, \cdots, 1)}_{n-2 \ times}$, where $n \geq 3$, we obtain

$$\Big(\sum_{i=2}^{n-1} \lambda_i\Big)^2 \leq \Big(\sum_{i=2}^{n-1} 1\Big)\Big(\sum_{i=2}^{n-1} \lambda_i^2\Big),$$

i.e.,

$$(\lambda_1 + \lambda_n)^2 \leq (n-2)(2m - \lambda_1^2 - \lambda_n^2).$$

Hence

$$\begin{aligned} (n-2)2m &\geq (\lambda_1 + \lambda_n)^2 + (n-2)(\lambda_1^2 + \lambda_n^2) \\ &= (\lambda_1 + \lambda_n)^2(n-1) - 2(n-2)\lambda_1\lambda_n. \end{aligned}$$

But

$$\frac{\lambda_1 + \lambda_n}{2} \geq \sqrt{\lambda_1 \lambda_n}$$

which implies that

$$-\lambda_1\lambda_n \geq -\Big(\frac{\lambda_1 + \lambda_n}{2}\Big)^2.$$

Hence

$$\begin{aligned} (n-2)2m &\geq (\lambda_1 + \lambda_n)^2(n-1) - 2(n-2)\tfrac{(\lambda_1+\lambda_n)^2}{4} \\ &= (\lambda_1 + \lambda_n)^2 \tfrac{n}{2}. \end{aligned}$$

Thus $\lambda_1 + \lambda_n \leq 2\sqrt{\frac{m(n-2)}{n}}$. □

We use the above lemma to find an upper bound for the width $\lambda_1 - \lambda_n$ of the spectrum:

Corollary 2.1. *For a graph G with $n \geq 3$ vertices and m edges,*

$$\lambda_1 - \lambda_n \leq 2\sqrt{2}\sqrt{\frac{m}{n}(n-2)}.$$

Proof. Applying the Cauchy-Schwarz inequality for the ordered pairs $(\lambda_1, -\lambda_n)$ and $(1, 1)$, we have

$$[1(\lambda_1) + 1(-\lambda_n)]^2 \leq (1+1)(\lambda_1^2 + \lambda_n^2)$$

implying that

$$\begin{aligned} \lambda_1 - \lambda_n &\leq \sqrt{2}\sqrt{\lambda_1^2 + \lambda_n^2} \\ &\leq \sqrt{2}(\lambda_1 + \lambda_n) \\ &\leq 2\sqrt{2}\sqrt{\tfrac{m}{n}(n-2)}. \end{aligned}$$

Thus the width of the spectrum of a graph G can be at most $2\sqrt{2}\sqrt{\frac{m}{n}(n-2)}$. □

Using $\lambda_1 + \lambda_n \le 2\sqrt{\frac{m}{n}(n-2)}$ and $\lambda_1 - \lambda_n \le 2\sqrt{2}\sqrt{\frac{m}{n}(n-2)}$, we find that

$$\lambda_1 \le (1+\sqrt{2})\sqrt{\frac{m}{n}(n-2)}$$

for $n \ge 3$ which is an upper bound for the largest eigenvalue.

§3. Upper bound for the energy of a graph

Theorem 3.1. *Let G be a graph with $n \ge 3$ vertices and m edges. If $n^2 \ge 4m$, then*

$$\mathcal{E}(G) \le \frac{2m}{n} + \sqrt{\frac{2m}{n}} + \sqrt{(n-2)\Big(2m - \frac{2m}{n} - \frac{4m^2}{n^2}\Big)}. \tag{3.1}$$

Equality holds iff G is $\frac{n}{2}K_2$.

Proof. Applying the Cauchy-Schwarz inequality to $(|\lambda_2|,\ |\lambda_3|,\ \cdots,\ |\lambda_{n-1}|)$ and $(\underbrace{1,\ 1,\ \cdots,\ 1}_{n-2\ times})$, we have

$$\Big(\sum_{i=2}^{n-1} |\lambda_i|\Big)^2 \le \Big(\sum_{i=2}^{n-1} 1\Big)\Big(\sum_{i=2}^{n-1} |\lambda_i|^2\Big)$$

and hence

$$(\mathcal{E}(G) - |\lambda_1| - |\lambda_n|)^2 \le (n-2)(2m - |\lambda_1|^2 - |\lambda_n|^2).$$

Therefore

$$\mathcal{E}(G) \le |\lambda_1| + |\lambda_n| + \sqrt{(n-2)(2m - |\lambda_1|^2 - |\lambda_n|^2)}.$$

Let now $|\lambda_1| = x$, $|\lambda_n| = y$. We maximize the function

$$f(x,y) = x + y + \sqrt{(n-2)(2m - x^2 - y^2)}.$$

Differentiating $f(x,y)$ partially wrt x and y, respectively, we obtain

$$f_x = 1 - \frac{x(n-2)}{\sqrt{(n-2)(2m-x^2-y^2)}}, \quad f_y = 1 - \frac{y(n-2)}{\sqrt{(n-2)(2m-x^2-y^2)}}$$

$$f_{xx} = -\frac{\sqrt{n-2}(2m-y^2)}{(2m-x^2-y^2)^{\frac{3}{2}}}, \qquad f_{yy} = -\frac{\sqrt{n-2}(2m-x^2)}{(2m-x^2-y^2)^{\frac{3}{2}}}$$

and

$$f_{xy} = -\frac{\sqrt{n-2}xy}{(2m-x^2-y^2)^{\frac{3}{2}}}.$$

For maxima or minima, consider the equations $f_x = 0$ and $f_y = 0$ which together imply that

$$x^2(n-1) + y^2 = 2m \quad \text{and} \quad y^2(n-1) + x^2 = 2m.$$

Solving the above equations, we obtain the values of x and y as $x = y = \sqrt{\frac{2m}{n}}$. At this point, the values of f_{xx}, f_{yy}, f_{xy} and $\Delta = f_{xx}f_{yy} - (f_{xy})^2$ are

$$f_{xx} = -\frac{\sqrt{n-2}(n-1)}{\sqrt{\frac{2m}{n}}(n-2)^{\frac{3}{2}}} \le 0, \quad f_{yy} = -\frac{\sqrt{n-2}(n-1)}{\sqrt{\frac{2m}{n}}(n-2)^{\frac{3}{2}}},$$

$$f_{xy} = -\frac{\sqrt{n-2}}{\sqrt{\frac{2m}{n}}(n-2)^{\frac{3}{2}}} \quad \text{and} \quad \Delta = \frac{n(n^2+3-3n)}{2m(n-2)^2} \geq 0.$$

Therefore $f(x,y)$ attains its maximum value at $x = y = \sqrt{\frac{2m}{n}}$. Thus the maximum value is equal to $f\Big(\sqrt{\frac{2m}{n}}, \sqrt{\frac{2m}{n}}\Big) = \sqrt{2mn}$. But $f(x,y)$ decreases in the intervals $\sqrt{\frac{2m}{n}} \leq x \leq \sqrt{m}$ and $0 \leq y \leq \sqrt{\frac{2m}{n}} \leq \sqrt{m}$. Since $n^2 \geq 4m$, we get that

$$\sqrt{\frac{2m}{n}} \leq \frac{2m}{n} \leq |\lambda_1| \leq \sqrt{m},\ 0 \leq |\lambda_n| \leq \sqrt{\frac{2m}{n}} \leq \sqrt{m}$$

also holds. Thus

$$f\Big(|\lambda_1|, |\lambda_n|\Big) \leq f\Big(\frac{2m}{n}, \sqrt{\frac{2m}{n}}\Big) \leq f\Big(\sqrt{\frac{2m}{n}}, \sqrt{\frac{2m}{n}}\Big)$$

and hence

$$\begin{aligned} \mathcal{E}(G) &\leq \tfrac{2m}{n} + \sqrt{\tfrac{2m}{n}} + \sqrt{(n-2)\Big(2m - \tfrac{2m}{n} - \tfrac{4m^2}{n^2}\Big)} \\ &\leq \sqrt{2mn}. \end{aligned}$$

For the graph $G \simeq \frac{n}{2}K_2$ $(n = 2m)$, $\mathcal{E}(G) = n$ and hence the equality holds. □

Now we show that the above bound is an improvisation of Koolen-Moulton bounds:
Take

$$g(x,y) = x + y + \sqrt{(n-1)(2m - x^2 - y^2)}.$$

Then clearly $f(x,y) \leq g(x,y)$ for all (x,y) in the given region of x and y. Further for $x = \frac{2m}{n}$, we have

$$f(\frac{2m}{n}, y) = \frac{2m}{n} + y + \sqrt{(n-2)(2m - \frac{4m^2}{n^2} - y^2)}.$$

But $f(\frac{2m}{n}, y)$ decreases in the interval $0 \leq y \leq \sqrt{2m - \frac{4m^2}{n^2}}$. Since $n^2 \geq 4m$,

$$0 \leq y \leq \sqrt{\frac{2m}{n}} \leq \sqrt{2m - \frac{4m^2}{n^2}}$$

also holds. Thus $f\Big(\frac{2m}{n}, \sqrt{\frac{2m}{n}}\Big) \leq f\Big(\frac{2m}{n}, 0\Big)$. But

$$f\Big(\frac{2m}{n}, 0\Big) \leq g\Big(\frac{2m}{n}, 0\Big) \quad \text{and} \quad g(\frac{2m}{n}, 0) = \frac{2m}{n} + \sqrt{(n-1)\Big(2m - \frac{4m^2}{n^2}\Big)}$$

which implies that $f\Big(\frac{2m}{n}, \sqrt{\frac{2m}{n}}\Big) \leq g\Big(\frac{2m}{n}, 0\Big)$ and finally

$$\sqrt{\frac{2m}{n}} + \sqrt{(n-2)\Big(2m - \frac{2m}{n} - \frac{4m^2}{n^2}\Big)} \leq \sqrt{(n-1)\Big(2m - \Big(\frac{2m}{n}\Big)^2\Big)}.$$

If $|\lambda_1| = |\lambda_n|$ then

$$\mathcal{E}(G) \leq 2\Big(\frac{2m}{n}\Big) + \sqrt{(n-2)\Big(2m - 2\Big(\frac{2m}{n}\Big)^2\Big)}.$$

As in the above proof when $y = x$,

$$f(x,x) = 2x + \sqrt{(n-2)(2m-2x^2)}.$$

But $f(x,x)$ decreases in the interval $\sqrt{\frac{2m}{n}} \leq x \leq \sqrt{m}$. Since $n^2 \geq 4m$,

$$\sqrt{\frac{2m}{n}} \leq \frac{2m}{n} \leq |\lambda_1| \leq \sqrt{m}$$

also holds. Thus we get $\mathcal{E}(G) \leq 2\Big(\frac{2m}{n}\Big) + \sqrt{(n-2)\Big(2m - 2\Big(\frac{2m}{n}\Big)^2\Big)}$. We claim that this bound is better than Koolen-Moulton bound.

Take

$$g(x,y) = x + y + \sqrt{(n-1)(2m - x^2 - y^2)}.$$

Then clearly $f(x,y) \leq g(x,y)$ for all (x,y) in the given region. Taking $x = \frac{2m}{n}$, we get

$$f(\frac{2m}{n}, y) = \frac{2m}{n} + y + \sqrt{(n-2)(2m - \frac{4m^2}{n^2} - y^2)}.$$

But $f(\frac{2m}{n}, y)$ decreases in the interval $0 \leq y \leq \sqrt{2m - \frac{4m^2}{n^2}}$. Since $n^2 \geq 4m$, the inequality

$$0 \leq y \leq \sqrt{\frac{2m}{n}} \leq \frac{2m}{n} \leq \sqrt{2m - \frac{4m^2}{n^2}}$$

also holds. Thus $f\Big(\frac{2m}{n}, \frac{2m}{n}\Big) \leq f\Big(\frac{2m}{n}, 0\Big)$. But $f\Big(\frac{2m}{n}, 0\Big) \leq g\Big(\frac{2m}{n}, 0\Big)$ and $g(\frac{2m}{n}, 0) = \frac{2m}{n} + \sqrt{(n-1)\Big(2m - \frac{4m^2}{n^2}\Big)}$ which together imply that $f\Big(\frac{2m}{n}, \frac{2m}{n}\Big) \leq g\Big(\frac{2m}{n}, 0\Big)$ and

$$\begin{aligned} \mathcal{E}(G) &\leq \tfrac{4m}{n} + \sqrt{(n-2)\Big(2m - 2\Big(\tfrac{2m}{n}\Big)^2\Big)} \\ &\leq \frac{2m}{n} + \sqrt{(n-1)\Big(2m - \Big(\frac{2m}{n}\Big)^2\Big)}. \end{aligned}$$

Theorem 3.2. *Let G be a graph with n vertices. Then*

$$\mathcal{E}(G) \leq \frac{n}{2}\Big[1 + \sqrt{\frac{n-2}{2}}\Big]. \tag{3.2}$$

Proof. By previous theorem, for $n^2 \geq 4m$, we have

$$\mathcal{E}(G) \leq 2\Big(\frac{2m}{n}\Big) + \sqrt{(n-2)\Big(2m - 2\Big(\frac{2m}{n}\Big)^2\Big)}.$$

We maximize the right hand side in terms of m. Let

$$g(x) = \frac{4x}{n} + \sqrt{(n-2)\Big(2x - 2\Big(\frac{2x}{n}\Big)^2\Big)}.$$

Then clearly

$$g'(x) = \frac{4}{n} + \sqrt{n-2}\,\frac{2 - \frac{16x}{n^2}}{2\sqrt{2x - 2(\frac{2x}{n})^2}}.$$

For maxima, from $g'(x) = 0$, we get

$$\frac{4}{n} = \frac{\sqrt{n-2}\left(\frac{8x}{n^2} - 1\right)}{\sqrt{2x - 2(\frac{2x}{n})^2}}$$

and hence we obtain $x = \frac{n^2}{8}$. At this point, the value of the function is obtained as

$$\begin{aligned} g\left(\frac{n^2}{8}\right) &= \frac{n}{2} + \sqrt{n-2}\sqrt{\frac{n^2}{4} - \frac{n^2}{8}} \\ &= \frac{n}{2}\left[1 + \sqrt{\frac{n-2}{2}}\right]. \end{aligned}.$$

□

Note that for all classes of graphs with $n^2 \geq 2m$, Koolen and Moulton bound is better than the bound in (3.1).

Clearly, the bound in (3.2) improves the Koolen and Moulton bound in terms of n:

$$\mathcal{E}(G) \leq \frac{n}{2}\left[1 + \sqrt{n}\right].$$

Illustrations

The following table illustrates that the upper bound (3.1) is better than Koolen and Moulton bounds (1.1) for graphs with $n^2 \geq 4m$.

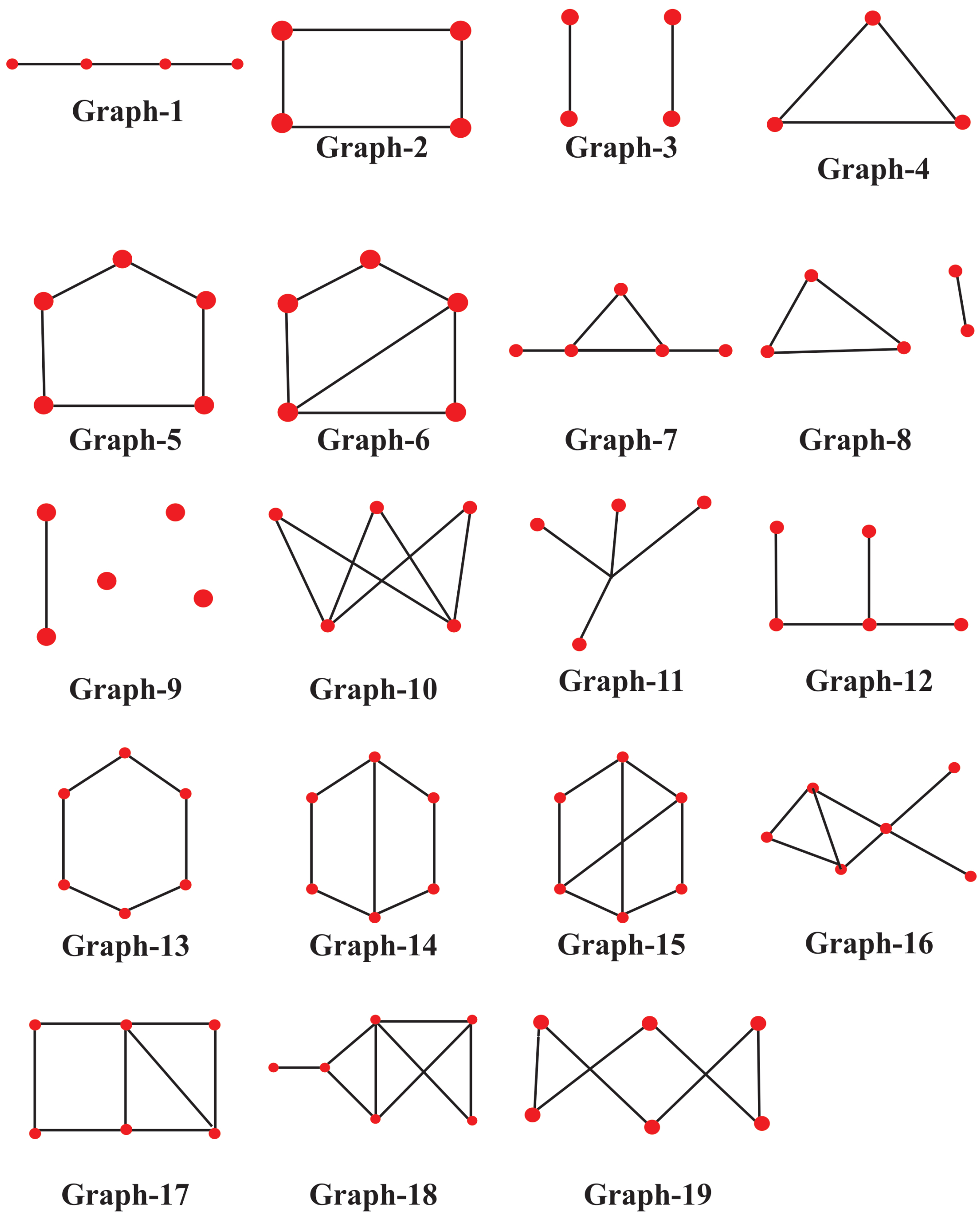
Graph-1
Graph-2
Graph-3
Graph-4
Graph-5
Graph-6
Graph-7
Graph-8
Graph-9
Graph-10
Graph-11
Graph-12
Graph-13
Graph-14
Graph-15
Graph-16
Graph-17
Graph-18
Graph-19

FIGURE-1

Graphs	m	n	Energy	K.M. Bound	Improved K.M. Bound	Improved K.M. Bound for $\mid \lambda_1 \mid = \mid \lambda_n \mid$
Graph 1	3	4	4.472136	4.854102	4.8460652	
Graph 2	4	4	4	5.4641016	5.4142136	4
Graph 3	2	4	4	4	4	4
Graph 4	3	4	4	4.854102	4.846052	
Graph 5	5	5	6.472136	6.8989795	6.8783152	
Graph 6	6	5	6.340173	7.3959984	7.3433059	
Graph 7	5	5	5.841693	6.8989795	6.8783152	
Graph 8	4	5	6	6.2647615	6.2590236	
Graph 9	1	5	2	3.112932	3.1109165	3.0449944
Graph 10	6	5	4.8990	7.3959984	7.3433059	6
Graph 11	4	5	4	6.2647615	6.2590236	6.1393877
Graph 12	4	5	5.2263	6.2647615	6.2590236	6.1393877
Graph 13	6	6	8	8.324555	8.313193	8
Graph 14	7	6	7.6568542	8.8738056	8.1943351	
Graph 15	8	6	6.9282	9.3333	9.2885363	8
Graph 16	7	6	7.4641016	8.8738056	8.8497351	
Graph 17	8	6	8.1826945	9.333	9.2885363	
Graph 18	8	6	8.0457827	9.333	9.2885363	
Graph 19	7	6	7.6569	8.8738056	8.8497351	8.1943351

Table-1.1

§4. Lower bounds for energy

Theorem 4.1. *Let G be a non-singular graph. Then*

$$\mathcal{E}(G) \geq n|detA|^{\frac{1}{n}}.$$

Equality holds iff G is $\frac{n}{2}K_2$ $(n = 2m)$.

Proof. For the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of G (or its adjacency matrix A), it is well known that $|\det(A)| = |\lambda_1\lambda_2 \cdots \lambda_n|$. Since G is non singular, we know that $|\det(A)| \neq 0$.

Applying the Cauchy-Schwarz inequality for n terms $a_i = \sqrt{|\lambda_i|}$ and $b_i = 1$ for all $i = 1,\ 2,\ \cdots,\ n$, we have

$$\sum_{i=1}^{n} \sqrt{|\lambda_i|} \leq \sqrt{\Big(\sum_{i=1}^{n} |\lambda_i|\Big)n}.$$

Hence we get

$$\sqrt{\mathcal{E}(G)} \geq \frac{\sum_{i=1}^{n}\sqrt{|\lambda_i|}}{\sqrt{n}}.$$

But

$$\frac{\sqrt{|\lambda_1|}+\sqrt{|\lambda_2|}+\cdots+\sqrt{|\lambda_n|}}{n} \geq \left(\sqrt{|\lambda_1\lambda_2\cdots\lambda_n|}\right)^{\frac{1}{n}}$$

implying that

$$\sqrt{\mathcal{E}(G)} \geq \frac{n\left(\sqrt{|\lambda_1\lambda_2\cdots\lambda_n|}\right)^{\frac{1}{n}}}{\sqrt{n}}.$$

Therefore $\mathcal{E}(G) \geq n|detA|^{\frac{1}{n}}$.

For the graph $G \simeq \frac{n}{2}K_2$ $(n = 2m), \det(A) = 1$ and hence the equality is true. □

Theorem 4.2. *Let G be a graph with $n \geq 1$ vertices and m edges and let $2m \geq n$. Then*

$$\mathcal{E}(G) \geq \frac{2m}{n} + \frac{(n-1)|\det(A)|^{\frac{1}{(n-1)}}}{\left(\frac{2m}{n}\right)^{\frac{1}{n-1}}}.$$

*Further the equality holds if **(i)** G is isomorphic to K_n **(ii)** G is isomorphic to $\frac{n}{2}K_2$ $(n = 2m)$.*

Proof. Using the Cauchy-Schwarz inequality for $\sqrt{|\lambda_2|}, \sqrt{|\lambda_3|}, \cdots, \sqrt{|\lambda_n|}$ and $(\underbrace{1, 1, \cdots, 1}_{n-2\ times})$, we have

$$\sum_{i=2}^{n}\sqrt{|\lambda_i|} \leq \sqrt{\left(\sum_{i=2}^{n}|\lambda_i|\right)(n-1)}$$

and equivalently

$$\sum_{i=2}^{n}\sqrt{|\lambda_i|} \leq \sqrt{\left(\mathcal{E}(G)-|\lambda_1|\right)(n-1)}$$

and hence

$$\sqrt{\mathcal{E}(G)-|\lambda_1|} \geq \frac{\sum_{i=2}^{n}\sqrt{|\lambda_i|}}{\sqrt{n-1}}.$$

But

$$\frac{\sqrt{|\lambda_2|}+\sqrt{|\lambda_3|}+\cdots+\sqrt{|\lambda_n|}}{n-1} \geq \left(\sqrt{|\lambda_2\lambda_3\cdots\lambda_n|}\right)^{\frac{1}{n-1}}$$

implying that

$$\sqrt{\mathcal{E}(G)-|\lambda_1|} \geq \frac{(n-1)\left(\sqrt{|\lambda_2\lambda_3\cdots\lambda_n|}\right)^{\frac{1}{(n-1)}}}{\sqrt{n-1}}.$$

Hence we get

$$\begin{aligned}\mathcal{E}(G) &\geq |\lambda_1| + (n-1)\left(|\lambda_2\lambda_3\cdots\lambda_n|\right)^{\frac{1}{n-1}} \\ &= |\lambda_1| + (n-1)\left(\frac{|\det(A)|}{|\lambda_1|}\right)^{\frac{1}{(n-1)}}\end{aligned}.$$

Let $|\lambda_1| = x$ and choose

$$f(x) = x + (n-1)\Big(\frac{|\det(A)|}{x}\Big)^{\frac{1}{(n-1)}}.$$

Then

$$f'(x) = 1 - \frac{|\det(A)|^{\frac{1}{(n-1)}}}{x^{\frac{n}{(n-1)}}}$$

and

$$f''(x) = \frac{n|\det(A)|^{\frac{1}{(n-1)}}}{(n-1)x^{\frac{(2n-1)}{(n-1)}}}.$$

For maxima or minima, the equation $f'(x) = 0$ gives the value $x = |\det(A)|^{\frac{1}{n}}$. At this point,

$$f''(x) = \frac{n}{(n-1)}|\det(A)|^{\frac{-1}{n}} \geq 0,\ n \neq 1.$$

Thus the function $f(x)$ is minimum at $x = |\det(A)|^{\frac{1}{n}}$ and the minimum value is equal to $f(|det(A)|^{\frac{1}{n}}) = n|det(A)|^{\frac{1}{n}}$. But

$$\begin{aligned} \tfrac{2m}{n} &= \tfrac{|\lambda_1|^2+|\lambda_2|^2+\cdots+|\lambda_n|^2}{n} \\ &\geq \tfrac{|\lambda_1|+|\lambda_2|+\cdots+|\lambda_n|}{n} \\ &\geq |\lambda_1\lambda_2\cdots\lambda_n|^{\frac{1}{n}}. \end{aligned}$$

This implies that $|det(A)|^{\frac{1}{n}} \leq \frac{2m}{n}$ since $2m \geq n$ and $\frac{2m}{n} \leq |\lambda_1|$. Therefore the function is increasing in the interval $|detA|^{\frac{1}{n}} \leq \frac{2m}{n} \leq |\lambda_1| \leq \sqrt{2m}$. Therefore

$$f(|\lambda_1|) \geq f\Big(\frac{2m}{n}\Big)$$

and

$$\mathcal{E}(G) \geq \frac{2m}{n} + \frac{(n-1)|\det(A)|^{\frac{1}{(n-1)}}}{\Big(\frac{2m}{n}\Big)^{\frac{1}{(n-1)}}}.$$

Also

(*i*) If G is isomorphic to K_n, then $|det(A)| = n-1$ and $\frac{2m}{n} = n-1$, and hence $\mathcal{E}(G) = 2(n-1)$.

(*ii*) If G is isomorphic to $\dfrac{n}{2}K_2$ $(n = 2m)$, then the eigenvalues are ± 1 (with multiplicity $\frac{n}{2}$ each) implying that $\mathcal{E}(G) = n$. □

Minimizing the lower bound $\mathcal{E}(G) \geq \dfrac{2m}{n} + \dfrac{(n-1)|\det(A)|^{\frac{1}{(n-1)}}}{\Big(\frac{2m}{n}\Big)^{\frac{1}{(n-1)}}}$ in terms of n gives $\mathcal{E}(G) \geq n|detA|^{\frac{1}{n}}$.

Theorem 4.3. *Let G be a graph with m edges and n (> 3) vertices such that $2m \geq n$. Then*

$$\mathcal{E}(G) \geq 2\Big(\frac{2m}{n}\Big) + \frac{(n-2)|\det(A)|^{\frac{1}{n-2}}}{\Big(\frac{2m}{n}\Big)^{\frac{2}{n-2}}}.$$

Equality holds iff G is isomorphic to $\frac{n}{2}K_2$ $(n = 2m)$ or $K_{n,n}$.

Proof. For $n-2$ entries of eigenvalues $(\sqrt{|\lambda_2|},\ \sqrt{|\lambda_3|},\ \cdots,\ \sqrt{|\lambda_{n-1}|})$ and $(\underbrace{1,1,\cdots,1}_{n-2\ times})$, apply the Cauchy-Schwarz inequality

$$\sum_{i=2}^{n-1}\sqrt{|\lambda_i|}\le\sqrt{\Big(\sum_{i=2}^{n-1}|\lambda_i|\Big)(n-2)}$$

and hence

$$\sum_{i=2}^{n-1}\sqrt{|\lambda_i|}\le\sqrt{\Big(\mathcal{E}(G)-|\lambda_1|-|\lambda_n|\Big)(n-2)}$$

implying that

$$\sqrt{\mathcal{E}(G)-|\lambda_1|-|\lambda_n|}\ge\frac{\sum_{i=2}^{n-1}\sqrt{|\lambda_i|}}{\sqrt{n-2}}.$$

But the arithmetic mean is greater than or equal to the geometric mean, we get

$$\sqrt{\mathcal{E}(G)-|\lambda_1|-|\lambda_n|}\ge\frac{(n-2)\Big(\sqrt{|\lambda_2\lambda_3\cdots\lambda_{n-1}|}\Big)^{\frac{1}{n-2}}}{\sqrt{n-2}}$$

and therefore we obtain

$$\mathcal{E}(G)\ge|\lambda_1|+|\lambda_n|+(n-2)\Big(\frac{|\det(A)|}{|\lambda_1||\lambda_n|}\Big)^{\frac{1}{n-2}}.$$

Put $|\lambda_1|=x$ and $|\lambda_n|=y$. We minimize the right side of the above function. Let

$$f(x,y)=x+y+(n-2)\Big(\frac{|\det(A)|}{xy}\Big)^{\frac{1}{n-2}}.$$

Then

$$f_x=1-|\det(A)|^{\frac{1}{n-2}}y(xy)^{\frac{-(n-1)}{n-2}},\quad f_y=1-|\det(A)|^{\frac{1}{n-2}}x(xy)^{\frac{-(n-1)}{n-2}},$$

$$f_{xx}=|\det(A)|^{\frac{1}{n-2}}\Big(\frac{n-1}{n-2}\Big)y^2(xy)^{\frac{-(2n-3)}{n-2}},\quad f_{yy}=|\det(A)|^{\frac{1}{n-2}}\Big(\frac{n-1}{n-2}\Big)x^2(xy)^{\frac{-(2n-3)}{n-2}}$$

and

$$f_{xy}=|\det(A)|^{\frac{1}{n-2}}\frac{(xy)^{\frac{-(n-1)}{n-2}}}{n-2}.$$

To find the maxima or minima, use the equations $f_x=0$ and $f_y=0$ which gives

$$xy^{\frac{1}{n-1}}=|\det(A)|^{\frac{1}{n-1}}\ \text{ and }\ yx^{\frac{1}{n-1}}=|\det(A)|^{\frac{1}{n-1}}.$$

Solving these equations gives $x=|det(A)|^{\frac{1}{n}}$ and $y=|det(A)|^{\frac{1}{n}}$. At this point, the values of f_{xx}, f_{yy}, f_{xy} and $\Delta=f_{xx}f_{yy}-(f_{xy})^2$ are

$$f_{xx}=f_{yy}=\Big(\frac{n-1}{n-2}\Big)|\det(A)|^{\frac{-1}{n}},$$

$$f_{xy}=\frac{1}{n-2}|\det(A)|^{\frac{-1}{n}}$$

and

$$\Delta=\Big(\frac{n}{n-2}\Big)|\det(A)|^{\frac{-2}{n}}\ge 0,$$

for all $n \neq 2$. The minimum value is then equal to

$$f\Big(|det(A)|^{\frac{1}{n}},\ |det(A)|^{\frac{1}{n}}\Big) = n|det(A)|^{\frac{1}{n}}.$$

But $|\det(A)|^{\frac{1}{n}} \leq \frac{2m}{n} \leq |\lambda_1| \leq \sqrt{2m}$ and $0 \leq |\lambda_n| \leq |\det(A)|^{\frac{1}{n}} \leq \frac{2m}{n} \leq \sqrt{2m}$. For $2m \geq n$, $f(x,y)$ increases in the intervals $|\det(A)|^{\frac{1}{n}} \leq \frac{2m}{n} \leq x \leq \sqrt{2m}$ and $0 \leq y \leq |\det(A)|^{\frac{1}{n}} \leq \frac{2m}{n} \leq \sqrt{2m}$, i.e., $f(x,y)$ increases in the intervals $|\det(A)|^{\frac{1}{n}} \leq \frac{2m}{n} \leq |\lambda_1| \leq \sqrt{2m}$ and $0 \leq |\lambda_n| \leq |\det(A)|^{\frac{1}{n}} \leq \frac{2m}{n} \leq \sqrt{2m}$. Thus

$$\begin{aligned} f\Big(|\lambda_1|, |\lambda_n|\Big) &\geq f\Big(\tfrac{2m}{n}, \tfrac{2m}{n}\Big) \\ &\geq f\Big(\tfrac{2m}{n}, |\det(A)|^{\frac{1}{n}}\Big) \\ &\geq f\Big(|\det(A)|^{\frac{1}{n}}, |\det(A)|^{\frac{1}{n}}\Big). \end{aligned}$$

Hence we get

$$\mathcal{E}(G) \geq 2\Big(\frac{2m}{n}\Big) + \frac{(n-2)|\det(A)|^{\frac{1}{n-2}}}{\Big(\frac{2m}{n}\Big)^{\frac{2}{n-2}}}.$$

Also

(i) If G is isomorphic to $\dfrac{n}{2}K_2$ $(n = 2m)$, then G has eigenvalues ± 1 (with multiplicity $\frac{n}{2}$ each) implying $\mathcal{E}(G) = n$.

(ii) If G is isomorphic to $K_{n,n}$, then G has eigenvalues $\pm n$ and 0 with multiplicity $2n-2$ giving $\mathcal{E}(G) = 2n$. □

Note that we can now prove that this bound improves the McClelland bound: Consider the function

$$\phi(m) = 2\Big(\frac{2m}{n}\Big) + (n-2)\frac{|\det(A)|^{\frac{1}{(n-2)}}}{\frac{2m}{n}^{\frac{2}{(n-2)}}} - \sqrt{2m + n(n-1)|\det(A)|^{\frac{2}{n}}}.$$

For a fixed n, we show that $\phi'(m) > 0$ and $\phi''(m) > 0$ by differentiating in terms of m:

$$\phi'(m) = \frac{4}{n} - \frac{4|\det(A)|^{\frac{1}{(n-2)}}}{n\Big(\frac{2m}{n}\Big)^{\frac{n}{n-2}}} - \frac{1}{\sqrt{2m + n(n-1)|\det(A)|^{\frac{2}{n}}}}.$$

Since $|\det(A)|^{\frac{1}{n}} \leq \dfrac{2m}{n}$, we get

$$\frac{|\det(A)|^{\frac{1}{n-2}}}{\Big(\frac{2m}{n}\Big)^{\frac{n}{n-2}}} \leq \frac{1}{n^2}$$

and as $2m \geq n$,

$$\frac{1}{\sqrt{2m + n(n-1)|\det(A)|^{\frac{1}{n}}}} \leq \frac{1}{n^2}.$$

Thus $\phi'(x) > 0$. Also

$$\phi''(m) = \frac{4|\det(A)|^{\frac{1}{(n-2)}}}{(n-2)\left(\frac{2m}{n}\right)^{\frac{2(n-1)}{n-2}}} + \frac{1}{(2m + n(n-1)|\det(A)|^{\frac{2}{n}})^{\frac{3}{2}}}.$$

Clearly $\phi''(m) > 0$. Thus $\phi(m)$ is an increasing function which implies that

$$2\left(\frac{2m}{n}\right) + (n-2)\frac{|\det(A)|^{\frac{1}{n-2}}}{\frac{2m}{n}^{\frac{2}{n-2}}} \geq \sqrt{2m + n(n-1)|\det(A)|^{\frac{1}{n}}}.$$

The above bound in Theorem 4.3 can be minimized by differentiating with respect to m to get the lower bound for $\mathcal{E}(G)$ in terms of n:

$$\mathcal{E}(G) \geq n|detA|^{\frac{1}{n}}.$$

Table−1.2 below gives us an improvisation of the lower bound for the graph classes in Figure−1

Graphs	m	n	Energy	McClelland Bound	Improved McClelland Bound
Graph 1	3	4	4.472136	4.2426407	4.3333
Graph 2	4	4	4	2.8284271	4
Graph 3	2	4	4	4	4
Graph 4	3	4	4	2.4494897	3
Graph 5	5	5	6.472136	6.0324256	6.3811016
Graph 6	6	5	6.340173	3.4641016	4.8
Graph 7	5	5	5.841693	3.1622777	4
Graph 8	4	5	6	5.8643123	5.9630236
Graph 10	6	5	4.8990	3.4641016	4.8
Graph 11	4	5	4	2.8284271	3.2
Graph 12	4	5	5.2263	2.8284271	3.2
Graph 13	6	6	8	7.7215304	8
Graph 14	7	6	7.6568542	6.6332496	7.2852813
Graph 15	8	6	6.9282	4	5.333333
Graph 16	7	6	7.4641016	3.7416574	4.6666667
Graph 17	8	6	8.1826945	6.78233	7.7828231
Graph 18	8	6	8.0457827	4	5.3333333
Graph 19	7	6	7.6569	6.6332496	7.2852813

Table-1.2

§5. Brief summary and conclusion

In this paper, we find some lower and upper bounds for the energy of graphs. Attempts are continuously being made by researchers to improve these bounds. The lower and upper bounds obtained in this paper are improved versions of McClelland and Koolen-Moulton bounds.

Acknowledgements The last author is supported by the Research Fund of Uludag University, (KUAP) F-2015/17.

References

[1] D. Cvetković and I. Gutman (eds.), *Applications of Graph Spectra*, Mathematical Institution, Belgrade, (2009)

[2] D. Cvetković and I. Gutman (eds.), *Selected Topics on Applications of Graph Spectra*, Mathematical Institute Belgrade, (2011)

[3] A. Graovac, I. Gutman and N. Trinajstić, *Topological Approach to the Chemistry of Conjugated Molecules*, Springer, Berlin, **4**, (1977)

[4] I. Gutman, *The energy of a graph*, Ber. Math-Statist. Sekt. Forschungszentrum Graz, **103**, 1-22, (1978)

[5] I. Gutman, *Algebraic Combinatorics and Applications*, in The energy of a graph: Old and New Results, edtd. by A. Betten, A. Kohnert, R. Laue, A. Wassermann, Springer, Berlin, 196-211, (2001)

[6] I. Gutman and O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin, (1986)

[7] H. Liu, M. Lu and F. Tian, Some upper bounds for the energy of graphs Journal of Mathematical Chemistry, **41 (1)**, 45-57, (2007)

[8] K. C. Das, S. A. Mojallal and I.Gutman, *Improving the McCelland inequality for energy*, MATCH Commun. Math. Comput. Chem., **70**, 663-668, (2013)

[9] J. H. Koolen, V. Moulton and I. Gutman, *Improving the McCelland inequality for π-electron energy*, Chem. Phys. Lett., **320**, 213-216, (2000)

[10] J. H. Koolen and V. Moulton, *Maximal energy of graphs*, Adv. Appl. Math, **26**, 47-52, (2001)

[11] J. H. Koolen and V. Moulton, *Maximal energy of bipartite graphs*, Graphs and Combinatorics, **19**, 131-135, (2003)

[12] B. J. McClelland, *Properties of the latent root of a matrix: The estimation of π-electron energies*, J. Chem. Phys., **54**, 640-643, (1971)

Scientia Magna
Vol. 13 (2018), No. 1, 63-71

Bipolar valued fuzzy G-subalgebras and bipolar valued fuzzy ideals of G-algebra

Deena Al-Kadi

Department of Mathematics and Statistic, Taif University
Taif, Saudi Arabia
E-mail: dak12le@hotmail.co.uk

Abstract In this paper, we study G-subalgebras and ideals of G-algebra based on the concept of bipolar valued fuzzy sets. We give characterizations for bipolar valued fuzzy G-subalgebras. We investigate the relation between a bipolar valued fuzzy G-subalgebras and a bipolar valued fuzzy ideal of G-algebra.

Keywords G-algebra, fuzzy sets, bipolar valued fuzzy sets.
2010 Mathematics Subject Classification 03G25, 06F35.

§1. Introduction

The concept of fuzzy sets was introduced by Zadeh [1] as a method for representing uncertainty. A fuzzy set A in a space of points X is characterized by a membership function which is denoted by $f_A(x)$ for any point x in X. This function is associated with a real number ranging between 0 and 1. The value of the function f_A at x represents the membership value of x in A. Extensions of fuzzy sets were proposed to treat imprecision such as interval valued fuzzy sets [2], intuitionistic fuzzy sets [3], L-fuzzy sets [4] and bipolar valued fuzzy sets [5], [6]. In [7], the authors investigates the similarities and differences between the representations of interval-valued fuzzy sets, intuitionistic fuzzy sets and bipolar-valued fuzzy sets. Interval valued fuzzy sets represents the membership degree by an interval value that reflects the uncertainty in assigning membership degrees. Intuitionistic fuzzy sets describes membership degree with a membership degree and a non-membership degree. In bipolar valued fuzzy sets, the membership degrees represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter property. The membership degree in a bipolar valued fuzzy set range between the interval $[0, 1]$ and $[-1, 0]$. If the membership degree is 0 this means that the elements are irrelevant to the corresponding property, if the membership degree is in the interval $(0, 1]$ then elements satisfy the property and if the membership lies in $[-1, 0]$ then the elements satisfy implicit counter property. Many studies have been done on bipolar fuzzy sets. For example, in [8] bipolar fuzzy structure of BG-subalgebras has been studied. Also, the bipolar fuzzy subalgebras and closed ideals of BCH-algebra and BCK/BCI-algebras has been introduced in [9], [10] respectively. More precisely, fuzzy extensions has been investigated in G-algebra. In [11] intuitionistic fuzzy G-subalgebras of G-algebras is considered and some characterization of intuitionistic fuzzy G-algebras are given. In [12] the L-fuzzification of G-subalgebras are considered and a characterization of L-fuzzy G-algebras is given.

In this paper, we complete the studies of fuzzy extensions in G-algebra. We study G-subalgebras and closed ideals of G-algebra based on bipolar valued fuzzy sets. The paper is organized as follows: In Section 2, we give basic definitions and propositions on G-algebra and fuzzy sets. In Section 3, we introduce bipolar valued fuzzy G-subalgebras and investigate some of its properties. Moreover, we give characterizations for bipolar valued fuzzy G-subalgebras. In Section 4, we introduce the notions of bipolar valued fuzzy ideals/closed ideals of G-algebra and the relations between bipolar valued fuzzy G-subalgebras and bipolar valued fuzzy ideals/closed ideals of G-algebra has been investigated.

§2. Preliminaries

Definition 2.1 [13, Definition 2.1] *A G-algebra is a system $(X, *, 0)$ where X is a non-empty set, $*$ a binary operation and 0 a constant satisfying the following axioms:*

(1) $x * x = 0$,

(2) $x * (x * y) = y$, *for all* $x, y \in X$.

Proposition 2.2 [13, Proposition 2.1] *If $(X, *, 0)$ is a G-algebra, then the following conditions hold:*

(1) $x * 0 = x$,

(2) $0 * (0 * x) = x$, *for any* $x \in X$.

Proposition 2.3 [13, Proposition 2.2] *Let $(X, *, 0)$ be a G-algebra. Then, the following conditions hold for any $x, y \in X$,*

(1) $(x * (x * y)) * y = 0$,

(2) $x * y = 0 \Longrightarrow x = y$,

(3) $0 * x = 0 * y \Longrightarrow x = y$.

Theorem 2.4 [13, Theorem 2.6] *Let X be a G-algebra. Then the following are equivalent, for all $x, y, z \in X$:*

(1) $(x * y) * z = (x * z) * y$.

(2) $(x * y) * (x * z) = z * y$.

Theorem 2.5 [13, Theorem 2.7] *Let X be a G-algebra. Then the following are equivalent, for all $x, y, z \in X$:*

(1) $(x * y) * (x * z) = z * y$.

(2) $(x * z) * (y * z) = x * y$.

Lemma 2.6 [13, Lemma 2.1] *Let $(X, *, 0)$ be a G-algebra. Then $a * x = a * y$ implies $x = y$, for any $a, x, y \in X$.*

Definition 2.7 [13, Definition 3.3] *A G-algebra $(X, *, 0)$ satisfying $(x * y) * (z * t) = (x * z) * (y * t)$ for any x, y, z and $t \in X$ is called a medial G-algebra.*

Lemma 2.8 [13, Lemma 3.1] *If $(X, *, 0)$ is a medial G-algebra then for any x, y and $z \in X$ the following holds:*

(1) $(x * y) * x = 0 * y$.

(2) $x * (y * z) = (x * y) * (0 * z)$.

(3) $x * (y * z) = (x * z) * y$.

A non-empty subset S of a G-algebra X is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$.

Definition 2.9 [1] *Let X be the collection of objects x then a fuzzy set A in X is defined as: $A = \{(x, f_A(x)) \mid x \in X\}$ where $f_A(x)$ is called the membership values of x in A and $f_A(x) \in [0, 1]$.*

Definition 2.10 [10] *Let X be non-empty set with objects x. A bipolar fuzzy set f in X is an object having the form $f = \{(x, f^+(x), f^-(x)) \mid x \in X\}$, where $f^+ : X \to [0, 1]$ and $f^- : X \to [-1, 0]$ are maps.*

If $f^+(x) \neq 0$ and $f^-(x) = 0$ then x is regarded as having only positive satisfaction degree to the property corresponding to the bipolar fuzzy set f. If $f^+(x) = 0$ and $f^-(x) \neq 0$ then x does not satisfy the property of f and satisfies the counter property of f. For an element x, it possible to have $f^+(x) \neq 0$ and $f^-(x) \neq 0$ which means that the membership function of the property overlaps of its counter property over some elements of X.

If $f_1 = \{(x, f_1^+(x), f_1^-(x))\}$ and $f_2 = \{(x, f_2^+(x), f_2^-(x))\}$ are two bipolar fuzzy sets on X then the intersection and the union of two bipolar fuzzy sets are given as follows:

$$\begin{aligned} f_1 \cap f_2 &= (f_1^+(x) \cap f_2^+(x), f_1^-(x) \cap f_2^-(x)) \\ &= \{\min\{f_1^+(x), f_2^+(x)\}, \max\{f_1^-(x), f_2^-(x)\}\}, \\ f_1 \cup f_2 &= (f_1^+(x) \cup f_2^+(x), f_1^-(x) \cup f_2^-(x)) \\ &= \{\max\{f_1^+(x), f_2^+(x)\}, \min\{f_1^-(x), f_2^-(x)\}\}. \end{aligned}$$

§3. Bipolar fuzzy G-subalgebra

In this section we introduce bipolar fuzzy G-subalgebras and we give characterizations of it with examples. For the rest of the paper we will denote the bipolar valued fuzzy set $f = \{(x, f^+(x), f^-(x)) \mid x \in X\}$ by $f = (f^+, f^-)$ and X for the system $(X, *, 0)$ for shortness.

Definition 3.1 *Let f be a fuzzy set in a G-algebra X. Then f is called a fuzzy G-subalgebra of X if, $f(x * y) \geq \min\{f(x), f(y)\}$, for all $x, y \in X$, where $f(x)$ is the membership value of x in X.*

Definition 3.2 *Let X be a G-algebra and let f be a bipolar fuzzy set in X. Then the f is a bipolar fuzzy G-subalgebra if, for all $x, y \in X$ the following are satisfied:*

(1) $f^+(x*y) \geq \min\{f^+(x), f^+(y)\}$,

(2) $f^-(x*y) \leq \max\{f^-(x), f^-(y)\}$.

Example 3.3 *Consider the G-algebra X where* $X = \{0, 1, 2\}$ *and* $*$ *is defined by Cayley Table 1. Let* $f = (f^+, f^-)$ *be a bipolar fuzzy set in X where* f^+ *and* f^- *are defined as follows:*

Table 1:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

$$f^+(x) = \begin{cases} 0.66, & x \in \{0, 2\}; \\ 0.51, & x = 1. \end{cases}$$

$$f^-(x) = \begin{cases} -0.57, & x \in \{0, 2\}; \\ -0.36, & x = 1. \end{cases}$$

Then f is not a bipolar fuzzy G-subalgebra of X as $f^+(0*2) = f^+(1) = 0.51$ *whereas* $\min\{f^+(0), f^+(2)\} = \min\{0.66, 0.66\} = 0.66$ *i.e.* $f^+(0*2) \ngeq \min\{f^+(0), f^+(2)\}$.

Example 3.4 *Consider the G-algebra X where* $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ *and* $*$ *is defined by Cayley Table 2. Let* $f = (f^+, f^-)$ *be bipolar fuzzy set in X where* f^+ *and* f^- *are defined by:*

Table 2:

$*$	0	1	2	3	4	5	6	7
0	0	2	1	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	2	1	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

$$f^+(x) = \begin{cases} 0.74, & x \in \{0, 1, 2, 3\}; \\ 0.52, & \textit{otherwise}. \end{cases}$$

$$f^-(x) = \begin{cases} -0.57, & x \in \{0,1,2,3\}; \\ -0.36, & \text{otherwise.} \end{cases}$$

Then by direct computations we find that f is a bipolar fuzzy G-subalgebra of X.

In the next theorems we give characterizations for a bipolar fuzzy set of a G-algebra to be a bipolar fuzzy G-subalgebra.

Theorem 3.5 *Let A be a non-empty subset of a G-algebra X and let $f = (f^+, f^-)$ be a bipolar fuzzy set in X defined by*

$$f^+(x) = \begin{cases} \gamma_1, & \text{if } x \in A; \\ \gamma_2, & \text{otherwise,} \end{cases}$$

$$f^-(x) = \begin{cases} \delta_1, & \text{if } x \in A; \\ \delta_2, & \text{otherwise,} \end{cases}$$

where $\gamma_1 \geq \gamma_2$ in $[0,1]$ and $\delta_1 \leq \delta_2$ in $[-1,0]$. Then f is a bipolar fuzzy G-subalgebra of X if and only if A is a G-subalgebra of X.

Proof. "$\Rightarrow$". Suppose that f is a bipolar fuzzy G-subalgebra of X. For $x, y \in A$ we have $f^+(x * y) \geq \min\{f^+(x), f^+(y)\} = \min\{\gamma_1, \gamma_1\} = \gamma_1$ and $f^-(x * y) \leq \max\{f^-(x), f^-(y)\} = \max\{\delta_1, \delta_1\}$. Hence, $x * y \in A$ and so A is a G-subalgebra.
"$\Leftarrow$". Suppose A is a G-subalgebra of X. We have the following cases:

Case(1) x and $y \in A$. If $x, y \in A$ then $x * y \in A$. Hence $f^+(x * y) = \gamma_1 = \min\{f^+(x), f^+(y)\}$ and $f^-(x * y) = \delta_1 = \max\{f^-(x), f^-(y)\}$.

Case(2) $x \notin A$ or $y \notin A$. If $x \notin A$ or $y \notin A$ then $f^+(x * y) \geq \gamma_2 = \min\{f^+(x), f^+(y)\}$ and $f^-(x * y) \leq \delta_2 = \max\{f^-(x), f^-(y)\}$. Hence, in both cases, f is a bipolar fuzzy G-subalgebra of X.

□

Theorem 3.6 *Let X be a medial G-algebra and A a non-empty subset of X. If $f = (f^+, f^-)$ is a bipolar fuzzy set in X defined by:*

$$f^+(x) = \begin{cases} \gamma_1, & \text{if for } a \in A,\ x * a = (0 * a) * (0 * x); \\ \gamma_2, & \text{otherwise,} \end{cases}$$

$$f^-(x) = \begin{cases} \delta_1, & \text{if for } a \in A,\ x * a = (0 * a) * (0 * x); \\ \delta_2, & \text{otherwise,} \end{cases}$$

where $x \in X$, $\gamma_1 \geq \gamma_2$ in $[0,1]$ and $\delta_1 \leq \delta_2$ in $[-1,0]$. Then f is a bipolar fuzzy G-subalgebra of X.

Proof. Let $x, y \in X$ and $a \in A$.Then $(x * y) * a = 0 * (0 * (x * y)) * a = (0 * a) * (0 * (x * y))$ as X is medial. We consider two cases.

Case(1) If $x * a = (0 * a) * (0 * x)$ and $y * a = (0 * a) * (0 * y)$ and so $f^+(x * y) = \gamma_1 = \min\{f^+(x), f^+(y)\}$ and $f^-(x * y) = \delta_1 = \max\{f^-(x), f^-(y)\}$.

Case(2) If $x*a \neq (0*a)*(0*x)$ or $y*a \neq (0*a)*(0*y)$. Then $f^+(x*y) \geq \gamma_2 = \min\{f^+(x), f^+(y)\}$ and $f^-(x * y) \leq \delta_2 = \max\{f^-(x), f^-(y)\}$.

Hence f is a bipolar fuzzy G-subalgebra of a medial G-algebra. □

Proposition 3.7 *Let $f = (f^+, f^-)$ be a bipolar fuzzy G-subalgebra in a G-algebra X. Then, for $x \in X$, $f^+(0)$ is an upper bound of $f^+(x)$ and $f^-(0)$ is a lower bound of $f^-(x)$.*

Proof. Let $x \in X$. Then, as f is a bipolar fuzzy G-subalgebra in X, we have $f^+(0) = f^+(x*x) \geq \min\{f^+(x), f^+(x)\} = f^+(x)$ and $f^-(0) = f^-(x*x) \leq \max\{f^-(x), f^-(x)\} = f^-(x)$. This proves the proposition. □

Theorem 3.8 *Let $f = (f^+, f^-)$ be a bipolar fuzzy G-subalgebra in X. If there exists a sequence x_n in X where n is positive integer such that $\lim_{x\to\infty} f^+(x_n) = 1$ and $\lim_{x\to\infty} f^-(x_n) = -1$, then $f^+(0) = 1$ and $f^-(0) = -1$.*

Proof. From Proposition 3.7, $f^+(0) \geq f^+(x)$, for all $x \in X$ and so $f^+(0) \geq f^+(x_n)$. Therefore, $1 \geq f^+(0) \geq \lim_{x\to\infty} f^+(x_n)$. As $\lim_{x\to\infty} f^+(x_n) = 1$ we have $f^+(0) = 1$. Similarly, we can show that $f^-(0) = -1$. □

Proposition 3.9 *Let $f = (f^+, f^-)$ be a bipolar fuzzy G-subalgebra in X. Then, for all $x \in X$, $f^+(0 * x) \geq f^+(x)$ and $f^-(0 * x) \leq f^-(x)$.*

Proof. For $x \in X$, we have $f^+(0 * x) \geq f^+(x)$ as $f^+(0 * x) \geq \min\{f^+(0), f^+(x)\} = \min\{f^+(x * x), f^+(x)\} \geq \min\{\min\{f^+(x), f^+(x)\}, f^+(x)\} = f^+(x)$. It can be shown that $f^-(0*x) \leq f^-(x)$ similarly. □

Next we show that the intersection of two bipolar fuzzy G-subalgebras is a bipolar fuzzy G-subalgebra.

Theorem 3.10 *Let $f_1 = (f_1^+, f_1^-)$ and $f_2 = (f_2^+, f_2^-)$ be two bipolar fuzzy G-subalgebras in a G-algebra X. Then $f_1 \cap f_2$ is a bipolar fuzzy G-subalgebra in X.*

Proof. Suppose that f_1 and f_2 are bipolar fuzzy G-subalgebras in X and let x and $y \in f_1 \cap f_2$. Then $x, y \in f_1$ and $x, y \in f_2$. We have $f_1 \cap f_2 = (f_1^+ \cap f_2^+, f_1^- \cap f_2^-)$ where

$$\begin{aligned} f_1^+ \cap f_2^+ &= \min\{f_1^+(x * y), f_2^+(x * y)\} \\ &\geq \min\{\min\{f_1^+(x), f_1^+(y)\}, \min\{f_2^+(x), f_2^+(y)\}\} \\ &= \min\{\min\{f_1^+(x), f_2^+(x)\}, \min\{f_1^+(y), f_2^+(y)\}\} \\ &= \min\{f_1^+ \cap f_2^+(x), f_1^+ \cap f_2^+(y)\}, \end{aligned}$$

$$\begin{aligned} f_1^- \cap f_2^- &= \max\{f_1^-(x * y), f_2^-(x * y)\} \\ &\leq \max\{\max\{f_1^-(x), f_1^-(y)\}, \max\{f_2^-(x), f_2^-(y)\}\} \\ &= \max\{\max\{f_1^-(x), f_2^-(x)\}, \max\{f_1^-(y), f_2^-(y)\}\} \\ &= \max\{f_1^- \cap f_2^-(x), f_1^- \cap f_2^-(y)\}. \end{aligned}$$

□

The union of two bipolar fuzzy G-subalgebras need not be a bipolar fuzzy G-subalgebra as shown in the next example.

Example 3.11 *Consider the G-algebra X where $X = \{0,1,2,3,4,5,6,7\}$ with Cayley Table 2. Let $f_1 = (f_1^+, f_1^-)$ and $f_2 = (f_2^+, f_2^-)$ be bipolar fuzzy sets in X where f_1^+, f_1^-, f_2^+ and f_2^- are defined as follows:*

$$f_1^+(x) = \begin{cases} 0.74, & x \in \{0,1,2,3\}; \\ 0.52, & \textit{otherwise}, \end{cases}$$

$$f_1^-(x) = \begin{cases} -0.57, & x \in \{0,3\}; \\ -0.11, & \textit{otherwise}. \end{cases}$$

$$f_2^+(x) = \begin{cases} 0.84, & x \in \{0,4\}; \\ 0.30, & \textit{otherwise}, \end{cases}$$

$$f_2^-(x) = \begin{cases} -0.30, & x \in \{0,1,2,3\}; \\ -0.20, & \textit{otherwise}. \end{cases}$$

Using Theorem 3.5, we know that f_1 and f_2 are bipolar fuzzy G-subalgebras. We have $f_1^+ \cup f_2^+(3 * 4) = \max\{f_1^+(3*4), f_2^+(3*4)\} = \max\{f_1^+(7), f_2^+(7)\} = \max\{0.52, 0.30\} = 0.52$ whereas $\min\{f_1^+ \cup f_2^+(3), f_1^+ \cup f_2^+(4)\} = \min\{0.74, 0.84\} = 0.74$ and hence $f_1^+ \cup f_2^+(3*4) = 0.52 \ngeq 0.74 = \min\{f_1^+ \cup f_2^+(3), f_1^+ \cup f_2^+(4)\}$. This proves that the union of two bipolar fuzzy G-subalgebras need not be a bipolar fuzzy G-subalgebra.

§4. Bipolar fuzzy ideal

In this section we introduce the bipolar fuzzy ideal and study some of its properties then we investigate the relation between bipolar fuzzy G-subalgebra and Bipolar fuzzy ideal/closed ideal.

Definition 4.1 *A subset I of a G-algebra X is called an ideal of X if:*

(1) $0 \in I$,

(2) $x * y \in I$ *and* $y \in I$ *implies* $x \in I$.

An ideal I is said to be a closed ideal of X if $x \in I$ implies $0 * x \in I$.

Definition 4.2 *A bipolar fuzzy set $f = (f^+, f^-)$ in a G-algebra X is called a bipolar fuzzy ideal of X if:*

(1) $f^+(0) \geq f^+(x)$ *and* $f^-(0) \leq f^-(x)$,

(2) $f^+(x) \geq \min\{f^+(x*y), f^+(y)\}$,

(3) $f^-(x) \leq \max\{f^-(x*y), f^-(y)\}$.

Definition 4.3 *A bipolar fuzzy set $f = (f^+, f^-)$ in a G-algebra X is called a bipolar fuzzy closed ideal of X if:*

(1) $f^+(0 * x) \geq f^+(x)$ *and* $f^-(0 * x) \leq f^-(x)$,

(2) $f^+(x) \geq \min\{f^+(x * y), f^+(y)\}$,

(3) $f^-(x) \leq \max\{f^-(x * y), f^-(y)\}$.

Proposition 4.4 *Let $f = (f^+, f^-)$ be a bipolar fuzzy ideal of a G-algebra X. If $x * y = 0$ then $f^+(x) \geq f^+(y)$ and $f^-(x) \leq f^-(y)$.*

Proof. The proof is direct. □

Theorem 4.5 *Every bipolar fuzzy ideal of a G-algebra X is a bipolar fuzzy G-subalgebra of X.*

Proof. Let $f = (f^+, f^-)$ be a bipolar fuzzy ideal of a G-algebra X. As $x * (x * y) * y = 0$ then from Proposition 4.4, $f^+(x * (x * y)) \geq f^+(y)$ and $f^-(x * (x * y)) \leq f^-(y)$. Hence, using Definition 4.2, $f^+(x*(x*y)) \geq f^+(y) \geq \min\{f^+(x*(x*y)*y), f^+(y)\} = \min\{f^+(0), f^+(y)\} \geq \min\{f^+(x), f^+(y)\}$. We also have $f^-(x * (x * y)) \leq f^-(y) \leq \max\{f^-(x * (x * y) * y), f^-(y)\} = \max\{f^-(0), f^-(y)\} \leq \max\{f^-(x), f^-(y)\}$. Therefore f is a bipolar fuzzy G-subalgebra. □

Proposition 4.6 *Every bipolar fuzzy G-subalgebra satisfying $f^+(x) \geq \min\{f^+(x*y), f^+(y)\}$ and $f^-(x) \leq \max\{f^-(x * y), f^-(y)\}$ is a bipolar fuzzy closed ideal.*

Proof. Direct to prove. □

Theorem 4.7 *Let $f = (f^+, f^-)$ be a bipolar fuzzy G-subalgebra of a medial G-algebra X such that for $x, y \in X$, $f^+(y * x) \geq f^+(x * y)$ and $f^-(y * x) \geq f^-(x * y)$ then f is a bipolar fuzzy closed ideal of X.*

Proof. As $x = x*0$ we have $f^+(x) = f^+(x*0) \geq f^+(0*x) \geq f^+((y*y)*x) \geq f^+((y*x)*y)$ as X is medial. Then $f^+((y * x) * y) \geq \min\{f^+(y * x), f^+(y)\} \geq \min\{f^+(x * y), f^+(y)\}$. Similarly $f^-(x) \leq \max\{f^-(x * y), f^-(y)\}$. Hence with Proposition 3.9, f is a bipolar fuzzy closed ideal of X. □

References

[1] L. A. Zadeh, Fuzzy sets, Information and control, 8 (1965), no.3, 338-353.

[2] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I. Information and Control, 18 (1975), 199-249.

[3] K. T. Atanassov, Intuitionistic fuzzy sets. Fuzzy sets and Systems, 20 (1986), no. 1, 87-96.

[4] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications, 18 (1967), no. 1, 145-174.

[5] W. Zhang, Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis, Fuzzy Information Processing Society Biannual Conference, Industrial Fuzzy Control and Intelligent Systems Conference, and the NASA Joint Technology Workshop on Neural Networks and Fuzzy Logic, IEEE. 1994, 305-309.

[6] W. Zhang, Bipolar Fuzzy Sets, IEEE INTERNATIONAL CONFERENCE ON FUZZY SYSTEMS, 1998, 835-840.

[7] KMy. Lee, KMi. Lee and K. J. Cios, Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets. International Conference on Computing and Information Technologies: Exploring Emerging Technologies: Montclair State University, NJ, USA. World Scientific, (2001), p 433.

[8] T. Senapati, Bipolar fuzzy structure of BG-subalgebras, The Journal of Fuzzy Mathematics, 23 (2015), no. 1, 209-220.

[9] , Y. B. Jun, and S. Z. Song, Subalgebras and closed ideals of BCH-algebras based on bipolar-valued fuzzy sets, Sci. Math. Jpn. (2008), 427-437.

[10] K. J. Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, Bull. Malays. Math. Sci. Soc, 32 (2009), no. 3, 361-373.

[11] C. Jana, T. Senapati, M. Bhowmik and M. Pal, On Intuitionistic Fuzzy G-subalgebras of G-algebras, Fuzzy Information and Engineering, 7 (2015), no. 2, 195-209.

[12] T. Senapati, C. Jana, M. Bhowmik and M. Pal, L-fuzzy G-subalgebras of G-algebras, Journal of the Egyptian Mathematical Society, 23 (2015), no. 2, 219-223.

[13] R. K. Bandrur and N. Rafii, On G-algebras, Scientia Magna International Book Series, 8 (2012), no. 3,1-7.

Scientia Magna
Vol. 13 (2018), No. 1, 72-79

On $\wedge_e$-sets and its Generalizations

M. Angayarkanni[1], P. Dhanasekaran[2] and A. Vadivel [3]

[1] Department of Mathematics, Kandaswamy Kandar's College,
P-Velur, Tamil Nadu-638 182, India
E-mail: angayarkanni66@rediffmail.com
[2] Department of Mathematics, Salem Sowdeswari College (SFC), Salem-10
E-mail: dhana_mathsquare@gmail.com
[3] Department of Mathematics, Government Arts College, (Autonomous)
Karur, Tamilnadu-639005, India
E-mail: avmaths@gmail.com

Abstract In this paper, we define the concept of $\wedge_e$-sets (resp. $\vee_e$-sets) of a topological space i.e., the intersection of e-open (resp. the union of e-closed) sets. We study the fundamental property of $\wedge_e$-sets (resp. $\vee_e$-sets) and investigate the topologies defined by these families of sets.

Keywords e-open sets, $\wedge_e$-sets, $\vee_e$-sets, g$\wedge_e$-sets, g$\vee_e$-sets and topology $\tau^{\wedge_e}$.
2010 Mathematics Subject Classification 54A05, 54D30, 54H05, 54G99.

§1. Introduction and preliminaries

In general topology, the arbitrary intersection of open sets is not open and the arbitrary union of closed sets is not closed. These properties motivated Maki [9] to introduce the concepts of $\wedge$-sets and $\vee$-sets in topological spaces. Several topologists like Miguel Caldas Cueva, Saeid Jafari, Govindappa Navalagi, Erdal Ekici, Noiri, Baker, Moshokoa and Julian Dontchev [3–7] have contributed more articles on the above sets.

Recently, In 2008, Erdal Ekici [8] introduced a new class of generalized open sets called e-open sets into the field of topology. This class is a subset of the class of semipreopen sets [2]. In this paper to introduce the concept of $\wedge_e$-sets (resp. $\vee_e$-sets) which is the intersection of e-open (resp. the union of e-closed) sets. We also investigate the notions of generalized $\wedge_e$-sets and generalized $\vee_e$-sets in a topological space (X, τ). Moreover, we present a new topology $\tau^{\wedge_e}$ on (X, τ) by utilizing the notions of $\wedge_e$-sets and $\vee_e$-sets. In this connection, we examine some of the properties of this new topology.

Throughout the present paper, the spaces X and Y mean topological spaces. For a subset A of a space X, $Cl(A)$ and $Int(A)$ represent the closure of A and the interior of A, respectively. A subset A of a space X is said to be regular open (regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$) [10]. The δ-interior of a subset A of X is the union of all regular open sets of X contained in A and it is denoted by $Int_\delta(A)$ [11]. A subset A is called δ-open if $A = Int_\delta(A)$. The complement of a δ-open set is called δ-closed. The δ-closure of a set A in a space (X, τ)

is defined by $\{x \in X : A \cap Int(Cl(B)) \neq \phi,\ B \in \tau \text{and} x \in B\}$ and it is denoted by $Cl_\delta(A)$. A subset A of a space $(X,\ \tau)$ is called e-open [8] if $A \subseteq ClInt_\delta(A)) \cup IntCl_\delta(A))$ and e-closed [8] if $ClInt_\delta(A)) \cap IntCl_\delta(A)) \subseteq A$. The e-closure of a set A, denoted by $eCl(A)$, is the intersection of all e-closed sets containing A. The e-interior of a set A denoted by $eInt(A)$, is the union of all e-open sets contained in A.

The family of all e-open (resp. e-closed) sets in (X, τ) will be denoted by $eO(X, \tau)$ (resp. $eC(X, \tau)$).

Proposition 1.1. [8]

(1) The union of any family of e-open sets is e-open.

(2) The intersection of an open and an e-open set is an e-open set.

Lemma 1.1. *[8] The e-closure $eCl(A)$ is the set of all $x \in X$ such that $O \cap A \neq \phi$ for every $O \in eO(X,\ x)$, where $eO(X,\ x) = \{U \mid x \in U,\ U \in eO(X,\ \tau)\}$.*

Definition 1.1. *[9] Let X be a space and $A \subseteq X$. Then $A^\vee = \bigcup\{F : F \subseteq A$ and F is closed $\}$ and $A^\wedge = \bigcap\{U : A \subseteq U$ and U is open $\}$. Moreover, A is said to be $\vee$-set if $A = A^\vee$ and A is said to be $\wedge$-set if $A = A^\wedge$.*

§2. $\wedge_e$-sets and $\vee_e$-sets

Definition 2.1. *Let A be a subset of a topological space $(X,\ \tau)$. We define the subsets $A^{\wedge_e}$ and $A^{\vee_e}$ as follows:*

$$A^{\wedge_e} = \bigcap\{O \setminus O \supseteq A,\ O \in eO(X,\ \tau)\} \quad \text{and} \quad A^{\vee_e} = \bigcup\{F \setminus F \subseteq A,\ F^c \in eO(X,\ \tau)\}.$$

Proposition 2.1. *Let A, B and $\{B_\lambda : \lambda \in \Omega\}$ be subsets of a topological space $(X,\ \tau)$. Then the following properties are valid:*

(i) $\phi^{\vee_e} = \phi$ *and* $\phi^{\wedge_e} = \phi$.

(ii) $X^{\vee_e} = X$ *and* $X^{\wedge_e} = X$.

(iii) $B \subseteq B^{\wedge_e}$.

(iv) If $A \subseteq B$, *then* $A^{\wedge_e} \subseteq B^{\wedge_e}$ *and* $A^{\vee_e} \subseteq B^{\vee_e}$.

(v) $B^{\vee_b} \subseteq B$.

(vi) $(B^{\wedge_e})^{\wedge_e} = B^{\wedge_e}$ *and* $(B^{\vee_e})^{\vee_e} = B^{\vee_e}$.

(vii) $\left[\bigcup_{\lambda \in \Omega} B_\lambda\right]^{\wedge_e} = \bigcup_{\lambda \in \Omega} B_\lambda^{\wedge_e}$.

(viii) If $A \in eO(X,\ \tau)$, *then* $A = A^{\wedge_e}$.

(ix) $(B^c)^{\wedge_e} = (B^{\vee_e})^c$.

(x) If $B \in eC(X,\ \tau)$, *then* $B = B^{\vee_e}$.

(xi) $\left[\bigcap_{\lambda\in\Omega} B_\lambda\right]^{\wedge_e} \subseteq \bigcap_{\lambda\in\Omega} B_\lambda^{\wedge_e}$, $\left[\bigcap_{\lambda\in\Omega} B_\lambda\right]^{\vee_e} \subseteq \bigcap_{\lambda\in\Omega} B_\lambda^{\vee_e}$.

(xii) $\left[\bigcup_{\lambda\in\Omega} B_\lambda\right]^{\vee_e} \supseteq \bigcup_{\lambda\in\Omega} B_\lambda^{\vee_e}$.

(xiii) $B^{\vee_e} \subseteq B^{\vee}$ *and* $B^{\wedge_e} \supseteq B^{\wedge}$.

Proof. The proofs of (i), (ii), (iii) and (v) are clear by Definition 2.1.

(iv) Suppose that $x \notin B^{\wedge_e}$. Then there exists a subset $O \in eO(X, \tau)$ such that $O \supseteq B$ with $x \notin O$. Since $B \supseteq A$, thus $x \notin A^{\wedge_e}$ and thus $A^{\wedge_e} \subseteq B^{\wedge_e}$. Similarly $A^{\vee_e} \subseteq B^{\vee_e}$.

(vi) Follows from (iii), (v) and Definition 2.1.

(vii) Suppose that there exists a point x such that $x \notin \left[\bigcup_{\lambda\in\Omega} B_\lambda\right]^{\wedge_e}$. Then, there exists a subset $O \in eO(X, \tau)$ such that $\bigcup_{\lambda\in\Omega} B_\lambda \subseteq O$ and $x \notin O$. Thus, for each $\lambda \in \Omega$ we have $x \notin B_\lambda^{\wedge_e}$. This implies that $x \notin \bigcup_{\lambda\in\Omega} B_\lambda^{\wedge_e}$.

Conversely, suppose that there exists a point $x \in X$ such that $x \notin \bigcup_{\lambda\in\Omega} B_\lambda^{\wedge_e}$. Then by Definition 2.1, there exists subsets $O_\lambda \in eO(X, \tau)$ (for each $\lambda \in \Omega$) such that $x \notin O_\lambda$, $B_\lambda \subseteq O_\lambda$.

Let $O = \bigcup_{\lambda\in\Omega} O_\lambda$. Then we have that $x \notin \bigcup_{\lambda\in\Omega} O_\lambda$, $\bigcup_{\lambda\in\Omega} B_\lambda \subseteq O$ and $O \in eO(X, \tau)$. This implies that $x \notin \left[\bigcup_{\lambda\in\Omega} B_\lambda\right]^{\wedge_e}$. Thus the proof of (vii) is complete.

(viii) By Definition 2.1 and since $A \in eO(X, \tau)$, we have $A^{\wedge_e} \subseteq A$. By (iii), we have that $A^{\wedge_e} = A$.

(ix) $\quad (B^{\vee_e})^c = \bigcap\{F^c / F^c \supseteq B^c,\ F^c \in eO(X, \tau)\} = (B^c)^{\wedge_e}$.

(x) If $B \in eC(X, \tau)$, then $B^c \in eO(X, \tau)$. By (vii) and (viii), $B^c = (B^c)^{\wedge_e} = (B^{\vee_e})^c$. Hence $B = B^{\vee_e}$.

(xi) Suppose that there exists a point x such that $x \notin \bigcap_{\lambda\in\Omega} B_\lambda^{\wedge_e}$. Then, there exists $\lambda \in \Omega$ such that $x \notin B_\lambda^{\wedge_e}$. Hence there exists $O \in eO(X, \tau)$ such that $O \supseteq B_\lambda$ and $x \notin O$. Thus $x \notin \left[\bigcap_{\lambda\in\Omega} B_\lambda\right]^{\wedge_e}$. This implies that $\left[\bigcap_{\lambda\in\Omega} B_\lambda\right]^{\wedge_e} \subseteq \bigcap_{\lambda\in\Omega} B_\lambda^{\wedge_e}$. Similarly, we can easily prove that $\left[\bigcap_{\lambda\in\Omega} B_\lambda\right]^{\vee_e} \subseteq \bigcap_{\lambda\in\Omega} B_\lambda^{\vee_e}$.

(xii)
$$\begin{aligned}
\left[\bigcup_{\lambda\in\Omega} B_\lambda\right]^{\vee_e} &= \left[\left(\left(\bigcup_{\lambda\in\Omega} B_\lambda\right)^c\right)^{\wedge_e}\right]^c \\
&= \left[\left(\bigcap_{\lambda\in\Omega} B_\lambda^c\right)^{\wedge_e}\right]^c \\
&\supseteq \left[\bigcap_{\lambda\in\Omega} (B_\lambda^c)^{\wedge_e}\right]^c \\
&= \left[\bigcap_{\lambda\in\Omega} \left(B_\lambda^{\vee_e}\right)^c\right]^c \\
&= \bigcup_{\lambda\in\Omega} B_\lambda^{\vee_e} \qquad \text{(By (viii) and (xi))}.
\end{aligned}$$

(xiii) Follows from Definition 1.1 and Definition 2.1. □

Definition 2.2. *In a topological space* (X, τ), *a subset* B *is a* $\wedge_e$*-set (resp.* $\vee_e$*-set of* (X, τ)*), if* $B = B^{\wedge_e}$ *(resp.* $B = B^{\vee_e}$*). By* $\wedge_e$ *(resp.* $\vee_e$*), we denote the family of all* $\wedge_e$*-sets (resp.* $\vee_e$*-sets) of* (X, τ).

Remark 2.1. *By Proposition 2.1 (vii) and (x), we have that :*

(i) If $B \in eO(X, \tau)$, *then* B *is a* $\wedge_e$*-set.*

(ii) If $B \in eC(X, \tau)$, *then* B *is a* $\vee_e$*-set.*

The converses of the above Remark 2.1 need not be true as shown by the following examples.

Example 2.1. *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$, *then*

(i) $\{a, b\}$ *is a* $\wedge_e$*-set in* (X, τ) *but it is not* $eO(X, \tau)$.

(ii) $\{c\}$ *is a* $\vee_e$*-set in* (X, τ) *but it is not* $eC(X, \tau)$.

Theorem 2.1.

(i) The subsets ϕ *and* X *are* $\wedge_e$*-sets and* $\vee_e$*-sets.*

(ii) Every union of $\wedge_e$*-sets (resp.*$\vee_e$*-sets) is a* $\wedge_e$*-sets (resp.*$\vee_e$*-sets).*

(iii) Every intersection of $\wedge_e$*-sets (resp.* $\vee_e$*-sets) is a* $\wedge_e$*-sets (resp.* $\vee_e$*-sets).*

(iv) A subset B *is a* $\wedge_e$*-set if and only if* B^c *is a*$\vee_e$*-set.*

Proof. (i) and (iv) are obvious.

(ii) Let $\{B_\lambda : \lambda \in \Omega\}$ be a family of $\wedge_e$-sets in a topological space (X, τ). Then by Definition 2.2 and Propostion 2.1(vii),

$$\bigcup_{\lambda\in\Omega} B_\lambda = \bigcup_{\lambda\in\Omega} B_\lambda^{\wedge_e} = [\bigcup_{\lambda\in\Omega} B_\lambda]^{\wedge_e}.$$

(iii) Let $\{B_\lambda : \lambda \in \Omega\}$ be a family of $\wedge_e$-sets in (X, τ). Then by Proposition 2.1 (xi) and Definition 2.2,

$$[\bigcap_{\lambda\in\Omega} B_\lambda]^{\wedge_e} \subseteq \bigcap_{\lambda\in\Omega} B_\lambda^{\wedge_e} = \bigcap_{\lambda\in\Omega} B_\lambda.$$

Hence by Proposition 2.1 (iii), $\bigcap_{\lambda\in\Omega} B_\lambda = [\bigcap_{\lambda\in\Omega} B_\lambda]^{\wedge_e}$. □

Remark 2.2 *By Theorem 2.1,* $\wedge_e$ *(resp.* $\vee_e$*) is a topology on* X *containing all e-open (resp. e-closed) sets. Clearly* $(X, \wedge_e)$ *and* $(X, \vee_e)$ *are Alexandroff spaces [1], i.e, arbitrary intersections of open sets are open.*

Definition 2.3. *A topological space* (X, τ) *is said to be e-*T_1 *if for each pair of distinct points* x *and* y *of* X, *there exists a e-open set* U_x *containing* x *but not* y *and an e-open set* U_y *containing* y *but not* x. *It is obvious that* (X, τ) *is e-*T_1 *if and only if for each* $x \in X$, *the singleton* $\{x\}$ *is e-closed.*

Theorem 2.2. *For a topological space* (X, τ), *the following properties are equivalent:*

(i) (X, τ) *is e-*T_1.

(ii) Every subset of X *is a* $\wedge_e$*-set.*

(iii) Every subset of X *is a* $\vee_e$*-set.*

Proof. It is obvious that (ii) $\Leftrightarrow$ (iii).

(i) $\Rightarrow$ (iii) : Let A be any subset of X. Since $A = \bigcup\{ \{x\} \mid x \in A\}$, A is the union of e-closed sets, hence a $\vee_e$-set.

(iii) $\Rightarrow$ (i) : Since by (iii), we have that every singleton is a union of e-closed sets, i.e. it is e-closed, then $(X,\ \tau)$ is a e-T_1 space. □

§3. $g\wedge_e$-sets and $g\vee_e$-sets

In this section, by using the $\wedge_e$-operator and $\vee_e$-operator, we introduce the classes of generalized $\wedge_e$-sets (= $g\wedge_e$-sets) and generalized $\vee_e$-sets (= $g\vee_e$-sets) as an analogy of the sets introduced by Maki [9].

Definition 3.1. *In a topological space* $(X,\ \tau)$*, a subset* B *is called a* $g\wedge_e$*-set of* $(X,\ \tau)$ *if* $B^{\wedge_e} \subseteq F$ *whenever* $B \subseteq F$ *and* F *is e-closed.*

Definition 3.2. *In a topological space* $(X,\ \tau)$*, a subset* B *is called a* $g\vee_e$*-set of* $(X,\ \tau)$ *if* B^c *is a* $g\wedge_e$*-set of* $(X,\ \tau)$*.*

Remark 3.1. *We shall see, however, that we obtain nothing new according to the following results.*

Proposition 3.1. *For a subset* B *of a topological space* $(X,\ \tau)$*, the following properties hold:*

(i) B *is a* $g\wedge_e$*-set if and only if* B *is a* $\wedge_e$*-set.*

(ii) B *is a* $g\vee_e$*-set if and only if* B *is a* $\vee_e$*-set.*

Proof. (i) Every $\wedge_e$-set is $g\wedge_e$-set. Now, let B be a $g\wedge_e$-set. Suppose that $x \in B^{\wedge_e}\backslash B$. Since for each $x \in X$, the singleton $\{x\}$ is e-open or e-closed. If $\{x\}$ is e-open, then $X\backslash\{x\}$ is e-closed. Since $B \subseteq X\backslash\{x\}$, we have $B^{\wedge_e} \subseteq X\backslash\{x\}$ which is a contradiction. If $\{x\}$ is e-closed, $X\backslash\{x\}$ is e-open and $B \subset X\backslash\{x\}$. Therefore, we have $B^{\wedge_e} \subset X\backslash\{x\}$. This is a contradiction. Hence $B^{\wedge_e} = B$ and B is a $\wedge_e$-set.

(ii) This is proved in a similar way. □

§4. The associated topology $\tau^{\wedge_e}$

In this section, we define a closure operator $C^{\wedge_e}$ and the associated topology $\tau^{\wedge_e}$ on the topological space $(X,\ \tau)$ by using the family of $\wedge_e$-sets.

Definition 4.1. *For any subset* B *of a topological space* $(X,\ \tau)$*, define*

$$C^{\wedge_e}(B) = \bigcap\{U\ :\ B \subseteq U,\ U \in \wedge_e\} \quad \textit{and} \quad Int^{\vee_e}(B) = \bigcup\{F\ :\ B \supseteq U,\ F \in \vee_e\}.$$

Proposition 4.1. *For any subset* B *of a topological space* $(X,\ \tau)$*,*

(a) $B \subseteq C^{\wedge_e}(B)$*.*

(b) $C^{\wedge_e}(B^c) = (Int^{\vee_e}(B))^c$.

(c) $C^{\wedge_e}(\phi) = \phi$.

(d) Let $\{B_\lambda : \lambda \in \Omega\}$ *be a family of* $(X,\ \tau)$. *Then* $\bigcup_{\lambda\in\Omega} C^{\wedge_e}(B_\lambda) = C^{\wedge_e}(\bigcup_{\lambda\in\Omega} B_\lambda)$.

(e) $C^{\wedge_e}(C^{\wedge_e}(B)) = C^{\wedge_e}(B)$.

(f) If $A \subseteq B$, *then* $C^{\wedge_e}(A) \subseteq C^{\wedge_e}(B)$.

(g) If B *is a* $\wedge_e$*-set, then* $C^{\wedge_e}(B) = B$.

(h) If B *is a* $\vee_e$*-set, then* $Int^{\vee_e}(B) = B$.

Proof. (a), (b) and (c): Clear.

(d) Suppose that there exists a point x such that $x \notin C^{\wedge_e}(\bigcup_{\lambda\in\Omega} B_\lambda)$. Then, there exists a subset $U \in \wedge_e$ such that $\bigcup_{\lambda\in\Omega} B_\lambda \subseteq U$ and $x \notin U$. Thus, for each $\lambda \in \Omega$ we have $x \notin C^{\wedge_e}(B_\lambda)$. This implies that $x \notin \bigcup_{\lambda\in\Omega} C^{\wedge_e}(B_\lambda)$.

Conversely, we suppose that there exists a point $x \in X$ such that $x \notin \bigcup_{\lambda\in\Omega} C^{\wedge_e}(B_\lambda)$. Then, there exists subsets $U_\lambda \in \wedge_e$ for all $\lambda \in \Omega$, such that $x \notin U_\lambda$, $B_\lambda \subseteq U_\lambda$. Let $U = \bigcup_{\lambda\in\Omega} U_\lambda$. From this and Proposition 2.1(vi), we have that $x \notin U$, $\bigcup_{\lambda\in\Omega} B_\lambda \subseteq U$ and $U \in \wedge_e$. Thus, $x \notin C^{\wedge_e}(\bigcup_{\lambda\in\Omega} B_\lambda)$.

(e) Suppose that there exists a point $x \in X$ such that $x \notin C^{\wedge_e}(B)$. Then, there exists a subset $U \in \wedge_e$ such that $x \notin U$ and $U \supseteq B$. Since $U \in \wedge_e$ we have $C^{\wedge_e}(B) \subseteq U$. Thus, we have $x \notin C^{\wedge_e}(C^{\wedge_e}(B))$. Therefore $C^{\wedge_e}(C^{\wedge_e}(B)) \subseteq C^{\wedge_e}(B)$. The converse containment relation is clear by (a).

(f) Clear.

(g) By (a) and Definition 4.1, the proof is clear.

(h) By Definitin 4.1, (g) and (b) hold. □

Then we have the following.

Theorem 4.1. *$C^{\wedge_e}$ is a Kuratowski closure operator on X.*

Definition 4.2. *Let $\tau^{\wedge_e}$ be the topology on X generated by $C^{\wedge_e}$ in the usual manner, i.e.,$\tau^{\wedge_e} = \{B : B \subseteq X,\ C^{\wedge_e}(B^c) = B^c\}$. We define a family $\rho^{\wedge_e}$ by $\rho^{\wedge_e} = \{B : B \subseteq X,\ C^{\wedge_e}(B) = B\}$.*

By Definition 4.2, $\rho^{\wedge_e} = \{B : B \subseteq X,\ B^c \in \tau^{\wedge_e}\}$.

Proposition 4.2. *Let $(X,\ \tau)$ be a topological space. Then,*

(a) $\tau^{\wedge_e} = \{B : B \subseteq X,\ Int^{\vee_e}(B) = B\}$.

(b) $\wedge_e = \rho^{\wedge_e}$.

(c) $\vee_e = \tau^{\wedge_e}$.

(d) If $eC(X,\ \tau) = \tau^{\wedge_e}$, *then every* $\wedge_e$*-set of* $(X,\ \tau)$ *is e-open (i. e.,* $eO(X,\ \tau) = \wedge_e$*).*

(e) If every $\wedge_e$-set of $(X,\ \tau)$ is e-open (i.e., $\wedge_e \subseteq eO(X,\ \tau)$), then $\tau^{\wedge_e} = \{B : B \subseteq X,\ B = B^{\vee_e}\}$.

(f) If every $\wedge_e$-set of $(X,\ \tau)$ is e-closed (i.e., $\wedge_e \subseteq eC(X,\ \tau)$), then $eO(X,\ \tau) = \tau^{\wedge_e}$.

Proof. (a) By Definition 4.2 and Proposition 4.1, if $A \subseteq X$ then $A \in \tau^{\wedge_e}$ if and only if $C^{\wedge_e}(A^c) = A^c$, if and only if $(Int^{\vee_e}(A))^c = A^c$, if and only if $Int^{\vee_e}(A) = A$ if and only if $A \in \{B : B \subseteq X, Int^{\vee_e}(B) = B\}$.

(b) Let B be a subset of X. By Proposition 2.1 (viii), $eO(X,\ \tau) \subseteq \wedge_e$ and

$$C^{\wedge_e}(B) = \bigcap\{U \mid B \subseteq U,\ U \in \wedge_e\} \subseteq \bigcap\{U \mid B \subseteq U,\ U \in eO(X,\ \tau)\} = B^{\wedge_e}.$$

Therefore, we have $C^{\wedge_e}(B) \subseteq B^{\wedge_e}$. Now suppose that $x \notin C^{\wedge_e}(B)$. There exists $U \in \wedge_e$ such that $B \subseteq U$ and $x \notin U$. Since $U \in \wedge_e$, $U = U^{\wedge_e} = \bigcap\{V \mid U \subseteq V \in eO(X,\ \tau)\}$ and hence there exists $V \in eO(X,\ \tau)$ such that $U \subseteq V$ and $x \notin V$. Thus $x \notin V$ and $B \subseteq V \in eO(X,\ \tau)$. This shows that $x \notin B^{\wedge_e}$. Therefore, $B^{\wedge_e} \subseteq C^{\wedge_e}(B)$ and hence $B^{\wedge_e} = C^{\wedge_e}(B)$, for any subset B of X. By the definitions of $\wedge_e$ and $\rho^{\wedge_e}$, we obtain $\wedge_e = \rho^{\wedge_e}$.

(c) Let $B \in \tau^{\wedge_e}$. Then $C^{\wedge_e}(B^c) = B^c$ and $B^c \in \rho^{\wedge_e}$. By (b) $B^c \in \wedge_e$ and $B^c = (B^c)^{\wedge_e}$. Therefore, by Proposition 2.1 (ix), $B^c = (B^{\vee_e})^c$ and $B = B^{\vee_e}$. This shows that $B \in \vee_e$. Consequently, we obtain $\tau^{\wedge_e} \subseteq \vee_e$. Quite similary, we obtain $\tau^{\wedge_e} \supseteq \vee_e$ and hence $\vee_e = \tau^{\wedge_e}$.

(d) Let B be any $\wedge_e$-set. i.e., $B \in \wedge_e$. By (b), $B \in \rho^{\wedge_e}$ thus, $B^c \in \tau^{\wedge_e}$. From the assumption, we have, $B^c \in eC(X,\ \tau)$ and hence $B \in eO(X,\ \tau)$.

(e) Let $A \subseteq X$ and $A \in \tau^{\wedge_e}$. Then by Definitions 4.1 and 4.2,

$$\begin{aligned} A^c &= C^{\wedge_e}(A^c) \\ &= \bigcap\{U\ :\ U \supseteq A^c, U \in \wedge_e\} \\ &= \bigcap\{U\ :\ U \supseteq A^c, U \in eO(X,\ \tau)\} \\ &= (A^c)^{\wedge_e}. \end{aligned}$$

Using Proposition 2.1 (ix), we have $A = A^{\vee_e}$. i. e., $A \in \{B : B \subseteq X, B = B^{\vee_e}\}$.

Conversely, if $A \in \{B\ :\ B \subseteq X, B = B^{\vee_e}\}$ then by Proposition 3.1 (ii), A is a $g\vee_e$-set. Thus $A \in \vee_e$. By using (c), $A \in \tau^{\wedge_e}$.

(f) Let $A \subseteq X$ and $A \in \tau^{\wedge_e}$. Then

$$A = (C^{\wedge_e}(A^c))^c = (\bigcap\{U : A^c \subseteq U,\ U \in \wedge_e\})^c = \bigcup\{U^c : U^c \subseteq A, U \in \wedge_e\}.$$

Conversely, if $A \in eO(X, \tau)$, then by (b), $A \in \wedge_e$. By assumption, $A \in eC(X,\ \tau)$. By using (c), $A \in \tau^{\wedge_e}$. □

Proposition 4.3. *If $eO(X,\ \tau) = \tau^{\wedge_e}$, then $(X,\ \tau^{\wedge_e})$ is a discrete space.*

Proof. Suppose that $\{x\}$ is not e-open in $(X,\ \tau)$. Then $\{x\}$ is e-closed in $(X,\ \tau)$. Thus $\{x\} \in \tau^{\wedge_e}$ by Proposition 4.2 (c). Suppose that $\{x\}$ is e-open in $(X,\ \tau)$, then $\{x\} \in eO(X,\ \tau) = \tau^{\wedge_e}$. Therefore, every singleton $\{x\}$ is $\tau^{\wedge_e}$-open and hence every subset of X is $\tau^{\wedge_e}$-open. □

Conclusion: The concepts of $\wedge_e$-sets and $\vee_e$-sets are used to characterize the e-closed and e-open sets. Also the basic operators namely $C^{\wedge_e}$ and $Int^{\vee_e}$ operators are studied using the above sets.

Acknowledgements

The authors would like to thank from the anonymous reviewers for carefully reading of the manuscript and giving useful comments, which will help us to improve the paper.

References

[1] P. Alexandroff, *Diskrete Raume*, Math. Sb., **2** (1937), 501-508.

[2] D. Andrijevic, *Semi-preopen sets*, Mat. Vesnik, **38** (1986), 24-32.

[3] C. W. Baker, M. Caldas, S. Jafari and S. P. Moshokoa, *More on almost contra λ-continuous functions*, Fasciculi Mathematici, **42** (2009), 17-28.

[4] M. Caldas, E. Ekici, S. Jafari and T. Noiri, *on the class of contra λ-continuous functions*, Ann. Unive. Sci. Budapest Sec. Math., **49** (2006), 75-86.

[5] M. Caldas and S. Jafari, *On some low separation axioms via λ-open and λ-closure operator*, Rend. Circ. Mat. Di Palermo, **54** (2) (2005), 195-208.

[6] M. Caldas, S. Jafari and G. Navalagi, *More on λ-closed sets in topological spaces*, Revista Colombiana de Matematicas, **41** (2) (2007), 355-369.

[7] E. Ekici, S. Jafari, M. Caldas and T. Noiri, *Weakly λ-continuous functions*, *Novi Sad J. Math.*, **38** (2) (2008), 47-56.

[8] E. Ekici, *On e-open sets, DP^*-sets and $DP\epsilon^*$-sets and decompositions of continuity*, Arabian Journal for Science and Engineering, **33** (2A)(2008), 269-282.

[9] H. Maki *Generalized $\wedge$-sets and the associated closure operator*, The special issue in Commenoration of Prof. Kazusada IKEDA' Retirement **1** (10)(1986), 139-146.

[10] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, TAMS, **41** (1937), 375-381.

[11] N. Velico, *H-closed topological spaces,* Amer. Math. Soc. Transl., **78**(2) (1968), 103-118.

Scientia Magna
Vol. 13 (2018), No. 1, 80-87

On rarely fuzzy regular semi continuous functions in fuzzy topological spaces

E. Elavarasan[1]

[1]Department of Mathematics,
Shree Raghavendra Arts and Science College, Keezhamoongiladi, Chidambaram-608102
(Affiliated to Thiruvalluvar University), Tamil Nadu, India.
E-mail: maths.aras@gmail.com

Abstract In this paper, we introduce the concepts of fuzzy rarely regular semi continuous functions in the sense of Šostak's is introduced. Some interesting properties and characterizations of them are investigated. Also, some applications to fuzzy compact spaces are established.

Keywords Rarely fuzzy regular semi continuous, FRS-compact space, rarely FRS-almost compact space and rarely FRS-T_2-spaces.
2010 Mathematics Subject Classification 54A40.

§1. Introduction

Šostak [11] introduced the notion of fuzzy topology as an extension of Chang [2] and Lowen's [8] fuzzy topology. Later on he has developed the theory of fuzzy topological spaces in [12] and [13]. Popa [9] introduced the notion of rarely continuity as a generalization of weak continuity [6] which has been further investigated by Long and Herrington [7] and Jafari [4, 5]. Recently Vadivel and Elavarasan [16] introduced the concept of r-fuzzy regular semi open and fuzzy regular semi continuous functions in fuzzy topological spaces in the sense of Šostak's. In this paper, we introduce the concepts of rarely fuzzy regular semi continuous functions in the sense of Šostak's [11] is introduced. Some interesting properties and characterizations of them are investigated. Also, some applications to fuzzy compact spaces are established.

§2. Preliminaries

Throughout this paper, let X be a nonempty set, $I = [0,\ 1]$ and $I_0 = (0,\ 1]$. For $\lambda \in I^X$, $\overline{\lambda}(x) = \lambda$ for all $x \in X$. For $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by $x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$ Let $Pt(X)$ be the family of all fuzzy points in X. A fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$. All other notations and definitions are standard, for all in the fuzzy set theory.

Definition 2.1. *[11] A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:*

(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,

(O2) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \tau(\mu_i)$, *for any* $\{\mu_i\}_{i\in\Gamma} \subset I^X$,

(O3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, *for any* $\mu_1,\ \mu_2 \in I^X$.

The pair $(X,\ \tau)$ is called a fuzzy topological space (for short, fts). A fuzzy set λ is called an r-fuzzy open (r-fo, for short) if $\tau(\lambda) \geq r$. A fuzzy set λ is called an r-fuzzy closed (r-fc, for short) set iff $\overline{1} - \lambda$ is an r-fo set.

Theorem 2.1. *[3] Let (X, τ) be a fts. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows: $C_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu,\ \tau(\overline{1} - \mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following statements:*

(C1) $C_\tau(\overline{0}, r) = \overline{0}$,

(C2) $\lambda \leq C_\tau(\lambda,\ r)$,

(C3) $C_\tau(\lambda,\ r) \vee C_\tau(\mu,\ r) = C_\tau(\lambda \vee \mu,\ r)$,

(C4) $C_\tau(\lambda,\ r) \leq C_\tau(\lambda,\ s)$ *if* $r \leq s$,

(C5) $C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$.

Theorem 2.2. *[3] Let (X, τ) be a fts. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows: $I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda,\ \tau(\mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator I_τ satisfies the following statements:*

(I1) $I_\tau(\overline{1}, r) = \overline{1}$,

(I2) $I_\tau(\lambda,\ r) \leq \lambda$,

(I3) $I_\tau(\lambda,\ r) \wedge I_\tau(\mu,\ r) = I_\tau(\lambda \wedge \mu,\ r)$,

(I4) $I_\tau(\lambda,\ r) \leq I_\tau(\lambda,\ s)$ *if* $s \leq r$,

(I5) $I_\tau(I_\tau(\lambda,\ r),\ r) = I_\tau(\lambda,\ r)$.

(I6) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)$ *and* $C_\tau(\overline{1} - \lambda, r) = \overline{1} - I_\tau(\lambda,\ r)$

Definition 2.2. *[10] Let $(X,\ \tau)$ be a fts, $\lambda \in I^X$ and $r \in I_0$. Then*

(1) a fuzzy set λ is called r-fuzzy regular open (for short, r-fro) if $\lambda = I_\tau(C_\tau(\lambda, r), r)$.

(2) a fuzzy set λ is called r-fuzzy regular closed (for short, r-frc) if $\lambda = C_\tau(I_\tau(\lambda, r), r)$.

Definition 2.3. *[16] Let $(X,\ \tau)$ be a fts and $\lambda \in I^X,\ r \in I_0$. Then*

(1) λ is called r-fuzzy regular semi open (for short, r-frso) if there exists r-fro set $\mu \in I^X$ and $\mu \leq \lambda \leq C_\tau(\mu, r)$.

(2) λ is called r-fuzzy regular semi closed (for short, r-frsc) if there exists r-frc set $\mu \in I^X$ and $I_\tau(\mu, r) \leq \lambda \leq \mu$.

(3) The r-fuzzy regular semi interior of λ, denoted by $RSI_\tau(\lambda, r)$, is defined by $RSI_\tau(\lambda, r) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda,\ \mu$ is r-frso $\}$.

(4) The r-fuzzy regular semi closure of λ, denoted by $RSC_\tau(\lambda, r)$ is defined by $RSC_\tau(\lambda, r) = \bigwedge\{\mu \in I^X \mid \mu \geq \lambda,\ \mu$ is r-frsc $\}$.

Definition 2.4. *[15] Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be a function and $r \in I_0$. Then f is called fuzzy regular continuous if $f^{-1}(\lambda)$ is r-fro set in I^X for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r$.*

Definition 2.5. *[16] Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's. Let $f : X \to Y$ be a mapping. Then f is said to be:*

(1) fuzzy regular semi irresolute (resp. fuzzy regular semi continuous) iff $f^{-1}(\mu)$ is r-frso for each r-frso set $\mu \in I^Y$ (resp. $\mu \in I^Y,\ \eta(\mu) \geq r$).

(2) fuzzy regular semi irresolute open (resp. fuzzy regular semi open) iff $f(\lambda)$ is r-frso in Y for each r-frso set $\lambda \in I^X$ (resp. $\lambda \in I^X,\ \tau(\lambda) \geq r$).

(3) fuzzy regular semi irresolute closed (resp. fuzzy regular semi closed) iff $f(\lambda)$ is r-frsc in Y for each r-frsc set $\lambda \in I^X$ (resp. $\lambda \in I^X,\ \tau(\overline{1} - \lambda) \geq r$).

(4) fuzzy regular semi irresolute homeomorphism iff f is bijective, f and f^{-1} are fuzzy regular semi irresolute.

Definition 2.6. *[1] Let (X, τ) be a fts and $r \in I_0$. For $\lambda \in I^X$, λ is called an r-fuzzy rare set if $I_\tau(\lambda, r) = \overline{0}$.*

Definition 2.7. *[1] Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f : (X,\ \tau) \to (Y,\ \eta)$ be a function. Then f is called*

(1) weakly continuous if for each $\mu \in I^Y$, where $\sigma(\mu) \geq r,\ r \in I_0,\ f^{-1}(\mu) \leq I_\tau(f^{-1}(C_\sigma(\mu, r)), r)$.

(2) rarely continuous if for each $\mu \in I^Y$, where $\sigma(\mu) \geq r,\ r \in I_0$, there exists an r-fuzzy rare set $\lambda \in I^Y$ with $\mu + C_\sigma(\lambda, r) \geq 1$ and $\rho \in I^X$, where $\tau(\rho) \geq r$ such that $f(\rho) \leq \mu \vee \lambda$.

Proposition 2.1. *[1] Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be any two fts's, $r \in I_0$ and $f : (X,\ \tau) \to (Y,\ \sigma)$ is fuzzy open and one-to-one, then f preserves r-fuzzy rare sets.*

§3. Rarely fuzzy regular semi continuous functions

Definition 3.1. *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be a fts's, and $f : (X,\ \tau) \to (Y,\ \eta)$ be a function. Then f is called*

(1) rarely fuzzy regular continuous [14] if for each $\mu \in I^Y$, where $\sigma(\mu) \geq r,\ r \in I_0$, there exists an r-fuzzy rare set $\lambda \in I^Y$ with $\mu + C_\sigma(\lambda, r) \geq 1$ and a r-fro set $\rho \in I^X$ such that $f(\rho) \leq \mu \vee \lambda$.

(2) rarely fuzzy regular semi continuous if for each $\mu \in I^Y$, where $\sigma(\mu) \geq r$, $r \in I_0$, there exists an r-fuzzy rare set $\lambda \in I^Y$ with $\mu + C_\sigma(\lambda, r) \geq 1$ and an r-frso set $\rho \in I^X$, such that $f(\rho) \leq \mu \vee \lambda$.

Remark 3.1.

(1) Every weakly continuous function is rarely continuous [1] but converse need not be true.

(2) Every fuzzy regular continuous function is fuzzy regular semi continuous but converse need not be true.

(3) Every rarely fuzzy regular continuous function is rarely fuzzy regular semi continuous but converse need not be true.

(4) Every fuzzy regular semi continuous function is rarely fuzzy regular semi continuous but converse need not be true.

(5) Every rarely fuzzy regular continuous function is rarely continuous but converse need not be true.

From the above definition and remarks it is not difficult to conclude that the following diagram of implications is true.

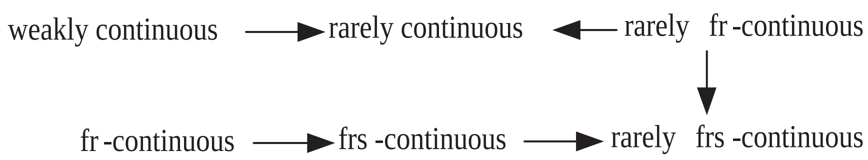

Diagram - I

Example 3.1. *Let $X = \{a, b, c\} = Y$, $\mu, \delta \in I^X, \lambda \in I^Y$ with $r \in I_0$ are defined as* $\lambda(a) = 0.5, \lambda(b) = 0.5, \lambda(c) = 0.6$; $\mu(a) = 0.4, \mu(b) = 0.5, \mu(c) = 0.6$; $\delta(a) = 0.4, \delta(b) = 0.5, \delta(c) = 0.4$. *We define smooth topologies $\tau, \sigma : I^X \to I$ as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{3} & \text{if } \lambda = \mu, \\ \frac{1}{3} & \text{if } \lambda = \delta, \\ 0 & \text{otherwise.} \end{cases} \qquad \sigma(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{3} & \text{if } \lambda = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = \frac{1}{3}$, then the function f is fuzzy regular semi continuous but not fuzzy regular continuous. Since the fuzzy set λ is r-fo set in Y, $f^{-1}(\lambda)$ is r-frso set, because there exists a r-fro set $\mu \in I^X$ such that $\mu \leq \lambda \leq C_\tau(\mu, r)$. But λ is not r-fro set.

Example 3.2. *Let $X = \{a, b, c\} = Y$. Define μ, $\delta \in I^X$, $\lambda_1 \in I^Y$ as follows:* $\mu(a) = 0.4, \mu(b) = 0.5, \mu(c) = 0.6$; $\delta(a) = 0.4, \delta(b) = 0.5, \delta(c) = 0.4$; $\lambda_1(a) = 0.8$, $\lambda_1(b) = 0.6$, $\lambda_1(c) =$

0.8. *Define the fuzzy topologies* $\tau,\ \sigma : I^X \to I$ *as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ \frac{1}{2} & \text{if } \lambda = \delta, \\ 0 & \text{otherwise,} \end{cases} \qquad \sigma(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $r = 1/2$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = a,\ f(b) = b,\ f(c) = c$ *and* $\lambda_1 \in I^Y$ *with* $\sigma(\lambda_1) \geq r,\ \lambda_2 \in I^Y$ *be an 1/2-fuzzy rare set defined by* $\lambda_2(a) = 0.6,\ \lambda_2(b) = 0.8,\ \lambda_2(c) = 0.8$ *and a r-frso set* $\lambda \in I^X$ *is defined by* $\lambda(a) = 0.5, \lambda(b) = 0.5, \lambda(c) = 0.6,\ f(\lambda) = (0.5,\ 0.5,\ 0.6) \leq \lambda_1 \vee \lambda_2 = (0.8,\ 0.8,\ 0.8)$. *Then f is rarely fuzzy regular semi continuous but not rarely fuzzy regular continuous, because* $\lambda \in I^X$ *is not r-fro set.*

Example 3.3. *In Example 3.2, f is fuzzy rarely continuous but not rarely fuzzy regular continuous. Since* $\lambda_1 \in I^Y$ *with* $\sigma(\lambda_1) \geq r,\ \lambda_2 \in I^Y$ *be an 1/2-fuzzy rare set defined by* $\lambda_2(a) = 0.6,\ \lambda_2(b) = 0.8,\ \lambda_2(c) = 0.8$ *and a r-fo set* μ *with* $\tau(\mu) \geq r,\ f(\mu) = (0.4,\ 0.5,\ 0.6) \leq \lambda_1 \vee \lambda_2 = (0.8,\ 0.8,\ 0.8)$ *and also r-fo set* δ *with* $\tau(\delta) \geq r,\ f(\delta) = (0.4,\ 0.5,\ 0.4) \leq \lambda_1 \vee \lambda_2 = (0.8,\ 0.8,\ 0.8)$.

Example 3.4. *In Example 3.2, f is rarely fuzzy regular semi continuous but not fuzzy regular semi continuous. Since* $\lambda_1 \in I^Y$ *with* $\sigma(\lambda_1) \geq r$*, there exist a r-fro set* $\mu \in I^X$ *such that* $\mu \leq \lambda_1 \nleq C_\tau(\mu, r),\ f^{-1}(\lambda_1)$ *is not r-fuzzy regular semiopen set in X.*

Definition 3.2. *Let* $(X,\ \tau)$ *and* $(Y,\ \sigma)$ *be a fts's, and* $f : (X,\ \tau) \to (Y,\ \eta)$ *be a function. Then f is called weakly fuzzy regular semi continuous if for each r-frso set* $\mu \in I^Y,\ r \in I_0$, $f^{-1}(\mu) \leq I_\tau(f^{-1}(C_\sigma(\mu, r)), r)$.

Definition 3.3. *A fts* (X, τ) *is said to be fuzzy* $RST_{1/2}$*-space if every r-frso set* $\lambda \in I^X,\ r \in I_0$ *is r-fro set.*

Theorem 3.1. *Let* $(X,\ \tau)$ *and* $(Y,\ \sigma)$ *be any two fuzzy topological spaces. If* $f : (X,\ \tau) \to (Y,\ \sigma)$ *is both fuzzy regular semi open, fuzzy regular semi irresolute and* $(X,\ \tau)$ *is fuzzy* $RST_{1/2}$ *space, then it is weakly fuzzy regular semi continuous.*

Proof. Let $\lambda \in I^X, r \in I_0$ with $\tau(\lambda) \leq r$. Since f is fuzzy regular semi open $f(\lambda) \in I^Y$ is r-frso. Also, since f is fuzzy regular semi irresolute, $f^{-1}(f(\lambda)) \in I^X$ is r-frso set. Since $(X,\ \tau)$ is fuzzy $RST_{1/2}$ space, every r-frso set is r-fro set and also every r-fro set is r-fo set, now, $\tau(f^{-1}(f(\lambda))) \geq r$. Consider $f^{-1}(f(\lambda)) \leq f^{-1}(C_\sigma(f(\lambda), r))$ from which $I_\tau(f^{-1}(f(\lambda)), r) \leq I_\tau(f^{-1}(C_\sigma(f(\lambda), r)), r)$. Since $\tau(f^{-1}(f(\lambda))) \geq r,\ f^{-1}(f(\lambda)) \leq I_\tau(f^{-1}(C_\sigma(f(\lambda), r)), r)$. Thus f is weakly fuzzy regular semi continuous. □

Definition 3.4. *Let* (X, τ) *be a fts. A FRS-open cover of* (X, τ) *is the collection* $\{\lambda_i \in I^X,\ \lambda_i$ *is r-frso,* $i \in J\}$ *such that* $\bigvee_{i \in J} \lambda_i = \overline{1}$.

Definition 3.5. *A fts* (X, τ) *is said to be FRS-compact space if every FRS-open cover of* (X, τ) *has a finite sub cover.*

Definition 3.6. *A fts* (X, τ) *is said to be rarely FRS-almost compact if every FRS-open cover* $\{\lambda_i \in I^X,\ \lambda_i$ *is r-frso,* $i \in J\}$ *of* (X, τ)*, there exists a finite subset* J_0 *of J such that* $\bigvee_{i \in J} \lambda_i \vee \rho_i = \overline{1}$ *where* $\rho_i \in I^X$ *are r-fuzzy rare sets.*

Theorem 3.2. *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be any two fts's, $r \in I_0$ and $f : (X,\ \tau) \to (Y,\ \sigma)$ be rarely fuzzy regular semi continuous. If $(X,\ \tau)$ is FRS-compact then $(Y,\ \sigma)$ is rarely FRS-almost compact.*

Proof. Let $\{\lambda_i \in I^Y,\ i \in J\}$ be FRS-open cover of (Y,σ). Then $\overline{1} = \vee_{i\in J}\lambda_i$. Since f is rarely fuzzy regular semi continuous, there exists an r-fuzzy rare sets $\rho_i \in I^Y$ such that $\lambda_i + C_\sigma(\rho_i, r) \geq \overline{1}$ and an r-frso set $\mu_i \in I^X$ such that $f(\mu_i) \leq \lambda_i \vee \rho_i$. Since (X,τ) is FRS-compact, every FRS-open cover of (X,τ) has a finite sub cover. Thus $\overline{1} \leq \vee_{i\in J_0}\mu_i$. Hence $\overline{1} = f(\overline{1}) = f(\vee_{i\in J_0}\mu_i) = \vee_{i\in J_0} f(\mu_i) \leq \vee_{i\in J_0}\lambda_i \vee \rho_i$. Therefore (Y,σ) is rarely FRS-almost compact. □

Theorem 3.3. *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be any two fts's, $r \in I_0$ and $f : (X,\ \tau) \to (Y,\ \sigma)$ be rarely fuzzy regular continuous. If $(X,\ \tau)$ is FRS-compact then $(Y,\ \sigma)$ is rarely FRS-almost compact.*

Proof. Since every rarely fuzzy regular continuous function is rarely fuzzy regular semi continuous, then proof follows immediately from the Theorem 3.2.. □

Theorem 3.4. *Let $(X,\ \tau)$, $(Y,\ \sigma)$ and $(Z,\ \eta)$ be any fts's, $r \in I_0$. If $f : (X,\ \tau) \to (Y,\ \sigma)$ be rarely fuzzy regular semi continuous, fuzzy regular semi open and $g : (Y,\ \sigma) \to (Z,\ \eta)$ is fuzzy open and one-to-one, then $g \circ f : (X,\ \tau) \to (Z,\ \eta)$ is rarely fuzzy regular semi continuous.*

Proof. Let $\lambda \in I^X$ with $\tau(\lambda) \geq r$. Since f is fuzzy regular semi open $f(\lambda) \in I^Y$ with $\sigma(f(\lambda)) \geq r$. Since f is rarely fuzzy regular semi continuous, there exists an r-fuzzy rare set $\rho \in I^Y$ with $f(\lambda) + C_\sigma(\rho, r) \geq \overline{1}$ and an r-frso set $\mu \in I^X$ such that $f(\mu) \leq f(\lambda) \vee \rho$. By the proposition 2.1., $g(\rho) \in I^Z$ is also an r-fuzzy rare set. Since $\rho \in I^Y$ is such that $\rho < \gamma$ for all $\gamma \in I^Y$ with $\sigma(\gamma) \geq r$, and g is injective, it follows that $(g \circ f)(\lambda) + C_\eta(g(\rho), r) \geq \overline{1}$. Then $(g \circ f)(\mu) = g(f(\mu)) \leq g(f(\lambda) \vee \rho) \leq g(f(\lambda)) \vee g(\rho) \leq (g \circ f)(\lambda) \vee g(\rho)$. Hence the result. □

Theorem 3.5. *Let $(X,\ \tau)$, $(Y,\ \sigma)$ and $(Z,\ \eta)$ be any fts's, $r \in I_0$. If $f : (X,\ \tau) \to (Y,\ \sigma)$ be fuzzy regular semi open, onto and $g : (Y,\ \sigma) \to (Z,\ \eta)$ be a function such that $g \circ f : (X,\ \tau) \to (Z,\ \eta)$ is rarely fuzzy regular semi continuous, then g is rarely fuzzy regular semi continuous.*

Proof. Let $\lambda \in I^X$ and $\mu \in I^Y$ be such that $f(\lambda) = \mu$. Let $(g \circ f)(\lambda) = \gamma \in I^Z$ with $\eta(\gamma) \geq r$. Since $(g \circ f)$ is fuzzy regular semi continuous, there exists a rare set $\rho \in I^Z$ with $\gamma + C_\eta(\rho, r) \geq \overline{1}$ and an r-frso set $\delta \in I^X$ such that $(g \circ f)(\delta) \leq \gamma \vee \rho$. Since f is fuzzy regular semi open, $f(\delta) \in I^Y$ is an r-frso set. Thus there exists a r-fuzzy rare set $\rho \in I^Z$ with $\gamma + C_\eta(\rho, r) \geq \overline{1}$ and an r-frso set $f(\delta) \in I^Y$ such that $g(f(\delta)) \leq \gamma \vee \rho$. Hence g is rarely fuzzy regular semi continuous. □

Theorem 3.6. *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be any two fuzzy topological spaces, $r \in I_0$. If $f : (X,\ \tau) \to (Y,\ \sigma)$ is rarely fuzzy regular semi continuous and $(X,\ \tau)$ is fuzzy $RST_{1/2}$-space, then f is rarely fr-continuous.*

Proof. The proof is trivial. □

Definition 3.7. *A fts (X,τ) is said to be rarely FRS-T_2-space if for each pair $\lambda,\ \mu \in I^X$ with $\lambda \neq \mu$ there exist r-frso sets $\rho_1,\ \rho_2 \in I^X$ with $\rho_1 \neq \rho_2$ and a r-fuzzy rare set $\gamma \in I^X$ with $\rho_1 + C_\tau(\gamma, r) \geq \overline{1}$ and $\rho_2 + C_\tau(\gamma, r) \geq \overline{1}$ such that $\lambda \leq \rho_1 \vee \gamma$ and $\mu \leq \rho_2 \vee \gamma$.*

Theorem 3.7. *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be any two fuzzy topological spaces, $r \in I_0$. If $f : (X,\ \tau) \to (Y,\ \sigma)$ is fuzzy regular semi open and injective and $(X,\ \tau)$ is rarely FRS-T_2 space, then $(Y,\ \sigma)$ is also a rarely FRS-T_2 space.*

Proof. $\lambda,\ \mu \in I^X$ with $\lambda \neq \mu$. Since f is injective, $f(\lambda) \neq f(\mu)$. Since $(X,\ \tau)$ is rarely FRS-T_2-space, there exist r-frso sets $\rho_1,\ \rho_2 \in I^X$ with $\rho_1 \neq \rho_2$ and a r-fuzzy rare set $\gamma \in I^X$ with $\rho_1 + C_\tau(\gamma, r) \geq \overline{1}$ and $\rho_2 + C_\tau(\gamma, r) \geq \overline{1}$ such that $\lambda \leq \rho_1 \vee \gamma$ and $\mu \leq \rho_2 \vee \gamma$. Since f is fuzzy regular semi open, $f(\rho_1),\ f(\rho_2) \in I^Y$ are r-frso sets with $f(\rho_1) \neq f(\rho_2)$. Since f is fuzzy regular semi open and one-to-one, $f(\gamma)$ is also an r-fuzzy rare set with $f(\rho_1) + C_\sigma(\gamma, r) \geq \overline{1}$ and $f(\rho_2) + C_\sigma(\gamma, r) \geq \overline{1}$ such that $f(\lambda) \leq f(\rho_1 \vee \gamma)$ and $f(\mu) \leq f(\rho_1 \vee \gamma)$. Thus $(Y,\ \sigma)$ is rarely FRS-T_2-space. □

Acknowledgements

The author would like to thank from the anonymous reviewers for carefully reading of the manuscript and giving useful comments, which will help us to improve the paper.

References

[1] B. Amudhambigai, M. K. Uma and E. Roja. On rarely $\widetilde{g}$-continuous functions in smooth fuzzy topological spaces. The Journal of Fuzzy Mathematics 20 (2) (2012), 433-442.

[2] C. L. Chang. Fuzzy topological spaces. J. Math. Anal. Appl. 24 (1968), 182-190.

[3] K. C. Chattopadhyay and S. K. Samanta. Fuzzy topology. Fuzzy Sets and Systems 54 (1993), 207-212.

[4] S. Jafari. A note on rarely continuous functions. Univ. Bacau. Stud. Cerc. St. Ser. Mat. 5 (1995), 29-34.

[5] S. Jafari. On some properties of rarely continuous functions. Univ. Bacau. Stud. Cerc. St. Ser. Mat. 7 (1997), 65-73.

[6] N. Levine. Decomposition of continuity in topological spaces. Amer. Math. Monthly 68 (1) (1961), 44-46.

[7] P. E. Long and L. L. Herrington. Properties of rarely continuous functions. Glasnik Math. 17 (37) (1982), 147-153.

[8] R. Lowen. Fuzzy topological spaces and fuzzy compactness. J. Math. Anal. Appl. 56 (1976), 621-633.

[9] V. Popa. sur certain decompositionde la continuite dans les espaces topologiques. Glasnik Mat. Setr III. 14 (34) (1979), 359-362.

[10] Seok Jong Lee and Eun Pyo Lee. Fuzzy r-regular open sets and fuzzy almost r-continuous maps. Bull. Korean Math. Soc. 39 (3) (2002), 441-453.

[11] A. P. Šostak. On a fuzzy topological structure. Rend. Circ. Matem. Palermo Ser II. 11 (1986), 89-103.

[12] A. P. Šostak. Basic structures of fuzzy topology. J. Math. Sci. 78 (6) (1996), 662-701.

[13] A. P. Šostak. On the neighborhood structure of a fuzzy topological spaces. Zh. Rodova Univ. Nis. Ser. Math. 4 (1990), 7-14.

[14] A. Vadivel and E. Elavarasan. On rarely generalized regular fuzzy continuous functions in fuzzy topological spaces. Sahand Communications in Mathematical Analysis 4 (1) (2016), 101-108.

[15] A. Vadivel and E. Elavarasan. Applications of r-generalized regular fuzzy closed sets. Annals of Fuzzy Mathematics and Informatics 12 (5) (2016), 719-738.

[16] A. Vadivel and E. Elavarasan. r-fuzzy regular semi open sets in smooth topological spaces. Sahand Communications in Mathematical Analysis 6 (1) (2017), 1-17.

Scientia Magna
Vol. 13 (2018), No. 1, 88-92

The mean value of exponential divisor function over square-full numbers

Dan Liu[1] and Ya-nan Xu[2]

[1]School of Mathematics and Statistics, Shandong Normal University, Shandong, China
[2]Modern Education Center, Shandong Police College, Shandong, China
E-mail: liudanprime@.163.com, 1071730135@qq.com

Abstract Let $n > 1$ be an integer. The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i}$, if $b_i \mid a_i$ for every $i \in 1, 2, \cdots, s$. Let $\tau^{(e)}(n)$ denote the exponential divisor function. In this paper, we will study the mean value of exponential divisor function over square-full numbers, that is

$$\sum_{\substack{n\leq x \\ n\ is\ square-full}} (\tau_3^{(e)}(n))^2 = \sum_{n\leq x} (\tau_3^{(e)}(n))^2 f_2(n),$$

where $f_2(n)$ is the characteristic function of square-full integers, i.e.

$$f_2(n) = \begin{cases} 1, & n \text{ is square-full}, \\ 0, & \text{otherwise}. \end{cases}$$

Keywords Dirichlet convolution; Asympototic formula; Exponential divisor function.

§1. Introduction

Many scholars are interested in researching the divisor problem, and they have obtained a large number of good results. However, there are many problems hasn't been solved. American-Romanian number theorist Florentin Smarandache [6] introduced hundreds of interesting sequences and arithmetical functions. In 1991, he published a book named *only problems*, *not solutions*!, and one problem is that, a number n is called simple number if the product of its proper divisors is less than or equal to n. Generally speaking, $n = p$, or $n = p^2$, or $n = p^3$, or pq, where p and q are distinct primes. The properties of this simple number sequence hasn't been studied yet. And other problems are introduced in this book, such as proper divisor products sequence and the largest exponent (of power p) which divides n, where $p \geq 2$ is an integer.

In the definition of exponential divisor: suppose $n > 1$ is an integer, and $n = \prod_i^t p_i^{a_i}$. If $d = \prod_i^t p_i^{b_i}$ satisfies $b_i \mid a_i, i = 1, 2, \cdots, t$, then d is called an exponential divisor of n, notation $d \mid_e n$. By convention $1 \mid_e 1$.

J.Wu [4] improved the above result got the following result:

$$\sum_{n\le x}\tau^{(e)}(n)=A(x)+Bx^{\frac{1}{2}}+O(x^{\frac{2}{9}}\log x),$$

where

$$A=\prod_p\left(1+\sum_{a=2}^{\infty}\frac{d(a)-d(a-1)}{p^a}\right),$$

$$B=\prod_p\left(1+\sum_{a=5}^{\infty}\frac{d(a)-d(a-1)-d(a-2)+d(a-3)}{p^{\frac{a}{2}}}\right).$$

M.V.Subbarao [2] also proved for some positive integer r,

$$\sum_{n\le x}(\tau^{(e)}(n))^r\sim A_r x,$$

where

$$A_r=\prod_p\left(1+\sum_{a=2}^{\infty}\frac{(d(a))^r-(d(a-1))^r}{p^a}\right).$$

L.Toth [3] proved

$$\sum_{n\le x}(\tau^{(e)}(n))^r=A_r(x)+x^{\frac{1}{2}}P_{2^r-2}(\log x)+O(x^{u_r+\varepsilon}),$$

where $P_{2^r-2}(t)$ is a polynomial of degree 2^r-2 in t, $u_r=\frac{2^{r+1}-1}{2^{r+1}+1}$.

Similarly to the generalization of $d_k(n)$ from $d(n)$, we define the function $\tau_k^{(e)}(n)$:

$$\tau_k^{(e)}(n)=\prod_{p_i^{a_i}\| n}d_k(a_i),k\ge 2,$$

Obviously when $k=2$, that is $\tau^{(e)}(n)$. $\tau_3^{(e)}(n)$ is obviously a multiplicative function. In this paper we investigate the case $k=3$, i.e. the properties of the function $\tau_3^{(e)}(n)$.

In this paper, we will study the asymptotic formula for the mean value of the function $(\tau_3^{(e)}(n))^2$ over square-full numbers.

Theorem 1.1. *We have the asymptotic formula*

$$\sum_{\substack{n\le x\\ n\ is\ square-full}}(\tau_3^{(e)}(n))^2=x^{\frac{1}{2}}R_8(\log x)+O(x^{\frac{35}{108}+\varepsilon}),$$

where $R_8(t)$ is a polynomial of degree 8 in t.

Notation. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

§2. Some lemmas

lemma 2.1. *Let*

$$\tau_3^{(e)}(n)=\prod_{p_i^{a_i}\| n}d_3(a_i),$$

then we have

$$\sum_{n\le x}(\tau_3^{(e)}(n))^2=\zeta^9(2s)G(s),$$

where the infinite series $G(s)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$ *is absolutely convergent for* $\Re s>\frac{1}{3}$.

Proof. By Euler's product formula, we can get

$$\begin{aligned}\sum_{\substack{n=1\\ n\ is\ square-full}}^{\infty}\frac{(\tau_3^{(e)}(n))^2}{n^s}&=\sum_{n=1}^{\infty}\frac{(\tau_3^{(e)}(n))^2f_2(n)}{n^s}\\ &=\prod_p\left(1+\frac{d_3^2(2)f_2(p^2)}{p^{2s}}+\frac{d_3^2(3)f_2(p^3)}{p^{3s}}+\frac{d_3^2(4)f_2(p^4)}{p^{4s}}+\cdots\frac{d_3^2(r)f_2(p^r)}{p^{rs}}\right)\\ &=\prod_p\left(1+\frac{3^2}{p^{2s}}+\frac{3^2}{p^{3s}}+\frac{6^2}{p^{4s}}+\frac{3^2}{p^{5s}}+\cdots\right)\\ &=\zeta^9(2s)\left(1+\frac{9}{p^{3s}}+\cdots\right)\\ &=\zeta^9(2s)G(s),\end{aligned}\tag{1}$$

where the infinite series $G(s):=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$ is absolutely convergent for $\Re s>\frac{1}{3}$. □

lemma2.2. *Suppose* $k\ge 2$ *is an integer. Then*

$$D_k(x)=\sum_{n\le x}d_k(n)=x\sum_{j=0}^{k-1}c_j(\log x)^j+O(x^{\alpha_k+\varepsilon}),$$

where c_j *is a calculable constant,* ε *is a sufficiently small positive constant,* α_k *is the infimum of numbers* α_k, *such that*

$$\Delta_k(x)=\sum_{n\le x}d_k(n)-xP_{k-1}(\log x)\ll x^{\alpha_k+\varepsilon},\tag{2}$$

and

$$\alpha_2\le\frac{131}{416},\quad \alpha_3\le\frac{43}{94},$$
$$\alpha_k\le\frac{3k-4}{4k},\quad 4\le k\le 8,$$

$$\begin{aligned}&\alpha_9\le\frac{35}{54},\quad \alpha_{10}\le\frac{41}{61},\ \alpha_{11}\le\frac{7}{10},\\ &\alpha_k\le\frac{k-2}{k+2},\quad 12\le k\le 25,\\ &\alpha_k\le\frac{k-1}{k+4},\quad 26\le k\le 50,\\ &\alpha_k\le\frac{31k-98}{32k},\quad 51\le k\le 57,\\ &\alpha_k\le\frac{7k-34}{7k},\quad k\ge 58.\end{aligned}$$

lemma2.3. *Let*

$$D(\underbrace{2,\cdots,2}_{k};x)=\sum_{n\leq x}d(\underbrace{2,\cdots,2}_{k};n),$$

then we have

$$D(\underbrace{2,\cdots,2}_{k};x)=x^{\frac{1}{2}}P_{k-1}(\log x)+O(x^{\alpha_k+\varepsilon}),$$

where, the definition of α_k is as above.

Proof. Recall that

$$d(\underbrace{2,\cdots,2}_{\text{k}};n)=\sum_{n=a_1^2\cdots a_k^2}1,$$

by hyperbolic summation formula, we have

$$D(\underbrace{2,\cdots,2}_{\text{k}};x)=\sum_{n\leq x}d(\underbrace{2,\cdots,2}_{\text{k}};n)=\sum_{m^2\leq x}d_k(m),$$

from lemma 2.2, we can get

$$D(\underbrace{2,\cdots,2}_{\text{k}};n)=x^{\frac{1}{2}}P_{k-1}(\log x)+O(x^{\alpha_k+\varepsilon}),$$

where α_k is as defined in lemma 2.2. □

lemma2.4. *Suppose $f(m),g(n)$ are arithmetical functions such that*

$$\sum_{m\leq x}f(m)=\sum_{j=1}^{J}x^{\alpha_j}P_j(\log x)+O(x^{\alpha}),\quad \sum_{n\leq x}\mid g(n)\mid=O(x^{\beta}),$$

where $\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_J>\alpha>\beta>0, P_j(t)$ is a polynomial in t, if $h(n)=\sum_{n=md}f(m)g(d)$, then

$$\sum_{n\leq x}h(n)=\sum_{j=1}^{J}x^{\alpha_j}Q_j(\log x)+O(x^{\alpha}),$$

where $Q_j(t)$ $j=1,\cdots,J$ is a polynomial in t.

§3. Proof of Theorem 1.1

Proof. From lemma 2.1, we have $G(s)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$ is absolutely convergent for $\Re s>\frac{1}{3}$, and then

$$\sum_{n\leq x}\mid g(n)\mid\ll x^{\frac{1}{3}+\epsilon}.$$

Let

$$F(s)=\zeta^9(2s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s},$$

where

$$f(n) = d(\underbrace{2, \cdots, 2}_{9}; n).$$

From lemma 2.3, we have

$$\sum_{n \le x} f(n) = \sum_{n \le x} d(\underbrace{2, \cdots, 2}_{9}; n) = x^{\frac{1}{2}} Q_8(\log x) + O(x^{\frac{35}{108}+\varepsilon}).$$

where $Q_8(\log x)$ is a polynomial in $\log x$ of degree 8, α_k is defined in lemma 2.2. From lemma 2.1, we have

$$(\tau_3^{(e)}(n))^2 f_2(n) = \sum_{n=ml} f(m)g(l).$$

From lemma 2.4, we complete the proof of Theorem 1.1. □

Acknowledgements

The authors express their gratitude to the referee for a careful reading of the manuscript and many valuable suggestions which highly improve the quality of this paper.

References

[1] A. Ivić, The Riemann zeta-function: theory and applications. Oversea Publishing House,2003.

[2] M.V.Subbarao, On some arithmetic convolutions. In: The Theory of Arithmetic Functions. Lecture Notes in Mathematics. Vol.251, Springer,1972,247-271.

[3] L. Tóth, An order result for the exponential divisor function. Publ. Math. Debrecen,2007,71(1-2):165-171.

[4] J. Wu, Problème de diviseurs exponentiels at entiers exponentiellement sans facteur carré [J]. Théor Nombres Bordeaux,7(1995),133-141.

[5] L. Zhang, M. Lü and W. Zhai, On the Smarandache ceil function and the Dirichlet divisor function. Sci. Magna,2008,4(4):55-57.

[6] F. Smarandache, Only problems, Not solutions! Chicago: Xiquan Publishing House,1993.

Scientia Magna
Vol. 13 (2018), No. 1, 93-104

Resonance between exponential functions and divisor function over arithmetic progressions

Xuejiao Liu [1] and Meimei Lv[2]

[1]Department of Mathematics and Statistics, Shandong Normal University, Shandong, China
[2]No. 34 Middle School, Jinan, Shandong, China
E-mail: xiaoshitiao@163.com,lvmeimei2001@126.com

Abstract We study the resonance phenomenon between divisor function and exponential functions of the form $e(\alpha n^{\beta})$, where $0 \neq \alpha \in R$ and $0 < \beta < 1$. An asymptotic formula is established for the nonlinear exponential sum

$$\sum_{\substack{n\sim X\\ n\equiv l \bmod q}} d(n)e(\alpha n^{\beta})$$

when $\beta = \frac{1}{2}$ and $|\alpha|$ is close to $\frac{2\sqrt{k}}{q}, k \in Z^{+}$.

Keywords divisor function, nonlinear exponential sum, arithmetic progression.
2010 Mathematics Subject Classification 11N37, 11L07, 11L05.

§1. Introduction and main results

Let $d(n)$ denote the number of divisors of n. The properties of divisor function $d(n)$ attract many researchers' interest, and they have got many generalization of the divisor problem (for example, see [3], [4]). However, there are many problems hasn't been solved. For example, F.Smarandache gave some unsolved problems in his book *only problems, not solutions*! (see [5]). Here we focus our attention on the resonance between divisor function and exponential functions.

In 1916, Hardy [6] studied the sum

$$S(X,t) = \sum_{1\leq n\leq X} n^{-\frac{1}{2}} d(n) e^{-it\sqrt{n}}$$

and showed that, if $t \neq 4\pi q^{1/2}$ for any positive integer q, then

$$S(X,t) = o(X^{\varepsilon})$$

and, if $t = 4\pi q^{1/2}$ for some integer q, then

$$S(X,t) = \frac{2(1+i)d(q)}{q^{\frac{1}{4}}} X^{\frac{1}{4}} + o(X^{\varepsilon})$$

as $X \to \infty$. The above result can be seen as the resonance phenomenon between divisor function and exponential functions. Recently, Sun and Wu [14] considered a similar problem and proved when $|\alpha|\beta X^\beta < \frac{\sqrt{X}}{2}$, then

$$\sum_{n\le X} d(n)e(\alpha n^\beta) \ll (|\alpha|\beta X^\beta)^{-1} X\log X,$$

and when $|\alpha|\beta X^\beta \ge \frac{\sqrt{X}}{2}$, then

$$\sum_{n\le X} d(n)e(\alpha n^\beta) \begin{cases} \ll |2\beta-1|^{-\frac{1}{2}}(|\alpha|\beta X^\beta)^{1+\varepsilon}, & \text{if} \quad \beta\ne\frac{1}{2},\\ = \kappa_{\alpha,q}\eta(\alpha,q)X^{\frac{3}{4}}d(q)q^{-\frac{1}{4}} + O\left(|\alpha|^{\frac{1}{2}}X^{\frac{1}{4}+\varepsilon} + |\alpha|^{-1}X^{\frac{1}{2}+\varepsilon}\right), & \text{if} \quad \beta=\frac{1}{2}, \end{cases}$$

where $\kappa_{\alpha,q} = 1$ or 0 according to if there exists a positive integer q satisfying

$$||\alpha| - 2\sqrt{q}| \le X^{-\frac{1}{2}}, \quad 1 \le |\alpha| < \sqrt{X},$$

and

$$\eta(\alpha,q) = \frac{1-\operatorname{sgn}(\alpha)i}{2}\int_1^2 u^{-\frac{1}{4}} e\left(\operatorname{sgn}(\alpha)\left(|\alpha|-2\sqrt{q}\right)\sqrt{uX}\right)du.$$

In 1973, Saburô Uchiyama [16] considered the sum

$$U(\alpha,X) = U(\alpha,X,q,l) = \sum_{\substack{1\le n\le X\\ n\equiv l(\bmod q)}} n^{-\frac{1}{2}}d(n)\mathrm{e}(\alpha\sqrt{n}) \quad (\alpha>0),$$

where q and l are integers with $q\ge 1, 0\le l<q$, and showed that, if $\alpha \ne \frac{2\sqrt{k}}{q}$ for any integer k, then

$$U(\alpha,X) = O_\alpha(\log X),$$

and if $\alpha = \frac{2\sqrt{k}}{q}$ for some integer k, then

$$U(\alpha,X) = \frac{2(1-i)\sigma(k;q,l)}{q^{\frac{3}{2}}k^{\frac{1}{4}}}X^{\frac{1}{4}} + O_\alpha(\log X),$$

provided that $\alpha \ge 4q^3$, where

$$\sigma(k;q,l) = \sum_{m|k} S\left(\frac{k}{m}, ml; q\right),$$

and $S(m,n;q)$ denotes the Kloosterman sum.

Motivated by the above results, we study the resonance phenomenon between divisor function and exponential functions over arithmetic progressions. To do this, using a different method we establish an asymptotic formula when $\beta = \frac{1}{2}$ and $|\alpha|$ is close to $\frac{2\sqrt{k}}{q}, k\in Z^+$, and obtain a new result when $\beta \ne \frac{1}{2}$. Our main instruments are Voronoi formula, the estimation of exponential sums and the weighted stationary phase. Due to the orthogonality of the additive characters, we can obtain the Kloosterman sum after applying the Voronoi summation formula, which let us can use Weil's bound to get the saving in the q-aspect.Our result is the following theorem.

Theorem 1.1. *Suppose $X>1$, $0<\beta<1$ and $0\ne\alpha\in R$. Suppose also $l,q\in N$ and $l\le q\le X^{\frac{1}{2}}$.*
(i) For $|\alpha|\beta X^\beta < \frac{\sqrt{X}}{2q}$, one has

$$\sum_{\substack{n\sim X\\ n\equiv l \bmod q}} d(n)e(\alpha n^\beta) \ll q^{\frac{1}{2}+\varepsilon}\left(|\alpha|\beta X^\beta\right)^{-1} X\log X.$$

(ii) For $|\alpha|\beta X^\beta \geq \frac{\sqrt{X}}{2q}$ *and* $\beta \neq \frac{1}{2}$*, one has*

$$\sum_{\substack{n\sim X\\ n\equiv l \bmod q}} d(n)e(\alpha n^\beta) \ll q^{\frac{1}{2}+\varepsilon}|2\beta-1|^{-\frac{1}{2}}\left(|\alpha|\beta X^\beta\right)^{1+\varepsilon}.$$

(iii) For $|\alpha|\beta X^\beta \geq \frac{\sqrt{X}}{2q}$ *and* $\beta = \frac{1}{2}$*, if* $|\alpha| < \frac{1}{q}$ *or* $|\alpha| \geq \frac{\sqrt{X}}{q}$ *one has*

$$\sum_{\substack{n\sim X\\ n\equiv l \bmod q}} d(n)e(\alpha n^\beta) \ll (q|\alpha|)^{\frac{1}{2}+\varepsilon}X^{\frac{1}{4}+\epsilon},$$

if $\frac{1}{q} \leq |\alpha| \leq \frac{\sqrt{X}}{q}$ *one has*

$$\begin{aligned}\sum_{\substack{n\sim X\\ n\equiv l \bmod q}} d(n)e(\alpha n^\beta) &= \frac{1}{q}\sum_{k|q}\mathcal{E}(\alpha,n_k)k^{-\frac{1}{2}}X^{\frac{3}{4}}d(n_k)S(-l,-n_k;k)n_k^{-\frac{1}{4}}\\ &\quad +O\left((q|\alpha|)^{\frac{1}{2}+\varepsilon}X^{\frac{1}{4}+\varepsilon}\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}(\alpha,n_k) &= \frac{\delta_k\epsilon_\alpha(1+i)}{2}\int_1^2 u^{-\frac{1}{4}}e\left(\operatorname{sgn}(\alpha)\left(|\alpha|-\frac{2\sqrt{n_k}}{k}\right)\sqrt{Xu}\right)du,\\ \epsilon_\alpha &= \begin{cases}-i, & \text{if} \quad \alpha>0,\\ 1, & \text{if} \quad \alpha<0,\end{cases}\end{aligned}$$

and $\delta_k = 1$ *or 0 according to whether there is a positive integer* n_k *for* $k|q$ *satisfying*

$$|k|\alpha| - 2\sqrt{n_k}| \leq X^{-\frac{1}{2}}$$

or not.

§2. Some lemmas

To prove Theorem 1.1, we need quote some lemmas. First, we introduce some notations. Let the Kloosterman sum be defined as

$$S(m,n;c) = \sideset{}{^*}\sum_{d \bmod c} \mathrm{e}\left(\frac{md+n\overline{d}}{c}\right),$$

where the sum is extended over a reduced set of residues modulo c and $\overline{d}$ denotes the inverse of d modulo c. Then the famous Weil's bound of Kloosterman sum gives

$$|S(m,n;c)| \leq (m,n,c)^{\frac{1}{2}}c^{\frac{1}{2}}d(c).$$

Let K_v and Y_v denote the standard K-Bessel function and Y-Bessel function, respectively. Then we have the following Voronoi formula (see [7]).

Lemma 2.1. *Suppose that* $0<a<b$, $(h,k)=1$, $f(x)\in C^1[a,b]$, *Then*

$$k\sideset{}{'}\sum_{a\leq n\leq b} d(n)f(n)e\left(\frac{nh}{k}\right) = G_3 + \sum_{n=1}^{\infty} d(n)e\left(-\frac{n\overline{h}}{k}\right)G_4(n) + \sum_{n=1}^{\infty} d(n)e\left(\frac{n\overline{h}}{k}\right)G_5(n), \quad (2.1)$$

where G_3, $G_4(y)$ and $G_5(y)$ are the following integral transforms:

$$G_3 = \int_a^b (\log x + 2\gamma - 2\log k) f(x)dx,$$

$$G_4(y) = -2\pi \int_0^\infty Y_0\left(4\pi\frac{\sqrt{xy}}{k}\right) f(x)dx,$$

$$G_5(y) = 4\int_0^\infty K_0\left(4\pi\frac{\sqrt{xy}}{k}\right) f(x)dx,$$

where $\sum_{a\le n\le b}{}'$ means that if a or b is an integer then the first or the last term in the sum (1.1) is halved, and γ is Euler's constant.

We need the following lemma for asymptotic expansions of the Bessel functions (see [1]).

Lemma 2.2. *Suppose that $z > 0$, Then*

$$K_0(z) = \sqrt{\frac{\pi}{2z}} e^{-z}\left\{1 + O\left(z^{-1}\right)\right\}, \tag{2.2}$$

$$Y_0(z) = \sqrt{\frac{2}{\pi z}}\left\{\sin\left(z - \frac{\pi}{4}\right) - \frac{1}{8z}\cos\left(z - \frac{\pi}{4}\right) + O\left(z^{-2}\right)\right\}. \tag{2.3}$$

We also need the following result (see [15, Lemmas 4.3 and 4.5]).

Lemma 2.3. *Let $G(x)$ and $F(x)$ be real function in $[a,b]$ with $G(x)/F'(x)$ monotonic. Suppose that $|G(x)| \le M$.*
(i) If $F'(x) \ge u > 0$ or $F'(x) \le -u < 0$, then

$$\int_a^b G(x)\, e\left(F(x)\right) dx \ll \frac{M}{u}.$$

(ii) If $F''(x) \ge v > 0$ or $F''(x) \le -v < 0$, then

$$\int_a^b G(x) e\left(F(x)\right) dx \ll \frac{M}{\sqrt{v}}.$$

§3. Proof of theorem 1.1

In this section, we give the proof of Theorem 1.1. Using the formula of the Ramanujan sum

$$\sum_{d|q}\sum_{\substack{a=1\\(a,d)=1}}^{d} e\left(\frac{an}{d}\right) = \begin{cases} q, & q \mid n,\\ 0, & q \nmid n,\end{cases}$$

we obtain

$$\sum_{\substack{n\sim X\\ n\equiv l \bmod q}} d(n) e\left(\alpha n^\beta\right) = \frac{1}{q}\sum_{k|q}\ \sideset{}{^*}\sum_{h \bmod k} e\left(-\frac{hl}{k}\right)\sum_{n\sim X} d(n) e\left(\frac{hn}{k}\right) e\left(\alpha n^\beta\right), \tag{3.1}$$

where the $\sum^*$ means the summation is restricted by $(h,k) = 1$.

Let $\Delta > 1$ and $0 \le \phi(x) \le 1$ be a C^∞ function supported on $[1,2]$, which is identically 1 on $[1+\Delta^{-1}, 2-\Delta^{-1}]$ and satisfies $\phi^{(r)}(x) \ll \Delta^r$ for $r \le 0$. Using the well-known bound $\sum_{n\le x} d(n) \ll x\log x$, we have

$$\sum_{n\sim X} d(n)e\left(\frac{hn}{k}\right)e\left(\alpha n^\beta\right) = \sum_{n=1}^{\infty} d(n)e\left(\frac{hn}{k}\right)W(n) + O\left(\frac{X}{\Delta}\log X\right), \tag{3.2}$$

where

$$W(x) = \phi\left(\frac{x}{X}\right)e\left(\alpha x^\beta\right).$$

Applying Lemma 2.1 with $f(n) = W(n)$we have

$$k\sum_{n=1}^{\infty} d(n)e\left(\frac{hn}{k}\right)W(n) = G_3 + \sum_{n=1}^{\infty} d(n)e\left(-\frac{n\bar{h}}{k}\right)G_4(n) + \sum_{n=1}^{\infty} d(n)e\left(\frac{n\bar{h}}{k}\right)G_5(n), \tag{3.3}$$

where

$$G_3 = \int_0^\infty (\log x + 2\gamma - 2\log k)\,W(x)\,dx,$$

$$G_4(y) = -2\pi\int_0^\infty Y_0\left(\frac{4\pi\sqrt{xy}}{k}\right)W(x)\,dx,$$

$$G_5(y) = 4\int_0^\infty K_0\left(\frac{4\pi\sqrt{xy}}{k}\right)W(x)\,dx.$$

Now, we estimate the contributions from these three terms $G_3, G_4(n)$ and $G_5(n)$ to (3.1), respectively. First, we estimate the contribution from the term G_3. Changing variables $x = Xt$ and applying (i) in Lemma 2.3, we have

$$\begin{aligned} G_3 &= X\int_1^2 (\log X + \log t + 2\gamma - 2\log k)\,\phi(t)e\left(\alpha(tX)^\beta\right)dt \\ &\ll X\frac{\log X}{|\alpha|\beta X^\beta} \\ &= \left(|\alpha|\beta X^\beta\right)^{-1} X\log X. \end{aligned}$$

Thus the contribution from G_3 to (3.1) is

$$\begin{aligned} &\frac{1}{q}\sum_{k|q}\ \sum_{h \bmod k}{}^{*}\, e\left(-\frac{hl}{k}\right)\frac{1}{k}G_3 \\ &\ll \frac{1}{q}\sum_{k|q}\ \sum_{h \bmod k}{}^{*}\, e\left(-\frac{hl}{k}\right)\frac{1}{k}\left(|\alpha|\beta X^\beta\right)^{-1} X\log X \\ &\ll q^{-1+\varepsilon}\left(|\alpha|\beta X^\beta\right)^{-1} X\log X. \end{aligned} \tag{3.4}$$

Second, we estimate the contribution from the term $G_5(n)$. Using (2.2) in Lemma 2.2 with $z = \frac{4\pi\sqrt{xy}}{k}$, we have

$$K_0\left(\frac{4\pi\sqrt{xy}}{k}\right) = \frac{1}{2\sqrt{2}}k^{\frac{1}{2}}(xy)^{-\frac{1}{4}}\exp\left(-\frac{4\pi\sqrt{xy}}{k}\right)\left\{1 + O\left(k(xy)^{-\frac{1}{2}}\right)\right\}.$$

Inserting it into $G_5(y)$, for $y \geq 1$ we obtain

$$\begin{aligned} G_5(y) &= \sqrt{2}k^{\frac{1}{2}}y^{-\frac{1}{4}}\int_0^{\infty} x^{-\frac{1}{4}}\exp\left(-\frac{4\pi\sqrt{xy}}{k}\right)\left\{1+O\left(k(xy)^{-\frac{1}{2}}\right)\right\}\phi\left(\frac{x}{X}\right)e\left(\alpha x^{\beta}\right)dx \\ &\ll k^{\frac{1}{2}}y^{-\frac{1}{4}}\int_X^{2X} x^{-\frac{1}{4}}\exp\left(-\frac{4\pi\sqrt{xy}}{k}\right)dx \\ &\ll k^{\frac{1}{2}}X^{\frac{3}{4}}y^{-\frac{1}{4}}\exp\left(-\frac{4\pi\sqrt{Xy}}{k}\right). \end{aligned}$$

Applying the Weil's bound of Kloosterman sum and using the well-known bound $\sum_{n\leq x} d(n)^2 \ll x\log^3 x$, we can derive that the contribution from $G_5(n)$ to (3.1) is

$$\begin{aligned} &\frac{1}{q}\sum_{k|q}\sum_{h \bmod k}{}^{*} e\left(-\frac{hl}{k}\right)\frac{1}{k}\sum_{n=1}^{\infty} d(n)e\left(\frac{n\overline{h}}{k}\right)G_5(n) \\ &= \frac{1}{q}\sum_{k|q}\frac{1}{k}\sum_{n=1}^{\infty} d(n)S\left(-l,n;k\right)G_5(n) \\ &\ll \frac{1}{q}\sum_{k|q}\frac{1}{k}\sum_{n=1}^{\infty} d(n)|S\left(-l,n;k\right)|k^{\frac{1}{2}}X^{\frac{3}{4}}n^{-\frac{1}{4}}\exp\left(-\frac{4\pi\sqrt{Xn}}{k}\right) \\ &\ll \frac{1}{q}\sum_{k|q}\frac{1}{k}\sum_{n=1}^{\infty} d(n)\left(l,n,k\right)^{\frac{1}{2}}d(k)kX^{\frac{3}{4}}n^{-\frac{1}{4}}\exp\left(-\frac{4\pi\sqrt{Xn}}{k}\right) \qquad (3.5) \\ &= \frac{1}{q}\sum_{k|q} d(k)X^{\frac{3}{4}}\sum_{n\leq k^2X^{-1+\varepsilon}}(l,n,k)^{\frac{1}{2}}d(n)n^{-\frac{1}{4}} + O\left(X^{-100}\right) \\ &\ll \frac{1}{q}\sum_{k|q} d(k)X^{\frac{3}{4}}\left(\sum_{n\leq k^2X^{-1+\varepsilon}}|d(n)|^2\right)^{\frac{1}{2}}\left(\sum_{n\leq k^2X^{-1+\varepsilon}}(l,n,k)\,n^{-\frac{1}{2}}\right)^{\frac{1}{2}} + O\left(X^{-100}\right) \\ &\ll q^{\frac{1}{2}+\varepsilon}X^{\varepsilon}. \end{aligned}$$

Last, we turn to estimate the contribution from the term $G_4(n)$. Using (2.3) in Lemma 2.2, we have

$$Y_0(z) = \frac{1-i}{2i\sqrt{\pi z}}\left(e^{iz}-ie^{-iz}\right) - \frac{1-i}{16\sqrt{\pi z^3}}\left(e^{iz}+ie^{-iz}\right) + O\left(z^{-\frac{5}{2}}\right),$$

Inserting it into $G_4(y)$ with $z = \frac{4\pi\sqrt{xy}}{k}$, we get

$$\begin{aligned} G_4(y) = &\frac{(1+i)k^{\frac{1}{2}}}{2}\int_0^{\infty}(xy)^{-\frac{1}{4}}\phi\left(\frac{x}{X}\right)e\left(\alpha x^{\beta}\right)\left(e\left(\frac{2\sqrt{xy}}{k}\right)-ie\left(-\frac{2\sqrt{xy}}{k}\right)\right)dx \\ &+\frac{k^{\frac{3}{2}}(1-i)}{2\cdot 32\pi}\int_0^{\infty}(xy)^{-\frac{3}{4}}\phi\left(\frac{x}{X}\right)e\left(\alpha x^{\beta}\right)\left(e\left(\frac{2\sqrt{xy}}{k}\right)+ie\left(-\frac{2\sqrt{xy}}{k}\right)\right)dx \\ &+O\left(k^{\frac{5}{2}}\int_0^{\infty}(xy)^{-\frac{5}{4}}\phi\left(\frac{x}{X}\right)dx\right). \end{aligned}$$

Changing variable $x = Xt^2$, we obtain

$$G_4(y) = (1+i)k^{\frac{1}{2}}X^{\frac{3}{4}}y^{-\frac{1}{4}}\int_0^{\infty} t^{\frac{1}{2}}\phi\left(t^2\right)e\left(\alpha X^{\beta}t^{2\beta}\right)\left(e\left(\frac{2\sqrt{Xy}}{k}t\right)-ie\left(-\frac{2\sqrt{Xy}}{k}\right)t\right)dt$$

$$
\begin{aligned}
&+\frac{k^{\frac{3}{2}}(1-i)}{32\pi}X^{\frac{1}{4}}y^{-\frac{3}{4}}\int_0^\infty t^{-\frac{1}{2}}\phi(t^2)e\left(\alpha X^\beta t^{2\beta}\right)\left(e\left(\frac{2\sqrt{Xy}}{k}t\right)+ie\left(-\frac{2\sqrt{Xy}}{k}\right)t\right)dt\\
&+O\left(2k^{\frac{3}{2}}X^{-\frac{1}{4}}y^{-\frac{5}{4}}\int_0^\infty t^{-\frac{5}{2}}\phi\left(t^2\right)dt\right)\\
=\ &G_{41}(y)+O\left(k^{\frac{3}{2}}X^{-\frac{1}{4}}y^{-\frac{5}{4}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
G_{41}(y)=&\ a_1k^{\frac{1}{2}}X^{\frac{3}{4}}y^{-\frac{1}{4}}\left(P_+\left(\frac{2\sqrt{Xy}}{k}\right)-iP_+\left(-\frac{2\sqrt{Xy}}{k}\right)\right)\\
&+a_2k^{\frac{3}{2}}X^{\frac{1}{4}}y^{-\frac{3}{4}}\left(P_-\left(\frac{2\sqrt{Xy}}{k}\right)+iP_-\left(-\frac{2\sqrt{Xy}}{k}\right)\right),
\end{aligned}
\tag{3.6}
$$

with

$$
a_1=1+i,\quad a_2=\frac{1-i}{32\pi},
$$

and

$$
P_\pm(w)=\int_0^\infty t^{\pm\frac{1}{2}}\phi\left(t^2\right)e\left(\alpha X^\beta t^{2\beta}+wt\right)dt. \tag{3.7}
$$

The contribution from O-term to (3.1) is

$$
\begin{aligned}
&\frac{1}{q}\sum_{k|q}\ \sideset{}{^*}\sum_{h \bmod k}e\left(-\frac{hl}{k}\right)\frac{1}{k}\sum_{n=1}^\infty d(n)e\left(\frac{n\overline{h}}{k}\right)k^{\frac{5}{2}}X^{-\frac{1}{4}}n^{-\frac{5}{4}}\\
&=\frac{1}{q}\sum_{k|q}\frac{1}{k}\sum_{n=1}^\infty d(n)S(-l,n;k)k^{\frac{1}{2}}k^2X^{-\frac{1}{4}}n^{-\frac{5}{4}}\\
&\ll q^{1+\varepsilon}X^{-\frac{1}{4}+\varepsilon}.
\end{aligned}
\tag{3.8}
$$

The integral $P_\pm(w)$ defined in (3.7) has been studied by Ren and Ye [13] and Sun and Wu [14]. Here, we just briefly recall their estimates. Due to the differences in parameters, we shall choose them carefully later to get the q-aspect saving. Let

$$
f(t)=\alpha X^\beta t^{2\beta}+\omega t.
$$

Then we have

$$
\begin{aligned}
f'(t)\quad &=2\alpha\beta X^\beta t^{2\beta-1}+\omega\\
&=\mathrm{sgn}(\alpha)\left(2|\alpha|\beta X^\beta t^{2\beta-1}+\mathrm{sgn}(\alpha)\mathrm{sgn}(\omega)|\omega|\right),
\end{aligned}
$$

$$
f''(t)=2\alpha\beta(2\beta-1)X^\beta t^{2\beta-2}.
$$

If $\alpha w>0$ or

$$
\alpha w<0\ \text{ but }\ |w|\notin\left[\frac{1}{2}|\alpha|\beta X^\beta,4|\alpha|\beta X^\beta\right], \tag{3.9}
$$

then for all $t\in[1,\sqrt{2}]$, we have

$$
f'(t)\gg\max\{|\alpha|\beta X^\beta,|w|\},\quad f''(t)\ll|\alpha|\beta X^\beta.
$$

Noting that $\phi(u)$ is supported on $[1,2]$, we have

$$P_{\pm}(w)=\int_{1}^{\sqrt{2}} t^{\pm\frac{1}{2}}\phi(t^2)e\left(\alpha X^{\beta}t^{2\beta}+wt\right)dt.$$

By partial integration we have, for ω satisfying (3.9),

$$P_{\pm}(w)\ll\frac{\Delta^{l-1}}{\max\left\{|\alpha|\beta X^{\beta},|\omega|\right\}^{l}},\quad l=1,2. \tag{3.10}$$

Let $\omega=\pm\frac{2\sqrt{Xy}}{k}$. Then $|\omega|\in\left[\frac{1}{2}|\alpha|\beta X^{\beta},4|\alpha|\beta X^{\beta}\right]$ means that $y\in\left[\frac{1}{16}\left(|\alpha|\beta k\right)^2X^{2\beta-1},4\left(|\alpha|\beta k\right)^2X^{2\beta-1}\right]$. For convenience, we write

$$I:=\left[\frac{1}{16}\left(|\alpha|\beta k\right)^2X^{2\beta-1},4\left(|\alpha|\beta k\right)^2X^{2\beta-1}\right].$$

For $y=n\notin I$, by (3.6) and (3.10), we have

$$G_{41}(y)\ll\left(k^{\frac{1}{2}}X^{\frac{3}{4}}y^{-\frac{1}{4}}+k^{\frac{3}{2}}X^{\frac{1}{4}}y^{-\frac{3}{4}}\right)R_1(X,y,k), \tag{3.11}$$

where

$$R_1(X,y,k)\ll\frac{\Delta^{l-1}}{\left(\frac{\sqrt{Xy}}{k}\right)^{l}},\quad l=1,2. \tag{3.12}$$

Therefore the contribution from $G_{41}(n)$ with $n\notin I$ to (3.1) is

$$\begin{aligned}&\frac{1}{q}\sum_{k|q}\ \sideset{}{^*}\sum_{h \bmod k} e\left(-\frac{hl}{k}\right)\frac{1}{k}\sum_{n\notin I}d(n)e\left(-\frac{n\overline{h}}{k}\right)G_{41}(n)\\ \ll&\frac{1}{q}\sum_{k|q}\frac{1}{k}\sum_{n=1}^{\infty}d(n)S(-l,-n;k)(k^{\frac{1}{2}}X^{\frac{3}{4}}n^{-\frac{1}{4}}+k^{\frac{3}{2}}X^{\frac{1}{4}}n^{-\frac{3}{4}})R_1(X,y,k).\end{aligned} \tag{3.13}$$

Let $Y=k^2\Delta^2X^{-1}$. Using (3.12) with

$$l=\begin{cases}1, & n\le Y,\\ 2, & n>Y,\end{cases}$$

we get that (3.13) is

$$\begin{aligned}&\ll\frac{1}{q}\sum_{k|q}\frac{1}{k}\sum_{n\le Y}d(n)|S(-l,-n;k)|\left(k^{\frac{1}{2}}X^{\frac{3}{4}}n^{-\frac{1}{4}}+k^{\frac{3}{2}}X^{\frac{1}{4}}n^{-\frac{3}{4}}\right)kX^{-\frac{1}{2}}n^{-\frac{1}{2}}\\ &+\frac{1}{q}\sum_{k|q}\frac{1}{k}\sum_{n>Y}d(n)|S(-l,-n;k)|\left(k^{\frac{1}{2}}X^{\frac{3}{4}}n^{-\frac{1}{4}}+k^{\frac{3}{2}}X^{\frac{1}{4}}n^{-\frac{3}{4}}\right)\Delta k^2X^{-1}n^{-1}\\ &=:\Sigma_1+\Sigma_2,\end{aligned} \tag{3.14}$$

For Σ_1, we have

$$\Sigma_1 = \frac{1}{q}\sum_{k|q}\frac{1}{k}\sum_{n\le Y}d(n)S(-l,-n;k)(k^{\frac{3}{2}}X^{\frac{1}{4}}n^{-\frac{3}{4}}+k^{\frac{5}{2}}X^{-\frac{1}{4}}n^{-\frac{5}{4}})$$

$$\begin{aligned}&\ll \frac{1}{q}\sum_{k|q} d(k)kX^{\frac{1}{4}}\sum_{n\le Y} d(n)(n,l,k)^{\frac{1}{2}}n^{-\frac{3}{4}}\\&\quad+\frac{1}{q}\sum_{k|q} d(k)k^2X^{-\frac{1}{4}}\sum_{n\le Y} d(n)(n,l,k)^{\frac{1}{2}}n^{-\frac{5}{4}}\\&\ll q^{\frac{1}{2}+\varepsilon}\Delta^{\frac{1}{2}+\varepsilon}.\end{aligned}$$

where we have used the well-known bound $\sum_{n\le x} d^2(n) \ll x\log^3 x$ and the Weil's bound for Kloosterman sum. Analogously, since the series $\sum_{n=1}^{\infty}\frac{d(n)^2}{n^{1+\varepsilon}}$ is convergent, so we obtain

$$\Sigma_2 \ll q^{\frac{1}{2}+\varepsilon}\Delta^{\frac{1}{2}+\varepsilon}.$$

Thus, the contribution of (3.14) is

$$\ll q^{\frac{1}{2}+\varepsilon}\Delta^{\frac{1}{2}+\varepsilon}. \tag{3.15}$$

Now we can get our first conclusion. If $|\alpha|\beta X^{\beta} < \frac{\sqrt{X}}{2q}$, then $I\cap Z^+ = \emptyset$, by (3.3) – (3.5), (3.8) and (3.15), choosing

$$\Delta = q^{-\frac{1}{3}}X^{\frac{2}{3}},$$

we have

$$\sum_{\substack{n\sim x\\ n\equiv l \bmod q}} d(n)e(\alpha n^{\beta}) \ll q^{\frac{1}{2}+\varepsilon}(|\alpha|\beta X^{\beta})^{-1}X\log X, \tag{3.16}$$

which proves (i) in Theorem 1.1.

For $y = n\in I$, noting the trivial estimate $\int_1^{\sqrt{2}} t^{\pm\frac{1}{2}}\phi\left(t^2\right)e\left(f(t)\right)dt \ll 1$, and then inserting it into (3.1), we obtain the contribution of the second term in the right of (3.6) is

$$\begin{aligned}&\ll \frac{1}{q}\sum_{k|q}\frac{1}{k}\sum_{n\in I} d(n)|S(-l,-n;k)|k^{\frac{3}{2}}X^{\frac{1}{4}}n^{-\frac{3}{4}}\\&\ll q^{\frac{1}{2}+\varepsilon}\left(|\alpha|\beta X^{\beta}\right)^{\frac{1}{2}+\varepsilon}.\end{aligned} \tag{3.17}$$

For the first term in the right of (3.6), we use (3.10) to bound the terms with $\alpha\omega > 0$ and obtain

$$P_+\left(\frac{2\sqrt{Xy}}{k}\right) - iP_+\left(-\frac{2\sqrt{Xy}}{k}\right) = \epsilon_\alpha\int_0^{\infty} t^{\frac{1}{2}}\phi(t^2)e\left(f_1(t)\right)dt + O(R_1\left(X,y,k\right)),$$

where

$$\epsilon_\alpha = \begin{cases} -i, & \alpha > 0,\\ 1, & \alpha < 0,\end{cases} \tag{3.18}$$

and

$$f_1(t) = f_1(t,y) = \operatorname{sgn}(\alpha)\left(|\alpha|X^{\beta}t^{2\beta} - \frac{2\sqrt{Xy}}{k}t\right). \tag{3.19}$$

Since the contribution from the O-term is absorbed by (3.17), we only need to estimate

$$\epsilon_\alpha\frac{a_1}{q}\sum_{k|q} k^{-\frac{1}{2}}X^{\frac{3}{4}}\sum_{n\in I} d(n)S(-l,-n;k)n^{-\frac{1}{4}}\int_1^{\sqrt{2}} t^{\frac{1}{2}}\phi(t^2)e\left(f_1(t,n)\right)dt. \tag{3.20}$$

To estimate (3.20), we consider two cases according to $\beta = \frac{1}{2}$ or not.

Case 1. If $\beta \neq \frac{1}{2}$, we have

$$f_1''(t) = \operatorname{sgn}(\alpha)|\alpha|(2\beta)\,(2\beta-1)\,X^{\beta}t^{2\beta-2} \gg |\alpha|\beta(2\beta-1)X^{\beta}, \quad \text{for } t \in [1,\sqrt{2}]$$

By partial integration and (ii) in Lemma 2.3, we have

$$\int_1^{\sqrt{2}} t^{\frac{1}{2}}\phi(t^2)e\left(f_1(t,n)\right)dt \ll \left(|\alpha|\beta(2\beta-1)X^{\beta}\right)^{-\frac{1}{2}}.$$

Therefore (3.20) is

$$\begin{aligned} &\ll \left(|\alpha|\beta(2\beta-1)X^{\beta}\right)^{-\frac{1}{2}} X^{\frac{3}{4}}\frac{1}{q}\sum_{k|q}k^{-\frac{1}{2}+\varepsilon}\sum_{n\in I}d(n)|S(-l,-n;k)|n^{-\frac{1}{4}} \\ &\ll q^{\frac{1}{2}+\varepsilon}|2\beta-1|^{-\frac{1}{2}}\left(|\alpha|\beta X^{\beta}\right)^{1+\varepsilon}, \end{aligned} \tag{3.21}$$

which proves (ii) in Theorem 1.1.

Case 2. If $\beta = \frac{1}{2}$, let

$$H_r = 2^{-r}k|\alpha|X^{\frac{1}{2}}, \quad 1 \le r \le r_0 = \left[\log_2\left(k|\alpha|X^{\frac{1}{2}}\right)\right]+1,$$

and define

$$A(H_r) := \left\{n : H_rX^{-\frac{1}{2}} < \left|k|\alpha|-2\sqrt{n}\right| \le 2H_rX^{-\frac{1}{2}}\right\}.$$

Then it is easy to get that

$$|A(H_r)| \le H_rk|\alpha|X^{-\frac{1}{2}}.$$

Furthermore for $n \in A(H_r)$, we have

$$|f_1'(t)| = \left||\alpha| - \frac{2\sqrt{n}}{k}\right|X^{\frac{1}{2}} > H_rk^{-1}.$$

Then using (i) in lemma 2.3, we obtain

$$\int_1^{\sqrt{2}} t^{\frac{1}{2}}\phi(t^2)e\left(f_1(t)\right)dt \ll H_r^{-1}k,$$

that is

$$\int_0^{\infty} t^{\frac{1}{2}}\phi(t^2)e\left(\operatorname{sgn}(\alpha)\left(k|\alpha|-2\sqrt{n}\right)\sqrt{X}t\right)dt \ll H_r^{-1}k.$$

Noting the trivial bound $d(n) \ll n^{\varepsilon}$, then in this case the contribution of (3.20) is

$$\begin{aligned} &\ll \frac{1}{q}\sum_{k|q}k^{\frac{1}{2}}X^{\frac{3}{4}+\varepsilon}\sum_{r=1}^{r_0}H_r^{-1}\sum_{n\in A(H_r)}d(n)|S(-l,-n;k)|n^{-\frac{1}{4}} \\ &\ll \frac{1}{q}\sum_{k|q}k^{\frac{1}{2}+\varepsilon}X^{\frac{3}{4}+\varepsilon}\sum_{r=1}^{r_0}H_r^{-1}\left(|\alpha|^2k^2\right)^{\varepsilon-\frac{1}{4}}k^{\frac{1}{2}+\varepsilon}|A(H_r)| \\ &\ll q^{\frac{1}{2}+\varepsilon}X^{\frac{1}{4}+\varepsilon}|\alpha|^{\frac{1}{2}+\varepsilon}. \end{aligned} \tag{3.22}$$

Let

$$I_0 = \left\{n : \left|k|\alpha|-2\sqrt{n}\right| \le X^{-\frac{1}{2}}\right\},$$

then it is easy to know that

$$|I_0| \le k|\alpha|X^{-\frac{1}{2}}.$$

Now, the remaining work is to estimate

$$\epsilon_\alpha \frac{a_1}{q} \sum_{k|q} k^{-\frac{1}{2}} X^{\frac{3}{4}} \sum_{n\in I\cap I_0} d(n) S\left(-l,-n;k\right) n^{-\frac{1}{4}} \int_1^{\sqrt{2}} t^{\frac{1}{2}} \phi(t^2) e\left(f_1(t,n)\right) dt. \tag{3.23}$$

If $|\alpha| \le \frac{1}{q}$, then $I_0 = \emptyset$. In this case, (3.23) vanishes.

If $|\alpha| \ge \frac{\sqrt{X}}{q}$, using the trivial bound $d(n) \ll n^\varepsilon$, the contribution of (3.23) is

$$\begin{aligned} &\frac{1}{q}\sum_{k|q} k^{-\frac{1}{2}} X^{\frac{3}{4}} \sum_{\left|k|\alpha|-2\sqrt{n}\right|\le X^{-\frac{1}{2}}} d(n)|S(-l,-n;k)|n^{-\frac{1}{4}} \\ &\ll \frac{1}{q}\sum_{k|q} k^{-\frac{1}{2}} X^{\frac{3}{4}} |I_0| k^{1+\varepsilon} \left(k^2|\alpha|^2\right)^{\varepsilon-\frac{1}{4}} \\ &\ll q^\varepsilon X^{\frac{1}{4}+\varepsilon} |\alpha|^{\frac{1}{2}+\varepsilon}. \end{aligned} \tag{3.24}$$

we prove the first part of (iii) in Theorem 1.1.

If $\frac{1}{q} \le |\alpha| \le \frac{\sqrt{X}}{q}$, then there exists at most one integer n, which we write $n = n_k$, satisfying

$$\left|k|\alpha| - 2\sqrt{n}\right| \le X^{-\frac{1}{2}}$$

for every $k|q$. Therefore (3.20) becomes

$$\begin{aligned} &\epsilon_\alpha \frac{a_1}{q} \sum_{k|q} \delta_k k^{-\frac{1}{2}} X^{\frac{3}{4}} d(n_k) S(-l,-n_k;k) n_k^{-\frac{1}{4}} \int_1^{\sqrt{2}} t^{\frac{1}{2}} \phi(t^2) e\left(f_1(t,n_k)\right) dt \\ &= \epsilon_\alpha \frac{a_1}{2q} \sum_{k|q} \delta_k k^{-\frac{1}{2}} X^{\frac{3}{4}} d(n_k) S(-l,-n_k;k) n_k^{-\frac{1}{4}} \int_1^2 u^{-\frac{1}{4}} e\left(f_1\left(u^{\frac{1}{2}}, n_k\right)\right) du \\ &+ O\left(k^{-1+\varepsilon}\Delta^{-1} X^{\frac{3}{4}} d(n_k)(n_k)^{-\frac{1}{4}}\right) \\ &= \frac{1}{q}\sum_{k|q} \varepsilon\left(\alpha, n_k\right) k^{-\frac{1}{2}} X^{\frac{3}{4}} d(n_k) S\left(-l,-n_k;k\right) n_k^{-\frac{1}{4}} + O(1), \end{aligned} \tag{3.25}$$

where

$$\mathcal{E}\left(\alpha, n_k\right) = \frac{\delta_k \epsilon_\alpha a_1}{2} \int_1^2 u^{-\frac{1}{4}} e\left(\operatorname{sgn}(\alpha)\left(|\alpha| - \frac{2\sqrt{n_k}}{k}\right)\sqrt{Xu}\right) du,$$

and $\delta_k = 1$ or 0 according to whether there is a positive integer n_k or not.

When $\beta = \frac{1}{2}$ and $\frac{1}{q} \le |\alpha| \le \frac{\sqrt{X}}{q}$, we note that $q|\alpha| \le \sqrt{X}$. Recalling $a_1 = 1 + i$ and

$$\epsilon_\alpha = \begin{cases} -i, & \alpha > 0, \\ 1, & \alpha < 0, \end{cases}$$

from (3.22) – (3.25), we can conclude that (3.20) is equal to

$$\frac{1}{q}\sum_{k|q} \mathcal{E}(\alpha, n_k) k^{-\frac{1}{2}} X^{\frac{3}{4}} d(n_k) S(-l,-n_k;k) n_k^{-\frac{1}{4}} + O\left(\left(q|\alpha|\right)^{\frac{1}{2}+\varepsilon} X^{\frac{1}{4}+\varepsilon}\right), \tag{3.26}$$

which proves the second part of (iii) in Theorem 1.1.

References

[1] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products. 7th ed. New York: Academic Press, 2007.

[2] J. L. Hafner, *Some remarks on odd Maass wave forms*, Math Z, **196** (1987), 129–132.

[3] L. Tóth, An order result for the exponential divisor function. Publ. Math. Debrecen,2007,71(1-2):165-171.

[4] J. Wu, Problème de diviseurs exponentiels at entiers exponentiellement sans facteur carré [J]. Théor Nombres Bordeaux,7(1995),133-141.

[5] F. Smarandache, Only problems, Not solutions! Chicago: Xiquan Publishing House,1993.

[6] G. H. Hardy, *On Dirichlet's divisor problem*, Proc. London Math. Soc, **15** (1916), 1–25.

[7] M. Jutila, A method in the theory of exponential sums, Tata Institute Lecture Notes 80, Springer (1987).

[8] H. H. Kim, *Functoriality for the exterior square of* GL_4 *and symmetric fourth of* GL_2, With appendix 1 by D. Ramakrishnan and appendix 2 by H. Kim & P. Sarnak, J. Amer. Math. Soc. **16** (2003), 139–183.

[9] E. Kowalski, P. Michel and J. Vanderkam, *Rankin-Selberg L-functions in the level aspect*, Duke Math. J. **114** (2002), 123–191.

[10] Y. K. Lau and G. S. Lü, *Sums of Fourier coefficients of cusp forms*, Quart. J. Math, **62** (2011), 687–716.

[11] J. Y. Liu and Y. B. Ye, *Subconvexity for Rankin-Selberg L-function of Maass forms*, Geom. Funct. Anal., **12** (2002), 135–149.

[12] S. D. Miller and W. Schmid, *The highly oscillatory behavior of automorphic distributions for* $SL(2)$, Lett Math Phys, **69** (2004), 265–286.

[13] X. M. Ren and Y. B. Ye *Resonance between automorphic forms and exponential functions*, Sci. China Math, **53** (2010), 2463–2472.

[14] Q. F. Sun and Y. Y. Wu, *Exponential sums involving Maass forms*, Front. Math. China, **9** (2014), no. 6, 1349–1366.

[15] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed. Qxford University Press, (2014).

[16] S. Uchiyama, *On some exponential sums involving the divisor function over arithmetical progressions*, Math. J. Okayama Univ, **16**(1974), issue 2 173–146.

Scientia Magna
Vol. 13 (2018), No. 1, 105-110

The average estimate of a hybrid arithmetic function

Shan Cui[1] and Ai Liu[2]

[1,2]Department of Mathematics and Statistics, Shandong Normal University
Ji'nan 250014, Shandong, P. R. China
E-mail: [1]1344532228@qq.com and [2]1330350753@qq.com

Abstract In this paper we consider the mean value of hybrid arithmetic function of the form $\lambda_f^6(n)\sigma^b(n)\phi^c(n)$, where $\lambda_f(n)$ is the n-th Fourier coefficients of the holomorphic cusp form f, $\sigma(n)$ is the sum-of-divisors function, $\phi(n)$ is the Euler's totient function. In detail, we prove that for any $\varepsilon > 0$,

$$\sum_{n\leq x}\lambda_f^6(n)\sigma^b(n)\phi^c(n) = x^{(b+c+1)}P_4(\log x) + O(x^{b+c+\frac{631}{652}+\varepsilon}),$$

where $b, c \in \mathbb{R}$.

Keywords Arithmetic function, Cusp form, Automorphic L-function, Fourier coefficients.
2010 Mathematics Subject Classification 11F30, 11F66.

§1. Introduction

Throughout this paper, let $k \geq 1$ be an even integer, and H_k^* be the set of all normalized Hecke primitive eigencuspform of weight k for the full modular group $SL_2(\mathbb{Z})$. Each $f \in H_k^*$ has a Fourier expansion at the cusp ∞

$$f(z) = \sum_{n\geq 1}\lambda_f(n)n^{\frac{k-1}{2}}e^{2\pi inz}.$$

Here $\lambda_f(n)$ is the eigenvalue of normalized Hecke operator T_n. Then $\lambda_f(n)$ is real and satisfies the multiplicative property. For any integers $m \geq 1$ and $n \geq 1$, we have

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)}\lambda_f\left(\frac{mn}{d^2}\right).$$

It is worth mentioning that the Fourier coefficients are very meaningful subject. In 1974, Deligne [2] proved the Ramanujan-Peterson conjecture

$$|\lambda_f(n)| \leq d(n),$$

where $d(n)$ is the Dirichlet divisor function.

The Hecke L-function of $f \in H_k^*$ is defined:

$$L(f,s) = \sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}, \qquad Re(s) > 1.$$

Rankin [7] proved that

$$\sum_{n\leq x}\lambda(n)\ll x^{1/3}(\log x)^{-\delta},$$

where $0<\delta<0.06$.

In this paper we estimate the sum of $\sum\limits_{n\leq x}\lambda_f^6(n)\sigma^b(n)\phi^c(n)$, where $\sigma(n)$ is sum of divisor function, $\phi(n)$ is Euler's totient function and $b,c\in\mathbb{R}$. We establish the following result.

Theorem 1. *Let $b,c\in\mathbb{R}$, then for any $\varepsilon>0$,*

$$S(x)=\sum_{n\leq x}\lambda_f^6(n)\sigma^b(n)\phi^c(n)=x^{b+c+1}P_4(\log x)+O\left(x^{b+c+\frac{631}{652}+\varepsilon}\right),$$

where $P_4(t)$ is a polynomial in t of degree 4 and O-constant depends on f.

§2.Preliminaries

This section is devoted to give some preliminary results for the proof of Theorem 1.

Lemma 2.1. *For any $\varepsilon>0$, $\frac{1}{2}\leq\sigma\leq1$, and $|t|\geq2$, we have*

$$\zeta(\sigma+it)\ll_\varepsilon(1+|\,t\,|)^{max\{\frac{13}{42}(1-\sigma),0\}},$$
$$L(sym^jf,\sigma+it)\ll_{f,\varepsilon}(1+|\,t\,|)^{max\{\frac{j+1}{2}(1-\sigma),0\}},$$
$$L(sym^jf\times sym^if,\sigma+it)\ll_{f,\varepsilon}(1+|\,t\,|)^{max\{\frac{(j+1)(i+1)}{2}(1-\sigma),0\}},$$

where $L\left(sym^jf,s\right)$ is the symmetric power L-function, and $L\left(sym^jf\times sym^if,s\right)$ is the Rankin-Selberg L-function.

Proof. See [1] and chapter 5 of the literature [4], respectively. □

Lemma 2.2. *For $i,j=1,2,3,4$, and for any $\varepsilon>0$, $T\geq T_0$ (where T_0 is sufficiently large), we have the estimate*

$$\int_T^{2T}\left|L\left(sym^jf,\frac{1}{2}+\varepsilon+it\right)\right|^2dt\ll_{f,\varepsilon}T^{\frac{j+1}{2}+\varepsilon},$$

$$\int_T^{2T}\left|L\left(sym^jf\times sym^if,\frac{1}{2}+\varepsilon+it\right)\right|^2dt\ll_{f,\varepsilon}T^{\frac{(j+1)(i+1)}{2}+\varepsilon}.$$

Proof. For the proof of Lemma 2.2, we can see section 1 of [6], respectively. □

Lemma 2.3. *Suppose $F(s)=\sum\limits_{n=1}^{\infty}\frac{a(n)}{n^s}$ converges absolutely for $\sigma>1$ and $a(n)\leq A(n)$, where $A(n)$ is monotonically increasing and $\sum\limits_{n=1}^{\infty}\frac{a(n)}{n^s}=O\left(\frac{1}{(\sigma-1)^\alpha}\right)$ with $\alpha>0$ as $\alpha\to1^+$. If $b>1$ and $x=N+\frac{1}{2}$ with $N\in\mathbb{N}$, then for $T\geq2$,*

$$\sum_{n\leq x}a(n)=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}F(s)\frac{x^s}{s}ds+O\left(\frac{x^b}{T(b-1)^\alpha}\right)+\left(\frac{xA(2x)\log x}{T}\right).$$

Proof. The proof of the Lemma is given in section 1.2.1 of [5]. □

Lemma 2.4. *Let $f \in H_k^*$ and $\lambda_f(n)$ denote its n-th normalized Fourier coefficient. Define*

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^6(n)\sigma^b(n)\phi^c(n)}{n^s} = L_1(s-b-c)H(s),$$

then $F(s)$ can be decomposed into

$$F(s) = L(sym^4 f \times sym^2 f, s-b-c)L^4(sym^4 f, s-b-c)L^8(sym^2 f, s-b-c)\zeta^5(s-b-c)H(s),$$

where $H(s)$ is analytic and bounded for $Re(s) > b+c+\frac{1}{2}$.

Proof. $F(s) = \sum\limits_{n=1}^{\infty} \frac{\lambda_f^6(n)\sigma^b(n)\phi^c(n)}{n^s}$ is the Dirichlet series of $\lambda_f^6(n)\sigma^b(n)\phi^c(n)$. Each of $\lambda_f(n)$, $\sigma(n)$ and $\phi(n)$ satisfies the multiplicative property, thus $\lambda_f^6(n)\sigma^b(n)\phi^c(n)$ is multiplicative. Hence we can write $F(s)$ as a product over primes $F(s) = \prod\limits_p f_p(s)$.

Therefore we obtain

$$\begin{aligned} f_p(s) &= \sum_{k=0}^{\infty} \frac{\lambda_f^6(p^k)\sigma^b(p^k)\phi^c(p^k)}{p^{ks}} \\ &= 1 + \frac{\lambda_f^6(p)\sigma^b(p)\phi^c(p)}{p^s} + \frac{\lambda_f^6(p^2)\sigma^b(p^2)\phi^c(p^2)}{p^{2s}} + \cdots . \end{aligned}$$

Referring to Deligne [6], for arbitrary prime p, we can write $\alpha_f(p)$ and $\beta_f(p)$ for

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \qquad |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1.$$

For $j \geq 1$, we have

$$\lambda_f(p)^j = \frac{\alpha_f(p)^{j+1} - \beta_f(p)^{j+1}}{\alpha_f(p) - \beta_f(p)} = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m.$$

Then,

$$\begin{aligned} f_p(s) &= 1 + \frac{(\alpha_f(p)+\beta_f(p))^6 (p+1)^b(p-1)^c}{p^s} + \frac{\left(\frac{\alpha_f^3(p)-\beta_f^3(p)}{\alpha_f(p)-\beta_f(p)}\right)^6 (p^2+p+1)^b(p^2-p)^c}{p^{2s}} + \cdots \\ &= 1 + \frac{(\alpha_f(p)+\beta_f(p))^6 (p+1)^b(p-1)^c}{p^s} \\ &\quad + \frac{\left(\alpha_f^2(p)+\alpha_f(p)\beta_f(p)+\beta_f^2(p)\right)^6 (p^2+p+1)^b(p^2-p)^c}{p^{2s}} + \cdots \\ &= 1 + \frac{(\alpha_f(p)+\beta_f(p))^6}{p^{s-b-c}} \left(1+\frac{1}{p}\right)^b \left(1-\frac{1}{p}\right)^c + O\left(p^{b+c-\sigma-1} + p^{2(b+c-\sigma)}\right). \end{aligned}$$

Therefore,

$$
\begin{aligned}
F(s) =& \prod_p \Bigg(1 + \frac{\alpha_f^6(p) + 6\alpha_f^4(p) + 15\alpha_f^2(p) + 20 + 15\beta_f^2(p) + 6\beta_f^4(p) + \beta_f^6(p)}{p^{s-b-c}} \\
& \times \left(1+\frac{1}{p}\right)^b \left(1-\frac{1}{p}\right)^c + O\left(p^{2(b+c-\sigma)}\right)\Bigg) \\
=& L(sym^4 f \times sym^2 f, s-b-c) \prod_p \Bigg(1 + \frac{4\alpha_f^4(p) + 12\alpha_f^2(p) + 17 + 12\beta_f^2(p) + 4\beta_f^4(p)}{p^{s-b-c}} \\
& \times \left(1+\frac{1}{p}\right)^b \left(1-\frac{1}{p}\right)^c + O\left(p^{2(b+c-\sigma)}\right)\Bigg) \\
=& L(sym^4 f \times sym^2 f, s-b-c) L^4(sym^4 f, s-b-c) L^8(sym^2 f, s-b-c) \\
& \times \prod_p \left(1 + \frac{5}{p^{s-b-c}} \left(1+\frac{1}{p}\right)^b \left(1-\frac{1}{p}\right)^c + O\left(p^{2(b+c-\sigma)}\right)\right) \\
=& L(sym^4 f \times sym^2 f, s-b-c) L^4(sym^4 f, s-b-c) L^8(sym^2 f, s-b-c) \zeta^5(s-b-c) \\
& \times \prod_p \left(\left(1+\frac{1}{p}\right)^b \left(1-\frac{1}{p}\right)^c + O\left(p^{2(b+c-\sigma)}\right)\right), \\
=& L(sym^4 f \times sym^2 f, s-b-c) L^4(sym^4 f, s-b-c) L^8(sym^2 f, s-b-c) \zeta^5(s-b-c) H(s),
\end{aligned}
$$

where $H(s)$ is absolutely convergent in $Re\ (s) > b + c + \frac{1}{2}$. □

§3. Proof of Theorem 1

Proof. We apply Lemma 2.3 to the sum $S(x) = \sum_{n \le x} \lambda_f^6(n)\sigma^b(n)\phi^c(n)$, let $f(n) = \lambda_f^6(n)\sigma^b(n)\phi^c(n)$. We know that $f(n) \le Bn^{b+c+\varepsilon}$, and B is a real constant depending on ε. Referring to Lemma 2.3, for $Re\ (s) > b + c + \frac{1}{2}$, we have

$$
\begin{aligned}
S(x) &= \frac{1}{2\pi i} \int_{b+c+1+\varepsilon-iT}^{b+c+1+\varepsilon+iT} \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T^{s-b-c-1}}\right) + O\left(\frac{xB(2x)^{b+c+\varepsilon}\log x}{T}\right) \\
&= \frac{1}{2\pi i} \int_{b+c+1+\varepsilon-iT}^{b+c+1+\varepsilon+iT} L_1(s-b-c)H(s)\frac{x^s}{s} ds + O\left(\frac{x^{b+c+1+\varepsilon}}{T}\right), \qquad (3.1)
\end{aligned}
$$

where

$$L_1(s-b-c) = L(sym^4 f \times sym^2 f, s-b-c) L^4(sym^4 f, s-b-c) L^8(sym^2 f, s-b-c) \zeta^5(s-b-c),$$

and T with $1 \le T \le x$ is a parameter to be specified later.

Our aim is to estimate the integral in (3.1). So we consider the closed contour Γ:

$$I = [b+c+1+\varepsilon-iT, b+c+1+\varepsilon+iT], \qquad II = [b+c+1+\varepsilon+iT, b+c+\frac{1}{2}+\varepsilon+iT],$$

$$III = [b+c+\frac{1}{2}+\varepsilon+iT, b+c+\frac{1}{2}+\varepsilon-iT], \qquad IV = [b+c+\frac{1}{2}+\varepsilon-iT, b+c+1+\varepsilon-iT].$$

Let

$$I_1 = \int_I L_1(s-b-c)H(s)\frac{x^s}{s}ds, \qquad I_2 = \int_{II} L_1(s-b-c)H(s)\frac{x^s}{s}ds,$$

$$I_3 = \int_{III} L_1(s-b-c)H(s)\frac{x^s}{s}ds, \qquad I_4 = \int_{IV} L_1(s-b-c)H(s)\frac{x^s}{s}ds.$$

By the residue theorem, we have

$$\begin{aligned}\frac{1}{2\pi i}I_1 &= \frac{1}{2\pi i}\int_\Gamma L_1(s-b-c)H(s)\frac{x^s}{s}ds - \frac{1}{2\pi i}(I_2+I_3+I_4)\\ &= x^{b+c+1}P_4(\log x) - \frac{1}{2\pi i}\left(I_2+I_3+I_4\right), \qquad (3.2)\end{aligned}$$

where $P_4(t)$ is the polynomial of degree 4 in t.

For I_2 and I_4, according to Lemma 2.1, we get

$$\begin{aligned}I_2+I_4 &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} L_1(\sigma+iT)\frac{x^{b+c+\sigma}}{b+c+\sigma+iT}d\sigma\\ &\ll x^{b+c}\int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} L_1(\sigma+iT)\frac{x^\sigma}{T}d\sigma\\ &\ll x^{b+c}\int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} T^{(\frac{15}{2}+\frac{5}{2}\times 4+\frac{3}{2}\times 8+\frac{13}{42}\times 5)(1-\sigma)+\varepsilon}\frac{x^\sigma}{T}d\sigma\\ &\ll x^{b+c}\max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} T^{\frac{631}{21}}\left(\frac{x}{T^{\frac{652}{21}}}\right)^\sigma\\ &\ll \frac{x^{b+c+1+\varepsilon}}{T}. \qquad (3.3)\end{aligned}$$

For I_3, using Lemmas 2.1-2.2 and the Cauchy inequality, we obtain

$$\begin{aligned}I_3 &\ll \int_1^T \left|L\left(sym^4f\times sym^2f,\frac{1}{2}+\varepsilon+it\right)L^4\left(sym^4f,\frac{1}{2}+\varepsilon+it\right)L^8\left(sym^2f,\frac{1}{2}+\varepsilon+it\right)\right.\\ &\quad \left.\times\zeta^5\left(\frac{1}{2}+\varepsilon+it\right)H_4(b+c+\frac{1}{2}+\varepsilon+it)\right|\frac{x^{b+c+\frac{1}{2}+\varepsilon}}{|b+c+\frac{1}{2}+\varepsilon+it|}dt + x^{b+c+\frac{1}{2}+\varepsilon}\\ &\ll x^{b+c+\frac{1}{2}+\varepsilon}\int_1^T \left|L\left(sym^4f\times sym^2f,\frac{1}{2}+\varepsilon+it\right)L^4\left(sym^4f,\frac{1}{2}+\varepsilon+it\right)\right.\\ &\quad \left.\times L^8\left(sym^2f,\frac{1}{2}+\varepsilon+it\right)\zeta^5\left(\frac{1}{2}+\varepsilon+it\right)\right|\frac{1}{t}dt + x^{b+c+\frac{1}{2}+\varepsilon}\\ &\ll x^{b+c+\frac{1}{2}+\varepsilon} + x^{b+c+\frac{1}{2}+\varepsilon}\log T\max_{1\le T_1\le T}\frac{1}{T_1}\left(\max_{\frac{T_1}{2}\le t\le T_1}\left|\zeta\left(\frac{1}{2}+\varepsilon+it\right)\right|^5\right)\\ &\quad\times\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(sym^4f\times sym^2f,\frac{1}{2}+\varepsilon+it\right)L^2\left(sym^2f,\frac{1}{2}+\varepsilon+it\right)L^4\left(sym^2f,\frac{1}{2}+\varepsilon+it\right)\right|^2dt\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_{\frac{T_1}{2}}^{T_1}\left|L^2\left(sym^2f,\frac{1}{2}+\varepsilon+it\right)L^4\left(sym^2f,\frac{1}{2}+\varepsilon+it\right)\right|^2dt\right)^{\frac{1}{2}}\\ &\ll x^{b+c+\frac{1}{2}+\varepsilon} + x^{b+c+\frac{1}{2}+\varepsilon}T^{\frac{305}{21}}\\ &\ll x^{b+c+\frac{1}{2}+\varepsilon}T^{\frac{305}{21}}. \qquad (3.4)\end{aligned}$$

According to (3.2)-(3.4), we have

$$\frac{1}{2\pi i}\int_{b+c+1+\varepsilon-iT}^{b+c+1+\varepsilon+iT} L_1(s)H(s)\frac{x^s}{s}ds = x^{b+c+1}P_4(\log x) + O\Big(x^{b+c+\frac{1}{2}+\varepsilon}T^{\frac{305}{21}}\Big) + O\bigg(\frac{x^{b+c+1+\varepsilon}}{T}\bigg).$$

Taking $T = x^{\frac{21}{652}}$, we get

$$S(x) = x^{b+c+1}P_4(\log x) + O\left(x^{b+c+\frac{631}{652}+\varepsilon}\right).$$

This completes the proof of Theorem 1.

□

Acknowledgements

This work is supported by Natural Science Foundation of China (Grant No. 11101249). The authors wish to thank to the referee for a careful reading of manuscript and many valuable suggestions which highly reprove the quality of this paper.

References

[1] J. Bourgain. Decoupling exponential sums and the Riemann zeta function. J. Amer. Math., Soc. 30 (2017), no. 1, 205 – 224.

[2] P. Deligne. La Conjecture de Weil, I, Inst. Hautes Études Sci. Publ. Math., 43 (1974), 273 – 307.

[3] A. Ivić. The Riemann Zeta-Function, New York. John Wiley & Sons. 1985.

[4] H. Iwaniec and E. Kowalski. Analytic Number Theory. Amer. Math. Soc., 2004.

[5] A. Karatsuba. Complex Analysis in Number Theory. CRC Press, Boca Raton, Florida, 1995.

[6] H. X. Lao. The cancellation of Fourier coefficient of cusp forms over different sparse sequences. Acta Math. Sin. (Engl. Ser), 29 (2013), 1963 – 1972.

[7] R. A. Rankin. Sums of cusp form coefficients, Automorphic forms and analytic number theory. Univ. Montreal, 1990, 115 – 121.

Scientia Magna
Vol. 13 (2018), No. 1, 111-120

The number of representation of the natural number by quaternary quadratic forms

Zhenxing Xie[1] and Xiaowei Yang[2]

[1,2]School of Mathematical and Statistics, Shandong Normal University
Ji'nan 250014, P. R. China
E-mail: [1]xzhenxing9109@163.com and [2]15054159906@163.com

Abstract Let $R(n)$ denote the number of representation of the natural number n by quaternary quadratic form $f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + 5(x_3^2 + x_4^2)$, where x_1, x_2, x_3, and x_4 are integers. In this paper we establish the asymptotic formulae of $\sum_{n\le x} R(n)$ and $\sum_{n\le x} R^2(n)$.

Keywords quaternary quadratic forms, Fourier coefficients, cusp forms.
2010 Mathematics Subject Classification 11F30, 11F70.

§1. Introduction

Fomenko [4] considered the problem on the distribution of integral points on

$$x_1^2 + \cdots + x_k^2 = y_1^2 + \cdots + y_k^2 \qquad (k \ge 2), \tag{1}$$

described by the asymptotic formula for $\sum_{n\le x} r_k^2(n)$, where $x_i, y_i \in \mathbb{Z}$, $1 \le i \le k$, $n \in \mathbb{N}^+$, and $r_k(n)$ denotes the number of representations of the natural number n as the sum of k squares of integer. Distribution of integral points on cones of the form (1) has a long history (see [4]). Fomenko [4] and Müller [10, 11], by the Rankin-Selberg convolution method, that for $k \ge 3$

$$\sum_{n\le x} r_k^2(n) = cx^{k-1} + \Delta_k(x),$$

with a certain constant $c = c(k) > 0$ and $\Delta_k(x)$ denotes the error term of this asymptotic formula, and the error term behaves as

$$\Delta_k(x) \ll x^{(k-1)\frac{4k-5}{4k-3}}.$$

Fomenko [5] improved the estimate for $k = 4$

$$\Delta_4(x) \ll x^2(\log x)^{\frac{5}{3}}.$$

In this paper we are interested in the representation number of n by quaternary quadratic form $f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + 5(x_3^2 + x_4^2)$. We denote the number of representations of the natural number n by

$$R(n) = R\left(n = f(x_1, x_2, x_3, x_4)\right).$$

We establish the following results.

Theorem 1. *For $x \geq 2$, we have*

$$\sum_{n\leq x} R(n) = \frac{\pi^2}{10}x^2 + O\left(x(\log x)^{\frac{2}{3}}\right).$$

Theorem 2. *For $x \geq 2$, we have*

$$\sum_{n\leq x} R^2(n) = C_1 x^3 + O\left(x^2(\log x)^{\frac{11}{3}}\right),$$

where C_1 is a constant.

§2. Preliminaries

In this section we will briefly recall some fundamental facts about holomorphic cusp forms and automorphic L-functions, and also give some lemmas which will be used in the proof of our results.

Let N be a positive integer, and ψ be a primitive Dirichlet character modulo N. Denote by $H_k(N)$ the set of all normailzed Hecke primitive eigencuspforms of even integer weight k for the congruence modular group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 (\mathrm{mod} N) \right\},$$

where $SL_2(\mathbb{Z})$ denote the full modular group.

For $f \in H_k(N)$, it has the Fourier expansion at the cusp ∞

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz) \qquad (\Im z > 0),$$

where $e(z) := \exp(2\pi i z)$. By the work of Deligne [2] on the Ramanujan-Petersson Conjecture

$$|a_f(n)| \leq d(n) n^{\frac{k-1}{2}}, \qquad n \geq 1,$$

where $d(n)$ is the divisor function.

The Hecke L-function attached to $f \in H_k(N)$ is defined for $\sigma = \Re s > (k+1)/2$ by

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}.$$

According to [1], for any prime number p, we have

$$L(f,s) = \prod_p \left(1 - \frac{a_f(p)}{p^s} + \frac{\psi(p)}{p^{2s-k+1}}\right)^{-1} = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1},$$

where α_p and β_p are two complex conjugates and satisfy $\alpha_p + \beta_p = a_f(p)$ and $\alpha_p \beta_p = \psi(p) p^{k-1}$.

Lemma 2.1. *Let χ be a primitive character modulo q.*

(i) For any $\varepsilon > 0$,

$$L(\sigma + it, \chi) \ll (q(|t| + 1))^{\max\left\{\frac{1}{3}(1-\sigma),0\right\}+\varepsilon}, \tag{2}$$

uniformly for $1/2 \leq \sigma \leq 1$ *and* $t \geq 1$ *with* $q \ll t^2$.

(ii) For $T \geq 2$, $q \geq 1$, *and* $|\sigma - 1/2| \leq (200 \log qT)^{-1}$,

$$\sum_{\chi \bmod q}^{*} \int_0^T |L(\sigma + it, \chi)|^4 dt \ll \varphi(q) T (\log qT)^4, \tag{3}$$

where $\sum\limits_{\chi \bmod q}^{*}$ *denotes the sums of primitive characters, and* $\varphi(n)$ *is the Euler function.*

Proof. (i) was proved in Heath-Brown [7], and (ii) was proved in Pan and Pan [12]. □

Lemma 2.2. *Let* $f \in H_k(N)$ *and* χ *be a primitive character modulo* q. *For any* $\varepsilon > 0$, *we have*

$$\int_T^{2T} |L(sym^2 f \otimes \chi, \sigma + it)|^2 dt \ll_{f,\varepsilon} (qT)^{3(k-\sigma)+\varepsilon} \tag{4}$$

uniformly for $k - 1/2 \leq \sigma \leq k$ *and* $T \geq 1$. *Moreover,*

$$L(sym^2 f \otimes \chi, \sigma + it) \ll_{f,\varepsilon} (q(|t| + 1))^{\max\left\{\frac{3}{2}(k-\sigma),0\right\}+\varepsilon} \tag{5}$$

uniformly for $k - 1 + \varepsilon \leq \sigma \leq k + \varepsilon$.

Proof. From Shimura [14] we learn that $L(sym^2 f \otimes \chi, s)$ satisfies a functional equation, so it is a general L-function introduced by Perelli [13]. Then from Theorem 4 in [13] we deduce the estimate (4). The convexity bound (5) can be obtained by standard arguments similar to Lemma 2.4 in Jiang and Lü [9]. □

Now we study the distributions of $a_f(n)$, $a_f(n)\sigma(n)$, and $a_f^2(n)$, where $a_f(n) = \lambda_f(n) n^{\frac{k-1}{2}}$ and $\sigma(n) = \sum\limits_{d|n} d$.

Lemma 2.3. *For* $f \in H_k(N)$ *and* $x \geq 2$, *we have*

$$\sum_{n \leq x} a_f(n) = O\left(x^{\frac{3k-1}{6}} (\log x)^{-0.1185}\right), \tag{6}$$

$$\sum_{n \leq x} a_f(n)\sigma(n) = O\left(x^{\frac{3k+5}{6}} (\log x)^{-0.1185}\right), \tag{7}$$

where the O*-constants depend on* f.

Proof. We know from Theorem 2 in Wu [18] that for $f \in H_k(N)$ and $x \geq 2$,

$$S_f(x) = \sum_{n \leq x} \lambda_f(n) = O\left(x^{\frac{1}{3}} (\log x)^{-0.1185}\right), \tag{8}$$

where the O-constant depends on f.

By Abel's partial summation formula and (8), we have

$$\sum_{n\le x} a_f(n) = S_f(x)x^{\frac{k-1}{2}} - \int_1^x S_f(u)\left(u^{\frac{k-1}{2}}\right)' du = O\left(x^{\frac{3k-1}{6}}(\log x)^{-0.1185}\right).$$

With the same method, it's easy to get

$$\sum_{n\le x} a_f(n)n = O\left(x^{\frac{3k+5}{6}}(\log x)^{-0.1185}\right). \tag{9}$$

By the Rankin-Selberg convolution method, Dirichlet convolution, and (9), we conclude that

$$\sum_{n\le x} a_f(n)\sigma(n) = O\left(x^{\frac{3k+5}{6}}(\log x)^{-0.1185}\right),$$

where the O-constant depends on f. This completes the proof. □

Lemma 2.4. *Let $f \in H_k(N)$, for any $\varepsilon > 0$ and $x \ge 2$, we have*

$$\sum_{n\le x} a_f^2(n) = C_2 x^k + O\left(x^{k-\frac{2}{5}+\varepsilon}\right),$$

where C_2 is a constant and the O-constant depends on f.

Proof. By the Rankin-Selberg convolution method in chapter 13 of [8], we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a_f^2(n)}{n^s} &= \prod_p \left(1-\frac{\alpha_f^2(p)\beta_f^2(p)}{p^{2s}}\right)\left(1-\frac{\alpha_f^2(p)}{p^s}\right)^{-1}\left(1-\frac{\alpha_f(p)\beta_f(p)}{p^s}\right)^{-2}\left(1-\frac{\beta_f^2(p)}{p^s}\right)^{-1}\\ &= \frac{L(sym^2 f, s)L(s+1-k,\psi)}{L(2s+2-2k,\psi^2)} =: L_1(s),\end{aligned}$$

Clearly $L^{-1}(2s+2-2k,\psi^2)$ is absolutely convergent and has free from zeros for $\Re s \ge (2k-1)/2+\varepsilon$, $L(sym^2 f, s)L(s+1-k,\psi)$ has one simple pole at $s = k$ in the whole s-plane.

By Perron's formula, we have

$$\sum_{n\le x} a_f^2(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} L_1(s)\frac{x^s}{s}ds + O\left(\frac{x^b}{T(b-1)^\alpha}\right) + O\left(\frac{xA(2x)\log x}{T}\right),$$

where T with $10 \le T \le x$ is a parameter to be specified later and $\alpha = 1$, $b = k+\varepsilon$, and $A(n) = n^{k-1+\varepsilon}$. Then we have

$$\sum_{n\le x} a_f^2(n) = \frac{1}{2\pi i}\int_{k+\varepsilon-iT}^{k+\varepsilon+iT} L_1(s)\frac{x^s}{s}ds + O\left(\frac{x^{k+\varepsilon}}{T}\right).$$

We move the line to $\Re s = (2k-1)/2+\varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned}\sum_{n\le x} a_f^2(n) &= \operatorname*{Res}_{s=k}\left\{L_1(s)\frac{x^s}{s}\right\} + O\left(\frac{x^{k+\varepsilon}}{T}\right)\\ &+ \frac{1}{2\pi i}\left(\int_{k+\varepsilon-iT}^{\frac{2k-1}{2}+\varepsilon-iT} + \int_{\frac{2k-1}{2}+\varepsilon-iT}^{\frac{2k-1}{2}+\varepsilon+iT} + \int_{\frac{2k-1}{2}+\varepsilon+iT}^{k+\varepsilon+iT}\right) L_1(s)\frac{x^s}{s}ds \\ &=: C_2x^k + I_1 + I_2 + I_3 + O\left(\frac{x^{k+\varepsilon}}{T}\right),\end{aligned} \tag{10}$$

where C_2 is a constant.

For I_2, using (3) and (4), we have

$$I_2 \ll x^{\frac{2k-1}{2}+\varepsilon}T^{\frac{3}{4}+\frac{1}{4}+\frac{1}{4}-1} + x^{\frac{2k-1}{2}+\varepsilon} \ll x^{\frac{2k-1}{2}+\varepsilon}T^{\frac{1}{4}}. \tag{11}$$

For the integrals over the horizontal segments, using (2) and (5), we have

$$I_1 + I_3 \ll \max_{\frac{2k-1}{2}+\varepsilon\leq\sigma\leq k+\varepsilon} T^{\frac{41k}{24}-1+\varepsilon}\left(\frac{x}{T^{\frac{41}{24}}}\right)^{\sigma} \ll \frac{x^{k+\varepsilon}}{T}. \tag{12}$$

Combining (10)–(12), we have

$$\sum_{n\leq x} a_f^2(n) = C_2x^k + O\left(x^{k-\frac{1}{2}+\varepsilon}T^{\frac{1}{4}}\right) + O\left(\frac{x^{k+\varepsilon}}{T}\right).$$

Taking $T = x^{\frac{2}{5}}$, we obtain

$$\sum_{n\leq x} a_f^2(n) = C_2x^k + O\left(x^{k-\frac{2}{5}+\varepsilon}\right).$$

The proof is complete. □

Finally, we give some facts about $\sigma(n)$.

Lemma 2.5. *Let $n = p_1^{m_1}p_2^{m_2}\cdots p_r^{m_r}$, we have*

$$\sigma(n) = \frac{1-p_1^{m_1+1}}{1-p_1}\frac{1-p_2^{m_2+1}}{1-p_2}\cdots\frac{1-p_r^{m_r+1}}{1-p_r}.$$

Proof. This fact was proved in Tichmarsh [16]. □

Lemma 2.6. *For $x \geq 2$,*

$$\sum_{n\leq x}\sigma(n) = \frac{\pi^2}{12}x^2 + O\left(x(\log x)^{\frac{2}{3}}\right). \tag{13}$$

And

$$\sum_{n\leq x}\sigma^2(n) = \frac{5\zeta(3)}{6}x^3 + O\left(x^2(\log x)^{\frac{5}{3}}\right). \tag{14}$$

Moreover

$$\sum_{\substack{n\leq x\\ t|n}}\sigma^2(n) = \frac{\pi^2A_2(t)}{18t}x^3 + O\left(x^2(\log x)^{\frac{5}{3}}S(t)\right), \tag{15}$$

where the O-constant is independent of x and t,

$$A_1(n) = \sum_{q|n}\frac{\varphi(q)}{q^2}, \qquad A_2(n) = \sum_{k|n}\frac{B_1(k;n)}{k} = O\left(d(n)\log n\right),$$

and for $k \mid n$

$$B_1(k;n) = \sum_{\substack{m=1 \\ (m,n/k)=1}} \frac{A_1(mn)}{n^2}, \qquad S(n) = \prod_{p|n}\left(1+\frac{2}{p-1}\right),$$

where $\varphi(n)$ is the Euler function.

Proof. These facts were proved in Walfisz [17] and Ramaiah and Suryanarayana [15], respectively. □

§3. Proof of Theorem 1

Proof. Elstrodt, Grunewald and Mennicke [3] proved the following result:

$$R(n) = \frac{4}{3}\left(\sigma_1(n,4) + 5\sigma_1\left(\frac{n}{5},4\right) + 2a(n)\right),$$

where $\sigma_1(a,k) := \sum\limits_{d|a, k\nmid d} d$, $\sigma_1(a,k) = 0$ if a is not a natural number, $a(n)$ is defined by

$$\eta^2(2z)\eta^2(10z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}, \qquad \Im z > 0.$$

Note that $\eta^2(2z)\eta^2(10z)$ is the normalized cusp form of weight 2, and $\eta(z)$ is defined in the half-plane $H = \{z : \Im z > 0\}$ by the equation

$$\eta(z) = e\left(\frac{z}{24}\right)\prod_{n=1}^{\infty}\left(1 - e(nz)\right).$$

By virtue of

$$\sigma_1(n,k) := \sum_{d|n, k\nmid d} d = \sum_{d|n} d - \sum_{d|n, k|d} d = \sigma(n) - k\sigma\left(\frac{n}{k}\right),$$

we obtain

$$R(n) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma\left(\frac{n}{4}\right) + \frac{20}{3}\sigma\left(\frac{n}{5}\right) - \frac{80}{3}\sigma\left(\frac{n}{20}\right) + \frac{8}{3}a(n).$$

Now $R(n)$ is divided into four cases.

Case 1 if $4 \nmid n$, $5 \nmid n$, $20 \nmid n$, then

$$R(n) = \frac{4}{3}\sigma(n) + \frac{8}{3}a(n). \tag{16}$$

Case 2 if $4 \mid n$, $5 \nmid n$, $20 \nmid n$, then

$$R(n) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma\left(\frac{n}{4}\right) + \frac{8}{3}a(n). \tag{17}$$

Case 3 if $4 \nmid n$, $5 \mid n$, $20 \nmid n$, then

$$R(n) = \frac{4}{3}\sigma(n) + \frac{20}{3}\sigma\left(\frac{n}{5}\right) + \frac{8}{3}a(n). \tag{18}$$

Case 4 if $4 \mid n$, $5 \mid n$, $20 \mid n$, then

$$R(n) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma\left(\frac{n}{4}\right) + \frac{20}{3}\sigma\left(\frac{n}{5}\right) - \frac{80}{3}\sigma\left(\frac{n}{20}\right) + \frac{8}{3}a(n). \tag{19}$$

From (16)–(19), we obtain

$$\begin{aligned}\sum_{n\le x} R(n) =& \frac{4}{3}\sum_{n\le x}\sigma(n) - \frac{16}{3}\sum_{\substack{n\le x\\4|n}}\sigma\left(\frac{n}{4}\right) + \frac{20}{3}\sum_{\substack{n\le x\\5|n}}\sigma\left(\frac{n}{5}\right)\\ &- \frac{80}{3}\sum_{\substack{n\le x\\4|n,5|n}}\sigma\left(\frac{n}{20}\right) + \frac{8}{3}\sum_{n\le x} a(n)\\ =:& S_1 - S_2 + S_3 - S_4 + S_5.\end{aligned} \tag{20}$$

By (13), we have

$$S_1 = \frac{4}{3}\sum_{n\le x}\sigma(n) = \frac{\pi^2}{9}x^2 + O\left(x(\log x)^{\frac{2}{3}}\right). \tag{21}$$

$$S_2 = \frac{16}{3}\sum_{k_1\le \frac{x}{4}}\sigma(k_1) = \frac{\pi^2}{36}x^2 + O\left(x(\log x)^{\frac{2}{3}}\right). \tag{22}$$

$$S_3 = \frac{20}{3}\sum_{k_2\le \frac{x}{5}}\sigma(k_2) = \frac{\pi^2}{45}x^2 + O\left(x(\log x)^{\frac{2}{3}}\right). \tag{23}$$

$$S_4 = \frac{80}{3}\sum_{k_3\le \frac{x}{20}}\sigma(k_3) = \frac{\pi^2}{180}x^2 + O\left(x(\log x)^{\frac{2}{3}}\right). \tag{24}$$

Taking $k = 2$ in (6), we have

$$S_5 = \frac{8}{3}\sum_{n\le x} a(n) = O\left(x^{\frac{5}{6}}(\log x)^{-0.1185}\right). \tag{25}$$

Combining (20)–(25), we obtain

$$\sum_{n\le x} R(n) = \frac{\pi^2}{10}x^2 + O\left(x(\log x)^{\frac{2}{3}}\right).$$

The proof of Theorem 1.1 is complete. □

§4. Proof of Theorem 2

Proof. From (16)–(20), it is easy to get

$$\begin{aligned}\sum_{n\le x} R^2(n) =& \frac{16}{9}\sum_{n\le x}\sigma^2(n) - \frac{128}{9}\sum_{\substack{n\le x\\4|n}}\sigma(n)\sigma\left(\frac{n}{4}\right) + \frac{160}{9}\sum_{\substack{n\le x\\5|n}}\sigma(n)\sigma\left(\frac{n}{5}\right)\\ &- \frac{640}{9}\sum_{\substack{n\le x\\4|n,5|n}}\sigma(n)\sigma\left(\frac{n}{20}\right) + \frac{64}{9}\sum_{n\le x}\sigma(n)a(n) + \frac{256}{9}\sum_{\substack{n\le x\\4|n}}\sigma^2\left(\frac{n}{4}\right)\end{aligned}$$

$$
\begin{aligned}
&-\frac{640}{9}\sum_{\substack{n\le x\\4|n,5|n}}\sigma\left(\frac{n}{4}\right)\sigma\left(\frac{n}{5}\right)+\frac{2560}{9}\sum_{\substack{n\le x\\4|n,5|n}}\sigma\left(\frac{n}{4}\right)\sigma\left(\frac{n}{20}\right)-\frac{256}{9}\sum_{\substack{n\le x\\4|n}}\sigma\left(\frac{n}{4}\right)a(n)\\
&+\frac{400}{9}\sum_{\substack{n\le x\\5|n}}\sigma^2\left(\frac{n}{5}\right)-\frac{3200}{9}\sum_{\substack{n\le x\\4|n,5|n}}\sigma\left(\frac{n}{5}\right)\sigma\left(\frac{n}{20}\right)+\frac{320}{9}\sum_{\substack{n\le x\\5|n}}\sigma\left(\frac{n}{5}\right)a(n)\\
&+\frac{6400}{9}\sum_{\substack{n\le x\\4|n,5|n}}\sigma^2\left(\frac{n}{20}\right)-\frac{1280}{9}\sum_{\substack{n\le x\\4|n,5|n}}\sigma\left(\frac{n}{20}\right)a(n)+\frac{64}{9}\sum_{n\le x}a^2(n)
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
=:\ &S_6-S_7+S_8-S_9+S_{10}+S_{11}-S_{12}+S_{13}\\
&-S_{14}+S_{15}-S_{16}+S_{17}+S_{18}-S_{19}+S_{20}.
\end{aligned}
$$

By (14), we have

$$S_6=\frac{16}{9}\sum_{n\le x}\sigma^2(n)=\frac{40\zeta(3)}{27}x^3+O\left(x^2(\log x)^{\frac{5}{3}}\right), \tag{27}$$

$$S_{11}=\frac{256}{9}\sum_{k_4\le\frac{x}{4}}\sigma^2(k_4)=\frac{10\zeta(3)}{27}x^3+O\left(x^2(\log x)^{\frac{5}{3}}\right), \tag{28}$$

$$S_{15}=\frac{400}{9}\sum_{k_5\le\frac{x}{5}}\sigma^2(k_5)=\frac{8\zeta(3)}{27}x^3+O\left(x^2(\log x)^{\frac{5}{3}}\right), \tag{29}$$

$$S_{18}=\frac{6400}{9}\sum_{k_6\le\frac{x}{20}}\sigma^2(k_6)=\frac{2\zeta(3)}{27}x^3+O\left(x^2(\log x)^{\frac{5}{3}}\right). \tag{30}$$

Using Lemma 2.5 and Lemma 2.6, we have

$$
\begin{aligned}
S_7&=\frac{128}{9}\sum_{\substack{n\le x\\4|n}}\sigma(n)\sigma\left(\frac{n}{4}\right)=\frac{128}{9}\sum_{\substack{k_7\le\frac{x}{4^\alpha}\\4\nmid k_7,\alpha\ge1}}\sigma(4^\alpha)\sigma(4^{\alpha-1})\sigma^2(k_7)\\
&=\frac{128}{9}\sum_{1\le\alpha\le\frac{\log x}{\log 4}}\sigma(4^\alpha)\sigma(4^{\alpha-1})\left(\sum_{k_7\le\frac{x}{4^\alpha}}\sigma^2(k_7)-\sum_{\substack{k_{24}\le\frac{x}{4^\alpha}\\4|k_7}}\sigma^2(k_7)\right)\\
&=C_3x^3+O\left(x^2(\log x)^{\frac{8}{3}}\right),
\end{aligned}
\tag{31}
$$

where C_3 is a constant.

Similarly, we have

$$S_8=\frac{160}{9}\sum_{\substack{n\le x\\5|n}}\sigma(n)\sigma\left(\frac{n}{5}\right)=C_4x^3+O\left(x^2(\log x)^{\frac{8}{3}}\right), \tag{32}$$

$$S_9=\frac{640}{9}\sum_{\substack{n\le x\\4|n,5|n}}\sigma(n)\sigma\left(\frac{n}{20}\right)=C_5x^3+O\left(x^2(\log x)^{\frac{11}{3}}\right), \tag{33}$$

$$S_{12}=\frac{640}{9}\sum_{\substack{n\le x\\4|n,5|n}}\sigma\left(\frac{n}{4}\right)\sigma\left(\frac{n}{5}\right)=C_6x^3+O\left(x^2(\log x)^{\frac{11}{3}}\right), \tag{34}$$

$$S_{13} = \frac{2560}{9} \sum_{\substack{n \le x \\ 4|n, 5|n}} \sigma\left(\frac{n}{4}\right) \sigma\left(\frac{n}{20}\right) = C_7 x^3 + O\left(x^2 (\log x)^{\frac{11}{3}}\right), \tag{35}$$

$$S_{16} = \frac{3200}{9} \sum_{\substack{n \le x \\ 4|n, 5|n}} \sigma\left(\frac{n}{5}\right) \sigma\left(\frac{n}{20}\right) = C_8 x^3 + O\left(x^2 (\log x)^{\frac{11}{3}}\right), \tag{36}$$

where C_4, C_5, C_6, C_7, and C_8 are constants.

Choosing $k = 2$ in (7), we have

$$S_{10} = \frac{64}{9} \sum_{n \le x} \sigma(n) a(n) = O\left(x^{\frac{11}{6}} (\log x)^{-0.1185}\right), \tag{37}$$

$$S_{14} = \frac{256}{9} \sum_{\substack{n \le x \\ 4|n}} \sigma\left(\frac{n}{4}\right) a(n) = O\left(x^{\frac{11}{6}} (\log x)^{-0.1185}\right), \tag{38}$$

$$S_{17} = \frac{320}{9} \sum_{\substack{n \le x \\ 5|n}} \sigma\left(\frac{n}{5}\right) a(n) = O\left(x^{\frac{11}{6}} (\log x)^{-0.1185}\right), \tag{39}$$

$$S_{19} = \frac{1280}{9} \sum_{\substack{n \le x \\ 4|n, 5|n}} \sigma\left(\frac{n}{20}\right) a(n) = O\left(x^{\frac{11}{6}} (\log x)^{-0.1185}\right). \tag{40}$$

On taking $k = 2$ in Lemma 2.4, we have

$$S_{20} = \frac{64}{9} \sum_{n \le x} a^2(n) = C_9 x^2 + O\left(x^{\frac{8}{5}+\varepsilon}\right), \tag{41}$$

where C_9 is a constant.

We get from (26)–(41) that

$$\sum_{n \le x} R^2(n) = C_1 x^3 + O\left(x^2 (\log x)^{\frac{11}{3}}\right),$$

where C_1 is a constant. The proof of Theorem 1.2 is complete. □

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No.11101249). The authors wish to thank the referee for a careful reading of the manuscript and many valuable suggestions which highly improve the quality of this paper.

References

[1] K. Chakraborty, A. Juyal, S. D. Kumar and B. Maji. An asymtotic expansion of a Lambert series associated to cusp forms. Int. J. Number Theory 14 (2018), 289 – 299.

[2] P. Deligne. La conjecture de Weil. I. Inst. Hautes Études Sci. Publ. Math. 43 (1974), 273 – 307.

[3] J. Elstrodt, F. Grunewald and J. Mennicke. Zeta-functions of binary Hermitian forms and special values of Eisentein series on three-dimensional hyperbolic space. Math. Ann. 277 (1987), no. 4, 655 – 708.

[4] O. M. Fomenko. Distribution of lattice points on surfaces of second order. Zap. Nauchn. Semin. 212 (1994), 164 – 195.

[5] O. M. Fomenko. On the distribution of integral points on cones. J. Math. Sci. 178 (2011), no. 2, 227 – 233.

[6] A. Good. The square mean of Dirichlet series associated with cusp forms. Mathematika. 29 (1982), no. 2, 278 – 295.

[7] D. R. Heath-Brown. Hybrid bounds for L-functions: a q-analogue of Van Corput's method and a t-analogue of Burgess' method. *Recent Progrees in Analytic Number Theory*, (Academic Press, London), 1 (1981), 121 – 126.

[8] H. Iwaniec. Topics in Classical Automorphic Forms. American Mathmatical Society RI, Rhode Island, 1997.

[9] Y. J. Jiang. and G. S. Lü. On the higher mean over arithmetic progressions of Fourier coefficients of cusp forms. Acta Arith. 166 (2014), 231 – 252.

[10] W. Mller. The mean square of Dirichlet series associated with automorphic forms. Monatsh. Math. 113 (1992), no. 2, 121 – 159.

[11] W. Mller. The Rankin-Selberg method for nonholomorphic automorphic forms. J. Number Theory 51 (1995), no. 1, 48 – 86.

[12] C. D. Pan. and C. B. Pan. Fundamentals of Analytic Number Theory(in Chinese), Science Press, Beijing, 1997.

[13] A. Perelli. General L-functions. Ann. Mat. Pura Appl 130 (1982), 287 – 306.

[14] G. Shimura. On the holomorphic of certain Dirchlet series. Proc. London Math. Soc. 31 (1975), 79 – 98.

[15] V. Sita Ramaiah and D. Suryanarayana. An order result involving the σ-function. Indian J. pure appl. Math. 12 (1981), no. 10, 1192 – 1200.

[16] E. C. Tichmarsh. The Theory of the Riemann Zeta-Function. 2nd ed. Oxford University Press, Oxford, 1986.

[17] A. Walfisz. Weylsche exponential summen in der neueren zahlentheorie. VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[18] J. Wu. Power sums of Hecke eigenvalues and application. Acta Arith. 137 (2009), no. 4, 333 – 344.

Scientia Magna
Vol. 13 (2018), No. 1, 121-125

A short interval result for the function $\kappa^{(e)}(n)$

Li Ma[1] and Rui Zhang[2]

[1]School of Mathematics and Statistics, Shandong Normal University
Jinan, China
E-mail: 15275171109@163.com
[2]School of Mathematics and Statistics, Shandong Normal University
Jinan, China
E-mail: zhangruimath@163.com

Abstract Let $n > 1$ be an intrger, $\kappa^{(e)}(n)$ denote the maximal e-squarefree e-divisor of n. In this paper, we shall establish a short interval result for the function $\kappa^{(e)}(n)$.

Keywords The maximal e-squarefree e-divisor function, Arithmetic function, Short interval.
2010 Mathematics Subject Classification 11N37.

§1. Introduction and preliminaries

Many scholars are interested in researching the divisor problem, and they have obtained a large number of good results. However, there are many problems hasn't been solved. For example, F.Smarandache gave some unsolved problems in his book *Only problems, Not solutions!* [3], and one problem is that, a number n is called simple number if the product of its proper divisors is lass than or equal to n. Generally speaking, $n = p$,or $n = p^2$, or $n = p^3$, or pq, where p and q are distinct primes. The properties of this simple number sequence hasn't been studied yet. And other problems are introduced in this book, such as proper divisor products sequence and the largest exponent (of power p) which divides n, where $p \geq 2$ is an integer.

Let $n > 1$ be an integer of canonical from $n = \prod_{i=1}^{s} p_i^{a^i}$. The integer $d = \prod_{i=1}^{s} p_i^{b^i}$ is called an exponential divisor of n if $b_i | a_i$ for every $i \in \{1, 2 \cdots s\}$, notation: $d|_e n$. By convention $1|_e 1$.

The integer $n > 1$ is called e-squarefree, if all exponents $a_1, \cdots a_s$ are squarefree. The integer 1 is also considered to be e-squarefree. Consider now the exponential squarefree exponential divisor (e-squarefree e-divisor) of n. Here $d = \prod_{i=1}^{s} p_i^{b^i}$ is called an e-squarefree e-divisor of $n = \prod_{i=1}^{s} p_i^{a^i} > 1$, if $b_1 | a_1 \cdots b_s | a_s$, $b_1 \cdots b_s$ are squarefree. Note that the integer 1 is e-squarefree but is not an e-divisor of $n > 1$. Let $\kappa^{(e)}(n)$ denote the maximal e-squarefree e-divisor of n. The function $\kappa^{(e)}(n)$ is called the maximal e-squarefree e-divisor function, which is multiplicative and if $n = \prod_{i=1}^{s} p_i^{a^i} > 1$, then (see[1])

$$\kappa^{(e)}(n) = p_1^{\kappa(a_1)} \cdots p_r^{\kappa(a_r)},$$

where $\kappa^{(e)}(n) = \prod_{p|n} p$. The function $\kappa^{(e)}(n)$ is multiplicative and $\kappa^{(e)}(p^a) = p^{\kappa(a)}$ for every prime power p^a. Hence for every prime p, $\kappa^{(e)}(p) = p$, $\kappa^{(e)}(p^2) = p^2$, $\kappa^{(e)}(p^3) = p^3$, $\kappa^{(e)}(p^4) = p^2 \cdots$

Many authors have investigated the properties of the function $\kappa^{(e)}(n)$, see [4] and [5]. Recently L. Tóth [1] proved that the estimate

$$\sum_{n\le x}\kappa^{(e)}(n) = \frac{1}{2}\prod_p(1+\sum_{a=4}^{\infty}\frac{p^{\kappa(a)}-p^{1+\kappa(a-1)}}{p^a})x^2 + O(x^{\frac{5}{4}}\delta(x)),$$

where

$$\delta(x) = \delta_A(x) := \exp(-A(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}),$$

A is a positive constant.

The aim of this short text is to study the short interval case and prove the following.

Theorem *If $x^{\frac{1}{9}+2\epsilon} < y \le x$, then*

$$\sum_{x<n\le x+y}\kappa^{(e)}(n) = \frac{G(2)}{\zeta(4)}\int_x^{x+y} tdt + O(yx^{-\frac{\epsilon}{2}}) + O(x^{\frac{1}{9}+\frac{3}{2}\epsilon}).$$

Natation Through out this paper, ϵ is always denotes a fixed but sufficiently small positive constant.

§2. Proof of the theorem

In order to prove our theorem, we need the following lemmas.

Lemma 1. *Suppose s is a complex number ($\Re s > 1$), then*

$$\sum_{n=1}^{\infty}\frac{\kappa^{(e)}(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(4s-4)}G(s),$$

where the Dirichlet series $G(s) = \sum_{n=1}^{\infty}\frac{g(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{6}{5}$.

Proof. Here $\kappa^{(e)}(n)$ is multiplicative and by Euler product formula we have for ($\Re s > 1$) that,

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\kappa^{(e)}(n)}{n^s} &= \prod_p(1+\frac{\kappa^{(e)}(p)}{p^s}+\frac{\kappa^{(e)}(p^2)}{p^{2s}}+\frac{\kappa^{(e)}(p^3)}{p^{3s}}+\frac{\kappa^{(e)}(p^4)}{p^{4s}}+\frac{\kappa^{(e)}(p^5)}{p^{5s}}+\cdots)\\
&= \prod_p(1+\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\frac{p^3}{p^{3s}}+\frac{p^2}{p^{4s}}+\frac{p^5}{p^{5s}}+\cdots)\\
&= \prod_p(1+\frac{1}{p^{s-1}}+\frac{1}{p^{2s-2}}+\frac{1}{p^{3s-3}}+\frac{\frac{1}{p^2}}{p^{4s-4}}+\frac{1}{p^{5s-5}}+\cdots)\\
&= \zeta(s-1)\prod_p(1+\frac{\frac{1}{p^2}-1}{p^{4s-4}}+\frac{1-\frac{1}{p^2}}{p^{5s-5}}+\cdots)\\
&= \frac{\zeta(s-1)}{\zeta(4s-4)}\prod_p(1+\frac{1}{p^{4s-2}}+\frac{1-\frac{1}{p^2}}{p^{5s-5}}+\cdots)
\end{aligned}$$

$$= \frac{\zeta(s-1)}{\zeta(4s-4)} G(s).$$

So we get $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ and by the properties of Dirichlet series, it is absolutely convergent for $\Re s > \frac{6}{5}$. □

Lemma 2.

$$\sum_{n \leq x} n = \frac{1}{2} x^2 + O(x).$$

Proof. This is easily from partial summation formula. □

Let $f(n)$, $h(n)$ be arithmetic functions defined by the following Dirichlet series (for $\Re s > 1$).

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s-1) G(s), \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^{4s}} = \zeta^{-1}(4s-4).$$

Lemma 3. *Let $f(n)$ be an arithmetic function defined by (1), then we have*

$$\sum_{n \leq x} f(n) = \frac{1}{2} x^2 G(2) + O(x).$$

Proof. From Lemma 1 the infinite series $\sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ converges absolutely for $\Re s > \frac{6}{5}$, it follows that

$$\sum_{n \leq x} g(n) \ll 1.$$

Therefore from the definition of $f(n)$ and Lemma 2, we obtain

$$\begin{aligned}
\sum_{n \leq x} f(n) &= \sum_{n \leq x} m \cdot g(k) \\
&= \sum_{k \leq x} g(k) \sum_{m \leq \frac{x}{k}} m \\
&= \sum_{k \leq x} g(k) \left[\frac{1}{2} \left(\frac{x}{k} \right)^2 + O(1) + O(x) \right] \\
&= \frac{1}{2} x^2 G(2) + O(x).
\end{aligned}$$

□

Lemma 4. *Let $k \geq 2$ be a fixed integer, $1 < y \leq x$ be large real numbers and*

$$B(x, y; k, \epsilon) := \sum_{\substack{x < nm^k \leq x+y \\ m > x^\epsilon}} 1.$$

Then we have

$$B(x, y; k, \epsilon) \ll yx^{-\epsilon} + x^{\frac{1}{2k+1}\log x}.$$

Proof. This Lemma is very important when studying the short interval distribution of l-free number, see [2]. □

Next we prove our Theorem. From Lemma 4 and the definition of $f(n)$, $h(n)$, we get

$$h(n) = d_{-1}(n)n^4 \ll n^{4+\epsilon},$$

and

$$\kappa^{(e)}(n) = \sum_{n=km^4} f(k)h(m).$$

So we have

$$Q(x+y) - Q(x) = \sum_{n < km^4 \leq x+y} f(k)h(m) = \sum_1 + O\left(\sum_2\right), \tag{2}$$

where

$$\sum_1 = \sum_{m \leq x^\epsilon} h(m) \sum_{\frac{x}{m^4} < k \leq \frac{x+y}{m^4}} f(k),$$

$$\sum_2 = \sum_{\substack{x < nm^k \leq x+y \\ m > x^\epsilon}} |f(k)h(m)|. \tag{3}$$

In view of Lemma3,

$$\begin{aligned} \sum_1 &= \sum_{m \leq x^\epsilon} h(m) \left(\frac{G(2)}{m^8} \int_x^{x+y} t dt + O(1) + O\left(\frac{x}{m^4}\right)\right) \\ &= \frac{G(2)}{\zeta(4)} \int_x^{x+y} t dt + O(yx^{-\frac{\epsilon}{2}}) + O(x^\epsilon). \end{aligned} \tag{4}$$

By Lemma 4, we have

$$\begin{aligned} \sum_2 &\ll x^{\epsilon^2} \sum_{x < km^4 \leq x+y} 1 \\ &\ll x^{\epsilon^2} \left(yx^{-\epsilon} + x^{\frac{1}{9}+\epsilon}\right) \\ &\ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{9}+\frac{3}{2}\epsilon}. \end{aligned}$$

(5)

Now from (2)-(5), we obtain

$$\sum_{x<n\leq x+y}\kappa^{(e)}(n)=\frac{G(2)}{\zeta(4)}\int_x^{x+y}tdt+O(yx^{-\frac{\epsilon}{2}})+O(x^{\frac{1}{9}+\frac{3}{2}\epsilon}).$$

Acknowledgements

The authors express their gratitude to the referee for a careful reading of the manuscript and many valuable suggestions which highly improve the quality of this paper.

References

[1] L.Tóth. On certain arithmetic functions involving exponential divisors, II. Annales Univ. Sci. Budapest. Sect. Comp. 27(2007),155-166.

[2] M.Filaseta and O.Trifonov. The distribution of square full numbers in short intervals, Acta Arith, 67(1994), 323-333.

[3] F.Smarandache. Only problems, Not solutions! Chicago: Xiquan Publishing House, 1993.

[4] L.Tóth. On certain arithmetic functions involving exponential divisors, Annales Univ. Sci. Budapest. Sect. Comp., 24 (2004), 285-294.

[5] V.Sita Ramaiah and D.Suryanarayana, On a method of Eckford Cohen, Boll. Un. Mat. Ital., (6) 1-B (1982), 1235-1251.

Scientia Magna
Vol. 13 (2018), No. 1, 126-132

The mean value of $t^{(e)}(n)$ over cube-full numbers

Rui Zhang[1] and Li Ma[2]

[1]School of Mathematics and Statistics, Shandong Normal University
Shandong Jinan, China
E-mail: zhangruimath@163.com
[2]School of Mathematics and Statistics, Shandong Normal University
Shandong Jinan, China
E-mail: 15275171109@163.com

Abstract Let $n > 1$ be an integer, the function $t^{(e)}(n)$ denote the number of e-squarefree e-divisors of n. In this paper, we will study the mean value of $t^{(e)}(n)$ over cube-full numbers, that is

$$\sum_{\substack{n \le x \\ n\ is\ cube-full}} t^{(e)}(n) = \sum_{n \le x} t^{(e)}(n) f_3(n)$$

.

Keywords exponentially squarefree, exponential divisors, Dirichlet convolution.
2010 Mathematics Subject Classification 11N37.

§1. Introduction and preliminaries

An integer $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is called k-full number if all the exponents $a_1 \ge k,\ a_2 \ge k,\ \cdots,\ a_r \ge k$, when $k = 3$. n is called cube-full integers, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full} , \\ 0, & \text{otherwise} . \end{cases}$$

Let $n > 1$ be an integer of canonical form $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. The integer $d = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$ is called an exponential divisor (e-divisor) of n, if $b_i | a_i$ for every $i \in 1,\ 2,\ \cdots,\ r$. The integer $n > 1$ is called exponentially squarefree (e-squarefree) if all the exponents $a_1,\ a_2,\ \cdots,\ a_r$ are squarefree. The integer 1 is also considered to be e-squarefree.

Many scholars are interested in researching the divisor problem and have obtained a large number of good results. But there are many problems hasn't been solved. For example, F.Smarandache gave some unsolved problems in his book *Only problems, Not solutions*! [5], and one problem is that the integer $d = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$ is called an e-squarefree e-divisor of $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} > 1$, if $b_1 | a_1,\ \cdots,\ b_r | a_r$ and $b_1,\ \cdots,\ b_r$ are squarefree. Note that the integer 1 is e-squarefree and it is not an e-divisor of $n > 1$. There is the exponential analogues of the functions representing the number of squarefree divisors of n (i.e. $\theta(n) = 2^{\omega(n)},\ where\ \omega(n) =$

r). Let

$$t^{(e)}(n) = 2^{\omega(a_1)} \cdots 2^{\omega(a_r)}$$

where $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} > 1$. The function $t^{(e)}(n)$ is multiplicative and $t^{(e)}(p^a) = 2^{\omega(a)}$ for every prime power p^a. Here for every prime p, $t^{(e)}(p) = 1$, $t^{(e)}(p^2) = t^{(e)}(p^3) = t^{(e)}(p^4) = t^{(e)}(p^5) = t^{(e)}(p^7) = 2$, $t^{(e)}(p^6) = 4$, $\cdots$.

L.Tóth [2] proved the following results:

(1) The Dirichlet series of $t^{(e)}(n)$ is of form

$$\sum_{n=1}^{\infty} \frac{t^{(e)}(n)}{n^s} = \zeta(s)\zeta(2s)V(s), \ \Re s > 1,$$

where $V(s) = \sum\limits_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{4}$.

(2)

$$\sum_{n \le x} t^{(e)}(n) = C_1 x + C_2 x^{\frac{1}{2}} + O(x^{\frac{1}{4}+\epsilon})$$

for every $\varepsilon > 0$, where C_1, C_2 are constants given by

$$C_1 := \prod_p \left(1 + \frac{1}{p^2} + \sum_{a=6}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)}}{p^a}\right),$$

$$C_2 := \zeta(\frac{1}{2}) \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-3)}}{p^{\frac{a}{2}}}\right).$$

(3)

$$\limsup_{n\to\infty} \frac{\log t^{(e)}(n) \log\log n}{\log n} = \frac{1}{2}\log 2.$$

The aim of this paper is to establish the following asymptotic formula for the mean value of the function $t^{(e)}(n)$ over cube-full numbers.

Theorem 1.1 *We have the asymptotic formula*

$$\sum_{\substack{n \le x \\ n \text{ is cube-full}}} t^{(e)}(n) = x^{\frac{1}{3}} Q_{1,1}(\log x) + x^{\frac{1}{4}} Q_{1,2}(\log x) + x^{\frac{1}{5}} Q_{1,3}(\log x) + x^{\frac{1}{6}} + O(x^{\sigma_0+\epsilon}),$$

where $Q_{1,k}(t)$, $k = 1, 2, 3$ *are polynomials of degree* 1 *in* t, $\sigma_0 = \frac{25173878}{166652796} = 0.151055839 \cdots$.

Natation Through out this paper, ϵ always denotes a fixed but sufficiently small positive constant.

§2. Some Lemmas

Lemma 2.1 *Let* $f(m)$, $g(n)$ *are arithmetical functions such that*

$$\sum_{m \le x} f(m) = \sum_{j=1}^{J} x^{\alpha_j} P_j(\log x) + O(x^{\alpha}),$$

$$\sum_{n\le x}|g(n)|=O(x^{\beta})\ ,$$

where $\alpha_1\ge\alpha_2\ge\alpha_3\ge\cdots\alpha_J>\alpha>\beta>0$. $P_j(t)$ *are polynomials in* t. *If* $h(n)=\sum_{n=md}f(m)g(d)$, *then*

$$\sum_{n\le x}h(n)=\sum_{j=1}^{J}x^{\alpha_j}Q_j(\log x)+O(x^{\alpha})\ ,$$

where $Q_j(t)$ *are polynomials in* t, $(j=1,\cdots,J)$.

Lemma 2.2 *The Dirichlet series of* $t^{(e)}(n)$ *is of form*

$$\sum_{\substack{n\le x\\ n\ is\ cube-full}}\frac{t^{(e)}(n)}{n^s}=\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\zeta(6s)G(s),\Re s>1$$

where $G(s)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$ *is absolutely convergent for* $\Re s>\frac{1}{7}$.

Proof.

$$\begin{aligned}
\sum_{\substack{n=1\\ n\ is\ cube-full}}^{\infty}\frac{t^{(e)}(n)}{n^s}&=\sum_{n=1}^{\infty}\frac{t^{(e)}(n)f_3(n)}{n^s}\\
&=\prod_p\left(1+\frac{t^{(e)}(p)f_3(p)}{p^s}+\frac{t^{(e)}(p^2)f_3(p^2)}{p^{2s}}+\frac{t^{(e)}(p^3)f_3(p^3)}{p^{3s}}+\frac{t^{(e)}(p^4)f_3(p^4)}{p^{4s}}+\cdots\right)\\
&=\prod_p\left(1+\frac{t^{(e)}(p^3)}{p^{3s}}+\frac{t^{(e)}(p^4)}{p^{4s}}+\frac{t^{(e)}(p^5)}{p^{5s}}+\cdots\right)\\
&=\prod_p\left(1+\frac{2^{\omega(3)}}{p^{3s}}+\frac{2^{\omega(4)}}{p^{4s}}+\frac{2^{\omega(5)}}{p^{5s}}+\cdots\right)\\
&=\prod_p\left(1+\frac{2}{p^{3s}}+\frac{2}{p^{4s}}+\frac{2}{p^{5s}}+\cdots\right)\\
&=\zeta(3s)\prod_p\left(1+\frac{1}{p^{3s}}+\frac{2}{p^{4s}}+\frac{2}{p^{5s}}+\cdots\right)\\
&=\zeta^2(3s)\prod_p\left(1+\frac{2}{p^{4s}}+\frac{2}{p^{5s}}+\frac{1}{p^{6s}}+\cdots\right)\\
&=\zeta^2(3s)\zeta(4s)\prod_p\left(1+\frac{1}{p^{4s}}+\frac{2}{p^{5s}}+\frac{1}{p^{6s}}+\cdots\right)\\
&=\zeta^2(3s)\zeta^2(4s)\prod_p\left(1+\frac{2}{p^{5s}}+\frac{1}{p^{6s}}-\frac{2}{p^{7s}}+\cdots\right)\\
&=\zeta^2(3s)\zeta^2(4s)\zeta(5s)\prod_p\left(1+\frac{1}{p^{5s}}+\frac{1}{p^{6s}}-\frac{2}{p^{7s}}+\cdots\right)\\
&=\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\prod_p\left(1+\frac{1}{p^{6s}}-\frac{2}{p^{7s}}+\cdots\right)
\end{aligned}$$

$$= \zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\zeta(6s)G(s)$$

where $G(s) = \sum\limits_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod\limits_p (1 - \frac{2}{p^{7s}} + \cdots)$ is absolutely convergent for $\Re s > \frac{1}{7}$, and

$$\sum_{n\le x} |g(n)| \ll x^{\frac{1}{7}+\epsilon} .$$

□

Lemma 2.3 *Let* $\frac{1}{2} \le \sigma \le 1,\ t \ge t_0 \ge 2,$ *we have*

$$\zeta(\sigma + it) \ll t^{\frac{1-\sigma}{3}} \log t.$$

Proof. See E.C.Titchmarsh[4]. □

Lemma 2.4 *Let* $\frac{1}{2} < \sigma < 1$, *define*

$$\begin{aligned}
m(\sigma) &= \frac{4}{3-4\sigma}, & \frac{1}{2} < \sigma \le \frac{5}{8},\\
m(\sigma) &= \frac{19}{6-6\sigma}, & \frac{35}{54} < \sigma \le \frac{41}{60}\\
m(\sigma) &= \frac{2112}{859-948\sigma}, & \frac{41}{60} < \sigma \le \frac{3}{4}\\
m(\sigma) &= \frac{12408}{4537-4890\sigma}, & \frac{3}{4} < \sigma \le \frac{5}{6}\\
m(\sigma) &= \frac{4324}{1031-1044\sigma}, & \frac{5}{6} < \sigma \le \frac{7}{8}\\
m(\sigma) &= \frac{98}{31-32\sigma}, & \frac{7}{8} < \sigma \le 0.91591\\
m(\sigma) &= \frac{24\sigma-9}{(4\sigma-1)(1-\sigma)}, & 0.91591 < \sigma \le 1-\epsilon
\end{aligned}$$

Proof. See A. Ivić[3]. □

Lemma 2.5

$$\sum_{n\le x} d(3,3,4,4,5,5,6;n) = x^{\frac{1}{3}}P_{1,1}(\log x) + x^{\frac{1}{4}}P_{1,2}(\log x) + x^{\frac{1}{5}}P_{1,3}(\log x) + x^{\frac{1}{6}} + O(x^{\sigma_0+\epsilon})$$

where $\sigma_0 = \frac{25173878}{166652796} = 0.151055839\ \cdots$, $P_{1,k}(t),\ k = 1,\ 2,\ 3$ *are polynomials of degree* 1 *in* t.

Proof. By perron's formula, we have

$$S(x) = \sum_{n\le x} \delta(n)d(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\zeta(6s)\frac{x^s}{s}ds + O(\frac{x^{\frac{1}{3}+\epsilon}}{T})$$

where $b=\frac{1}{3}+\epsilon,\ T=x^c,\ c$ is a very large number of fixed numbers. $\frac{1}{7}<\sigma_0<\frac{1}{6}$. According to the residue theorem, we have

$$S(x)=x^{\frac{1}{3}}P_{1,1}(\log x)+x^{\frac{1}{4}}P_{1,2}(\log x)+x^{\frac{1}{5}}P_{1,3}(\log x)+x^{\frac{1}{6}}+I_1+I_2+I_3+O(1)\ ,$$

$$I_1=\frac{1}{2\pi i}\int_{b-it}^{\sigma_0-it}\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\zeta(6s)\frac{x^s}{s}ds$$

$$I_2=\frac{1}{2\pi i}\int_{\sigma_0-it}^{\sigma_0+it}\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\zeta(6s)\frac{x^s}{s}ds$$

$$I_3=\frac{1}{2\pi i}\int_{\sigma_0+it}^{b+it}\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\zeta(6s)\frac{x^s}{s}ds$$

Since $\sigma_0>\frac{7}{43}+\delta,\ (s=\sigma+iT)$, and from Lemma 2.3, we have,

$$\begin{aligned}I_1+I_3 &\ll \int_{\sigma_0}^{\frac{1}{3}+\epsilon}|\zeta(3\sigma+3iT)|^2|\zeta(4\sigma+4iT)|^2|\zeta(5\sigma+5iT)|^2|\zeta(6\sigma+6iT)|x^\sigma T^{-1}d\sigma\\
&\ll T^{-1}\left(\int_{\sigma_0}^{\frac{1}{6}}+\int_{\frac{1}{6}}^{\frac{1}{5}}+\int_{\frac{1}{5}}^{\frac{1}{4}}+\int_{\frac{1}{4}}^{\frac{1}{3}}+\int_{\frac{1}{3}}^{\frac{1}{3}+\epsilon}\right)\\
&\times|\zeta(3\sigma+3iT)|^2|\zeta(4\sigma+4iT)|^2|\zeta(5\sigma+5iT)|^2|\zeta(6\sigma+6iT)|x^\sigma d\sigma\\
&\ll T^{-1+\epsilon}\int_{\sigma_0}^{\frac{1}{6}}T^{\frac{2(1-3\sigma)}{3}+\frac{2(1-4\sigma)}{3}+\frac{2(1-5\sigma)}{3}+\frac{(1-6\sigma)}{3}}x^\sigma d\sigma\\
&+T^{-1+\epsilon}\int_{\frac{1}{6}}^{\frac{1}{5}}T^{\frac{2(1-3\sigma)}{3}+\frac{2(1-4\sigma)}{3}+\frac{2(1-5\sigma)}{3}}x^\sigma d\sigma+T^{-1+\epsilon}\int_{\frac{1}{5}}^{\frac{1}{4}}T^{\frac{2(1-3\sigma)}{3}+\frac{2(1-4\sigma)}{3}}x^\sigma d\sigma\\
&+T^{-1+\epsilon}\int_{\frac{1}{4}}^{\frac{1}{3}}T^{\frac{2(1-3\sigma)}{3}}x^\sigma d\sigma+T^{-1+\epsilon}\int_{\frac{1}{3}}^{\frac{1}{3}+\epsilon}x^\sigma d\sigma\\
&\ll x^{\frac{1}{7}}T^{-\delta+\epsilon}+x^{\frac{1}{6}}T^{-\frac{4}{3}+\epsilon}+x^{\frac{1}{5}}T^{-\frac{3}{5}+\epsilon}+x^{\frac{1}{4}}T^{-\frac{5}{6}+\epsilon}+x^{\frac{1}{3}}T^{-1+\epsilon}+x^{\frac{1}{3}}T^{-1}\\
&\ll x^{\frac{1}{3}+\epsilon}T^{-\delta+\epsilon}\ ,\end{aligned}$$

where δ is very small normal number, $\delta>\epsilon$.

$$I_2\ll x^{\sigma_0}\left(1+\int_1^T|\zeta(3\sigma+3iT)|^2|\zeta(4\sigma+4iT)|^2|\zeta(5\sigma+5iT)|^2|\zeta(6\sigma+6iT)|t^{-1}dt\right).$$

According to the partial integral formula, we have

$$I_4=\int_1^T|\zeta(3\sigma+3iT)|^2|\zeta(4\sigma+4iT)|^2|\zeta(5\sigma+5iT)|^2|\zeta(6\sigma+6iT)|dt\ll T^{1+\epsilon}.$$

If $p_i>0,\ (i=1,2,3,4)$ are real number, and $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}=1$, by Hölder inequality, we have

$$I_4\ll\left(\int_1^T|\zeta(3\sigma_0+3iT)|^{2p_1}dt\right)^{\frac{1}{p_1}}\left(\int_1^T|\zeta(4\sigma_0+4iT)|^{2p_2}dt\right)^{\frac{1}{p_2}}\left(\int_1^T|\zeta(5\sigma_0+5iT)|^{2p_3}dt\right)^{\frac{1}{p_3}}$$

$$\left(\int_1^T |\zeta(6\sigma_0+6iT)|^{p_4}dt\right)^{\frac{1}{p_4}} .$$

So, we have to prove

$$\int_0^T |\zeta(3\sigma_0+3iT)|^{2p_1}dt \ll T^{1+\epsilon} ,$$
$$\int_0^T |\zeta(4\sigma_0+4iT)|^{2p_2}dt \ll T^{1+\epsilon} ,$$
$$\int_0^T |\zeta(5\sigma_0+5iT)|^{2p_3}dt \ll T^{1+\epsilon} ,$$
$$\int_0^T |\zeta(6\sigma_0+6iT)|^{p_4}dt \ll T^{1+\epsilon} .$$

Let $m(3\sigma_0)=2p_1,\ m(4\sigma_0)=2p_2,\ m(5\sigma_0)=2p_3,\ m(6\sigma_0)=p_4$, since $\frac{2}{m(3\sigma_0)}+\frac{2}{m(4\sigma_0)}+\frac{2}{m(5\sigma_0)}+\frac{1}{m(6\sigma_0)}=1$, and from Lemma 2.4, we have $\sigma_0=\frac{25173878}{166652796}=0.151055839\cdots$. □

§3. Proof of Theorem 1.1

Let

$$\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\zeta(6s)G(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}, \qquad \Re s>1,$$
$$\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\zeta(6s)=\sum_{n=1}^{\infty}\frac{d(3,3,4,4,5,5,6;n)}{n^s},$$

such that

$$f(n)=\sum_{n=md} d(3,3,4,4,5,5,6;m)g(d). \tag{3.1}$$

From Lemma 2.5 and the definition of $d(3,3,4,4,5,5,6;m)$, we get

$$\sum_{m\le x} d(3,3,4,4,5,5,6;m)=x^{\frac13}P_{1,1}(\log x)+x^{\frac14}P_{1,2}(\log x)+x^{\frac15}P_{1,3}(\log x)+x^{\frac16}+O(x^{\sigma_0+\epsilon}) , \tag{3.2}$$

where $P_{1.k}(t),\ (k=1,2,3)$ are polynomials of degree 1 in t. In addition, we have

$$\sum_{n\le x}|g(n)|=O(x^{\frac17+\epsilon}) . \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), and applying Lemma 2.1, we have

$$\sum_{n\le x} f(n)=x^{\frac13}Q_{1,1}(\log x)+x^{\frac14}Q_{1,2}(\log x)+x^{\frac15}Q_{1,3}(\log x)+x^{\frac16}+O(x^{\sigma_0+\epsilon}) , \tag{3.4}$$

where $Q_{1.k}(t),\ (k=1,2,3)$ are polynomials of degree 1 in t.

From Lemma 2.2, we have

$$t^{(e)}(n)f_3(n)=\sum_{n=md} d(3,3,4,4,5,5,6;m)g(d)=f(n).$$

Then we complete the proof of Theorem 1.1.

Acknowledgements

The authors express their gratitude to the referee for a careful reading of the paper and many valuable suggestions which highly improve the quality of this paper.

References

[1] M.V. Subbarao, On some arithmetic convolutions. In: The Theory of Arithmetic Functions. Lecture Notes in Mathematics. Vol, 251, Springer, 1972, 247-271.

[2] L.Tóth, On certain arithmetic functions involving exponential divisors, II. Annales Univ. Sci. Budapest. Sect. Comp. 27(2007),155-166.

[3] A.Ivić, The Riemann zeta-function:theory and applications. John Wiley and Sons, New York, 1985, 387-393.

[4] E.C. Titchmarsh, The theory of the Riemann zeta-function [M]. Oxford: Clarendon Press, 1951.

[5] F.Smarandache, Only problems, Not solutions! Chicago: Xiquan Publishing House, 1993.

Scientia Magna
Vol. 13 (2018), No. 1, 133-140

Asymptotics for double sum involving Fourier coefficient of cusp forms

Ai Liu [1] and Linli Wei [2]

[1,2]Department of Mathematics and Statistics, Shandong Normal University
Ji'nan 250014, Shandong, P. R. China
E-mail: [1]1330350753@qq.com and [2]2576599154@qq.com

Abstract Let $\lambda_f(n)$ be the n-th normalized Fourier coefficient of a holomorphic Hecke eigenform $f(z) \in S_k(\Gamma)$. In this paper, we study the asymptotics for double sum involving Fourier coefficient $\lambda_f^i(n)$ of cusp forms, where $i = 2, 4, 6, 8$.

Keywords Abel's summation formula, Cusp form, Fourier coefficients.
2010 Mathematics Subject Classification 11F30, 11N37.

§1. Introduction

Automorphic forms and automorphic L-functions are important tools in modern number theory. Let $S_k(\Gamma)$ be the space of holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = SL(2, Z)$. Let $f(z)$ be an eigenfunction of all the Hecke operators belonging to $S_k(\Gamma)$. Then the Hecke eigenform $f(z)$ has the following Fourier expansion at the cusp ∞

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z},$$

where we normalize $f(z)$ such that $a_f(1) = 1$. Instead of $a_f(n)$, one often considers the normalized Fourier coefficient

$$\lambda_f(n) = \frac{a_f(n)}{n^{\frac{k-1}{2}}}.$$

We know that $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right),$$

where $m \geq 1$ and $n \geq 1$ are any integers.

The Fourier coefficients of cusp forms are interesting objects. In 1974, P. Deligne [3] proved the Ramanujan-Petersson conjecture

$$| \lambda_f(n) | \leq d(n), \quad n \geq 1,$$

where $d(n)$ is the divisor function.

In 1990, Rankin [4] considered the sum of the Fourier coefficients of cusp forms, and proved that

$$\sum_{n\leq x}\lambda_f(n)\ll x^{\frac{1}{3}}(\log x)^{-\delta},$$

where $0<\delta<0.06$.

Rankin [5] and Selberg [6] studied the average behavior of $\lambda_f^2(n)$ over natural numbers and showed that, for any $\varepsilon>0$

$$\sum_{n\leq x}\lambda_f^2(n)=cx+O_{f,\varepsilon}\left(x^{\frac{3}{5}+\varepsilon}\right).$$

Subsequently, Lü [8]- [9] further studied the higher moments of $\lambda_f(n)$, and proved that

$$\sum_{n\leq x}\lambda_f^4(n)=cx\log x+c'x+O_{f,\varepsilon}\left(x^{\frac{7}{8}+\varepsilon}\right),$$

$$\sum_{n\leq x}\lambda_f^6(n)=xP_4(\log x)+O_{f,\varepsilon}\left(x^{\frac{31}{32}+\varepsilon}\right),$$

$$\sum_{n\leq x}\lambda_f^8(n)=xP_{13}(\log x)+O_{f,\varepsilon}\left(x^{\frac{127}{128}+\varepsilon}\right),$$

where $P_i(x)$ is a polynomial of degree i in x.

In this paper, we will study asymptotics for double sums which involve the Fourier coefficient of cusp forms.

Theorem 1.1 *Let $k\in N$, then*

(i) *for any Riemann integrable function $f:[0,1]\to R$, the following equality holds*

$$\lim_{x\to\infty}\frac{1}{xP_1(\log x)}\sum_{ij\leq x}f\left(\frac{ij}{x}\right)\lambda_f^2(i)\lambda_f^2(j)=\int_0^1 f(t)dt,$$

(ii) *if $w:[0,1]\to R$ is Riemann integrable, $v_1:[0,1]\to[0,1],\cdots,v_k:[0,1]\to[0,1]$ are all continuous functions, with $v_1(1)\cdots v_k(1)\neq 0$, the following equality holds*

$$\lim_{x\to\infty}\frac{1}{xP_1(\log x)}\sum_{ij\leq x}w\left(\frac{ij}{x}v_1\left(\frac{\log_1(ij)}{\log_1 x}\right)\cdots v_k\left(\frac{\log_k(ij)}{\log_k x}\right)\right)\lambda_f^2(i)\lambda_f^2(j)$$
$$=\int_0^1 w(t\cdot v_1\cdots v_k)dt,$$

(iii) *if $v_0:[0,1]\to R$ is Riemann integrable, $v_1:[0,1]\to R,\cdots,v_k:[0,1]\to R$ are all continuous functions, with $v_1(1)\cdots v_k(1)\neq 0$, the following equality holds*

$$\lim_{x\to\infty}\frac{1}{xP_1(\log x)}\sum_{ij\leq x}v_0\left(\frac{ij}{x}\right)v_1\left(\frac{\log_1(ij)}{\log_1 x}\right)\cdots v_k\left(\frac{\log_k(ij)}{\log_k x}\right)\lambda_f^2(i)\lambda_f^2(j)$$
$$=v_1(1)\cdots v_k(1)\int_0^1 v_0(t)dt,$$

where $P_1(x)$ is a polynomial of degree 1 in x.

Theorem 1.2 *Let $k\in N$, then*

(i) *for any Riemann integrable function $f : [0,1] \to R$, the following equality holds*

$$\lim_{x\to\infty} \frac{1}{xP_3(\log x)} \sum_{ij\le x} f\left(\frac{ij}{x}\right) \lambda_f^4(i)\lambda_f^4(j) = \int_0^1 f(t)dt,$$

(ii) *if $w : [0,1] \to R$ is Riemann integrable, $v_1 : [0,1] \to [0,1], \cdots, v_k : [0,1] \to [0,1]$ are all continuous functions, with $v_1(1)\cdots v_k(1) \neq 0$, the following equality holds*

$$\lim_{x\to\infty} \frac{1}{xP_3(\log x)} \sum_{ij\le x} w\left(\frac{ij}{x} v_1\left(\frac{\log_1(ij)}{\log_1 x}\right)\cdots v_k\left(\frac{\log_k(ij)}{\log_k x}\right)\right) \lambda_f^4(i)\lambda_f^4(j)$$
$$= \int_0^1 w(t\cdot v_1\cdots v_k)dt,$$

(iii) *if $v_0 : [0,1] \to R$ is Riemann integrable, $v_1 : [0,1] \to R, \cdots, v_k : [0,1] \to R$ are all continuous functions, with $v_1(1)\cdots v_k(1) \neq 0$, the following equality holds*

$$\lim_{x\to\infty} \frac{1}{xP_3(\log x)} \sum_{ij\le x} v_0\left(\frac{ij}{x}\right) v_1\left(\frac{\log_1(ij)}{\log_1 x}\right)\cdots v_k\left(\frac{\log_k(ij)}{\log_k x}\right) \lambda_f^4(i)\lambda_f^4(j)$$
$$= v_1(1)\cdots v_k(1)\int_0^1 v_0(t)dt,$$

where $P_3(x)$ is a polynomial of degree 3 in x.

Theorem 1.3 *Let $k \in N$, then*

(i) *for any Riemann integrable function $f : [0,1] \to R$, the following equality holds*

$$\lim_{x\to\infty} \frac{1}{xP_9(\log x)} \sum_{ij\le x} f\left(\frac{ij}{x}\right) \lambda_f^6(i)\lambda_f^6(j) = \int_0^1 f(t)dt,$$

(ii) *if $w : [0,1] \to R$ is Riemann integrable, $v_1 : [0,1] \to [0,1], \cdots, v_k : [0,1] \to [0,1]$ are all continuous functions, with $v_1(1)\cdots v_k(1) \neq 0$, the following equality holds*

$$\lim_{x\to\infty} \frac{1}{xP_9(\log x)} \sum_{ij\le x} w\left(\frac{ij}{x} v_1\left(\frac{\log_1(ij)}{\log_1 x}\right)\cdots v_k\left(\frac{\log_k(ij)}{\log_k x}\right)\right) \lambda_f^6(i)\lambda_f^6(j)$$
$$= \int_0^1 w(t\cdot v_1\cdots v_k)dt,$$

(iii) *if $v_0 : [0,1] \to R$ is Riemann integrable, $v_1 : [0,1] \to R, \cdots, v_k : [0,1] \to R$ are all continuous functions, with $v_1(1)\cdots v_k(1) \neq 0$, the following equality holds*

$$\lim_{x\to\infty} \frac{1}{xP_9(logx)} \sum_{ij\le x} v_0\left(\frac{ij}{x}\right) v_1\left(\frac{log_1(ij)}{log_1 x}\right)\cdots v_k\left(\frac{log_k(ij)}{log_k x}\right) \lambda_f^6(i)\lambda_f^6(j)$$
$$= v_1(1)\cdots v_k(1)\int_0^1 v_0(t)dt,$$

where $P_9(x)$ is a polynomial of degree 9 in x.

Theorem 1.4 *Let $k \in N$, then*

(i) *for any Riemann integrable function $f : [0,1] \to R$, the following equality holds*

$$\lim_{x\to\infty} \frac{1}{xP_{27}(\log x)} \sum_{ij\le x} f\left(\frac{ij}{x}\right) \lambda_f^8(i)\lambda_f^8(j) = \int_0^1 f(t)dt,$$

(ii) *if $w : [0,1] \to R$ is Riemann integrable, $v_1 : [0,1] \to [0,1], \cdots, v_k : [0,1] \to [0,1]$ are all continuous functions, with $v_1(1)\cdots v_k(1) \ne 0$, the following equality holds*

$$\lim_{x\to\infty} \frac{1}{xP_{27}(\log x)} \sum_{ij\le x} w\left(\frac{ij}{x} v_1\left(\frac{\log_1(ij)}{\log_1 x}\right)\cdots v_k\left(\frac{\log_k(ij)}{\log_k x}\right)\right) \lambda_f^8(i)\lambda_f^8(j)$$
$$= \int_0^1 w(t\cdot v_1\cdots v_k)dt,$$

(iii) *if $v_0 : [0,1] \to R$ is Riemann integrable, $v_1 : [0,1] \to R, \cdots, v_k : [0,1] \to R$ are all continuous functions, with $v_1(1)\cdots v_k(1) \ne 0$, the following equality holds*

$$\lim_{x\to\infty} \frac{1}{xP_{27}(\log x)} \sum_{ij\le x} v_0\left(\frac{ij}{x}\right) v_1\left(\frac{\log_1(ij)}{\log_1 x}\right)\cdots v_k\left(\frac{\log_k(ij)}{\log_k x}\right) \lambda_f^8(i)\lambda_f^8(j)$$
$$= v_1(1)\cdots v_k(1)\int_0^1 v_0(t)dt,$$

where $P_{27}(x)$ is a polynomial of degree 27 in x.

§2.Preliminaries

In this section, we are devoted to give some preliminary results for the proofs of Theorems.

Lemma 2.1 *Let $h : (0,\infty) \to R$ be a function, satisfying exists $x_0 > 0$ with $h(x) > 0$ for all $x \ge x_0$, and h is differentiable on (x_0,∞) and $\lim\limits_{x\to\infty} \frac{xh'(x)}{h(x)} = 1$. Let $k \in N$, $A, B \subseteq N$ and $v : A\times B \to [0,\infty)$ be such that its double summatory function is equivalent to h. Then*

(i) *for any Riemann integrable function $f : [0,1] \to R$ the following equality holds*

$$\lim_{x\to\infty} \frac{1}{h(x)} \sum_{\substack{ij\le x\\ (i,j)\in A\times B}} f\left(\frac{ij}{x}\right) v(i,j) = \int_0^1 f(t)dt,$$

(ii) *if $w : [0,1] \to R$ is Riemann integrable, $v_1 : [0,1] \to [0,1], \cdots, v_k : [0,1] \to [0,1]$ are all continuous functions, with $v_1(1)\cdots v_k(1) \ne 0$, the following equality holds*

$$\lim_{x\to\infty} \frac{1}{h(x)} \sum_{\substack{ij\le x\\ (i,j)\in A\times B}} w\left(\frac{ij}{x} v_1\left(\frac{\log_1(ij)}{\log_1 x}\right)\cdots v_k\left(\frac{\log_k(ij)}{\log_k x}\right)\right) v(i,j)$$
$$= \int_0^1 w(t\cdot v_1\cdots v_k)dt,$$

(iii) *if $v_0 : [0,1] \to R$ is Riemann integrable, $v_1 : [0,1] \to R, \cdots, v_k : [0,1] \to R$ are all continuous functions, with $v_1(1)\cdots v_k(1) \neq 0$, the following equality holds*

$$\lim_{x\to\infty} \frac{1}{h(x)} \sum_{\substack{ij\le x\\ (i,j)\in A\times B}} v_0\left(\frac{ij}{x}\right) v_1\left(\frac{\log_1(ij)}{\log_1 x}\right)\cdots v_k\left(\frac{\log_k(ij)}{\log_k x}\right) v(i,j)$$
$$= v_1(1)\cdots v_k(1)\int_0^1 v_0(t)dt.$$

Proof. Statements (**i**)-(**iii**) follow Corollary 2 in M. Bănescu and D. Popa [1]. □

Lemma 2.2 *The following evaluations hold*

$$\sum_{ij\le x} \lambda_f^2(i)\lambda_f^2(j) = xP_1(\log x) + O\left(x^{\frac{4}{5}+\varepsilon}\right), \tag{2.1}$$

$$\sum_{ij\le x} \lambda_f^4(i)\lambda_f^4(j) = xP_3(\log x) + O\left(x^{\frac{15}{16}+\varepsilon}\right), \tag{2.2}$$

$$\sum_{ij\le x} \lambda_f^6(i)\lambda_f^6(j) = xP_9(\log x) + O\left(x^{\frac{63}{64}+\varepsilon}\right), \tag{2.3}$$

$$\sum_{ij\le x} \lambda_f^8(i)\lambda_f^8(j) = xP_{27}(\log x) + O\left(x^{\frac{255}{256}+\varepsilon}\right), \tag{2.4}$$

where $P_i(x)$ is a polynomial of degree i in x.

Proof. By the hyperbola method of Dirichlet, we have

$$\sum_{ij\le x} \lambda_f^2(i)\lambda_f^2(j) = 2\sum_{i\le\sqrt{x}} \lambda_f^2(i) F\left(\frac{x}{i}\right) - \left(F(\sqrt{x})\right)^2, \tag{2.5}$$

where $F(x) = \sum\limits_{i\le x}\lambda_f^2(i) = cx + R(x)$ with $R(x) = O\left(x^{\frac{3}{5}+\varepsilon}\right)$. According to Abel's summation formula (see Apostol [10]), we have

$$\begin{aligned}\sum_{i\le x}\frac{\lambda_f^2(i)}{i} &= \int_1^x \frac{1}{u} d\left(\sum_{t\le u}\lambda_f^2(t)\right)\\ &= \frac{1}{x}\sum_{t\le x}\lambda_f^2(t) + \int_1^x \left(cu + O\left(u^{\frac{3}{5}+\varepsilon}\right)\right)\frac{1}{u^2}du\\ &= c + O\left(x^{-\frac{2}{5}+\varepsilon}\right) + c\log x + \int_1^x O\left(u^{-\frac{7}{5}+\varepsilon}\right)du.\end{aligned} \tag{2.6}$$

Denote $O\left(u^{-\frac{7}{5}+\varepsilon}\right)$ by $f(u)$, we obtain

$$\int_1^x O\left(u^{-\frac{7}{5}+\varepsilon}\right)du = \int_1^x f(u)du = \int_1^\infty f(u)du - \int_x^\infty f(u)du.$$

We learn that $\mid f(u) \mid\le cu^{-\frac{7}{5}+\varepsilon}$, and $\int_1^\infty f(u)du$ is convergent. There exists a $M>0$, such that

$$\int_1^x O\left(u^{-\frac{7}{5}+\varepsilon}\right)du = M + O\left(x^{-\frac{2}{5}+\varepsilon}\right). \tag{2.7}$$

Further (2.6) gives

$$\sum_{i\le x}\frac{\lambda_f^2(i)}{i}=c\log x+c'+O\left(x^{-\frac{2}{5}+\varepsilon}\right).$$

We deduce

$$\begin{aligned}\sum_{i\le\sqrt{x}}\lambda_f^2(i)F\left(\frac{x}{i}\right)&=\sum_{i\le\sqrt{x}}\lambda_f^2(i)\left(c\frac{x}{i}+O\left(x^{\frac{3}{5}+\varepsilon}\right)\right)\\&=cx\sum_{i\le\sqrt{x}}\frac{\lambda_f^2(i)}{i}+O\left(x^{\frac{3}{5}+\varepsilon}\sum_{i\le\sqrt{x}}\frac{\lambda_f^2(i)}{\frac{3}{5}+\varepsilon}\right)\qquad(2.8)\\&=cx(\log x)+c'x+O\left(x^{\frac{4}{5}+\varepsilon}\right)+R_1(x),\end{aligned}$$

where $R_1(x)=O\left(x^{\frac{3}{5}+\varepsilon}\sum_{i\le\sqrt{x}}\frac{\lambda_f^2(i)}{i^{\frac{3}{5}+\varepsilon}}\right)$.

Using the Abel's summation formula, we have

$$\begin{aligned}\sum_{i\le x}\frac{\lambda_f^2(i)}{i^{\frac{3}{5}+\varepsilon}}&=\int_1^x u^{-\frac{3}{5}-\varepsilon}d\left(\sum_{t\le u}\lambda_f^2(t)\right)\\&=cx^{\frac{2}{5}+\varepsilon}+O\left(x^{\varepsilon}\right)+\frac{3}{5}\int_1^x\left(cu^{-\frac{3}{5}-\varepsilon}+O\left(u^{-1}\right)\right)du\\&\ll x^{\frac{2}{5}+\varepsilon}.\end{aligned}$$

Thus $R_1(x)=O\left(x^{\frac{3}{5}+\varepsilon}\sum_{i\le\sqrt{x}}\frac{\lambda_f^2(i)}{i^{\frac{3}{5}+\varepsilon}}\right)=O\left(x^{\frac{4}{5}}\right)$, and (2.8) gives

$$\sum_{i\le\sqrt{x}}\lambda_f^2(i)F\left(\frac{x}{i}\right)=cx\log x+O\left(x\right).\qquad(2.9)$$

In addition

$$\left(F(\sqrt{x})\right)^2=cx+O\left(x^{\frac{4}{5}+\varepsilon}\right).\qquad(2.10)$$

From (2.9) and (2.10) , we obtain (2.1). Similarly, we get (2.2) - (2.4). □

§3. Proof of Theorems

In this section, we only give the proof of Theorem 1.1. The proofs of remaining Theorems are similar to that of Theorem 1.1. In order to avoid repetition, we omit them.

Proof. Let $h(x)=xP_1(\log x)$, where $P_1(x)$ is a polynomial of degree 1 in x. Then there exists $x_0>0$ with $h(x)>0$ for all $x>x_0$, and h is differentiable on (x_0,∞). Further

$$h'(x)=P_1(\log x)+c,$$

thus

$$\lim_{x\to\infty}\frac{xh'(x)}{h(x)}=\lim_{x\to\infty}\frac{xP_1(\log x)+x}{xP_1(\log x)}=1.$$

According to Lemma 2.2, the double summatory function of $v(i,j) = \lambda_f^2(i)\lambda_f^2(j)$ is equivalent to h, namely,

$$\sum_{\substack{ij\leq x\\ i,j\in N}} \lambda_f^2(i)\lambda_f^2(j) \sim xP_1(\log x), \qquad (x\to\infty).$$

From Lemma 2.1, we get

(i) for any Riemann integrable function $f:[0,1]\to R$, the following equality holds

$$\lim_{x\to\infty}\frac{1}{xP_1(\log x)}\sum_{ij\leq x} f\left(\frac{ij}{x}\right)\lambda_f^2(i)\lambda_f^2(j) = \int_0^1 f(t)dt,$$

(ii) if $w:[0,1]\to R$ is Riemann integrable, $v_1:[0,1]\to[0,1],\cdots,v_k:[0,1]\to[0,1]$ are all continuous functions, with $v_1(1)\cdots v_k(1)\neq 0$, the following equality holds

$$\lim_{x\to\infty}\frac{1}{xP_1(\log x)}\sum_{ij\leq x} w\left(\frac{ij}{x}v_1\left(\frac{\log_1(ij)}{\log_1 x}\right)\cdots v_k\left(\frac{\log_k(ij)}{\log_k x}\right)\right)\lambda_f^2(i)\lambda_f^2(j)$$
$$=\int_0^1 w(t\cdot v_1\cdots v_k)dt,$$

(iii) if $v_0:[0,1]\to R$ is Riemann integrable, $v_1:[0,1]\to R,\cdots,v_k:[0,1]\to R$ are all continuous functions, with $v_1(1)\cdots v_k(1)\neq 0$, the following equality holds

$$\lim_{x\to\infty}\frac{1}{xP_1(\log x)}\sum_{ij\leq x} v_0\left(\frac{ij}{x}\right)v_1\left(\frac{\log_1(ij)}{\log_1 x}\right)\cdots v_k\left(\frac{\log_k(ij)}{\log_k x}\right)\lambda_f^2(i)\lambda_f^2(j)$$
$$=v_1(1)\cdots v_k(1)\int_0^1 v_0(t)dt.$$

This completes the proof of Theorem 1.1. □

§4. Applications of Theorems

In this section we give the applications of theorems.

Corollary *For any function $f:[0,1]\to R$ such that $t\to tf(t)$ is Riemann integrable, the following equalities hold*

$$\lim_{x\to\infty}\frac{\sum_{ij\leq x} f\left(\frac{ij}{x}\right)\lambda_f^2(i)\lambda_f^2(j)}{xP_1(\log x)} = \int_0^1 tf(t)dt,$$

$$\lim_{x\to\infty}\frac{\sum_{ij\leq x} f\left(\frac{ij}{x}\right)\lambda_f^4(i)\lambda_f^4(j)}{xP_3(\log x)} = \int_0^1 tf(t)dt,$$

$$\lim_{x\to\infty}\frac{\sum_{ij\leq x} f\left(\frac{ij}{x}\right)\lambda_f^6(i)\lambda_f^6(j)}{xP_9(\log x)} = \int_0^1 tf(t)dt,$$

$$\lim_{x\to\infty}\frac{\sum_{ij\leq x} f\left(\frac{ij}{x}\right)\lambda_f^8(i)\lambda_f^8(j)}{xP_{27}(\log x)} = \int_0^1 tf(t)dt.$$

Proof. The statement follows from Corollary 7 (**ii**) in M. Bănescu and D. Popa [2]. □

Acknowledgements

This work is supported by Shandong Provincial Natural Science Foundation (Grant No. ZR2018MA003). The authors wish to thank to the referee for a careful reading of manuscript and many valuable suggestions which highly reprove the quality of this paper.

References

[1] M. Bănescu and D. Popa. New extensions of some classical theorems in number theory. J. Number Theory 133 (2013), no. 13, 3771 – 3795.

[2] M. Bănescu and D. Popa. Asymptotic evaluations for some double sums in number theory. J. Number Theory 11 (2015), no. 7, 2073 – 2085.

[3] P. Deligne. La conjecture be Weil. I. Inst Hautes Etudes Sci. Publ. Math., 43 (1974), 29 – 39.

[4] R. A. Rankin. Sums of cusp form coefficients. Automorphic forms and analytic number theory. Univ. Montreal, 1990, 115 – 121.

[5] R. A. Rankin. Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithemtical functions II. The order of the Fourier coefficients of the integral modular forms. Proc. Cambridge Phil. Soc., 35 (1939), 357 – 372.

[6] A. Selberg. Bemerkungen über eine Dirichletsche Rehie, die mit der Modulformen nahe verbunden ist. Arch. Math. Naturvid. 43 (1940), 47 – 50.

[7] J. B. Friedlander, H. Iwaniec. Summation formula for coefficients of L-functions. Canad. J. Math., 57 (2005), 494 – 505.

[8] G. S. Lü. Average behaviour of Fourier coefficients of cusp forms. Proc. Amer. Math. Soc., 137 (2009), 1961 – 1969.

[9] G. S. Lü. The sixth and eight moments of Fourier coefficientso of cusp form. J. Number Theory 129 (2009), 2790 – 2800.

[10] T. M. Aposol. Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics, Springer, 1998.

Scientia Magna
Vol. 13 (2018), No. 1, 141-153

A survey on Smarandache notions in number theory III: Smarandache LCM function

Xiaolin Chen

School of Mathematics, Northwest University
Xi'an 710127, China
E-mail: xlchen@stumail.nwu.edu.cn

Abstract In this paper we give a survey on recent results on Smarandache LCM function.

Keywords Smarandache notion, Smarandache LCM function, sequence, mean value.
2010 Mathematics Subject Classification 11A07, 11B50, 11L20, 11N25.

§1. Definition and simple properties

For any positive integer n, the famous Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$.

Some elementary properties of $SL(n)$ can be found in [20].

M. Le [12]. *For any positive integer n, let $S(n)$ be the Smarandache function. Every positive integer n satisfying*

$$SL(n) = S(n), \qquad S(n) \neq n$$

can be expressed as

$$n = 12 \qquad or \qquad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1, p_2, \cdots, p_r, p$ are distinct primes and $\alpha_1, \alpha_2, \cdots, \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$ $(i = 1, 2, \cdots, r)$.

Q. Wu [26]. *Conjecture. There is no any positive integer $n \geq 2$ such that*

$$\sum_{d \mid n} \frac{1}{SL(d)}$$

is an integer, where $\sum_{d \mid n}$ denotes the summation over all positive divisors of n.

1) *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, where $p_1 < p_2 < \cdots < p_r$. If $\alpha_1 = 1$, then the conjecture is true.*

2) *For any integer $n > 1$, if $SL(n)$ is a prime, then the conjecture is true.*

3) *Let p be a prime and α be any positive integer. If $n = p^\alpha$, then the conjecture is true.*

4) *If n is a square-free number ($n > 1$, and any prime $p \mid n \Longrightarrow p^2 \nmid n$), then the conjecture is true.*

W. Zhu [42]. 1) *For any integer $n > 1$, if*

$$\sum_{d|n} \frac{1}{SL(d)}$$

is an integer, where $\sum_{d|n}$ denotes the summation over all positive divisors of n, then n is a Square-full number (for any prime p, $p \mid n \Longrightarrow p^2 \mid n$).

2) *For any odd number $n > 1$, if*

$$\sum_{d|n} \frac{1}{SL(d)}$$

is an integer, where $\sum_{d|n}$ denotes the summation over all positive divisors of n, then n is a Cubic-full number (for any prime p, $p \mid n \Longrightarrow p^3 \mid n$).

3) *For any integer $n > 1$ with $(n, 6) = 1$, if*

$$\sum_{d|n} \frac{1}{SL(d)}$$

is an integer, where $\sum_{d|n}$ denotes the summation over all positive divisors of n, then n is a 5-full number (for any prime p, $p \mid n \Longrightarrow p^5 \mid n$).

Conjecture. For any positive integer n,

$$\sum_{d|n} \frac{1}{SL(d)}$$

is an integer, where $\sum_{d|n}$ denotes the summation over all positive divisors of n, if and only if $n = 1, 36$.

L. Wu and S. Yang [25]. 1) *Let p be a prime and $n = 4p^\alpha$. Then $n = 36$ if and only if*

$$\sum_{d|n} \frac{1}{SL(d)}$$

is an integer, where $\sum_{d|n}$ denotes the summation over all positive divisors of n.

2) *Let $n = 4p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, where $p_1 < p_2 < \cdots < p_r$. If $r \geq 2$ and $p_1 \geq 5$, then $\sum_{d|n} \frac{1}{SL(d)}$ is not an integer, where $\sum_{d|n}$ denotes the summation over all positive divisors of n.*

G. Feng [7]. 1) *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in \mathbb{N} \\ SL(n) \leq x}} 1 = 2^{\frac{x}{\ln x}\left[1 + O\left(\frac{\ln\ln x}{\ln x}\right)\right]},$$

where $\mathbb{N}$ denotes the set of all positive integers.

2) *For any real number $x > 1$, let $\pi(x)$ denotes the number of all primes $p \leq x$, then we have the limite formula*

$$\lim_{x\longrightarrow\infty}\left[\sum_{\substack{n\in\mathbb{N}\\ SL(n)\leq x}} 1\right]^{\frac{1}{\pi(x)}} = 2.$$

L. Zhang, X. Zhao and J. Han [34]. *Let $p \geq 17$ be a prime. Then*

$$SL(2^p+1) \geq 10p+1, \qquad SL(2^p-1) \geq 10p+1.$$

§2. Mean values of the Smarandache LCM function

X. Du [6]. *Let k be any fixed positive integer. Then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} SL(n) = Ax^2 + c_1\frac{x^2}{\ln x} + c_2\frac{x^2}{\ln^2 x} + \cdots + c_k\frac{x^2}{\ln^k x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where $A = \frac{1}{2}\sum_p \frac{1}{p^2-1}$, and c_i $(i = 1, 2, \cdots, k)$ are computable constants.

Z. Lv [19]. 1) *Let $k \geq 2$ be a fixed integer. Then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ are computable constants.

2) *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Y. Liu and J. Li [15]. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \ln SL(n) = x\ln x + O(x).$$

L. Cheng [5]. *For any positive integer n, the arithmetical function $\overline{\Omega}(n)$ is defined as*

$$\overline{\Omega}(n) = \begin{cases} 0, & \text{if } n = 1, \\ \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r, & \text{if } n > 1 \text{ and } n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \end{cases}$$

Let $k \geq 2$ be any fixed positive integer. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} \overline{\Omega}(n) SL(n) = \sum_{i=1}^{k}\frac{c_i\cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where c_i $(i = 1, 2, \cdots, k)$ *are computable constants.*

Y. Zhao [39]. *For any positive integer* n*, the arithmetical function* $\overline{\Omega}(n)$ *is defined as*

$$\overline{\Omega}(n) = \begin{cases} 0, & \textit{if} \;\; n = 1, \\ \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r, & \textit{if} \;\; n > 1 \;\; \textit{and} \;\; n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \end{cases}$$

Let $k \geq 2$ *be any fixed positive integer. For any real* $x \geq 2$ *we have the asymptotic formula*

$$\sum_{n \leq x} \left(SL(n) - \overline{\Omega}(n) \right)^2 = \frac{4}{5} \cdot \zeta\left(\frac{5}{2}\right) \cdot \frac{x^{\frac{5}{2}}}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^{\frac{5}{2}}}{\ln^i x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ *is the Riemann zeta function and* c_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

W. Lu and L. Gao [18]. *For any positive integer* n*, the arithmetical function* $\overline{\Omega}(n)$ *is defined as*

$$\overline{\Omega}(n) = \begin{cases} 0, & \textit{if} \;\; n = 1, \\ \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r, & \textit{if} \;\; n > 1 \;\; \textit{and} \;\; n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \end{cases}$$

Let $k \geq 2$ *be any fixed positive integer. For any real* $x \geq 2$ *and* $\beta > 1$ *we have the asymptotic formula*

$$\sum_{n \leq x} \left(SL(n) - \overline{\Omega}(n) \right)^\beta = \frac{2}{2\beta + 1} \cdot \zeta\left(\frac{2\beta + 1}{2}\right) \cdot \frac{x^{\frac{2\beta+1}{2}}}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^{\frac{2\beta+1}{2}}}{\ln^i x} + O\left(\frac{x^{\frac{2\beta+1}{2}}}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ *is the Riemann zeta function and* c_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

X. Li [16]. 1) *Let* k *be any fixed positive integer. For any real number* $x > 1$*, we have the asymptotic formula*

$$\sum_{n \leq x} P(n) SL(n) = x^3 \cdot \sum_{i=1}^{k} \frac{c_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $P(n)$ *denotes the largest prime divisor of* n*, and* c_i $(i = 1, 2, \cdots, k)$ *are computable constants.*

2) *Let* k *be any fixed positive integer. For any real number* $x > 1$*, we have the asymptotic formula*

$$\sum_{n \leq x} p(n) SL(n) = x^3 \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $p(n)$ *denotes the smallest prime divisor of* n*,* b_i $(i = 1, 2, \cdots, k)$ *are computable constants and* $b_1 = \frac{1}{3}$*.*

Y. Xue [27]. *Let* $P(n)$ *denote the largest prime divisor of* n*, and let* $p(n)$ *denote the smallest prime divisor of* n*. For any real number* $x > 1$ *and any positive integer* k*, we have the asymptotic formula*

$$\sum_{n \leq x} (P(n) - p(n)) SL(n) = \zeta(3) \cdot x^3 \cdot \sum_{i=1}^{k} \frac{c_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ is the Riemann zeta function, $c_1 = \frac{1}{3}$, c_i $(i = 2, 3, \cdots, k)$ are computable constants.

J. Chen [4]. *Let $P(n)$ denote the largest prime divisor of n. For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} (SL(n) - P(n))^2 = \frac{2\zeta\left(\frac{5}{2}\right)x^{\frac{5}{2}}}{5\ln x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right).$$

where $\zeta(s)$ is the Riemann zeta function.

L. Zhang and X. Zhao [33]. *Let $P(n)$ denote the largest prime divisor of n, and let $k > 1$ be any fixed positive integer. For any real $x \geq 1$ and $\beta \geq 1$ we have*

$$\sum_{n\leq x} (SL(n) - P(n))^\beta = \frac{2\zeta\left(\frac{2\beta+1}{2}\right)x^{\frac{2\beta+1}{2}}}{(2\beta+1)\ln x} + \sum_{i=2}^{k}\frac{c_i x^{\frac{2\beta+1}{2}}}{\ln^i x} + O\left(\frac{x^{\frac{2\beta+1}{2}}}{\ln^2 x}\right).$$

where $\zeta(s)$ is the Riemann zeta function and c_i $(i = 2, 3, \cdots, k)$ are computable constants.

G. Lu [17]. *Let $d(n) = \sum_{d|n} 1$ denote the Dirichlet divisor function, and let $k \geq 2$ be any fixed positive integer. For any real $x \geq 2$ we have the asymptotic formula*

$$\sum_{n\leq x} d(n)SL(n) = \frac{\pi^4}{36}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 1, 2, \cdots, k)$ are computable constants.

J. Peng, J. Guo, B. Li and X. Tuo [22]. *Let $d(n) = \sum_{d|n} 1$ denote the Dirichlet divisor function, and let $k \geq 2$ be any fixed positive integer. For any real $x > 1$ we have the asymptotic formula*

$$\sum_{n\leq x} d(n)SL(n) = \frac{\pi^4}{36}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ are computable constants.

J. Fu and H. Liu [8]. *Define $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, $\alpha \geq 1$. Let $k \geq 2$ be any fixed positive integer. For any real $x \geq 2$ we have the asymptotic formula*

$$\sum_{n\leq x} \sigma_\alpha(n)SL(n) = \frac{\zeta(\alpha+2)\zeta(2)x^{\alpha+2}}{(2+\alpha)\ln x} + \sum_{i=2}^{k}\frac{c_i x^{\alpha+2}}{\ln^i x} + O\left(\frac{x^{\alpha+2}}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ is the Riemann zeta function and c_i $(i = 1, 2, \cdots, k)$ are computable constants.

W. Huang [10]. *For any positive integer n, define*

$$F(k,n) = \begin{cases} 0, & \text{if } n = 1, \\ \alpha_1 p_1^k + \alpha_2 p_2^k + \cdots + \alpha_r p_r^k, & \text{if } n > 1 \text{ and } n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}. \end{cases}$$

Let N be any fixed natural number. For any real $x \geq 1$ we have the asymptotic formula

$$\sum_{n\leq x} \left(SL^k(n) - F(k,n)\right)^2 = x^{k+1}\sum_{i=1}^{N}\frac{c_i}{\ln^{i+1} x} + O\left(\frac{x^{k+1}}{\ln^{N+2} x}\right),$$

where c_i $(i = 1, 2, \cdots, k)$ *are computable constants and* $c_1 = \dfrac{\pi^2}{6}$.

Q. Zhao and L. Gao [36]. *For any positive integer* n*, the prime factor sum function* $\overline{w}(n)$ *is defined as*

$$\overline{w}(n) = \begin{cases} 1, & \text{if } n = 1, \\ p_1 + p_2 + \cdots + p_r, & \text{if } n > 1 \text{ and } n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \end{cases}$$

For any real $x \geq 2$ *we have the asymptotic formula*

$$\sum_{n \leq x} SL(n)\overline{w}(n) = D\frac{x^3}{\ln x} + O\left(\frac{x^3}{\ln^2 x}\right),$$

where $D = \dfrac{1}{3} \sum\limits_{n \leq \sqrt{x}} \dfrac{1}{n^3}$ *is a computable constant.*

Q. Zhao, L. Gao and L. Ai [37]. *For any positive integer* n*, the prime factor sum function* $\overline{w}(n)$ *is defined as*

$$\overline{w}(n) = \begin{cases} 1, & \text{if } n = 1, \\ p_1 + p_2 + \cdots + p_r, & \text{if } n > 1 \text{ and } n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \end{cases}$$

For any real $x \geq 2$ *we have the asymptotic formula*

$$\sum_{n \leq x} \left(SL(n) - \overline{w}(n)\right)^2 = \frac{2\zeta\left(\frac{5}{2}\right) x^{\frac{5}{2}}}{5 \ln x} + \sum_{i=2}^{k} \frac{c_i x^{\frac{5}{2}}}{\ln^i x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ *is the Riemann zeta function and* c_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

M. Zhu [41]. *Let* $k > 2$ *be any positive integer. For any real* $x > 1$ *we have the asymptotic formula*

$$\sum_{n \leq x} \Lambda(n) SL(n) = x^2 \sum_{i=1}^{k} \frac{c_i}{\ln^{i-1} x} + O\left(\frac{x^2}{\ln^k x}\right),$$

where c_i $(i = 1, 2, 3, \cdots, k)$ *are computable constants,* $c_1 = \dfrac{1}{2}$*, and*

$$\Lambda(n) = \begin{cases} \ln p, & \text{if } n = p^\alpha,\ p \text{ is a prime, } \alpha \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

.

B. Li, J. Guo and J. Peng [13]. *Define a function* $SM(n)$ *as follows:*

$$SM(n) = \begin{cases} 1, & \text{if } n = 1, \\ \max\{\alpha_1 p_1, \alpha_2 p_2, \cdots, \alpha_r p_r\}, & \text{if } n > 1 \text{ and } n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \end{cases}$$

Let $k \geq 2$ *be any fixed positive integer. For any real* $x \geq 2$ *we have the asymptotic formula*

$$\sum_{n \leq x} \left(SL(n) - SM(n)\right)^2 = \frac{2}{5}\zeta\left(\frac{5}{2}\right) \frac{x^{\frac{5}{2}}}{\ln x} + \sum_{i=2}^{k} \frac{c_i x^{\frac{5}{2}}}{\ln^i x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ are computable constants and $\zeta(s)$ is the Riemann zeta function.

Y. Yang and G. Ren [32]. *Define a function $SM(n)$ as follows:*

$$SM(n) = \begin{cases} 1, & \text{if } n = 1, \\ \max\{\alpha_1 p_1, \alpha_2 p_2, \cdots, \alpha_r p_r\}, & \text{if } n > 1 \text{ and } n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \end{cases}$$

For any real number x we have the asymptotic formula

$$\sum_{n \le x} (SL(n) - SM(n))^2 = \frac{2}{5}\zeta\left(\frac{5}{2}\right)\frac{x^{\frac{5}{2}}}{\ln x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta function.

X. Wang and L. Gao [24]. *For any positive integer n, the famous pseudo-Smarandache function $Z(n)$ is defined by*

$$Z(n) = \min\left\{m : \ n \Big| \frac{m(m+1)}{2}, \ m \in \mathbb{N}\right\}.$$

Let $k \ge 2$ be any fixed positive integer. For any real $x \ge 2$ we have the asymptotic formula

$$\sum_{n \le x} Z(n) SL(n) = \frac{\zeta(3)\, x^3}{3 \ln x} + \sum_{i=2}^{k} \frac{c_i x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ is the Riemann zeta function and c_i $(i = 1, 2, \cdots, k)$ are computable constants.

L. Zheng and L. Gao [40]. *Let $S(n)$ denote the Smarandache function and let $\sigma_\alpha(n)$ denote the divisor function $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, $\alpha \ge 1$. Let k be any fixed positive integer. For any real or complex number α and real number $x \ge 3$ we have the asymptotic formula*

$$\sum_{n \le x} \sigma_\alpha(n) (SL(n) - S(n))^2 = \frac{2\zeta\left(\alpha + \frac{5}{2}\right)\zeta\left(\frac{5}{2}\right) x^{\alpha + \frac{5}{2}}}{(2\alpha + 5)\ln x} + \sum_{i=2}^{k} \frac{c_i x^{\alpha+\frac{5}{2}}}{\ln^i x} + O\left(\frac{x^{\alpha+\frac{5}{2}}}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ is the Riemann zeta function and c_i $(i = 2, 3, \cdots, k)$ are computable constants.

§3. Mean values of the Smarandache LCM function over sequences

H. Liu and S. Lv [14]. *Define $Z(n) = \min\left\{k : n \le \frac{k(k+1)}{2}\right\}$. Let $k \ge 2$ be any fixed positive integer. For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n \le x} SL(Z(n)) = \frac{\pi^2}{18} \cdot \frac{(2x)^{\frac{3}{2}}}{\ln \sqrt{2x}} + \sum_{i=2}^{k} \frac{c_i (2x)^{\frac{3}{2}}}{\ln^i \sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ are computable constants.

J. Chai and L. Gao [1]. *Define $U(n) = \min\{k : n \le k(2k-1), k \in \mathbb{N}\}$. Let $k \ge 2$ be any fixed positive integer. For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n \le x} SL(U(n)) = \frac{\pi^2}{144} \cdot \frac{(2x)^{\frac{3}{2}}}{\ln \sqrt{2x}} + \sum_{i=2}^{k} \frac{c_i (2x)^{\frac{3}{2}}}{\ln^i \sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where c_i $(i = 1, 2, \cdots, k)$ are computable constants.

J. Chai, L. Gao and L. Tuo [2]. *Let $P(n)$ denote the largest prime divisor of n, and let $k \geq 2$ be any fixed positive integer. For any real number $x \geq 3$, we have the asymptotic formula*

$$\sum_{n\leq x}\left(SL(a_k(n)) - (k-1)P(n)\right)^2 = \frac{2\zeta\left(\frac{5}{2}\right)}{5}\cdot\frac{x^{\frac{5}{2}}}{\ln x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right) + O\left(\frac{k^2 x^{\frac{4}{3}}}{\ln x}\right),$$

where $\zeta(s)$ is the Riemann zeta function, and $a_k(n)$ denotes the k-th power complements of n.

W. Huang [9]. *Define*

$$\begin{aligned} u_r(n) &= \min\left\{m + \frac{1}{2}m(m-1)(r-2) : n \leq m + \frac{1}{2}m(m-1)(r-2), r \in \mathbb{N}, r \geq 3\right\}, \\ v_r(n) &= \max\left\{m + \frac{1}{2}m(m-1)(r-2) : n \geq m + \frac{1}{2}m(m-1)(r-2), r \in \mathbb{N}, r \geq 3\right\}. \end{aligned}$$

Let k be any fixed positive integer. For any real number $x > 1$, we have the asymptotic formula

$$\begin{aligned} \sum_{n\leq x} SL(u_r(n)) &= \frac{\pi^2}{18(r-2)^3}\cdot\frac{(2(r-2)x)^{\frac{3}{2}}}{\ln\sqrt{\frac{2x}{r-2}}} + \sum_{i=2}^{k}\frac{c_i(2(r-2)x)^{\frac{3}{2}}}{\ln^i\sqrt{\frac{2x}{r-2}}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right), \\ \sum_{n\leq x} SL(v_r(n)) &= \frac{\pi^2}{18(r-2)^3}\cdot\frac{(2(r-2)x)^{\frac{3}{2}}}{\ln\sqrt{\frac{2x}{r-2}}} + \sum_{i=2}^{k}\frac{c_i(2(r-2)x)^{\frac{3}{2}}}{\ln^i\sqrt{\frac{2x}{r-2}}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right), \end{aligned}$$

where c_i $(i = 1, 2, \cdots, k)$ are computable constants.

Z. Lai [11]. *Define*

$$p_d(n) = \prod_{d|n} d, \qquad q_d(n) = \prod_{\substack{d|n \\ d<n}} d.$$

Let $k \geq 2$ be any fixed positive integer. For any integer $x \geq 1$, we have the asymptotic formula

$$\begin{aligned} \sum_{n\leq x} SL\left(p_d(n)\right) &= \frac{\pi^4 x^2}{72\ln x} + \sum_{i=2}^{k}\frac{c_i x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right), \\ \sum_{n\leq x} SL\left(q_d(n)\right) &= \frac{(\pi^4 - 12\pi^2)x^2}{72\ln x} + \sum_{i=2}^{k}\frac{d_i x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right), \end{aligned}$$

where c_i $(i = 2, 3, \cdots, k)$, d_i $(i = 2, 3, \cdots, k)$ are computable constants.

X. Zhao, L. Zhang, H. Xu and R. Guo [38]. *Let $\mathcal{A}$ denote the set of the simple numbers. For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{\substack{n\leq x \\ n\in\mathcal{A}}} SL^k(n) = \frac{Bx^{k+1}}{(k+1)\ln x} + \frac{Cx^{k+1}}{(k+1)^2\ln^2 x} + O\left(\frac{x^{k+1}}{\ln^3 x}\right),$$

$$\sum_{\substack{n\leq x \\ n\in\mathcal{A}}} \frac{1}{SL(n)} = D\ln\ln x + \frac{E\sqrt{x}\ln\ln x}{\ln x} + O\left(\frac{\sqrt{x}}{\ln x}\right),$$

where k is a nonnegative real number, and B, C, D, E are computable constants.

§4. Other functions and sequences related to the Smarandache LCM function

X. Pan [21]. *Define* $L(n) = [1, 2, \cdots, n]$. *For any positive integer* n, *we have the asymptotic formula*

$$\left(\frac{L(n^2)}{\prod\limits_{p\le n^2} p}\right)^{\frac{1}{n}} = e + O\left(\exp\left(-c\frac{(\ln n)^{\frac{3}{5}}}{(\ln\ln n)^{\frac{1}{5}}}\right)\right),$$

where $\prod\limits_{p\le n^2}$ *denotes the production over all prime* $p \le n^2$.

For any positive integer n, the famous Smarandache LCM dual function is defined by

$$SL^*(n) = \max\left\{k \in \mathbb{N} :\ [1, 2, \cdots, k] \mid n\right\}.$$

C. Tian and N. Yuan [23]. 1) *For any real number* $s > 1$, *the series* $\sum\limits_{n=1}^{\infty} \frac{SL^*(n)}{n^s}$ *is absolutely convergent, and*

$$\sum_{n=1}^{\infty} \frac{SL^*(n)}{n^s} = \zeta(s) \sum_{\alpha=1}^{\infty} \sum_{p} \frac{(p^\alpha - 1)(p^s - 1)}{[1, 2, \cdots, p^\alpha]^s},$$

where $\zeta(s)$ *is the Riemann zeta function,* $\sum\limits_{p}$ *denotes the summation over all primes.*

2) *For any real number* $x > 1$, *we have the asymptotic formula*

$$\sum_{n\le x} SL^*(n) = cx + O\left(\ln^2 x\right),$$

where $c = \sum\limits_{\alpha=1}^{\infty} \sum\limits_{p} \frac{(p^\alpha - 1)(p - 1)}{[1, 2, \cdots, p^\alpha]}$ *is a constant.*

B. Chen [3]. *Let* $w(n)$ *denote all the different prime factor numbers of* n. *For any positive integer* n,

$$\prod_{d|n} SL^*(d) + 1 = 2^{w(n)}$$

has solutions if and only if $n = p^\alpha$, *where* $\alpha \ge 1$ *and* $p \ge 3$ *is a prime.*

For any positive integer n, the dual function of Smarandache LCM function is defined by

$$\overline{SL}(n) = \begin{cases} 1, & \text{if } n = 1, \\ \min\{p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_r^{\alpha_r}\}, & \text{if } n > 1 \text{ and } n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \end{cases}$$

X. Yan [28]. *For any real number* $x > 1$, *we have the asymptotic formula*

$$\sum_{n\le x} \frac{\overline{SL}(n)}{SL(n)} = \frac{x}{\ln x} + O\left(\frac{x(\ln\ln x)^2}{\ln^2 x}\right).$$

H. Zhao and Z. Ye [35]. 1) *Let k be any fixed positive integer. For any real number $x>1$, we have the asymptotic formula*

$$\sum_{n\leq x}\overline{SL}(n)=\sum_{i=1}^{k}\frac{c_i x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),$$

where c_i $(i=2,3,\cdots,k)$ are computable constants, and $c_1=\frac{1}{2}$.

2) *Let $p(n)$ denote the smallest prime divisor of n, and let k be any fixed positive integer. For any real number $x>1$, we have the asymptotic formula*

$$\sum_{n\leq x}\left(\overline{SL}(n)-p(n)\right)^2=\sum_{i=1}^{k}\frac{c_i x^{\frac{5}{2}}}{\ln^i x}+O\left(\frac{x^{\frac{5}{2}}}{\ln^{k+1}x}\right),$$

where c_i $(i=2,3,\cdots,k)$ are computable constants, and $c_1=\frac{2}{5}$.

X. Yan [29]. *Let $p(n)$ denote the smallest prime divisor of n, and let k be any fixed positive integer. For any real number $x>1$, we have the asymptotic formula*

$$\sum_{n\leq x}\left(\overline{SL}(n)-p(n)\right)^2=\sum_{i=2}^{k}\frac{c_i x^{\frac{5}{2}}}{\ln^i x}+O\left(\frac{x^{\frac{5}{2}}}{\ln^{k+1}x}\right),$$

where c_i $(i=1,2,\cdots,k)$ are computable constants, and $c_1=\frac{4}{5}$.

Y. Yang [30]. *Let $p(n)$ denote the smallest prime divisor of n, and let k be any fixed positive integer. For any real number $x>1$, we have the asymptotic formula*

$$\sum_{n\leq x}p(n)\ln\overline{SL}(n)=\frac{x^2}{2}+\sum_{i=1}^{k-1}\frac{c_i x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^k x}\right),$$

where c_i $(i=1,2,\cdots,k-1)$ are computable constants.

Y. Yang [31]. 1) *Define $Z(n)=\min\left\{k: n\leq\frac{k(k+1)}{2}\right\}$. For any real number $x>1$, we have the asymptotic formula*

$$\sum_{n\leq x}SL(Z(n))\cdot\overline{SL}(Z(n))=\frac{2x^2}{\ln 2x}+O\left(\frac{x^2}{\ln^2 x}\right).$$

2) *Define $Z(n)=\min\left\{k: n\leq\frac{k(k+1)}{2}\right\}$. For any real number $x>1$, we have the asymptotic formula*

$$\sum_{n\leq x}\frac{\overline{SL}(Z(n))}{SL(Z(n))}=\frac{2x}{\ln 2x}+O\left(\frac{x\ln\ln x}{\ln^2 x}\right).$$

References

[1] Jingxia Chai and Li Gao. On the hybrid mean value of the Smarandache function. Journal of Gansu Sciences 22 (2010), no. 4, 40 - 42. (In Chinese with English abstract).

[2] Jingxia Chai, Li Gao and Lei Tuo. On the hybrid mean value of the Smarandache LCM function and the k-th power complements. Journal of Yanan University (Natural Science Edition) 29 (2010), no. 3, 10 - 12. (In Chinese with English abstract).

[3] Bin Chen. On the equation of the Smarandache dual LCM function. Journal of Weinan Normal University 27 (2012), no. 2, 21 - 40.

[4] Jianbin Chen. Value distribution of the F. Smarandache LCM function. Scientia Magna 3 (2007), no. 2, 15 - 18.

[5] Lin Cheng. On the mean value of the Smarandache LCM function. Scientia Magna 3 (2007), no. 4, 41 - 44.

[6] Xiaoying Du. On the mean value of the Smarandache LCM function $SL(n)$. Scientia Magna 2 (2006), no. 2, 85 - 90.

[7] Guoping Feng. On the F. Smarandache LCM function. Scientia Magna 3 (2007), 5 - 8.

[8] Jing Fu and Hua Liu. On a hybrid mean value of the F. Smarandache LCM function and the divisor function. Journal of Inner Monggolia Normal University (Natural Science Edition) 39 (2010), no. 6, 560 - 562. (In Chinese with English abstract).

[9] Wei Huang. Hybrid mean value estimate for two Smarandache LCM functions. Basic Sciences Journal of Textile Universities 24 (2011), no. 3, 390 - 393. (In Chinese with English abstract).

[10] Wei Huang. On hybrid mean value distribution involving Smarandache LCM functions. Journal of Xi'an University of Arts and Science (Natural Science Edition) 15 (2012), no. 1, 1 - 3. (In Chinese with English abstract).

[11] Zhanchi Lai. On hybrid mean value formula involving of Smarandache LCM function and Smarandache sequences. Journal of Anhui University of Science and Technology (Natural Science Edition) 35 (2015), no. 4, 16 - 18. (In Chinese with English abstract).

[12] Maohua Le. An equation concerning the Smarandache LCM function. Smarandache Notions Journal 14 (2004), 186 - 188.

[13] Bo Li, Jinbao Guo and Juan Peng. On the average variance of the Smarandache LCM function. Henan Science 31 (2013), no. 4, 422 - 424. (In Chinese with English abstract).

[14] Hua Liu and Songtao Lv. A composite function involving the F. Smarandache function. Jiangxi Science 27 (2009), no. 3, 325 - 327. (In Chinese with English abstract).

[15] Yanyan Liu and Jianghua Li. On the F. Smarandache LCM function and its mean value. Scientia Magna 3 (2007), no. 4, 52 - 55.

[16] Xiaoyan Li. On the mean value of the Smarandache LCM function. Scientia Magna 3 (2007), no. 3, 58 - 62.

[17] Guoliang Lu. On the hybrid mean value of the F. Smarandache LCM function and the Dirichlet divisor function. Pure and Applied Mathematics 23 (2007), no. 3, 315 - 318. (In Chinese with English abstract).

[18] Weiyang Lu and Li Gao. The high hybrid mean value of the Smarandache LCM function and the arithmetical function $\overline{\Omega}(n)$. Journal of Yanan University (Natural Science Edition) 36 (2017), no. 1, 13 - 20. (In Chinese with English abstract).

[19] Zhongtian Lv. On the F. Smarandache LCM function and its mean value. Scientia Magna 3 (2007), no. 1, 22 - 25.

[20] Amarnath Murthy. Some notions on least common multiples. Smarandache Notions Journal 12 (2001), 307 - 309.

[21] Xiaowei Pan. A new limit theorem involving the Smarandache LCM sequence. Scientia Magna 2 (2006), no. 2, 20 - 23.

[22] Juan Peng, Jinbao Guo, Bo Li and Xiaoquan Tuo. A hybrid mean value formula involing Smarandache function. Journal of Yanan University (Natural Science Edition) 32 (2013), no. 1, 8 - 10. (In Chinese with English abstract).

[23] Chengliang Tian and Na Yuan. On the Smarandache LCM dual function. Scientia Magna 3 (2007), no. 2, 25 - 28.

[24] Xihan Wang and Li Gao. On the hybrid mean value of the Pseudo-Smarandache function and F. Smarandache LCM function. Journal of Yanan University (Natural Science Edition) 35 (2016), no. 4, 21 - 23. (In Chinese with English abstract).

[25] Li Wu and Shichun Yang. A conjecture involving the Smarandache LCM function. Journal of Southwest University for Nationalities (Natural Science Edition) 37 (2011), no. 5, 696 - 698. (In Chinese with English abstract).

[26] Qibin Wu. A conjecture involving the F. Smarandache LCM function. Scientia Magna 3 (2007), 26 - 28.

[27] Ranrong Xue. On the F. Smarandache LCM function $SL(n)$. Scientia Magna 3 (2007), no. 4, 69 - 73.

[28] Xiaoxia Yan. Hybrid mean value problem of the Smarandache LCM function and its dual function. Journal of Inner Monggolia Normal University (Natural Science Edition) 39 (2010), no. 3, 229 - 231. (In Chinese with English abstract).

[29] Xiaoxia Yan. On the mean square value of the dual function of Smarandache LCM function and the smallest prime divisor function. Basic Sciences Journal of Textile Universities 23 (2010), no. 3, 323 - 325. (In Chinese with English abstract).

[30] Yanting Yang. The hybrid mean values involving the dual function of Smarandache LCM function. Journal of Xianyang Normal University 29 (2014), no. 2, 9 - 14. (In Chinese with English abstract).

[31] Yanting Yang. The hybrid mean values involving Smarandache functions. Henan Science 34 (2016), no. 11, 1789 - 1793. (In Chinese with English abstract).

[32] Yanting Yang and Ganglian Ren. On the hybrid mean value of Smarandache LCM function and Smarandache function $SM(n)$. Journal of Natural Science of Heilongjiang University 30 (2013), no. 3, 318 - 320. (In Chinese with English abstract).

[33] Lixia Zhang and Xiqing Zhao. On the β-th hybrid mean value of the Smarandache LCM function. Journal of Hubei University (Natural Science Edition) 38 (2016), no. 4, 315 - 317. (In Chinese with English abstract).

[34] Lixia Zhang, Xiqing Zhao and Jianqin Han. A lower bound estimate for the Smarandache LCM function. Henan Science 33 (2015), no. 8, 1291 - 1293. (In Chinese with English abstract).

[35] Hongxing Zhao and Zhenglin Ye. The value distribution of a new F. Smarandache function. Mathematics in Practice and Theory 38 (2008), no. 5, 154 - 157. (In Chinese with English abstract).

[36] Qin Zhao and Li Gao. On hybrid mean value of the F. Smarandache LCM function and the prime factor sum function. Henan Science 30 (2012), no. 1, 15 - 17. (In Chinese with English abstract).

[37] Qin Zhao, Li Gao and Lun Ai. Mean value of the F. Smarandache LCM function. Henan Science 34 (2016), no. 1, 7 - 10. (In Chinese with English abstract).

[38] Xiqing Zhao, Lixia Zhang, Hongxin Xu and Rui Guo. On the mean value properties of the sequence of simple numbers for the Smarandache LCM function. Henan Science 34 (2016), no. 3, 305 - 309. (In Chinese with English abstract).

[39] Yuane Zhao. On the mean square error of the Smarandache LCM function. Pure and Applied Mathematics 24 (2008), no. 1, 71 - 74. (In Chinese with English abstract).

[40] Lu Zheng and Li Gao. On the mean value of the Smarandache function and the divisor function. Journal of Yanan University (Natural Science Edition) 36 (2017), no. 2, 14 - 17. (In Chinese with English abstract).

[41] Min Zhu. The mean value on a composite function involving F. Smarandache LCM function. Journal of Southwest University for Nationalities (Natural Science Edition) 39 (2013), no. 4, 564 - 566. (In Chinese with English abstract).

[42] Weiyi Zhu. A conjecture involving the F. Smarandache LCM function. Acta Mathematica Sinica, Chinese Series 51 (2008), no. 5, 955 - 958. (In Chinese with English abstract).

Scientia Magna
Vol. 13 (2018), No. 1, 154-166

A survey on Smarandache notions in number theory IV: Smarandache double factorial function

Yuchan Qi

School of Mathematics, Northwest University
Xi'an 710127, China
E-mail: ycqimath@163.com

Abstract In this paper we give a survey on recent results on Smarandache double factorial function.
Keywords Smarandache double factorial function, sequence, mean value.
2010 Mathematics Subject Classification 11A07, 11B50, 11L20, 11N25.

§1. Definition and simple properties

For any positive integer n, the Smarandache double factorial function $Sdf(n)$ is defined as the smallest positive integer m such that $n \mid m!!$, where

$$m!! = \begin{cases} 2 \times 4 \times \cdots \times m, & \text{if } 2 \mid m, \\ 1 \times 3 \times \cdots \times m, & \text{if } 2 \nmid m. \end{cases} \tag{1.1}$$

By the definition of $Sdf(n)$, it is easy to show that $Sdf(1) = 1$ and $Sdf(n) > 1$ if $n > 1$. M. Le, Q. Yang, T. Wang, H. Li gave many properties for the value of $Sdf(n)$.

M. Le [7]. 1. *If $2 \nmid n$ and*

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

is the factorization of n, where $p_1, p_2, \ldots, p_k$ are distinct odd primes and $a_1, a_2, \ldots, a_k$ are positive integers, then

$$Sdf(n) = \max\left(Sdf(p^{a_1}), Sdf(p_2^{a_2}), \ldots, Sdf(p_k^{a_k})\right).$$

2. *If $2 \mid n$ and*

$$n = 2^a n_1,$$

where a, n_1 are positive integers with $2 \nmid n_1$, then

$$Sdf(n) \le \max\left(Sdf(2^a), 2Sdf(n_1)\right).$$

3. *Let p be a prime and let a be a positive integer. Then we have*

$$p \mid Sdf(p^a).$$

4. *Let p be the smallest prime divisor of n. Then we have*

$$Sdf(n) \geq p.$$

5. *The equality*

$$Sdf(n) = n$$

holds if and only if n satisfies one of the following conditions:

(i) $n = 1, 9$.

(ii) $n = p$, *where p is a prime.*

(iii) $n = 2p$, *where p is a prime.*

6. *Let p be a prime, and let $N(p)$ denotes the number of solutions x of the equation*

$$Sdf(x) = p, \ x \in \mathbb{N}.$$

For any positive integer t, let $p(t)$ denotes the $t-$th odd prime. if $p = p(t)$, then

$$N(p) = \prod_{i=1}^{t-1} (a(i) + 1),$$

where

$$a(i) = \sum_{m=1}^{\infty} \left(\left[\frac{p-2}{(p(i))^m} \right] - \left[\frac{(p-3)/2}{(p(i))^m} \right] \right), \ i = 1, 2, \ldots, t-1.$$

7. *Let a, b be two positive integers. Then we have*

$$Sdf(ab) \leq \begin{cases} Sdf(a) + Sdf(b), & \text{if} \ \ 2 \mid a \ \ \text{and} \ \ 2 \mid b, \\ Sdf(a) + 2Sdf(b), & \text{if} \ \ 2 \mid a \ \ \text{and} \ \ 2 \nmid b, \\ 2Sdf(a) + 2Sdf(b) - 1, & \text{if} \ \ 2 \nmid a \ \ \text{and} \ \ 2 \nmid b. \end{cases}$$

8. *For any positive integer m, n, k, all the solutions (m, n, k) of the equation*

$$Sdf(mn) = m^k \times Sdf(m)$$

are given in the following four classes:

(i) $m = 1, n$ *and k are positive integers.*

(ii) $n = 1, k = 1, m = 1, 9, p$ *or $2p$, where p is a prime.*

(iii) $m = 2, k = 1, n$ *is 2 or an odd integer with $n \geq 1$.*

(iv) $m = 3, k = 1, n = 3$.

9. *For any positive integer n, k with $n > 1, k > 1$. The equation*

$$(Sdf(n))^k = k \times Sdf(nk)$$

has only the solutions $(n, k) = (2, 4)$ and $(3, 3)$.

10. *The equation*

$$Sdf(n)! = Sdf(n!)$$

has only the solutions $n = 1, 2, 3$.

11. *For any positive integer n, we have*

$$Sdf(n!) \leq \begin{cases} n, & if \quad n = 1, 2, \\ 2n, & if \quad n > 2. \end{cases}$$

Q. Yang, T. Wang and H. Li [20]. 1. *For any positive integer n, we have*

$$Sdf(n!) \leq \begin{cases} n, & if \quad n = 1, 2, \\ 2(n-1), & if \quad n = 2^\alpha, \alpha > 1, \\ 2n, & otherwise \quad . \end{cases}$$

2. *The equality*

$$Sdf(n!) = n$$

holds if and only if $n = 1, 2, 3$.

3. *For any positive integer n, we have*

$$\frac{n}{Sdf(n!)} \leq 1 + \frac{1}{n}.$$

4. *For any $\varepsilon > 0$, there exists no positive integer n such that* $\dfrac{Sdf(n!)}{n} < \varepsilon$.

5. *For any positive integer n, we have*

$$Sdf\,(Sdf(n!)) = \begin{cases} Sdf(n), & if \quad n = 1, 2, \\ Sdf(2(n-1)), & if \quad n = 2^\alpha, \alpha > 1, \\ Sdf(2n), & otherwise \quad . \end{cases}$$

The properties for $Sdf(2n)$ have also been studied.

6. *If $2 \nmid n$, and*

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

where $p_1, p_2, \ldots, p_k$ are distinct odd primes and $a_1, a_2, \ldots, a_k$ are positive integers, then

$$Sdf(2n) = 2 \max\left(Sdf(p^{a_1}), Sdf(p_2^{a_2}), \ldots, Sdf(p_k^{a_k})\right).$$

7. *If $2 \mid n$ and*

$$n = 2^a n_1,$$

where a, n_1 are positive integers with $2 \nmid n_1$, then

$$Sdf(2n) \leq \max\left(2Sdf(2^a), 4Sdf(n_1)\right).$$

8. *Let p be the smallest prime divisor of n. Then we have*

$$Sdf(2n) \geq 2p.$$

9. *the equation*

$$Sdf(2n) = 2n$$

holds if and only if $n = 1$ or n be a prime.

Many problems and conjectures proposed by Russo have been stuied by M. Le, it is as follows.

M. Le [7]. 1. *For any positive integer n, we have*

$$\frac{n}{Sdf(n)} \leq \frac{n}{8} + 2.$$

2. *If $n = (2r)!!$, where r is a positive integer with $r \geq 20$, then*

$$Sdf(n) < n^{0.1}.$$

By the above inequality, the following conclusion has been proved.

3. *If $n = (2r)!!$, where r is a positive integer with $r \geq 20$, then the inequalities*

$$\frac{Sdf(n)}{n} > \frac{1}{n^{0.73}}, \quad \frac{1}{n \times Sdf(n)} < \frac{1}{n^{5/4}}, \quad \frac{1}{n} + \frac{1}{Sdf(n)} < \frac{1}{n^{1/4}}$$

are false.

4. *For any positive integer ε, there exist some n such that*

$$\frac{Sdf(n)}{n} < \varepsilon.$$

5. *The difference $|Sdf(n+1) - Sdf(n)|$ is unbounded.*

6. *The following infinite series satisfy*

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{Sdf(n)} = \infty.$$

7. *The following infinite product satisfy*

$$P = \prod_{n=1}^{\infty} \frac{1}{Sdf(n)} = 0.$$

§2. Mean values of the Smarandache double factorial function

C. Dumitrscu and V. Seleacu [1]. *For any real number $x > 1$, and any fixed positive integer k, then*

$$\sum_{n \leq x} (Sdf(n) - P(n))^2 = \frac{\zeta(3)}{24} \frac{x^3}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\log^{k+1} x}\right),$$

where $P(n)$ is the largest prime factor of n, $\zeta(n)$ denotes the Riemann Zeta-function. $c_i (i = 1, 2, \ldots, k)$ are computable constants.

J. Gao and H. Liu [4]. 1. *If $x > 2$, then for any positive integer k we have*

$$\sum_{n \leq x} \Lambda_1(n) Sdf(n) = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x}\right) + O\left(\frac{x^2}{\log^k x}\right),$$

where

$$\Lambda_1(n) = \begin{cases} \log p, & \textit{if } n \textit{ is a prime } p, \\ 0, & \textit{otherwise} \end{cases}.$$

and $a_m (m = 1, 2, \ldots, k-1)$ *are computable constants.*

2. *If* $x > 2$, *then for any positive integer* k *we have*

$$\sum_{n \le x} \Lambda(n) Sdf(n) = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right),$$

where $\Lambda(n)$ *is the Mangoldt function.*

Z. Xu [18]. *Let* $P(n)$ *denotes the largest prime factor of* n, $S(n)$ *denotes the smallest positive integer* m *such that* $n \mid m!$ *(* $S(n)$ *is called the Smarandache function), then for any real number* $x > 1$, *we have the asymptotic formula*

$$\sum_{n \le x} (S(n) - P(n))^2 = \frac{2\zeta(\frac{3}{2}) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x} \right),$$

where $\zeta(n)$ *denotes the Riemann Zeta-function.*

J. Wang [12]. 1. *For any real number* $x > 1$ *and any fixed positive integer* k, *we have the asymptotic formula*

$$\sum_{n \le x} (Sdf(n) - P(n))^2 = \frac{\zeta(3)}{24} \frac{x^3}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x} \right),$$

where all c_i *are computable constants.*

2. *For any real number* $x > 1$ *and any fixed positive integer* k, *we have the asymptotic formula*

$$\sum_{n \le x} (Sdf(n) - S(n))^2 = \frac{\zeta(3)}{24} \frac{x^3}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x} \right).$$

H. Shen [10]. 1. *For any real number* $x > 2$ *and any fixed positive integer* k, *we have the asymptotic formula*

$$\sum_{n \le x} Sdf(n) = \frac{x \ln x}{\ln \ln x} + O\left(\frac{x \ln x}{(\ln \ln x)^2} \right).$$

J. Ge [5]. *For any real number* $x > 1$, *we have the asymptotic formula*

$$\sum_{n \le x} \left(Sdf(n) - \frac{3 + (-1)^n}{2} P(n) \right)^2 = \frac{8}{3} \cdot \zeta(\frac{3}{2}) \cdot \frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x} \right),$$

where all c_i *are computable constants.*

X. Wang [13]. *For any positive integer $k \geq 4$, there exist infinite number of positive integer arrays $(m_1, m_2, \ldots, m_k)$ satisfy the following equation*

$$Sdf(\prod_{i=1}^{k} m_i) = \sum_{i=1}^{k} Sdf(m_i).$$

B. Zhang [22]. 1. *For any real number $x > 1$ and any fixed positive integer $k \geq 2, r$, we have the asymptotic formula*

$$\sum_{n\leq x} (Sdf(n) - P(n))^k = \frac{\zeta(k+1)}{8(k+1)} \cdot \frac{x^{k+1}}{\ln x} + \sum_{i=2}^{r} \frac{c_i x^{k+1}}{\ln^i x} + O\left(\frac{x^{k+1}}{\ln^{k+1} x}\right).$$

2. *For any real number $x > 1$ and any fixed positive integer $k \geq 2, r$, we have the asymptotic formula*

$$\sum_{n\leq x} (Sdf(n) - S(n))^k = \frac{\zeta(k+1)}{8(k+1)} \cdot \frac{x^{k+1}}{\ln x} + \sum_{i=2}^{r} \frac{c_i x^{k+1}}{\ln^i x} + O\left(\frac{x^{k+1}}{\ln^{k+1} x}\right).$$

X. Fan, X. Zhu and X. Yan [2]. 1. *For any real number $x \geq 2$ we have the asymptotic formula*

$$\sum_{n\leq x} \ln Sdf(n) = x \ln x + O(x).$$

2. *For any positive integer $n > 1$, we have*

$$\frac{\sum_{k=2}^{n} \frac{\ln Sdf(k)}{\ln k}}{n} = 1 + O(\frac{1}{\ln n}).$$

3. *For any positive integer $n > 1$, we have*

$$\lim_{n\to\infty} \frac{\sum_{k=2}^{n} \frac{\ln Sdf(k)}{\ln k}}{n} = 1.$$

4. *For any positive integer $n > 1$, we have*

$$\frac{Sdf(n)}{\theta(n)} = O(\frac{1}{\ln n}) = 1,$$

where $\theta(n) = \sum_{k\leq n} \ln Sdf(k)$.

5. *For any positive integer $n > 1$, we have*

$$\lim_{n\to\infty} \frac{Sdf(n)}{\theta(n)} = 0.$$

M. Zhu [24]. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} Sdf(n) = \frac{7\pi^2}{24} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

X. Fan and X. Yan [3]. *For any real number $x \geq 2$ and any fixed positive integer k, we have the asymptotic formula*

$$\sum_{n\leq x} Sdf(n) = \frac{5\pi^2}{48}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{e_i x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where $e_i (i = 1, 2, \ldots, k)$ are computable constants.

The famous pseudo-Smarandache function $Z(n)$ is defined as the smallest positive integer m such that $n \leq m(m+1)/2$, that is $Z(n) = \min\{m : m \in \mathbb{N}, n \leq m(m+1)/2\}$.

W. Lu, L. Gao and H. Hao [8]. *For any real number $x > 1$ and any fixed positive integer $k \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} Sdf(Z(n)) = \frac{\pi^2}{18}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln\sqrt{2x}} + \sum_{i=2}^{k}\frac{a_i(2x)^{\frac{3}{2}}}{\ln^i\sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where $a_i (i = 1, 2, \ldots, k)$ are computable constants. Specifically, when $k = 1$, then for any real number $x > 1$ we have

$$\sum_{n\leq x} Sdf(Z(n)) = \frac{\pi^2}{18}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln\sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right).$$

L. Wang [11]. *The function $U(n)$ is defined as the smallest positive integer k such that $U(n) = \min\{k : k \in \mathbb{N}, n \leq k(2k-1)\}$. For any real number $x > 1$ and any fixed positive integer $j \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} Sdf(U(n)) = \frac{\pi^2}{144}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln\sqrt{2x}} + \sum_{i=2}^{j}\frac{b_i(2x)^{\frac{3}{2}}}{\ln^i\sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{j+1} x}\right),$$

where $b_i (i = 1, 2, \ldots, k)$ are computable constants.

W. Huang [6]. 1. *The famous pseudo-Smarandache function $U_*(n)$ and $U(n)$ are defined as*

$$U_*(n) = \min\left\{m : m \in \mathbb{N}^+, n \leq m + \frac{1}{2}m(m-1)\cdot(r-2), r \in \mathbb{N}^+, r \geq 3\right\},$$
$$U(n) = \max\left\{m : m \in \mathbb{N}^+, n \geq m + \frac{1}{2}m(m-1)\cdot(r-2), r \in \mathbb{N}^+, r \geq 3\right\}.$$

For any real number $x > 1$ and any fixed positive integer $k \geq 2, r \geq 3$, we have the following asymptotic formula

$$\sum_{n\leq x} Sdf(U_*(n)) = \frac{5\pi^2}{72\sqrt{r-2}}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln\sqrt{2x}} + \sum_{i=2}^{k}\frac{d_i(2x)^{\frac{3}{2}}}{\ln^i\sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$
$$\sum_{n\leq x} Sdf(U(n)) = \frac{5\pi^2}{72\sqrt{r-2}}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln\sqrt{2x}} + \sum_{i=2}^{k}\frac{e_i(2x)^{\frac{3}{2}}}{\ln^i\sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where $d_i, e_i (i = 1, 2, \ldots, k)$ are computable constants.

2. *For $r = 6$ we have*

$$\sum_{n\le x} Sdf(U_*(n)) = \frac{5\pi^2}{144}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln\sqrt{2x}} + \sum_{i=2}^{k}\frac{f_i(2x)^{\frac{3}{2}}}{\ln^i\sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1}x}\right),$$

$$\sum_{n\le x} Sdf(U(n)) = \frac{5\pi^2}{144}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln\sqrt{2x}} + \sum_{i=2}^{k}\frac{h_i(2x)^{\frac{3}{2}}}{\ln^i\sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1}x}\right),$$

where $f_i, h_i (i = 1, 2, \ldots, k)$ are computable constants.

3. *The pseudo-Smarandache function $Z_*(n)$ and $Z(n)$ are defined as*

$$Z_*(n) = \max\left\{m : m\in\mathbb{N}^+, n \le m + \frac{m(m-1)}{2}\right\}.$$

$$Z(n) = \min\left\{m : m\in\mathbb{N}^+, n \ge m + \frac{m(m-1)}{2}\right\}.$$

For any real number $x > 1$ we have the following asymptotic formula

$$\sum_{n\le x} Sdf(Z_*(n)) = \frac{5\pi^2}{18}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln\sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

$$\sum_{n\le x} Sdf(Z(n)) = \frac{5\pi^2}{18}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln\sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right).$$

W. Lu and L. Gao [9]. 1. *For any real number $x \ge 2$ and any fixed positive integer $k \ge 2, r$, when $\beta > 1$ we have the asymptotic formula*

$$\sum_{n\le x}(Sdf(n) - P(n))^{\beta} = \frac{\zeta(\beta+1)}{2^{\beta+1}(\beta+1)}\cdot\frac{x^{\beta+1}}{\ln x} + \sum_{i=2}^{k}\frac{a_i x^{\beta+1}}{\ln^i x} + O\left(\frac{x^{\beta+1}}{\ln^{k+1}x}\right).$$

where all a_i are computable constants.

2. *Let k be any fixed positive integer , $k \ge 2$, then for any real number or complex number α and any real number $x \ge 2$, when $\beta > 1$ we have*

$$\sum_{n\le x}\delta_\alpha(n)\,(Sdf(n) - P(n))^{\beta} = \frac{(1+2^\alpha)\zeta(\beta+1)\zeta(\beta+1-\alpha)}{2^{\beta+1}(\beta+1)}\cdot\frac{x^{\beta+1}}{\ln x} + \sum_{i=2}^{k}\frac{b_i x^{\beta+1}}{\ln^i x} + O\left(\frac{x^{\beta+1}}{\ln^{k+1}x}\right).$$

where $\delta_\alpha(n)$ is the divisor function, and all b_i are computable constants.

§3. Mean values of the Smarandache triple factorial function

For any positive integer n, the Smarandache triple factorial function $d3_f(n)$ is defined to be the smallest integer such that $d3_f(n)!!!$ is a multiple of n.

Q. You [21]. 1. *If $x \ge 2$, then for any positive integer k we have*

$$\sum_{n\le x}\Lambda_1(n)d3_f(n) = x^2\left(\frac{1}{2} + \sum_{m=1}^{k-1}\frac{a_m}{\log^m x}\right) + O\left(\frac{x^2}{\log^k x}\right),$$

where

$$\Lambda_1(n) = \begin{cases} \log p, & \text{if } n \text{ is a prime } p, \\ 0, & \text{otherwise} \quad . \end{cases}$$

and $a_m(m = 1, 2, \ldots, k-1)$ *are computable constants.*

2. *If* $x \geq 2$*, then for any positive integer* k *we have*

$$\sum_{n \leq x} \Lambda(n) d3_f(n) = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right),$$

where $\Lambda(n)$ *is the Mangoldt function.*

§4. The dual function of the Smarandache double factorial function, and other generalizations

The analogue of Smarandache double factorial function is defined as

$$Sdf_1(2x) = \min\{2m \in \mathbb{N} : 2x \leq (2m)!!\}, x \in (1, \infty),$$

$$Sdf_1(2x+1) = \min\{2m+1 \in \mathbb{N} : (2x+1) \leq (2m+1)!!\}, x \in (1, \infty),$$

which is defined on a subset of real numbers. Clearly $Sdf_1(n) = m$ if $x \in ((m-1)!!, m!!]$ for $m \geq 2$, therefore this function is defined for $x \geq 1$.

M. Zhu [23]. *For any real number* $x \geq 2$*, we have*

$$\sum_{n \leq x} Sdf_1(n) = \frac{2x \ln x}{\ln \ln x} + O\left(\frac{x (\ln x)(\ln \ln \ln x)}{(\ln \ln x)^2} \right).$$

N. Yuan [19]. 1. *The Smarandache* $t-$*factorial function* $S_k(t, n)$ *is defined as the smallest positive integer* m *such that* $m!_t$ *is divisible by* n^k*, that is*

$$S_k(t, n) = \min\{m \in \mathbb{N} : n^k \mid m!_t\},$$

where $m!_t$ *denotes*

$$m!_t = m \times (m-t) \times \cdots \times (t+i) \times i, m \equiv i (\text{mod } t), i = 0, 1, \ldots, t-1.$$

For any real number $x \geq 2$ *and any positive integer* t*, we have the following asymptotic formula*

$$\sum_{n \leq x} S_k(t, n) = \begin{cases} \frac{[t(5k-3)+3]\pi^2}{24} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x} \right), & \text{if } 2 \mid t, \\ \frac{[t(k-1)+1]\pi^2}{12} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x} \right), & \text{if } 2 \nmid t. \end{cases}$$

2. *For any real number* $x \geq 2$ *and any positive integer* k*, we have the following asymptotic formula*

$$\sum_{n \leq x} S_k(1, n) = \sum_{n \leq x} S_k(n) = \frac{k\pi^2}{12} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x} \right).$$

3. *For any real number $x \geq 2$ we have*

$$\sum_{n\leq x} S_1(2,n) = \sum_{n\leq x} Sdf(n) = \frac{7\pi^2}{24}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

The dual function of the Smarandache double factorial function $S^{**}(n)$ is defined as

$$S^{**}(n) = \begin{cases} \max\{2m : (2m)!! \mid n, m \in \mathbb{N}^+\}, & n \text{ be an even number} , \\ \max\{2m-1 : (2m-1)!! \mid n, m \in \mathbb{N}^+\}, & n \text{ be an odd number} , \end{cases}$$

where

$$\begin{aligned} (2m)!! &= 2\times 4\times \cdots \times (2m), \\ (2m-1)!! &= 1\times 3\times \cdots \times (2m-1). \end{aligned}$$

Y. Wang [14]. 1. *For any complex number s, when Re $s > 1$, we have $\sum_{n=1}^{\infty} \frac{\Lambda(n)S^{**}(n)}{n^s}$ is a convergent series, and*

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)S^{**}(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} + \frac{\ln 2}{2^s}\left(1+\frac{2}{2^{2s}}\right)\left(1-\frac{1}{2^s}\right)^{-1} + \frac{2\ln 3}{3^s}\left(1-\frac{1}{3^s}\right)^{-1},$$

where $\zeta(s)$ denotes the Riemann Zeta-function, $\zeta'(s)$ is the differential coefficient of $\zeta(s)$.

2. *For any complex number s, when Re $s > 1$, we have*

$$\lim_{s\to 1}\left((s-1)\sum_{n=1}^{\infty}\frac{\Lambda(n)S^{**}(n)}{n^s}\right) = 1,$$

let $s = 2, 4$, we have the following equations

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{\Lambda(n)S^{**}(n)}{n^2} &= \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{\ln n}{n^2} + \frac{3\ln 2}{8} + \frac{\ln 3}{4}, \\ \sum_{n=1}^{\infty}\frac{\Lambda(n)S^{**}(n)}{n^4} &= \frac{90}{\pi^4}\sum_{n=1}^{\infty}\frac{\ln n}{n^4} + \frac{43\ln 2}{640} + \frac{\ln 3}{40}. \end{aligned}$$

Y. Wang [15]. 1. *For any real number $x > 1$, we have*

$$\sum_{n\leq x}(S^{**}(n))^3 = \left(14e^{1/2} - \frac{3}{2} + 7\sum_{m=1}^{\infty}\frac{1}{(2m+1)!!}\right)x + O\left(\left(\frac{\ln x}{\ln\ln x}\right)^4\right).$$

2. *For any real number $x > 1$ and $k \geq 0$, we have*

$$\sum_{n\leq x} n^k(S^{**}(n))^3 = \frac{1}{k+1}\left(14e^{1/2} - \frac{3}{2} + 7\sum_{m=1}^{\infty}\frac{1}{(2m+1)!!}\right)x^{k+1} + O\left(x^k\left(\frac{\ln x}{\ln\ln x}\right)^4\right).$$

3. *For any real number $x > 1$ and $k > 1$, we have*

$$\sum_{n\leq x}\frac{(S^{**}(n))^3}{n^k} = \frac{1}{1-k}\left(14e^{1/2} - \frac{3}{2} + 7\sum_{m=1}^{\infty}\frac{1}{(2m+1)!!}\right)x^{1-k} + O\left(x^{-k}\left(\frac{\ln x}{\ln\ln x}\right)^4\right).$$

Y. Wang [16]. *For any real number $x > 1$, we have*

$$\sum_{n\le x}(S^{**}(n))^2 = \frac{13}{2}x + O\left(\left(\frac{\ln x}{\ln\ln x}\right)^3\right).$$

Y. Wang and T. Wang [17]. 1. *For any real number $x > 1$ and any fixed number $l \ge 0$, we have the asymptotic formula*

$$\sum_{n\le x} n^l (S^{**}(n))^4 = \frac{1}{2(1+l)}\left(105 + 64\mathrm{e}^{1/2} + 32\sum_{m=1}^{\infty}\frac{1}{(2m+1)!!}\right)x^{1+l} + O\left(x^l\left(\frac{\ln x}{\ln\ln x}\right)^5\right).$$

2. *For any real number $x > 1$ and any fixed number $l > 0 (l \neq 1)$, we have*

$$\sum_{n\le x}\frac{(S^{**}(n))^4}{n^l} = \frac{1}{2(1-l)}\left(105 + 64\mathrm{e}^{1/2} + 32\sum_{m=1}^{\infty}\frac{1}{(2m+1)!!}\right)x^{1-l} + O\left(x^{-l}\left(\frac{\ln x}{\ln\ln x}\right)^5\right).$$

3. *For any real number $x > 1$ we have*

$$\sum_{n\le x}(S^{**}(n))^4 = \frac{1}{2}\left(105 + 64\mathrm{e}^{1/2} + 32\sum_{m=1}^{\infty}\frac{1}{(2m+1)!!}\right)x + O\left(\left(\frac{\ln x}{\ln\ln x}\right)^5\right).$$

4. *For any real number $x > 1$, we have*

$$\sum_{n\le x}\frac{(S^{**}(n))^4}{n^{\frac{1}{2}}} = \left(105 + 64\mathrm{e}^{1/2} + 32\sum_{m=1}^{\infty}\frac{1}{(2m+1)!!}\right)x^{\frac{1}{2}} + O\left(x^{-\frac{1}{2}}\left(\frac{\ln x}{\ln\ln x}\right)^5\right).$$

References

[1] C Dumitrscu and V Seleacu. Some notions and questions in Number Theory. Erhus University Press, Glendale, 1994.

[2] Xuhui Fan. On two questions of Smarandache double factorial function. Jounal of Kunming University of Science and Technology (Natural Science Edition) 36 (2011), no. 1, 71 - 74. (In Chinese with English abstract).

[3] Xuhui Fan and Xinrong Yan. On the mean value of the Smarandache double factorial function. Journal of Air Force Engineering University (Natural Science Edition) 14 (2013), no. 4, 88 - 90. (In Chinese with English abstract).

[4] Jing Gao and Huaning Liu. On the mean value of Smarandache double factorial function. Smarandache Notions Journal 14 (2004), 193 - 195.

[5] Jian Ge. On the value distribution problems of the Smarandache double-factorial function. Pure and Applied Mathematics 24 (2008), no. 3, 473 - 476. (In Chinese with English abstract).

[6] Wei Huang. On the hybrid mean value of the Smarandache double factorial function and the approximate pseudo-Smarandache function. Journal of Southwest University for Nationalities (Natural Science Edition) 43 (2017), no. 2, 167 - 171. (In Chinese with English abstract).

[7] Maohua Le. On the Smarandache double factorial function. Smarandache Notions Journal 13 (2002), 209 - 228.

[8] Weiyang Lu, Li Gao and Hongfei Hao. On the hybrid mean value of the Smarandache double factorial function and the pseudo-Smarandache function. Jiangxi Science 32 (2014), no. 2, 189 - 191. (In Chinese with English abstract).

[9] Weiyang Lu, Li Gao. On the hybrid mean value of the Smarandache double-factorial function. Journal of Jishou University (Natural Science Edition) 38 (2017), no. 3, 4 - 7. (In Chinese with English abstract).

[10] Hong Shen. A new arithmetical function and its value distribution. Pure and Applied Mathematics 23 (2007), no. 2, 235 - 238. (In Chinese with English abstract).

[11] Long Wang. On the hybrid mean value of the Smarandache double factorial function. Journal of Yan'an Vocational & Technical Institute 28 (2014), no. 4, 95 - 96. (In Chinese).

[12] Jianping Wang. On the value distribution properties of the Smarandache double-factorial function. Scientia Magna 3 (2007), no. 4, 111 - 114.

[13] Xiaoying Wang. On certain equations involving the Smarandache double factorial function. Scientia Magna 4 (2008), no. 1, 56 - 59.

[14] Yang Wang. An identity of the Smarandache double factorial function. Journal of Nanyang Normal University 11 (2012), no. 12, 11 - 13. (In Chinese with English abstract).

[15] Yang Wang. On the cubic mean value and weighted mean value of the Smarandache double factorial dual function. Mathematics in practice and theory 43 (2013), no. 16, 285 - 289. (In Chinese with English abstract).

[16] Yang Wang. On the quadratic mean value of the Smarandache double factorial dual function. Journal of Nanyang Normal University 12 (2013), no. 9, 1 - 3. (In Chinese with English abstract).

[17] Yang Wang and Ting Wang. On the 4-th weighted mean value of the Smarandache double factorial dual function. Journal of Inner Mongolia Normal University (Natural Science Edition) 44 (2015), no. 2, 149 - 151. (In Chinese with English abstract).

[18] Zhengfeng Xu. The value distribution property of the Smarandache function. Acta Mathematica Sinica (Chinese Series) 49 (2006), no. 5, 1009 - 1012.

[19] Na Yuan. A new arithmetical function and its asymptotic formula. Scientia Magna 3 (2005), no. 1, 33 - 38.

[20] Qianli Yang, Tao Wang and Hailong Li. A question about Smarandache double factorial function. Journal of Xi'an University of Engineering Science and Technology 20 (2006), no. 4, 494 - 496. (In Chinese with English abstract).

[21] Qiying You. On Smarandache triple factorial function. Scientia Magna 1 (2010), no. 1, 89 - 92.

[22] Bo Zhang. Double factorial function and its hybrid mean value asymptotic formula. Science Technology and Engineering 10 (2010), no. 18, 4463 - 4465. (In Chinese with English abstract).

[23] Minhui Zhu. Mean value of the additive analogue of Smarandache function. Scientia Magna 1 (2005), no. 1, 179 - 182.

[24] Minhui Zhu. On the mean value of the Smarandache double factorial function. Journal of Air Force Engineering University 1 (2013), no. 1, 197 - 201.

Scientia Magna
Vol. 13 (2018), No. 1, 167-181

A survey on Smarandache notions in number theory V: Smarandache simple function, summands function, power function and exponent function

Yafei Song

School of Mathematics, Northwest University
Xi'an 710127, China
E-mail: songyf02@163.com

Abstract In this paper we give a survey on recent results on Smarandache simple function, summands function, power function and exponent function.

Keywords Smarandache simple function, summands function, power function, exponent function.
2010 Mathematics Subject Classification 11A07, 11B50, 11L20, 11N25.

§1. Smarandache simple function

For any prime p and any positive integer k, let $S_p(k)$ denote the smallest positive integer such that $p \mid S_p(k)$. Then $S_p(k)$ is called the Smarandache simple function of p and k.

M. Le [13]. *For any p and k, we have $p \mid S_p(k)$ and $k(p-1) < S_p(k) \le kp$.*

M. Zhu [30]. *On the subset of real numbers, we define additive analogues of Smarandache simple function $S_p(x)$ as follows:*

$$S_p(x) = \min\{m \in N : p^x \le m!\}, x \in (1, \infty),$$

$$S_p^*(x) = \max\{m \in N : m! \le p^x\}, x \in [1, \infty).$$

1. *For any real number $x \ge 2$, we have*

$$S_p(x) = \frac{x \ln p}{\ln x} + O\left(\frac{x \ln \ln x}{\ln^2 x}\right).$$

Obviously, we have

$$S_p(x) = \begin{cases} S_p^*(x) + 1, & \text{if } x \in (m!, (m+1)!) \quad (m \ge 1), \\ S_p^*(x), & \text{if } x = (m+1)! \quad (m \ge 1). \end{cases}$$

2. *For any real number $x \ge 2$, we have*

$$S_p^*(x) = \frac{x \ln p}{\ln x} + O\left(\frac{x \ln \ln x}{\ln^2 x}\right).$$

M. Zhu [31]. *Define additive analogues of Smarandache simple function*

$$\overline{S}_p(n) = \min\{m \in N^+ : p^n \leq m!!\}, \quad n \in (1, \infty),$$

$$\overline{S}^*_p(n) = \max\{m \in N^+ : m!! \leq p^n\}, \quad n \in (1, \infty).$$

1. *Let p be a fixed prime, for any real number $x \geq 1$, we have*

$$\sum_{n \leq x} \sigma_\alpha(\overline{S}_p(n)) = \begin{cases} \frac{\pi^2}{3} \frac{x^2 \ln p}{\ln^2 x} \ln\left(\frac{2x \ln p}{\ln x}\right) + O\left(\frac{x^2}{\ln^2 x}\right), & \text{if } \alpha = 1, \\ \frac{\zeta(\alpha+1)}{\alpha+1} \frac{2^{\alpha+1} x^{\alpha+1} \ln^\alpha p}{\ln^{\alpha+1} x} \ln\left(\frac{2x \ln p}{\ln x}\right) + O\left(\frac{x^{\alpha+1}}{\ln^{\alpha+1} x}\right), & \text{if } \alpha \neq 1. \end{cases}$$

2. *Let p be a fixed prime, for any real number $x \geq 1$, we have*

$$\sum_{n \leq x} \sigma_\alpha(\overline{S}^*_p(n)) = \begin{cases} \frac{\pi^2}{3} \frac{x^2 \ln p}{\ln^2 x} \ln\left(\frac{2x \ln p}{\ln x}\right) + O\left(\frac{x^2}{\ln^2 x}\right), & \text{if } \alpha = 1, \\ \frac{\zeta(\alpha+1)}{\alpha+1} \frac{2^{\alpha+1} x^{\alpha+1} \ln^\alpha p}{\ln^{\alpha+1} x} \ln\left(\frac{2x \ln p}{\ln x}\right) + O\left(\frac{x^{\alpha+1}}{\ln^{\alpha+1} x}\right), & \text{if } \alpha \neq 1. \end{cases}$$

H. Liu [14]. *On a subset of real numbers, we define additive analogues of Smarandache simple function $p(x)$ as follows:*

$$p(x) = \min\{m \in N_+ : p^x \leq m!\}, \quad x \in (1, \infty),$$

$$p^*(x) = \max\{m \in N_+ : m! \leq p^x\}, \quad x \in [1, \infty).$$

1. *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} \sigma_\alpha(p(n)) = \begin{cases} \frac{\zeta(\alpha+1)}{\alpha+1} \frac{x^{\alpha+1} \ln^\alpha p}{\ln^{\alpha+1} x} \left[\ln\left[\frac{x \ln p}{\ln x}\right] - \frac{1}{\alpha+1}\right] + O\left[\frac{x^\alpha}{\ln^{\alpha-1} x}\right], & \text{if } \alpha > 1, \\ \frac{\pi^2}{12} \frac{x^2 \ln p}{\ln^2 x} \left[\ln\left[\frac{x \ln p}{\ln x}\right] - \frac{1}{2}\right] + O(x \ln x), & \text{if } \alpha = 1, \\ \frac{\zeta(\alpha+1)}{\alpha+1} \frac{x^{\alpha-1} \ln^\alpha p}{\ln^{\alpha+1} x} \left[\ln\left[\frac{x \ln p}{\ln x}\right] - \frac{1}{\alpha+1}\right] + O\left[\frac{x}{\ln x}\right], & \text{if } 0 < \alpha < 1. \end{cases}$$

where $\zeta(s)$ is the Riemann zeta-function, $\sigma_\alpha(n)$ is the divisor function.

2. *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} \sigma_\alpha(p^*(n)) = \begin{cases} \frac{\zeta(\alpha+1)}{\alpha+1} \frac{x^{\alpha+1} \ln^\alpha p}{\ln^{\alpha-1} x} \left[\ln\left[\frac{x \ln p}{\ln x}\right] - \frac{1}{\alpha+1}\right] + O\left[\frac{x^\alpha}{\ln^{\alpha-1} x}\right], & \text{if } \alpha > 1, \\ \frac{\pi^2}{12} \frac{x^2 \ln p}{\ln^2 x} \left[\ln\left[\frac{x \ln p}{\ln x}\right] - \frac{1}{2}\right] + O(x \ln x), & \text{if } \alpha = 1, \\ \frac{\zeta(\alpha+1)}{\alpha+1} \frac{x^{\alpha+1} \ln^\alpha p}{\ln^{\alpha-1} x} \left[\ln\left[\frac{x \ln p}{\ln x}\right] - \frac{1}{\alpha+1}\right] + O\left[\frac{x}{\ln x}\right], & \text{if } 0 < \alpha < 1. \end{cases}$$

H. Liu [15]. 1. *Let p be a fixed prime, then for any real number $x \geq 1$, we have*

$$\sum_{n \leq x} d(p(n)) = x(\ln x - 2 \ln \ln x) + O(x \ln p).$$

2. *Let p be a fixed prime, then for any real number $x \geq 1$, we have*

$$\sum_{n \leq x} d(p^*(n)) = x(\ln x - 2 \ln \ln x) + O(x \ln p).$$

H. Liu [16]. 1. *Let p be a fixed prime, for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} d(p(n)) = x(\ln x - \ln\ln x) + o(x).$$

2. *Let p be a fixed prime, then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} d(p^*(n)) = x(\ln x - \ln\ln x) + o(x).$$

H. Liu and M. Zhu [17]. 1. *Let p be a fixed prime, then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} d(p(n)) = x(\ln x - 2\ln\ln x) + O(x).$$

2. *Let p be a fixed prime, then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} d(p^*(n)) = x(\ln x - 2\ln\ln x) + O(x).$$

§2. Smarandache summands function

For any positive integer n and fixed integer $k > 1$, define Smarandache summands function $S(n,k)$, $AS(n,k)$ as follows:

$$S(n,k) = \sum_{\substack{|n-ki|\leq n \\ i=0,1,2\cdots}} (n - ik),$$

$$AS(n,k) = \sum_{\substack{|n-ki|\leq n \\ i=0,1,2\cdots}} |n - ik|.$$

J. Wang [25]. 1. *For any complex number s with $Re(s) > 2$, then*

$$\sum_{n=1}^{\infty} \frac{S(n,4)}{n^s} = \frac{1}{2}\left[1 - \frac{1}{2^{s-1}}\right]\zeta(s-1) + \frac{1}{2}\left[1 - \frac{1}{2^{s-1}}\right]\zeta(s),$$

where $\zeta(s)$ is Riemann zeta-function.

2. *For any complex number s with $Re(s) > 3$, then*

$$\sum_{n=1}^{\infty} \frac{S(n^2,8)}{n^s} = \frac{1}{4}\left[1 - \frac{1}{2^{s-2}}\right]\zeta(s-2) + \frac{3}{4}\left[1 - \frac{1}{2^s}\right]\zeta(s).$$

Specifically, if $s = 4$, then

$$\sum_{n=1}^{\infty} \frac{S(n^2,8)}{n^4} = \frac{\pi^2}{32} + \frac{5\pi^4}{604}.$$

3. *For any complex number s with $Re(s) > 3$, then*

$$\sum_{n=1}^{\infty} \frac{S(n^2,16)}{n^s} = \left[\frac{1}{8} + \frac{1}{2^{s-1}}\right]\left[1 - \frac{1}{2^{s-2}}\right]\zeta(s-2) + \left[\frac{7}{8} + \frac{1}{2^{s-1}}\right]\left[1 - \frac{1}{2^s}\right]\zeta(s).$$

Specifically, if $s = 4$, *then*

$$\sum_{n=1}^{\infty} \frac{S(n^2, 14)}{n^4} = \frac{\pi^2}{32} + \frac{\pi^4}{96}.$$

4. *For any complex number* s *with* $Re(s) > 3$, *then*

$$\sum_{n=1}^{\infty} \frac{S(n^2, 6)}{n^s} = \frac{1}{3}\left[1 - \frac{1}{3^{s-2}}\right]\zeta(s-2) + \frac{2}{3}\left[1 - \frac{1}{3^s}\right]\zeta(s).$$

Specifically, if $s = 4$, *then*

$$\sum_{n=1}^{\infty} \frac{S(n^2, 6)}{n^4} = \frac{4\pi^2}{81} + \frac{16\pi^4}{2187}.$$

5. *Let* $p > 2$ *be a prime, for any complex number* s *with* $Re(s) > p$, *then*

$$\sum_{n=1}^{\infty} \frac{S(n^{p-1}, p)}{n^s} = \frac{2}{p}\left[1 - \frac{1}{3^{s-p+1}}\right]\zeta(s-p+1) + \left[1 - \frac{2}{p}\right]\left[1 - \frac{1}{p^s}\right]\zeta(s).$$

Y.Zhao [32]. 1. *Let* $k > 1$ *be a fixed integral number, then for any integral number* $x > 1$, *we have*

$$\sum_{n \le x} S(n, k) = \frac{1}{4}\left[1 - \frac{3 + (-1)^k}{2k}\right] x^2 + R(x, k),$$

where $|R(x, k)| \le \frac{7}{8}k^2 + \frac{5k}{8}x$.

2. *Let* $k > 1$ *be a fixed integral number, then for any integral number* $x > 1$, *we have*

$$\sum_{n \le x} AS(n, k) = \frac{1}{3k}x^3 + \frac{1}{4}\left[1 + \frac{7 + (-1)^k}{2k}\right] x^2 + R_1(x, k),$$

where $|R_1(x, k)| \le \frac{7}{8}k^2 + \frac{7}{8}kx + \frac{x}{6k} + \frac{x}{2}$.

J. Chen [1]. 1. *For any complex number* s, *if* $Re(s) > 3$, *then*

$$\sum_{n=1}^{\infty} \frac{AS(n, 2)}{n^s} = 2\zeta(s-2) + \frac{1}{2^{s-1}}\zeta(s-1),$$

where $\zeta(s)$ *is Riemann zeta-function.*

2. *For any complex number* s, *if* $Re(s) > 5$, *then*

$$\sum_{n=1}^{\infty} \frac{AS(n^2, 8)}{n^s} = \frac{1}{8}\zeta(s-4) + \left[\frac{3}{4} + \frac{1}{2^s}\right]\zeta(s-2) + \left[\frac{1}{8} + \frac{1}{2^{s-1}}\right]\left[1 - \frac{1}{2^s}\right]\zeta(s).$$

Specifically, if $s = 6$, *then*

$$\sum_{n=1}^{\infty} \frac{AS(n^2, 8)}{n^4} = \frac{\pi^2}{48} + \frac{49\pi^4}{5760} + \frac{7\pi^6}{43008}.$$

3. *For any complex number* s, *if* $Re(s) > 5$, *then*

$$\sum_{n=1}^{\infty} \frac{AS(n^2, 4)}{n^s} = \frac{1}{4}\zeta(s-4) + \left[\frac{1}{2} + \frac{1}{2^{s-1}}\right]\zeta(s-2) + \frac{1}{4}\left[1 - \frac{1}{2^s}\right]\zeta(s).$$

Specifically, if $s = 6$, *then*

$$\sum_{n=1}^{\infty} \frac{AS(n^2,4)}{n^4} = \frac{\pi^2}{24} + \frac{17\pi^4}{2880} + \frac{\pi^6}{3840}.$$

4. *For any complex number* s, *if* $Re(s) > 5$, *then*

$$\sum_{n=1}^{\infty} \frac{AS(2n^2,6)}{n^s} = \frac{2}{3}\zeta(s-4) + \left[\frac{2}{3} + \frac{4}{3^{s-1}}\right]\zeta(s-2) + \frac{2}{3}\left[1 - \frac{1}{3^s}\right]\zeta(s).$$

Specifically, if $s = 6$, *then*

$$\sum_{n=1}^{\infty} \frac{AS(2n^2,6)}{n^4} = \frac{\pi^2}{9} + \frac{83\pi^4}{10935} + \frac{1456\pi^6}{2066751}.$$

5. *Let* $p > 2$ *be a prime, for any complex number* s *with* $Re(s) > p$, *then*

$$\sum_{n=1}^{\infty} \frac{AS(n^{p-1},p)}{n^s} = \frac{1}{p}\left[1 - \frac{1}{p^{s-2p+2}}\right]\zeta(s-2p+2) + \left[1 - \frac{2}{p}\right]\left[1 - \frac{1}{p^{s-p-1}}\right]\zeta(s-p+1) + \frac{1}{p}\left[1 - \frac{1}{p^s}\right]\zeta(s).$$

W. Huang [9]. *Let* $n, k \geq 1$ *are two integers,* $m \geq 0$ *is a fixed integer. Then we difine Smarandache summands function* $AS(n, m, k)$ *as follows:*

$$AS(n,m,k) = \sum_{i=0}^{\left[\frac{m+n}{k}\right]} |n - ki|.$$

Let $k > 1$ *and* $m \geq 0$ *are fixed integral numbers, for any arbitrary integral number* $x > 1$, *then we have*

$$\sum_{n \leq x} AS(n,m,k) = \frac{x^3}{6k} + \frac{x^3}{4} + R(x,k),$$

where $|R(x,k)| \leq \frac{5k^2}{12} + \frac{km}{4} + (\frac{m^2}{2k} + m + \frac{3k}{4} - \frac{m}{2k} - \frac{1}{2} - \frac{1}{12k})x$.

§3. Smarandache power function

Define Smarandache power function $SP(n)$ as follows:

$$SP(n) = \min\{m : n \mid m^m, m \in N\}.$$

Z. Xu [27]. 1. *For any real number* $x \geq 1$, *we have*

$$\sum_{n \leq x} SP(n) = \frac{1}{2}x^2 \prod_p \left(1 - \frac{1}{p(p+1)}\right) + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where $\prod_p$ *denotes the product of all prime,* ϵ *is an arbitrary positive number.*

2. *For any real number* $x \geq 1$, *we have*

$$\sum_{n \leq x} \phi(SP(n)) = \frac{1}{2}x^2 \prod_p \left(1 - \frac{2}{p(p+1)}\right) + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where ϕ is Euler function.

3. *For any real number $x \geq 1$, we have*

$$\sum_{n\leq x} d(SP(n)) = \frac{6x\ln x}{\pi^2} + \left(\frac{12\gamma - 6}{\pi^2} - \frac{72\zeta'(2)}{\pi^4}\right) x + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where $d(n)$ and $\zeta(s)$ denote Dirichlet divisor function and Riemann zeta-function, respectively, γ is Euler constant.

H. Zhou [33]. *For any complex number s with $Re(s) > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mu(n)}{(SP(n^k))^s} = \begin{cases} \frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)}, & if \;\; k = 1, 2, \\ \frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)} - \frac{2^s-1}{4^s}, & if \;\; k = 3, \\ \frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)} - \frac{2^s-1}{4^s} + \frac{3^s-1}{9^s}, & if \;\; k = 4, 5, \end{cases}$$

where $\mu(n)$ denotes the möbius function.

Specifically, taking $s = 2$ and 4, we deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mu(n)}{(SP(n^k))^2} = \begin{cases} \frac{10}{\pi^2}, & if \;\; k = 1, 2, \\ \frac{10}{\pi^2} - \frac{3}{16}, & if \;\; k = 3, \\ \frac{10}{\pi^2} - \frac{3}{16} + \frac{8}{81}, & if \;\; k = 4, 5. \end{cases}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mu(n)}{(SP(n^k))^4} = \begin{cases} \frac{102}{\pi^4}, & if \;\; k = 1, 2, \\ \frac{102}{\pi^4} - \frac{15}{256}, & if \;\; k = 3, \\ \frac{102}{\pi^4} - \frac{15}{256} + \frac{80}{6561}, & if \;\; k = 4, 5. \end{cases}$$

Y. He [10]. *We have*

$$\sum_{n\leq x} \frac{\mu(n)}{SP(n)} = x + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

$$\sum_{n\leq x} \frac{\mu(n)}{SP^k(n)} = x^{1-k} + O\left(x^{\frac{1}{2}-k+\epsilon}\right).$$

B. Cheng [2]. *We define the Dirichlet divisor function for the Smaracdache power sequence as following: $SD(n) = \sigma_k(SP(n))$, where $\sigma_k(n) = \sum_{d|n} d^k$ is the divisor functions and $k > 0$.*

Let k be any real number with $k > 0$, then for any real number $x \geq 1$, we have

$$\sum_{n\leq x} SD(n) = \frac{\zeta(k+1)\cdot x^{k+1}}{\zeta(2)\cdot(k+1)} + O\left(x^{k+\frac{1}{2}+\epsilon}\right),$$

where $SD(n) = \sigma_k(SP(n))$, $\zeta(s)$ is the Riemann zeta-function, and ϵ denotes any fixed positive number.

Specifically, taking $k = 1$, then

$$\sum_{n\le x} Ss(n) = \frac{1}{2}x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where $Ss(k) = \sigma(SP(k))$ denotes the Dirichlet divisor function for the Smaracdache power sequence.

Y. Yang and M. Fang [28]. *For the Smarandache power function $SP(k)$, we have $\lim_{n\to\infty} \frac{S_1}{S_2} = 0$, where $S_1 = \sum_{k=1}^{\infty} \left(\frac{1}{\phi(SP(k))}\right)^2$, $S_2 = \left(\sum_{k=1}^{\infty} \frac{1}{\phi(SP(k))}\right)^2$.*

W. Huang and J. Zhao [11]. 1. *For any random real number $x \ge 3$ and given real number k $(k > 0)$, we have*

$$\sum_{n\le x} (SP(n))^k = \frac{\zeta(k+1)}{(k+1)\zeta(2)} x^{k+1} \prod_p \left(1 - \frac{1}{p^k(p+1)}\right) + O\left(x^{k+\frac{1}{2}+\epsilon}\right),$$

$$\sum_{n\le x} \frac{(SP(n))^k}{n} = \frac{\zeta(k+1)}{k\zeta(2)} x^k \prod_p \left(1 - \frac{1}{p^k(p+1)} + O\left(x^{k+\frac{1}{2}+\epsilon}\right)\right),$$

where $\zeta(k)$ is the Riemann zeta-function, ϵ denotes any fixed positive number, and $\prod_p$ denotes the product over all primes.

2. *For any random real number $x \ge 3$ and given real number $k^{'} > 0$, we have*

$$\sum_{n\le x} (SP(n))^{\frac{1}{k'}} = \frac{6k^{'}\zeta\left(\frac{1+k'}{k'}\right)}{(k'+1)\pi^2} x^{\frac{k'+1}{k'}} \prod_p \left(1 - \frac{1}{(1+p)p^{\frac{1}{k'}}}\right) + O\left(x^{\frac{k'+2}{2k'}+\epsilon}\right).$$

Specifically, we have

$$\sum_{n\le x} (SP(n))^{\frac{1}{2}} = \frac{4\zeta(\frac{3}{2})}{\pi^2} x^{\frac{3}{2}} \prod_p \left(1 - \frac{1}{p^{\frac{1}{2}}(p+1)} + O\left(x^{1+\epsilon}\right)\right),$$

$$\sum_{n\le x} (SP(n))^{\frac{1}{3}} = \frac{9\zeta(\frac{4}{3})}{2\pi^2} x^{\frac{4}{3}} \prod_p \left(1 - \frac{1}{p^{\frac{1}{3}}(p+1)} + O\left(x^{\frac{5}{6}+\epsilon}\right)\right).$$

3. *For any random real number $x \ge 3$, and $k = 1, 2, 3$. We have*

$$\sum_{n\le x} (SP(n)) = \frac{1}{2}x^2 \prod_p \left(1 - \frac{1}{p(p+1)} + O\left(x^{\frac{3}{2}+\epsilon}\right)\right),$$

$$\sum_{n\le x} (SP(n))^2 = \frac{6\zeta(3)}{3\pi^2} x^3 \prod_p \left(1 - \frac{1}{p^2(p+1)} + O\left(x^{\frac{5}{2}+\epsilon}\right)\right),$$

$$\sum_{n\le x} (SP(n))^3 = \frac{\pi^2}{60} x^4 \prod_p \left(1 - \frac{1}{p^3(p+1)} + O\left(x^{\frac{7}{2}+\epsilon}\right)\right).$$

4. *For any random real number $x \geq 3$, We have*

$$\sum_{n\leq x}\phi((SP(n))^k)=\frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p\left(1-\frac{1}{p^k(p+1)}+O\left(x^{k+\frac{1}{2}+\epsilon}\right)\right),$$

where $\phi(n)$ is the Euler function.

5. *For any random real number $x \geq 3$, We have*

$$\sum_{n\leq x}d((SP(n))^k)=B_0x\ln^k x+B_1x\ln^{k-1}x+B_2x\ln^{k-2}x+\cdots+B_{k-1}x\ln x+B_kx+O(x^{\frac{1}{2}+\epsilon}),$$

where $d(n)$ is Dirichlet divisor function and $B_0, B_1, B_2, \cdots, B_{k-1}, B_k$ is computable constant.

P. Ren, Y. Wang and S. Deng [22]. 1. *For any real number $x \geq 3$ and fixed real number k, l $(k > 0, l \geq 0)$, we have*

$$\sum_{n\leq x}n^l(SP(n))^k=\frac{\zeta(k+1)}{(k+l+1)\zeta(2)}x^{k+l+1}\prod_p\left(1-\frac{1}{p^k(1+p)}\right)+O\left(x^{k+l+\frac{1}{2}+\epsilon}\right),$$

$$\sum_{n\leq x}n^l\frac{(SP(n))^k}{n^l}=\frac{\zeta(k+1)}{(k-l+1)\zeta(2)}x^{k-l+1}\prod_p\left(1-\frac{1}{p^k(1+p)}\right)+O\left(x^{k-l+\frac{1}{2}+\epsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, ϵ denotes any fixed positive number, and $\prod_p$ denotes the product over all primes.

2. *For any real number $x \geq 3$ and fixed real number $k' > 0$, we have*

$$\sum_{n\leq x}(SP(n))^{\frac{1}{k'}}=\frac{6k'\zeta\left(\frac{k'+1}{k'}\right)}{(k'+1)\pi^2}x^{\frac{k'+1}{k'}}\prod_p\left(1-\frac{1}{p^{\frac{1}{k'}}(1+p)}\right)+O\left(x^{\frac{k'+2}{2k'}+\epsilon}\right).$$

Specifically, we have

$$\sum_{n\leq x}(SP(n))^{\frac{1}{3}}=\frac{9\zeta\left(\frac{4}{3}\right)}{2\pi^2}x^{\frac{4}{3}}\prod_p\left(1-\frac{1}{p^{\frac{1}{3}}(1+p)}\right)+O\left(x^{\frac{5}{6}+\epsilon}\right),$$

$$\sum_{n\leq x}(SP(n))^{\frac{1}{2}}=\frac{4\zeta\left(\frac{3}{2}\right)}{\pi^2}x^{\frac{3}{2}}\prod_p\left(1-\frac{1}{p^{\frac{1}{2}}(1+p)}\right)+O\left(x^{1+\epsilon}\right).$$

3. *For any real number $x \geq 3$ and fixed real number, we have*

$$\sum_{n\leq x}n^l(SP(n))=\frac{1}{(l+2)}x^{l+2}\prod_p\left(1-\frac{1}{p(1+p)}\right)+O\left(x^{l+\frac{3}{2}+\epsilon}\right),$$

$$\sum_{n\leq x}n^l(SP(n))^2=\frac{6\zeta(3)}{(l+3)\pi^2}x^{l+3}\prod_p\left(1-\frac{1}{p^2(1+p)}\right)+O\left(x^{l+\frac{5}{2}+\epsilon}\right),$$

$$\sum_{n\leq x}n^l(SP(n))^3=\frac{\pi^2}{15(l+4)}x^{l+4}\prod_p\left(1-\frac{1}{p^3(1+p)}\right)+O\left(x^{l+\frac{7}{2}+\epsilon}\right).$$

L. Gao and Y. Ma [6]. *If the product of all true divisors of n less than or equal to n, then n is called simple number. We have*

$$\sum_{\substack{n\in\Lambda\\ n\le x}} \frac{1}{S(SP(n))} = \frac{2x\ln\ln x}{\ln x} + D_1\frac{x}{\ln x} + o\left(\frac{x\ln\ln x}{\ln^2 x}\right),$$

$$\sum_{\substack{n\in\Lambda\\ n\le x}} S(SP(n) = \frac{2x\ln\ln x}{\ln x} + D_2\frac{x}{\ln x} + o\left(\frac{x\ln\ln x}{\ln^2 x}\right),$$

where Λ *denotes the set of simple numbers,* $S(n)$ *is Smarandache function.*

§4. Smarandache exponent function

For any fixed prime p and positive integer n, Smarandache exponent function $e_p(n)$ as follows:

$$e_p(n) = \max\{\alpha : p^\alpha \mid n\}.$$

H. Liu and W. Zhang [18]. *Let p be a prime,* $e_p(n)$ *denotes the largest exponent (of power p) which divides n,* $\alpha(n,p) = \sum_{k\le n} e_p(k)$. *For any prime p and any fixed positive integer n, we have*

$$\sum_{p\le n}\alpha(n,p) = n\ln\ln n + cn + c_1\frac{n}{\ln n} + c_2\frac{n}{\ln^2 n} + \cdots + c_k\frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n}\right),$$

where k is any fixed positive integer, $c_i(i = 1, 2, \cdots,)$ *are some computable constants.*

C. Lv [3]. *Let p be a prime,* $e_p(n)$ *denotes the largest exponent of power p which divides n. Let p be a prime,* $m \ge 0$ *be an integer. Then for any real number* $x \ge 1$*, we have*

$$\sum_{n\le x} e_p^m(n) = \frac{p-1}{p}a_p(m)x + O(\log^{m+1} x),$$

where $a_P(m)$ *is a computable constant.*

Specifically, taking $m = 1, 2, 3$*, for any real number* $x \ge 1$*, we have*

$$\sum_{n\le x} e_p(n) = \frac{p}{p-1}x + O(\log^2 x),$$

$$\sum_{n\le x} e_p^2(n) = \frac{p+1}{(p-1)^2}x + O(\log^3 x),$$

$$\sum_{n\le x} e_p^3(n) = \frac{p^2+4p+1}{(p-1)^3}x + O(\log^4 x).$$

C. Lv [4]. *Let p be a prime,* $\phi(n)$ *is the Euler totient function. Then for any real number* $x \ge 1$*, we have*

$$\sum_{n\le x} e_p(n)\phi(n) = \frac{3p}{(p^2-1)\pi^2}x^2 + O(x^{\frac{3}{2}+\epsilon}).$$

N. Gao [7]. 1. *Let p and q are two primes, then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} p^{e_q(b(n))} = \frac{q^2+p^2q+p}{q^2+q+1}x + O(x^{\frac{1}{2}+\epsilon}),$$

where ϵ is any fixed positive number.

2. *Let q be a prime, then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} p^{e_q(b(n))} = qx + O(x^{\frac{1}{2}+\epsilon}),$$

where ϵ is any fixed positive number.

X. Wang [26]. *Let p be a prime, $m \geq 1$ be an integer, then for any real number $x > 1$, we have*

$$\sum_{n^m\leq x} ((n+1)^m - n^m)e_p(n) = \frac{1}{p-1}\frac{m}{m+1}x + O(x^{1-\frac{1}{m}}).$$

T. Zhang [34]. *Let n be any positive integer, $P_d(n)$ denotes the product of all positive divisors of n. That is, $P_d(n) = \prod_{p} d$ and let p be a prime, $a_p(n)$ denotes the largest exponent (of power p) such that $p^{a_p(n)} \mid n$.*

Let p be a prime, then for any real number $x \geq 1$, we have

$$\sum_{n\leq x} a_p(P_d(n)) = \frac{x\ln x}{p(p-1)} + \frac{(p-1)^3(2\gamma-1)-(2p^4+4p^3+p^2-2p+1)\ln p}{p(p-1)^4}x + O(x^{\frac{1}{2}+\epsilon}),$$

where γ is the Euler constant, and ϵ denotes any fixed positive number.

Specifically, taking p = 2, 3,for any real number $x \geq 1$, we have

$$\sum_{n\leq x} a_2(P_d(n)) = \frac{1}{2}x\ln x + \frac{2\gamma - 65\ln 2 - 1}{2}x + O(x^{\frac{1}{2}+\epsilon}),$$

$$\sum_{n\leq x} a_3(P_d(n)) = \frac{1}{6}x\ln x + \frac{8\gamma - 137\ln 3 - 4}{24}x + O(x^{\frac{1}{2}+\epsilon}).$$

N. Gao [8]. *Let p and q are primes with $q \geq p$. Then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} p^{e_q(n)} = \begin{cases} \frac{q-1}{q-p}x + O(x^{\frac{1}{2}+\epsilon}), & \textit{if } q > p, \\ \frac{p-1}{p\ln p}x\ln x + \left(\frac{p-1}{p\ln p}(\gamma-1) + \frac{p+1}{2p}\right)x + O(x^{\frac{1}{2}+\epsilon}), & \textit{if } q = p. \end{cases}$$

where ϵ is any fixed positive number, γ is the Euler constant.

X. Pan and P. Zhang [20]. 1. *For any prime p and complex number s with $Re(s) > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{e_p(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

where $\zeta(s)$ is the Riemann zeta-function.

2. *For any prime p, we have*

$$e_p(m) = \sum_{\substack{n\in N \\ S_p(n)=m}} 1.$$

Z. Qu [21]. *For any real number $x \geq 1$ and fixed integer $t \geq 1$, if $t > 1$, then*

$$\sum_{n\leq x} e_p(n)\sigma_t(n) = \frac{(p^{t+1}+p-2)\zeta(t+1)}{(p-1)(p^{t+1}-1)}x^{t+1} + O(x^t \ln x),$$

if $t = 1$, then

$$\sum_{n\leq x} e_p(n)\sigma_t(n) = \frac{(p+2)\pi^2}{6(p^2-1)}x^2 + O(x\ln^3 x).$$

W. Huang [12]. *Let p, q be primes, $k \geq 2$ be a fixed positive integer, n be an arbitrary positive integer, $A_k(n)$ be k-th complement number of n, $\zeta(s)$ be Riemann zeta-function. For any real number $x \geq 1$, we have*

$$\sum_{n\leq x} p^{e_q(A_k(n))} = \frac{q^{k-1}+p^{k-1}q^{k-2}+p^{k-2}q^{k-3}+\cdots+p}{q^{k-1}+q^{k-2}+q^{k-3}+\cdots+1}x + O(x^{\frac{1}{2}+\epsilon}),$$

where ϵ is an arbitrary positive integer.

Specifically, if $k = 2$ or $k = 3$, let p, q be primes, for any real number $x \geq 1$, we have

$$\sum_{n\leq x} p^{e_q(A_2(n))} = \frac{p+q}{q+1}x + O(x^{\frac{1}{2}+\epsilon}),$$

$$\sum_{n\leq x} p^{e_q(A_3(n))} = \frac{q^2+p^2q+p}{q^2+q+1}x + O(x^{\frac{1}{2}+\epsilon}).$$

G. Ren [23]. 1. *Let $q \geq 3$ be any fixed positive integer, then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} e_q(n) = \frac{x}{q-1} + O(\log x).$$

2. *If $q \geq 3$ be any fixed positive integer, $k \geq 2$ is an integer, then we have*

$$\sum_{n\leq x} e_q^k(n) = \frac{q-1}{q}B_q(k)x + O(\log^{k+1} x),$$

where $B_q(k)$ is given by the recursion formulas: $B_q(0) = \frac{1}{q}$,

$$B_q(k) = \frac{1}{q}\left(\binom{k}{1}B_q(k-1) + \binom{k}{2}B_q(k-2) + \cdots + \binom{k}{k-1}B_q(1) + B_q(0) + 1\right).$$

Specifically, taking $q = p_1p_2$ in Theorem 1, where p_1, p_2 are two fixed distinct primes, for any real number $x \geq 1$, we have

$$\sum_{n\leq x} e_{p_1p_2}(n) = \frac{x}{p_1p_2-1} + O(\log x).$$

Y. Liu and P. Gao [19]. *For any positive integer n, we have $n = u^k v$, where v is a k-power free number. Let $b_k(n)$ be the k-power free part of n. Let p be a prime, k be any fixed positive integer. Then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} e_p(b_k(n)) = \left(\frac{p^k - p}{(p^k-1)(p-1)} - \frac{k-1}{p^k-1}\right) x + O(x^{\frac{1}{2}+\epsilon}),$$

where ϵ denotes any fixed positive number.

Specifically, taking $k = 2$, for any real number $x \geq 1$, we have

$$\sum_{n\leq x} e_p(b_k(n)) = \frac{1}{p+1} x + O(x^{\frac{1}{2}+\epsilon}).$$

G. Ren [24]. *Let p and q be two distinct primes, then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} e_{pq}(n) = \frac{x}{pq-1} + O(x^{\frac{1}{2}+\epsilon}),$$

where ϵ is any fixed positive number.

W. Zhang [35]. *For any fixed prime p and any real number $x \geq 1$, we have*

$$\sum_{n\leq x} a_p(S_p(n)) = \frac{p+1}{(p-1)^2} x + O(\ln^3 x).$$

Specifically, taking $p = 2, 3$, for any real number $x \geq 1$, we have

$$\sum_{n\leq x} a_2(S_2(n)) = 3x + O(\ln^3 x),$$

$$\sum_{n\leq x} a_3(S_3(n)) = x + O(\ln^3 x).$$

J. Zhao [36]. *Let p be a prime, then for any real number $x \geq 1$, we have*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} e_p(n) = C(p,k) a_p(k) x^{\frac{1}{k}} + O(x^{\frac{1}{2k}+\epsilon}),$$

where

$$C(p,k) = \frac{6k}{\pi^2}\left(1 - \frac{1}{p - p^{\frac{k-1}{k}}} + 1\right)\prod_q \left(1 + \frac{1}{(q+1)(q^{\frac{1}{k}-1})}\right),$$

$a_p(k)$ is a computable positive constant, $\epsilon > 0$ is any fixed real number, and $\prod_q$ denotes the product over all prime q.

Q. Yang and M. Yang [29]. *Define for any two primes p and q with $(p,q) = 1$, let $e_{pq}(n)$ denotes the largest exponent of power pq which divides n. That is*

$$e_{pq}(n) = \max\{\alpha \ : \ (pq)^\alpha \mid n, \alpha \in N^+\}.$$

Let p and q are two primes with $(p,q)=1$, then for any real number $x \geq 1$, we have

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} e_{pq}(n) = C_{p,q} k x^{\frac{1}{k}} + O(x^{\frac{1}{2k}+\epsilon}),$$

where

$$C_{p,q} = \frac{(p-1)(q-1)}{pq}\sum_{n=1}^{\infty}\frac{n}{(pq)^n}$$

is a computable positive constant, and ϵ denotes any fixed positive number.

J. Du [5]. *Define arithmetical function $b_{pq}(n) = \sum_{t|n} e_{pq}\left(\frac{n}{t}\right) e_{pq}(t)$. Let p and q are two distinct primes, then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} b_{pq}(n) = \frac{x\ln x}{(pq-1)^2} + \frac{1-2\gamma-pq+2pq\gamma-2pq\ln(pq)}{(pq-1)^3}x + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where ϵ is any fixed positive number, and γ is the Euler constant.

References

[1] Jiao Chen. One kind Dirichlet series involving the Smarandache summands function. Journal of Southwest China Normal University(Natural Science Edition) 36(2011), no. 1, 39 - 43. (In Chinese with English abstract).

[2] Bin Cheng. On the mean value of the Dirichlet divisor function for the Smarandache power sequence. Scientia Magna 4(2008), no. 1, 39 - 43.

[3] Lv Chuan. A number theoretic function and its mean value. Research on Smarandache problems in number theory, 33 - 35, Hexis, Phoenix, AZ, 2004.

[4] Lv Chuan. On the mean value of an arithmetical function. Research on Smarandache problems in number theory, 89 - 92, Hexis, Phoenix, AZ, 2004.

[5] Jianli Du. A number theoretic function and its mean value. Research on Smarandache problems in number theory II, 115 - 117.

[6] Li Gao and Yafeng Ma. The average issue involving the Smarandache power function, Basic Sciences Journal of Textile Universities 29(2016), no. 1, 8 - 10. (In Chinese with English abstract).

[7] Nan Gao. A hybrid number theoretic function and its mean value. Research on Smarandache problems in number theory, 107 - 109, Hexis, Phoenix, AZ, 2004.

[8] Nan Gao. A number theoretic function and its mean value. Research on Smarandache problems in number theory, 49 - 52, Hexis, 2004.

[9] Wei Huang. On the mean value of generalized Smarandache summands function. Journal of Jishou University(Natural Science Edition) 33(2012), no. 2, 7 - 9. (In Chinese with English abstract).

[10] Yanfeng He. On the hybrid mean value formula involving Smarandache power function. Journal of Yanan University(Natural Science Edition) 27(2008), no. 1, 1 - 2. (In Chinese with English abstract).

[11] Wei Huang and Jiaolian Zhao. On the note of the Smarandache power function. Scientia Magna 6(2010), no. 3, 93 - 98.

[12] Wei Huang. The hybrid mean value of involving the Smarandache exponent function $e_p(n)$. Journal of Baoji University of Arts and Sciences (Natural Science) 32(2012), no. 3, 4 - 5,27. (In Chinese with English abstract).

[13] Maohua Le. On smarandache simple functions. Smarandache Notions Journal 10(1999), no. (1-3), 161 - 162.

[14] Hua Liu. On the divisor function $\sigma_\alpha(n)$ involving the F.Smarandache simple function. Basic Sciences Journal of Textile Universities 20(2007), no. 4, 361 - 363. (In Chinese with English abstract).

[15] Hua Liu. On the F.Smarandache simple function. Scientia Magna 2(2006), no. 2, 98 - 100. (In Chinese with English abstract).

[16] Hua Liu. The mean value on F.Smarandache function. Journal of ShangQiu Teachers College 27(2011), no. 3, 24 - 25. (In Chinese with English abstract).

[17] Hua Liu and Min Zhu. The mean value on F.Smarandache simple function. Journal of Southwest University for Nationalities · Natural Science Edition 37(2011), no. 2, 216 - 217. (In Chinese with English abstract).

[18] Hongyan Liu and Wenpeng Zhang. A number theoretic function and its mean value property. Smarandache Notions Journal 13(2002), 155 - 159.

[19] Yanni Liu and Peng Gao. Mean value of a new arithmetic function. Scientia Magna 1(2014), no. 1, 187 - 189.

[20] Xiaowei Pan and Pei Zhang. An identity involving the function $e_p(n)$. Scientia Magna 3(2007), no. 3, 26 - 29.

[21] Zhilian Qu. Hybrid mean value of the Smarandache exponent function $e_p(n)$ and the function $\sigma_t(n)$. Natural Science Journal of Hainan University 29(2011), no. 1, 4 - 7. (In Chinese with English abstract).

[22] Peng Ren, Yang Wang and Shuxian Deng. Some notes on the Smarandache power function, Science and Technology Review 28(2010), no. 17, 50-53. (In Chinese with English abstract).

[23] Ganglian Ren. A number theoretic function and its mean value. Scientia Magna 1(2014), no. 1, 53 - 56.

[24] Ganglian Ren. A number theoretic function and its mean value. Scientia Magna 1(2014), no. 1, 53 - 56.

[25] Jianping Wang. One kind Dirichlet series involving the Smarandache Summands function. Journal of Shaanxi Normal University(Natural Science Edition) 38(2010), no. 5, 14 - 17. (In Chinese with English abstract).

[26] Xiaoying Wang. On the mean value of an arithmetical function. 77-79, Hexis, Phoenix, AZ, 2004 .

[27] Zhefeng Xu. On the mean value of the Smarandache power function. Acta Mathematica Sinica 49(2006), no. 1, 77 - 80. (In Chinese with English abstract).

[28] Yanting Yang and Min Fang. On the Smarandache totient function and the Smarandache power sequence. Scienta Magna 5(2009), no. 1, 124 - 127.

[29] Qianli Yang and Mingshun Yang. An arithmetical function and the perfect k-th power numbers. Research on Smarandache problems in number theory II, 91 - 94.

[30] Minhui Zhu. The additive analogue of sumarandache simple function. Research on Smarandache Problems in Number Theory, 39 - 41, Hexis, Phoenix, AZ, 2004.

[31] Huimin Zhu. On the hybrid mean value involving the F.Smarandache simple function and the Dirichlet divisor function . Journal of Inner Mongolia Normal University(Natural Science Edition) 39(2010), no. 5, 441 - 443. (In Chinese with English abstract).

[32] Yuane Zhao. On the mean value of the Smarandache Summands. Journal of Southwest China Normal University(Natural Science Edition) 36(2011), no. 1, 44 - 47. (In Chinese with English abstract).

[33] Huanqin Zhou. An infinite series involving the Smarandache power function $SP(n)$. Scientia Magna 2(2006), no. 3, 109 - 112.

[34] Tianping Zhang. An arithmetic function and the divisor product sequence. Research on Smarandache problems in number theory, 21 - 26, Hexis, Phoenix, AZ, 2004.

[35] Wenpeng Zhang. An arithmetic function and the primitive number of power P. Research on Smarandache problems in number theory(collected papers), 1 - 4, Hexis, 2004.

[36] Jiantang Zhao. An arithmetical function and the k-full number sequences. Research on Smarandache problems in number theory II, 131 - 134.

SCIENTIA MAGNA

An international journal

ISSN 1556- 6706