SCIENTIA MAGNA

An international journal

Edited by School of Mathematics
Northwest University, P.R.China

Vol. 12, No. 1, 2017 **ISSN 1556-6706**

SCIENTIA MAGNA

An international journal

Edited by

School of Mathematics

Northwest University

Xi'an, Shaanxi, China

Information for Authors

Scientia Magna is a peer-reviewed, open access journal that publishes original research articles in all areas of mathematics and mathematical sciences. However, papers related to Smarandache's problems will be highly preferred.

The submitted manuscripts may be in the format of remarks, conjectures, solved/unsolved or open new proposed problems, notes, articles, miscellaneous, etc. Submission of a manuscript implies that the work described has not been published before, that it is not under consideration for publication elsewhere, and that it will not be submitted elsewhere unless it has been rejected by the editors of Scientia Magna.

Manuscripts should be submitted electronically, preferably by sending a PDF file to ScientiaMagna@hotmail.com.

On acceptance of the paper, the authors will also be asked to transmit the TeX source file. PDF proofs will be e-mailed to the corresponding author.

Contents

Scientia Magna
Vol. 12 (2017), No. 1, 1-6

On the mean value of exponential divisor function

Jing Huang[1] and Ping Song[2]

[1]School of Mathematics and Statistics, Shandong Normal University
Shandong Jinan, China
E-mail: huangjingsdnu@163.com
[2]School of Mathematics and Statistics, Shandong Normal University
Shandong Jinan, China
E-mail: songpingsdnu@163.com

Abstract Let $n > 1$ be an integer. The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i}$, if $b_i \mid a_i$ for every $i \in 1, 2, \cdots, s$. Let $\tau^{(e)}(n)$ denote the exponential divisor function. In this paper, we study the sum $D(1, \underbrace{2, \cdots, 2}_{\text{k}}; x) = \sum_{n\leq x} d(1, \underbrace{2, \cdots, 2}_{\text{k}}; n)$ and get the asymptotic formula for it, where $d(1, \underbrace{2, \cdots, 2}_{\text{k}}; n) = \sum_{n=ab_1^2\cdots b_k^2} 1$. We get the mean value for the exponential divisor function, which improves the previous result.

Keywords Dirichlet convolution; asympototic formula; exponential divisor function.

1 Introduction

Many scholars are interested in researching the divisor problem, and they have obtained a large number of good results. However, there are many problems hasn't been solved. For example, F. Smarandache gave some unsolved problems in his book *only problems, not solutions!*, and one problem is that, a number n is called simple number if the product of its proper divisors is less than or equal to n. Generally speaking, $n = p$, or $n = p^2$, or $n = p^3$, or pq, where p and q are distinct primes. The properties of this simple number sequence hasn't been studied yet. And other problems are introduced in this book, such as proper divisor products sequence and the largest exponent (of power p) which divides n, where $p \geq 2$ is an integer.

In this the definition of exponential divisor: suppose $n > 1$ is an integer, and $n = \prod_i^t p_i^{a_i}$. If $d = \prod_i^t p_i^{b_i}$ satisfies $b_i \mid a_i, i = 1, 2, \cdots, t$, then d is called an exponential divisor of n, notation $d \mid_e n$. By convention $1 \mid_e 1$.

J. Wu [4] improved the above result got the following result:

$$\sum_{n\leq x} \tau^{(e)}(n) = A(x) + Bx^{\frac{1}{2}} + O(x^{\frac{2}{9}} \log x),$$

where

$$A = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{d(a) - d(a-1)}{p^a}\right),$$

$$B = \prod_p \left(1 + \sum_{a=5}^{\infty} \frac{d(a) - d(a-1) - d(a-2) + d(a-3)}{p^{\frac{a}{2}}}\right).$$

M. V. Subbarao [2] also proved for some positive integer r,

$$\sum_{n \le x} (\tau^{(e)}(n))^r \sim A_r x,$$

where

$$A_r = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{(d(a))^r - (d(a-1))^r}{p^a}\right).$$

L. Toth [3] proved

$$\sum_{n \le x} (\tau^{(e)}(n))^r = A_r(x) + x^{\frac{1}{2}} P_{2^r-2}(\log x) + O(x^{u_r+\varepsilon}),$$

where $P_{2^r-2}(t)$ is a polynomial of degree $2^r - 2$ in t, $u_r = \frac{2^{r+1}-1}{2^{r+1}+1}$.

Similarly to the generalization of $d_k(n)$ from $d(n)$, we define the function $\tau_k^{(e)}(n)$:

$$\tau_k^{(e)}(n) = \prod_{p_i^{a_i} \| n} d_k(a_i), k \ge 2,$$

Obviously when $k = 2$, that is $\tau^{(e)}(n)$. $\tau_3^{(e)}(n)$ is obviously a multiplicative function. In this paper we investigate the case $k = 3$, i.e. the properties of the function $\tau_3^{(e)}(n)$.

In this paper, we will study the asymptotic formula for the mean value of the r-th power of the function $\tau_3^{(e)}(n)$, where $r > 1$ is an integer.

Theorem 1.1. *For every integer $r > 1$, then we have*

$$\sum_{n \le x} (\tau_3^{(e)}(n))^r = A_r x + x^{\frac{1}{2}} R_{3^r-2}(\log x) + O(x^{b_r+\varepsilon}),$$

for every $\varepsilon > 0$, where $b_r := \frac{1}{3-\alpha_{3^r-1}}$, α_k is as defined in Lemma 2.2, the $O-$term is related to r, $R_{3^r-2}(x)$ is a polynomial of degree $3^r - 2$ and

$$A_r := \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{(d_3(a))^r - (d_3(a-1))^r}{p^a}\right),$$

where $d_3(n) = \sum_{n=m_1 m_2 m_3} 1.$

2 Some lemmas

In this section, we give some lemmas which will be used in the proof of our theorem. Lemma 2.2 and Lemma 2.3. can be found in [1] and [5].

Lemma 2.1. *For $r > 1$, then we have*

$$\sum_{n=1}^{\infty} \frac{(\tau_3^{(e)}(n))^r}{n^s} = \zeta(s)\zeta^{3^r-1}(2s)V(s),$$

where the infinite series $V(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{4}$.

Proof. By Euler's product formula, we can get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(\tau_3^{(e)}(n))^r}{n^s} &= \prod_p\left(1+\frac{(\tau_3^{(e)}(p))^r}{p^s}+\frac{(\tau_3^{(e)}(p^2))^r}{p^{2s}}+\frac{(\tau_3^{(e)}(p^3))^r}{p^{3s}}+\frac{\tau_3^{(e)}(p^4))^r}{p^{4s}}+\cdots\right)\\ &= \prod_p\left(1+\frac{d_3^r(1)}{p^s}+\frac{d_3^r(2)}{p^{2s}}+\frac{d_3^r(3)}{p^{3s}}+\frac{d_3^r(4)}{p^{4s}}+\frac{d_3^r(5)}{p^{5s}}+\cdots\right)\\ &= \prod_p\left(1+\frac{1}{p^s}+\frac{3^r}{p^{2s}}+\frac{3^r}{p^{3s}}+\frac{6^r}{p^{4s}}+\frac{3^r}{p^{5s}}+\cdots\right)\\ &= \zeta(s)\left(1+\frac{3^r-1}{p^{2s}}+\frac{3^r}{p^{4s}}+\cdots\right)\\ &= \zeta(s)\zeta^{3^r-1}(2s)V(s),\end{aligned} \tag{2.1}$$

where the infinite series $V(s):=\sum_{n=1}^{\infty}\frac{v(n)}{n^s}$ is absolutely convergent for $\Re s>\frac{1}{4}$. □

Lemma 2.2. *Suppose $k\geq 2$ is an integer. Then*

$$D_k(x)=\sum_{n\leq x}d_k(n)=x\sum_{j=0}^{k-1}c_j(\log x)^j+O(x^{\alpha_k+\varepsilon}),$$

where c_j is a calculable constant, ε is a sufficiently small positive constant, α_k is the infimum of numbers α_k, such that

$$\Delta_k(x)=\sum_{n\leq x}d_k(n)-xP_{k-1}(\log x)\ll x^{\alpha_k+\varepsilon}, \tag{2.2}$$

and

$$\begin{aligned}&\alpha_2\leq\frac{131}{416},\quad \alpha_3\leq\frac{43}{94},\\ &\alpha_k\leq\frac{3k-4}{4k},\quad 4\leq k\leq 8,\\ &\alpha_9\leq\frac{35}{54},\quad \alpha_{10}\leq\frac{41}{61},\ \alpha_{11}\leq\frac{7}{10},\\ &\alpha_k\leq\frac{k-2}{k+2},\quad 12\leq k\leq 25,\\ &\alpha_k\leq\frac{k-1}{k+4},\quad 26\leq k\leq 50,\\ &\alpha_k\leq\frac{31k-98}{32k},\quad 51\leq k\leq 57,\\ &\alpha_k\leq\frac{7k-34}{7k},\quad k\geq 58.\end{aligned}$$

Lemma 2.3. *Suppose $f(m), g(n)$ are arithmetical functions such that*

$$\sum_{m\leq x}f(m)=\sum_{j=1}^{J}x^{\alpha_j}P_j(\log x)+O(x^{\alpha}),\quad \sum_{n\leq x}\mid g(n)\mid=O(x^{\beta}),$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_J > \alpha > \beta > 0, P_j(t)$ *is a polynomial in* t*, if* $h(n) = \sum_{n=md} f(m)g(d)$*, then*

$$\sum_{n\leq x} h(n) = \sum_{j=1}^{J} x^{\alpha_j} Q_j(\log x) + O(x^{\alpha}),$$

where $Q_j(t)$ $\{j = 1, \cdots, J\}$ *is a polynomial in* t.

3 The mean value of $d(1, 2, \cdots, 2; n)$

Theorem 3.1. *Suppose* $k \geq 2$ *is an integer, then*

$$D(1, \underbrace{2, \cdots, 2}_{k}; x) = \sum_{n\leq x} d(1, \underbrace{2, \cdots, 2}_{k}; n) = \zeta^k(2)x + x^{\frac{1}{2}} Q_{k-1}(\log x) + O(x^{\frac{1}{3-\alpha_k}+\varepsilon}).$$

Proof. Recall that $d(1, \underbrace{2, \cdots, 2}_{\mathrm{k}}; n) = \sum_{n=ab_1^2\cdots b_k^2} 1$, by hyperbolic summation formula, we have

$$\begin{aligned} D(1, \underbrace{2, \cdots, 2}_{\mathrm{k}}; x) &= \sum_{n\leq x} d(1, \underbrace{2, \cdots, 2}_{\mathrm{k}}; n) = \sum_{m^2 l\leq x} d_k(m) \\ &= \sum_{m\leq y} d_k(m) \sum_{m^2 l\leq x} 1 + \sum_{l\leq z} \sum_{m^2 l\leq x} d_k(m) - \sum_{m\leq y} d_k(m) \sum_{l\leq z} 1 \qquad (3.1) \\ &:= S_1 + S_2 - S_3, \end{aligned}$$

where y, z are parameters that will be determined later, and satisfy that $y^2 z = x, 1 \leq y \leq x$. Now, we deal with S_1, S_2 and S_3, separately

$$\begin{aligned} S_1 &= \sum_{m\leq y} d_k(m) \sum_{m^2 l\leq x} 1 = \sum_{m\leq y} d_k(m)[\frac{x}{m^2}] \\ &= x \sum_{m\leq y} \frac{d_k(m)}{m^2} + O\left(\sum_{m\leq y} d_k(m)\right) \qquad (3.2) \\ &= \zeta^k(2)x - x \sum_{m>y} \frac{d_k(m)}{m^2} + O(y^{1+\varepsilon}). \end{aligned}$$

Using Lemma 2.2 and partial summation formula, we have

$$\begin{aligned} \sum_{m>y} \frac{d_k(m)}{m^2} &= \int_{y^+}^{\infty} \frac{1}{t^2} d\left(\sum_{m\leq t} d_k(m)\right) \\ &= \int_{y^+}^{\infty} \frac{1}{t^2} d\left(t \sum_{j=0}^{k=1} c_j (\log t)^j + O(t^{\alpha_k+\varepsilon})\right) \\ &= \sum_{j=0}^{k=1} c_j \int_{y^+}^{\infty} \frac{1}{t^2} d(\log t)^j + O(y^{-2+\alpha_k+\varepsilon}) \\ &= \sum_{j=0}^{k=1} c_j y^{-1}[(\log y)^j + 2j(\log y)^{j-1} + 2j(j-1)(\log y)^{j-2} + \cdots + 2j(j-1)\cdots 1] \\ &+ O(y^{-2+\alpha_k+\varepsilon}). \end{aligned}$$

Since $y=\sqrt{\frac{x}{z}}$, we have $\log y=\frac{1}{2}(\log x-\log z)$, inserting this into (3.2), we can get

$$S_1=\zeta_k(2)x-S_{11}-S_{12}+O(y^{1+\varepsilon}+xy^{-2+\alpha_k+\varepsilon}), \tag{3.3}$$

where

$$S_{11}=x^{\frac{1}{2}}z^{\frac{1}{2}}\sum_{j=1}^{k-1}\frac{c_j}{2^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i(\log z)^i,$$

$$S_{12}=2x^{\frac{1}{2}}z^{\frac{1}{2}}\sum_{j=1}^{k-1}\frac{c_j}{2^j}\sum_{i=0}^{j}\frac{j!}{i!2^{j-i}}\sum_{s=0}^{i}C_i^s(\log x)^{i-s}(-1)^s(\log z)^s.$$

By Lemma 2.2, we get

$$\begin{aligned}S_2&=\sum_{l\le z}\sum_{m\le\sqrt{\frac{x}{l}}}d_k(m)=\sum_{l\le z}\left(\sqrt{\frac{x}{l}}\sum_{j=1}^{k-1}c_j\left(\log\sqrt{\frac{x}{l}}\right)^j+O\left(\left(\sqrt{\frac{x}{l}}\right)^{\alpha_k+\varepsilon}\right)\right)\\&=x^{\frac{1}{2}}\sum_{j=0}^{k-1}\frac{c_j}{2^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i\sum_{l\le z}l^{-\frac{1}{2}}(\log l)^i+O(xy^{-2+\alpha_k+\varepsilon}),\end{aligned} \tag{3.4}$$

where

$$\sum_{l\le z}l^{-\frac{1}{2}}(\log l)^i=\int_{1^-}^{z}t^{-\frac{1}{2}}(\log t)^i d[t]=\int_{1^-}^{z}t^{-\frac{1}{2}}(\log t)^i dt+\int_{1^-}^{z}t^{-\frac{1}{2}}(\log t)^i d\Delta(t). \tag{3.5}$$

We can easily get that $\Delta(t)=O(1)$. Using partial integral formula, we have

$$\int_{1^-}^{z}t^{-\frac{1}{2}}(\log t)^i d\Delta(t)=\omega_i+O(z^{-\frac{1}{2}+\varepsilon}), \tag{3.6}$$

where ω_i is a constant. We can also obtain that

$$\int_{1^-}^{z}t^{-\frac{1}{2}}(\log t)^i dt=2z^{\frac{1}{2}}(\log z)^i-2^2iz^{\frac{1}{2}}(\log z)^{i-1}+\cdots+(-1)^{i+1}2^{i+1}i!. \tag{3.7}$$

Combing (3.4)-(3.7), we have

$$S_2=x^{\frac{1}{2}}\widetilde{Q}_{k-1}(\log x)+S_{21}+S_{22}+O(xy^{-2+\alpha_k+\varepsilon}), \tag{3.8}$$

where

$$\widetilde{Q}_{k-1}(\log x)=\sum_{j=0}^{k-1}\frac{c_j}{2^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i(\omega_i-(-1)^i2^{i+1}i!),$$

$$S_{21}=2x^{\frac{1}{2}}z^{\frac{1}{2}}\sum_{j=0}^{k-1}\frac{c_j}{2^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i(\log z)^i,$$

$$S_{22}=2x^{\frac{1}{2}}z^{\frac{1}{2}}\sum_{j=0}^{k-1}\frac{c_j}{2^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i\sum_{s=0}^{i-1}(-1)^{s-i}2^{i-s}\frac{i!}{s!}(\log z)^s.$$

For S_3, we have

$$\begin{aligned} S_3 &= \sum_{m\le y} d_k(m) \sum_{l\le z} 1 = zy \sum_{j=0}^{k-1} c_j (\log y)^j + O(y^{\alpha_k+\varepsilon} z) + O(y^{1+\varepsilon}) \\ &= yz \sum_{j=0}^{k-1} c_j (\log y)^j + O(y^{\alpha_k+\varepsilon} z + y^{1+\varepsilon}). \end{aligned} \tag{3.9}$$

Inserting $y = \sqrt{\frac{x}{z}}$, and $\log y = \frac{1}{2}(\log x - \log z)$ into (3.7), then

$$S_3 = S_{31} + O(y^{\alpha_k+\varepsilon} z + y^{1+\varepsilon}), \tag{3.10}$$

where

$$S_{31} = \frac{1}{2} z^{\frac{1}{2}} \sum_{j=0}^{k-1} \frac{c_j}{2^j} \sum_{i=0}^{j} C_j^i (\log x)^{j-i} (-1)^i (\log z)^i.$$

Note that $C_j^i = \frac{i!}{j!(i-j)!}$. After some simplification we can easily get that $S_{11} + S_{31} = S_{21}, S_{12} = S_{22}$. Taking $y = \frac{x}{3-\alpha_k}, z = \frac{x(1-\alpha_k)}{3-\alpha_k}$, then Theorem 3.1 is proved. □

Acknowledgements

The authors express their gratitude to the referee for a careful reading of the manuscript and many valuable suggestions which highly improve the quality of this paper.

References

[1] A. Ivić. The Riemann zeta-function: theory and applications. Oversea Publishing House, 2003.

[2] M. V. Subbarao. On some arithmetic convolutions. In: The Theory of Arithmetic Functions. Lecture Notes in Mathematics. Vol. 251, Springer, 1972, 247-271.

[3] L. Tóth. An order result for the exponential divisor function. Publ. Math. Debrecen 71 (2007), no. 1-2, 165 - 171.

[4] J. Wu. Problème de diviseurs exponentiels at entiers exponentiellement sans facteur carré. J. Théor Nombres Bordeaux 7 (1995), no. 1, 133C141.

[5] L. Zhang, M. Lü and W. Zhai. On the Smarandache ceil function and the Dirichlet divisor function. Sci. Magna 4 (2008), no. 4, 55 - 57.

Scientia Magna
Vol. 12 (2017), No. 1, 7-13

The mean value of $P^*(n)$ over cube-full numbers

Ping Song[1] and Jing Huang[2]

[1]School of Mathematics and Statistics, Shandong Normal University
Shandong Jinan, China
E-mail: songpingsdnu@163.com
[2]School of Mathematics and Statistics, Shandong Normal University
Shandong Jinan, China
E-mail: huangjingsdnu@163.com

Abstract Let $n > 1$ be an integer, $P^*(n)$ be the unitary analogue of the gcd-sum function. In this paper, we consider the mean value of $P^*(n)$ over cube-full numbers, that is

$$\sum_{\substack{n \leq x \\ n\ is\ cube-full}} P^*(n) = \sum_{n \leq x} P^*(n) f_3(n),$$

where $f_3(n)$ is the characteristic function of cube-full integers, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full}, \\ 0, & \text{otherwise}. \end{cases}$$

Keywords divisor problem, Dirichlet convolution method, mean value.

An integer $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r}$ is called k-full number if all the exponents $a_1 \geq k, a_2 \geq k, \cdots, a_r \geq k$. When $k = 3$, n is called cube-full integers, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full}, \\ 0, & \text{otherwise}. \end{cases}$$

In 1972, M. V. Subbarao [4] gave the definition of the exponential divisor, i.e. $n > 1$ is an integer and $n = \prod_{i=1}^{r} p_i^{a_i}, d = \prod_{i=1}^{r} p_i^{c_i}$, if $c_i \mid a_i, i = 1, 2, \cdots, r$, then d is an exponential divisor of n. We denote $d \mid_e n$. Two integers $n, m > 1$ have common exponential divisors if they have the same prime factors and in this case, i.e. for $n = \prod_{i=1}^{r} p_i^{a_i}, m = \prod_{i=1}^{r} p_i^{b_i}, a_i, b_i \geq 1\ (1 \leq i \leq r)$, the greatest common exponential divisor of n and m is $(n, m)_e = \prod_{i=1}^{r} p_i^{(a_i, b_i)}$. Here $(1, 1)_e = 1$ by convention and $(1, m)_e$ does not exist for $m > 1$.

The integers $n, m > 1$ are called exponentially coprime, if they have the same prime factors and $(a_i, b_i) = 1$ for every $1 \leq i \leq r$, with the notation of above. In this case $(n, m)_e = S_r(n) = S_r(m)$. The function $S_r(n) = P_1 * \cdots * P_r$ can be found in the unsolved problem 63 (see [3]).

1 and $m > 1$ are not exponentially coprime. Let

$$P^*(n) = \sum_{k=1}^{n} (k, n)_*,$$

where $(k,n)_* := \max\{d \in \mathbb{N} : d \mid k, d \parallel n\}$, which was introduced by Tóth [5]. The function $P^*(n)$ is also multiplicative and $P^*(p^a) = 2p^a - 1$ for every prime power p^a $(a \geq 1)$.

Many authors have investigated the properties of the function $P^*(n)$, see [6] and [1]. Recently, L. Tóth [6] proved the following result:

$$\sum_{n \leq x} P^*(n) = \frac{\alpha}{2\zeta(2)} x^2 \log x + \beta x^2 + O(x^{\frac{3}{2}} \log x),$$

where $\alpha = \prod_p (1 - 1/(p+1)^2) \approx 0.775883, \alpha, \beta$ are constants.

The aim of this paper is to establish the following asymptotic formula for the mean value of the function $P^*(n)$ over cube-full numbers.

Theorem 0.1. *We have the asymptotic formula*

$$\sum_{n \leq x} P^*(n) = \frac{1}{4} x^{\frac{4}{3}} R_{1,1}(\log x) + \frac{1}{5} x^{\frac{5}{4}} R_{1,2}(\log x)$$
$$+ \frac{1}{6} x^{\frac{6}{5}} R_{1,3}(\log x) + O(x^{\frac{7}{6}} \exp(-D(\log x)^{\frac{3}{5}} (\log\log x)^{-\frac{1}{5}}),$$

where $R_{1,k}(t), k = 1, 2$ are polynomials of degree 1 in t, $D > 0$ is an absolute constant.

Notation. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

1 Some lemmas

Lemma 1.1. *Let $f(m), g(n)$ are arithmetical functions such that*

$$\sum_{m \leq x} f(m) = \sum_{j=1}^{J} x^{\alpha_j} P_j(\log x) + O(x^{\alpha}),$$
$$\sum_{n \leq x} \mid g(n) \mid = O(x^{\beta}),$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_J > \alpha > \beta > 0, P_j(t)$ are polynomials in t. If $h(n) = \sum_{n=md} f(m)g(d)$ then

$$\sum_{n \leq x} h(n) = \sum_{j=1}^{J} x^{\alpha_j} Q_j(\log x) + O(x^{\alpha}),$$

where $Q_j(t)$ are polynomials in t, $(j = 1, \cdots J)$.

Proof. This is Theorem 14.1 of Ivić [2]. □

Lemma 1.2. *Let $f(n)$ be an arithmetical function for which*

$$\sum_{n\leq x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O(x^a),$$
$$\sum_{n\leq x} | f(n) |= O(x^{a_1}(\log x)^r),$$

where $a_1 \geq a_2 \geq \cdots \geq a_l > \frac{1}{c} > a \geq 0, r \geq 0, P_j(t)$ are polynomials in t of degrees not exceeding r, $j = 1, \cdots J$, and $c \geq 1, b \geq 1$ are fixed integers. Suppose for $\Re s > 1$ that

$$\sum_{n=1}^{\infty} \frac{\mu_d(n)}{n^s} = \frac{1}{\zeta^b(s)},$$

if

$$h(n) = \sum_{d^c|n} \mu_b(d) f(n/d^c),$$

then

$$\sum_{n\leq x} h(n) = \sum_{j=1}^{l} x^{a_j} R_j(\log x) + E_c(x),$$

where $R_j(t)$ are polynomials in t of degrees not exceeding r, $(j = 1, \cdots l)$, and for some $D > 0$,

$$E_c(x) \ll x^{\frac{1}{c}} \exp(-D(\log x)^{\frac{3}{5}} (\log\log x)^{-\frac{1}{5}}).$$

Proof. See Theorem 14.2 of Ivić [2]. □

Lemma 1.3. *Let $P'(n) = \frac{P^*(n)}{n}, \Re s > 1$, we have*

$$\sum_{\substack{n=1 \\ n \text{ is cube-full}}}^{\infty} \frac{P'(n)}{n^s} = \frac{\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)}{\zeta(6s)} G(s),$$

where the Dirichlet series $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{7}$.

Proof.

$$\sum_{\substack{n=1 \\ n \text{ is cube-full}}}^{\infty} \frac{P'(n)}{n^s} = \sum_{n=1}^{\infty} \frac{P'(n)f_3(n)}{n^s}$$
$$= \prod_p \left(1 + \frac{P'(p^3)f_3(p^3)}{p^{3s}} + \frac{P'(p^4)f_3(p^4)}{p^{4s}} + ... + \frac{P'(p^r)f_3(p^r)}{p^{rs}}\right)$$
$$= \prod_p \left(1 + \frac{2}{p^{3s}} + \frac{2}{p^{4s}} + \frac{2}{p^{5s}} + \frac{2}{p^{6s}} + \frac{2}{p^{7s}} - \frac{1}{p^{3+3s}} - \frac{1}{p^{4+4s}} + \cdots\right)$$
$$= \zeta(3s) \prod_p \left(1 + \frac{1}{p^{3s}} + \frac{2}{p^{4s}} + \frac{2}{p^{5s}} + \cdots\right)$$

$$
\begin{aligned}
&= \zeta^2(3s)\prod_p\left(1+\frac{2}{p^{4s}}+\frac{2}{p^{5s}}-\frac{1}{p^{6s}}-\frac{2}{p^{7s}}+\cdots\right)\\
&= \zeta^2(3s)\zeta(4s)\prod_p\left(1+\frac{1}{p^{4s}}+\frac{2}{p^{5s}}-\frac{1}{p^{6s}}-\frac{2}{p^{7s}}+\cdots\right)\\
&= \zeta^2(3s)\zeta^2(4s)\prod_p\left(1++\frac{2}{p^{5s}}-\frac{1}{p^{6s}}-\frac{2}{p^{7s}}+\cdots\right)\\
&= \zeta^2(3s)\zeta^2(4s)\zeta(5s)\prod_p\left(1++\frac{1}{p^{5s}}-\frac{1}{p^{6s}}-\frac{2}{p^{7s}}+\cdots\right)\\
&= \zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\prod_p\left(1-\frac{1}{p^{6s}}-\frac{2}{p^{7s}}+\cdots\right)\\
&= \frac{\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)}{\zeta(6s)}G(s),
\end{aligned}
$$

where $G(s) := \sum_{n=1}^{\infty}\frac{g(n)}{n^s} = \prod_p(1-\frac{2}{p^{7s}}+\cdots)$, which is absolutely convergent for $\Re s > \frac{1}{7}$, and

$$\sum_{n\le x} \mid g(n) \mid \ll x^{\frac{1}{7}+\epsilon}.$$

Lemma 1.4.

$$\sum_{m\le x} d(3,3,4,4,5,5;m) = x^{\frac{1}{3}}P_{1,1}(\log x) + x^{\frac{1}{4}}P_{1,2}(\log x) + x^{\frac{1}{5}}P_{1,3}(\log x) + O(x^{\sigma_0+\epsilon}),$$

where $\sigma_0 = \frac{100699}{665232} = 0.15137426\cdots, P_{1,k}(t)$ *are polynomials of degree 1 in* $t, k = 1, 2.$

Proof. By perron's formula, we have

$$
\begin{aligned}
S(x) &= \sum_{n\le x}\delta(n)d(n)\\
&= \frac{1}{2\pi i}\int_{b-iT}^{b+iT}\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\frac{x^s}{s}ds + O(\frac{x^{\frac{1}{3}+\epsilon}}{T}),
\end{aligned}
$$

where $b = \frac{1}{3}+\epsilon, T = x^c, c$ is a very large number of fixed numbers. $\frac{1}{7} < \sigma_0 < \frac{1}{6}$. According to Residue's theorem, we have

$$
\begin{aligned}
S(x) &= x^{\frac{1}{3}}(P_3(\log x)) + x^{\frac{1}{4}}(P_4(\log x)) + x^{\frac{1}{5}}(P_5(\log x)) + I_1 + I_2 + I_3 + O(1),\\
I_1 &= \frac{1}{2\pi i}\int_{b-iT}^{\sigma_o-iT}\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\frac{x^s}{s}ds,\\
I_2 &= \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\frac{x^s}{s}ds,\\
I_3 &= \frac{1}{2\pi i}\int_{\sigma_0+iT}^{b+iT}\zeta^2(3s)\zeta^2(4s)\zeta^2(5s)\frac{x^s}{s}ds.
\end{aligned}
$$

Since $\sigma_0 > \frac{7}{43}$, we have

$$\begin{aligned} I_1 + I_3 &\ll \int_{\sigma_0}^{\frac{1}{3}+\epsilon} \mid \zeta(3\sigma + i3T) \mid^2 \mid \zeta(4\sigma + i4T) \mid^2 \mid \zeta(5\sigma + i5T) \mid^2 x^\sigma T^{-1} d\sigma \\ &\ll T^{-1} \left(\int_{\sigma_0}^{\frac{1}{5}} + \int_{\frac{1}{5}}^{\frac{1}{4}} + \int_{\frac{1}{4}}^{\frac{1}{3}} + \int_{\frac{1}{3}}^{\frac{1}{3}+\epsilon} \right) \\ &\quad \mid \zeta(3\sigma + i3T) \mid^2 \mid \zeta(4\sigma + i4T) \mid^2 \mid \zeta(5\sigma + i5T) \mid^2 x^\sigma d\sigma \\ &\ll x^{\frac{1}{6}} T^{-\delta+\epsilon} + x^{\frac{1}{5}} T^{-\frac{3}{5}+\epsilon} + x^{\frac{1}{4}} T^{-\frac{5}{6}+\epsilon} + x^{\frac{1}{3}} T^{-1+\epsilon} + x^{\frac{1}{3}+\epsilon} T^{-1} \\ &\ll x^{\frac{1}{3}+\epsilon} T^{-\delta+\epsilon}, \end{aligned}$$

where δ is very small normal number, $\delta > \epsilon$,

$$I_2 \ll x^{\sigma_0}(1 + \int_1^T (\mid \zeta(3\sigma + i3T) \mid^2 \mid \zeta(4\sigma + i4T) \mid^2 \mid \zeta(5\sigma + i5T) \mid^2 t^{-1} dt).$$

According to the partial integral formula, only the formula is proved,

$$I_4 = \int_1^T (\mid \zeta(3\sigma + i3T) \mid^2 \mid \zeta(4\sigma + i4T) \mid^2 \mid \zeta(5\sigma + i5T) \mid^2 dt) \ll T^{1+\epsilon}.$$

If $p_i > 0 (i = 1, 2, 3)$ are real number, and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, by Hölder inequality, we have

$$I_4 \ll (\int_1^T \mid \zeta(3\sigma + i3T) \mid^{2p_1} dt)^{\frac{1}{p_1}} (\int_1^T \mid \zeta(4\sigma + i4T) \mid^{2p_2} dt)^{\frac{1}{p_2}} (\int_1^T \mid \zeta(5\sigma + i5T) \mid^{2p_3} dt)^{\frac{1}{p_3}}.$$

So, we have to prove

$$\begin{aligned} \int_1^T \mid \zeta(3\sigma_0 + i3t) \mid^{2p_1} dt &\ll T^{1+\epsilon}, \\ \int_1^T \mid \zeta(4\sigma_0 + i4t) \mid^{2p_2} dt &\ll T^{1+\epsilon}, \\ \int_1^T \mid \zeta(5\sigma_0 + i5t) \mid^{2p_3} dt &\ll T^{1+\epsilon}. \end{aligned}$$

Let $m(3\sigma_0) = 2p_1, m(4\sigma_0) = 2p_2, m(5\sigma_0) = 2p_3$, since $\frac{2}{m(3\sigma_0)} + \frac{2}{m(4\sigma_0)} + \frac{2}{m(5\sigma_0)} = 1$, we have $\sigma_0 = \frac{100699}{665232} = 0.15137426\cdots$

2 Proof of Theorem 0.1

Let

$$\begin{aligned} \zeta^2(3s)\zeta^2(4s)\zeta^2(5s)G(s) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \Re s > 1, \\ \zeta^2(3s)\zeta^2(4s)\zeta^2(5s) &= \sum_{n=1}^{\infty} \frac{d(3,3,4,4,5,5;n)}{n^s}, \end{aligned}$$

such that

$$f(n)=\sum_{n=md} d(3,3,4,4,5,5;m)g(d). \tag{2.1}$$

From Lemma1.4 and the definition of $d(3,3,4,4,5,5;m)$ we get

$$\sum_{m\le x} d(3,3,4,4,5,5;m)=x^{\frac{1}{3}}P_{1,1}(\log x)+x^{\frac{1}{4}}P_{1,2}(\log x)+x^{\frac{1}{5}}P_{1,3}(\log x)+O(x^{\sigma_0+\epsilon}), \tag{2.2}$$

where $P_{1,k}(t)$ are polynomials of degree 1 in $t, k=1,2$.

In addition we have

$$\sum_{n\le x} \mid g(n)\mid = O(x^{\frac{1}{7}+\epsilon}). \tag{2.3}$$

Combining (2.1),(2.2) and (2.3), and applying Lemma 1.2, we have

$$\sum_{n\le x} f(n)=x^{\frac{1}{3}}Q_{1,1}(\log x)+x^{\frac{1}{4}}Q_{1,2}(\log x)+x^{\frac{1}{5}}Q_{1,3}(\log x)+O(x^{\sigma_0+\epsilon}), \tag{2.4}$$

where $Q_{1,1}(t), Q_{1,2}(t)$ are polynomials of degrees 1 in t, then we can get

$$\sum_{n\le x} \mid f(n)\mid \ll x^{\frac{1}{3}}\log x. \tag{2.5}$$

Since $\frac{1}{\zeta(6s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{6s}}, \Re s>\frac{1}{6}$, from Lemma 1.4 and (3.1) we have the relation

$$P'(n)f_2(n)=\sum_{n=md^6} f(m)\mu(d). \tag{2.6}$$

From (2.4), (2.5) and (2.6), in view of Lemma 1.3, we can get

$$\begin{aligned}\sum_{\substack{n\le x\\ n\ is\ cube-full}} P'(n) &= x^{\frac{1}{3}}R_{1,1}(\log x)+x^{\frac{1}{4}}R_{1,2}(\log x)+x^{\frac{1}{5}}R_{1,3}(\log x)\\ &+O(x^{\frac{1}{6}}\exp(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}).\end{aligned} \tag{2.7}$$

From the definition of $P'(n)$ and Abel's summation formula, we can easily get

$$\begin{aligned}\sum_{\substack{n\le x\\ n\ is\ cube-full}} P^*(n) &= \sum_{\substack{n\le x\\ n\ is\ cube-full}} P'(n)n\\ &=\int_1^x td\left(\sum_{\substack{n\le x\\ n\ is\ cube-full}} P'(n)\right)\\ &=\frac{1}{4}x^{\frac{4}{3}}R_{1,1}(\log x)+\frac{1}{5}x^{\frac{5}{4}}R_{1,2}(\log x)+\frac{1}{6}x^{\frac{6}{5}}R_{1,3}(\log x)\\ &+O(x^{\frac{7}{6}}\exp(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}),\end{aligned}$$

where $R_{1,k}=1,2$ are polynomials of degree 1 in $t, D>0$ is an absolute constant.

Then, we complete the proof of Theorem 0.1.

Acknowledgements

The authors express their gratitude to the referee for a careful reading of the manuscript and many valuable suggestions which highly improve the quality of this paper.

References

[1] S. Chen and W. Zhai. Reciprocals of the Gcd-Sum functions. J. Integer Seq. 14 (2011), no. 8, Article 11.8.3,13 pp.

[2] A. Ivić. The Riemann zeta-function: theory and applications. John Wiley and Sons, New York, 1985, 387 - 393.

[3] F. Smarandache. Only problems, Not solutions! Xiquan Publishing House, Chicago, 1993.

[4] M. V. Subbarao. On some arithmetic convolutions. In: The Theory of Arithmetic Functions. Lecture Notes in Mathematics. Vol. 251, Springer, 1972, 247-271.

[5] L. Tóth. The unitary analogue of Pillai's arithmetical function. Collect. Math. 40 (1989), no. 1, 19C30.

[6] L. Tóth. The unitary analogue of Pillai's arithmetical functionII. Notes Number Theory Discrete Math. 2 (1996), no. 2, 40-46.

Scientia Magna
Vol. 12 (2017), No. 1, 14-22

Some convex functions for Hermite-Hadamard integral inequalities

Opeyemi Omotoyinbo[1] and Adesanmi Mogbademu[2]

[1] Department of Mathematics, University of Lagos,
Akoka, Lagos, Nigeria.
E-mail: opetoyinbo@yahoo.com
[2]Department of Mathematics, University of Lagos,
Akoka, Lagos, Nigeria
E-mail: amogbademu@unilag.edu.ng

Abstract In this paper, two different classes of convex functions were employed to obtain some new integral inequalities of Hermite-Hadamard type. Our method of proofs is of independent interest. Moreover, few applications of our results to special means were considered at the end.

Keywords Hermite-Hadamard, integral inequalities, m-Godunova-Levin function, m-MT-convex function.
2010 Mathematics Subject Classification 26D07, 26D10, 26D99.

§1. Introduction and preliminaries

Theorem 1.1. *Let $f : I \subseteq R \to R$ be a convex function defined on the interval $I = [a, b]$ of the real numbers and let $a, b \in [c, d]$ where $a < b$. Then,*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The above double inequality, which we can say is the first fundamental result for convex functions with a natural geometric applications, is known in literature as the Hermite-Hadamard inequality. Furthermore, the inequality has several updates for different types of convex functions in literature (See [1], [8], [9], [10], [12] and [13]).

Definition 1.1.[2] *A function $f : I \to R$ is said to be convex, if for every $x, y \in I$ and $t \in [0, 1]$, we have*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \tag{1.2}$$

Definition 1.2.[5] *A function $f : [0, b] \to R$ is said to be m-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$, and $t \in [0, 1]$, we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \tag{1.3}$$

Remark 1.1. *For $m = 1$, we recapture the concept of convex functions defined on $[0, b]$ and for $m = 0$, the concept of starshapped function defined on $[0, b]$ is obtained. Recall that $f : [0, b] \to R$ is starshapped if*

$$f(tx) \leq tf(x), \tag{1.4}$$

for all $t \in [0, 1]$ and $x \in [0, b]$.

Definition 1.3.[4] *A function $f : I \to R$ is Godunova-Levin or said to belong to the class $Q(I)$ if f is nonnegative for all $x, y \in I$ and $t \in (0, 1)$ satisfies the inequality*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}. \tag{1.5}$$

Definition 1.4.[12] *A function $f : I \subseteq R \to R$ is said to belong to the class $MT(I)$ if f is nonnegative $\forall x, y \in I$ and $t \in (0, 1)$ satisfies the inequality*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.6}$$

Recently, Omotoyinbo and Mogbademu [9] introduced and defined two new classes of convex functions as follows:

Definition 1.5.[9] *A function $f : I \subseteq R \to R$ is m-Godunova-Levin or said to belong to the class $m - Q(I)$ if f is nonnegative for all $x, y \in I$ and $t \in (0, 1)$ with $m \in (0, 1]$ satisfies the inequality*

$$f(tx + m(1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{m(1-t)}. \tag{1.7}$$

Definition 1.6.[9] *A function $f : I \subseteq R \to R$ is said to belong to the class $m - MT(I)$ if f is nonnegative $\forall x, y \in I$ and $t \in (0, 1)$ with $m \in [0, 1]$ satisfies the inequality*

$$f(tx + m(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.8}$$

Dragomir et al. [3] obtained the following two new inequalities of Hadamard-type for a class of Godunova-Levin functions.

Theorem 1.2.[3] *Let $f \in Q(I), a, b \in I$ with $a < b$ and $f \in L_1[a, b]$. Then, one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a} \int_a^b f(x)dx$$

and

$$\frac{1}{b-a} \int_a^b p(x)f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.9}$$

where $p(x) = \frac{(b-a)(x-a)}{(b-a)^2}, x \in [a, b]$.

In [12], Tunç and Yildirim improved on the work of Dragomir et al. [3], and obtained the following two new inequalities of Hermite-Hadamard type for the class of MT-convex functions.

Theorem 1.3.[3] *Let $f \in MT(I), a, b \in I$ with $a < b$ and $f \in L_1[a,b]$. Then, one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\frac{2}{b-a}\int_a^b \tau(x)f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.10}$$

where $\tau(x) = \frac{\sqrt{(b-a)(x-a)}}{b-a}, x \in [a,b]$.

Theorem 1.4.[12] *Let $f \in MT(I)$, $f \in L_1[a,b]$, where $a, b \in I$ and $a < b$. Then,*

$$\frac{\pi}{2} f\left(\frac{a+b}{2}\right) \leq f(a) + f(b). \tag{1.11}$$

Recently, Tunç et al. [13], proved the following theorems:

Theorem 1.5.[13] *Let $f : [a,b] \subseteq R \to R$ nonnegative MT-convex function and $f \in L_1[a,b]$. Then,*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{\pi}{4}(f(a)+f(b)). \tag{1.12}$$

Theorem 1.6.[13] *Let $f, g \in [a,b] \to R$ two nonnegative MT-convex functions and $f, g \in L_1[a,b]$. Then,*

$$\frac{8}{3} fg\left(\frac{a+b}{2}\right) \leq M(a,b) + N(a,b), \tag{1.13}$$

where $M(a,b) = f(a)g(a) + f(b)g(b), N(a,b) = f(a)g(b) + f(b)g(a)$.

More recently, the authors [8] obtained the following two inequalities of Hermite-Hadamard type for classes of Godunova-Levin and MT-convex functions. The results obtained are independent of Tunç et al. [13].

Theorem 1.7.[8] *Let $f, g : [a,b] \to R$ be two functions with $a, b \in [0, \infty)$, where $a < b, I = [a,b]$ and $f, g \in L_1[a,b]$. If $f \in Q(I)$ and $g \in MT(I)$. Then, the following inequality holds;*

$$\frac{32}{15\pi} fg\left(\frac{a+b}{2}\right) \leq M(a,b) + N(a,b), \tag{1.14}$$

where $M(a,b) = f(a)g(a) + f(b)g(b), N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 1.8.[8] *Let $a, b \in [0, \infty), a < b, I = [a,b]$ with $f, g : [a,b] \to R$ be two functions and $f, g \in L_1[a,b]$. If $f \in Q(I)$ and $g \in MT(I)$. Then, the following inequality holds*

$$\frac{1}{b-a}\int_a^b \mu(x)f(x)g(x)dx \leq \frac{1}{2}\left(\frac{3\pi}{128}(f(a)g(a)+f(b)g(b)) + \frac{5\pi}{128}(f(a)g(b)+f(b)g(a))\right), \tag{1.15}$$

where $\mu(x) = \frac{(b-x)^2(x-a)^2}{(b-a)^4}, x \in [a,b]$.

In this paper, using an analytical approach, we established some new inequalities involving two different kinds of convex functions: m-Godunova-Levin and m-MT-convex functions. Our results generalize some well-known results in this area of research.

§2. Main results

Theorem 2.1. *Let* $f, g : [a, b] \subseteq R \to R$ *be two nonnegative* m-MT*-convex functions and* $f, g \in L_1[a, b]$ *where* $a, b \in I$ *and* $a < b$. *Then,*

$$\frac{2}{3} mfg\left(\frac{a+b}{2}\right) \le \left(\frac{m+1}{2}\right)(M(a,b)+N(a,b)),$$

where $M(a, b) = f(a)g(a) + f(b)g(b), N(a, b) = f(a)g(b) + f(b)g(a)$.

Proof. Since f and g are m-Q(I)-convex, we can write

$$\begin{aligned} f\left(\tfrac{a+b}{2}\right) &\le \tfrac{f(a)}{t} + \tfrac{f(b)}{m(1-t)} \\ &\le \tfrac{1}{2}\left(\tfrac{1}{t} + \tfrac{1}{m(1-t)}\right)(f(a)+f(b)), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} g\left(\tfrac{a+b}{2}\right) &\le \tfrac{g(a)}{t} + \tfrac{g(b)}{m(1-t)} \\ &\le \tfrac{1}{2}\left(\tfrac{1}{t} + \tfrac{1}{m(1-t)}\right)(g(a)+g(b)). \end{aligned} \tag{2.2}$$

Multiplying (2.1) and (2.2) together to obtain

$$fg\left(\frac{a+b}{2}\right) \le \frac{1}{4}\left(\frac{t+m(1-t)}{t(1-t)}\right)(f(a)+f(b))(g(a)+g(b)). \tag{2.3}$$

Observe that inequality (2.3) becomes

$$4mt(1-t)fg\left(\frac{a+b}{2}\right) \le (f(a)+f(b))(g(a)+g(b))(t+m(1-t))dt. \tag{2.4}$$

Integrating both sides of (2.4) over $[0, 1]$, we get

$$4mfg\left(\frac{a+b}{2}\right)\int_0^1 t(1-t)dx \le (f(a)+f(b))(g(a)+g(b))\int_0^1 (t+m(1-t))dt. \tag{2.5}$$

Further simplification of (2.5) completes the proof. □

Theorem 2.2. *Let* $g \in m - MT(I), a, b \in I$ *with* $a < b$ *and* $f \in L_1[a, b]$. *Then,*

$$f\left(\frac{a+mb}{2}\right) \le \frac{1}{2}\left(\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-a}\int_a^b f(x)dx\right). \tag{2.6}$$

Proof. Since $f \in m$-MT-convex, and $\forall x, y \in I$,

$$f(tx+m(1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{2.7}$$

with $t = \frac{1}{2}$ in (2.7),

$$f\left(\frac{a+mb}{2}\right) \le \frac{f(x)+mf(y)}{2}, \tag{2.8}$$

Now, set $x = ta + m(1-t)b$ and $y = m(1-t)a + tb$, then (2.8) becomes

$$f\left(\frac{a+mb}{2}\right) \le \frac{1}{2}\left(f(ta+m(1-t)b) + f(m(1-t)a+tb)\right). \tag{2.9}$$

By integrating both sides of (2.9) over $[0,1]$, we obtain

$$f\left(\frac{a+mb}{2}\right) \leq \frac{1}{2}\left(\int_0^1 f(ta+m(1-t)b)dt + \int_0^1 f(m(1-t)a+tb)dt\right). \tag{2.10}$$

It is easy to see that

$$\int_0^1 f(ta+m(1-t)b)dt = \frac{1}{mb-a}\int_a^{mb} f(x)dx$$

and

$$\int_0^1 f(m(1-t)a+tb)dt = \frac{1}{b-a}\int_{ma}^{b} f(x)dx. \tag{2.11}$$

Thus, substituting (2.11) into (2.10),the result follows immediately. □

Remarks 2.1. *(i) If we choose $m=1$ in (2.6), we obtain*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx,$$

which is the first part of Hadamard's inequality (1.1) and Theorem 1.3 of Tunç and Yildirim[12].

(ii) If $g \in m-Q(I)$, then Theorem 2.2 also holds.

Theorem 2.3. *Let $f : [a,b] \subseteq R \to R$ be a nonnegative m-MT-convex function and $f \in L_1([a,b])$ with $a,b \in I$ and $a<b$. Then*

$$\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-a}\int_{ma}^{b} f(x)dx \leq \frac{\pi}{4}(m+1)(f(a)+f(b)).$$

Proof. Since $f \in m-MT(I)$,

$$f(ta+m(1-t)b) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(a) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(b), \tag{2.12}$$

and

$$f(tb+m(1-t)a) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(b) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(a). \tag{2.13}$$

Adding (2.12) and (2.13), gives

$$f(ta+m(1-t)b) + f(tb+m(1-t)a) \leq \frac{1}{2}\left(\frac{t}{\sqrt{t}\sqrt{1-t}} + \frac{m(1-t}{\sqrt{t}\sqrt{1-t}}\right)(f(a)+f(b))$$

$$\leq \frac{1}{2}(f(a)+f(b))\int_0^1 (t^{\frac{1}{2}}(1-t)^{\frac{1}{2}} + mt^{\frac{1}{2}}(1-t)^{\frac{1}{2}})dt. \tag{2.14}$$

On substituting $x = ta+m(1-t)b$ (i.e. $dx = (a-mb)dt$) in (2.14), then the result follows immediately. □

Remarks 2.2. *If $m = 1$ in Theorem 2.3, we have*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{\pi}{4}(f(a)+f(b)),$$

which is Theorem 2.3 of Tunç et al. [13].

Theorem 2.4. *Let $f, g : [a,b] \subseteq R \to R$ be two nonnegative m-MT-convex functions and $f, g \in L_1([a,b])$ with $a, b \in I$ and $a < b$. Then*

$$8fg\left(\frac{a+b}{2}\right) \leq (m^2+m+1)(M(a,b)+N(a,b)).$$

Proof. Since f and g are m-MT-convex functions, we can write

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(a) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(b) \\ &\leq \frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{m\sqrt{1-t}}{2\sqrt{t}}\right)(f(a)+f(b)), \qquad (2.15) \\ g\left(\frac{a+b}{2}\right) &\leq \frac{\sqrt{t}}{2\sqrt{1-t}}g(a) + \frac{m\sqrt{1-t}}{2\sqrt{t}}g(b) \\ &\leq \frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{m\sqrt{1-t}}{2\sqrt{t}}\right)(g(a)+g(b)). \qquad (2.16) \end{aligned}$$

By multiplying (2.15) and (2.16) ,we obtain

$$\begin{aligned} fg\left(\frac{a+b}{2}\right) &\leq \left(\frac{1}{4}\left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{m\sqrt{1-t}}{2\sqrt{t}}\right)\right)^2 (f(a)+f(b))(g(a)+g(b)) \\ &\leq \frac{1}{16}\left(\frac{m^2(1-t)^2+2mt(1-t)+t^2}{t(1-t)}\right)(f(a)+f(b))(g(a)+g(b)). \qquad (2.17) \end{aligned}$$

It can be easily seen that (2.17) gives

$$16t(1-t)fg\left(\frac{a+b}{2}\right) \leq (m^2(1-t)^2+2mt(1-t)+t^2)(f(a)+f(b))(g(a)+g(b)). \qquad (2.18)$$

Integrating both sides of (2.18) over $[0,1]$, the proof is completed. □

Remarks 2.3. *If $m = 1$ in Theorem 2.4, we obtain*

$$\frac{8}{3}fg\left(\frac{a+b}{2}\right) \leq M(a,b)+N(a,b),$$

where $M(a,b) = f(a)g(a)+f(b)g(b), N(a,b) = f(a)g(b)+f(b)g(a)$, which is Theorem 2.4 of Tunç et al. [13].

Theorem 2.5. *If $m = 1$ in Theorem 2.4, we obtain*

$$\frac{8}{3}fg\left(\frac{a+b}{2}\right) \leq M(a,b)+N(a,b),$$

where $M(a,b) = f(a)g(a)+f(b)g(b), N(a,b) = f(a)g(b)+f(b)g(a)$, which is Theorem 2.4 of Tunç et al. [13].

Proof. Since $f \in m - Q(I)$ and g is $m - MT$-convex, we have

$$f(ta + m(1-t)b) \leq \frac{1}{t}f(a) + \frac{1}{m(1-t)}f(b), \tag{2.19}$$

$$g(ta + m(1-t)b) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}g(a) + \frac{m\sqrt{1-t}}{2\sqrt{t}}g(b). \tag{2.20}$$

Multiplying (2.19) and (2.20), we get

$$f(ta + m(1-t)b)g(ta + m(1-t)b)$$

$$\leq \frac{1}{2}\left(\frac{\sqrt{t}\sqrt{1-t}}{t(1-t)}f(a)g(a) + \frac{m\sqrt{t}\sqrt{1-t}}{t^2}f(a)g(b) + \frac{\sqrt{t}\sqrt{1-t}}{m(1-t)^2}f(b)g(a) + \frac{\sqrt{t}\sqrt{1-t}}{t(1-t)}f(b)g(b)\right). \tag{2.21}$$

If both sides of inequality (2.21) are multiplied by $mt^2(1-t)^2$, we have

$$m\int_0^1 t^2(1-t)^2\left(f(ta + m(1-t)b)(g(ta + m(1-t)b)\right)dt$$

$$\leq \frac{1}{2}(mf(a)g(a)\int_0^1 t(1-t)\sqrt{t}\sqrt{1-t}dt + mf(a)g(b)\int_0^1 (1-t)^2\sqrt{t}\sqrt{1-t}dt$$

$$+f(b)g(a)\int_0^1 t^2\sqrt{t}\sqrt{1-t}dt + mf(b)g(b)\int_0^1 t(1-t)\sqrt{t}\sqrt{1-t}dt). \tag{2.22}$$

Substituting $x = ta + (1-t)b$ and simplifying completely, inequality (2.22) gives:

$$\frac{m(mb-x)^2(x-a)^2}{(mb-a)^5}\int_a^b f(x)g(x)dx$$

$$\leq \frac{1}{2}\left(\frac{3m\pi}{128}(f(a)g(a) + f(b)g(b)) + \frac{5\pi}{128}(mf(a)g(b) + f(b)g(a))\right).$$

The proof is completed. □

Remarks 2.4. *If we set $m = 1$ in Theorem 2.5 above, we obtain Theorem 2.1 of Omotoyinbo and Mogbademu* [8].

§3. Applications

We now consider few applications of our results to the following special means of real numbers.

The Arithmetic Mean: $A = A(a,b) = \frac{(a+b)}{2},\ a,b \leq 0.$

The Geometric Mean: $G = G(a,b) = \sqrt{ab},\ a,b \leq 0.$

The Harmonic Mean: $H = H(a,b) = \frac{2ab}{a+b},\ a,b \leq 0.$

The following propositions hold:

Proposition 3.1. *If we set $m = 1$ in Theorem 2.5 above, we obtain Theorem 2.1 of Omotoyinbo and Mogbademu* [8].

Proof. If we set $m = 1$ in Theorem 2.5 and choose $x = \frac{a+b}{2}$, we obtain

$$\frac{(\frac{b-a}{2})^2 + (\frac{b-a}{2})^2}{(b-a)^4} \left(\frac{1}{b-a}\right) \frac{1}{(\frac{a+b}{2})^{2k}} \int_a^b dx \leq \frac{1}{256} \left(3\pi \left(\frac{1}{a^{2k}} + \frac{1}{b^{2k}}\right) + 5\pi \left(\frac{1}{a^k b^k} + \frac{1}{b^k a^k}\right)\right). \tag{3.1}$$

Simplifying (3.1), we have

$$\frac{1}{(\frac{a+b}{2})^{2k}} \leq \frac{1}{16} \left(\frac{3\pi(a^{2k} + b^{2k}) + 5\pi.2(ab)^k}{(ab)^{2k}}\right).$$

On further simplification, we get

$$\left(\frac{2ab}{a+b}\right)^{2k} \leq \frac{1}{16} \left(3\pi(a^{2k} + b^{2k}) + 5\pi.2(ab)^k\right). \tag{3.2}$$

Substituting into inequality (3.2), the relation

$$0 \leq (a+b)^2 \Longrightarrow ab \leq \frac{a^2 + b^2}{2} \Longrightarrow (ab)^p \leq \frac{a^{2k} + b^{2k}}{2},$$

we obtain

$$\left(\frac{2ab}{a+b}\right)^{2k} \leq \pi \left(\frac{a^{2k} + b^{2k}}{2}\right),$$

implying that

$$(H(a,b))^{2k} \leq \pi A(a^{2k}, b^{2k}).$$

Hence, the result is completed. □

Proposition 3.2. *Let $f(x) = g(x) = x^k$, then*

$$(A(a,b))^{2k} \leq \gamma G^2(a^k, b^k),$$

where $\gamma > 0, k \in (0, \frac{1}{1000})$.

Proof. If we set $m = 1$ in Theorem 2.5 and choose $x = \frac{a+b}{2}$, we obtain

$$\frac{(\frac{b-a}{2})^2 + (\frac{b-a}{2})^2}{(b-a)^4} \left(\frac{1}{b-a}\right) \left(\frac{a+b}{2}\right)^{2k} \int_a^b dx \leq \frac{1}{256} \left(3\pi(a^{2k} + b^{2k}) + 5\pi(2a^k b^k)\right). \tag{3.3}$$

Simplifying (3.3), we have;

$$\frac{1}{16} \left(\frac{a+b}{2}\right)^{2k} \leq \frac{1}{256} \left(3\pi(a^{2k} + b^{2k}) + 5\pi(2a^k b^k)\right). \tag{3.4}$$

By recalling and substituting the following standard equalities in (3.4)

$$A(a,b) = \frac{a+b}{2} \Longrightarrow (A(a,b))^{2k} = \left(\frac{a+b}{2}\right)^{2k},$$

$$A(a^{2p}, b^{2p}) = \frac{a^{2k} + b^{2k}}{2} \Longrightarrow 2A(a^{2k}, b^{2k}) = a^{2k} + b^{2k},$$

$$G(a,b) = \sqrt{ab} \Longrightarrow G^2(a^k, b^k) = a^k b^k,$$

we have

$$\begin{aligned}(A(a,b))^{2k} &\leq \frac{1}{16}\left(3\pi.2A(a^{2k},b^{2k}+5\pi.2G^2(a^k,b^k)\right)\\ &\leq \frac{1}{16}\left(3\pi(A(a,b))^{2k}+10\pi G^2(a^k,b^k)\right)\\ &\leq \gamma G^2(a^k,b^k),\end{aligned}$$

where $\gamma = \frac{10\pi}{16-3\pi} > 0$. Hence, the proof is completed. □

References

[1] M. A. Ardic and M. E. Ozdemir. On some inequalities for different kinds of convexity. J. Ineq. and Appl. 3 (2012), 1C6.

[2] M. K. Bakula, J. Peari and M. Ribicic. Companion inequalities for m-convex and (, m)-Convex functions. J. Ineq. for Pure and Appl. Math. 7 (2006), no. 5, 194-205.

[3] S. S. Dragomir, J. Pecaric$\hat{c}$ and L. E. Persson. Some inequalities of Hadamard type. Soochow J. Math. 21 (1995), 241 - 335.

[4] S. S. Dragomir, and C. E. M Pearce. On Jensen inequality for a class of functions of Godunova and Levin. Period. Math. Hunger. 33 (1996), no. 2, 93 - 100.

[5] S. S. Dragomir and G. Toader. Some inequalities for m-convex functions. Studia Univ. Babes-Bolyai Math. 38 (1993), no. 1, 21 - 28.

[6] E. K. Godunova and V. I. Levin. Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkii. Vycisli-tel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva, 1985, 138 - 142.

[7] J. Hadamard. Etude Sur les properties des fontions entieres et en particular dune function consideree par Riemann. J. Math. Pures Appl. 58 (1893), 171 - 215.

[8] O. Omotoyinbo and A. Mogbademu. On some Hadamard inequality for Godunova-Levin and MT-Convex functions. J. the Nig. Assoc. of Mathematical Phys. 25 (2013), 215 - 222.

[9] O. Omotoyinbo and A. Mogbademu. Some new Hermite-Hadamard integral inequalities for convex functions. Int. J. of Sci. and Innovat. Tech. 1 (2014), no. 1, 1 - 12.

[10] O. Omotoyinbo, A. Mogbademu and P. Olanipekun. Integral inequalities of Hermite-Hadamard type for λ-MT-convex function. J. of Mathematical Sci. and Appl. e-notes 4 (2016), no. 2, 14 - 22.

[11] M. E. Ozdemir, A. Ekinci and A. O. Akdemir. Some new integral inequalites for several kinds of convex functions. Arxiv.org/pdf/1202.2003.

[12] M. Tunç and H. Yildirim. On MT-convexity. http://arxiv.org/pdf/1205.5453.

[13] M. Tunç, Y. Subas and I. Karabayir. On some Hadamard type inequalities for MT-convex functions. Int. J. Open Problems Comp. Math. 6 (2013), no. 2, 102 - 113.

Scientia Magna
Vol. 12 (2017), No. 1, 23-31

Study of the Jacobson radical for a certain class of entire Dirichlet series

Niraj Kumar[1] and Garima Manocha[2]

[1]Department of Mathematics, Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110078, India
E-mail: nirajkumar2001@hotmail.com
[2]School of Business Administration, Bhagwan Parshuram Institute of Technology
PSP-4, Dr KN Katju Marg, Sector 17 Rohini, Delhi-110089, India
E-mail: garima89.manocha@gmail.com

Abstract In the present paper, a class Y of entire functions represented by Dirichlet series which satisfies certain conditions is proved to be a Γ-ring. Further the properties of the Jacobson radical of Y are then studied.

Keywords Dirichlet series, Gamma ring, Semi-Simple Gamma ring, Jacobson radical.
2010 Mathematics Subject Classification 30B50, 17D20, 16N20.

§1. Introduction

Let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad s = \sigma + it \quad (\sigma, t \in \mathbb{R}). \tag{1}$$

If $a_n{}'s \in \mathbb{C}$ and $\lambda_n{}'s \in \mathbb{R}$ satisfy the condition $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n \ldots; \lambda_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty \tag{2}$$

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = K < \infty \tag{3}$$

then from [1] the Dirichlet series (1) represents an entire function.

Let X denote the set of all entire Dirichlet series (1) and Y be the set of series (1) for which $e^{c_1 n\lambda_n}(n!)^{c_2}|a_n|$ is bounded where $c_1, c_2 \geq 0$ and are simultaneously not zero. Let Γ be the set of series (1) for which $e^{c_1 n\lambda_n}|a_n|$ is bounded. Then by [1] every element of Y and Γ represents entire function. Clearly $\Gamma \subset Y \subset X$. By putting $c_1 = 1$, $c_2 = 0$, one obtains the condition of paper [2] that is $e^{n\lambda_n}|a_n|$ is bounded whereas $c_1 = 0$, $c_2 = 1$, gives the condition of paper [3] which is $(n!)\,|a_n|$ is bounded. If

$$x(s) = \sum_{n=1}^{\infty} x_n e^{\lambda_n s},\ \ f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s},\ \ \alpha(s) = \sum_{n=1}^{\infty} \alpha_n e^{\lambda_n s} \tag{4}$$

where $x(s) \in X$, $f(s) \in Y$ and $\alpha(s) \in \Gamma$, define the binary operations i.e. addition and multiplication in X x Γ x Y as-

$$x(s) + \alpha(s) + f(s) = \sum_{n=1}^{\infty} p_n \; e^{\lambda_n s},$$

$$x(s).\alpha(s).f(s) = \sum_{n=1}^{\infty} q_n \; e^{\lambda_n s},$$

where

$$p_n = x_n + \alpha_n + a_n \quad \text{and} \quad q_n = x_n.\alpha_n.a_n.$$

The following definitions are required to prove main results.

Definition 1. *Let M and Γ be two additive abelian groups. If there exists a mapping M X Γ X $M \to M$ such that for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ the conditions*

1. $(x + y)\alpha z = x\alpha z + y\alpha z$,
2. $x(\alpha + \beta)z = x\alpha z + x\beta z$,
3. $x\alpha(y + z) = x\alpha y + x\alpha z$,
4. $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied then M is called a Γ-ring.

An additive subgroup I of M is a left (right) ideal of M if $M\Gamma I \subset I$ ($I\Gamma M \subset I$). If I is both a left and a right ideal of M then I is a two-sided ideal or simply an ideal of M.

Definition 2. *A Γ-ring M is right primitive if*

1. *The right operator ring R of M is a right primitive ring.*
2. $M\Gamma x = 0$ *implies* $x = 0$.

M is a two-sided primitive Γ-ring if it is both left and right primitive. A Γ-ring M is said to be primitive if it has a Γ-faithful irreducible module.

Definition 3. *The additive group N is said to be a Γ-ring M-module if there is a Γ-mapping from N X Γ X $M \to N$ by $(n, \gamma, m) \to n\gamma m$ such that*

1. $n\gamma(a + b) = n\gamma a + n\gamma b$,
2. $(n_1 + n_2)\gamma a = n_1\gamma a + n_2\gamma a$,
3. $(n\gamma a)\delta b = n\gamma(a\delta b)$

for all $n, n_1, n_2 \in N$, $a, b \in M$ and $\gamma, \delta \in \Gamma$. For the sake of brevity drop Γ-ring in a Γ-ring M-module and refer it merely as an M-module.

A submodule of an M-module N is an additive subgroup S of N such that $S\Gamma M \subseteq S$. N is said to be an irreducible M-module if $N\Gamma M \neq (0)$ and if the only submodules of N are (0) and N.

A generalization of the concept of Γ-rings was done by Barnes in [4] where analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ-rings was obtained. Luh in [5] and [6] discussed results on primitive Γ-ring. Kyuno defined the simplicial radical of a Γ-ring M with both left and right unities to be the intersection of its maximal ideals in [9]. Luh in [10] extended the notions of simplicity and complete primeness to Γ-rings. Kyuno in [11] introduced the Γ-ring M-module and defined the Jacobson radical along with the ideas of irreducible modules. For all notions relevant to ring theory refer [7] and [8]. Very recently Kumar and Manocha in [13] considered the set of all Entire Dirichlet series which formed a

Γ-ring and established various results on prime one-sided ideals and socles for this set. Various results have been proved for different classes of entire Dirichlet series where few of them may be found in [14–17].

The purpose of the present paper is to introduce the notion of a Γ-ring Y-module and define the Jacobson radical $J(Y)$ along with the ideas of irreducible modules. The properties of $J(Y)$ and its relation with $J(R)$ is then studied where R denotes the right operator ring of Γ-ring Y. Later semi-simplicity is defined by $J(Y) = (0)$. Also the relation between semi-simple Y and semi-simple R is studied for the class Y of entire functions represented by Dirichlet series.

Clearly X and Y form Γ-ring. Let G be a free abelian group generated by the set of all ordered pairs $(\alpha(s), f(s))$ where $f(s) \in Y$ and $\alpha(s) \in \Gamma$. Let T be a subgroup of elements $\sum_i m_i(\alpha_i(s), f_i(s)) \in G$ where m_i are integers such that $\sum_i m_i.\{a(s).\alpha_i(s).f_i(s)\} = 0$ for all $a(s) \in Y$. Denote by R the factor group G/T and by $[\alpha(s), f(s)]$ the coset $(\alpha(s), f(s)) + T$. Clearly every element in R can be expressed as a finite sum $\sum_i [\alpha_i(s), f_i(s)]$. Also for all $f_1(s), f_2(s) \in Y$ and $\beta(s) \in \Gamma$

$$[\alpha(s), f_1(s)] + [\beta(s), f_1(s)] = [\alpha(s) + \beta(s), f_1(s)]$$

$$[\alpha(s), f_1(s)] + [\alpha(s), f_2(s)] = [\alpha(s), f_1(s) + f_2(s)].$$

Define the multiplication in R by

$$\sum_i [\alpha_i(s), f_i(s)].\sum_j [\beta_j(s), g_j(s)] = \sum_{i,j} [\alpha_i(s), f_i(s).\beta_j(s).g_j(s)]$$

Then R forms a ring. Furthermore Y is a right R-module with the definition

$$a(s).\sum_i [\alpha_i(s), f_i(s)] = \sum_i \{a(s).\alpha_i(s).f_i(s)\} \text{ for all } a(s) \in Y.$$

The ring R is called the right operator ring of Γ-ring Y. Similarly the left operator ring L of Y can also be defined.

§2. Main Results

In this section main results are proved.

The additive group X is said to be a Γ-ring Y-module if there exists a Γ-mapping (Γ-composition) from X x Γ x $Y \to X$ by $\{x(s), \alpha(s), f(s)\} \to x(s).\alpha(s).f(s) \in X$ where $x(s)$, $\alpha(s)$ and $f(s)$ are as given by (4). Now let $x_1(s), x_2(s) \in X$, $\beta(s) \in \Gamma$, $f_1(s), f_2(s) \in Y$ such that

$$x_1(s) = \sum_{n=1}^{\infty} x_{n_1} e^{\lambda_n s}, \;\; x_2(s) = \sum_{n=1}^{\infty} x_{n_2} e^{\lambda_n s},$$

$$f_1(s) = \sum_{n=1}^{\infty} a_{n_1} e^{\lambda_n s}, \;\; f_2(s) = \sum_{n=1}^{\infty} a_{n_2} e^{\lambda_n s},$$

and

$$\beta(s) = \sum_{n=1}^{\infty} \beta_n e^{\lambda_n s}.$$

Clearly X is a Y-module. Also X is a Γ-faithful Y-module if $X\Gamma.f(s) = (0)$ forces $f(s) = (0)$. For a Y-module X we define $A_Y(X) = \{f(s) \in Y | X\Gamma.f(s) = (0)\}$.

Lemma 2.1. *If X is a Y-module then $A_Y(X)$ is a two-sided ideal of Y. Moreover X is a Γ-faithful $Y/A_Y(X)$-module.*

Proof. $A_Y(X)$ being a right ideal of Y is obvious from the axioms for a Y-module. Now we are required to show that $A_Y(X)$ is a left ideal of Y for which,

$$X\Gamma(Y\Gamma A_Y(X)) = (X\Gamma Y)\Gamma A_Y(X) \subseteq X\Gamma A_Y(X) = (0)$$

which implies

$$Y\Gamma A_Y(X) \subseteq A_Y(X).$$

Thus $A_Y(X)$ is a two-sided ideal of Y. Now make of X a $Y/A_Y(X)$-module as for $x(s) \in X$, $\alpha(s) \in \Gamma$ and $f(s)+A_Y(X) \in Y/A_Y(X)$, the action $x(s).\alpha(s).(f(s)+A_Y(X)) = x(s).\alpha(s).f(s)$. If $f_1(s) + A_Y(X) = f_2(s) + A_Y(X)$ implies $f_1(s) - f_2(s) \in A_Y(X)$ which further implies that $x(s).\alpha(s).(f_1(s) - f_2(s)) = (0)$. Thus $x(s).\alpha(s).f_1(s) = x(s).\alpha(s).f_2(s)$. The action of $Y/A_Y(X)$ on X is well-defined. Finally show that X is a Γ-faithful $Y/A_Y(X)$-module by $x(s).\alpha(s).(f(s) + A_Y(X)) = (0)$ implies $x(s).\alpha(s).f(s) = (0)$. Thus $f(s) \in A_Y(X)$ which implies that only the zero element of $Y/A_Y(X)$ annihilates all of X which completes the proof. □

Lemma 2.2. *X is an irreducible Y-module if and only if X is an irreducible R-module.*

Proof. Let X be an irreducible Y-module which implies $X\Gamma Y = X$. Now make an R-module from X by defining $\sum_i [\alpha_i(s), f_i(s)] \in R$ for $x(s) \in X$. The composition

$$x(s).\sum_i [\alpha_i(s), f_i(s)] = x(s).\sum_i (\alpha_i(s), f_i(s)) = \sum_i x(s).\alpha_i(s).f_i(s).$$

If

$$\sum_i (\alpha_i(s), f_i(s)) + T = \sum_j (\beta_j(s), g_j(s)) + T$$

implies

$$\sum_i (\alpha_i(s), f_i(s)) - \sum_j (\beta_j(s), g_j(s)) \in T.$$

Since

$$XT = (X\Gamma Y)T = X\Gamma(0) = (0)$$

implies

$$x(s).\left\{\sum_i (\alpha_i(s), f_i(s)) - \sum_j (\beta_j(s), g_j(s))\right\} = 0$$

which further implies

$$x(s).\sum_i (\alpha_i(s), f_i(s)) = x(s).\sum_j (\beta_j(s), g_j(s)).$$

Thus the composition from X x $R \to X$ is well defined. Let X' be an additive subgroup of X such that $X'R \subseteq X'$. Since $X'R = X'[\Gamma, Y] = X'\Gamma Y$ which implies $X'\Gamma Y \subseteq X'$. Therefore X' is a submodule of a Y-module X. Since X is irreducible implies X' must be X or (0). Thus X is an irreducible R-module.

Conversely let X be an irreducible R-module. Define $x(s).\gamma(s).f(s) = x(s)[\gamma(s), f(s)]$ a similar argument as in the proof above will show that X is an irreducible Y-module. Thus the proof is completed. □

Let R be the right operator ring of a Γ-ring Y. A right ideal η of R is said to be regular if there is $q(s) \in R$ such that $p(s) - q(s).p(s) \in \eta$ for all $p(s) \in R$.

Lemma 2.3. *Let R be the right operator ring of a Γ-ring Y. If X is an irreducible Y-module then X is isomorphic as an R-module to R/η for some maximal regular right ideal η of R. Conversely for every maximal regular right ideal η of R, R/η is an irreducible R-module.*

Proof. Let X be an irreducible Y-module. Since $A = \{x(s) \in X | x(s)\Gamma Y = (0)\}$ is a submodule of X and is not X it must be (0). Equivalently if $x(s) \neq 0$ is in X then $x(s)\Gamma Y \neq (0)$. However $x(s)\Gamma Y$ is a submodule of X hence $x(s)\Gamma Y = X$. By Lemma 2.2, X is a R-module and so we define $\phi : R \to X$ by $\phi(r(s)) = x(s).r(s)$ for every $r(s) \in R$. Clearly ϕ is a homomorphism of R into X as R-modules. Since $x(s).R = x(s)\Gamma Y = X$. Thus ϕ is surjective. Finally $Ker\phi = \{r(s) \in R | x(s).r(s) = 0\}$ is a right ideal η. Thus by standard homomorphism theorem X is isomorphic to R/η as a R-module. Any right ideal of R which properly contains η maps into a submodule of X. Hence η is a maximal right ideal in R. Since $x(s).R = X$ there exists an element $q(s) \in R$ such that $x(s).q(s) = x(s)$. Therefore for any $p(s) \in R$ we have $x(s).q(s).p(s) = x(s).p(s)$ which implies $x(s).(p(s) - q(s).p(s)) = 0$. This implies $p(s) - q(s).p(s) \in \eta$. Converse can be shown easily hence we omit the proof. This completes the proof. □

The Jacobson Radical of a Γ-ring Y written as $J(Y)$ is the set of all elements of Y which annihilate all the irreducible Y-modules. We note that $J(Y) = \bigcap A_Y(X)$ where intersection runs over all irreducible Y-modules X. Since $A_Y(X)$ is a two-sided ideal of Y by Lemma 2.1, thus $J(Y)$ is also a two-sided ideal of Y.

In ordinary ring theory, for the right operator ring R of a Γ-ring Y we have $J(R) = \bigcap A_R(X)$, where intersection runs over all irreducible R-modules X and $A_R(X) = \{r(s) \in R | X.r(s) = (0)\}$.

Theorem 2.1. *If Y is a Γ-ring and R is the right operator ring of Y then $J(Y) = J(R)^*$ and $J(R) = J(Y)^{*'}$.*

Proof.

$$\begin{aligned} A_Y(X)^{*'} &= \{r(s) \in R | Y.r(s) \subseteq A_Y(X)\} \\ &= \{r(s) \in R | X\Gamma Y.r(s) = (0)\} \\ &= \{r(s) \in R | X.r(s) = (0)\} \\ &= A_R(X). \end{aligned}$$

Also

$$\begin{aligned} A_R(X)^* &= \{f(s) \in Y | [\Gamma, f(s)] \subseteq A_R(X)\} \\ &= \{f(s) \in Y | X\Gamma f(s) = (0)\} \\ &= A_Y(X). \end{aligned}$$

By these facts and Lemma 2.2, we have

$$J(Y)^{*'} = \{\bigcap A_Y(X)\}^{*'} = \bigcap A_Y(X)^{*'} = \bigcap A_R(X) = J(R).$$

Similarly

$$J(R)^* = \{\bigcap A_R(X)\}^* = \bigcap A_R(X)^* = \bigcap A_Y(X) = J(Y).$$

Hence the proof is completed. □

A Γ-ring Y is said to be semi-simple if $J(Y) = (0)$.

Theorem 2.2. *If a Γ-ring Y is semi-simple then the right operator ring R of Y is also semi-simple.*

Proof. Let $(0)_Y$ be the zero ideal in Y and $(0)_R$ be the zero ideal in R. $J(Y)^{*'} = (0)_Y{}^{*'} = \{r(s) \in R | Y.r(s) = (0)_Y\} = (0)_R$, for Y is a faithful R-module. Hence by Theorem 2.1, we have $J(R) = (0)_R$. This completes the proof of the theorem. □

Theorem 2.3. *Let a Γ-ring Y be a Γ-faithful Y-module that is $Y\Gamma f(s) = (0)$ implies $f(s) = (0)$. If R is semi-simple then Y is semi-simple.*

Proof. Since Y is Γ-faithful, $J(R)^* = (0)_R{}^* = \{f(s) \in Y | [\Gamma, f(s)] = (0)_R\} = \{f(s) \in Y | Y\Gamma f(s) = (0)_R\} = (0)_Y$. This implies $J(Y) = (0)_Y$. Hence the theorem is proved. □

For the right operator ring R of a Γ-ring Y define $(\eta : R) = \{p(s) \in R | R.p(s) \subseteq \eta\}$ where η is the right ideal of R.

Lemma 2.4. *$A_R(X) = (\eta : R)$ is the largest two-sided ideal of R which lies in η where η is a maximal regular right ideal of R and X denotes R/η.*

Proof. If $p(s) \in A_R(X)$ then $X.p(s) = (0)$ implies $(r(s) + \eta).p(s) = \eta$ for all $r(s) \in R$. Then $R.p(s) \subseteq \eta$. Hence $A_R(X) \subseteq (\eta : R)$. Similarly $(\eta : R) \subseteq A_R(X)$ implies $A_R(X) = (\eta : R)$. Since η is regular there is $q(s) \in R$ with $p(s) - q(s).p(s) \in \eta$ for all $p(s) \in R$. In particular if $p(s) \in (\eta : R)$ then since $q(s).p(s) \in R.p(s) \subseteq \eta$ we get $p(s) \in \eta$. Thus the proof is completed. □

By Lemma 2.3 and Lemma 2.4, $A_Y(X) = (\eta : R)^*$ and so by the definition of $J(Y)$ one gets

Theorem 2.4. *$J(Y) = \bigcap (\eta : R)^*$ where η runs over all the maximal regular right ideals of R and where $(\eta : R)$ is the largest two-sided ideal of R lying in η.*

Definition 4. *An element $f(s)$ of a Γ-ring Y is said to be right-quasi-regular (abbreviated as rqr) if for any $\alpha(s) \in \Gamma$ the element $[\alpha(s), f(s)]$ of the right operator ring R of Y is right-quasi-regular in the usual sense. That is to say $f(s)$ is rqr if for any $\alpha(s) \in \Gamma$ there exists $\sum_{i=1}^{n}[\alpha_i(s), g_i(s)]$ in R such that*

$$[\alpha(s), f(s)] + \sum_{i=1}^{n}[\alpha_i(s), g_i(s)] - [\alpha(s), f(s)]\sum_{i=1}^{n}[\alpha_i(s), g_i(s)] = 0$$

that is

$$g(s).\alpha(s).f(s) + \sum_{i=1}^{n} g(s).\alpha_i(s).g_i(s) - \sum_{i=1}^{n}(g(s).\alpha_{(s)}.f(s)).\alpha_i(s).g_i(s) = 0$$

for all $g(s) \in Y$.

Theorem 2.5. *$J(Y)$ is a right-quasi-regular ideal and contains all right-quasi-regular ideals of Y.*

Proof. Ordinary ring theory shows that $J(R)$ is rqr ideal of R and contains all the rqr right ideals of R ([12] p.12). Clearly as shown before $J(Y) = J(R)^* = \{f(s) \in Y | [\Gamma, f(s)] \subseteq J(R)\}$. If $f(s) \in J(Y)$ then for any $\alpha(s) \in \Gamma$, $[\alpha(s), f(s)] \in J(R)$ implies $[\alpha(s), f(s)]$ is rqr that is $f(s)$ is rqr. Let X be a rqr ideal of Y. Thus it remains to show that $[\alpha(s), X] \subseteq J(R)$ where $\alpha(s) \in \Gamma$. If $x(s) \in X$ then $x(s)$ is rqr implies $[\alpha(s), x(s)]$ is rqr. Since X is a right ideal of Y,

$$[\alpha(s), X][\Gamma, Y] = [\alpha(s), X\Gamma Y] \subseteq [\alpha(s), X]$$

and hence $[\alpha(s), X]$ is a right ideal of R. Thus $[\alpha(s), X]$ is a rqr right ideal of R. This implies $[\alpha(s), X] \subseteq J(R)$ which completes the proof of the theorem. □

Lemma 2.5. *A Γ-ring Y is isomorphic to a subdirect sum of Γ-rings S_i, $i \in U$ if and only if for each $i \in U$ there exists in Y a two-sided ideal K_i such that $Y/K_i \cong S_i$, moreover $\bigcap_{i \in U} K_i = (0)$.*

Theorem 2.6. *A Γ-ring Y is primitive if and only if the right operator ring R is primitive and $Y\Gamma.f(s) = (0)$ forces $f(s) = (0)$.*

Proof. Let Y be a primitive Γ-ring and X be a Γ-faithful irreducible Y-module. By Lemma 2.2 X is an irreducible R-module. If $X.r(s) = (0)$ implies $X\Gamma Y.r(s) = (0)$ and so $Y.r(s) = (0)$ which further implies $r(s) = (0)$. Thus X is faithful. If $Y\Gamma.f(s) = (0)$ we get $(X\Gamma Y)\Gamma.f(s) = (0)$ implies $X\Gamma.f(s) = (0)$ and hence $f(s) = (0)$.

Conversely let X be a faithful irreducible R-module. By Lemma 2.2, X is an irreducible Y-module. To show that X is Γ-faithful we assume that $X\Gamma.f(s) = (0)$ which implies $X[\Gamma, f(s)] = (0)$ thus $[\Gamma, f(s)] = (0)$. Hence $Y\Gamma.f(s) = (0)$ implies $f(s) = (0)$. Thus the proof is completed. □

Theorem 2.7. *A Γ-ring Y is primitive if and only if there exists a maximal regular right ideal η in R such that $(\eta : R)^* = (0)$ where R denotes the right operator ring of Y. A primitive Γ-ring is semi-simple.*

Proof. Let Y be a primitive Γ-ring and X be a Γ-faithful irreducible Y-module. By Lemma 2.3, there exists a maximal regular right ideal η in R such that X is isomorphic to R/η as an R-module. Lemma 2.4 shows that $(\eta : R)^* = A_Y(X)$. Since X is Γ-faithful $A_Y(X) = (0)$. Thus $(\eta : R)^* = (0)$. Now let η be a maximal regular right ideal of R. Put $X = R/\eta$. Since $A_Y(X) = (\eta : R)^* = (0)$. Finally $J(Y) = \bigcap(\eta : R)^*$ where η runs over all maximal regular right ideals of R. Thus $J(Y) = (0)$. Hence Y is semi-simple. This completes the proof of the theorem. □

Theorem 2.8. *A Γ-ring Y is semi-simple if and only if it is isomorphic to a subdirect sum of primitive Γ-rings.*

Proof. Let Y be a semi-simple Γ-ring. By Theorem 2.4, $J(Y) = \bigcap(\eta : R)^*$ where η runs over all maximal regular right ideals of R. Since Y is semi-simple implies $\bigcap(\eta : R)^* = (0)$. By Lemma 2.5, Y is isomorphic to a subdirect sum of the $Y/(\eta : R)^*$. By Lemma 2.1 and 2.4, $Y/(\eta : R)^*$ is primitive. Therefore Y is isomorphic to a subdirect sum of primitive Γ-rings.

Conversely suppose that Y is isomorphic to a subdirect sum of the rings $Y_\phi = Y/K_\phi$. Therefore $\bigcap K_\phi = (0)$. If the rings Y_ϕ are all primitive they are semi-simple. Since $J(Y)$ maps into a quasi-regular right ideal of Y_ϕ. Hence $J(Y) \subseteq K_\phi$ which implies $J(Y) \subseteq \bigcap K_\phi = (0)$ proving that Y is semi-simple. Thus the proof is completed. □

References

[1] S. Mandelbrojt. Dirichlet Series: Principles and Methods. D. Riedel Pub. Co., Dordrecht, Holland, 1972.

[2] R. K. Srivastava. On the space of integral functions represented by Dirichlet series. Prog. of Math. 21 (1987), no. 1, 29-40.

[3] R. K. Srivastava. On a paper of Bhattacharya and Manna [1]. Internal Report. IC/93/417, 1993.

[4] W. E. Barnes. On the Γ-rings of Nobusawa. Pacific J. Math. 18 (1966), 411-422.

[5] J. Luh. On primitive Γ-rings with minimal one-sided ideals. Osaka J. Math. 5 (1968), 165-173.

[6] J. Luh. The structure of primitive gamma rings. Osaka J. Math. 7 (1970), 267-274.

[7] N. Jacobson. Structure of rings. Amer. Math. Soc. Colloquium Publ. 37. Providence, 1964.

[8] F. A. Szasz. Radicals of Rings. Wiley, Chichester, 1981.

[9] S. Kyuno. A gamma ring with the left and right unities. Math. Japon. 24 (1979), 191-193.

[10] J. Luh. On the theory of simple Γ-rings. Michigan Math. J. 16 (1969), 65-75.

[11] S. Kyuno. On the semi-simple gamma rings. Tohoku Math. J. 29 (1977), 217-225.

[12] I. N. Herstein. Noncommutative rings. The Carrus Math. Monographs. no. 15, 1968.

[13] N. Kumar and G. Manocha. A study on prime one-sided ideals, socles of Entire Dirichlet series as a gamma ring. Scientia Magna. 11 (2016), no.1, 49-54.

[14] N. Kumar and G. Manocha, On a class of entire functions represented by Dirichlet series. J. Egypt. Math. Soc. 21 (2013), no. 1, 21-24.

[15] N. Kumar and G. Manocha, A class of entire Dirichlet series as an FK-space and a Frechet space. Acta Math. Sci. 33B (2013), no. 6, 1571-1578.

[16] N. Kumar and G. Manocha, Certain results on a class of entire functions represented by Dirichlet series having complex frequencies. Acta Univ. M. Belii, Ser. Math. 23 (2015), 95-100.

[17] N. Kumar, L. Chutani and G. Manocha, Certain results on a class of Entire Dirichlet series in two variable, Scientia Magna. 2016 (Accepted).

Scientia Magna
Vol. 12 (2017), No. 1, 32-45

Weyl's theorem for m-quasi N-class A_k operators

D. Senthilkumar[1] and R. Murugan[2*]

[1]Department of Mathematics, Barathiar University
Government Arts College(Autonomous),Coimbatore, India
E-mail: senthilsenkumhari@gmail.com
[2]Department of Mathematics, Barathiar University
Government Arts College(Autonomous),Coimbatore, India
E-mail: rmuruganmaths@gmail.com

Abstract In this paper we studied m-quasi N-class A_k operator, where k is positive integer, which coincides with m-quasi N-class A operator for $k = 1$. We prove that if T is m-quasi N-class A_k operator then T is finite ascent, we prove that T is an isoloid and Weyl's theorem holds for T and $f(T)$, where f is an analytic function in a neighborhood of the spectrum of T. We also show that Aluthge Transformaion of m-quasi N-class A_k operators.

Keywords N-class A_k operators, quasi N-class A_k Operators, Paranormal Operators, Weyl's Theorem.
2010 Mathematics Subject Classification 47B38, 47B65.

§1. Introduction

Let $T \in B(H)$ be the Banach algebra of all bounded linear operators on a non-zero complex Hilbert space H. By an operator T, We mean an element form $B(H)$. If T lies in $B(H)$, then T^* denotes the adjoint of T in $B(H)$. An operator T is called paranormal if $\left\|T^2x\right\| \geq \|Tx\|^2$ for every unite vector $x \in H$. An operator T belongs to class A, if $\left|T^2\right| \geq |T|^2$. An operator T is called n-perinormal for positive integer n such that $n \geq 2$, if $T^{*n}T^n \geq (T^*T)^n$.An operator T is called k-paranornormal for positive integer k, if $\left\|T^{k+1}x\right\| \geq \|Tx\|^{k+1}$ for every unit vector in $x \in H$. For $0 < p < 1$, an operator T is said to be p-hyponormal if $(T^*T) \geq (TT^*)^p$ if $p = 1$, T is called hyponormal. An operator T is called log-hyponormal if T is invertible and $log\,(T^*T) \geq log\,(TT^*)$. An operator T is said to be class $A(k)$ for $k > 0$, if $\left(T^*\,|T|^{2k}\,T\right)^{\frac{1}{k+1}} \geq |T|^2$. An operator T is called normaloid if $r(T) = \|T\|$, where $r(T) = sup\,\{|\lambda| : \lambda \in \sigma(T)\}$ and isoloid if every isoloid point of $\sigma(T)$ is an eigen values of T. We defined an operator $T \in B(H)$ as N-class A_k if $|T|^2 \leq N\left(\left|T^{k+1}\right|\right)^{\frac{2}{k+1}}$ for a positive integer k. If $k = 1$, then N-class A_k coincides with N-class A operator. We have shown that p-hyponormal operators and log-hyponormal operators are class A_k operators, for every positive integer k and class A_k operators are k-paranormal operators.

In this paper, we introduced a new class of operator is called m-quasi N-class A_k operators for each positive integers k, m and N, which is superclasses of class A_k operators and prove that weyl's holds for m-quasi N-class A_k operators.

class $A \subset$ class $A_k \subset$ N-class $A_k \subset$ quasi N-class $A_k \subset$ m-quasi N-class A_k

§2. Definition and examples

In this section m-quasi N-class A_k operators are defined and show that an example. It is shown that powers and inverse of an invertible class A operator are m-quasi class A_k for all positive integers k, m and N.

Definition 2.1. An operator $T \in B(H)$ is said to be m-quasi N-class A_k for some positive integers k, m and N if

$$T^{*m}\left[|T|^2 - N\left(\left|T^{k+1}\right|\right)^{\frac{2}{k+1}}\right]T^m \leq 0$$

.

Proposition 2.2. An operator $T \in B(H)$ is defined to be m-quasi N-class A_k for some positive integers k, m and N if

1. $k = 1$ the m-quasi N-class A_k operator coincides m-quasi N-class A.
2. $k = 1, m = 1$ the m-quasi N-class A_k operator coincides quasi N-class A.
3. $m = 1$ the m-quasi N-class A_k operator coincides quasi N-class A_k.
4. $k = 1, N = 1$ the m-quasi N-class A_k operator coincides m-quasi class A.
5. $N = 1$ the m-quasi N-class A_k operator coincides m-quasi class A_k.
6. $m = 1, N = 1$ the m-quasi N-class A_k operator coincides quasi class A_k.
7. $k = 1, m = 1, N = 1$ the m-quasi N-class A_k operator coincides quasi class A.

Example 2.3. Suppose taht H is the direct sum of a denumerable number of copies of two dimensional Hilbert space $R \times R$ and A and B two positive operators on $R \times R$. For any fixed positive integer n, define an operator $T = T_{A,B,n}$ on H as follows:

$$T\left(x_1, x_2, x_3, ...x_n\right) = \left(0, A\left(x_1\right), A\left(x_2\right), ..A\left(x_n\right), B\left(x_{n+1}\right)..\right)$$

Its adjoint T^* is given by

$$T^*\left(x_1, x_2, x_3, ...x_n\right) = \left(0, A\left(x_1\right), A\left(x_2\right), ..A\left(x_n\right), B\left(x_{n+1}\right)..\right)$$

For $n \geq k$, $T = T_{A,B,n}$ is qusi N -class A_k if and only if A and B satisfies

$$NA^m\left(A^{k+1-i}B^{2i}A^{k+1-i}\right)^{\frac{2}{k+1}}A^m \geq A^{*(2+m)}$$

for $i = 1, 2, ..k$. If $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $T = T_{A,B,n}$ is of quasi N-class A_2.

Since $S \geq 0$ implies $T^{*m}ST^m \geq 0$, the following result is trivial. The convers is true, if T is invertible.

Thoeorem 2.4. If $T \in B(H)$ is N-class A_k , for some positive integer $k \geq 1$, then T is also quasi N-class A_k.

Thoeorem 2.5. If $T \in B(H)$ is N-class A_k , for some positive integer $k \geq 1$, then T is also quasi N-class A_k.

Thoeorem 2.6. If $T \in B(H)$ is N-class A_k operator for some positive integer $k \geq 1$, then T^* is N-class A_k operator.
Form Theorem 2.5 and 2.6, we get the following results.

Thoeorem 2.7. Let T be an invertible m - quasi N-class A_k operator then,
1. T is m-qusi N-class A_k operator for every positive integer k.
2. m-qusi N -class$A_1 \subseteq$ qusi N -class A_2 m-qusi N -class $A_3 \subseteq$.......
3. For all positive integer n, T^n is m-quasi N-class A_k operator for every integer k.
4. T^{-1} is m-quasi N-class A_k operator for every positive integer k.

Thoeorem 2.8. [15] If $A, B \in B(H)$ satisfy $A \geq 0$ and $\|B\| \leq 1$, then $(B^*AB)^{\delta} \geq B^*A^{\delta}B$ for all $\delta \in (0, 1]$.

Thoeorem 2.9. [15] If A is a positive operator, then the following inequalities hold for all $x \in H$
1. $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r \|x\|^2 (1 - r)$ for all $0 < r \leq 1$.
2. $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r \|x\|^2 (1 - r)$ for $r > 1$.

Thoeorem 2.10. If $T \in B(H)$ is m-quasi N-classA_k operator if and only if
$\left\|T^{m+1}x\right\|^2 \leq N \left\|T^{k+1+m}x\right\|^{\frac{2}{k+1}} \|T^m x\|^{\frac{2k}{k+1}}$ for all $x \in H$.

Proof. From by the definition of m-quasi N-classA_k operator for every $x \in H$.

$$T^{*m}\left[N\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2\right]T^m \geq 0$$

$$0 \leq T^{*m}\left[N\left|T^{k+1}\right|^{\frac{2}{k+1}}\right]T^m - T^{*m}\left[|T|^2\right]T^m$$

$$0 \leq \left\langle T^{*m}N\left|T^{k+1}\right|^{\frac{2}{k+1}}T^m x, x\right\rangle - \left\langle T^{*m}|T|^2 T^m x, x\right\rangle$$

$$0 \leq \left\langle N\left|T^{k+1}\right|^{\frac{2}{k+1}}T^m x, T^m x\right\rangle - \left\langle |T|^2 T^m x, T^m x\right\rangle$$

$$0 \leq N\left\langle \left|T^{k+1}\right|^{\frac{2}{k+1}}T^m x, T^m x\right\rangle^{\frac{1}{k+1}}\|T^m x\|^{\frac{2k}{k+1}} - \left\langle T^{m+1}x, T^{m+1}x\right\rangle$$

$$0 \geq \left\|T^{m+1}x\right\|^2 - N\left\|T^{k+1+m}x\right\|^{\frac{2}{k+1}}\|T^m x\|^{\frac{2k}{k+1}}$$

$$\left\|T^{m+1}x\right\|^2 \leq N\left\|T^{k+1+m}x\right\|^{\frac{2}{k+1}}\|T^m x\|^{\frac{2k}{k+1}}.$$

□

Thoeorem 2.11. If T is m-quasi N-classA_k operator for a positive integers k, m and N then T is $(m+1)$-hyponormal.

Proof. From by the definition of m-quasi N-classA_k operator for every $x \in H$.

$$T^{*m}\left[N\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2\right]T^m \geq 0$$
$$T^{*m}\left(T^*T\right)T^m \leq T^{*m}\left(T^{m+1}T^{m+1}\right)^{\frac{1}{k+1}}T^m$$
$$T^{*(m+1)}T^{(m+1)} \leq N\left(T^{*(k+1+m)}T^{(k+1+m)}\right)^{\frac{1}{k+1}}$$
$$\left(T^*T\right)^{m+1} \geq \left(TT^*\right)^{m+1}.$$

Therefore T is $(m+1)$-hyponormal.

□

§3. Aluthge Transformation of m-quasi N-classA_k operators

In this section it is shown that if T is m-quasi N-classA_k operator, Then T^* is also m-quasi N-classA_k operators, if$T = U|T|$ be the polar decomposition of T then $\widetilde{T}$ is m-quasi N-classA_k operators and if $\widetilde{T}$ is m-quasi N-classA_k, then $\widetilde{T}^{(*)}$ is also m-quasi N-classA_k operators are proved.

Defintion 3.1. [1] Let $T = U|T|$ be the polar decomposition of an operator T, then $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Althuge transformation.

Thoeorem 3.2. [19] If T is a bounded linear operator on Hilbert space then we know that,
(i). $T = U|T| = |T^*|U$ is the polar decomposition of an operator T.
(ii). $T^* = U^*|T^*| = |T|U^*$ is the polar decomposition of an operator T.

Thoeorem 3.3. If T is N-class$A(k)$operator then T is N-classA_k operator.

Thoeorem 3.4. If T is m-quasi N-classA_k operator then T^* is m-quasi N-classA_k operator.

Proof. From by the definition of m-quasi N-classA_k operator for every $x \in H$.

$$T^{*m}\left[N\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2\right]T^m \geq 0$$
$$\left(T^*T\right)^{m+1} \leq N\left\{\left(T^*T\right)^{k+1+m}\right\}^{\frac{1}{k+1}}$$
$$\left(T^*T\right)^{*(m+1)} \leq N\left\{\left(T^*T\right)^{*(k+1+m)}\right\}^{\frac{1}{k+1}}$$
$$T^{*(m+1)}T^{(m+1)} \leq N\left(T^{*(k+1+m)}T^{(k+1+m)}\right)^{\frac{1}{k+1}}$$
$$T^{*m}\left[N\left|T^{*(k+1)}\right|^{\frac{2}{k+1}} - |T^*|^2\right]T^m \geq 0$$
$$T^{*m}\left[N\left|T^{*(k+1)}\right|^{\frac{2}{k+1}} - |T^*|^2\right]T^m \geq 0.$$

Therefore T^* is m-quasi N-class A_k operators. □

Thoeorem 3.5. If T is m-quasi N-classA_k operator then T^{-1} is m-quasi N-classA_k operator.

Thoeorem 3.6. Let T is m-quasi N-classA_k operator for a positive integers k, m and N, S is a unitary operator then $C = TS$ is m-quasi N-classA_k operator.

Proof. From by the definition of m-quasi N-classA_k operator for every $x \in H$.

$$T^{*m}\left[N\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2\right]T^m \geq 0$$

$$(T^*T)^{m+1} \leq N\left\{(T^*T)^{k+1+m}\right\}^{\frac{1}{k+1}}$$

$$(C^*C)^{(m+1)} \leq N\left\{(C^*C)^{(k+1+m)}\right\}^{\frac{1}{k+1}}$$

$$((TS)^*(TS))^{(m+1)} \leq N\left\{((TS)^*(TS))^{(k+1+m)}\right\}^{\frac{1}{k+1}}$$

$$(S^*T^*ST)^{(m+1)} \leq N\left\{(S^*T^*TS)^{(k+1+m)}\right\}^{\frac{1}{k+1}}$$

$$T^{*m}|T|^2T^m \leq NT^{*m}\left|T^{k+1}\right|^{\frac{2}{k+1}}T^m.$$

Therefore $C = TS$ is m-quasi N-classA_k operators. □

Thoeorem 3.7. Let $T = U|T| \in B(H)$ be the polar decomposition of m-quasi N-classA_k operator for a positive integers k, m and N, then $\widetilde{T}$ is m-quasi N-classA_k operators.

Proof. From by the definition of m - quasi N - classA_k operator for every $x \in H$.

$$T^{*m}\left[N\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2\right]T^m \geq 0$$

$$T^{*m}(T^*T)T^m \leq NT^{*m}\left(T^{m+1}T^{m+1}\right)^{\frac{1}{k+1}}T^m$$

$$T^{*(m+1)}T^{(m+1)} \leq N\left(T^{*(k+1+m)}T^{(k+1+m)}\right)^{\frac{1}{k+1}}$$

$$(T^*T)^{m+1} \leq N\left\{(T^*T)^{(k+1+m)}\right\}^{\frac{1}{k+1}}$$

$$(U^*|T^*|U|T|)^{m+1} \leq N\left\{(U^*|T^*|U|T|)^{(k+1+m)}\right\}^{\frac{1}{k+1}}U$$

$$\left(|T|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}}\right)^{m+1} \leq N\left\{\left(|T|^{\frac{1}{2}}U^*|T^*|U|T|^{\frac{1}{2}}\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}}U$$

$$\left(|T|^{\frac{1}{2}}U^*|T|U|T^*|^{\frac{1}{2}}\right)^{m+1} \leq N\left\{\left(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}}U$$

$$U^*\left(|T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right)^{m+1}U \leq NU^*\left\{\left(|T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}}U$$

$$\left(\widetilde{T}^*\widetilde{T}\right)^{m+1} \leq N\left\{\left(\widetilde{T}^*\widetilde{T}\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}}$$

$$\widetilde{T}^{*m}\left|\widetilde{T}\right|^2\widetilde{T}^m \leq N\widetilde{T}^{*m}\left|\widetilde{T}^{k+1}\right|^{\frac{2}{k+1}}\widetilde{T}^m.$$

Therefore $\widetilde{T}$ is m-quasi N-classA_k operators. □

Thoeorem 3.8. Let $T = U|T| \in B(H)$ be the polar decomposition of m-quasi N-classA_k operator for a positive integers k, m and N, then $\widetilde{T}^*$ is m-quasi N-classA_k operators.

Proof. From theorem 3.7.Using $|T| = U^*|T^*|U$ we get $\widetilde{T}^*$ is m-quasi N-classA_k operators. □

§4. ∗ - Aluthge Transformation of m-quasi N-classA_k operators

In this section to proved T is ∗ - Aluthge Transformation of m-quasi N-classA_k operator is adjoint of ∗ - Aluthge Transformation of m-quasi N-classA_k operator and if T is adjoint of ∗ - Aluthge Transformation of m-quasi N-classA_k operator is adjoint Aluthge Transformation of m-quasi N-classA_k operators, $\widetilde{T}_{(s,t)}$ is m-quasi N-classA_k operator are discussed.

T. Yamazaki [19] has defined the following ∗ - Aluthge and adjoint of ∗ - Aluthge transformation and powers of p - hponormal operators.

Definition 4.1. [19] Let $T = U|T|$ be the polar decomposition of an operator T, then ∗-Althuge transformation T is $\widetilde{T}^{(*)} = |T^*|^{\frac{1}{2}} U |T^*|^{\frac{1}{2}}$.

Definition 4.2. [19] Let $T = U|T|$ be the polar decomposition of an operator T. Then adjoint of ∗ - Aluthge transformation T is $\left(\widetilde{T}^{(*)}\right)^* = |T^*|^{\frac{1}{2}} U^* |T^*|^{\frac{1}{2}}$.

Thoeorem 4.3. [19] If T is a bounded linear operator on a Hilbert space, Then we know that
(i). $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is the Aluthge transformation then adjoint of aluthge transformation $\widetilde{T}^*$ is given by $\widetilde{T}^{(*)} = |T|^{\frac{1}{2}} U^* |T|^{\frac{1}{2}}$.
(ii). $\widetilde{T}^{(*)} = \left(\widetilde{T}^*\right)^* = |T^*|^{\frac{1}{2}} U |T^*|^{\frac{1}{2}}$ is the ∗ - aluthge transformation then adjoint of ∗ - aluthge transformation $\left(\widetilde{T}^{(*)}\right)^* = |T^*|^{\frac{1}{2}} U^* |T^*|^{\frac{1}{2}}$.

Thoeorem 4.4. Let $T = U|T|$ be the polar decomposition of an operator T, then Althuge transformation $\widetilde{T}_{(s,t)}$ is defined has $\widetilde{T}_{(s,t)} = |T|^s U |T|^t$ for an s, t such that $S \geq 0$ and $t \geq 0$.

Thoeorem 4.5. Let $\widetilde{T}$ is m - quasi N - class A_k operator for a positive integers m, k and N then $\left(\widetilde{T}^{(*)}\right)^*$ is m - quasi N - class A_k operator.

Proof. From by the definition of m - quasi N - classA_k operator for every $x \in H$.

$$T^{*m}\left[N\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2\right]T^m \geq 0$$

$$(T^*T)^{m+1} \leq N\left\{(T^*T)^{(k+1+m)}\right\}^{\frac{1}{k+1}}$$

$$U^*\left(|T|^{\frac{1}{2}} U^* |T^*| U |T^*|^{\frac{1}{2}}\right)^{m+1} U \leq NU^*\left\{\left(|T|^{\frac{1}{2}} U^* |T^*| U |T|^{\frac{1}{2}}\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}} U$$

$$U^*\left(|T|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}\right)^{m+1}U \leq NU^*\left\{\left(|T|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}}U$$

$$U^*\left(\left(\widetilde{T}^{(*)}\right)^*\widetilde{T}^{(*)}\right)^{m+1}U \leq U^*\left\{\left(\left(\widetilde{T}^{(*)}\right)^*\widetilde{T}^{(*)}\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}}U$$

$$\left(\widetilde{T}^{(*)}\right)^{*m}\left|\left(\widetilde{T}^{(*)}\right)^*\right|^2\left(\widetilde{T}^{(*)}\right)^m \leq N\left(\widetilde{T}^{(*)}\right)^{*m}\left|\left(\widetilde{T}^{(*)}\right)^{*k+1}\right|^{\frac{2}{k+1}}\left(\widetilde{T}^{(*)}\right)^m.$$

Therefore $\left(\widetilde{T}^{(*)}\right)^*$ is m-quasi N-classA_k operators. □

Thoeorem 4.6. Let $\widetilde{T}$ is m - quasi N - class A_k operator for a positive integers m, k and N then $\widetilde{T}^{(*)}$ is m - quasi N - class A_k operator.

Proof. From the Theorem 4.5 using $|T^*| = U^*|T^*|U$ we get $\widetilde{T}^{(*)}$ is m - quasi N - classA_k operators. □

Thoeorem 4.7. Let $\widetilde{T}^{(*)}$ is m - quasi N - class A_k operator for a positive integers m, k and N then $\widetilde{T}^*$ is m - quasi N - class A_k operator.

Proof. From the definition of m - quasi N - classA_k operator for every $x \in H$.

$$(T^*T)^{m+1} \leq N\left\{(T^*T)^{(k+1+m)}\right\}^{\frac{1}{k+1}}$$

$$\left(\widetilde{T}\widetilde{T}^*\right)^{m+1} \leq N\left\{\left(\widetilde{T}\widetilde{T}^*\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}}$$

$$U^*\left(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right)^{m+1}U \leq NU^*\left\{\left(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}}U$$

$$U^*\left(\widetilde{T}^*\widetilde{T}\right)^{m+1}U \leq NU^*\left\{\left(\widetilde{T}^*\widetilde{T}\right)^{k+m+1}\right\}^{\frac{1}{k+1}}U$$

$$\widetilde{T}^{*m}|T|^2\widetilde{T}^m \leq N\widetilde{T}^{*m}\left|T^{k+1}\right|^{\frac{2}{k+1}}\widetilde{T}^m.$$

Therefore $\widetilde{T}^*$ is m-quasi N-classA_k operators. □

Thoeorem 4.8. Let $\widetilde{T}^{(*)}$ is m - quasi N - class A_k operator for a positive integers m, k and N then $\widetilde{T}$ is m - quasi N - class A_k operator.

Proof. From the Theorem 4.7 using $|T^*| = U^*|T|U$ we get $\widetilde{T}$ is m - quasi N - classA_k operators. □

Thoeorem 4.9. Let $\widetilde{T}^{(*)}$ is m - quasi N - class A_k operator for a positive integers m, k and N then $\widetilde{T}$ is m - quasi N - class A_k operator.

Proof. From the Theorem 4.6 ans 4.7 then we get $\widetilde{T}$ is m - quasi N - classA_k operators. □

Thoeorem 4.10. Let $\widetilde{T}^{(*)}$ is m - quasi N - class A_k operator for a positive integers m, k and N then $\widetilde{T}^*$ is m - quasi N - class A_k operator.

Proof. From the Theorem 4.9 using $|T^*| = U^* |T| U$ then we get $\widetilde{T}^*$ is m - quasi N - classA_k operators. □

Thoeorem 4.11. Let $T = U |T|$ be the polar decomposition of an operator for a positive integers k, m and N for $0 < p \leq 1$, then $\widetilde{T}_{s,t} = |T|^s U |T|^t$ is $2\{(1+m) + min(s,t)\}$ quasi N - class A_k operator for $s, t > 0$ such that $max(s,t) \geq p$.

Proof. From by the definition of m - quasi N - classA_k operator for every $x \in H$.

$$T^{*(m+1)}T^{(m+1)} \leq N \left(T^{*(k+1+m)}T^{(k+1+m)}\right)^{\frac{1}{k+1}}$$

$$\left(\widetilde{T}^*_{s,t}\widetilde{T}_{s,t}\right)^{m+1} \leq N \left\{\left(\widetilde{T}^*_{s,t}\widetilde{T}_{s,t}\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}}$$

$$\left(|T|^t U^* |T|^s |T|^s U |T|^t\right)^{m+1} \leq N \left\{\left(|T|^t U^* |T|^s |T|^s U |T|^t\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}} U$$

$$\left(U^* |T^*|^t |T|^{2s} |T^*|^t U\right)^{m+1} \leq N \left\{\left(U^* |T^*|^t |T|^{2s} |T^*|^t U\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}} U$$

$$U^* \left(\beta^{\frac{t}{2}} A^s \beta^{\frac{t}{2}}\right)^{m+1} U \leq NU^* \left\{\left(\beta^{\frac{t}{2}} A^s \beta^{\frac{t}{2}}\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}} U$$

$$U^* \beta^{2\{(1+m)+min(s,t)\}} U \leq NU^* \left\{\left(\beta^{2\{(1+m)+min(s,t)\}}\right)^{(k+1+m)}\right\}^{\frac{1}{k+1}} U$$

$$\widetilde{T}^{*m} |T|^{2\{(1+m)+min(s,t)\}} \widetilde{T}^m \leq N\widetilde{T}^{*m} \left|T^{k+1}\right|^{2\{(1+m)+min(s,t)\}\frac{2}{k+1}} \widetilde{T}^m.$$

Therefore $2\{(1+m) + min(s,t)\}$ is m-quasi N-classA_k operators. □

§5. Matrix Representation

In this section Matrix representation of an operator is used to study various properties of an operator. $T = \begin{bmatrix} A & S \\ 0 & 0 \end{bmatrix}$ for class A operator with respect to direct sum of closure of range of T and kernel of T^*. We extened this to m - qausi N-class A_k operator.

Thoeorem 5.1. Let T be a m - qausi N-class A_k operator for a positive integers k, m and N with no dense range and T has the following representation $T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}$ on $H = \overline{ran(T^m)} \oplus ran(T^{*m})$, then T_1 is m - quasi N-class A_k operator on $\overline{ran(T^m)}$ and T_3 is nilpotent. furthermore $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Consider the matrix representation of T with respect to decomposition $H = \overline{ran(T)} \oplus ker(T)$ $T = \begin{bmatrix} A & S \\ 0 & 0 \end{bmatrix}$ let P be the orthogonal projection of H onto $ran(T)$ then $T_1 = PTP =$

TP since T is m - quasi N - class A_k operator we have,

$$P\left[N\left(|T|^{k+1}\right)^{\frac{2}{k+1}} - |T|^2\right]P \geq 0$$

$$\begin{aligned}
P\left[N\left(|T|^{k+1}\right)^{\frac{2}{k+1}}\right]P &= P\left[N\left((T^*T)^{k+1}\right)^{\frac{1}{k+1}}\right]P \\
&= P\left[N\left(T^{*k+1}T^{k+1}\right)^{\frac{1}{k+1}}\right]P \\
&\leq N\left[PT^{*k+1}T^{k+1}P\right]^{\frac{1}{k+1}} \\
&= \begin{bmatrix} \left|T_1^{k+1}\right|^{\frac{2}{k+1}} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \left|T_1^{k+1}\right|^{\frac{2}{k+1}} & 0 \\ 0 & 0 \end{bmatrix} \\
&\geq P\left(\left|T_1^{k+1}\right|\right)^{\frac{2}{k+1}} P \\
&\geq |T|^2 \\
&= \begin{bmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{bmatrix} \qquad (1)
\end{aligned}$$

Hence On $\overline{ran\,(T^m)}$. Also for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in H$

$$\begin{aligned}
\langle T_3^m x_2, x_3\rangle &= \langle T^m(I-P)x, (I-P)x\rangle \\
&= \langle (I-P)x, T^{*m}(I-P)x\rangle \\
&= 0 \\
T_3 &= 0. \qquad (2)
\end{aligned}$$

since $\sigma(T_1) \cup \sigma(T_2) = \sigma(T) \cup \tau$, where τ is the union of the holds in $\sigma(T)$ which happen to be subset of $\sigma(T_1) \cap \sigma(T_2)$,and $\sigma(T_1) \cap \sigma(T_2)$ has no interior points therefore we have $\sigma(T) = \sigma(T_1) \cup \{0\}$. □

Since A_k operators are isoloid, The following results follows immediately

Corollary 5.2. Let $T \in B(H)$ be m - quasi N - class A_k operator for a positive integers k, m, N and T not have dense range. If T has the following representation $T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}$ on $H = \overline{ran(T)} \oplus kerT^*$, then T_1 is isoloid.

§6. SVEP of m - quasi N - class A_k operators

In this section, it is proved m - quasi N - class A_k operators are isoloids, they finite ascent and SVEP.

Thoeorem 6.1. If T is m - quasi N-class A_k operator for some positive integers k, m and N then T is an isoloid.

Proof. $T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}$ on $H = \overline{ran(T^m)} \oplus kerT^{*m}$ and λ_0 be an isolated point of $\sigma(T)$. Then either $\lambda_0 = 0$ or $0 \neq \lambda_0 \in iso\sigma(T_1)$. Since T_1 is isoloid, if $\lambda_0 \in iso\sigma(T_1)$, then $\lambda_0 \in \sigma_p(T_1)$ and hence $\lambda_0 \in \sigma_p(T)$ On the contrary, if $\lambda_0 = 0$ and $\lambda_0 \notin \sigma_p(T_1)$, then T_1 is invertible. Since $dimkerT_3 \neq 0$, there exists $x \neq 0$ in $kerT_3$ and for any $x \neq 0$ in H. $T\left(-T_1^{-1}T_2x \oplus x\right) = 0$. Hence $-T_1^{-1}T_2x \oplus x \in kerT$ and $\lambda_0 \in \sigma_p(T_1)$. Hence in both cases, λ_0 is an eigenvalue of T. Therefore T is isoloid. □

Thoeorem 6.2. If T is m - quasi N-class A_k operator for some positive integers k, m and N, for $\lambda \in C$, $\sigma(T) = \lambda$ then $T = \lambda$.

Thoeorem 6.3. If T is m - quasi N-class A_k operator for some positive integers k, m and N, for $\lambda \in C$, $\sigma(T) = \lambda$ then $T = \lambda$ if $\lambda \neq 0$ and $T - \lambda$ is nilpotent, if $\lambda = 0$.

Proof. If $\lambda = 0$,$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}$ on $H = \overline{ran(T)^m} \oplus kerT^{*m}$, where T_1 is N-class A_k operator and $\sigma(T) = \sigma(T_1) \cup 0$. Hence $\lambda(T_1) =$. Hence by theorem 6.2, $T_1 = 0$. Hence $T^m = 0$. Hence $T - \lambda$ is nilpotent. Assume that $\lambda \neq 0$. Then T is an invertible m - quasi N-class A_k operator and hence N-classA_k with $\sigma(T) = \lambda$. Then again by theorem 6.2 $T = \lambda$. □

Thoeorem 6.4. If T is quasi N-class A_k operator for a positive integers k, m , N and M is an invariant subspace of T, then the restriction $T_{|M}$ is N-class A_k.

Thoeorem 6.5. If T is quasi N-class A_k operator for a positive integers k, m , N and $0 \neq \lambda \in \sigma_p(T)$ and T is of the form $T = \begin{bmatrix} \lambda & T_2 \\ 0 & T_3 \end{bmatrix}$ on H $ker(T - \lambda) \oplus ker(T - \lambda)^{\perp}$, then 1. $T_2 = 0$, 2. T_3 is m - quasi N - class A_k.

Proof. let P be the orthogonal projection of H onto $ker(T - \lambda)$. since T is m - quasi N - class A_k, T satisfies,

$$
\begin{aligned}
P\left[N\left(|T|^{k+1}\right)^{\frac{2}{k+1}} - |T|^2\right]P & \geq 0 \\
P\left[N\left(|T|^{k+1}\right)^{\frac{2}{k+1}}\right]P & = P\left[N\left((T^*T)^{k+1}\right)^{\frac{1}{k+1}}\right]P \\
& = P\left[N\left(T^{*k+1}T^{k+1}\right)^{\frac{1}{k+1}}\right]P \\
& \leq N\left[PT^{*k+1}T^{k+1}P\right]^{\frac{1}{k+1}} \\
& \geq P\left(\left|T_1^{k+1}\right|\right)^{\frac{2}{k+1}} P
\end{aligned}
$$

$$\begin{aligned} &\geq P\left|T\right|^2 P \\ &= \begin{bmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} NP\left|T^{k+1}\right|^{\frac{2}{k+1}} P &= \begin{bmatrix} |T|^2 & 0 \\ 0 & 0 \end{bmatrix} \\ &= P\left|\lambda\right|^2 P \end{aligned}$$

Therefore $N\left|T^{k+1}\right|^{\frac{2}{k+1}}$ is the form

$$N\left|T^{k+1}\right|^{\frac{2}{k+1}} = \begin{bmatrix} |\lambda|^2 & A \\ A^* & B \end{bmatrix} \begin{bmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{bmatrix} = P\left|T^{k+1}\right|^2 P = PN\left|T^{k+1}\right|^{\frac{2}{k+1}} P, A = 0$$

Therefore,

$$N\left|T^{k+1}\right|^{\frac{2}{k+1}} = \begin{bmatrix} |\lambda|^2 & 0 \\ 0 & B \end{bmatrix}$$

and

$$N\left|T^{k+1}\right|^2 = \begin{bmatrix} |\lambda|^{2(k+1)} & 0 \\ 0 & B^{k+1} \end{bmatrix}$$

This implise that

$$\lambda^k T_2 + ... + T_2 T_3^k = 0. \tag{3}$$

and

$$B = N\left|T_3^{k+1}\right|^{\frac{2}{k+1}}$$

Therefore

$$0 \leq T^{*m}\left(N\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2\right)T^m = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}$$

Where $X = 0$, $Y = -\lambda^{-m+1}T_2T_3^m$ and

$$\begin{aligned} Z = &-\left(\lambda^{-m-1}T_2^* + ... + T_3^{*(m-1)}T_2^*\right)\overline{\lambda}T_2T3^m \\ &- \lambda T_3^{*m}T_2^*\left(\lambda^{m-1}T_2 + ... + T_2T_3^{m-1}\right) \\ &- T_3^{*m}\left|T_2\right|^2 T_3^m + T_3^{*m}\left(N\left|T_3^{k+1}\right|^{\frac{2}{k+1}} - |T_3|^2\right)T_3^m \end{aligned}$$

A matrix of the form $\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \geq 0$ if and only if $X \geq 0$, $Z \geq 0$ and $Y = X^{\frac{1}{2}}WZ^{\frac{1}{2}}$, for some contraction W. Therefore , $T_2T_3 = 0$. This together with $\lambda^k T_2 + \lambda^{k-1}T_2T_3 + ... + T_2T_3^k = 0$ gives that $T_2 = 0$ and T_3 is m-quasi N - class A_k operators. □

Corollary 6.6. If T is m - quasi N - class A_k operator for some positive integers k, m and N, $0 \neq \lambda \in \sigma_p(T)$, then T is of the form $T = \begin{bmatrix} \lambda & 0 \\ 0 & T_3 \end{bmatrix}$ on $(T-\lambda) \oplus \overline{ran(T-\lambda)^*}$, where T_3 is m - quasi N - class A_k and $ker(T-\lambda) = 0$.

Corollary 6.7. If T is m - quasi N - class A_k operator for some positive integers k, m and N, then $(T-\lambda)x = 0$ for $\lambda \neq 0$ and $x \in H$, then $(T-\lambda)^* x = 0$.

Proof. If $\lambda \in iso\sigma(T)$, the spectral projection E_λ of T with respect to λ is defined by $E_\lambda = \frac{1}{2\pi i}\int_{\partial D}(z-T)^{-1}dz$ where D is a closed disk with centre at λ and radius small enough such that $D \cap \sigma(T) = \lambda$. Then $E_\lambda^2 = E_\lambda$, $E_\lambda T = TE_\lambda$, $\sigma\left(T_{|E_\lambda H}\right) = \lambda$ and $ker(T-\lambda) \subset E_\lambda H$. □

Theorem 6.8. If T is m - quasi N - class A_k operator for some positive integers k, m and N, then T is of finite ascent.

Proof. If $\lambda \neq 0$, $ker(T-\lambda) \subseteq ker(T-\lambda)^*$. Hence if $x \in ker(T-\lambda)^2$, then $\|(T-\lambda)x\|^2 = \langle (T-\lambda)^*(T-\lambda)x, x\rangle = 0$ which implies $x \in ker(T-\lambda)$. Therefore $ker(T-\lambda)^2 = ker(T-\lambda)$. If $\lambda = 0$, by let $0 \neq x \in kerT^{k+2}$. $x \in kerT^2 \subset kerT^{k+1}$. Therefore $kerT^{k+2} = kerT^{k+1}$. Hence T is finite asecent. □

§7.Weyl's theorem for m - quasi N - class A_k operator

In this section it is shown that weyl's theorem holds for quasi N-class A_k operators ,quasi N-class A_k operators have index less than or equal to zero , spectral mapping theorem for weyl's spectrum holds and also that weyl's theorem holds for any function of quasi N-class A_k operators, which is analytic in a neighborhood of the spectrum of quasi N-class A_k operators.

Theorem 7.1. For given operators $A, B, C \in B(H)$, there is equality $\omega(A) \cup \omega(B) = \omega(M_c) \cup_\tau$, where $M_c = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and τ is the union of certain holes in $\omega(M_c)$ which happen to be a subset of $\omega(A) \cap \omega(B)$.

Theorem 7.2. If T is a N-class A_k operator for a positive integer k, then $f(\omega(T)) = \omega(f(T))$ for every $f \in H(\sigma(T))$.

Theorem 7.3. Suppose $A \in B(H)$ and $B \in B(H)$ are isoloid. If weyl's theorem holds for A and B, and if $\omega(A) \cap \omega(B)$ has no interior points, then weyl's theorem holds for $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

Theorem 7.4. If either $SP(A)$ or $SP(B)$ has no Pseudhoholes and if A is an isoloid operator for which weyl's theorem holds then for every $C \in B(K, H)$,weyl's theorem holds for

$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ = weyl's theorem holds for $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$.

Theorem 7.5. If $T \in B(H)$, then the following are equivalent
1. $ind(T - \lambda I)ind(T - \lambda\mu) \geq 0$ for each pair $\lambda, \mu \in C - \sigma_e(T)$
2. $f(\omega(T)) = \omega(f(T))$ for every $f \in H(\sigma(T))$.

Theorem 7.6. If $T \in B(H)$ is isoloid, then $f(\sigma(T) - \pi_{00}(T)) = \sigma(f((T)) - \pi_{00}f(T))$, for every $f \in H(\sigma(T))$.

Theorem 7.7. If T is m - quasi N-class A_k operator for some positive integers k, m and N, then weyl's theorem holds for T.

Proof. By theorem 3.1 $T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}$ on $H = \overline{ran(T)} \oplus ker(T^*)$ with $ker(T_1) = \{0\}$ then T_1 is quasi N-class A_k operator on $\overline{ran(T)}$ and T_3. Then by corollary 3.2 and by [2] (Theorem 3.3), $x \in kerT^2 \subset kerT^{k+1}$. Hence $kerT^{k+2} = kerT^{k+1}$. Hence T is finite ascent. □

Theorem 7.8. If T is m - quasi N-class A_k operator some positive integers k, m and N, then $ind(T - \lambda I) \leq 0$ for all complex numbers λ.

Proof. If T is of finite ascent by Theorem 6.8 $ind(T - \lambda) \neq 0$ for all complex number λ.

□

Theorem 7.9. If T m - quasi N-class A_k operator for some positive integers k, m and N, then $f(\sigma(T) - \pi_{00}(T)) = \sigma(f((T)) - \pi_{00}f(T))$, for every $f \in H(\sigma(T))$. the following result is trivial.

Theorem 7.10. If T is a m - quasi N-class A_k operator for some positive integers k, m and N, then $f(\omega(T)) = \omega(f(T))$ for every $f \in H(\sigma(T))$.

Theorem 7.11. If $T \in B(H)$ is m - quasi N-classA_k for a positive integers k, m and N, then weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Proof. By Theorem 7.10, Theorem 7.8 and Theorem 6.4, for every $f \in H(\sigma(T))$,

$$f(\sigma(T) - \pi_{00}(T)) = \sigma(f((T)) - \pi_{00}f(T)) = f(\omega(T)) = \omega(f(T)).$$

Hence weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T))$. □

Acknowledgements

This work is supported by Senthilkumar. The authors wish to thank Yamazaki.

References

[1] A. Aluthge. On p-hyponormal operators for $0 < p < 1$. Integral Equation operator theorey 13 (1990), 307 - 315.

[2] A. Aluthge. Properties of p-hyponormal operators. unpublished dissertation, Vanderbilt University, Nashville, TN, 1990.

[3] T. Ando. Operators with a norm condition. Acta Sci. Math. (Szeged) 33 (1972), 169 - 178.

[4] L. A. Coburn. Weyls's theorem for nonnormal operators. Michigan Math. J. 13 (1966), 285 - 288.

[5] B. P. Duggal. Weyl's theorem for totally hereditarily normaloid operators. Rendi. Cirolo Mat. Palermo LIII (2004).

[6] J. K. Han, H. Y. Lee and W. Y. Lee. Invertible completions of upper triangular operator matrices. Proc. Amer. Math. Soc. 128 (1999), 119 - 123.

[7] F. Hansen. An equality. Math. Ann. 246 (1980), 249 - 250.

[8] R. E. Harte and W. Y. Lee. Another note on Weyl's theorem. Trans. Amer. Math. Soc. 349 (1997), 2115 - 2124.

[9] H. Heuser. Functional Analysis. Marcel Dekker, New York, 1982.

[10] I. H. Jeon and I. H. Kim. On operators satisfying $T^* |T|^2 T \geq T^* |T|^2 T$. Lin. Alg. Appl. 418 (2006), 854 - 862.

[11] W. Y. Lee. Weyl's theorem for operator matrices. Integr. Equ. Oper. Theory, 32 (1998), 319 - 331.

[12] W. Y. Lee. Weyl spectra of operator matrices. Proc. Amer. Math. Soc. 129 (2001), 131 - 138.

[13] W. Y. Lee and S. H. Lee. A spectral mapping theorem for the Weyl spectrum. Glasgow Math. J. 38 (1996), no.1, 61 - 64.

[14] C. A. McCharthy. C_p. Israel J.Math. 17 (2006), no.2, 125 - 134.

[15] C. M. Pearcy. Some recent developments in operator theory. CBMS 36, AMS, Providence, (1978).

[16] S. Panayappan, N. Jayanthi and D. Sumathi. Weyl's theoerem and tensor product for class Ak operators. Pure Mathematical Sciences 1 (2012), no.1, 13 - 23.

[17] A. Uchiyama. Weyl's theorem for class A operators. Mathematical Inequalities and Applications 4 (2001), No.1, 143 - 150.

[18] T. Yamazaki. On powers of N-class$A(K)$ operators including p-hyponormal and log-hyponormal operators. Math. Inequal. Appl. 3 (2000), 97 - 104.

[19] T. Yamazaki. Paralleisms between Aluthge transformation and powres of operators. Acta Sci. Math. (szeged) 67 (2001), 809 - 820.

Scientia Magna
Vol. 12 (2017), No. 1, 46-56

Almost contra generalized α regular-continuous functions in topological spaces

S. Sekar[1] and G. Kumar[2]

[1]Department of Mathematics, Government Arts College (Autonomous)
Salem – 636 007, Tamil Nadu, India
E-mail: sekar_nitt@rediffmail.com
[2]Department of Mathematics, A.E.T. College
Attur – 636 108, Tamil Nadu, India
E-mail: kumargmaths@gmail.com

Abstract In this paper, the authors introduce a new class of functions called almost contra generalized α regular-continuous function (briefly almost contra gαr-continuous) in topological spaces. Some characterizations and several properties concerning almost contra gαr-continuous functions are obtained.

Keywords $g\alpha r$-closed sets, $g\alpha r$-closed map, $g\alpha r$-continuous map, contra $g\alpha r$-continuity.
2010 Mathematics Subject Classification 54C05, 54C08, 54C10.

§1. Introduction

In 2002, Jafari and Noiri introduced and studied a new form of functions called contra-pre continuous functions. The purpose of this paper is to introduce and study almost contra $g\alpha r$-continuous functions via the concept of $g\alpha r$-closed sets. Also, properties of almost contra $g\alpha r$-continuity are discussed. Moreover, we obtain basic properties and preservation theorems of almost contra $g\alpha r$-continuous functions and relationships between almost contra $g\alpha r$-continuity and $g\alpha r$-regular graphs.

Through out this paper (X, τ) and (Y, σ) represent the non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $A \subseteq X$, the closure of A and interior of A will be denoted by $cl(A)$ and $int(A)$ respectively, union of all $g\alpha r$-open sets X contained in A is called $g\alpha r$-interior of A and it is denoted by $g\alpha rint(A)$, the intersection of all $g\alpha r$-closed sets of X containing A is called $g\alpha r$-closure of A and it is denoted by $g\alpha rcl(A)$ [13].

§2. Preliminaries

Definition 2.1. *Let a subset A of a topological space (X, τ), is called*
1) a α-open set [8] if $A \subseteq int(cl(int(A)))$.
2) a generalised-closed set (briefly g-closed) [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

3) a weakly-closed set(briefly w-closed) [10] if $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi open.*
4) a generalized $*$*-closed set (briefly g*$*$*-closed) [12] if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is g-open in* X.
5) a generalized α*-closed set (briefly g*α*-closed) [7] if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* α *open in* X.
6) an α *generalized-closed set (briefly* α*g-closed) [6] if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.
7) a generalized b- closed set (briefly gb- closed) [1] if $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X. *8) a semi generalized b-closed set (briefly sgb- closed) [4] if* $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi open in* X.
9) a generalized α*b- closed set (briefly g*α*b- closed) [11] if* $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* α *open in* X.
10) a regular generalized b- closed set (briefly rgb- closed) [8] if $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.
11) a generalized pre regular-closed set (briefly gpr-closed) [3] if $pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.
12) a generalized α *regular-closed set (briefly g*α*r-closed) [9] if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.

Definition 2.2. *A function* $f : (X, \tau) \to (Y, \sigma)$*, is called*
1) almost contra continuous [15] if $f^{-1}(V)$ *is closed in* (X, τ) *for every regular-open set* V *of* (Y, σ).
2) almost contra gα*-continuous [7] if* $f^{-1}(V)$ *is g-closed in* (X, τ) *for every regular-open set* V *of* (Y, σ).
3) almost contra α*g-continuous [6] if* $f^{-1}(V)$ *is* α*g-closed in* (X, τ) *for every regular-open set* V *of* (Y, σ).
4) almost contra gpr-continuous [3] if $f^{-1}(V)$ *is gpr-closed in* (X, τ) *for every regular-open set* V *of* (Y, σ).
5) almost contra gb-continuous [2] if $f^{-1}(V)$ *is gb-closed in* (X, τ) *for every regular-open set* V *of* (Y, σ).
6) almost contra rgb-continuous [14] if $f^{-1}(V)$ *is rgb-closed in* (X, τ) *for every regular-open set* V *of* (Y, σ).

§3. Almost contra generalized α regular-continuous functions

In this section, we introduce almost contra generalized α regular-continuous functions and investigate some of their properties.

Definition 3.1. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is called almost contra generalized* α *regular - continuous if* $f^{-1}(V)$ *is g*α*r - closed in* (X, τ) *for every regular open set* V *in* (Y, σ).

Example 3.2. *Let* $X = Y = \{a, b, c\}$ *with* $\tau = \{X, \varphi, \{b\}, \{c\}, \{b, c\}\}$ *and* $\sigma = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}.\{a, c\}\}$. *Define a function* $f : (X, \tau) \to (Y, \sigma)$ *by* $f(a) = b$, $f(b) = c$, $f(c) = a$. *Clearly* f *is almost contra g*α*r - continuous.*

Theorem 3.3. *If $f : X \to Y$ is contra $g\alpha r$ - continuous then it is almost contra $g\alpha r$ - continuous.*

Proof. Obvious, because every regular open set is open set. □

Remark 3.4. *Converse of the above theorem need not be true in general as seen from the following example.*

Example 3.5. *Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{b\}, \{c\}, \{b, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Then f is almost contra $g\alpha r$ - continuous function but not contra $g\alpha r$ - continuous, because for the open set $\{a\}$ in Y and $f^{-1}\{a\} = \{c\}$ is not $g\alpha r$ - closed in X.*

Theorem 3.6. *i) Every almost contra $g\alpha$ - continuous function is almost contra $g\alpha r$ - continuous function.*
ii) Every almost contra αg - continuous function is almost contra $g\alpha r$ - continuous function.
iii) Every almost contra $g\alpha r$ - continuous function is almost contra gpr - continuous function.
iv) Every almost contra g - continuous function is almost contra $g\alpha r$ - continuous function.
v) Every almost contra w - continuous function is almost contra $g\alpha r$ - continuous function.
vi) Every almost contra g^* - continuous function is almost contra $g\alpha r$ - continuous function.
vii) Every almost contra $g\alpha r$ - continuous function is almost contra rgb - continuous function.

Remark 3.7. *Converse of the above statements is not true as shown in the following example.*

Example 3.8. *i) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \varphi, \{a, b\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$, $f(b) = b$, $f(c) = a$. Clearly f is almost contra $g\alpha r$ - continuous but f is not almost contra $g\alpha$ - continuous. Because $f^{-1}(\{a, c\}) = \{a, c\}$ is not $g\alpha$ - closed in (X, τ) where $\{a, c\}$ is regular - open in (Y, σ).*
ii) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{b\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Clearly f is almost contra $g\alpha r$ - continuous but f is not almost contra αg - continuous. Because $f^{-1}(\{a, c\}) = \{a, b\}$ is not αg - closed in (X, τ) where $\{a, c\}$ is regular - open in (Y, σ).
iii) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is almost contra $g\alpha r$ - continuous but f is not almost contra gpr - continuous. Because $f^{-1}(\{b\}) = \{a\}$ is not g - closed in (X, τ) where $\{b\}$ is regular - open in (Y, σ).
iv) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is almost contra $g\alpha r$ - continuous but f is not almost contra g - continuous. Because $f^{-1}(\{b\}) = \{a\}$ is not g - closed in (X, τ) where $\{b\}$ is regular - open in (Y, σ).
v) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is almost contra w - continuous but f is not almost contra $g\alpha r$ - continuous. Because $f^{-1}(\{b, c\}) = \{a, b\}$ is not $g\alpha r$ - closed in (X, τ) where $\{b, c\}$ is regular - open in (Y, σ).
vi) Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is almost contra

$g\alpha r$ - continuous but f is not almost contra g^* - continuous. Because $f^{-1}(\{a,c\}) = \{b,c\}$ is not g^* - closed in (X,τ) where $\{a,c\}$ is regular - open in (Y,σ).

vii) Let $X = Y = \{a,b,c\}$ with $\tau = \{X,\varphi,\{a\},\{c\},\{a,c\}\}$ and $\sigma = \{Y,\varphi,\{a\},\{b\},\{a,b\},\{a,c\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Clearly f is almost contra rgb - continuous but f is not almost contra $g\alpha r$ - continuous. Because $f^{-1}(\{a,c\}) = \{a,b\}$ is not $g\alpha r$ - closed in (X,τ) where $\{a,c\}$ is regular - open in (Y,σ).

Theorem 3.9. *The following are equivalent for a function $f : X \to Y$,*

1) f is almost contra $g\alpha r$ - continuous.

2) for every regular closed set F of Y, $f^{-1}(F)$ is $g\alpha r$ - open set of X.

3) for each $x \in X$ and each regular closed set F of Y containing $f(x)$, there exists $g\alpha r$ - open U containing x such that $f(U) \subset F$.

4) for each $x \in X$ and each regular open set V of Y not containing $f(x)$, there exists $g\alpha r$ - closed set K not containing x such that $f^{-1}(V) \subset K$.

Proof. (i) $\Rightarrow$ (2) : Let F be a regular closed set in Y, then $Y - F$ is a regular open set in Y. By (1), $f^{-1}(Y - F) = X - f^{-1}(F)$ is $g\alpha r$ - closed set in X. This implies $f^{-1}(F)$ is $g\alpha r$ - open set in X. Therefore, (2) holds.

(2) $\Rightarrow$ (1) : Let G be a regular open set of Y. Then $Y - G$ is a regular closed set in Y. By (2), $f^{-1}(Y - G)$ is $g\alpha r$ - open set in X. This implies $X - f^{-1}(G)$ is $g\alpha r$ - open set in X, which implies $f^{-1}(G)$ is $g\alpha r$ - closed set in X. Therefore, (1) hold.

(2) $\Rightarrow$ (3) : Let F be a regular closed set in Y containing $f(x)$, which implies $x \in f^{-1}(F)$. By (2), $f^{-1}(F)$ is $g\alpha r$ - open in X containing x. Set $U = f^{-1}(F)$, which implies U is $g\alpha r$ - open in X containing x and $f(U) = f(f^{-1}(F)) \subset F$. Therefore (3) holds.

(3) $\Rightarrow$ (2) : Let F be a regular closed set in Y containing $f(x)$, which implies $x \in f^{-1}(F)$. From (3), there exists $g\alpha r$ - open U_x in X containing x such that $f(U_x) \subset F$. That is $U_x \subset f^{-1}(F)$. Thus $f^{-1}(F) = \{\cup U_x : x \in f^{-1}(F)$, which is union of $g\alpha r$ - open sets. Therefore, $f^{-1}(F)$ is $g\alpha r$ - open set of X.

(3) $\Rightarrow$ (4) : Let V be a regular open set in Y not containing $f(x)$. Then $Y - V$ is a regular closed set in Y containing $f(x)$. From (3), there exists a $g\alpha r$ - open set U in X containing x such that $f(U) \subset Y - V$ .This implies $U \subset f^{-1}(Y - V) = X - f^{-1}(V)$. Hence, $f^{-1}(V) \subset X - U$. Set $K = X - V$, then K is $g\alpha r$ - closed set not containing x in X such that $f^{-1}(V) \subset K$.

(4) $\Rightarrow$ (3) : Let F be a regular closed set in Y containing $f(x)$. Then $Y - F$ is a regular open set in Y not containing $f(x)$. From (4), there exists $g\alpha r$ - closed set K in X not containing x such that $f^{-1}(Y - F) \subset K$. This implies $X - f^{-1}(F) \subset K$. Hence, $X - K \subset f^{-1}(F)$, that is $f(X - K) \subset F$. Set $U = X - K$, then U is $g\alpha r$ - open set containing x in X such that $f(U) \subset F$. □

Theorem 3.10. *The following are equivalent for a function $f : X \to Y$,*

1) f is almost contra $g\alpha r$ - continuous.

2) $f^{-1}(Int(Cl(G)))$ is $g\alpha r$ - closed set in X for every open subset G of Y.

3) $f^{-1}(Cl(Int(F)))$ is $g\alpha r$ - open set in X for every closed subset F of Y.

Proof. (1) $\Rightarrow$ (2) : Let G be an open set in Y. Then $Int(Cl(G))$ is regular open set in Y. By (1), $f^{-1}(Int(Cl(G)) \in g\alpha r - C(X)$.

(2) $\Rightarrow$ (1) : Proof is obvious.
(1) $\Rightarrow$ (3) : Let F be a closed set in Y. Then $Cl(Int(G))$ is regular closed set in Y. By (1), $f^{-1}(Cl(Int(G)) \in g\alpha r - O(X)$.
(3) $\Rightarrow$ (1) : Proof is obvious. □

Definition 3.11. *A function $f : X \rightarrow Y$ is said to be R - map if $f^{-1}(V)$ is regular open in X for each regular open set V of Y.*

Definition 3.12. *A function $f : X \rightarrow Y$ is said to be perfectly continuous if $f^{-1}(V)$ is clopen in X for each open set V of Y.*

Theorem 3.13. *For two functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, let $g \circ f : X \rightarrow Z$ be a composition function. Then, the following properties hold.*
i) If f is almost contra $g\alpha r$ - continuous and g is an R - map, then $g \circ f$ is almost contra $g\alpha r$ - continuous.
ii) If f is almost contra $g\alpha r$ - continuous and g is perfectly continuous, then $g \circ f$ is contra $g\alpha r$ - continuous.
iii) If f is contra $g\alpha r$ - continuous and g is almost continuous, then $g \circ f$ is almost contra $g\alpha r$ - continuous.

Proof. i) Let V be any regular open set in Z. Since g is an R - map, $g^{-1}(V)$ is regular open in Y. Since f is almost contra $g\alpha r$ - continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $g\alpha r$ - closed set in X. Therefore $g \circ f$ is almost contra $g\alpha r$ - continuous.
ii) Let V be any regular open set in Z. Since g is perfectly continuous, $g^{-1}(V)$ is clopen in Y. Since f is almost contra $g\alpha r$ - continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $g\alpha r$ - open and $g\alpha r$ - closed set in X. Therefore $g \circ f$ is $g\alpha r$ continuous and contra $g\alpha r$ - continuous.
iii) Let V be any regular open set in Z. Since g is almost continuous, $g^{-1}(V)$ is open in Y. Since f is almost contra $g\alpha r$ - continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $g\alpha r$ - closed set in X. Therefore $g \circ f$ is almost contra $g\alpha r$ - continuous. □

Theorem 3.14. *Let $f : X \rightarrow Y$ be a contra $g\alpha r$ - continuous and $g : Y \rightarrow Z$ be $g\alpha r$ - continuous. If Y is $Tg\alpha r$ - space, then $g \circ f : X \rightarrow Z$ is an almost contra $g\alpha r$ - continuous.*

Proof. Let V be any regular open and hence open set in Z. Since g is $g\alpha r$ - continuous $g^{-1}(V)$ is $g\alpha r$ - open in Y and Y is $Tg\alpha r$ - space implies $g^{-1}(V)$ open in Y. Since f is contra $g\alpha r$ - continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $g\alpha r$ - closed set in X. Therefore, $g \circ f$ is an almost contra $g\alpha r$ - continuous. □

Theorem 3.15. *If $f : X \rightarrow Y$ is surjective strongly $g\alpha r$ - open (or strongly $g\alpha r$ - closed) and $g : Y \rightarrow Z$ is a function such that $g \circ f : X \rightarrow Z$ is an almost contra $g\alpha r$ - continuous, then g is an almost contra $g\alpha r$ - continuous.*

Proof. Let V be any regular closed (resp. regular open) set in Z. Since $g \circ f$ is an almost contra $g\alpha r$ - continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $g\alpha r$ - open (resp. $g\alpha r$ - closed) in X. Since f is surjective and strongly $g\alpha r$ - open (or strongly $g\alpha r$ - closed), $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is $g\alpha r$ - open(or $g\alpha r$ - closed). Therefore g is an almost contra $g\alpha r$ - continuous. □

Definition 3.16. *A function $f : X \to Y$ is called weakly $g\alpha r$ - continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in g\alpha r - O(X; x)$ such that $f(U) \subset cl(V)$.*

Theorem 3.17. *If a function $f : X \to Y$ is an almost contra $g\alpha r$ - continuous, then f is weakly $g\alpha r$ - continuous function.*

Proof. Let $x \in X$ and V be an open set in Y containing $f(x)$. Then $cl(V)$ is regular closed in Y containing $f(x)$. Since f is an almost contra $g\alpha r$ - continuous function by Theorem 3.13(2), $f^{-1}(cl(V))$ is $g\alpha r$ - open set in X containing x. Set $U = f^{-1}(cl(V))$, then $f(U) \subset f(f^{-1}(Cl(V))) \subset cl(V)$. This shows that f is weakly $g\alpha r$ - continuous function. □

Definition 3.18. *A space X is called locally $g\alpha r$ - indiscrete if every $g\alpha r$ - open set is closed in X.*

Theorem 3.19. *If a function $f : X \to Y$ is almost contra $g\alpha r$ - continuous and X is locally $g\alpha r$ - indiscrete space, then f is almost continuous.*

Proof. Let U be a regular open set in Y. Since f is almost contra $g\alpha r$ - continuous $f^{-1}(U)$ is $g\alpha r$ - closed set in X and X is locally $g\alpha r$ - indiscrete space, which implies $f^{-1}(U)$ is an open set in X. Therefore f is almost continuous. □

Lemma 3.20. *Let A and X_0 be subsets of a space X. If $A \in g\alpha r - O(X)$ and $X_0 \in \tau^{\alpha}$, then $A \cap X_0 \in g\alpha r - O(X_0)$.*

Theorem 3.21. *If $f : X \to Y$ is almost contra $g\alpha r$ - continuous and $X_0 \in \tau^{\alpha}$ then the restriction $f/X_0 : X_0 \to Y$ is almost contra $g\alpha r$ - continuous.*

Proof. Let V be any regular open set of Y. By Theorem, we have $f^{-1}(V) \in g\alpha r - O(X)$ and hence $(f/X_0)^{-1}(V) = f^{-1}(V) \cap X_0 \in g\alpha r - O(X_0)$. By Lemma 3.20, it follows that f/X_0 is almost contra $g\alpha r$ - continuous. □

Theorem 3.22. *If $f : X \to \prod Y_\lambda$ is almost contra $g\alpha r$ - continuous, then $P_\lambda \circ f : X \to Y_\lambda$ is almost contra $g\alpha r$ - continuous for each $\lambda \in \nabla$, where P_λ is the projection of $\prod Y_\lambda$ onto Y_λ.*

Proof. Let Y_λ be any regular open set of Y. Since P_λ is continuous open, it is an R - map and hence $(P_\lambda)^{-1} \in RO(\prod Y_\lambda)$.
By theorem, $f^{-1}(P_\lambda^{-1}(V)) = (P_\lambda \circ f)^{-1} \in g\alpha r - O(X)$. Hence $P_\lambda \circ f$ is almost contra $g\alpha r$ - continuous. □

§4. $g\alpha r$ - regular graphs and strongly contra $g\alpha r$ - closed graphs

Definition 4.1. *A graph G_f of a function $f : X \to Y$ is said to be $g\alpha r$ - regular (strongly contra $g\alpha r$ - closed) if for each $(x, y) \in (X \times Y) \backslash G_f$, there exist a $g\alpha r$ - closed set U in X containing x and $V \in R - O(Y)$ such that $(U \times V) \cap G_f = \varphi$.*

Theorem 4.2. *If $f : X \to Y$ is almost contra $g\alpha r$ - continuous and Y is T_2, then G_f is $g\alpha r$ - regular in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)\backslash G_f$. It is obvious that $f(x) \neq y$. Since Y is T_2, there exists $V, W \in RO(Y)$ such that $f(x) \in V$, $y \in W$ and $V \cap W = \varphi$. Since f is almost contra $g\alpha r$ - continuous, $f^{-1}(V)$ is a $g\alpha r$ - closed set in X containing x. If we take $U = f^{-1}(V)$, we have $f(U) \subset V$. Hence, $f(U) \cap W = \varphi$ and G_f is $g\alpha r$ - regular. □

Theorem 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ the graph function defined by $g(x) = (x, f(x))$ for every $x \in X$. Then f is almost $g\alpha r$ - continuous if and only if g is almost $g\alpha r$ - continuous.*

Proof. Necessary : Let $x \in X$ and $V \in g\alpha r - O(Y)$ containing $f(x)$. Then, we have $g(x) = (x, f(x)) \in R - O(X \times Y)$. Since f is almost $g\alpha r$ - continuous, there exists a $g\alpha r$ - open set U of X containing x such that $g(U) \subset X \times Y$. Therefore, we obtain $f(U) \subset V$. Hence f is almost $g\alpha r$ continuous.

Sufficiency : Let $x \in X$ and w be a regular open set of $X \times Y$ containing $g(x)$. There exists $U_1 \in RO(X, \tau)$ and $V \in RO(Y, \sigma)$ such that $(x, f(x)) \in (U_1 \times V) \subset W$. Since f is almost $g\alpha r$ - continuous, there exists $U_2 \in g\alpha r - O(X, \tau)$ such that $x \in U_2$ and $f(U_2) \subset V$. Set $U = U_1 \cap U_2$. We have $x \in U_x \in g\alpha r - O(X, \tau)$ and $g(U) \subset (U_1 \times V) \subset W$. This shows that g is almost $g\alpha r$ - continuous. □

Theorem 4.4. *If a function $f : X \to Y$ be a almost contra $g\alpha r$ - continuous and almost continuous, then f is regular set - connected.*

Proof. Let $V \in RO(Y)$. Since f is almost contra $g\alpha r$ - continuous and almost continuous, $f^{-1}(V)$ is $g\alpha r$ - closed and open. So $f^{-1}(V)$ is clopen. It turns out that f is regular set - connected. □

§5. Connectedness

Definition 5.1. *A space X is called $g\alpha r$ - connected if X cannot be written as a disjoint union of two non - empty $g\alpha r$ - open sets.*

Theorem 5.2. *If $f : X \to Y$ is an almost contra $g\alpha r$ - continuous surjection and X is $g\alpha r$ - connected, then Y is connected.*

Proof. Suppose that Y is not a connected space. Then Y can be written as $Y = U_0 \cup V_0$ such that U_0 and V_0 are disjoint non - empty open sets. Let $U = int(cl(U_0))$ and $V = int(cl(V_0))$. Then U and V are disjoint nonempty regular open sets such that $Y = U \cup V$. Since f is almost contra $g\alpha r$ - continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are $g\alpha r$ - open sets of X. We have $X = f^{-1}(U) \cup f^{-1}(V)$ such that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint. Since f is surjective, this shows that X is not $g\alpha r$ - connected. Hence Y is connected. □

Theorem 5.3. *The almost contra $g\alpha r$ - continuous image of $g\alpha r$ - connected space is connected.*

Proof. Let $f : X \to Y$ be an almost contra $g\alpha r$ - continuous function of a $g\alpha r$ - connected space X onto a topological space Y. Suppose that Y is not a connected space. There exist non - empty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen in Y. Since f is almost contra $g\alpha r$ - continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are $g\alpha r$ - open in X. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are non - empty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that X is not $g\alpha r$ - connected. This is a contradiction and hence Y is connected. □

Definition 5.4. *A topological space X is said to be $g\alpha r$ - ultra connected if every two non - empty $g\alpha r$ - closed subsets of X intersect.*

A topological space X is said to be hyper connected if every open set is dense.

Theorem 5.5. *If X is $g\alpha r$ - ultra connected and $f : X \to Y$ is an almost contra $g\alpha r$ - continuous surjection, then Y is hyper connected.*

Proof. Suppose that Y is not hyperconnected. Then, there exists an open set V such that V is not dense in Y. So, there exist non - empty regular open subsets $B_1 = int(cl(V))$ and $B_2 = Y - cl(V)$ in Y. Since f is almost contra $g\alpha r$ - continuous, $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are disjoint $g\alpha r$ - closed. This is contrary to the $g\alpha r$ - ultra - connectedness of X. Therefore, Y is hyperconnected. □

§6. Separation axioms

Definition 6.1. *A topological space X is said to be $g\alpha r - T_1$ space if for any pair of distinct points x and y, there exist a $g\alpha r$ - open sets G and H such that $x \in G$, $y \notin G$ and $x \notin H$, $y \in H$.*

Theorem 6.2. *If $f : X \to Y$ is an almost contra $g\alpha r$ - continuous injection and Y is weakly Hausdorff, then X is $g\alpha r - T_1$.*

Proof. Suppose Y is weakly Hausdorff. For any distinct points x and y in X, there exist V and W regular closed sets in Y such that $f(x) \in V$, $f(y) \notin V$, $f(y) \in W$ and $f(x) \notin W$. Since f is almost contra $g\alpha r$ - continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $g\alpha r$ - open subsets of X such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $y \in f^{-1}(W)$ and $x \notin f^{-1}(W)$. This shows that X is $g\alpha r - T_1$. □

Corollary 6.3. *If $f : X \to Y$ is a contra $g\alpha r$ - continuous injection and Y is weakly Hausdorff, then X is $g\alpha r - T_1$.*

Definition 6.4. *A topological space X is called Ultra Hausdorff space, if for every pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X containing x and y, respectively.*

Definition 6.5. *A topological space X is said to be $g\alpha r - T_2$ space if for any pair of distinct points x and y, there exist disjoint $g\alpha r$ - open sets G and H such that $x \in G$ and $y \in H$.*

Theorem 6.6. *If $f : X \to Y$ is an almost contra $g\alpha r$ - continuous injective function from space X into a Ultra Hausdorff space Y, then X is $g\alpha r - T_2$.*

Proof. Let x and y be any two distinct points in X. Since f is an injective $f(x) \neq f(y)$ and Y is Ultra Hausdorff space, there exist disjoint clopen sets U and V of Y containing $f(x)$ and $f(y)$ respectively. Then $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $g\alpha r$ - open sets in X. Therefore X is $g\alpha r - T_2$. □

Definition 6.7. *A topological space X is called Ultra normal space, if each pair of disjoint closed sets can be separated by disjoint clopen sets.*

Definition 6.8. *A topological space X is said to be $g\alpha r$ - normal if each pair of disjoint closed sets can be separated by disjoint $g\alpha r$ - open sets.*

Theorem 6.9. *If $f : X \to Y$ is an almost contra $g\alpha r$ - continuous closed injection and Y is ultra normal, then X is $g\alpha r$ - normal.*

Proof. Let E and F be disjoint closed subsets of X. Since f is closed and injective $f(E)$ and $f(F)$ are disjoint closed sets in Y . Since Y is ultra normal there exists disjoint clopen sets U and V in Y such that $f(E) \subset U$ and $f(F) \subset V$. This implies $E \subset f^{-1}(U)$ and $F \subset f^{-1}(V)$. Since f is an almost contra $g\alpha r$ - continuous injection, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $g\alpha r$ - open sets in X. This shows X is $g\alpha r$ - normal. □

Theorem 6.10. *If $f : X \to Y$ is an almost contra $g\alpha r$ - continuous and Y is semi - regular, then f is $g\alpha r$ - continuous.*

Proof. Let $x \in X$ and V be an open set of Y containing $f(x)$. By definition of semi - regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset V$. Since f is almost contra $g\alpha r$ - continuous, there exists $U \in g\alpha r - O(X, x)$ such that $f(U) \subset G$. Hence we have $f(U) \subset G \subset V$. This shows that f is $g\alpha r$ - continuous function. □

§7. Compactness

Definition 7.1. *A space X is said to be:*
(1) $g\alpha r$ - compact if every $g\alpha r$ - open cover of X has a finite subcover.
(2) $g\alpha r$ - closed compact if every $g\alpha r$ - closed cover of X has a finite subcover.
(3) Nearly compact if every regular open cover of X has a finite subcover.
(4) Countably $g\alpha r$ - compact if every countable cover of X by $g\alpha r$ - open sets has a finite subcover.
(5) Countably $g\alpha r$ - closed compact if every countable cover of X by $g\alpha r$ - closed sets has a finite sub cover.
(6) Nearly countably compact if every countable cover of X by regular open sets has a finite sub cover.
(7) $g\alpha r$ - Lindelof if every $g\alpha r$ - open cover of X has a countable sub cover.
(8) $g\alpha r$ - Lindelof if every $g\alpha r$ - closed cover of X has a countable sub cover.
(9) Nearly Lindelof if every regular open cover of X has a countable sub cover.
(10) S - Lindelof if every cover of X by regular closed sets has a countable sub cover.
(11) Countably S - closed if every countable cover of X by regular closed sets has a finite sub -

cover.
(12) S - closed if every regular closed cover of x has a finite sub cover.

Theorem 7.2. *Let $f : X \to Y$ be an almost contra $g\alpha r$ - continuous surjection. Then, the following properties hold:*
(1) If X is $g\alpha r$ - closed compact, then Y is nearly compact.
(2) If X is countably $g\alpha r$ - closed compact, then Y is nearly countably compact.
(3) If X is $g\alpha r$ - Lindelof, then Y is nearly Lindelof.

Proof. (1) Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of Y. Since f is almost contra $g\alpha r$ - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is $g\alpha r$ - closed cover of X. Since X is $g\alpha r$ - closed compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{(V_\alpha) : \alpha \in I_0\}$ which is finite sub cover of Y, therefore Y is nearly compact.
(2) Let $\{V_\alpha : \alpha \in I\}$ be any countable regular open cover of Y . Since f is almost contra $g\alpha r$ - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable $g\alpha r$ - closed cover of X. Since X is countably $g\alpha r$ - closed compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{(V_\alpha) : \alpha \in I_0\}$ is finite subcover for Y . Hence Y is nearly countably compact.
(3) Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of Y. Since f is almost contra $g\alpha r$ - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is $g\alpha r$ - closed cover of X. Since X is $g\alpha r$ - Lindelof, there exists a countable subset I_0 of I such that $X = \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{(V_\alpha) : \alpha \in I_0\}$ is finite sub cover for Y . Therefore, Y is nearly Lindelof. □

Theorem 7.3. *Let $f : X \to Y$ be an almost contra $g\alpha r$ - continuous surjection. Then, the following properties hold:*
(1) If X is $g\alpha r$ - compact, then Y is S - closed.
(2) If X is countably $g\alpha r$ - closed, then Y is is countably S - closed.
(3) If X is $g\alpha r$ - Lindelof, then Y is S - Lindelof.

Proof. (1) Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y. Since f is almost contra $g\alpha r$ - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is $g\alpha r$ - open cover of X. Since X is $g\alpha r$ - compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{V_\alpha : \alpha \in I_0\}$ is finite sub cover for Y. Therefore, Y is S - closed. (2) Let $\{V_\alpha : \alpha \in I\}$ be any countable regular closed cover of Y. Since f is almost contra $g\alpha r$ - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable $g\alpha r$ - open cover of X. Since X is countably $g\alpha r$ - compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{V_\alpha : \alpha \in I_0\}$ is finite sub cover for Y. Hence, Y is countably S - closed. (3) Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y. Since f is almost contra $g\alpha r$ - continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is $g\alpha r$ - open cover of X. Since X is $g\alpha r$ - Lindelof, there exists a countable sub - set I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \cup\{V_\alpha : \alpha \in I_0\}$ is finite sub cover for Y. Hence, Y is S - Lindelof. □

Acknowledgements

The authors gratefully acknowledge the Dr. G. Balaji, Professor of Mathematics and Head, Department of Science and Humanities, Al-Ameen Engineering College, Erode - 638 104, for encouragement and support. The authors also heartfelt thank to Dr. M. Vijayarakavan, Associate Professor, Department of Mathematics, VMKV Engineering College, Salem - 636 308, Tamil Nadu, India, for his kind help and suggestions.

References

[1] Ahmad Al - Omari and Mohd. Salmi Md. Noorani. On Generalized b - closed sets. Bull. Malays. Math. Sci. Soc(2) 32 (2009), no. 1, 19 - 30.

[2] Ahmad Al-Omari and Noorani AMd. Some properties of contra b-continuous and almost contra b-continuous functions. European J. of Pure and App. Math. 2 (2009), 213 - 30.

[3] M. Caldas and S. Jafari. On some applications of b - open sets in topological spaces, Kochi J. Math. 24 (1998), no. 4, 681 - 688.

[4] G. Dimaio and T. Noiri. On s-closed spaces. Indian J. pure appl.math. 18 (1987), no. 3, 226 - 233.

[5] J. Dontchev. Contra continuous functions and strongly S-closed spaces. Int. Math. Sci. 19 (1996), 303 - 310.

[6] D. Iyappan and N. Nagaveni. On semi generalized b - closed set, Nat. Sem. On Mat & Comp. Sci, Jan (2010), Proc.6.

[7] N. Levine. Generalized closed sets in topology. Tend Circ., Mat. Palermo (2). 19 (1970), 89 - 96.

[8] N. Levine. Semi - open sets and semi - continuity in topological spaces. Amer. Math. Monthly, 70 (1963), 36 - 41.

[9] K. Mariappa and S. Sekar. On regular generalized b-closed set. Int. Journal of Math. Analysis. 7 (2013), no. 13, 613 - 624.

[10] Metin Akdag and Alkan Ozkan. Some properties of Contra gb-continuous functions. Journal of New results in Science, 1 (2012), 40 - 49.

[11] O. Njastad, On some classes of nearly open sets, Pacific J Math., 15 (1965), 961 - 970.

[12] T. Noiri. Almost continuous functions. Kyunpook Math. J. 28 (1988), 71 - 81

[13] S. Sekar and G. Kumar. On $g\alpha r$ - closed set in Topological Spaces, International Journal of Pure and Applied Mathematics. 108 (2016), no. 4, 791 - 800.

[14] S. Sekar and K. Mariappa. Almost contra regular generalized b- continuous functions. International Journal of Pure and Applied Mathematics. 97 (2014), no. 2, 161 - 176.

[15] M. K. Singal and A. R. Singal. Almost continuous mappings. Yokohama Math J. 16 (1968), 63 - 73.

Scientia Magna
Vol. 12 (2017), No. 1, 57-65

Contra semi generalized star b- continuous functions in topological spaces

S. Sekar[1] and B. Jothilakshmi[2]

[1]Department of Mathematics, Government Arts College (Autonomous)
Salem – 636 007, Tamil Nadu, India
E-mail: sekar_nitt@rediffmail.com
[2]Department of Mathematics, Government Arts College (Autonomous)
Coimbatore – 641 045, Tamil Nadu, India
E-mail: balujothi87@gmail.com

Abstract In this paper, the authors introduce a new class of functions called contra semi generalized star b-continuous function (briefly contra sg^*b-continuous) in topological spaces. Some characterizations and several properties concerning contra semi generalized star b-continuous functions are obtained.

Keywords sg^*b-open set, sg^*b-continuity, gp^*-continuity, contra gp^*-continuity.
2010 Mathematics Subject Classification 54C05, 54C08, 54C10.

§1. Introduction

In 1970, Dontchev [13] introduced the notions of contra continuous function. A new class of function called contra b-continuous function introduced by Nasef [6]. In 2009, Omari and Noorani [1] have studied further properties of contra b-continuous functions. In this paper, we introduce the concept of contra sg^*b-continuous function via the notion of sg^*b-open set and study some of the applications of this function. We also introduce and study two new spaces called sg^*b-Hausdorff spaces, sg^*b-normal spaces and obtain some new results.

Through out this paper (X, τ) and (Y, σ) represent the non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $A \subseteq X$, the closure of A and interior of A will be denoted by $cl(A)$ and $int(A)$ respectively, union of all sg^*b-open sets X contained in A is called sg^*b-interior of A and it is denoted by sg^*b-$int(A)$, the intersection of all sg^*b-closed sets of X containing A is called sg^*b-closure of A and it is denoted by sg^*b-$cl(A)$.

§2. Preliminaries

Definition 2.1. *Let a subset A of a topological space (X, τ), is called*
1) a pre-open set [10] if $A \subseteq int(cl(A))$.
2) a semi-open set [8] if $A \subseteq cl(int(A))$.

3) a α -open set [12] if $A \subseteq int(cl(int(A)))$.
4) a α generalized closed set (briefly αg- closed) [9] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
5) a generalized $*$ closed set (briefly g^*-closed) [13] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$ open in X.
6) a generalized b- closed set (briefly gb- closed) [1] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
7) a generalized semi-pre closed set (briefly gsp - closed) [7] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
8) a semi generalized closed set (briefly sg- closed) [3] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X.
9) a generalized pre regular closed set (briefly gpr-closed) [6] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.
10) a semi generalized b- closed set (briefly sgb- closed) [7] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X.
11) a $\ddot{g}$ - closed set [12] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg open in X.
12) a semi generalized star b - closed set (briefly sg^*b - closed) [14] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg open in X.

Definition 2.2. *A function $f : (X, \tau) \to (Y, \sigma)$, is called*
1) a contra continuous [5] if $f^{-1}(V)$ is closed in (X, τ) for every open set V of (Y, σ).
2) a contra $\ddot{g}$-continuous [2] if $f^{-1}(V)$ is b-closed in (X, τ) for every open set V of (Y, σ).
3) a contra pre-continuous [10] if $f^{-1}(V)$ is pre-closed in (X, τ) for every open set V of (Y, σ).
4) a contra semi-continuous [8] if $f^{-1}(V)$ is semi-closed in (X, τ) for every open set V of (Y, σ).
5) a contra gpr-continuous [6] if $f^{-1}(V)$ is gpr-closed in (X, τ) for every open set V of (Y, σ).
6) a contra gsp-continuous [4] if $f^{-1}(V)$ is gsp-closed in (X, τ) for every open set V of (Y, σ).
7) a contra gb-continuous [11] if $f^{-1}(V)$ is gb-closed in (X, τ) for every open set V of (Y, σ).
8) a contra sg-continuous [15] if $f^{-1}(V)$ is sg-closed in (X, τ) for every open set V of (Y, σ).

§3. On Contra semi generalized star b - continuous functions

In this section, we introduce contra semi generalized star b - ontinuous functions and investigate some of their properties.

Definition 3.1. *A function $f : (X, \tau) \to (Y, \sigma)$ is called contra semi generalized star b - continuous if $f^{-1}(V)$ is sg^*b - closed in (X, τ) for every open set V in (Y, σ).*

Example 3.2. *Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \varphi, \{a, b\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$, $f(b) = b$, $f(c) = a$. Clearly f is contra sg^*b - continuous.*

Definition 3.3. *Let A be a subset of a space (X, τ).*
(i) The set $\cap\{F \subset X : A \subset F, F$ is $sg^*b - closed\}$ is called the sg^*b - closure of A and it is denoted by $sg^*b - cl(A)$.
(ii) The set $\cup\{G \subset X : G \subset A, G$ is $sg^*b - open\}$ is called the sg^*b - interior of A and it is

denoted by $sg^*b - int(A)$.

Lemma 3.4. *For* $x \in X$, $x \in sg^*b - cl(A)$ *if and only if* $U \cap A \neq \phi$ *for every* sg^*b *- open set* U *containing* x.

Proof. Necessary part : Suppose there exists a sg^*b - open set U containing x such that $U \cup A = \varphi$. Since $A \subset X - U$, $sg^*b - cl(A) \subset X - U$. This implies $x \notin sg^*b - cl(A)$. This is a contradiction. Sufficiency part : Suppose that $x \notin sg^*b - cl(A)$. Then $\exists$ a sg^*b - closed subset F containing A such that $x \notin F$. Then $x \in X - F$ is sg^*b - open, $(X - F) \cap A = \varphi$. This is contradiction. □

Lemma 3.5. *The following properties hold for subsets* A, B *of a space* X*:*
(i) $x \in ker(A)$ *if and only if* $A \cap F \neq \phi$ *for any* $F \in (X, x)$.
(ii) $A \subset ker(A)$ *and* $A = ker(A)$ *if* A *is open in* X.
(iii) If $A \subset B$, *then* $ker(A) \subset ker(B)$.

Theorem 3.6. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a map. The following conditions are equivalent:*
(i) f *is contra* sg^*b *- continuous,*
(ii) The inverse image of each closed in (Y, σ) *is* sg^*b *- open in* (X, τ),
(iii) For each $x \in X$ *and each* $F \in C(Y, f(x))$, *there exists* $U \in sg^*b - O(X)$, *such that* $f(U) \subset F$,
(iv) $f(sg^*b - cl(X)) \subset ker(f(A))$, *for every subset* A *of* X,
(v) $sg^*b - cl(f^{-1}(B)) \subset f^{-1}(ker(B))$, *for every subset* B *of* Y.

Proof. (i) $\Leftrightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious.
(iii) $\to$ (ii) : Let F be any closed set of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in sg^*b - O(X, x)$ such that $f(U_x) \subset F$.
Hence we obtain $f^{-1}(F) = \bigcup \{U_x \backslash x \in f^{-1}(F)\} \in sg^*b - O(X, x)$. Thus the inverse of each closed set in (Y, σ) is sg^*b - open in (X, τ).
(ii) $\Rightarrow$ (iv) : Let A be any subset of X. Suppose that $y \notin ker f(A))$. By lemma there exists $F \in C(Y, y)$ such that $f(A) \cap F = \varphi$. Then, we have $A \cap f^{-1}(F) = \varphi$ and $sg^*b - cl(A) \cap f^{-1}(F) = \varphi$.
Therefore, we obtain $f\big(sg^*b - cl(A)\big) \cap F = \varphi$ and $y \notin f\big(sg^*b - cl(A)\big)$. Hence we have $f\big(sg^*b - cl(X)\big) \subset ker\big(f(A)\big)$.
(iv) $\Rightarrow$ (v): Let B be any subset of Y. By (iv) and Lemma, We have $f\Big(sg^*b - cl\big(f^{-1}(B)\big)\Big) \subset \Big(ker\Big(f\big(f^{-1}(B)\big)\Big)\Big) \subset ker(B)$ and
$sg^*b - cl\big(f^{-1}(B)\big) \subset f^{-1}\big(ker(B)\big)$.
(v) $\Rightarrow$ (i): Let V be any open set of Y. By lemma we have
$sg^*b - cl\big(f^{-1}(V)\big) \subset f^{-1}\big(ker(V)\big) = f^{-1}(V)$ and $sg^*b - cl\big(f^{-1}(V)\big) = f^{-1}(V)$. It follows that $f^{-1}(V)$ is sg^*b - closed in X. We have f is contra sg^*b - continuous. □

Definition 3.7. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is called* sg^*b *- continuous if the preimage of every open set of* Y *is* sg^*b *- open in* X.

Remark 3.8. *The following two examples will show that the concept of* sg^*b *- continuity and contra* sg^*b *- continuity are independent from each other.*

Example 3.9. *Let* $X = Y = \{a, b, c\}$ *with* $\tau = \{X, \varphi, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$ *and* $\sigma = \{Y, \varphi, \{b\}, \{b, c\}\}$. *Define a function* $f : (X, \tau) \to (Y, \sigma)$ *by* $f(a) = b$, $f(b) = c$, $f(c) = a$.

Clearly f is contra sg^*b - continuous but f is not sg^*b - continuous. Because $f^{-1}(\{b,c\}) = \{a,b\}$ is not sg^*b - open in (X,τ) where $\{b,c\}$ is open in (Y,σ).

Example 3.10. *Let $X = Y = \{a,b,c\}$ with $\tau = \{X, \varphi, \{c\}, \{a,c\}\}$ and $\sigma = \{Y, \varphi, \{a,b\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is sg^*b - continuous but f is not contra sg^*b - continuous. Because $f^{-1}(\{a,b\}) = \{a,c\}$ is not contra sg^*b - closed in (X,τ) where $\{a,b\}$ is open in (Y,σ).*

Theorem 3.11. *If a function $f : (X,\tau) \to (Y,\sigma)$ is contra sg^*b - continuous and (Y,σ) is regular then f is sg^*b - continuous.*

Proof. Let x be an arbitrary point of (X,τ) and V be an open set of (Y,σ) containing $f(x)$. Since (Y,σ) is regular, there exists an open set W of (Y,σ) containing $f(x)$ such that $cl(W) \subset V$. Since f is contra sg^*b - continuous, by Theorem there exists $U \in sg^*b - O(X,x)$ such that $f(U) \subset cl(W)$. Then $f(U) \subset cl(W) \subset V$. Hence f is sg^*b - continuous. □

Theorem 3.12. *Every contra - continuous function is contra sg^*b - continuous function.*

Proof. Let V be an open set in (Y,σ). Since f is contra - continuous function, $f^{-1}(V)$ is b - closed in (X,τ). Every closed set is sg^*b - closed. Hence $f^{-1}(V)$ is sg^*b - closed in (X,τ). Thus f is contra sg^*b - continuous function. □

Remark 3.13. *The converse of theorem need not be true as shown in the following example.*

Example 3.14. *Let $X = Y = \{a,b,c\}$ with $\tau = \{X, \varphi, \{b\}, \{c\}, \{b,c\}\}$ and $\sigma = \{Y, \varphi, \{c\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = a$, $f(b) = c$, $f(c) = b$. Clearly f is contra sg^*b - continuous but f is not contra continuous. Because $f^{-1}(\{c\}) = \{b\}$ is not closed in (X,τ) where $\{c\}$ is open in (Y,σ).*

Theorem 3.15. *(i) Every contra $\ddot{g}$-continuous function is contra sg^*b-continuous function.*
(ii) Every contra semi-continuous function is contra sg^*b-continuous function.
(iii) Every contra α - continuous function is contra sg^*b-continuous function.
(iv) Every contra pre-continuous function is contra gb-continuous function.
(v) Every contra αg-continuous function is contra gsp-continuous function.
(vi) Every contra sg^*b-continuous function is contra gsp-continuous function.
(vii) Every contra sg^*b-continuous function is contra gb-continuous function.
(viii) Every contra sg-continuous function is contra sg^*b-continuous function.

Remark 3.16. *Converse of the above statements is not true as shown in the following example.*

Example 3.17. *(i) Let $X = Y = \{a,b,c\}$ with $\tau = \{X, \varphi, \{a,b\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{a,c\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is contra sg^*b - continuous but f is not contra $\ddot{g}$-continuous. Because $f^{-1}(\{a,c\}) = \{b,c\}$ is not $\ddot{g}$-closed in (X,τ) where $\{a,c\}$ is open in (Y,σ).*
(ii) Let $X = Y = \{a,b,c\}$ with $\tau = \{X, \varphi, \{a,c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{a,b\}\{a,c\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Clearly f is contra sg^*b-continuous but f is not contra semi-continuous. Because $f^{-1}(\{a,c\}) = \{b,c\}$ is not semi-closed

in (X,τ) where $\{a,c\}$ is open in (Y,σ).

(iii) Let $X = Y = \{a,b,c\}$ with $\tau = \{X,\varphi,\{b\},\{a,b\}\}$ and $\sigma = \{Y,\varphi,\{b\},\{b,c\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is contra sg^*b-continuous but f is not contra α-continuous. Because $f^{-1}(\{b,c\}) = \{a,b\}$ is not α-closed in (X,τ) where $\{b,c\}$ is open in (Y,σ).

(iv) Let $X = Y = \{a,b,c\}$ with $\tau = \{X,\varphi,\{a\},\{b\},\{a,b\}\}$ and $\sigma = \{Y,\varphi,\{b\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Clearly f is contra gb - continuous but f is not contra sg^*b - continuous. Because $f^{-1}(\{b\}) = \{a\}$ is not pre - closed in (X,τ) where $\{b\}$ is open in (Y,σ).

(v) Let $X = Y = \{a,b,c\}$ with $\tau = \{X,\varphi,\{a,c\}\}$ and $\sigma = \{Y,\varphi,\{a\},\{a,b\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$. Clearly f is contra sg^*b - continuous but f is not contra αg-continuous. Because $f^{-1}(\{a\}) = \{a\}$ is not αg-closed in (X,τ) where $\{a\}$ is open in (Y,σ).

(vi) Let $X = Y = \{a,b,c\}$ with $\tau = \{X,\varphi,\{a\},\{a,c\}\}$ and $\sigma = \{Y,\varphi,\{b,c\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is contra gsp - continuous but f is not contra g^* - continuous. Because $f^{-1}(\{a,c\}) = \{a,b\}$ is not sg^*b - closed in (X,τ) where $\{a,c\}$ is open in (Y,σ).

(vii) Let $X = Y = \{a,b,c\}$ with $\tau = \{X,\varphi,\{a\},\{a,b\}\}$ and $\sigma = \{Y,\varphi,\{a\},\{b\},\{a,b\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ be $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is contra gb - continuous but f is not contra sg^*b - continuous. Because $f^{-1}(\{a,b\}) = \{a,c\}$ is not sg^*b - closed in (X,τ) where $\{a,b\}$ is open in (Y,σ).

(viii) Let $X = Y = \{a,b,c\}$ with $\tau = \{X,\varphi,\{b\},\{a,b\}\}$ and $\sigma = \{Y,\varphi,\{a\},\{a,b\},\{a,c\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Clearly f is contra sg^*b - continuous but f is not contra sg - continuous. Because $f^{-1}(\{a,c\}) = \{b,c\}$ is not sgb - closed in (X,τ) where $\{a,c\}$ is open in (Y,σ).

Remark 3.18. *The concept of contra sg^*b - continuous and contra sgb - continuous are independent as shown in the following examples.*

Example 3.19. *Let $X = Y = \{a,b,c\}$ with $\tau = \{X,\varphi,\{b\},\{a,b\}\}$ and $\sigma = \{Y,\varphi,\{a,c\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = a$, $f(b) = c$, $f(c) = b$. Clearly f is contra sg^*b-continuous but f is not contra sgb-continuous. Because $f^{-1}(\{a,c\}) = \{a,b\}$ is not sgb-closed in (X,τ) where $\{a,c\}$ is open in (Y,σ).*

Example 3.20. *Let $X = Y = \{a,b,c\}$ with $\tau = \{X,\varphi,\{a\},\{c\},\{a,c\},\{b,c\}\}$ and $\sigma = \{Y,\varphi,\{a\},\{a,b\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = c$, $f(b) = b$, $f(c) = a$. Clearly f is contra sgb-continuous but f is not contra sg^*b - continuous. Because $f^{-1}(\{a,b\}) = \{a,c\}$ is not sg^*b - closed in (X,τ) where $\{a,b\}$ is open in (Y,σ).*

Definition 3.21. *A space (X,τ) is said to be (i) sg^*b - space if every sg^*b - open set of X is open in X, (ii) locally sg^*b - indiscrete if every sg^*b - open set of X is closed in X.*

Theorem 3.22. *If a function $f : X \to Y$ is contra sg^*b - continuous and X is sg^*b - space then f is contra continuous.*

Proof. Let $V \in O(Y)$. Then $f^{-1}(V)$ is sg^*b - closed in X. Since X is sg^*b - space, $f^{-1}(V)$ is closed in X. Hence f is contra continuous. □

Theorem 3.23. *Let X be locally sg^*b - indiscrete. If $f : X \to Y$ is contra sg^*b - continuous, then it is continuous.*

Proof. Let $V \in O(Y)$. Then $f^{-1}(V)$ is sg^*b - closed in X. Since X is locally sg^*b - indiscrete space, $f^{-1}(V)$ is open in X. Hence f is continuous. □

Definition 3.24. *A function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by G_f.*

Definition 3.25. *The graph G_f of a function $f : X \to Y$ is said to be contra sg^*b - closed if for each $(x, y) \in (X \times Y) - G_f$ there exists $U \in sg^*b - O(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G_f$.*

Theorem 3.26. *If a function $f : X \to Y$ is contra sg^*b - continuous and Y is Urysohn, then G_f is contra sg^*b - closed in the product space $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - G_f$. Then $y \neq f(x)$ and there exist open sets H_1, H_2 such that $f(x) \in H_1,\ y \in H_2$ and $cl(H_1) \cap cl(H_2) = \varphi$. From hypothesis, there exists $V \in sg^*b - O(X, x)$ such that $f(V) \subset cl(H_1)$. Therefore, we have $f(V) \cap cl(H_2) = \varphi$. This shows that G_f is contra sg^*b - closed in the product space $X \times Y$. □

Theorem 3.27. *If $f : X \to Y$ is sg^*b - continuous and Y is T_1, then G_f is contra sg^*b - closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - G_f$. Then $y \neq f(x)$ and there exist open set V of Y such that $f(x) \in V$ and $y \notin V$. Since f is sg^*b - continuous, there exists $U \in sg^*b - O(X, x)$ such that $f(U) \subset V$. Therefore, we have $f(U) \cap (Y - V) = \varphi$ and $(Y - V) \in sg^*b - C(Y, y)$. This shows that G_f is contra sg^*b - closed in $X \times Y$. □

Theorem 3.28. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$, the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is contra sg^*b - continuous, then f is contra sg^*b - continuous.*

Proof. Let U be an open set in Y, then $X \times U$ is an open set in $X \times Y$. Since g is contra sg^*b - continuous. It follows that $f^{-1}(U) = g^{-1}(X \times U)$ is an sg^*b - closed in X. Hence f is sg^*b - continuous. □

Theorem 3.29. *If $f : X \to Y$ is a contra sg^*b - continuous function and $g : Y \to Z$ is a continuous function, then $g \circ f : X \to Z$ is contra sg^*b - continuous.*

Proof. Let $V \in O(Y)$. Then $g^{-1}(V)$ is open in Y. Since f is contra sg^*b - continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is sg^*b - closed in X. Therefore, $g \circ f : X \to Z$ is contra sg^*b - continuous. □

Theorem 3.30. *Let $p : X \times Y \to Y$ be a projection. If A is sg^*b - closed subset of X, then $p^{-1}(A) = A \times Y$ is sg^*b - closed subset of $X \times Y$.*

Proof. Let $A \times Y \subset U$ and U be a regular open set of $X \times Y$. Then $U = V \times Y$ for some regular open set V of X. Since A is sg^*b - closed in X, $bcl(A)$ and so $bcl(A) \times Y \subset V \times Y = U$. Therefore $bcl(A \times Y) \subset U$. Hence $A \times Y$ is sg^*b - closed sub set of $X \times Y$. □

§4. Applications

Definition 4.1. *A topological space (X, τ) is said to be sg^*b - Hausdorff space if for each pair of distinct points x and y in X there exists $U \in sg^*b - O(X, x)$ and $V \in sg^*b - O(X, y)$ such that $U \cap V = \varphi$.*

Example 4.2. *Let $X = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Let x and y be two distinct points of X, there exists an sg^*b-open neighbourhood of x and y respectively such that $\{x\} \cap \{y\} = \varphi$. Hence (X, τ) is sg^*b-Hausdorff space.*

Theorem 4.3. *If X is a topological space and for each pair of distinct points x_1 and x_2 in X, there exixts a function f of X into Uryshon topological space Y such that $f(x_1) \neq f(x_2)$ and f is contra sg^*b-continuous at x_1 and x_2, then X is sg^*b-Hausdorff space.*

Proof. Let x_1 and x_2 be any distinct points in X. By hypothesis, there is a Uryshon space Y and a function $f : X \to Y$ such that $f(x_1) \neq f(x_2)$ and f is contra sg^*b-continuous at x_1 and x_2. Let $y_i = f(x_i)$ for $i = 1, 2$ then $y_1 \neq y_2$. Since Y is Uryshon, there exists open sets U_{y_1} and U_{y_2} containing y_1 and y_2 respectively in Y such that $cl(U_{y_1}) \cap cl(U_{y_2}) = \varphi$. Since f is contra sg^*b-continuous at x_1 and x_2, there exists and sg^*b-open sets V_{x_1} and V_{x_2} containing x_1 and x_2 respectively in X such that $f(V_{xi}) \subset cl(U_{yi})$ for $i = 1, 2$. Hence we have $(V_{x_1}) \cap (V_{x_2}) = \varphi$. Therefore X is sg^*b - Hausdorff space. □

Corollary 4.4. *If f is contra sg^*b - continuous injection of a topological space X into a Uryshon space Y then X is sg^*b-Hausdorff.*

Proof. Let x_1 and x_2 be any distinct points in X. By hypothesis, f is contra sg^*b-continuous function of X into a Uryshon space Y such that $f(x_1) \neq f(x_2)$, because f is injective. Hence by theorem, X is sg^*b - Hausdorff. □

Definition 4.5. *A topological space (X, τ) is said to be sg^*b - normal if each pair of non - empty disjoint closed sets in (X, τ) can be separated by disjoint sg^*b - open sets in (X, τ).*

Definition 4.6. *A topological space (X, τ) is said to be ultra normal if each pair of non - empty disjoint closed sets in (X, τ) can be separated by disjoint clopen sets in (X, τ).*

Theorem 4.7. *If $f : X \to Y$ is a contra sg^*b - continuous function, closed, injection and Y is Ultra normal, then X is sg^*b - normal.*

Proof. Let U and V be disjoint closed subsets of X. Since f is closed and injective, $f(U)$ and $f(V)$ are disjoint subsets of Y. Since Y is ultra normal, there exists disjoint clopen sets A and B such that $f(U) \subset A$ and $f(V) \subset B$. Hence $U \subset f^{-1}(A)$ and $V \subset f^{-1}(B)$. Since f is contra sg^*b - continuous and injective, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint sg^*b - open sets in X. Hence X is sg^*b - normal. □

Definition 4.8. *A topological space X is said to be sg^*b - connected if X is not the union of two disjoint non - empty sg^*b - open sets of X.*

Theorem 4.9. *A contra sg^*b - continuous image of a sg^*b - connected space is connected.*

Proof. Let $f : X \to Y$ be a contra sg^*b - continuous function of sg^*b - connected space X onto a topological space Y. If possible, let Y be disconnected. Let A and B form disconnectedness of Y. Then A and B are clopen and $Y = A \cup B$ where $A \cap B = \varphi$. Since f is contra sg^*b - continuous, $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are non - empty sg^*b - open sets in X. Also $f^{-1}(A) \cap f^{-1}(B) = \varphi$. Hence X is non - sg^*b - connected which is a contradiction. Therefore Y is connected. □

Theorem 4.10. *Let X be sg^*b - connected and Y be T_1. If $f : X \to Y$ is a contra sg^*b - continuous, then f is constant.*

Proof. Since Y is T_1 space $v = \{f^{-1}(y) : y \in Y\}$ is a disjoint sg^*b - open partition of X. If $|v| \geq 2$, then X is the union of two non empty sg^*b - open sets. Since X is sg^*b - connected, $|v| = 1$. Hence f is constant. □

Theorem 4.11. *If $f : X \to Y$ is a contra sg^*b - continuous function from sg^*b - connected space X onto space Y, then Y is not a discrete space.*

Proof. Suppose that Y is discrete. Let A be a proper non - empty open and closed subset of Y. Then $f^{-1}(A)$ is a proper non - empty sg^*b - clopen subset of X, which is a contradiction to the fact X is sg^*b - connected. □

Acknowledgements

The authors gratefully acknowledge the Dr. G. Balaji, Professor of Mathematics and Head, Department of Science and Humanities, Al-Ameen Engineering College, Erode - 638 104, for encouragement and support. The authors also heartfelt thank to Dr. M. Vijayarakavan, Associate Professor, Department of Mathematics, VMKV Engineering College, Salem - 636 308, Tamil Nadu, India, for his kind help and suggestions.

References

[1] Ahmad Al - Omari and Mohd. Salmi Md. Noorani. On Generalized b - closed sets. Bull. Malays. Math. Sci. Soc(2) 32 (2009), no. 1, 19 - 30.

[2] D. Andrijevic. Semi - pre open sets. Mat.Vesnik. 38 (1986), no. 1, 24 - 32.

[3] P. Bhattacharya and B. K. Lahiri. Semi - generalized closed sets on topology, Indian J.Maths. 29 (1987), no. 3, 375 - 382.

[4] J. Dontchev. On generalized semi - pre open sets. Mem. Fac. Sci. Kochi. Univ. ser. A. math 16 (1995), 35.

[5] J. Dontchev. Contra continuous functions and strongly S-closed spaces. Int. Math. Sci. 19 (1996), 303 - 310.

[6] Y. Gnanambal. On generalized pre - regular closed sets in topological spaces. Indian J. Pure Appl. Math. 28 (1997), 351 - 360.

[7] D. Iyappan and N. Nagaveni. On semi generalized b - closed set, Nat. Sem. On Mat & Comp.Sci, Jan (2010), Proc.6.

[8] N. Levine. Semi - open sets and semi - continuity in topological spaces. Amer. Math. Monthly, 70 (1963), 36 - 41.

[9] H. Maki, R. Devi and K.Balachandran. Associated topologies of generalized α - closed sets and α - generalized closed sets, Mem. Fac. Sci. Kochi. Univ. Ser. A. Math. 15 (1994), 51 - 63.

[10] A. S. Mashor Abd.El, M. E. Monsef and S. N. Ei - Deeb. On Pre continuous and weak pre - continuous mapping, Proc. Math. Phys. Soc. Egypt. 53 (1982), 47 - 53.

[11] Metin Akdag and Alkan Ozkan. Some properties of Contra gb-continuous functions. Journal of New results in Science, 1 (2012), 40 - 49.

[12] O. Njastad, On some classes of nearly open sets, Pacific J Math., 15 (1965), 961 - 970.

[13] N. Nagaveni. Studies on generalized on homeomorphisms in topological spaces. Ph.D Thesis, Bharathiar University, Coimbatore 1999.

[14] O. Ravi and S. Ganesan. g - closed sets in Topology, International Journal of Computer Science & Emerging Technologies. 2 (2011), no. 3, 330 - 337.

[15] O. Ravi, M. Lellis Thivagar and R. Latha. Properties of contra sg continuous maps. Gen. Math. Notes. 4 (2011), no. 1, 70 - 84.

[16] S. Sekar and G. Kumar. On $g\alpha r$ - closed set in Topological Spaces, International Journal of Pure and Applied Mathematics. 108 (2016), no. 4, 791 - 800.

Scientia Magna
Vol. 12 (2017), No. 1, 66-77

On some generalised I-convergent sequence spaces of interval numbers

Vakeel. A. Khan† and Mohd Shafiq‡

† ‡ Department of Mathematics, Aligarh Muslim University, Aligarh-2-2002 INDIA
E-mail: vakhanmaths@gmail.com

Abstract In this article we introduce and study some spaces of I-convergent sequences of interval numbers with the help of a sequence $\mathcal{F} = (f_k)$ of modulii and a bounded sequence $p = (p_k)$ of positive real numbers. We study some topological and algebraic properties, prove the decomposition theorem and study some inclusion relations on these spaces.

Keywords Interval numbers, ideal, filter, I-convergent sequence, solid and monotone space, Banach space, modulus function.
2000 Mathematics Subject Classification: 41A10, 41A25, 41A36, 40A30.

§1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively. Let ℓ_∞, c and c_0 be denote the Banach spaces of bounded, convergent and null sequences respectively with norm $\|x\| = \sup_k \mid x_k \mid$. We denote $\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$, the space of all real or complex sequences.

Any subspace λ of the linear space ω of sequences is called a sequence space. A sequence space λ with linear topology is called a K-space provided each of maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous, for all $i \in \mathbb{N}$. A space λ is called an FK-space provided λ is complete linear metric space. An FK-space whose topology is normable is called a BK-space.

It is an admitted fact that the real and complex numbers are playing a vital role in the world of mathematics. Many mathematical structures have been constructed with the help of these numbers. In recent years, since 1965 fuzzy numbers and interval numbers also managed their place in the world of mathematics and credited into account some alike structures. Interval arithmetic was first suggested by P. S. Dwyer [5] in 1951. Further development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by R. E. Moore [15] in 1959 and Moore and Yang [16] and others and have developed applications to differential equations.

Recently, Chiao [4] introduced sequences of interval numbers and defined usual convergence of sequences of interval numbers. Şengönül and Eryılmaz [27] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete.

A set (closed interval) of real numbers x such that $a \leq x \leq b$ is called an interval number.[4] A real interval can also be considered as a set. Thus, we can investigate some properties of interval numbers for instance, arithmetic properties or analysis properties. Let us denote the set of all real valued closed intervals by $I\mathbb{R}$. Any element of $I\mathbb{R}$ is called a closed interval and it is denoted by $\bar{A} = [x_l, x_r]$. $I\mathbb{R}$ is a quasilinear space under the algebraic operations and a partial order relation for $I\mathbb{R}$ found in [27, 30] and any subspace of $I\mathbb{R}$ is called quasilinear subspace (see [27, 30]).

The set of all interval numbers $I\mathbb{R}$ is a complete metric space defined by

$$d(\bar{A}_1, \bar{A}_2) = \max\{\mid x_{1_l} - x_{2_l} \mid, \mid x_{1_r} - x_{2_r} \mid\}, \text{(see [16, 27])}, \tag{1}$$

where x_l and x_r are first and last points of $\bar{A}$, respectively.

In a special case, $\bar{A}_1 = [a, a]$, $\bar{A}_2 = [b, b]$, we obtain the usual metric of $\mathbb{R}$ with

$$d(\bar{A}_1, \bar{A}_2) = \mid a - b \mid .$$

Let us define transformation f from $\mathbb{N}$ to $I\mathbb{R}$ by $k \rightarrow f(k) = \bar{\mathcal{A}}$, $\bar{\mathcal{A}} = (\bar{A}_k)$. The function f is called sequence of interval numbers, where $\bar{A}_k$ is the k^{th} term of the sequence $(\bar{A}_k)$.

Let us denote the set of sequences of interval numbers with real terms by

$$\omega(\bar{\mathcal{A}}) = \{\bar{\mathcal{A}} = (\bar{A}_k) : \bar{A}_k \in I\mathbb{R}\}. \tag{2}$$

The algebraic properties of $\omega(\bar{\mathcal{A}})$ can be found in [4, 27].

The following definitions were given by Şengönül and Eryılmaz in [27]. A sequence $\bar{\mathcal{A}} = (\bar{A}_k) = ([x_{k_l}, x_{k_r}])$ of interval numbers is said to be convergent to an interval number $\bar{A}_0 = [x_{0_l}, x_{0_r}]$ if for each $\epsilon > 0$, there exists a positive integer n_0 such that $d(\bar{A}_k, \bar{A}_0) < \epsilon$, for all $k \geq n_0$ and we denote it as $\lim_k \bar{A}_k = \bar{A}_0$.

Thus, $\lim_k \bar{A}_k = \bar{A}_0 \Leftrightarrow \lim_k x_{k_l} = x_{0_l}$ and $\lim_k x_{k_r} = x_{0_r}$ and it is said to be Cauchy sequence of interval numbers if for each $\epsilon > 0$, there exists a positive integer k_0 such that $d(\bar{A}_k, \bar{A}_m) < \epsilon$, whenever $k,\ m \geq k_0$.

Let us denote the space of all convergent, null and bounded sequences of interval numbers by $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ and $\ell_\infty(\bar{\mathcal{A}})$, respectively. The sets $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ and $\ell_\infty(\bar{\mathcal{A}})$ are complete metric spaces with the metric

$$\widehat{d}(\bar{A}_k, \bar{B}_k) = \sup_k \max\{|x_{k_l} - y_{k_l}|, |x_{k_r} - y_{k_r}|\} \text{(see [27])}. \tag{3}$$

If we take $\bar{B}_k = \bar{O}$ in (3) then, the metric $\widehat{d}$ reduces to

$$\widehat{d}(\bar{A}_k, \bar{O}) = \sup_k \max\{|x_{k_l}|, |x_{k_r}|\}. \tag{4}$$

In this paper, we assume that a norm $||\bar{A}_k||$ of the sequence of interval numbers $(\bar{A}_k)$ is the distance from $(\bar{A}_k)$ to $\bar{O}$ and satisfies the following properties: $\forall \bar{A}_k, \bar{B}_k \in \lambda(\bar{\mathcal{A}})$ and $\forall \alpha \in \mathbb{R}$

(N_1) $\forall \bar{A}_k \in \lambda(\bar{\mathcal{A}}) - \{\bar{O}\}, ||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})} > 0$,

(N_2) $||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})} = 0 \Leftrightarrow \bar{A}_k = \bar{O}$,

(N_3) $||\bar{A}_k + \bar{B}_k||_{\lambda(\bar{\mathcal{A}})} \leq ||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})} + ||\bar{B}_k||_{\lambda(\bar{\mathcal{A}})}$,

(N_4) $||\alpha\bar{A}_k||_{\lambda(\bar{\mathcal{A}})} = |\alpha|||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})}$, where $\lambda(\bar{\mathcal{A}})$ is a subset of $\omega(\bar{\mathcal{A}})$.

Let $\bar{\mathcal{A}} = (\bar{A}_k) = ([x_{k_l}, x_{k_r}])$ be the element of $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ or $\ell_\infty(\bar{\mathcal{A}})$. Then, in the light of above discussion, the classes of sequences $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ and $\ell_\infty(\bar{\mathcal{A}})$ are normed interval spaces normed by

$$\parallel \bar{\mathcal{A}} \parallel = \sup_k \max\{\mid x_{k_l} \mid, \mid x_{k_r} \mid\} \text{ (see [27])}. \tag{5}$$

Throughout, $\bar{O} = [0, 0]$ and $\bar{I} = [1, 1]$ represent zero and identity interval numbers according to addition and multiplication, respectively.

As a generalisation of usual convergence for the sequences of real or complex numbers, the concept of statistical convergent was first introduced by Fast [6] and also independently by Buck [3] and Schoenberg [26]. Later on, it was further investigated from a sequence space point of view and linked with the Summability Theory by Fridy [7], Šalát [23], Tripathy [28] and many others. The statistical convergence has been extended to interval numbers by Esi as follows in [1, 2].

Let us suppose that $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}})$. If, for every $\epsilon > 0$,

$$\lim_k \frac{1}{k}|\{n \in \mathbb{N} : \| \bar{A}_n - \bar{A}_0 \| \geq \epsilon,\ n \leq k\}| = 0. \tag{6}$$

Then, the sequence $\bar{\mathcal{A}} = (\bar{A}_k)$ is said to be statistically convergent to an interval number $\bar{A}_0$, where vertical lines denote the cardinality of the enclosed set. That is, if $\delta(A(\epsilon)) = 0$, where $A(\epsilon) = \{k \in \mathbb{N} : \| \bar{A}_k - \bar{A}_0 \| \geq \epsilon\}$.

The notion of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salăt and Wilczyński [13,14]. Later on, it was studied by Šalát, Tripathy and Ziman[24,25], Esi and Hazarika [1], Tripathy and Hazarika [29], Khan *et al* [8,9,10], Mursaleen and Sunil [17] and many others.

Definition 1.1. Let $\mathbb{N}$ be the set of natural numbers. Then, a family of sets $I \subseteq 2^{\mathbb{N}}$ (power set of $\mathbb{N}$) is said to be an ideal if

(i) I is additive. That is, $\forall A, B \in I \Rightarrow A \cup B \in I$,

(ii) I is hereditary. That is $\forall A \in I$and $B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $\pounds(I) \subseteq 2^{\mathbb{N}}$ is said to be filter on $\mathbb{N}$ if and only if

(i) $\Phi \notin \pounds(I)$,

(ii) $\forall\ A,\ B \in \pounds(I)$ we have $A \cap B \in \pounds(I)$,

(iii) $\forall\ A \in \pounds(\mathrm{I})$ and $A \subseteq B \Rightarrow B \in \pounds(I)$.

An Ideal $I \subseteq 2^{\mathbb{N}}$ is called non-trivial if $I \neq 2^{\mathbb{N}}$.

A non-trivial ideal $I \subseteq 2^{\mathbb{N}}$ is called admissible if $\{\{x\} : x \in \mathbb{N}\} \subseteq I$.

Let us suppose that I be an ideal. Then, a sequence $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) \subset \omega(\bar{\mathcal{A}})$.

(i) is said to be I-convergent to an interval number $\bar{A}_0$ if for every $\epsilon > 0$,

$$\{k \in \mathbb{N} : \| \bar{A}_k - \bar{A}_0 \| \geq \epsilon\} \in I.$$

In this case, we write $I - \lim \bar{A}_k = \bar{A}_0$. If $\bar{A}_0 = \bar{O}$. Then, the sequence $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}})$ is said to be I-null. In this case, we write $I - \lim \bar{A}_k = \bar{O}$.

(ii) is said to be I-Cauchy, if for every $\epsilon > 0$, there exists a number $m = m(\epsilon)$ such that

$$\{k \in \mathbb{N} : \| \bar{A}_k - \bar{A}_m \| \geq \epsilon\} \in I.$$

(iii) is said to be I-bounded, if there exists some $M > 0$ such that

$$\{k \in \mathbb{N} : \| \bar{A}_k \| \geq M\} \in I.$$

We know that for each ideal I, there is a filter $\pounds(I)$ corresponding to I. That is, $\pounds(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$.

Definition 1.2. A sequence space $\lambda(\bar{\mathcal{A}})$ of interval numbers is

(iv) said to be solid (normal), if $(\alpha_k \bar{A}_k) \in \lambda(\bar{\mathcal{A}})$, whenever $(\bar{A}_k) \in \lambda(\bar{\mathcal{A}})$ and for any sequence (α_k) of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$,

(v) said to be symmetric, if $(\bar{A}_{\pi(k)}) \in \lambda(\bar{\mathcal{A}})$, whenever $\bar{A}_k \in \lambda(\bar{\mathcal{A}})$, where π is a permutation on $\mathbb{N}$,

(vi) said to be sequence algebra, if $(\bar{A}_k) * (\bar{B}_k) = (\bar{A}_k.\bar{B}_k) \in \lambda(\bar{\mathcal{A}})$ whenever $(\bar{A}_k),\ (\bar{B}_k) \in \lambda(\bar{\mathcal{A}})$,

(vii) said to be convergence free, if $(\bar{B}_k) \in \lambda(\bar{\mathcal{A}})$ whenever $(\bar{A}_k) \in \lambda(\bar{\mathcal{A}})$ and $\bar{A}_k = \bar{O}$ implies $\bar{B}_k = \bar{O}$, for all k.

Definition 1.3. Let $K = \{k_1 < k_2 < k_3 ...\} \subset \mathbb{N}$. The K-step space of the $\lambda(\bar{\mathcal{A}})$ is a sequence space $\mu_K^{\lambda(\bar{\mathcal{A}})} = \{(\bar{A}_{k_n}) \in \omega(\bar{\mathcal{A}}) : (\bar{A}_k) \in \lambda(\bar{\mathcal{A}})\}$.

Definition 1.4. A canonical pre-image of a sequence $(\bar{A}_{k_n}) \in \mu_K^{\lambda(\bar{\mathcal{A}})}$ is a sequence $(\bar{B}_k) \in \omega(\bar{\mathcal{A}})$ defined by

$$\bar{B}_k = \begin{cases} \bar{A}_k, & \text{if } k \in K, \\ \bar{O}, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space $\mu_K^{\lambda(\bar{\mathcal{A}})}$ is a set of canonical preimages of all elements in $\mu_K^{\lambda(\bar{\mathcal{A}})}$. That is, $\bar{\mathcal{B}}$ is in the canonical preimage of $\mu_K^{\lambda(\bar{\mathcal{A}})}$ iff $\bar{\mathcal{B}}$ is the canonical preimage of some $\bar{\mathcal{A}} \in \mu_K^{\lambda(\bar{\mathcal{A}})}$.

Definition 1.5. A sequence space $\lambda(\bar{\mathcal{A}})$ is said to be monotone, if it contains the canonical preimages of its step space.

Definition 1.6. A function $f : [0, \infty) \longrightarrow [0, \infty)$ is called a modulus function if

(1) $f(t) = 0$ if and only if $t = 0$, $f(t + u) \leq f(t) + f(u)$ for all $t,\ u \geq 0$,

(3) f is increasing,

(4) f is continuous from the right at zero.

A modulus function f is said to satisfy Δ_2-Condition for all values of u if there exists a constant $K > 0$ such that $f(Lu) \leq \mathrm{KL}f(u)$ for all values of $L > 1$. The idea of modulus function was introduced by Nakano in 1953, (See [19] , Nakano, 1953).

Ruckle [20−22] used the idea of a modulus function f to construct the sequence space

$$X(f) = \Big\{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\Big\} = \Big\{x = x_k : \big(f(|\ x_k\ |)\big) \in X\Big\}. \tag{7}$$

After then, E. Kolk [11,12] gave an extension of $X(f)$ by considering a sequence of moduli $\mathcal{F} = (f_k)$ and defined the sequence space

$$X(\mathcal{F}) = \Big\{x = (x_k) : \big(f_k(|x_k|)\big) \in X\Big\}. \tag{8}$$

Mursaleen and Noman [17] introduced the notion of λ-convergent and λ-bounded sequences. We extended this concept to the sequence of interval numbers as follows.

Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity. That is,

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 ..., \lambda_k \to \infty \text{ as } k \to \infty. \tag{9}$$

The sequence $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}})$ is λ-convergent to an interval number $\bar{A}_0$, called the λ-limit of $\bar{\mathcal{A}}$, if $\Lambda_m(\bar{\mathcal{A}}) \to \bar{A}_0$, as $m \to \infty$, where

$$\Lambda_m(\bar{\mathcal{A}}) = \frac{1}{\lambda_m}\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1})\bar{A}_k,\ k \in \mathbb{N}.$$

Here and in the sequel, we shall use the convention that any term with a negative subscript is equal to naught. For example, $\lambda_{-1} = 0$.

In particular, $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}})$ is said to be λ-null, if $\Lambda_m(\bar{\mathcal{A}}) \to 0$, as $m \to \infty$.

The sequence $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}})$ is λ-bounded, if $\sup_m \parallel \Lambda_m(\bar{\mathcal{A}}) \parallel < \infty$. It can be seen that if $\lim_m \bar{A}_m = \bar{A}$ in the ordinary sense of convergence of interval numbers, then

$$\lim_m \left(\frac{1}{\lambda_m}\left(\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1}) \parallel \bar{A}_k - \bar{A} \parallel\right)\right) = 0. \tag{10}$$

This implies that

$$\lim_m \parallel \Lambda_m(\bar{\mathcal{A}}) - \bar{A} \parallel = \lim_m \parallel \frac{1}{\lambda_m}\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1})(\bar{A}_k - \bar{A}) \parallel = 0, \tag{11}$$

which yields that $\lim_m \Lambda_m(\bar{\mathcal{A}}) = \bar{A}$ and hence $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}})$ is λ-convergent to $\bar{A}$.

Let us denote the classes of I-convergent, I-null, bounded I-convergent and bounded I-null sequences of interval numbers with $\mathcal{C}^I(\bar{A})$, $\mathcal{C}^I_\circ(\bar{A})$, $\mathcal{M}^I_{\mathcal{C}}(\bar{A})$ and $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{A})$, respectively.

We need the following popular inequalities throughout the paper.

Let $p = (p_k)$ be the bounded sequence of positive reals numbers. For any complex λ, whenever $H = \sup_k(p_k) < \infty$, we have $| \lambda |^{p_k} \leq \max(1, | \lambda |^H)$. Also, whenever $H = \sup_k(p_k)$, we have $| a_k + b_k |^{p_k} \leq C(| a_k |^{p_k} + | b_k |^{p_k})$, where $C = \max(1, 2^{H-1})$.

For any modulus function f, we have the inequalities $| f(x) - f(y) | \leq f(x - y)$ and $f(nx) \leq nf(x)$, for all $x,\ y \in [0, \infty]$.

Now, we give some important Lemmas.

Lemma.1.1. Every solid space is monotone.

Lemma.1.2. Let $K \in \pounds(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$, where $\pounds(I) \subseteq 2^{\mathbb{N}}$ filter on $\mathbb{N}$.

Lemma.1.3. If $I \subseteq 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap \mathbb{N} \notin I$.

§2. Main results

Let us give a most important definition for this paper:

Definition 2.1.[30] Let $\bar{X}$ be a space of interval numbers. A function $g : \bar{X} \longrightarrow \mathbb{R}$ is called paranorm on $\bar{X}$, if for all $A,\ B \in \bar{X}$,

(P_1) $g(A) = 0$, if $A = \bar{0}$,

(P_2) $g(A) \geq 0$,

(P_3) $g(-A) = g(A)$,

(P_4) $g(A + B) \leq g(A) + g(B)$,

(P_5) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and (A_n), $A_0 \in \bar{X}$ with $g(A_n) \to g(A_0)$ $(n \to \infty)$, then $g(\lambda_n A_n) \to g(\lambda A_0) \to 0$ $(n \to \infty)$,

(P_6) If $A \leq B$, then $g(A) \leq g(B)$.

In this article, we introduce and study the following classes of sequences:

$$\mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) = \Big\{\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \big\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) - \bar{A} \parallel \big)^{p_k} \geq \epsilon\big\} \in I, \text{ for some } \bar{A}\Big\}, \tag{12}$$

$$\mathcal{C}_0^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) = \Big\{\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \big\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) \parallel \big)^{p_k} \geq \epsilon\big\} \in I\Big\}, \tag{13}$$

$$\ell_\infty^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) = \Big\{\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \exists\, K > 0 \text{ s.t.} \big\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) \parallel \big)^{p_k} \geq K\big\} \in I\Big\}, \tag{14}$$

$$\ell_\infty(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) = \Big\{\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \sup_k f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) \parallel \big)^{p_k} < \infty\Big\}. \tag{15}$$

We also denote $\mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) = \ell_\infty(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \cap \mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ and $\mathcal{M}_{\mathcal{C}_\circ}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) = \ell_\infty(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \cap \mathcal{C}_0^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$, where $p = (p_k)$ is a bounded sequence of positive real numbers, $\mathcal{F} = (f_k)$ is a sequence of moduli and $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) \subset \omega(\bar{\mathcal{A}})$ is a bounbed sequence of interval numbers.

If we take $p = (p_k) = 1$, for all $k \in \mathbb{N}$, we have

$$\mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}) = \Big\{\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \big\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) - \bar{A} \parallel \big) \geq \epsilon\big\} \in I, \text{ for some } \bar{A}\Big\}, \tag{16}$$

$$\mathcal{C}_0^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}) = \Big\{\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \big\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) \parallel \big) \geq \epsilon\big\} \in I\Big\}, \tag{17}$$

$$\ell_\infty^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}) = \Big\{\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \big\{k \in \mathbb{N} : \exists\, K > 0 s.t. f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) \parallel \big) \geq K\big\} \in I\Big\}, \tag{18}$$

$$\ell_\infty(\bar{\mathcal{A}}, \Lambda, \mathcal{F}) = \Big\{\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \sup_k f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) \parallel \big) < \infty\Big\}. \tag{19}$$

Theorem 2.2. Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be the bounded sequence of positive real numbers. Then, the classes of sequences $\mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ and $\mathcal{M}_{\mathcal{C}_\circ}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ are paranormed spaces, paranormed by $g(\bar{\mathcal{A}}) = g((\bar{A}_k)) = \sup_k f_k\big(\parallel \bar{\Lambda}_k(A_k) \parallel \big)^{\frac{p_k}{M}}$, where $M = \max\{1, \sup_k p_k\}$.

Proof. Let $\bar{\mathcal{A}} = (\bar{A}_k)$, $\bar{\mathcal{B}} = (\bar{B}_k) \in \mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$.

(P_1) It is Clear that $g(\bar{\mathcal{A}}) = 0$, if $\bar{\mathcal{A}} = \bar{\theta}$.

(P_2) It is also obvious that $g(\bar{\mathcal{A}}) \geq 0$.

(P_3) $g(\bar{\mathcal{A}}) = g(-\bar{\mathcal{A}})$ is obvious.

(P_4) Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using Minkowski's inequality, we have

$$g(\bar{\mathcal{A}} + \bar{\mathcal{B}}) = g(\bar{A}_k + \bar{B}_k) = \sup_k f_k\big(\parallel \Lambda_k(\bar{A}_k + \bar{B}_k)) \parallel \big)^{\frac{p_k}{M}}$$

$$= \sup_k f_k\big(\parallel \Lambda_k(\bar{A}_k) + \Lambda_k(\bar{B}_k)) \parallel \big)^{\frac{p_k}{M}}$$

$$\leq \sup_k f_k\big(\parallel \Lambda_k(\bar{A}_k) \parallel^{\frac{p_k}{M}} + \parallel \Lambda_k(\bar{B}_k)) \parallel \big)^{\frac{p_k}{M}}$$

$$\leq \sup_k f\big(\parallel \bar{A}_k \parallel \big)^{\frac{p_k}{M}} + \sup_k f\big(\parallel \bar{B}_k \big) \parallel^{\frac{p_k}{M}}$$

$$= g(\bar{\mathcal{A}}) + g(\bar{\mathcal{B}}).$$

Therefore, $g(\bar{\mathcal{A}} + \bar{\mathcal{B}}) \leq g(\bar{\mathcal{A}}) + g(\bar{\mathcal{B}})$, for all $\bar{\mathcal{A}},\ \bar{\mathcal{B}} \in \mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$.

(P_5) Let (λ_k) be a sequence of scalars with $(\lambda_k) \to \lambda\ (k \to \infty)$ and $(\bar{A}_k),\ \bar{A}_0 \in \mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ with $g(\bar{A}_k) \to g(\bar{A}_0),\ (k \to \infty)$. Note that $g(\lambda\bar{\mathcal{A}}) \leq \max\{1, |\ \lambda\ |\}g(\bar{\mathcal{A}})$.

Then, since the inequality $g(\bar{A}_k) \leq g(\bar{A}_k - \bar{A}_0) + g(\bar{A}_0)$ holds by subadditivity of g, the sequence $\{g(\bar{A}_k)\}$ is bounded.

Therefore, $|\ g(\lambda_k\bar{A}_k) - g(\lambda\bar{A}_0)\ | = |\ g(\lambda_k\bar{A}_k) - g(\lambda\bar{A}_k) + g(\lambda\bar{A}_k) - g(\lambda\bar{A}_0)\ | \leq |\ \lambda_k - \lambda\ |^{\frac{p_k}{M}}|\ g\big(\lambda_k\bar{A}_k\big)\ | + |\ \lambda\ |^{\frac{p_k}{M}}|\ g(\bar{A}_k) - g(\bar{A}_0)\ | \to 0$, as $(k \to \infty)$. That is to say that scalar multiplication is continuous.

(P_6) Since each $f_k,\ k \in \mathbb{N}$ is an increasing function, it is clear that $g(\bar{\mathcal{A}}) \leq g(\bar{\mathcal{B}}$, if $\bar{\mathcal{A}} \subseteq \bar{\mathcal{B}}$. Hence $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ is a paranormed space. For $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$, the result is similar.

Theorem 2.3. The set $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ is closed subspace of $\ell_\infty(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$.

Proof. Let $\bar{\mathcal{A}}^{(n)} = (\bar{A}^{(n)}_k)$ be a Cauchy sequence in $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ such that $\bar{A}^{(n)}_k \to \bar{A}$. We show that $\bar{A} \in \mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$. Since $\bar{\mathcal{A}}^{(n)} = (\bar{A}^{(n)}_k) \in \mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$. Then, there exists $\bar{A}_n$ such that $\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(n)}) - \bar{A}_n \parallel \big)^{p_k} \geq \epsilon\} \in I$. We need to show that

(1) $(\bar{A}_n)$ converges to $\bar{A}_0$.

(2) If $U = \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) - \bar{A}_0 \parallel \big)^{p_k} < \epsilon\}$, then $U^c \in I$.

(1) Since $\bar{\mathcal{A}}^{(n)} = (\bar{A}^{(n)}_k)$ is Cauchy sequence in $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \Rightarrow$ for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sup_k f\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(n)}) - \Lambda_k(\bar{\mathcal{A}}^{(q)}) \parallel \big)^{\frac{p_k}{M}} < \frac{\epsilon}{3}$, for all $n,\ q \geq k_0$ where $M = \max\{1, \sup_k p_k\}$. For $\epsilon > 0$, we have

$$B_{nq} = \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(n)}) - \Lambda_k(\bar{\mathcal{A}}^{(q)}) \parallel \big)^{p_k} < (\frac{\epsilon}{3})^M\},$$

$$B_q = \{k \in \mathbb{N} : f\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(q)}) - \bar{A}_q \parallel \big)^{p_k} < (\frac{\epsilon}{3})^M\},$$

$$B_n = \left\{k \in \mathbb{N} : f\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(n)}) - \bar{A}_n \parallel \big)^{p_k} < (\frac{\epsilon}{3})^M\right\}.$$

Then, $B^c_{nq},\ B^c_q, B^c_n \in I$.

Let $B^c = B^c_{nq} \cup B^c_q \cup B^c_n$, where $B = \{k \in \mathbb{N} : f_k\big(\parallel \bar{A}_q - \bar{A}_n \parallel \big)^{p_k} < \epsilon\}$. Then, $B^c \in I$. We choose $k_0 \in B^c$. Then, for each $n,\ q \geq k_0$, we have $\big\{k \in \mathbb{N} : f_k\big(\parallel \bar{A}_q - \bar{A}_n \parallel \big)^{p_k} < \epsilon\big\} \supseteq \Big[\{k \in \mathbb{N} : f_k\big(\parallel \bar{A}_q - \Lambda_k(\bar{\mathcal{A}}^{(q)}) \parallel \big)^{p_k} < (\frac{\epsilon}{3})^M\} \cap \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(n)}) - \Lambda_k(\bar{\mathcal{A}}^{(q)}) \parallel \big)^{p_k} < (\frac{\epsilon}{3})^M\} \cap \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(n)}) - \bar{A}_n \parallel \big)^{p_k} < (\frac{\epsilon}{3})^M\}\Big]$. Then, $(\bar{A}_n)$ is a Cauchy sequence of interval numbers, so there exists some interval number $\bar{A}_0$ such that $\bar{A}_n \to \bar{A}_0$ as $n \to \infty$.

(2) Let $0 < \delta < 1$ be given. Then, we show that, if $U = \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) - \bar{A}_0 \parallel \big)^{p_k} < \delta\}$, then, $U^c \in I$. Since $\bar{\mathcal{A}}^{(n)} = (\bar{A}^{(n)}_k) \to \bar{A}$ then, there exists $q_0 \in \mathbb{N}$ such that

$$P = \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(q_0)}) - \Lambda_k(\bar{\mathcal{A}}) \parallel \big)^{p_k} < (\frac{\delta}{3D})^M\} \tag{20}$$

implies $P^c \in I$, where $D = \max\{1, 2^{H-1}\}$, $H = \sup_k p_k \geq 0$. The number q_0 can be chosen that together with (20), we have $Q = \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{A}_{q_0}) - \bar{A}_0 \parallel\big)^{p_k} < (\frac{\delta}{3D})^M\}$ such that $Q^c \in I$. Since $\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(q_0)}) - \Lambda_k(\bar{A}_{q_0}) \parallel\big)^{p_k} \geq \delta\} \in I$. Then, we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where $S = \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(q_0)}) - \Lambda_k(\bar{A}_{q_0}) \parallel\big)^{p_k} < (\frac{\delta}{3D})^M\}$.

Let $U^c = P^c \cup Q^c \cup S^c$, where $U = \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) - \bar{A}_0 \parallel\big)^{p_k} < \delta\}$.

Therefore, for each $k \in U^c$, we have

$$\begin{aligned}
&\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}) - \bar{A}_0 \parallel\big)^{p_k} < \delta\} \\
&\supseteq [\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(q_0)}) - \Lambda_k(\bar{\mathcal{A}}) \parallel\big)^{p_k} < (\frac{\delta}{3D})^M\} \\
&\cap \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{\mathcal{A}}^{(q_0)}) - \Lambda_k(\bar{A}_{q_0}) \parallel\big)^{p_k} < (\frac{\delta}{3D})^M\} \\
&\cap \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{A}_{q_0}) - \bar{A}_0 \parallel\big)^{p_k} < (\frac{\delta}{3D})^M\}]. \qquad (21)
\end{aligned}$$

Then, the result follows from (21). Since the inclusions $\mathcal{M}^I_{\mathcal{C}}(\bar{A}, \Lambda, \mathcal{F}, p) \subset \ell_\infty(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ and $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \subset \ell_\infty(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ are strict so in view of Theorem 2.3 we have the following result.

Theorem 2.4. The spaces $\mathcal{M}^I_{\mathcal{C}}(\bar{A}, \Lambda, \mathcal{F}, p)$ and $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ are nowhere dense subsets of $\ell_\infty(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$.

Theorem 2.5. The spaces $\mathcal{C}^I_0(\bar{A}, \Lambda, \mathcal{F}, p)$ and $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ are both solid and monotone.

Proof. We shall prove the result for $\mathcal{C}^I_0(\bar{A}, \Lambda, \mathcal{F}, p)$. For $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$, the result follows similarly.

For, let $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I_0(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ and (α_k) be a sequence of scalars with $|\ \alpha_k\ | \leq 1$, for all $k \in \mathbb{N}$. Since $|\ \alpha_k\ |^{p_k} \leq \max\{1, |\ \alpha_k\ |^G\} \leq 1$, for all $k \in \mathbb{N}$, we have

$$f_k\big(\parallel \alpha_k \Lambda_k(\bar{A}_k) \parallel\big)^{p_k} \leq f_k\big(\parallel \Lambda_k(\bar{A}_k) \parallel\big)^{p_k}, \text{ for all k} \in \mathbb{N},$$

which further implies that

$$\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{A}_k) \parallel\big)^{p_k} \geq \epsilon\} \supseteq \{k \in \mathbb{N} : f_k\big(\parallel \alpha_k \Lambda_k(\bar{A}_k) \parallel\big)^{p_k} \geq \epsilon\}.$$

Thus, $\alpha_k(\bar{A}_k) \in \mathcal{C}^I_0(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$. Therefore, the space $\mathcal{C}^I_0(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ is solid and hence by Lemma 1.1, it is monotone.

Theorem 2.6. Let $G = \sup_k p_k < \infty$ and I be an admissible ideal. Then, the following are equivalent.

(a) $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$;

(b) there exists $\bar{\mathcal{B}} = (\bar{B}_k) \in \mathcal{C}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ such that $\bar{A}_k = \bar{B}_k$, for a.a.k.r. I;

(c) there exists $\bar{\mathcal{B}} = (\bar{B}_k) \in \mathcal{C}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ and $\bar{\mathcal{C}} = \bar{C}_k \in \mathcal{C}^I_0(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ such that $\bar{A}_k = \bar{B}_k + \bar{C}_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{B}_k) - \bar{A} \parallel\big)^{p_k} \geq \epsilon\} \in I$;

(d) there exists a subset $K = \{k_1 < k_2 < k_3 ...\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim_{n \to \infty} f_k\big(\parallel \Lambda_n(\bar{A})_{k_n} - \bar{A} \parallel\big)^{p_{k_n}} = 0$.

Proof. (a) implies (b)

Let $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$. Then, there exists interval number $\bar{A}$ such that the set $\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{A}_k) - \bar{A} \parallel \big)^{p_k} \geq \epsilon\} \in I$.

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that $\{k \leq m_t : f_k\big(\parallel \Lambda_k(\bar{A}_k) - \bar{A} \parallel \big)^{p_k} \geq t^{-1}\} \in I$. Define a sequence $\bar{\mathcal{B}} = (\bar{B}_k)$ as $\bar{B}_k = \bar{A}_k$, for all k $\leq m_1$. For $m_t < k \leq m_{t+1}$, $t \in \mathbb{N}$,

$$\bar{B}_k = \begin{cases} \bar{A}_k, & \text{if } f_k\big(\parallel \Lambda_k(\bar{A}_k) - \bar{A} \parallel \big)^{p_k} < t^{-1}, \\ \bar{A}, & \text{otherwise.} \end{cases}$$

Then, $\bar{\mathcal{B}} = (\bar{B}_k) \in \mathcal{C}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ and from the inclusion $\{k \leq m_t : \bar{A}_k \neq \bar{B}_k\} \subseteq \{k \leq m_t : f_k\big(\parallel \Lambda_k(\bar{A}_k) - \bar{A} \parallel \big)^{p_k} \geq \epsilon\} \in I$. we get $\bar{A}_k = \bar{B}_k$ for a.a.k.r. I.

(b) implies (c)

For $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$, then, there exists $\bar{\mathcal{B}} = (\bar{B}_k) \in \mathcal{C}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ such that $\bar{A}_k = \bar{B}_k$, for a.a.k.r. I. Let $K = \{k \in \mathbb{N} : \bar{A}_k \neq \bar{B}_k\}$, then $K \in I$.

Define $\bar{\mathcal{C}} = \bar{C}_k$ as follows:

$$\bar{C}_k = \begin{cases} \bar{A}_k - \bar{B}_k, \text{if } k \in K, \\ O, \qquad \text{if } k \notin K. \end{cases}$$

Then, $\bar{\mathcal{C}} = (\bar{C}_k) \in \mathcal{C}_0^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ and $\bar{\mathcal{B}} = (\bar{B}_k) \in \mathcal{C}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$.

(c) implies (d)

Suppose (c) holds. Let $\epsilon > 0$ be given. Let $P_1 = \{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{C}_k) \parallel)^{p_k} \geq \epsilon\} \in I$ and $K = P_1^c = \{k_1 < k_2 < k_3 < k_4 ...\} \in \pounds(I)$. Then, we have $\lim\limits_{n\to\infty} f_k\big(\parallel \Lambda_k(\bar{A}_{k_n}) - \bar{A} \parallel \big)^{p_{k_n}} = 0$.

(d) implies (a)

Let $K = \{k_1 < k_2 < k_3 < k_4 ...\} \in \pounds(I)$ and $\lim\limits_{n\to\infty} f_k\big(\parallel \Lambda_k(\bar{A}_{k_n}) - \bar{A} \parallel)^{p_{k_n}} = 0$. Then, for any $\epsilon > 0$, and Lemma (II), we have $\{k \in \mathbb{N} : f_k\big(\parallel \Lambda_k(\bar{A}_k) - \bar{A} \parallel \big)^{p_k} \geq \epsilon\} \subseteq K^c \cup \{k \in K : f_k\big(\parallel \Lambda_k(\bar{A}_k) - \bar{A} \parallel \big)^{p_k} \geq \epsilon\}$. Thus, $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$.

Theorem 2.7. Let $\mathcal{F} = (f_k)$ and $\mathcal{G} = (g_k)$ be two sequences of modulus functions and for each $k \in \mathbb{N}$, (f_k) and (g_k) satisfying Δ_2-Condition and $p = (p_k) \in \ell_\infty$ be a bounded sequence of positive real numbers. Then,

(a) $\mathcal{X}(\bar{\mathcal{A}}, \Lambda, \mathcal{G}, p) \subseteq \mathcal{X}(\bar{\mathcal{A}}, \Lambda, \mathcal{F} \circ \mathcal{G}, p)$,

(b) $\mathcal{X}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \cap (\bar{\mathcal{A}}, \Lambda, \mathcal{G}, p) \subseteq \mathcal{X}(\bar{\mathcal{A}}, \Lambda, \mathcal{F} + \mathcal{G}, p)$, for $\mathcal{X}= \mathcal{C}^I$, $\mathcal{C}_\circ^I$, $\mathcal{M}_\mathcal{C}^I$ and $\mathcal{M}_{\mathcal{C}_\circ}^I$.

Proof. (a) Let $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}_\circ^I(\bar{\mathcal{A}}, \Lambda, \mathcal{G}, p)$ be any arbitrary element. Then, the set

$$\left\{k \in \mathbb{N} : g_k\Big(\parallel \Lambda_k(\bar{A}_k) \parallel \Big)^{p_k} \geq \epsilon\right\} \in I. \tag{22}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_k(t) < \epsilon$, $0 \leq t \leq \delta$. Let us denote

$$\bar{B}_k = g_k\Big(\parallel \Lambda_k(\bar{A}_k) \parallel \Big)^{p_k}, \tag{23}$$

and consider $\lim\limits_k f_k\big(\bar{B}_k\big) = \lim\limits_{\bar{B}_k \leq \delta, k\in\mathbb{N}} f_k\big(\bar{B}_k\big) + \lim\limits_{\bar{B}_k > \delta, k\in\mathbb{N}} f_k\big(\bar{B}_k\big)$. Now, since f_k for each k $\in \mathbb{N}$ is an modulus function, we have

$$\lim_{\bar{B}_k \leq \delta, k\in\mathbb{N}} f_k\big(\bar{B}_k\big) \leq f_k\big(2\big) \lim_{\bar{B}_k \leq \delta, k\in\mathbb{N}} (\bar{B}_k). \tag{24}$$

For $\bar{B}_k > \delta$, we have $\bar{B}_k < \frac{\bar{B}_k}{\delta} < 1+\frac{\bar{B}_k}{\delta}$. Now, since each f_k is non-decreasing and modulus, it follows that

$$f_k(\bar{B}_k) < f_k(1+\frac{\bar{B}_k}{\delta}) < \frac{1}{2}f_k(2)+\frac{1}{2}f_k(\frac{2\bar{B}_k}{\delta}).$$

Again, since each $f_k,\ k \in \mathbb{N}$ satisfies $\Delta_2 -$ Condition, we have

$$f_k\big(\bar{B}_k\big) < \frac{1}{2}K\frac{(\bar{B}_k)}{\delta}f_k\big(2\big)+\frac{1}{2}K\frac{(\bar{B}_k)}{\delta}f_k(2).$$

Thus, $f_k\big(\bar{B}_k\big) < K\frac{(\bar{B}_k)}{\delta}f_k\big(2\big)$. Hence,

$$\lim_{\bar{B}_k>\delta, k\in\mathbb{N}} f_k\big(\bar{B}_k\big) \le \max\{1, (K\delta^{-1}f_k\big(2\big))^H\} \lim_{\bar{B}_k>\delta, k\in\mathbb{N}} (\bar{B}_k),\ H = \max\{1, \sup_k p_k\}. \tag{25}$$

Therefore, from (23), (24) and (25), we have $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I_\circ(\bar{\mathcal{A}}, \Lambda, \mathcal{F}\circ\mathcal{G}, p)$. Thus, $\mathcal{C}^I_\circ(\bar{\mathcal{A}}, \Lambda, \mathcal{G}, p) \subseteq\ \mathcal{C}^I_\circ(\bar{\mathcal{A}}, \Lambda, \mathcal{F}\circ\mathcal{G}, p)$. Hence, $\mathcal{X}(\bar{\mathcal{A}}, \Lambda, \mathcal{G}, p) \subseteq \mathcal{X}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}\circ\mathcal{G}, p)$ for $\mathcal{X}= \mathcal{C}^I_\circ$. For $\mathcal{X}= \mathcal{C}^I$, $\mathcal{M}^I_\mathcal{C}$ and $\mathcal{M}^I_{\mathcal{C}_\circ}$ the inclusions can be established similarly.

(b) Let $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I_\circ(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \cap \mathcal{C}^I_\circ(\bar{\mathcal{A}}, \Lambda, \mathcal{G}, p)$. Let $\epsilon > 0$ be given. Then, the sets

$$\Big\{k \in \mathbb{N} : f_k\Big(\parallel \Lambda_k(\bar{A}_k) \parallel\Big)^{p_k} \ge \epsilon\Big\} \in I, \tag{26}$$

and

$$\Big\{k \in \mathbb{N} : g_k\Big(\parallel \Lambda_k(\bar{A}_k) \parallel\Big)^{p_k} \ge \epsilon\Big\} \in I. \tag{27}$$

Therefore, from (26) and (27), we have $\Big\{k \in \mathbb{N} : \mathcal{F}+\mathcal{G}\Big(\parallel \Lambda_k(\bar{A}_k) \parallel\Big)^{p_k} \ge \epsilon\Big\} \in I$. Thus, $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I_\circ(\bar{\mathcal{A}}, \Lambda, \mathcal{F}+\mathcal{G}, p)$ Hence, $\mathcal{C}^I_\circ(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \cap \mathcal{C}^I_\circ(\bar{\mathcal{A}}, \Lambda, \mathcal{G}, p) \subseteq \mathcal{C}^I_\circ(\bar{\mathcal{A}}, \Lambda, \mathcal{F}+\mathcal{G}, p)$. For $\mathcal{X}= \mathcal{C}^I$, $\mathcal{M}^I_\mathcal{C}$ and $\mathcal{M}^I_{\mathcal{C}_\circ}$, the inclusions are similar. For $g_k(x) = x$ and $f_k(x) = f(x)$, $\forall\ x \in [0, \infty)$, we have the following corollary.

Corollary 2.8. $\mathcal{X}(\bar{\mathcal{A}}, \Lambda, p) \subseteq \mathcal{X}(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$, for $\mathcal{X}= \mathcal{C}^I$, $\mathcal{C}^I_\circ$, $\mathcal{M}^I_\mathcal{C}$ and $\mathcal{M}^I_{\mathcal{C}_\circ}$.

Theorem 2.9. Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions. Then, the inclusions $\mathcal{C}^I_0(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \subset \mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \subset \ell^I_\infty(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ hold.

Proof. Let $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{G}, p)$ be any arbitrary element. Then, there exists some interval number $\bar{A}$ such that the set $\big\{k \in \mathbb{N} : f_k(\parallel \Lambda_k(\bar{A}_k) - \bar{A} \parallel)^{p_k} \ge \epsilon\big\} \in I$. Since, each f_k is modulus, we have

$$f_k(\parallel \Lambda_k(\bar{A}_k) \parallel)^{p_k} = f_k(\parallel \Lambda_k(\bar{A}_k) - \bar{A} + \bar{A} \parallel)^{p_k} \le f_k(\parallel \Lambda_k(\bar{A}_k) - \bar{A} \parallel)^{p_k} + f_k(\parallel \bar{A} \parallel)^{p_k}.$$

Taking supremum over k on both sides, we get $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell^I_\infty(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$. The inclusion $\mathcal{C}^I_0(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \subset \mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ is obvious. Hence $\mathcal{C}^I_0(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \subset \mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p) \subset \ell^I_\infty(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$.

Theorem 2.10. The spaces $\mathcal{C}^I_0(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ and $\mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ are sequence algebra.

Proof. Let $\bar{\mathcal{A}} = (\bar{A}_k),\ \bar{\mathcal{B}} = (\bar{B}_k) \in \mathcal{C}^I_0(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$. Then, the sets

$$\Big\{k \in \mathbb{N} : f_k\Big(\parallel \Lambda_k(\bar{A}_k) \parallel\Big)^{p_k} \ge \epsilon\Big\} \in I, \tag{28}$$

and

$$\Big\{k \in \mathbb{N} : f_k\Big(\parallel \Lambda_k(\bar{B}_k) \parallel\Big)^{p_k} \ge \epsilon\Big\} \in I. \tag{29}$$

Therefore, from (28) and (29), we have $\left\{k \in \mathbb{N} : f_k\Big(\parallel \Lambda_k(\bar{A}_k.\bar{B}_k) \parallel \Big)^{p_k} \geq \epsilon\right\} \in I$. Thus, $\bar{\mathcal{A}} \cdot \bar{\mathcal{B}} \in \mathcal{C}_0^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$. Hence, $\mathcal{C}_0^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ is a sequence algebra. Similarly, we can prove that $\mathcal{C}^I(\bar{\mathcal{A}}, \Lambda, \mathcal{F}, p)$ is a sequence algebra.

Acknowledgements

The authors would like to record their gratitude to the reviewer for his careful reading and making some useful corrections which improved the presentation of the paper.

References

[1] E. Ayhan and B. Hazarika. Some I-convergent of double Λ-interval sequences defined by Orlicz function. Global J. of Mathematical Analysis 1 (2013), No. 3, 110 - 116.

[2] E. Ayhan. λ-sequence spaces of interval numbers. Appl. Math. Inf. 8 (2014), No. 3, 1099 - 1102.

[3] R. C. Buck. Generalized asymptotic density. Amer. J. Math. 75 (1953), 335 - 346.

[4] K. P. Chiao. Fundamental properties of interval vector max-norm. Tamsui Oxford Journal of Mathematical Sciences 18 (2002), No. 2, 219 - 233.

[5] P. S. Dwyer. Linear computation. New York, Wiley, 1951.

[6] H. Fast. Sur la convergence statistique. Colloq. Math. 2 (1951), 241 - 244.

[7] J. A. Fridy. On statistical convergence. Analysis 5 (1985), 301 - 313.

[8] V. A. Khan, Mohd Shafiq and K. Ebadullah. On paranorm I-convergent sequence spaces of interval numbers. J. of Nonlinear Analysis and Optimisation (Theory and Application) 5 (2014), No. 1, 103 - 114.

[9] V. A. Khan. Ahyan Esi and Mohd Shafiq. On paranorm BV_σ I-convergent sequence spaces defined by an Orlicz function. Global J. Mathematical Analysis 2 (2014), No. 2, 28 - 43.

[10] V. A. Khan, Suthep Suantai and K. Ebadullah. On some I-convergent sequence spaces defined by a sequence of moduli. J. Nonlinear Analysis and Optimization 3 (2012), No. 2, 145 - 152.

[11] E. Kolk. On strong boundedness and summability with respect to a sequence of moduli. Acta Comment. Univ. Tartu. 960 (1993), 41 - 50.

[12] E. Kolk. Inclusion theorems for some sequence spaces defined by a sequence of modulii. Acta Comment. Univ. Tartu. 970 (1994), 65 - 72.

[13] P. Kostyrko, M. Mačaj and T. Šalát. Statistical convergence and I-convergence. Real Analysis Exchange.

[14] P. Kostyrko, T. Šalát and W. Wilczyński. I-convergence. Raal Analysis Exchange 26 (2000), No. 2, 669 - 686.

[15] R. E. Moore. Automatic error analysis in digital computation. LSMD-48421, Lockheed Missiles and Space Company, 1959.

[16] R. E. Moore and C. T. Yang. Interval analysis I. LMSD-285875, Lockheed Missiles and Space Company, Palo Alto, Calif., 1959.

[17] M. Mursaleen and K. Noman Abdullah. On the spaces of λ-convergent and bounded sequences. Thai Journal of Mathematics 8 (2010), No. 2, 311 - 329.

[18] M. Mursaleen and K. Sharma Sunil. Spaces of Ideal Convergent sequences, (2014), Article ID 134534, 6 pages. http://dx.doi. org/10.1155/2014/134534.

[19] H. Nakano. Concave modulars. J. Math. Soc. Japan 5 (1953), 29 - 49.

[20] W. H. Ruckle. On perfect symmetric BK-spaces. Math. Ann. 175 (1968), 121 - 126.

[21] W. H. Ruckle. Symmetric coordinate spaces and symmetric bases. Canad. J. Math. 19 (1967), 828 - 838.

[22] W. H. Ruckle. FK-spaces in which the sequence of coordinate vectors is bounded. Canad. J. Math. 25 (1973), No. 5, 973 - 975.

[23] T. Šalát. On statistical convergent sequences of real numbers. Math. Slovaca 30 (1980).

[24] T. Šalát, B. C. Tripathy and M. Ziman. On some properties of I-convergence. Tatra Mt. Math. Publ. 28 (2004), 279 - 286.

[25] T. Šalát, B. C. Tripathy and M. Ziman. On I-convergence field. Ital. J. Pure Appl. Math. 17 (2005), 45 - 54.

[26] I. J. Schoenberg. The integrability of certain functions and related summability methods. Amer. Math. Monthly 66 (1959), 361 - 375.

[27] M. Sengönül and A. Eryilmaz. On the sequence spaces of interval numbers. Thai J. of Mathematics 8 (2010), No. 3, 503 - 510.

[28] B. C. Tripathy. On statistical convergence. Proc. Estonian Acad. Sci. Phy. Math. Analysis (1998), 299 - 303.

[29] B. C. Tripathy and B. Hazarika. Paranorm I-convergent sequence spaces. Math. Slovaca 59 (2009), No. 4, 485 - 494.

[30] Yilmaz Ylimaz. Sümeyye Cakan and Sahika Aytekin. Topological quasilinear spaces. Hindawi Publishing Corporation, Abstract and Applied Analysis, 2012.

Scientia Magna
Vol. 12 (2017), No. 1, 78-91

Cartan involution and geometry of semi-simple Lie groups

M. Aliabadi[1] and B. Hassanzadeh[2*]

[1]Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago, 851 S. Morgan St, Chicago, IL 60607, USA
E-mail: maliab2@uic.edu
[2]Department of Mathematics, Azarbaijan University of shahid madani
Tabriz, Iran
E-mail: babakmath777@gmail.com

Abstract In this paper we study geometrical properties of the semi-simple Lie groups by Cartan involution and Cartan decomposition. Using Cartan involution and a bilinear symmetric form, we investigate the Hermitian Covariant derivative $\bar{\nabla}$, the covariant derivative and their relationship. Then we study left invariant and bi-invariant almost complex structure when $\bar{\nabla}J = 0$. The relation between the Hermitian covariant derivative and covariant derivative is studied by Cartan involution while it was a special automorphism.

Keywords Cartan involution, semi-simple Lie algebra, Hermitian covariant derivative.
2010 Mathematics Subject Classification 42A20, 42A32.

§1. Introduction

Let G be a semi-simple Lie group, $\mathfrak{g}$ be its Lie algebra and J be an almost complex structure. A subgroup(submanifold) $\bar{G}$ is called holomorphic if $J(T_e\bar{G}) \subset T_e\bar{G}$, e is identity element of $\bar{G}$, where $T_e\bar{G}$ denotes the tangent space to $\bar{G}$ at the point e. $\bar{G}$ is called totally real if $J(T_e\bar{G}) \subset T_e\bar{G}^{\perp}$ for identity element $e \in \bar{G}$, where $T_e\bar{G}^{\perp}$ denotes the normal space to $\bar{G}$ at the point e. A generalization of holomorphic and totally real subgroups, slant submanifolds were introduced by B. Y. Chen in [1]. We recall that the subgroup $\bar{G}$ is called slant [1] if for $e \in \bar{G}$ and $X \in T_e\bar{G}$, the angle between JX and $T_e\bar{G}$ is a constant $\theta(X) \in [0, \frac{\pi}{2}]$, i.e., it does not depend on the choice of point of $\bar{G}$ and $X \in T_e\bar{G}$. It follows that invariant and totally real subgroup with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively. A slant subgroup which is neither invariant nor totally real is called a proper slant subgroup [4].

In this paper we study the differential geometry of semi-simple Lie groups with Cartan involution. It helps us to decompose the semi-simple Lie algebra of Lie group that is called Cartan decomposition. Here we tried to use geometric and Lie algebraic tools for study of semi-simple Lie groups, like almost complex structure and Cartan involution. The aim of the present paper is to study the left invariant and bi-invariant almost complex structure's behavior

[2*]corresponding author

regarding Hermitian bilinear, symmetric form and Hermitian covariant derivative when Cartan involution have special relationship with almost complex structure. Furthermore, we had shown that the relationship between almost complex structure and algebraic homomorphisms affect the subalgebras. We believe that involutions are powerful tools for studying the Lie group, as are useful in non-commutative geometry, as they are developing in Harmonic Analysis we can use that results in differential geometry. This is the first paper of applications of involutions in the semi-simple Lie groups and in the next step we will show how they are connected algebra and geometry by left and right transformations instead of Exponential Functions.

§2. Preliminaries

Some definitions and basic concepts.

First, we note that throughout this paper $F = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. A *Lie algebra* over F is pair $(\mathfrak{g}, [.,.])$, where $\mathfrak{g}$ is a vector space over F and

$$[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is an F-bilinear map satisfying the following properties

i)$[X, Y] = -[Y, X]$,
ii)$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$,

the latter is the Jacobi identity. In this paper for any $X, Y \in \mathfrak{g}$ we have

$$[X, Y] = \nabla_X Y - \nabla_Y X,$$

where ∇ is covariant derivative. A Lie subalgebra of a Lie algebra is a vector space that is closed under bracket.

Let G be a Lie group(smooth manifold equipped with positive definite bilinear, symmetric form) and $e \in G$ be identity element, then $\mathfrak{g} = T_eG$ is called Lie algebra of Lie group.($dim\mathfrak{g} = dimT_eG = dimG$)

First we say a Lie group H of a Lie group G is a subgroup which is also a submanifold. The following theorem shows the relationship between Lie subgroups and Lie algebras.

Theorem 2.1. [3] Let G be a Lie group.
(a) If H is a Lie subgroup of G, then $\mathfrak{h} \simeq T_eH \subset T_eG \simeq \mathfrak{g}$ g is a Lie subalgebra.
(b) If $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra, there exists a unique connected Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$.

Let G be a Lie group equipped with a bilinear symmetric form, by using only these identities and combining a few permutations of variables obtain the formula

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}\{&X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &- \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle\}.\end{aligned} \quad (1)$$

But in this case three terms of it vanish since $\langle X, Y\rangle$ is constant [2].

Definition 2.2. Let G be a connected Lie group. The subgroup

$$Z(G) = \{x \in G : xy = yx, \forall y \in G\},$$

is called the *center* of G It is a Lie subgroup with corresponding Lie subalgebra

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, Y] = 0, \forall Y \in \mathfrak{g}\}.$$

In present paper we study semi-simple Lie groups and will review geometric features of these groups. First we give required concepts.

A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called an *ideal* if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. The Lie algebra of a normal Lie subgroup of G is necessarily an ideal. A Lie algebra $\mathfrak{g}$ is called *simple* if it has no nontrivial ideals (that is 0 and $\mathfrak{g}$ are the only ideals in $\mathfrak{g}$). It is called *semi-simple* if it is a direct sum of simple Lie algebras or contains no nonzero solvable ideals. Note that this in particular implies that the center $Z(\mathfrak{g}) = 0$. A Lie group is called simple (respectively semi-simple) if its Lie algebra is simple (respectively semi-simple).

If $(G, \langle , \rangle)$ be a Lie group equipped by left invariant inner product and $\mathfrak{g}$ is the Lie algebra of G and H is a Lie subgroup with Lie algebra $\mathfrak{h}$, we defined

$$\mathfrak{h}^{\perp} = \{X \in \mathfrak{h} \mid \langle X, Y\rangle = 0; \forall Y \in \mathfrak{g}\}.$$

The next theorem expresses useful information about semi-simple Lie algebras.

Theorem 2.2. [3] Let $\mathfrak{g}$ be a semi-simple Lie algebra.
(i) If I is ideal of $\mathfrak{g}$, then $\mathfrak{g} = I \oplus I^{\perp}$.
(ii) If $\mathfrak{g}$ be a semi-simple Lie algebra, then any subideal of $\mathfrak{g}$ is ideal of $\mathfrak{g}$.
(iii) If $\mathfrak{g}$ is semi-simple, any ideal of $\mathfrak{g}$ is semi-simple.

Next two states are so important throughout of this paper.

Let $\langle , \rangle$ be a left invariant bilinear symmetric form on a connected Lie group G. This form will also be right invariant if and only if ad(X) is skew-adjoint for every $X \in \mathfrak{g}$. therefore if $\langle , \rangle$ be a left invariant form on a connected Lie group G, then ad(X) is skew-adjoint, in this case

$$\nabla_X Y = \frac{1}{2}[X, Y],$$

whenever ∇ is covariant derivative. Throughout this paper Lie groups are connected.

§3. Cartan involution

Let $\mathfrak{g}$ be a Lie algebra of Lie group G. An automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\sigma^2 = id_{\mathfrak{g}}$ is called involution. Such an involution yields a decomposition into eigenspaces to the eigenvalues +1 and -1. An involution θ of a semi-simple Lie algebra $\mathfrak{g}$ such that the symmetric bilinear form

$$B_\theta(X, Y) = -B(X, \theta Y),$$

is positive definite and is called Cartan involution.

Definition 3.1. A *Cartan decomposition*, is a vector space decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$, where $\mathfrak{t}$ is a subalgebra, $\mathfrak{p}$ is a subspace, $[\mathfrak{t}, \mathfrak{t}] \subseteq \mathfrak{t}$, $[\mathfrak{t}, \mathfrak{p}] \subseteq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{t}$, and B_θ is negative definite on $\mathfrak{t}$ and positive definite on $\mathfrak{p}$.

For B_θ we can consider:
(1) B is negative definite on $\mathfrak{t}$, then $\theta X = X$, for any $X \in \mathfrak{t}$.
(2)B is positive definite on $\mathfrak{p}$, then $\theta V = -V$, for any $V \in \mathfrak{p}$.
We have a decomposition of the from $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. for all $X, X^{'} \in \mathfrak{t}$ and $Y, Y^{'} \in \mathfrak{p}$, furthermore; we have

$$[\mathfrak{t}, \mathfrak{t}] \subseteq \mathfrak{t} \Rightarrow \theta[X, X^{'}] = [X, X^{'}] = [\theta X, \theta X^{'}],$$
$$[\mathfrak{t}, \mathfrak{p}] \subseteq \mathfrak{p} \Rightarrow \theta[X, Y] = -[X, Y] = [X, -Y] = [\theta X, \theta Y],$$
$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{t} \Rightarrow \theta[Y, Y^{'}] = [Y, Y^{'}] = [-Y, -Y^{'}] = [\theta Y, \theta Y^{'}].$$

Thus $\mathfrak{t}$ is a Lie subalgebra, while any subalgebra of $\mathfrak{p}$ is commutative. B_θ is symmetric, bilinear, positive definite form. In the following theorem we consider behavior of $\mathfrak{t}$ and $\mathfrak{p}$ with automorphisms.

Theorem 3.1. Let $\sigma \in aut\mathfrak{g}$ be an arbitrary element such that $B_\theta(\sigma X, \sigma Y) = B_\theta(X, Y)$, then

i)$\sigma(\mathfrak{t}) \subseteq \mathfrak{t}$,
ii)$\sigma(\mathfrak{p}) \subseteq \mathfrak{p}$.

Proof. Let $X, Y \in \mathfrak{t}$ and $V, W \in \mathfrak{p}$ are arbitrary elements. If $\sigma(\mathfrak{p}) \subseteq \mathfrak{t}$, then

$$\sigma([X, V]) = [\sigma(X), \sigma(V)],$$

left hand is an element of $\mathfrak{t}$, then from right hand we conclude $\sigma(\mathfrak{t})$ must be an element of $\mathfrak{t}$. On the other hand, assuming $\sigma(\mathfrak{t}) \subseteq \mathfrak{p}$, contradicts $\sigma([X, Y]) = [\sigma(X), \sigma(Y)]$. Furthermore, if $\sigma(\mathfrak{p}) \subseteq \mathfrak{p}$, then

$$\sigma([X, V]) = [\sigma(X), \sigma(V)],$$

which implies $\sigma(\mathfrak{t}) \subseteq \mathfrak{t}$ and if $\sigma(\mathfrak{t}) \subseteq \mathfrak{t}$, using

$$\sigma([V, W]) = [\sigma(V), \sigma(W)],$$

we cannot determine $\sigma(\mathfrak{p})$. Therefore $\sigma(\mathfrak{t}) \subseteq \mathfrak{t}$. If $\sigma(\mathfrak{p}) \subseteq \mathfrak{t}$, then

$$B_\theta([V, X], Y) = B_\theta([\sigma(V), \sigma(X)], \sigma(Y)).$$

Thus $\sigma(\mathfrak{p}) \subseteq \mathfrak{p}$. □

Suppose B_θ is a bi-invariant form, then

$$B_\theta(X, [V, W]) = B_\theta(W, [X, V]) = B_\theta(V, [W, X]), \tag{2}$$

for all $X, Y, V, W \in \mathfrak{g}$. Some straightforward computations show that

$$R(X, Y)Z = \frac{1}{4}[[X, Y], Z]. \tag{3}$$

Now we can see

$$\nabla R(X,Y;V,W)=0,$$

for all $X,Y,V,W\in\mathfrak{g}$, therefore semi-simple Lie groups with bi-invariant form are symmetric spaces. Applying ∇ to R, we have

$$\nabla_V R(X,Y)Z+\nabla_Z R(X,Y)V=\frac{1}{2}R(X,Y)[Z,V],$$

for all $X,Y,Z,V\in\mathfrak{g}$. The theorem explain features of semi-simple Lie groups when B_θ is bi-invariant.

Theorem 3.2. Let G be a semi-simple Lie group equipped with bi-invariant form B_θ and $\mathfrak{h}$ is a subset of $\mathfrak{p}$ so that it's a Lie subalgebra. Then
(1)$ad\mathfrak{t}(\mathfrak{h})\in\mathfrak{p}$,
(2)$ad\mathfrak{t}(\mathfrak{p})\in\mathfrak{h}$,
(3)$B_\theta([U,V],[X,Y])=B_\theta([X,V],[U,Y])=0$, for all $X,Y\in\mathfrak{t}$ and $U\in\mathfrak{h}$, $V\in\mathfrak{p}$,
(4)$\mathfrak{h}\subset\mathfrak{p}$,(proper subset)
(5)$[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{t}$, (proper subset).

Proof. From definition we know $\mathfrak{h}$ is an abelian subalgebra and from (2) we get

$$B_\theta([X,U],W)=0,$$

for all $U,W\in\mathfrak{h}$ and $X\in\mathfrak{t}$, now proof of (1) is trivial. Let $U\in\mathfrak{h}$, $V\in\mathfrak{p}$ and $X\in\mathfrak{t}$, then

$$B_\theta(X,[V,U])=B_\theta(U,[X,V])=B_\theta(V,[U,X])\neq 0.$$

Hence $[\mathfrak{t},\mathfrak{p}]\in\mathfrak{h}$ and proof of (2) is completed. For third part from Jacobi identity we get

$$[U,[X,V]]=[[U,X],V]+[X,[U,V]]=0. \tag{4}$$

Thus $[[X,U],V]=[X,[U,V]]$ and we have

$$B_\theta([U,[X,V]],Y)=-B_\theta([X,V],[U,Y])=0, \tag{5}$$

and

$$B_\theta([[X,U],V],Y)=B_\theta([X,U],[V,Y])=-B_\theta([X,Y],[U,V])=0.$$

Let $\mathfrak{p}=\mathfrak{h}$ in this case from (5) we conclude that $[U,X]\in\mathfrak{h}$, from (1) proof of (4) is trivial. Finally; suppose $X,Y\in\mathfrak{t}$ and Y is an arbitrary element, then there are $V,W\in\mathfrak{t}$ such that $X=[V,W]$, then

$$B_\theta(X,Y)=B_\theta([V,W],Y)=B_\theta(V,[W,Y])=0.$$

Therefore $X\in Rad\mathfrak{g}$ and the proof is completed. □

As regards Lie group G is a smooth manifold we introduce to almost complex structure. Let G be a real 2n-dimensional Lie group. An *almost complex structure* on G is defined by a linear map $J : \mathfrak{g} \to \mathfrak{g}$ satisfying:

$$J^2 = -I,$$

the pair $(\mathfrak{g}, J)$ is called an almost complex Lie algebra[1]. If a Lie group has almost complex Lie algebra, it's called almost complex Lie group. The Nijenhuis tensor N of an almost complex structure J is defined by

$$N(X,Y) = J[X,Y] - [JX,Y] - [X,JY] - J[JX,JY],$$

for any $X, Y \in \mathfrak{g}$.

Let J be a left invariant almost complex structure, also if B_θ be a left invariant form, if for any $X, Y \in \mathfrak{g}$, we have

$$B_\theta(X,Y) = B_\theta(JX,JY), \tag{6}$$

B_θ is called *Hermitian* form. Using (6) we have

$$B_\theta(JX,Y) = B_\theta(J^2X,JY) = -B_\theta(X,JY),$$

which means that J is skew-symmetric.

If Nijenhuis tensor vanishes, then there exists a complex structure on G and J is an almost complex structure which is induced from the complex structure on G. An almost complex structure J on a Lie group G is called bi-invariant if it is invariant with respect to both, left and right translations on G, On corresponding Lie algebra $\mathfrak{g}$ of G this condition is equivalent to $ad \circ J = J \circ ad$, i. e.

$$[X,JY] = J[X,Y].$$

for all $X, Y \in \mathfrak{g}$. Immediately we conclude that $[JX,JY] = -[X,Y]$ [6]. In this case $\mathfrak{k}$ and $\mathfrak{p}$ are invariant and it's obvious that Nijenhuis tensor is vanishes, i. e. J is a complex structure.

An almost Hermitian Lie group G is called *nearly Kahler* Lie group if

$$(\nabla_X J)X = 0,$$

also, it is called *Kahler* Lie group if

$$(\nabla_X J)Y = 0.$$

for all $X, Y \in \mathfrak{g}$. If G is a Kahler Lie group and J is a bi-invariant almost complex structure, then

$$R(X,Y)JZ = JR(X,Y)Z. \tag{7}$$

Hence, we compute

$$B_\theta(R(X,Y)JZ,W) = B_\theta(R(X,Y)JW,Z).$$

We define the Holomorphic curvature by

$$H(X) = B_\theta(R(X, JX)X, JX).$$

and it's trivial $H(X) = 0$, for any $X \in \mathfrak{g}$. If J and B_θ are left invariant, then we call the (G, B_θ, J) is left invariant almost complex Lie group and also for bi-invariant. If φ be an automorphism, from below equation

$$\varphi[\mathfrak{t}, \mathfrak{t}] = [\varphi\mathfrak{t}, \varphi\mathfrak{t}],$$

we conclude $\varphi : \mathfrak{t} \to \mathfrak{t}$, i.e. $\mathfrak{t}$ is invariant by any automorphism and now from

$$B_\theta(\varphi X, \varphi V) = B_\theta(X, V),$$

we realized that $\varphi : \mathfrak{p} \to \mathfrak{p}$ or $\varphi(\mathfrak{p}) = 0$, for any $X \in \mathfrak{t}$ and $V \in \mathfrak{p}$. In this case from the first theorem of homomorphism of Lie algebra, we get

$$\frac{\mathfrak{g}}{\mathfrak{p}} \preceq \mathfrak{t}$$

and we have $V + \mathfrak{g} \in \mathfrak{t}$, therefore $\varphi(\mathfrak{p}) \neq 0$ and $\varphi : \mathfrak{p} \to \mathfrak{p}$, in a similar way it can be shown that $\varphi(\mathfrak{t}) \neq 0$. For an almost Hermitian (G, J, B_θ) we define the $J - Ric$ *tensor* as

$$Ric^J(X, Y) = traceZ \to -J(R(Z, X)JY)). \tag{8}$$

If G is a Kahler manifold then Ric^J and Ric coincide. By simple calculations, we get

$$Ric(X, Y) = \frac{1}{4}\Sigma_i B_\theta([X, e_i], [Y, e_i]) = \frac{1}{4}B_\theta(X, Y), \tag{9}$$

whenever $\{e_i\}_{i=1}^{2n}$ is an orthonormal basis. Now from $Ric(X, Y) = \frac{S}{2n}B_\theta(X, Y)$ we have $S = \frac{n}{2}$ or $\frac{1}{4}dimG$. We calculate the *Weyl curvature tensor*, but first we need definition of *Kulkarni-Nomizu product.*

$$\begin{aligned}(h \odot k)(X, Y, Z, V) = {} & h(X, Z)k(Y, V) + h(Y, V)k(X, Z) \\ & - h(X, V)k(Y, Z) - h(Y, Z)k(X, V),\end{aligned} \tag{10}$$

for all $X, Y, Z, V \in \mathfrak{g}$. Then we have the Weyl tensor

$$\begin{aligned}W(X, Y, Z, V) = {} & R(X, Y, Z, V) - \frac{1}{2n-2}B_\theta(X, V) \odot Ric^\circ(Y, Z) \\ & - \frac{S}{4n(2n-1)}B_\theta(X, V) \odot B_\theta(Y, Z).\end{aligned} \tag{11}$$

Also, $Ric^\circ = ric - \frac{S}{n}B_\theta$, therefore

$$Ric^\circ(Y, Z) = -\frac{1}{4}B_\theta(Y, Z).$$

Then

$$W(X, Y, Z, V) = \frac{1}{4}B_\theta([X, Y], [Z, V]) - \frac{12}{(n-1)(2n-1)}B_\theta(X, V) \odot B_\theta(Y, Z).$$

From the algebraic *Bianchi identity* we define map b with

$$b(R)(X,Y,Z,V)=\frac{1}{3}\{R(X,Y,Z,V)+R(Y,Z,X,V)+R(Z,X,Y,V)\},$$

for all $X,Y,Z,V\in\mathfrak{g}$. In this case we will have

$$\begin{aligned}b(R)(X,Y,Z,V)=\frac{1}{3}\{&B_\theta([[X,Y],Z],V)+B_\theta([[Y,Z],X],V)\\&+B_\theta([[Z,X],Y],V)\}=0.\end{aligned}$$

Now we give some calculations which are frequently used in this paper.

Suppose $X\in\mathfrak{t}$ and $Y\in\mathfrak{p}$ are arbitrary elements of $\mathfrak{g}$ such that $[X,Y]=V$ and there is $Z\in\mathfrak{t}$ such that $[Z,[X,Y]]=W$, then $B_\theta(V,W)=0$. In the other case let $X,Y\in\mathfrak{t}$, $V,W\in\mathfrak{p}$ and $Z,U\in\mathfrak{n}$, $\mathfrak{n}$ is a subspace of $\mathfrak{p}$, such that $V=[Z,X]$ and $W=[U,Y]$, then

$$\begin{aligned}B_\theta(V,W)&=B_\theta([Z,X],[U,Y])=B_\theta(X,[[U,Y],Z]),\\&=B_\theta(X,[U,[Y,Z]])=B_\theta([U,X],[Z,Y]).\end{aligned}$$

Then, if we put $V^{'}=[U,X]$, $W^{'}=[Z,Y]$, we get

$$B_\theta(V,W)=B_\theta(V^{'},W^{'}).$$

Now let $X,Y\in\mathfrak{t}$ and $V,W,W^{'}\in\mathfrak{p}$ such that $X=[V,W]$ and $Y=[V,W^{'}]$, then using Jacobi identity, we find

$$\begin{aligned}[W^{'},X]&=[W^{'},[V,W]]=-[Y,W]+[V,[W^{'},W]],\\ [W,Y]&=[W,[V,W^{'}]]=-[X,W^{'}]+[V,[W,W^{'}]],\\ [[W^{'},W],V]&=[W,Y]-[X,W^{'}].\end{aligned}$$

If $W,W^{'}\in\mathfrak{h}$, where $\mathfrak{h}$ is a subalgebra of $\mathfrak{p}$, then

$$[X,W^{'}]=-[Y,W],$$

using theorem 3.2 and Jacobi identity we have

$$[\mathfrak{t},[\mathfrak{h},\mathfrak{h}^\perp]]=[[\mathfrak{t},\mathfrak{h}],\mathfrak{h}^\perp]+[\mathfrak{h},[\mathfrak{t},\mathfrak{h}^\perp]]=[\mathfrak{h}^\perp,\mathfrak{h}^\perp]+[\mathfrak{h},\mathfrak{h}].$$

Let $X=[V,V^{'}]$ and $Y=[W,W^{'}]$, for all $V,W\in\mathfrak{h}$ and $V^{'},W^{'}\in\mathfrak{h}^\perp$, then

$$[X,Y]=[X,[W,W^{'}]]=[[X,W],W^{'}],$$

and we conclude $[X,Y]\in[\mathfrak{h}^\perp,\mathfrak{h}^\perp]$. Let $[X,Y]=[V,W]=Z$, for all $X,Y\in\mathfrak{t}$ and $V,W\in\mathfrak{p}$, then by straightforward calculations we have

$$B_\theta(Z,Z)=B_\theta([X,V],[Y,W])-B_\theta([X,W],[Y,V]).$$

For Hermitian forms we can provide the definition of *Hermitian covariant derivative* by

$$\bar{\nabla}_XY=\nabla_XY+\frac{1}{2}(\nabla_XJ)Y, \tag{12}$$

for all $X,Y\in\mathfrak{g}$ [5]. It is obvious that if J is a bi-invariant almost complex structure, then $\bar{\nabla}=\nabla$ and if J is a left invariant almost complex structure, some calculation gives the following results.

(1)$\bar{\nabla}_X Y - \bar{\nabla}_Y X = [X,Y] + \frac{1}{2}J[X,Y] + \frac{1}{2}\nabla_X JY - \frac{1}{2}\nabla_Y JX$,
(2)$\bar{\nabla}_X X = \nabla_X X + \frac{1}{2}\nabla_X JX - \frac{1}{2}J\nabla_X X$,
(3)$(\bar{\nabla}_X J)Y = (\nabla_X J)Y - J\nabla_X JY$,
(4)$\bar{\nabla}_X JY = \nabla_X JY - \frac{1}{2}\nabla_X Y - \frac{1}{2}\nabla_X JY$,

for all $X, Y \in \mathfrak{g}$.

Curvature tensor Corresponding to the definition of the Hermitian covariant derivative in (12) we define curvature tensor by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z,$$

for all $X, Y, Z \in \mathfrak{g}$. With the relatively long calculations we get the following fact.

$$\begin{aligned}\bar{R}(X,Y)Z = &\frac{5}{4}R(X,Y)Z - \frac{1}{2}JR(X,Y)Z + \frac{1}{2}R(X,Y)JZ \\ &- \frac{1}{4}JR(X,Y)JZ + \frac{1}{4}\nabla_{[X,Y]}Z + \frac{1}{4}J\nabla_{[X,Y]}JZ \\ &+ \frac{1}{4}\nabla_X J\nabla_Y JZ + \frac{1}{4}J\nabla_X J\nabla_Y Z - \frac{1}{4}\nabla_Y J\nabla_X JZ \\ &- \frac{1}{4}J\nabla_Y J\nabla_X Z.\end{aligned}$$

Furthermore, using similar calculations we find

$$\begin{aligned}\bar{K}(X,Y) = &\frac{5}{4}K(X,Y) + \frac{1}{2}R(X,Y,Y,JX) + \frac{1}{2}R(X,Y,JY,X), \\ &+ \frac{1}{4}R(X,Y,JY,JX) + \frac{1}{4}B_\theta(\nabla_{[X,Y]}Y,X) - \frac{1}{4}B_\theta(\nabla_{[X,Y]}JY,JX), \\ &+ \frac{1}{4}B_\theta(\nabla_X J\nabla_Y JY,X) - \frac{1}{4}B_\theta(\nabla_X J\nabla_Y Y,JX) - \frac{1}{4}B_\theta(\nabla_Y J\nabla_X JY,X), \\ &+ \frac{1}{4}B_\theta(\nabla_Y J\nabla_X Y,JX),\end{aligned}$$

for all orthonormal $X, Y \in \mathfrak{g}$. In the following lemma we investigate the relation between the Hermitian covariant derivative and the sectional curvature.

Theorem 3.3. Let (G, B_θ, J) be a semi-simple almost Hermitian left invariant almost complex Lie group. Assume that $\bar{\nabla}_X Y = 0$ and $[X,Y] = 0$, for all $X, Y \in \mathfrak{g}$. Then the following statements are equivalent
i)$J(\mathfrak{t}) \subset \mathfrak{t}$,
ii)$J(\mathfrak{p}) \subset \mathfrak{p}$.

Proof. Let $\bar{\nabla}_X Y = 0$, then

$$\nabla_X Y = -\frac{1}{2}(\nabla_X J)Y, \tag{13}$$

since B_θ is left invariant form we get

$$\nabla_X Y = -\frac{1}{2}\nabla_X JY + \frac{1}{2}J\nabla_X Y \tag{14}$$

and applying J on (14) we get

$$J\nabla_X Y = -\frac{1}{2}J\nabla_X JY - \frac{1}{2}\nabla_X Y, \tag{15}$$

using (12) we have

$$\bar{\nabla}_X JY = \nabla_X JY - \frac{1}{2}\nabla_X Y - \frac{1}{2}J\nabla_X JY. \tag{16}$$

Combining (14) and (16) we obtain

$$\bar{\nabla}_X JY = \frac{5}{4}\nabla_X JY - \frac{1}{4}J\nabla_X Y - \frac{1}{2}J\nabla_X JY. \tag{17}$$

Also, combining (15) and (16) we get

$$\bar{\nabla}_X JY = \nabla_X JY + J\nabla_X Y. \tag{18}$$

Comparing (17) and (18) and brief calculations, we have

$$(\nabla_X J)Y = 4J\nabla_X Y + 2J\nabla_X JY, \tag{19}$$

using (13) and (19) and brief calculations, we get

$$\nabla_X Y = -2J\nabla_X Y - J\nabla_X JY. \tag{20}$$

By changing X and Y we find

$$\nabla_Y X = -2J\nabla_Y X - J\nabla_Y JX. \tag{21}$$

By subtracting the (20) and (21) we obtain

$$[X, Y] = -2J[X, Y] - J[JX, Y]. \tag{22}$$

Let $J(\mathfrak{t}) \subset \mathfrak{t}$ and $J(\mathfrak{p}) \subset \mathfrak{t}$, from (22) we have

$$[\mathfrak{p}, \mathfrak{t}] = -2J[\mathfrak{p}, \mathfrak{t}] - J[J\mathfrak{p}, \mathfrak{t}]. \tag{23}$$

It's trivial left hand is an element of $\mathfrak{p}$ and right hand is an element of $\mathfrak{t}$, therefore $J(\mathfrak{p}) \subset \mathfrak{p}$, conversely, is proven in similarly way. □

Lemma 3.1. Let G be an almost Hermitian Lie group with a left invariant almost complex structure. If $(\bar{\nabla}_X J)Y = 0$, then

$$\nabla_X Y - J\nabla_X Y = -\nabla_X JY - J\nabla_X JY,$$

for all $X, Y \in \mathfrak{g}$.

Proof. Using (13) and some calculations the proof is trivial. □

Lemma 3.2. Let G be an almost Hermitian Lie group with a left invariant almost complex structure. Then

$$\bar{\nabla}_X Y - \nabla_X Y = \frac{1}{2}\{(\bar{\nabla}_X J)Y - J(\nabla_X J)Y\},$$

for all $X, Y \in \mathfrak{g}$.

Proof. From (12) we have

$$\begin{aligned}(\bar\nabla_X J)Y &= \bar\nabla_X JY - J\bar\nabla_X Y,\\ &= (\nabla_X J)Y - J(\nabla_X J)Y. \end{aligned} \tag{24}$$

From (12) and (24) we get

$$\bar\nabla_X Y = \nabla_X Y + \frac{1}{2}(\bar\nabla_X J)Y - \frac{1}{2}J(\nabla_X J)Y,$$

and the proof is trivial. □

§4. Semi-simple Lie group with special automorphisms

In this section a type of automorphisms are studied that called special automorphisms which they have a delicate relationship with almost complex structure, at the first step main definition.

Definition 4.1. The automorphism σ is called *special automorphism* if

$$J\sigma X = -\sigma JX,$$

for any $X \in \mathfrak{g}$.

Throughout this section automorphisms are assumed to be special. Since a Cartan involution is an automorphism, then for $X \in \mathfrak{t}$ we have

$$J\theta X = JX \quad \textit{and} \quad J\theta X = -\theta JX,$$

therefore $JX = -\theta JX$ and JX must be an element of $\mathfrak{p}$. Also, if $X \in \mathfrak{p}$, then

$$J\theta X = -JX \quad \textit{and} \quad J\theta X = -\theta JX.$$

Thus $-JX = -\theta JX$ and we conclude $JX \in \mathfrak{t}$. Ultimately, if θ as Cartan involution is special automorphism then $\mathfrak{t}$ and $\mathfrak{p}$ are anti-invariant. Let J be a left invariant almost complex structure. By straightforward calculations we get the following facts about Nijenhuis tensor,
(1)$N(\theta X, \theta Y) = -\theta N(X, Y)$, for all $X, Y \in \mathfrak{g}$,
(2)$N(\theta X, Y) = N(X, \theta Y)$, for all $X, Y \in \mathfrak{g}$ or $X, Y \in \mathfrak{p}$,
(3)$N(\theta X, Y) = -N(X, \theta Y)$, for all $X \in \mathfrak{g}$ $Y \in \mathfrak{p}$.

Moreover, for the Hermitian covariant derivative we have,
(1)$\bar\nabla_{\theta X}\theta Y = \bar\nabla_X Y$, for all $X, Y \in \mathfrak{g}$ or $X, Y \in \mathfrak{p}$,
(2)$\bar\nabla_{\theta X}\theta Y = -\bar\nabla_X Y$, for all $X \in \mathfrak{g}$, $Y \in \mathfrak{p}$,
(3)$\theta(\bar\nabla_X Y) = \bar\nabla_X Y - (\nabla_X J)Y$, for all $X, Y \in \mathfrak{g}$.

Following lemma shows the relationship between the θ and the Hermitian covariant derivative.

Lemma 4.1. Let (G, J, B_θ) be a left invariant almost Hermitian semi-simple Lie group and θ be a special automorphism as a Cartan involution, then

$$\theta\bar\nabla_X Y = \bar\nabla_{\theta X}\theta Y - (\nabla_{\theta X} J)\theta Y$$

for all $X, Y \in \mathfrak{g}$.

Proof. We have

$$\theta(\nabla_X J)Y = \theta\nabla_X JY - \theta J\nabla_X Y = -\nabla_{\theta X} J\theta Y + J\nabla_{\theta X}\theta Y = -(\nabla_{\theta X} J)\theta Y. \quad (25)$$

From (12) and (25) the proof is trivial. □

Lemma 4.2. Let (G, J, B_θ) be a left invariant semi-simple almost complex Lie group such that Cartan involution be a special automorphism. Assume $\bar{\nabla}_X Y = 0$, then $\nabla_X Y = 0$, for all $X, Y \in \mathfrak{g}$.

Proof. From the assumptions we have

$$2\nabla_X Y + J\nabla_X Y = \nabla_X JY. \quad (26)$$

Using the anti invariancy of $\mathfrak{k}$ and $\mathfrak{p}$ and applying θ to (26) we find

$$2\nabla_X Y - J\nabla_X Y = -\nabla_X JY \quad (27)$$

From (26) and (27) we have $\nabla_X Y = 0$, for all $X, Y \in \mathfrak{p}$. Now let $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, applying θ to (26) we obtain

$$-2\nabla_X Y + J\nabla_X Y = \nabla_X JY. \quad (28)$$

Comparing (26) and (28) we have

$$J\nabla_X Y = \nabla_X JY, \quad (29)$$

thus $\bar{\nabla}_X Y = \nabla_X Y$ and the proof is complete. □

Corollary 4.1. Let (G, J, B_θ) be a left invariant semi-simple almost complex Lie group such that Cartan involution be a special automorphism. Assume $\bar{\nabla}_X Y = 0$, then
(1)$(\bar{\nabla}_X J)Y = \nabla_X JY + J\nabla_X JY$, for all $X, Y \in \mathfrak{k}$,
(2)$(\bar{\nabla}_X J)Y = 0$, for all $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$.

Proof. We have

$$\begin{aligned}(\bar{\nabla}_X J)Y &= \bar{\nabla}_X JY - J\bar{\nabla}_X Y,\\ &= \nabla_X JY + \frac{1}{2}(\nabla_X J)Y,\\ &= \nabla_X JY - \frac{1}{2}J\nabla_X JY.\end{aligned}$$

for (1) and (2) the proof is trivial. □

Theorem 4.1. Let (G, J, B_θ) be a left invariant semi-simple almost complex Lie group such that Cartan involution be a special automorphism. Assume $(\bar{\nabla}_X J)Y = 0$, for all $X, Y \in \mathfrak{g}$, then $\mathfrak{g}$ is a nearly Kahler Lie algebra.

Proof. Using the definition of $\bar{\nabla}$ and assumptions we have

$$\bar{\nabla}_X JY = J\bar{\nabla}_X Y,$$

then

$$\nabla_X JY - \frac{1}{2}\nabla_X Y - \frac{1}{2}J\nabla_X JY = J\nabla_X Y + \frac{1}{2}J\nabla_X JY - \frac{1}{2}\nabla_X Y, \tag{30}$$

therefore

$$\nabla_X Y = -\nabla_X JY + J\nabla_X Y + J\nabla_X JY. \tag{31}$$

Applying θ to (31) we have

$$\nabla_X Y = \nabla_X JY - J\nabla_X Y + J\nabla_X JY. \tag{32}$$

Comparing (31) and (32) we obtain

$$[X,Y] = -2\nabla_X JY + 2J\nabla_X Y = -2(\nabla_X J)Y, \tag{33}$$

we put X=Y and obtain $(\nabla_X J)X = 0$, now the proof is complete. □

Lemma 4.3. Let (G, J, B_θ) be a left invariant semi-simple almost complex Lie group such that Cartan involution be a special automorphism and $\nabla_X JY = -J\nabla_X Y$. Then

$$(\bar{\nabla}_X J)Y = 2\nabla_X JY,$$

for all $X, Y \in \mathfrak{g}$.

Proof. We have

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)Y = \nabla_X Y + \nabla_X JY. \tag{34}$$

This implies

$$\begin{aligned}(\bar{\nabla}_X J)Y &= \bar{\nabla}_X JY - J\bar{\nabla}_X Y,\\ &= \nabla_X JY - \nabla_X Y - J\nabla_X Y - J\nabla_X JY,\\ &= -2J\nabla_X Y.\end{aligned} \tag{35}$$

The proof is complete. □

Some problems for study

Let (G, J, B_θ) be a left invariant, semi-simple, almost complex Lie group.

(1) It would be interesting to research the case that

$$ad^2(X)V = -\|X\|^2 V$$

for all $X \in \mathfrak{t}$ and $V \in \mathfrak{p}$.

(2) So it would be interesting to find properties of G while J is abelian almost complex structure

$$[X, Y] = [JX, JY]$$

for all $X, Y \in \mathfrak{g}$.

(3) It would be interesting to consider case $J\theta = \theta J$.

(4) For Hermitian case we can bring the definition of J-Hermitian covariant derivative by

$$\bar{\nabla}_X Y = \nabla_X JY + \frac{1}{2}(\nabla_X J)JY$$

for all $X, Y \in \mathfrak{g}$. So it would be interesting to study G with $\bar{\nabla}$ and B_θ.

(5) It will be so interesting to study G, while covariant derivative

$$\nabla_X JY = -J\nabla_X Y$$

for all $X, Y \in \mathfrak{g}$.

(6) It would be interesting to study slant, semi-slant, hemi-slant subgroups of semi-simple Lie groups [1].

References

[1] B. Y. Chen. Katholieke Universitieit Leuven, Leuven, 1990.

[2] S. Helgason. Differential geometry and symmetric spaces. Academic Press, New York, 1962.

[3] E. J. Humphreys. Introduction to Lie algebras and representation theory. Springer-Verlag, New york, Hiedlberg, Berlin, 1972.

[4] B. Sahin. Slant submersions from almost Hermitian manifolds. Bull. Math. Soc. Sci. Math. Roumanie 54(102) (2011), no. 1, 93C105.

[5] D. Salamon. Spin geometry and Seiberg-Witten invariants. (web version from 13 September 1996).

[6] M. Teofilova. Lie Groups as Kähler manifolds with Killing Norden metric. C. R. Acad. Bulgare Sci. 65 (2012), no. 6, 733C742.

Scientia Magna
Vol. 12 (2017), No. 1, 92-106

On fuzzy maximal regular semi-open sets and maps in fuzzy topological spaces

A. Vadivel[1] and B. Vijayalakshmi[2]

[1]Department of Mathematics, Government Arts College (Autonomous)
Karur, Tamilnadu-639005, India
E-mail: avmaths@gmail.com
[2]Department of Mathematics, Government Arts College
Chidambaram, Tamilnadu-608102, India
E-mail: mathvijaya2006au@gmail.com

Abstract The purpose of this paper is to introduce the notion of fuzzy maximal regular-semi-open sets, fuzzy minimal regular-semi closed sets, fuzzy maximal regular continuous, fuzzy maximal regular semi-continuous, fuzzy maximal regular irresolute, fuzzy maximal regular semi-irresolute functions. Some basic properties and characterization theorems are also to be investigated.

Keywords fuzzy maximal regular semi-open set, fuzzy minimal regular semi-closed set, fuzzy maximal regular semi-continuous, fuzzy maximal regular semi-irresolute functions, fuzzy maximal regular semi-connectedness.
2010 Mathematics Subject Classification 54A40, 54C05, 03E72.

§1. Introduction and preliminaries

The concept of a fuzzy subset was introduced and studied by L. A. Zadeh [16] in the year 1965. The subsequent research activities in this area and related areas have found applications in many branches of Science and Engineering. C. L. Chang [4] introduced and studied fuzzy topological spaces in 1968 as a generalization of topological spaces. Many researchers like R. H. Warren ([15], [14],) K. K. Azad ([1], [2]) G. Balasubramanian and P. Sundaram [3], S. R. Malghan and S. S. Benchalli ([7], [8]), M. N. Mukerjee and B. Ghosh [10], A. Mukherjee [9], A. N. Zahren [17], J. A. Goguen [6] and many others have contributed to the development of fuzzy topological spaces.

In 2001-2003, Nakaoka and Oda ([11], [12] and [13]) introduced minimal open sets and maximal open sets, which are subclasses of open sets. In 1978, Cameron [5] introduced regular semi-open set which is weaker than regular open set and regular closed set in topological spaces. K. K. Azad [1] defined fuzzy regular open sets in fuzzy topological spaces in the year 1981 and study their related fuzzy regular continuity in fuzzy topological spaces. Thereafter, Mathematicians gave in several papers in different and interesting new open sets. In 1994, A. N. Zahren [17] introduced the concept of fuzzy regular semi-open sets in fuzzy topological spaces. In this paper, we introduce and study the notion of fuzzy maximal regular semi-open sets and fuzzy minimal regular semi-closed sets in fts's. In section 3, we also introduce a new class of mappings viz., fuzzy maximal regular semi-continuous, fuzzy maximal regular semi-irresolute functions, fuzzy maximal regular semi-connectedness and establish interrelationship among them and some of their properties, characterization theorems and some applications in details. As for basic preliminaries some definitions and results are given for ready references.

Throughout this paper, X, Y, and Z mean fuzzy topological space (fts, for short) in Chang's sense. For a fuzzy set λ of a fts X, the notion I^X, $\lambda^c = 1_X - \lambda$, $Cl(\lambda)$, $Int(\lambda)$, $FM_aRInt(\lambda)$, $FM_iRCl(\lambda)$ will respectively stand for the set of all fuzzy subsets of X, the complement, fuzzy closure, fuzzy interior, fuzzy maximal regular interior, fuzzy minimal regular closure of λ. By 1_ϕ (or 0_X or ϕ) and 1_X (or X) we will mean the fuzzy null set and fuzzy whole set with constant membership function 0 (zero function) and 1 (unit function) respectively. A fuzzy point x_t for $t \in I_0$ is an element of I^X such that $x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$

The set of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$. A fuzzy set λ is quasi-coincident with μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If λ is not quasi-coincident with μ, we denoted $\lambda \bar{q} \mu$.

Definition 1.1. *[1] A fuzzy subset λ of fuzzy space (X, τ) is said to be*

(1) fuzzy regular open set if $Int(Cl(\lambda)) = \lambda$.

(2) fuzzy regular closed set if $Cl(Int(\lambda)) = \lambda$ or if $1_X - \lambda$ is fuzzy regular open set in X.

The class of all fuzzy regular open and fuzzy regular closed sets are respectively denoted by $FRO(X)$ and $FRC(X)$.

Lemma 1.1. *[1] If a fuzzy topological space (fts, for short) (X, τ) is product related to another fts (Y, σ) then for $\lambda \in I^X$ and $\mu \in I^Y$, $Cl(\lambda \times \mu) = Cl(\lambda) \times Cl(\mu)$.*

Lemma 1.2. *[1] If $f : (X, \tau_i) \to (Y, \sigma_i)$ fuzzy mappings and λ_i be fuzzy set of Y_i, $(i = 1, 2)$. Then, $(f_1 \times f_2)^{-1}(\lambda_1 \times \lambda_2) = f_1^{-1}(\lambda_1) \times f_2^{-1}(\lambda_2)$.*

Definition 1.2. *[17] A fuzzy set λ of a fts X is said to be fuzzy regular semi-open set in fts X if there exists a fuzzy regular open set μ in X such that $\mu \leq \lambda \leq Cl(\mu)$ (or $\lambda \leq Cl(FRInt(\mu))$ and its complement $1_X - \lambda$ is called fuzzy regular semi-closed set of X.*

We denote the class of all fuzzy regular semi-open sets and fuzzy regular semi-closed sets in fts X by $FRSO(X)$ and $FRSC(X)$.

§2. Fuzzy maximal regular semi-open sets and fuzzy minimal regular semi-closed sets

In this section, we introduce the notion of fuzzy maximal regular open (resp. maximal regular semi open) sets and fuzzy minimal regular closed (resp. minimal regular semi closed) sets and study their properties. Some fundamental theorems and their applications are also studied.

Definition 2.1. *A non empty proper fuzzy regular open set λ of any fuzzy space (X, τ) is said to be fuzzy maximal regular open set if any fuzzy regular open set which contains λ is* *either* λ *or* 1_X.

Definition 2.2. *A non empty proper fuzzy regular closed set* β *of any fuzzy space* (X,τ) *is said to be fuzzy minimal regular closed set if any fuzzy regular closed set contained in* β *is either* 1_ϕ *or* β *or equivalently, if* β^c *is fuzzy maximal regular open set in* (X,τ).

The family of all fuzzy maximal regular open set and fuzzy minimal regular closed sets are respectively, denoted by $FM_aRO(X)$ and $FM_iRC(X)$.

Definition 2.3. *A non empty proper fuzzy subset* $\lambda \in I^X$ *of any fts* (X,τ) *is said to be fuzzy maximal regular semi-open set in* X *if there exists a fuzzy maximal regular open set* λ_1 *such that* $\lambda_1 \leq \lambda \leq Cl(\lambda_1)$ *or if* $\lambda \leq Cl(FM_aRInt(\lambda))$.

Definition 2.4. *A non empty proper fuzzy subset* $\beta \in I^X$ *of any fts* (X,τ) *is said to be fuzzy minimal regular semi-closed set in* X *if there exists a fuzzy minimal regular closed set* β_1 *such that* $Int(\beta_1) \leq \beta \leq \beta_1$ *or if* $FM_iRCl(Int(\lambda)) \leq \lambda$.

Or equivalently, the complement (i.e., $1_X - \beta$) of β is fuzzy maximal regular semi-open set in X.

Or equivalently, the complement of a fuzzy maximal regular semi-open set is called a fuzzy minimal regular semi-closed set in X.

The family of all fuzzy maximal regular semi-open and fuzzy minimal regular semi-closed sets are respectively denoted by $FM_aRSO(X)$ and $FM_iRSC(X)$.

Example 2.1. *Let* $X = \{a,b,c\}$ *and* $\tau = \{1_\phi, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, 1_X\}$, *where*

$\lambda_1(a) = 0,\ \lambda_1(b) = 1,\ \lambda_1(c) = 1;$

$\lambda_2(a) = 1,\ \lambda_2(b) = 1,\ \lambda_2(c) = 0;$

$\lambda_3(a) = 1,\ \lambda_3(b) = 0,\ \lambda_3(c) = 1;$

$\lambda_4(a) = 1,\ \lambda_4(b) = 0,\ \lambda_4(c) = 0;$

$\lambda_5(a) = 0,\ \lambda_5(b) = 0,\ \lambda_5(c) = 1;$

$\lambda_6(a) = 0,\ \lambda_6(b) = 1,\ \lambda_6(c) = 0.$

Then (X,τ) *forms a fts.* $FRO(X) = \{1_\phi, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, 1_X\}$.

$$FM_aRO(X) = \{\lambda_1, \lambda_2, \lambda_3\} = FM_aRSO(X) \text{ and}$$
$$FM_iRC(X) = \{\lambda_4, \lambda_5, \lambda_6\} = FM_aRCO(X).$$

Theorem 2.1.

(1) Union of arbitrary member of fuzzy maximal regular semi-open (resp. fuzzy maximal regular open) sets is either fuzzy maximal regular semi-open (resp. fuzzy maximal regular open) set or fuzzy whole set 1_X.

(2) Intersection of arbitrary member of fuzzy minimal regular semi-closed (resp. fuzzy minimal regular closed) sets is either fuzzy minimal regular semi-closed (resp. fuzzy minimal regular closed) set or fuzzy null set 1_ϕ.

Proof. (1) Let $\{\lambda_\alpha : \alpha \in \Lambda\}$ be an arbitrary collection of fuzzy maximal regular semi-open (resp. fuzzy maximal regular open) sets in X. Then, for each α, there exists a $\mu_\alpha \in FM_aRSO(X)$ (resp. $\mu_\alpha \in FM_aRO(X)$) such that $\mu_\alpha \leq \lambda_\alpha \leq Cl(\mu_\alpha)$. Taking union of all such relations, we have

$$\bigvee \mu_\alpha \leq \lambda_\alpha \leq \bigvee Cl(\mu_\alpha).$$

$\Rightarrow \bigvee \mu_\alpha \leq \bigvee \lambda_\alpha \leq \bigvee Cl(\mu_\alpha) \leq Cl(\bigvee \mu_\alpha)$ [Since, $\bigvee Cl(\mu_\alpha) \leq Cl(\bigvee \mu_\alpha)$]

$$\Rightarrow \bigvee \mu_\alpha \leq \bigvee \lambda_\alpha \leq Cl(\bigvee \mu_\alpha). \quad (1)$$

But, $\bigvee \mu_\alpha = 1_X$ or $\mu_i = \mu_j$, $i, j \in \Lambda$.

If $\bigvee \mu_\alpha = 1_X$, then (1) $\Rightarrow 1_X \leq \bigvee \lambda_\alpha \leq Cl(1_X) \Rightarrow \bigvee \lambda_\alpha = 1_X$.

If $\mu_i = \mu_j = \mu$, $i, j \in \Lambda$, then (1) $\Rightarrow \mu \leq \bigvee \lambda_\alpha \leq Cl(\mu) \Rightarrow \bigvee \mu_\alpha \in FM_aRSO(X)$. (resp. $\bigvee \mu_\alpha \in FM_aRO(X)$).

(2) Let $\{\beta_\alpha : \alpha \in \Lambda\}$ be any collection of fuzzy minimal regular semi-open (resp. fuzzy minimal regular open) sets in X. Then, ${\beta_\alpha}^c \in FM_aRSO(X)$
(resp. ${\beta_\alpha}^c \in FM_aRO(X)$), for each α and for each α, there exists a $\gamma_\alpha \in FM_aRSO(X)$ (resp. $\gamma_\alpha \in FM_aRO(X)$) such that $\gamma_\alpha \leq \beta_\alpha \leq Cl(\gamma_\alpha)$ for each α. Taking union of all such relations, we have

$\bigvee \gamma_\alpha \leq \bigvee \beta_\alpha^c \leq \bigvee Cl(\gamma_\alpha)$

$\Rightarrow \bigvee \gamma_\alpha \leq \bigvee \beta_\alpha^c \leq \bigvee Cl(\gamma_\alpha) \leq Cl(\bigvee \gamma_\alpha)$ [Since, $\bigvee Cl(\mu_\alpha) \leq Cl(\bigvee \mu_\alpha)$]

$\Rightarrow \bigvee \gamma_\alpha \leq \bigvee \beta_\alpha^c \leq Cl(\bigvee \gamma_\alpha)$.

$\Rightarrow \bigvee \beta_\alpha^c \in FM_aRSO(X)$. (resp. $\bigvee \beta_\alpha^c \in FM_aRO(X)$).

But $\bigvee \gamma_\alpha = 1_X$ or $\gamma_i = \gamma_j$, $i,\ j \in \Lambda$.

If $\bigvee \gamma_\alpha = 1_X$ then (i) $1_X \leq \bigvee \beta_\alpha^c \leq Cl(1_X) \Rightarrow \bigvee \beta_\alpha^c = 1_X$.

If $\gamma_i = \gamma_j = \gamma$, $i,\ j \in \Lambda$, then (i) $\Rightarrow \gamma \leq \bigvee \beta_\alpha^c \leq Cl(\gamma)$

$\Rightarrow \bigvee \beta_\alpha^c = (\bigwedge \beta_\alpha)^c \in FM_aRSO(X)$(resp.$(\bigwedge \beta_\alpha)^c \in FM_aRO(X)$)

$\Rightarrow \bigwedge \beta_\alpha \in FM_iRSC(X)$. (resp. $\bigwedge \beta_\alpha \in FM_iRC(X)$). □

Remark 2.1.

(1) Intersection of finite or infinite number of fuzzy maximal regular semi-open (resp. maximal regular open) sets may not fuzzy maximal regular semi-open (resp. maximal regular open) sets in X.

(2) Union of finite or infinite number of fuzzy minimal regular semi-closed (resp. minimal regular closed) sets may not fuzzy minimal regular semi-closed (resp. minimal regular closed) sets in X.

Example 2.2. *In Example 2.1, λ_1 and λ_2 are fuzzy maximal regular open set in X. But $\lambda_1 \wedge \lambda_2 = \lambda_6$ which is not fuzzy maximal regular open set in X. Also, λ_4 and λ_5 are fuzzy minimal regular closed set in X. But $\lambda_4 \vee \lambda_5 = \lambda_3$ which is not fuzzy minimal regular closed set in X. Also, we see that λ_1 and λ_2 are fuzzy maximal regular semi-open set in X. But $\lambda_1 \wedge \lambda_2 = \lambda_6$ which is not fuzzy maximal regular semi-open set in X. Also, λ_4 and λ_5 are fuzzy minimal regular semi-closed sets in X. But $\lambda_4 \vee \lambda_5 = \lambda_3$ which is not fuzzy minimal regular semi-closed set in X.*

Remark 2.2. *Every fuzzy maximal regular open (resp. fuzzy minimal regular closed) set is fuzzy maximal regular semi-open (resp. fuzzy minimal regular semi-closed) set but the converse is false which is shown by the following example.*

Example 2.3. *Let $X = \{a, b, c\}$ and $\tau = \{0_X, \lambda_1, \lambda_2, \lambda_3, \lambda_4, 1_X\}$ where*

$\lambda_1(a) = 0.1,\ \lambda_1(b) = 0.4,\ \lambda_1(c) = 0.6;$

$\lambda_2(a) = 0.7,\ \lambda_2(b) = 0.2,\ \lambda_2(c) = 0.4$;
$\lambda_3(a) = 0.7,\ \lambda_3(b) = 0.4,\ \lambda_3(c) = 0.6$;
$\lambda_4(a) = 0.1,\ \lambda_4(b) = 0.2,\ \lambda_4(c) = 0.4$.

Fuzzy closed sets in X are $\tau^c = \{0_X, \delta_1, \delta_2, \delta_3, \delta_4, 1_X\}$ *where*
$\delta_1(a) = 0.9,\ \delta_1(b) = 0.6,\ \delta_1(c) = 0.4$;
$\delta_2(a) = 0.3,\ \delta_2(b) = 0.8,\ \delta_2(c) = 0.6$;
$\delta_3(a) = 0.3,\ \delta_3(b) = 0.6,\ \delta_3(c) = 0.4$;
$\delta_4(a) = 0.9,\ \delta_4(b) = 0.8,\ \delta_4(c) = 0.6$.

Then (X, τ) *forms a fts. Here,* λ_1, λ_2, λ_3, λ_4 *are fro sets in* X. *Here,* λ_3 *is the only fuzzy maximal regular open set in* X. *This implies* $\lambda_3 \in FM_aRO(X)$. *Now, since* $\lambda_3 \leq \lambda \leq Cl(\lambda_3) = \delta_4$, *we take* $\lambda = (0.8_a, 0.4_b, 0.6_c)$. *This implies that* λ *is fuzzy maximal regular semi-open set in* X, *which is neither fuzzy regular open nor fuzzy maximal regular open set in* X. *Similarly,* δ_3 *is fuzzy minimal regular closed set in* X *implies that* $\delta_3 \in FM_iRC(X)$. *Now, since* $Int(\delta_3) \leq \delta \leq \delta_3$, *we take* $\delta = (0.3_a, 0.2_b, 0.4_c)$. *This implies that* δ *is fuzzy minimal regular semi-closed set in* X, *which is neither fuzzy regular closed nor fuzzy minimal regular closed set in* X.

Corollary 2.1. *Let* X *be a fuzzy topological space, then*

(1) $FM_aRO(X) \subset FM_aRSO(X)$.

(2) *If* $\lambda \in FM_aRSO(X)$ *and* $\lambda \leq \lambda_1 \leq Cl(\lambda)$, *then* $\lambda_1 \in FM_aRSO(X)$.

Definition 2.5.

(1) *The union of all fuzzy maximal regular semi-open sets of* X *contained in a fuzzy set* μ *is known as fuzzy maximal regular semi-interior of* μ *and is denoted by* $FM_aRSInt(\mu)$ *i. e.,* $FM_aRSInt(\mu) = sup\{\lambda_j : \lambda_j \leq \mu \text{ and } \lambda_j \in FM_aRSO(X)\ ;\ j \in \Lambda, \text{ an arbitrary index set.}\}$

(2) *The intersection of all fuzzy minimal regular semi-closed sets of* X *containing a fuzzy set* μ *is called fuzzy minimal regular semi-closure of* μ *and is denoted by* $FM_iRSCl(\mu)$ *i. e.,* $FM_iRSCl(\mu) = inf\{\gamma_k^c\text{: } \gamma_k^c \geq \mu \text{ and } \gamma_k^c \in FM_aRSC(X)\ ;\ k \in \Lambda, \text{ an index set. }\}$

(3) *A fuzzy set* α *is called fuzzy maximal regular neighborhood (shortly,* FM_aR*-nbd) of a fuzzy point* x_p *in* X *if there exists a* $\beta \in FM_aRO(X)$ *such that* $x_p \in \beta \leq \alpha$.

(4) *A fuzzy set* α *is called fuzzy maximal regular semi-neighborhood (shortly,* FM_aRS*-nbd) of a fuzzy point* x_p *in* X *if there exists a* $\beta \in FM_aRSO(X)$ *such that* $x_p \in \beta \leq \alpha$.

(5) *A fuzzy set* α *is called fuzzy maximal regular-quasi-neighborhood (shortly,* FM_aRQ*-nbd) of a fuzzy point* x_p *in* X *if there exists a* $\beta \in FM_aRO(X)$ *such that* $x_p q \beta \leq \alpha$.

(6) *A fuzzy set* α *is called fuzzy maximal regular semi quasi-neighborhood (shortly,* FM_aRSQ*-nbd) of a fuzzy point* x_p *in* X *if there exists a* $\beta \in FM_aRSO(X)$ *such that* $x_p q \beta \leq \alpha$.

(7) *A fuzzy point* x_p *is said to be fuzzy maximal regular limit (or cluster) point of a fuzzy set* $\lambda \in I^X$ *of a fuzzy space* X *if for every* FM_aR*-open nbd* μ *of* x_p, $(\mu - x_p) \wedge \lambda \neq 0_X$.

(8) *A fuzzy point* x_p *is said to be fuzzy maximal regular semi-limit (or cluster) point of a fuzzy set* $\lambda \in I^X$ *of a fuzzy space* X *if for every* FM_aR*-semi open nbd* μ *of* x_p, $(\mu - x_p) \wedge \lambda \neq 0_X$.

Theorem 2.2.

(1) For a fuzzy set $\lambda \in I^X$, a fuzzy singleton $x_r \in FM_iRSCl(\lambda)$, iff $\lambda q \eta$ for each fuzzy maximal regular semi-open set η of a fts X such that $x_r q \eta$.

(2) For a fuzzy set $\lambda \in I^X$, a fuzzy singleton $x_r \in FM_iRCl(\lambda)$ iff every fuzzy maximal regular open quasi neighborhood η of x_r, is quasi-coincident with λ.

Proof. (1) Let $x_r \in FM_iRSCl(\lambda)$ and let $\lambda q \eta$ for each fuzzy maximal regular semi-open set of a fts X such that $x_r q \eta$. Then $\lambda \leq \eta^c$ such that $\lambda > \eta^c(x)$. Since η^c is fuzzy minimal regular semi-closed set, so $FM_iRSCl(\lambda) \leq \eta^c < r$, for $x \in X$. Hence $x_r \notin FM_iRSCl(\lambda)$ which contradicts the given hypothesis. Thus $\lambda q \eta$ for each fuzzy maximal regular semi-open set η of X such that $x_r q \eta$.

Conversely, suppose that $\lambda q \eta$ for each fuzzy maximal regular semi-open set of X with $x_r q \eta$ and let $x_r \notin FM_iRSCl(\lambda)$. Then $r > FM_iRSCl(\lambda) \Rightarrow 1 - r < FM_aRSInt(1_X - \lambda) = V$ (say) $\Rightarrow r + V > 1$ with $V \leq \lambda^c \Rightarrow x_r q V$ such that $\lambda q V$ for fuzzy maximal regular semi-open set η of X which is a contradiction to the given hypothesis. Hence $x_r \in FM_iRSCl(\lambda)$.

(2) Similar to the proof of (1). □

Theorem 2.3. *If λ_1 is a fuzzy maximal regular semi-open set of X and $\lambda_1 \leq \lambda \leq Cl(\lambda_1)$, then λ is a fuzzy maximal regular semi-open set of X.*

Proof. Since λ_1 is fuzzy maximal regular semi-open set in X, so there exists a fuzzy maximal regular open set U such that $U \leq \lambda_1 \leq Cl(U)$. Then $U \leq \lambda_1 \leq \lambda \leq Cl(\lambda_1) \leq Cl(U)$. Hence $U \leq \lambda \leq Cl(U)$. Thus λ is a fuzzy maximal regular semi-open set of X. □

Theorem 2.4. *If β_1 is fuzzy minimal regular semi-closed sets in X and $Int(\beta_1) \leq \beta \leq \beta_1$, then β is also fuzzy minimal regular semi-closed in X.*

Proof. Let β_1 be a fuzzy minimal regular semi-closed set of X. Then there exists a minimal regular closed set H in X such that $Int(H) \leq \beta_1 \leq H$. Hence $Int(H) \leq Int(\beta_1) \leq \beta \leq \beta_1 \leq H$. This implies that $Int(H) \leq \beta \leq H$. Hence, β is a fuzzy minimal regular semi-closed set of X. □

Theorem 2.5. *A fuzzy set $\lambda \in I^X$ is fuzzy maximal regular semi-open set in X iff $Cl(\lambda) = Cl(FM_aRInt(\lambda))$.*

Proof. Let λ be fuzzy maximal regular semi-open set in X. So by definition of fuzzy maximal regular semi-open set $\lambda \leq Cl(FM_aRInt(\lambda))$. Taking closure on both sides, we have,

$$Cl(\lambda) \leq Cl(FM_aRInt(\lambda)) \tag{2}$$

Also, we have, $FM_aRInt(\lambda)) \leq \lambda$

$$\Rightarrow Cl(FM_aRInt(\lambda)) \leq Cl(\lambda). \tag{3}$$

Hence from (2) and (3), we have $Cl(\lambda) = Cl(FM_aRInt(\lambda))$.

Conversely, let us suppose that

$$Cl(\lambda) = Cl(FM_aRInt(\lambda)) \tag{4}$$

Now, we have $FM_aRInt(\lambda) \leq \lambda \leq Cl(\lambda) \Rightarrow FM_aRInt(\lambda) \leq \lambda \leq Cl(FM_aRInt(\lambda))$ [using (3)] $\Rightarrow FM_aR$-open set $\leq \lambda \leq Cl(FM_aR$-open set$) \Rightarrow \lambda$ is fuzzy maximal regular semi-open set in X. □

Theorem 2.6. *A fuzzy subset β of X is fuzzy minimal regular semi-closed iff there exists a fuzzy minimal regular closed set β_1 in X such that $Int\beta_1 \leq \beta \leq \beta_1$.*

Proof. Suppose, β is a fuzzy minimal regular semi-closed set in X. By definition, β^c is fuzzy maximal regular semi-open set in X. Therefore, there exists a fuzzy maximal regular open set λ such that $\lambda \leq \beta^c \leq Cl(\lambda)$, which implies

$$Int(\lambda^c) = (Cl(\lambda))^c \leq \beta \leq \lambda^c.$$

Take $\beta_1 = \lambda^c$, so that β_1 is a minimal regular closed set, such that $Int\beta_1 \leq \beta \leq \beta_1$.

Conversely, suppose that there exists a fuzzy minimal regular closed set β_1 in X such that $Int\beta_1 \leq \beta \leq \beta_1 \Rightarrow \beta_1^c \leq \beta^c \leq (Int\beta_1)^c = Cl(\beta_1^c)$. Therefore, there exists a maximal regular open set $\lambda = \beta_1^c$ such that $\lambda_1 \leq \beta^c \leq Cl(\lambda)$.

$\Rightarrow \beta^c$ is a fuzzy maximal regular semi-open set in X and hence β is fuzzy minimal regular semi-closed set in X. □

Lemma 2.1. *For any fuzzy set $\lambda \in I^X$ of a fuzzy space (X, τ),*

(1) $1_X - FM_aRSInt(\lambda) = FM_iRSCl(1_X - \lambda)$.

(2) $1_X - FM_iRSCl(\lambda) = FM_aRSInt(1_X - \lambda)$.

Proof. The proofs are easy and follow from Definition 2.5. □

The following theorem can be easily verified.

Theorem 2.7. *For any fuzzy set $\lambda, \mu \in I^X$ of a fts (X, τ), the following properties hold:*

(1) μ is fuzzy maximal regular semi-open in X iff $\mu = FM_aRSInt(\mu)$.

(2) μ is fuzzy minimal regular semi-closed in X iff $\mu = FM_iRSCl(\mu)$.

(3) $\lambda \leq \mu \Rightarrow FM_iRSCl(\lambda) \leq FM_iRSCl(\mu)$ and $FM_aRSInt(\lambda) \leq FM_aRSInt(\mu)$.

(4) $FM_iRSCl(FM_iRSCl(\mu)) = FM_iRSCl(\mu)$.

(5) $FRSCl(\mu) \geq FM_iRSCl(\mu)$.

(6) $FRSInt(\mu) \leq FM_aRSInt(\mu)$

(7) $FM_aRSInt(\lambda) \in FM_aRSO(X)$

(8) $FM_iRSCl(\lambda) \in FM_iRSC(X)$.

(9) μ is fuzzy minimal regular semi-closed in X iff $\mu = FM_iRSCl(\mu)$.

Theorem 2.8. *If a fts (X, τ) is product related to an another fts (Y, σ) and $\lambda \in FM_aRO(X)$, $\mu \in FM_aRO(Y)$. Then $\lambda \times \mu \in FM_aRO(X \times Y)$.*

Proof. Obvious. □

Theorem 2.9. *If a fts (X, τ) is product related to an another fts (Y, σ) and $\lambda \in FM_aRSO(X)$, $\mu \in FM_aRSO(Y)$. Then $\lambda \times \mu \in FM_aRSO(X \times Y)$.*

Proof. Since, $\lambda \in FM_aRSO(X)$, $\mu \in FM_aRSO(Y)$. So, there exists $\gamma \in FM_aRO(X)$ and $\eta \in FM_aRO(Y)$ such that $\gamma \le \lambda \le Cl(\gamma)$ and $\eta \le \mu \le Cl(\eta)$ which implies that

$$\gamma \times \eta \le \lambda \times \mu \le Cl(\gamma) \times Cl(\eta). \tag{5}$$

Since (X, τ) is product related to (Y, σ). So, by Lemma 1.1.[1], we must have, $Cl(\gamma) \times Cl(\eta) = Cl(\gamma \times \eta)$ and then (5) $\Rightarrow \gamma \times \eta \le \lambda \times \mu \le Cl(\gamma \times \eta)$. Since by Theorem 2.8, $\gamma \times \eta$ is fuzzy maximal regular open set in the product space $(X \times Y, \tau \times \sigma)$. Hence $\lambda \times \mu$ is fuzzy maximal regular semi-open set in the product fts $(X \times Y, \tau \times \sigma)$ i.e., $\lambda \times \mu \in FM_aRSO(X \times Y)$. □

§3. Fuzzy maximal regular continuous (resp. fuzzy maximal regular semi-continuous) and fuzzy maximal regular irresolute (resp. fuzzy maximal regular semi-irresolute) functions

In this section, we introduce some new notions of fuzzy mappings viz., fuzzy maximal regular continuous, fuzzy maximal regular semi-continuous, fuzzy maximal regular irresolute and fuzzy maximal regular semi-irresolute functions. We also establish some of their characterization theorems and show some interrelationship among these new class of functions.

Definition 3.1. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy maximal regular continuous (shortly, FM_aR-cont.) iff for each $\lambda \in FO(Y)$, $f^{-1}(\lambda) \in FM_aRO(X)$.*

Example 3.1. *Let $X = Y = \{a, b\}$ and $\tau = \{0_X, \lambda_1, \lambda_2, 1_X\}$, $\sigma = \{0_X, \lambda_1, 1_X\}$ where λ_1 and λ_2 are defined as*

$\lambda_1(a) = \frac{3}{4}$, $\lambda_1(b) = \frac{1}{4}$;

$\lambda_2(a) = \frac{1}{4}$, $\lambda_2(b) = \frac{1}{4}$;

Then (X, τ) and (Y, σ) forms a fts. $FRO(X) = \{\lambda_1, \lambda_2\}$. Consider the function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(x) = x$, $\forall x \in X$. Here λ_1 is the only non-empty proper fuzzy open set in Y and also λ_1 is fuzzy maximal regular open set in Y such that

$$f^{-1}(\lambda_1(x)) = \lambda_1(f(x)) = \lambda_1(x) \in FM_aRO(X).$$

Thus f is fuzzy maximal regular continuous function on X.

Theorem 3.1. *For a mapping $f : (X, \tau) \to (Y, \sigma)$ the following statements are equivalent:*

(1) f is FM_aR-continuous function.

(2) for every fuzzy point x_r of X and for every fuzzy neighborhood η of $f(x_r)$ in (Y, σ), there exists a fuzzy maximal regular open neighborhood γ of x_r in (X, τ) such that $f(\gamma) \le \eta$.

(3) $f^{-1}(\beta) \in FM_iRC(X)$, $\forall \beta \in FC(Y)$.

Proof. We need to prove the following implications: (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4).

(1) $\Rightarrow$ (2): Let f be FM_aR-continuous function and let $x_r \in X$ and η be any fuzzy neighborhood of $f(x_r)$ in Y. Then there exists a $\mu \in \sigma$ such that $f(x_r) \le \mu \le \eta \Rightarrow x_r \in f^{-1}(\mu) \le f^{-1}(\eta)$. As f is FM_aR-continuous and $\mu \in \sigma$. So, $f^{-1}(\mu) \in FM_aRO(X)$ so that $\gamma = f^{-1}(\eta)$ is a fuzzy maximal regular neighborhood of x_r in X such that $f(\gamma) = \eta \le \eta$.

(2) $\Rightarrow$ (3): Let (2) is true for the function f and let β be a closed set in Y and $x_r \in f^{-1}(\beta^c)$ $\Rightarrow f(x_r) \in \beta^c$. Since β^c is open neighborhood of $f(x_r)$, so, by hypothesis, there exists a fuzzy maximal regular neighborhood V of x_r in X such that $f(V) \le \beta^c$ so that $V \le f^{-1}(\beta^c)$. Since, V is fuzzy maximal regular neighborhood of x_r, there exists a fuzzy maximal regular open set δ such that $x_r \in \delta \le f^{-1}(\beta^c) \Rightarrow \bigvee\{x_r\} \le \bigvee\{\delta\} \le \bigvee\{f^{-1}(\beta^c)\} \Rightarrow f^{-1}(\beta^c) \le \bigvee\{\delta\} \le \bigvee f^{-1}(\beta^c)$ $\Rightarrow (f^{-1}(\beta))^c = \bigvee\{\delta\} = \delta$ or $1_X \in FM_aRO(X) \Rightarrow f^{-1}(\beta) \in FM_iRC(X)$.

(3) $\Rightarrow$ (4): Let (3) is true and $\lambda \in FO(Y)$. Then $f^{-1}(\lambda) = (f^{-1}(\lambda^c)^c) = (f^{-1}(\lambda^c))^c$. Since, $\lambda^c \in FC(Y)$, by hypothesis, we have, $f^{-1}(\lambda^c) \in FM_iRC(X)$ and hence $(f^{-1}(\lambda^c))^c = f^{-1}(\lambda) \in FM_aRO(X)$ showing that f is FM_aR-continuous function on X. □

Theorem 3.2. *For FM_aR-continuous mapping f from a fts (X, τ) into another fts (Y, σ) the following statements hold:*

(1) $f(FM_aRInt(\lambda)) \ge Intf(\mu)$, for every fuzzy set μ in X.

(2) $FM_aRInt(f^{-1}(\lambda)) \ge f^{-1}(Int(\lambda))$, for every fuzzy set $\lambda \in Y$ and for onto map f.

(3) $f(FM_iRCl(\mu)) \le Clf(\mu)$, for every fuzzy set μ in X.

(4) $FM_iRCl(f^{-1}(\lambda)) \le f^{-1}(Cl(\lambda))$, for every fuzzy set λ in Y and for onto map f.

Proof. (1) Since, $Int(f(\mu))$ is fuzzy open set in Y and f is FM_aR-continuous, $f^{-1}(Intf(\mu)) \in FM_aRO(X)$. As we know that $f(\mu) \ge Intf(\mu) \Rightarrow \mu \ge f^{-1}(Intf(\mu)) \Rightarrow FM_aRInt(\mu) \ge f^{-1}(Intf(\mu))$ so that $f(FM_aRInt(\mu)) \ge Intf(\mu)$.

(2) Since, $f^{-1}(\lambda)$ is a fuzzy set in X, so for $\mu = f^{-1}(\lambda)$ (1) must hold, i.e.,

$$f(FM_aRInt(f^{-1}(\lambda))) \ge Int(f(f^{-1}(\lambda))) = Int(\lambda),$$

as f is onto mapping. Hence, $FM_aRInt(f^{-1}(\lambda)) \ge f^{-1}(Int(\lambda))$.

(3) Since, $Cl(f(\mu))$ is fuzzy closed set in Y and f is FM_aR-continuous, $f^{-1}(Cl(f(\mu)) \in FM_iRC(X)$. Now, $f(\mu) \le Cl(f(\mu)) \Rightarrow \mu \le f^{-1}(Cl(f(\mu)))$

$\Rightarrow FM_iRCl(\mu) \le f^{-1}(Cl(f(\mu)))$. Thus $f(FM_iRCl(\mu)) \le Cl[f(\mu)]$.

(4) Since, $f^{-1}(\lambda) \in I^X, \forall \lambda \in I^Y$, so for $\mu = f^{-1}(\lambda)$ we have from (3) $f(FM_iRCl(f^{-1}(\lambda))) \ge Cl(f(f^{-1}(\lambda))) = Cl(\lambda)$. [Being f an onto map.]
Hence, $FM_aRInt(f^{-1}(\lambda)) \ge f^{-1}[Cl(\lambda)]$. □

Definition 3.2. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy maximal regular irresolute (shortly, FM_aR-irresolute) iff for each $\lambda \in FM_aRO(Y)$, $f^{-1}(\lambda) \in FM_aRO(X)$.*

Example 3.2. *Consider the function $f : (X, \tau) \to (X, \sigma)$ defined by $f(x) = x$, $\forall x \in X$, where (X, τ) defined in Example 2.3 and (X, σ) is defined as $\sigma = \{0_X, \lambda_3, 1_X\}$. Here, λ_3 is the only non empty proper fuzzy maximal regular open set in Y such that $f^{-1}(\lambda_3(x)) = \lambda_3(f(x)) = \lambda_3(x) \in FM_aRO(X)$. Thus f is fuzzy maximal regular irresolute function on X.*

Definition 3.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy maximal regular semi-irresolute iff for each $\lambda \in FM_aRSO(Y)$, $f^{-1}(\lambda) \in FM_aRSO(X)$.*

Example 3.3. *Consider the function $f : (X, \tau) \to (X, \sigma)$ defined by $f(x) = x$, $\forall x \in X$, where (X, τ) and (X, σ) are defined as $X = Y = \{a, b, c\}$ and $\tau = \{0_X, \lambda_1, \lambda_2, 1_X\}$, $\sigma = \{0_X, \lambda_2, 1_X\}$ where*

$\lambda_1(a) = 0.3,\ \lambda_1(b) = 0.4,,\ \lambda_1(c) = 0.5;$

$\lambda_2(a) = 0.6,\ \lambda_2(b) = 0.5,;\ \lambda_2(c) = 0.5;$

Here λ_2 is the only non-empty proper fuzzy maximal regular semi open set in Y such that

$$f^{-1}(\lambda_2(x)) = \lambda_2(f(x)) = \lambda_2(x) \in FM_aRSO(X).$$

Thus f is fuzzy maximal regular semi-irresolute function on X.

Theorem 3.3. *If $f : (X, \tau_1) \to (Y, \tau_2)$ be FM_aR-continuous function and $g : (Y, \tau_2) \to (Z, \tau_3)$ be fuzzy continuous function. Then $g \circ f : (X, \tau_1) \to (Z, \tau_3)$ is also FM_aR-continuous function.*

Proof. Let $\lambda \in FO(Z)$. Now $(g \circ f)^{-1}(\lambda) = (f^{-1} \circ g^{-1})(\lambda) = (f^{-1}(g^{-1}(\lambda)))$. Since g is fuzzy continuous, $g^{-1}(\lambda)$ is fuzzy open and then $(g \circ f)^{-1}(\lambda) = (f^{-1}($ fuzzy open in $Y))$. But f being FM_aR-continuous, $(g \circ f)^{-1}(\lambda) \in FM_aRO(X)$. This shows that $g \circ f$ is FM_aR-continuous function. □

Theorem 3.4. *If $f : (X, \tau_1) \to (Y, \tau_2)$ be FM_aR-irresolute function and $g : (Y, \tau_2) \to (Z, \tau_3)$ be FM_aR-continuous function. Then $g \circ f : (X, \tau_1) \to (Z, \tau_3)$ is also FM_aR-continuous function.*

Proof. Let $\lambda \in FO(Z)$. Now $(g \circ f)^{-1}(\lambda) = (f^{-1} \circ g^{-1})(\lambda) = (f^{-1}(g^{-1}(\lambda)))$. Since g is FM_aR-continuous, $g^{-1}(\lambda)$ is FM_aR-open and then $(g \circ f)^{-1}(\lambda) = (f^{-1}(FM_aR$-open set in $Y))$. But f being FM_aR-irresolute, $(g \circ f)^{-1}(\lambda) \in FM_aRO(X)$. This shows that $g \circ f$ is FM_aR-continuous function. □

Theorem 3.5. *Composition of two FM_aR-irresolute function is again a FM_a-irresolute function.*

Proof. Straight forward. □

Definition 3.4. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy maximal regular semi-continuous (shortly, FM_aRS-continuous) iff for each $\lambda \in FO(Y)$, $f^{-1}(\lambda) \in FM_aRSO(X)$.*

Example 3.4. *Let $f : (X, \tau) \to (X, \sigma)$ be a function defined by $f(x) = x$, $\forall x \in X$, where (X, τ) and (Y, σ) are defined in Example 3.3. Since for $\lambda_2 \in \sigma$,*

$$f^{-1}(\lambda_2(x)) = \lambda_2(f(x)) = \lambda_2(x) \in FM_aRSO(X),$$

Thus f is fuzzy maximal regular semi continuous function on X.

Theorem 3.6. *Let X_i, Y_i, $(i = 1, 2)$ be fts such that X_1 is product related to X_2 and $f_i : (X_i, \tau_i) \to (Y_i, \sigma_i)$ $(i = 1, 2)$ fuzzy maximal regular semi-continuous function. Then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is also fuzzy maximal regular semi-continuous function on $X_1 \times X_2$.*

Proof. Let $\lambda \in FO(Y_1)$, $\mu \in FO(Y_2)$. Then $\lambda \times \mu \in FO(Y_1 \times Y_2)$. Using Lemma 1.2 [1], we have $(f_1 \times f_2)^{-1}(\lambda \times \mu) = f_1^{-1}(\lambda) \times f_2^{-1}(\mu)$. Since, f is fuzzy maximal regular semi-continuous function on X_i. So, $f_i^{-1}(\lambda)$ is fuzzy maximal regular semi-open set on X_i. Again, since X_1 is product related to X_2. So,

$$(f_1 \times f_2)^{-1}(\lambda \times \mu) = f_1^{-1}(\lambda) \times f_2^{-1}(\mu) \in FM_aRSO(X)$$

and hence $f_1 \times f_2$ is fuzzy maximal regular semi-continuous function on $X_1 \times X_2$. □

Theorem 3.7. *Let X_i, Y_i, $(i = 1, 2, ..., n)$ be fts such that X_i is product related to X_j $(i \neq j)$ and $f_i : (X_i, \tau_i) \to (Y_i, \sigma_i)$ $(i = 1, 2, ..., n)$ fuzzy maximal regular semi-continuous function. Then $\prod_{i=1}^{n} f_i : \prod_{i=1}^{n} X_i \to \prod_{i=1}^{n} Y_i$ is also fuzzy maximal regular semi-continuous function on $\prod_{i=1}^{n} X_i$.*

Proof. Obvious. □

Theorem 3.8. *Let $f : X \to Y$ be a function defined by $f(x) = y$, $\forall x \in X$ and $g : X \to X \times Y$ a graph of the map f defined by $g(x) = (x, f(x))$, $\forall x \in X$. If g is fuzzy maximal regular semi-continuous, then so is f.*

Proof. Let $\mu \in FO(Y)$. Then for $1_X \in FO(X)$, $1_X \times \mu$ is a fuzzy open set in $X \times Y$. Since g is a graph of the map f, So $g(x) = (x, y) = (x, f(x))$, $\forall x \in X$. Now $\forall x \in X$ we have,

$$\begin{aligned} g^{-1}(1_X \times \mu)(x) &= (1_X \times \mu)(g(x)) \\ &= (1_X \times \mu)(x, f(x)) \\ &= min\{1_X(x), \mu(f(x))\} \\ &= 1_X(x) \wedge f^{-1}(\mu)(x) \\ &= (1_X \wedge f^{-1}(\mu))(x) \\ &= f^{-1}(\mu)(x). \end{aligned}$$

Since g is fuzzy maximal regular semi-continuous, so $g^{-1}(1_X \times \mu) = f^{-1}(\mu) \in FM_aRSO(X)$, $\forall \mu \in FO(Y)$. Hence f is fuzzy maximal regular semi-continuous function on X. □

Theorem 3.9.

(1) *Every fuzzy maximal regular continuous function is fuzzy maximal regular semi-continuous function. But the converse may not be true which can be seen from the following example.*

Consider the function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(x) = x$, $\forall x \in X$, where (X, τ) defined in Example 2.2 and (Y, σ) is defined as $Y = \{a, b, c\}$, $\sigma = \{0_Y, \delta, 1_Y\}$, where

$\delta(a) = 0.8$, $\delta(b) = 0.4$, $\delta(c) = 0.6$. *Here, δ is the only non empty proper fuzzy open set in Y. Since,*

$$f^{-1}(\delta(x)) = \delta(f(x)) = \delta(x)$$

which is a FM_aRSO set in X but not FM_aR-Open set in X. Thus f is fuzzy maximal regular semi-continuous but not fuzzy maximal regular continuous function on X.

(2) Every fuzzy maximal regular-irresolute function is fuzzy maximal regular semi-irresolute function.

Proof. (1) Proof follows from Corollary 2.1 [1], i.e., from the fact that "every fuzzy maximal regular open set is fuzzy maximal regular semi-open set in a fts (X, τ)".

(2) Obvious. □

Definition 3.5.

(1) A collection $\mathcal{M}$ is said to be fuzzy maximal regular open cover (shortly, FM_aR-open cover) of a fuzzy set $\mu \in I^X$ iff $\mathcal{M}$ covers μ and each member of $\mathcal{M}$ is fuzzy maximal regular open set in X i.e., $\mu \leq sup\{\mu_\alpha \in FM_aRO(X) : \mu_\alpha \in \mathcal{M}, \forall \alpha \in \Lambda\}$.

(2) A collection $\mathcal{M}$ is said to be fuzzy maximal regular semi-open cover (shortly, FM_aRS-open cover) of a fuzzy set $\mu \in I^X$ iff $\mathcal{M}$ covers μ and each member of $\mathcal{M}$ is fuzzy maximal regular semi-open in X. i.e., $\mu \leq sup\{\mu_\alpha \in FM_aRSO(X) : \mu_\alpha \in \mathcal{M}, \forall \alpha \in \Lambda\}$.

Definition 3.6.

(1) A fuzzy set $\lambda \in I^X$ of a fts (X, τ) is said to be fuzzy maximal regular compact (shortly, FM_aR-compact) iff each FM_aR-open cover $\mathcal{M}$ of λ has a finite subcover $\mathcal{M}_0$ which also covers λ.

(2) A fuzzy set $\lambda \in I^X$ of a fts (X, τ) is said to be fuzzy maximal regular semi-compact (shortly, FM_aR-compact) iff each FM_aR-semi-open cover $\mathcal{M}$ of λ has a finite subcover $\mathcal{M}_0$ which also covers λ.

Theorem 3.10.

(1) Fuzzy maximal regular continuous image of a FM_aR-compact set is fuzzy compact.

(2) Fuzzy maximal regular semi-continuous image of a FM_aRS-compact set is fuzzy compact.

Proof. (1) Let $f : X \to Y$ be fuzzy maximal regular continuous and $\beta \in I^X$, a FM_aR-compact set of a fts X and $P = \{\mu_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of $f(\beta)$ such that $f(\beta) \leq SupP \Rightarrow \beta \leq f^{-1}(f(\beta)) \leq f^{-1}(Sup\{\mu_\alpha : \alpha \in \Lambda\}) = Sup\{f^{-1}(\mu_\alpha) : \alpha \in \Lambda\}$. Then $Q = \{f^{-1}(\mu_\alpha) : \alpha \in \Lambda\}$ is a fuzzy cover of β. Since f is fuzzy maximal regular continuous function, $f^{-1}(\mu_\alpha) \in FM_aRO(X)$, $\forall \alpha \in \Lambda$, an arbitrary index set and then Q is FM_aR-open cover of β. Since β is FM_aR-compact, there exixts a finite sub-cover $Q = \{f^{-1}(\mu_\alpha) : \alpha = 1, 2, 3, ..., n\}$ of Q such that $\beta \leq Sup\{f^{-1}(\mu_\alpha) : \alpha = 1, 2, ..., n\}$. Since each $f^{-1}(\mu_\alpha)$ is distinct, FM_aR-open set in X. So, by Theorem 2.7 [6], $Sup\{f^{-1}(\mu_\alpha) : \alpha = 1, 2, ..., n\} = 1_X$ so that

$\alpha \leq 1_X \Rightarrow f(\alpha) \leq f(1_X) = 1_Y$. This shows that $P = \{1_Y\}$ is the existing finite subcover of α. Hence, $f(\beta)$ is compact set in Y.

(2) Same as the proof of (1). □

Theorem 3.11.

(1) If $f : X \to Y$ is fuzzy maximal regular irresolute function and $\lambda \in I^X$, a FM_aR-compact set of X, then $f(\lambda)$ is FM_aR-compact set in Y.

(2) If $f : X \to Y$ is fuzzy maximal regular semi-irresolute function and $\lambda \in I^X$, a FM_aR-semi-compact set of X, then $f(\lambda)$ is FM_aR-compact set in Y.

Proof. (1) Let λ be a FM_aR-compact set of X and $Q = \{\mu_\alpha : \alpha \in \Lambda\}$ be a FM_aR-open cover of $f(\lambda)$ such that $f(\lambda) \leq SupQ$. Then, $P = \{f^{-1}(\mu_\alpha) : \alpha \in \Lambda\}$ is a cover of λ. Since f is fuzzy maximal regular irresolute function, each $f^{-1}(\mu_\alpha) \in FM_aRO(X)$, $\forall \alpha \in \Lambda =$ arbitrary index set and then P is FM_aR-open cover of λ. Since, λ is FM_aR-compact, there exists a finite subcover $P_0 = \{f^{-1}(\mu_\alpha) : \alpha = 1, 2, 3, ..., n\}$ of P such that

$$\lambda \leq Sup\{f^{-1}(\mu_\alpha) : \alpha = 1, 2, 3, ..., n\}.$$

Since, each $f^{-1}(\mu_\alpha)$ is distinct FM_aR-open set in X. So, by Theorem 2.7 [6], $Sup\{f^{-1}(\mu_\alpha) : \alpha = 1, 2, 3, ..., n\} = 1_X$ so that $\alpha \leq 1_X \Rightarrow f(\alpha) \leq f(1_X) = 1_Y$. This shows that $Q_0 = \{1_Y\}$ is existing finite FM_aR-open subcover of Q. Hence, $f(\lambda)$ is FM_aR-compact set in Y.

(2) Same as the proof of (1). □

Definition 3.7.

(1) Two non-empty fuzzy sets λ and μ of a fuzzy space (X, τ) are said to be fuzzy maximal regular separated (in short, FM_aR-separated) if $FM_aRCl(\lambda)\overline{q}\mu$ and $FM_aRCl(\mu)\overline{q}\lambda$.

(2) Two non-empty fuzzy sets λ and μ of a fuzzy space (X, τ) are said to be fuzzy maximal regular semi separated (in short, FM_aRS-separated) if $FM_aRSCl(\lambda)\overline{q}\mu$ and $FM_aRSCl(\mu)\overline{q}\lambda$.

(3) A fuzzy subset β is said to be fuzzy maximal regular connected (shortly, FM_aR-connected) iff β cannot be expressed as the union of two FM_aR-separated sets λ and μ of X.

(4) A fts X is said to be fuzzy maximal regular connected (shortly, FM_aR-connected) iff X cannot be expressed as the union of two non-empty disjoint FM_aR-open sets λ and μ i.e $X \neq \lambda \vee \mu$, where $\lambda, \mu \in FM_aRO(X)$.

(5) A fuzzy subset β is said to be fuzzy maximal regular semi connected (shortly, FM_aRS-connected) iff β cannot be expressed as the union of two FM_aR-semi separated sets λ and μ of X.

(6) A fts X is said to be fuzzy maximal regular semi connected (shortly, FM_aR-semi-connected) iff X cannot be expressed as the union of two non-empty disjoint FM_aR-semi-open sets λ and μ i.e $X \neq \lambda \vee \mu$, where $\lambda, \mu \in FM_aRS$-$O(X)$.

Example 3.5 *By easy computations, in Example 2.3, it follows that $FM_aRCl(\lambda_3)q\lambda_2$ and $FM_aRCl(\lambda_2)q\lambda_3$. Thus λ_2 and λ_3 are not fuzzy maximal regular separated. (i.e.) FM_aR-separated. Also $\beta \neq \lambda_2 \vee \lambda_3 \Rightarrow \beta$ is FM_aR-connected. In a similar manner, λ_2 and λ_3 are not fuzzy maximal regular semi separated. (i.e.) FM_aRS-separated.*

Theorem 3.12. *A fuzzy subset $\lambda \in I^X$ of a fts (X, τ) is FM_aR-connected (resp. FM_aR-semi-connected) iff X cannot be expressed as the union of two non-empty disjoint FM_aR-closed sets (FM_aR-semi-closed sets.)*

Proof. Follows from Definition 3.7. □

Theorem 3.13.

(1) *If $f : X \to Y$ is FM_aR-continuous surjection map and X is FM_aR-connected, then Y is fuzzy connected.*

(2) *If $f : X \to Y$ is FM_aR-semi-continuous surjection map and X is FM_aR-semi-connected, then Y is fuzzy connected.*

Proof. (1) Suppose that $f(X) = Y$ is not fuzzy connected space. Then, there exists non-empty fuzzy open sets λ and μ such that $f(X) = \lambda \vee \mu \Rightarrow$ Both λ and μ are fuzzy clopen sets in Y. Then $X = f^{-1}(\lambda) \vee f^{-1}(\mu)$. Since f is FM_aR-continuous and λ and μ are non-empty disjoint fuzzy closed sets, $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are also non-empty disjoint and $\in FM_aRC(X)$. This shows that X is not FM_aR-connected which is a contradiction to the given hypothesis. Hence Y is fuzzy connected.

(2) Same as the proof of (1). □

Theorem 3.14.

(1) *If $f : X \to Y$ is FM_aR-irresolute surjection map and X is FM_aR-connected, then Y is FM_aR-connected.*

(2) *If $f : X \to Y$ is FM_aR-semi-irresolute surjection map and X is FM_aR-semi-connected, then Y is FM_aR-semi-connected.*

Proof. (1) Suppose that $f(X) = Y$ is not FM_aR-connected space. Then, there exists non-empty fuzzy open sets λ and μ such that $f(X) = \lambda \vee \mu \Rightarrow$ Both λ and μ are FM_aR-open as well as FM_iR-closed sets in Y. Then $X = f^{-1}(\lambda) \vee f^{-1}(\mu)$. Since λ and μ are non-empty disjoint FM_iR-closed sets and f is FM_aR-irresolute surjection, $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are also non-empty disjoint and $\in FM_iR$-C(X) such that $X = f^{-1}(\lambda) \vee f^{-1}(\mu)$. This shows from Theorem 3.12 that X is not FM_aR-connected which is a contradiction to the given hypothesis that X is FM_aR-connected. Hence,Y is FM_aR-connected.

(2) Similar to the proof of (1). □

Conclusion: In this paper, we have introduced fuzzy maximal regular semi-open sets, fuzzy minimal regular semi-closed sets, fuzzy maximal regular semi-continuous, fuzzy maximal regular semi-irresolute functions and fuzzy maximal regular semi-connectedness in fts's. Also, we have studied some basic properties and characterization theorems. Finally, we have given some counter examples to show that these types of sets and mappings are not equivalent. These results will help to extend some generalized closed sets, mappings, compactness and hence it will help to improve fuzzy bitopological and smooth topological spaces.

Acknowledgements

The authors would like to thank from the anonymous reviewers for carefully reading of the manuscript and giving useful comments, which will help to improve the paper.

References

[1] K. K. Azad. On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity. J. Math. Anal. Appl. 82 (1981), No. 1, 14 - 32.

[2] K. K. Azad. Fuzzy Hausdorff spaces and fuzzy perfect mappings. J. Math. Anal. Appl. 82 (1981), 297 - 305.

[3] G. Balasubramanian and P. Sundaram. On some generalizations of fuzzy continuous functions. Fuzzy Sets and Systems 86 (1997), No. 1, 93 - 100.

[4] C. L. Chang. Fuzzy topological spaces. J. Math. Anal. Appl. 24 (1968), 182 - 190.

[5] D. E. Cameron. Properties of s-closed spaces. Proc. Amer. Math. Soc. 72 (1978), 581 - 586.

[6] J. A. Goguen. The fuzzy Tychonoff theorem. J. Math. Anal. Appl. 43 (1973), No. 3, 734 - 742.

[7] S. R. Malghan and S. S. Benchalli. Open maps, closed maps and local compactness in fuzzy topological spaces. J. Math. Anal. Appl. 99 (1984), No. 2, 338 - 349.

[8] S. R. Malghan and S. S. Benchalli. On fuzzy topological spaces. Glas. Mat. Ser. III 16(36) (1981), no. 2, 313C325.

[9] A. Mukerjee. On fuzzy completely semi continuous and weakly completely semi continuous functions. Indian J. Pure Appl. Math. 29 (1998), No. 2, 191 - 197.

[10] M. N. Mukherjee and B. Ghosh. Some stronger forms of fuzzy continuous mappings on fuzzy topological spaces. Fuzzy Sets and Systems 38 (1990), 375 - 387.

[11] F. Nakoda and N. Oda. Some properties of maximal open sets. Int. J. Math. Math. Sci. 21 (2003), 1331 - 1340.

[12] F. Nakoda and N. Oda. Some applications of minimal open sets. Int. J. Math. Math. Sci. 27 (2001), No. 8, 471 - 476.

[13] F. Nakaoka and N. Oda. On Minimal closed sets. Proceedings of Topological spaces, Theory and its Applications 5 (2003), 19 - 21.

[14] R. H. Warrren. Continuity of mappings on fuzzy topological spaces. Notices Amer. Math. Soc. 21 (1974), A–451.

[15] R. H. Warren. Neighborhoods, bases and continuity in topological spaces. Rocky Mountain J. Math. 8 (1978), No.3, 459 - 470.

[16] L. A. Zadeh. Fuzzy sets. Information and Control. 8 (1965), 338 - 353.

[17] A. N. Zahren. Fuzzy regular semi-open sets and S-closed spaces. J. Fuzzy Math. 2 (1994), 579 - 586.

Scientia Magna
Vol. 12 (2017), No. 1, 107-114

On intuitionistic fuzzy e-compactness

D. Sobana[1], V. Chandrasekar[2] and A. Vadivel[3]

[1] [2] Department of Mathematics, Kandaswamy Kandar's College
P-velur, Tamil Nadu-638 182, India
E-mail : slmsobana@gmail.com
E-mail : vckkc3895@gmail.com
[3]Department of Mathematics, Government Arts College
Karur, Tamil Nadu-639005, India
E-mail: avmaths@gmail.com

Abstract In this paper, we have introduced and study the concept of intuitionistic fuzzy e-compactness. Several preservation properties and some characterizations concerning intuitionistic fuzzy e-compactness have been obtained.

Keywords Intuitionistic fuzzy topology, intuitionistic fuzzy e-open set, intuitionistic fuzzy e-compact space.
2010 Mathematics Subject Classification 54A40, 54A99, 03E72, 03E99.

§1. Introduction and preliminaries

After the introduction of fuzzy sets by Zadeh [11], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. Using the notion of intuitionistic fuzzy sets, Coker [4] introduced the notion of fuzzy topological spaces. In [10] Vadivel, introduced the notions of intuitionistic fuzzy e-open sets and intuitionistic fuzzy e-continuity. In this paper, we introduce and study the concept of intuitionistic fuzzy e-compactness. Several preservation properties and some characterizations concerning intuitionistic fuzzy e-compactness have been obtained.

Before entering in to our work, we recall the following notations, definitions and results of intuitionistic fuzzy sets as given by Atanassov [2]. Coker [4] and Seenivasan [9]. Throughout this paper, (X, τ), (Y, σ), and (Z, η) (or simply X, Y and Z) are always means an intuitionistic fuzzy topological spaces on which no separation axioms are assumed unless otherwise mentioned.

First we shall present the fundamental definitions and results which will be used in the sequel.

Definition 1.1. *[2] Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS, for short) A is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$.*

Obviously, every fuzzy set A on a nonempty set X is an IFS having the form $A = \{\langle x, \mu_A(x), 1 - \nu_A(x)\rangle : x \in X\}$.

Definition 1.2. *[2] Let X be a nonempty set and the IFS's A and B be in the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \mu_B(x)\rangle : x \in X\}$ and let $A = \{A_j : j \in J\}$ be an arbitrary family of IFS's in X, then*

(i) $A \leq B$ *iff* $\forall x \in X[\mu_A(x) \leq \nu_A(x)$ *and* $\nu_A(x) \geq \gamma_A(x)]$;

(ii) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$;

(iii) $\wedge A_j = \{\langle x, \wedge\mu_{A_j}(x), \vee\nu_{A_j}(x)\rangle : x \in X\}$;

(iv) $\vee A_j = \{\langle x, \vee\mu_{A_j}(x), \wedge\nu_{A_j}(x)\rangle : x \in X\}$;

(v) $\overline{\vee A_j} = \vee\overline{A}_j$, $\overline{\wedge A_j} = \wedge\overline{A}_j$;

(vi) $1_\sim = \{\langle x, 1, 0\rangle : x \in X\}$ *and* $0_\sim = \{\langle x, 0, 1\rangle : x \in X\}$;

(vii) $\overline{\overline{A}} = A$, $\overline{0_\sim} = 1_\sim$ *and* $\overline{1_\sim} = 0_\sim$.

Definition 1.3. *[4] Let X and Y be two nonempty sets and $f : X \to Y$ be a function.*

(i) If $B = \{\langle y, \mu_B(y), \nu_B(y)\rangle : y \in X\}$ is an IFS in Y, then the preimage of B under f denoted and defined by $f^{-1}(B) = \{\langle f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x)\rangle : x \in X\}$;

(ii) If $A = \{\langle x, \lambda_A(x), \nu_A(x)\rangle : x \in X\}$ is an IFS in X, then the image of A under f denoted and defined by $f(A) = \{\langle y, f(\lambda_A)(y), f_-(\nu_A)(y) : y \in Y\rangle\}$ where $f_-(\nu_A) = 1 - f(1 - \nu_A)$.

Definition 1.4. *[5] Let $A = \langle x, \mu_A, \nu_A\rangle$ and $B = \langle x, \mu_B, \nu_B\rangle$ be two IFS's in X. Then, A and B are said to be quasi-coincident, denoted by AqB, iff there exists an element $x \in X$ such that $\mu_A(x) > \nu_B(x)$ or $\nu_A(x) < \mu_B(x)$. The negation of AqB will be denoted by $A\overline{q}B$.*

Definition 1.5. *[4] An intuitionistic fuzzy topology (IFT, for short) on a non-empty set X is a family Ψ of IFS's in X satisfying the following axioms.*

(i) $0_\sim, 1_\sim \in \Psi$,

(ii) $A_1 \wedge A_2 \in \Psi$ *for every* $A_1, A_2 \in \Psi$,

(iii) $\vee A_j \in \Psi$ *for every* $\{A_j : j \in J\} \subseteq \Psi$.

In this case the pair (X, Ψ) is called an intuitionistic fuzzy topological space (IFTS, for short) and each IFS in Ψ is known as an intuitionistic fuzzy open set (IFOS, for short) in X.

Definition 1.6. *[4] The complement $\overline{A}$ of IFOS A in IFTS(X, Ψ) is called an intuitionistic fuzzy closed set (IFCS, for short).*

Definition 1.7. *[4] Let (X, Ψ) be an IFTS and $A = \langle x, \mu_A(x), \nu_A(x)\rangle$ be an IFS in X. Then the fuzzy interior and fuzzy closure of A are denoted and defined by $cl(A) = \wedge\{K : K$ is an IFCS in X and $A \leq K\}$ and $int(A) = \vee\{G : G$ is an IFOS in X and $G \leq A\}$.*

Definition 1.8. *[10] Let A be an IFS in an IFTS(X, Ψ). A is called*

(i) an intuitionistic fuzzy e-open set (IFeOS, for short) in X if $A \leq clint_\delta(A) \vee intcl_\delta(A)$,

(ii) *an intuitionistic fuzzy e-closed set (IFeCS, for short) in X if $A \geq clint_\delta(A) \wedge intcl_\delta(A)$.*

Definition 1.9. *[10] Let (X, Ψ) be an IFTS and $A = \langle x, \mu_A, \nu_A \rangle$ be an IFS in X. Then the intuitionistic fuzzy e-closure and intuitionistic fuzzy e-interior are defined and denoted by $cl_e(A) = \wedge\{K : K$ is an IFeCS in X and $A \leq K\}$ and $int_e(A) = \vee\{G : G$ is an IFeOS in X and $G \leq A\}$. It is clear that A is an IFeCS (IFeOS) in X iff $A = cl_e(A)(A = int_e(A))$.*

Definition 1.10. *Let f be a mapping from an IFTS X into an IFTS Y. The mapping f is called*

(i) *an intuitionistic fuzzy e-continuity [10] if $f^1(B)$ is an IFGOS in X for each IFOS B in Y.*

(ii) *an intuitionistic fuzzy e-irresolute [10] if $f^1(B)$ is an IFGOS in X for each IFOS B in Y.*

§2. Intuitionistic fuzzy e-compact spaces

Definition 2.1. *Let X be an IFTS. A family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ of IFOS's in X satisfies the condition $\bigcup\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\} = 1\!\!1$ is called an intuitionistic fuzzy open cover of X.*

A finite subfamily of an intuitionistic fuzzy open cover $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ which is also an intuitionistic fuzzy open cover of X is called a finite subcover of $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$.

An IFTS X is called intuitionistic fuzzy compact if and only if every intuitionistic fuzzy open cover has a finite subcover.

Definition 2.2. *Let A be an IFS in an IFTS X. A family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ of IFOS's in X satisfies the condition $A \subseteq \bigcup\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ is called an intuitionistic fuzzy open cover of A.*

A finite subfamily of an intuitionistic fuzzy open cover $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ of A which is also an intuitionistic fuzzy open cover of A is called a finite subcover of $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$.

An IFTS $A = \langle x, \mu_A, \nu_A\rangle$ in an IFTS X is called intuitionistic fuzzy compact if and only if every intuitionistic fuzzy open cover of A have a finite subcover.

Definition 2.3. *Let X be an IFTS. A family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ of IFeOS's in X satisfies the condition $\bigcup\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\} = 1\!\!1$ is called an intuitionistic fuzzy e-open cover of X.*

A finite subfamily of an intuitionistic fuzzy e-open cover $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ which is also an intuitionistic fuzzy e-open cover of X is called a finite subcover of $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$.

Definition 2.4. *Let X be an IFTS. A family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ of IFeCS's in X has the finite intersection property if every finite sub-family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i = 1, 2, ..., n\}$ satisfies the condition $\bigcap_{k=1}^{n}\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\} \neq 0\!\!\!0$.*

Definition 2.5. *An IFTS X is called intuitionistic fuzzy e-compact if and only if every intuitionistic fuzzy e-open cover has a finite subcover.*

Theorem 2.1. *An IFTS X is intuitionistic fuzzy e-compact if and only if every family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ of IFSPCS's with the finite intersection property has a nonempty intersection.*

Proof. Suppose X is intuitionistic fuzzy e-compact and $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ is any family of IFSPCS's in X such that $\bigcap\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\} = 0_\sim$. Therefore $\bigcap\{\langle \mu_{G_i}\rangle \,|i \in I\} = 0_\sim$ and $\bigcup\{\langle \nu_{G_i}\rangle \,|i \in I\} = 1_\sim$. Then $\bigcup\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\} = 1_\sim$, so $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ is a intuitionistic fuzzy e-open cover of X. Since X is intuitionistic fuzzy e-compact there is a finite subcover $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i = 1, 2, ..., n\}$. Then $\bigcup_{k=1}^{n}\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,| = 1_\sim\}$. Hence $\bigvee\{\langle \nu_{G_{i}(x)}\rangle \,|i = 1, 2, ..., n\} = 1_\sim$ and $\bigwedge\{\langle \mu_{G_i}(x)\rangle \,|i = 1, 2, ..., n\} = 0_\sim$. Finally $\bigcap_{k=1}^{n}\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,| = 0_\sim\}$. We have proved that if X is intuitionistic fuzzy e-compact space, then given any family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ of IFSPCS's whose intersection is empty, the intersection of some finite subfamily is empty. Conversely, let X has the finite intersection property. It means that if the intersection of any family of IFSPCS's is empty, the intersection of each finite subfamily is empty. Suppose $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ is any intuitionistic fuzzy e-open cover of X. Then $\bigcup\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\} = 1_\sim$. Therefore,

$$\bigvee\{\langle \mu_{G_i}(x)\rangle \,|i \in I\} = 1_\sim \textit{ and } \bigwedge\{\langle \nu_{G_i}(x)\rangle \,|i \in I\} = 0_\sim.$$

Hence $\bigcap\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\} = 0_\sim$, so $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ is a family of IFSPCS's whose intersection is empty. According to the assumption, we can find finite subfamily $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i = 1, 2, ..., n\}$ such that $\bigcap_{k=1}^{n}\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,| = 0_\sim\}$, so $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i = 1, 2, ..., n\}$ is a finite subcover of $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$. Therefore, X is intuitionistic fuzzy e-compact. □

Theorem 2.2. *Let $f : X \to Y$ be an intuitionistic fuzzy e-irresolute mapping from an IFTS X onto IFTS Y. If X is intuitionistic fuzzy e-compact, then Y is intuitionistic fuzzy e-compact, as well.*

Proof. Let $\{\langle y, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ be any intuitionistic fuzzy e-open cover of Y. Then

$$\bigcup\{\langle y, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\} = 1_\sim.$$

From the relation $f^{-1}(\bigcup\{\langle y, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}) = 1_\sim$ follows that $\bigcup\{\langle y, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\} = 1_\sim$, so $\{f^{-1}(\{\langle y, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ is a intuitionistic fuzzy e-open cover of X. Since X is intuitionistic fuzzy e-compact, there exists a finite subcover $\{f^{-1}(\langle x, \mu_{G_i}, \nu_{G_i}\rangle)|i = 1, 2, ..., n\}$. Therefore

$$\bigcup\{f^{-1}(\langle y, \mu_{G_i}, \nu_{G_i}\rangle)|i = 1, 2, ..., n\} = 1_\sim.$$

Hence $f(\bigcup\{f^{-1}(\langle y, \mu_{G_i}, \nu_{G_i}\rangle)|i = 1, 2, ..., n\}) = 1_\sim$, so $\bigcup\{f(f^{-1}(\langle y, \mu_{G_i}, \nu_{G_i}\rangle))|i = 1, 2, ..., n\} = 1_\sim$. From $\bigcup\{(\langle y, \mu_{G_i}, \nu_{G_i}\rangle)|i = 1, 2, ..., n\} = 1_\sim$ follows that Y is intuitionistic fuzzy e-compact. □

Theorem 2.3. *Let $f : X \to Y$ be an intuitionistic fuzzy e-continuous mapping from an IFTS X onto IFTS Y. If X is intuitionistic fuzzy e-compact, then Y is fuzzy compact.*

Proof. It is similar to the proof of the Theorem 2.2. □

Definition 2.6. *Let A be an IFS in an IFTS X. A family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ of IFeOS's in X satisfies the condition $A \subseteq \bigcup\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ is called intuitionistic fuzzy e-open cover of A.*

A finite subfamily of a intuitionistic fuzzy e-open cover $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ of A which is also a intuitionistic fuzzy e-open cover of A is called a finite subcover of $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$.

Definition 2.7. *An IFS $A = \langle x, \mu_A, \nu_A\rangle$ in an IFTS X is called intuitionistic fuzzy e-compact if and only if every intuitionistic fuzzy e-open cover of A has a finite subcover.*

Theorem 2.4. *An IFS $A = \langle x, \mu_A, \nu_A\rangle$ in an IFTS X is intuitionistic fuzzy e-compact if and only if for each family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ of IFGOS's with properties $\mu_A \leq \vee\{\mu_{G_i}|i \in I\}$ and $1 - \nu_A \leq \vee\{1 - \nu_{G_i}|i \in I\}$ there exists a finite subfamily $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i = 1, 2, ..., n\}$ such that $\mu_A = \vee\{\mu_{G_i}|i = 1, 2, ..., n\}$ and $1 - \nu_A = \vee\{1 - \nu_{G_i}|i = 1, 2, ..., n\}$.*

Proof. Suppose $A = \langle x, \mu_A, \nu_A\rangle$ is a intuitionistic fuzzy e-compact set in IFTS X and

$$\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$$

be any family of IFeCS's in X satisfies the condition $\mu_A \leq \vee\{\mu_{G_i}|i \in I\}$ and $1 - \nu_A \leq \vee\{1 - \nu_{G_i}|i \in I\}$. Then $1 - \nu_A \leq 1 - \wedge\{\nu_{G_i}|i \in I\}$, so $\nu_A \geq \wedge\{\nu_{G_i}|i \in I\}$. Hence $A \subseteq \bigcup\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$. According to the assumption there exists finite subfamily

$$\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i = 1, 2, ..., n\}$$

such that $A \subseteq \bigcup\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i = 1, 2, ..., n\}$. It follows that $\mu_A = \vee\{\mu_{G_i}|i = 1, 2, ..., n\}$ and $1 - \nu_A = \vee\{1 - \nu_{G_i}|i = 1, 2, ..., n\}$.

Conversely, let $A = \langle x, \mu_{G_i}, \nu_{G_i}\rangle$ be any IFS in IFTS X and let $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ be any family of IFeCS's in X satisfies the condition $\mu_A \leq \vee\{\mu_{G_i}|i \in I\}$ and $1-\nu_A \leq \vee\{1-\nu_{G_i}|i \in I\}$. From $1 - \nu_A \leq 1 - \wedge\{\nu_{G_i}|i \in I\}$, so $\nu_A \geq \wedge\{\nu_{G_i}|i \in I\}$ follows that $\mu_A \geq \wedge\{\nu_{G_i}|i \in I\}$, so

$$A \subseteq \bigcup\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}.$$

Hence $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ is a intuitionistic fuzzy e-open cover of IFS A. According to the assumption there exists finite subfamily $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i = 1, 2, ..., n\}$ such that $\mu_A = \vee\{\mu_{G_i}|i = 1, 2, ..., n\}$ and $1 - \nu_A \leq \vee\{1 - \nu_{G_i}|i = 1, 2, ..., n\}$. From $\mu_A \leq \vee\{\mu_{G_i}|i = 1, 2, ..., n\}$ and $\nu_A \geq \wedge\{\mu_{G_i}|i = 1, 2, ..., n\}$ we obtain that $A \subseteq \bigcup\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle \,|i = 1, 2, ..., n\}$. Therefore, A is intuitionistic fuzzy e-compact. □

Remark2.1. *From the definition above it is not difficult to conclude that every intuitionistic fuzzy e-compact in an IFTS is fuzzy compact.*

Theorem 2.5. *Let $f : X \to Y$ be an intuitionistic fuzzy e-irresolute mapping from an IFTS X onto IFTS Y. If A is intuitionistic fuzzy e-compact, then $f(A)$ is intuitionistic fuzzy e-compact.*

Proof. Let $\{\langle y, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$ be any intuitionistic fuzzy e-open cover of $f(A)$. Then $f(A) \subseteq \bigcup\{\langle y, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\}$. From the relation $A \subseteq f^{-1}(\bigcup\{\langle y, \mu_{G_i}, \nu_{G_i}\rangle \,|i \in I\})$ follows that $A \subseteq \bigcup\{f^{-1}(\langle y, \mu_{G_i}, \nu_{G_i}\rangle)|i \in I\}$, so $\{f^{-1}(\langle y, \mu_{G_i}, \nu_{G_i}\rangle)|i \in I\}$ is an intuitionistic fuzzy e-open cover of A. Since A is intuitionistic fuzzy e-compact, there exists a finite subcover $\{f^1(\langle y, \mu_{G_i}, \nu_{G_i}\rangle)|i = 1, 2, ..., n\}$. Therefore $A \subseteq \bigcup\{f^{-1}(\langle y, \mu_{G_i}, \nu_{G_i}\rangle)|i = 1, 2, ..., n\}$. Hence

$$f(A) \quad \subseteq \quad f(\bigcup\{f^1(\langle y, \mu_{G_i}, \nu_{G_i}\rangle)|i = 1, 2, ..., n\})$$

$$\begin{aligned} &= \bigcup\{f(f^1(\langle y, \mu_{G_i}, \nu_{G_i}\rangle))|i = 1, 2, ..., n\} \\ &= \bigcup\{(\langle y, \mu_{G_i}, \nu_{G_i}\rangle)|i = 1, 2, ..., n\} \end{aligned}$$

so $f(A)$ is intuitionistic fuzzy e-compact. □

Theorem 2.6. *Let $f : X \to Y$ be an intuitionistic fuzzy e-continuous mapping from an IFTS X onto IFTS Y. If A is intuitionistic fuzzy e-compact, then $f(A)$ is fuzzy compact.*

Definition 2.8. *An IFTS X is called intuitionistic fuzzy e-Lindelöf (fuzzy Lindelöf) if and only if every intuitionistic fuzzy e-open (fuzzy open) cover of X has a countable subcover.*

Definition 2.9. *An IFTS $A = \langle x, \mu_A, \nu_A\rangle$ in an IFTS X is called intuitionistic fuzzy e-Lindelöf (fuzzy Lindelöf) if and only if every intuitionistic fuzzy e-open (fuzzy open) cover of X has a countable subcover.*

Definition 2.10. *An IFTS X is called countable intuitionistic fuzzy e-compact (countably fuzzy compact) if and only if every countable intuitionistic fuzzy e-open (fuzzy open) cover of X has a finite subcover.*

Definition 2.11. *An IFTS $A = \langle x, \mu_A, \nu_A\rangle$ in an IFTS X is called countable intuitionistic fuzzy e-compact (countably fuzzy compact) if and only if every countable intuitionistic fuzzy e-open (fuzzy open) cover of A has a finite subcover.*

Remark2.2. *From the definitions above we may conclude that*

(i) Every intuitionistic fuzzy e-Lindelöf of IFTS is fuzzy Lindelöf;

(ii) Every countably intuitionistic fuzzy e-compact of IFTS is countably fuzzy compact;

(iii) Every countably intuitionistic fuzzy e-compact of IFTS is intuitionistic fuzzy e-compact.

Theorem 2.7. *If an IFTS X is both intuitionistic fuzzy e-Lindelöf and countably intuitionistic fuzzy e-compact, then it is intuitionistic fuzzy e-compact.*

Theorem 2.8. *If an IFS A in an IFTS X is both intuitionistic fuzzy e-Lindelöf and fuzzy countably intuitionistic fuzzy e-compact, then A is intuitionistic fuzzy e-compact.*

Theorem 2.9. *Let X be an intuitionistic fuzzy e-Lindelöf IFTS. Then X is countably intuitionistic fuzzy e-compact if and only if X is intuitionistic fuzzy e-compact.*

Proof. In the Remark 2.2, it is mentioned that if X is intuitionistic fuzzy e-compact, then it is countably intuitionistic fuzzy e-compact. Conversely, let $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle |i \in I\}$ be any intuitionistic fuzzy e-open cover of X. Since X is intuitionistic fuzzy e-Lindelöf, there exists a countable subcover $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle |i = 1, 2, ...\}$ of $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle |i \in I\}$. Therefore

$$\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle |i = 1, 2, ...\}$$

is countably intuitionistic fuzzy e-open cover of X, so there exists subcover $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle |i = 1, 2, ...n\}$ of $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle |i = 1, 2, ...\}$. Hence X is intuitionistic fuzzy e-compact. □

Theorem 2.10. *Let an IFeOS A be intuitionistic fuzzy e-Lindelöf in an IFTS. Then A is countably intuitionistic fuzzy e-compact if and only if A is intuitionistic fuzzy e-compact.*

Proof. The proof is similar to the proof of the previous theorem. □

Theorem 2.11. *Let $f : X \to Y$ be an intuitionistic fuzzy e-irresolute mapping from an IFTS X onto IFTS Y. If X is intuitionistic fuzzy e-Lindelöf (countably intuitionistic fuzzy e-compact), then Y is intuitionistic fuzzy e-Lindelöf (countably intuitionistic fuzzy e-compact), as well.*

Proof. It is similar to the proof of the Theorem 2.2. □

Theorem 2.12. *Let $f : X \to Y$ be an intuitionistic fuzzy e-continuous mapping from an IFTS X onto IFTS Y. If X is intuitionistic fuzzy e-Lindelöf (countably intuitionistic fuzzy e-compact), then Y is fuzzy Lindelöf (countably fuzzy compact).*

Proof. It is similar to the proof of the Theorem 2.3. □

Theorem 2.13. *Let $f : X \to Y$ be an intuitionistic fuzzy e-irresolute mapping from an IFTS X onto IFTS Y. If A is intuitionistic fuzzy e-Lindelöf (countably intuitionistic fuzzy e-Lindelöf (countably intuitionistic fuzzy e-compact)), then $f(A)$ is intuitionistic fuzzy e-Lindelöf (countably intuitionistic fuzzy e-compact), as well.*

Proof. It is similar to the proof of the Theorem 2.5. □

Theorem 2.14. *Let $f : X \to Y$ be an intuitionistic fuzzy e-continuous mapping from an IFTS X onto IFTS Y. If A is intuitionistic fuzzy e-Lindelöf (countably intuitionistic fuzzy e-compact), then $f(A)$ is fuzzy Lindelöf (countably fuzzy compact).*

Proof. It is similar to the proof of the Theorem 2.6. □

Acknowledgements

The authors would like to thank from the anonymous reviewers for carefully reading of the manuscript and giving useful comments, which will help to improve the paper.

References

[1] K. Atanassov. Intuitionistic fuzzy sets. VII ITKR's Session, Sofia, (1983) (in Bulgarian).

[2] K. Atanassov. Intuitionistic fuzzy sets. Fuzzy Sets and Systems 20 (1986), 87 - 96.

[3] D. Coker. An introduction to fuzzy subspaces in intuitionistic fuzzy topological spaces. J. Fuzzy Math. 4 (1976), No. 2, 749 - 764.

[4] D. Coker. An introduction to intuitionistic fuzzy topological spaces. Fuzzy Sets and Systems 88 (1997), 81 - 89.

[5] D. Coker and M. Demirci. On intuitionistic fuzzy points. NIFS 1 (1995), 79 - 84.

[6] D. Coker and A. Hayder Es. On fuzzy compactness in intuitionistic fuzzy topological spaces. J. Fuzzy Math. 3 (1995), No. 4, 899 - 909.

[7] H. Gurcay, D. Coker and A. H. Es. On fuzzy continuity in intuitionistic fuzzy topological spaces. J. Fuzzy Math. 5 (1997), No. 2, 365 - 378.

[8] Y. B. Jun, J. O. Kang and S. Z. Song. Intuitionistic fuzzy irresolute and continuous mappings. Far East J. Math. Sci. 17 (2005), No. 2, 201 - 216.

[9] V. Seenivasan and K. Kamala. Fuzzy e-continuity and fuzzy e-open sets. Ann. Fuzzy Math. Inform. 8 (2014), No. 1, 141 - 148.

[10] A. Vadivel and D. Sobana. On fuzzy e-open sets and fuzzy e-continuity in intuitionistic fuzzy topological spaces. accepted in Sahand communications in mathematical Analysis.

[11] L. A. Zadeh. Fuzzy Sets. Inform and Control. 9 (1965), 338 - 353.

Scientia Magna
Vol. 12 (2017), No. 1, 115-119

Zagreb and multiplicative Zagreb indices of r-subdivision graphs of double graphs

Muge Togan, Aysun Yurttas and Ismail Naci Cangul

Department of Mathematics, Uludag University
Faculty of Arts and Science, Gorukle 16059 Bursa/Turkey
E-mail(s): capkinm@uludag.edu.tr, ayurttas@uludag.edu.tr, ncangul@gmail.com

Abstract Let G be a null, path, cycle, star, complete or a tadpole graph. In this paper, the first and second Zagreb and multiplicative Zagreb indices of subgraphs of the double graphs of G are obtained.

Keywords Zagreb indices, Zagreb coindices, topological indices, subdivision graph, double graph.
2010 Mathematics Subject Classification 05C10, 05C30.

§1. Introduction

Let $G = (V, E)$ be a simple graph with $\mid V(G) \mid= n$ vertices and $\mid E(G) \mid= m$ edges. For a vertex $v \in V(G)$, we denote the degree of v by $d_G(v)$. A vertex with degree one is called a pendant vertex. As usual, we denote by $N_n, P_n, C_n, S_n, K_n, K_{t,s}$ and $T_{t,s}$ the null, path, cycle, star, complete, bipartite and tadpole graphs, respectively.

The subdivision graph $S(G)$ of a graph G is the graph obtained from G by replacing each of its edges by a path of length 2, or equivalently by inserting an additional vertex into each edge of G.

Several topological graph indices have been defined and studied by many mathematicians and chemists as most graphs are generated from molecules by replacing atoms with vertices and bonds with edges. Two of the most important topological graph indices are called first and second Zagreb indices denoted by $M_1(G)$ and $M_2(G)$, respectively:

$$M_1(G) = \sum_{u\in V(G)} d_G^2(u) \quad \text{and} \quad M_2(G) = \sum_{u,v\in E(G)} d_G(u)d_G(v). \tag{1}$$

They were first defined 45 years ago by Gutman and Trinajstic, [7], and are referred to due to their uses in QSAR and QSPR. In 2010, Todeschini and Consonni, [8], have introduced the multiplicative variants of these additive graph invariants by

$$\Pi_1(G) = \prod_{u\in V(G)} d_G^2(u) \text{ and } \Pi_2(G) = \prod_{u,v\in E(G)} d_G(u)d_G(v) \tag{2}$$

and called them multiplicative Zagreb indices. Zagreb indices and multiplicative Zagreb coindices of graphs have been studied in [3] and some bounds related to those are obtained. Similarly, these multiplicative Zagreb indices are calculated for main graph operations in [1].

For a graph G with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$, we take another copy of G with vertices labelled by $\{v_1, v_2, ..., v_n\}$ where v_i corresponds to v_i for each i. If we connect v_i to the neighbours of v_i for each i, we obtain a new graph called the double graph of G. It is denoted by $D(G)$. Double graphs were first introduced by Indulal and Vijayakumari, [10], in the study of equienergetic graphs. Later Munarini et al. [9], calculated the double graphs of N_n and $K_{t,s}$ as N_{2n} and $K_{2m,2n}$, respectively. These subdivision graphs were recently studied by Togan, Yurttas and Cangul in [4] where ten types of Zagreb indices and coindices including first and second Zagreb indices and multiplicative Zagreb indices were calculated. In [2], the Zagreb indices of the line graphs of the subdivision graphs are studied. Also in [11], the resistance distance and the Kirchhoff index in double graphs were studied and the closed-form formulas for resistance distances and the Kirchhoff index of double graphs were derived. Here, we first calculate double graphs of other simple graph types such as cycle graphs C_n, path graphs P_n, star graphs S_n, complete graphs K_n, tadpole graphs $T_{t,s}$. For a graph G, the subdivision graph of G denoted by $S(G)$ is defined by adding one vertex to each existing edge.

Similarly the r-subdivision graph of G denoted by $S^r(G)$ is defined by adding r vertices to each edge by Togan, Yurttas and Cangul in [5]. In this paper we shall calculate the first and second Zagreb indices and multiplicative Zagreb indices of all these graphs. We also give some relations between these numbers. Similar calculations were made in [6] for subdivision graphs of double graphs. For convenience, we shall denote the number of vertices and edges of G, $D(G)$, $S^r(G)$ and $S^r(D(G))$ by $n, m, n^d, m^d, n(S_r), m(S_r)$ and $n^d(S_r), m^d(S_r)$, respectively. Obviously,

Lemma 1. *With the above notation, we have*

a) $n^d = 2n$,

b) $m^d = 4m$,

c) $n(S_r) = n + rm$,

d) $m(S_r) = (r + 1)m$,

e) $n^d(S_r) = rm^d + n^d = 2n + 4mr$,

f) $m^d(S_r) = (r + 1)m^d = 4m(r + 1)$.

Proof. a) $n^d = 2n$ by definition.

b) $m^d = 2m + \sum_{i=1}^{n} d_{v_i} = 2m + 2m = 4m$.

c) and d) follows by the definition of r-subgraph.

e) $n^d(S_r) = rm^d + n^d$, by definition. Also by a) and b), $n^d(S_r) = 2n + 4mr$.

f) $m^d(S_r) = (r + 1)m^d$ by definition, and $m^d(S_r) = 4m(r + 1)$ by b). □

§2. First and Second Zagreb and Multiplicative Zagreb Indices of Some r-Subgraphs of Double Graphs

For a null graph N_n, one can not obtain a subdivision graph by adding a new vertex so our result will be given for other graph types.

Theorem 2. *Let* $m, n, m(S_r), n(S_r), m^d(S_r), n^d(S_r)$ *be the number of edges and vertices of* $G, S^r(G)$ *and* $S^r(D(G))$*, respectively. Then the first and second Zagreb indices of r-subdivision* *graph of double graphs of path, cycle, star, complete and tadpole graphs is given as follows:*

$$M_1(S^r(D(G))) = \begin{cases} 16(nr - r + 1) + 32(n-2) & \text{if } G = P_n, n \geq 2 \\ 32n + 16nr & \text{if } G = C_n, n > 2 \\ 8(n-1)\left[1 + 2r + (n-1)\right] & \text{if } G = S_n, n \geq 2 \\ 8n(n-1)\left[(n-1) + r\right] & \text{if } G = K_n, n \geq 2 \\ 16\left[(s+t)(r+2) + 1\right] & \text{if } G = T_{t,s}, t \geq 3, s \geq 1 \end{cases}$$

and

$$M_2(S^r(D(G))) = \begin{cases} 32 + 16(n-1)(r-1) + 64(n-2) & \text{if } G = P_n, n \geq 2 \\ 64n + 16n(r-1) & \text{if } G = C_n, n > 2 \\ 8(n-1)\left[2n + r - 1\right] & \text{if } G = S_n, n \geq 2 \\ 8n(n-1)(2n + r - 3) & \text{if } G = K_n, n \geq 2 \\ 16\left[(s+t)(r^2+3) + 2\right] & \text{if } G = T_{t,s}, t \geq 3, s \geq 1. \end{cases}$$

Proof. We prove the theorem for star graphs. Similar methods can be used for others. Let G be a star graph S_n. Its r-subgraph has $n(S_r) = mr + n = (n-1)r + n$ and $m(S_r) = (r+1)(n-1)$ and for a star graph, r -subgraph of its double graph has $n^d(S_r) = 4r(n-1)+2n = 4r(n-1)+2n$ and $m^d(S_r) = 4(r+1)(n-1)$. In $S^r(D(S_n))$, we have two vertices with degree $2(n-1)$ in the centers of stars, $r(n-1)$ vertices of degree 2 on each edge of star and $2(n-1)$ vertices of degree 2 at the end points of star.

So if we use the definition of $M_1(G)$, we have

$$\begin{aligned} M_1(S^r(D(S_n))) &= 2^2 \cdot 2 \cdot (n-1) + 2^2 \cdot 4r \cdot (n-1) + 2 \cdot [2(n-1)]^2 \\ &= 8(n-1) + 8r(n-1) + 8(n-1)^2 \\ &= 8(n-1) \cdot [1 + r + (n-1)]\,. \end{aligned}$$

There are 3 types of entries in $M_2(S^r(D(S_n)))$:

i) u is an endpoint (pendant vertice) and v is a newly added vertex of degree 2 in $S^r(D(G))$: For each u, there are $m = n-1$ added vertices which forms an edge with u so each vertex pair adds $(2 \cdot 2) \cdot 4(n-1)$ to $M_2(S^r(D(S_n)))$.

ii) u is the central vertex with degree $2(n-1)$ and v is a newly added vertex of degree 2 which forms an edge together with u. So a total of $2 \cdot 2(n-1) \cdot 4(n-1)$ is added.

iii) Both u and v are middle vertices (of degree 2) and $uv \in E(G)$: There are $r-1$ vertex pair in each edge of star so $2 \cdot 2 \cdot [2(n-1)(r-1) + 2(n-1)]$ is added to $M_2(S^r(D(S_n)))$. Finally adding all these together we get the desired result:

$$\begin{aligned} M_2(S^r(D(S_n))) &= (2 \cdot 2) \cdot 2(n-1) + [2(n-1) \cdot 2] \cdot 4(n-1) \\ &\quad +2 \cdot 2 \cdot [2(n-1)(r-1) + 2(n-1)] \\ &= 8(n-1) + 16(n-1)^2 + 8r(n-1) \end{aligned}$$

$$= \quad 8(n-1)(2n+r-1).$$

□

Theorem 3. *The first and second multiplicative Zagreb indices of the r -th subdivision of double graphs of several graph types are given by*

$$\Pi_1(S^r(D(G))) = \begin{cases} 2^{8[1+r(n-1)+(n-2)]} & \text{if } G = P_n, n \geq 2 \\ 2^{8n(r+1)} & \text{if } G = C_n, n > 2 \\ 2^{4(n-1)(1+2r)}\left[2(n-1)\right]^4 & \text{if } G = S_n, n \geq 2 \\ \left[2(n-1)\right]^{4n} 2^{4nr(n-1)} & \text{if } G = K_n, n \geq 2 \\ 2^{8[(t+s)(r+1)-1]}3^4 & \text{if } G = T_{t,s}, t \geq 3, s \geq 1 \end{cases}$$

and

$$\Pi_2(S^r(D(G))) = \begin{cases} 2^{2[(r-1)(n-1)+4(3n-4)]} & \text{if } G = P_n, n \geq 2 \\ 2^{8n(r+2)} & \text{if } G = C_n, n > 2 \\ 2^{8(n-1)(r+1)}(n-1)^{4(n-1)} & \text{if } G = S_n, n \geq 2 \\ (n-1)^{4n(n-1)}2^{4n(n-1)(r+1)} & \text{if } G = K_n, n \geq 2 \\ 2^{8[(t+s)(r+2)-2]}3^{12} & \text{if } G = T_{t,s}, t \geq 3, s \geq 1. \end{cases}$$

Proof. We prove the theorem for complete graphs. Similar methods can be used for others. Let G be a complete graph K_n. Its r-subgraph has $n(S_r) = mr + n$ and $m(S_r) = (r+1)m$ and for a complete graph, r-subgraph of its double graph has $n^d(S_r) = 2n + rm(S_r) = 2n + 2nr(n-1)$ and $m^d(S_r) = (r+1)m^d = 4(r+1)(n-1)$. In $S^r(D(K_n))$ we have $2n$ vertices with degree $2(n-1)$ and $r\left[2m + n(n-1)\right]$ vertices of degree 2.

So if we use the definition of $\Pi_1(G)$, we have

$$\Pi_1(S^r(D(K_n))) = \left[2(n-1)\right]^{4n} \cdot 2^{4nr(n-1)}.$$

There are 2 types of entries in $\Pi_2(S^r(D(K_n)))$:

i) u is an endpoint with degree $2(n-1)$ and v is a newly added vertex of degree 2 in $S^r(D(K_n))$: For each u, there are $2(n-1)$ added vertices v forming an edge with u, so that each vertex pair adds $2 \cdot 2(n-1) \cdot [2n \cdot 2(n-1)]$ to $\Pi_2(S^r(D(K_n)))$.

ii) Both u and v are middle vertices (of degree 2) which form an edge: There are $r-1$ vertex pairs in each edge of a complete graph so $(2 \cdot 2) \cdot (r-1) \cdot [2m + n(n-1)]$ is added to $\Pi_2(S^r(D(K_n)))$. Finally adding all these together, we get the desired result:

$$\begin{aligned} \Pi_2(S^r(D(K_n))) &= \left[2 \cdot 2(n-1)\right]^{2n \cdot 2(n-1)} \cdot (2 \cdot 2)^{(r-1) \cdot [2m+n(n-1)]} \\ &= \left[4(n-1)\right]^{4n(n-1)} .4^{2n(n-1)(r-1)} \\ &= 2^{4n(n-1)(r+1)} n^{4n(n-1)}. \end{aligned}$$

□

Acknowledgements

This work was supported by the research fund of Uludag University project no: F-2015/17.

References

[1] K. Ch. Das, A. Yurttas, M. Togan, I. N. Cangul and A. S. Cevik. The multiplicative Zagreb indices of graph operations. J. Inequal. Appl. 90, (2013), 1 - 14.

[2] P. S. Ranjini, V. Lokesha and I. N. Cangul. On the Zagreb indices of the line graphs of the subdivision graphs. Appl. Math. Comput. 218 (2011), 699 - 702.

[3] K. C. Das, N. Akgunes, M. Togan, A. Yurttas, I. N. Cangul and A. S. Cevik. On the first Zagreb index and multiplicative Zagreb coindices of graphs. An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 24 (2016), no. 1, 153 - 176.

[4] M. Togan, A. Yurttas and I. N. Cangul. All versions of Zagreb indices and coindices of subdivision graphs of certain graph types. Adv. Stud. Contemp. Math. 26 (1) (2016), 227 - 236.

[5] M. Togan, A. Yurttas and I. N. Cangul. Some formulae and inequalities on several Zagreb indices of r-subdivision graphs. Enlightments of Pure and Applied Mathematics 1 (1) (2015), 29 - 45.

[6] A. Yurttas, M. Togan and I. N. Cangul. Zagreb indices and multiplicative Zagreb indices of subdivision graphs of double graphs. Adv. Stud. Contemp. Math. 26 (3) (2016), 407 - 416.

[7] I. Gutman and N. Trinajstic. Graph theory and molecular orbitals III, Total π-electron energy of alternant hydrocarbons. Chem. Phys. Lett. 17 (1972), 535 - 538.

[8] R. Todeschini and V. Consonni. New local vertex invariant and molecular descriptors based on functions of the vertex degrees. MATCH 64 (2010), 359 - 372.

[9] E. Munarini, C. P. Cippo, A. Scagliola and N. Z. Salvi. Double graphs. Discrete Math. 308 (2008), 242 - 254.

[10] G. Indulal and A. Vijayakumar. On a Pair of Equienergetic Graphs. Commun. Math. Comput. Chem. 55 (2006), 83 - 90.

[11] Q. Huang, H. Chen and Q. Deng. Resistance distances and the Kirchhoff index in double graphs. J. Appl. Math. Comput. J. Appl. Math. Comput. 50 (2016), no. 1-2, 1C14.

Scientia Magna
Vol. 12 (2017), No. 1, 120-131

Third Hankel determinant for certain subclass of p-valent functions associated with generalized Ruscheweyh derivative operator

S. M. Patil[1] and S. M. Khairnar[2]

[1]Department of Applied Sciences,
SSVPS B.S. Deore College of Engineering,
Deopur, Dhule, Maharashtra, India
E-mail: sunitashelar1973@gmail.com
[2]Professor & Head, Department of Engineering Sciences,
Dr.D.Y.Patil School of Engineering, Lohegaon,
Pune-412105, Maharashtra, INDIA.
E-mail: smkhairnar2007@gmail.com

Abstract The objective of this paper is to obtain an upper bound to the $H_3(P)$ Hankel determinant for certain subclass of p-valent functions associated with Ruscheweyh derivative operator by using Toeplitz determinant.

Keywords Analytic function, p-valent function, upper bound, Hankel determinant, positive real function, Toeplitz determinant.
2010 Mathematics Subject Classification 30C45.

§1. Introduction, Definition and Motivation

Let S_p (p is fixed integer ≥ 1) denote the class of all analytic functions f(z) of the form,

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1}$$

defined on the open unit disk,

$$\mathbb{U} : \{z \in \mathbb{C} : |z| < 1\} \tag{2}$$

and let $S_1 = S$. A function $f(z) \in S_p$ is said to be p-valent starlike function ($\frac{f(z)}{z} \neq 0$), if it satisfies the condition,

$$\Re\left\{\frac{zf'(z)}{pf(z)}\right\} > 0 \quad (z \in \mathbb{U}) \tag{3}$$

Let f be an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z\text{-}\zeta)^{-\lambda}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

The generalized fractional derivative of order λ is defined,

$$J_{0,z}^{\lambda,\mu,\nu} f(z) = \begin{cases} \frac{1}{\Gamma 1-\lambda} \frac{d}{dz}\Big\{ z^{\lambda-\mu} \int_0^2 (z-\zeta)^{\lambda} \,.\, zF_1 & \\ (\mu-\lambda, 1-\nu; 1-\lambda; 1-\frac{\zeta}{2}) f(\zeta) d\zeta \Big\}, & 0 \leq \lambda < 1 \\ \frac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\nu} f(z) (n \leq \lambda < n+1, n \in N), & (k > max \\ & \{0, \mu-\nu \\ & -1\} - 1) \end{cases} \tag{4}$$

Provided further that,

$$f(z) = 0(|z|^k) \quad (z \ni 0) \tag{5}$$

It follows at once from the above definition that,

$$J_{0,z}^{\lambda\lambda,\nu} f(z) = D_z^{\lambda} f(z) \quad (0 \leq \lambda < 1) \tag{6}$$

Furthermore in terms of gamma function we have,

$$J_{0,z}^{\lambda,\mu,\nu} = \frac{\Gamma(\rho+1)\Gamma(\rho-\mu+\nu+2)}{\Gamma(\rho-\mu+1)\Gamma(\rho-\lambda+\nu+2)} z^{\rho-\mu} \quad (0 \leq \lambda < 1, \\ \rho > max\{0, \mu-\nu-1\} - 1) \tag{7}$$

In recent paper Goyal and Goyal[23] defined as generalized Ruscheweyh derivatives,

$$J_p^{\lambda,\mu} f_{n,p}, \quad \mu > -1 \quad as \tag{8}$$

$$\begin{aligned} J_p^{\lambda,\mu} f_{n,p}(z) &= \frac{\Gamma(\mu-\lambda+\nu+2)}{\Gamma(\nu+2)\Gamma(\mu+1)} z^p J_{0,z}^{\lambda,\mu,\nu} (z^{\mu-p} f_{n,p}(z)) \\ &= z^p + \sum_{k=n+p}^{\infty} a_k \beta_p^{\lambda,\mu}(k) z^k \end{aligned} \tag{9}$$

where,

$$\beta_p^{\lambda,\mu}(k) = \frac{\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)} \tag{10}$$

for $\lambda = \mu$, this generalized Ruscheweyh derivatives get reduced Ruscheweyh derivative of f(z) of order λ.

$$\begin{aligned} D_p^{\lambda} f(z) &= \frac{z^p}{\Gamma\lambda+1} \frac{d^{\lambda}}{dz^{\lambda}} (z^{\lambda-p} f(z)) \\ &= z^p + \sum_{k=p+1}^{\infty} a_k \beta_k(\lambda) z^k \end{aligned} \tag{11}$$

where,

$$\begin{aligned} \beta_k(\lambda) &= \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)\Gamma(k-p+1)} \\ &= \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!} \end{aligned} \tag{12}$$

The qth Hankel determinant for q $\geq$ 1 and n $\geq$ 1 is stated by Noonan and Thomas [21] as,

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

This determinant has been considered by several authors in the literature, for example Noor [22] determined the rate of growth of H$_q$(n) as n $\rightarrow \infty$ for the function f given by (1.1) with bounded boundary rotation. Ehrenbarg [6] studied the Hankel determinant for exponential polynomials. It is well known [5] that for f $\in$ S and given by (1.1). The sharp inequality $|a_3 - a_2^2| \leq 1$ holds. This corresponds to the Hankel determinant with q = 2 and n = 1.

Fekete - Szego [8] then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and f $\in$ S for a given family f of the function in A, the sharp upper bound for the nonlinear functional $|a_2a_4 - a_3^2|$ is popularly known as the second Hankel determinant for various subclass of analytic functions were obtained by different researchers including Janteng et al [10], Mishra and Gochhayat [19] and Murugusundaramoorthy and Magesh [20]. Recently Trailokya Panigrahi and Krishna et al obtain the sharp bound in the case of q = 2 and n = p+1 denoted by H$_2$(p+1) given by $|a_{p+1}a_{p+3} - a_{p+2}^2|$.

For our discussion in this paper, we consider the Hankel determinant in the case q = 3and n = p denoted by H$_3$(p) given by,

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix}$$

for f $\in$ S$_p$, a$_p$ = 1 so that we have,

$$H_3(p) = a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2) - a_{p+3}(a_{p+3} - a_{p+1}a_{p+2}) + a_{p+4}(a_{p+2} - a_{p+1}^2) \quad (13)$$

and by applying triangle inequality, we obtain

$$\begin{aligned} |H_3(p)| \leq |a_{p+2}||a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}||a_{p+1}a_{p+2} - a_{p+3}| \\ + |a_{p+4}||a_{p+2} - a_{p+1}^2| \end{aligned} \quad (14)$$

Motivated by the results obtained by D. V. Krishna et. al[1].And the result obtained by Babalola [3] and different researchers in this direction finding the sharp bound to Hankel determinant H$_3$(P) for the class RT.

In this paper, we obtain an upper bound to the functional $|a_{p+1}a_{p+2} - a_{p+3}|$ and hence for H$_3$(p), for the function f given in (1.1) belonging to the class S$_p^*(\lambda)$ is defined as follows.

Definition 1.1. *A function f(z) $\in S_p$ is said to be in the class $S_p^*(\lambda)$ if it satisfies the condition,*

$$\Re\left\{\frac{zD_p^\lambda f(z)}{PD_p^\lambda f(z)}\right\} > 0 \tag{15}$$

where,

$$D_p^\lambda f(z) = z^p + \sum_{k=p+1}^{\infty} \beta_k(\lambda) a_k z^k$$

$$\beta_k(\lambda) = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}$$

§2. Preliminary Lemmas

Lemma 2.1. *If the function $\rho \in p$ is given by the series,*

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots$$

then the following sharp estimate holds,

$$|p_k| \le 2 \qquad k = 1, 2, \dots$$

Lemma 2.2. *If the function $\rho \in p$ is given by the series then,*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

$$4c_3 = c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)((1-|x|^2)z)$$

for some x,z, $|x| \le 1$, $|z| \le 1$.

Lemma 2.3. *The power series of p given in (2.1) converges in δ in to function p if and only if the Toeplitz determinant*

$$D_n = \begin{vmatrix} 2 & C_1 & C_2 & \dots & C_n \\ C-1 & 2 & C_1 & \dots & C_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C-n & C-n+1 & \dots & \dots & 2 \end{vmatrix}$$

Where, n =1,2,3,... and $C_k = \bar{C}_k$ $\forall$ non-negative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} p_k p_0(e^{it_k z})$$

p_k >0 , t_k real and $t_k \neq t_j$ for $k \neq j$ in this case $D_n >$ 0 for n $<$ m - 1 and D_n =0 for n $\ge$ m

§3. Main Results

Theorem 3.1. *If f(z) $\in S_p^*(\lambda)$ with $p \in \mathbb{N}$ then,*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{4p^2}{(\lambda+p)^2(\lambda+p+1)^2} \tag{16}$$

Proof. Let D_p^λ f(z) = $z^p + \sum_{k=P+1}^{\infty} \beta_k(\lambda) a_k z^k$ be in the class $S_p^*(\lambda)$ from definition (1) [1,1] there exist an analytic function p $\in$ P in the unit disk $\mathbb{U}$ with p(0) = 1 and $\Re\{p(z)\} > 0$ such that,

$$\frac{z(D_p^\lambda f(z))'}{pD_p^\lambda f(z)} = p(z) \tag{17}$$

$$z(D_p^\lambda f(z))' = p((p(z)))D_p^\lambda f(z) \tag{18}$$

$$z\left\{pz^{p-1} + \sum_{k=P+1}^{\infty} \beta_k(\lambda)a_k k^{k-1}\right\} = p[1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots] \left[z^p + \sum_{k=p+1}^{\infty} \beta_k(\lambda)a_k z^k\right] \tag{19}$$

$$pz^p + \sum_{k=p+1}^{\infty} \beta_k(\lambda)a_k k z^k = [p + pc_1 z + pc_2 z^2 + pc_3 z^3 + \ldots] \left[z^p + \sum_{k=p+1}^{\infty} \beta_k(\lambda)a_k z^k\right] \tag{20}$$

Equating the coefficients of z^{p+1}, z^{p+2}, z^{p+3}, and z^{p+4} we get,

$$a_{p+1} = \frac{pc_1}{\beta_{p+1}(\lambda)} \tag{21}$$

$$a_{p+2} = \frac{pc_2 + p^2c_1^2}{2\beta_{p+2}(\lambda)} \tag{22}$$

$$a_{p+3} = \frac{2pc_3 + 3p^2c_1c_2 + p^3c_1^3}{6\beta_{p+3}(\lambda)} \tag{23}$$

$$a_{p+4} = \frac{6pc_4 + 6p^3c_2c_1^2 + 3p^2c_2^2 + 8p_1^2c_1c_3 + p^4c_1^4}{24\beta_{p+4}(\lambda)} \tag{24}$$

Considering the second Hankel functional $|a_{p+1}a_{p+2} - a_{p+2}^2|$ for the function f $\in S_p^*(\lambda)$ and substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from above relation we have,

$$
\begin{aligned}
|a_{p+1}a_{p+3}-a_{p+2}^2| &= \Bigg|\frac{pc_1}{\beta_{p+1}(\lambda)}\left(\frac{2pc_3+3p^2c_1c_2+p^3c_1^3}{6\beta_{p+3}(\lambda)}\right)- \\
&\left(\frac{pc_2+p^2c_1^2}{2\beta_{p+2}(\lambda)}\right)^2\Bigg| \\
&= \Bigg|\frac{p^2c_1c_3+3p^3c_1^2c_2+p^4c_1^4}{6\beta_{p+1}(\lambda)\beta_{p+3}(\lambda)}-\frac{1}{4(\beta_{p+2}(\lambda))^2} \\
&\left[p^2c_2^2+2p^3c_1^2c_2+p^4c_1^4\right]\Bigg| \qquad (25)\\
&= \Bigg|\frac{p^2c_1c_3+3p^2c_1^2c_2+p^4c_1^4}{(\lambda+p)^2(p+\lambda+1)(p+\lambda+2)}- \\
&\frac{p^2c_2^2+2p^3c_1^2c_2+p^4c_1^4}{(\lambda+p)^2(\lambda+p+1)^2}\Bigg| \\
&= A(p,\lambda)\big|p^2c_1c_3+3p^2c_1^2c_2+p^4c_1^4-B(p,\lambda) \\
&\big[p^2c_2^2+2p^3c_1^2c_2+p^4c_1^4\big]\big|
\end{aligned}
$$

$$
\begin{aligned}
where, A(p,\lambda) &= \frac{1}{(\lambda+p)^2(\lambda+p+1)(\lambda+p+2)} \\
B(p,\lambda) &= \frac{\lambda+p+2}{\lambda+p+1}
\end{aligned} \qquad (26)
$$

By applying lemma,

$$
\begin{aligned}
&= A(p,\lambda)\Bigg\{\Bigg|p^2c_1\left[\frac{c_1^3}{4}+\frac{c_1(4-c_1^2)x}{2}-\frac{c_1(4-c_1^2)x^2}{4}+\frac{(4-c_1^2)}{2}(1-|x|^2z)\right] \\
&+3p^2c_1^2\left[\frac{c_1^2+x(4-c_1^2)}{2}\right]+p^4c^4-\beta(p,\lambda)\left\{p^2\left(\frac{c_1^2+x(4-c)^2}{2}\right)^2\right. \qquad (27)\\
&\left.+2p^3c_1\left(\frac{c_1^2+x(4-c_1^2)}{2}\right)+p^4c_1^4\right\}\Bigg|\Bigg\} \\
&|z|\le 1, \quad |x|=\rho, \quad c_1=c\in[0,2]
\end{aligned}
$$

$$
\begin{aligned}
&\le A(p,\lambda)\Bigg\{\frac{p^2c^4}{4}+\frac{p^2c^2(4-c)^2\rho}{2}-\frac{p^2c^2(4-c^2)\rho^2}{4}+ \\
&\frac{p^2c^2(4-c^2)}{2}(1-\rho^2)+\frac{3p^2c^4}{2}+\frac{3}{2}p^2c^2\rho(4-c^2)+p^4c^4-B(p,\lambda) \qquad (28)\\
&\left[\frac{p^2}{4}\big(c^4+2c^2\rho(4-c^2)+\rho^2(4-c^2)^2\big)+p^3c^3+p^3c\rho(4-c^2)+p^4c^4\right]\Bigg\} \\
&= F(c,\rho)
\end{aligned}
$$

$$\begin{aligned}F'(c,\rho) &= A(p,\lambda)\Bigg\{\frac{p^2c^2(4-c^2)}{2} - \frac{p^2c^2(4-c^2)\rho}{2} - p^2c^2(4-c^2)\rho \\ &\quad + \frac{3}{2}p^2c^2(4-c^2) - \frac{B(p,\lambda)p^2}{4}\big[2c^2(4-c^2) + 2\rho(4-c^2)^2 \\ &\quad + p^3c(4-c^2)\big]\Bigg\} \\ F'(c,\rho) &> 0\end{aligned} \tag{29}$$

$\therefore F'(c,\rho)$ is an increasing function of ρ and hence it can not have maximum value of any point in the interior of closed square[0,2] $\times$ [0,1] after the fixed c $\in$ [0,2]. $\rho = 1$, F(c,1) = G(c).

$$\begin{aligned}G(c) &\le A(p,\lambda)\Bigg\{\frac{p^2c^4}{4} + \frac{p^2c^2(4-c^2)}{2} - \frac{p^2c^2(4-c^2)}{4} + \frac{3p^2c^4}{2} \\ &\quad + \frac{3}{2}p^2c^2(4-c^2) + p^4c^4 + B(p,\lambda)\Bigg[\frac{p^2c^4}{4} + \frac{p^2c^2}{2}(4-c^2) \\ &\quad + \frac{p^2}{4}(4-c^2)^2 + p^3c^3 + p^2c(4-c^2) + p^4c^4\Bigg]\Bigg\}\end{aligned} \tag{30}$$

put c = 0 and $\rho = 1$ the upper bound of (25)

$$\begin{aligned}&\le A(p,\lambda)\Bigg\{B(p\lambda)\Bigg[\frac{p^2}{4}\times 16\Bigg]\Bigg\} \\ &\le 4p^2A(p,\lambda)B(p,\lambda) \\ &\le \frac{4p^2}{(\lambda+p)^2(\lambda+p+1)^2}\end{aligned} \tag{31}$$

□

Corollary 3.2. *If p = 1, $\lambda = 0$ then $|a_2a_4 - a_3^2| \le 1$ with coinside the result Janteng et al.*

Theorem 3.3. *If f(z) $\in S_p^*(\lambda)$ then,*

$$|a_{p+1}a_{p+2} - a_{p+3}| \le \frac{4p}{(\lambda+p)(\lambda+p+1)(\lambda+p+2)} \tag{32}$$

Proof. From equation (21),(22)and (23)

$$\begin{aligned}|a_{p+1}a_{p+2}-a_{p+3}| &= \left|\frac{pc_1}{\beta_{p+1}(\lambda)}\times\frac{pc_2+p^2c_1}{\beta_{p+2}(\lambda)}-\frac{2pc_3+3p^2c_1c_2+p^3c_1^3}{6\beta_{p+3}(\lambda)}\right| \\ &= \left|\frac{p^2c_1c_2+p^3c_1^2}{(\lambda+p)^2)(\lambda+p+1)}-\right. \\ &\left.\frac{2pc_3+3p^2c_1c_2+p^3c_1^3}{(\lambda+p)(\lambda+p+1)(\lambda+p+2)}\right| \\ &= c(p,\lambda)\Big|p^2c_1c_2+p^3c_1^2-D(p,\lambda) \\ &\left[2pc_3+3p^2c_1c_2+p^3c_1^3\right]\Big|\end{aligned} \tag{33}$$

$$C(p,\lambda)=\frac{1}{(\lambda+p)^2(\lambda+p+1)} \quad \& \quad D(p,\lambda)=\frac{\lambda+p}{(\lambda+p+2)}$$

Applying lemma (2.2)

$$\begin{aligned}&= c(p,\lambda)\left\{\left|p^2c_1\left[\frac{c_1^2+x(4-c_1^2)}{2}\right]+p^3c_1^2-D(p,\lambda)\left[2p\left[\frac{c_1^3}{4}\right.\right.\right.\right. \\ &+\frac{2c_1(4-c_1^2)x}{4}-\frac{c_1(4-c_1^2)x^2}{4}+\left.\frac{2(4-c_1^2)(1-|x|^2z)}{4}\right] \\ &\left.\left.\left.+3p^2c_1\left[\frac{c_1^2+x(4-c_1^2)}{2}\right]+p^3c_1^3\right]\right|\right\} \\ &|z|\le 1, \quad |x|=\rho, \quad c_1=c\in[0,2] \\ &\le c(p,\lambda)\left\{\frac{p^2c^4}{2}+\frac{p^2c\rho(4-c^2)}{2}+p^3c^2-D(p,\lambda)\left[\frac{pc^3}{2}\right.\right. \\ &+pc(4-c^2)\rho-\frac{pc(4-c^2)\rho^2}{2}+p(4-c^2)(1-\rho^2)+\frac{3}{2}p^2 \\ &\left.\left.\left[c^3+\rho(4-c^2)\right]p^3c^3\right]\right\} \\ &\le F(c,\rho)\end{aligned} \tag{34}$$

$F'(c,\rho) < 0 \Rightarrow F'(c,\rho)$ is an increasing function $c \in [0,2]$. If $\rho = 1$, $F(c,1) = G(c)$.

$$\begin{aligned}G(c) &\le c(p,\lambda)\left\{\frac{p^2c^4}{2}+\frac{p^2c(4-c^2)}{2}+p^3c^2+D(p,\lambda)\left[\frac{pc^3}{2}+\right.\right. \\ &pc(4-c^2)-\frac{p(c-2)(4-c^2)}{2}+p(4-c^2)+\frac{3p^2}{2}\left[c^2+(4-c^2)\right] \\ &\left.\left.p^3c_1^3\right]\right\} \qquad G(c)\le G(0)\end{aligned} \tag{35}$$

upper bounds for (25)

$$\begin{aligned} |a_{p+1}a_{p+2} - a_{p+3}| &\leq c(p,\lambda)\Big\{D(p,\lambda)4p\Big\} \\ &\leq \frac{4p}{(\lambda+p)(\lambda+p+1)(\lambda+p+2)} \end{aligned} \tag{36}$$

Theorem agree with the following result due to Babalola [3]. □

Corollary 3.4. *If p = 1, λ = 0 then,*

$$|a_2a_4 - a_3| \leq \frac{2}{3} \tag{37}$$

Hence we have G(c) $\leq$ G(0) $= \frac{2}{3}$, c $\in$ [0,1). This is less than 2, which is the case when c $\in$ [1,2]. Thus the maximum of function, $|a_2a_3 - a_4|$ corresponds to $\rho = 1$ and c = 2. We put $\rho =$ 1 and c = 2 in (28).

$$\begin{aligned} |a_{p+1}a_{p+2} - a_{p+3}| &= c(p,\lambda)\Bigg|\frac{p^2}{2}(16) + p^3 4 - D(p,\lambda)\Bigg[4p+ \\ &\frac{3}{2}p^2(8) + 8p^3\Bigg]\Bigg| \\ &= c(p,\lambda)\Bigg|8p^2 + 4p^3 - D(p,\lambda)\Bigg[4p + 12p^2 + 8p^3\Bigg]\Bigg| \\ &If \quad p = 1 \quad and \quad \lambda = 0 \quad then, \\ &\leq \frac{1}{2}\Bigg|12 - \frac{1}{3}(24)\Bigg| \\ &\leq \frac{1}{2}\Bigg|12 - 8\Bigg| \\ &\leq 2 \end{aligned} \tag{38}$$

$\therefore |a_2a_4 - a_3| \leq 2$ with the result due to Babalola and Gagandeep et al.

Theorem 3.5. *If f(z)* $\in S_p^*(\lambda)$ *then,*

$$\begin{aligned}
|a_{p+2} - a_{p+1}^2| &= \left|\frac{pc_2 + p^2c_1^2}{2(\lambda+p)(\lambda+p+1)} - \frac{p^2c_1^2}{(\lambda+p)^2}\right| \\
&= \left|\frac{pc_2}{2(\lambda+p)(\lambda+p+1)} + \frac{p^2c_1^2(\lambda+p) - 2p^2c_1^2(\lambda+p+1)}{2(\lambda+p)^2(\lambda+p+1)}\right| \\
&= \left|\frac{p\left[\frac{c_1^2+x(4-c_1^2)}{2}\right] + p^2c_1}{2(\lambda+p)(\lambda+p+2)} - \frac{p^2c_1^2}{(\lambda+p)^2}\right| \\
&= \frac{p}{2(\lambda+p)(\lambda+p+1)}\left|c_2 - \frac{p^2c_1^2}{2}\left[c_1(\lambda+p)\right]\right| \\
&= \left|\frac{pc_1^2 + px(4-c_1^2) + 2p^2c_1}{2(\lambda+p)(\lambda+p+1)} - \frac{p^2c_1^2}{(\lambda+p)^2}\right| \\
|x| &= \rho \\
&\le \left|\frac{pc_1^2 + p\rho(4-c^2) + 2p^2c}{2(\lambda+p)(\lambda+p+1)} - \frac{p^2c^2}{(\lambda+p)^2}\right| \\
\rho &= 1 \quad and \quad c = 0 \\
&= \frac{2p}{(\lambda+p)(\lambda+p+1)}
\end{aligned} \tag{39}$$

Corollary 3.6. *If* $\lambda = 0$, $p = 1$ *then,*

$$|a_3 - a_2^2| \le 1$$

By using theorem (3.1), (3.3) and (3.5) and using inequality $|a_k| \le$ k, then we get $|H_3(1)| \le 16$, results consider with the result of Babalola [3] and Jetange [12].

References

[1] D. V. Krishna, B. Venkateswarlu and T. Ramreddy. Third Hankel determinant for certain subclass of p-valent functions. Complex Var. Elliptic Equ. 60 (2015), no. 9, 1301C1307.

[2] A. Abubaker and M. Darus. Hankel determinant for a class of analytic functions involving a generalized linear differential operator. Int. J. Pure Appl. Math. 69 (2011), no. 4, 429C435.

[3] K. O. Babalola. On $H_3(1)$ Hankel determinant for some classes of univalent functions. Inequality Theory and Applications 6 (2010), 1 - 7.

[4] Gagandeep Singh and Gurcharanjit Singh. On third Hankel determinant for a subclass of analytic functions. Open Science Journal of Mathematics and Application 3 (2015), no. 6, 152 - 155.

[5] P. L. Duren. Univalent Functions. Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer-Verlag, New York, 1983.

[6] R. Ehrenborg. The Hankel determinant of exponential polynomials. Amer. Math. Monthly 107 (2000), no. 6, 557 - 560.

[7] S. Ruscheweyh. New criteria for univalent functions. Proc. Amer. Math. Soc. 49 (1975), 109 - 115.

[8] M. Fekete and G. Szego. Eine Bemerkung uber ungerade schlichte Funktionen. J. London Math. Soc 8 (1933), 85 - 89.

[9] S. Shama, S. R. Kulkarni and M. Jay. On a class of univalent functions defined by Ruscheweyh derivatives. Kynugpook Math J. 43 (2003), 579 - 585.

[10] A. Janteng, S. A. Halim and M. Darus. Coefficient inequality for a function whose derivative has positive real part. J. Inequal. Pure Appl. Math. 7 (2006), no. 2, 1 - 5.

[11] A. Janteng, S. A. Halim and M. Darus. Hankel determinant for starlike and convex functions. Int. J. Math. Anal. (Ruse) 1 (2007), no. 13-16, 619C625.

[12] A. Janteng, S. A. Halim and M. Darus. Estimate on the second Hankel functional for functions whsoe derivative has a positive real part. J. Quality Measurement Anal. 4 (2008), 189 - 195.

[13] D. V. Krishna and T. Ramreddy. Hankel determinant for p-valent starlike and convex functions of order α. Novi Sad J. Math. 42 (2012), no. 2, 89 - 102.

[14] D. V. Krishna and T. Ramreddy. Hankel determinant for starlike and convex functions of order alpha. Tbilisi Math. J. 5 (2012), no. 1, 65 - 76.

[15] D. V. Krishna and T. Ramreddy. Coefficient inequality for certain subclass of p-valent functions. Palestine J. Math. 4 (2015), no. 1, 223 - 228.

[16] R. J. Libera and E. J. Zlotkiewicz. Early coefficient of the inverse of a regular convex function. Proc. Amer. Math. Soc. 85 (1982), no. 2, 225 - 230.

[17] R. J. Libera and E. J. Zlotkiewicz. Coefficients bounds for the inverse of a function with derivative in P. Proc. Amer. Math. Soc. 87 (1983), no. 2, 251 - 257.

[18] T. H. Mac Gregor. Functions whose derivative have a positive real part. Trans. Amer. Math. Soc. 104 (1962), no. 3, 532 - 537.

[19] A. K. Mishra and P. Gochhayat. Second Hankel determinant for a class of analytic functions defined by fractional derivative. Int. J. Math. Math. Sci., 2008, Art. ID 153280.

[20] G. Murugusundaramoorthy and N. Magesh. Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant. Bull. Math. Anal. Appl. 1 (2009), no. 3, 85 - 89.

[21] J. W. Noonan and D. K. Thomas. On the second Hankel determinant of areally mean p-valent functions. Trans. Amer. Math. Soc. 223 (1976), no. 2, 337 - 346.

[22] K. I. Noor. Hankel determinant problem for the class of functions with bounded boundary rotation. Rev. Roum. Math. Pures Et Appl. 28 (1983), no. 8, 731 - 739.

[23] S. P. Goyal and R. Goyal. On a class of multivalent functions defined by generalized Ruscheweyh derivatives involving a genaral fractional derivative operator. J. Indian Acad. Math. 27 (2005), 439 - 456.

Scientia Magna
Vol. 12 (2017), No. 1, 132-144

A survey on Smarandache notions in number theory I: Smarandache function

Huaning Liu

School of Mathematics, Northwest University
Xi'an 710127, China
E-mail: hnliu@nwu.edu.cn

Abstract In this paper we give a survey on recent results on Smarandache function.

Keywords Smarandache notion, Smarandache function, sequence, mean value.
2010 Mathematics Subject Classification 11A07, 11B50, 11L20, 11N25.

§1. Definition and simple properties

For any positive integer n, the famous Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is,

$$S(n) = \min\{m: \ n \mid m!, \ n \in \mathbb{N}\}. \tag{1.1}$$

Many people studied the lower bound of $S(n)$.

M. Le [16]. *Let $p > 2$ be a prime. Then $S\left(2^{p-1}(2^p-1)\right) \geq 2p+1$.*

J. Su [35]. *Let $p \geq 5$ be a prime. Then $S\left(2^{p-1}(2^p-1)\right) \geq 6p+1$.*

J. Su and S. Shang [36]. *Let $p \geq 7$ be a prime. Then $S(2^p+1) \geq 6p+1$.*

M. Liang [23]. *Let $p > 7$ be a prime. Then $S(2^p \pm 1) \geq 8p+1$.*

T. Wen [40]. *Let $p \geq 17$ be a prime. Then $S(2^p \pm 1) \geq 10p+1$.*

C. Shi [33]. *Let $p \geq 17$ be a prime. Then $S(2^p \pm 1) \geq 14p+1$.*

X. Wang [38]. *For any $m \in \mathbb{N}$, let $p \geq 9m^2(\log m+1)^3$ be a prime. Then $S(2^p-1) \geq 2mp+1$.*

F. Li and C. Yang [20]. *Let a and b be distinct positive integers, and let $p \geq 17$ be a prime. Then $S\left(a^p+b^p\right) \geq 8p+1$.*

P. Shi and Z. Liu [34]. *Let a and b be distinct positive integers, and let $p \geq 17$ be a prime. Then $S\left(a^p+b^p\right) \geq 10p+1$.*

L. Gao, H. Hao and W. Lu [6]. *Let a and b be positive integers with $a > b$, and let $p \geq 17$ be a prime. Then $S\left(a^p-b^p\right) \geq 8p+1$.*

J. Wang [37]. *Let $F_n = 2^{2^n}+1$ be the Fermat number. Then $S(F_n) \geq 8 \cdot 2^n + 1$ for $n \geq 3$.*

M. Zhu [55]. *Let $F_n = 2^{2^n}+1$ be the Fermat number. Then $S(F_n) \geq 12 \cdot 2^n + 1$ for $n \geq 3$.*

M. Liu and Y. Jin [26]. *Let $F_n = 2^{2^n}+1$ be the Fermat number. Then $S(F_n) \geq 4(4n+9) \cdot 2^n + 1$ for $n \geq 4$.*

M. Bencze [2]. *For positive integer sequences* $m_1, \cdots, m_n$*, we have*

$$S\left(\prod_{k=1}^{n} m_k\right) \leq \sum_{k=1}^{n} S(m_k).$$

M. Le [17]. *There are infinite many* $n \in \mathbb{N}$ *such that* $S(n) \leq S(n - S(n))$.

The distribution properties have also been studied.

W. Zhu [56]. *Let* $m = p_1^{T_1} p_2^{T_2} \cdots p_k^{T_k}$*, where* $p_1, p_2, \cdots, p_k$ *are distinct primes. For any* $n \in \mathbb{N}$*, we have*

$$S(m^n) = n \cdot \max_{1 \leq i \leq k} \{(p_i - 1)T_i\} + O\left(\frac{m}{\ln m} \ln n\right).$$

M. Le [15]. *For any distinct positive integers* k *and* n, $\log_{k^n} S\left(n^k\right)$ *is never a positive integer.*

F. Du [4]. 1. *Assume that* $n = p_1 p_2 \cdots p_k$*, where* $p_1, p_2, \cdots, p_k$ *are distinct primes. Then* $\sum_{d|n} \frac{1}{S(d)}$ *can not be an integer.*

2. *Suppose that* $n = p^T$*, where* $p > 2$ *is a prime and* $T \leq p$*. Then* $\sum_{d|n} \frac{1}{S(d)}$ *can not be an integer.*

3. *Let* $n = p_1^{T_1} p_2^{T_2} \cdots p_{k-1}^{T_{k-1}} \cdot p_k$*, where* $p_1, p_2, \cdots, p_k$ *are distinct primes. If* $S(n) = p_k$*, then* $\sum_{d|n} \frac{1}{S(d)}$ *can not be an integer.*

L. Huan [9]. 1. *Assume that* $n = p_1 p_2 \cdots p_k$*, where* $p_1, p_2, \cdots, p_k$ *are distinct primes. Then we have*

$$\prod_{d|n} S(d) = p_1 \cdot p_2^2 \cdots p_{k-1}^{2^{k-2}} p_k^{2^{k-1}}.$$

B. Liu and X. Pan [25]. *For any positive integer* n*, the formula*

$$\frac{S(2)S(4)\cdots S(2n)}{S(1)S(3)\cdots S(2n-1)}$$

is an integer if and only if $n = 1$.

A. Zhang [49]. *For integer* $n > 1$*, we have*

$$\frac{1}{n} \left|\{m : 1 \leq m \leq n, S(m) \text{ is a prime}\}\right| = 1 + O\left(\frac{1}{\ln n}\right).$$

W. Xiong [43]. *Define*

$$ES(n) = \left|\{a : 1 \leq a \leq n, 2 \mid S(a)\}\right|, \qquad OS(n) = \left|\{a : 1 \leq a \leq n, 2 \nmid S(a)\}\right|.$$

Then for integer $n > 1$*, we have*

$$\frac{ES(n)}{OS(n)} = O\left(\frac{1}{\ln n}\right).$$

Q. Liao and W. Luo [24]. *Let p be a prime and α be a positive integer.*

1) *For any positive integer r and $\alpha = p^r$, we have*

$$S(p^\alpha) = p^{r+1} - p^r + p.$$

2) *For any positive integer r, $t \in [1, r]$ and $\alpha = p^r - t$, we have*

$$S(p^\alpha) = p^{r+1} - p^r.$$

3) *For any positive integer r, $t \in [r+1, p^r - p^{r-1}]$ and $\alpha = p^r - t$.*

(I) *If*

$$\alpha = p^r - r - \sum_{i=1}^{n-1} (-1)^{i-1}(p^{k_i} - k_i) + (-1)^n p^{k_n}$$

with

$$k_i < p^{k_i - 1}(p-1) - 1, \qquad 1 \le i \le n-1,$$

then we have

$$S(p^\alpha) = (p-1)\left(p^r + \sum_{i=1}^{n} (-1)^i p^{k_i}\right) + (-1)^n p.$$

(II) *If*

$$\alpha = p^r - r - \sum_{i=1}^{n-1} (-1)^{i-1}(p^{k_i} - k_i) + (-1)^n \left(p^{k_n} - t\right)$$

with $t \in [1, k_n]$ and

$$k_i < p^{k_i - 1}(p-1) - 1, \qquad 1 \le i \le n-1,$$

then

$$S(p^\alpha) = (p-1)\left(p^r + \sum_{i=1}^{n} (-1)^i p^{k_i}\right).$$

Q. Liao and W. Luo [24]. *Let $\phi(n)$ be the Euler function and let $\sigma(n)$ be the sum of the different positive factors for n.*

1) *For any positive integer k, there are no any prime p and positive integer m coprime with p, such that $\phi(pm) = S(p^k)$ and $S(p^k) \ge S(m^k)$.*

2) *For any positive integer k, if there are some prime p and positive integer m coprime with p, such that $\phi(p^2 m) = S(p^{2k})$ and $S(p^{2k}) \ge S(m^k)$, then $p = 2k+1$ or $2 \ne p \le k$. Furthermore,*

(I) *If $2k+1 = p$, then*

$$(p, m) = (2k+1, 1), \quad (2k+1, 2), \quad (2, 3).$$

(II) *If $2 \le p \le k$, then $k \ge 3$ and*

$$\begin{cases} 2 \le \phi(m) \le \frac{2k^2+k-1}{3}, & k \equiv 2(\text{mod} 3), \\ 2 \le \phi(m) \le \frac{2k^2+k}{3}, & \textit{otherwise}. \end{cases}$$

3) *For any positive integer k, if there are some prime p and positive integer m coprime with p, such that $\phi(p^\alpha m) = S(p^{\alpha k})$ and $S(p^{\alpha k}) \geq S(m^k)$. Then $\alpha k + 1 > p^{\alpha-3}(p^2-1)$ and $1 \leq \phi(m) \leq q$, where*

$$\alpha k + 1 = qp^{\alpha-3}(p^2-1) + r, \qquad 0 \leq r < p^{\alpha-3}(p^2-1).$$

4) *For any positive integer k, there exist some prime p and positive integer m coprime with p, such that $\phi(p^3 m) = S(p^{3k})$ and $S(p^{3k}) \geq S(m^k)$, $m = 1, 2$.*

Q. Liao and W. Luo [24]. 1) *For any prime p, there is no any positive integer α such that $\dfrac{\sigma(p^\alpha)}{S(p^\alpha)}$ is a positive integer.*

2) *Let p be an odd prime, $\alpha \geq 1$ and $n = 2^\alpha p$.*

(I) *If $\sum_{i=1}^{\infty}\left[\dfrac{p}{2^i}\right] \geq \alpha$ and $\dfrac{\sigma(n)}{S(n)}$ is a positive integer, then $2^{\alpha+1} \equiv 1 (\mathrm{mod}\ p)$.*

(II) *If $\sum_{i=1}^{\infty}\left[\dfrac{p}{2^i}\right] < \alpha$ and $\dfrac{\sigma(n)}{S(n)}$ is a positive integer, then*

$$\frac{\sigma(n)}{S(n)} = m\frac{2^{\alpha+1}-1}{d} \qquad \text{and} \qquad p = m\frac{S(2^\alpha)}{d} - 1,$$

where $d = \left(2^{\alpha+1}-1, S(2^\alpha)\right)$ and $0 < m \leq d$.

§2. Mean values of the Smarandache function

C. Yang and D. Liu [45]. *Define $\sigma(n) = \sum_{d|n} d$. For any real $x \geq 3$ we have*

$$\sum_{n\leq x} \sigma\left(S(n)\right) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Y. Wang [39]. *For any real $x \geq 2$ we have the asymptotic formula*

$$\sum_{n\leq x}\frac{S(n)}{n} = \frac{\pi^2}{6}\cdot\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

W. Yao [48]. *Let $\Lambda(n)$ be the Mangoldt function. For any real $x \geq 1$ we have*

$$\sum_{n\leq x}\Lambda(n)S(n) = \frac{x^2}{4} + O\left(\frac{x^2\log\log x}{\log x}\right).$$

B. Shi [31]. *Let k be any fixed positive integer. For any real $x \geq 1$ we have*

$$\sum_{n\leq x}\Lambda(n)S(n) = x^2\sum_{i=0}^{k}\frac{c_i}{\log^i x} + O\left(\frac{x^2}{\log^{k+1} x}\right),$$

where c_i $(i = 0, 1, \cdots, k)$ are constants, and $c_0 = 1$.

Z. Lv [28]. *Let k be any fixed positive integer. For any real $x > 2$ we have the asymptotic formula*

$$\sum_{n\le x}\left(S(n)-S\left(S(n)\right)\right)^2=\frac{2}{3}\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}\sum_{i=1}^{k}\frac{c_i}{\log^i x}+O\left(\frac{x^{\frac{3}{2}}}{\log^{k+1}x}\right),$$

where $\zeta(s)$ is the Riemann zeta function, c_i $(i = 1, 2, \cdots, k)$ are computable constants, and $c_1 = 1$.

J. Ge [7]. *The Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$. Let k be any fixed positive integer. For any real $x > 2$ we have the asymptotic formula*

$$\sum_{n\le x}\left(SL(n)-S(n)\right)^2=\frac{2}{3}\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}\sum_{i=1}^{k}\frac{c_i}{\log^i x}+O\left(\frac{x^{\frac{3}{2}}}{\log^{k+1}x}\right),$$

where $\zeta(s)$ is the Riemann zeta function, c_i $(i = 1, 2, \cdots, k)$ are computable constants.

X. Fan and C. Zhao [5]. *Let $d(n)$ be the divisor function. For any real $x \ge 2$ we have*

$$\sum_{n\le x}S(n)d(n)=\frac{\pi^4}{36}\cdot\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right).$$

Z. Lv [29]. *Let $k \ge 2$ be any fixed positive integer. For any real $x > 1$ we have*

$$\sum_{n\le x}S(n)d(n)=\frac{\pi^4}{36}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ are computable constants.

M. Zhu [54]. *Define $\sigma_\alpha(n)=\sum_{d|n}d^\alpha$, $\alpha \ge 1$. Let $k \ge 2$ be any fixed positive integer. For any real $x > 1$ we have*

$$\sum_{n\le x}S(n)\sigma_\alpha(n)=\frac{\zeta(\alpha+2)\zeta(2)}{2+\alpha}\cdot\frac{x^{\alpha+2}}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^{\alpha+2}}{\ln^i x}+O\left(\frac{x^{\alpha+2}}{\ln^{k+1}x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ are computable constants.

H. Zhou [53]. *Let $k \ge 1$ be any fixed positive integer. For any complex s with* Re $s > 1$ *we have*

$$\sum_{n=1}^{\infty}\frac{\Lambda(n^k)}{S^s(n^k)}=-\zeta(s)\frac{\zeta'(ks)}{\zeta(ks)}.$$

Y. Guo [8]. *Define a function $F(n)$ as follows:*

$$F(n)=\begin{cases}0, & \text{if } n=1,\\ \alpha_1p_1+\alpha_2p_2+\cdots+\alpha_rp_r, & \text{if } n>1 \text{ and } n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}.\end{cases}$$

Let $k \ge 1$ be any fixed positive integer. For any real $x > 1$ we have

$$\sum_{n\le x}\left(F(n)-S(n)\right)^2=\sum_{i=1}^{k}\frac{c_i\cdot x^2}{\ln^{i+1}x}+O\left(\frac{x^2}{\ln^{k+2}x}\right),$$

where c_i $(i = 1, 2, \cdots, k)$ *are computable constants, and* $c_1 = \frac{\pi^2}{6}$.

C. Shi [32]. *For any positive integer* k, *the Smarandache* kn*-digital sequence* $a(k, n)$ *is defined as all positive integers which can be partitioned into two groups such that the second part is* k *times bigger that the first. For* $1 \le k \le 9$ *and real* $x > 1$ *we have*

$$\sum_{n \le x} \frac{S(n)}{a(k,n)} = \frac{3\pi^2}{20k} \ln \ln x + O(1).$$

C. Yang, C. Li and D. Liu [44]. *For any real* $x \ge 2$ *we have*

$$\sum_{n \le x} S^2(n) = \frac{\zeta(3)x^3}{3 \ln x} + O\left(\frac{x^3}{\ln^2 x}\right),$$

$$\sum_{n \le x} \frac{S^2(n)}{n} = \frac{\zeta(3)x^2}{3 \ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

W. Huang [11]. *Let* $k \ge 1$ *be any fixed integer. For any real* $x \ge 2$ *we have*

$$\sum_{n \le x} S^k(n) = \frac{\zeta(k+1)}{k+1} \cdot \frac{x^{k+1}}{\ln x} + O\left(\frac{x^{k+1}}{\ln^2 x}\right),$$

$$\sum_{n \le x} \frac{S^k(n)}{n} = \frac{2\zeta(k+1)}{k+1} \cdot \frac{x^k}{\ln x} + O\left(\frac{x^k}{\ln^2 x}\right).$$

C. Li, C. Yang and D. Liu [19]. *Let* $P(n)$ *denote the largest prime factor of* n. *For any real* $x \ge 2$ *we have*

$$\sum_{n \le x} \frac{S(n)}{P(n)} = x \ln 2 + \frac{6x^{\frac{2}{3}}}{\ln x} + O\left(\frac{x^{\frac{2}{3}}}{\ln^2 x}\right).$$

M. Yang [46]. *For any real* $x \ge 2$ *we have*

$$\sum_{n \le x} \frac{S(n)}{SL(n)} = x + O\left(\frac{x \ln \ln x}{\ln x}\right),$$

$$\sum_{n \le x} \frac{P(n)}{SL(n)} = x + O\left(\frac{x \ln \ln x}{\ln x}\right).$$

L. Li, J. Hao and R. Duan [22]. *For any real* $x \ge 1$ *we have*

$$\sum_{n \le x} \ln S(n) = x \ln x + O(x).$$

Z. Liu and P. Shi [27]. *For any real* $x \ge 3$ *and* $\beta > 1$ *we have*

$$\sum_{n \le x} (S(n) - P(n))^\beta = \frac{2\zeta\left(\frac{\beta+1}{2}\right) x^{\frac{\beta+1}{2}}}{(\beta+1) \ln x} + O\left(\frac{x^{\frac{\beta+1}{2}}}{\ln^2 x}\right).$$

W. Huang [12]. *For $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, we define $\varpi(n) = p_1 + p_2 + \cdots + p_k$. For any real $x \geq 2$ we have*

$$\sum_{n \leq x} S(n)\varpi(n) = B\frac{x^3}{\ln x} + O\left(\frac{x^3}{\ln^2 x}\right),$$

where B is computable constant.

G. Chen [3]. *Define $H(n) = \sum_{[r,s]=n} S(r)S(s)$. Let $k \geq 1$ be any fixed positive integer. For any real $x > 1$ we have*

$$\sum_{n \leq x} H(n) = \sum_{i=1}^{k} \frac{d_i \cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where d_i $(i = 1, 2, \cdots, k)$ are computable constants, and $d_1 = \frac{1}{3} \cdot \frac{\zeta^3(3)}{\zeta(6)}$.

Q. Yang [47]. *For any real $\delta \leq 1$, the series*

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{\delta}}$$

diverges.

For any real $\epsilon > 0$, the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon S(n)}}$$

converges.

§3. Mean values of the Smarandache function over sequences

W. Zhang and Z. Xu [50]. *Let $a(n)$ denote the square complements of n. For any real $x \geq 3$ we have the asymptotic formula*

$$\sum_{n \leq x} S(a(n)) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

H. Li and X. Zhao [21]. *Let $r_k(n)$ denote the integer part of k-th root of n. For any real $x \geq 3$ we have*

$$\sum_{n \leq x} S(r_k(n)) = \frac{\pi^2}{6(k+1)} \cdot \frac{x^{1+\frac{1}{k}}}{\ln x} + O\left(\frac{x^{1+\frac{1}{k}}}{\ln^2 x}\right).$$

J. Ma [30]. *Define $L(n) = [1, 2, \cdots, n]$. For any real $x \geq 1$ we have*

$$\sum_{n \leq x} S\left(L(n)\right) = \frac{1}{2}x^2 + O\left(x^{\frac{23}{18}+\epsilon}\right).$$

Q. Wu [41]. *Define* $Z(n) = \min\left\{k : n \leq \frac{k(k+1)}{2}\right\}$. *Let* $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n\leq x} S\left(Z(n)\right) = \frac{\pi^2}{18}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln 2x} + \sum_{i=2}^{k}\frac{c_i(2x)^{\frac{3}{2}}}{\ln^i 2x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

H. Zhao [51]. *Let* $a_k(n)$ *denote the* k*-th power complements of* n. *For any real* $x \geq 3$ *we have*

$$\sum_{n\leq x}\left(S(a_k(n)) - (k-1)P(n)\right)^2 = \frac{2\zeta\left(\frac{3}{2}\right)}{3}\cdot\frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right).$$

W. Huang [10]. *Define* $u(n) = \min\left\{k : n \leq k(2k-1)\right\}$. *Let* $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n\leq x} S\left(u(n)\right) = \frac{\pi^2}{144}\cdot\frac{(2x)^{\frac{3}{2}}}{\ln\sqrt{2x}} + \sum_{i=2}^{k}\frac{c_i(2x)^{\frac{3}{2}}}{\ln^i\sqrt{2x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

Q. Zhao and L. Gao [52]. *Define* $W(n) = \min\left\{k : n \leq k(3k+1)\right\}$. *Let* $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n\leq x} S\left(W(n)\right) = \frac{\pi^2}{486}\cdot\frac{(3x)^{\frac{3}{2}}}{\ln\sqrt{3x}} + \sum_{i=2}^{k}\frac{b_i(3x)^{\frac{3}{2}}}{\ln^i\sqrt{3x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where b_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

W. Huang and J. Zhao [14]. *Define*

$$\begin{aligned}
u_r(n) &= \min\left\{m + \frac{1}{2}m(m-1)(r-2) : n \leq m + \frac{1}{2}m(m-1)(r-2), r\in\mathbb{N}, r\geq 3\right\},\\
v_r(n) &= \max\left\{m + \frac{1}{2}m(m-1)(r-2) : n \geq m + \frac{1}{2}m(m-1)(r-2), r\in\mathbb{N}, r\geq 3\right\}.
\end{aligned}$$

Let $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\begin{aligned}
\sum_{n\leq x} S(u_r(n)) &= \frac{\pi^2}{18(r-2)^3}\cdot\frac{(2(r-2)x)^{\frac{3}{2}}}{\ln\sqrt{2(r-2)x}} + \sum_{i=2}^{k}\frac{c_i(2(r-2)x)^{\frac{3}{2}}}{\ln^i\sqrt{2(r-2)x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),\\
\sum_{n\leq x} S(v_r(n)) &= \frac{\pi^2}{18(r-2)^3}\cdot\frac{(2(r-2)x)^{\frac{3}{2}}}{\ln\sqrt{2(r-2)x}} + \sum_{i=2}^{k}\frac{c_i(2(r-2)x)^{\frac{3}{2}}}{\ln^i\sqrt{2(r-2)x}} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),
\end{aligned}$$

where c_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

W. Huang [13]. *Define* $a(n) = n - u_r(n)$ *and* $b(n) = v_r(n) - n$. *Let* $k \geq 1$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n\leq x} S(n)a(n) = \frac{8\sqrt[4]{2}\pi^2}{63(r-2)^{\frac{5}{4}}}\cdot\frac{x^{\frac{7}{4}}}{\ln 2x} + O\left(\frac{x^{\frac{7}{4}}}{\ln^2 2x}\right),$$

$$\sum_{n\le x} S(n)b(n) = \frac{8\sqrt[4]{2}\pi^2}{63(r-2)^{\frac{5}{4}}}\cdot\frac{x^{\frac{7}{4}}}{\ln 2x}+O\left(\frac{x^{\frac{7}{4}}}{\ln^2 2x}\right).$$

R. Xie, L. Gao and Q. Zhao [42]. *Define* $q_d(n)=\prod_{\substack{d\mid n\\ d<n}}$. *Let* $k\ge 1$ *be any fixed positive integer. For any real* $x>1$ *we have*

$$\sum_{n\le x}\left(S\left(q_d(n)\right)-\left(\frac{1}{2}d(n)-1\right)P(n)\right)^2=\sum_{i=1}^{k}c_i\frac{x^{\frac{3}{2}}}{\ln^i x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1}x}\right),$$

where c_i $(i=1,2,\cdots,k)$ *are computable constants, and*

$$c_1=\frac{3}{2}\cdot\frac{\zeta^4\left(\frac{3}{2}\right)}{\zeta(3)}-2\zeta^2\left(\frac{3}{2}\right)+\frac{2}{3}\zeta\left(\frac{3}{2}\right).$$

B. Li, J. Guo and H. Dong [18]. *Define*

$$U(n)=\begin{cases}1, & \text{if } n=1,\\ \max\limits_{1\le i\le r}\{\alpha_1p_1,\alpha_2p_2,\cdots,\alpha_rp_r\}, & \text{if } n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}.\end{cases}$$

Let $k\ge 2$ *be any fixed positive integer. For any real* $x\ge 3$ *we have*

$$\sum_{n\le x}\left(S\left(a_k(n)\right)-(k-1)\,U(n)\right)^2=\frac{2}{3}\zeta\left(\frac{3}{2}\right)k^2\cdot\frac{x^{\frac{3}{2}}}{\ln x}+O\left(\frac{x^{\frac{11}{6}}}{\ln^2 x}\right).$$

J. Bai and W. Huang [1]. *Let* $\mathcal{A}$ *denote the set of the simple numbers. Let* $k\ge 2$ *be any fixed positive integer. For any real* $x\ge 2$ *we have*

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}} S^k(n)=\frac{Bx^{k+1}}{(k+1)\ln x}+\sum_{i=2}^{k}\frac{C_ix^{k+1}}{\ln^i x}+O\left(\frac{x^{k+1}}{\ln^{k+1}x}\right),$$

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}}\frac{1}{S(n)}=D\ln\ln x+\frac{E\sqrt{x}\ln\ln x}{\ln x}+O\left(\frac{\sqrt{x}}{\ln x}\right),$$

where $B,D,E,$ C_i $(i=2,3,\cdots,k)$ *are computable constants.*

References

[1] Jiazhi Bai and Wei Huang. On the mean value properties of the sequence of simple numbers for the Smarandache function $S(n)$. Henan Science 35 (2017), no. 4, 521 - 525. (In Chinese with English abstract).

[2] Mihaly Bencze. An inequality for the Smarandache function. Smarandache Notions Journal 10 (1999), 160.

[3] Guohui Chen. On the least common multiple product of Smarandache function. Journal of Southwest University (Natural Science Edition) 35 (2013), no. 10, 63 - 66. (In Chinese with English abstract).

[4] Fengying Du. On a conjecture of the Smarandache function $S(n)$. Pure and Applied Mathematics 23 (2007), no. 2, 205 - 213. (In Chinese with English abstract).

[5] Xuhui Fan and Chunxiang Zhao. On the hybrid mean value of the Smarandache function and the Dirichlet divisor function. Pure and Applied Mathematics 24 (2008), no. 4, 662 - 665. (In Chinese with English abstract).

[6] Li Gao, Hongfei Hao and Weiyang Lu. A lower bound estimate for Smarandache function on sequence $a^p - b^p$. Journal of Yanan University (Natural Science Edition) 33 (2014), no. 3, 1 - 3. (In Chinese with English abstract).

[7] Jian Ge. Mean value of F. Smarandache LCM function. Scientia Magna 3 (2007), no. 2, 109 - 112.

[8] Yanchun Guo. A new additive function and the F. Smarandache function. Scientia Magna 5 (2009), no. 1, 128 - 132.

[9] Le Huan. On the properties of Smarandache function and its related functions. Journal of Southwest University (Natural Science Edition) 35 (2013), no. 4, 67 - 70. (In Chinese with English abstract).

[10] Wei Huang. One hybrid mean value formula involing Smarandache function. Science Technology and Engineering 9 (2009), no. 16, 4750 - 4752. (In Chinese with English abstract).

[11] Wei Huang. Smarandache involving function and its asymptotic formula. Journal of Jishou University (Natural Science Edition) 32 (2011), no. 5, 9 - 10. (In Chinese with English abstract).

[12] Wei Huang. Hybrid mean value of F. Smarandache function and the prime factor sum function. Journal of Chongqing University of Posts and Telecommunications (Natural Science Edition) 24 (2012), no. 6, 804 - 806. (In Chinese with English abstract).

[13] Wei Huang. Mean-value estimate for two Smarandache hybrid functions. Journal of Jishou University (Natural Science Edition) 37 (2016), no. 3, 1 - 3. (In Chinese with English abstract).

[14] Wei Huang and Jiaolian Zhao. Mean-value estimate for two Smarandache hybrid functions. Pure and Applied Mathematics 26 (2010), no. 6, 890 - 894. (In Chinese with English abstract).

[15] Maohua Le. The integral values of $\log_{k^n} S\left(n^k\right)$. Smarandache Notions Journal 11 (2000), 71 - 72.

[16] Maohua Le. A lower bound for $S\left(2^{p-1}\left(2^p - 1\right)\right)$. Smarandache Notions Journal 12 (2001), 217 - 218.

[17] Maohua Le. An inequality concerning the Smarandache function. Journal of Foshan University (Natural Science Edition) 26 (2008), no. 3, 1 - 2.

[18] Bo Li, Jinbao Guo and Hailang Dong. On the average variance of the Smarandache composite function. Journal of Yanan University (Natural Science Edition) 32 (2013), no. 2, 4 - 6. (In Chinese with English abstract).

[19] Chao Li, Cundian Yang and Duansen Liu. On the average value distribution of the Smarandache function. Journal of Gansu Sciences 22 (2010), no. 3, 24 - 27. (In Chinese with English abstract).

[20] Fenjü Li and Changyu Yang. A lower bound estimate problem for the Smarandache function. Journal of Northwest University (Natural Science Edition) 41 (2011), no. 3, 377 - 379. (In Chinese with English abstract).

[21] Hailong Li and Xiaopeng Zhao. On the Smarandache function and the k-th roots of a positive integer. Research on Smarandache problems in number theory. Vol. I, 119C122, Hexis, Phoenix, AZ, 2004.

[22] Ling Li, Jun Hao and Rui Duan. On a conjecture of F. Smarandache function. Journal of Changchun Normal University (Natural Science Edition) 30 (2011), no. 4, 11 - 12. (In Chinese with English abstract).

[23] Ming Liang. A lower bound for the values of Smarandache function of Mersenne numbers. Journal of Guangdong University of Petrochemical Technology 24 (2014), no. 4, 47 - 50. (In Chinese with English abstract).

[24] Qunying Liao and Wenli Luo. The explicit formula for the Smarandache function and solutions of related equations. Journal of Sichuan Normal University (Natural Science Edition) 40 (2017), no. 1, 1 - 10. (In Chinese with English abstract).

[25] Baoli Liu and Xiaowei Pan. On a problem related to function $S(n)$. Scientia Magna 3 (2007), no. 4, 84 - 86.

[26] Miaohua Liu and Yingji Jin. A lower bound for the values of Smarandache function of Fermat numbers. Mathematics in Practice and Theory 45 (2015), no. 8, 283 - 286. (In Chinese with English abstract).

[27] Zhuo Liu and Peng Shi. On the β-th hybrid mean value of the Smarandache function. Basic Sciences Journal of Textile Universities 25 (2012), no. 3, 335 - 338. (In Chinese with English abstract).

[28] Zhongtian Lv. On the F. Smarandache function and its mean value. Scientia Magna 3 (2007), no. 2, 104 - 108.

[29] Zhongtian Lv. On a hybrid mean value of the F. Smarandache function and the divisor function. Basic Sciences Journal of Textile Universities 20 (2007), no. 3, 234 - 236. (In Chinese with English abstract).

[30] Jinping Ma. A hybrid mean value involving the Smarandache function. Journal of Zhengzhou University (Natural Science Edition) 39 (2007), no. 1, 31 - 32. (In Chinese with English abstract).

[31] Baohuai Shi. The hybrid mean value of the Smarandache function and the Mangoldt function. Scientia Magna 2 (2006), no. 4, 98 - 100.

[32] Chan Shi. On the hybrid mean value of the Smarandache kn-digital sequence and Smarandache function. Scientia Magna 6 (2010), no. 4, 20 - 23.

[33] Chan Shi. A lower bound estimate for the Smarandache function. Journal of Shangluo University 28 (2014), no. 2, 9 - 10. (In Chinese with English abstract).

[34] Peng Shi and Zhuo Liu. A new lower bound estimate for Smarandache function on sequence a^p+b^p. Journal of Southwest China Normal University (Natural Science Edition) 38 (2013), no. 8, 10 - 14. (In Chinese with English abstract).

[35] Juanli Su. A lower bound estimate for the Smarandache function. Basic Sciences Journal of Textile Universities 22 (2009), no. 1, 133 - 134. (In Chinese with English abstract).

[36] Juanli Su and Songye Shang. A new lower bound estimate for the Smarandache function. Pure and Applied Mathematics 24 (2008), no. 4, 706 - 708. (In Chinese with English abstract).

[37] Jinrui Wang. On the Smarandache function and the Fermat number. Scientia Magna 4 (2008), no. 2, 25 - 28.

[38] Xiaohan Wang. The lower bound for Smarandache functions values of Mersenne numbers. Journal of Northwest University (Natural Science Edition) 44 (2014), no. 3, 367 - 369. (In Chinese with English abstract).

[39] Yongxing Wang. On the Smarandache function. Research on Smarandache problems in number theory. Vol. II, 103C106, Hexis, Phoenix, AZ, 2005.

[40] Tianding Wen. A lower bound estimate for the Smarandache function. Pure and Applied Mathematics 26 (2010), no. 3, 413 - 416. (In Chinese with English abstract).

[41] Qibin Wu. A composite function involving the Smarandache function. Pure and Applied Mathematics 23 (2007), no. 4, 463 - 466. (In Chinese with English abstract).

[42] Rui Xie, Li Gao and Qin Zhao. On the mean value of a F. Smarandache function and the greatest prime divisor function. Henan Science 29 (2011), no. 9, 1024 - 1026. (In Chinese with English abstract).

[43] Wenjing Xiong. On the parity of the Smarandache function. Pure and Applied Mathematics 24 (2008), no. 2, 363 - 366. (In Chinese with English abstract).

[44] Cundian Yang, Chao Li and Duansen Liu. The properties of the Smarandache function. Journal of Gansu Sciences 22 (2010), no. 1, 24 - 25. (In Chinese with English abstract).

[45] Cundian Yang and Duansen Liu. On the mean value of a new arithmetical function. Research on Smarandache problems in number theory. Vol. II, 75C77, Hexis, Phoenix, AZ, 2005.

[46] Mingshun Yang. On the hybrid mean value of the Smarandache function and the Smarandache LCM function. Journal of Northwest University (Natural Science Edition) 40 (2010), no. 5, 772 - 773. (In Chinese with English abstract).

[47] Qianli Yang. Some arithmetical properties of the Smarandache series. Scientia Magna 10 (2014), no. 1, 64 - 67.

[48] Weili Yao. On the Smarandache function and its hybrid mean value. Scientia Magna 1 (2005), no. 1, 79 - 81.

[49] Ailing Zhang. On a problem of the Smarandache function. Pure and Applied Mathematics 24 (2008), no. 2, 385 - 387. (In Chinese with English abstract).

[50] Wenpeng Zhang and Zhefeng Xu. On the Smarandache function and square complements. Scientia Magna 1 (2005), no. 1, 1 - 3.

[51] Hongxing Zhao. On the F. Smarandache function and the k-th power complements. Journal of Northwest University (Natural Science Edition) 37 (2007), no. 6, 948 - 950. (In Chinese with English abstract).

[52] Qin Zhao and Li Gao. On the mean value of the Smarandache function. Henan Science 30 (2012), no. 2, 153 - 155. (In Chinese with English abstract).

[53] Huanqin Zhou. On the Smarandache function and the Riemann zeta function. Pure and Applied Mathematics 24 (2008), no. 1, 41 - 44. (In Chinese with English abstract).

[54] Min Zhu. On a hybrid mean value of the Smarandache function and the divisor function. Jiangxi Science 30 (2012), no. 6, 714 - 715. (In Chinese with English abstract).

[55] Minghui Zhu. On the Smarandache function and the Fermat number. Journal of Northwest University (Natural Science Edition) 40 (2010), no. 4, 583 - 585. (In Chinese with English abstract).

[56] Weiyi Zhu. On the asymptotic property of the F. Smarandache function $S(m^n)$. Pure and Applied Mathematics 23 (2007), no. 1, 1 - 3. (In Chinese with English abstract).

Scientia Magna
Vol. 12 (2017), No. 1, 145-153

A survey on Smarandache notions in number theory II: pseudo-Smarandache function

Huaning Liu

School of Mathematics, Northwest University
Xi'an 710127, China
E-mail: hnliu@nwu.edu.cn

Abstract In this paper we give a survey on recent results on pseudo-Smarandache function.

Keywords Smarandache notion, pseudo-Smarandache function, sequence, mean value.
2010 Mathematics Subject Classification 11A07, 11B50, 11L20, 11N25.

§1. Definition and simple properties

According to [11], the pseudo-Smarandache function $Z(n)$ is defined by

$$Z(n) = \min\left\{m: \ n\left|\frac{m(m+1)}{2}\right.\right\}.$$

Some elementary properties can be found in [11] and [1].

R. Pinch [20]. *For any given $L > 0$ there are infinitely many values of n such that $\frac{Z(n+1)}{Z(n)} > L$, and there are infinitely many values of n such that $\frac{Z(n-1)}{Z(n)} > L$.*

For any integer $k \geq 2$, the equation $\frac{n}{Z(n)} = k$ has infinitely many solutions n.

The ration $\frac{Z(2n)}{Z(n)}$ is not bounded.

Fix $\frac{1}{2} < \beta < 1$ and integer $t \geq 5$. The number of integers n with $e^{t-1} < n < e^t$ such that $Z(n) < n^{\beta}$ is at most $196t^2e^{\beta t}$.

The series $\sum_{n=1}^{\infty} \frac{1}{Z(n)^{\alpha}}$ is convergent for any $\alpha > 1$.

Some explicit expressions of $Z(n)$ for some particular cases of n were given by Abdullah-Al-Kafi Majumdar.

A. A. K. Majumdar [18]. *If $p \geq 5$ is a prime, then*

$$Z(2p) = \begin{cases} p-1, & \text{if } \ 4 \mid p-1, \\ p, & \text{if } \ 4 \mid p+1, \end{cases}$$

$$Z(3p) = \begin{cases} p-1, & \text{if } \ 3 \mid p-1, \\ p, & \text{if } \ 3 \mid p+1, \end{cases}$$

$$Z(4p) = \begin{cases} p-1, & if\ \ 8 \mid p-1, \\ p, & if\ \ 8 \mid p+1, \\ 3p-1, & if\ \ 8 \mid 3p+1, \\ 3p, & if\ \ 8 \mid 3p+1, \end{cases}$$

$$Z(6p) = \begin{cases} p-1, & if\ \ 12 \mid p-1, \\ p, & if\ \ 12 \mid p+1, \\ 2p-1, & if\ \ 4 \mid 3p+1, \\ 2p, & if\ \ 4 \mid 3p-1. \end{cases}$$

A. A. K. Majumdar [18]. *If* $p \geq 7$ *is a prime, then*

$$Z(5p) = \begin{cases} p-1, & if\ \ 10 \mid p-1, \\ p, & if\ \ 10 \mid p+1, \\ 2p-1, & if\ \ 5 \mid 2p-1, \\ 2p, & if\ \ 5 \mid 2p+1. \end{cases}$$

If $p \geq 11$ *is a prime, then*

$$Z(7p) = \begin{cases} p-1, & if\ \ 7 \mid p-1, \\ p, & if\ \ 7 \mid p+1, \\ 2p-1, & if\ \ 7 \mid 2p-1, \\ 2p, & if\ \ 5 \mid 2p+1, \\ 3p-1, & if\ \ 7 \mid 3p-1, \\ 3p, & if\ \ 7 \mid 3p+1. \end{cases}$$

If $p \geq 13$ *is a prime, then*

$$Z(11p) = \begin{cases} p-1, & if\ \ 11 \mid p-1, \\ p, & if\ \ 11 \mid p+1, \\ 2p-1, & if\ \ 11 \mid 2p-1, \\ 2p, & if\ \ 11 \mid 2p+1, \\ 3p-1, & if\ \ 11 \mid 3p-1, \\ 3p, & if\ \ 11 \mid 3p+1, \\ 4p-1, & if\ \ 11 \mid 4p-1, \\ 4p, & if\ \ 11 \mid 4p+1, \\ 5p-1, & if\ \ 11 \mid 5p-1, \\ 5p, & if\ \ 11 \mid 5p+1. \end{cases}$$

A. A. K. Majumdar [18]. *Let p and q be two primes with $q > p \geq 5$. Then*

$$Z(pq) = \min\{qy_0 - 1, px_0 - 1\},$$

where

$$\begin{aligned} y_0 &= \min\{y : x, y \in \mathbb{N}, qy - px = 1\}, \\ x_0 &= \min\{x : x, y \in \mathbb{N}, px - qy = 1\}. \end{aligned}$$

A. A. K. Majumdar [18]. *If $p \geq 3$ is a prime, then $Z(2p^2) = p^2 - 1$. If $p \geq 5$ is a prime, then $Z(3p^2) = p^2 - 1$.*

If $p \geq 3$ is a prime and $k \geq 3$ is an integer, then

$$\begin{aligned} Z(2p^k) &= \begin{cases} p^k, & \text{if } 4 \mid p - 1 \text{ and } k \text{ is odd}, \\ p^k - 1, & \text{otherwise}, \end{cases} \\ Z(3p^k) &= \begin{cases} p^k, & \text{if } 3 \mid p + 1 \text{ and } k \text{ is odd}, \\ p^k - 1, & \text{otherwise}. \end{cases} \end{aligned}$$

S. Gou and J. Li [2]. *The equation $Z(n) = Z(n+1)$ has no positive integer solutions.*

For any given positive integer M, there exists a positive integer s such that

$$|Z(s) - Z(s+1)| > M.$$

Y. Zheng [29]. *For any given positive integer M, there are infinitely many positive integers n such that*

$$|Z(n+1) - Z(n)| > M.$$

M. Yang [27]. *Suppose that n has primitive roots. Then $Z(n)$ is a primitive root modulo n if and only if $n = 2, 3, 4$.*

W. Lu, L. Gao, H. Hao and X. Wang [17]. *Let $p \geq 17$ be a prime. Then we have*

$$Z(2^p + 1) \geq 10p, \qquad Z(2^p - 1) \geq 10p.$$

L. Gao, H. Hao and W. Lu [?]. *Let $p \geq 17$ be a prime, and let a, b be distinct positive integers. Then we have*

$$Z(a^p + b^p) \geq 10p.$$

Y. Ji [10]. *Let r be a positive integer. Suppose that $r \neq 1, 2, 3, 5$. Then*

$$Z(2^r + 1) \geq \frac{1}{2}\left(-1 + \sqrt{2^{r+3} \cdot 5 + 41}\right).$$

Assume that $r \neq 1, 2, 4, 12$. *Then*

$$Z\left(2^r - 1\right) \geq \frac{1}{2}\left(-1 + \sqrt{2^{r+3} \cdot 3 - 23}\right).$$

§2. Mean values of the pseudo-Smarandache function

Y. Lou [16]. *For any real* $x > 1$ *we have*

$$\sum_{n \leq x} \ln Z(n) = x \ln x + O(x).$$

W. Huang [9]. *For any integer* $n > 1$ *we have*

$$\frac{\sum\limits_{k=2}^{n} \frac{\ln Z(k)}{\ln k}}{n} = 1 + O\left(\frac{1}{\ln n}\right), \qquad \frac{Z(n)}{\sum\limits_{k \leq n} \ln Z(k)} = O\left(\frac{1}{\ln n}\right).$$

L. Cheng [4]. *Let* $p(n)$ *denote the smallest prime divisor of* n, *and let* k *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n \leq x} \frac{p(n)}{Z(n)} = \frac{x}{\ln x} + \sum_{i=2}^{k} \frac{a_i x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

X. Wang, L. Gao and W. Lu [23]. *Define*

$$\overline{\Omega}(n) = \begin{cases} 0, & \text{if } n = 1, \\ \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r, & \text{if } n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \end{cases}$$

Let $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n \leq x} Z(n)\overline{\Omega}(n) = \frac{\zeta(3) x^3}{3 \ln x} + \sum_{i=2}^{k} \frac{a_i x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where a_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

H. Hao, L. Gao and W. Lu [8]. *Let* $d(n)$ *denote the divisor function, and let* $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n \leq x} Z(n) d(n) = \frac{\pi^4}{36} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{a_i x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where a_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

X. Wang, L. Gao and W. Lu [24]. *Define*

$$D(n) = \min\left\{m : \ m \in \mathbb{N}, n \mid \prod_{i=1}^{m} d(i)\right\}.$$

Let $k \geq 2$ be any fixed positive integer. For any real $x > 1$ we have

$$\sum_{n \leq x} Z(n) \ln D(n) = \frac{\zeta(3) \ln 2}{3} \cdot \frac{x^3}{\ln x} + \sum_{i=2}^{k} \frac{a_i x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where a_i $(i = 2, 3, \cdots, k)$ are computable constants.

§3. The dual of the pseudo-Smarandache function, the near pseudo-Smarandache function, and other generalizations

According to [21], the dual of the pseudo-Smarandache function is defined by

$$Z_*(n) = \max\left\{m \in \mathbb{N} : \ \frac{m(m+1)}{2} \mid n\right\}.$$

D. Liu and C. Yang [15]. *Let $\mathcal{A}$ denote the set of simple numbers. For any real $x \geq 1$ we have*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}}} Z_*(n) = C_1 \frac{x^2}{\ln x} + C_2 \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right),$$

where C_1, C_2 are computable constants.

X. Zhu and L. Gao [30]. *We have*

$$\sum_{n=1}^{\infty} \frac{Z_*(n)}{n^\alpha} = \zeta(\alpha) \sum_{m=1}^{\infty} \frac{2m}{m^\alpha (m+1)^{2\alpha}}.$$

The near pseudo Smarandache function $K(n)$ is defined as

$$K(n) = \sum_{i=1}^{n} i + k(n),$$

where $k(n) = \min\left\{k : \ k \in \mathbb{N}, n \mid \sum_{i=1}^{n} i + k\right\}$. Some recurrence formulas satisfied by $K(n)$ were derived in [19].

H. Yang and R. Fu [26]. *For any real $x \geq 1$ we have*

$$\begin{aligned}
\sum_{n \leq x} d\left(K(n) - \frac{n(n+1)}{2}\right) &= \frac{3}{4} x \log x + Ax + O\left(x^{\frac{1}{2}} \log^2 x\right), \\
\sum_{n \leq x} \phi\left(K(n) - \frac{n(n+1)}{2}\right) &= \frac{93}{28\pi^2} x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right),
\end{aligned}$$

where $\phi(n)$ denotes the Euler function, A is a computable constant, and $\epsilon > 0$ is any real number.

Y. Zhang [28]. *For any real number $s > \frac{1}{2}$, the series*

$$\sum_{n=1}^{\infty} \frac{1}{K^s(n)}$$

is convergent, and

$$\sum_{n=1}^{\infty}\frac{1}{K(n)}=\frac{2}{3}\ln 2+\frac{5}{6},\qquad \sum_{n=1}^{\infty}\frac{1}{K^2(n)}=\frac{11}{108}\pi^2-\frac{22+2\ln 2}{27}.$$

Y. Li, R. Fu and X. Li [14]. *We have*

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in\mathcal{A}}}K(n) &= \frac{x^2\ln\ln x}{3\ln x}+B\frac{x^2}{\ln x}+\frac{2x^2\ln\ln x}{9\ln^2 x}+O\left(\frac{x^2}{\ln^2 x}\right),\\ \sum_{\substack{n\le x\\ n\in\mathcal{A}}}\frac{1}{K(n)} &= \frac{2}{3}(\ln\ln x)^2+D\ln\ln x+E+O\left(\frac{\ln\ln x}{\ln x}\right).\end{aligned}$$

L. Gao, R. Xie and Q. Zhao [5]. *Define*

$$p_d(n)=\prod_{d\mid n}d,\qquad q_d(n)=\prod_{\substack{d\mid n\\ d<n}}d.$$

For any real $x>1$ we have

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in\mathcal{A}}}K\left(p_d(n)\right) &= \frac{x^5}{5\ln x}\ln\ln x+A_1\frac{x^5}{\ln x}+\frac{x^5}{25\ln^2 x}\ln\ln x+O\left(\frac{x^5}{\ln^2 x}\right),\\ \sum_{\substack{n\le x\\ n\in\mathcal{A}}}K\left(q_d(n)\right) &= \frac{x^3}{3\ln x}\ln\ln x+A_2\frac{x^3}{\ln x}+\frac{x^3}{9\ln^2 x}\ln\ln x+O\left(\frac{x^3}{\ln^2 x}\right),\end{aligned}$$

where A_1, A_2 are computable constants.

Other generalizations on the near pseudo-Smarandache function have been given. For example, define

$$Z_3(n)=\min\left\{m:\ m\in\mathbb{N}, n\mid\frac{m(m+1)(m+2)}{6}\right\}.$$

The elementary properties were studied in [6] and [7].

Y. Wang [25]. *Define*

$$U_t(n)=\min\left\{k:\ 1^t+2^t+\cdots+n^t+k=m, n\mid m,k,t,m\in\mathbb{N}\right\}.$$

For any real number $s>1$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{U_1^s(n)} &= \zeta(s)\left(2-\frac{1}{2^s}\right),\\ \sum_{n=1}^{\infty}\frac{1}{U_2^s(n)} &= \zeta(s)\left(1+\frac{1}{5^s}-\frac{1}{6^s}+2\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\right),\\ \sum_{n=1}^{\infty}\frac{1}{U_3^s(n)} &= \zeta(s)\left(1+\left(1-\frac{1}{2^s}\right)^2\right).\end{aligned}$$

M. Tong [22]. *Define*

$$Z_0(n) = \begin{cases} \min\{m: \ m \in \mathbb{N}, n \mid m(m+1)\}, & \text{if } 2 \mid n, \\ \min\{m: \ m \in \mathbb{N}, n \mid m^2\}, & \text{if } 2 \nmid n. \end{cases}$$

For any real $x > 1$, we have

$$\sum_{n \le x} Z_0(2n-1) = \frac{3\zeta(3)}{\pi^2} x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

X. Li [12]. *Define*

$$C(n) = \min\left\{a+b: \ a, b \in \mathbb{N}, n \mid \frac{a(a+1)}{2} + b\right\}.$$

For any real $x > 1$, we have

$$\begin{aligned} \sum_{n \le x} C(n) &= \sqrt{2} x^{\frac{3}{2}} + O(x), \\ \sum_{n \le x} \frac{1}{C(n)} &= \ln 2 \cdot \sqrt{2x} + O(\ln x), \\ \sum_{n \le x} d(C(n)) &= \frac{1}{2} x \ln x + x\left(2\gamma + \frac{5}{2}\ln 2 - \frac{3}{2}\right) + O\left(x^{\frac{3}{4}}\right), \end{aligned}$$

where γ is the Euler constant.

Y. Li [13]. *Define*

$$D(n) = \max\left\{ab: \ a, b \in \mathbb{N}, n = \frac{a(a+1)}{2} + b\right\}.$$

For any real $x > 1$, we have

$$\begin{aligned} \sum_{n \le x} D(n) &= \frac{4\sqrt{6}}{45} x^{\frac{5}{2}} + O\left(x^2\right), \\ \sum_{n \le x} \frac{C(n)}{D(n)} &= \frac{9\sqrt{3}}{4} \ln x + O(1). \end{aligned}$$

References

[1] Charles Ashbacher. Pluckings from the tree of Smarandache sequences and functions. American Research Press (ARP), Lupton, AZ, 1998.

[2] Su Gou and Jianghua Li. On the pseudo-Smarandache function. Scientia Magna 3 (2007), no. 4, 81C83.

[3] Li Gao, Hongfei Hao and Weiyang Lu. A lower bound estimate for pseudo-Smarandache function. Henan Science 32 (2014), no. 5, 707 - 710. (In Chinese with English abstract).

[4] Lin Cheng. On the mean value of the pseudo-Smarandache function. Scientia Magna 3 (2007), no. 3, 97C100.

[5] Li Gao, Rui Xie and Qin Zhao. Two arithmetical functions invloving near pseudo Smarandache noptions and their asymptotic formulas. Journal of Yanan University (Natural Science Edition) 30 (2011), no. 2, 1 - 3. (In Chinese with English abstract).

[6] Shoupeng Guo. A generalization of the pseudo Smarandache function. Journal of Guizhou University (Natural Science Edition) 27 (2010), no. 1, 6 - 7. (In Chinese with English abstract).

[7] Shoupeng Guo. Elementary properties on the generalization of the pseudo Smarandache function. Journal of Changchun University 20 (2010), no. 8, 4 - 5. (In Chinese with English abstract).

[8] Hongfei Hao, Li Gao and Weiyang Lu. On the hybrid mean value of the pseudo-Smarandache function and the Dirichlet divisor function. Journal of Yanan University (Natural Science Edition) 34 (2015), no. 2, 46 - 48. (In Chinese with English abstract).

[9] Wei Huang. On two questions of the pseudo Smarandache function $Z(n)$. Journal of Jishou University (Natural Science Edition) 35 (2014), no. 5, 10 - 12. (In Chinese with English abstract).

[10] Yongqiang Ji. Upper bounds and lower bounds for the pseudo-Smarandache function. Mathematics in Practice and Theory 46 (2016), no. 1, 275 - 279. (In Chinese with English abstract).

[11] Kenichiro Kashihara. Comments and topics on Smarandache notions and problems. Erhus University Press, Vail, AZ, 1996.

[12] Xihan Li. A new pseudo-Smarandache function and its mean value. Journal of Xi'an Polytechnic University 26 (2012), no. 1, 105 - 107. (In Chinese with English abstract).

[13] Yijun Li. On the mean value of a new Smarandache function. Journal of Inner Mongolia Noemal University (Natural Science Edition) 41 (2012), no. 3, 244 - 246. (In Chinese with English abstract).

[14] Yuying Li, Ruiqin Fu and Xuegong Li. Calculation of the mean value of two approximate pseudo-Smarandache function. Journal of Xi'an Shiyou University (Natural Science Edition) 25 (2010), no. 5, 99 - 102. (In Chinese with English abstract).

[15] Duansen Liu and Cundian Yang. On a dual of the pseudo Smarandache function and its asymptotic formula. Research on Smarandache problems in number theory, 123C127, Hexis, Phoenix, AZ, 2004.

[16] Yuanbing Lou. On the pseudo Smarandache function. Scientia Magna 3 (2007), no. 4, 48C50.

[17] Weiyang Lu, Li Gao, Hongfei Hao and Xihan Wang. A lower bound estimate for the pseudo-Smarandache function. Journal of Shaanxi University of Science and Technology 32 (2014), no. 6, 180 - 183. (In Chinese with English abstract).

[18] Abdullah-Al-Kafi Majumdar. A note on the pseudo-Smarandache function. Scientia Magna 2 (2006), no. 3, 1C25.

[19] Abdullah-Al-Kafi Majumdar. A note on the near pseudo Smarandache function. Scientia Magna 4 (2008), no. 4, 104C111.

[20] Richard Pinch. Some properties of the pseudo-Smarandache function. Scientia Magna 1 (2005), no. 2, 167 - 172.

[21] József Sándor. On a dual of the pseudo-Smarandache function. Smarandache Notions Journal 13 (2002), no. 1-3, 18C23.

[22] Minna Tong. A new pseudo-Smarandache function and its mean value. Basic Sciences Journal of Textile Universities 26 (2013), no. 1, 18 - 20. (In Chinese with English abstract).

[23] Xihan Wang, Li Gao and Weiyang Lu. A hybrid mean value of the pseudo-Smarandache function. Natural Science Journal of Hainan University 33 (2015), no. 2, 97 - 99. (In Chinese with English abstract).

[24] Xihan Wang, Li Gao and Weiyang Lu. The hybrid mean value of the pseudo-Smarandache function. Henan Science 33 (2015), no. 10, 1682 - 1685. (In Chinese with English abstract).

[25] Yu Wang. Some identities involving the near pseudo Smarandache function. Scientia Magna 3 (2007), no. 2, 44C49.

[26] Hai Yang and Ruiqin Fu. On the mean value of the near pseudo Smarandache function. Scientia Magna 2 (2006), no. 2, 35C39.

[27] Mingshun Yang. On a problem of the pseudo Smarandache function. Pure and Applied Mathematics 24 (2008), no. 3, 449 - 451. (In Chinese with English abstract).

[28] Yongfeng Zhang. On the near pseudo Smarandache function. Scientia Magna 3 (2007), no. 1, 98C101.

[29] Yani Zheng. On the pseudo Smarandache function and its two conjectures. Scientia Magna 3 (2007), no. 4, 74C76.

[30] Xiaoyan Zhu and Li Gao. An equation involving Smarandache function. Journal of Yanan University (Natural Science Edition) 28 (2009), no. 2, 5 - 6. (In Chinese with English abstract).

SCIENTIA MAGNA

An international journal

ISSN 1556- 6706