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Φ-variational stability for discontinuous system

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Abstract Bounded Φ-variation functions are development and generalization of bounded variation functions in the usual sense. The concept of Henstock-Kurzweil integral is an effective tool in dealing with highly infinite oscillation functions. In this paper, the concept of Φ-variational stability is defined by using Φ-function theory, the Φ-variational stability of the bounded Φ-variation solution to discontinuous system is discussed, and the Ljapunov type theorems for Φ-variational stability and asymptotically Φ-variational stability of the bounded Φ-variation solution are established. These results are essential generalization of variation stability of bounded variation solution of this system, and the certain foundation is laid in the research of highly infinite oscillation functions.

Keywords Discontinuous system, Henstock-Kurzweil integral, bounded Φ-variation solution, Φ-variational stability, Ljapunov function.
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§1. Introduction

Consider the general discontinuous system

$$x' = f(t, x), \tag{$*$}$$

where $x = (x_1, x_2, ..., x_n)^T, x' = \frac{dx}{dt}, f : G \to R^n$, G is an open region in R^{n+1}. $(*)$ is called discontinuous system if f is a function with some discontinuity. In [1] and [2], the existence, uniqueness and stability for the solution of Caratheodory system and Filippov system were obtained by using the Lebesgue integral and the solutions obtained are absolute continuous functions. The Henstock-Kurzweil integral was introduced by Henstock and Kurzweil independently during 1957-1958, the existence, uniqueness and variational stability for the bounded variation solutions of a class of discontinuous system were discussed by using the Henstock-Kurzweil integral in [3, 4].

The functions of bounded Φ-variation were introduced by Musiclak and Orlicz [5,6], which are generalization and development of functions of bounded variation in usual sense. The

[1]This work is supported by the Young Teachers' Scientific Research Projects of Tianshui normal university (TSY 201202).

existence theorem of bounded Φ-variation solutions for a class of discontinuous system is established in [7]. In this paper, the concept of Φ-variational stability of bounded Φ-variation solutions for discontinuous system is given and the stability is discussed through making use of Ljapunov function. The Ljapunov type theorems for Φ-variational stability and asymptotically Φ-variational stability of the bounded Φ-variation solutions are established. These results are essential generalization of variation stability of bounded variation solution to discontinuous system in [4].

§2. Preliminaries and definitions

Definition 2.1.[3,4,7,8] A function $x : [a, b] \to R^n$ is called Henstock-Kurzweil integrable over $[a, b]$, if there is a $A \in R^n$ such that given $\varepsilon > 0$, there is a positive function $\delta : [a, b] \to (0, +\infty)$ such that for any $\delta(t)-$ fine partition $D : a = t_0 < t_1 < \cdots < t_k = b$ and $\{\xi_1, \xi_2, \cdots, \xi_k\}$ satisfying $\xi_j \in [t_{j-1}, t_j] \subset [\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)]$, we have

$$\| \sum_{j=1}^{k} x(\xi_j)(t_j - t_{j-1}) - A\| < \varepsilon.$$

A is called the Henstock-Kurzweil integral of $x(t)$ over $[a, b]$ and will be denoted by $\int_a^b x(t)dt$.

Definition 2.2.[3,4,7,8] *A* function $x : [a, b] \to R^n$ is called H-K-Stieltjes integral over $[a, b]$, function $y : [a, b] \to R^n$, if there is a $A \in R^n$ such that given $\varepsilon > 0$, there is a positive function $\delta : [a, b] \to (0, +\infty)$ such that for any $\delta(t)-$ fine partition $D : a = t_0 < t_1 < \cdots < t_k = b$ and $\{\xi_1, \xi_2, \cdots, \xi_k\}$ satisfying $\xi_j \in [t_{j-1}, t_j] \subset [\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)]$, we have

$$\| \sum_{j=1}^{k} x(\xi_j)(y(t_j) - y(t_{j-1})) - A\| < \varepsilon.$$

A is called the H-K-Stieltjes integral of $x(t)$ over $[a, b]$ and will be denoted by $\int_a^b x(t)dy(t)$.

Let $\Phi(u)$ denote a continuous and increasing function defined for $u \geq 0$ with $\Phi(0) = 0$, $\Phi(u) > 0$ for $u > 0$, and satisfying the following conditions:

(Δ_1) There exist $u_0 \geq 0$ and $a > 0$ such that $\Phi(2u) \leq a\Phi(u)$ for $0 < u \leq u_0$;

(Δ_2) $\Phi(u)$ is a convex function.

Let $[a, b] \subset R, -\infty < a < b < +\infty$. We consider the function $x : [a, b] \to R^n$, $x(t)$ is of bounded Φ-variation over $[a, b]$ if for any partition $\pi : a = t_0 < t_1 < \cdots < t_m = b$, we have $V_\Phi(x; [a, b]) = \sup_\pi \sum_{i=1}^{m} \Phi(\|x(t_i) - x(t_{i-1})\|) < +\infty$, $V_\Phi(x; [a, b])$ is called Φ-variation of $x(t)$ over $[a, b]$. We always assume $\Phi(u)$ satisfying (Δ_1) and (Δ_2).

Given $c > 0$, we denote $B_c = \{x \in R^n; \|x\| < c\}$. Let $I \subset (a, b) \subset R$ be an interval with $-\infty < a < b < +\infty$, and set $G = B_c \times I$.

Definition 2.3.[3,4,7] *A* Caratheodory function $f : G \to R^n$ belongs to the class $\mathcal{V}_\Phi(G, h, \omega)$, if

(1) There exists a positive function $\delta : [a, b] \to (0, +\infty)$, satisfying $\tau \in [u, v] \subset [\tau - \delta(\tau), \tau + \delta(\tau)] \subset I$ and $x \in \bar{B}$for every $[u, v]$, we have

$$\| f(x, \tau)(v - u) \| \leq \Phi(|h(v) - h(u)|).$$

(2) Satisfying $\tau \in [u, v] \subset [\tau - \delta(\tau), \tau + \delta(\tau)] \subset I$ and $x, y \in \bar{B}$ for every $[u, v]$, we have

$$\| f(x, \tau) - f(y, \tau) \| (v - u) \leq \omega(\| x - y \|)\Phi(|h(v) - h(u)|),$$

where $h : [a, b] \to R$ is an increasing function and continuous from the left on the interval $[a, b]$, $\omega : [0, +\infty) \to R$ is a continuous and increasing function with $\omega(0) = 0$.

(3) The function $f(\psi(t), t)$ is $H - K$ integrable on $[\alpha, \beta]$ for every step function $\psi(t)$ on $[\alpha, \beta] \subset [a, b]$.

Lemma 2.1.[3,4,7] Assume that $f : G \to R^n$ satisfies the condition (1) of Definition 2.3, if $x : [\alpha, \beta] \to R^n, [\alpha, \beta] \subset (a, b)$ is such that $(x(t), t) \in G$ for every $t \in [\alpha, \beta]$ and if the Kurzweil integral $\int_\alpha^\beta f(x(t), t)$ exists, then for every pair $s_1, s_2 \in [\alpha, \beta]$ the inequality $\| \int_\alpha^\beta f(x(t), t)dt \| \leq V_\Phi(h; [s_1, s_2])$ holds.

Corollary 2.1.[3,4,7] Assume that $f : G \to R^n$ satisfies the condition (1) of Definition 2.3, if $x : [\alpha, \beta] \to R^n$, $[\alpha, \beta] \subset (a, b)$ is a solution of $(*)$, then x is of bounded Φ-variation and $V_\Phi(x; [\alpha, \beta]) \leq \Phi(V_\Phi(h; [\alpha, \beta])) < +\infty$. Moreover every point in $[\alpha, \beta]$ at which the function h is continuous from the left is a left continuity point of the solution $x : [\alpha, \beta] \to R^n$.

Lemma 2.2.[4,8] Assume that $-\infty < a < b < +\infty$ and that $f, g : [a, b] \to R$ are functions which are continuous from the left in $[a, b]$. If for every $\sigma \in [a, b]$ there exists $\delta(\sigma) > 0$ such that for every $\eta \in (0, \delta(\sigma))$ the inequality $f(\sigma + \eta) - f(\sigma) \leq g(\sigma + \eta) - g(\sigma)$ holds, then $f(s) - f(a) \leq g(s) - g(a)$ for all $s \in [a, b]$.

If $f(0, t) = 0$ for every $t \in [0, +\infty)$ then the function x given by $x(t) = 0$ for $t \geq 0$ is a solution of $(*)$ on the whole half-axis $[0, +\infty)$.

Definition 2.4. The solution $x \equiv 0$ of $(*)$ is called Φ-variationally stable if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $y : [t_0, t_1] \to B_c$, $0 \leq t_0 < t_1 < +\infty$ is a function of bounded Φ-variation on $[t_0, t_1]$, continuous from the left on $(t_0, t_1]$ with $\Phi(\|y(t_0)\|) < \delta$ and $V_\Phi((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, t_1]) < \delta$, then we have $\Phi(\|y(t)\|) < \varepsilon$.

Definition 2.5. The solution $x \equiv 0$ of $(*)$ is called Φ-variationally attracting if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$ there is a $T = T(\varepsilon) \geq 0$ and $\gamma = \gamma(\varepsilon) > 0$ such that if $y : [t_0, t_1] \to B_c$, $0 \leq t_0 < t_1 < +\infty$ is a function of bounded Φ-variation on $[t_0, t_1]$, continuous from the left on $(t_0, t_1]$ with $\Phi(\|y(t_0)\|) < \delta_0$ and $V_\Phi((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, t_1]) < \gamma$, then $\Phi(\|y(t)\|) < \varepsilon$ for all $t \in [t_0, t_1] \bigcap [t_0 + T(\varepsilon), +\infty)$ and $t_0 \geq 0$.

Definition 2.6. The solution $x \equiv 0$ of $(*)$ is called Φ-variationally asymptotically stable if it is Φ-variationally stable and Φ-variationally attracting.

§3. Main results

Theorem 3.1. Assume that $V : [0, +\infty) \times R^n \to R$ is such that for every $x \in R^n$ the function $V(\cdot, x) : [0, +\infty) \to R$ is continuous from the left in $(0, +\infty)$ and

(1) The inequality

$$|V(t, x) - V(t, y)| \leq L\Phi(\|x - y\|) \tag{1}$$

holds for $x, y \in R^n$, $t \in [0, +\infty)$ with a constant $L > 0$;

(2) There exists a real function $H : R^n \to R$ such that for every solution $x : (\alpha,\beta) \to R^n$ of the discontinuous system $(*)$ on $(\alpha,\beta) \subset [0,+\infty)$, we have

$$\lim_{\eta\to 0+} \sup \frac{V(t+\eta, x(t+\eta)) - V(t,x(t))}{\eta} \le H(x(t)) \tag{2}$$

for $t \in (\alpha,\beta)$.

If $y : [t_0,t_1] \to R^n$, $0 \le t_0 < t_1 < +\infty$ is continuous from the left on $(t_0,t_1]$ and of bounded Φ-variation on $[t_0,t_1]$, then the inequality

$$V(t_1, y(t_1)) \le V(t_0, y(t_0)) + NV_\Phi((y(s) - \int_{t_0}^{s} f(y(t),t)dt); [t_0,t_1]) + R(t_1 - t_0) \tag{3}$$

holds, where $R = \sup_{t\in[t_0,t_1]} H(x(t)), N = L\chi(a')$, ($\chi(a')$ was defined by Theorem 1.2 in [5].

Proof. Let $y : [t_0,t_1] \to R^n, 0 \le t_0 < t_1 < +\infty$ be a function of bounded Φ-variation on $[t_0,t_1]$ and continuous from the left on $(t_0,t_1]$ and let $\sigma \in [t_0,t_1]$ be an arbitrary point. It is clear that the function $V(t,y(t)) : [t_0,t_1] \times R^n \to R$ is continuous from the left on $(t_0,t_1]$.

Assume that $x : [\sigma, \sigma + \eta_1(\sigma)] \to R^n$ is a solution of bounded Φ-variation of $(*)$ on the interval $[\sigma, \sigma+\eta_1(\sigma)]$, $\eta_1 > 0$ with the initial condition $x(\sigma) = y(\sigma)$. The existence of such a solution is guaranteed by Theorem 3.3 in [7]. By (1), we have

$$V(\sigma+\eta, y(\sigma+\eta)) - V(\sigma+\eta, x(\sigma+\eta)) \le L\Phi(\|y(\sigma+\eta) - x(\sigma+\eta)\|)$$

$$= L\Phi(\|y(\sigma+\eta) - y(\sigma) - \int_{\sigma}^{\sigma+\eta} f(x(t),t)dt\|)$$

for every $\eta \in [0, \eta_1(\sigma)]$.

By (2), we obtain

$$\begin{aligned} & V(\sigma+\eta, y(\sigma+\eta)) - V(\sigma, x(\sigma)) \\ & = V(\sigma+\eta, y(\sigma+\eta)) - V(\sigma+\eta, x(\sigma+\eta)) + V(\sigma+\eta, x(\sigma+\eta)) - V(\sigma, x(\sigma)) \\ & \le L\Phi(\|y(\sigma+\eta) - y(\sigma) - \textstyle\int_{\sigma}^{\sigma+\eta} f(x(t),t)dt\|) + \eta H(x(\sigma)) \\ & \le L\Phi(\|y(\sigma+\eta) - y(\sigma) - \textstyle\int_{\sigma}^{\sigma+\eta} f(x(t),t)dt\|) + \eta R + \eta\varepsilon, \end{aligned}$$

where $\varepsilon > 0$ is arbitrary and $\eta \in (0, \eta_2(\sigma))$ with $\eta_2(\sigma) \le \eta_1(\sigma)$, $\eta_2(\sigma) > 0$ is sufficiently small.

Denote

$$\Psi(s) = y(s) - \int_{t_0}^{s} f(y(t),t)dt, s \in [t_0,t_1],$$

the function $\Psi : [t_0,t_1] \to R^n$ is of bounded Φ-variation and continuous from the left on $(t_0,t_1]$. By Theorem 1.2, Theorem 1.11 and Theorem 1.17 in [5], we have

$$\begin{aligned} & V(\sigma+\eta, y(\sigma+\eta)) - V(\sigma, x(\sigma)) \\ & \le L\Phi(\|y(\sigma+\eta) - y(\sigma) - \textstyle\int_{\sigma}^{\sigma+\eta} f(y(t),t)dt\| + \|\textstyle\int_{\sigma}^{\sigma+\eta}[f(y(t),t) - f(x(t),t)]dt\|) \\ & \quad +\eta R + \eta\varepsilon \le L\chi(a')\{\Phi(\|y(\sigma+\eta) - y(\sigma) - \textstyle\int_{\sigma}^{\sigma+\eta} f(y(t),t)dt\|) \\ & \quad +\Phi(\|\textstyle\int_{\sigma}^{\sigma+\eta}[f(y(t),t) - f(x(t),t)]dt\|)\} + \eta R + \eta\varepsilon \end{aligned}$$

$$\leq L\chi(a')V_\Phi((y(s) - \int_\sigma^s f(y(t),t)dt); [\sigma, \sigma+\eta]) + \eta R + \eta\varepsilon$$

$$+L\chi(a')\Phi(\| \int_\sigma^{\sigma+\eta}[f(y(t),t) - f(x(t),t)]dt\|)$$

$$\leq NV_\Phi(\Psi; [\sigma, \sigma+\eta]) + \eta R + \eta\varepsilon + N\Phi(\| \int_\sigma^{\sigma+\eta}[f(y(t),t) - f(x(t),t)]dt\|)$$

$$\leq N(V_\Phi(\Psi; [, \sigma+\eta]) - V_\Phi(\Psi; [, \sigma])) + \eta R + \eta\varepsilon$$

$$+N\Phi(\| \int_\sigma^{\sigma+\eta}[f(y(t),t) - f(x(t),t)]dt\|) \tag{4}$$

for every $\eta \in (0, \eta_2(\sigma))$.

Since $f \in \mathcal{V}_\Phi(G, h, \omega)$ and by Definition 2.1, there exists a positive function $\delta(\tau) : [\sigma, \sigma + \eta] \to (0, +\infty)$ for $\varepsilon > 0$ such that any $\delta(\tau)-$fine partition $D = \{(\tau_j; [\alpha_{j-1}, \alpha_j]), j = 1, 2, \cdots, m\}$ on $[\sigma, \sigma+\eta]$, we have

$$\| \int_\sigma^{\sigma+\eta}[f(y(t),t) - f(x(t),t)]dt\|$$

$$\leq \| \int_\sigma^{\sigma+\eta}[(y(t),t) - f(x(t),t)]dt - \sum_{j=1}^m [f(y(\tau_j), \alpha_j) - f(x(\tau_j), \alpha_j)$$

$$-f(y(\tau_j), \alpha_{j-1}) + f(x(\tau_j), \alpha_{j-1})]\|$$

$$+\| \sum_{j=1}^m [f(y(\tau_j), \alpha_j) - f(x(\tau_j), \alpha_j) - f(y(\tau_j), \alpha_{j-1}) + f(x(\tau_j), \alpha_{j-1})]\|$$

$$< \frac{\varepsilon}{2} + \sum_{j=1}^m \omega(\| y(\tau_j) - x(\tau_j) \|)\Phi(| h(\alpha_j) - h(\alpha_{j-1}) |). \tag{5}$$

By Theorem 1.3 and Theorem 1.17 in [5], we have

$$\sum_{j=1}^m \omega(\| y(\tau_j) - x(\tau_j) \|)\Phi(| h(\alpha_j) - h(\alpha_{j-1}) |)$$

$$= \sum_{j=1}^m \omega(\| y(\tau_j) - x(\tau_j) \|)V_\Phi(h; [\alpha_{j-1}, \alpha_j])$$

$$\leq \sum_{j=1}^m \omega(\| y(\tau_j) - x(\tau_j) \|)[V_\Phi(h; [\sigma, \alpha_j]) - V_\Phi(h; [\sigma, \alpha_{j-1}])].$$

Denote $U(t) = V_\Phi(h; [\sigma, t])$ for $\sigma \leq t \leq \sigma+\eta$. By definitions of $\Phi(u)$ and h, $U(t)$is a function of non-navigate and nondecreasing on $[\sigma, \sigma+\eta]$ and continuous from the left on $(\sigma, \sigma+\eta]$, then by (5) we have

$$\sum_{j=1}^m \omega(\| y(\tau_j) - x(\tau_j) \|)\Phi(| h(\alpha_j) - h(\alpha_{j-1}) |)$$

$$\leq \sum_{j=1}^m \omega(\| y(\tau_j) - x(\tau_j) \|)(U(\alpha_j) - U(\alpha_{j-1}))$$

$$\leq\| \sum_{j=1}^m \omega(\| y(\tau_j) - x(\tau_j) \|)(U(\alpha_j) - U(\alpha_{j-1})) - \int_\sigma^{\sigma+\eta} \omega(\| y(t) - x(t) \|)dU(t) \|$$

$$+ \int_\sigma^{\sigma+\eta} \omega(\| y(t) - x(t) \|)dU(t) < \frac{\varepsilon}{2} + \int_\sigma^{\sigma+\eta} \omega(\| y(t) - x(t) \|)dU(t). \tag{6}$$

By (5), (6), Theorem 1.16 in [8] and arbitrary $\varepsilon > 0$, we have

$$\| \int_\sigma^{\sigma+\eta}[f(y(t),t) - f(x(t),t)]dt\| \leq \int_\sigma^{\sigma+\eta} \omega(\| y(t) - x(t) \|)dU(t)$$

$$= \lim_{\alpha\to 0+}[\int_\sigma^{\sigma+\alpha} \omega(\| y(t) - x(t) \|)dU(t) + \int_{\sigma+\alpha}^{\sigma+\eta} \omega(\| y(t) - x(t) \|)dU(t)]$$

$$= \omega(\| y(\sigma) - x(\sigma) \|)(U(\sigma+) - U(\sigma)) + \lim_{\alpha\to 0+} \int_{\sigma+\alpha}^{\sigma+\eta} \omega(\| y(t) - x(t) \|)dU(t)$$

$= \lim_{\alpha\to 0+} \int_{\sigma+\alpha}^{\sigma+\eta} \omega(\| y(t) - x(t) \|) dU(t)$

$\leq \sup_{\rho\in[\sigma,\sigma+\eta]} \omega(\| y(\rho) - x(\rho) \|) \lim_{\alpha\to 0+}(U(\sigma+\eta) - U(\sigma+\alpha))$

$= \sup_{\rho\in[\sigma,\sigma+\eta]} \omega(\| y(\rho) - x(\rho) \|)(U(\sigma+\eta) - U(\sigma+)).$ (7)

Let $\rho \in [\sigma, \sigma+\eta_2(\sigma)]$, we have $y(\rho) - x(\rho) = y(\rho) - y(\sigma) - \int_\sigma^\rho f(x(t), t)dt$, and therefore

$$\lim_{\rho\to\sigma+}(y(\rho) - x(\rho)) = y(\sigma+) - y(\sigma) - \lim_{\rho\to\sigma+}(f(x(\sigma), \rho) - f(x(\sigma), \sigma))$$

$$= y(\sigma+) - y(\sigma) - (f(x(\sigma), \sigma+) - f(x(\sigma), \sigma)) = \Psi(\sigma+) - \Psi(\sigma),$$

and also

$$\lim_{\rho\to\sigma+} \|y(\rho) - x(\rho)\| = \|\Psi(\sigma+) - \Psi(\sigma)\|. \tag{8}$$

For every $\varepsilon > 0$ denote

$$\beta = \frac{\varepsilon}{N(U(t_1) - U(t_0) + 1)} > 0 \tag{9}$$

and assume that $r = r(\beta) > 0$ is such that $\omega(r) < \beta$. Further, we choice $\alpha \in (0, \frac{r}{2})$.

Since (8) holds, there is an $\eta_3(\sigma) \in (0, \eta_2(\sigma))$ such that

$$\|y(\rho) - x(\rho)\| \leq \|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha,$$

and also

$$\omega(\|y(\rho) - x(\rho)\|) \leq \omega(\|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha) \tag{10}$$

for $\rho \in (\sigma, \sigma+\eta_3(\sigma))$.

Denote $P(\beta) = \{\sigma \in [t_0, t_1]; \|\Psi(\sigma+) - \Psi(\sigma)\| \geq \frac{r}{2}\}$, since Ψ is of bounded Φ-variation on $[t_0, t_1]$, the set $P(\beta)$ is finite and we denote by $n(\beta)$ the number of elements of $P(\beta)$.

If $\sigma \in [t_0, t_1] \setminus P(\beta)$ and $\rho \in (\sigma, \sigma+\eta_3(\sigma))$ then by (10) we have

$$\omega(\|y(\rho) - x(\rho)\|) \leq \omega(\|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha) \leq \omega(\frac{r}{2} + \alpha) < \omega(\frac{r}{2} + \frac{r}{2}) = \omega(r) < \beta,$$

and by (7) also

$$\| \int_\sigma^{\sigma+\eta} [f(y(t), t) - f(x(t), t)]dt \| \leq \beta(U(\sigma+\eta) - U(\sigma+)) \tag{11}$$

whenever $\eta \in (0, \eta_3(\sigma))$.

If $\sigma \in [t_0, t_1] \bigcap P(\beta)$ then there exists $\eta_4(\sigma) \in (0, \eta_3(\sigma))$ such that for $\eta \in (0, \eta_4(\sigma))$ we have

$$U(\sigma+\eta) - U(\sigma+) = | U(\sigma+\eta) - U(\sigma+) | < \frac{\beta}{(n(\beta)+1)\omega(\|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha)}$$

and $(\sigma, \sigma+\eta_4(\sigma)) \bigcap P(\beta) = \emptyset$. Hence (7) and (10) yield

$$\| \int_\sigma^{\sigma+\eta} [f(y(t), t) - f(x(t), t)]dt\|$$

$$\leq \omega(\|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha)\frac{\beta}{(n(\beta)+1)\omega(\|\Psi(\sigma+) - \Psi(\sigma)\| + \alpha)} = \frac{\beta}{n(\beta)+1} \tag{12}$$

for every $\eta \in (\sigma, \sigma + \eta_4(\sigma))$.

Denote $\widetilde{U}_\alpha(t) = \frac{\beta}{n(\beta)+1}\sum_{\sigma\in P(\beta)} I_\sigma(t)$ for $t \in [t_0, t_1]$ where $I_\sigma(t) = 0$ for $t \leq \sigma$ and $I_\sigma(t) = 1$ for $t > \sigma$. The function $\widetilde{U}_\alpha : [t_0, t_1] \to R$ is nondecreasing and continuous from the left and

$$\widetilde{U}_\alpha(t_1) - \widetilde{U}_\alpha(t_0) = \frac{\beta}{n(\beta)+1}n(\beta) < \beta. \tag{13}$$

The points of discontinuity of the function $\widetilde{U}_\alpha$ are clearly only the points belonging to $P(\beta)$, and for $t \in P(\beta)$ we have

$$\widetilde{U}_\alpha(t+) - \widetilde{U}_\alpha(t) = \frac{\beta}{n(\beta)+1}, \quad t \in [t_0, t_1].$$

Using the function $\widetilde{U}_\alpha$, we can set $U_\alpha(t) = \beta U_c(t) + \widetilde{U}_\alpha(t)$ for $t \in [t_0, t_1]$ where by U_c denote the continuous part of the function U, then the function U_α is nondecreasing and continuous from the left on $[t_0, t_1]$ and by (9) and (13) we obtain

$$\begin{aligned} U_\alpha(t_1) - U_\alpha(t_0) &= \beta(U_c(t_1) - U_c(t_0)) + \widetilde{U}_\alpha(t_1) - \widetilde{U}_\alpha(t_0) \\ &< \beta[U(t_1) - U(t_0) + 1] = \frac{\varepsilon}{N}. \end{aligned} \tag{14}$$

If $\sigma \in [t_0, t_1] \setminus P(\beta)$ then set $\delta(\sigma) = \eta_3(\sigma) > 0$ and if $\sigma \in [t_0, t_1] \bigcap P(\beta)$ then set $\delta(\sigma) = \eta_4(\sigma) > 0$. By (11), (12) and definition of U_α we obtain the inequality

$$\|\int_\sigma^{\sigma+\eta}[f(y(t), t) - f(x(t), t)]dt\| \leq U_\alpha(\sigma + \eta) - U_\alpha(\sigma)$$

for $\sigma \in [t_0, t_1], \eta \in [0, , \delta(\sigma)]$, then by Theorem 1.3 and Theorem 1.17 in [5], we have $\Phi(\|\int_\sigma^{\sigma+\eta}[f(y(t), t) - f(x(t), t)]\|) \leq \Phi(U_\alpha(\sigma+\eta) - U_\alpha(\sigma)) = V_\Phi(U_\alpha; [\sigma, \sigma+\eta]) \leq V_\Phi(U_\alpha; [t_0, \sigma+\eta]) - V_\Phi(U_\alpha; [t_0, \sigma])$, therefor by (4) we have

$$\begin{aligned} &V(\sigma + \eta, y(\sigma + \eta)) - V(\sigma, x(\sigma)) \\ &\leq N(V_\Phi(\Psi; [t_0, \sigma + \eta]) - V_\Phi(\Psi; [t_0, \sigma]) + V_\Phi(U_\alpha; [t_0, \sigma + \eta]) - V_\Phi(U_\alpha; [t_0, \sigma])) + \eta R + \eta\varepsilon \\ &= G(\sigma+\eta) - G(\sigma), \end{aligned} \tag{15}$$

for $\sigma \in [t_0, t_1], \eta \in [0, \delta(\sigma)]$ where $G(t) = NV_\Phi(\Psi; [t_0, t]) + R(t-t_0) + \varepsilon(t-t_0) + NV_\Phi(U_\alpha; [t_0, t])$, $G$is of bounded Φ-variation on $[t_0, t_1]$ and continuous from the left on $(t_0, t_1]$. By (14), (15) and Lemma 2.2, we obtain

$$\begin{aligned} &V(t_1, y(t_1)) - V(t_0, y(t_0)) \leq G(t_1) - G(t_0) \\ &= NV_\Phi(\Psi; [t_0, t_1]) + R(t_1 - t_0) + \varepsilon(t_1 - t_0) + NV_\Phi(U_\alpha; [t_0, t_1]) \\ &< NV_\Phi(\Psi; [t_0, t_1]) + R(t_1 - t_0) + \varepsilon(t_1 - t_0) + N\Phi(\tfrac{\varepsilon}{N}), \end{aligned}$$

Since ε can be arbitrary small, we obtain the result given in (3) from this inequality.

Theorem 3.2. Assume that $V : [0, +\infty) \times \bar{B}_a \to R, 0 < a < c$ is such that for every $x \in \bar{B}_a = \{y \in R^n : \|y\| \leq a\}$ the function $V(\cdot, x)$ is continuous from the left and

(1) Assume that the function $V(t,x)$ is positive definite, i.e. there exists a continuous and increasing real function $\upsilon:[0,+\infty)\to R$ Such that $\upsilon(\rho)=0\Longleftrightarrow\rho=0$;

(2) for all $(t,x)\in[0,+\infty)\times\bar{B}_a$, we have

$$V(t,x)\geq\upsilon(\Phi(\|x\|)) \tag{16}$$

and

$$V(t,0)=0. \tag{17}$$

For any $x,y\in\bar{B}_a$ and $L>0$ is a constant, we have

$$|V(t,x)-V(t,y)|\leq L\Phi(\|x-y\|). \tag{18}$$

If the function $V(t,x(t))$ is nonincreasing and of bounded Φ-variation along every solution x of $(*)$ then the solution $x\equiv 0$ of $(*)$ is Φ-variationally stable.

Proof. Since we assume that the function $V(t,x(t))$ is nonincrease and of bounded Φ-variation whenever $x:[\alpha,\beta]\to R^n$ is a solution of $(*)$, we have

$$\lim_{\eta\to 0^+}\sup\frac{V(t+\eta,x(t+\eta))-V(t,x(t))}{\eta}\leq 0 \tag{19}$$

for $t\in[\alpha,\beta]$.

Let $\varepsilon>0$ be given and let $y:[t_0,t_1]\to R^n$, $0\leq t_0<t_1<+\infty$ be of bounded Φ-variation on $[t_0,t_1]$ and continuous from the left in $(t_0,t_1]$. Since the function V satisfies the assumptions (2) of Theorem 3.1 with $H\equiv 0$, We obtain by (3), (17) and (18) the inequality

$$\begin{aligned} V(r,y(r)) &\leq V(t_0,y(t_0))+NV_\Phi((y(s)-\textstyle\int_{t_0}^s f(y(t),t)dt);[t_0,r]) \\ &\leq L\Phi(\|y(t_0)\|)+NV_\Phi((y(s)-\textstyle\int_{t_0}^s f(y(t),t)dt);[t_0,r]) \\ &\leq N\Phi(\|y(t_0)\|)+NV_\Phi((y(s)-\textstyle\int_{t_0}^s f(y(t),t)dt);[t_0,r]) \end{aligned} \tag{20}$$

holds for every $r\in[t_0,t_1]$.

Define $\alpha(\varepsilon)=\inf_{r\leq\varepsilon}\upsilon(r)$, then $\alpha(\varepsilon)>0$ for $\varepsilon>0$ and $\lim_{\varepsilon\to 0+}\alpha(\varepsilon)=0$. Further choose $\delta(\varepsilon)>0$ such that $2N\delta(\varepsilon)<\alpha(\varepsilon)$. If function y satisfying $\Phi(\|y(t_0)\|)<\delta(\varepsilon)$ and $V_\Phi((y(s)-\int_{t_0}^s f(y(t),t)dt);[t_0,r])<\delta(\varepsilon)$, then by (20), we have

$$V(r,y(r))\leq 2N\delta(\varepsilon)<\alpha(\varepsilon), r\in[t_0,t_1]. \tag{21}$$

If there exists a $\widetilde{t}\in[t_0,t_1]$ such that $\Phi(\|y(\widetilde{t})\|)\geq\varepsilon$ and by (16) we get the inequality $V(\widetilde{t},y(\widetilde{t}))\geq\upsilon(\Phi(\|y(\widetilde{t})\|))\geq\inf_{r\leq\varepsilon}\upsilon(r)=\alpha(\varepsilon)$, which contradicts (21). Hence $\Phi(\|y(t)\|)<\varepsilon$ for all $t\in(t_0,t_1]$ and by Definition 2.7 the solution $x\equiv 0$ of $(*)$ is Φ-variationally stable.

Theorem 3.3. Let $V:[0,+\infty)\times\bar{B}_a\to R$ be a function with the properties given in Theorem 3.2. If for every solution $x:[t_0,t_1]\to\bar{B}_a$ of bounded Φ-variation of $(*)$ the inequality

$$\lim_{\eta\to 0+}\sup\frac{V(t+\eta,x(t+\eta))-V(t,x(t))}{\eta}\leq -H(x(t)) \tag{22}$$

holds for every $t\in[t_0,t_1]$, where $H:R^n\to R$ is continuous, $H(0)=0, H(x)>0$ for $x\neq 0$, then the solution $x\equiv 0$ of $(*)$ is Φ-variationally-asymptotically stable.

Proof. From (22) it is clear that the function $V(t, x(t))$ is of bounded Φ-variation and nonincreasing along every solution $x(t)$ of $(*)$ and therefore by Theorem 3.2, the solution $x \equiv 0$ is Φ-variationally stable.

It remains to show that the solution $x \equiv 0$ is Φ-variationally attracting. From the Φ-variational stability of the solution $x \equiv 0$ of $(*)$, there is a $\delta_0 \in (0, \Phi(a))$ such that if $y : [t_0, t_1] \to R^n$ is of bounded Φ-variation on $[t_0, t_1]$ where $0 \le t_0 < t_1 < +\infty, y$ is continuous from the left on $(t_0, t_1]$ and such that $\Phi(\|y(t_0)\|) < \delta_0$ and $V_\Phi((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, t_1]) < \delta_0$, then $\Phi(\|y(t)\|) < \Phi(a)$ for $t \in [t_0, t_1]$, by the definition of $\Phi(u)$, we have $\|y(t)\| < a$, i.e for every $t \in (t_0,\ t_1]$ we obtain $y(t) \in \bar{B}_a$. Let $\varepsilon > 0$ be arbitrary, from the Φ-variational stability of the solution $x \equiv 0$ we obtain that there is a $\delta(\varepsilon) > 0$ such that for every $y : [t_0, t_1] \to R^n$ of bounded Φ-variation on $[t_0, t_1]$ where $0 \le t_0 < t_1 < +\infty$, y is continuous from the left on $(t_0, t_1]$ and such that $\Phi(\|y(t_0)\|) < \delta(\varepsilon)$ and $V_\Phi((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, t_1]) < \delta(\varepsilon)$, we have $\Phi(\|y(t)\|) < \varepsilon$ for $t \in [t_0, t_1]$. Define $\gamma(\varepsilon) = \min(\delta_0, \delta(\varepsilon))$ and $T(\varepsilon) = -N\frac{\delta_0+\gamma(\varepsilon)}{R} > 0$ where $R = \sup\{-H(x) : \gamma(\varepsilon) \le \Phi(\|x\|) < \varepsilon\} = -\inf\{H(x) : \gamma(\varepsilon) \le \Phi(\|x\|) < \varepsilon\} < 0$ and assume that $y : [t_0, t_1] \to R^n$ of bounded Φ-variation on $[t_0, t_1]$ where $0 \le t_0 < t_1 < +\infty, y$ is continuous from the left on $(t_0, t_1]$ and such that $\Phi(\|y(t_0)\|) < \delta_0$ and

$$V_\Phi((y(s) - \int_{t_0}^s f(y(t), t)dt); [t_0, t_1]) < \gamma(\varepsilon). \tag{23}$$

Assume that $T(\varepsilon) < t_1 - t_0, i.e. t_0 + T(\varepsilon) < t_1$. We show that there exists a $t^* \in [t_0, t_0 + T(\varepsilon)]$ such that $\Phi(\|y(t^*)\|) < \gamma(\varepsilon)$. Assume the contrary, i.e. $\Phi(\|y(s)\|) \ge \gamma(\varepsilon)$ for every $s \in [t_0, t_0 + T(\varepsilon)]$. Theorem 3.1 yields

$$\begin{aligned}
&V(t_0 + T(\varepsilon), y(t_0 + T(\varepsilon))) - V(t_0, y(t_0)) \\
&\le NV_\Phi((y(s) - \textstyle\int_{t_0}^s f(y(t), t)dt); [t_0, t_0 + T(\varepsilon)]) + RT(\varepsilon) \\
&< N\gamma(\varepsilon) + R\frac{-N(\delta_0+\gamma(\varepsilon))}{R} = -N\delta_0.
\end{aligned}$$

Hence $V(t_0+T(\varepsilon), y(t_0+T(\varepsilon))) \le V(t_0, y(t_0)) - N\delta_0 \le L\Phi(\|y(t_0)\|) - N\delta_0 \le N\Phi(\|y(t_0)\|) - N\delta_0 < (N - N)\delta_0 = 0$, and this contradicts the inequality $V(t_0 + T(\varepsilon), y(t_0 + T(\varepsilon))) \ge \upsilon(\Phi(\|y(t_0 + T(\varepsilon))\|)) \ge \upsilon(\gamma(\varepsilon)) > 0$. Hence necessarily there is a $t^* \in [t_0, t_0 + T(\varepsilon)]$ such that $\Phi(\|y(t^*)\|) < \gamma(\varepsilon)$. And by (23), we have $\Phi(\|y(t)\|) < \varepsilon$ for $t \in [t^*, t_1]$. Consequently, $\Phi(\|y(t)\|) < \varepsilon$ for $t > t_0 + T(\varepsilon)$. Therefore, the solution $x \equiv 0$ is a Φ-variationally attracting solution of $(*)$.

Remark 3.1. If the function $\Phi(u)$ is defined in last section such that $0 < \frac{\Phi(u)}{u} < +\infty$ then by Theorem 1.15 in paper [5], we have $BV_\Phi[\alpha, \beta] = BV[\alpha, \beta]$ where $BV_\Phi[\alpha, \beta]$ and $BV[\alpha, \beta]$ denote the classes of the functions bounded Φ-variation and the functions bounded variation on $[\alpha, \beta]$ respectively in usual sense. Hence, the main results of this paper are equivalent to the results of variational stability in [4].

If $\lim_{u\to 0^+} \frac{\Phi(u)}{u} = 0$ then by Theorem 1.15 in [5], we have $BV[\alpha, \beta] \subset BV_\Phi[\alpha, \beta]$. Such as if $\Phi(u) = u^p(1 < p < +\infty)$ we have $\lim_{u\to 0^+} \frac{\Phi(u)}{u} = \frac{u^p}{u} = 0$. Therefore these results are essential generalization of variation stability of bounded variation solution to discontinuous system in [4].

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Note on right circulant matrices with Pell and Pell-Lucas sequences

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Abstract In this paper, we shall construct right circulant matrices from the Pell and Pell-Lucas sequences. Furthermore, we derive the eigenvalues and Euclidean norm of these matrices. We also give some bounds on their spectral norms.

Keywords Determinant, eigenvalue, Euclidean norm, Pell sequence, Pell-Lucas sequence, right circulant matrix, spectral norm.
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§1. Introduction

The Pell and Pell-Lucas sequence satisfy the following recurrence relations, respectively:

$$P_n = 2P_{n-1} + P_{n-2}, P_0 = 0, P_1 = 1;$$
$$Q_n = 2Q_{n-1} + Q_{n-2}, Q_0 = Q_1 = 2.$$

Their n^{th} are given by

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}, \quad (1)$$
$$Q_n = \alpha^n + \beta^n. \quad (2)$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

Bozkurt [1] constructed the right circulant matrices with Pell and Pell-Lucas sequences. These matrices take the following froms, resepectively:

$$RCIRC_n(\vec{P}) = \begin{pmatrix} P_0 & P_1 & P_2 & ... & P_{n-2} & P_{n-1} \\ P_{n-1} & P_0 & P_1 & ... & P_{n-3} & P_{n-2} \\ P_{n-2} & P_{n-1} & P_0 & ... & P_{n-4} & P_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ P_2 & P_3 & P_4 & ... & P_0 & P_1 \\ P_1 & P_2 & P_3 & ... & P_{n-1} & P_0 \end{pmatrix} \quad (3)$$

$$RCIRC_n(\vec{Q}) = \begin{pmatrix} Q_0 & Q_1 & Q_2 & \dots & Q_{n-2} & Q_{n-1} \\ Q_{n-1} & Q_0 & Q_1 & \dots & Q_{n-3} & Q_{n-2} \\ Q_{n-2} & Q_{n-1} & Q_0 & \dots & Q_{n-4} & Q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_2 & Q_3 & Q_4 & \dots & Q_0 & Q_1 \\ Q_1 & Q_2 & Q_3 & \dots & Q_{n-1} & Q_0 \end{pmatrix} \quad (4)$$

He presented the determinants and the ineverses of these matrices.

§2. Preliminary results

The following will be used to prove some of the main results.

Lemma 2.1.

$$\sum_{k=0}^{n-1} \left[\alpha^k - \beta^k\right] \omega^{-mk} = \frac{\alpha^n - \beta^n + (\alpha^{n-1} - \beta^{n-1})\omega^{-m} - (\alpha - \beta)}{\omega^{-2m} + 2\omega^{-m} - 1}, \quad (5)$$

where $\omega = e^{2\pi i/n}$.

Proof.

$$\begin{aligned} \sum_{k=0}^{n-1} \left[\alpha^k - \beta^k\right] \omega^{-mk} &= \frac{1-\alpha^n}{1-\alpha\omega^{-m}} + \frac{1-\beta^n}{1-\beta\omega^{-m}} \\ &= \frac{(1-\alpha^n)(1-\beta\omega^{-m}) - (1-\beta^n)(1-\alpha\omega^{-m})}{1 - 2\omega^{-m} - \omega^{-2m}} \\ &= \frac{\alpha^n - \beta^n + (\alpha^{n-1} - \beta^{n-1})\omega^{-m} - (\alpha - \beta)}{\omega^{-2m} + 2\omega^{-m} - 1}. \end{aligned}$$

Lemma 2.2.

$$\sum_{k=0}^{n-1} \left[\alpha^k - \beta^k\right]^2 = \frac{Q_{2n} + Q_{2n-2}}{4} - (-1)^n. \quad (6)$$

Proof.

$$\begin{aligned} \sum_{k=0}^{n-1} \left[\alpha^k - \beta^k\right]^2 &= \sum_{k=0}^{n-1} \left[\alpha^{2k} - 2(\alpha\beta)^k + \beta^{2k}\right] \\ &= \frac{1-\alpha^{2n}}{1-\alpha^2} + \frac{1-\beta^{2n}}{1-\beta^2} - 2\left(\frac{1-(-1)^n}{2}\right) \\ &= \frac{(1-\alpha^{2n})(1-\beta^2) + (1-\beta^{2n})(1-\alpha^2)}{(1-\alpha^2)(1-\beta^2)} + (-1)^n - 1 \\ &= \frac{-4 - (\alpha^{2n} + \beta^{2n}) - (\alpha^{2n-2} + \beta^{2n-2})}{-4} + (-1)^n - 1 \\ &= \frac{Q_{2n} + Q_{2n-2}}{4} - (-1)^n. \end{aligned}$$

Lemma 2.3.

$$\sum_{k=0}^{n-1} \left[\alpha^k + \beta^k\right]^2 = \frac{Q_{2n} + Q_{2n-2}}{4} + 2 - (-1)^n. \quad (7)$$

Proof.

$$\begin{aligned}\sum_{k=0}^{n-1}\left[\alpha^k+\beta^k\right]^2 &= \sum_{k=0}^{n-1}\left[\alpha^{2k}+2(\alpha\beta)^k+\beta^{2k}\right]\\ &= \frac{1-\alpha^{2n}}{1-\alpha^2}+\frac{1-\beta^{2n}}{1-\beta^2}+2\left(\frac{1-(-1)^n}{2}\right)\\ &= \frac{(1-\alpha^{2n})(1-\beta^2)+(1-\beta^{2n})(1-\alpha^2)}{(1-\alpha^2)(1-\beta^2)}+1-(-1)^n\\ &= \frac{-4-(\alpha^{2n}+\beta^{2n})-(\alpha^{2n-2}+\beta^{2n-2})}{-4}+1-(-1)^n\\ &= \frac{Q_{2n}+Q_{2n-2}}{4}+2-(-1)^n.\end{aligned}$$

For the rest of the paper, we will use $|A|$, $||A||_E$ and $||A||_2$ to denote the determinant, euiclidean norm and spectral norm of matrix A, respectively.

§3. Main results

Theorem 3.1. The eigenvalues of $RCIRC_n(\vec{P})$ are given by

$$\lambda_m=\frac{(P_{n-1}-1)\omega^{-m}+P_n}{\omega^{-2m}+2\omega^{-m}-1}. \tag{8}$$

Proof.

$$\begin{aligned}\lambda_m &= \sum_{k=0}^{n-1}P_k\omega^{-mk}\\ &= \sum_{k=0}^{n-1}\frac{[\alpha^k-\beta^k]\omega^{-mk}}{2\sqrt{2}}\\ &= \frac{\alpha^n-\beta^n+(\alpha^{n-1}-\beta^{n-1})\omega^{-m}-(\alpha-\beta)}{2\sqrt{2}(\omega^{-2m}+2\omega^{-m}-1)}\\ &= \frac{(P_{n-1}-1)\omega^{-m}+P_n}{\omega^{-2m}+2\omega^{-m}-1}.\end{aligned}$$

Theorem 3.2. The eigenvalues of $RCIRC_n(\vec{Q})$ are given by

$$\mu_m=\frac{Q_{n+1}+2(\omega^{-m}-1)}{\omega^{-2m}+2\omega^{-m}-1}. \tag{9}$$

Proof.

$$\begin{aligned}
\mu_m &= \sum_{k=0}^{n-1} Q_k \omega^{-mk} \\
&= \sum_{k=0}^{n-1} [\alpha^k + \beta^k] \omega^{-mk} \\
&= \frac{1-\alpha^n}{1-\alpha\omega^{-m}} + \frac{1-\beta^n}{1-\beta\omega^{-m}} \\
&= \frac{(1-\alpha^n)(1-\beta\omega^{-m}) + (1-\beta^n)(1-\alpha\omega^{-m})}{1-2\omega^{-m}-\omega^{-2m}} \\
&= \frac{2-Q_n-2\omega^{-m}-Q_{n-1}\omega^{-m}}{1-2\omega^{-m}-\omega^{-2m}} \\
&= \frac{Q_{n+1}+2(\omega^{-m}-1)}{\omega^{-2m}+2\omega^{-m}-1}.
\end{aligned}$$

Theorem 3.3.

$$\left\|RCIRC_n(\vec{P})\right\|_E = \frac{\sqrt{2n(Q_{2n}+Q_{2n-2})-(-1)^n 64}}{8}. \tag{10}$$

Proof.

$$\begin{aligned}
\left\|RCIRC_n(\vec{P})\right\|_E &= \sqrt{n\sum_{k=0}^{n-1}\left[\frac{\alpha^k-\beta^k}{2\sqrt{2}}\right]^2} \\
&= \sqrt{\frac{n}{8}\sum_{k=0}^{n-1}[\alpha^k-\beta^k]^2} \\
&= \sqrt{\frac{n}{8}\left[\frac{Q_{2n}+Q_{2n-2}}{4}-(-1)^n\right]} \\
&= \frac{\sqrt{2n(Q_{2n}+Q_{2n-2})-(-1)^n 64}}{8}.
\end{aligned}$$

Theorem 3.4.

$$\left\|RCIRC_n(\vec{Q})\right\|_E = \frac{\sqrt{n(Q_{2n}+Q_{2n-2})+8-(-1)^n 4}}{2}. \tag{11}$$

Proof.

$$\begin{aligned}
\left\|RCIRC_n(\vec{Q})\right\|_E &= \sqrt{n\sum_{k=0}^{n-1}[\alpha^k+\beta^k]^2} \\
&= \sqrt{n\left[\frac{Q_{2n}+Q_{2n-2}}{4}+2-(-1)^n\right]} \\
&= \frac{\sqrt{n(Q_{2n}+Q_{2n-2})+8-(-1)^n 4}}{2}.
\end{aligned}$$

Theorem 3.5.

$$\frac{P_{n-1}-P_n-1}{4} \leq \left\|RCIRC_n(\vec{P})\right\|_2 \leq \frac{P_{n+1}-1}{2}. \tag{12}$$

Proof. Note that

$$\left\|RCIRC_n(\vec{P})\right\|_2 = \max\{|\lambda_m|\},$$

hence, we have the following inequality

$$\begin{aligned} \frac{|(P_{n-1}-1)\omega^{-m}| - P_n}{|\omega^{-2m}+2\omega^{-m}-1|} &\leq \left\|RCIRC_n(\vec{P})\right\|_2 \leq \frac{|(P_{n-1}-1)\omega^{-m}| + P_n}{|\omega^{-2m}+2\omega^{-m}-1|} \\ \frac{P_{n-1}-P_n-1}{4} &\leq \left\|RCIRC_n(\vec{P})\right\|_2 \leq \frac{P_{n-1}-1+P_n}{2} \\ \frac{P_{n-1}-P_n-1}{4} &\leq \left\|RCIRC_n(\vec{P})\right\|_2 \leq \frac{P_{n+1}-1}{2}. \end{aligned}$$

Theorem 3.6.

$$\frac{Q_{n-1}}{4} \leq \left\|RCIRC_n(\vec{Q})\right\|_2 \leq \frac{Q_{n+1}+4}{2}. \tag{13}$$

Proof. Similar as the previous theorem 3.5.

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Evaluation of a summation formula confederated with contiguous relation

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Abstract The main aim of the present paper is to evaluate a summation formula confederated with contiguous relation and recurrence relation.

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§1. Introduction and preliminaries

Definition 1.1. Generalized Gaussian hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{lc} a_1, a_2, \cdots, a_A \ ; & \\ & z \\ b_1, b_2, \cdots, b_B \ ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}, $$

or

$$ {}_AF_B\left[\begin{array}{lc} (a_A) \ ; & \\ & z \\ (b_B) \ ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{lc} (a_j)_{j=1}^A \ ; & \\ & z \\ (b_j)_{j=1}^B \ ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!}, \tag{1} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers.

Definition 1.2. Contiguous relation [E. D. p. 51(10), Andrews p. 363(9. 16)] is defined as follows:

$$ (a-b)\ {}_2F_1\left[\begin{array}{lc} a,\ b\ ; & \\ c \ ; & z \end{array}\right] = a\ {}_2F_1\left[\begin{array}{lc} a+1,\ b\ ; & \\ c \ ; & z \end{array}\right] - b\ {}_2F_1\left[\begin{array}{lc} a,\ b+1\ ; & \\ c \ ; & z \end{array}\right]. \tag{2} $$

Definition 1.3. Recurrence relation of gamma function is defined as follows:

$$ \Gamma(z+1) = z\ \Gamma(z). \tag{3} $$

Definition 1.4. Legendre duplication formula [Bells & Wong p. 26(2.3.1)] is defined as follows:

$$\sqrt{\pi}\ \Gamma(2z) = 2^{(2z-1)}\ \Gamma(z)\ \Gamma\left(z+\frac{1}{2}\right), \tag{4}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{5}$$

$$= \frac{2^{(a-1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\Gamma(a)}. \tag{6}$$

Definition 1.5. Bailey summation theorem [Prud, p. 491(7.3.7.8)] is defined as follows:

$${}_2F_1\left[\begin{matrix} a, 1-a & ; & \frac{1}{2} \\ c & ; & \end{matrix}\right] = \frac{\Gamma(\frac{c}{2})\ \Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} = \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-1}\ \Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})}. \tag{7}$$

§2. Main results of summation formulae

$${}_2F_1\left[\begin{matrix} a\ ,\ -a-49 & ; & \frac{1}{2} \\ c & ; & \end{matrix}\right]$$

$$= \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c+49}}\Bigg[\frac{4001594832981264113679943021363200000 0}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-74697042588755634816870901715435520000a}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{45701551565700312458589664488136704000a^2}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-12557326585358016296749495447708262400a^3}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1607017301962727354472466577244395520a^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-66334092838839671097859298211598848a^5 - 4173963485546119223032377440607360a^6}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{354268334898336692529886398612096a^7 + 6493428631831030022245150741472a^8}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-716075598615688398506581518048a^9 - 13409906223325457338650846040a^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{695228979926866432431858936a^{11} + 20318017073982774702824162a^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1810815297559131274363 68a^{13} - 1275819888730941598282 0a^{14}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1138097578781122295384a^{15}+19481152877787037182a^{16}+46874133673870272a^{17}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{257645168805200a^{18}-1896116954424a^{19}-31420958338a^{20}-136837008a^{21}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-32980a^{22}+1176a^{23}+2a^{24}+10287372370222143901447317752905728000c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-14395305433770223515104403804232089600ac}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{6978518665667790332981570878266408960a^{2}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1525408782428633734350085301678112768a^{3}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{147018436784177047800874770446431027 2a^{4}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-25186819706236967969752199080869888a^{5}c-4916413813362186868890467613499392a^{6}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{180523758084624041877205846241280a^{7}c+94446500034526258986536455864 32a^{8}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-29007375200718772891099104460 8a^{9}c-13790932677896490825858473472a^{10}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1321682546654629416569894 40a^{11}c+114782104518034681741278 72a^{12}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{82419396059301592042752a^{13}c-3441243829617943295232a^{14}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-68832268710004800000a^{15}c-1208441744614 24128a^{16}c+8743439744911872a^{17}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{96443587688448a^{18}c+222439176960a^{19}c-2314760448a^{20}c-16144128a^{21}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-29952a^{22}c+10775935227221392570782056875819008000 0c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-11960479169473324898967275309786726400 0ac^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{4691558167395406291786464429186613248 0a^{2}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-8179716567466648414966719773717397504a^{3}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{572414151475230486442581108591747072a^{4}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{422512281576207955969323781277491 2a^{5}c^{2}-2107743828904816275105349039067648a^{6}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{21777324127000959865037942782208a^{7}c^{2}+3811313237604914006353415570432a^{8}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-17208144178264485968744984832a^{9}c^{2}-4072656512763355130085794976a^{10}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-30448585606100284800091824a^{11}c^{2}+19712858519658091308495 52a^{12}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{379093929414388823354 72a^{13}c^{2}-1368141128677077520 48a^{14}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1001833203725386032 0a^{15}c^{2}-8669552182140220 8a^{16}c^{2}+228479601333984a^{17}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{7725554252416a^{18}c^{2}+40796996240a^{19}c^{2}+25577552a^{20}c^{2}-336336a^{21}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-624a^{22}c^{2}+644233981779638031449985433353584640 00c^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-5830015020289933701656524157513367552 0ac^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1869196098231847508520633214367799705 6a^{2}c^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-2586065877925151796735934133773159014 4a^{3}c^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{122171345804374971207330799098003456a^{4}c^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{45335408559195109914547343351808 00a^{5}c^{3}-46763196943157976969006959792947 2a^{6}c^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-46973271715179808827267052830 72a^{7}c^{3}+72296954218102218084313200230 4a^{8}c^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{90832027307779504040007168 00a^{9}c^{3}-52957857198852051152374886 4a^{10}c^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-11402466632175022106486784a^{11}c^{3}+1081694995567662907791 36a^{12}c^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{50632225886219116339 20a^{13}c^{3}+3200080509295887360 0a^{14}c^{3}-49899558090590208 0a^{15}c^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-8205717988435968a^{16}c^{3}-29314507622400a^{17}c^{3}+167330042880a^{18}c^{3}+1506785280a^{19}c^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{3075072a^{20}c^{3}+253471389730583592603800261288263680 00c^{4}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-18978543797716490715225178573727858688ac^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{4991684025479138391935744923474722816a^2c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-541099749641591095487701589399519232a^3c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{14606668941153562310429072394294067 2a^4c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1292549598384288914173152912040960a^5c^4-59385691383614919077242368788992a^6c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1892451113166353101369696786944a^7c^4+72169632318929524275249528576a^8c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{2228687725107369556056237120a^9c^4-28472710924167755430485664a^{10}c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-140943036034035471376940 8a^{11}c^4-5501770617352422372960a^{12}c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{29767490708509666 1120a^{13}c^4+399533263161100 6400a^{14}c^4+1153916414851200a^{15}c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-295200075538368a^{16}c^4-1932687556800a^{17}c^4-1947065120a^{18}c^4+15695680a^{19}c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{32032a^{20}c^4+709953511885433632320287486900 3845632c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-4432336230110420896646668265044574208ac^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{955466413384038667615798431427067904a^2c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-79380397141399923587473758714593280a^3c^5+6438059316216642848934951242956 80a^4c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{209465328810891152290757390991360a^5c^5-4028319000276505780447170035712a^6c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-290915966729411309455672688640a^7c^5+312414553311136661480030 2080a^8c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{241803413658639276317153280a^9c^5+5844453787756576250818 56a^{10}c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-8784479855673207189504 0a^{11}c^5-106780577205134536704 0a^{12}c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{6128657556733071360a^{13}c^5+189548697075978240a^{14}c^5+958658985123840a^{15}c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-3202625986560a^{16}c^{5}-40683202560a^{17}c^{5}-92252160a^{18}c^{5}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1487119300880904242605217024921567232c^{6}-7767775536173890825348200016914677776ac^{6}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{136593392051968205997829269713846272a^{2}c^{6}-8388092183067964659019677414162432a^{3}c^{6}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-91340670507632683968759660462080a^{4}c^{6}+22224973282782547275305692944384a^{5}c^{6}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-35529674554603169040172140544a^{6}c^{6}-26210702433049018598332609536a^{7}c^{6}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-103200723777727230528706560a^{8}c^{6}+14895636537534753758389632a^{9}c^{6}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{182403094584283413160320a^{10}c^{6}-2663633385875852662272a^{11}c^{6}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-62256983715951664640a^{12}c^{6}-174817816321564416a^{13}c^{6}+4084756093786368a^{14}c^{6}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{34430420344320a^{15}c^{6}+48609200640a^{16}c^{6}-282522240a^{17}c^{6}-640640a^{18}c^{6}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{240838218499636521398013613710508032c^{7}-105287696394810237973591256098406400ac^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{14976654351475036084045130810523648a^{2}c^{7}-638818690035905266292950472589312a^{3}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-20960541592500617722444448268288a^{4}c^{7}+1615109961509085989321865953280a^{5}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{21023698439479542356321501184a^{6}c^{7}-1497276217131824539913158656a^{7}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-23179379774042918881787904a^{8}c^{7}+515692831831999968706560a^{9}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{12163046240440205180928a^{10}c^{7}-6024819137364099072a^{11}c^{7}-1837489800964866048a^{12}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-12946877731307520a^{13}c^{7}+23007583272960a^{14}c^{7}+495947612160a^{15}c^{7}+1265172480a^{16}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{30871602486909275910830889875013632c^{8}-11272571429148914722533062085705728ac^{8}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1280968945576182794260760676794368a^{2}c^{8}-33589734750958810846109982916608a^{3}c^{8}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-2206717841266713417081546031104a^{4}c^{8}+79728366271118889239432724480a^{5}c^{8}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{23060692978907338635472650 24a^6c^8 - 5226332557745315889627 9552a^7c^8}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-156166841944776701927 1168a^8c^8 + 633574574033038589 9520a^9c^8}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{433099863114577072128a^{10}c^8 + 2445053402814038016a^{11}c^8 - 26209896905127936a^{12}c^8}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-303502718238720a^{13}c^8 - 568422666240a^{14}c^8 + 2583060480a^{15}c^8 + 6589440a^{16}c^8}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{31856420852890313428354688845086 72c^9 - 967550638958541171077959970193408ac^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{862997747349788295511715830824 96a^2c^9 - 99894552165826120455303462912 0a^3c^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1536176900857106154931303219 20a^4c^9 + 2411197000642734061356318720a^5c^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{137452472767000072564506624a^6c^9 - 7850646412473773811302 40a^7c^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-610092571995894566609280a^8c^9 - 25603280954760757248 0a^9c^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{876959110851644620 8a^{10}c^9 + 887856711873331 20a^{11}c^9 - 46103447470080a^{12}c^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-3278764769280a^{13}c^9 - 9559080960a^{14}c^9 + 267889258438085483657974117302272c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-67263084939581397145810722881536ac^{10} + 4593733502680143944975661400064a^2c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{853933914061004374915783065 6a^3c^{10} - 7676662187554452553435054080a^4c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{19441815522855857897013248a^5c^{10} + 5387422458233145326796800a^6c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{20474586985880613634048a^7c^{10} - 1490183150114131066880a^8c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-14543250526116888576a^9c^{10} + 83327898738180096a^{10}c^{10} + 1503081400877056a^{11}c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{3662123745280a^{12}c^{10} - 13661519872a^{13}c^{10} - 39829504a^{14}c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{18518169236138399504184033411072c^{11} - 381236030578207260652722978 8160ac^{11}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{192340688684970209357454114816a^2c^{11} + 28551978080169379943216578 56a^3c^{11}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-2818730373110250740027722816a^4c^{11} - 2077996489024813179863040a^5c^{11}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{143506954311464043675648a^6c^{11} + 15417102765831204437248a^7c^{11}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-213663678515097108480a^8c^{11} - 341036842572840960a^9c^{11} - 235238901940224a^{10}c^{11}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{12876603457536a^{11}c^{11} + 43797970944a^{12}c^{11} + 1058340146221982392995043868672c^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-176775637161072795796969095168ac^{12} + 62358496447082809859673948816a^2c^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{17911200346282002286038220 8a^3c^{12} - 754963155868841274048512 0a^4c^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-121866570019913213214720a^5c^{12} + 25000096634399709921 28a^6c^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{446132951017718415 36a^7c^{12} - 113644325380325376a^8c^{12} - 4446228579287040a^9c^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-14131384909824a^{10}c^12 + 44710428672a^{11}c^{12} + 152076288a^{12}c^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{5016913535585306767082599219 2c^{13} - 67093635189595793973659566 08ac^{13}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1506143913205423274396221 44a^2c^{13} + 703916330593389656408064 0a^3c^{13}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-14095386433207721263104 0a^4c^{13} - 3711753500792413224960a^5c^{13}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{235302765199129313 28a^6c^{13} + 762927749215027200a^7c^{13} + 1587081243525120a^8c^{13}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-31366085345280a^9c^{13} - 128024838144a^{10}c^{13} + 19742506088915430176372817 92c^{14}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-2079126741057199088504668 16ac^{14} + 2433971414854765171965952a^2c^{14}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{19741967100056549720064 0a^3c^{14} - 15692229947231463014 40a^4c^{14}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-73662625200783360000a^5c^{14} - 2804668312466227 2a^6c^{14} + 800613105008640 0a^7c^{14}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{33986276229120a^8c^{14} - 93351444480a^9c^{14} - 381026304a^{10}c^{14}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{64395185753779071029870592c^{15} - 5229548355031289718374400ac^{15}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1478862665087535022080 0a^2c^{15}+40801388574453792768 00a^3c^{15}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-18387388143658598 40a^4c^{15}-98430957958201344 0a^5c^{15}-396364898893824 0a^6c^{15}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{47795939573760a^7c^{15}+243856834560a^8c^{15}+17337542691785568860 24192c^{16}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-10569999821576431520 9728ac^{16}-46277697446159319040 0a^2c^{16}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{62047273223138574336a^3c^{16}+274014892392579072a^4c^{16}-8585594583121920a^5c^{16}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-51289950781440a^6c^{16}+124468592640a^7c^{16}+635043840a^8c^{16}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{38253074247649509507072c^{17}-16902482622562990817 28ac^{17}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-17030905158269140992a^2c^{17}+677374598861291520a^3c^{17}+5103823135703040a^4c^{17}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-44281532252160a^5c^{17}-301234913280a^6c^{17}+683995510182533660672c^{18}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-20883614573480902656ac^{18}-300907292140765184a^2c^{18}+5031796612792320a^3c^{18}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{47159301898240a^4c^{18}-102503546880a^5c^{18}-697303040a^6c^{18}+9749520870090473472c^{19}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-192157335441899520ac^{19}-3363268429086720a^2c^{19}+22788156948480a^3c^{19}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{232532213760a^4c^{19}+108101378949251072c^{20}-1238931957874688ac^{20}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-24121180160000a^2c^{20}+47475326976a^3c^{20}+484442112a^4c^{20}+898096988946432c^{21}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-4991691522048ac^{21}-101871255552a^2c^{21}+5257174188032c^{22}-9453961216ac^{22}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-192937984a^2c^{22}+19327352832c^{23}+33554432c^{24}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-174640699428026688529844729138380800000a}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{203146678984534054497568403939819520000a^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-851387551758203986531005042420080640 00a^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{16061280659531143673149881316338278400a^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-1222084280207838423985096126241433600a^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-12283691196452377116468673849996800a^6+5784460407216745090330911562252800a^7}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-64163356321096906750752842736000a^8-13694126502362695192734160137600a^9}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{70187206573178654636297887200a^{10}+19908972323990635614823432800a^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{173269159811839104008127000a^{12}-13877737266129829896585600a^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-321996506691917154610800a^{14}+1564939330380426184800a^{15}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{137080053389848041000a^{16}+1531503700614518400a^{17}-5877737170852800a^{18}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-269958340159200a^{19}-2165643711000a^{20}-2271561600a^{21}+62053200a^{22}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{352800a^{23}+600a^{24}+174640699428027308978246462377820160000c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-532864667571457665192973079183032320000ac}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{402516691346265375314108597722509312000a^2c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-125889625791092176675310990146311475200a^3c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{17789322238914085698760878421723806720a^4c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-839642652359017269802420497774739200a^5c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-43737799664111358026071194502939200a^6c+4327054136964868650145069269816000a^7c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{65710116059613413534021275073520a^8c-8866517878966272728311508397200a^9c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-150823836484592478601551835500a^{10}c+8809059086367927385757986500a^{11}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{245730574725150062467057345a^{12}c-2419735295216118363229200a^{13}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-1588841966143295784490 50a^{14}c - 1380872827191951285500a^{15}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{247109495415617278 95a^{16}c + 584963403786946800a^{17}c + 3188363454621400a^{18}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-23871357856500a^{19}c - 392868365505a^{20}c - 1709061200a^{21}c - 409650a^{22}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{14700a^{23}c + 25a^{24}c + 3297179885869259534826210739617792 00000c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-5936101158632511603798388528077864960 00ac^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{3323393925076453245713424484789911552 00a^2c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-802920012764038929130362613869477888 00a^3c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{843575225397876564934340549060444160 0a^4c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-182800700903842751424899605150310400a^5c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-27864925923991726587974486723020800a^6c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{1153485575395825211251677531417600a^7c^2 + 54557531398454979304743054796800a^8c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-1866238092614775212796148761600a^9c^2 - 82875721352461554443602089600a^{10}c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{896036369449822368051211200a^{11}c^2 + 71047302433059217158129600a^{12}c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{483566295038792118753600a^{13}c^2 - 21705016205569712558400a^{14}c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-427069008643285008000a^{15}c^2 - 7040371396275360 00a^{16}c^2 + 548291207816352 00a^{17}c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{601725264940800a^{18}c^2 + 1380827448000a^{19}c^2 - 14486472000a^{20}c^2 - 100900800a^{21}c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-187200a^{22}c^2 + 27623217969281010921420403742500454 4000c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-36101648874575190083538600870213058 5600ac^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{157654746786562756857346716177587896320a^2c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-298118457544233404063389010588061696 00a^3c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{226104046910181204666259398090946 5600a^4c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{74679625665711304914829246136064 00a^5c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-828881013150028380197370017138 3040a^6c^3+105753830113991285004211505699200a^7c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{15276314790358723950186005280000a^8c^3-898396565353087809817469 47200a^9c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-167268839120576396051892736 80a^{10}c^3-1152019617036769967129826 00a^{11}c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{82577278293808323459070 00a^{12}c^3+1554701330922017480202 00a^{13}c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-6032340056610198474 00a^{14}c^3-4175166234961521800 0a^{15}c^3-3587699236595188 00a^{16}c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{968095828299600a^{17}c^3+32206016242400a^{18}c^3+169856687000a^{19}c^3+106306200a^{20}c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-1401400a^{21}c^3-2600a^{22}c^3+138505441369671545827613807716230758400c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-141033120649691397441988498324822425600ac^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{49191424454401897055805697928999731200a^2c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-7286784027262375583445490685229465600a^3c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{374067588927556175588296387795353600a^4c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{120668794153929899594355694863360 00a^5c^4-14183678015826818468999742236160 00a^6c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-117248853407902707239264420352 00a^7c^4+222557707020330258107630215680 0a^8c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{25892642522406895816692384000a^9c^4-166040358724251089262475200 0a^{10}c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-3471800010562428441536640 0a^{11}c^4+3490588924214633361120 00a^{12}c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{15758070846864650304000a^{13}c^4+98029228370495136000a^{14}c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-1569321907433280000a^{15}c^4-25609443449990400a^{16}c^4-91184052960000a^{17}c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{523867344000a^{18}c^4+4708704000a^{19}c^4+9609600a^{20}c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{471031371808289637262586891990823075 84c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-3860691689473860715273202521244958 7200ac^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{108729098458568389059451441606213632 00a^2c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-125064616278784530493118404933447680 0a^3c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{37484137237073610387082179113287680a^4c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{2960056649806039350518792618176000a^5c^5-147835927587805127993477304710400a^6c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-4407777576011344380877953734400a^7c^5+181587707284941430929852497280a^8c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{5386052499736584815450220000a^9c^5-73426510897677856161406800a^{10}c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-3490081968585840976348000a^{11}c^5-129783575583981021903 20a^{12}c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{746347714949116280000a^{13}c^5+9937225737538642400a^{14}c^5+24546217327632 00a^{15}c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-738607265716320a^{16}c^5-4828187364000a^{17}c^5-4859654800a^{18}c^5+39239200a^{19}c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{80080a^{20}c^5+1164351228433807821493229120913408 0000c^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-781184626528971398846410658603335680 0ac^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{178244733534652734852004010506321920 0a^2c^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-15618599123769688290439984910499840 0a^3c^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{1628516051137267451572910992588800a^4c^6+4131306550846887788264184317952 00a^5c^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-86822922882861576321775785984 00a^6c^6-5843379961554099916428352512 00a^7c^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{683922590897145304463216640\,0a^8c^6+496205711149910054823129600a^9c^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{10412913876826096411776 00a^{10}c^6-18291127709822555427840 0a^{11}c^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-219780434207816401920 0a^{12}c^6+12956443561424563200a^{13}c^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{394556420664806400a^{14}c^6+1989973649664000a^{15}c^6-6690587904000a^{16}c^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-84756672000a^17c^6-192192000a^18c^6+21854340177546728938275545674678272 00c^7}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-121057637219399655634901851003944960 0ac^7}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{22335611472046158369537238916726784 0a^2c^7}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-144178401151700570770409940717568 00a^3c^7-127654808190213133895037229670400a^4c^7}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{38362342001061342556663535539200a^5c^7-1009436273128009524146965452 80a^6c^7}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-459302955961680584847160704 00a^7c^7-15851970345894971269363200 0a^8c^7}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{2649122095420665779634720 0a^9c^7+318516097490735070738240a^{10}c^7}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-479697614876759959680 0a^{11}c^7-11073571309073004160 0a^{12}c^7}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-3071186841314592 00a^{13}c^7+7303674051273600a^{14}c^7+61439842464000a^{15}c^7}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{86692320000a^{16}c^7-504504000a^{17}c^7-1144000a^{18}c^7}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{3208416832846810634304909748469760 00c^8-1472616856633496629782429938221056 00ac^8}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{2182858878772001376446975162449920 0a^2c^8-978764458188092993647066703462400a^3c^8}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-2993778251686921668471413637120 0a^4c^8+2477913960753858299794587648000a^5c^8}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{30306732837796421151215616000a^6c^8-2324406168836210117372313600a^7c^8}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-35079004867678815190118400a^8c^8+810488727074359517184000a^9c^8}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{18840273819769239552000a^{10}c^{8}-110754128041869312 00a^{11}c^{8}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-2870202835480780800a^{12}c^{8}-20168019615744000a^{13}c^{8}+36128581632000a^{14}c^{8}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{774918144000a^{15}c^{8}+1976832000a^{16}c^{8}+3762385409469978840909284783423488 0c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-14310486246486860625946551595827200ac^{9}+1685406368191632966927765317222400a^{2}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-467740533092605541799028834304 00a^{3}c^{9}-2907367898999056189584016384000a^{4}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{110290157599761586095108096000a^{5}c^{9}+30914383083249156189031424 00a^{6}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-7300024192873466551777 2800a^{7}c^{9}-2137975103648864229235200a^{8}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{91012053239545518080 00a^{9}c^{9}+59978705829538862080 0a^{10}c^{9}+3364607652652646400a^{11}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-364788800440064 00a^{12}c^{9}-421246938112000a^{13}c^{9}-788646144000a^{14}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{3587584000a^{15}c^{9}+9152000a^{16}c^{9}+3578017643742734189270453452800000c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-11248222081527569326298300940288 00ac^{10}+1035295887912484738770092949504 00a^{2}c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-131121510852167107057790484480 0a^{3}c^{10}-185621293383786411465336422400a^{4}c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{30600779722705876219920384 00a^{5}c^{10}+16853674053705885046210560 0a^{6}c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-101634389903093340733440 0a^{7}c^{10}-75771265876540543795200a^{8}c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-312238666563644620800a^{9}c^{10}+10967326380122112000a^{10}c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{1106870001549312 00a^{11}c^{10}-58633012838400a^{12}c^{10}-409845596160 0a^{13}c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-11948851200a^{14}c^{10}+278986102350768311768576111411200c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-7212488871245470287359345950720 0ac^{11}+506555358997858161922454061056 0a^{2}c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{53299599493544386893840384 00a^{3}c^{11}-853557562657738768361226240 0a^{4}c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{2510064914868328356331 5200a^5c^{11}+60595565342660840193228 80a^6c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{221554919919789094912 00a^7c^{11}-1690530802659461734400a^8c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-164153148919818240 00a^9c^{11}+95043876381941760a^{10}c^{11}+1706929285734400a^{11}c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{4157702348800a^{12}c^{11}-15524454400a^{13}c^{11}-45260800a^{14}c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{17973019590141218234189217792000c^{12}-37929347080341924651881988096 00ac^{12}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{196285879020447610271642419200a^2c^{12}+278621484366175192169840640 0a^3c^{12}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-28980156376006060061491200 0a^4c^{12}-206470530647117807616000 0a^5c^{12}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{148846508105105620992000a^6c^{12}+1585317292830248140800a^7c^{12}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-222994548198309888800a^8c^{12}-3543972128686080 00a^9c^{12}-241573183488000a^{10}c^{12}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{13413128601600a^{11}c^{12}+456228864 00a^{12}c^{12}+961450433202482165846785392640c^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-163998227183988503762121523200ac^{13}+59246484104751176164442112 00a^2c^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{1665155459158403508961280 00a^3c^{13}-7213269573432725603942400a^4c^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-1150379098828347924480 00a^5c^{13}+24046316054539563008 00a^6c^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{42663988630042214400a^7c^{13}-110264137559654400a^8c^{13}-4272497037312000a^9c^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-13576756838400a^{10}c^{13}+42990796800a^{11}c^{13}+146227200a^{12}c^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{428191083274361164215091200 00c^{14}-5828749054688624947023052800ac^{14}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{13399822166001268618690560 0a^2c^{14}+6158133813035578687488000a^3c^{14}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-125789151018504683520000a^4c^{14}-3283349863865057280000a^5c^{14}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{21132740770489958400a^6c^{14}+679691867258880000a^7c^{14}+1409416298496000a^8c^{14}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-28005433344000a^9c^{14}-114307891200a^{10}c^{14}+15882258439075030034808832 00c^{15}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-16976202093012413768007680 0ac^{15}+20433784815355878231244 80a^2c^{15}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{162454135634000379904000a^3c^{15}-1316763562211776921600a^4c^{15}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-6109945649376952320 0a^5c^{15}-21663400701788160a^6c^{15}+6667626921984000a^7c^{15}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{28300728729600a^8c^{15}-77792870400a^9c^{15}-317521920a^{10}c^{15}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{489676636408659144867840 00c^{16}-402599292302830243676160 0ac^{16}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{121520461185500577792 00a^2c^{16}+316365033451715297280 0a^3c^{16}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-1616946504361574400a^4c^{16}-767423554781184000a^5c^{16}-3085932036096000a^6c^{16}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{37340577792000a^7c^{16}+190513152000a^8c^{16}+12493487768713664502169 60c^{17}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-769395103355778629632 00ac^{17}-3290332494852849664 00a^2c^{17}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{4543438157176504320 0a^3c^{17}+199713870087782400a^4c^{17}-6309093310464000a^5c^{17}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-37687050240000a^6c^{17}+91521024000a^7c^{17}+466944000a^8c^{17}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{261804562936391270400 00c^{18}-11661869693962027008 00ac^{18}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-11694085683491635200a^2c^{18}+469496995774464000a^3c^{18}+3535117221888000a^4c^{18}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-30751064064000a^5c^{18}-209190912000a^6c^{18}+445504624400741171200c^{19}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-13687619181241958400ac^{19}-196960707910041600a^2c^{19}+3308414369792000a^3c^{19}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{31005671424000a^4c^{19}-67436544000a^5c^{19}-458752000a^6c^{19}+6054208717455360000c^{20}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-119879948147097600ac^{20}-2097585900748800a^2c^{20}+14242598092800a^3c^{20}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{145332633600a^4c^{20}+64105204122910720c^{21}-737026192179200ac^{21}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-14349002342400a^2c^{21}+28259123200a^3c^{21}+288358400a^4c^{21}+509356277760000c^{22}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$+\frac{-2836188364800ac^{22}-57881395200a^2c^{22}+2855482163200c^{23}-5138022400ac^{23}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}+$$

$$\left.+\frac{-104857600a^2c^{23}+10066329600c^{24}+16777216c^{25}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+50}{2})}\right]. \tag{8}$$

§3. Derivation of summation formulae

Derivation of main result (8):

Substituting $b=-a-49, z=\frac{1}{2}$ in given result (2), we get

$$(2a+49)\ {}_2F_1\left[\begin{array}{ccc} a\ , & -a-49 & ;\ \frac{1}{2} \\ c & & ; \end{array}\right]$$

$$=a\ {}_2F_1\left[\begin{array}{ccc} a+1\ , & -a-49 & ;\ \frac{1}{2} \\ c & & ; \end{array}\right]+(a+49)\ {}_2F_1\left[\begin{array}{ccc} a\ , & -a-48 & ;\ \frac{1}{2} \\ c & & ; \end{array}\right].$$

Now using same parallel method which is used in [6], we can prove the main result.

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Coefficient bounds for certain subclasses

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Abstract In this paper, we introduce some subclasses of analytic functions and determine the sharp upper bounds of the functional $|a_2a_4 - a_3^2|$ for the functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belonging to these classes in the unit disc $E = \{z :| z |< 1\}$.

Keywords Analytic functions, starlike functions, convex functions, alpha-convex functions, functions whose derivative has a positive real part, Bazilevic functions, Hankel determinant.

§1. Introduction and preliminaries

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

in the unit disc $E = \{z :| z |< 1\}$.

Let S be the class of functions $f(z) \in A$ and univalent in E.

Let $M_\alpha(\alpha \geq 0)$ be the class of functions in A which satisfy the conditions

$$\frac{f(z)f'(z)}{z} \neq 0$$

and

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right] > 0. \tag{2}$$

The class M_α was introduced by Mocanu [15] and functions of this class are called α-convex functions. Obviously $M_0 \equiv S^*$, the class of starlike functions and $M_1 \equiv K$, the class of convex functions. Miller, Mocanu and Reade [14] have shown that α-convex functions are starlike in E, and for $\alpha \geq 1$, all α-convex functions are convex in E. Therefore α-convex functions are also called α-starlike functions. Concept of α-convex functions gives a continuous parametrization between starlike functions and convex functions.

$H_\alpha(\alpha \geq 0)$ be the class of functions in A which satisfy the condition

$$Re\left[(1-\alpha)f'(z) + \alpha\frac{(zf'(z))'}{f'(z)}\right] > 0. \tag{3}$$

This class was introduced by Al-Amiri and Reade [1]. In particular $H_0 \equiv R$, the class of functions whose derivative has a positive real part and studied by Macgregor [12]. Also $H_1 \equiv K$.

$B_\alpha(\alpha \geq 0)$ is the class of functions in A which satisfy the condition

$$Re\left[f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right] > 0. \quad (4)$$

The class B_α was introduced by Singh [19] and studied further by Thomas [21] and El-Ashwah and Thomas [3]. Functions of this class are called Bazilevic functions. Particularly $B_0 \equiv S^*$ and $B_1 \equiv R$.

In 1976, Noonan and Thomas [16] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example, Noor [17] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions given by Eq.(1) with bounded boundary. Ehrenborg [2] studied the Hankel determinant of exponential polynomials and the Hankel transform of an integer sequence is defined and some of its properties discussed by Layman [9]. Also Hankel determinant for different classes was studied by various authors including Hayman [5], Pommerenke [18], Janteng et al.[6,7,8] and recently by Mehrok and Singh [13].

Easily, one can observe that the Fekete-Szegö functional is $H_2(1)$. Fekete and Szegö [4] then further generalised the estimate of $|a_3 - \mu a_2^2|$ where μ is real and $f \in S$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$,

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

For μ complex, Szynal [21] obtained the estimates for $|a_3 - \mu a_2^2|$ for the class M_α. Al-Amiri and Reade [1] obtained the estimates for $|a_3 - \mu a_2^2|$ for the class H_α and Singh [19] obtained the estimates for $|a_3 - \mu a_2^2|$ for the class B_α.

In this paper, we seek upper bound of the functional $|a_2a_4 - a_3^2|$ for the functions belonging to the classes M_α, H_α and B_α. Results due to various authors follow as special cases.

§2. Main result

Let P be the family of all functions p analytic in E for which $Re(p(z)) > 0(z \in E)$ and

$$p(z) = 1 + p_1z + p_2z^2 + ... \quad (5)$$

Lemma 2.1. If $p \in P$, then $|p_k| \leq 2(k = 1, 2, 3, ...)$.

This result is due to Pommerenke [18].

Lemma 2.2. If $p \in P$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some x and z satisfying $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0, 2]$.

This result was proved by Libera and Zlotkiewiez [10,11].

Theorem 2.1. If $f \in M_\alpha$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{(1+2\alpha)^2}\left[\frac{3\alpha(1+\alpha)^3}{(1+3\alpha)(2+15\alpha+24\alpha^2+7\alpha^3)} + 1\right]. \tag{6}$$

Proof. As $f \in M_\alpha$, so from (2)

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = p(z). \tag{7}$$

On expanding and equating the coefficients of z, z^2 and z^3 in (7), we obtain

$$a_2 = \frac{p_1}{1+\alpha}, \tag{8}$$

$$a_3 = \frac{p_2}{2(1+2\alpha)} + \frac{(1+3\alpha)p_1^2}{2(1+2\alpha)(1+\alpha)^2} \tag{9}$$

and

$$a_4 = \frac{p_3}{3(1+3\alpha)} + \frac{(1+5\alpha)p_1p_2}{2(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(4+6\alpha+17\alpha^2)p_1^3}{6(1+2\alpha)(1+3\alpha)(1+\alpha)^3}. \tag{10}$$

Using (8), (9) and (10), it yields

$$|a_2a_4 - a_3^2| = \frac{1}{C(\alpha)}\left|\begin{array}{r} 4(1+2\alpha)^2(1+\alpha)^3p_1(4p_3) + 12(1+2\alpha)(1+5\alpha)(1+\alpha)^2p_1^2(2p_2) \\ +8(1+2\alpha)(1+6\alpha+17\alpha^2)^2p_1^4 - 3(1+3\alpha)((1+\alpha)^2(2p_2) + 2(1+3\alpha)p_1^2)^2 \end{array}\right|. \tag{11}$$

where $C(\alpha) = 48(1+3\alpha)(1+2\alpha)^2(1+\alpha)^4$.

Using lemma 2.1 and lemma 2.2 in (11) and replacing p_1 by p, it can be easily established that

$$|a_2a_4 - a_3^2| \leq \frac{1}{C(\alpha)}\left[\begin{array}{r} [-4(1+2\alpha)(1+\alpha)^2 - 12(1+2\alpha)(1+5\alpha)(1+\alpha)^2 - 8(1+2\alpha)(1+6\alpha+17\alpha^2) \\ +3(1+3\alpha)(3+8\alpha+\alpha^2)]p^4 + [8(1+2\alpha)^2(1+\alpha)^3 + 12(1+2\alpha)(1+5\alpha) \\ -6(1+3\alpha)(3+8\alpha+\alpha^2)(1+\alpha)^2]p^2(4-p^2)\delta + (1+\alpha)^3(2-p)[6(1+\alpha)(1+3\alpha) \\ -(1+4\alpha+7\alpha^2)](4-p^2)p^2\delta^2 + 8(1+2\alpha)^2(1+\alpha)^3p(4-p^2) \end{array}\right],$$

where $\delta = |x| \leq 1$.

Therefore

$$|a_2a_4 - a_3^2| \leq \frac{1}{C(\alpha)}F(\delta),$$

where

$F(\delta) = [-4(1+2\alpha)(1+\alpha)^2 - 12(1+2\alpha)(1+5\alpha)(1+\alpha)^2 - 8(1+2\alpha)(1+6\alpha+17\alpha^2) + 3(1+3\alpha)(3+8\alpha+\alpha^2)]p^4 + [8(1+2\alpha)^2(1+\alpha)^3 + 12(1+2\alpha)(1+5\alpha) - 6(1+3\alpha)(3+8\alpha + \alpha^2)(1+\alpha)^2]p^2(4-p^2)\delta + (1+\alpha)^3(2-p)[6(1+\alpha)(1+3\alpha) - (1+4\alpha+7\alpha^2)](4-p^2)p^2\delta^2 + 8(1+2\alpha)^2(1+\alpha)^3 p(4-p^2).$

As $F'(\delta) > 0$, so $F(\delta)$ is an increasing function in [0, 1] and therefore Max$F(\delta) = F(1)$.

Consequently

$$|a_2a_4 - a_3^2| \leq \frac{1}{C(\alpha)}G(p), \tag{12}$$

where $G(p) = F(1)$.

So

$$G(p) = -A(\alpha)p^4 + B(\alpha)p^2 + 48(1+3\alpha)(1+\alpha)^4,$$

where

$A(\alpha) = 4\alpha(1+\alpha)(2+15\alpha+24\alpha^2+7\alpha^3)$ and $B(\alpha) = 48\alpha(1+\alpha)^4$.

Now $G^{'}(p) = -4A(\alpha)p^3 + 2B(\alpha)p$ and $G^{''}(p) = -12A(\alpha)p^2 + 2B(\alpha)$.

$G^{'}(p) = 0$ gives $p[2A(\alpha)p^2 - B(\alpha)] = 0$.

$G^{''}(p)$ is negative at $p = \sqrt{\frac{6(1+\alpha)^3}{(2+15\alpha+24\alpha^2+7\alpha^3)}} = p^{'}$.

So Max$G(p) = G(p^{'})$.

Hence from (12), we obtain (6).

The result is sharp for $p_1 = p^{'}$, $p_2 = p_1^2 - 2$ and $p_3 = p_1(2-p_1^2)$.

For $\alpha = 0$ and $\alpha = 1$ respectively, theorem 2.1 gives the following results due to Janteng et al.[8].

Corollary 2.1. If $f(z) \in S^*$, then

$$|a_2a_4 - a_3^2| \leq 1.$$

Corollary 2.2. If $f(z) \in K$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$

Theorem 2.2. If $f \in H_\alpha$, then

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{4}{9(1+\alpha)^2} & \text{if } 0 \leq \alpha \leq \frac{5}{17}, \\ \frac{(17\alpha-5)^2}{144(1+2\alpha)(1+20\alpha+7\alpha^2-4\alpha^3)} + \frac{4}{9(1+\alpha)^2} & \text{if } \frac{5}{17} \leq \alpha \leq 1. \end{cases} \tag{13}$$

Proof. Since $f \in H_\alpha$, so from (3)

$$(1-\alpha)f'(z) + \alpha\frac{(zf'(z))^{'}}{f'(z)} = p(z). \tag{14}$$

On expanding and equating the coefficients of z, z^2 and z^3 in (14), we obtain

$$a_2 = \frac{p_1}{2}, \tag{15}$$

$$a_3 = \frac{p_2 + \alpha p_1^2}{3(1+\alpha)} \tag{16}$$

and

$$a_4 = \frac{p_3}{4(1+2\alpha)} + \frac{3\alpha p_1 p_2}{4(1+\alpha)(1+2\alpha)} + \frac{\alpha(2\alpha-1)p_1^3}{4(1+\alpha)(1+2\alpha)}. \tag{17}$$

Using (15), (16) and (17), it yields

$$|a_2a_4 - a_3^2| = \frac{1}{C(\alpha)}\left|\begin{array}{r} 9(1+\alpha)^3 p_1(4p_3) + 54(1+\alpha)p_1^2(2p_2) \\ +36\alpha(2\alpha-1)(1+\alpha)p_1^4 - 8(1+2\alpha)(2p_2+2\alpha p_1^2)^2 \end{array}\right|, \tag{18}$$

where $C(\alpha) = 288(1+3\alpha)(1+2\alpha)(1+\alpha)^2$.

Using lemma 2.1 and lemma 2.2 in (18) and replacing p_1 by p, we obtain

$$|a_2a_4 - a_3^2| \le \frac{1}{C(\alpha)}\left[\begin{array}{r} (1-12\alpha+3\alpha^2+8\alpha^3)p^4 + 2(1+13\alpha+4\alpha^2)p^2(4-p^2)\delta \\ +(2-p)[16(1+2\alpha)-(1+2\alpha+9\alpha^2)p^2](4-p^2)\delta^2 + 18(1+\alpha)^2 p(4-p^2) \end{array}\right],$$

where $\delta = |x| \le 1$.

Therefore

$$|a_2a_4 - a_3^2| \le \frac{1}{C(\alpha)}F(\delta),$$

where

$$F(\delta) = (1-12\alpha+3\alpha^2+8\alpha^3)p^4 + 2(1+13\alpha+4\alpha^2)p^2(4-p^2)\delta + (2-p)[16(1+2\alpha)-(1+2\alpha+9\alpha^2)p^2](4-p^2)\delta^2 + 18(1+\alpha)^2 p(4-p^2)$$

is an increasing function.

Therefore Max$F(\delta) = F(1)$. Consequently

$$|a_2a_4 - a_3^2| \le \frac{1}{C(\alpha)}G(p), \tag{19}$$

where $G(p) = F(1)$.

So

$$G(p) = -A(\alpha)p^4 + B(\alpha)p^2 + 128(1+2\alpha),$$

where $A(\alpha) = 2(1+20\alpha+17\alpha^2-4\alpha^3)$ and $B(\alpha) = 4(1+\alpha)(17\alpha-5)$.

Case I. For $0 \le \alpha \le \frac{5}{17}$, $B(\alpha) < 0$.

So $G(p)$ is maximum at $p = 0$ and it follows the first result of (13).

In this case, the result is sharp for $p_1 = 0$, $p_2 = -2$ and $p_3 = 0$.

Case II. For $\frac{5}{17} \le \alpha \le 1$, as in theorem 2.1, $G(p)$ is maximum for $p = \sqrt{\frac{(1+\alpha)(17\alpha-5)}{(1+20\alpha+7\alpha^2-4\alpha^3)}} = p'$, so Max$G(p) = G(p')$.

In this case, the result is sharp for $p_1 = p'$, $p_2 = p_1^2 - 2$ and $p_3 = p_1(p_1^2 - 3)$.

Hence the theorem. For $\alpha = 0$ in theorem 2.2, we obtain the following results due to Janteng et al.[6].

Corollary 2.3. If $f(z) \in R$, then

$$|a_2a_4 - a_3^2| \le \frac{4}{9}.$$

Putting $\alpha = 1$ in theorem 2.2, we get the following results due to Janteng et al.[8].

Corollary 2.4. If $f(z) \in K$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$

On the same lines, we can easily prove the following theorem:

Theorem 2.3. If $f \in B_\alpha$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{(2+\alpha)^2}.$$

The result is sharp for $p_1 = 0$, $p_2 = -2$ and $p_3 = 0$.

For $\alpha = 0$, theorem 2.3 gives the following result due to Janteng et al.[8].

Corollary 2.5. If $f(z) \in S^*$, then

$$|a_2a_4 - a_3^2| \leq 1.$$

For $\alpha = 0$, theorem 2.3 gives the following result due to Janteng et al.[6].

Corollary 2.6. If $f(z) \in R$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$

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Certain regression models involving univariate maximal and minimal survival functions on mixed censored survival data

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Abstract Mixed censored survival data arise from taking the maximum or minimum of two-component or even multi-component systems. Tied to this is the need to estimate maximal or minimal survival functions from related marginal survival functions. In this study we exploit the fact that the transformation of survival time dynamics of one component to another can be considered to result from variations in shift (mean), concentration (mode), spread (variance), degree of symmetry (skewness) and peakedness (kurtosis) as well as the underlying relative proportion of the censored data. Consequently, we propose some regression models that can serve for prediction purposes. The models are evaluated using a combination of some real data from the literature combined with some Weibull simulated data of varying levels of censoring and parameters. To the extent that the performance measures are excellent, the suggested models can be effective for prediction analysis.

Keywords Maximal survival function, minimial survival function, mixed censored survival data, univariate.
2000 Mathematics Subject Classification: 62N01, 62N02.

§1. Introduction and preliminaries

Mixed censored survival data arise from taking the maximum or minimum of two-component systems or even multi-components. Tied to this is the need to estimate maximal or minimal survival functions from related marginal survival functions. Bivariate survival data for two-component systems pervade many fields of scientific research. For example, medical researchers have obtained data on the infection times of the left and right kidneys of patients, and times to HIV infection of sex partners. Clayton (1978) applied regression model for censored survival data to epidemiological studies of chronic disease incidence. A related model for association in bivariate survivorship time distributions was proposed for the analysis of familial tendency in disease incidence.[3] The possible extension of the model to general multivariate survisorship distribution was indicated. Oakes (1982) discussed a reparameterization of a model introduced by Clayton in bivariate life-tables and the inference for the parameter governing the association was considered when the marginal distributions are specified up to Lehnmann alternatives.[5]

The information matrix was derived explicitly and it was shown that the parameterization is moderately successful in introducing orthogonality between the associated parameters. Demographers have collected data on the two survival times of individual twins. Economists interested in modeling the joint retirement decisions of married couples have collected data of times to retirement for both the wife and the husband of each surveyed household. One of the common features shared by taking the maximum or minimum of these data is a mixture of the two duration times, calling for a joint distribution that allows a probabillity mass on the 45^0 line in the sample space. The exploitation of facts of the transformation of survival time dynamics of one component to another which will result to variations in shift (mean), concentration (mode), spread (variance), degree of symmetry (skewness) and peakedness (kurtosis) as well as the underlying relative proportion of the censored data is the purpose of this study. Specifically, we propose some regression models that can serve for prediction purpose.

In [4], the likelihood function of the estimation from censored survival data was given and several examples are seen. The type of estimate studied here can be briefly indicated as follows. Suppose from a two-component system a random sample of N paired subjects are taken such that $T_{11}, T_{12}, \ldots, T_{1N}$ and $T_{21}, T_{22}, \ldots, T_{2N}$ are two random variables. Taking the maximum and minimum between the observed lifetimes separately will result to two sets of mixed samples. The sample distribution functions $\hat{F}_{Y_2}(t)$ and $\hat{F}_{Y_1}(t)$ are naturally defined as that which assigns a probability of $\frac{1}{N}$ to each of the given values, so that $\hat{F}_{Y_2}(t)$ equals $\frac{1}{N}$ times the number of sample values less than the argument t_{Y_2} and $\hat{F}_{Y_1}(t)$ equals $\frac{1}{N}$ times the number of sample values less than the argument t_{Y_1}. Besides describing the sample, they are also non-parametric estimates of the mixed population distributions. When observations are incomplete the corresponding estimates are still step functions with discontinuities at the ages of observed deaths, but can no longer be obtained as a mere description of the sample. The two samples are incomplete in the sense that some items in the mixtures have given not random samples $T_{11}, T_{12}, \ldots, T_{1N}$ and $T_{21}, T_{22}, \ldots, T_{2N}$ of random variables T_1 and T_2 itself (called the lifetimes), but the observed lifetimes $Z_i = (Z_{1i}, Z_{2i})$ where $Z_{1i} = \min(T_{1i}, C_{1i})$ and $Z_{2i} = \min(T_{2i}, C_{2i})$; $i = 1, 2, \ldots, N$. Here C_{1i} and C_{2i}, called limits of observation (i.e. censored times), are constants or values of the other random variables, which are assumed to be independent of T_{1i} and T_{2i} respectively (unless otherwise stated). For each item it is known whether one has

$$T_{1i} \leq C_{1i},\ Z_{1i} = T_{1i} \quad \text{and} \quad T_{2i} \leq C_{2i},\ \ Z_{2i} = T_{2i}, \quad \text{(deaths)},$$

or

$$T_{1i} > C_{1i},\ Z_{1i} = C_{1i} \quad \text{and} \quad T_{2i} > C_{2i}, Z_{2i} = C_{2i}, \quad \text{(losses)}.$$

Ordinarily the T_{1i}, T_{2i}, C_{1i} and C_{2i} are so defined as to be necessarily non-negative. The mixed censored survival data is such that an item in the mixture belongs to one population with certain probability and to the other with another probability. This results into a division of mutually exclusive classes of deaths and losses. A loss by definition always precludes the desired knowledge of $\vec{T}_i = (T_{1i}, T_{2i})$. On the other hand, a death does not preclude the knowledge of the corresponding $\vec{C}_i = (C_{1i}, C_{2i})$.

Notation 1.1. Let T_{1i} and C_{1i} represent the failure and censoring time for the ith subject of the first component. Also let T_{2i} and C_{2i} represent the failure and censoring time for the ith subject of the second component. Let the bivariate vector failure time $\vec{T_i}$ have cumulative distribution function $F_i = (F_{1i}, F_{2i})$, probability density function $f_i = (f_{1i}, f_{2i})$ and survivor function $\bar{F}_i = (\bar{F}_{1i}, \bar{F}_{2i})$. Also let the bivariate censorsing life time $\vec{C_i}$ have cumulative distribution function, cummulative distribution function $G_i = (G_{1i}, G_{2i})$, probability density function $g_i = (g_{1i}, g_{2i})$ and survivor function $\bar{G}_i = (\bar{G}_{1i}, \bar{G}_{2i})$. Let Y_i be the corresponding vector of covariates. Let $T_i = (T_{1i}, T_{2i})$ and $C_i = (C_{1i}, C_{2i})$ be independently conditional on Y_i. On each of N individuals, we observe the group $(Y_i, \delta_{1i}, \delta_{2i}, \gamma_i)$ where $Z_i = (Z_{1i}, Z_{2i}) =$ denotes the follow up time for the paired entity on individual i under study.

$\delta_{1i} =$ denotes a censoring indicator for the first member of an entity on individual i.

$\delta_{2i} =$ denotes a censoring indicator for the second member of an entity on individual i.

$\gamma_i =$ the binary indicator used to indicate which of the following up time is considered on individual i.

That is,

$$\gamma_i = \begin{cases} 1, & \text{if } Z_{1i} \leq Z_{2i}, \\ \\ 0, & \text{if } Z_{1i} > Z_{2i}. \end{cases}$$

$L_{\max}(Y, \delta_1, \delta_2, \gamma) =$ the likelihood function for the maximal survival function and $L_{min}(Y, \delta_1, \delta_2, \gamma) =$ the likelihood function for the minimal survival function.

Observing the maximum between the follow up time for the paired entity on the ith individual (i.e. $Y_i = \max(Z_{1i}, Z_{2i})$, we have the likelihood function, for the N i.i.d groups $(Y, \delta_1, \delta_2, \gamma)$, for the maximum survival function as

$$\begin{aligned} L_{\max}(Y, \delta_1, \delta_2, \gamma) \quad = \quad & \prod_{i=1}^{n} \{(g_{1i}(y)\bar{F}_{1i}(y))^{[(1-\delta_{1i})(1-\gamma_i)]} \times (g_{2i}(y)\bar{F}_{2i}(y))^{[(1-\delta_{2i})\gamma_i]} \\ & \times (f_{2i}(y)\bar{G}_{2i}(y))^{[\delta_{2i}\gamma_i]} \times (f_{1i}(y)\bar{G}_{1i}(y))^{[\delta_{1i}(1-\gamma_i)]} \\ & \times \left(\int_0^y g_{2i}(\tau)\bar{F}_{2i}(\tau)d\tau\right)^{[(1-\delta_{2i})(1-\gamma_i)]} \times \left(\int_0^y g_{1i}(\tau)\bar{F}_{1i}(\tau)d\tau\right)^{[(1-\delta_{1i})\gamma_i]} \\ & \times \left(\int_0^y f_{2i}(\tau)\bar{G}_{2i}(\tau)d\tau\right)^{[\delta_{2i}(1-\gamma_i)]} \times \left(\int_0^y f_{1i}(\tau)\bar{G}_{1i}(\tau)d\tau\right)^{[\delta_{1i}\gamma_i]} \}. \quad (1) \end{aligned}$$

In the same light, observation of the minimum follow up time for the paired entity also on the ith individual (i.e. $Y_i = \min(Z_{1i}, Z_{2i})$) results in the likelihood function for the minimal survival function given by

$$\begin{aligned} L_{\min}(Y, \delta_1, \delta_2, \gamma) \quad = \quad & \prod_{i=1}^{n} \{(g_{1i}(y)\bar{F}_{1i}(y))^{[(1-\delta_{1i})\gamma_i]} \times (g_{2i}(y)\bar{F}_{2i}(y))^{[(1-\delta_{2i})(1-\gamma_i)]} \\ & \times (f_{2i}(y)\bar{G}_{2i}(y))^{[\delta_{2i}(1-\gamma_i)]} \times (f_{1i}(y)\bar{G}_{1i}(y))^{[\delta_{1i}\gamma_i]} \\ & \times \left(\int_y^\infty g_{2i}(\tau)\bar{F}_{2i}(\tau)d\tau\right)^{[(1-\delta_{2i})\gamma_i]} \times \left(\int_y^\infty g_{1i}(\tau)\bar{F}_{1i}(\tau)d\tau\right)^{[(1-\delta_{1i})(1-\gamma_i)]} \\ & \times \left(\int_y^\infty f_{2i}(\tau)\bar{G}_{2i}(\tau)d\tau\right)^{[\delta_{2i}\gamma_i]} \times \left(\int_y^\infty f_{1i}(\tau)\bar{G}_{1i}(\tau)d\tau\right)^{[\delta_{1i}(1-\gamma_i)]} \}. \quad (2) \end{aligned}$$

Defined for various censored and failure situations.

Notation 1.2.

$$l_{i0} = g_i(y)\bar{F}_i(y) = Pr\{C_i = y,\ T_i > y\};\ \ i = 1, 2,$$
$$l_{i1} = f_i(y)\bar{G}_i(y) = Pr\{T_i = y,\ C_i > y\};\ i = 1, 2.$$
$$L_{ij}(y) = \int_0^y l_{ij}(\tau)d\tau \text{ and } \bar{L}_{ij}(y) = \int_y^\infty l_{ij}(\tau)d\tau;\ \ i = 1, 2,\ j = 0, 1.$$

It is clear that the terms of $L_{\max}(Y, \delta_1, \delta_2, \gamma)$ and $L_{\min}(Y, \delta_1, \delta_2, \gamma)$ can be expressed as

$$\begin{aligned} L_{\max}(Y, \delta_1, \delta_2, \gamma) &= \prod_{i=1}^n \{(l_{10}(y))^{[(1-\delta_{1i})(1-\gamma_i)]} \times (l_{20}(y))^{[(1-\delta_{2i})\gamma_i]} \\ &\times (l_{21}(y))^{[\delta_{2i}\gamma_i]} \times (l_{11}(y))^{[\delta_{1i}(1-\gamma_i)]} \times (L_{10}(y))^{[(1-\delta_{1i})\gamma_i]} \\ &\times (L_{20}(y))^{[(1-\delta_{2i})(1-\gamma_i)]} \times (L_{21}(y))^{[\delta_{2i}(1-\gamma_i)]} \times (L_{11}(y))^{[\delta_{1i}\gamma_i]}\}, \quad (3) \\ L_{\min}(Y, \delta_1, \delta_2, \gamma) &= \prod_{i=1}^n \{(l_{10}(y))^{[(1-\delta_{1i})\gamma_i]} \times (l_{20}(y))^{[(1-\delta_{2i})(1-\gamma_i)]} \\ &\times (l_{21}(y))^{[\delta_{2i}(1-\gamma_i)]} \times (l_{11}(y))^{[\delta_{1i}(1-\gamma_i)]} \times (\bar{L}_{10}(y))^{[(1-\delta_{1i})\gamma_i]} \\ &\times (\bar{L}_{20}(y))^{[(1-\delta_{2i})(1-\gamma_i)]} \times (\bar{L}_{21}(y))^{[\delta_{2i}(1-\gamma_i)]} \times (\bar{L}_{11}(y))^{[\delta_{1i}\gamma_i]}\}. \quad (4) \end{aligned}$$

Dividing (3) by (4), we have (5) as

$$\begin{aligned} \frac{L_{\max}(Y, \delta_1, \delta_2, \gamma)}{L_{\min}(Y, \delta_1, \delta_2, \gamma)} &= \left\{ \left(\frac{l_{11}(y)}{L_{11}(y)}\right)^{[\delta_1(1-\gamma)]} \times \left(\frac{L_{20}(y)}{l_{20}(y)}\right)^{[(1-\delta_2)(1-\gamma)]} \right. \\ &\times \left(\frac{l_{20}(y)}{\bar{L}_{20}(y)}\right)^{[(1-\delta_2)\gamma]} \times \left(\frac{L_{11}(y)}{l_{11}(y)}\right)^{[\delta_1\gamma]} \times \left(\frac{l_{10}(y)}{\bar{L}_{10}(y)}\right)^{[(1-\delta_1)(1-\gamma)]} \\ &\times \left. \left(\frac{L_{21}(y)}{l_{21}(y)}\right)^{[\delta_2(1-\gamma)]} \times \left(\frac{l_{21}(y)}{\bar{L}_{21}(y)}\right)^{[\delta_2\gamma]} \times \left(\frac{L_{10}(y)}{l_{10}(y)}\right)^{[(1-\delta_2)\gamma]} \right\}. \quad (5) \end{aligned}$$

We introduce the following notations

$$m_{ij} = \frac{l_{ij}(y)}{\bar{L}_{ij}(y)};\ \ i = 1, 2;\ j = 0, 1,$$
$$n_{ij} = \frac{l_{ij}(y)}{L_{ij}(y)};\ \ i = 1, 2;\ \ j = 0, 1.$$

Remark 1.1. In light of this notations, likelihood functions here in this section, the ratio of the likelihood function of the the maximal and minimal survival functions will be given in Sections 2. It is established that any regression of maximal survival functions on minimal survival functions is essentially a relation of certain instantaneous censored and failure rates. The validation of these models (which will be seen in Section 2) was such that for fixed first component data of proportion $p = \frac{58}{70}$ of censoring to Weibull distribuition (β, η), the second component data was generated for varied levels of censoring of proportion $p = \frac{58}{70}, \frac{35}{70}$, and $\frac{12}{70}$ with mean parameters λ and $.75\lambda$. We thus generally have 6 combinations identified as $(a, b, c, d) \equiv$ (mean of real first component data, proportion of censored points in first component data, mean of simulated second component data, proportion of censored points in the second component data). The marginal, maximal and minimal survival function for the two components was obtained and appropriate regression analysis was carried out. The results will be presented also in Section 2. Discussions and conclusions will be made. The study concludes that the eight models developed in this study gave good hypothesis and can reliably predict minimal survival function and maximal survival function from marginal survival functions and recommends that models such as the ones discussed herewith, in particular Models 2.7, 2.4 and 2.8 (the best performing models) be adopted for studying failure times in phenomena that can be modelled in terms of components.

§2. Main results

Using these notations in (5), we have the ratio of the likelihood of the maximal and minimal survival functions given by

$$\frac{L_{\max}(Y,\delta_1,\delta_2,\gamma)}{L_{\min}(Y,\delta_1,\delta_2,\gamma)} = \left(\frac{(m_{10}(y))^{(1-\delta_1)}(m_{11}(y))^{\delta_1}}{(n_{20}(y))^{(1-\delta_2)}(n_{21}(y))^{\delta_2}}\right)^{1-\gamma}\left(\frac{(m_{20}(y))^{(1-\delta_2)}(m_{21}(y))^{\delta_2}}{(n_{10}(y))^{(1-\delta_1)}(n_{11}(y))^{\delta_1}}\right)^{\gamma}. \qquad (6)$$

(6) breaks down, for specific values of δ_1,δ_2,γ into the following ratio for likelihoods of maximum and minimal survival functions e.g.

$$\frac{L_{\max}(y|\delta_1=0,\ \delta_2=0,\ \gamma=0)}{L_{\min}(y|\delta_1=0,\ \delta_2=0,\ \gamma=0)} = \frac{m_{10}(y)}{n_{20}(y)}.$$

It is noted that $m_{ij}(y)$ and $n_{ij}(y)$, $i=1,2;\ j=0,1$ can be likened to instantaneous event rates. (6) shows that the ratio of the likelihoods of the maximal and minimal survival function can be expressed as ratio of instantaneous censored and failure rates. Any regression of maximal survival functions on minimal survival functions is essentially a relation of certain instantaneous censored and failure rates.

For the purpose of exploying this relation, it is necessary we state some results that suggest appropriate models of interest.

Theorem 2.1. Let X_1,X_2 be stochastically independent random variables with distribution function $F_1(\cdot)$ and $F_2(\cdot)$ respectively. Also, let maximum $Y_2=\max(X_1,X_2)$ and the minimum $Y_1=\min(X_1,X_2)$ be ordered statistics respectively having the distribution functions $G_2(\cdot)$ and $G_1(\cdot)$ respectively. Then for Y_2 and Y_1 we have that $G_2(t)=F_1(t)F_2(t)$ and $G_1(t)=1-((1-F_1(t))(1-F_2(t))$.

Proof. $G_2(t)=Pr(Y_2<t)=Pr(X_1<t;X_2<t)=Pr(X_1<t)Pr(X_2<t)$ (by stochastic independence)

$$=F_1(t)F_2(t). \qquad (7)$$

Similarly

$$G_1(t)=1-Pr(Y_1>t)=1-Pr(X_1>t,\ X_2>t)=1-(1-F_1(t))(1-F_2(t)). \qquad (8)$$

Model 2.1. The (8) suggests the regression model $\ln\bar{F}_{Y_{2i}}(t)=\alpha_1\ln\bar{F}_{1i}(t)+\alpha_2\ln\bar{F}_{2i}(t)+\ln\varepsilon_i$, where $\varepsilon_i\sim N(0,1)$ and $\ln=\log_e$, $\bar{F}_{Y_1}(t)=Pr(Y_1>t)=\bar{F}_{\min(X_1,X_2)}(t)$, $\bar{F}_1(t)=Pr(X_1>t)$ and $\bar{F}_2(t)=Pr(X_2>t)$.

Model 2.2. The (7) suggests the regression model $\ln \bar{F}_{Y_{1i}}(t) = \beta_1 \ln \bar{F}_{1i}(t) + \beta_2 \ln \bar{F}_{2i}(t) + \ln \tau_i$, where $\tau_i \sim N(0,1)$ and $\ln = \log_e$, $\bar{F}_{Y_2}(t) = Pr(Y_2 > t) = \bar{F}_{\max(X_1,X_2)}(t)$, $\bar{F}_1(t) = Pr(X_1 > t)$ and $\bar{F}_2(t) = Pr(X_2 > t)$.

Theorem 2.2. If $\bar{F}_x(t)$ is a survivor function, then $\bar{F}_Y(t) = \frac{\beta \bar{F}_x(t)}{1-(1-\beta)\bar{F}_x(t)}, t > 0$ is a proper survivor function for all $0 < \beta \leq 1$.

Proof. This follows trivially from the definition.

From this relation, we have that

$$\frac{\beta(1 - \bar{F}_Y(t))}{\bar{F}_Y(t)} = \frac{1 - \bar{F}_X(t)}{\bar{F}_X(t)}. \tag{9}$$

(9) can be assumed to hold for

1. $\bar{F}_Y(t) \equiv \bar{F}_{\max(X_1,X_2)}(t)$ against $\bar{F}_X(t) \equiv \bar{F}_{X_1}(t)$,
2. $\bar{F}_Y(t) \equiv \bar{F}_{\max(X_1,X_2)}(t)$ against $\bar{F}_X(t) \equiv \bar{F}_{X_2}(t)$,
3. $\bar{F}_Y(t) \equiv \bar{F}_{\min(X_1,X_2)}(t)$ against $\bar{F}_X(t) \equiv \bar{F}_{X_1}(t)$,
4. $\bar{F}_Y(t) \equiv \bar{F}_{\min(X_1,X_2)}(t)$ against $\bar{F}_X(t) \equiv \bar{F}_{X_2}(t)$.

In view of all these we formulate the following set of regression models.

Model 2.3.

$$\frac{1 - \bar{F}_{Y_{2i}}(t)}{\bar{F}_{Y_{2i}}(t)} = \gamma_1 \frac{(1 - \bar{F}_{1i}(t))}{\bar{F}_{1i}(t)} + \gamma_2 \frac{(1 - \bar{F}_{2i}(t))}{\bar{F}_{2i}(t)} + \tau_i,$$

where $\tau_i \sim N(0,1)$ and $\bar{F}_{Y_2}(t) \equiv \bar{F}_{\max(X_1,X_2)}(t)$.

Model 2.4.

$$\frac{1 - \bar{F}_{Y_{1i}}(t)}{\bar{F}_{Y_{1i}}(t)} = \alpha_1 \frac{(1 - \bar{F}_{1i}(t))}{\bar{F}_{1i}(t)} + \alpha_2 \frac{(1 - \bar{F}_{Y_{2i}}(t))}{\bar{F}_{2i}(t)} + \varepsilon_i,$$

where $\varepsilon_i \sim N(0,1)$ and $\bar{F}_{Y_1}(t) \equiv \bar{F}_{\min(X_1,X_2)}(t)$.

Theorem 2.3. Assume that the random variable X is a symmetric triangular distribution with the probability density function symmetric about h given by

$$h(x) = \begin{cases} \frac{a+(x-h)}{a^2}, & h - a < x < h, \\ \\ \frac{a-(x-h)}{a^2}, & h < x < h + a, \end{cases}$$

then the corresponding cumulative distribution is given by

$$H(x) = \begin{cases} \frac{1}{2a^2}(x - (h - a))^2, & h - a < x < h, \\ \\ \frac{1}{2a^2}(a^2 + 2a(x - h) - (x - h)^2), & h < x < h + a. \end{cases}$$

Proof. See [1].

Theorem 2.4. Let $f_0(x)$ be a unimodal probability density function, $H(x)$ a cumulative distribution function having probability density function $h(x)$, and assume both $f_0(x)$ and $h(x)$ are symmetric about zero.

Let $f(x) = \frac{2(1+\alpha x^2)}{1+\alpha k} f_0(x) H(w(x)), \ \ -\infty < x < \infty$ and $k = \int_{-\infty}^{\infty} x^2 f_0(x)dx$, then $f(x)$ is a probability density function for any odd function $w(x)$ and any $\alpha > 0$.

Proof. See [6].

Corollary 2.1. In the particular case that $f_0(x)$ in Theorem 2.4 is a probability density function $w(x) = x$, $h(x) = \frac{1}{2}(1 - \frac{|x|}{a})_{I_{-a,a}(x)}$, so that

$$H(x) = \begin{cases} \frac{1}{2a^2}(a^2 + 2ax + x^2), & -a < x < 0, \\ \\ \frac{1}{2a^2}(a^2 + 2ax - x^2), & 0 < x < a. \end{cases}$$

The function

$$f(x) = \begin{cases} \frac{1+\alpha x^2}{\alpha^2(1+\alpha k)} f_0(x)(a^2 + 2ax + x^2), & -a < x < 0, \\ \\ \frac{1+\alpha x^2}{a^2(1+\alpha k)} f_0(x)(a^2 + 2ax - x^2), & 0 < x < a, \end{cases}$$

is also a probability density function. [1]

Remark 2.1. We note as follows.

1. the relationship in Corollary 2.1 can be simply expressed as $f(x) = \sum_{i=1}^{2} a_i x^i f_0(x)$ for certain co-efficient $a_i, \ i = 1, 2$.
2. The practical effect of Corollary 2.1 is to generate a possibly asymmetric and/or bimodal density distribution from a symmetric and unimodal density.
3. A survival function may essentially be geometrically different from another survival function by such characteristics (measures) as shift (mean), concentration (mode), spread (variance), asymmetry (skewness) and peakedness (kurtosis) all of which can be captured in the mechanism of the transformation in Models 2.5 and 2.6.

Model 2.5.

$$\bar{F}_{Y_{2i}}(t) = \sum_{i=1}^{2} a_i t^i \bar{F}_{1i}(t) + \sum_{i=1}^{2} b_i t^i \bar{F}_{2i}(t) + \sum_{i=1}^{2} m_i t^i \bar{F}_{1i}(t)\bar{F}_{2i}(t) + \varepsilon_i, \ \text{ where } \ \varepsilon_i \sim N(0,1).$$

Model 2.6.

$$\bar{F}_{Y_{1i}}(t) = \sum_{i=1}^{2} c_i t^i \bar{F}_{1i}(t) + \sum_{i=1}^{2} d_i t^i \bar{F}_{2i}(t) + \sum_{i=1}^{2} n_i t^i \bar{F}_{1i}(t)\bar{F}_{2i}(t) + \tau_i, \ \text{ where } \ \tau_i \sim N(0,1).$$

Theorem 2.5. If $\bar{F}(t)$ is a survival function, so also is, if $\bar{F}^*(t)$ defined by

$$\bar{F}^*(t) = e^{[-(\frac{1}{\bar{F}(t)})^{\theta-1}+1]}, \ \theta > 1.$$

Proof. This follows from the definition of survival function. In view of Theorem 2.5 we postulate that $\bar{F}_{\max}(t)$ can be expressed in terms of $\bar{F}_1(t)$ and $\bar{F}_2(t)$ as follows

$$\bar{F}_{\max}(t) = \left(e^{[-(\frac{1}{\bar{F}(t)})^{(\theta_1-1)}+1]} \times e^{[-(\frac{1}{\bar{F}(t)})^{(\theta_2-1)}+1]}\right). \tag{10}$$

Taking logarithm of (10), we have

$$\ln \bar{F}_{\max}(t) = 2 - (\bar{F}_1(t))^{1-\theta_1} - (\bar{F}_2(t))^{1-\theta_2},$$

where $\theta_1, \theta_2 > 1$ and $\ln = \log_e$.

We can thus develop a regression model as such

Model 2.7.

$$\ln \bar{F}_{Y_{2i}}(t) = \beta_0 + \alpha_1(\bar{F}_{1i}(t))^{-\frac{1}{4}} + \alpha_2(\bar{F}_{1i}(t))^{-\frac{3}{4}} + \alpha_3(\bar{F}_{2i}(t))^{-\frac{1}{4}} + \alpha_4(\bar{F}_{2i}(t))^{-\frac{3}{4}} + \varepsilon_i, \text{ where } \varepsilon_i \sim N(0,1).$$

Model 2.8.

$$\ln \bar{F}_{Y_{1i}}(t) = \beta_0 + \gamma_1(\bar{F}_{1i}(t))^{-\frac{1}{4}} + \gamma_2(\bar{F}_{1i}(t))^{-\frac{3}{4}} + \gamma_3(\bar{F}_{2i}(t))^{-\frac{1}{4}} + \gamma_4(\bar{F}_{2i}(t))^{-\frac{3}{4}} + \tau_i, \text{ where } \tau_i \sim N(0,1).$$

These sets of models were validated using a real data for first marginal component and simulated data for second marginal component. The parameters are generally somewhat correlated, it may thus not be advisable to independently choose the parameters in a modelling situation. For example, the maximum likelihood estimate for the real data assuming Weibull [7] distribution for time to failure is given as $\hat{\beta} = 1.823823334$ and $\hat{\eta} = 5666.376110$. In the light of these properties a set of data with some shape parameter (β) but double the scale parameter (η) will have double the mean. This point proved useful in the simulation exercise. The maximum likelihood estimates of the Weibull distribution for the real data were obtained and thus its mean (λ, say). Considered to be of special interest is the extent to which these sets of models were validated to different mix (proportion) of censored data and for different closeness or balance of the marginal components. The proportion p of censored data were varied for different amounts $p = 58/70, 35/70$ and $12/70$. In the same way different combinations of the marginal components were considered. Thus, we had cases (λ, λ) and $(\lambda, 75\lambda)$. This is to say that the distribution of the first marginal component is equal with that of the second marginal component in terms of the mean of the Weibull distribution and where there is a shift in the sense that the second marginal component is only .75 of the mean. This elucidated situations when censored points for the second component was high ($p = 58/70$) just like the original data; censored points were even ($p = 35/70$) and censored points were sparse ($p = 12/70$). In summary, each of the regression models were validated with six generated samples of the second marginal component against the real data from the literature adopted for the first marginal component. Among the statistical tools that were employed as appropriate for the validation of the performances of these models were the adjusted R^2 (adj. R^2) and error sum of squares (SSE). The necessary results are given on Table 2.1 for various levels of censoring i.e., highly, evenly and sparsely censored as well as for equal and shift in mean. Models 2.1, 2.3, 2.5 and 2.7 connect maximal survival functions to the marginal survival functions while Models 2.2, 2.4, 2.6 and 2.8 connect minimal survival functions to marginal survival functions.

Table 2.1. Adjusted R^2 and Residual Sum of Squares Values for the Maximal and Minimal Response function by level of censoring of Simulated and Shift of Mean for the Models relative to the real data (from combination $(\lambda, 58/70; \lambda, 58/70)$ to combination $(\lambda, 58/70; .75\lambda, 12/70)$.

Intensity (Mean)	Response Variable	Highly-censored		Evenly-censored		Sparsely-censored	
		Adj. R^2	SSE	Adj. R^2	SSE	Adj. R^2	SSE
Equal Mean	Model 2.1.	.980	.003	.979	.018	.879	.034
	Model 2.3.	.977	.004	.964	.054	.968	.191
	Model 2.5.	.909	.446	.889	.446	.896	.378
	Model 2.7.	.992	.000	.997	.001	1.000	.000
	Model 2.2.	.985	.003	.951	.018	.998	.010
	Model 2.4.	.985	.004	.965	.029	.977	.034
	Model 2.6.	.904	.470	.901	.468	.912	.391
	Model 2.8.	.996	.000	.994	.001	.999	.000
Shifted Mean	Model 2.1.	.989	.001	.937	.008	.968	.014
	Model 2.3.	.992	.001	.979	.011	.971	.019
	Model 2.5.	.929	.370	.964	.160	.948	.184
	Model 2.7.	.993	.000	.963	.004	.935	.009
	Model 2.2.	.998	.001	.805	27.814	.993	.083
	Model 2.4.	.999	.001	.761	1.844	.999	.333
	Model 2.6.	.916	.355	.951	1776.725	.920	.213
	Model 2.8.	.999	.000	.998	.003	.989	.071

The study have been able to achieve the formulation of regression models based on the original thought that every survival function is a transformation of another survival function by such characteristics (measure) as shift (mean), concentration (mode), spread (variance), degree of symmetry (skewness) and peakedness (kurtosis). Based on the analysis of the data used, the following conclusions were reached.

(1) all the sets of regression models can reliably predict minimal survival functions and maximal survival functions from marginal survival functions.

(2) irrespective of the level of censoring and shift in mean, Models 2.4. and 2.8. perform excellently in prediction of minimal response functions and the best overall is Model 2.8 with adjusted R^2 in the range (.989 - .999). Model 2.8 compared with Model 2.6 in the range (.901 - .951). Similarly, in terms of residual sum of squares Model 2.8 is in the range (.000 - .071) and Model 2.6 is in the range (.213 - 1776.725) irrespective of whether it is equal or shifted mean.

However for the prediction of maximal response function, the performance of Model 2.7 in relation to the others is less pronounced for shifted mean compared with when we have equal mean. This is especially so when using the criterion of the adjusted R^2. It is noted that Models 2.5 and 2.6 which makes allowance for bimodality have not performed well in relation to the others but it performed best where the proportion of censoring when the two components are closely equally censored (i.e. in the case where the data is approaching bimodal).

It is recommended that models such as the ones informed by (9) and (10), in particular 2.7, 2.4 and 2.8 (the best performing models) be adopted for studying failure times in phenomena that can be modelled in terms of components. This is particularly so in the transformation when the survival time dynamics of one component to another component can be considered to be shift (mean), concentration (mode), spread (variance), degree of symmetry (skewness) and peakedness (kurtosis).

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Fekete-Szegö Inequality for a Subclass of p-valently Alpha Convex Functions

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Abstract In the present paper, we obtain sharp upper bound of the functional $|a_{p+2} - \mu a_{p+1}^2|$ (μ real) for a subclass of p-valently alpha convex functions and deduce several results in the form of corollaries and remarks.

Keywords Subordination, bounded functions, p-valent and univalent functions, starlike functions, convex functions and α-convex functions.
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§1. Introduction

By U, we denote the class of bounded analytic functions $w(z)$ in the unit disc $E = \{z :| z |< 1\}$ and of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \; z \in E, \tag{1}$$

which satisfying the conditions $w(0) = 0, |w(z)| < 1$.

It is well known that

$$|c_1| \le 1, |c_2| \le 1 - |c_1|^2. \tag{2}$$

Let $f(z)$ and $F(z)$ be two analytic functions in the unit disc E, then $f(z)$ is said to be subordinate to $F(z)$ if there exists a function $w(z) \in U$ such that $f(z) = F(w(z))$ and we write as $f(z) \prec F(z)$.

In case the function F is univalent, the above subordination is equivalent to $f(0) = F(0)$ and $f(E) \subset F(E)$.

Let A_p(p is a positive integer) denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \tag{3}$$

which are analytic in the unit disc E.

By A, we denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{4}$$

analytic in the unit disc E.

S_p^* represents the class of p-valently starlike functions $f(z)$ in A_p and satisfying the condition

$$Re\left[\frac{zf'(z)}{f(z)}\right] > 0,\ z \in E. \tag{5}$$

$S_p^*(A,B)$ represents the class of functions $f(z)$ in A_p which satisfy the condition

$$\frac{zf'(z)}{f(z)} \prec p\frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1,\ z \in E. \tag{6}$$

Clearly $S_p^*(1,-1) \equiv S_p^*$, which shows that $S_p^*(A,B)$ is a subclass of S_p^*. To avoid repetition, we lay down, once for all that $-1 \leq B < A \leq 1$ and $z \in E$.

S^* is the class of analytic univalent starlike functions $f(z)$ in A such that

$$Re\left[\frac{zf'(z)}{f(z)}\right] > 0. \tag{7}$$

$S^*(A,B)$ is the subclass of univalent starlike functions $f(z)$ in A such that

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}. \tag{8}$$

Janowski [6] introduced the class $S^*(A,B)$ and proved some results by the method of variational techniques. Goel and Mehrok [2,3] studied the class $S^*(A,B)$ using subordination principle.

The class K_p consists of functions $f(z)$ in A_p for which

$$Re\left[\frac{(zf'(z))'}{f'(z)}\right] > 0. \tag{9}$$

Clearly $f(z) \in K_p$ implies that $\frac{zf'(z)}{p} \in S_p^*$.

$K_p(A,B)$ is the subclass of functions $f(z)$ in A_p for which

$$\frac{(zf'(z))'}{f'(z)} \prec p\frac{1+Az}{1+Bz}. \tag{10}$$

K stands for the class of convex univalent functions $f(z)$ in A with the condition

$$Re\left[\frac{(zf'(z))'}{f'(z)}\right] > 0. \tag{11}$$

$K(A,B)$ is the subclass of K of functions $f(z)$ in A with the condition

$$\frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}. \tag{12}$$

Let $M_p(\alpha)(\alpha \geq 0)$ denote the class of functions $f(z)$ in A_p which satisfy the conditions $\frac{f(z).f'(z)}{z^{2p-1}} \neq 0$ and

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right] > 0. \tag{13}$$

$M_p(\alpha; A, B)(\alpha \geq 0)$ denotes the subclass of $M_p(\alpha)$ of functions $f(z)$ in A_p and satisfying the conditions $\frac{f(z).f'(z)}{z^{2p-1}} \neq 0$ and

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} \prec p\frac{1+Az}{1+Bz}. \tag{14}$$

$M(\alpha)$ denotes the class of functions $f(z)$ in A and satisfying the conditions $\frac{f(z).f'(z)}{z} \neq 0$ and

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right] > 0. \tag{15}$$

The class $M(\alpha)$ was introduced by Mocanu [9]. Miller et al. [8] have shown that all α-convex functions are starlike in E and for $\alpha \geq 1$, all α-convex functions are convex in E.

$M(\alpha; A, B)(\alpha \geq 0)$ is the subclass of $M(\alpha)$ of functions $f(z)$ in A and satisfying the conditions $\frac{f(z).f'(z)}{z} \neq 0$ and

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} \prec p\frac{1+Az}{1+Bz}. \tag{16}$$

The class $M(\alpha; A, B)$ was studied by Goel and Mehrok [4,5]. Fekete and Szegö [1] made an early study for the bounds of $|a_3 - \mu a_2^2|$ (μ real) when $f(z)$ is analytic univalent in E. The well known result due to them states that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic and univalent in E, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu,\ \mu \leq 0; \\ 1 + 2exp(\frac{-2\mu}{1-\mu}),\ 0 \leq \mu \leq 1; \\ 4\mu - 3,\ \mu \geq 1. \end{cases}$$

The following observations are obvious:

(i) $M_p(\alpha; 1, -1) \equiv M_p(\alpha)$.

(ii) $M_1(\alpha; A, B) \equiv M(\alpha; A, B)$.

(iii) $M_p(0; A, B) \equiv S_p^*(A, B)$.

(iv) $M_p(0; 1, -1) \equiv S_p^*$.

(v) $M_1(0; A, B) \equiv S^*(A, B)$.

(vi) $M_1(0; 1, -1) \equiv S^*$.

(vii) $M_p(1; A, B) \equiv K_p(A, B)$.

(viii) $M_p(1; 1, -1) \equiv K_p$.

(ix) $M_1(1; A, B) \equiv K(A, B)$.

(x) $M_1(1; 1, -1) \equiv K$.

§2. Main results

Theorem 2.1. Let $f(z) \in M_p(\alpha; A, B)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{(A-B)^2 p^4}{(p+\alpha)^2}(\lambda - \mu), \ \mu \leq \lambda - \nu; \\ \\ \frac{(A-B)p^2}{2(p+2\alpha)}, \ \lambda - \nu \leq \mu \leq \lambda + \nu; \\ \\ \frac{(A-B)^2 p^4}{(p+\alpha)^2}(\mu - \lambda), \ \mu \geq \lambda + \nu. \end{cases} \tag{17}$$

where

$$\lambda = \frac{p(A-B)(p^2 + 2p\alpha + \alpha) - B(p+\alpha)^2}{2p^2(A-B)(p+2\alpha)} \tag{18}$$

and

$$\nu = \frac{(p+\alpha)^2}{2p^2(A-B)(p+2\alpha)}. \tag{19}$$

Results are sharp.

Proof. By definition of $M_p(\alpha; A, B)$, we have

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = p\frac{1+Aw(z)}{1+Bw(z)}.$$

Expanding the series, we get

$$\begin{aligned} &(1-\alpha)[p + a_{p+1}z + (2a_{p+2} - a_{p+1}^2)z^2 + ...] + \alpha\left[p + \frac{(p+1)}{p}a_{p+1}z\right] \\ &+\alpha\left[\frac{2(p+2)}{p}a_{p+2} - \frac{(p+1)^2}{p^2}a_{p+1}^2\right]z^2 + ... \\ = &\ p[1 + (A-B)c_1 z + (A-B)(c_2 - Bc_1^2)z^2 + ...]. \end{aligned} \tag{20}$$

Equating the coefficients of z and z^2 in (20), we obtain

$$a_{p+1} = \frac{(A-B)p^2 c_1}{(p+\alpha)} \tag{21}$$

and

$$a_{p+2} = \frac{(A-B)p^2 c_2}{2(p+2\alpha)} + \frac{(A-B)^2 p^3 (p^2 + 2p\alpha + \alpha)c_1^2}{2(p+2\alpha)(p+\alpha)^2} - \frac{Bp(A-B)c_1^2}{2(p+2\alpha)}. \tag{22}$$

From equations (21) and (22), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{(A-B)p^2 c_2}{2(p+2\alpha)} + \frac{(A-B)^2 p^4}{(p+\alpha)^2}(\lambda - \mu)c_1^2, \tag{23}$$

where λ is defined in (18).

Applying triangular inequality in (23), it yields

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2|c_2|}{2(p+2\alpha)} + \frac{(A-B)^2 p^4}{(p+\alpha)^2}|\lambda - \mu||c_1|^2. \tag{24}$$

Using (2) in (24), we obtain

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2}{2(p+2\alpha)} + \frac{(A-B)^2p^4}{(p+\alpha)^2}[|\lambda - \mu| - \nu], \tag{25}$$

where ν is defined in (19).

Case I. $\mu \leq \lambda$.

From (2.9), we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2}{2(p+2\alpha)} + \frac{(A-B)^2p^4}{(p+\alpha)^2}[(\lambda - \nu) - \mu]. \tag{26}$$

If $\mu \leq (\lambda - \nu)$, from (26)

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| &\leq \frac{(A-B)p^2}{2(p+2\alpha)} + \frac{(A-B)^2p^4}{(p+\alpha)^2}(\lambda - \mu - \nu) \\ &= \frac{(A-B)^2p^4}{(p+\alpha)^2}(\lambda - \mu). \end{aligned}$$

If $\lambda - \nu \leq \mu \leq \lambda$, again from (26)

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2}{2(p+2\alpha)}.$$

Case II. $\mu \geq \lambda$.

From (2.9), we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2}{2(p+2\alpha)} + \frac{(A-B)^2p^4}{(p+\alpha)^2}[\mu - (\lambda + \nu)]. \tag{27}$$

If $\mu \leq (\lambda + \nu)$, from (27)

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2}{2(p+2\alpha)}.$$

If $\mu \geq (\lambda + \nu)$, again from (27)

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)^2p^4}{(p+\alpha)^2}(\mu - \lambda).$$

For $\alpha = 0$, equality sign hold in first and third inequalities of (17) for the function $f_1(z)$ defined by

$$f_1(z) = \begin{cases} z^p(1 + B\delta z)^{\frac{p(A-B)}{B}}, \ B \neq 0; \\ \\ z^p e^{(pA\delta z)}, \ B = 0, |\delta| = 1. \end{cases} \tag{28}$$

and equality sign hold in second inequality of (17) for the function $f_0(z)$ defined by

$$f_0(z) = \begin{cases} z^p(1 + B\delta z^2)^{\frac{p(A-B)}{2B}}, \ B \neq 0; \\ \\ z^p e^{\frac{(pA\delta z^2)}{2}}, \ B = 0, |\delta| = 1. \end{cases} \tag{29}$$

For $\alpha > 0$, equality sign hold in first and third inequalities of (17) for the function $f_{1(\alpha)}(z)$ defined by

$$f_{1(\alpha)}(z) = \begin{cases} \left[\frac{1}{\alpha}\left(\int_0^z t^{\frac{p}{\alpha}-1}(1+B\delta t)^{\frac{p(A-B)}{B\alpha}}dt\right)\right]^\alpha, \ B \neq 0; \\ \left[\frac{1}{\alpha}\left(\int_0^z t^{\frac{p}{\alpha}-1}e^{\frac{(pA\delta t)}{\alpha}}dt\right)\right]^\alpha, \ B = 0, |\delta| = 1. \end{cases} \tag{30}$$

and equality sign hold in second inequality of (17) for the function $f_{0(\alpha)}(z)$ defined by

$$f_{0(\alpha)}(z) = \begin{cases} \left[\frac{1}{\alpha}\left(\int_0^z t^{\frac{p}{\alpha}-1}(1+B\delta t^2)^{\frac{p(A-B)}{2B\alpha}}dt\right)\right]^\alpha, \ B \neq 0; \\ \left[\frac{1}{\alpha}\left(\int_0^z t^{\frac{p}{\alpha}-1}e^{\frac{(pA\delta t^2)}{2\alpha}}dt\right)\right]^\alpha, \ B = 0, |\delta| = 1. \end{cases} \tag{31}$$

Remark 2.1. Choosing $p = 1$ in the Theorem 2.1, we get the results proved by Goel and Mehrok [4].

Choosing $A = 1, B = -1$ in the Theorem 2.1, we have

Corollary 2.1. If $f \in M_p(\alpha)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{4p^4}{(p+\alpha)^2}(\lambda_1 - \mu), \ \mu \leq \lambda_1 - \nu_1; \\ \frac{p^2}{p+2\alpha}, \ \lambda_1 - \nu_1 \leq \mu \leq \lambda_1 + \nu_1; \\ \frac{4p^4}{(p+\alpha)^2}(\mu - \lambda_1), \ \mu \geq \lambda_1 + \nu_1. \end{cases} \tag{32}$$

where

$$\lambda_1 = \frac{2p(p^2 + 2p\alpha + \alpha) + (p+\alpha)^2}{4p^2(p+2\alpha)}$$

and

$$\nu_1 = \frac{(p+\alpha)^2}{4p^2(p+2\alpha)}.$$

For $\alpha = 0$, the extremal function for the first and third inequalities of (32) is obtained by putting $A = 1, B = -1$ in (28) and extremal function for the second inequality of (30) is obtained by putting $A = 1, B = -1$ in (29).

For $\alpha > 0$, the extremal function for the first and third inequalities of (32) is obtained by putting $A = 1, B = -1$ in (30) and extremal function for the second inequality of (32) is obtained by putting $A = 1, B = -1$ in (31).

Choosing $\alpha = 0$ in the Theorem 2.1, we have

Corollary 2.2. If $f \in S_p^*(A, B)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{p(A-B)}{2}[(A-B)p - B] - p^2(A-B)^2\mu, \ \mu \leq \frac{(A-B)p-(1+B)}{2(A-B)p}; \\ \frac{p(A-B)}{2}, \ \frac{(A-B)p-(1+B)}{2(A-B)p} \leq \mu \leq \frac{(A-B)p+(1-B)}{2(A-B)p}; \\ p^2(A-B)^2\mu - \frac{p(A-B)}{2}[(A-B)p - B], \ \mu \geq \frac{(A-B)p+(1-B)}{2(A-B)p}. \end{cases} \tag{33}$$

The first and third inequalities of (33) are sharp for the function $f_1(z)$ defined in (28) and the second inequality is sharp for the function $f_0(z)$ defined in (29).

Remark 2.2. By taking $p = 1$ in the Corollary 2.2, we get the estimates of $|a_3 - \mu a_2^2|$ for the class $S^*(A, B)$.

Remark 2.3. Taking $A = 1$, $B = -1$ in the Corollary 2.2, we obtain the following result: If $f \in S_p^*$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} p(2p+1) - 4p^2\mu,\ \mu \leq \frac{1}{2}; \\ \\ p,\ \frac{1}{2} \leq \mu \leq \frac{p+1}{2p}; \\ \\ 4p^2\mu - p(2p+1),\ \mu \geq \frac{p+1}{2p}. \end{cases}$$

Remark 2.4. Taking $p = 1, A = 1, B = -1$ in the Corollary 2.2, we obtain the following result: If $f \in S^*$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu,\ \mu \leq \frac{1}{2}; \\ \\ 1,\ \frac{1}{2} \leq \mu \leq 1; \\ \\ 4\mu - 3,\ \mu \geq 1. \end{cases}$$

This result is due to Keogh and Merkes [7].

Putting $\alpha = 1$ in the Theorem 2.1, we get

Corollary 2.3. If $f \in K_p(A, B)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{(A-B)[(A-B)p-B]p^2}{2(p+2)} - \frac{(A-B)^2p^4}{(p+1)^2}\mu,\ \mu \leq \frac{(p+1)^2[(A-B)p-(1+B)]}{2(A-B)p^2(p+2)}; \\ \\ \frac{(A-B)p^2}{2(p+2)},\ \frac{(p+1)^2[(A-B)p-(B+1)]}{2(A-B)p^2(p+2)} \leq \mu \leq \frac{(p+1)^2[(A-B)p+(1-B)]}{2(A-B)p^2(p+2)}; \\ \\ \frac{(A-B)^2p^4}{(p+1)^2}\mu - \frac{(A-B)[(A-B)p-B]p^2}{2(p+2)},\ \mu \geq \frac{(p+1)^2[(A-B)p+(1-B)]}{2(A-B)p^2(p+2)}. \end{cases} \quad (34)$$

The first and third inequalities of (34) are sharp for the function defined by

$$f_1(z) = \begin{cases} \int_0^z pt^{p-1}(1 + B\delta t)^{\frac{p(A-B)}{B}} dt,\ B \neq 0; \\ \\ \int_0^z pt^{p-1}e^{pA\delta t}dt,\ B = 0, |\delta| = 1. \end{cases}$$

and the second inequality of (34) is sharp for the function defined by

$$f_0(z) = \begin{cases} \int_0^z pt^{p-1}(1 + B\delta t^2)^{\frac{p(A-B)}{2B}} dt,\ B \neq 0; \\ \\ \int_0^z pt^{p-1}e^{\frac{pA\delta t^2}{2}} dt,\ B = 0, |\delta| = 1. \end{cases}$$

Remark 2.5. Letting $p = 1$ in Corollary 2.3, we obtain the estimates of $|a_3 - \mu a_2^2|$ for the class $K(A, B)$.

Remark 2.6.

Letting $A = 1, B = -1$ in Corollary 2.3, we have the result: If $f \in K_p$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{p^2(2p+1)}{p+2} - \frac{4p^4}{(p+1)^2}\mu, \ \mu \leq \frac{(p+1)^2}{2p(p+2)}; \\ \\ \frac{p^2}{p+2}, \ \frac{(p+1)^2}{2p(p+2)} \leq \mu \leq \frac{(p+1)^3}{2p^2(p+2)}; \\ \\ \frac{4p^4}{(p+1)^2}\mu - \frac{p^2(2p+1)}{p+2}, \ \mu \geq \frac{(p+1)^3}{2p^2(p+2)}. \end{cases}$$

Remark 2.7. Taking $p = 1, A = 1, B = -1$ in the Corollary 2.3, we obtain the following result: If $f \in K$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu, \ \mu \leq \frac{2}{3}; \\ \\ \frac{1}{3}, \ \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\ \\ \mu - 1, \ \mu \geq \frac{4}{3}. \end{cases}$$

This result is due to Keogh and Merkes [7].

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The mean value of function $q_k^{(e)}(n)$ over cube-full number

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Abstract Let $n > 1$ be an integer, $q_k^{(e)}(n)$ is a multiplicative function. In this paper, we shall study the mean value of exponential divisor function $q_k^{(e)}(n)$ over cube-full number, that is

$$\sum_{\substack{n\le x\\ n\ is\ cube-full}} q_k^{(e)}(n) = \sum_{n\le x} q_k^{(e)}(n) f_3(n),$$

where $f_3(n)$ is the characteristic function of cube-full integers, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full;} \\ 0, & \text{otherwise.} \end{cases}$$

Keywords cube-full number, divisor problem, Dirichlet convolution, residue theorem, mean value.

§1. Introduction

Let $n > 1$ be an integer of canonical form $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. The integer n is called a k-full number if $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $a_1 \ge k, a_2 \ge k, \cdots, a_r \ge k$. Let $f_k(n)$ be the characteristic function of k-full integers, i.e.

$$f_k(n) = \begin{cases} 1, & n \text{ is } k\text{-full;} \\ 0, & \text{otherwise.} \end{cases}$$

We call an integer $n = \prod_{i=1}^{r} p_i^{a_i}$ exponentially k-free if all the exponents $a_i (1 \le i \le r)$ are k-free, i.e. are not divisible by the k-th power of any prime ($k \ge 2$). Let $q_k^{(e)}(n)$ denote the characteristic function of exponentially k-free integers.

Obviously the function $q_k^{(e)}(n)$ is multiplicative, and for every prime power p^a, there are $q_k^{(e)}(p) = q_k^{(e)}(p^2) = q_k^{(e)}(p^3) = \ldots = q_k^{(e)}(p^{2^k - 1}) = 1, q_k^{(e)}(p^{2^k}) = 0$.

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Many authors have studied the properties of the exponential divisor function $q_k^{(e)}(n)$. L. Toth [2] proved the following result:

$$\sum_{n\le x} q_k^{(e)}(n) = D_k x + O(x^{1/2^k}\delta(x)),$$

where

$$D_k = \prod_p \Big(1 + \sum_{a=2^k}^{\infty} \frac{q_k(a)-q_k(a-1)}{p^a}\Big),$$

$q_k(n)$ denoting the characteristic function of k-free integers.

In this paper, we shall study the mean value of exponential divisor function $q_k^{(e)}(n)$ over cube-full number, that is

$$\sum_{\substack{n\le x\\ n\ is\ cube-full}} q_k^{(e)}(n) = \sum_{n\le x} q_k^{(e)}(n) f_3(n),$$

where $f_3(n)$ is the characteristic function of cube-full integers, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full;} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.1. For some $D > 0$,

$$\sum_{\substack{n\le x\\ n\ is\ cube-full}} q_k^{(e)}(n) = \frac{\zeta(\frac{4}{3})\zeta(\frac{5}{3})G(\frac{1}{3})}{\zeta(\frac{8}{3})} x^{\frac{1}{3}} + \frac{\zeta(\frac{3}{4})\zeta(\frac{5}{4})G(\frac{1}{4})}{\zeta(2)} x^{\frac{1}{4}} + \frac{\zeta(\frac{3}{5})\zeta(\frac{4}{5})G(\frac{1}{5})}{\zeta(\frac{8}{5})} x^{\frac{1}{5}}$$

$$+ O\Big(x^{\frac{1}{8}} \exp\big(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\big)\Big),$$

where $G(s) = \prod_p \big(1 - \frac{1}{p^{9s}} + \cdots\big)$ is absolutely convergent for $Res > \frac{1}{9}$.

Notations 1.1. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

§2. Proof of the theorem

In order to prove our theorem, we need the following lemmas.

Lemma 2.1. [3]

$$\Delta(3,4,5;x) \ll x^{\frac{22}{177}} \log^3 x.$$

Proof. The proof of this bound depends on the theory of two-dimensional exponent pairs.

Lemma 2.2. Suppose $f(n)$ is arithmetical function, and satisfy

$$\sum_{n\le x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O(x^a),$$

$$\sum_{n\le x} |f(n)| = O(x^{a_1}\log^r x),$$

here $a_1 \geq a_2 \geq \cdots \geq a_l > 1/c > a \geq 0, r \geq 0, P_1(t), \cdots, P_l(t)$ are polynomials on t whose degree are not exceed r, and $c \geq 1$, $b \geq 1$ are fixed integers.

Suppose for $Res > 1$, have

$$\sum_{n=1}^{\infty} \frac{\mu_b(n)}{n^s} = \frac{1}{\zeta^b(s)}.$$

If

$$h(n) = \sum_{d^c|n} \mu_b(d) f(n/d^c),$$

then

$$\sum_{n\leq x} h(n) = \sum_{j=1}^{l} x^{a_j} R_j(\log x) + E_c(x),$$

here $R_1(t), \cdots, R_l(t)$ are polynomials on t whose degree are not exceed r, and for some $D > 0$, have

$$E_c(x) \ll x^{1/c} \exp\Big(\big(-D(\log x)^{3/5}(\log\log x)^{-1/5}\big)\Big).$$

Proof. If $b = 1$, see theorem 14.2 in A. Ivić [2]. If $b \geq 2$, it can be proved in the same method.

Lemma 2.3. Let $s = \sigma + it$ is a complex number, then we have

$$\sum_{\substack{n=1 \\ n\ is\ cube-full}}^{\infty} \frac{q_k^{(e)}(n)}{n^s} = \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} G(s), \tag{1}$$

where the Dirichlet series $G(s) = \prod_p \left(1 - \frac{1}{p^{9s}} + \cdots\right)$ is absolutely convergent for $Res > \frac{1}{9}$.

Proof. Let

$$F(s) := \sum_{\substack{n=1 \\ n\ is\ cube-full}}^{\infty} \frac{q_k^{(e)}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{q_k^{(e)}(n) f_3(n)}{n^s}, (Res > 1)$$

here

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full;} \\ 0, & \text{otherwise.} \end{cases}$$

From $q_k^{(e)}(n)$ is multiplicative we have for $Res > 1$ that,

$$\begin{aligned}
F(s) &= \sum_{\substack{n=1 \\ n \text{ is cube-full}}}^{\infty} \frac{q_k^{(e)}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{q_k^{(e)}(n) f_3(n)}{n^s} \\
&= \prod_p \left(1 + \frac{q_k^{(e)}(p) f_3(p)}{p^s} + \frac{q_k^{(e)}(p^2) f_3(p^2)}{p^{2s}} + \frac{q_k^{(e)}(p^3) f_3(p^3)}{p^{3s}} + \cdots + \frac{q_k^{(e)}(p^{2^k-1}) f_3(p^{2^k-1})}{p^{(2^k-1)s}}\right) \\
&= \prod_p \left(1 + \frac{1}{p^{3s}} + \frac{1}{p^{4s}} + \frac{1}{p^{5s}} + \cdots + \frac{1}{p^{(2^k-1)s}}\right) \\
&= \zeta(3s) \prod_p \left(1 + \frac{1}{p^{4s}} + \frac{1}{p^{5s}} - \frac{1}{p^{2^k s}} - \frac{1}{p^{(2^k+1)s}} - \frac{1}{p^{(2^k+2)s}}\right) \\
&= \zeta(3s)\zeta(4s) \prod_p \left(1 + \frac{1}{p^{5s}} - \frac{1}{p^{8s}} - \frac{1}{p^{9s}} - \frac{1}{p^{2^k s}} - \frac{1}{p^{(2^k+1)s}} + \cdots\right) \\
&= \zeta(3s)\zeta(4s)\zeta(5s) \prod_p \left(1 - \frac{1}{p^{8s}} - \frac{1}{p^{9s}} + \cdots\right) \\
&= \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} \prod_p \left(1 - \frac{1}{p^{9s}} + \cdots\right) \\
&= \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} G(s),
\end{aligned}$$

where $G(s) = \prod_p \left(1 - \frac{1}{p^{9s}} + \cdots\right)$, and it is absolutely convergent for $\sigma > \frac{1}{9} + \epsilon$.

Next we prove our theorem.

From lemma 2.3, we have known

$$F(s) := \sum_{\substack{n=1 \\ n \text{ is cube-full}}}^{\infty} \frac{q_k^{(e)}(n)}{n^s} = \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} G(s),$$

where $G(s)$ is absolutely convergent for $\sigma > \frac{1}{9} + \epsilon$.

Define

$$G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

$$H(s) := \zeta(3s)\zeta(4s)\zeta(5s)G(s) := \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\sum_{n=ml} d(3,4,5;m)g(l)}{n^s}, (Res > 1)$$

here $h(n) = \sum_{n=ml} d(3,4,5;m)g(l)$. Then we can get

$$\sum_{n \le x} h(n) = \sum_{ml \le x} d(3,4,5;m)g(l) = \sum_{l \le x} g(l) \sum_{m \le \frac{x}{l}} d(3,4,5;m).$$

From Residue theorem and lemma 2.1, we can get

$$\begin{aligned}
\sum_{n \le x} d(3,4,5;n) &= \zeta(\frac{4}{3})\zeta(\frac{5}{3})x^{\frac{1}{3}} + \zeta(\frac{3}{4})\zeta(\frac{5}{4})x^{\frac{1}{4}} + \zeta(\frac{3}{5})\zeta(\frac{4}{5})x^{\frac{1}{5}} + \Delta(3,4,5;x) \\
&= \zeta(\frac{4}{3})\zeta(\frac{5}{3})x^{\frac{1}{3}} + \zeta(\frac{3}{4})\zeta(\frac{5}{4})x^{\frac{1}{4}} + \zeta(\frac{3}{5})\zeta(\frac{4}{5})x^{\frac{1}{5}} + O(x^{\frac{22}{177}}). \qquad (2)
\end{aligned}$$

Then from (2) and Abel integral formula, we have the relation:

$$\begin{aligned}\sum_{n\le x}h(n)&=\sum_{l\le x}g(l)\Big[\zeta(\frac{4}{3})\zeta(\frac{5}{3})(\frac{x}{l})^{\frac13}+\zeta(\frac{3}{4})\zeta(\frac{5}{4})(\frac{x}{l})^{\frac14}+\zeta(\frac{3}{5})\zeta(\frac{4}{5})(\frac{x}{l})^{\frac15}+O((\frac{x}{l})^{\frac{22}{177}})\Big]\\&=\zeta(\frac{4}{3})\zeta(\frac{5}{3})x^{\frac13}\sum_{l\le x}\frac{g(l)}{l^{1/3}}+\zeta(\frac{3}{4})\zeta(\frac{5}{4})x^{\frac14}\sum_{l\le x}\frac{g(l)}{l^{1/4}}+\zeta(\frac{3}{5})\zeta(\frac{4}{5})x^{\frac15}\sum_{l\le x}\frac{g(l)}{l^{1/5}}+O\Big(x^{\frac{22}{177}}\sum_{l\le x}\frac{|g(l)|}{l^{22/177}}\Big)\\&=\zeta(\frac{4}{3})\zeta(\frac{5}{3})x^{\frac13}\sum_{l=1}^{\infty}\frac{g(l)}{l^{1/3}}+\zeta(\frac{3}{4})\zeta(\frac{5}{4})x^{\frac14}\sum_{l=1}^{\infty}\frac{g(l)}{l^{1/4}}+\zeta(\frac{3}{5})\zeta(\frac{4}{5})x^{\frac15}\sum_{l=1}^{\infty}\frac{g(l)}{l^{1/5}}+O\Big(x^{\frac13}\sum_{l>x}\frac{|g(l)|}{l^{1/3}}\Big)\\&\quad+O\Big(x^{\frac14}\sum_{l>x}\frac{|g(l)|}{l^{1/4}}\Big)+O\Big(x^{\frac15}\sum_{l>x}\frac{|g(l)|}{l^{1/5}}\Big)+O\Big(x^{\frac{22}{177}}\sum_{l\le x}\frac{|g(l)|}{l^{22/177}}\Big)\\&=\zeta(\frac{4}{3})\zeta(\frac{5}{3})G(\frac{1}{3})x^{\frac13}+\zeta(\frac{3}{4})\zeta(\frac{5}{4})G(\frac{1}{4})x^{\frac14}+\zeta(\frac{3}{5})\zeta(\frac{4}{5})G(\frac{1}{5})x^{\frac15}+O(x^{\frac19}).\end{aligned}$$

From lemma 2.2 and Perron's formula [4], we can get:

$$\begin{aligned}\sum_{\substack{n\le x\\ n\ is\ cube-full}}q_k^{(e)}(n)&=\frac{\zeta(\frac43)\zeta(\frac53)G(\frac13)}{\zeta(\frac83)}x^{\frac13}+\frac{\zeta(\frac34)\zeta(\frac54)G(\frac14)}{\zeta(2)}x^{\frac14}+\frac{\zeta(\frac35)\zeta(\frac45)G(\frac15)}{\zeta(\frac85)}x^{\frac15}\\&\quad+O\Big(x^{\frac18}\exp\big(-D(\log x)^{\frac35}(\log\log x)^{-\frac15}\big)\Big).\end{aligned}$$

Here $D>0$, and $G(s)=\prod_p\big(1-\frac{1}{p^{9s}}+\cdots\big)$ is absolutely convergent for $Res>\frac19$.

Now our theorem is proved.

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Some arithmetical properties of the Smarandache series

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Abstract The Smarandache function $S(n)$ is defined as the minimal positive integer m such that $n|m!$. The main purpose of this paper is to study the analyze converges questions for some series of the form $\sum_{n=1}^{\infty} \frac{1}{S(n)^{\delta}}$, i.e., we proved the series $\sum_{n=1}^{\infty} \frac{1}{S(n)^{\delta}}$ diverges for any $\delta \leq 1$, and $\sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon S(n)}}$ converges for any $\epsilon > 0$.

Keywords Smarandache function, smarandache series, converges.

§1. Introduction and results

For every positive integer n, let $S(n)$ be the minimal positive integer m such that $n|m!$, i.e.,

$$S(n) = \min\{m : m \in \mathbb{N}, n|m!\}.$$

This function is known as Smarandache function [1]. Easily, one has $S(1) = 1, S(2) = 2, S(3) = 3, S(4) = 4, S(5) = 5, S(6) = 3, S(7) = 7, S(8) = 4, S(9) = 6, S(10) = 5, \cdots$.

Use the standard factorization of $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $p_1 < p_2 < \cdots < p_k$, it's trivial to have

$$S(n) = \max_{1 \leq i \leq k} \{S(p_i^{\alpha_i})\}.$$

Many scholars have studied the properties of $S(n)$, for example, M. Farris and P. Mitchell [2] show the boundary of $S(p^{\alpha})$ as

$$(p-1)\alpha + 1 \leq S(p^{\alpha}) \leq (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Z. Xu [3] noticed the following interesting relationship formula

$$\pi(x) = -1 + \sum_{n=2}^{[x]} \left[\frac{S(n)}{n} \right],$$

by the fact that $S(p) = p$ for p prime and $S(n) < n$ except for the case $n = 4$ and $n = p$, where $\pi(x)$ denotes the number of prime up to x, and $[x]$ the greatest integer less or equal to x. Those and many other interesting results on Smarandache function $S(n)$, readers may refer to [2]-[6].

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Let p be a fixed prime and $n \in \mathbb{N}$, the primitive numbers of power p, denoted by $S_p(n)$, is defined by

$$S_p(n) = \min\{m : m \in \mathbb{N}, p^n | m!\} = S(p^n).$$

Z. Xu [3] obtained the identity between Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \sigma > 1$ and an infinite series involving $S_p(n)$ as

$$\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

and he also obtained some other asymptotic formulae for $S_p(n)$. F. Luca [4] proved the series $\sum_{n=1}^{\infty} \frac{1}{S(n)^{S(n)^\delta}}$ converges for all $\delta \geq 1$ and diverges for all $\delta < 1$, and the series $\sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon \log n}}$ converges for any $\varepsilon > 0$.

In this note, we studied the analyze converges problems for the infinite series involving $S(n)$. That is, we shall prove the following conclusions:

Theorem 1.1. For any $\delta \leq 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^\delta}$$

diverges.

Theorem 1.2. For any $\varepsilon > 0$, the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{\varepsilon S(n)}}$$

converges.

§2. Some lemmas

To complete the proof of theorems, we need two Lemmas.

Lemma 2.1. Let p be any fixed prime. Then for any real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1}\left(\ln x + \gamma + \frac{p \ln p}{p-1}\right) + O(x^{-\frac{1}{2}} + \epsilon),$$

where γ is the Euler constant, ϵ denotes any fixed positive numbers.

Proof. See Theorem 2 of [3].

Lemma 2.2.[7] Let $\epsilon > 0$ and $d(n)$ denotes the divisor function of positive integer n. Then

$$d(n) = O(n^\epsilon) \leq C_\epsilon n^\epsilon,$$

where the o-constant C_ϵ depends on ϵ.

Proof. The proof follows [7] by writing $n = \prod_{p|n} p^\alpha$, the standard factorization of n. Then

$$p^{\alpha\epsilon} \geq 2^{\alpha\epsilon} = e^{\alpha\epsilon \ln 2} \geq \alpha\epsilon \ln 2 \geq \frac{1}{2}(a+1)\epsilon \ln 2.$$

If $p^\epsilon \geq 2$, then $p^{\alpha\epsilon} \geq 2^\alpha \geq \alpha + 1$. Therefore,

$$\frac{d(n)}{n^\epsilon} = \prod_{p|n} \frac{\alpha+1}{p^{\alpha\epsilon}} = \prod_{\substack{p|n \\ p^\epsilon<2}} \frac{\alpha+1}{p^{\alpha\epsilon}} \prod_{\substack{p|n \\ p^\epsilon\geq 2}} \frac{\alpha+1}{p^{\alpha\epsilon}} \geq \prod_{\substack{p|n \\ p^\epsilon<2}} \frac{\alpha+1}{\frac{1}{2}(a+1)\epsilon\ln 2} \prod_{\substack{p|n \\ p^\epsilon\geq 2}} \frac{\alpha+1}{\alpha+1}.$$

The last item in above inequality is $\prod_{\substack{p|n \\ p^\epsilon<2}} \frac{2}{\epsilon\ln 2}$, which is less than $\prod_{p^\epsilon<2} \frac{2}{\epsilon\ln 2} = C_\epsilon$, say, the o-constant C_ϵ depends on ϵ.

§3. Proof of theorems

Proof of Theorem 1.

We may treat the case $\delta = 1$ first. By Lemma 1 and the notation $S_p(n) = S(p^n)$, we have

$$\sum_{n=1}^{\infty} \frac{1}{S(p^n)} = \lim_{x\to+\infty} \sum_{\substack{n=1 \\ S_p(n)\leq x}}^{\infty} \frac{1}{S_p(n)} = \infty.$$

Obviously, for $\delta \leq 1$, $\sum\limits_{n=1}^{\infty} \frac{1}{S(n)^\delta}$ diverges follows easily by the trivial inequality:

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^\delta} \geq \sum_{n=1}^{\infty} \frac{1}{S(n)} \geq \sum_{n=1}^{\infty} \frac{1}{S(p^n)},$$

complete the proof.

Proof of Theorem 2.

It certainly suffices to assume that $\epsilon \leq 1$. We rewrite series $\sum\limits_{n=1}^{\infty} \frac{1}{S(n)^{\varepsilon S(n)}}$ as

$$\sum_{k=1}^{\infty} \frac{u(k)}{k^{\varepsilon k}},$$

where $u(k) = \sharp\{n : S(n) = k\}$. For every positive integer n such that $S(n) = k$ is a divisor of $k!$, i.e. $u(k) \leq d(k!)$. By Lemma 2 and the inequality bellow

$$(k!)^2 = \prod_{j=1}^{k} j(k+1-j) < \prod_{j=1}^{k} \left(\frac{k+1}{2}\right)^2 = \left(\frac{k+1}{2}\right)^{2k}.$$

we have

$$u(k) \leq d(k!) \leq C_\epsilon (k!)^\epsilon < C_\epsilon \left(\frac{k+1}{2}\right)^{\epsilon k}.$$

where C_ϵ means that the constant depending on ϵ.

Therefore, recalling that the properties of the sequence $\left(1+\frac{1}{k}\right)^k$, we have

$$\sum_{k=1}^{\infty} \frac{u(k)}{k^{\varepsilon k}} \leq C_\epsilon \sum_{k=1}^{\infty} \frac{1}{k^{\varepsilon k}} \left(\frac{k+1}{2}\right)^{\epsilon k} = C_\epsilon \sum_{k=1}^{\infty} \frac{1}{2^{\varepsilon k}} \left(\frac{k+1}{k}\right)^{\epsilon k} < C_1 \sum_{k=1}^{\infty} \frac{1}{2^{\varepsilon k}},$$

for some constant C_1, it follows that series $\sum_{k=1}^{\infty} \frac{u(k)}{k^{\varepsilon k}}$ is bounded above by

$$C_1 \sum_{k=1}^{\infty} \frac{1}{2^{\varepsilon k}} = \frac{C_1}{2^{\varepsilon} - 1},$$

completing the proof.

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Transmuted Weibull-geometric distribution and its applications

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Abstract A generalization of the Weibull geometric distribution so-called the transmuted Weibull geometric distribution is proposed and studied. Various structural properties including explicit expressions for the moments, moment generating function of the new distribution are derived. The estimation of the model parameters is performed by maximum likelihood method. We hope that the new distribution proposed here will serve as an alternative model to the other models which are available in the literature for modeling positive real data in many areas.

Keywords Weibull-geometric distribution, moments, order statistics, transmutation map, maximum likelihood estimation, reliability function.
2000 Mathematics Subject Classification: 62E15, 62F30.

§1. Introduction and motivation

The Weibull distribution is one of the most commonly used lifetime distribution in modeling lifetime data. In practice, it has been shown to be very flexible in modeling various types of lifetime distributions with monotone failure rates but it is not useful for modeling the bathtub shaped and the unimodal failure rates which are common in reliability and biological studies. Several distributions have been proposed in the literature to model lifetime data. Adamidis and Loukas [2] introduced the two-parameter exponential-geometric (EG) distribution with decreasing failure rate. Kus [9] introduced the exponential-Poisson distribution (following the same idea of the EG distribution) with decreasing failure rate and discussed various of its properties. Marshall and Olkin [17] presented a method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Adamidis et al. [1] proposed the extended exponential-geometric (EEG) distribution which generalizes the EG distribution and discussed various of its statistical properties along with its reliability features. The hazard function of the EEG distribution can be monotone decreasing, increasing or constant. Barreto-Souza et al. [6] introduced a Weibull-geometric (WG) distribution following from compounded a Weibull distribution with a geometric distribution and obtained a new lifetime distribution with more flexibility than the Weibull distribution and study some of its properties. Suppose that $\{Y_i\}_{i=1}^{Z}$ are independent and identically distributed (iid) random variables following the Weibull distribution $W(\alpha,\theta)$ with scale parameter $\alpha > 0$, shape parameter $\theta > 0$ and probability density function

$$g(x;\alpha,\theta) = \theta\alpha^{\theta}x^{\theta-1}e^{-(\alpha x)^{\theta}}, x > 0,$$

and N a discrete random variable having a geometric distribution with probability function $P(n;p) = (1-p)p^{n-1}$ for $n \in N$ and $p \in (0,1)$. Let $X = \min\{Y_i\}_{i=1}^{N}$. The marginal pdf of X is given by

$$f(x;p,\alpha,\theta) = \theta\alpha^{\theta}(1-p)x^{\theta-1}e^{-(\alpha x)^{\theta}}\left[1-pe^{-(\alpha x)^{\theta}}\right]^{-2}, x > 0, \tag{1}$$

which defines the WG distribution (see Barreto-Souza et al. [6]. It is evident that (1) is much more flexible than the Weibull distribution. The EG distribution is a special case of the WG distribution for $\theta = 1$. When p approaches zero, the WG distribution leads to the Weibull $W(\alpha,\theta)$ distribution. The corresponding cumulative distribution function and hazard rate function are given by

$$F(x;p,\alpha,\theta) = \frac{1-e^{-(\alpha x)^{\theta}}}{1-pe^{-(\alpha x)^{\theta}}}, x > 0, \tag{2}$$

and

$$h(x;p,\alpha,\theta) = \theta\alpha^{\theta}x^{\theta-1}\left[1-pe^{-(\alpha x)^{\theta}}\right]^{-1}, \tag{3}$$

respectively.

In this article we defined the family of transmuted Weibull-geometric distribution. The main feature of this model is that a transmuted parameter is introduced in the subject distribution which provides greater flexibility in the form of new distributions. Using the quadratic rank transmutation map studied by Shaw et al. [18], we develop the four parameter transmuted Weibull-geometric $TWG(x;p,\alpha,\theta,\lambda)$. We provide a comprehensive description of mathematical properties of the subject distribution with the hope that it will attract wider applications in reliability, engineering and in other areas of research. The concept of transmuted explained in the following subsection.

§2. Transmutation map

In this subsection we demonstrate transmuted probability distribution. Let F_1 and F_2 be the cumulative distribution functions, of two distributions with a common sample space. The general rank transmutation as given in Shaw et al. [18] is defined as

$$G_{R12}(u) = F_2(F_1^{-1}(u)) \text{ and } G_{R21}(u) = F_1(F_2^{-1}(u)).$$

Note that the inverse cumulative distribution function also known as quantile function is defined as

$$F^{-1}(y) = \inf_{x\in R}\{F(x) \geq y\} \text{ for } y \in [0,1].$$

The functions $G_{R12}(u)$ and $G_{R21}(u)$ both map the unit interval $I = [0,1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $G_{Rij}(0) = 0$ and $G_{Rij}(0) = 1$. A quadratic Rank Transmutation Map (QRTM) is defined as

$$G_{R12}(u) = u + \lambda u(1-u), |\lambda| \leq 1, \tag{4}$$

from which it follows that the cdf's satisfy the relationship

$$F_2(x) = (1+\lambda)F_1(x) - \lambda F_1(x)^2, \tag{5}$$

which on differentiation yields,

$$f_2(x) = f_1(x)\left[(1+\lambda) - 2\lambda F_1(x)\right] \tag{6}$$

where $f_1(x)$ and $f_2(x)$ are the corresponding pdfs associated with cdf $F_1(x)$ and $F_2(x)$ respectively. An extensive information about the quadratic rank transmutation map is given in Shaw et al. [18] that at $\lambda = 0$ we have the distribution of the base random variable.

Many authors dealing with the generalization of some well- known distributions. Aryal and Tsokos [4] defined the transmuted generalized extreme value distribution and they studied some basic mathematical characteristics of transmuted Gumbel probability distribution and it has been observed that the transmuted Gumbel can be used to model climate data. Also Aryal and Tsokos [3] presented a new generalization of Weibull distribution called the transmuted Weibull distribution . Recently, Aryal [18] proposed and studied the various structural properties of the transmuted Log- Logistic distribution, Muhammad Khan and King [10] introduced the transmuted modified Weibull distribution which extends recent development on transmuted Weibull distribution by Aryal et al. [3] and they studied the mathematical properties and maximum likelihood estimation of the unknown parameters. Merovci introduced the transmuted Rayleigh distribution, transmuted generalized Rayleigh distribution, and transmuted Lindley distribution and they studied the mathematical properties and maximum likelihood estimation of the unknown parameters. Elbatal [7] introduced transmuted modified inverse Weibull distribution.

§3. Transmuted Weibull geometric distribution

In this section we studied the transmuted Weibull-geometric (TWG) distribution. Now using (1) and (2) we have the cdf of transmuted Weibull-geometric (TWG) distribution

$$F_{TWG}(x;p,\alpha,\theta,\lambda) = \frac{1-e^{-(\alpha x)^\theta}}{1-pe^{-(\alpha x)^\theta}}\left[1+\lambda-\lambda\left(\frac{1-e^{-(\alpha x)^\theta}}{1-pe^{-(\alpha x)^\theta}}\right)\right], \tag{7}$$

where λ is the transmuted parameter. The corresponding probability density function (pdf) of the transmuted Weibull-geometric is given by

$$\begin{aligned} f_{TWG}(x;p,\alpha,\theta,\lambda) &= f_{WG}(x)\left[(1+\lambda) - 2\lambda F_{WG}(x)\right] \\ &= \theta\alpha^\theta(1-p)x^{\theta-1}e^{-(\alpha x)^\theta}\left[1-pe^{-(\alpha x)^\theta}\right]^{-2} \\ &\times\left\{(1+\lambda) - 2\lambda\left(\frac{1-e^{-(\alpha x)^\theta}}{1-pe^{-(\alpha x)^\theta}}\right)\right\}, \end{aligned} \tag{8}$$

respectively.

Figures (1) and (2) illustrates some of the possible shapes of the pdf and cdf of the TWG distribution for selected values of the parameters θ, p, α and λ, respectively.

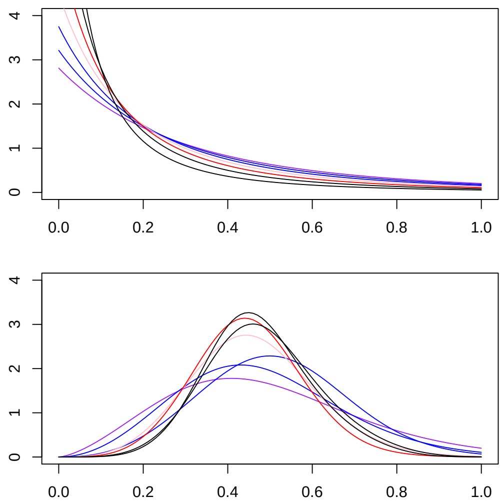

Figure 1: The pdf's of various TWG distributions for values of parameters:fig a) $\theta = 1; \alpha = 1.5 \lambda = 0.5$, fig b) $\theta = 2.5, 3, 3.5, 4, 4.5, 5, 5.5; \lambda = 0.5, 0.6, 0.7, 0.8, 0.9, 0.4, 0.3$ and for $p = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, \alpha = 1.5$; with color shapes purple, blue, plum, pink, red, black and darkcyan, respectively.

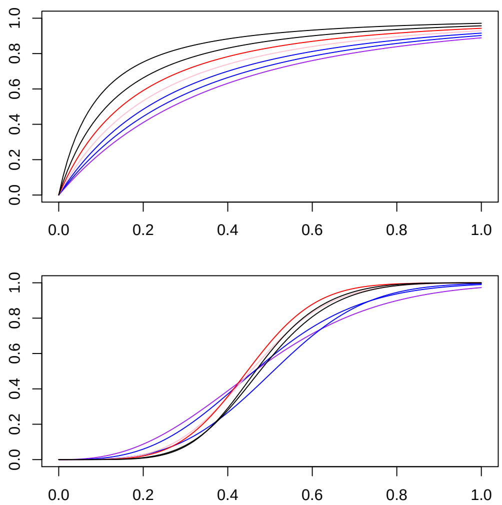

Figure 2: The cdf's of various TWG distributions for values of parameters::fig a) $\theta = 1; , \alpha = 1.5 \lambda = 0.5$, fig b) $\theta = 2.5, 3, 3.5, 4, 4.5, 5, 5.5; \lambda = 0.5, 0.6, 0.7, 0.8, 0.9, 0.4, 0.3$ and for $p = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, \alpha = 1.5$; with color shapes purple, blue, plum, pink, red, black and darkcyan, respectively.

The reliability function (RF) of the transmuted Weibull-geometric distribution is denoted by $R_{TWG}(x)$ also known as the survivor function and is defined as

$$\begin{aligned} R_{TWG}(x) &= 1 - F_{TWG}(x) \\ &= 1 - \frac{1-e^{-(\alpha x)^\theta}}{1-pe^{-(\alpha x)^\theta}}\left[1+\lambda-\lambda\left(\frac{1-e^{-(\alpha x)^\theta}}{1-pe^{-(\alpha x)^\theta}}\right)\right]. \end{aligned} \tag{9}$$

It is important to note that $R_{TWG}(x) + F_{TWG}(x) = 1$. One of the characteristic in reliability analysis is the hazard rate function (HF) defined by

$$h_{TWG}(x) = \frac{f_{TWG}(x)}{1-F_{TWG}(x)}. \tag{10}$$

It is important to note that the units for $h_{TWG}(x)$ is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters. The cumulative hazard function of the transmuted Weibull-geometric distribution is denoted by $H_{TWG}(x)$ and is defined as

$$H_{TWG}(x) = -\ln\left|\frac{1-e^{-(\alpha x)^\theta}}{1-pe^{-(\alpha x)^\theta}}\left[1+\lambda-\lambda\left(\frac{1-e^{-(\alpha x)^\theta}}{1-pe^{-(\alpha x)^\theta}}\right)\right]\right|. \tag{11}$$

It is important to note that the units for $H_{TWG}(x)$ is the cumulative probability of failure per unit of time, distance or cycles. we can show that. For all choice of parameters the distribution has the decreasing patterns of cumulative instantaneous failure rates.

Figure 3 illustrates some of the possible shapes of the hazard function of the TWG distribution for selected values of the parameters θ, p, α and λ, respectively.

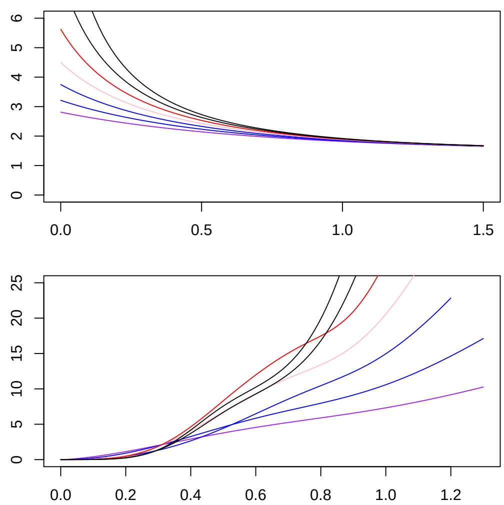

Figure 3: The hazard functions of various TWG distributions for values of parameters:fig a $\theta = 1;, \alpha = 1.5\lambda = 0.5$, fig b: $\theta = 2.5, 3, 3.5, 4, 4.5, 5, 5.5; \lambda = 0.5, 0.6, 0.7, 0.8, 0.9, 0.4, 0.3$ and for $p = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, , \alpha = 1.5$; with color shapes purple, blue, plum, pink, red, black and darkcyan, respectively.

§4. Statistical properties

This section is devoted to studying statistical properties of the (TWG) distribution.

4.1 Moments

In this subsection we discuss the r_{th} moment for (TWG) distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem 3.1. If X has $TWG(x;\phi), \phi = (p, \alpha, \theta, \lambda)$ then the r_{th} moment of X is given by the following

$$\mu_r^{'}(x) = (1+\lambda)\sum_{j=0}^{\infty}\frac{(j+1)(1-p)p^j\Gamma(\frac{r}{\theta}+1)}{\alpha^r(j+1)^{\frac{r}{\theta}}} - \lambda\sum_{j=0}^{\infty}\frac{(1-p)\Gamma(\frac{r}{\theta}+1)}{\alpha^r}\left[\frac{(j+2)}{(j+1)^{\frac{r}{\theta}}} - \frac{(j+1)}{(j+2)^{\frac{r}{\theta}}}\right]. \quad (12)$$

Proof. Let X be a random variable with density function (8). The r_{th} ordinary moment of the (TWG) distribution is given by

$$\mu_r^{'}(x) = E(X^{r)} = \int_0^{\infty} x^r f(x,\Phi)dx$$
$$= \theta\alpha^{\theta}(1-p)\left\{(1+\lambda)\int_0^{\infty} x^{r+\theta-1}e^{-(\alpha x)^{\theta}}\left[1-pe^{-(\alpha x)^{\theta}}\right]^{-2} - 2\lambda\int_0^{\infty} x^{r+\theta-1}e^{-(\alpha x)^{\theta}}\left(1-e^{-(\alpha x)^{\theta}}\right)\left(1-pe^{-(\alpha x)^{\theta}}\right)^{-3}\right\}. \quad (13)$$

If $|z| < 1$ and $k > 0$, we have the series representation

$$(1-z)^{-k} = \sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}z^j. \quad (14)$$

Expanding the term $\left(1-pe^{-(\alpha x)^{\theta}}\right)^{-3}$ as in (14) yields

$$\mu_r^{'}(x) = \theta\alpha^{\theta}(1-p)\left\{(1+\lambda)\sum_{j=0}^{\infty}(j+1)p^j\int_0^{\infty} x^{r+\theta-1}e^{-(j+1)(\alpha x)^{\theta}}dx - \lambda\sum_{j=0}^{\infty}(j+1)(j+2)p^j\int_0^{\infty} x^{r+\theta-1}e^{-(j+1)(\alpha x)^{\theta}}\left(1-e^{-(\alpha x)^{\theta}}\right)dx\right\}$$
$$= (1+\lambda)\sum_{j=0}^{\infty}\frac{(j+1)(1-p)p^j\Gamma(\frac{r}{\theta}+1)}{\alpha^r(j+1)^{\frac{r}{\theta}}} - \lambda\sum_{j=0}^{\infty}\frac{(1-p)\Gamma(\frac{r}{\theta}+1)}{\alpha^r} \times\left[\frac{(j+2)}{(j+1)^{\frac{r}{\theta}}} - \frac{(j+1)}{(j+2)^{\frac{r}{\theta}}}\right]. \quad (15)$$

Which completes the proof.

Based on the first four moments of the (TWG) distribution, the measures of skewness $A(\Phi)$ and kurtosis $k(\Phi)$ of the (TWG) distribution can obtained as

$$A(\varphi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^{\frac{3}{2}}}$$

and

$$k(\varphi) = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^2}.$$

4.2 Moment Generating function

In this subsection we derived the moment generating function of (TWG) distribution.

Theorem 3.2. If X has (TWG) distribution, then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left\{ (1+\lambda) \sum_{j=0}^{\infty} \frac{(j+1)(1-p)p^j \Gamma(\frac{r}{\theta}+1)}{\alpha^r (j+1)^{\frac{r}{\theta}}} \right.$$
$$\left. -\lambda \sum_{j=0}^{\infty} \frac{(1-p)\Gamma(\frac{r}{\theta}+1)}{\alpha^r} \left[\frac{(j+2)}{(j+1)^{\frac{r}{\theta}}} - \frac{(j+1)}{(j+2)^{\frac{r}{\theta}}} \right] \right\}. \tag{16}$$

Proof. We start with the well known definition of the moment generating function given by

$$\begin{aligned} M_X(t) &= \int_0^{\infty} e^{tx} f_{TWG}(x,\phi)dx \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!}\ x^r f_{TWG}(x,\phi)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r^{'} \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \left\{ (1+\lambda) \sum_{j=0}^{\infty} \frac{(j+1)(1-p)p^j \Gamma(\frac{r}{\theta}+1)}{\alpha^r (j+1)^{\frac{r}{\theta}}} \right. \\ &\quad \left. -\lambda \sum_{j=0}^{\infty} \frac{(1-p)\Gamma(\frac{r}{\theta}+1)}{\alpha^r} \left[\frac{(j+2)}{(j+1)^{\frac{r}{\theta}}} - \frac{(j+1)}{(j+2)^{\frac{r}{\theta}}} \right] \right\}. \end{aligned} \tag{17}$$

Which completes the proof.

§5. Distribution of the order statistics

In this section, we derive closed form expressions for the pdfs of the r_{th} order statistic of the (TWG) distribution, also, the measures of skewness and kurtosis of the distribution of the r_{th} order statistic in a sample of size n for different choices of $n; r$ are presented in this section. Let $X_1, X_2, \ldots, X_n$ be a simple random sample from (TWG) distribution with pdf and cdf given by (7) and (8), respectively. Let $X_1, X_2, \ldots, X_n$ denote the order statistics obtained from this sample. We now give the probability density function of $X_{r:n}$, say $f_{r:n}(x, \Phi)$ and the moments of $X_{r:n}$, $r = 1, 2, \ldots, n$. Therefore, the measures of skewness and kurtosis of the distribution of the $X_{r:n}$ are presented. The probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x,\varphi) = \frac{1}{B(r, n-r+1)} [F(x,\varphi)]^{r-1} [1 - F(x,\varphi)]^{n-r} f(x,\varphi), \tag{18}$$

where $F(x,\varphi)$ and $f(x,\varphi)$ are the cdf and pdf of the (TWG) distribution given by (7), (8), respectively, and $B(.,.)$ is the beta function. The pdf of the r_{th} order statistic for a transmuted weibull geometric distribution is given by

$$\begin{aligned} f_{r:n}(x,\varphi) = {} & \frac{1}{B(r,n-r+1)} \theta\alpha^{\theta}(1-p)x^{\theta-1}e^{-(\alpha x)^{\theta}} \left[1 - pe^{-(\alpha x)^{\theta}}\right]^{-2} \left\{(1+\lambda) - 2\lambda\left(\frac{1-e^{-(\alpha x)^{\theta}}}{1-pe^{-(\alpha x)^{\theta}}}\right)\right\} \\ & \times \left\{\frac{1-e^{-(\alpha x)^{\theta}}}{1-pe^{-(\alpha x)^{\theta}}}\left[1+\lambda-\lambda\left(\frac{1-e^{-(\alpha x)^{\theta}}}{1-pe^{-(\alpha x)^{\theta}}}\right)\right]\right\}^{r-1} \\ & \times \left\{1 - \frac{1-e^{-(\alpha x)^{\theta}}}{1-pe^{-(\alpha x)^{\theta}}}\left[1+\lambda-\lambda\left(\frac{1-e^{-(\alpha x)^{\theta}}}{1-pe^{-(\alpha x)^{\theta}}}\right)\right]\right\}^{n-r}. \end{aligned}$$

Therefore, and the pdf of the smallest order statistic $X(1)$ is

$$\begin{aligned} f_{1:n}(x,\varphi) = {} & n\theta\alpha^{\theta}(1-p)x_{(1)}^{\theta-1}e^{-(\alpha x_{(1)})^{\theta}} \left[1 - pe^{-(\alpha x_{(1)})^{\theta}}\right]^{-2} \\ & \times \left\{(1+\lambda) - 2\lambda\left(\frac{1-e^{-(\alpha x_{(1)})^{\theta}}}{1-pe^{-(\alpha x_{(1)})^{\theta}}}\right)\right\} \\ & \times \left\{1 - \frac{1-e^{-(\alpha x_{(1)})^{\theta}}}{1-pe^{-(\alpha x_{(1)})^{\theta}}}\left[1+\lambda-\lambda\left(\frac{1-e^{-(\alpha x_{(1)})^{\theta}}}{1-pe^{-(\alpha x_{(1)})^{\theta}}}\right)\right]\right\}^{n-1}, \end{aligned}$$

and the pdf of the largest order statistic $X(n)$ is given by

$$\begin{aligned} f_{n:n}(x,\varphi) = {} & n\theta\alpha^{\theta}(1-p)x_{(n)}^{\theta-1}e^{-(\alpha x_{(n)})^{\theta}} \left[1 - pe^{-(\alpha x_{(n)})^{\theta}}\right]^{-2} \\ & \times \left\{(1+\lambda) - 2\lambda\left(\frac{1-e^{-(\alpha x_{(n)})^{\theta}}}{1-pe^{-(\alpha x_{(n)})^{\theta}}}\right)\right\} \\ & \left\{\frac{1-e^{-(\alpha x_{(n)})^{\theta}}}{1-pe^{-(\alpha x_{(n)})^{\theta}}}\left[1+\lambda-\lambda\left(\frac{1-e^{-(\alpha x_{(n)})^{\theta}}}{1-pe^{-(\alpha x_{(n)})^{\theta}}}\right)\right]\right\}^{n-1}. \end{aligned}$$

§6. Least squares and Weighted least squares estimators

In this section we provide the regression based method estimators of the unknown parameters of the transmuted Weibull geometric distribution, which was originally suggested by Swain et al. [19] to estimate the parameters of beta distributions. It can be used some other cases also. Suppose $Y_1, \ldots, Y_n$ is a random sample of size n from a distribution function $G(.)$ and suppose $Y_{(i)}$; $i = 1, 2, \ldots, n$ denotes the ordered sample. The proposed method uses the distribution of

$G(Y_{(i)})$. For a sample of size n, we have

$$E\left(G(Y_{(j)})\right)=\frac{j}{n+1}, V\left(G(Y_{(j)})\right)=\frac{j(n-j+1)}{(n+1)^2(n+2)}$$
$$\text{and } Cov\left(G(Y_{(j)}),G(Y_{(k)})\right)=\frac{j(n-k+1)}{(n+1)^2(n+2)}; \text{for } j<k,$$

see Johnsonet al. [8]. Using the expectations and the variances, two variants of the least squares methods can be used.

Method 1. (Least Squares Estimators) Obtain the estimators by minimizing

$$\sum_{j=1}^{n}\left(G(Y_{(j)}-\frac{j}{n+1}\right)^2, \tag{6.1}$$

with respect to the unknown parameters. Therefore in case of TWG distribution the least squares estimators of p, α,θ,and λ, say, $\widehat{p}_{LSE}$, $\widehat{\alpha}_{LSE}$, $\widehat{\theta}_{LSE}$ and $\widehat{\lambda}_{LSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}\left[\frac{1-e^{-(\alpha x_j)^\theta}}{1-pe^{-(\alpha x_j)^\theta}}\left[1+\lambda-\lambda\left(\frac{1-e^{-(\alpha x_j)^\theta}}{1-pe^{-(\alpha x_j)^\theta}}\right)\right]-\frac{j}{n+1}\right]^2$$

with respect to θ,p and λ.

Method 2. (Weighted Least Squares Estimators) The weighted least squares estimators can be obtained by minimizing

$$\sum_{j=1}^{n}w_j\left(G(Y_{(j)}-\frac{j}{n+1}\right)^2, \tag{19}$$

with respect to the unknown parameters, where

$$w_j=\frac{1}{V\left(G(Y_{(j)})\right)}=\frac{(n+1)^2(n+2)}{j(n-j+1)}.$$

Therefore, in case of TWG distribution the weighted least squares estimators of p, α,θ and λ, say, $\widehat{p}_{WLSE}$, $\widehat{\alpha}_{WLSE}$, $\widehat{\theta}_{WLSE}$ and $\widehat{\lambda}_{wLSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}w_j\left[\frac{1-e^{-(\alpha x_j)^\theta}}{1-pe^{-(\alpha x_j)^\theta}}\left[1+\lambda-\lambda\left(\frac{1-e^{-(\alpha x_j)^\theta}}{1-pe^{-(\alpha x_j)^\theta}}\right)\right]-\frac{j}{n+1}\right]^2,$$

with respect to the unknown parameters only.

§7. Estimation and inference

In this section, we determine the maximum likelihood estimates ($MLEs$) of the parameters of the (TWG) distribution from complete samples only. Let $X_1,X_2,\ldots,X_n$ be a random sample of size n from $TWG(x;\phi),\phi=(p,\alpha,\theta,\lambda)$. The likelihood function for the vector of parameters can be written as

$$
\begin{aligned}
Lf(x_{(i)},\phi) &= \Pi_{i=1}^{n} f(x_{(i)},\phi) \\
&= \left(\theta\alpha^{\theta}(1-p)\right)^{n}\ \Pi_{i=1}^{n}(x_i)^{\theta-1}\ e^{-\sum\limits_{i=1}^{n}(\alpha x_i)^{\theta}}\ \Pi_{i=1}^{n}\left[1-pe^{-(\alpha x_i)^{\theta}}\right]^{-2} \\
&\times \Pi_{i=1}^{n}\left\{(1+\lambda)-2\lambda\left(\frac{1-e^{-(\alpha x_i)^{\theta}}}{1-pe^{-(\alpha x_i)^{\theta}}}\right)\right\}.
\end{aligned}
\tag{20}
$$

Taking the log-likelihood function for the vector of parameters $\phi=(p,\alpha,\theta,\lambda)$ we get

$$
\begin{aligned}
\log\ L &= n\log\theta + n\theta\log\alpha + n\log(1-p) + (\theta-1)\sum_{i=1}^{n}\log(x_i) - \theta\sum_{i=1}^{n}(\alpha x_i) - 2 \\
&\sum_{i=1}^{n}\log\left[1-pe^{-(\alpha x_i)^{\theta}}\right] + \sum_{i=1}^{n}\log\left\{(1+\lambda)-2\lambda\left(\frac{1-e^{-(\alpha x_i)^{\theta}}}{1-pe^{-(\alpha x_i)^{\theta}}}\right)\right\}.
\end{aligned}
\tag{21}
$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (21). The components of the score vector are given by

$$
\begin{aligned}
\frac{\partial\log\ L}{\partial\theta} &= \frac{n}{\theta} + n\log\alpha + \sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}(\alpha x_i) - 2\sum_{i=1}^{n}\frac{pe^{-(\alpha x_i)^{\theta}}(\alpha x_i)^{\theta}\log(\alpha x_i)}{\left[1-pe^{-(\alpha x_i)^{\theta}}\right]} \\
&+ \sum_{i=1}^{n}\frac{1}{\left\{(1+\lambda)-2\lambda\left(\frac{1-e^{-(\alpha x_i)^{\theta}}}{1-pe^{-(\alpha x_i)^{\theta}}}\right)\right\}}\left[\frac{(1-p)e^{-(\alpha x_i)^{\theta}}(\alpha x_i)^{\theta}\log(\alpha x_i)}{\left[1-pe^{-(\alpha x_i)^{\theta}}\right]^{2}}\right] \\
&= 0,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\frac{\partial\log\ L}{\partial\alpha} &= \frac{n\theta}{\alpha} - \theta\sum_{i=1}^{n}(x_i) - 2\sum_{i=1}^{n}\frac{\theta pe^{-(\alpha x_i)^{\theta}}x_i(\alpha x_i)^{\theta-1}}{\left[1-pe^{-(\alpha x_i)^{\theta}}\right]} \\
&- 2\lambda\sum_{i=1}^{n}\frac{1}{\left\{(1+\lambda)-2\lambda\left(\frac{1-e^{-(\alpha x_i)^{\theta}}}{1-pe^{-(\alpha x_i)^{\theta}}}\right)\right\}}\left[\frac{(1-p)\theta e^{-(\alpha x_i)^{\theta}}x_i(\alpha x_i)^{\theta-1}}{\left[1-pe^{-(\alpha x_i)^{\theta}}\right]^{2}}\right] = 0,
\end{aligned}
\tag{23}
$$

‘

$$
\begin{aligned}
\frac{\partial\log\ L}{\partial p} &= \frac{-n}{1-p} + 2\sum_{i=1}^{n}\frac{e^{-(\alpha x_i)^{\theta}}}{\left[1-pe^{-(\alpha x_i)^{\theta}}\right]} \\
&- 2\lambda\sum_{i=1}^{n}\frac{1}{\left\{(1+\lambda)-2\lambda\left(\frac{1-e^{-(\alpha x_i)^{\theta}}}{1-pe^{-(\alpha x_i)^{\theta}}}\right)\right\}}\left[\frac{e^{-(\alpha x_i)^{\theta}}(1-e^{-(\alpha x_i)^{\theta}})}{\left[1-pe^{-(\alpha x_i)^{\theta}}\right]^{2}}\right] = 0,
\end{aligned}
\tag{24}
$$

and

$$
\frac{\partial\log\ L}{\partial\lambda} = \sum_{i=1}^{n}\frac{1-2\left(\frac{1-e^{-(\alpha x_i)^{\theta}}}{1-pe^{-(\alpha x_i)^{\theta}}}\right)}{\left\{(1+\lambda)-2\lambda\left(\frac{1-e^{-(\alpha x_i)^{\theta}}}{1-pe^{-(\alpha x_i)^{\theta}}}\right)\right\}} = 0.
\tag{25}
$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (22)-(25) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters.

194	413	90	74	55	23	97	50	359	50	130	487	102	15	14	10	57
320	261	51	44	9	254	493	18	209	41	58	60	48	56	87	11	102
12	5	100	14	29	37	186	29	104	7	4	72	270	283	7	57	33
100	61	502	220	120	141	22	603	35	98	54	181	65	49	12	239	14
18	39	3	12	5	32	9	14	70	47	62	142	3	104	85	67	169
24	21	246	47	68	15	2	91	59	447	56	29	176	225	77	197	438
43	134	184	20	386	182	71	80	188	230	152	36	79	59	33	246	1
79	3	27	201	84	27	21	16	88	130	14	118	44	15	42	106	46
230	59	153	104	20	206	5	66	34	29	26	35	5	82	5	61	31
118	326	12	54	36	34	18	25	120	31	22	18	156	11	216	139	67
310	3	46	210	57	76	14	111	97	62	26	71	39	30	7	44	11
63	23	22	23	14	18	13	34	62	11	191	14	16	18	130	90	163
208	1	24	70	16	101	52	208	95								

§8. Application

The first data set given in table 1 consist of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes. The pooled data with 213 observations were first analyzed by Proschan [20] and discussed further by Adamidis et al. [20] and Kus [9].

In order to compare the two distribution models, we consider criteria like -2ℓ, AIC (Akaike information criterion)and AICC (corrected Akaike information criterion) for the data set. The better distribution corresponds to smaller -2ℓ, AIC and AICC values:

$$\mathrm{AIC} = 2k - 2\ell \qquad \text{and} \qquad \mathrm{AICC} = \mathrm{AIC} + \frac{2k(k+1)}{n-k-1},$$

where k is the number of parameters in the statistical model, n the sample size and ℓ is the maximized value of the log-likelihood function under the considered model.

Table 1 shows parameters $MLEs$, $-\log(L)$, AIC and $K-S$ values to each one of the two fitted distributions for data set.The values in Tables 1, indicate that the TWG is a strong competitor to other distribution used here for fitting data set. A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. fig4). The fitted density for the TWG model is closer to the empirical histogram than the fits of the WG model.

Table 1: The ML estimates, Log-likelihood, AIC and K-S for data set

Model	ML Estimates	-LL	K-S	AIC
TWG	$\hat{\theta} = 1.027$	1178.264	0.0343	2364.528
	$\hat{p} = 0.844$			
	$\hat{\alpha} = 0.545 \cdot 10^{-2}$			
	$\hat{\lambda} = -0.762$			
WG	$\hat{\theta} = 1.337$	1181.947	0.0398	2369.894
	$\hat{p} = 0.999$			
	$\hat{\alpha} = 1.729 \cdot 10^{-5}$			

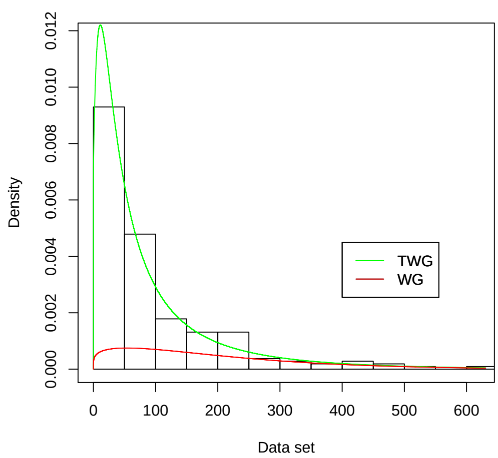

Figure 4: Estimated densities of the models for data set.

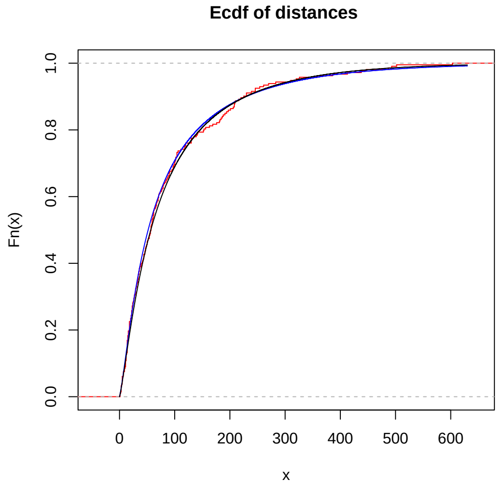

Figure 5: Empirical, cdf for TWG and WG for data set .

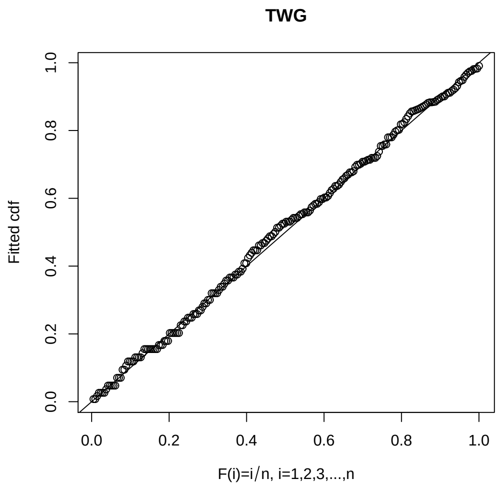

Figure 6: PP plots for fitted TWG for data set.

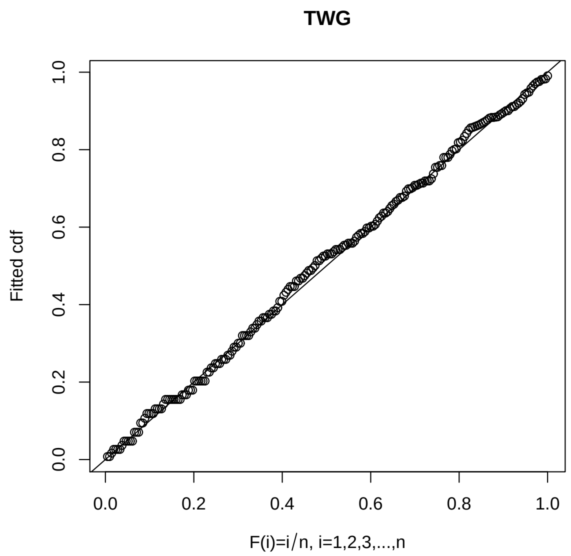

Figure 7: PP plots for fitted WG for data set.

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A sheaf construction on the primary-like spectrum of modules

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Abstract Let M be a unitary module over a commutative ring R with identity . Let $Spec_L(M)$ denote the primary-like spectrum of M which has been equipped with the Zariski topology $\mathcal{T}$. In this paper, we obtain an R-module $\mathcal{O}_{Spec_L(M)}(\mathcal{U})$ for each open set $\mathcal{U}$ of the topological space $(Spec_L(M), \mathcal{T})$. We show that $\mathcal{O}_{Spec_L(M)}$ is a sheaf of R-modules over $Spec_L(M)$.

Keywords Primary-like submodule, Zariski topology, Sheaf of a module.
2000 Mathematics Subject Classification: 13C99, 13A99.

§1. Introduction and preliminaries

All rings in this article are commutative with identity and modules are unitary. For a submodule N of an R-module M we denote the annihilator of the factor module M/N by $(N : M)$, i.e., $(N : M) = \{r \in R \mid rM \subseteq N\}$. We call M faithful if $Ann(M) = (0 : M) = 0$.

A submodule P of an R-module M is said to be p-prime if $P \neq M$ and for $p = (P : M)$, whenever $rm \in P$ (where $r \in R$ and $m \in M$), then $m \in P$ or $r \in p$. The collection of all prime submodules of M is denoted by $Spec(M)$.[8] For a submodule N of M the radical of N, denoted by rad$N$, is the intersection of all prime submodules of M containing N, unless no such primes exist, in which case rad$N = M$.[9] For an ideal I of R, the radical of I is denoted by $\sqrt{I}$.

A proper submodule Q of M is primary-like provided that $rm \in Q$, for $r \in R$ and $m \in M$, implies $r \in (Q : M)$ or $x \in \mathrm{rad}\, Q$.[2] If $\sqrt{(Q : M)} = p$ is a prime ideal, then Q is also called a p-primary-like submodule of M.[2]

An R-module M is said to be primeful if either $M = 0$ or for every $p \in V(Ann(M))$, there exists $P \in Spec(M)$ such that $(P : M) = p$.[5] If M/N is a primeful R-module, we say that N satisfies the primeful property. In this case $(\mathrm{rad}\, N : M) = \sqrt{(N : M)}$.[5] It is easily seen that if Q is a primary-like submodule of the R-module M satisfying the primeful property, then $(Q : M)$ is a primary ideal of R and so Q is a p-primary-like submodule of M, where $p = \sqrt{(Q : M)} = (\mathrm{rad}\, Q : M)$.

The primary-like spectrum $Spec_L(M)$ is defined to be the set of all primary-like submodules of M satisfying the primeful property.[2] $Spec_L(M)$ may be empty for some non-zero modules M. For example $Spec_L(\mathbb{Q}) = \emptyset$ while $Spec(\mathbb{Q}) = \{0\}$, where $\mathbb{Q}$ is the module of rational numbers over the ring of integers $\mathbb{Z}$. Throughout the rest of this paper, we assume that $Spec_L(M)$ is non-empty.

In the literature, there are many different generalizations of the Zariski topology for modules over commutative rings. For example, Lu defined a Zariski topology on $Spec(M)$ whose closed sets are $V(N) = \{P \in Spec(M) \mid (N : M) \subseteq (P : M)\}$ for any submodule N of M.[7] As a new generalization of the Zariski topology, we introduce the Zariski topology on $Spec_L(M)$ for any R-module M in which closed sets are varieties $\nu(N) = \{Q \in Spec_L(M) \mid \sqrt{(N : M)} \subseteq \sqrt{(Q : M)}\}$ of all submodules N of M. In the following lemma we show that, if $\eta(M)$ denotes the collection of all subsets $\nu(N)$ of $Spec_L(M)$, then $\eta(M)$ satisfies the axioms for the closed subsets of a topological space on $Spec_L(M)$, called Zariski topology and denoted by $\mathcal{T}$.

Lemma 1.1. Let M be an R-module. Then for submodules N, N' and $\{N_i \mid i \in I\}$ of M we have

(1) $\nu(0) = Spec_L(M)$ and $\nu(M) = \emptyset$.

(2) $\bigcap_{i\in I} \nu(N_i) = \nu(\sum_{i\in I}(N_i : M)M)$.

(3) $\nu(N) \cup \nu(N') = \nu(N \cap N')$.

Proof. (1) and (3) are trivial. (2) follows from the following implications:

$$\begin{aligned}
Q \in \bigcap_{i\in I} \nu(N_i) &\Rightarrow \sqrt{(Q : M)} \supseteq \sqrt{(N_i : M)}\ \forall i \in I \\
&\Rightarrow \sqrt{(Q : M)} \supseteq (N_i : M)\ \forall i \in I \\
&\Rightarrow \sqrt{(Q : M)} \supseteq \sum_{i\in I}(N_i : M) \\
&\Rightarrow \sqrt{(Q : M)}M \supseteq \sum_{i\in I}(N_i : M)M \\
&\Rightarrow (\sqrt{(Q : M)}M : M) \supseteq (\sum_{i\in I}(N_i : M)M : M) \\
&\Rightarrow ((\operatorname{rad} Q : M)M : M) \supseteq (\sum_{i\in I}(N_i : M)M : M) \\
&\Rightarrow (\operatorname{rad} Q : M) \supseteq (\sum_{i\in I}(N_i : M)M : M) \\
&\Rightarrow \sqrt{(Q : M)} \supseteq \sqrt{(\sum_{i\in I}(N_i : M)M : M)} \\
&\Rightarrow Q \in \nu(\sum_{i\in I}(N_i : M)M).
\end{aligned}$$

For the reverse inclusion we have

$$\begin{aligned}
Q \in \nu(\sum_{i\in I}(N_i : M)M) \quad &\Rightarrow \quad \sqrt{(Q:M)} \supseteq \sqrt{(\sum_{i\in I}(N_i:M)M : M)}\\
&\Rightarrow \quad \sqrt{(Q:M)} \supseteq (\sum_{i\in I}(N_i:M)M : M)\\
&\Rightarrow \quad \sqrt{(Q:M)} \supseteq ((N_i:M)M : M) \quad \forall i \in I\\
&\Rightarrow \quad \sqrt{(Q:M)} \supseteq (N_i : M) \quad \forall i \in I\\
&\Rightarrow \quad \sqrt{(Q:M)} \supseteq \sqrt{(N_i : M)} \quad \forall i \in I\\
&\Rightarrow \quad Q \in \bigcap_{i\in I} \nu(N_i)
\end{aligned}$$

It is well-known that for any commutative ring R, there is a sheaf of rings on $Spec(R)$, called the structure sheaf, denoted by $O_{Spec(R)}$, defined as follows: for each prime ideal p of R, let R_p be the localization of R at p. For an open set U of $Spec(R)$, we define $O_{Spec(R)}(U)$ to be the set of functions $s : U \to \coprod_{p\in U} R_p$, such that $s(p) \in R_p$, for each $p \in U$, and such that s is locally a quotient of elements of R: to be precise, we require that for each $p \in U$, there is a neighborhood V of p, contained in U, and elements $a, f \in R$, such that for each $q \in V$, $f \notin q$, and $s(q) = \frac{a}{f}$ in R_q(see for example [3], for definition and basic properties of the sheaf $O_{Spec(R)}$).

Let M be a primeful faithful R-module. In [10], the author obtained an R-module $O_{Spec(M)}(U)$ for each open set U in $Spec(M)$ (with respect to the Zariski topology) such that $O_{Spec(M)}$ is a sheaf of modules over $Spec(M)$ which is a generalization of the structure sheaf of rings. As a new generalization, we obtain an R-module $\mathcal{O}_{Spec_L(M)}(\mathcal{U})$ for any open set $\mathcal{U}$ of $Spec_L(M)$ (with respect to the Zariski topology) to construct a sheaf $\mathcal{O}_{Spec_L(M)}$ of R-modules over $Spec_L(M)$ (Theorem 2.5).

§2. A sheaf of modules

For each $P \in Spec(M)$, we put $\nu_P = \{X_r \mid P \in X_r\}$, where $X_r = Spec(M) - V(rM)$. Also for each $r \in R$ we consider the R-module of fractions $M_r = \{\frac{m}{r^n} \mid m \in M, n \in \mathbb{N}\}$. In [10], it is proved that if M is a faithful primeful R-module and $X_s \subseteq X_r$, then there exists an R-module homomorphism $\rho^*_{X_s,X_r} : M_r \to M_s$ defined by $\frac{m}{r^n} \mapsto \frac{a^n m}{s^{nk}}$ for some positive integer k and $a \in R$. Moreover the inductive limit $lim_{X_r\in\nu_P} M_r$ of the direct system $\{M_r, \rho^*_{X_s,X_r}\}$ has been introduced as a $lim_{D_r\in\nu_p} R_r$-module where $(P : M) = p$ and $\nu_p = \{D_r \mid p \in D_r\}$. In particular, it has been shown that if M is a faithful primeful R-module, then $lim_{X_r\in\nu_P} M_r \cong M_p$, where $p = (P : M)$.[10] In this section, we obtain an inductive limit of a new direct system similarly. In particular, we investigate sufficient conditions under which these limits are isomorphic as R_p-modules, where $p = \sqrt{(Q : M)}$ for some $Q \in Spec_L(M)$ (Corollary 2.2). We will use $\overline{R}$, $X^{\overline{R}}$ and $\mathcal{X}$ to represent $R/Ann(M)$, $Spec(R/Ann(M))$ and $Spec_L(M)$, respectively.

Lemma 2.1. Let r be a non-nilpotent element of a ring R, S be the multiplicatively closed subset $\{1, r, r^2, r^3, \cdots, r^t, \cdots\}$ and M be an R-module. Let $Q \in \mathcal{X}$. Then $\sqrt{(rM : M)} \nsubseteq \sqrt{(Q : M)}$ if and only if $\sqrt{(Q : M)} \cap S = \emptyset$.

Proof. Suppose that $\sqrt{(rM:M)} \nsubseteq \sqrt{(Q:M)}$. Therefore $r \notin \sqrt{(Q:M)}$. Since $\sqrt{(Q:M)}$ is a prime ideal of R, then $1, r^2, r^3, \cdots \notin \sqrt{(Q:M)}$. Thus $\sqrt{(Q:M)} \cap S = \emptyset$. Conversely, assume $\sqrt{(Q:M)} \cap S = \emptyset$ and $\sqrt{(rM:M)} \subseteq \sqrt{(Q:M)}$. Hence $r \in \sqrt{(Q:M)} \cap S$, a contradiction. Thus $\sqrt{(rM:M)} \nsubseteq \sqrt{(Q:M)}$.

For each $r \in R$, we set $\mathcal{X}_r = \mathcal{X} - \nu(rM)$ and $D_{\overline{r}} = X^{\overline{R}} - V(\overline{R}\overline{r})$. It is easily seen that $\mathcal{X}_{0_R} = \emptyset$, $\mathcal{X}_{1_R} = \mathcal{X}$.

Lemma 2.2. Let r be a non-nilpotent element of a ring R and M an R-module. Then $\mathcal{X}_r = \{Q \in \mathcal{X} \mid r \notin \sqrt{(Q:M)}\}$.

Proof.

$$\begin{aligned}\nu(rM) &= \{Q \in \mathcal{X} \mid \sqrt{(Q:M)} \supseteq \sqrt{(rM:M)}\}\\ &= \{Q \in \mathcal{X} \mid \sqrt{(Q:M)} \supseteq (rM:M)\}\\ &= \{Q \in \mathcal{X} \mid \sqrt{(Q:M)} \cap S \neq \emptyset\}, \text{ by Lemma 2.1}\\ &= \{Q \in \mathcal{X} \mid r \in \sqrt{(Q:M)}\}.\end{aligned}$$

Thus $\mathcal{X}_r = \mathcal{X} - \nu(rM) = \{Q \in \mathcal{X} \mid r \notin \sqrt{(Q:M)}\}$.

Theorem 2.1. Let M be an R-module and $Q \in \mathcal{X}$. Let S be a multiplicatively closed subset of R such that $S \cap \sqrt{(Q:M)} = \emptyset$. Then $S^{-1}Q$ is a primary-like submodule of $S^{-1}R$-module $S^{-1}M$ satisfying the primeful property.

Proof. It is easy to see that $m/1 \in S^{-1}M \backslash S^{-1}Q$ for each $m \in M \backslash Q$ and so $S^{-1}Q \neq S^{-1}M$. Suppose $(r/s)(x/t) \in S^{-1}Q$ and $r/s \notin (S^{-1}Q : S^{-1}M)$ for $r/s \in S^{-1}R$ and $x/t \in S^{-1}M$. Since $S^{-1}((Q:M)) \subseteq (S^{-1}Q : S^{-1}M)$, then $r \notin (Q:M)$. Thus there exist $u, w \in S$, $q \in Q$ such that $wurx = wstq$. It follows that $x \in \operatorname{rad} Q$ since Q is primary-like. Thus $x/t \in S^{-1}(\operatorname{rad} Q) \subseteq \operatorname{rad}(S^{-1}Q)$, by [9]. Thus $S^{-1}Q$ is a primary-like submodule of $S^{-1}M$. Let $S^{-1}p$ be a prime ideal of $S^{-1}R$ containing $(S^{-1}Q : S^{-1}M)$. Hence $p \cap S = \emptyset$ and,

$$\begin{aligned}S^{-1}((Q:M)) &\subseteq (S^{-1}Q : S^{-1}M) \subseteq S^{-1}p\\ &\Rightarrow S^{-1}(\sqrt{(Q:M)}) \subseteq S^{-1}p\\ &\Rightarrow \sqrt{(Q:M)} \subseteq p.\end{aligned}$$

Since Q satisfies the primeful property, there exists a p-prime submodule P of M containing Q. By [8], since $p \cap S = \emptyset$, $S^{-1}P$ is an $S^{-1}p$-prime submodule of $S^{-1}M$ containing $S^{-1}Q$, i.e., $S^{-1}Q$ satisfies the primeful property.

Combining Lemma 2.2 and Theorem 2.1, we can easily deduce the following:

Corollary 2.1. Let r be a non-nilpotent element of R and S be the multiplicatively closed subset $\{1, r, r^2, r^3, \cdots\}$. If $Q \in \nu(rM)$, then $S^{-1}Q \in Spec_L(S^{-1}M)$.

For any R-module M, we define the map $\phi : \mathcal{X} \to X^{\overline{R}}$ given by $\phi(Q) = \overline{\sqrt{(Q:M)}}$. The following results offers a useful piece of information for the map ϕ.

Proposition 2.1. Let M be an R-module. Then $\phi^{-1}(V^{\overline{R}}(\overline{I})) = \nu(IM)$, for every ideal $I \in V(Ann(M))$. Therefore the map ϕ is continuous for the Zariski topology $\mathcal{T}$ on $\mathcal{X}$.

Proof. Suppose $Q \in \phi^{-1}(V^{\overline{R}}(\overline{I}))$. Then $\phi(Q) \in V^{\overline{R}}(\overline{I})$ and so $\sqrt{(Q:M)} \supseteq I$. Hence $\sqrt{(Q:M)} \supseteq \sqrt{(IM:M)}$. Thus $Q \in \nu(IM)$. The argument is reversible and so ϕ is continuous.

Theorem 2.2. Let M be an R-module and the map ϕ be surjective. Then $\phi(\nu(N)) = V^{\overline{R}}(\overline{(N:M)})$ and $\phi(\mathcal{X} - \nu(N)) = X^{\overline{R}} - V^{\overline{R}}(\overline{(N:M)})$ for every submodule N of M, i.e., ϕ is both closed and open.

Proof. As we have seen in Proposition 2.1, ϕ is a continuous map such that $\phi^{-1}(V^{\overline{R}}(\overline{I})) = \nu(IM)$, for every ideal $I \in V(Ann(M))$. Hence for every submodule N of M we have $\phi^{-1}(V^{\overline{R}}(\overline{(N:M)})) = \nu((N:M)M) = \nu(N)$. Thus $\phi(\nu(N)) = \phi \circ \phi^{-1}(V^{\overline{R}}(\overline{(N:M)})) = V^{\overline{R}}(\overline{(N:M)})$ as ϕ is surjective. Similarly, $\phi(\mathcal{X} - \nu(N)) = \phi(\phi^{-1}(X^{\overline{R}}) - \phi^{-1}(V^{\overline{R}}(\overline{(N:M)}))) = X^{\overline{R}} - V^{\overline{R}}(\overline{(N:M)})$.

Lemma 2.3. Let M be an R-module. Then $\phi(\mathcal{X}_r) \subseteq D_{\overline{r}}$ and the equality holds if ϕ is surjective.

Proof. By Proposition 2.1, $\phi^{-1}(D_{\overline{r}}) = \phi^{-1}(X^{\overline{R}} - V(\overline{R}\overline{r})) = \mathcal{X} - \nu(rM) = \mathcal{X}_r$. Hence $\phi(\mathcal{X}_r) \subseteq D_{\overline{r}}$. The equality follows form Theorem 2.2.

Lemma 2.4. Let $r, s \in R$. Then the following hold.

(1) $\mathcal{X}_{rs} = \mathcal{X}_r \cap \mathcal{X}_s$.

(2) $\mathcal{X}_{r^n} = \mathcal{X}_r$ for all $n \in N$.

(3) If r is nilpotent, then $\mathcal{X}_r = \emptyset$.

Proof. (1) by Proposition 2.1, $\mathcal{X}_{rs} = \phi^{-1}(D_{\overline{r}\overline{s}})$. Hence $\mathcal{X}_{rs} = \phi^{-1}(D_{\overline{r}}) \cap \phi^{-1}(D_{\overline{s}}) = \mathcal{X}_r \cap \mathcal{X}_s$.

(2) Follows from (1). (3) Since r is nilpotent, $r^n = 0$ for some $n \in N$. Hence by (2), $\mathcal{X}_r = \mathcal{X}_{r^n} = \mathcal{X}_0 = \emptyset$.

For the remainder of this article, we assume that every R-module M is faithful and the map ϕ is surjective except Lemmas 2.8 and 2.9 and Theorem 2.4.

Lemma 2.5. Let $r, s \in R$ and M be an R-module. If $\mathcal{X}_s \subseteq \mathcal{X}_r$, then $s \in \sqrt{<r>}$.

Proof. Suppose $\mathcal{X}_s \subseteq \mathcal{X}_r$. Hence $\phi(\mathcal{X}_s) \subseteq \phi(\mathcal{X}_r)$. Since ϕ is surjective, $D_{\overline{s}} \subseteq D_{\overline{r}}$, by Lemma 2.3. Therefore $D_s \subseteq D_r$ since M is faithful. Thus $s \in \sqrt{<r>}$.

Lemma 2.6. Let $r, s \in R$ and M be an R-module. If $\mathcal{X}_s \subseteq \mathcal{X}_r$, then there exists an R-module homomorphism $\varphi_{r,s} : M_r \to M_s$.

Proof. Since $\mathcal{X}_s \subseteq \mathcal{X}_r$, by Lemma 2.5 $s^n = tr$ for some positive integer n and $t \in R$. Hence $\varphi_{r,s} : M_r \to M_s$ given by $\frac{m}{r^k} \mapsto \frac{t^k m}{s^{nk}}$. It is easy to see that $\varphi_{r,s}$ is well-defined and an R-module homomorphism.

Lemma 2.7. Let $r, s, t \in R$ and M be an R-module. If $\mathcal{X}_t \subseteq \mathcal{X}_s \subseteq \mathcal{X}_r$, then $\varphi_{s,t} \circ \varphi_{r,s} = \varphi_{r,t}$.

Proof. By Lemma 2.5, $t^{n_1} = t_1 s$ and $s^{n_2} = t_2 r$ for some positive integers n_1, n_2 and $t_1, t_2 \in R$. Hence $t^{n_1 n_2} = t_1^{n_2} s^{n_2} = t_1^{n_2} t_2 r$. Now we have $\varphi_{r,t}(\frac{m}{r^n}) = \frac{t_1^{n n_2} t_2^n m}{t^{n n_1 n_2}} = \varphi_{s,t} \circ \varphi_{r,s}(\frac{m}{r^n})$.

Let M be an R-module. Then for $Q \in \mathcal{X}$ we set $\digamma_Q = \{\mathcal{X}_r \mid Q \in \mathcal{X}_r\}$. Define an order in $\digamma_Q$ as follows. For $\mathcal{X}_r$ and $\mathcal{X}_s$, write $\mathcal{X}_r \leq \mathcal{X}_s$ when $\mathcal{X}_s \subseteq \mathcal{X}_r$. For arbitrary elements $\mathcal{X}_{t_1}$ and $\mathcal{X}_{t_2}$ in $\digamma_Q$, we have $\mathcal{X}_{t_1} \cap \mathcal{X}_{t_2} = \mathcal{X}_{t_1 t_2}$. Then we get $\mathcal{X}_{t_1} \leq \mathcal{X}_{t_1 t_2}$ and $\mathcal{X}_{t_2} \leq \mathcal{X}_{t_1 t_2}$ even though there may not be an order between $\mathcal{X}_{t_1}$ and $\mathcal{X}_{t_1 t_2}$. For an element $\mathcal{X}_r$ in $\digamma_Q$ there corresponds an R-module M_r by Corollary 2.1 and that for $\mathcal{X}_r \leq \mathcal{X}_s$, there is an induced R-module homomorphism $\varphi_{r,s} : M_r \to M_s$. Then we defined the inductive limit $Lim_{\mathcal{X}_r \in \digamma_Q} M_r$ of $\{M_r, \varphi_{r,s}\}$ as follows:

Let $\Sigma = \{(\frac{m}{r^n}, \mathcal{X}_r) \mid \mathcal{X}_r \in \digamma_Q\}$ and define $(\frac{m}{r^{n_1}}, \mathcal{X}_r) \sim (\frac{m'}{s^{n_2}}, \mathcal{X}_s)$ if and only if there exists $\mathcal{X}_t \in \digamma_N$ such that $\mathcal{X}_r \leq \mathcal{X}_t$ and $\mathcal{X}_s \leq \mathcal{X}_t$ satisfying $\varphi_{r,t}(\frac{m}{r^{n_1}}) = \varphi_{s,t}(\frac{m'}{s^{n_2}})$. Then $Lim_{\mathcal{X}_r \in \digamma_Q} M_r$ is the quotient set $\Sigma/\sim$.

For $[(a, \mathcal{X}_r)], [(b, \mathcal{X}_s)] \in Lim_{\mathcal{X}_r \in \digamma_Q} M_r$, let $t = rs$. Then $\mathcal{X}_r \leq \mathcal{X}_t$ and $\mathcal{X}_s \leq \mathcal{X}_t$. Hence $[(a, \mathcal{X}_r)] = [(\varphi_{r,t}(a), \mathcal{X}_t)]$ and $[(b, \mathcal{X}_s)] = [(\varphi_{s,t}(b), \mathcal{X}_t)]$. Then we define

$$[(a, \mathcal{X}_r)] + [(b, \mathcal{X}_s)] = [(\varphi_{r,rs}(a) + \varphi_{s,rs}(b), \mathcal{X}_{rs})] \text{ and}$$
$$[(t, D_r)][(m, \mathcal{X}_s)] = [(\rho_{D_{rs},D_r}(t)\varphi_{s,rs}(m), \mathcal{X}_{rs})].$$

where $\phi^{-1}(D_r) = \mathcal{X}_r$ and $\rho_{D_{rs},D_r} : R_r \to R_{rs}$.

Proposition 2.2. Let M be an R-module. If $\rho_{D_{rs},D_r}(a) = \rho_{D_{trs},D_r}(a)$ for all $r, s, t, a \in R$, then $Lim_{\mathcal{X}_r \in \digamma_Q} M_r$ is a $lim_{D_r \in \nu_p} R_r$-module, where $p = \sqrt{(Q : M)}$ and $\nu_p = \{D_r \mid p \in D_r\}$.

Proof. It sufficient we consider the action $lim_{D_r \in \nu_p} R_r$-module $Lim_{\mathcal{X}_r \in \digamma_Q} M_r$ by $[(t, D_r)]$ $[(m, \mathcal{X}_s)] = [(\rho_{D_{rs},D_r}(t)\varphi_{s,rs}(m), \mathcal{X}_{rs})]$.

Theorem 2.3. Let M be an R-module, $Q \in \mathcal{X}$ and $p = \sqrt{(Q : M)}$. Then there exists an R_p-module isomorphism

$$Lim_{\mathcal{X}_r \in \digamma_Q} M_r \cong M_p.$$

Proof. Assume $\mu : M_p \to Lim_{\mathcal{X}_r \in \digamma_Q} M_r$ by $\mu(\frac{m}{r}) = [(\frac{m}{r}, \mathcal{X}_r)]$. It is easy to see that μ is well-defined. We show that μ is an R_p-module homomorphism.

$$\mu(\frac{m_1}{r_1} + \frac{m_2}{r_2}) = \mu(\frac{m_1 r_2 + m_2 r_1}{r_1 r_2}) = [(\frac{m_1 r_2 + m_2 r_1}{r_1 r_2}, \mathcal{X}_{r_1 r_2}]$$

and

$$\mu(\frac{m_1}{r_1}) + \mu(\frac{m_2}{r_2}) = [(\frac{m_1}{r_1}, \mathcal{X}_{r_1})] + [(\frac{m_2}{r_2}, \mathcal{X}_{r_2})] = [(\varphi_{r_1,r_1r_2}(\frac{m_1}{r_1}) + \varphi_{r_2,r_1r_2}(\frac{m_2}{r_2}), \mathcal{X}_{r_1r_2})]$$
$$= [(\frac{m_1 r_2}{r_1 r_2} + \frac{m_2 r_1}{r_1 r_2}, \mathcal{X}_{r_1 r_2})] = [(\frac{m_1 r_2 + m_2 r_1}{r_1 r_2}, \mathcal{X}_{r_1 r_2})].$$

Suppose $\frac{m}{r} \in M_p$ and $\frac{s}{t} \in R_p$. Then $\mu(\frac{s}{t}.\frac{m}{r}) = \mu(\frac{sm}{tr}) = [(\frac{sm}{tr}, \mathcal{X}_{tr})]$. Since $lim_{D_r \in \nu_p} R_r \simeq R_p$. Hence $\frac{s}{t}.\mu(\frac{m}{r}) = [(\frac{s}{t}, D_t)][(\frac{m}{r}, \mathcal{X}_r)] = [(\frac{sr}{tr}, D_{tr})][(\frac{mt}{rt}, \mathcal{X}_{rt})] = [(\frac{sr}{tr}.\frac{mt}{rt}, \mathcal{X}_{tr})] = [(\frac{sm}{rt}, \mathcal{X}_{rt})]$. Thus μ is an R_p-module homomorphism.

We can define $\theta : Lim_{\mathcal{X}_r \in \digamma_Q} M_r \to M_p$ by $\theta([(\frac{m}{r^n}, \mathcal{X}_r)]) = \frac{m}{r^n}$. It is clear that θ is well-defined. We show that θ is an R_p-module homomorphism. We can easily show that $\theta((\frac{m_1}{r_1}, \mathcal{X}_{r_1}) + (\frac{m_2}{r_2}, \mathcal{X}_{r_2})) = \theta((\frac{m_1}{r_1}, \mathcal{X}_{r_1})) + \theta((\frac{m_2}{r_2}, \mathcal{X}_{r_2}))$ for every $r_1, r_2 \in R \backslash p$ and $m_1, m_2 \in M$. Assume $[(\frac{m_1}{r_1}, \mathcal{X}_{r_1})] \in Lim_{\mathcal{X}_r \in \digamma_Q} M_r$ and $\frac{s}{t} \in R_p$. Therefore $\theta(\frac{s}{t}.[(\frac{m_1}{r_1}, \mathcal{X}_{r_1})]) = \theta([(\frac{s}{t}, D_t)][(\frac{m_1 t}{r_1 t}, \mathcal{X}_{r_1 t})]) = \theta([(\frac{m_1 s}{r_1 t}, \mathcal{X}_{r_1 t})]) = \frac{m_1 s}{r_1 t} = \frac{s}{t}.\theta([(\frac{m_1}{r_1}, \mathcal{X}_{r_1})])$. Suppose $\frac{m}{r} \in M_p$. Hence $(\theta \circ \mu)(\frac{m}{r}) = \frac{m}{r}$. On the other hand, let $[(\frac{s}{r^n}, \mathcal{X}_r)] \in Lim_{\mathcal{X}_r \in \digamma_Q} M_r$. Then $(\mu \circ \theta)([(\frac{s}{r^n}, \mathcal{X}_r)]) = \mu(\frac{s}{r^n}) = [(\frac{s}{r^n}, \mathcal{X}_{r^n})] = [(\frac{s}{r^n}, \mathcal{X}_r)]$ since $\mathcal{X}_{r^n} = \mathcal{X}_r$. Consequently, μ and θ are isomorphisms.

The saturation of a submodule N of an R-module M with respect to a prime ideal p of R is the contraction of N_p in M and designated by $S_p(N)$. It is known that $S_p(N) = \{m \in M \mid rm \in N \text{ for some } r \in R \backslash p\}$. [6]

Lemma 2.8. Let $Q \in \mathcal{X}$. Then $S_p(Q) \subseteq \operatorname{rad} Q$ for every $p \in V((Q : M))$. In particular, if $S_p(Q)$ is a prime submodule of M for some $p \in V((Q : M))$, then $S_p(Q) = \operatorname{rad} Q$.

Proof. Straightforward.

Lemma 2.9. Let $Q \in \mathcal{X}$. Consider the following statements.

(1) $radQ$ is a p-prime submodule of M.

(2) $radQ$ is a p-primary-like submodule of M.

Then (1) $\Leftrightarrow$ (2).
Furthermore, if $(Q : M) = p$ is a radical ideal of R and

(3) Q is a p-primary-like submodule of M,

then $(1) - (3)$ are equivalent.

Proof. (3) $\Rightarrow$ (1) Since $S_p(Q) \subseteq \operatorname{rad} Q$, then $S_p(Q) \neq M$. Thus by [6] and Lemma 2.8, $\operatorname{rad} Q$ is prime. The verification of the other implications is straightforward.

Theorem 2.4. Let M be an R-module and $Q \in \mathcal{X}$. Then for each the following cases $\operatorname{rad} Q$ is prime.

(1) R is a zero-dimensional ring.

(2) For each $Q \in \mathcal{X}$ and prime ideal $p = \sqrt{(Q : M)}$, $(S_p(Q) : M)$ is a radical ideal.

(3) For each $Q \in \mathcal{X}$ and prime ideal $p = \sqrt{(Q : M)}$, $S_p(Q) \neq M$.

(4) M is a multiplication module.

(5) R is a Noetherian domain and $Q \in \mathcal{X}$ is contained in only finitely many prime submodules of M.

Proof. (1) Suppose $Q \in \mathcal{X}$. Since R is zero-dimensional, $\sqrt{(Q : M)} = (P : M)$ for all prime submodules P containing Q. So $p = \sqrt{(Q : M)} = (\operatorname{rad} Q : M)$ is a prime ideal of R. Now, if $rm \in \operatorname{rad} Q$ and $m \notin \operatorname{rad} Q$, there is a prime submodule P containing Q such that $rm \in P$ and $m \notin P$ and so $r \in (P : M) = \sqrt{(Q : M)} = (\operatorname{rad} Q : M)$. Thus $\operatorname{rad} Q$ is prime.

(2) $p = \sqrt{(Q : M)} \subseteq \sqrt{(S_p(Q) : M)} \subseteq (\operatorname{rad} Q : M) = \sqrt{(Q : M)} = p$. It follows that $\sqrt{(S_p(Q) : M)} = p$. Now since $(S_p(Q) : M)$ is a radical ideal, we have $(S_p(Q) : M) = p$. It follows from [6] and Lemma 2.8, $radQ$ is a prime submodule.

(3) Suppose $S_p(Q) \neq M$. By [6], $S_p(Q)$ is a prime submodule of M. It follows from Lemma 2.8, $\operatorname{rad} Q$ is a prime submodule of M.

(4) Since for every $Q \in \mathcal{X}$, $\operatorname{rad} Q$ is proper and $(radQ : M)$ is prime, $\operatorname{rad} Q$ is prime by [1].

(5) By Lemma 2.9 we may assume that $(Q : M) \neq 0$. If P is a prime submodule containing Q, then $0 \subset \sqrt{(Q : M)} \subseteq (P : M)$ is a chain of prime ideals of R. If the later containment is proper, by [4] there are infinitely many prime ideals p with $(Q : M) \subset p \subset (P : M)$ and so we have infinitely prime submodules P containing Q, a contradiction. Hence we have $\sqrt{(Q : M)} = (P : M)$, for all prime submodules P containing N. Now, if $rm \in \operatorname{rad} Q$ and $m \notin \operatorname{rad} Q$, there is a prime submodule P containing Q such that $rm \in P$ and $m \notin P$ and so that $r \in (P : M) = \sqrt{(Q : M)} = (\operatorname{rad} Q : M)$.

Corollary 2.2. Let M be an R-module, $Q \in \mathcal{X}$ and $p = \sqrt{(Q : M)}$. Then there exists an R_p-module isomorphism

$$Lim_{\mathcal{X}_r \in \digamma_Q} M_r \cong lim_{X_r \in \nu_{\operatorname{rad} Q}} M_r.$$

Proof. Combine Theorem 2.3 and [10].

Lemma 2.10. If $\mathcal{X}_r = \cup_{\lambda\in\Lambda}\mathcal{X}_{r_\lambda}$ and for $m \in M_r$, $\varphi_{r,r_\lambda}(m) = 0$, $\lambda \in \Lambda$, then $m = 0$.

Proof. Suppose $m = \frac{m'}{r^n} \in M_r$. If $m = 0$, then there exists a positive integer n' such that $r^{n'}m' = 0$. Put $I = \{s \in R \mid sm' = 0\}$. Then I is an ideal of R. Therefore, $m = 0$ if and only if $r \in \sqrt{I}$. Therefore $r \in p$ for every prime ideal p of R containing I. Since ϕ is surjective, $p = \sqrt{(Q : M)}$ for some $Q \in \mathcal{X}$. Suppose $m \neq 0$ in M_r. Then there exists a prime ideal p of R containing I such that $r \notin p$, where $p = \sqrt{(Q : M)}$. Hence $Q \in \mathcal{X}_r$. Since $\mathcal{X}_r = \cup_{\lambda\in\Lambda}\mathcal{X}_{r_\lambda}$, there exists r_λ such that $Q \in \mathcal{X}_{r_\lambda}$ and so $r_\lambda \notin p$. Since $\varphi_{r,r_\lambda}(m) = 0$ the image of m in M_{r_λ} is 0. Therefore $\frac{m'a^{n'}}{r_\lambda^{nn'}} = \frac{0}{r_\lambda}$, where $a \in R$ and $r_\lambda \notin p$. Then $r_\lambda^n a^{n'}m' = 0$. Hence there exists $b = r_\lambda^n a^{n'} \in R\backslash p$ such that $bm' = 0$. Thus $b \in I \subseteq p$, a contradiction.

Lemma 2.11. Let $\mathcal{X} = \cup_{\lambda\in\Lambda}\mathcal{X}_{r_\lambda}$. Then the ideal generated by $\{r_\lambda\}$ equals to R.

Proof. Suppose $\mathcal{X} = \cup_{\lambda\in\Lambda}\mathcal{X}_{r_\lambda}$. Since ϕ is surjective, $Spec(R) = \phi(\mathcal{X}) = \phi(\cup_{\lambda\in\Lambda}\mathcal{X}_{r_\lambda}) = \cup_{\lambda\in\Lambda}(\phi(\mathcal{X}_{r_\lambda})) = \cup_{\lambda\in\Lambda}D_{r_\lambda}$ by Lemma 2.3. Hence $Spec(R) = \cup_{\lambda\in\Lambda}(Spec(R) - V(Rr_\lambda)) = Spec(R) - \cap_{\lambda\in\Lambda}V(Rr_\lambda) = Spec(R) - V(\sum_{\lambda\in\Lambda} Rr_\lambda)$. Therefore $V(\sum_{\lambda\in\Lambda} Rr_\lambda) = \emptyset$. Thus $\sum_{\lambda\in\Lambda} Rr_\lambda = R$.

Proposition 2.3. Given that $\mathcal{X}_r = \cup_{\lambda\in\Lambda}\mathcal{X}_{r_\lambda}$ and that, for $m_\lambda \in M_{r_\lambda}$, $\lambda \in \Lambda$, and for arbitrary $\lambda, \lambda' \in \Lambda$, $\varphi_{r_\lambda,r_\lambda r_{\lambda'}}(m_\lambda) = \varphi_{r_{\lambda'},r_\lambda r_{\lambda'}}(m_{\lambda'})$, then there exists $m \in M_r$ satisfying $m_\lambda = \varphi_{1,r_\lambda}(m)$, $\lambda \in \Lambda$.

Proof. Let $\sigma_r : R \to R_r$ define by $s \to \frac{sr}{s}$ for all $s \in R$. Let us write $\sigma_r(s)$ simply as $\overline{s}$. When $\mathcal{X}_t \subseteq \mathcal{X}_r$, then $t \in \sqrt{<r>}$ by Lemma 2.5. Therefore, there exists an R-module isomorphism from M_t to $(M_r)_{\overline{t}}$. Let $\overline{M} = M_r$ and $\overline{\mathcal{X}} = Spec_L(\overline{M})$ and let $\overline{r_\lambda}$ be the element in $\overline{R}$ determined by r_λ. Then $\mathcal{X}_r = \overline{\mathcal{X}}$ and $\mathcal{X}_{r_\lambda} = \overline{\mathcal{X}_{r_\lambda}}$. Therefore, we can assume $r = 1$ without loss of generality. If we let $r = 1$, our assumption $\mathcal{X} = \cup_{\lambda\in\Lambda}\mathcal{X}_{r_\lambda}$. From Lemma 2.11, one can choose finitely many $r_1, r_2, \cdots, r_n$ among $\{r_\lambda\}_{\lambda\in\Lambda}$ such that $\mathcal{X} = \overset{n}{\underset{i=1}{\cup}}\mathcal{X}_{r_i}$. From the hypothesis, we have $\varphi_{r_j,r_jr_i}(m_j) = \frac{r_i^n m_j'}{(r_ir_j)^n} = \varphi_{r_i,r_jr_i}(m_i) = \frac{r_j^n m_i'}{(r_ir_j)^n}$. Therefore, one can suppose a non-negative integer t_{ji} such that $(r_jr_i)^{t_{ji}}(r_i^n m_j' - r_j^n m_i') = 0$, $1 \leq j < i \leq n$. Then let $k > n + t_{ji}$ for all $1 \leq j < i \leq n$. Since for every $1 \leq h \leq n$ we can write $m_h = \frac{m_h'}{r_h^n} = \frac{r_h^{k-n}m_h'}{r_h^k}$. We set $m_h'' = r_h^{k-n}m_h'$. Hence $m_h = \frac{m_h''}{r_h^k}$. Also we have $m_j''r_i^k - m_i''r_j^k = (r_j^{k-n}m_j')r_i^k - (r_i^{k-n}m_i')r_j^k = (r_jr_i)^k r_j^{-n}m_j' - (r_jr_i)^k r_i^{-n}m_i' = (r_jr_i)^{n+t_{ji}}(r_jr_i)^s r_j^{-n}m_j' - (r_jr_i)^{n+t_{ji}}(r_jr_i)^s r_i^{-n}m_i' = (r_jr_i)^{t_{ji}}[(r_jr_i)^s r_i^n m_j' - (r_jr_i)^s r_j^n m_i'] = (r_jr_i)^s\ [(r_jr_i)^{t_{ji}}(r_i^n m_j' - r_j^n m_i')] = 0$, $1 \leq j < i \leq n$. Since $\mathcal{X}_{r_i} = \mathcal{X}_{r_i^k}$, then $\mathcal{X} = \cup_{i=1}^n \mathcal{X}_{r_i}^k$. Hence by Lemma 2.11, there exists $a_i \in R$ such that $\sum_{i=1}^n a_i r_i^k = 1$. Assume $m = \sum_{i=1}^n a_i m_i'' \in M$. Hence $r_j^k m = \sum_{i=1}^n a_i r_j^k m_i'' = \sum_{i=1}^n a_i r_i^k m_j'' = m_j''$. Thus $\varphi_{1,r_j}(m) = \frac{m_j''}{r_j^k} = m_j$. On the other hand, for arbitrary $\lambda \in \Lambda$, we put $b_\lambda = m_\lambda \varphi_{1,r_\lambda}(m)$. Then for any j we have $\varphi_{r_\lambda,r_jr_\lambda}(b_\lambda) = \varphi_{r_\lambda,r_jr_\lambda}(m_\lambda) - \varphi_{1,r_jr_\lambda}(m) = \varphi_{r_j,r_jr_\lambda}(m_j) - \varphi_{r_\lambda,r_jr_\lambda}(m) = \varphi_{r_j,r_jr_\lambda}(\varphi_{r_\lambda,r_jr_\lambda}(m)) - \varphi_{r_\lambda,r_jr_\lambda}(m) = \varphi_{r_\lambda,r_jr_\lambda}(m) - \varphi_{r_\lambda,r_jr_\lambda}(m) = 0$. Hence $b_\lambda = 0$, by Lemma 2.10. Thus $m_\lambda = \varphi_{1,r_\lambda}(m)$ for $\lambda \in \Lambda$.

Let M be an R-module. We will define the sheaf $\mathcal{O}_\mathcal{X}$ of R-modules over $(\mathcal{X}, \mathcal{T})$. Let $\mathcal{U}$ be an arbitrary open set of $\mathcal{X}$. Define $\mathcal{O}_\mathcal{X}(\mathcal{U})$ abruptly as a subset of $\prod_{Q\in\mathcal{U},\sqrt{Q:M}=p} M_p$ where p is a prime ideal of R:

$$\mathcal{O}_{\mathcal{X}}(\mathcal{U}) = \left\{ \begin{array}{r} \{m_p\} \in \prod\limits_{Q\in\mathcal{U},\sqrt{Q:M}=p} M_p : \text{If an open covering} \{\mathcal{X}_{r_\lambda}\}_{\lambda\in\Lambda} \text{of } \mathcal{U} \text{ and} \\ m_\lambda \in M_{r_\lambda} \text{are chosen properly, then for } Q \in \mathcal{X}_\lambda, \text{the germ } m_\lambda \\ \text{at } Q \text{ coincides with } m_p \text{where } \sqrt{(Q:M)} = p \end{array} \right\}.$$

Lemma 2.12. Let $m_\lambda \in M_{r_\lambda}$ and $m_{\lambda'} \in M_{r_{\lambda'}}$. Then $\varphi_{r_\lambda, r_\lambda r_{\lambda'}}(m_\lambda) = \varphi_{r_{\lambda'}, r_\lambda r_{\lambda'}}(m_{\lambda'})$.

Proof. The germ of m_λ at $Q \in \mathcal{X}_{r_\lambda r_{\lambda'}}$ equaling the germ of $m_{\lambda'}$ at Q means that the inductive limit; namely, for a certain $\mathcal{X}_{h_Q} \subseteq \mathcal{X}_{r_\lambda r_{\lambda'}}$ containing Q, $(*)$ $\varphi_{r_\lambda, r_{\lambda''}}(m_\lambda) = \varphi_{r_{\lambda'}, r_{\lambda''}}(m_{\lambda'})$. Then for each $Q \in \mathcal{X}_{r_\lambda r_{\lambda'}}$, choose an open set $\mathcal{X}_{h_Q}$ satisfying $(*)$. We get $\mathcal{X}_{r_\lambda r_{\lambda'}} = \bigcup_{Q\in\mathcal{X}_{r_\lambda r_{\lambda'}}} \mathcal{X}_{h_Q}$. Then, from $(*)$ for all $Q \in \mathcal{X}_{r_\lambda r_{\lambda'}}$ we have $\varphi_{r_{\lambda'}, h_Q}(\varphi_{r_\lambda, h_Q}(m_\lambda)) = \varphi_{r_{\lambda'}, h_Q}(\varphi_{r_{\lambda'}, h_Q}(m_{\lambda'}))$. Therefore $\varphi_{r_{\lambda'}, h_Q}(\varphi_{r_\lambda, h_Q}(m_\lambda) - \varphi_{r_{\lambda'}, h_Q}(m_{\lambda'})) = 0$. Thus $\varphi_{r_\lambda, r_{\lambda'}}(m_\lambda) = \varphi_{r_\lambda, r_{\lambda'}}(m_{\lambda'})$ by Lemma 2.10.

Let X be a topological space, and for an open subset U of X, let $F(U)$ be an R-module. Then F is said to be a sheaf if the following conditions are satisfied.[3]

(1) For an open set $V \subseteq U$ of X, there exists an R-module homomorphism $\rho^*_{V,U} : F(U) \to F(V)$ such that:

  (i) $\rho^*_{U,U} = id_{F(U)}$,

  (ii) for open sets $W \subseteq V \subseteq U$ of X, $\rho^*_{W,U} = \rho^*_{W,V} \circ \rho^*_{V,U}$.

(2) Let U be an open set in X such that U is a union of open sets, i.e., $U = \cup_{j\in J} U_j$.

  (i) If $m \in F(U)$ satisfies $\rho^*_{U_j,U}(m) = 0$, $j \in J$, then $m = 0$.

  (ii) If $m_j \in F(U_j)$, $j \in J$, satisfies $\rho^*_{U_i\cap U_j, U_j}(m_j) = \rho^*_{U_i\cap U_j, U_i}(m_i)$, $i, j \in J$, then there exists $m \in F(U)$ $m_j = \rho^*_{U_j,U}(m)$.

Let M be an R-module. It is easily seen that $\mathcal{O}_{\mathcal{X}}(\mathcal{U})$ with addition and multiplication as $\{m_p\} + \{m'_p\} = \{m_p + m'_p\}$ and $r\{m_p\} = \{rm_p\}$, where $r \in R$, $m, m' \in M$ and p is prime ideal of R, is an R-module. Furthermore, for open sets $\mathcal{V} \subseteq \mathcal{U}$ we define $\varphi_{\mathcal{U},\mathcal{V}} : \mathcal{O}_{\mathcal{X}}(\mathcal{U}) \to \mathcal{O}_{\mathcal{X}}(\mathcal{V})$ given by $\{m_p\}_{Q\in\mathcal{U}} \mapsto \{m'_{p'}\}_{Q'\in\mathcal{V}}$, where $p = \sqrt{(Q:M)}$ and $p' = \sqrt{(Q':M)}$.

Theorem 2.5. Let M be an R-module. Then $\mathcal{O}_{\mathcal{X}}$ is a sheaf of R-modules over $\mathcal{X}$ with the Zariski topology $\mathcal{T}$ and $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) = M$.

Proof. It is easy to see that the condition (1) of the above definition hole. Now we check the condition (2). Suppose that $\mathcal{U} = \cup_{i\in I}\mathcal{U}_i$ and $m \in \mathcal{O}_{\mathcal{X}}(\mathcal{U})$ such that $\varphi_{\mathcal{U},\mathcal{U}_i}(m) = 0$, $i \in I$. Since $m \in \mathcal{O}_{\mathcal{X}}(\mathcal{U})$, then $m = \{m_p\} \in \prod_{Q\in\mathcal{U},\sqrt{Q:M}=p} M_p$. Therefore $0 = \varphi_{\mathcal{U},\mathcal{U}_i}(m) = \{m_p\}_{Q\in\mathcal{U}_i}$. Hence for every $Q \in \mathcal{U}_i$ we have $m_p = 0$. Since $\mathcal{U} = \cup_{i\in I}\mathcal{U}_i$, then for every $Q \in \mathcal{U}$ we have $m_p = 0$. Thus $m = 0$. Assume $m_i \in \mathcal{O}_{\mathcal{X}}(\mathcal{U}_i)$ and $\varphi_{\mathcal{U}_i,\mathcal{U}_i\cap\mathcal{U}_j}(m_i) = \varphi_{\mathcal{U}_j,\mathcal{U}_i\cap\mathcal{U}_j}(m_j)$, $i, j \in I$. Put $m_i = \{m_p^{(i)}\} \in \prod_{Q\in\mathcal{U},\sqrt{Q:M}=p} M_p$. Therefore $\{m_p^{(i)}\}_{Q\in\mathcal{U}_i\cap\mathcal{U}_j} = \varphi_{\mathcal{U}_i,\mathcal{U}_i\cap\mathcal{U}_j}(m_i) = \varphi_{\mathcal{U}_j,\mathcal{U}_i\cap\mathcal{U}_j}(m_j) = \{m_p^{(j)}\}_{Q\in\mathcal{U}_i\cap\mathcal{U}_j}$. Hence for each $Q \in \mathcal{U}_i \cap \mathcal{U}_j$ we have $m_p^{(i)} = m_p^{(j)}$. Now, let $m = \{m_p\} \in \prod_{Q\in\mathcal{U},\sqrt{Q:M}=p} M_p$. For every $Q \in \mathcal{U}_i$ we put $m_p = m_p^{(i)}$. Hence $m \in \mathcal{O}_{\mathcal{X}}(\mathcal{U})$ and

$\varphi_{\mathcal{U},\mathcal{U}_i}(m) = \{m_p\}_{Q\in\mathcal{U}_i} = m_i$. Thus the condition (2) holds. It is clear that $\mathcal{O}_{\mathcal{X}}(\mathcal{X}_r) = \mathcal{O}_{\mathcal{X}}(\mathcal{X}) = M$ for $r = 1$.

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Soft neutrosophic semigroup and their generalization

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Abstract Soft set theory is a general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. In this paper we introduced soft neutrosophic semigroup,soft neutosophic bisemigroup, soft neutrosophic N-semigroup with the discuissionf of some of their characteristics. We also introduced a new type of soft neutrophic semigroup, the so called soft strong neutrosophic semigoup which is of pure neutrosophic character. This notion also foound in all the other corresponding notions of soft neutrosophic thoery. We also given some of their properties of this newly born soft structure related to the strong part of neutrosophic theory.

Keywords Neutrosophic semigroup, neutrosophic bisemigroup, neutrosophic N-semigroup, soft set, soft semigroup, soft neutrosophic semigroup, soft neutrosophic bisemigroup, soft neutrosophic N-semigroup.

§1. Introduction and preliminaries

Florentine Smarandache for the first time introduced the concept of neutrosophy in 1995, which is basically a new branch of philosophy which actually studies the origin, nature, and scope of neutralities. The neutrosophic logic came into being by neutrosophy. In neutrosophic logic each proposition is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. Neutrosophic logic is an extension of fuzzy logic. In fact the neutrosophic set is the generalization of classical set, fuzzy conventional set, intuitionistic fuzzy set, and interval valued fuzzy set. Neutrosophic logic is used to overcome the problems of impreciseness, indeterminate, and inconsistencies of date etc. The theory of neutrosophy is so applicable to every field of algebra. W. B. Vasantha Kandasamy and Florentin Smarandache introduced neutrosophic fields, neutrosophic rings,neutrosophic vector spaces,neutrosophic groups,neutrosophic bigroups and neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, and neutrosophic

N-semigroups, neutrosophic loops, nuetrosophic biloops, and neutrosophic N-loops, and so on. Mumtaz ali et al. introduced nuetrosophic LA-semigroups.

Molodtsov introduced the theory of soft set. This mathematical tool is free from parameterization inadequacy, syndrome of fuzzy set theory, rough set theory, probability theory and so on. This theory has been applied successfully in many fields such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration, and probability. Recently soft set theory attained much attention of the researchers since its appearance and the work based on several operations of soft set introduced in [2, 9, 10]. Some properties and algebra may be found in [1] . Feng et al. introduced soft semirings in [5]. By means of level soft sets an adjustable approach to fuzzy soft set can be seen in [6]. Some other concepts together with fuzzy set and rough set were shown in [7, 8].

This paper is about to introduced soft nuetrosophic semigroup, soft neutrosophic group, and soft neutrosophic N-semigroup and the related strong or pure part of neutrosophy with the notions of soft set theory. In the proceeding section, we define soft neutrosophic semigroup, soft neutrosophic strong semigroup, and some of their properties are discussed. In the next section, soft neutrosophic bisemigroup are presented with their strong neutrosophic part. Also in this section some of their characterization have been made. In the last section soft neutrosophic N-semigroup and their corresponding strong theory have been constructed with some of their properties.

§2. Definition and properties

Definition 2.1. Let S be a semigroup, the semigroup generated by S and I i.e. $S \cup I$ denoted by $\langle S \cup I \rangle$ is defined to be a neutrosophic semigroup where I is indeterminacy element and termed as neutrosophic element.

It is interesting to note that all neutrosophic semigroups contain a proper subset which is a semigroup.

Example 2.1. Let $Z = \{$the set of positive and negative integers with zero$\}$, Z is only a semigroup under multiplication. Let $N(S) = \{\langle Z \cup I \rangle\}$ be the neutrosophic semigroup under multiplication. Clearly $Z \subset N(S)$ is a semigroup.

Definition 2.2. Let $N(S)$ be a neutrosophic semigroup. A proper subset P of $N(S)$ is said to be a neutrosophic subsemigroup, if P is a neutrosophic semigroup under the operations of $N(S)$. A neutrosophic semigroup $N(S)$ is said to have a subsemigroup if $N(S)$ has a proper subset which is a semigroup under the operations of $N(S)$.

Theorem 2.1. Let $N(S)$ be a neutrosophic semigroup. Suppose P_1 and P_2 be any two neutrosophic subsemigroups of $N(S)$ then $P_1 \cup P_2$ (i.e. the union) the union of two neutrosophic subsemigroups in general need not be a neutrosophic subsemigroup.

Definition 2.3. A neutrosophic semigroup $N(S)$ which has an element e in $N(S)$ such that $e * s = s * e = s$ for all $s \in N(S)$, is called as a neutrosophic monoid.

Definition 2.4. Let $N(S)$ be a neutrosophic monoid under the binary operation $*$. Suppose e is the identity in $N(S)$, that is $s * e = e * s = s$ for all $s \in N(S)$. We call a proper subset P of $N(S)$ to be a neutrosophic submonoid if

1. P is a neutrosophic semigroup under $*$.
2. $e \in P$, i.e., P is a monoid under $*$.

Definition 2.5. Let $N(S)$ be a neutrosophic semigroup under a binary operation $*$. P be a proper subset of $N(S)$. P is said to be a neutrosophic ideal of $N(S)$ if the following conditions are satisfied.

1. P is a neutrosophic semigroup.
2. For all $p \in P$ and for all $s \in N(S)$ we have $p * s$ and $s * p$ are in P.

Definition 2.6. Let $N(S)$ be a neutrosophic semigroup. P be a neutrosophic ideal of $N(S)$, P is said to be a neutrosophic cyclic ideal or neutrosophic principal ideal if P can be generated by a single element.

Definition 2.7. Let $(BN(S), *, o)$ be a nonempty set with two binary operations $*$ and o. $(BN(S), *, o)$ is said to be a neutrosophic bisemigroup if $BN(S) = P1 \cup P2$ where atleast one of $(P1, *)$ or $(P2, o)$ is a neutrosophic semigroup and other is just a semigroup. $P1$ and $P2$ are proper subsets of $BN(S)$, i.e. $P1 \subsetneq P2$.

If both $(P1, *)$ and $(P2, o)$ in the above definition are neutrosophic semigroups then we call $(BN(S), *, o)$ a strong neutrosophic bisemigroup. All strong neutrosophic bisemigroups are trivially neutrosophic bisemigroups.

Example 2.2. Let $(BN(S), *, o) = \{0, 1, 2, 3, I, 2I, 3I, S(3), *, o\} = (P_1, *) \cup (P_2, o)$ where $(P_1, *) = \{0, 1, 2, 3, I, 2I, 3I, *\}$ and $(P_2, o) = (S(3), o)$. Clearly $(P_1, *)$ is a neutrosophic semigroup under multiplication modulo 4. (P_2, o) is just a semigroup. Thus $(BN(S), *, o)$ is a neutrosophic bisemigroup.

Definition 2.8. Let $(BN(S) = P1 \cup P2; o, *)$ be a neutrosophic bisemigroup. A proper subset $(T, o, *)$ is said to be a neutrosophic subbisemigroup of $BN(S)$ if

1. $T = T1 \cup T2$ where $T1 = P1 \cap T$ and $T2 = P2 \cap T$.
2. At least one of $(T1, o)$ or $(T2, *)$ is a neutrosophic semigroup.

Definition 2.9. Let $(BN(S) = P_1 \cup P_2, o, *)$ be a neutrosophic strong bisemigroup. A proper subset T of $BN(S)$ is called the strong neutrosophic subbisemigroup if $T = T_1 \cup T_2$ with $T_1 = P_1 \cap T$ and $T_2 = P_2 \cap T$ and if both $(T_1, *)$ and (T_2, o) are neutrosophic subsemigroups of $(P_1, *)$ and (P_2, o) respectively. We call $T = T_1 \cup T_2$ to be a neutrosophic strong subbisemigroup, if atleast one of $(T_1, *)$ or (T_2, o) is a semigroup then $T = T_1 \cup T_2$ is only a neutrosophic subsemigroup.

Definition 2.10. Let $(BN(S) = P_1 \cup P_2 *, o)$ be any neutrosophic bisemigroup. Let J be a proper subset of $B(NS)$ such that $J_1 = J \cap P_1$ and $J_2 = J \cap P_2$ are ideals of P_1 and P_2 respectively. Then J is called the neutrosophic bi-ideal of $BN(S)$.

Definition 2.11. Let $(BN(S), *, o)$ be a strong neutrosophic bisemigroup where $BN(S) = P_1 \cup P_2$ with $(P_1, *)$ and (P_2, o) be any two neutrosophic semigroups. Let J be a proper subset of $BN(S)$ where $I = I_1 \cup I_2$ with $I_1 = J \cap P_1$ and $I_2 = J \cap P_2$ are neutrosophic ideals of the neutrosophic semigroups P_1 and P_2 respectively. Then I is called or defined as the strong neutrosophic bi-ideal of $B(N(S))$.

Union of any two neutrosophic bi-ideals in general is not a neutrosophic bi-ideal. This is true of neutrosophic strong bi-ideals.

Definition 2.12. Let $\{S(N), *_1, \ldots, *_N\}$ be a non empty set with N-binary operations defined on it. We call $S(N)$ a neutrosophic N-semigroup (N a positive integer) if the following conditions are satisfied.

1. $S(N) = S_1 \cup \ldots \cup S_N$ where each S_i is a proper subset of $S(N)$ i.e. $S_i \subsetneq S_j$ or $S_j \subsetneq S_i$ if $i \neq j$.

2. $(S_i, *_i)$ is either a neutrosophic semigroup or a semigroup for $i = 1, 2, \ldots, N$.

If all the N-semigroups $(Si, *i)$ are neutrosophic semigroups (i.e. for $i = 1, 2, \ldots, N$) then we call $S(N)$ to be a neutrosophic strong N-semigroup.

Example 2.3. Let $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$ be a neutrosophic 4-semigroup where

$S_1 = \{Z_{12}$, semigroup under multiplication modulo 12$\}$.

$S_2 = \{0, 1, 2, 3, I, 2I, 3I$, semigroup under multiplication modulo 4$\}$, a neutrosophic semigroup.

$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \langle R \cup I \rangle \right\}$, neutrosophic semigroup under matrix multiplication and $S_4 = \langle Z \cup I \rangle$, neutrosophic semigroup under multiplication.

Definition 2.13. Let $S(N) = \{S_1 \cup S_2 \cup \ldots \cup S_N, *_1, \ldots, *_N\}$ be a neutrosophic N-semigroup. A proper subset $P = \{P_1 \cup P_2 \cup \ldots \cup P_N, *_1, *_2, \ldots, *_N\}$ of $S(N)$ is said to be a neutrosophic Nsubsemigroup if $P_i = P \cap S_i, i = 1, 2, \ldots, N$ are subsemigroups of S_i in which atleast some of the subsemigroups are neutrosophic subsemigroups.

Definition 2.14. Let $S(N) = \{S_1 \cup S_2 \cup \ldots \cup S_N, *_1, \ldots, *_N\}$ be a neutrosophic strong N-semigroup. A proper subset $T = \{T_1 \cup T_2 \cup \ldots \cup T_N, *_1, \ldots, *_N\}$ of $S(N)$ is said to be a neutrosophic strong sub N-semigroup if each $(T_i, *_i)$ is a neutrosophic subsemigroup of $(S_i, *_i)$ for $i = 1, 2, \ldots, N$ where $T_i = T \cap S_i$.

If only a few of the $(Ti, *i)$ in T are just subsemigroups of $(Si, *i)$ (i.e. $(Ti, *i)$ are not neutrosophic subsemigroups then we call T to be a sub N-semigroup of $S(N)$.

Definition 2.15. Let $S(N) = \{S_1 \cup S_2 \cup \ldots \cup S_N, *_1, \ldots, *_N\}$ be a neutrosophic N-semigroup. A proper subset $P = \{P_1 \cup P_2 \cup \ldots \cup P_N, *_1, \ldots, *_N\}$ of $S(N)$ is said to be a neutrosophic N-subsemigroup, if the following conditions are true,

i. P is a neutrosophic sub N-semigroup of $S(N)$.

ii. Each $P_i = P \cap S_i, i = 1, 2, \ldots, N$ is an ideal of S_i.

Then P is called or defined as the neutrosophic N-ideal of the neutrosophic N-semigroup $S(N)$.

Definition 2.16. Let $S(N) = \{S_1 \cup S_2 \cup \ldots \cup S_N, *_1, \ldots, *_N\}$ be a neutrosophic strong N-semigroup. A proper subset $J = \{I_1 \cup I_2 \cup \ldots \cup I_N\}$ where $I_t = J \cap S_t$ for $t = 1, 2, \ldots, N$ is said to be a neutrosophic strong N-ideal of $S(N)$ if the following conditions are satisfied.

1. Each is a neutrosophic subsemigroup of $S_t, t = 1, 2, \ldots, N$ i.e. It is a neutrosophic strong N-subsemigroup of $S(N)$.

2. Each is a two sided ideal of S_t for $t = 1, 2, \ldots, N$.

Similarly one can define neutrosophic strong N-left ideal or neutrosophic strong right ideal of $S(N)$.

A neutrosophic strong N-ideal is one which is both a neutrosophic strong N-left ideal and N-right ideal of $S(N)$.

Throughout this subsection U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A \subset E$. Molodtsov [12] defined the soft set in the following manner:

Definition 2.17. A pair (F, A) is called a soft set over U where F is a mapping given by $F : A \longrightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-elements of the soft set (F, A), or as the set of e-approximate elements of the soft set.

Example 2.4. Suppose that U is the set of shops. E is the set of parameters and each parameter is a word or senctence. Let E={high rent, normal rent, in good condition, in bad condition}. Let us consider a soft set (F, A) which describes the attractiveness of shops that Mr. Z is taking on rent. Suppose that there are five houses in the universe $U = \{h_1, h_2, h_3, h_4, h_5\}$ under consideration, and that $A = \{e_1, e_2, e_3\}$ be the set of parameters where

e_1 stands for the parameter high rent.

e_2 stands for the parameter normal rent.

e_3 stands for the parameter in good condition.

Suppose that

$F(e_1) = \{h_1, h_4\}$.

$F(e_2) = \{h_2, h_5\}$.

$F(e_3) = \{h_3, h_4, h_5\}$.

The soft set (F, A) is an approximated family $\{F(e_i), i = 1, 2, 3\}$ of subsets of the set U which gives us a collection of approximate description of an object. Thus, we have the soft set (F, A) as a collection of approximations as below:

$(F, A) = \{$high rent $= \{h_1, h_4\}$, normal rent $= \{h_2, h_5\}$, in good condition $= \{h_3, h_4, h_5\}\}$.

Definition 2.18. For two soft sets (F, A) and (H, B) over U, (F, A) is called a soft subset of (H, B) if

1. $A \subseteq B$.

2. $F(e) \subseteq G(e)$, for all $e \in A$.

This relationship is denoted by $(F, A) \tilde{\subset} (H, B)$. Similarly (F, A) is called a soft superset of (H, B) if (H, B) is a soft subset of (F, A) which is denoted by $(F, A) \tilde{\supset} (H, B)$.

Definition 2.19. Two soft sets (F, A) and (H, B) over U are called soft equal if (F, A) is a soft subset of (H, B) and (H, B) is a soft subset of (F, A).

Definition 2.20. (F, A) over U is called an absolute soft set if $F(e) = U$ for all $e \in A$ and we denote it by U.

Definition 2.21. Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by$(F, A) \cap_R (G, B) = (H, C)$ where (H, C) is defined as $H(c) = F(c) \cap G(c)$ for all $c \in C = A \cap B$.

Definition 2.22. The extended intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $e \in C$, $H(e)$ is defined as

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cap G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(F, A) \cap_{\varepsilon} (G, B) = (H, C)$.

Definition 2.23. The resticted union of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $e \in C$, $H(e)$ is defined as the soft set $(H, C) = (F, A) \cup_R (G, B)$ where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$.

Definition 2.24. The extended union of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $e \in C$, $H(e)$ is defined as

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(F, A) \cup_{\varepsilon} (G, B) = (H, C)$.

Definition 2.25. A soft set (F, A) over S is called a soft semigroup over S if $(F, A) \overset{\curlywedge}{\circ} (F, A) \subseteq (F, A)$.

It is easy to see that a soft set (F, A) over S is a soft semigroup if and only if $\phi \neq F(a)$ is a subsemigroup of S.

Definition 2.26. A soft set (F, A) over a semigroup S is called a soft left (right) ideal over S, if $(S, E) \subseteq (F, A), ((F, A) \subseteq (S, E))$.

A soft set over S is a soft ideal if it is both a soft left and a soft right ideal over S.

Proposition 2.1. A soft set (F, A) over S is a soft ideal over S if and only if $\phi \neq F(a)$ is an ideal of S.

Definition 2.27. Let (G, B) be a soft subset of a soft semigroup (F, A) over S, then (G, B) is called a soft subsemigroup (ideal) of (F, A) if $G(b)$ is a subsemigroup (ideal) of $F(b)$ for all $b \in A$.

§3. Soft neutrosophic semigroup

Definition 3.1. Let $N(S)$ be a neutrosophic semigroup and (F, A) be a soft set over $N(S)$. Then (F, A) is called soft neutrosophic semigroup if and only if $F(e)$ is neutrosophic subsemigroup of $N(S)$, for all $e \in A$.

Equivalently (F, A) is a soft neutrosophic semigroup over $N(S)$ if $(F, A) \overset{\curlywedge}{\circ} (F, A) \subseteq (F, A)$, where $\tilde{N}_{(N(S),A)} \neq (F, A) \neq \overset{\sim}{\phi}$.

Example 3.1. Let $N(S) = \langle Z^+ \cup \{0\}^+ \cup \{I\} \rangle$ be a neutrosophic semigroup under $+$. Consider $P = \langle 2Z^+ \cup I \rangle$ and $R = \langle 3Z^+ \cup I \rangle$ are neutrosophic subsemigroup of $N(S)$. Then clearly for all $e \in A$, (F, A) is a soft neutrosophic semigroup over $N(S)$, where $F(x_1) = \{\langle 2Z^+ \cup I \rangle\}$, $F(x_2) = \{\langle 3Z^+ \cup I \rangle\}$.

Theorem 3.1. A soft neutrosophic semigroup over $N(S)$ always contain a soft semigroup over S.

Proof. The proof of this theorem is straight forward.

Theorem 3.2. Let (F, A) and (H, A) be two soft neutrosophic semigroups over $N(S)$. Then their intersection $(F, A) \cap (H, A)$ is again soft neutrosophic semigroup over $N(S)$.

Proof. The proof is staight forward.

Theorem 3.3. Let (F, A) and (H, B) be two soft neutrosophic semigroups over $N(S)$. If $A \cap B = \phi$, then $(F, A) \cup (H, B)$ is a soft neutrosophic semigroup over $N(S)$.

Remark 3.1. The extended union of two soft neutrosophic semigroups (F, A) and (K, B) over $N(S)$ is not a soft neutrosophic semigroup over $N(S)$.

We take the following example for the proof of above remark.

Example 3.2. Let $N(S) = \langle Z^+ \cup I\rangle$ be the neutrosophic semigroup under $+$. Take $P_1 = \{\langle 2Z^+ \cup I\rangle\}$ and $P_2 = \{\langle 3Z^+ \cup I\rangle\}$ to be any two neutrosophic subsemigroups of $N(S)$. Then clearly for all $e \in A$, (F, A) is a soft neutrosophic semigroup over $N(S)$, where $F(x_1) = \{\langle 2Z^+ \cup I\rangle\}, F(x_2) = \{\langle 3Z^+ \cup I\rangle\}$.

Again Let $R_1 = \{\langle 5Z^+ \cup I\rangle\}$ and $R_2 = \{\langle 4Z^+ \cup I\rangle\}$ be another neutrosophic subsemigroups of $N(S)$ and (K, B) is another soft neutrosophic semigroup over $N(S)$, where $K(x_1) = \{\langle 5Z^+ \cup I\rangle\}, K(x_3) = \{\langle 4Z^+ \cup I\rangle\}$.

Let $C = A \cup B$. The extended union $(F, A) \cup_\varepsilon (K, B) = (H, C)$ where $x_1 \in C$, we have $H(x_1) = F(x_1) \cup K(x_1)$ is not neutrosophic subsemigroup as union of two neutrosophic subsemigroup is not neutrosophic subsemigroup.

Proposition 3.1. The extended intersection of two soft neutrosophic semigroups over $N(S)$ is soft neutrosophic semigruop over $N(S)$.

Remark 3.2. The restricted union of two soft neutrosophic semigroups (F, A) and (K, B) over $N(S)$ is not a soft neutrosophic semigroup over $N(S)$.

We can easily check it in above example.

Proposition 3.2. The restricted intersection of two soft neutrosophic semigroups over $N(S)$ is soft neutrosophic semigroup over $N(S)$.

Proposition 3.3. The AND operation of two soft neutrosophic semigroups over $N(S)$ is soft neutrosophic semigroup over $N(S)$.

Proposition 3.4. The OR operation of two soft neutosophic semigroup over $N(S)$ may not be a soft nuetrosophic semigroup over $N(S)$.

Definition 3.2. Let $N(S)$ be a neutrosophic monoid and (F, A) be a soft set over $N(S)$. Then (F, A) is called soft neutrosophic monoid if and only if $F(e)$ is neutrosophic submonoid of $N(S)$, for all $x \in A$.

Example 3.3. Let $N(S) = \langle Z \cup I\rangle$ be a neutrosophic monoid under $+$. Let $P = \langle 2Z \cup I\rangle$ and $Q = \langle 3Z \cup I\rangle$ are neutrosophic submonoids of $N(S)$. Then (F, A) is a soft neutrosophic monoid over $N(S)$, where $F(x_1) = \{\langle 2Z \cup I\rangle\}, F(x_2) = \{\langle 3Z \cup I\rangle\}$.

Theorem 3.4. Every soft neutrosophic monoid over $N(S)$ is a soft neutrosophic semigroup over $N(S)$ but the converse is not true in general.

Proof. The proof is straightforward.

Proposition 3.5. Let (F, A) and (K, B) be two soft neutrosophic monoids over $N(S)$. Then

1. Their extended union $(F, A) \cup_{\varepsilon} (K, B)$ over $N(S)$ is not soft neutrosophic monoid over $N(S)$.

2. Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $N(S)$ is soft neutrosophic monoid over $N(S)$.

3. Their restricted union $(F, A) \cup_{R} (K, B)$ over $N(S)$ is not soft neutrosophic monoid over $N(S)$.

4. Their restricted intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $N(S)$ is soft neutrosophic monoid over $N(S)$.

Proposition 3.6. Let (F, A) and (H, B) be two soft neutrosophic monoid over $N(S)$. Then

1. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic monoid over $N(S)$.

2. Their OR operation $(F, A) \vee (H, B)$ is not soft neutrosophic monoid over $N(S)$.

Definition 3.3. Let (F, A) be a soft neutrosophic semigroup over $N(S)$, then (F, A) is called Full-soft neutrosophic semigroup over $N(S)$ if $F(x) = N(S)$, for all $x \in A$. We denote it by $N(S)$.

Theorem 3.5. Every Full-soft neutrosophic semigroup over $N(S)$ always contain absolute soft semigroup over S.

Proof. The proof of this theorem is straight forward.

Definition 3.4. Let (F, A) and (H, B) be two soft neutrosophic semigroups over $N(S)$. Then (H, B) is a soft neutrosophic subsemigroup of (F, A), if

1. $B \subset A$.

2. $H(a)$ is neutrosophic subsemigroup of $F(a)$, for all $a \in B$.

Example 3.4. Let $N(S) = \langle Z \cup I \rangle$ be a neutrosophic semigroup under $+$. Then (F, A) is a soft neutrosophic semigroup over $N(S)$, where $F(x_1) = \{\langle 2Z \cup I \rangle\}, F(x_2) = \{\langle 3Z \cup I \rangle\}, F(x_3) = \{\langle 5Z \cup I \rangle\}$.

Let $B = \{x_1, x_2\} \subset A$. Then (H, B) is soft neutrosophic subsemigroup of (F, A) over $N(S)$, where $H(x_1) = \{\langle 4Z \cup I \rangle\}, H(x_2) = \{\langle 6Z \cup I \rangle\}$.

Theorem 3.6. A soft neutrosophic semigroup over $N(S)$ have soft neutrosophic subsemigroups as well as soft subsemigroups over $N(S)$.

Proof. Obvious.

Theorem 3.7. Every soft semigroup over S is always soft neutrosophic subsemigroup of soft neutrosophic semigroup over $N(S)$.

Proof. The proof is obvious.

Theorem 3.8. Let (F, A) be a soft neutrosophic semigroup over $N(S)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic subsemigroups of (F, A) then

1. $\cap_{i \in I} (H_i, B_i)$ is a soft neutrosophic subsemigroup of (F, A).

2. $\wedge_{i \in I} (H_i, B_i)$ is a soft neutrosophic subsemigroup of $\wedge_{i \in I} (F, A)$.

3. $\cup_{i \in I} (H_i, B_i)$ is a soft neutrosophic subsemigroup of (F, A) if $B_i \cap B_j = \phi$, for all $i \neq j$.

Proof. Straightforward.

Definition 3.5. A soft set (F, A) over $N(S)$ is called soft neutrosophic left (right) ideal over $N(S)$ if $N(S) \overset{\curlywedge}{\circ} (F, A) \subseteq (F, A)$, where $\tilde{N}_{(N(S),A)} \neq (F, A) \neq \widetilde{\phi}$ and $N(S)$ is Full-soft neutrosophic semigroup over $N(S)$.

A soft set over $N(S)$ is a soft neutrosophic ideal if it is both a soft neutrosophic left and a soft neutrosophic right ideal over $N(S)$.

Example 3.5. Let $N(S) = \langle Z \cup I \rangle$ be the neutrosophic semigroup under multiplication. Let $P = \langle 2Z \cup I \rangle$ and $Q = \langle 4Z \cup I \rangle$ are neutrosophic ideals of $N(S)$. Then clearly (F, A) is a soft neutrosophic ideal over $N(S)$, where $F(x_1) = \{\langle 2Z \cup I \rangle\}, F(x_2) = \{\langle 4Z \cup I \rangle\}$.

Proposition 3.7. (F, A) is soft neutrosophic ideal if and only if $F(x)$ is a neutrosophic ideal of $N(S)$, for all $x \in A$.

Theorem 3.9. Every soft neutrosophic ideal (F, A) over $N(S)$ is a soft neutrosophic semigroup but the converse is not true.

Proposition 3.8. Let (F, A) and (K, B) be two soft neutrosophic ideals over $N(S)$. Then

1. Their extended union $(F, A) \cup_\varepsilon (K, B)$ over $N(S)$ is soft neutrosophic ideal over $N(S)$.

2. Their extended intersection $(F, A) \cap_\varepsilon (K, B)$ over $N(S)$ is soft neutrosophic ideal over $N(S)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $N(S)$ is soft neutrosophic ideal over $N(S)$.

4. Their restricted intersection $(F, A) \cap_\varepsilon (K, B)$ over $N(S)$ is soft neutrosophic ideal over $N(S)$.

Proposition 3.9.

1. Let (F, A) and (H, B) be two soft neutrosophic ideal over $N(S)$.

2. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic ideal over $N(S)$.

3. Their OR operation $(F, A) \vee (H, B)$ is soft neutrosophic ideal over $N(S)$.

Theorem 3.10. Let (F, A) and (G, B) be two soft semigroups (ideals) over S and T respectively. Then $(F, A) \times (G, B)$ is also a soft semigroup (ideal) over $S \times T$.

Proof. The proof is straight forward.

Theorem 3.11. Let (F, A) be a soft neutrosophic semigroup over $N(S)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic ideals of (F, A) then

1. $\cap_{i \in I} (H_i, B_i)$ is a soft neutrosophic ideal of (F, A).

2. $\wedge_{i \in I} (H_i, B_i)$ is a soft neutrosophic ideal of $\wedge_{i \in I} (F, A)$.

3. $\cup_{i \in I} (H_i, B_i)$ is a soft neutrosophic ideal of (F, A).

4. $\vee_{i \in I} (H_i, B_i)$ is a soft neutrosophic ideal of $\vee_{i \in I} (F, A)$.

Definition 3.6. A soft set (F, A) over $N(S)$ is called soft neutrosophic principal ideal or soft neutrosophic cyclic ideal if and only if $F(x)$ is a principal or cyclic neutrosophic ideal of $N(S)$, for all $x \in A$.

Proposition 3.10. Let (F, A) and (K, B) be two soft neutrosophic principal ideals over $N(S)$. Then

1. Their extended union $(F, A) \cup_\varepsilon (K, B)$ over $N(S)$ is not soft neutrosophic principal ideal over $N(S)$.

2. Their extended intersection $(F, A) \cap_\varepsilon (K, B)$ over $N(S)$ is soft neutrosophic principal ideal over $N(S)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $N(S)$ is not soft neutrosophic principal ideal over $N(S)$.

4. Their restricted intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $N(S)$ is soft neutrosophic principal ideal over $N(S)$.

Proposition 3.11. Let (F, A) and (H, B) be two soft neutrosophic principal ideals over $N(S)$. Then

1. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic principal ideal over $N(S)$.

2. Their OR operation $(F, A) \vee (H, B)$ is not soft neutrosophic principal ideal over $N(S)$.

§3. Soft neutrosophic bisemigroup

Definition 3.1. Let $\{BN(S), *_1, *_2\}$ be a neutrosophic bisemigroup and let (F, A) be a soft set over $BN(S)$. Then (F, A) is said to be soft neutrosophic bisemigroup over $BN(G)$ if and only if $F(x)$ is neutrosophic subbisemigroup of $BN(G)$ for all $x \in A$.

Example 3.1. Let $BN(S) = \{0, 1, 2, I, 2I, \langle Z \cup I\rangle, \times, +\}$ be a neutosophic bisemigroup. Let $T = \{0, I, 2I, \langle 2Z \cup I\rangle, \times, +\}, P = \{0, 1, 2, \langle 5Z \cup I\rangle, \times, +\}$ and $L = \{0, 1, 2, Z, \times, +\}$ are neutrosophic subbisemigroup of $BN(S)$. The (F, A) is clearly soft neutrosophic bisemigroup over $BN(S)$, where $F(x_1) = \{0, I, 2I, \langle 2Z \cup I\rangle, \times, +\}, F(x_2) = \{0, 1, 2, \langle 5Z \cup I\rangle, \times, +\}, F(x_3) = \{0, 1, 2, Z, \times, +\}$.

Theorem 3.1. Let (F, A) and (H, A) be two soft neutrosophic bisemigroup over $BN(S)$. Then their intersection $(F, A) \cap (H, A)$ is again a soft neutrosophic bisemigroup over $BN(S)$.

Proof. Straightforward.

Theorem 3.2. Let (F, A) and (H, B) be two soft neutrosophic bisemigroups over $BN(S)$ such that $A \cap B = \phi$, then their union is soft neutrosophic bisemigroup over $BN(S)$.

Proof. Straightforward.

Proposition 3.1. Let (F, A) and (K, B) be two soft neutrosophic bisemigroups over $BN(S)$. Then

1. Their extended union $(F, A) \cup_{\varepsilon} (K, B)$ over $BN(S)$ is not soft neutrosophic bisemigroup over $BN(S)$.

2. Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $BN(S)$ is soft neutrosophic bisemigroup over $BN(S)$.

3. Their restricted union $(F, A) \cup_{R} (K, B)$ over $BN(S)$ is not soft neutrosophic bisemigroup over $BN(S)$.

4. Their restricted intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $BN(S)$ is soft neutrosophic bisemigroup over $BN(S)$.

Proposition 3.2. Let (F, A) and (K, B) be two soft neutrosophic bisemigroups over $BN(S)$. Then

1. Their AND operation $(F, A) \wedge (K, B)$ is soft neutrosophic bisemigroup over $BN(S)$.

2. Their OR operation $(F, A) \vee (K, B)$ is not soft neutrosophic bisemigroup over $BN(S)$.

Definition 3.2. Let (F, A) be a soft neutrosophic bisemigroup over $BN(S)$, then (F, A) is called Full-soft neutrosophic bisemigroup over $BN(S)$ if $F(x) = BN(S)$, for all $x \in A$. We denote it by $BN(S)$.

Definition 3.3. Let (F, A) and (H, B) be two soft neutrosophic bisemigroups over $BN(S)$. Then (H, B) is a soft neutrosophic subbisemigroup of (F, A), if

1. $B \subset A$.

2. $H(x)$ is neutrosophic subbisemigroup of $F(x)$, for all $x \in B$.

Example 3.2. Let $BN(S) = \{0, 1, 2, I, 2I, \langle Z \cup I\rangle, \times, +\}$ be a neutosophic bisemigroup. Let $T = \{0, I, 2I, \langle 2Z \cup I\rangle, \times, +\}, P = \{0, 1, 2, \langle 5Z \cup I\rangle, \times, +\}$ and $L = \{0, 1, 2, Z, \times, +\}$ are neutrosophic subbisemigroup of $BN(S)$. The (F, A) is clearly soft neutrosophic bisemigroup over $BN(S)$, where $F(x_1) = \{0, I, 2I, \langle 2Z \cup I\rangle, \times, +\}, F(x_2) = \{0, 1, 2, \langle 5Z \cup I\rangle, \times, +\}, F(x_3) = \{0, 1, 2, Z, \times, +\}$.

Then (H, B) is a soft neutrosophic subbisemigroup of (F, A), where $H(x_1) = \{0, I, \langle 4Z \cup I\rangle, \times, +\}, H(x_3) = \{0, 1, 4Z, \times, +\}$.

Theorem 3.3. Let (F, A) be a soft neutrosophic bisemigroup over $BN(S)$ and $\{(H_i, B_i); i \in I\}$ be a non-empty family of soft neutrosophic subbisemigroups of (F, A) then

1. $\cap_{i \in I}(H_i, B_i)$ is a soft neutrosophic subbisemigroup of (F, A).

2. $\wedge_{i \in I}(H_i, B_i)$ is a soft neutrosophic subbisemigroup of $\wedge_{i \in I}(F, A)$.

3. $\cup_{i \in I}(H_i, B_i)$ is a soft neutrosophic subbisemigroup of (F, A) if $B_i \cap B_j = \phi$, for all $i \neq j$.

Proof. Straightforward.

Theorem 3.4. (F, A) is called soft neutrosophic biideal over $BN(S)$ if $F(x)$ is neutrosophic biideal of $BN(S)$, for all $x \in A$.

Example 3.3. Let $BN(S) = (\{\langle Z \cup I\rangle, 0, 1, 2, I, 2I, +, \times\}(\times$ under multiplication modulo 3)). Let $T = \{\langle 2Z \cup I\rangle, 0, I, 1, 2I, +, \times\}$ and $J = \{\langle 8Z \cup I\rangle, \{0, 1, I, 2I\}, +\times\}$ are ideals of $BN(S)$. Then (F, A) is soft neutrosophic biideal over $BN(S)$, where $F(x_1) = \{\langle 2Z \cup I\rangle, 0, I, 1, 2I, +, \times\}, F(x_2) = \{\langle 8Z \cup I\rangle, \{0, 1, I, 2I\}, +\times\}$.

Theorem 3.5. Every soft neutrosophic biideal (F, A) over $BS(N)$ is a soft neutrosophic bisemigroup but the converse is not true.

Proposition 3.3. Let (F, A) and (K, B) be two soft neutrosophic biideals over $BN(S)$. Then

1. Their extended union $(F, A) \cup_\varepsilon (K, B)$ over $BN(S)$ is not soft neutrosophic biideal over $BN(S)$.

2. Their extended intersection $(F, A) \cap_\varepsilon (K, B)$ over $BN(S)$ is soft neutrosophic biideal over $BN(S)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $BN(S)$ is not soft neutrosophic biideal over $BN(S)$.

4. Their restricted intersection $(F, A) \cap_\varepsilon (K, B)$ over $BN(S)$ is soft neutrosophic biideal over $BN(S)$.

Proposition 3.4. Let (F, A) and (H, B) be two soft neutrosophic biideal over $BN(S)$. Then

1. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic biideal over $BN(S)$.

2. Their OR operation $(F, A) \vee (H, B)$ is not soft neutrosophic biideal over $BN(S)$.

Theorem 3.6.

Let (F, A) be a soft neutrosophic bisemigroup over $BN(S)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic biideals of (F, A) then

1. $\cap_{i \in I}(H_i, B_i)$ is a soft neutrosophic biideal of (F, A).

2. $\wedge_{i \in I}(H_i, B_i)$ is a soft neutrosophic biideal of $\wedge_{i \in I}(F, A)$.

§4. Soft neutrosophic strong bisemigroup

Definition 4.1. Let (F, A) be a soft set over a neutrosophic bisemigroup $BN(S)$. Then (F, A) is said to be soft strong neutrosophic bisemigroup over $BN(G)$ if and only if $F(x)$ is neutrosophic strong subbisemigroup of $BN(G)$ for all $x \in A$.

Example 4.1. Let $BN(S) = \{0, 1, 2, I, 2I, \langle Z \cup I \rangle, \times, +\}$ be a neutrosophic bisemigroup. Let $T = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}$ and $R = \{0, 1, I, \langle 4Z \cup I \rangle, \times, +\}$ are neutrosophic strong subbisemigroups of $BN\,(S)$. Then (F, A) is soft neutrosophic strong bisemigroup over $BN\,(S)$, where $F\,(x_1) = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}, F\,(x_2) = \{0, I, 1, \langle 4Z \cup I \rangle, \times, +\}$.

Theorem 4.1. Every soft neutrosophic strong bisemigroup is a soft neutrosophic bisemigroup but the converse is not true.

Proposition 4.1. Let (F, A) and (K, B) be two soft neutrosophic strong bisemigroups over $BN(S)$. Then

1. Their extended union $(F, A) \cup_\varepsilon (K, B)$ over $BN(S)$ is not soft neutrosophic strong bisemigroup over $BN(S)$.

2. Their extended intersection $(F, A) \cap_\varepsilon (K, B)$ over $BN(S)$ is soft neutrosophic stong bisemigroup over $BN(S)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $BN(S)$ is not soft neutrosophic stong bisemigroup over $BN(S)$.

4. Their restricted intersection $(F, A) \cap_\varepsilon (K, B)$ over $BN(S)$ is soft neutrosophic strong bisemigroup over $BN(S)$.

Proposition 4.2. Let (F, A) and (K, B) be two soft neutrosophic strong bisemigroups over $BN(S)$. Then

1. Their AND operation $(F, A) \wedge (K, B)$ is soft neutrosophic strong bisemigroup over $BN(S)$.

2. Their OR operation $(F, A) \vee (K, B)$ is not soft neutrosophic strong bisemigroup over $BN(S)$.

Definition 4.2. Let (F, A) and (H, B) be two soft neutrosophic strong bisemigroups over $BN(S)$. Then (H, B) is a soft neutrosophic strong subbisemigroup of (F, A), if

1. $B \subset A$.

2. $H(x)$ is neutrosophic strong subbisemigroup of $F(x)$, for all $x \in B$.

Example 4.2. Let $BN(S) = \{0, 1, 2, I, 2I, \langle Z \cup I \rangle, \times, +\}$ be a neutrosophic bisemigroup. Let $T = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}$ and $R = \{0, 1, I, \langle 4Z \cup I \rangle, \times, +\}$ are neutrosophic strong subbisemigroups of $BN\,(S)$. Then (F, A) is soft neutrosophic strong bisemigroup over $BN\,(S)$, where $F\,(x_1) = \{0, I, 2I, \langle 2Z \cup I \rangle, \times, +\}, F\,(x_2) = \{0, I, \langle 4Z \cup I \rangle, \times, +\}$.

Then (H, B) is a soft neutrosophic strong subbisemigroup of (F, A), where $H\,(x_1) = \{0, I, \langle 4Z \cup I \rangle\,, \times, +\}$.

Theorem 4.2. Let (F, A) be a soft neutrosophic strong bisemigroup over $BN\,(S)$ and $\{(H_i, B_i)\,; i \in I\}$ be a non empty family of soft neutrosophic strong subbisemigroups of (F, A) then

1. $\cap_{i \in I}\,(H_i, B_i)$ is a soft neutrosophic strong subbisemigroup of (F, A).

2. $\wedge_{i \in I}\,(H_i, B_i)$ is a soft neutrosophic strong subbisemigroup of $\wedge_{i \in I}\,(F, A)$.

3. $\cup_{i\in I}(H_i, B_i)$ is a soft neutrosophic strong subbisemigroup of (F, A) if $B_i \cap B_j = \phi$, for all $i \neq j$.

Proof. Straightforward.

Definition 4.3. (F, A) over $BN(S)$ is called soft neutrosophic strong biideal if $F(x)$ is neutosophic strong biideal of $BN(S)$, for all $x \in A$.

Example 4.3. Let $BN(S) = (\{\langle Z \cup I\rangle, 0, 1, 2, I, 2I\}, +, \times(\times$ under multiplication modulo 3)). Let $T = \{\langle 2Z \cup I\rangle, 0, I, 1, 2I, +, \times\}$ and $J = \{\langle 8Z \cup I\rangle, \{0, 1, I, 2I\}, +\times\}$ are neutrosophic strong ideals of $BN(S)$. Then (F, A) is soft neutrosophic strong biideal over $BN(S)$, where $F(x_1) = \{\langle 2Z \cup I\rangle, 0, I, 1, 2I, +, \times\}, F(x_2) = \{\langle 8Z \cup I\rangle, \{0, 1, I, 2I\}, +\times\}$.

Theorem 4.3. Every soft neutrosophic strong biideal (F, A) over $BS(N)$ is a soft neutrosophic bisemigroup but the converse is not true.

Theorem 4.4. Every soft neutrosophic strong biideal (F, A) over $BS(N)$ is a soft neutrosophic strong bisemigroup but the converse is not true.

Proposition 4.3. Let (F, A) and (K, B) be two soft neutrosophic strong biideals over $BN(S)$. Then

1. Their extended union $(F, A)\cup_{\varepsilon}(K, B)$ over $BN(S)$ is not soft neutrosophic strong biideal over $BN(S)$.

2. Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $BN(S)$ is soft neutrosophic strong biideal over $BN(S)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $BN(S)$ is not soft neutrosophic strong biideal over $BN(S)$.

4. Their restricted intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $BN(S)$ is soft neutrosophic stong biideal over $BN(S)$.

Proposition 4.4. Let (F, A) and (H, B) be two soft neutrosophic strong biideal over $BN(S)$. Then

1. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic strong biideal over $BN(S)$.

2. Their OR operation $(F, A) \vee (H, B)$ is not soft neutrosophic strong biideal over $BN(S)$.

Theorem 4.5. Let (F, A) be a soft neutrosophic strong bisemigroup over $BN(S)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic strong biideals of (F, A) then

1. $\cap_{i\in I}(H_i, B_i)$ is a soft neutrosophic strong biideal of (F, A).

2. $\wedge_{i\in I}(H_i, B_i)$ is a soft neutrosophic strong biideal of $\wedge_{i\in I}(F, A)$.

§5. Soft neutrosophic N-semigroup

Definition 5.1. Let $\{S(N), *_1, \ldots, *_N\}$ be a neutrosophic N-semigroup and (F, A) be a soft set over $\{S(N), *_1, \ldots, *_N\}$. Then (F, A) is termed as soft neutrosophic N-semigroup if and only if $F(x)$ is neutrosophic sub N-semigroup, for all $x \in A$.

Example 5.1. Let $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$ be a neutrosophic 4-semigroup where

$S_1 = \{Z_{12}$, semigroup under multiplication modulo 12$\}$.

$S_2 = \{0, 1, 2, 3, I, 2I, 3I$, semigroup under multiplication modulo 4$\}$, a neutrosophic semigroup.

$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle R \cup I \rangle \right\}$, neutrosophic semigroup under matrix multiplication.

$S_4 = \langle Z \cup I \rangle$, neutrosophic semigroup under multiplication. Let $T = \{T_1 \cup T_2 \cup T_3 \cup T_4, *_1, *_2, *_3, *_4\}$ is a neutosophic sub 4-semigroup of $S(4)$, where $T_1 = \{0, 2, 4, 6, 8, 10\} \subseteq Z_{12}$, $T_2 = \{0, I, 2I, 3I\} \subset S_2$, $T_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \subset S_3$, $T4 = \{\langle 5Z \cup I \rangle\} \subset S_4$, the neutrosophic semigroup under multiplication. Also let $P = \{P_1 \cup P_2 \cup P_3 \cup P_4, *_1, *_2, *_3, *_4\}$ be another neutrosophic sub 4-semigroup of $S(4)$, where $P_1 = \{0, 6\} \subseteq Z_{12}$, $P_2 = \{0, 1, I\} \subset S_2$, $P_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \subset S_3$, $P_4 = \{\langle 2Z \cup I \rangle\} \subset S_4$. Then (F, A) is soft neutrosophic 4-semigroup over $S(4)$, where

$$F(x_1) = \{0, 2, 4, 6, 8, 10\} \cup \{0, I, 2I, 3I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \cup \{\langle 5Z \cup I \rangle\},$$

$$F(x_2) = \{0, 6\} \cup \{0, 1, I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \cup \{\langle 2Z \cup I \rangle\}.$$

Theorem 5.1. Let (F, A) and (H, A) be two soft neutrosophic N-semigroup over $S(N)$. Then their intersection $(F, A) \cap (H, A)$ is again a soft neutrosophic N-semigroup over $S(N)$.

Proof. Straightforward.

Theorem 5.2. Let (F, A) and (H, B) be two soft neutrosophic N-semigroups over $S(N)$ such that $A \cap B = \phi$, then their union is soft neutrosophic N-semigroup over $S(N)$.

Proof. Straightforward.

Proposition 5.1. Let (F, A) and (K, B) be two soft neutrosophic N-semigroups over $S(N)$. Then

1. Their extended union $(F, A) \cup_{\varepsilon} (K, B)$ over $S(N)$ is not soft neutrosophic N-semigroup over $S(N)$.

2. Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $S(N)$ is soft neutrosophic N-semigroup over $S(N)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $S(N)$ is not soft neutrosophic N-semigroup over $S(N)$.

4. Their restricted intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $S(N)$ is soft neutrosophic N-semigroup over $S(N)$.

Proposition 5.2. Let (F, A) and (K, B) be two soft neutrosophic N-semigroups over $S(N)$. Then

1. Their AND operation $(F, A) \wedge (K, B)$ is soft neutrosophic N-semigroup over $S(N)$.

2. Their OR operation $(F, A) \vee (K, B)$ is not soft neutrosophic N-semigroup over $S(N)$.

Definition 5.2. Let (F, A) be a soft neutrosophic N-semigroup over $S(N)$, then (F, A) is called Full-soft neutrosophic N-semigroup over $S(N)$ if $F(x) = S(N)$, for all $x \in A$. We denote it by $S(N)$.

Definition 5.3. Let (F, A) and (H, B) be two soft neutrosophic N-semigroups over $S(N)$. Then (H, B) is a soft neutrosophic sub N-semigroup of (F, A), if

1. $B \subset A$.
2. $H(x)$ is neutrosophic sub N-semigroup of $F(x)$, for all $x \in B$.

Example 5.2. Let $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$ be a neutrosophic 4-semigroup where

$S_1 = \{Z_{12}$, semigroup under multiplication modulo 12$\}$.

$S_2 = \{0, 1, 2, 3, I, 2I, 3I$, semigroup under multiplication modulo 4$\}$, a neutrosophic semigroup.

$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle R \cup I \rangle \right\}$, neutrosophic semigroup under matrix multiplication.

$S_4 = \langle Z \cup I \rangle$, neutrosophic semigroup under multiplication. Let $T = \{T_1 \cup T_2 \cup T_3 \cup T_4, *_1, *_2, *_3, *_4\}$ is a neutosophic sub 4-semigroup of $S(4)$, where $T_1 = \{0, 2, 4, 6, 8, 10\} \subseteq Z_{12}$, $T_2 = \{0, I, 2I, 3I\} \subset S_2$, $T_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \subset S_3$, $T_4 = \{\langle 5Z \cup I \rangle\} \subset S_4$, the neutrosophic semigroup under multiplication. Also let $P = \{P_1 \cup P_2 \cup P_3 \cup P_4, *_1, *_2, *_3, *_4\}$ be another neutrosophic sub 4-semigroup of $S(4)$, where $P_1 = \{0, 6\} \subseteq Z_{12}$, $P_2 = \{0, 1, I\} \subset S_2$, $P_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \subset S_3$, $P_4 = \{\langle 2Z \cup I \rangle\} \subset S_4$. Also let $R = \{R_1 \cup R_2 \cup R_3 \cup R_4, *_1, *_2, *_3, *_4\}$ be a neutrosophic sub 4-semigroup os $S(4)$ where $R_1 = \{0, 3, 6, 9\}, R_2 = \{0, I, 2I\}, R_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle 2Z \cup I \rangle \right\}, R_4 = \{\langle 3Z \cup I \rangle\}$. Then (F, A) is soft neutrosophic 4-semigroup over $S(4)$, where

$$
\begin{aligned}
F(x_1) &= \{0, 2, 4, 6, 8, 10\} \cup \{0, I, 2I, 3I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \cup \{\langle 5Z \cup I \rangle\}, \\
F(x_2) &= \{0, 6\} \cup \{0, 1, I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \cup \{\langle 2Z \cup I \rangle\}, \\
F(x_3) &= \{0, 3, 6, 9\} \cup \{0, I, 2I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle 2Z \cup I \rangle \right\} \cup \{\langle 3Z \cup I \rangle\}.
\end{aligned}
$$

Clearly (H, B) is a soft neutrosophic sub N-semigroup of (F, A), where

$$
\begin{aligned}
H(x_1) &= \{0, 4, 8\} \cup \{0, I, 2I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \cup \{\langle 10Z \cup I \rangle\}, \\
H(x_3) &= \{0, 6\} \cup \{0, I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle 4Z \cup I \rangle \right\} \cup \{\langle 6Z \cup I \rangle\}.
\end{aligned}
$$

Theorem 5.3. Let (F, A) be a soft neutrosophic N-semigroup over $S(N)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic sub N-semigroups of (F, A) then

1. $\cap_{i\in I}(H_i, B_i)$ is a soft neutrosophic sub N-semigroup of (F, A).

2. $\wedge_{i\in I}(H_i, B_i)$ is a soft neutrosophic sub N-semigroup of $\wedge_{i\in I}(F, A)$.

3. $\cup_{i\in I}(H_i, B_i)$ is a soft neutrosophic sub N-semigroup of (F, A) if $B_i \cap B_j = \phi$, for all $i \neq j$.

Proof. Straightforward.

Definition 5.4. (F, A) over $S(N)$ is called soft neutrosophic N-ideal if $F(x)$ is neutosophic N-ideal of $S(N)$, for all $x \in A$.

Theorem 5.4. Every soft neutrosophic N-ideal (F, A) over $S(N)$ is a soft neutrosophic N-semigroup but the converse is not true.

Proposition 5.3. Let (F, A) and (K, B) be two soft neutrosophic N-ideals over $S(N)$. Then

1. Their extended union $(F, A) \cup_\varepsilon (K, B)$ over $S(N)$ is not soft neutrosophic N-ideal over $S(N)$.

2. Their extended intersection $(F, A) \cap_\varepsilon (K, B)$ over $S(N)$ is soft neutrosophic N-ideal over $S(N)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $S(N)$ is not soft neutrosophic N-ideal over $S(N)$.

4. Their restricted intersection $(F, A) \cap_\varepsilon (K, B)$ over $S(N)$ is soft neutrosophic N-ideal over $S(N)$.

Proposition 5.4. Let (F, A) and (H, B) be two soft neutrosophic N-ideal over $S(N)$. Then

1. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic N-ideal over $S(N)$.

2. Their OR operation $(F, A) \vee (H, B)$ is not soft neutrosophic N-ideal over $S(N)$.

Theorem 5.5. Let (F, A) be a soft neutrosophic N-semigroup over $S(N)$ and $\{(H_i, B_i);\ i \in I\}$ is a non empty family of soft neutrosophic N-ideals of (F, A) then

1. $\cap_{i\in I}(H_i, B_i)$ is a soft neutrosophic N-ideal of (F, A).

2. $\wedge_{i\in I}(H_i, B_i)$ is a soft neutrosophic N-ideal of $\wedge_{i\in I}(F, A)$.

§6. Soft neutrosophic strong N-semigroup

Definition 6.1. Let $\{S(N), *_1, \ldots, *_N\}$ be a neutrosophic N-semigroup and (F, A) be a soft set over $\{S(N), *_1, \ldots, *_N\}$. Then (F, A) is called soft neutrosophic strong N-semigroup if and only if $F(x)$ is neutrosophic strong sub N-semigroup, for all $x \in A$.

Example 6.1. Let $S(N) = \{S_1 \cup S_2 \cup S_3 \cup S_4, *_1, *_2, *_3, *_4\}$ be a neutrosophic 4-semigroup where

$S_1 = \langle Z_6 \cup I\rangle$, a neutrosophic semigroup.

$S_2 = \{0, 1, 2, 3, I, 2I, 3I$, semigroup under multiplication modulo 4$\}$, a neutrosophic semigroup.

$S_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle R \cup I\rangle \right\}$, neutrosophic semigroup under matrix multiplication.

$S_4 = \langle Z \cup I \rangle$, neutrosophic semigroup under multiplication. Let $T = \{T_1 \cup T_2 \cup T_3 \cup T_4, *_1, *_2, *_3, *_4\}$ is a neutosophic strong sub 4-semigroup of $S(4)$, where $T_1 = \{0, 3, 3I\} \subseteq \langle Z_6 \cup I \rangle$, $T_2 = \{0, I, 2I, 3I\} \subset S_2$, $T_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \subset S_3$, $T_4 = \{\langle 5Z \cup I \rangle\} \subset S_4$, the neutrosophic semigroup under multiplication. Also let $P = \{P_1 \cup P_2 \cup P_3 \cup P_4, *_1, *_2, *_3, *_4\}$ be another neutrosophic strong sub 4-semigroup of $S(4)$, where $P_1 = \{0, 2I, 4I\} \subseteq \langle Z_6 \cup I \rangle$, $P_2 = \{0, 1, I\} \subset S_2$, $P_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \subset S_3$, $P_4 = \{\langle 2Z \cup I \rangle\} \subset S_4$. Then (F, A) is soft neutrosophic strong 4-semigroup over $S(4)$, whereThen (F, A) is soft neutrosophic 4-semigroup over $S(4)$, where

$$F(x_1) = \{0, 3, 3I\} \cup \{0, I, 2I, 3I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Q \cup I \rangle \right\} \cup \{\langle 5Z \cup I \rangle\},$$

$$F(x_2) = \{0, 2I, 4I\} \cup \{0, 1, I\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \langle Z \cup I \rangle \right\} \cup \{\langle 2Z \cup I \rangle\}.$$

Theorem 6.1. Every soft neutrosophic strong N-semigroup is trivially a soft neutrosophic N-semigroup but the converse is not true.

Proposition 6.1. Let (F, A) and (K, B) be two soft neutrosophic strong N-semigroups over $S(N)$. Then

1. Their extended union $(F, A) \cup_\varepsilon (K, B)$ over $S(N)$ is not soft neutrosophic strong N-semigroup over $S(N)$.
2. Their extended intersection $(F, A) \cap_\varepsilon (K, B)$ over $S(N)$ is soft neutrosophic strong N-semigroup over $S(N)$.
3. Their restricted union $(F, A) \cup_R (K, B)$ over $S(N)$ is not soft neutrosophic strong N-semigroup over $S(N)$.
4. Their restricted intersection $(F, A) \cap_\varepsilon (K, B)$ over $S(N)$ is soft neutrosophic strong N-semigroup over $S(N)$.

Proposition 6.2. Let (F, A) and (K, B) be two soft neutrosophic strong N-semigroups over $S(N)$. Then

1. Their *AND* operation $(F, A) \wedge (K, B)$ is soft neutrosophic strong N-semigroup over $S(N)$.
2. Their *OR* operation $(F, A) \vee (K, B)$ is not soft neutrosophic strong N-semigroup over $S(N)$.

Definition 6.2. Let (F, A) and (H, B) be two soft neutrosophic strong N-semigroups over $S(N)$. Then (H, B) is a soft neutrosophic strong sub N-semigroup of (F, A), if

1. $B \subset A$.
2. $H(x)$ is neutrosophic strong sub N-semigroup of $F(x)$, for all $x \in B$.

Theorem 6.2.

1. Let (F, A) be a soft neutrosophic strong N-semigroup over $S(N)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic stong sub N-semigroups of (F, A) then
2. $\cap_{i \in I} (H_i, B_i)$ is a soft neutrosophic strong sub N-semigroup of (F, A).

3. $\wedge_{i \in I}(H_i, B_i)$ is a soft neutrosophic strong sub N-semigroup of $\wedge_{i \in I}(F, A)$.

4. $\cup_{i \in I}(H_i, B_i)$ is a soft neutrosophic strong sub N-semigroup of (F, A) if $B_i \cap B_j = \phi$, for all $i \neq j$.

Proof. Straightforward.

Definition 6.3. (F, A) over $S(N)$ is called soft neutrosophic strong N-ideal if $F(x)$ is neutosophic strong N-ideal of $S(N)$, for all $x \in A$.

Theorem 6.3. Every soft neutrosophic strong N-ideal (F, A) over $S(N)$ is a soft neutrosophic strong N-semigroup but the converse is not true.

Theorem 6.4. Every soft neutrosophic strong N-ideal (F, A) over $S(N)$ is a soft neutrosophic N-semigroup but the converse is not true.

Proposition 6.3. Let (F, A) and (K, B) be two soft neutrosophic strong N-ideals over $S(N)$. Then

1. Their extended union $(F, A) \cup_{\varepsilon} (K, B)$ over $S(N)$ is not soft neutrosophic strong N-ideal over $S(N)$. 2. Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $S(N)$ is soft neutrosophic strong N-ideal over $S(N)$.

3. Their restricted union $(F, A) \cup_R (K, B)$ over $S(N)$ is not soft neutrosophic strong N-ideal over $S(N)$.

4. Their restricted intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $S(N)$ is soft neutrosophic strong N-ideal over $S(N)$.

Proposition 6.4. Let (F, A) and (H, B) be two soft neutrosophic strong N-ideal over $S(N)$. Then

1. Their AND operation $(F, A) \wedge (H, B)$ is soft neutrosophic strong N-ideal over $S(N)$.

2. Their OR operation $(F, A) \vee (H, B)$ is not soft neutrosophic strong N-ideal over $S(N)$.

Theorem 6.5. Let (F, A) be a soft neutrosophic strong N-semigroup over $S(N)$ and $\{(H_i, B_i); i \in I\}$ is a non empty family of soft neutrosophic strong N-ideals of (F, A) then

1. $\cap_{i \in I}(H_i, B_i)$ is a soft neutrosophic strong N-ideal of (F, A).

2. $\wedge_{i \in I}(H_i, B_i)$ is a soft neutrosophic strong N-ideal of $\wedge_{i \in I}(F, A)$.

Conclusion

This paper is an extension of neutrsphic semigroup to soft semigroup. We also extend neutrosophic bisemigroup, neutrosophic N-semigroup to soft neutrosophic bisemigroup, and soft neutrosophic N-semigroup. Their related properties and results are explained with many illustrative examples, the notions related with strong part of neutrosophy also established within soft semigroup.

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Some remarks on fuzzy σ-Baire spaces

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Abstract In this paper we investigate several characterizations of fuzzy σ-Baire spaces and study the conditions under which a fuzzy topological space becomes a fuzzy σ-Baire space.

Keywords Fuzzy dense set, fuzzy σ-nowhere dense set, fuzzy Baire space, fuzzy σ-Baire space, fuzzy almost resolvable space, fuzzy submaximal space, fuzzy Volterra space, fuzzy weakly Volterra space.
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§1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by L. A. Zadeh[17] in 1965. This concept provides a natural foundation for treating mathematically the fuzzy phenomena, which exist pervasively in our real world. The first notion of fuzzy topological spaces had been defined by C. L. Chang [3] in 1968 and this paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed.

The concepts of Baire spaces have been studied extensively in classical topology [4],[5],[7],[18]. The concept of Baire spaces in fuzzy setting was introduced and studied by G. Thangaraj and S. Anjalmose in [9]. The concept of σ-nowhere dense set was introduced and studied by Jiling Cao and Sina Greenwood [6]. The concept of fuzzy σ-Baire space was introduced and studied by G. Thangaraj and E. Poongothai in [11]. In this paper, we investigate several characterizations of fuzzy σ-Baire spaces and study the inter-relations between fuzzy σ-Baire spaces, fuzzy Baire spaces, fuzzy almost resolvable spaces, fuzzy Volterra spaces and fuzzy weakly Volterra spaces.

§2. Preliminaries

By a fuzzy topological space we shall mean a non - empty set X together with a fuzzy topology T (in the sense of CHANG) and denote it by (X, T).

Definition 2.1. Let λ and μ be any two fuzzy sets in (X, T). Then we define $\lambda \vee \mu : X \to [0, 1]$ as follows : $(\lambda \vee \mu)(x) = Max\{\lambda(x), \mu(x)\}$. Also we define $\lambda \wedge \mu : X \to [0, 1]$ as follows : $(\lambda \wedge \mu)(x) = Min\{\lambda(x), \mu(x)\}$.

For a family $\{\lambda_i/i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined respectively as $\psi(x) = sup_i\{\lambda_i(x), x \in X\}$ and $\delta(x) = inf_i\{\lambda_i(x), x \in X\}$.

Definition 2.2. Let (X, T) be a fuzzy topological space. For a fuzzy set λ of X, the interior $int(\lambda)$ and the closure $cl(\lambda)$ of X are defined respectively as

(i) $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$,

(ii) $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.1.[1] For any fuzzy set λ in a fuzzy topological space (X, T),

(i) $1 - cl(\lambda) = int(1 - \lambda)$,

(ii) $1 - int(\lambda) = cl(1 - \lambda)$.

Definition 2.3.[8] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy dense if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.4.[8] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $intcl(\lambda) = 0$.

Definition 2.5.[2] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy F_σ-set in (X, T) if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where $1 - \lambda_i \in T$ for $i \in I$.

Definition 2.6.[2] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy G_δ-set in (X, T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in T$ for $i \in I$.

Definition 2.7.[11] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy σ-nowhere dense if λ is a fuzzy F_σ-set in (X, T) such that $int(\lambda) = 0$.

Definition 2.8.[9] A fuzzy topological space (X, T) is called a fuzzy Baire space if $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$ where λ_i's are fuzzy nowhere dense sets in (X, T).

Definition 2.9.[8] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where λ_i's are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category.

Definition 2.10.[11] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy σ-first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where λ_i's are fuzzy σ-nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy σ-second category.

Definition 2.11.[11] Let λ be a fuzzy σ-first category set in a fuzzy topological space (X, T). Then $1 - \lambda$ is called a fuzzy σ-residual set in (X, T).

Lemma 2.2.[1] For a family of $\{\lambda_\alpha\}$ of fuzzy sets of a fuzzy topological space X, $\vee cl(\lambda_\alpha) \leq cl(\vee\lambda_\alpha)$. In case is a finite set, $\vee cl(\lambda_\alpha) = cl(\vee\lambda_\alpha)$. Also $\vee int(\lambda_\alpha) \leq int(\vee\lambda_\alpha)$.

§3. Fuzzy σ-Baire spaces

Definition 3.1.[11] Let (X, T) be a fuzzy topological space. Then (X, T) is called a fuzzy σ-Baire space if $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where λ_i's are fuzzy σ-nowhere dense sets in (X, T).

Theorem 3.1.[11] Let (X, T) be a fuzzy topological space. Then the following are equivalent:

(1) (X, T) is a fuzzy σ-Baire space.

(2) $int(\lambda) = 0$ for every fuzzy σ-first category set λ in (X, T).

(3) $cl(\mu) = 1$ for every fuzzy σ-residual set μ in (X, T).

Definition 3.2.[16] A fuzzy topological space (X, T) is called a fuzzy almost resolvable space if $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where the fuzzy sets λ_i's in (X, T) are such that $int(\lambda_i) = 0$. Otherwise (X, T) is called a fuzzy almost irresolvable space.

Definition 3.3. A fuzzy topological space (X, T) is called a fuzzy σ-first category space if the fuzzy set 1_X is a fuzzy σ-first category set in (X, T). That is, $1_X = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are fuzzy σ-nowhere dense sets in (X, T). Otherwise (X, T) will be called a fuzzy σ-second category space in (X, T).

Proposition 3.1. If the fuzzy topological space (X, T) is a fuzzy σ-first category space, then (X, T) is a fuzzy almost resolvable space.

Proof. Let the fuzzy topological space (X, T) be a fuzzy σ-first category space. Then we have $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where λ_i's are fuzzy σ-nowhere dense sets in (X, T). Since λ_i is a fuzzy σ-nowhere dense set in (X, T), λ_i is a fuzzy F_σ-set in (X, T) and $int(\lambda_i) = 0$. Hence $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where $int(\lambda_i) = 0$ and therefore (X, T) is a fuzzy almost resolvable space.

Theorem 3.2.[11] If the fuzzy topological space (X, T) is a fuzzy σ-Baire space, then (X, T) is a fuzzy σ-second category space.

Proposition 3.2. If the fuzzy topological space (X, T) is a fuzzy σ-Baire space, then (X, T) is a fuzzy almost irresolvable space.

Proof. Let (X, T) be a fuzzy σ-Baire space. Since every fuzzy σ-Baire space is a fuzzy σ-second category space, (X, T) is not a fuzzy σ-first category space. Then $\vee_{i=1}^{\infty}(\lambda_i) \neq 1$, where λ_i's are fuzzy σ-nowhere dense sets in (X, T). Now λ_i's are fuzzy σ-nowhere dense sets in (X, T), implies that $int(\lambda_i) = 0$, $(i = 1\ to\ \infty)$. Hence $\vee_{i=1}^{\infty}(\lambda_i) \neq 1$, where $int(\lambda_i) = 0$ and therefore (X, T) is a fuzzy almost irresolvable space.

Definition 3.4.[13] A fuzzy topological space (X, T) is called a fuzzy strongly irresolvable space if $clint(\lambda) = 1$ for each fuzzy dense set λ in (X, T).

Proposition 3.3. If the fuzzy topological space (X, T) is a fuzzy strongly irresolvable, fuzzy σ-Baire space and λ is a fuzzy σ-first category set in (X, T), then λ is a fuzzy nowhere dense set in (X, T).

Proof. Let λ be a fuzzy σ-first category set in (X, T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are fuzzy σ-nowhere dense sets in (X, T). Since (X, T) is a fuzzy σ-Baire space, by theorem 3.1, $int(\lambda) = 0$ in (X, T). Then we have $1 - int(\lambda) = 1$. This implies that $cl(1 - \lambda) = 1$. Since (X, T) is a fuzzy strongly irresolvable space, for the fuzzy dense set $1 - \lambda$ in (X, T), we have $clint(1 - \lambda) = 1$. Then $1 - intcl(\lambda) = 1$ and hence $intcl(\lambda) = 0$. Therefore λ is a fuzzy nowhere dense set in (X, T).

Proposition 3.4. If λ_i's are the fuzzy σ-first category sets in a fuzzy strongly irresolvable, fuzzy σ-Baire space (X, T), then $\vee_{i=1}^{\infty}(\lambda_i)$ is a fuzzy first category set in (X, T).

Proof. Let λ_i's be the fuzzy σ-first category sets in a fuzzy strongly irresolvable, fuzzy σ-Baire space (X, T). Then, by Proposition 3.3, λ_i's are fuzzy nowhere dense sets in (X, T) and hence $\vee_{i=1}^{\infty}(\lambda_i)$ is a fuzzy first category set in (X, T).

Proposition 3.5. If $int\big(\vee_{i=1}^{\infty}(\lambda_i)\big) = 0$, where λ_i's are the fuzzy σ-first category sets in a fuzzy strongly irresolvable, fuzzy σ-Baire space (X,T), then (X,T) is a fuzzy Baire space.

Proof. Let λ_i's be the fuzzy σ-first category sets in a fuzzy strongly irresolvable, fuzzy σ-Baire space (X,T). Then, by Proposition 3.4, $\vee_{i=1}^{\infty}(\lambda_i)$ is a fuzzy first category set in (X,T). Then $\vee_{i=1}^{\infty}(\lambda_i) = \vee_{j=1}^{\infty}(\mu_j)$, where the fuzzy sets μ_j's are fuzzy nowhere dense sets in (X,T). Now $int\big(\vee_{i=1}^{\infty}(\lambda_i)\big) = 0$, implies that $int\big(\vee_{j=1}^{\infty}(\mu_j)\big) = 0$, where the fuzzy sets μ_j's are fuzzy nowhere dense sets in (X,T). Hence the fuzzy topological space (X,T) is a fuzzy Baire space.

Definition 3.5.[2] A fuzzy topological space (X,T) is called a fuzzy submaximal space if for each fuzzy set λ in (X,T) such that $cl(\lambda) = 1$, then $\lambda \in T$ in (X,T).

Theorem 3.3.[13] If the fuzzy topological space (X,T) is a fuzzy submaximal space, then (X,T) is a fuzzy strongly irresolvable space.

Proposition 3.6. If the fuzzy topological space (X,T) is a fuzzy submaximal, fuzzy σ-Baire space and λ is a fuzzy σ-first category set in (X,T), then λ is a fuzzy nowhere dense set in (X,T).

Proof. Let λ be a fuzzy σ-first category set in a fuzzy submaximal, fuzzy σ-Baire space. Since (X,T) is a fuzzy submaximal space, by theorem 3.3, (X,T) is a fuzzy strongly irresolvable space. Then (X,T) is a fuzzy strongly irresolvable, fuzzy σ-Baire space. Since λ is a fuzzy σ-first category set in (X,T), by proposition 3.3, λ is a fuzzy nowhere dense set in (X,T).

Proposition 3.7. If λ_i's are the fuzzy σ-first category sets in a fuzzy submaximal, fuzzy σ-Baire space (X,T), then $\vee_{i=1}^{\infty}(\lambda_i)$ is a fuzzy first category set in (X,T).

Proof. Let λ_i's be the fuzzy σ-first category sets in a fuzzy submaximal, fuzzy σ-Baire space (X,T). Then, by proposition 3.6, λ_i's are fuzzy nowhere dense sets in (X,T) and hence $\vee_{i=1}^{\infty}(\lambda_i)$ is a fuzzy first category set in (X,T).

Proposition 3.8. If $int\big(\vee_{i=1}^{\infty}(\lambda_i)\big) = 0$, where λ_i's are the fuzzy σ-first category sets in a fuzzy submaximal, fuzzy σ-Baire space (X,T), then (X,T) is a fuzzy Baire space.

Proof. Let λ_i's be the fuzzy σ-first category sets in a fuzzy submaximal, fuzzy σ-Baire space (X,T). Then, by proposition 3.7, $\vee_{i=1}^{\infty}(\lambda_i)$ is a fuzzy first category set in (X,T). Then $\vee_{i=1}^{\infty}(\lambda_i) = \vee_{j=1}^{\infty}(\mu_j)$,where the fuzzy sets μ_j's are fuzzy nowhere dense sets in (X,T). Now $int\big(\vee_{i=1}^{\infty}(\lambda_i)\big) = 0$, implies that $int\big(\vee_{j=1}^{\infty}(\mu_j)\big) = 0$, where the fuzzy sets μ_j's are fuzzy nowhere dense sets in (X,T). Hence (X,T) is a fuzzy Baire space.

Definition 3.6.[14] A fuzzy topological space (X,T) is called a fuzzy Volterra space if $cl(\wedge_{i=1}^{N}(\lambda_i)) = 1$, where λ_i's are fuzzy dense and fuzzy G_δ-sets in (X,T).

Definition 3.7.[14] Let (X,T) be a fuzzy topological space. Then (X,T) is called a fuzzy weakly Volterra space if $cl\Big(\wedge_{i=1}^{N}(\lambda_i)\Big) \neq 0$, where λ_i's are fuzzy dense and fuzzy $G_\delta-$sets in (X,T).

Proposition 3.9. If λ_i's $(i = 1\ to\ N)$ are fuzzy σ-nowhere dense sets in a fuzzy topological space (X,T) and $int\big(\vee_{i=1}^{N}(\lambda_i)\big) = 0$, then (X,T) is a fuzzy Volterra space.

Proof. Let λ_i's $(i = 1\ to\ N)$ be fuzzy σ-nowhere dense sets in (X,T). Then λ_i's are fuzzy F_σ-sets with $int(\lambda_i) = 0$ $(i = 1\ to\ N)$. Now $1 - int(\lambda_i) = 1$. Then, we have $cl(1 - \lambda_i) = 1$. That is, $(1 - \lambda_i)$'s are fuzzy dense sets in (X,T). Since λ_i's are fuzzy F_σ-sets, $(1 - \lambda_i)$'s are fuzzy G_δ-sets in (X,T). Hence $(1 - \lambda_i)$'s are fuzzy dense and fuzzy G_δ-sets in (X,T). Now,

$cl\big(\wedge_{i=1}^{N}(1-\lambda_i)\big) = cl\big(1-[\vee_{i=1}^{N}(\lambda_i)]\big) = 1 - int\big(\vee_{i=1}^{N}(\lambda_i)\big) = 1 - 0 = 1$. Hence (X,T) is a fuzzy Volterra space.

Theorem 3.4.[12] In a fuzzy topological space (X,T), a fuzzy set λ is a fuzzy σ-nowhere dense set in (X,T) if and only if $1-\lambda$ is a fuzzy dense and fuzzy G_δ-set in (X,T).

Theorem 3.5.[15] If the fuzzy topological space (X,T) is a fuzzy almost irresolvable space, then (X,T) is a fuzzy weakly Volterra space.

Theorem 3.6.[15] If the fuzzy topological space (X,T) is a fuzzy σ-second category space, then (X,T) is a fuzzy weakly Volterra space.

Proposition 3.10. If a fuzzy topological space (X,T) is not a fuzzy weakly Volterra space, then (X,T) is a fuzzy σ-first category space.

Proof. Let (X,T) be a fuzzy non-weakly Volterra space. Then, we have $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 0$, where λ_i's are fuzzy dense and fuzzy G_δ-sets in (X,T). Since λ_i's are fuzzy dense and fuzzy G_δ-sets in (X,T), by theorem 3.4, $(1-\lambda_i)$'s are fuzzy σ-nowhere dense sets in (X,T). Let μ_j's ($j = 1$ *to* ∞) be fuzzy σ-nowhere dense sets in (X,T) in which the first N fuzzy σ-nowhere dense sets be $(1-\lambda_i)$'s. Now $\vee_{i=1}^{N}(1-\lambda_i) \le \vee_{j=1}^{\infty}(\mu_j)$. Then $1-\wedge_{i=1}^{N}(\lambda_i) \le \vee_{j=1}^{\infty}(\mu_j)$ and hence $int[1-\wedge_{i=1}^{N}(\lambda_i)] \le int[\vee_{j=1}^{\infty}(\mu_j)]$. This implies that $1 - cl[\wedge_{i=1}^{N}(\lambda_i)] \le int[\vee_{j=1}^{\infty}(\mu_j)]$. Then $1 - int[\vee_{j=1}^{\infty}(\mu_j)] \le cl[\wedge_{i=1}^{N}(\lambda_i)]$. Now $1 - [\vee_{j=1}^{\infty}(\mu_j)] \le 1 - int[\vee_{j=1}^{\infty}(\mu_j)] \le cl[\wedge_{i=1}^{N}(\lambda_i)]$. Since $cl[\wedge_{i=1}^{N}(\lambda_i)] = 0$, $1 - [\vee_{j=1}^{\infty}(\mu_j)] = 0$. Then, $[\vee_{j=1}^{\infty}(\mu_j)] = 1$, where μ_j's ($j = 1$ *to* ∞) are fuzzy σ-nowhere dense sets in (X,T). Hence (X,T) is a fuzzy σ-first category space.

Proposition 3.11. If a fuzzy topological space (X,T) is a fuzzy weakly Volterra space, then (X,T) is not a fuzzy σ-Baire space.

Proof. Let (X,T) be a fuzzy weakly Volterra space. Then, we have $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) \ne 0$ where λ_i's are fuzzy dense and fuzzy G_δ-sets in (X,T). Since λ_i's are fuzzy dense and fuzzy G_δ-sets in (X,T), by theorem 3.4, $(1-\lambda_i)$'s are fuzzy σ-nowhere dense sets in (X,T). Let μ_j's ($j = 1$ *to* ∞) be fuzzy σ-nowhere dense sets in (X,T) in which the first N fuzzy σ-nowhere dense sets be $(1-\lambda_i)$'s. Now $\vee_{i=1}^{N}(1-\lambda_i) \le \vee_{j=1}^{\infty}(\mu_j)$. Then $1-\wedge_{i=1}^{N}(\lambda_i) \le \vee_{j=1}^{\infty}(\mu_j)$ and hence $int[1-\wedge_{i=1}^{N}(\lambda_i)] \le int[\vee_{j=1}^{\infty}(\mu_j)]$. This implies that $1 - cl[\wedge_{i=1}^{N}(\lambda_i)] \le int[\vee_{j=1}^{\infty}(\mu_j)]$. Since $cl[\wedge_{i=1}^{N}(\lambda_i)] \ne 0$, $int[\vee_{j=1}^{\infty}(\mu_j)] \ne 0$ where μ_j's ($j = 1$ *to* ∞) are fuzzy σ-nowhere dense sets in (X,T). Hence (X,T) is not a fuzzy σ-Baire space.

Remarks. The inter-relation between fuzzy σ-Baire spaces, fuzzy almost irresolvable spaces, fuzzy weakly Volterra spaces and fuzzy σ-second category spaces can be summarized as follows :

Fuzzy σ-Baire spaces

$\downarrow$

Fuzzy almost irresolvable spaces

$\swarrow \qquad \searrow$

Fuzzy σ-second category spaces   Fuzzy weakly Volterra spaces

But, Fuzzy σ-second category spaces $\nrightarrow$ Fuzzy σ-Baire spaces

and

Fuzzy weakly Volterra spaces $\nrightarrow$ Fuzzy σ-Baire spaces.

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