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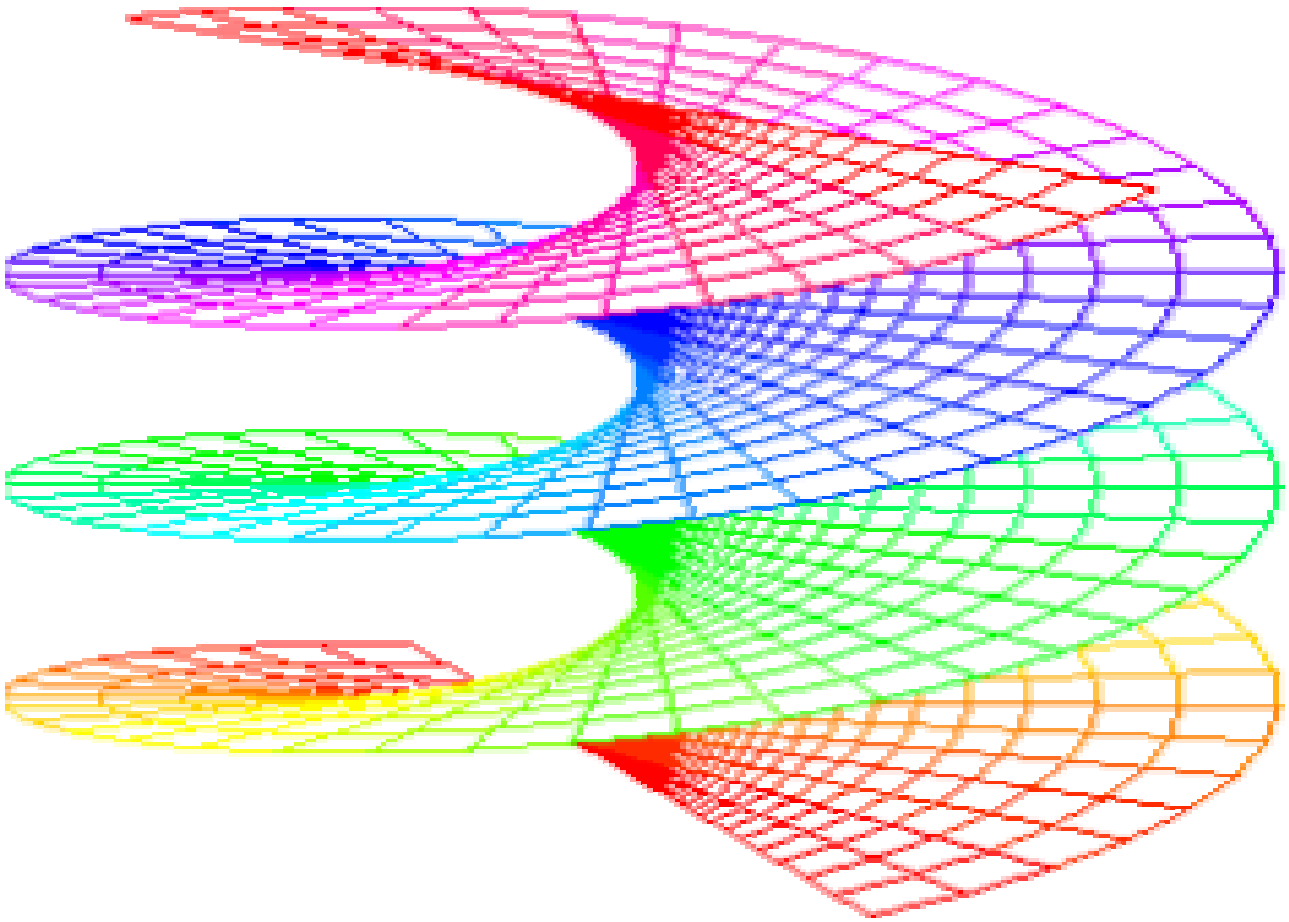

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Contents

ON THE SMARANDACHE FUNCTION AND SQUARE COMPLEMENTS *

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Abstract The main purpose of this paper is using the elementary method to study the mean value properties of the Smarandache function, and give an interesting asymptotic formula.

Keywords: Smarandache function; Square complements; Asymtotic formula.

§1. Introduction

Let n be an positive integer, if $a(n)$ is the smallest integer such that $na(n)$ is a perfect square number, then we call $a(n)$ as the square complements of n. The famous Smarandache function $S(n)$ is defined as following:

$$S(n) = \min\{m : m \in N, n|m!\}.$$

In problem 27 of [1], Professor F. Smarandache let us to study the properties of the square complements. It seems no one know the relation between this sequence and the Smaradache function before. In this paper, we shall study the mean value properties of the Smarandache function acting on the square complements, and give an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

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Theorem. For any real number $x \geq 3$, we have the asymptotic formula

$$\sum_{n\leq x} S(a(n)) = \frac{\pi^2 x^2}{12\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

§2. Proof of the theorem

To complete the proof of the theorem, we need some simple Lemmas. For convenience, we denote the greatest prime divisor of n by $p(n)$.

Lemma 1. If n is a square free number or $p(n) > \sqrt{n}$, then $S(n) = p(n)$.

Proof. (i) n is a square free number. Let $n = p_1 p_2 \cdots p_r p(n)$, then

$$p_i | p(n)!, \qquad i = 1, 2, \cdots, r.$$

So $n|p(n)!$, but $p(n) \dagger (p(n)-1)!$, so $n \dagger (p(n)-1)!$, that is, $S(n) = p(n)$;

(ii) $p(n) > \sqrt{n}$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p(n)$, so we have

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} < \sqrt{n}$$

then

$$p_i^{\alpha_i} | p(n)!, \qquad i = 1, 2, \cdots, r.$$

So $n|p(n)!$, but $p(n) \dagger (p(n)-1)!$, so $S(n) = p(n)$.

This proves Lemma 1.

Lemma 2. Let p be a prime, then we have the asymptotic formula

$$\sum_{\sqrt{x}\leq p\leq x} p = \frac{x^2}{2\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Proof. Let $\pi(x)$ denotes the number of the primes up to x. Noting that

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right),$$

from the Abel's identity [2], we have

$$\begin{aligned}
\sum_{\sqrt{x}\leq p\leq x} p &= \pi(x)x - \pi(\sqrt{x})\sqrt{x} - \int_{\sqrt{x}}^{x} \pi(t)dt \\
&= \frac{x^2}{\ln x} - \frac{1}{2}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right) \\
&= \frac{x^2}{2\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).
\end{aligned}$$

This proves Lemma 2.

Now we prove the theorem. First we have

$$\begin{aligned}
\sum_{n\leq x} S(a(n)) &= \sum_{m^2 n\leq x} S(n)|\mu(n)| \\
&= \sum_{m\leq\sqrt{x}} \sum_{n\leq\frac{x}{m^2}} S(n)|\mu(n)|. \qquad (1)
\end{aligned}$$

To the inner sum, using the above lemmas we get

$$\begin{aligned}
&\sum_{n\le\frac{x}{m^2}} S(n)|\mu(n)| \\
= &\sum_{\substack{np\le\frac{x}{m^2}\\ p\ge\sqrt{np}}} p|\mu(n)| + O\left(x^{\frac{3}{2}}\right) \\
= &\sum_{\substack{np\le\frac{x}{m^2}\\ p\ge\sqrt{\frac{x}{m^2}}}} p|\mu(n)| + O\left(x^{\frac{3}{2}}\right) \\
= &\sum_{n\le\sqrt{\frac{x}{m^2}}} |\mu(n)| \sum_{\sqrt{\frac{x}{m^2}}\le p\le\frac{x}{nm^2}} p + O\left(x^{\frac{3}{2}}\right) \\
= &\sum_{n\le\ln^2 x} \frac{|\mu(n)|x^2}{2n^2m^4\ln\frac{x}{nm^2}} + \sum_{\ln^2 x<n\le\sqrt{\frac{x}{m^2}}} \frac{|\mu(n)|x^2}{2n^2m^4\ln\frac{x}{nm^2}} + O\left(\frac{x^2}{m^4\ln^2 x}\right) \\
= &\frac{\zeta(2)x^2}{2\zeta(4)m^4\ln x} + O\left(\frac{x^2}{m^4\ln^2 x}\right). \qquad (2)
\end{aligned}$$

Combining (1) and (2), we have

$$\begin{aligned}
\sum_{n\le x} S(a(n)) &= \frac{\zeta(2)x^2}{2\zeta(4)\ln x}\sum_{m\le\sqrt{x}}\frac{1}{m^4} + O\left(\frac{x^2}{\ln^2 x}\sum_{m\le\sqrt{x}}\frac{1}{m^4}\right) \\
&= \frac{\zeta(2)x^2}{2\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).
\end{aligned}$$

Noting that $\zeta(2)=\frac{\pi^2}{6}$, so we have

$$\sum_{n\le x} S(a(n)) = \frac{\pi^2x^2}{12\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

This completes the proof of Theorem.

Reference

[1] F. Smaradache, Only Problems, Not Solutions, Xiquan Publishing House, Chicago, 1993.

[2] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976, pp. 77.

[3] Pan Chengdong and Pan Chengbiao, Element of the Analytic Number Theory, Science Press, Beijing, 1991.

ON THE INTEGER PART OF THE K-TH ROOT OF A POSITIVE INTEGER *

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Abstract For any positive integer m, let $a(m)$ denotes the integer part of the k-th root of m. That is, $a(m) = \left[m^{1/k}\right]$. In this paper, we study the asymptotic properties of

$$\sigma_{-\alpha}(f(a(m))),$$

where $0 < \alpha \leq 1$ be a fixed real number, $\sigma_{-\alpha}(n) = \sum_{l|n} \frac{1}{l^{\alpha}}$, $f(x)$ be a polynomial with integer coefficients. An asymptotic formula is obtained.

Keywords: Integer part sequence; k-th root; Mean value; Asymptotic formula.

§1. Introduction

For any positive integer m, let $a(m)$ denotes the integer part of the k-th root of m. That is, $a(m) = \left[m^{1/k}\right]$. For example, $a(1) = 1$, $a(2) = 1$, $a(3) = 1$, $a(4) = 1, \cdots, a(2^k) = 2$, $a(2^k + 1) = 2, \cdots, a(3^k - 1) = 2$, $a(3^k) = 3, \cdots$. In problem 80 of reference [1], Professor F. Smarandach asked us to study the asymptotic properties of the sequence $\{a(m)\}$. About this problem, it seems that none had studied it, at least we have not seen related paper before. In this paper, we shall use the elementary method to study the asymptotic properties

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of $\sigma_{-\alpha}(f(a(m)))$, and give an interesting asymptotic formula. That is, we shall prove the following:

Theorem. *Let $0 < \alpha \leq 1$ be a fixed real number, $f(x)$ be a polynomial with integer coefficients. Then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{m\leq x} \sigma_{-\alpha}(f(a(m))) = C_f(\alpha)x + O\left(x^{1-\alpha/k+\varepsilon}\right),$$

where

$$\sigma_{-\alpha}(n) = \sum_{l|n} \frac{1}{l^\alpha}, \quad C_f(\alpha) = \sum_{d=1}^{\infty} P_f(d)d^{-1-\alpha}, \quad P_f(d) = \sum_{f(n)\equiv 0(\bmod d), 0<n\leq d} 1,$$

and ε denotes any fixed positive number.

§2. Several lemmas

To complete the proof of the theorem, we need the following two simple lemmas:

Lemma 1. *Let $0 < \alpha \leq 1$ be a fixed real number, $f(x)$ be a polynomial with integer coefficients. Then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x, f(n)\neq 0} \sigma_{-\alpha}(f(a(n))) = C_f(\alpha)x + O\left(x^{1-\alpha}\ln^\gamma x\right),$$

where γ is a certain constant, and

$$C_f(\alpha) = \sum_{d=1}^{\infty} P_f(d)d^{-1-\alpha}, \qquad P_f(d) = \sum_{f(n)\equiv 0(\bmod d), 0<n\leq d} 1.$$

Proof. (See reference [2]).

Lemma 2. *Let M be a fixed positive integer, $f(x)$ be a polynomial with integer coefficients. Then we have*

$$\sum_{t=1}^{M} t^{k-1}\sigma_{-\alpha}(f(t)) = \frac{C_f(\alpha)}{k}M^k + O\left(M^{k-\alpha}\ln^\gamma M\right).$$

Proof. Let $A(y) = \sum_{t\leq y} \sigma_{-\alpha}(f(t))$, by Abel's identity (see Theorem 4.2 of [3]) we have

$$\sum_{t=1}^{M} t^{k-1}\sigma_{-\alpha}(f(t))$$

$$\begin{aligned}
&= M^{k-1}A(M) - A(1) - (k-1)\int_1^M y^{k-2}\,A(y)dy \\
&= M^{k-1}\left(C_f(\alpha)M + O\left(M^{1-\alpha}\ln^{\gamma} M\right)\right) \\
&\quad -(k-1)\int_1^M y^{k-2}\left(C_f(\alpha)y + O\left(y^{1-\alpha}\ln^{\gamma} y\right)\right)dy \\
&= C_f(\alpha)M^k + O\left(M^{k-\alpha}\ln^{\gamma} M\right) - \frac{C_f(\alpha)(k-1)}{k}M^k + O\left(M^{k-\alpha}\ln^{\gamma} M\right) \\
&= \frac{C_f(\alpha)}{k}M^k + O\left(M^{k-\alpha}\ln^{\gamma} M\right).
\end{aligned}$$

This completes the proof of Lemma 2.

§3. Proof of Theorem

In this section, we shall complete the proof of Theorem. For any real number $x \geq 1$, let M be a fixed positive integer such that

$$M^k \leq x < (M+1)^k.$$

Let a_0 denotes the constant term of $f(x)$, from the definition of $a(m)$ and Lemma 2, we have

$$\begin{aligned}
&\sum_{m\leq x}\sigma_{-\alpha}(f(a(m))) \\
&= \sum_{t=1}^{M}\sum_{(t-1)^k\leq m<t^k}\sigma_{-\alpha}(f(a(m))) + \sum_{M^k\leq m\leq x}\sigma_{-\alpha}(f(a(m))) \\
&= \sum_{t=1}^{M-1}\sum_{t^k\leq m<(t+1)^k}\sigma_{-\alpha}(f(t)) + \sigma_{-\alpha}(a_0) + \sum_{M^k\leq m\leq x}\sigma_{-\alpha}(f(M)) \\
&= \sum_{t=1}^{M-1}\left((t+1)^k - t^k\right)\sigma_{-\alpha}(f(t)) + O\left(\sum_{M^k\leq m\leq (M+1)^k}\sigma_{-\alpha}(f(M))\right) \\
&= \sum_{t=1}^{M-1}\left(C_k^1 t^{k-1} + C_k^2 t^{k-2} + \cdots + 1\right)\sigma_{-\alpha}(f(t)) \\
&\quad +O\left(\sum_{M^k\leq m\leq (M+1)^k}\sigma_{-\alpha}(f(M))\right) \\
&= k\sum_{t=1}^{M} t^{k-1}\sigma_{-\alpha}(f(t)) + O\left(M^{k-1}\sigma_{-\alpha}(f(t))\right) \\
&= C_f(\alpha)M^k + O\left(M^{k-\alpha+\varepsilon}\right),
\end{aligned}$$

where we have used the fact that $\sigma_{-\alpha}(n) \ll n^{\varepsilon}$.

On the other hand, note that the estimate

$$\begin{aligned} 0 \leq x - M^k &< (M+1)^k - M^k = kM^{k-1} + C_k^2 M^{k-2} + \cdots + 1 \\ &= M^{k-1}\left(k + C_k^2 \frac{1}{M} + \frac{1}{M^{k-1}}\right) \ll x^{\frac{k-1}{k}}. \end{aligned}$$

Now combining the above, we can immediately get the asymptotic formula

$$\sum_{m \leq x} \sigma_{-\alpha}(f(a(m))) = C_f(\alpha)x + O\left(x^{1-\alpha/k+\varepsilon}\right).$$

This completes the proof of Theorem.

References

[1] F. Smarandache, Only Problems, Not Solutions. Chicago: Xiquan Publ. House, 1993.

[2] G. Babaev, N. Gafurov and D. Ismoliov, Some asymptotic formulas connected with divisors of polynomials, Trudy Mat. Inst. Steklov **163** (1984), 10-18.

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SMARANDACHE "CHOPPED" N^N AND $N + 1^{N-1}$

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Florentin Smarandache has posed many problems that deal with perfect powers. See [1] for example. Perfect powers of the form N^N are aesthetically pleasing because of their symmetry. But in my opinion they would be more agreeable if their number of decimal digits (their "length" in base-10 representation) were equal to N. In this note we will considere numbers of the form N^N and $N - 1^{N+1}$ that have been "chopped off" to have N decimal digits. We will refer to these numbers as Smarandache Chopped N^N numbers, and Smarandache Chopped $N - 1^{N+1}$ numbers; and we will investigate them to see if 1) they are prime, 2) they are automorphic.

§1 Smarandache Chopped N^N Numbers

There are only three numbers of the form N^N that do not need to be chopped. That is, their decimal length is already equal to N: 11 = 1, 88 = 16777216, and 99 = 387420489. It is easy to see that there will be no more naturally equal to N. For example, 613613 has 1709 digits, 12341234 has 3815 digits; as we progress the decimal lengths continue to increase.

Definition: Smarandache Chopped N^N numbers are numbers formed from the first N digits of N^N. We will call this sequence $SC(n)$:

$$\begin{array}{rl} n = & 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \\ SC(n) = & 1, \quad x, \quad x, \quad x, \quad x, \quad x, \quad x, \quad 1677216, \quad 387420489, \\ n = & 10, \quad 11, \quad 12, \\ SC(n) = & 1000000000, \quad 28531167061, \quad 891610044825, \\ n = & 13, \cdots \\ SC(n) = & 3028751065922, \cdots \end{array}$$

For $n = 2$ through 7, $SC(n)$ is not defined, since those values lack one digit of being the proper length. Now we shall consider whether any terms of the

$SC(n)$ sequence are prime, and automorphic. A prime number surely requires no definition here, but perhaps an automorphic number[2] does. The term automorphic is usually applied to squares, but here we broaden the definition a bit. An automorphic number is a positive integer defined by some function, f, whose functional value terminates with the digits of n. For example, if $f(n) = n^2$, then 76 is automorphic because $76^2 = 5776$ ends with 76.

Concerning the question of which Smarandache Chopped N^N numbers are prime, a computer program was written, and $SC(65)$ and $SC(603)$ were discovered and proved to be prime. No more were found up to $n = 3000$. Question: Are there infinitely many SC primes?

Concerning the question of which Smarandache Chopped N^N numbers are automorphic, a computer program was written, and when $n = 1, 9, 66$, and 6051, $SC(n)$ is automorphic. No more were found up to $n = 20000$. Question: Are there infinitely many SC automorphic numbers?

Here is $SC(66)$ to demonstrate that it is automorphic:

$$SC(66) = 122998480353523742535746057982495245384860995389682130228631906566$$

§2 Smarandache Chopped $N - 1^{N+1}$ Numbers

Numbers formed from the first N digits of $N - 1^{N+1}$ also have an intriguing symmetry. There are only three numbers of the form $N - 1^{N+1}$ that do not need to be chopped: $0^2 = 0$, $6^8 = 1679616$, and $7^9 = 40353607$. It is easy to see that there will be no more that are naturally equal to N. We will call this sequence $SC2(n)$.

$$\begin{array}{rccccccccc} n = & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, \\ SC2(n) = & 0, & x, & x, & x, & x, & x, & 1679616, & 40353607, \\ n = & & & 9, & & 10, & & 11, \cdots \\ SC2(n) = & & & 107374182, & & 3138105960, & & 10000000000, \cdots \end{array}$$

Primes: A program was written, and $SC2(44)$, $SC2(64)$, and $SC2(1453)$ were discovered and proved to be prime. No more were found up to $n = 3000$. Question: Are there infinitely many $SC2$ primes?

Automorphics: A program was written, and $SC2(9416)$ was the only term discovered to be automorphic. No more were found up to $n = 20000$. Question: Are there infinitely many $SC2$ automorphic numbers?

§3 Additional Questions

1. Do these sequences, $SC(n)$ and $SC2(n)$, defy basic analysis because of their "chopped" property?

2. What other properties do the $SC(n)$ and $SC2(n)$ sequences have?

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[1] F. Smarandache, Only Problems, Not Solutions, Xiquan Publishing House, Chicago, 1993.

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THE 57-TH SMARANDACHE'S PROBLEM II *

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Abstract For any positive integer n, let r be the positive integer such that: the set $\{1, 2, \cdots, r\}$ can be partitioned into n classes such that no class contains integers x, y, z with $x^y = z$. In this paper, we use the elementary methods to give a sharp lower bound estimate for r.

Keywords: Smarandache-type multiplicative functions; Mangoldt function; Hybrid mean value.

§1. Introduction

For any positive integer n, let r be a positive integer such that: the set $\{1, 2, \cdots, r\}$ can be partitioned into n classes such that no class contains integers x, y, z with $x^y = z$. In [1], Schur asks us to find the maximum r. About this problem, Liu Hongyan [2] obtained that $r \geq n^{m+1}$, where m is any integer with $m \leq n + 1$.

In this paper, we use the elementary methods to improve Liu Hongyan's result. That is, we shall prove the following:

Theorem. *For sufficiently large integer n, let r be a positive integer such that: the set $\{1, 2, \cdots, r\}$ can be partitioned into n classes such that no class contains integers x, y, z with $x^y = z$. Then we have*

$$r \geq \left(n^{n!} + 2\right)^{n^{n!}+1} - 1.$$

*This work is supported by the N.S.F.(60472068) and the P.N.S.F. of P.R.China.

§2. Proof of the Theorem

In this section, we complete the proof of the theorem.

Let $r = \left(n^{n!}+2\right)^{n^{n!}+1} - 1$ and partition the set $\{1, 2, \cdots, \left(n^{n!}+2\right)^{n^{n!}+1} - 1\}$ into n classes as follows:

$$\begin{array}{lllll}
\text{Class 1:} & 1, & n^{n!}+1, & n^{n!}+2, & \cdots, \quad \left(n^{n!}+2\right)^{n^{n!}+1}-1. \\
\text{Class 2:} & 2, & n+1, & n+2, & \cdots, \quad n^2. \\
\vdots & & & & \\
\text{Class k:} & k, & n^{(k-1)!}+1, & n^{(k-1)!}+2, & \cdots, \quad n^{k!}. \\
\vdots & & & & \\
\text{Class n:} & n, & n^{(n-1)!}+1, & n^{(n-1)!}+2, & \cdots, \quad n^{n!}.
\end{array}$$

It is obvious that Class $k(k \geq 2)$ contains no integers x, y, z with $x^y = z$. In fact for any integers $x, y, z \in$ Class k, $k = 2, 3, \cdots, n$, we have

$$x^y \geq \left(n^{(k-1)!}+1\right)^k > n^{k!} \geq z.$$

Similarly, Class 1 also contains no integers x, y, z with $x^y = z$.

This completes the proof of the theorem.

Reference

[1] F. Smarandache, Only Problems, Not Solutions, Xiquan Publishing House, Chicago, 1993.

[2] Liu Hongyan and Zhang Wenpeng. A note on the 57-th Smarandache's problem. Smarandache Notions Journal **14** (2004), 164-165.

PERFECT POWERS IN SMARANDACHE N- EXPRESSIONS

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Abstract The main purpose of this paper is to study the concept of Smarandache n-expressions (but with a slight modification), its perfect powers, give conjectures, and proposed future studies.

Keywords: Smarandache n-expressions, Smarandache 5-expressions, Smarandache 2-expressions, perfect powers in Smarandache type expressions.

§1. Introduction

In [1] M. Perez & E. Burton, documented that J. Castillo [2], asked how many primes are there in the Smarandache n-expressions:

$$x_1^{x_2} + x_2^{x_3} + \cdots + x_n^{x_1} \qquad (1)$$

where $n > 1, x_1, x_2, \cdots, x_n > 1$, and $gcd(x_1, x_2, \cdots, x_n) = 1$
In this paper, with only slight modification of (1), we got (2) namely;

$$a^{x_1} + a^{x_2} + \cdots + a^{x_n} \qquad (2)$$

where $a > 1, x_1, x_2, \cdots, x_n \geq 0$, and $gcd(a, x_1, x_2, \cdots, x_n) = 1$
I will study the following cases of equation (2).

§2. Case1 of $5-$Expressions

$$3^{x_1} + 3^{x_2} + 3^{x_3} + 3^{x_4} + 3^{x_5} = k^2, \qquad (3)$$

The solution of (3) is:
$x_1 = 2m, x_2 = 2m + 1, x_3 = 2m + 2, x_4 = 2m + 3, x_5 = 2m + 4$, and $k = (11)3^m$.

Proof.

$$\begin{aligned} 3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5} &= 3^{2m}+3^{2m+1}+3^{2m+2}+3^{2m+3}+3^{2m+4} \\ &= 3^{2m}(1+3^1+3^2+3^3+3^4) \\ &= 3^{2m}(121) \\ &= k^2. \end{aligned}$$

Examples:

$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}$	k^2
$3^0+3^1+3^2+3^3+3^4$	11^2
$3^2+3^3+3^4+3^5+3^6$	33^2
$3^4+3^5+3^6+3^7+3^8$	99^2
$3^6+3^7+3^8+3^9+3^{10}$	297^2
$3^8+3^9+3^{10}+3^{11}+3^{12}$	891^2
$3^{10}+3^{11}+3^{12}+3^{13}+3^{14}$	2673^2

The first terms and the nth terms of the sequence are:

$$121, 1089, 9801, 88209, 793881, \cdots, (11)^2(9)^{n-1}, \cdots \tag{4}$$

Where the square roots are:

$$11, 33, 99, 297, 891, \cdots, (11)(3)^{(n-1)}, \cdots \tag{5}$$

Notice that there is no prime numbers in (5), (excluding 11).
The sum of (5) is $\frac{11(3^n)-1}{2}$, and there is no limit, since $\frac{11(3^n)-1}{2}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture: if p, q, r, s, t are primes numbers, then the equation $3^p+3^q+3^r+3^s+3^t=k^2$ has no solution.

§3. Case2 of 5-Expressions

$$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}=k^2+k^2+k^2 \tag{6}$$

The solution of (3) is: $x_1 = 2m+1, x_2 = 2m+2, x_3 = 2m+3, x_4 = 2m+4, x_5 = 2m+5$, and $k = 11(3)^{\frac{2m+1}{2}}$.

Proof.

$$\begin{aligned} 3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5} &= 3^{2m+1}+3^{2m+2}+3^{2m+3}+3^{2m+4}+3^{2m+5} \\ &= 3^{2m+1}(1+3^1+3^2+3^3+3^4) \\ &= 3^{2m+1}(121) \\ &= 3k^2. \end{aligned}$$

Examples:

$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}$	$3k^2$
$3^1+3^2+3^3+3^4+3^5$	$11^2+11^2+11^2$
$3^3+3^4+3^5+3^6+3^7$	$33^2+33^2+33^2$
$3^5+3^6+3^7+3^8+3^9$	$99^2+99^2+99^2$
$3^7+3^8+3^9+3^{10}+3^{11}$	$297^2+297^2+297^2$
$3^9+3^{10}+3^{11}+3^{12}+3^{13}$	$891^2+891^2+891^2$
$3^{11}+3^{12}+3^{13}+3^{14}+3^{15}$	$2673^2+2673^2+2673^2$

The first terms and nth terms of the sequence are:

$$(3)121, (3)1089, (3)9801, (3)88209, (3)793881\cdots(11)^2(3)(9)^{n-1}, \cdots \quad (7)$$

The sum of (7) is $\frac{11^2(3)(9^n-1)}{8}$, and there is no limit, since $\frac{11^2(3)(9^n-1)}{8}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture: if p, q, r, s, t are primes numbers, then the equation $3^p+3^q+3^r+3^s+3^t=3k^2$ has no solution.

§4. Case3 of 5-Expressions

$$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}=(11)^2(61)3^{x_1}, \quad (8)$$

The solution of (8) is:
$x_1=2m+1, x_2=2m+3, x_3=2m+5, x_4=2m+7, x_5=2m+9$, and $k=(61^{\frac{1}{2}})(11)(3^{\frac{2m+1}{2}})$.

Proof.

$$\begin{aligned} 3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5} &= 3^{2m+1}+3^{2m+3}+3^{2m+5}+3^{2m+7}+3^{2m+9} \\ &= (3^{2m+1})(11)^2(61). \end{aligned}$$

Examples:

$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}$	$(3^{2m+1})(11)^2(61)$
$3^1+3^3+3^5+3^7+3^9$	$(3)(11)^2(61)$
$3^3+3^5+3^7+3^9+3^{11}$	$(3)^3(11)^2(61)$
$3^5+3^7+3^9+3^{11}+3^{13}$	$(3)^5(11)^2(61)$
$3^7+3^9+3^{11}+3^{13}+3^{15}$	$(3)^7(11)^2(61)$
$3^9+3^{11}+3^{13}+3^{15}+3^{17}$	$(3)^9(11)^2(61)$
$3^{11}+3^{13}+3^{15}+3^{17}+3^{19}$	$(3)^{11}(11)^2(61)$

The first terms and nth terms of the sequence are:

$$(3)(11)^2(61), (3)^3(11)^2(61), (3)^5(11)^2(61), (3)^7(11)^2(61), (3)^9(11)^2(61),$$
$$(3)^{11}(11)^2(61), \cdots, (3)(61)(11)^2(9)^{n-1}, \cdots \quad (9)$$

The sum of (9) is $\frac{11^2(3)(61)(9^n-1)}{8}$, and there is no limit, since $\frac{11^2(3)(61)(9^n-1)}{8}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture: if p, q, r, s, t are primes numbers, then the equation $3^p+3^q+3^r+3^s+3^t = 3^{2m+1}(11)^2(61)$ has no solution.

§5. Case4 of 5-Expressions

$$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5} = (11)^2(61)3^{x_1}, \tag{10}$$

The solution of (10) is:
$x_1 = 2m, x_2 = 2m+2, x_3 = 2m+4, x_4 = 2m+6, x_5 = 2m+8$, and $k = (61^{\frac{1}{2}})(11)(3^m)$.

Proof.

$$\begin{aligned} 3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5} &= 3^{2m}+3^{2m+2}+3^{2m+4}+3^{2m+6}+3^{2m+8} \\ &= 3^{2m}(11)^2(61). \end{aligned}$$

Examples:

$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}$	$3^{2m}(11)^2(61)$
$3^2+3^4+3^6+3^8+3^{10}$	$(3)^2(11)^2(61)$
$3^4+3^6+3^8+3^{10}+3^{12}$	$(3)^4(11)^2(61)$
$3^6+3^8+3^{10}+3^{12}+3^{14}$	$(3)^6(11)^2(61)$
$3^8+3^{10}+3^{12}+3^{14}+3^{16}$	$(3)^8(11)^2(61)$
$3^{10}+3^{12}+3^{14}+3^{16}+3^{18}$	$(3)^{10}(11)^2(61)$
$3^{12}+3^{14}+3^{16}+3^{18}+3^{20}$	$(3)^{12}(11)^2(61)$

The first terms and nth terms of the sequence are:

$$(3)^2(11)^2(61), (3)^4(11)^2(61), (3)^6(11)^2(61), (3)^8(11)^2(61),$$
$$\cdots (61)(11)^2(3)^2(9)^{n-1}, \cdots \tag{11}$$

The sum of (11) is $\frac{(3^2)11^2(61)(9^n-1)}{8}$, and there is no limit, since $\frac{(3^2)11^2(61)(9^n-1)}{8}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture: if p, q, r, s, t are primes numbers, then the equation $3^p+3^q+3^r+3^s+3^t = 3^{2m}(11)^2(61)$ has no solution.

§6. Case5 of 2-Expressions

$$3^x+3^y = z^2, \tag{12}$$

The solution of (12) is $x = 2m, y = 2m + 1$, and $z = 2(3)^m$
Proof. $3^x + 3^y = 3^{2m} + 3^{2m+1} = 3^{2m}(1 + 3) = z^2$
Examples:

$3^x + 3^y$	z^2
$3^2 + 3^3$	6^2
$3^4 + 3^5$	18^2
$3^6 + 3^7$	54^2
$3^8 + 3^9$	162^2
$3^{10} + 3^{11}$	486^2
$3^{12} + 3^{13}$	1458^2

The first terms and nth terms of the sequence are:
$36, 324, 2916, 26244, 236196, \cdots, (6)^2(9)^(n-1), \cdots$ (13)
Where the square roots are:

$$6, 18, 54, 162, 486, 1458, \cdots, (6)(3)(n-1), \cdots \quad (14)$$

The sum of the first n terms of the sequence (14) is given by the following formula.

$$\frac{6 - 6(3)^n}{1 - 3} = 3(3^n - 1).$$

and there is no limit, since $3(3^n - 1)$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture:
1) The equation $3^x + 3^y = z^2$ has one solution in prime numbers, if x, and y are prime numbers, namely $(x, y) = (2, 3)$.
2) The equation $3^{x^2} + 3^{y^2} = z^2$ has unique solution, if x, and y are prime numbers, namely $(x, y, z) = (3, 2, 162)$.

§7. Case6 of 2-Expressions

$$3^x + 3^y = 3z^2, \quad (15)$$

The solution of (15) is $x = 2m + 1, y = 2m + 2$, and $z = 2(3)^{\frac{2m+1}{2}}$
Proof. $3^x + 3^y = 3^{2m+1} + 3^{2m+2} = 3^{2m}(3 + 9) = 3^{2m}(12) = 3z^2$
Examples:

$3^x + 3^y$	$3z^2$
$3^1 + 3^2$	$12 = 2^2 + 2^2 + 2^2$
$3^3 + 3^4$	$108 = 6^2 + 6^2 + 6^2$
$3^5 + 3^6$	$972 = 18^2 + 18^2 + 18^2$
$3^7 + 3^8$	$8748 = 54^2 + 54^2 + 54^2$
$3^9 + 3^{10}$	$78732 = 162^2 + 162^2 + 162^2$
$3^{11} + 3^{12}$	$708588 = 486^2 + 486^2 + 486^2$
$3^{13} + 3^{14}$	$6377292 = 1458^2 + 1458^2 + 1458^2$

The first terms and nth terms of the sequence are:
$12, 108, 972, 8748, 78732, \cdots, 12(9)n-1, \cdots$ (16)
The sum of the first n terms of the sequence (16) is given by the following formula.

$$\frac{12-12(9)^n}{1-9}=\frac{3(9^n-1)}{2}.$$

and there is no limit, since $\frac{3(9^n-1)}{2}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.
Conjecture: The equation $3^x+3^y=12(3)^{2m}$ has no solution, if x, and y are prime numbers.

§8. Case7 of 2-Expressions

$$3^x+3^y=(10)3^{2m+1}, \tag{17}$$

The solution of (17) is $x=2m+1, y=2m+3$.
Proof: $3^x+3^y=3^{2m+1}+3^{2m+3}=3^{2m+1}(1+9)=3^{2m+1}(10)$

Examples:

3^x+3^y	$10(3)^{2m+1}$
3^1+3^3	30
3^3+3^5	270
3^5+3^7	2430
3^7+3^9	21870
3^9+3^{11}	196830
$3^{11}+3^{13}$	1771470
$3^{13}+3^{15}$	15943230

The first terms and the nth terms of the sequence are:

$$30, 270, 2430, 21870, 196830, \cdots, 30(9)n-1, \cdots \tag{18}$$

The sum of the first n terms of the sequence (18) is given by the following formula.

$$\frac{30-30(9)^n}{1-9}=\frac{15(9^n-1)}{4}.$$

and there is no limit, since $\frac{15(9^n-1)}{4}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.
Conjecture: The equation $3^x+3^y=10(3)^{2m+1}$ has infinitely many solutions, if x, and y are prime numbers.

§9. Case8 of 2-Expressions

$$3^x + 3^y = 3^{2m}(10), \tag{19}$$

The solution of (19) is $x = 2m, y = 2m + 2$.
Proof. $3^x + 3^y = 3^{2m} + 3^{2m+2} = 3^{2m}(1 + 9) = 3^{2m}(10)$

Examples:

$3^x + 3^y$	$10(3)^{2m}$
$3^2 + 3^4$	90
$3^4 + 3^6$	810
$3^6 + 3^8$	7290
$3^8 + 3^{10}$	65610
$3^{10} + 3^{12}$	590490
$3^{12} + 3^{14}$	5314410
$3^{14} + 3^{16}$	47829690

The first terms and the nth terms of the sequence are:

$$90, 810, 7290, 65610, 590490, \cdots, 90(9)^{n-1}, \cdots \tag{20}$$

The sum of the first n terms of the sequence (20) is given by the following formula.

$$\frac{90 - 90(9)^n}{1 - 9} = \frac{45(9^n - 1)}{4}.$$

and there is no limit, since $\frac{45(9^n-1)}{4}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.
Conjecture: The equation $3^x + 3^y = 10(3)^{2m}$ has infinitely many solutions, if x, and y are prime numbers.

§10. Case9 of 2-Expressions

$$3^x - 3^y = 2(3)^y, \tag{21}$$

The solution of (21) is $x = 6m - 2, y = 6m - 3$.
Proof. $3^x - 3^y = 3^{6m-2} - 3^{6m-3} = 2(3)^{6m-3}$

Examples:

$3^x - 3^y$	$2(3)^{6m-3}$
$3^4 - 3^3$	$2(3)^3$
$3^{10} - 3^9$	$2(3)^9$
$3^{16} - 3^{15}$	$2(3)^{15}$
$3^{22} - 3^{21}$	$2(3)^{21}$
$3^{28} - 3^{27}$	$2(3)^{27}$
$3^{34} - 3^{33}$	$2(3)^{33}$
$3^{40} - 3^{39}$	$2(3)^{39}$

The first terms and the nth terms of the sequence are:

$$2(3)^3, 2(3)^9, 2(3)^{15}, 2(3)^{21}, 2(3)^{27}, \cdots, 2(3)^3(729)^{n-1}, \cdots \tag{22}$$

The sum of the first n terms of the sequence (20) is given by the following formula.

$$\frac{54 - 54(729)^n}{1 - 729} = \frac{27(729^n - 1)}{364}.$$

and there is no limit, since $\frac{27(729^n-1)}{364}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture: The equation $3^x - 3^y = 2(3)^y$ has no solutions, if x, and y are prime numbers.

§11. Case10 of 2-Expressions

$$3^x + 3^y = 4(3)^y, \tag{23}$$

The solution of (23) is $x = 6m - 2, y = 6m - 3$.

Proof. $3^x + 3^y = 3^{6m-2} + 3^{6m-3} = 4(3)^{6m-3}$

Examples:

$3^x + 3^y$	$4(3)^{6m-3}$
$3^4 + 3^3$	$4(3)^3$
$3^{10} + 3^9$	$4(3)^9$
$3^{16} + 3^{15}$	$4(3)^{15}$
$3^{22} + 3^{21}$	$4(3)^{21}$
$3^{28} + 3^{27}$	$4(3)^{27}$
$3^{34} + 3^{33}$	$4(3)^{33}$
$3^{40} + 3^{39}$	$4(3)^{39}$

The first terms and the nth terms of the sequence are:

$$4(3)3, 4(3)9, 4(3)15, 4(3)21, 4(3)27, \cdots, 4(3)3(729)n - 1, \cdots \tag{24}$$

The sum of the first n terms of the sequence (24) is given by the following formula.

$$\frac{108 - 108(729)^n}{1 - 729} = \frac{27(729^n - 1)}{182}.$$

and there is no limit, since $\frac{27(729^n-1)}{182}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture: The equation $3^x + 3^y = 4(3)^y$ has no solutions, if x, and y are prime numbers.

§12. Case11 of 2-Expressions

$$2^x + 2^y = z^2, \tag{25}$$

The solution of (25) is $x = 2m - 2, y = 2m + 1$, and $z = 3(2)^{m-1}$.
Proof. $2^x + 2^y = 2^{2m-2} + 2^{2m+1} = 2^{2m}(2^{-2} + 2^1) = 9(2)^{2m-2} = z^2$
Examples:

$2^x + 2^y$	Z^2
$2^0 + 2^3$	3^2
$2^2 + 2^5$	6^2
$2^4 + 2^7$	12^2
$2^6 + 2^9$	24^2
$2^8 + 2^{11}$	48^2
$2^{10} + 2^{13}$	96^2
$2^{12} + 2^{15}$	192^2

The first terms and the nth terms of the sequence are:

$$9, 36, 144, 576, 2304 \cdots, (9)(4)n - 1, \cdots \tag{26}$$

Where the square roots are:

$$3, 6, 12, 24, 48, 96 \cdots, (3)(2)(n-1), \cdots \tag{27}$$

The sum of the first n terms of the sequence (27) is given by the following formula.

$$\frac{3 - 3(2)^n}{1 - 2} = 3(2^n - 1).$$

and there is no limit, since $3(2^n - 1)$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.
Conjecture: The equation $2^x + 2^y = z^2$ has one solution if x, and y are prime numbers, i.e. $(x, y) = (2, 5)$.

Future Studies

The Smarandache n-expressions suggest that there may be future interesting n-expressions yet to be revealed.

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ON THE M-TH POWER RESIDUE OF N *

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Abstract For any positive integer n, let $a_m(n)$ denote the m-th power residue of n. In this paper, we use the elementary method to study the asymptotic properties of $\log(a_m(n!))$, and give an interesting asymptotic formula for it.

Keywords: m-th power residue of n; Chebyshev's function; Asymptotic formula.

§1. Introduction

Let $m > 2$ be a fixed integer. For any positive integer n, we define $a_m(n)$ as the m-th power residue of n (See reference [1]). That is, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers, then $a_m(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$, where $\beta_i = \min(\alpha_i, m-1)$. Let p be a prime, and for any real number $x > 1$, $\theta(x) = \sum_{p \le x} \log p$ denotes the Chebyshev's function of x. In this paper, we will use the elementary methods to study the asymptotic properties of $\log(a_m(n!))$, and give an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. *Let $m > 1$ be a fixed positive integer. Then for any positive integer n, we have the asymptotic formula:*

$$\log(a_m(n!)) = n\left(\sum_{a=1}^{m-1}\frac{1}{a}\right) + O\left(n\exp\left(\frac{-A\log^{\frac{3}{5}} n}{(\log\log n)^{\frac{1}{5}}}\right)\right),$$

where A is a fixed positive constant.

§2. Proof of the theorem

Before the proof of Theorem, a lemma will be useful.

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Lemma. *Let p be a prime. Then for any real number $x \geq 2$, we have the asymptotic formula:*

$$\theta(x) = x + O\left(x \exp\left(\frac{-A\log^{\frac{3}{5}} x}{(\log\log x)^{\frac{1}{5}}}\right)\right),$$

where A is a positive constant.

Proof. See reference [2] or [3].

Now we use this Lemma to complete the proof of Theorem. In fact, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers. Suppose that $m \ll n$, if $(m-1)p \leq n < mp$, then $p^{m-1} \parallel n!$. From the definition of $a_m(n)$, we can write

$$a_m(n!) = \prod_{\frac{n}{2}<p\leq n} p \prod_{\frac{n}{3}<p\leq\frac{n}{2}} p^2 \cdots \prod_{\frac{n}{m-1}<p\leq\frac{n}{m-2}} p^{m-2} \prod_{p\leq\frac{n}{m-1}} p^{m-1}.$$

By taking the logistic computation in the two sides, we have

$$\begin{aligned}
& \log(a_m(n!)) \\
= \; & \log\left(\prod_{\frac{n}{2}<p\leq n} p \prod_{\frac{n}{3}<p\leq\frac{n}{2}} p^2 \cdots \prod_{\frac{n}{m-1}<p\leq\frac{n}{m-2}} p^{m-2} \prod_{p\leq\frac{n}{m-1}} p^{m-1}\right) \\
= \; & \theta(n) - \theta\left(\frac{n}{2}\right) + 2\left(\theta\left(\frac{n}{2}\right) - \theta\left(\frac{n}{3}\right)\right) + \cdots \\
& + (m-2)\left(\theta\left(\frac{n}{m-2}\right) - \theta\left(\frac{n}{m-1}\right)\right) + (m-1)\theta\left(\frac{n}{m-1}\right) \\
= \; & \theta(n) + \theta\left(\frac{n}{2}\right) + \cdots + \theta\left(\frac{n}{m-1}\right).
\end{aligned}$$

Then, combining Lemma, we can get the asymptotic formula:

$$\begin{aligned}
\log(a_m(n!)) &= n + \frac{n}{2} + \cdots + \frac{n}{m-1} + O\left(n\exp\left(\frac{-A\log^{\frac{3}{5}} n}{(\log\log n)^{\frac{1}{5}}}\right)\right) \\
&= n\left(1 + \frac{1}{2} + \cdots + \frac{1}{m-1}\right) + O\left(n\exp\left(\frac{-A\log^{\frac{3}{5}} n}{(\log\log n)^{\frac{1}{5}}}\right)\right) \\
&= n\left(\sum_{a=1}^{m-1}\frac{1}{a}\right) + O\left(n\exp\left(\frac{-A\log^{\frac{3}{5}} n}{(\log\log n)^{\frac{1}{5}}}\right)\right).
\end{aligned}$$

This completes the proof of Theorem.

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GENERALIZATION OF THE DIVISOR PRODUCTS AND PROPER DIVISOR PRODUCTS SEQUENCES

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Abstract Let n be a positive integer, $p_d(n)$ denotes the product of all positive divisors of n, $q_d(n)$ denotes the product of all proper divisors of n. In this paper, we study the properties of the sequences of $\{p_d(n)\}$ and $\{q_d(n)\}$, and prove that the generalized results for the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$.

Keywords: Divisor and proper divisor product; Generalization ; Sequence.

§1. Introduction and results

Let n be a positive integer, $p_d(n)$ denotes the product of all positive divisors of n. That is, $p_d(n) = \prod_{d|n} d$. For example, $p_d(1) = 1$, $p_d(2) = 2$, $p_d(3) = 3$, $p_d(4) = 8, \cdots, p_d(p) = p, \cdots$. $q_d(n)$ denotes the product of all proper divisors of n. That is, $q_d(n) = \prod_{d|n,\ d<n} d$. For example, $q_d(1) = 1$, $q_d(2) = 1$, $q_d(3) = 1$, $q_d(4) = 2, \cdots$. In problem 25 and 26 of [1], Professor F. Smarandache asked us to study the properties of the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$. About this problem, Liu Hongyan and Zhang Wenpeng in [2] have studied it and proved the Makowsiki & Schinzel conjecture in [3] hold for $\{p_d(n)\}$ and $\{q_d(n)\}$. One of them is that for any positive integer n, we have the inequality:

$$\sigma(\phi(p_d(n))) \geq \frac{1}{2} p_d(n), \tag{1}$$

where $\sigma(n)$ is the divisor sum function, $\phi(n)$ is the Euler's function.

In this paper, as the generalization of [2], we will consider the properties of the sequences of $\{p_d(n)\}$ and $\{q_d(n)\}$ for k-th divisor sum function, and give two more general results. That is, we shall prove the following:

Theorem 1. Let $n = p^\alpha$, p be a prime and α be a positive integer. Then for any fixed positive integer k, we have the inequality

$$\sigma_k(\phi(p_d(n))) \geq \frac{1}{2^k} p_d^k(n),$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the k-th divisor sum function.

Theorem 2. Let $n = p^\alpha$, p be a prime and α be a positive integer. Then for any fixed positive integer k, we have the inequality

$$\sigma_k(\phi(q_d(n))) \geq \frac{1}{2^k} q_d^k(n).$$

§2. Proof of the theorems

In this section, we shall complete the proof of the theorem. First we need two Lemmas as following:

Lemma 1. For any positive integer n, then we have the identity $p_d(n) = n^{\frac{d(n)}{2}}$ and $q_d(n) = n^{\frac{d(n)}{2}-1}$,
where $d(n) = \sum_{d|n} 1$ is the divisor function.

Proof. (See Reference [2] Lemma 1).

Lemma 2. For any positive integer n, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ with $\alpha_i \geq 2$ $(1 \leq i \leq s)$, $p_j (1 \leq j \leq s)$ are some different primes with $p_1 < p_2 \cdots p_s$. Then for any fixed positive integer k, we have the estimate

$$\sigma_k(\phi(n)) \geq \phi^k(n) \cdot \prod_{p|n} \left(1 + \frac{1}{p^k}\right).$$

Proof. From the properties of the Euler's function we have

$$\begin{aligned} \phi(n) &= \phi(p_1^{\alpha_1})\phi(p_2^{\alpha_2}) \cdots \phi(p_s^{\alpha_s}) \\ &= p_1^{\alpha_1-1} p_2^{\alpha_2-1} \cdots p_s^{\alpha_s-1}(p_1-1)(p_2-1)\cdots(p_s-1). \end{aligned} \tag{2}$$

Here, let $(p_1-1)(p_2-1)\cdots(p_s-1) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s} q_1^{r_1} q_2^{r_2} \cdots q_t^{r_t}$, where $\beta_i \geq 0$, $1 \leq i \leq s; r_j \geq 1, 1 \leq j \leq t$ and $q_1 < q_2 < \cdots < q_t$ are different primes. Note that $\sigma_k(p^\alpha) = 1^k + p^k + \cdots + p^{k\alpha} = \frac{p^{k(\alpha+1)}-1}{p^k-1}$, for any $k > 0$.

Then for (2), we deduce that

$$\begin{aligned} \sigma_k(\phi(n)) &= \sigma_k(p_1^{\alpha_1+\beta_1-1} p_2^{\alpha_2+\beta_2-1} \cdots p_s^{\alpha_s+\beta_s-1} q_1^{r_1} q_2^{r_2} \cdots q_t^{r_t}) \\ &= \prod_{i=1}^{s} \frac{p_i^{k(\alpha_i+\beta_i)}-1}{p_i^k-1} \prod_{j=1}^{t} \frac{q_j^{k(r_j+1)}-1}{q_j^k-1} \\ &= p_1^{k(\alpha_1+\beta_1)} p_2^{k(\alpha_2+\beta_2)} \cdots p_s^{k(\alpha_s+\beta_s)} q_1^{kr_1} q_2^{kr_2} \cdots q_t^{kr_t} \end{aligned} \tag{3}$$

$$\times \prod_{i=1}^{s} \frac{1-\frac{1}{p_i^{k(\alpha_i+\beta_i)}}}{p_i^k-1} \prod_{j=1}^{t} \frac{1-\frac{1}{q_j^{k(r_j+1)}}}{1-\frac{1}{q_j^k}}$$

$$= n^k \cdot \prod_{i=1}^{s}\left(1-\frac{1}{p_i}\right)^k \prod_{i=1}^{s} \frac{1-\frac{1}{p_i^{k(\alpha_i+\beta_i)}}}{1-\frac{1}{p_i^x}} \prod_{j=1}^{t} \frac{1-\frac{1}{q_j^{k(r_j+1)}}}{1-\frac{1}{q_j^k}}.$$

Because

$$\phi(n) = n \cdot \prod_{p|n}\left(1-\frac{1}{p}\right), \tag{4}$$

then from (3) and (4) we get

$$\begin{aligned}
\sigma_k(\phi(n)) &= n^k \cdot \frac{\phi^k(n)}{n^k} \cdot \prod_{i=1}^{s} \frac{1-\frac{1}{p_i^{k(\alpha_i+\beta_i)}}}{1-\frac{1}{p_i^x}} \prod_{j=1}^{t} \frac{1-\frac{1}{q_j^{k(r_j+1)}}}{1-\frac{1}{q_j^k}} \\
&= \phi^k(n) \cdot \prod_{i=1}^{s}(1+\frac{1}{p_i^k}+\cdots+\frac{1}{p_i^{k(\alpha_i+\beta_i)-1}}) \\
&\quad \times \prod_{j=1}^{t}(1+\frac{1}{q_j^k}+\cdots+\frac{1}{q_j^{k(r_j+1)-1}}) \\
&\geq \phi^k(n) \cdot \prod_{p|n}\left(1+\frac{1}{p^k}\right).
\end{aligned}$$

This completes the proof of Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of Theorem 1. Here we will debate this problem in two cases:

(i) If n is a prime, then $d(n) = 2$. So from Lemma 1 we have

$$P_d(n) = n^{\frac{d(n)}{2}} = n. \tag{5}$$

Noting that $\phi(n) = n-1$, then from (5) we immediately get

$$\sigma_k(\phi(P_d(n))) = \sigma_k(n-1) = \sum_{d|(n-1)} d^k \geq (n-1)^k \geq \frac{1}{2^k} \cdot n^k = \frac{1}{2^k} P_d^k(n).$$

(ii) If $n = p^\alpha$, p be a prime and $\alpha > 1$ be any positive integer. Then $d(n) = \alpha+1$. So that

$$P_d(n) = n^{\frac{d(n)}{2}} = p^{\frac{\alpha(\alpha+1)}{2}}. \tag{6}$$

Using Lemma 2 and (6), we can easily deduce that

$$\sigma_k(\phi(P_d(n))) = \sigma_k(\phi(p^{\frac{\alpha(\alpha+1)}{2}}))$$

$$\begin{aligned}
&\geq \phi^k(p^{\frac{\alpha(\alpha+1)}{2}}) \prod_{p_1|p^{\frac{\alpha(\alpha+1)}{2}}} \left(1+\frac{1}{p_1^k}\right) \\
&= p^{\frac{k\alpha(\alpha+1)}{2}} \cdot \left(1-\frac{1}{p}\right)^k \cdot \left(1+\frac{1}{p^k}\right) \\
&\geq p^{\frac{k\alpha(\alpha+1)}{2}} \cdot \left(1-\frac{1}{p}\right)^k \\
&\geq p^{\frac{k\alpha(\alpha+1)}{2}} \cdot \frac{1}{2^k} = \frac{1}{2^k} P_d^k(n).
\end{aligned}$$

This completes the proof of Theorem 1.

Similarly, we can easily prove Theorem 2. That is,

(i) If n is a prime, then $d(n) = 2$. So from Lemma 1 we have

$$q_d(n) = n^{\frac{d(n)}{2}-1} = 1, \tag{7}$$

hence

$$\sigma_k(\phi(q_d(n))) = \sigma_k(1) = 1 \geq \frac{1}{2^k} q_d^k(n).$$

(ii) If $n = p^\alpha$, p be a prime and $\alpha > 1$ be any positive integer. Then $d(n) = \alpha + 1$, so that

$$q_d(n) = n^{\frac{d(n)}{2}-1} = p^{\frac{\alpha(\alpha-1)}{2}}. \tag{8}$$

Using Lemma 2 and (8), we have

$$\sigma_k(\phi(q_d(n))) = \sigma_k(\phi(p^{\frac{\alpha(\alpha-1)}{2}})) \geq \frac{1}{2^k} q_d^k(n).$$

This completes the proof of Theorem 2.

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THE SCIENCE OF LUCKY SCIENCES

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Abstract A lucky science as defined by Smarandache is whereby the correct result to a mathematical equation is achieved by erroneous methods ([1]). For example, if asked 10+10, we might say 20, which happens to be correct in all positive integer bases except for base 2, in which case the answer should have been 100. As we probably did the sum in base 10, we have been lucky (however we would have been unlucky in base 2). This paper questions under which circumstances we may be lucky or unlucky.

§1. Introduction

A few more examples of lucky science in action.

In Smarandache Notions Journal 14 ([2]), an example is given of a lucky differentiation. If $g(x) = x^x$, then $g'(x) = nx^{n-1}$, so $g'(x) = n^n$. Then $f(x) = e^x$ is of this form, so $f'(e) = e^e$, which happens to be correct, but the method used is only valid for this example with x=e, i.e. given the vast majority of functions, this method fails to produce the correct example.

Another example given in [2] is $16/64 = 1/4$ - simply cancel the 6's.

But this does not work for $26/76$, or practically anything else.

Let a numerator is given by $n_1 \cdots n_a$, and a denominator by $d_1 \cdots d_b$, and $A = \{1, \cdots, a\}$ and $B = \{1, \cdots, b\}$, and A_6 is a subset of A such that a_i is 6, and similarly for B_6.

Then when does the cancelling of the sixes work? More generally when does cancelling of any given integer/integer set work? And even more generally, when does any erroneous method work?

§2. Smarandache Function

The Smarandache function $S(k)$ is defined as the lowest value such that k divides $S(k)!$ ([3]).

If we glibly say $S(k) = k$ for all k, this is our lucky method. We might even have a 'proof' of it! And we check that $S(1) = 1, S(2) = 2, S(3) =$

$3, S(4) = 4$ and $S(5) = 5$. So we can assume our method is good, and declare it to a bewildered professor who says 'but $S(6) = 3$'.

What went wrong? Our lucky method failed to be a truthful interpretation of the question, and hence it failed. However if in testing our hypothesis we considered only primes (every integer is a unique factorization of the primes after all), we would be correct.

So we can define a few terms;

Let E be a mathematical problem.

Let L be a lucky method on E, and let C be a correct method on E

Let $L(x)$ be the set of x such that $L(x)$ equals $C(x)$, i.e. the set of x for which the lucky method produces the correct result, and $L'(x)$ to be the set of x such that $L(x)$ does not equal $C(x)$, i.e. the set of x for which the lucky method fails. In the example in this section, E is the Smarandache numbers, and $L(x) = \{1, 4, primes\}$.

Examples revisited

§3. Differentiation

The derivation given in the introduction hardly ever works. If we consider $g'(x) = nx^{n-1}$, then we have differentiated with respect to x. x here is a real variable, and due to the normal criteria of continuity, g' is an accepted result.

$f(x) = e^x$ is a different function to $g(x)$, and this is the first step we make in determining L. x is still a real variable, but now it is an exponent, and so has been transformed, and hence behaves differently. Also, e is not just a number, it is a function;

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

From here we see that $d/dx\{xk/k!\} = kx^{k-1}/k! = (k-1)x^{k-1}$, and so $f'(x) = f(x) = e^e$.

Now $f(e) = e^e$, and $g(e) = e^e$, and $f'(e) = g'(e) = e^e$.

So L contains $\{e\}$.

But look at the region around e, i.e. between $e-d$ and $e+d$ for some (small) d.

$$g'(e+d) = (e+d)(e+d)^{e+d-1} = (e+d)^{e+d}$$

.

However;

$$f'(e+d) = e^{e+d}$$

which is greater than $g'(e+d)$.

Similarly $f'(e-d)$ is always greater than $g'(e-d)$.

So when is $f'(x)$ equal to $g'(x)$?

Answer : When $e^x = nx^{n-1}$, i.e. $x = \{e\}$.

So if E is determine the differential of e^x, C is f, L is g, and $L(x) = \{e\}$.

§4. Bases

The question raised in the abstract of this paper is interesting - in which bases does a simple addition sum remain valid.

$20 + 20 = 40$ is true in base 5 and above, $27 + 31 = 60$ is valid is base 8 only.

Let us define E as to determine whether a sum $S = S_1 + S_2 = z$ is valid in a base. C is then the usual definition of addition. $L(S = z)$ is the set of bases such that the sum $S = z$ is valid.

Given any sum, base k addition is always valid - the 27 above in base 7 is another way of writing 30_7 (but in base 7 the correct answer is 61).

We construct a table of $27 + 31$ in the bases 2 to 10:

Base	*Answer*
2	1110
3	212
4	130
5	113
6	102
7	61
8	60
9	59
10	59
11	59

So $L(27 + 31 = 59) = \{9, 10, 11, 12, \cdots\}$

And $L(27 + 31 = z)$ for z greater than 59 is defined according to the table above (e.g. $L(27 + 31 = 60) = \{8\}$).

We can easily say that if S is valid in base k and base $k + 1$, then it is valid for all further bases, as in this case all the problematic carries have been absorbed by the base.

If we let K be the lowest base that $S = z$ is true and has no carries, we can define $L(S = z)$ as $\{K, K + 1, K + 2, \cdots\}$.

Let $Z = z$ in this case, and so furthermore, for each z greater than z, $L(s = z)$ is either empty or a single point.

§5. Fractions

This is the hardest problem yet to analyze.

Let E be the problem of reducing a fraction to it's simplest form. Then C is the problem of factoring the numerator and denominator, and removing common prime factors.

L_D is defined as cancelling a set of digits D from both the numerator and denominator, and $L_D(r)$ is the set such that the rational r is produced both by C and by L_D.

Let's see if we can construct such a number. Let's start with the obvious $1/2$.

We require the numerator to be twice that of the denominator. Trivially, let $D = \{3, 6\}$, then $1[3]/2[6]$ cancels to $1/2$, e.g. $13/26$, $1333/2666$ or $331/662$.

But for the case of D a single integer this is impossible.

Proof. Let n be the numerator of any such fraction. Then we may generalize n as;

$$n = \sum_{\substack{i=0 \\ i \notin J}}^{N} d10^i + \sum_{j \in J} 10^j$$

for some $J-$ this holds the position of the 1's.

$$m = \sum_{\substack{i=0 \\ i \notin K}}^{M} d10^i + 2 \sum_{k \in K} 10^k$$

for some $K-$ this holds the position of the 2's.

We now require $n/m = 1/2$, or $2n = m$.

Note that d must equal 6, but $6 + 6$ produces a carry, and as the next component in the sum is either $1 + 1$ or $6 + 6$, we end up with a 3.

Hence $L_D(1/2)$ is the empty set for $|D| = 1$.

So why does 16/64 work?

Potential fractions for $1/4$ can be expanded as above, but we find a solution quickly.

If $(10 + d)/10d + 4 = 1/4$, then $6d = 36$, so $d = 6$.

Hence $L_6(1/4)$ contains 16/64.

General solutions to these equations is beyond the scope of this paper.

§6. Summary

Any erroneous method may produce correct answers for specific numbers. The science of lucky sciences develops this hit-and-miss scene into a mathematical system.

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SMARANDACHE SEQUENCE OF UNHAPPY NUMBERS

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Abstract The main purpose of this paper is to introduce new concepts of Smarandache numbers, namely Smarandache Sequence of Unhappy Numbers, and give definition, theorem, and ask open problems.

Keywords: Happy Number, Unhappy Number, Smarandache Sequence of Unhappy Numbers, Reversed Smarandache Sequence of Unhappy Numbers.

1.1 Definition. Iterating the process of summing the squares of the decimal digits of a number and if the process terminates in 4, then the original number is called Unhappy Number (UN).

Examples:

1) $2 \to 4$.

2) $3 \to 9 \to 8165 \to 61 \to 35 \to 34 \to 25 \to 29 \to 85 \to 89 \to 145 \to 42 \to 20 \to 4$.

3) $99 \to 162 \to 41 \to 17 \to 50 \to 25 \to 29 \to 95 \to 106 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$. Hence, 2, 3, and 99 are unhappy numbers.

1.3 The sequence of Unhappy Numbers (UN). The proposed sequence of the UN is;

$$UN = 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 15, 16, 17, 18, 20, 21, \cdots.$$

Note that UN is a counterpart of the sequence of Happy Numbers (HN), for more details about HN see [1].

1.4 Theorem. HN UN = N, where HN, the set of Happy Numbers, and UN, the set of Unhappy Numbers, and N, the set of Natural numbers.

That's to say that the natural numbers may be classified to happy or unhappy, there are no other choice.

proof of the theorem. Consider the order subsets HN, and UN of N (i.e. UN N, and HN N). Where

$$UN = 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 15, 16, 17, 18, 20, 21, \cdots,$$

$$HN = 1, 7, 10, 13, 19, 23, 28, 31, 32, 44, 49, 68, 70, 79, 82, 86, 91, 94, \cdots,$$

and

$$N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \cdots.$$

Now UN and HN are well- ordered sets, since 2, 1 are the first elements respectively of UN, and HN.

Now

$$HN \bigcup UN = 1, 7, 10, 13, 19, 23, 28, 31, 32, \cdots; 2, 3, 4, 5, 6, 8, 9, 11, 12, \cdots$$

i.e. the union ordered from left to right, and it is well-ordered.

Thus, HN $\bigcup$ UN = N (since N is well-ordered set).

1.5 Interesting Note: Iterating the process of summing the squares of the decimal digits of a number in both sequence UN, and HN then the process terminates in 4, and in 1 respectively, i.e. 4 and 1 are two squares, so $4 - 1 = 3, 4 + 1 = 5$, and $3^2 + 4^2 = 5^2$, hence the first Pythagorean triples may have relationship with happy and unhappy numbers.

1.6 Smarandache Unhappy Sequence (SUS). SUS is the sequence formed from concatenation of numbers in UN sequence, i.e. $SUS = \{2, 23, 234, 2345, 23456, 234568, 2345689 \cdots\}$. Problems:

1)$2, 23$ are prime numbers; how many terms of SUS are primes?

2)23 is a happy number; how many terms of SUS are happy numbers?

3)234 is an unhappy number; how many terms of SUS are unhappy numbers?

4)$2, 3, 4, 5, 6$ are consecutive unhappy numbers; how many consecutive terms of SUS are unhappy numbers?

1.7 Reversed Smarandache Unhappy Sequence (RSUS).

$$RSUS = 2, 32, 432, 5432, 65432, 865432, 9865432, \cdots.$$

It is obvious that there are no such prime numbers (excluding 2) Problems:

1)32 is a happy number; how many terms of RSUS are happy numbers?

2)432 is an unhappy number; how many terms of RSUS are unhappy numbers?

Reference

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ON M-TH POWER FREE PART OF AN INTEGER

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Abstract In this paper, we using the elementary method to study the convergent property of one class Dirichlet series involving a special sequences, and give an interesting identity for it.

Keywords: m-th power free part; Infinity series; Identity.

§1. Introduction and results

For any positive integer n and $m \geq 2$, we define $C_m(n)$ as the m-th power free part of n. That is,

$$C_m(n) = \min\{n/d^m : d^m \mid n, \quad d \in N\}.$$

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ is the prime powers decomposition of n, then we have: $C_m(n_1^m n_2) = C_m(n_2)$, and

$$C_m(n) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}, \quad if \quad \alpha_i \leq m-1.$$

Now for any positive integer k, we also define arithmetic function $\delta_k(n)$ as follows:

$$\delta_k(n) = \begin{cases} \max\{d \in N \quad | \quad d|n, (d,k) = 1\}, & if \quad n \neq 0, \\ 0, & if \quad n = 0. \end{cases}$$

Let $\mathcal{A}$ denotes the set of all positive integers n satisfy the equation $C_m(n) = \delta_k(n)$. That is, $\mathcal{A} = \{n \in N, C_m(n) = \delta_k(n)\}$. In this paper, we using the elementary method to study the convergent property of the Dirichlet series involving the set $\mathcal{A}$, and give an interesting identity for it. That is, we shall prove the following conclusion:

Theorem. *Let $m \geq 2$ be a fixed positive integer. Then for any real number $s > 1$, we have the identity*

$$\sum_{\substack{n=1 \\ n \in \mathcal{A}}}^{\infty} \frac{1}{n^s} = \frac{\zeta(s)}{\zeta(ms)} \prod_{p|k} \frac{1 - \frac{1}{p^s}}{(1 - \frac{1}{p^{ms}})^2},$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_p$ denotes the product over all primes..

Note that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and $\zeta(6) = \pi^6/945$, from our Theorem we may immediately deduce the following:

Corollary. *Let $\mathcal{B} = \{n \in N, C_2(n) = \delta_k(n)\}$ and $\mathcal{C} = \{n \in N, C_3(n) = \delta_k(n)\}$, then we have the identities:*

$$\sum_{\substack{n=1\\ n\in\mathcal{B}}}^{\infty} \frac{1}{n^2} = \frac{15}{\pi^2} \prod_{p|k} \frac{p^6}{(p^2+1)(p^4-1)}$$

and

$$\sum_{\substack{n=1\\ n\in\mathcal{C}}}^{\infty} \frac{1}{n^2} = \frac{305}{2\pi^4} \prod_{p|k} \frac{p^{10}}{(p^4+p^2+1)(p^6-1)}.$$

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. First, we define the arithmetical function $a(n)$ as follows:

$$a(n) = \begin{cases} 1, & if \quad n \in \mathcal{A}, \\ 0, & otherwise. \end{cases}$$

For any real number $s > 0$, it is clear that

$$\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} < \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent if $s > 1$, thus $\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s}$ is also convergent if $s > 1$. Now we find the set $\mathcal{A}$. From the definition of $C_m(n)$ and $\delta_k(n)$ we know that $C_m(n)$ and $\delta_k(n)$ both are multiplicative functions. So in order to find all solutions of the equation $C_m(n) = \delta_k(n)$, we only discuss the case $n = p^\alpha$. If $n = p^\alpha$, $(p, k) = 1$, then the equation $C_m(p^\alpha) = \delta(p^\alpha)$ has solution if and only if $1 \le \alpha \le m-1$. If $n = p^\alpha$, $p \mid k$, then the equation $C_m(p^\alpha) = \delta(p^\alpha)$ have solutions if and only if $m \mid \alpha$. Thus, by the Euler product formula (see [1]), we have

$$\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s} = \prod_p \left(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \cdots + \frac{a(p^{m-1})}{p^{(m-1)s}} + \cdots\right)$$

$$
\begin{aligned}
&= \prod_{p\dagger k}\left(1+\frac{a(p)}{p^s}+\frac{a(p^2)}{p^{2s}}+\cdots+\frac{a(p^{m-1})}{p^{(m-1)s}}\right)\\
&\quad\times\prod_{p|k}\left(1+\frac{a(p)}{p^{ms}}+\frac{a(p^2)}{p^{2ms}}+\frac{a(p^3)}{p^{3ms}}+\cdots\right)\\
&= \prod_{p\dagger k}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots+\frac{1}{p^{(m-1)s}}\right)\\
&\quad\times\prod_{p|k}\left(1+\frac{1}{p^{ms}}+\frac{1}{p^{2ms}}+\frac{1}{p^{3ms}}+\cdots\right)\\
&= \frac{\zeta(s)}{\zeta(ms)}\prod_{p|k}\frac{1-\frac{1}{p^s}}{(1-\frac{1}{p^{ms}})^2},
\end{aligned}
$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_p$ denotes the product over all primes.

This completes the proof of Theorem.

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ON TWO NEW ARITHMETIC FUNCTIONS AND THE K-POWER COMPLEMENT NUMBER SEQUENCES *

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Abstract The main purpose of this paper is to study the asymptotic property of the k-power complement numbers (where $k \geq 2$ is a fixed integer), and obtain some interesting asymptotic formulas.

Keywords: k-power complement number; Asymptotic formula; Arithmetic function.

§1. Introduction

Let $k \geq 2$ is a fixed integer, for each integer n, let $C(n)$ denotes the smallest integer such that $n \times C(n)$ is a perfect k-power, $C(n)$ is called k-power complement number of n. In problem 29 of reference [1], Professor F. Smarandache asked us to study the properties of the k-power complement number sequences. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, we define two arithmetic function $D(n)$ and $I(n)$ similar to the derivative and integral function in mathematical analysis as follows:

$$D(n) = D\left(p_1^{\alpha_1}\right) D\left(p_2^{\alpha_2}\right) \cdots D\left(p_s^{\alpha_s}\right), \qquad D\left(p^{\alpha}\right) = \alpha p^{\alpha-1}$$

and

$$I(n) = I\left(p_1^{\alpha_1}\right) I\left(p_2^{\alpha_2}\right) \cdots I\left(p_s^{\alpha_s}\right), \qquad I\left(p^{\alpha}\right) = \frac{1}{\alpha+1} p^{\alpha+1}.$$

In this paper, we use the analytic method to study the asymptotic properties of the functions $D(n)$ and $I(n)$ for the k-power complement number sequences, and obtain some interesting asymptotic formulas. That is, we shall prove the following conclusions:

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Theorem 1. For any real number $x \geq 1$, we have the asymptotic formula

$$\begin{aligned}
&\sum_{n\leq x}\frac{1}{D(C(n))}\\
=\ &\frac{6(k-1)\zeta\left(\frac{k}{k-1}\right)\cdot x^{\frac{1}{k-1}}}{\pi^2}\prod_p\left(1+\frac{p}{p+1}\sum_{i=1}^{k-2}\frac{1}{(k-i)p^{k-1-i}\cdot p^{\frac{i}{k-1}}}\right)\\
&+O\left(x^{\frac{2k-1}{2k(k-1)}+\varepsilon}\right),
\end{aligned}$$

where ε denotes any fixed positive number.

Theorem 2. For any real number $x \geq 1$, we have the asymptotic formula

$$\begin{aligned}
&\sum_{n\leq x}I(C(n))d(C(n))\\
=\ &\frac{6\zeta(k(k+1))\cdot x^{k+1}}{(k+1)\pi^2}\prod_p\left(1+\frac{p}{p+1}\left(\sum_{i=2}^{k}\frac{p^{k+1-i}}{p^{(k+1)i}}-\frac{1}{p^{k(k+1)}}\right)\right)\\
&+O\left(x^{k+\frac{1}{2}+\varepsilon}\right),
\end{aligned}$$

where $d(n)=\sum_{d|n}1$ is the divisor function.

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems. Let

$$f(s)=\sum_{n=1}^{\infty}\frac{1}{D(C(n))n^s}.$$

Because $D(n)$ and $C(n)$ are all multiplicative function, so from the Euler product formula [2] and the definition of $D(n)$ and $C(n)$ we have

$$\begin{aligned}
&f(s)\\
=\ &\prod_p\left(1+\frac{1}{D(C(p))p^s}+\frac{1}{D(C(p^2))p^{2s}}+\cdots\right)\\
=\ &\prod_p\left(1+\left(\sum_{i=1}^{k-1}\frac{1}{(k-i)p^{k-1-i}\cdot p^{is}}+\frac{1}{p^{ks}}\right)\left(1+\frac{1}{p^{ks}}+\frac{1}{p^{2ks}}+\cdots\right)\right)\\
=\ &\zeta(ks)\prod_p\left(1+\frac{1}{p^{(k-1)s}}+\sum_{i=1}^{k-2}\frac{1}{(k-i)p^{k-1-i}\cdot p^{is}}\right)\\
=\ &\frac{\zeta(ks)\zeta((k-1)s)}{\zeta((2k-2)s)}\prod_p\left(1+\frac{p^{(k-1)s}}{p^{(k-1)s}+1}\sum_{i=1}^{k-2}\frac{1}{(k-i)p^{k-1-i}\cdot p^{is}}\right),
\end{aligned}$$

where $\zeta(s)$ is Riemann zeta function. Obviously, we have the inequality

$$|\frac{1}{D(C(n))}| \leq 1, \qquad \left|\sum_{n=1}^{\infty} \frac{1}{D(C(n))n^{\sigma}}\right| < \frac{1}{\sigma - \frac{1}{k-1}},$$

where $\sigma > \frac{1}{k-1}$ is the real part of s. So by Perron formula [3]

$$\begin{aligned} & \sum_{n\leq x} \frac{a(n)}{n^{s_0}} \\ = \; & \frac{1}{2i\pi}\int_{b-iT}^{b+iT} f(s+s_0)\frac{x^s}{s}ds + O\left(\frac{x^b B(b+\sigma_0)}{T}\right) \\ & + O\left(x^{1-\sigma_0}H(2x)\min(1,\frac{\log x}{T})\right) + O\left(x^{-\sigma_0}H(N)\min(1,\frac{x}{||x||})\right), \end{aligned}$$

where N is the nearest integer to x, $\|x\| = |x - N|$. Taking $s_0 = 0$, $b = 1 + \frac{1}{k-1}$, $T = x^{1+\frac{1}{2k(k-1)}}$, $H(x) = 1$, $B(\sigma) = \frac{1}{\sigma - \frac{1}{k-1}}$, we have

$$\begin{aligned} & \sum_{n\leq x} \frac{1}{D(C(n))} \\ = \; & \frac{1}{2i\pi}\int_{1+\frac{1}{k-1}-iT}^{1+\frac{1}{k-1}+iT} \frac{\zeta(ks)\zeta((k-1)s)}{\zeta((2k-2)s)}R(s)\frac{x^s}{s}ds \\ & + O\left(x^{\frac{2k-1}{2k(k-1)}+\varepsilon}\right), \end{aligned}$$

where

$$R(s) = \prod_p \left(1 + \frac{p^{(k-1)s}}{p^{(k-1)s}+1}\sum_{i=1}^{k-2}\frac{1}{(k-i)p^{k-1-i}\cdot p^{is}}\right).$$

To calculate the main term

$$\frac{1}{2i\pi}\int_{1+\frac{1}{k-1}-iT}^{1+\frac{1}{k-1}+iT} \frac{\zeta(ks)\zeta((k-1)s)x^s}{\zeta((2k-2)s)s}R(s)ds,$$

we move the integral line from $s = 1 + \frac{1}{k-1} \pm iT$ to $s = \frac{1}{k} + \frac{1}{2k(k-1)} \pm iT$. This time, the function

$$f_1(s) = \frac{\zeta(ks)\zeta((k-1)s)}{\zeta((2k-2)s)}R(s)\frac{x^s}{s}$$

have a simple pole point at $s = \frac{1}{k-1}$ with residue $\frac{(k-1)\zeta(\frac{k}{k-1})\cdot x^{\frac{1}{k-1}}}{\zeta(2)}R(\frac{1}{k-1})$. So we have

$$\frac{1}{2i\pi}\left(\int_{1+\frac{1}{k-1}-iT}^{1+\frac{1}{k-1}+iT} + \int_{1+\frac{1}{k-1}+iT}^{\frac{1}{k}+\frac{1}{2k(k-1)}+iT} + \int_{\frac{1}{k}+\frac{1}{2k(k-1)}+iT}^{\frac{1}{k}+\frac{1}{2k(k-1)}-iT} + \int_{\frac{1}{k}+\frac{1}{2k(k-1)}-iT}^{1+\frac{1}{k-1}-iT}\right)$$

$$\frac{\zeta(ks)\zeta((k-1)s)x^s}{\zeta((2k-2)s)s}R(s)ds$$

$$= \frac{(k-1)\zeta\left(\frac{k}{k-1}\right)\cdot x^{\frac{1}{k-1}}}{\zeta(2)}\prod_p\left(1+\frac{p}{p+1}\sum_{i=1}^{k-2}\frac{1}{(k-i)p^{k-1-i}\cdot p^{\frac{i}{k-1}}}\right).$$

We can easy get the estimate

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\left(\int_{1+\frac{1}{k-1}+iT}^{\frac{1}{k}+\frac{1}{2k(k-1)}+iT}+\int_{\frac{1}{k}+\frac{1}{2k(k-1)}-iT}^{1+\frac{1}{k-1}-iT}\right)\frac{\zeta(ks)\zeta((k-1)s)x^s}{\zeta((2k-2)s)s}R(s)ds\right| \\
\ll &\int_{\frac{1}{k}+\frac{1}{2k(k-1)}}^{1+\frac{1}{k-1}}\left|\frac{\zeta(k(\sigma+iT))\zeta((k-1)(\sigma+iT))}{\zeta((2k-2)(\sigma+iT))}R(s)\frac{x^{1+\frac{1}{k-1}}}{T}\right|d\sigma \\
\ll &\frac{x^{1+\frac{1}{k-1}}}{T}=x^{\frac{1}{k}+\frac{1}{2k(k-1)}}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{\frac{1}{k}+\frac{1}{2k(k-1)}+iT}^{\frac{1}{k}+\frac{1}{2k(k-1)}-iT}\frac{\zeta(ks)\zeta((k-1)s)x^s}{\zeta((2k-2)s)s}R(s)ds\right| \\
\ll &\int_0^T\left|\frac{\zeta(1+\frac{1}{2(k-1)}+ikt)\zeta(\frac{2k-1}{2k}+i(k-1)t)}{\zeta(\frac{2k-1}{k}+i(2k-2)t)}\frac{x^{\frac{1}{k}+\frac{1}{2k(k-1)}}}{t}\right|dt \\
\ll &x^{\frac{1}{k}+\frac{1}{2k(k-1)}+\varepsilon}.
\end{aligned}$$

Note that $\zeta(2)=\frac{\pi^2}{6}$, we have

$$\begin{aligned}
\sum_{n\le x}\frac{1}{D(C(n))} &= \frac{6(k-1)\zeta\left(\frac{k}{k-1}\right)\cdot x^{\frac{1}{k-1}}}{\pi^2}\prod_p\left(1+\frac{p}{p+1}\sum_{i=1}^{k-2}\frac{1}{(k-i)p^{k-1-i}\cdot p^{\frac{i}{k-1}}}\right) \\
&\quad +O\left(x^{\frac{2k-1}{2k(k-1)}+\varepsilon}\right).
\end{aligned}$$

This completes the proof of Theorem 1.

Let

$$g(s)=\sum_{n=1}^{\infty}I(C(n))d(C(n))$$

from the definition of $I(n)$ and $C(n)$, we can also have

$$\begin{aligned}
g(s) &= \prod_p\left(1+\frac{I(C(p))d(C(p))}{p^s}+\frac{I(C(p^2))d(C(p^2))}{p^{2s}}+\cdots\right) \\
&= \prod_p\left(1+\left(\frac{p^k}{p^s}+\frac{p^{k-1}}{p^{2s}}+\cdots+\frac{p}{p^{ks}}\right)\left(1+\frac{1}{p^{ks}}+\frac{1}{p^{2ks}}+\cdots\right)\right)
\end{aligned}$$

$$= \zeta(ks)\prod_p\left(1-\frac{1}{p^{ks}}+\frac{p^k}{p^s}+\frac{p^{k-1}}{p^{2s}}+\cdots+\frac{p}{p^{ks}}\right)$$

$$= \frac{\zeta(ks)\zeta(s-k)}{\zeta(2s-2k)}\prod_p\left(1+\frac{p^{s-k}}{p^{s-k}+1}\left(\sum_{i=2}^{k}\frac{p^{k+1-i}}{p^{is}}-\frac{1}{p^{ks}}\right)\right).$$

Now by Perron formula [3] and the method of proving Theorem 1, we can also obtain Theorem 2.

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SMARANDACHE REPLICATING DIGITAL FUNCTION NUMBERS

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Abstract In 1987, Mike Keith introduced "repfigits" (replicating Fibonacci-like digits) [1]. In this paper two generalizations of repfigits are presented in which Smarandache-type functions are applied to the digits of n. Some conjectures and unsolved questions are then proposed.

Repfigits (replicating Fibonacci-like digits) are positive integers N such that in a sequence generated with the $n-$digits of N, and then continuing the sequence by summing the previous n terms, N eventually appears. For example, 3684 is a repfigit since it occurs in the sequence

$$3, 6, 8, 4, 21, 39, 72, 136, 268, 515, 991, 1910, 3684, \cdots$$

One generalization of repfigits is revrepfigits [2], in which the reversal of N occurs in a sequence generated in the same manner as given in the definition of repfigits. For example, 8166 is a revrepfigit since the sequence

$$8, 1, 6, 6, 21, 34, 67, 128, 250, 479, 924, 1781, 3434, 6618, \cdots$$

contains the reversal of 8166.

In this paper, two other generalizations of repfigits are made. These do away with the aesthetic aspect of the original repfigits, since we will not be concerned with all of the digits of a number to begin our sequences, only with three functions that operate on the base-10 representations of numbers.

SRDS Numbers. First, we will define some functions. Let $sd(n)$ denote the smallest digit of n, $ld(n)$ denote the largest digit of n, and $digsum(n)$ denote the digital sum of n, respectively. Examples: $sd(12345) = 1$ since 1 is the smallest digit of 12345, $ld(12345) = 5$ since 5 is the largest digit of 12345, and $digsum(12345) = 15$ since $1 + 2 + 3 + 4 + 5 = 15$. The digital sum function was mentioned in [3] and many papers in SNJ have dealt with it.

Definition: A Smarandache replicating digital sum number $SRDS$ is a number $N > 9$ such that when a sequence is formed by the recursion

$$SRDS(n) = SRDS(n-1) + SRDS(n-2) + SRDS(n-3),$$

where $SRDS(1) = sd(n), SRDS(2) = ld(n)$,and $SRDS(3) = digsum(n)$, then N occurs somewhere in the sequence.

For example, 8464 is a $SRDS$ number because it appears in the sequence

$$4, 8, 22, 34, 64, 120, 218, 402, 740, 1360, 2502, 4602, 8464, \cdots$$

Notice that the first term is the smallest digit of 8464, the second term is the largest digit of 8464, and the third term is the digital sum of 8464.

A computer program was written to search for $SRDS$ numbers, and the following were found.

18, 37, 53, 142, 284, 583, 4232, 4477, 5135, 7662, 8464, 9367, 15169, 22500, 24192,
28553, 40707, 46245, 49611, 59841, 199305, 213977, 228649, 232072, 302925, 398406,
771809, 1127617, 2280951, 2875059, 3174997, 7082341, 10217260, 14137273,...

Conjecture: There are infinitely many $SRDS$ numbers.

Unsolved questions: What is the level of algorithmic complexity for finding $SRDS$ numbers when using a brute-force method? What is the most efficient way to find these numbers? Are there infinitely many prime $SRDS$ numbers? Are $SRDS$ numbers more plentiful than repfigits?

SRDP Numbers.

Our second generalization is very similar to the $SRDS$ numbers. The only difference is that we will use the function $digprod(n)$ for the third term instead of $digsum(n)$, where $digprod(n)$ denotes the product of the nonzero digits of n. Example, $digprod(7605) = 210$ because $7 \times 6 \times 5 = 210$.

Definition: A Smarandache replicating digital product number (SRDP) is a number $N > 9$ such that when a sequence is formed by the recursion

$$SRDP(n) = SRDP(n-1) + SRDP(n-2) + SRDP(n-3),$$

where $SRDP(1) = sd(n), SRDP(2) = ld(n)$, and $SRDP(3) = digprod(n)$, then N occurs somewhere in the sequence. For example, 1941 is a $SRDP$ number because it appears in the sequence

$$1, 9, 36, 46, 91, 173, 310, 574, 1057, 1941, \cdots$$

Notice that the first term is the smallest digit of 1941, the second term is the largest digit of 1941, and the third term is the digital product of 1941.

A computer program was written to search for $SRDP$ numbers, and the following sequence was found.

13, 19, 29, 39, 44, 49, 54, 59, 64, 69, 74, 79, 84, 89, 94, 99, 284, 996, 1908, 1941,
2588, 3374, 3489, 10856, 34088, 39756, 125519, 140490, 240424, 244035, 317422,
420742, 442204, 777994, 1759032,...

Conjecture: There are infinitely many $SRDP$ numbers.

Unsolved questions: What is the level of algorithmic complexity for finding $SRDP$ numbers using a brute-force method? What is the most efficient way to find these numbers? Are there infinitely many prime $SRDP$ numbers? Are there more $SRDP$ numbers than $SRDS$ numbers?

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ON THE M-POWER RESIDUES NUMBERS SEQUENCE

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Abstract The main purpose of this paper is to study the distribution properties of m-power residues numbers, and give two interesting asymptotic formulae.

Keywords: m-power residues numbers; Mean value; Asymptotic formula.

§1. Introduction and results

For any given natural number $m \geq 2$, and any positive integer $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, we call $a_m(n) = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot \cdots \cdot p_r^{\beta^r}$ a m-power residue number, where $\beta_i = \min(m-1, \alpha_i), 1 \leq i \leq r$. In reference [1], Professor F. Smarandache asked us to study the properties of the m-power residue numbers sequence. Yet we still know very little about it.

Now we define two new number-theoretic functions $U(n)$ and $V(n)$ as following,

$$U(1) = 1, U(n) = \prod_{p|n} p,$$

$$V(1) = 1, V(n) = V\left(p_1^{\alpha_1}\right) \cdots U\left(p_r^{\alpha_r}\right) = (p^{\alpha_1} - 1) \cdots (p^{\alpha_r} - 1),$$

where n is any natural number with the form $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Obviously they are both multiplicative functions. In this paper, we shall use the analytic method to study the distribution properties of this sequence, and obtain two interesting asymptotic formulae. That is, we have the following two theorems:

Theorem 1. *Let $\mathcal{A}$ denotes the set of all m-power residues numbers, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}} U(n) = \frac{3x^2}{\pi^2}\prod_p\left(1+\frac{1}{p^3+p^2-p-1}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

where ε denotes any fixed positive number.

Theorem 2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}} V(n) = \frac{x^2}{2}\prod_p\left(1-\frac{1}{p^m}+\frac{1-p^m}{p^{m+2}+p^{m+1}}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1, let

$$f(s) = \sum_{\substack{n\in\mathcal{A}\\ n\leq x}} \frac{U(n)}{n^s}.$$

From the Euler product formula [2] and the definition of $U(n)$ we have

$$\begin{aligned} f(s) &= \prod_p\left(1+\frac{U(p)}{p^s}+\frac{U(p^2)}{p^{2s}}+\cdots+\frac{U(p^{m-1})}{p^{(m-1)s}}+\frac{U(p^m)}{p^{ms}}+\frac{U(p^{m+1})}{p^{(m+1)s}}\cdots\right)\\ &= \prod_p\left(1+\frac{1}{p^{s-1}}+\frac{1}{p^{2s-1}}+\cdots+\frac{1}{p^{(m-1)s-1}}+\frac{1}{p^{ms-1}}+\frac{1}{p^{(m+1)s-1}}+\cdots\right)\\ &= \prod_p\left(1+\frac{1}{p^{s-1}}+\frac{1}{p^{2s-1}(1-\frac{1}{p^s})}\right)\\ &= \frac{\zeta(s-1)}{\zeta(2(s-1))}\prod_p\left(1+\frac{p^s}{(p^s-1)(p^{2s-1}+p^s)}\right), \end{aligned}$$

where $\zeta(s)$ is the Riemann-zeta function. Obviously, we have inequality

$$|U(n)| \leq n, \qquad \left|\sum_{n=1}^{\infty}\frac{U(n)}{n^\sigma}\right| < \frac{1}{\sigma-2},$$

where $\sigma > 2$ is the real part of s. So by Perron formula [3]

$$\begin{aligned} \sum_{n\leq x}\frac{U(n)}{n^{s_0}} &= \frac{1}{2i\pi}\int_{b-iT}^{b+iT} f(s+s_0)\frac{x^s}{s}ds + O\left(\frac{x^bB(b+\sigma_0)}{T}\right)\\ &\quad +O\left(x^{1-\sigma_0}H(2x)\min(1,\frac{\log x}{T})\right)+O\left(x^{-\sigma_0}H(N)\min(1,\frac{x}{||x||})\right), \end{aligned}$$

where N is the nearest integer to x, $\|x\| = |x - N|$. Taking $s_0 = 0$, $b = 3$, $T = x^{\frac{3}{2}}$, $H(x) = x$, $B(\sigma) = \frac{1}{\sigma-2}$, we have

$$\sum_{n\leq x} U(n) = \frac{1}{2i\pi}\int_{3-iT}^{3+iT} \frac{\zeta(s-1)}{\zeta(2(s-1))} R(s)\frac{x^s}{s}ds + O(x^{\frac{3}{2}+\varepsilon}),$$

where

$$R(s) = \prod_p \left(1 + \frac{1}{p^3+p^2-p-1}\right).$$

To estimate the main term

$$\frac{1}{2i\pi}\int_{3-iT}^{3+iT} \frac{\zeta(s-1)}{\zeta(2(s-1))} R(s)\frac{x^s}{s}ds,$$

we move the integral line from $s = 3 \pm iT$ to $s = \frac{3}{2} \pm iT$. This time, the function

$$f(s) = \frac{\zeta(s-1)x^s}{\zeta(2(s-1))s} R(s)$$

has a simple pole point at $s = 2$ with residue $\frac{x^2}{2\zeta(2)}R(2)$. So we have

$$\begin{aligned}&\frac{1}{2i\pi}\left(\int_{3-iT}^{3+iT} + \int_{3+iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT} + \int_{\frac{3}{2}-iT}^{3-iT}\right) \frac{\zeta(s-1)x^s}{\zeta(2(s-1))s} R(s)ds\\ &= \frac{x^2}{2\zeta(2)}\prod_p \left(1 + \frac{1}{p^3+p^2-p-1}\right).\end{aligned}$$

We can easily get the estimate

$$\begin{aligned}&\left|\frac{1}{2\pi i}\left(\int_{3+iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}-iT}^{3-iT}\right) \frac{\zeta(s-1)x^s}{\zeta(2(s-1))s} R(s)ds\right|\\ &\ll \int_{\frac{3}{2}}^{3} \left|\frac{\zeta(\sigma-1+iT)}{\zeta(2(\sigma-1+iT))} R(s)\frac{x^3}{T}\right| d\sigma \ll \frac{x^3}{T} = x^{\frac{3}{2}}\end{aligned}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT} \frac{\zeta(s-1)x^s}{\zeta(2(s-2))s} R(s)ds\right| \ll \int_0^T \left|\frac{\zeta(1/2+it)}{\zeta(1+2it)}\frac{x^{\frac{3}{2}}}{t}\right| dt \ll x^{\frac{3}{2}+\varepsilon}.$$

Note that $\zeta(2) = \frac{\pi^2}{6}$, from the above we have

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}} U(n) = \frac{3x^2}{\pi^2}\prod_p \left(1 + \frac{1}{p^3+p^2-p-1}\right) + O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. Let

$$g(s) = \sum_{\substack{n \in \mathcal{A} \\ n \leq x}} \frac{V(n)}{n^s}.$$

From the Euler product formula [2] and the definition of $V(n)$, we also have

$$\begin{aligned}
& g(s) \\
= & \prod_p \left(1 + \frac{V(p)}{p^s} + \frac{V(p^2)}{p^{2s}} + \cdots + \frac{V(p^{m-1})}{p^{(m-1)s}} + \frac{V(p^m)}{p^{ms}} + \frac{V(p^{m+1})}{p^{(m+1)s}} + \cdots\right) \\
= & \prod_p \left(1 + \frac{p-1}{p^s} + \frac{p^2-1}{p^{2s}} + \cdots + \frac{p^{m-1}-1}{p^{(m-1)s}} + \frac{p^m-1}{p^{ms}} + \frac{p^{m+1}-1}{p^{(m+1)s}} + \cdots\right) \\
= & \prod_p \left(\frac{1 - \frac{1}{p^{m(s-1)}}}{1 - \frac{1}{p^{s-1}}} - \left(\frac{1 - \frac{1}{p^{s-1}}}{1 - \frac{1}{p^s}}\right)\left(\frac{p^{m-1}}{p^{ms}} - \frac{1}{p^s}\right)\right) \\
= & \zeta(s-1) \prod_p \left(1 - \frac{1}{p^{m(s-1)}} + \frac{(p^{m-1} - p^{(m-1)s})(p^s - p)}{p^{ms}(p^s - 1)}\right).
\end{aligned}$$

By Perron formula [3], and the method of proving Theorem 1, we can also obtain the result of Theorem 2.

This completes the proof of the theorems.

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SMARANDACHE REVERSE POWER SUMMATION NUMBERS

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Abstract A computer program was written and a search through the first $1000SRPS$ numbers yielded several useful results.

Consider the sequence: $1^1 = 1, 1^2 + 2^1 = 3, 1^3 + 2^2 + 3^1 = 8, 1^4 + 2^3 + 3^2 + 4^1 = 22, \cdots$

The formula for these numbers is

$$\sum_{k=1}^{n}(n-k+1)^k$$

which produces the sequence:
$1, 3, 8, 22, 65, 209, 732, 2780, 11377, 49863, 232768, 1151914, 6018785,$
$33087205, 190780212, 1150653920, 7241710929, 47454745803,$
$323154696184, 2282779990494, 16700904488705, 126356632390297,$
$987303454928972, \cdots$

We shall call these values the Smarandache Reverse Power Summation numbers $(SRPS)$, since the symmetry in their definition is reminiscent of other Smarandache classes of numbers, such as the sequences listed in [1], [2], and [3].

The purpose of this note is to define the $SRPS$ sequence, and to make an attempt at determining what types of numbers it contains.

A computer program was written and a search through the first $1000SRPS$ numbers yielded the following results:

Only the trivial square $SRPS(1) = 1$was found. Are there any nontrivial square $SRPS$ numbers? The author conjectures: no.

Two primes, $SRPS(2) = 3$, and

$$SRPS(34) = 4065902334349345653147857 9$$

were found. However, the author conjectures that there are more prime SRPS numbers, but probably not infinitely many.

The trivial triangular numbers $SRPS(1) = 1$ and $SRPS(2) = 3$ were found. Are there any nontrivial triangular $SRPS$ numbers?

When $n = 1, 2, 3, 6, 7, 16, 33$, and 99, $SRPS(n)$ is a Harshad number (a number that is divisible by the sum of its own digits). For example,

$$SRPS(16) = 1150653920$$

has a digital sum of 32, and $1150653920/32 = 35957935$. The author conjectures that there are infinitely many $SRPS$ Harshad numbers.

When $n = 1, 2, 3$, and 4, $SRPS(n)$ is a palindrome. Will there ever be any more palindromic $SRPS$ numbers?

When $n = 4, 5, 6, 9, 12, 13$, and 62, $SRPS(n)$ is a semiprime (a number that is the product of exactly two primes). For example, $SRPS(13) = 6018785 = 5 \times 1203757$. The author conjectures that there are infinitely many semiprime $SRPS$ numbers. (Note that due to the difficulty of factorization, only the first $67 SRPS$ numbers were checked instead of the first 1000.)

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SOME SMARANDACHE IDENTITIES

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Abstract The purpose of this article is to presents 23 Smarandache Identities (SI) (or Facts) with second, three, four, and five degrees. These SI have been obtained by the help of Maple 8(Programming language, see [1]).

§. Introduction

Smarandache values can be obtained by the flowing function: $S(n) = \min\{m \geq 1 : n \mid m!\}$. For example $S(92) = 23$, $S(2115) = 47$, and $S(37) = 37$. For more details visit [2].

$$SI.1 S(92)^2 + S(2115)^2 = S(37)^2 + S(37)^2$$
$$SI.2 S(68)^2 + S(1155)^2 = S(13)^2 + S(13)^2$$
$$SI.3 S(1020)^2 + S(260099)^2 = S(53)^2 + S(53)^2$$
$$SI.4 S(1336)^2 + S(446223)^2 = S(197)^2 + S(197)^2$$
$$SI.5 S(2068)^2 + S(1069155)^2 = S(37)^2 + S(37)^2$$
$$SI.6 S(1324)^2 + S(438243)^2 = S(5)^2 \times S(13)^3 + S(5)^2 \times S(13)^3$$
$$SI.7 S(240)^2 + S(14399)^2 = S(5) \times S(29)^2$$
$$SI.8 S(900)^2 + S(202499)^2 = S(5)^2 \times S(17)^3 + S(5)^2 \times S(17)^3$$
$$SI.9 S(620)^2 + S(96099)^2 = S(13)^2 \times S(17)^3 + S(13)^2 \times S(17)^3$$
$$SI.10 S(52)^2 + S(675)^2 = S(5)^2 + S(5)^2$$
$$SI.11 S(1428)^2 + S(509795)^2 = S(5)^4 + S(5)^4$$
$$SI.12 S(3)^2 + S(4)^2 = S(5)^2$$
$$SI.13 S(12)^5 + S(4)^5 = S(2)^{11}$$
$$SI.14 S(24)^5 + S(8)^5 = S(2)^{11}$$
$$SI.15 S(96)^5 + S(32)^5 = S(2)^{16}$$
$$SI.16 S(192)^5 + S(64)^5 = S(2)^{16}$$
$$SI.17 S(288)^5 + S(96)^5 = S(2)^{16}$$

$$SI.18 S(13440)^5 + S(40320)^5 = S(2)^{16}$$
$$SI.19 S(20480)^5 + S(61440)^5 = S(2)^{21}$$
$$SI.20 S(28672)^5 + S(86016)^5 = S(2)^{21}$$
$$SI.21 S(1)^3 + S(2)^3 + S(3)^3 = S(9)^2$$
$$SI.22 S(5)^3 + S(4)^3 + S(3)^3 = S(9)^3$$
$$SI.23 S(5)^3 + S(4)^3 + S(6)^3 = S(9)^3$$

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ON THE INTEGER PART OF A POSITIVE INTEGER'S K-TH ROOT

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Abstract The main purpose of this paper is using the elementary method and analytic method to study the asymptotic properties of the integer part of the k-th root positive integer, and give two interesting asymptotic formulae.

Keywords: k-th root; Integer part; Asymptotic formula.

§1. Introduction And Results

For any positive integer n, let $s(n)$ denote the integer part of k-th root of n. For example, $s(1) = 1$, $s(2) = 1$, $s(3) = 1$, $s(4) = 1, \cdots$, $s(2^k) = 1$, $s(2^k + 1) = 1, \cdots$, $s(3^k) = 1, \cdots$. In problem 80 of [1], Professor F.Smarandache asked us to study the properties of the sequence $s(n)$. About this problem, it seems that none had studied it, at least we have not seen related paper before. In this paper, we use the elementary method and analytic method to study the asymptotic properties of this sequence, and obtain two interesting asymptotic formulae. That is, we shall prove the following:

Theorem 1. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n \le x} \Omega(s(n)) = x \ln \ln x + (A - \log k)\, x + O\left(\frac{x}{\ln x}\right),$$

where $\Omega(n)$ denotes the total number of prime divisors of n, A is a constant.

Theorem 2. *Let m be a fixed positive integer and $\varphi(n)$ be the Euler totient function, then for any real number $x \ge 1$, we have the asymptotic formula*

$$\sum_{n \le x} \varphi((s(n), m)) = h(m)x + (k+1)h(m) + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

where $(s(n), m)$ denotes the greatest common divisor of $s(n)$ and m, $h(m) = \frac{\varphi(m)}{m} \prod\limits_{p^\alpha \| m} (1 + \alpha - \frac{\alpha - 1}{p})$, and ε is any positive number.

§2. Some Lemmas

To complete the proof of the theorems, we need the following two simple lemmas.

Lemma 1. *For any real number $x > 1$, then we have*

$$\sum_{n\le x}\Omega(n) = x\log\log x + Ax + O\left(\frac{x}{\log x}\right),$$

where $A = \gamma + \sum\limits_{p}(\log(1-\frac{1}{p})+\frac{1}{p}) + \sum\limits_{p}\frac{1}{p(p-1)}$, γ is the Euler constant.

Proof. See reference[2].

Lemma 2. *Let m be a fixed positive integer and $\varphi(n)$ be the Euler totient function, then for any real number $x \ge 1$, we have the asymptotic formula*

$$\sum_{n\le x}\varphi((m,n)) = x\cdot h(m) + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where (m,n) denotes the greatest common divisor of m and n, $h(m) = \frac{\varphi(m)}{m}\prod\limits_{p^\alpha\|m}(1+\alpha-\frac{\alpha-1}{p})$, and ε is any positive number.

Proof. Let

$$F(s) = \sum_{n=1}^{\infty}\frac{\varphi((m,n))}{n^s},$$

then from the Euler Product formula [3] and the multiplicative property of $\varphi(m,n)$, we may get

$$\begin{aligned} F(s) &= \prod_{p}\left(1+\frac{\varphi((m,p))}{p^s}+\frac{\varphi((m,p^2))}{p^{2s}}+\cdots\right)\\ &= \prod_{p^\alpha\|m}\left(1+\frac{\varphi((m,p))}{p^s}+\cdots+\frac{\varphi((m,p^{\alpha-1}))}{p^{(\alpha-1)s}}+\frac{\varphi((m,p^\alpha))}{p^{\alpha s}}(\frac{1}{1-\frac{1}{p^s}})\right)\\ &\quad\times\prod_{p\dagger m}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right)\\ &= \zeta(s)\prod_{p^\alpha\|m}\left(\left(1+\frac{\varphi((m,p))}{p^s}+\cdots+\frac{\varphi((m,p^{\alpha-1}))}{p^{(\alpha-1)s}}\right)\left(1-\frac{1}{p^s}\right)+\frac{\varphi((m,p^\alpha))}{p^{\alpha s}}\right), \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

Obviously, we have inequality

$$|\varphi((m,n))| < K, \qquad \left|\sum_{n=1}^{\infty}\frac{\varphi((m,n))}{n^\sigma}\right| < \frac{K}{\sigma-1},$$

where $\sigma > 1$ is the real part of s. So by Perron formula [4], taking $b = 2, T = x^{\frac{1}{2}}, H(x) = K, B(\sigma) = \frac{K}{\sigma-1}$, then we have

$$\sum_{n\le x}\varphi((m,n)) = \frac{1}{2\pi i}\int_{2-iT}^{2+iT}\zeta(s)R(s)\frac{x^s}{s}ds + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where

$$R(s)=\prod_{p^{\alpha}\|m}\left(\left(1+\frac{\varphi((m,p))}{p^{s}}+\cdots+\frac{\varphi((m,p^{\alpha-1}))}{p^{(\alpha-1)s}}\right)\left(1-\frac{1}{p^{s}}\right)+\frac{\varphi((m,p^{\alpha}))}{p^{\alpha s}}\right).$$

To estimate the main term

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\zeta(s)R(s)\frac{x^{s}}{s}ds$$

we move the integral line from $s=2\pm iT$ to $s=1/2\pm iT$. This time, we have a simple pole point at $s=1$ with residue $R(1)x$. That is

$$\frac{1}{2\pi i}\left(\int_{2-iT}^{2+iT}+\int_{2+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}+\int_{\frac{1}{2}-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^{s}}{s}ds=R(1)x.$$

Taking $T=x^{3/2}$, we can easily get the estimate

$$\begin{aligned}&\left|\frac{1}{2\pi i}\left(\int_{2+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^{s}}{s}ds\right|\\&\ll\int_{\frac{1}{2}}^{2}\left|\zeta(\sigma+iT)R(s)\frac{x^{2}}{T}\right|d\sigma\ll\frac{x^{2}}{T}=x^{\frac{1}{2}},\end{aligned}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\zeta(s)R(s)\frac{x^{s}}{s}ds\right|\ll\int_{0}^{T}\left|\zeta(\frac{1}{2}+it)R(s)\frac{x^{\frac{1}{2}}}{t}\right|dt\ll x^{\frac{1}{2}+\varepsilon}.$$

Noting that

$$\begin{aligned}R(1)&=\prod_{p^{\alpha}\|m}\left(\left(1+\frac{\varphi((m,p))}{p}+\cdots+\frac{\varphi((m,p^{\alpha-1}))}{p^{(\alpha-1)}}\right)\left(1-\frac{1}{p}\right)+\frac{\varphi((m,p^{\alpha}))}{p^{\alpha}}\right)\\&=\frac{\varphi(m)}{m}\prod_{p^{\alpha}\|m}(1+\alpha-\frac{\alpha-1}{p}).\end{aligned}$$

So we have the asympotic formula

$$\sum_{n\leq x}\varphi((m,n))=x\cdot h(m)+O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $h(m)=\frac{\varphi(m)}{m}\prod\limits_{p^{\alpha}\|m}(1+\alpha-\frac{\alpha-1}{p})$. This completes the proof of Lemma 2.

§3. Proof of Theorems

In this section, we will complete the proof of Theorems. First we prove Theorem 1. For any real number $x > 1$, let M be a fixed positive integer with $M^k \leq x \leq (M+1)^k$, from the definition of $s(n)$ we have

$$\begin{aligned}
\sum_{n\leq x}\Omega(s(n)) &= \sum_{t=1}^{M}\sum_{(t-1)^k\leq n<t^k}\Omega(s(n))+\sum_{M^k\leq n<x}\Omega(s(n)) \\
&= \sum_{t=1}^{M-1}\sum_{t^k\leq n<(t+1)^k}\Omega(s(n))+\sum_{M^k\leq n\leq x}\Omega(M) \\
&= \sum_{t=1}^{M-1}[(t+1)^k-t^k]\Omega(t)+O\left(\sum_{M^k\leq n<(M+1)^k}\Omega(M)\right) \\
&= k\sum_{t=1}^{M}t^{k-1}\Omega(t)+O\left(M^{k-1}\log M\right),
\end{aligned}$$

where we have used the estimate $\Omega(M) \ll \log n$.

Let $B(y) = \sum\limits_{n\leq y}\Omega(n)$, then by Able's identity and Lemma 1, we can easily deduce that

$$\begin{aligned}
&\sum_{t=1}^{M}t^{k-1}\Omega(t) = M^{k-1}B(M)-(k-1)\int_2^M y^{k-2}B(y)dy \\
= \;& M^{k-1}(M\log\log M+AM)-(k-1)\int_2^M(y^{k-1}\log\log y+Ay^{k-1})dy \\
&+O\left(\frac{M^k}{\log M}\right) \\
= \;& M^k\log\log M+AM^k-\frac{k-1}{k}(M^k\log\log M+AM^k)+O\left(\frac{M^k}{\log M}\right) \\
= \;& \frac{1}{k}M^k\log\log M+\frac{1}{k}AM+O\left(\frac{M^k}{\log M}\right).
\end{aligned}$$

Therefore, we can obtain the asymptotic formula

$$\sum_{n\leq x}\Omega(s(n)) = M^k\log\log M+AM+O\left(\frac{M^k}{\log M}\right).$$

On the other hand, we also have the estimate

$$0\leq x-M^k<(M+1)^k-M^k\ll x^{\frac{k-1}{k}}.$$

Now combining the above, we may immediately obtain the asymptotic formula

$$\sum_{n\leq x}\Omega(s(n))=x\log\log x+(A-\log k)\,x+O\left(\frac{x}{\log x}\right).$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. For any fixed positive integer m, we have

$$\begin{aligned}
&\sum_{n\leq x}\varphi((s(n),m))=\sum_{n\leq x}\varphi(([n^{\frac{1}{k}}],m))\\
=&\sum_{1\leq i<2^k}\varphi(([i^{\frac{1}{k}}],m))+\cdots+\sum_{N\leq i<(N+1)^k}\varphi(([i^{\frac{1}{k}}],m))+O\left(N^{\varepsilon}\right)\\
=&\sum_{j\leq N}[(j+1)^k-j^k]\varphi((j,m))+O\left(N^{\varepsilon}\right).
\end{aligned}$$

From Lemma 2, we can let

$$\begin{aligned}
A(N) &= \sum_{j\leq N}\varphi((j,m))=N\cdot h(m)+O\left(N^{\frac{1}{2}+\varepsilon}\right),\\
f(j) &= [(j+1)^k-j^k],
\end{aligned}$$

Then by Able's identity, we can easily obtain

$$\begin{aligned}
&\sum_{j\leq N}[(j+1)^k-j^k]\varphi((j,m))\\
=&\ A(N)f(N)-A(1)f(1)-\int_1^N A(t)f'(t)dt\\
=&\ [N\cdot h(m)+O\left(N^{\frac{1}{2}+\varepsilon}\right)][(N+1)^k-N^k]\\
&-\int_1^N[t\cdot h(m)+O\left(t^{\frac{1}{2}+\varepsilon}\right)]\cdot k[(t+1)^{k-1}-t^{k-1}]dt\\
=&\ k\cdot N^k h(m)+O\left(k\cdot N^{k-\frac{1}{2}+\varepsilon}\right)-(k-1)h(m)(n^k-1)\\
=&\ (k+N^k+1)h(m)+O\left(k\cdot N^{k-\frac{1}{2}+\varepsilon}\right)\\
=&\ h(m)x+(k+1)h(m)++O\left(x^{1-\frac{1}{2k}+\varepsilon}\right).
\end{aligned}$$

This completes the proof of Theorem 2.

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SMARANDACHE FRIENDLY CUBE NUMBERS

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Abstract The main purpose of this paper is to introduce new concepts of Smarandache numbers, namely Smarandache Friendly Cube Numbers, and give definitions, curious note, theorem, conjectures, proposed future studies, and ask open problems.

Keywords: Smarandache Friendly Triple Cube Numbers; Smarandache Friendly Pairs Cube Numbers.

1.1 Definition. The positive integers of ordered triple (m, n, k) are called Smarandache Friendly Triple Cube Numbers, denoted by $SFTCN_{(m,n,k)}$, if the following conditions satisfy:

1) The sum of its digits (i.e. m, n, and k) is cube.

2) The second integer (n) is formed by summing the digits of the first integer (m) after cubing it, and the integer (n) must be the reverse order of the first number (m).

3) The third integer (k) is obtained by cubing the second integer (n) and summing its digits, and (k) must equal the sum of its digits after cubing (k).

1.2 Example: $(53, 35, 26)$ is $SFTCN_{(53,35,26)}$, note the following conditions:

1) The sum of the digits of (53) is $5 + 3 = 8$(cube).

2) $(53)^3 = 148877$, then $1 + 4 + 8 + 8 + 7 + 7 = 35$, the digits sum is also cube $(3 + 5 = 8)$, and 35 is formed form the backorder of 53.

3) $(35)^3 = 42875$, then $4 + 2 + 8 + 7 + 5 = 26$, the sum digits of 26 are $2 + 6 = 8$, which is cube, and $26^3 = 17576$, $1 + 7 + 5 + 7 + 6 = 26$.

The proposed sequence of the

$$SFTCN_{(m,n,k)} := \{(10, 1, 1), (53, 35, 26), (62, 26, 26), (80, 8, 8), \cdots.$$

1.3 Conjectures:

1) The $SFTCN_{(m,n,k)}$ contains infinitely many triples.

2) The $SFTCN_{(m,n,k)}$ contains infinitely many triples that satisfy Transitive property, but there are exceptions such as the triple $(53, 35, 26)$.

1.4 Open problems:

1) What is the general formula of $SFTCN_{(m,n,k)}$?

2) What is the procedure that can verify $SFTCN_{(m,n,k)}$ (by using computer programming such as Maple, or Mathematica)?

3) How many triples prime are there in $SFTCN_{(m,n,k)}$?

4) What is the density of $SFTCN_{(m,n,k)}$?

5) Is there a relationship between $SFTCN_{(m,n,k)}$, and other Smarandache sequence (such as Smarandache cube-digital sequence [2])?

6) Are there such integers that satisfy $SFTCN_{(m,n,17)}$,and $SFTCN_{(m,n,71)}$?

2.1 Definition: Any two positive integers satisfy the following two conditions (are called Smarandache Friendly Pairs Cube Numbers, denoted by $SFPCN_{(m,n)}$):

1) The sum of its digits is cube.

2) The sum of its digits after cubing, equal itself.

2.2 Theorem. $SFPCN_{(m,n)}$ satisfies the Reflexive property.

Proof. suppose $m =_{ci}\ 10^i +_{ci-1}\ 10^{i-1} + \cdots +_{c1}\ 10 +_{c0}$, i.e. the decimal form of m.

Also, $n =_j\ 10^j +_{cj-1}\ 10^{j-1} + \cdots +_{c1}\ 10 +_{c0}$, i.e. the decimal form of n. Now by definition (condition 2) we must have

$$\left({}_{ci}10^i +_{ci-1}\ 10^{i-1} + \cdots +_{c1}\ 10 +_{c0}\right)^3 = \left({}_j10^j +_{cj-1}\ 10^{j-1} + \cdots +_{c1}\ 10 +_{c0}\right)^3,$$

for all i and j. Hence, $m^3 = n^3$, i.e. $SFPCN_{(m,n)} = SFPCN_{(n,m)}$, which is the Reflexive property.

2.3 Example: Consider $SFPC_{(8,8)}$, then we have

1) 8 cube.

2) $8^3 = 512$, the sum of digits $5 + 1 + 2 = 8$, hence $(8, 8)$ is $SFPCN_{(8,8)}$.

Thus, the proposed sequence of the

$$SFPCN_{(m,n)} := \{(1, 1), (8, 8), (17, 17), (26, 26), \cdots\},$$

noting that 71, and 62 are not $SFPCN$.

2.4 Curious note: the number 27 is not $SFPCN_{(27,27)}$, but $27^3 = 19683$, i.e. $1 + 9 + 6 + 8 + 3 = 27$, in addition, the number 18 has this property; can you find another ones?

2.5 Conjecture: $SFPCN_{(m,n)}$ is a special case from $SFTCN_{(m,n,k)}$.

2.6 Open problems:

1) Is the sequence of the proposed $SFPCN_{(m,n)}$, finite or infinite?

2) What is the general formula of $SFPCN_{(m,n)}$?

3) What is the formula that connects $SFPCN_{(m,n)}$ and $SFTCN_{(m,n,k)}$?

4) Is there a relationship between $SFPCN_{(m,n)}$, $SFTCN_{(m,n,k)}$, and Smarandache Sequence of Happy Cube Numbers [1]?

3.1 Definition: The ordered pair of integers (m, c) is called Fixed Smarandache Friendly Pairs Cube Numbers (FSFPCN) if the following conditions satisfy:

1) The sum of its digits (m, and c) is cube.

2) c is a constant formed from the digits of m after cubing .

3.2 Examples: $(35, 26), (26, 26), (44, 26), (62, 26), (71, 26)$, and so on.

3.3 Open problems:

1) Are there other examples for different constants that satisfy $(FSFPCN)$?

2) Is there a relationship between

$$SFPCN_{(m,n)}, SFTCN_{(m,n,k)} \quad \text{and} \quad FSFPCN_{(m,c)}?$$

4.1 Proposed future studies: The author invites the researcher for more studies about the following new concepts:

4.1.1 Smarandache Friendly Pairs of 4-th powers Numbers (such as, $7^4 = 2401$, the sum of its digits $= 7$, also $22^4 = 234256$, so $2+3+4+2+5+6 = 22$, also $25^4 = 390625$, then $3 + 9 + 0 + 6 + 2 + 5 = 25$, $36^4 = 1679616$, hence $1 + 6 + 7 + 9 + 6 + 1 + 6 = 36$, and so on).

4.1.2 Smarandache Friendly Pairs of 5-th powers Numbers (such as, $28^5 = 17210368$, so $1 + 7 + 2 + 1 + 0 + 3 + 6 + 8 = 28$, i.e. the sum of its digits $= 28$, also $35^5 = 52521875$, hence $5 + 2 + 5 + 2 + 1 + 8 + 7 + 5 = 35$, and so on).

4.1.3 Smarandache Friendly Pairs of n-th powers Numbers.

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SOME EXPRESSIONS OF THE SMARANDACHE PRIME FUNCTION

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Abstract The main purpose of this paper is using elementary arithmetical functions to give some expressions of the Smarandache Prime Function $P(n)$.

In this article we gave some expressions of the Smarandache Prime Function $P(n)$ (see reference [1]), using elementary arithmetical functions. The Smarandache Prime Function is the complementary of the Prime Characteristic Function:

$$P(n) = \begin{cases} 0 & if \quad n \quad is \quad a \quad prime, \\ 1 & if \quad n \quad is \quad a \quad composite. \end{cases}$$

Expression 1.

$$P(n) = 1 \left\lfloor \frac{lcm(1,2,\cdots,n)}{n \cdot lcm(1,2,\cdots,n-1)} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ is the floor function (see reference [2]).

Proof. We consider three cases:

Case 1: If $n = p$ with p prime, then we have

$$lcm(1,2,\cdots,p) = lcm(lcm(1,2,\cdots p-1),p) = p \cdot lcm(1,2,\cdots,p-1)$$

Therefore we have: $P(n) = 0$.

Case 2: If $n = p^{\alpha}$ with p is prime and α is a positive integer greater than one, we may have

$$\begin{aligned} & \left\lfloor \frac{lcm(1,2,\cdots,n)}{n \cdot lcm(1,2,\cdots,n-1)} \right\rfloor \\ = & \left\lfloor \frac{lcm(1,2,\cdots,p^{\alpha})}{n \cdot lcm(1,2,\cdots,p^{\alpha}-1)} \right\rfloor \\ = & \left\lfloor \frac{lcm(lcm(1,2,\cdots,p^{\alpha-1},\cdots,p^{\alpha}-1),p^{\alpha}}{n \cdot lcm(1,2,\cdots,p^{\alpha}-1)} \right\rfloor \end{aligned}$$

$$= \left\lfloor \frac{p \cdot lcm(1,2,\cdots,p^{\alpha-1},\cdots,p^{\alpha}-1)}{n \cdot (1,2,\cdots,p^{\alpha}-1)} \right\rfloor$$
$$= \left\lfloor \frac{p}{n} \right\rfloor = 0.$$

So we have: $P(n) = 1$.

Case 3: If $n = a \cdot b$ with $gcd(a,b) = 1$ and $a, b > 1$. We can suppose $a < b$, then we have

$$lcm(1,2,\cdots,a,\cdots,b,\cdots,n)$$
$$= lcm(1,2,\cdots,a,\cdots,b,\cdots,n-1,a \cdot b)$$
$$= lcm(1,2,\cdots,a,\cdots,b,\cdots,n-1)$$

and therefore we have:

$$\begin{aligned} P(n) &= 1 - \left\lfloor \frac{lcm(1,2,\cdots,n)}{n \cdot lcm(1,2,\cdots,n-1)} \right\rfloor \\ &= 1 - \left\lfloor \frac{1}{n} \right\rfloor = 1 - 0 = 1 \end{aligned}$$

With this the expression one is proven.

Expression 2. [3],[4]

$$P(n) = -\left\lfloor \frac{2 - \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor}{n} \right\rfloor$$

Proof. We consider $d(n) = \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor$ is the number of divisors of n because:

$$\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor = \begin{cases} 1 & if \quad i \quad divides \quad n, \\ 0 & if \quad i \quad not \quad divide \quad n. \end{cases}$$

If $n = p$ prime we have $d(n) = 2$ and therefore $P(n) = 0$.
If n is composite we have $d(n) > 2$ and therefore:

$$-1 < \frac{2 - d(n)}{n} < 0 \Longrightarrow P(n) = 1.$$

Expression 3.
We can also prove the following expression:

$$P(n) = 1 - \left\lfloor \frac{1}{n} \cdot GCD\left(\binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n-1}\right)\right\rfloor,$$

where $\binom{n}{i}$ is the binomial coefficient.

Can the reader prove this last expression?

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AN IMPROVED ALGORITHM FOR CALCULATING THE SUM-OF-FACTORIALS FUNCTION

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Abstract The sum of factorials function, also known as the left factorial function, is defined as $!n = 0! + 1! + \cdots + (n-1)!$. These have been used by Smarandache and Kurepa to define the Smarandache-Kurepa Function (see reference [1], [2]). This paper presents an effective method for calculating $!n$, and implements the Smarandache-Kurepa function by using one new method.

1. Introduction

We define $!n$ as $0! + 1! + \cdots + (n-1)!$.
A simple PARI/GP program to calculate these values is below:
$soff(n) = \sum_{i=0}^{n-1} i!$
Then,
for($i = 0, 10$,print1(","soff(i))) gives the desired output;
$0, 1, 2, 4, 10, 34, 154, 874, 5914, 46234, 409114,$
which is A003422 at OEIS [3].

2. A new method

If we write out what the sum of factorials function is doing, we can write:

$$1+$$
$$1+$$
$$1.2+$$
$$1.2.3+$$
$$1.2.3.4+$$
$$1.2.3.4.5+$$

and so on.

If we now read down the columns, we see that this can be written as:

$$1 + 1[1 + 2[1 + 3[1 + 4[1 + \cdots .$$

This is because we have an opening 1 from 0!. Then 1 is a factor of all the remaining factorials. However 1 is the only factor of 1 of the factorials, namely 1!, so we have

$$1 + 1[1 + \cdots .$$

Having removed the 1!, 2 is now a factor is all remaining factorials, and is the final factor in 2!, hence

$$1 + 1[1 + 2[1 + \cdots$$

and so on.

$!n$ requires inputs from 0! to $(n-1)!$, and hence we are required to stop the nested recursion by $n-1$. e.g. for !5, we have

$$1 + 1[1 + 2[1 + 3[1 + 4[1]]]].$$

We can validate this:

$$1 + 1[1 + 2[1 + 3[1 + 4[1]]]]$$

$$= 1 + 1[1 + 2[1 + 3[5]]]$$

$$= 1 + 1[1 + 2[16]]$$

$$= 1 + 1[33]$$

$$= 34.$$

3. Code for new method

We can see how the new method decreases execution time, the original method presented performs $O(k2)$ multiplications and $O(k)$ additions. This method performs $O(k)$ multiplications and $O(k)$ additions.

PARI/GP code for the routine is below:

qsoff(n) = local(r); $r = n$; forstep($i = n-2, 1, 4-1, r* = i; r++$); r

4. Implementing the Smarandache-Kurepa function

We need only consider primes, and the sk variable needs only range from 1 to $p-1$ (if !1 to $!p$ are not divisible by p, then $!(p+k)$ will never be as all new terms have p as a factor).

For prime ($p = 2, 500$, for ($sk = 1, p$, if (qsoff(sk)

This is obviously wasteful, we are calculating qsoff(sk) very repetitively. The code below stores the qsoff() values in a vector.

v=vector(500, i, qsoff(i)); forprime ($p = 2500$, for ($sk = 1$, p, if (v[sk]

The following output is produced:

$2, -, 4, 6, 6, -, 5, 7, 7, -, 12, 22, 16, -, -, -, -, 55, -, 54, 42, -, -, 24, -, -, 25,$

$-, -, 86, -, 97, -, 133, -, -, 64, 94, 72, 58, -, -, 49, 69, 19, -, 78, -, 14, -, 208,$

$167, -, 138, 80, 59, -, -, -, -, 63, 142, 41, -, 110, 22, 286, 39, -, 84, -, -, 215, 80,$

$14, 305, -, 188, 151, 53, 187, -, 180, -, -, -, -, 44, 32, 83, 92, -, 300, 16, -.$

5. Additional relations

The basic pattern created in this paper also allows for the rapid calculation of other Smarandache-like functions based on the sum of factorials function.

For example, we could define $SSF(n)$ as the sum of squares factorial, e.g. $SSF(10) = 0! + 1! + 4! + 9!$, and the corresponding general expansion is

$$1 + 1[1 + 2.3.4[1 + 5.6.7.8.9[1 + \cdots .$$

Or we can define the sum of factorials squared function as

$$0!2 + 1!2 + 2!2 + \cdots .$$

In this case, the expansion is

$$1 + 1[1 + 4[1 + 9[1 + \cdots .$$

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ON THE SMARANDCHE FUNCTION AND ITS HYBRID MEAN VALUE

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Abstract For any positive integer n, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined as the smallest $m \in N^+$ with $n|m!$. In this paper, we study the asymptotic property of a hybrid mean value of the Smarandache function and the Mangoldt function, and give an interesting hybrid mean value formula for it.

Keywords: the Smarandche function; the Mangoldt function; Mean value.

§1. Introduction

For any positive integer n, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined as the smallest $m \in N^+$ with $n|m!$. From the definition of $S(n)$, one can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime power factorization of n, then

$$S(n) = \max_{1 \le i \le k} S(p_i^{\alpha_i}).$$

About the arithmetical properties of $S(n)$, many people had studied it before (see reference [2]). In this paper, we study the asymptotic property of a hybrid mean value of the Smarandache function and the Mangoldt function, and give an interesting hybrid mean value formula for it. That is, we shall prove the following:

Theorem. For any real number $x \ge 1$, we have the asymptotic formula

$$\sum_{n \le x} \wedge(n) S(n) = \frac{x^2}{4} + O\left(\frac{x^2 \log\log x}{\log x}\right),$$

where $\wedge(n)$ is the Mangoldt function defined by

$$\wedge(n) = \begin{cases} \log p, & \text{if } n = p^{\alpha} (\alpha \ge 1); \\ 0, & \text{otherwise.} \end{cases}$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. Firstly, we need following:

Lemma. For any prime p and any positive integer α, we have

$$S(p^\alpha) = (p-1)\alpha + O\left(\frac{p\log\alpha}{\log p}\right).$$

Proof. From Theorem 1.4 of reference [3], we can obtain the estimate.

Now we use the above Lemma to complete the proof of the theorem. From the definition of $\wedge(n)$, we have

$$\begin{aligned}
&\sum_{n\le x}\wedge(n)S(n)\\
=&\sum_{p^\alpha\le x}S(p^\alpha)\log p\\
=&\sum_{p\le x}\sum_{\alpha\le\frac{\log x}{\log p}}\log p\left((p-1)\alpha+O\left(\frac{p\log\alpha}{\log p}\right)\right)\\
=&\sum_{p\le x}(p-1)\log p\sum_{\alpha\le\frac{\log x}{\log p}}\alpha+O\left(\sum_{p\le x}p\sum_{\alpha\le\frac{\log x}{\log p}}\log\alpha\right).
\end{aligned}$$

Applying Euler's summation formula, we can get

$$\sum_{\alpha\le\frac{\log x}{\log p}}\alpha=\frac{1}{2}\frac{\log^2 x}{\log^2 p}+O\left(\frac{\log x}{\log p}\right),$$

and

$$\sum_{\alpha\le\frac{\log x}{\log p}}\log\alpha=\frac{\log x}{\log p}\log\frac{\log x}{\log p}-\frac{\log x}{\log p}+O\left(\log\frac{\log x}{\log p}\right).$$

Therefore we have

$$\sum_{n\le x}\wedge(n)S(n)=\frac{1}{2}\log^2 x\sum_{p\le x}\frac{p}{\log p}+O\left(\log x\log\log x\sum_{p\le x}\frac{p}{\log p}\right). \quad (1)$$

If $x>0$ let $\pi(x)$ denote the number of primes not exceeding x, and let

$$a(n)=\begin{cases}1, & \text{if } n \text{ is a prime;}\\ 0, & \text{otherwise.}\end{cases}$$

then $\pi(x)=\sum_{p\le x}a(n)$. Note the asymptotic formula

$$\pi(x)=\frac{x}{\log x}+O\left(\frac{x}{\log^2 x}\right),$$

and from Abel's identity, we have

$$\begin{aligned}
&\sum_{p\le x}\frac{p}{\log p}\\
=\ &\sum_{n\le x}a(n)\frac{n}{\log n}\\
=\ &\pi(x)\frac{x}{\log x}-\pi(2)\frac{2}{\log 2}-\int_2^x\pi(t)d\left(\frac{t}{\log t}\right)\\
=\ &\frac{x}{\log x}\left(\frac{x}{\log x}+O\left(\frac{x}{\log^2 x}\right)\right)-\int_2^x\left(\frac{t}{\log t}+O\left(\frac{t}{\log^2 t}\right)\right)d\left(\frac{t}{\log t}\right)\\
=\ &\frac{1}{2}\frac{x^2}{\log^2 x}+O\left(\frac{x^2}{\log^3 x}\right). \qquad (2)
\end{aligned}$$

Combining (1) and (2), we have

$$\begin{aligned}
&\sum_{n\le x}\wedge(n)S(n)\\
=\ &\frac{1}{4}x^2+O\left(\frac{x^2}{\log x}\right)+O\left(\log x\log\log x\frac{x^2}{\log^2 x}\right)\\
=\ &\frac{1}{4}x^2+O\left(\frac{x^2\log\log x}{\log x}\right).
\end{aligned}$$

This completes the proof of the theorem.

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ON THE 83-TH PROBLEM OF F. SMARANDACHE

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Abstract For any positive integer n, let $m_q(n)$ denote the integer part of k-th root of n. That is, $m_q(n) = \left[n^{\frac{1}{k}}\right]$. In this paper, we study the properties of the sequences $\{m_q(n)\}$, and give an interesting asymptotic formula.

Keywords: Integer part; Mean value; Asymptotic formula.

§1. Introduction

For any positive integer n, let $m_q(n)$ denote the integer part of k-th root of n. That is, $m_q(n) = \left[n^{\frac{1}{k}}\right]$. For example, $m_q(1) = 1$, $m_q(2) = 1$, $m_q(3) = 1$, $m_q(4) = 1, \cdots$, $m_q(2^k) = 2$, $m_q(2^k + 1) = 2, \cdots$, $m_q(3^k) = 3, \cdots$. In problem 83 of [1], Professor F. Smarandache asked us to study the properties of the sequence $\{m_q(n)\}$. About this problem, it seems that none had studied it, at least we have not seen related paper before. In this paper, we use the elementary methods to study the properties of this sequence, and give an interesting asymptotic formula. That is, we shall prove the following:

Theorem. *m is any fixed positive integer, α is a real number . For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\le x} \sigma_\alpha((m_q(n), m)) = \frac{(2k-1)\sigma_{1-\alpha}(m)}{m^{1-\alpha}}x + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, ε is any fixed positive number.

When $\alpha = 0, 1$, we have

Corollary. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\le x} d((m_q(n), m)) = \frac{(2k-1)\sigma(m)}{m}x + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

$$\sum_{n\le x} \sigma((m_q(n), m)) = (2k-1)d(m)x + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

where $d(n)$ is divisor function, $\sigma(n)$ is divisor sum function.

§2. One lemma

To prove the theorem, we need the following lemma.

Lemma. *m is any fixed positive integer, α is a real number. For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x}\sigma_\alpha((n,m)) = \frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}}x + O\left(x^{\frac{1}{2k}+\varepsilon}\right),$$

where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, ε is any fixed positive number.

Proof. Let

$$g(s) = \sum_{n=1}^{\infty}\frac{\sigma_\alpha((m,n))}{n^s},$$

For m is a fixed number, $f(n) = (m,n)$ is a multiplicative function.we can proof that $\sigma_\alpha((m,n))$ is a multiplicative function too.

From the Euler product formula, we have

$$\begin{aligned}
g(s) &= \prod_p\left(1+\frac{\sigma_\alpha(f(p))}{p^s}+\frac{\sigma_\alpha(f(p^2))}{p^{2s}}+\cdots\right)\\
&= \prod_{p\dagger m}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right)\\
&\quad\times\prod_{p^\beta\| m}\left(1+\frac{1+p^\alpha}{p^s}+\cdots+\frac{\sum\limits_{i=0}^{\beta}(p^i)^\alpha}{p^{\beta s}}+\frac{\sum\limits_{i=0}^{\beta}(p^i)^\alpha}{p^{(\beta+1)s}}+\cdots\right)\\
&= \prod_{p\dagger m}\frac{1}{1-\frac{1}{p^s}}\prod_{p^\beta\| m}\left(1+\frac{1+p^\alpha}{p^s}+\cdots+\frac{\sum\limits_{i=0}^{\beta-1}(p^i)^\alpha}{p^{\beta s}}+\frac{\sum\limits_{i=0}^{\beta}(p^i)^\alpha}{p^{\beta s}}\frac{1}{1-\frac{1}{p^s}}\right)\\
&= \zeta(s)\prod_{p^\beta\| m}\left(1+\frac{1}{p^{s-\alpha}}+\frac{1}{p^{2(s-\alpha)}}+\cdots+\frac{1}{p^{\beta(s-\alpha)}}\right).
\end{aligned}$$

And for

$$|\sigma_\alpha((m,n))| < K = H(x), \qquad \left|\sum_{n=1}^{\infty}\frac{\sigma_\alpha((m,n))}{n^\sigma}\right| < \frac{K}{\sigma-1} = B(\sigma)$$

where K is a constant only about m and α, $\alpha > 1$ is real part of s. So we let $s_0 = 0$, $b = 2$, $T = x^{3/2}$. When x is a half odd, we let $N = x - 1/2$, $\|x\| = |x - N|$. By Perron formula, we have

$$\sum_{n\leq x}\sigma_\alpha(f(n))=\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\zeta(s)R(s)\frac{x^s}{s}ds+O(x^{1/2+\varepsilon}).$$

Where

$$R(s)=\prod_{p^\beta\|m}\left(1+\frac{1}{p^{s-\alpha}}+\frac{1}{p^{2(s-\alpha)}}+\cdots+\frac{1}{p^{\beta(s-\alpha)}}\right)$$

To estimate the main term

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\zeta(s)R(s)\frac{x^s}{s}ds,$$

we move the integral line from $2\pm iT$ to $1/2\pm iT$. This time, the function

$$\zeta(s)R(s)\frac{x^s}{s}$$

have a simple pole point at $s=1$, so we have

$$\frac{1}{2\pi i}\left(\int_{2-iT}^{2+iT}+\int_{2+iT}^{1/2+iT}+\int_{1/2+iT}^{1/2-iT}+\int_{1/2-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^s}{s}ds=R(1)x.$$

Taking $T=x^{\frac{3}{2}}$, we have

$$\begin{aligned}&\left|\frac{1}{2\pi i}\left(\int_{2+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^s}{s}ds\right|\\ \ll&\int_{\frac{1}{2}}^{2}\left|\zeta(\sigma+iT)R(s)\frac{x^2}{T}\right|d\sigma\\ \ll&\frac{x^2}{T}=x^{\frac{1}{2}};\end{aligned}$$

And we can easy get the estimate

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\zeta(s)R(s)\frac{x^s}{s}ds\right|\ll\int_0^T\left|\zeta(\frac{1}{2}+it)R(s)\frac{x^{\frac{1}{2}}}{t}\right|dt\ll x^{\frac{1}{2}+\varepsilon};$$

For

$$R(1)=\prod_{p^\beta\|m}\left(1+\frac{1}{p^{1-\alpha}}+\frac{1}{p^{2(1-\alpha)}}+\cdots+\frac{1}{p^{\beta(1-\alpha)}}\right)=\frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}}$$

We can have

$$\sum_{n\leq x}\sigma_\alpha(f(n))=\frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}}x+O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

This completes the proof of Lemma.

§3. Proof of the theorem

In this section, we shall complete the proof of Theorem. For any real number $x \geq 1$, let N be a fixed positive integer such that

$$N^k \leq x < (N+1)^k.$$

from the definition of $m_q(n)$ we have

$$\begin{aligned}
\sum_{n\leq x}\sigma_\alpha((m_q(n),m)) &= \sum_{n\leq x}\sigma_\alpha(([n^{\frac{1}{k}}],m)) \\
&= \sum_{1^k\leq i<2^k}\sigma_\alpha(([i^{\frac{1}{k}}],m)) + \sum_{2^k\leq i<3^k}\sigma_\alpha(([i^{\frac{1}{k}}],m)) \\
&+ \cdots + \sum_{N^k\leq i\leq x<(N+1)^k}\sigma_\alpha(([i^{\frac{1}{k}}],m)) + O(N^\varepsilon)t) \\
&= (2^k-1)\sigma_\alpha((1,m)) + (3^k-2^k)\sigma_\alpha((2,m)) \\
&+ \cdots + [(N+1)^k - N^k]\sigma_\alpha((N,m)) + O(N^\varepsilon) \\
&= \sum_{j\leq N}[(j+1)^k - j^k]\sigma_\alpha((j,m)) + O(N^\varepsilon),
\end{aligned}$$

where ε is any fixed positive number.

Let $A(N) = \sum_{j\leq N}\sigma_\alpha((j,m))$. From Lemma, we have

$$A(N) = \sum_{j\leq N}\sigma_\alpha((j,m)) = \frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}}N + O\left(N^{\frac{1}{2}+\varepsilon}\right),$$

And letting $f(j) = [(j+1)^k - j^k]$. By Abel's identity, we have

$$\begin{aligned}
&\sum_{j\leq N}[(j+1)^k - j^k]\sigma_\alpha((j,m)) \\
= \;& A(N)f(N) - A(1)f(1) - \int_1^N A(t)f'(t)dt \\
= \;& [\frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}}N + O\left(N^{\frac{1}{2}+\varepsilon}\right)][(N+1)^k - N^k \\
& -A(1)f(1) - \int_1^N [\frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}}t + O\left(t^{\frac{1}{2}+\varepsilon}\right)] \\
& [k(t+1)^{k-1} - kt^{k-1}]dt
\end{aligned}$$

From the binomial theorem, we have

$$\sum_{j\leq N}[(j+1)^k - j^k]\sigma_\alpha((j,m)) = \frac{(2k-1)\sigma_{1-\alpha}(m)}{m^{1-\alpha}}N^k + O\left(N^{k-\frac{1}{2}+\varepsilon}\right)$$

So

$$\begin{aligned}\sum_{n\leq x}\sigma_\alpha((m_q(n),m)) &= \sum_{n\leq x}\sigma_\alpha(([n^{\frac{1}{k}}],m))\\ &= \sum_{j\leq N}[(j+1)^k-j^k]\sigma_\alpha((j,m))+O(N^\varepsilon)\\ &= \frac{(2k-1)\sigma_{1-\alpha}(m)}{m^{1-\alpha}}x+O\left(x^{1-\frac{1}{2k}+\varepsilon}\right).\end{aligned}$$

This completes the proof of Theorem.

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ON SMARANDACHE TRIPLE FACTORIAL FUNCTION

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Abstract For any positive integer n, the Smarandache triple factorial function $d3_f(n)$ is defined to be the smallest integer such that $d3_f(n)!!!$ is a multiple of n. In this paper, we study the hybrid mean value of the Smarandache triple factorial function and the Mangoldt function, and give a sharp asymptotic formula.

Keywords: Triple factorial numbers; Hybrid mean value; Asymptotic formula.

§1. Introduction

According to [1], for any positive integer n, the Smarandache triple factorial function $d3_f(n)$ is defined to be the smallest integer such that $d3_f(n)!!!$ is a multiple of n. About this problem, we know very little. The problem is interesting because it can help us to calculate the Smarandache function.

In this paper, we study the hybrid mean value of the Smarandache triple factorial function and the Mangoldt function, and give a sharp asymptotic formula. That is, we shall prove the following theorems.

Theorem 1. *If $x \geq 2$, then for any positive integer k we have*

$$\sum_{n\leq x}\Lambda_1(n)d3_f(n) = x^2\left(\frac{1}{2}+\sum_{m=1}^{k-1}\frac{a_m}{\log^m x}\right)+O\left(\frac{x^2}{\log^k x}\right),$$

where

$$\Lambda_1(n) = \begin{cases} \log p, & \text{if } n \text{ is a prime } p\,; \\ 0, & \text{otherwise}, \end{cases}$$

and $a_m(m = 1, 2, \cdots, k-1)$ are computable constants.

Theorem 2. *If $x \geq 2$, then for any positive integer k we have*

$$\sum_{n\leq x}\Lambda(n)d3_f(n) = x^2\left(\frac{1}{2}+\sum_{m=1}^{k-1}\frac{a_m}{\log^m x}\right)+O\left(\frac{x^2}{\log^k x}\right),$$

where $\Lambda(n)$ is the Mangoldt function.

§2. One lemma

To complete the proofs of the theorems, we need the following lemma.

Lemma. *For any positive integer α, if $p \geq (3\alpha - 2)$ we have*

$$d3_f(p^\alpha) = (3\alpha - 2)p.$$

Proof. Since

$$[(3\alpha - 2)p]!!! = (3\alpha - 2)p \cdots (3\alpha - 3)p \cdots p,$$

so $p^\alpha \mid [(3\alpha - 2)p]!!!$. On the other hand, if $p \geq (3\alpha - 2)$, then $(3\alpha - 2)p$ is the smallest integer such that $[(3\alpha - 2)p]!!!$ is a multiple of p^α. Therefore $d3_f(p^\alpha) = (3\alpha - 2)p$.

§3. Proof of the theorems

In this section, we complete the proofs of the theorems. Let

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is prime;} \\ 0, & \text{otherwise.} \end{cases}$$

then for any positive integer k we have

$$\sum_{n \leq x} a(n) = \pi(x) = \frac{x}{\log x}\left(1 + \sum_{m=1}^{k-1} \frac{m!}{\log^m x}\right) + O\left(\frac{x}{\log^{k+1} x}\right).$$

By Abel's identity we have

$$\begin{aligned}
& \sum_{n \leq x} \Lambda_1(n) d3_f(n) \\
= & \sum_{p \leq x} p \log p = \sum_{n \leq x} a(n) n \log n \\
= & \pi(x) \cdot x \log x - \int_2^x \pi(t)\,(\log t + 1)\,dt \\
= & x^2\left(1 + \sum_{m=1}^{k-1} \frac{m!}{\log^m x}\right) + O\left(\frac{x^2}{\log^k x}\right) \\
& - \int_2^x \left(t + \frac{t}{\log t} + t \sum_{m=1}^{k-1} \frac{m!}{\log^m t} + \frac{t}{\log t} \sum_{m=1}^{k-1} \frac{m!}{\log^m t}\right. \\
& \left. + O\left(\frac{t\,(\log t + 1)}{\log^{k+1} t}\right)\right) dt \\
= & x^2\left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x}\right) + O\left(\frac{x^2}{\log^k x}\right),
\end{aligned}$$

where $a_m (m = 1, 2, \cdots, k-1)$ are computable constants. Therefore

$$\sum_{p \le x} p \log p = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right).$$

So we have

$$\sum_{n \le x} \Lambda_1(n) d3_f(n) = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right).$$

This proves Theorem 1.

It is obvious that $d3_f(p^\alpha) \le (3\alpha - 2)p$. From Lemma 1 we have

$$\begin{aligned} & \sum_{n \le x} \Lambda(n) d3_f(n) \\ = & \sum_{p^\alpha \le x} \log p \left[(3\alpha - 2)p \right] + \sum_{\substack{p^\alpha \le x \\ p < (3\alpha - 2)}} \log p \left[d3_f(p^\alpha) - (3\alpha - 2)p \right]. \end{aligned}$$

Note that

$$\begin{aligned} & \sum_{p^\alpha \le x} (3\alpha - 2)p \log p - \sum_{p \le x} p \log p \\ = & \sum_{\alpha \le \frac{\log x}{\log p}} \sum_{p \le x^{1/\alpha}} p \log p (2\alpha - 1) - \sum_{p^\alpha \le x} p \log p \\ = & \sum_{2 \le \alpha \le \frac{\log x}{\log p}} \sum_{p \le x^{1/\alpha}} p \log p (3\alpha - 2) \\ \ll & \sum_{2 \le \alpha \le \frac{\log x}{\log p}} \alpha x^{2/\alpha} \log x^{1/\alpha} \ll x \log^3 x \end{aligned}$$

and

$$\begin{aligned} \sum_{\substack{p^\alpha \le x \\ p < (3\alpha - 2)}} \log p \left[d_f(p^\alpha) - (3\alpha - 2)p \right] & \ll \sum_{\alpha \le \frac{\log x}{\log 2}} \sum_{p < (3\alpha - 2)} \alpha p \log p \\ & \ll \sum_{\alpha \le \frac{\log x}{\log 2}} (3\alpha - 2)^2 \alpha \log(3\alpha - 2) \ll \log^3 x, \end{aligned}$$

so we have

$$\sum_{n \le x} \Lambda(n) d3_f(n) = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right).$$

This completes the proof of Theorem 2.

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ON K-FACTORIALS AND SMARANDACHEIALS

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Abstract F. Smarandache defines a k-factorial as $n(n-k)(n-2k)\cdots$, terminating when $n-xk$ is positive and $n-(x+1)k$ is 0 or negative. Smarandacheials extend this definition into the negative numbers such that the factorial terminates when $|n-xk|$ is less than or equal to n and $|n-(x+1)k|$ is greater than n. This paper looks at some relations between these numbers.

Keywords: Smarandache function; Additive Analogue; Mean value formula.

k-factorials

We begin by looking at the k-factorial, represented by k exclamation marks after the variable.

The k-factorial is merely the full factorial $n!$ with some of the terms omitted.

e.g. the factorial for $n = 8$;

$n! = 8! = 8.7.6.5.4.3.2.1$

If we look at the 2-factorial $n!!$, we have

$n!! = 8.6.4.2$

and we see that $7, 5, 3$ and 1 are not present.

Similarly the 3-factorial $n!!!$ gives

$n!!! = 8.5.2$

and $7, 6, 4, 2$ and 1 are not present.

In the first case we have 7!! omitted, so we may write $n!! = n! \mid (n-1)!!$

For the 3-factorial, there are two sequences present, 7!!! and 6!!!, so

$n!!! = n! \mid [(n-1)!!!(n-2)!!!]$

Using the notation $n!_k$ for a k-factorial, we can easily obtain the general formula

$$n!_k = \frac{n!}{\prod_{i=1}^{k-1}(n-k)!_k}$$

A PARI/GP program to implement this is

{

$k$factorial(n, k)=local(result);

result=$n!$;

for ($i = 1, k-1$,result/=$k$factorial$(n-i, k)$);

result

}
although this is highly ineffective and not recommended for use.
To access the k-factorial function use
{
$k$factorial(n,k)=local(res);
res=vector$(n,i$,if $(i <= k,i,0))$;
for $(i = k+1,n$,res[i]=i*res[$i-k$]);
res[n];
}

This code stores 1 to k in a vector in positions 1 to k. Then each progressive term is calculated form the k-th previous entry and the current one. The above code actually calculates $n!_k$ for all n from 1 to n.

Even simpler, and the quickest yet is
{
kf(n,k)=local(r,c);
c=n
if (c==0,c=k);
r=c;
while (c<n,c+=k;r*=c);
r
}

The r variable holds the result, and the c variable is a counter. c is set to n mod k, and then incremented until it is n. r is multiplied by c at each stage.

We can also see that if $gcd(n,k) = k$, then $n!_k = k^{\frac{n}{k}}(\frac{n}{k})!$, so in this case we have

$$k^{\frac{n}{k}}(\frac{n}{k})!\prod_{i=1}^{k-1}(n-k)!_k = n!$$

If $gcd(n,k)! = k$, then the k-factorial seems more difficult to define. We address this problem shortly.

Smarandacheials

In extending the k-factorial to the negative integers, we need to further define the parameters involved.

If we let n be the starting number, and k be the decrease, then we also need to define m as n mod k, and then $m' = k - m$.

If m is greater than or equal to m', we can see that $SM(n,k) = \pm[n!_k * (n-(m-m'))!_k]$.

If m is less than m', then we have $SM(n,k) = \pm[n!_k*(n-(m-m')-k)!_k]$.

The plus/minus sign is not known yet - this is developed later in this paper.

This result follows because m represents the last integer from $n!_k$, and so $k-m$ will be the first negative integer from$(n-x)!_k$, and so we determine x.

If m is greater than or equal to m', then the difference $n-(n-x)$ must be the difference between m and m', so $x = m - m'$.

If $m < m'$, then we have a problem. The basic idea still works, however the negative factorial will rise to a higher level than the original n, and this is not

allowed. So the adjustment from subtracting k cuts the last negative integer out of the equation.

To combine these, define m^* as the smallest positive value of $(m-m')$ mod k, and now we may write;

$$SM(n,k) = \pm[n!_k^*(n-m^*)!_k].$$

For an example, consider $SM(13,5)$. Then $m = 3$ and $m' = 2$, and $m^* = (3-2) \bmod 5 = 1$, so we get;

$$\begin{aligned} SM(13,5) &= \pm 13!_5{}^*(13-1)!_5 \\ &= \pm 13!_5{}^*(12)!_5 \\ &= \pm 13.8.3^*12.7.2 \end{aligned}$$

However, for$SM(12,5), m^* = (2-3) \bmod 5 = 4$

$$\begin{aligned} &= \pm 12!_5{}^*(12-4)!_5 \\ &= \pm 12!_5{}^*8!_5 \\ &= \pm 12.7.2^*8.3 \end{aligned}$$

The sign is then simply $(-1)^\wedge$(number of terms in second Smarandacheial).

Number of terms in $n!_k$

Given $n!_k$, the expansion of the expression is;

$n(n-k)\cdots(n-ak)$

So there are $a+1$ terms.

Using a simple example, e.g. for $k = 5$, we can construct a table of the number of terms;

n	5	6	7	8	9	10	11
no. of terms	1	2	2	2	2	2	3

n	12	13	14	15	16	17	18
no. of terms	3	3	3	3	4	4	4

From this we see that there are ceil$(\frac{n}{k})$ terms.

Therefore a full expression for the Smarandacheial function is

$$SM(n,k) = (-1)^\wedge \left[\frac{n-m^*}{k}\right] n!_k(n-m^*)!_k$$

$n!_k$ **for** gcd$(n,k) < k$

In this case, we have no easy relation. We can however spot an interesting and deep relation with these k-factorials - their relation to a neighbouring $n!_k$ with $gcd(n,k) = k$.

To demonstrate this connection, we will examine $15!_5$.

This is $15.10.5 = 750$.

Now $16!_5$ is 16.11.6.1. There seems to be nothing else we can do.
However, we can write this as;

$$\left(\frac{16}{15}\right)\left(\frac{11}{10}\right)\left(\frac{6}{5}\right)15!_5$$

Still nothing, but then we see that $\frac{16}{15} = 1 + \frac{1}{15}$, and so on, and so we get;

$$\left(1+\frac{1}{15}\right)\left(1+\frac{1}{10}\right)\left(1+\frac{1}{5}\right)15!_5$$

If we expand the brackets, hey presto (I have skipped a few steps here)

$$16!_5 = 15!_5 + 15.10 + 15.5 + 10.5 + 15 + 10 + 5 + 1$$

This is generalizable into

$$(n+x)!_k = \left[n!_k + \sum_{d\in S}\frac{x^{d_i}n!_k}{d}\right]x$$

where x is less than k, S is the distinct power set of components of $n!_k$ (e.g. for $15!_5, S = 15, 10, 5$), and d_i is the number of elements of S involved in d.
For example, $18!_5$ gives

$$18!_5 = [15!_5 + 3(15.10 + 15.5 + 10.5) + 9(15 + 10 + 5) + 27]^*3$$

$18!_6 = 5616$, and the sum on the RHS is
$750 + 825 + 270 + 27 = 1872$, and $1872^*3 = 5616$.

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A NOTE ON EXPONENTIAL DIVISORS AND RELATED ARITHMETIC FUNCTIONS

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§1. Introduction

Let $n > 1$ be a positive integer, and $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ its prime factorization. A number $d \mid n$ is called an Exponential divisor (or e-divisor, for short) of n if $d = p_1^{b_1} \cdots p_r^{b_r}$ with $b_i \mid a_i (i = \overline{1,r})$. This notion has been introduced by E.G. Straus and M.V. Subbarao[1]. Let $\sigma_e(n)$, resp. $d_e(n)$ denote the sum, resp. number of e-divisors of n, and let $\sigma_e(1) = d_e(1) = 1$, by convention. A number n is called e-perfect, if $\sigma_e(n) = 2n$. For results and References involving e-perfect numbers, and the arithmetical functions $\sigma_e(n)$ and $d_e(n)$, see [4]. For example, it is well-known that $d_e(n)$ is multiplicative, and

$$d_e(n) = d(a_1) \cdots d(a_r), \tag{1}$$

where $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is the canonical form of n, and $d(a)$ denotes the number of (ordinary) divisors of a.
The e-totient function $\varphi_e(n)$, introduced and studied in [4] is multiplicative, and one has

$$\varphi_e(n) = \varphi(a_1) \cdots \varphi(a_r), \tag{2}$$

where φ is the classical Euler totient function.
Let $\sigma(a)$ denote the sum of (ordinary) divisors of a. The product of e-divisors of n, denoted by $T_e(n)$ has the following expression (see [9]):

$$T_e(n) = p_1^{\sigma(a_1)d(a_2)\cdots d(a_r)} \cdots p_r^{\sigma(a_r)d(a_1)\cdots d(a_r-1)} \tag{3}$$

A number n is called multiplicatively e-perfect if $T_e(n) = n^2$. Based on (3), in [9] we have proved that n is multiplicatively e-perfect iff n can be written as $n = p^m$, where $\sigma(m) = 2m$, and p is a prime. Two notions of exponentially-harmonic numbers have been recently introduced by the author in [11]. Finally, we note that for a given arithmetic function $f : N^* \to N^*$, in [5], [6] we have introduced the minimun function of f by

$$F_f(n) = \min\{k \geq 1 : n \mid f(k)\} \tag{4}$$

Various particular cases, including $f(k) = \varphi(k)$, $f(k) = \sigma(k)$, $f(k) = d(k), f(k) = S(k)$ (Smarandache function), $f(k) = T(k)$ (product of ordinary divisors), have been studiedrecently by the present author. He also studied the duals of these functions (when these have sense) defined by

$$F_f^*(n) = \max\{k \geq 1 : f(k) \mid n\} \tag{5}$$

See e.g. [10] and the References therein.

§2. Main notions and Results

The aim of this note is to introduce certain new arithmetic functions, related to the above considered notions.
Since for the product of ordinary divisors of n one can write

$$T(n) = n^{d(n)/2}, \tag{6}$$

trying to obtain a similar expression for $T_e(n)$ of the product of e-divisors of n, by (3) the following can be written:

Theorem 1.

$$T_e(n) = (t(n))^{d_e(n)/2}, \tag{7}$$

where $d_e(n)$ is the number of exponential divisors of n, given by (1); while the arithmetical function $t(n)$ is given by $t(1) = 1$

$$t(n) = p_1^{2\frac{\sigma(a_1)}{d(a_1)}} \cdots p_r^{2\frac{\sigma(a_r)}{d(a_r)}} \tag{8}$$

$n = p_1^{a_1} \cdots p_r^{a_r}$ being the prime factorization of $n > 1$.

Proof. This follows easily by relation (3), and the definition of $t(n)$ given by (8).

Remark For multiplicatively perfect numbers given by $T(n) = n^2$, see [7]. For multiplicatively deficient numbers, see [8].

Remark that

$$d_e(n) \leq d(n) \tag{9}$$

for all n, with equality only for $n = 1$. Indeed, by $d(a) < a + 1$ for $a \geq 2$, via (1) this is trivial.
On the other hand, the inequality

$$t(n) \leq n \tag{10}$$

is not generally valid. Let e.g. $n = p_1^{q_1} \cdots p_r^{q_r}$, where all q_i $(i = \overline{1,r})$ are primes. Then, by (8) $t(n) = p_1^{q_1+1} \cdots p_r^{q_r+1} = (p_1 \cdots p_r)n > n$. However, there is a particular case, when (10) is always true, namely suppose that $\omega(a_i) \geq 2$ for all $i = \overline{1,r}$ (where $\omega(a)$ denotes the number of distinct prime

factors of a). In [3] it is proved that if $\omega(a) \geq 2$, then $\frac{\sigma(a)}{d(a)} < \frac{a}{2}$. This gives (10) with strict inequality, if the above conditions are valid.
Without any condition one can prove:

Theorem 2. *For all* $n \geq 1$,

$$T_e(n) \leq T(n), \tag{11}$$

with equality only for $n = 1$ *and* $n = prime$.

Proof. The inequality to be proved becomes

$$\left(p_1^{\frac{\sigma(a_1)}{d(a_1)}} \cdots p_r^{\frac{\sigma(a_r)}{d(a_r)}}\right)^{d(a_1)\cdots d(a_r)} \leq (p_1^{a_1} \cdots p_r^{a_r})^{(a_1+1)\cdots(a_r+1)/2} \tag{12}$$

We will prove that

$$\frac{\sigma(a_1)}{d(a_1)} d(a_1) \cdots d(a_r) \leq \frac{a_1(a_1+1)\cdots(a_r+1)}{2}$$

with equality only if $r = 1$ and $a_1 = 1$. Indeed, it is known that (see [2]) $\frac{\sigma(a_1)}{d(a_1)} \leq \frac{a_1+1}{2}$, with equality only for $a_1 = 1$ and $a_1 = prime$. On the other hand, $d(a_1) \cdots d(a_r) \leq a_1(a_2+1)\cdots(a_r+1)$ is trivial by $d(a_1) \leq a_1$, $d(a_2) < a_2 + 1, \cdots, d(a_r) < a_r + 1$, with equality only for $a_1 = 1$ and $r = 1$. Thus (12) follows, with equality for $r = 1$, $a_1 = 1$, so $n = p_1 = prime$ for $n > 1$.
Remark In [4] it is proved that

$$\varphi_e(n) d_e(n) \geq a_1 \cdots a_r \tag{13}$$

Now, by (2), $d_e(n) \geq \frac{a_1}{\varphi(a_1)} \cdots \frac{a_r}{\varphi(a_r)} \geq 2^r$ if all a_i $(i = \overline{1,r})$ are even, since it is well-known that $\varphi(a) \leq \frac{a}{2}$ for $a =$even. Since $d(n) = (a_1+1)\cdots(a_r+1) \leq 2^{a_1} \cdots 2^{a_r} = 2^{a_1+\cdots+a_r} = 2^{\Omega(n)}$ (where $\Omega(n)$ denotes the total number of prime divisors of n), by (9) one can write:

$$2^{\omega(n)} \leq d_e(n) \leq 2^{\Omega(n)} \tag{14}$$

if all a_i are even, i.e. when n is a perfect square (right side always).
Similarly, in [4] it is proved that

$$\varphi_e(n) d_e(n) \geq \sigma(a_1) \cdots \sigma(a_r) \tag{15}$$

when all a_i $(i = \overline{1,r})$ are odd. Let all $a_i \geq 3$ be odd. Then, since $\sigma(a_i) \geq a_i + 1$ (with equality only if $a_i = prime$), (15) implies

$$\varphi_e(n) d_e(n) \geq d(n), \tag{16}$$

which is a converse to inequality (9).
Let now introduce the arithmetical function $t_1(n) = p_1^{2\sqrt{a_1}} \cdots p_r^{2\sqrt{a_r}}$, $t_1(1) = 1$ and let $\gamma(n) = p_1 \cdots p_r$ denote the "core" of n (see [2]). Then:

Theorem 3.

$$t_1(n) \geq t(n) \geq n\gamma(n) \quad for \ \ all \ \ n \geq 1. \tag{17}$$

Proof. This follows at once by the known double-inequality

$$\sqrt{a} \leq \frac{\sigma(a)}{d(a)} \leq \frac{a+1}{2}, \tag{18}$$

with equality for $a = 1$ on the left side, and for $a = 1$ and $a = prime$ on the right side. Therefore, in (17) one has equality when n is squarefree, while on the right side if n is squarefree, or $n = p_1^{q_1} \cdots p_r^{q_r}$ with all q_i $(i = \overline{1,r})$ primes. Clearly, the functions $t_1(n)$, $t(n)$ and $\gamma(n)$ are all multiplicative.
Finally, we introduce the minimun exponential totient function by (4) for $f(k) = \varphi_e(k)$:

$$E_e(n) = \min\{k \geq 1 : n \mid \varphi_e(k)\}, \tag{19}$$

where $\varphi_e(k)$ is the e-totient function given by (2). Let

$$E(n) = \min\{k \geq 1 : n \mid \varphi(k)\} \tag{20}$$

be the Euler minimum function (see [10]). The following result is true:

Theorem 4.

$$E_e(n) = 2^{E(n)} \quad for \ \ n > 1. \tag{21}$$

Proof. Let $k = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$. Then $k \geq 2^{\alpha_1 + \cdots + \alpha_s} \geq 2^s$. Let s be the least integer with $n \mid \varphi(s)$ (i.e. $s = E(n)$ by (20)). Clearly $\varphi_e(2^s) = \varphi(s)$, so $k = 2^s$ is the least $k \geq 1$ with property $n \mid \varphi_e(k)$. This finishes the proof of (21). For properties of $E(n)$, see [10].
Remark It is interesting to note that the "maximum e-totient", i.e.

$$E_e^*(n) = \max\{k \geq 1 : \varphi_e(k) \mid n\} \tag{22}$$

is not well defined. Indeed, e.g. for all primes p one has $\varphi_e(p) = 1 \mid n$, and $E_e^*(p) = +\infty$, so $E_e^*(n)$ given by (22) is not an arithmetic function.

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SMARANDACHE MULTIPLICATIVE FUNCTION

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Abstract The main purpose of this paper is using the elementary method to study the mean value properties of the Smarandache multiplicative function, and give an interesting asymptotic formula for it.

Keywords: Smarandache multiplicative function; Mean Value; Asymptotic formula.

§1. Introduction

For any positive integer n, we define $f(n)$ as a Smarandache multiplicative function, if $f(ab) = \max(f(a), f(b))$, $(a, b) = 1$. Now for any prime p and any positive integer α, we taking $f(p^\alpha) = \alpha p$. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, then

$$f(n) = \max_{1 \le i \le k} \{f(p_i^{\alpha_i})\} = \max_{1 \le i \le k} \{\alpha_i p_i\}.$$

Now we define $P_d(n)$ as another new arithmetical function. We let

$$P_d(n) = \prod_{d|n} d = n^{\frac{d(n)}{2}}, \qquad (1)$$

where $d(n) = \sum_{d|n} 1$ is the Dirichlet divisor function.

It is clear that $f(P_d(n))$ is a new Smarandache multiplicative function. About the arithmetical properties of $f(n)$, it seems that none had studied it before. This function is very important, because it has many similar properties with the Smarandache function $S(n)$. The main purpose of this paper is to study the mean value properties of $f(P_d(n))$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \ge 2$, we have the asymptotic formula*

$$\sum_{n \le x} f(P_d(n)) = \frac{\pi^4}{72} \frac{x^2}{\ln x} + C \cdot \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right),$$

where $C = \dfrac{5\pi^4}{288} + \dfrac{1}{2}\sum\limits_{n=1}^{\infty}\dfrac{d(n)\ln n}{n^2}$ *is a constant.*

§2. Proof of the Theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. For convenience, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the prime powers factorization of n, and $P(n)$ be the greatest prime factor of n, that is, $P(n) = \max\limits_{1\le i\le k}\{p_i\}$. Then we have

Lemma. *For any positive integer* n*, if there exists* $P(n)$ *such that* $P(n) > \sqrt{n}$*, then we have the identity*

$$f(n) = P(n).$$

Proof. From the definition of $P(n)$ and the condition $P(n) > \sqrt{n}$, we get

$$f(P(n)) = P(n). \qquad (2)$$

For other prime divisors p_i of n $(1 \le i \le k \quad and \quad p_i \ne P(n))$, we have

$$f(p_i^{\alpha_i}) = \alpha_i p_i.$$

Now we will debate the upper bound of $f(p_i^{\alpha_i})$ in three cases:

(I) If $\alpha_i = 1$, then $f(p_i) = p_i \le \sqrt{n}$.

(II) If $\alpha_i = 2$, then $f(p_i^2) = 2p_i \le 2\cdot n^{\frac{1}{4}} \le \sqrt{n}$.

(III) If $\alpha_i \ge 3$, then $f(p_i^{\alpha_i}) = \alpha_i\cdot p_i \le \alpha_i\cdot n^{\frac{1}{2\alpha_i}} \le n^{\frac{1}{2\alpha_i}}\cdot\frac{\ln n}{\ln p_i} \le \sqrt{n}$, where we use the fact that $\alpha \le \frac{\ln n}{\ln p}$, if $p^\alpha | n$.

Combining (I)-(III), we can easily obtain

$$f(p_i^{\alpha_i}) \le \sqrt{n}. \qquad (3)$$

From (2) and (3), we deduce that

$$f(n) = \max_{1\le i\le k}\{f(p_i^{\alpha_i})\} = f(P(n)) = P(n).$$

This completes the proof of Lemma.

Now we use the above Lemma to complete the proof of Theorem. First we define two sets A and B as following:

$$A = \{n | n \le x, P(n) \le \sqrt{n}\} \quad and \quad B = \{n | n \le x, P(n) > \sqrt{n}\}.$$

Using the Euler summation formula, we may get

$$\sum_{n\in A} f(P_d(n)) \ll \sum_{n\in A} P(n)d(n) \ll \sum_{n\le x}\sqrt{x}d(n) \ll x^{\frac{3}{2}}\ln x. \qquad (4)$$

For another part of the summation, since $P(n) = p$, we can assume that $n = pl$, where $p > l$ and $(p, l) = 1$. Note that

$$P_d(n) = n^{\frac{d(n)}{2}} = (pl)^{\frac{d(lp)}{2}} = (pl)^{d(l)}$$

and

$$f(P_d(n)) = f((pl)^{d(l)}) = f(p^{d(l)}) = d(l)p,$$

we have

$$\begin{aligned}
& \sum_{n\in B} f(P_d(n) \\
= & \sum_{\substack{pl\le x \\ p>\sqrt{pl}}} d(l)p = \sum_{\substack{pl\le x \\ p>l}} d(l)p = \sum_{p\le x} p \sum_{\substack{l\le \frac{x}{p} \\ l<p}} d(l) \\
= & \sum_{\sqrt{x}\le p\le x} p \sum_{l<\frac{x}{p}} d(l) + \sum_{p\le\sqrt{x}} p \sum_{l<p} d(l) \\
= & \sum_{p\le x} p \sum_{l<\frac{x}{p}} d(l) + O\left(\sum_{p\le\sqrt{x}} p \sum_{l<p} d(l)\right) + O\left(\sum_{p\le\sqrt{x}} p \sum_{l<\frac{x}{p}} d(l)\right) \\
= & \sum_{p\le\sqrt{x}} p \sum_{l<\frac{x}{p}} d(l) + \sum_{l\le\sqrt{x}} d(l) \sum_{p<\frac{x}{l}} p - \left(\sum_{p\le\sqrt{x}} p\right)\left(\sum_{l\le\sqrt{x}} d(l)\right) \\
& + O\left(\sum_{p\le\sqrt{x}} p \sum_{l<\frac{x}{p}} d(l)\right) + O\left(\sum_{p\le\sqrt{x}} p \sum_{l<\frac{x}{p}} d(l)\right), \qquad (5)
\end{aligned}$$

where we have used Theorem 3.17 of [3]. Note that the asymptotic formula (see Theorem 3.3 of [3])

$$\sum_{n\le x} d(n) = x\ln x + (2\gamma - 1)x + O\left(\sqrt{x}\right) \ll x\ln x, \zeta(2) = \frac{\pi^2}{6}$$

(where γ is the Euler constant) and

$$\pi(x) = \frac{x}{\ln x} + \frac{x}{\ln^2 x} + \frac{2x}{\ln^3 x} + O\left(\frac{x}{\ln^4 x}\right),$$

we have

$$\sum_{p\le x} p \sum_{l<\frac{x}{p}} d(l) = \sum_{p\le x} p\left[\frac{x}{p}\ln\frac{x}{p} + (2\gamma - 1)\frac{x}{p} + O\left(\sqrt{\frac{x}{p}}\right)\right] \ll x^{\frac{3}{2}} \qquad (6)$$

$$\sum_{p\le\sqrt{x}} p\sum_{l<p} d(l) \ll \sum_{p\le\sqrt{x}} p^2\ln p \ll x^{\frac{3}{2}} \qquad (7)$$

$$\sum_{p\le\sqrt{x}} p\sum_{l<\frac{x}{p}} d(l) \ll \sum_{p\le\sqrt{x}} p\times\frac{x}{p}\ln\frac{x}{p} \ll x^{\frac{3}{2}}. \qquad (8)$$

and

$$\left(\sum_{p\le\sqrt{x}} p\right)\left(\sum_{l\le\sqrt{x}} d(l)\right) \ll x^{\frac{3}{2}} \qquad (9)$$

Applying Abel's identity (Theorem 4.2 of [3]) we also have

$$\begin{aligned}\sum_{l\le\sqrt{x}} d(l)\sum_{p<\frac{x}{l}} p &= \sum_{l\le\sqrt{x}} d(l)\left[\frac{x}{l}\pi(\frac{x}{l}) - \int_2^{\frac{x}{l}}\pi(y)dy\right]\\ &= \sum_{l\le\sqrt{x}} d(l)\left[\frac{1}{2}\frac{x^2}{l^2\ln\frac{x}{l}} + \frac{5}{8}\frac{x^2}{l^2\ln^2\frac{x}{l}} + O\left(\frac{x^2}{l^2\ln^3 x}\right)\right]\\ &= \frac{\pi^4}{72}\frac{x^2}{\ln x} + C\cdot\frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right), \qquad (10)\end{aligned}$$

where $C = \dfrac{5\pi^4}{288} + \dfrac{1}{2}\sum\limits_{n=1}^{\infty}\dfrac{d(n)\ln n}{n^2}$ is a constant.

Combining (5), (6), (7),(8),(9) and (10) we may immediately deduce the asymptotic formula

$$\sum_{n\le x} f(P_d(n)) = \frac{\pi^4}{72}\frac{x^2}{\ln x} + C\cdot\frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right).$$

This completes the proof of Theorem.

Note. Substitute to

$$\sum_{n\le x} d(n) = x\ln x + (2\gamma-1)x + O\left(\sqrt{x}\right) \ll x\ln x, \zeta(2) = \frac{\pi^2}{6}$$

and

$$\pi(x) = \frac{x}{\ln x} + \frac{x}{\ln^2 x} + \frac{2x}{\ln^3 x} + O\left(\frac{x}{\ln^4 x}\right),$$

we can get a more accurate asymptotic formula for $\sum\limits_{n\le x} f(P_d(n))$.

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TWO FORMULAS FOR SMARANDACHE LCM RATIO SEQUENCES

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Abstract In this paper, a reduction formula for Smarandache LCM ratio sequences SLR(6)and SLR(7) are given.

Keywords: Smarandache LCM ratio sequences; Reduction formula.

§1. Introduction

Let $(x_1, x_2, ..., x_t)$ and $[x_1, x_2, ..., x_t]$ denote the greatest common divisor and the least common multiple of any positive integers $x_1, x_2, ..., x_t$ respectively. Let r be a positive integer with $r > 1$. For any positive integer n, let $T(r, n) = \frac{[n,n+1,...,n+r-1]}{[1,2,...r]}$, then the sequences $SLR(r) = \{T(r, n)\}_\infty$ is called Smarandache LCM ratio sequences of degree r. In reference [1], Maohua Le studied the properties of $SLR(r)$, and gave two reduction formulas for SLR(3) and SLR(4). In this paper, we will study the calculating problem of $SLR(6)$ and $SLR(7)$, and prove the following:

Theorem 1. For any positive integer n, we have the following result:
If $n \equiv 0, 15 \bmod 20$, then

$$SLR(6) = \frac{1}{7200}n(n+1)(n+2)(n+3)(n+4)(n+5);$$

If $n \equiv 1, 2, 6, 9, 13, 14, 17, 18 \bmod 20$, then

$$SLR(6) = \frac{1}{720}n(n+1)(n+2)(n+3)(n+4)(n+5);$$

If $n \equiv 5, 10 \bmod 20$, then

$$SLR(6) = \frac{1}{3600}n(n+1)(n+2)(n+3)(n+4)(n+5);$$

If $n \equiv 3, 4, 7, 8, 11, 12, 16, 19 \bmod 20$, then

$$SLR(6) = \frac{1}{1440}n(n+1)(n+2)(n+3)(n+4)(n+5).$$

Theorem 2. For any positive integer n, we have the following
If $n \equiv 0, 24, 30, 54 \bmod 60$, then

$$SLR(7) = \frac{1}{302400}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6);$$

If $n \equiv 1, 13, 17, 37, 41, 53 \bmod 60$, then

$$SLR(7) = \frac{1}{5040}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6);$$

If $n \equiv 2, 8, 16, 22, 26, 28, 32, 38, 46, 52, 56, 58 \bmod 60$, then

$$SLR(7) = \frac{1}{20160}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6);$$

If $n \equiv 3, 27, 51 \bmod 60$, then

$$SLR(7) = \frac{1}{30240}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6);$$

If $n \equiv 4, 10, 14, 20, 34, 40, 44, 50 \bmod 60$, then

$$SLR(7) = \frac{1}{100800}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6);$$

If $n \equiv 5, 25, 29, 49 \bmod 60$, then

$$SLR(7) = \frac{1}{25200}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6);$$

If $n \equiv 6, 12, 18, 36, 42, 48 \bmod 60$, then

$$SLR(7) = \frac{1}{60480}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6);$$

If $n \equiv 7, 11, 23, 31, 43, 47 \bmod 60$, then

$$SLR(7) = \frac{1}{10080}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6);$$

If $n \equiv 9, 45 \bmod 60$, then

$$SLR(7) = \frac{1}{75600}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6);$$

If $n \equiv 15, 39 \bmod 60$, then

$$SLR(7) = \frac{1}{151200}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6);$$

If $n \equiv 19, 55, 59, 35 \bmod 60$, then

$$SLR(7) = \frac{1}{50400}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6);$$

If $n \equiv 21, 33, 57 \bmod 60$, then

$$SLR(7) = \frac{1}{15120}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

§2. Proof of the theorem

To complete the proof of Theorem, we need following several simple Lemmas.

Lemma 1. For any positive integer a and b, we have $(a,b)[a,b] = ab$.

Lemma 2. For any positive integer s with $s < t$, we have

$$(x_1, x_2, ..., x_t) = ((x_1, ..., x_s), (x_{s+1}, ..., x_t))$$

and

$$[x_1, x_2, ..., x_t] = [[x_1, ..., x_s], [x_{s+1}, ..., x_t]].$$

Lemma 3. For any positive integer n, we have

$$T(4,n) = \begin{cases} \frac{1}{24}n(n+1)(n+2)(n+3), & \text{if } n \equiv 1,2 \bmod 3; \\ \frac{1}{72}n(n+1)(n+2)(n+3), & \text{if } n \equiv 0 \bmod 3. \end{cases}$$

Lemma 4. For any positive integer n, we have

$$T(5,n) = \begin{cases} \frac{1}{1440}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 0,8 \bmod 12; \\ \frac{1}{120}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 1,7 \bmod 12; \\ \frac{1}{720}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 2,6 \bmod 12; \\ \frac{1}{360}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 3,5,9,11 \bmod 12; \\ \frac{1}{480}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 4 \bmod 12; \\ \frac{1}{240}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 10 \bmod 12. \end{cases}$$

The proof of Lemma 1 and Lemma 2 can be found in [3], Lemma 3 was proved in [1]. Lemma 4 was proved in [4].

In the following, we shall use these Lemmas to complete the proof of Theorem 1. In fact, from the properties of the least common multiple of any positive integers, we know that

$$\begin{aligned} &[n, n+1, n+2, n+3, n+4, n+5] = [[n, n+1, n+2, n+3, n+4], n+5] \\ &= \frac{[n, n+1, n+2, n+3, n+4](n+5)}{([n, n+1, n+2, n+3, n+4], n+5)}. \end{aligned} \tag{1}$$

Note that $[1,2,3,4,5,6] = 60$, $[1,2,3,4,5] = 60$ and

$$([n, n+1, n+2, n+3, n+4], n+5)$$

$$= \begin{cases} 5, & \text{if } n \equiv 0, 20, 30, 50 \bmod 60; \\ 6, & \text{if } n \equiv 1, 13, 31, 49 \bmod 60; \\ 1, & \text{if } n \equiv 2, 6, 8, 12, 14, 18, 24, 26, 32, 38, 42, 44, 48, 54, 56 \bmod 60; \\ 4, & \text{if } n \equiv 3, 11, 23, 27, 39, 47, 51, 59 \bmod 60; \\ 3, & \text{if } n \equiv 4, 16, 22, 28, 34, 46, 52, 58 \bmod 60; \\ 10, & \text{if } n \equiv 5, 45 \bmod 60; \\ 2, & \text{if } n \equiv 9, 17, 21, 29, 33, 41, 53, 57 \bmod 60; \\ 15, & \text{if } n \equiv 10, 40 \bmod 60; \\ 20, & \text{if } n \equiv 15, 35 \bmod 60; \\ 12, & \text{if } n \equiv 7, 19, 31, 43 \bmod 60; \\ 30, & \text{if } n \equiv 25 \bmod 60; \\ 60, & \text{if } n \equiv 55 \bmod 60. \end{cases} \tag{2}$$

Now Theorem 1 follows from Lemma 3.

The proof of Theorem 2 is similar to the proof of Theorem 1. From the properties of the least common multiple of any positive integers, we know that

$$\begin{aligned} &[n, n+1, n+2, n+3, n+4, n+5, n+6] \\ &= [[n, n+1, n+2, n+3, n+4, n+5], n+6] \\ &= \frac{[n, n+1, n+2, n+3, n+4, n+5](n+6)}{([n, n+1, n+2, n+3, n+4, n+5], n+6)}. \end{aligned} \tag{3}$$

Note that $[1, 2, 3, 4, 5, 6, 7] = 420$, and

$$([n, n+1, n+2, n+3, n+4, n+5], n+6)$$

$$= \begin{cases} 6, & \text{if } n \equiv 0, 12, 36, 48 \bmod 60; \\ 1, & \text{if } n \equiv 1, 5, 7, 11, 13, 17, 23, 25, 31, 35, 37, 41, 43, 47, 53, 55 \bmod 60; \\ 4, & \text{if } n \equiv 2, 10, 22, 26, 38, 46, 50, 58 \bmod 60; \\ 3, & \text{if } n \equiv 3, 15, 21, 27, 33, 45, 51, 57 \bmod 60; \\ 10, & \text{if } n \equiv 4, 44 \bmod 60; \\ 12, & \text{if } n \equiv 6, 18, 30, 42 \bmod 60; \\ 2, & \text{if } n \equiv 8, 16, 20, 28, 32, 40, 52, 56 \bmod 60; \\ 15, & \text{if } n \equiv 9, 39 \bmod 60; \\ 20, & \text{if } n \equiv 14, 34 \bmod 60; \\ 5, & \text{if } n \equiv 19, 29, 49, 59 \bmod 60; \\ 30, & \text{if } n \equiv 24 \bmod 60; \\ 60, & \text{if } n \equiv 54 \bmod 60. \end{cases} \tag{4}$$

Now Theorem 2 follows from Theorem 1.

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THE 97-TH PROBLEM OF F.SMARANDACHE *

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Abstract The main purpose of this paper is using the analytic method to study the n-ary sieve sequence, and solved one conjecture about this sequence.

Keywords: Quad 97-th problem of F.Smarandache; n-ary sieve sequence; Conjecture.

§1. Introduction and results

In 1991, American-Romanian number theorist Florentin Smarandache introduced hundreds of interesting sequences and arithmetical functions, and presented 105 unsolved arithmetical problems and conjectures about these sequences and functions in book [1]. Already many researchers studied these sequences and functions from this book, and obtained important results. Among these problems, the 97-th unsolved problem is:

Let n be any positive integer with $n \geq 2$, starting to count on the natural numbers set at any step from 1:

— delete every n-th number;

— delete from the remaining ones, every (n^2)-th number;

$\cdots\cdots$;

and so on: delete from the remaining ones, every (n^k)-th number, $k = 1, 2, 3, \cdots$.

For this special sequence, there are two conjectures:

(1) there are an infinity of primes that belong to this sequence;

(2) there are an infinity of number of this sequence which are not prime.

In this paper, we shall use the analytic method to study the n-ary sieve sequence, and solved conjecture (2). That is, we have the following conclusion:

Theorem. For any positive integer $n \geq 2$, the conjecture (2) of n-ary sequence is true.

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§2. Proof of Theorem

In this section, we shall complete the proof of Theorem. For any fixed real number $x \geq 1$ and positive integer k, let $\mathcal{A}_k(x)$ denotes the number of remaining ones after deleting (n^k)-th number from the interval $[1, x]$. In the interval $[1, x]$, for any $n \in [1, x]$, first we delete n-th number from the interval $[1, x]$, then the number of remaining ones is

$$\mathcal{A}_1(x) = [x] - \left[\frac{x}{n}\right],$$

where $[x]$ denotes the greatest integer which is not exceeding x, and $x - 1 \leq [x] \leq x + 1$.

Note that

$$\mathcal{A}_1(x) = [x] - \left[\frac{x}{n}\right] \leq x + 1 - \frac{x}{n} = x\left(1 - \frac{1}{n}\right) + 1, \tag{1}$$

if we delete every (n^2)-th number from the remaining ones, then the number of remaining ones is

$$\mathcal{A}_2(x) = [x] - \left[\frac{x}{n}\right] - \left[\frac{[x] - [\frac{x}{n}]}{n^2}\right].$$

From (1), we have the inequality

$$\begin{aligned} & [x] - \left[\frac{x}{n}\right] - \left[\frac{[x] - [\frac{x}{n}]}{n^2}\right] \qquad (2)\\ \leq & \left[x\left(1 - \frac{1}{n}\right) + 1\right] - \left[\frac{x\left(1 - \frac{1}{n}\right) + 1}{n^2}\right] \\ \leq & x\left(1 - \frac{1}{n}\right) + 2 - \frac{x\left(1 - \frac{1}{n}\right) + 1}{n^2} \\ = & x\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n^2}\right) + \left(2 - \frac{1}{n^2}\right) \\ \leq & x\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n^2}\right) + 2. \end{aligned}$$

$\cdots\cdots$, and so on: if we delete every (n^k)-th number, from the remaining ones, we also have the inequality

$$\mathcal{A}_k(x) = x\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n^2}\right)\cdots\left(1 - \frac{1}{n^k}\right) + k. \tag{3}$$

Similarly, we can also deduce that

$$x\left(1 - \frac{1}{n}\right) - 1 = x - 1 - \frac{x}{n} \leq \mathcal{A}_1(x) = [x] - \left[\frac{x}{n}\right], \tag{4}$$

$$x\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)-2\leq \mathcal{A}_2(x)=[x]-\left[\frac{x}{n}\right]-\left[\frac{[x]-\left[\frac{x}{n}\right]}{n^2}\right], \qquad (5)$$

$\cdots\cdots$, and so on:

$$x\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\cdots\left(1-\frac{1}{n^k}\right)-k\leq \mathcal{A}_k(x). \qquad (6)$$

Combining (5) and (6), we have the asymptotic formula

$$\mathcal{A}_k(x)=x\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\cdots\left(1-\frac{1}{n^k}\right)+O(k). \qquad (7)$$

Note that $k\ll \ln x$, so we have

$$\mathcal{A}_k(x)=x\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\cdots\left(1-\frac{1}{n^k}\right)+O(\ln x). \qquad (8)$$

Let $\pi(x)$ denotes the number of the primes up to x, then we have (see reference [2])

$$\pi(x)=\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right). \qquad (9)$$

Note that $\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\cdots\left(1-\frac{1}{n^k}\right)$ is convergence if $k\to+\infty$, so

$$\mathcal{A}_k(x)-\pi(x)\to+\infty, \qquad \text{if} \quad x\to+\infty.$$

That is, there are an infinity of number of this sequence which are not prime.

This completes the proof of Theorem.

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ON TWO SUBSETS OF GENERALIZED SMARANDACHE PALINDROMES

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Abstract Two subsets of generalized Smarandache palindromes are constructed to determine some of their properties. New sequences, conjectures, and unsolved questions are given.

Palindromes are positive integers that read the same way forward and backward. Example: 53135 is a palindrome.

Generalized Smarandache Palindromes (GSPs) are positive integers of the form

$$r_1 r_2 \cdots r_n r_n \cdots r_2 r_1 \quad \text{or} \quad r_1 r_2 \cdots r_{n-1} r_n r_{n-1} \cdots r_2 r_1$$

where all $r_1, r_2, \cdots, r_n$ are integers consisting of one or more decimal digits ([1], [2]). Example: 2145645621 is a GSP because we can split it into groups such as (21)(456)(456)(21). Note that it must be possible to split any GSP into at least two groups to avoid making every integer a GSP.

In this paper we will consider two simple subsets of GSPs. These subsets will involve numbers of the form $R_1 R_2 R_2 R_1$ where $R_1 = n$ (with $n > 9$ in one case), and $R_2 = f(n)$, where f is a simple function, or $R_1 = f(n)$ and $R_2 = n$. The digital sum of n will be the first function used, which we will denote by ds(n). Note that regular palindromic numbers will not be considered, only "pure" GSPs.

Definition:A Smarandache digital sum GSP (SDG) is a number of the form $R_1 R_2 R_2 R_1$ where $R_1 = n$, with $n > 9$, and $R_2 = ds(n)$. For example, $SDG(13) = 134413$ because $ds(13) = 1 + 3 = 4$, and thus we concatenate 13_4_4_13.

Because we are not concerned with regular palindromic GSPs we will exclude from our formula all integers n such that their digital sum is also palindromic, since these are the only numbers that yield regular palindromes from the concatenation: $n\_ds(n)\_ds(n)\_n$.

A computer program was written to construct SDG numbers and exclude any regular palindromes. The following sequence was produced.

Sequence 1: 101110, 123312, 134413, 145514, 156615, 167716, 178817, 189918, 19101019, 202220, 213321, 235523, 246624, 257725, 268826,

279927, 28101028, 29111129, 303330, 314431, 325532, 347734, 358835, 369936, 37101037, 38111138, ...

The most natural place to start our investigation of the properties of SDGs is by asking if any are prime numbers.

A computer search produced the following n such that SDG(n) is prime.
Sequence 2: 17, 23, 43, 61, 71, 157, 167, 169, 193, 199, 269, 283, 307, 377, 379, 409, 469, 491, 497, 509, 523, 553, 559, 563, 587, 617, 631, 637, 677, 709, 737, 767, 839, 869, 871, 913, 947, 971, 983, 1003, 1039, 1051, 1061, 1067, 1069, 1073, 1081, 1093, 1121, 1123, 1147, 1241, 1243, 1267, 1303, 1369, 1409, 1441, 1451, ...

SDG primes appear to be plentiful. One of the larger primes found was 99877404099877, which can be split like this (99877)(40)(40)(99877) – among other ways – to show that it is a GSP.

It is well known that a regular palindrome > 11 must consist of an odd number of digits for it to be prime. (For a proof, see [3].) This is not the case for SDGs, however. In fact, it is easy to see that due to the way our subset of GSPs are constructed, they will always consist of an even number of decimal digits.

Note that the majority of the values in Sequence 2 are primes themselves, although some are composites.

Conjecture: There are infinitely many prime and composite values n such that SDG(n) is prime.

Unsolved question: Will there be more composite n or more prime n such that SDG(n) is prime?

Three computer searches were conducted to determine whether there were any square, triangular, or Fibonacci SDG values. None were found for all $n < 10^5$.

Unsolved question: Are there any square, triangular, or Fibonacci numbers in Sequence 1?

We now note a fascinating curiosity concerning SDG numbers. But before doing so we need another definition. Peter Wallrodt defined "brilliant numbers" as numbers consisting of two prime factors of the same length in decimal representation [4]. For example, $99973 = 257 * 389$, is a brilliant number since both of its prime factors have 3 digits. Brilliant numbers play a crucial role in testing prime factoring programs.

A computer program was written to search for brilliant SDG numbers. Below are the first 37 values of n (out of 85 found) such that SDG(n) is brilliant.
Sequence 3: 13, 149, 253, 547, 881, 1177, 1247, 1271, 1987, 2359, 3053, 3251, 3371, 4831, 4867, 4937, 5551, 7099, 10187, 10351, 10861, 10883, 11579, 11639, 11717, 11963, 12241, 12347, 12589, 13199, 13871, 14339, 14699, 14861, 14963, 15149, 15287, ...

Conjecture: There are infinitely many brilliant SDG numbers.

It is somewhat surprising that there are so many brilliant SDG numbers. Here is one of the larger ones: $32677252532677 = 3401213 \times 9607529$.

Unsolved question: What is it about the form of SDGs which make them highly susceptible to being brilliant numbers?

Our second subset of GSPs is provided to illustrate the large region of unexplored territory they represent. It will also involve another function, which we will define as $H(n) = ld(n)sd(n)$, where $ld(n)$ is the largest digit of n, and $sd(n)$ is the smallest digit of n, respectively. For example, $H(345) = 125$ because $5^3 = 125$.

Definition: A Smarandache digital power GSP (SDPG) is a number of the form $R_1R_2R_2R_1$ where $R_1 = H(n)$, and $R_2 = n$. For example, SDGP$(24) =$ 16242416 because $H(24) = 42 = 16$, and thus we concatenate $16\_24\_24\_16$.

A computer program was written to construct SDPG numbers and exclude any regular palindromes. The following sequence was produced.

Sequence 4: 273327, 25644256, 3125553125, 466566646656, 82354377823543, 167772168816777216, 38742048999387420489, 110101, 212122, 313133, 414144, 515155, 616166, 717177, 818188, 919199, 120201, 221212, 923239, 16242416, 25252525, 36262636, 49272749, 64282864, 81292981, 130301, 331313, 932329, 27333327, ...

Sequence 4 is more erratic than Sequence 1. And it is interesting that even though we switched the order of concatenation of n and $f(n)$ with this subset of GSPs, we shall see that they still share many of the same properties with SDGs. We close with some data and conjectures concerning Sequence 4.

The sequence n such that SDGP(n) is prime begins: 13, 23, 40, 59, 70, 89, 90, 229, 292, 293, 329, 392, 529, 692, 796, 949, 964, 982, 1000, 1002, 1017, 1018, 1024, 1033, 1035, 1063, 1068, 1069, 1074, ...

Conjecture: Sequence is infinite.

Computer searches revealed that there are no square, triangular, or Fibonacci SDGPs for all $n < 10^4$.

Conjecture: None exist.

The sequence of n such that SDGP(n) is brilliant begins: 30, 1003, 1006, 1054, 1327, 1707, 2070, 2076, 2077, 2089, 2208, 2250, 2599, 2620, 2701, 3004, 3007, 3037, 3505, 3700, 3807, 3820, 3909, 4045, ...

Conjecture: Sequence is infinite.

What other functions would be interesting to introduce into the construction of GSPs?

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THE SMARANDACHE FACTORIAL SEQUENCE *

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Abstract The main purpose of this paper is using the elementary method to study the asymptotic properties of the Smarandache factorial sequence, and give an interesting asymptotic formula.

Keywords: Factorial part; Mean value; Asymptotic formula.

§1. Introduction and result

According to reference [1], for any positive integer n, let $F(n)$ denotes the inferior factorial part of n. That is, $F(n)$ denotes the largest factorial less than or equal to n. For example, $F(1)$=1,$F(2)$=2,$F(3)$=2,$F(4)$=2,$F(5)$=2,$F(6)$=3, $\cdots$. On the other hand, $f(n)$ is called the superior factorial part of n if $f(n)$ is the smallest factorial greater than or equal to n. For example, $f(1) = 1, f(2) = 2, f(3) = 3, f(4) = 3, f(5) = 3, f(6) = 3, \cdots$ are all superior factorial part. In reference [1], Professor F. Smarandache asked us to study the properties of the factorial part. About this problem, it seems that none had studied it, at least we have not seen such a paper before. In this paper, we use the elementary method to study the mean value properties of the factorial part, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. Let $x \geq 1$, $\{a(n)\}$ denotes the set of $F(n)$, Then we have the asymptotic formula

$$\sum_{\substack{n=1 \\ a(n)\leq x}}^{\infty} \frac{1}{a(n)} = \frac{\ln^2 x}{2(\ln\ln x)^2} + O\left(\frac{\ln^2 x \ln\ln\ln x}{(\ln\ln x)^3}\right).$$

§2. A Lemma

To complete the proof of the theorem, we need the lemma

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Lemma. For any $x \geq 1$ and any fixed positive integer $n > 2$, let $n! \leq x < (n+1)!$, then we have the asymptotic formula

$$n = \frac{\ln x}{\ln \ln x} + O\left(\frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2}\right).$$

Proof. First noting the formula $n! \leq x < (n+1)!$, we take logarithm on both sides, then we have

$$\sum_{t=1}^{n} \ln t \leq \ln x < \sum_{t=1}^{n+1} \ln t.$$

Taking $f(t) = \ln t$ in Euler's summation formula [2] we obtain:

$$n \ln n - n + O(1) \leq \ln x \leq n \ln n - n + \ln n + O(1).$$

That is,

$$\ln x = n \ln n - n + O(\ln n). \tag{1}$$

From (1), we have $n = \frac{\ln x}{\ln n} + \frac{n}{\ln n} + O(1)$ and take logarithm on both sides, we easily get the main term of $\ln n$, that is: $\ln n = \ln \ln x + O(\ln \ln \ln x)$. So we get the asymptotic formula

$$n = \frac{\ln x}{\ln n} + O\left(\frac{n \ln n}{\ln^2 n}\right) = \frac{\ln x}{\ln \ln x} + O\left(\frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2}\right).$$

This completes the proof of the lemma.

§3. Proof of the theorem

In this section, we complete the proof of the theorem. Let $a(n)$ denotes the set of all the inferior factorial part, from the above lemma, we may have

$$\begin{aligned}
\sum_{\substack{n=1 \\ a(n) \leq x}}^{\infty} \frac{1}{a(n)} &= \sum_{n \leq m} \frac{nn!}{n!} = \sum_{n \leq m} n = \frac{m(m+1)}{2} \\
&= \frac{1}{2}\left(\frac{\ln x}{\ln \ln x} + O\left(\frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2}\right)\right)^2 + O\left(\frac{\ln x}{\ln \ln x}\right) \\
&= \frac{\ln^2 x}{2(\ln \ln x)^2} + O\left(\frac{\ln^2 x \ln \ln \ln x}{(\ln \ln x)^3}\right).
\end{aligned}$$

And then, we can use the same method to get the same result on the superior factorial part. This completes the proof of the theorem.

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THE SMARANDACHE MULTIPLICATIVE FUNCTION

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Abstract For any positive integer n, we define $f(n)$ as a Smarandache multiplicative function, if $f(ab) = \max(f(a), f(b))$, $(a, b) = 1$. Now for any prime p and any positive integer α, we take $f(p^\alpha) = \alpha p$. It is clear that $f(n)$ is a Smarandache multiplicative function. In this paper, we study the mean value properties of $f(n)$, and give an interesting mean value formula for it.

Keywords: Smarandache multiplicative function; Mean Value; Asymptotic formula.

§1 Introduction and results

For any positive integer n, we define $f(n)$ as a Smarandache multiplicative function, if $f(ab) = \max(f(a), f(b))$, $(a, b) = 1$. Now for any prime p and any positive integer α, we take $f(p^\alpha) = \alpha p$. It is clear that $f(n)$ is a new Smarandache multiplicative function, and if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, then

$$f(n) = \max_{1 \le i \le k} \{f(p_i^{\alpha_i})\} = \max_{1 \le i \le k} \{\alpha_i p_i\}. \tag{1}$$

About the arithmetical properties of $f(n)$, it seems that none had studied it before. This function is very important, because it has many similar properties with the Smarandache function $S(n)$ (see reference [1][2]). The main purpose of this paper is to study the mean value properties of $f(n)$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. For any real number $x \ge 2$, we have the asymptotic formula

$$\sum_{n \le x} f(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

§2 Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. For convenience, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime powers factorization of n, and $P(n)$ be the greatest prime factor of n, that is, $P(n) = \max_{1 \le i \le k} \{p_i\}$. Then we have

Lemma. For any positive integer n, if there exists $P(n)$ such that $P(n) > \sqrt{n}$, then we have the identity

$$f(n) = P(n).$$

Proof. From the definition of $P(n)$ and the condition $P(n) > \sqrt{n}$, we get

$$f(P(n)) = P(n). \tag{2}$$

For other prime divisors p_i of n ($1 \leq i \leq k$ and $p_i \neq P(n)$), we have

$$f(p_i^{\alpha_i}) = \alpha_i p_i.$$

Now we will debate the upper bound of $f(p_i^{\alpha_i})$ in three cases:

(I) If $\alpha_i = 1$, then $f(p_i) = p_i \leq \sqrt{n}$.

(II) If $\alpha_i = 2$, then $f(p_i^2) = 2p_i \leq 2 \cdot n^{\frac{1}{4}} \leq \sqrt{n}$.

(III) If $\alpha_i \geq 3$, then $f(p_i^{\alpha_i}) = \alpha_i \cdot p_i \leq \alpha_i \cdot n^{\frac{1}{2\alpha_i}} \leq n^{\frac{1}{2\alpha_i}} \cdot \frac{\ln n}{\ln p_i} \leq \sqrt{n}$, where we use the fact that $\alpha \leq \frac{\ln n}{\ln p}$ if $p^{\alpha} | n$.

Combining (I)-(III), we can easily obtain

$$f(p_i^{\alpha_i}) \leq \sqrt{n}. \tag{3}$$

From (2) and (3), we deduce that

$$f(n) = \max_{1 \leq i \leq k} \{f(p_i^{\alpha_i})\} = f(P(n)) = P(n).$$

This completes the proof of Lemma.

Now we use the above Lemma to complete the proof of the theorem. First we define two sets $\mathcal{A}$ and $\mathcal{B}$ as following:

$$\mathcal{A} = \{n | n \leq x, P(n) \leq \sqrt{n}\}, \qquad \mathcal{B} = \{n | n \leq x, P(n) > \sqrt{n}\}.$$

Using the Euler summation formula (see reference [3]), we may get

$$\begin{aligned} \sum_{n \in \mathcal{A}} f(n) &\ll \sum_{n \leq x} \sqrt{n} \ln n \\ &= \int_1^x \sqrt{t} \ln t dt + \int_1^x (t - [t])(\sqrt{t} \ln t)' dt + \sqrt{x} \ln x (x - [x]) \\ &\ll x^{\frac{3}{2}} \ln x. \end{aligned} \tag{4}$$

Similarly, from the Abel's identity we also have

$$\begin{aligned} \sum_{n \in \mathcal{B}} f(n) &= \sum_{\substack{n \leq x \\ P(n) > \sqrt{n}}} P(n) = \sum_{n \leq \sqrt{x}} \sum_{n \leq p \leq \frac{x}{n}} p \\ &= \sum_{n \leq \sqrt{x}} \sum_{\sqrt{x} \leq p \leq \frac{x}{n}} p + O\left(\sum_{n \leq \sqrt{x}} \sum_{n \leq p \leq \frac{x}{n}} \sqrt{x} \right) \\ &= \sum_{n \leq \sqrt{x}} \left(\frac{x}{n} \pi\left(\frac{x}{n}\right) - \sqrt{x} \pi(\sqrt{x}) - \int_{\sqrt{x}}^{\frac{x}{n}} \pi(s) ds \right) + O\left(x^{\frac{3}{2}} \ln x \right), \end{aligned} \tag{5}$$

where $\pi(x)$ denotes all the numbers of prime which is not exceeding x. Note that

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right),$$

from (5) we have

$$\begin{aligned}\sum_{\sqrt{x}\le p\le \frac{x}{n}} &= \frac{x}{n}\pi\left(\frac{x}{n}\right) - \sqrt{x}\pi(\sqrt{x}) - \int_{\sqrt{x}}^{\frac{x}{n}} \pi(s)ds \\ &= \frac{1}{2}\cdot\frac{x^2}{n^2\ln x/n} - \frac{1}{2}\cdot\frac{x}{\ln\sqrt{x}} + O\left(\frac{x^2}{n^2\ln^2 x/n}\right) \\ &\quad + O\left(\frac{x}{\ln^2\sqrt{x}}\right) + O\left(\frac{x^2}{n^2\ln^2 x/n} - \frac{x}{\ln^2\sqrt{x}}\right). \qquad (6)\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n\le\sqrt{x}} \frac{x^2}{n^2\ln x/n} &= \sum_{n\le\ln^2 x} \frac{x^2}{n^2\ln x/n} + O\left(\sum_{\ln^2 x\le n\le\sqrt{x}} \frac{x^2}{n^2\ln x}\right) \\ &= \frac{\pi^2}{6}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right), \qquad (7)\end{aligned}$$

and

$$\sum_{n\le\sqrt{x}} \frac{x^2}{n^2\ln^2 x/n} = O\left(\frac{x^2}{\ln^2 x}\right). \qquad (8)$$

From (4), (5), (6), (7) and (8) , we may immediately deduce that

$$\begin{aligned}\sum_{n\le x} f(n) &= \sum_{n\in\mathcal{A}} f(n) + \sum_{n\in\mathcal{B}} f(n) \\ &= \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).\end{aligned}$$

This completes the proof of the theorem.

Note. If we use the asymptotic formula

$$\pi(x) = \frac{x}{\ln x} + \frac{c_1 x}{\ln^2 x} + \cdots + \frac{c_m x}{\ln^m x} + O\left(\frac{x}{\ln^{m+1} x}\right)$$

to substitute

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$$

in (5) and (6), we can get a more accurate asymptotic formula for $\sum_{n\le x} f(n)$.

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ON CONSECUTIVE VALUES OF THE SMARANDACHE FUNCTION

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Abstract Implementing Jon Perry's algorithm to efficiently compute S(n), a search was conducted to find solutions for |S(n+1) - S(n)| = 1 for all $n \leq 225$, and 264 solutions were found.

In 1996 L. Tutescu conjectured that S(n) is never equal to S(n+1) for any value of n (see [1]), where S(n) denotes the Smarandache function of least m such that n evenly divides m!. Eric Weisstein has confirmed Tutescu's conjecture up to n=109 (see [2]).

With Tutescu's conjecture in mind one can examine the sequence of values given by the Smarandache function and observe that some of them are "very close" to being equal, where "very close" is defined as S(n) and S(n+1) being different by only 1. For example, S(9) = 6, and S(10) = 5. Thus, a natural question arises: Are there infinitely many positive integers n such that |S(n+1) - S(n)| = 1, where |x| denotes the absolute value of x? As far as the author knows this question has not been investigated previously.

Implementing Jon Perry's algorithm in [3] to efficiently compute S(n), a search was conducted to find solutions for |S(n+1) - S(n)| = 1 for all $n \leq 225$, and 264 solutions were found. The first 40 values are listed here:

1, 2, 3, 4, 9, 15, 35, 63, 99, 175, 195, 483, 1443, 2057, 2115, 2299, 3363, 3843, 5082, 5475, 6723, 7865, 11235, 11913, 12005, 22747, 24963, 26978, 27555, 31683, 37635, 41514, 46255, 51075, 62464, 68643, 76704, 77283, 89375, 95874, ...

Because there seems to be no legitimate reason these solutions should suddenly stop, we can state the following

Conjecture 1: *There are infinitely many positive integers n such that |S(n+1) - S(n)| = 1, where S(n) denotes the Smarandache function, and |x| denotes the absolute value of x.*

Unfortunately, the author is unaware of how to proceed with a proof. However, we can examine some of the larger solutions, make a few observations, and pose further questions.

Table 1

n n+1 S(n) S(n+1) Factorization of n Factorization of n+1

26978 26979 47 46 2*7*41*47 3*17*232 27555 27556 167 166 3*5*11*167 22*83*83 ... ... 4791500 4791501 74 73 22*53*7*372 33*11*13*17*73 4866435 4866436 2207 2206 32*5*72*2207 22*11032

Notice that the values 26978, 26979, 27555, and 27556 all have the same number of prime factors counting multiplicity.

Question 1: Are there infinitely many n such that |S(n+1) - S(n)| = 1 while n and n+1 both have the same number of prime factors?

Notice that 104975 = 52*13*17*19, and 104976 = 24*38 have 5 and 12 prime factors, respectively. This observation leads to another question.

Question 2: By how much can the number of prime factors of n and n+1 differ when |S(n+1) - S(n)| = 1?

Referring again to Table 1, we see that either n, n+1, or both are squareful (i.e., not squarefree). Will there ever be an n>2 such that |S(n+1) - S(n)| = 1 and n and n+1 are both squarefree?

A computer search reveals that there are an abundance of consecutive n's such that n and n+1 are both squarefree, while |S(n+1) - S(n)| = 2.

Here are the first few values of this sequence:

5, 14, 65, 77, 434, 902, 1769, 1829, 2665, 9590, 12121, 12921, 25877, 26058, 26105, 28542, 28633, 39902, 55390, 58705, 60377, 73185, 87989, 88409, 98106, 101170, 106490, 109213, 116653, 119685, 123710, 137309, 143877, 145705, 145858, 145885, 162734, 168817, 182001, 191270, ...

Which again leads to a conjecture.

Conjecture 2: *There are infinitely many values of n such that n and n+1 are both squarefree and |S(n+1) - S(n)| = 2.*

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ON THE 82-TH SMARANDACHE'S PROBLEM

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Abstract The main purpose of this paper is using the elementary method to study the asymptotic properties of the integer part of the k-th root positive integer, and give two interesting asymptotic formulae.

Keywords: k-th root; Integer part; Asymptotic formula.

§1. Introduction And Results

For any positive integer n, let $s_k(n)$ denote the integer part of k-th root of n. For example, $s_k(1) = 1$, $s_k(2) = 1$, $s_k(3) = 1$, $s_k(4) = 1, \cdots$, $s_k(2^k) = 2$, $s_k(2^k + 1) = 2, \cdots$, $s_k(3^k) = 3, \cdots$. In problem 82 of [1], Professor F.Smarandache asked us to study the properties of the sequence $s_k(n)$. About this problem, some authors had studied it, and obtained some interesting results. For instance, the authors [5] used the elementary method to study the mean value properties of $S(s_k(n))$, where Smarandache function $S(n)$ is defined as following:

$$S(n) = \min\{m : m \in N, n \mid m!\}.$$

In this paper, we use elementary method to study the asymptotic properties of this sequence in the following form: $\sum\limits_{n\leq x} \frac{\varphi(s_k(n))}{s_k(n)}$ and $\sum\limits_{n\leq x} \frac{1}{\varphi(s_k(n))}$, where $x \geq 1$ be a real number, $\varphi(n)$ be the Euler totient function, and give two interesting asymptotic formulae. That is, we shall prove the following:

Theorem 1. *For any real number $x > 1$ and any fixed positive integer $k > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \frac{\varphi(s_k(n))}{s_k(n)} = \frac{6}{\pi^2}x + O\left(x^{1-\frac{1}{k}-\varepsilon}\right),$$

where ε is any real number.

Theorem 2. *For any real number $x > 1$ and any fixed positive integer $k > 1$, we have the asymptotic formula*

$$\sum_{n\leq x}\frac{1}{\varphi(s_k(n))}=\frac{k\zeta(2)\zeta(3)}{(k-1)\zeta(6)}x^{1-\frac{1}{k}}+A+O\left(x^{1-\frac{2}{k}}\log x\right),$$

where $A=\gamma\sum\limits_{n=1}^{\infty}\frac{\mu^2(n)}{n\varphi(n)}-\sum\limits_{n=1}^{\infty}\frac{\mu^2(n)\log n}{n\varphi(n)}$.

§2. Proof of Theorems

In this section, we will complete the proof of Theorems. First we come to prove Theorem 1. For any real number $x > 1$, let M be a fixed positive integer with $M^k \leq x \leq (M+1)^k$, from the definition of $s_k(n)$ we have

$$\begin{aligned}\sum_{n\leq x}\frac{\varphi(s_k(n))}{s_k(n)} &= \sum_{t=1}^{M}\sum_{(t-1)^k\leq n<t^k}\frac{\varphi(s_k(n))}{s_k(n)}+\sum_{M^k\leq n<x}\frac{\varphi(s_k(n))}{s_k(n)}\\ &= \sum_{t=1}^{M-1}\sum_{t^k\leq n<(t+1)^k}\frac{\varphi(s_k(n))}{s_k(n)}+\sum_{M^k\leq n\leq x}\frac{\varphi(M)}{M}\\ &= \sum_{t=1}^{M-1}[(t+1)^k-t^k]\frac{\varphi(t)}{t}+O\left(\sum_{M^k\leq n<(M+1)^k}\frac{\varphi(M)}{M}\right)\\ &= k\sum_{t=1}^{M}t^{k-1}\frac{\varphi(t)}{t}+O\left(M^{k-1-\varepsilon}\right), \qquad (1)\end{aligned}$$

where we have used the estimate $\frac{\varphi(n)}{n}\ll n^{-\varepsilon}$.

Note that(see reference [3])

$$\sum_{n\leq x}\frac{\varphi(n)}{n}=\frac{6}{\pi^2}x+O\left((\log x)^{\frac{2}{3}}(\log\log x)^{\frac{4}{3}}\right). \qquad (2)$$

Let $B(y)=\sum\limits_{t\leq y}\frac{\varphi(t)}{t}$, then by Abel's identity (see Theorem 4.2 of [2]) and (2), we can easily deduce that

$$\begin{aligned}\sum_{t=1}^{M}t^{k-1}\frac{\varphi(t)}{t} &= M^{k-1}B(M)-B(1)-(k-1)\int_1^M y^{k-2}B(y)dy\\ &= M^{k-1}\left(\frac{6}{\pi^2}M+O\left((\log M)^{\frac{2}{3}}(\log\log M)^{\frac{4}{3}}\right)\right)\\ &- (k-1)\int_1^M\left(y^{k-2}\left(\frac{6}{\pi^2}y+O\left((\log y)^{\frac{2}{3}}(\log\log y)^{\frac{4}{3}}\right)\right)dy\right.\\ &= \frac{6}{k\pi^2}M^k+O\left((\log M)^{\frac{2}{3}}(\log\log M)^{\frac{4}{3}}\right). \qquad (3)\end{aligned}$$

Applying (1) and (3) we can obtain the asymptotic formula

$$\sum_{n\leq x}\frac{\varphi(s_k(n))}{s_k(n)}=\frac{6}{\pi^2}M^k+O\left(M^{k-1-\varepsilon}\right). \tag{4}$$

On the other hand, note that the estimate

$$0\leq x-M^k<(M+1)^k-M^k\ll x^{\frac{k-1}{k}} \tag{5}$$

Now combining (4) and (5) we can immediately obtain the asymptotic formula

$$\sum_{n\leq x}\frac{\varphi(s_k(n))}{s_k(n)}=\frac{6}{\pi^2}x+O\left(x^{1-\frac{1}{k}-\varepsilon}\right).$$

This proves Theorem 1.

Similarly, note that(see reference [4])

$$\sum_{n\leq x}\frac{1}{\varphi(n)}=\frac{\zeta(2)\zeta(3)}{\zeta(6)}\log x+A+O\left(\frac{\log x}{x}\right),$$

where $A=\gamma\sum\limits_{n=1}^{\infty}\frac{\mu^2(n)}{n\varphi(n)}-\sum\limits_{n=1}^{\infty}\frac{\mu^2(n)\log n}{n\varphi(n)}$. We can use the same method to obtain the result of Theorem 2.

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ON A NEW CLASS OF SMARANDACHE PRIME NUMBERS

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Abstract The purpose of this note is to report on the discovery of some new prime numbers that were built from factorials, the Smarandache Consecutive Sequence, and the Smarandache Reverse Sequence.

Consider numbers of the form $n! \times S_m(n) + 1$, where $S_m(n)$ gives the Smarandache Consecutive Sequence [1]:$1, 12, 123, 1234, 12345, \cdots$ and $n!$ is the factorial of n. If any of these numbers pass a simple, probable primality test, then proving them prime will be easy since $n! \times S_m(n)$ will contain a lot of small prime factors. That is, we will be able to use software programs that implement the classical tests to prove that numbers of this form are prime. See [2] for an introduction to primality proving using the classical tests.

The purpose of this note is to report on the discovery of some new prime numbers that were built from factorials, the Smarandache Consecutive Sequence, and the Smarandache Reverse Sequence[3].

Using the freely available primality-testing program PrimeFormGW[4], a computer search was performed for primes of the form $n! \times S_m(n) + 1$, and when $n = 1, 3, 6, 31$, and 302, $n! \times S_m(n) + 1$ is prime. No more primes were found up through n = 2000. The Brillhart-Lehmer-Selfridge test implemented in PrimeFormGW was used to prove primality. See [5] for more information on this test.

When $n = 9, 17$, and 25, $n! \times S_m(n) - 1$ is prime. No more primes were found up through $n = 2000$.

When $n = 1, 2, 10, 17, 18, 33, 63, 127, 482, 528, 1042, 1506$, and 1609, $n! \times S_{mr}(n) + 1$ is prime, where $S_{mr}(n)$ gives the Smarandache Reverse Sequence: $1, 21, 321, 4321, \cdots$ No more primes were found up through $n = 2000$. The largest value found, $1609! \times S_m(1609) + 1$, has $9,791$ digits.

When $n = 2, 4, 7, 14, 247, 341$, and 1799, $n! \times S_{mr}(n) - 1$ is prime. No more primes were found up through 2000. The largest prime found, $1799! \times S_{mr}(1799) - 1$has $11,165$ digits, qualifying it as a ąřgigantic primeąś since it has more than 10000 decimal digits[6]. Here is the PFGW primality certificate for this number:

PFGW Version 20041001. Win Stable (v1.2 RC1b) [FFT v23.8]

Primality testing $1799! \times S_{mr}(1799) - 1$ [$N + 1$, Brillhart - Lehmer - Selfridge].

Running $N + 1$ test using discriminant 1811, base $1 + \sqrt{1811}$.

Running $N + 1$ test using discriminant 1811, base $2 + \sqrt{1811}$.

Calling Brillhart-Lehmer-Selfridge with factored part 33.44 percent, $1799! \times S_{mr}(1799) - 1$ is prime! (133.1834s+0.0695s)

Questions: Why are there more primes of the form $n! \times S_{mr}(n) \pm 1$ than there are of the form $n! \times S_m(n) \pm 1$? Are there infinitely many primes of the forms mentioned in this note? When will mathematics be able to handle questions such as the preceding one?

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ON THE ODD SIEVE SEQUENCE

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Abstract The odd sieve sequence is the sequence, which is composed of all odd numbers that are not equal to the difference of two primes. In this paper, we use analytic method to study the mean value properties of this sequence, and give two interesting asymptotic formulae.

Keywords: The odd sieve sequence; Mean value; Asymptotic formula.

S1.Introduction

The odd sieve sequence is the sequence, which is composed of all odd numbers that are not equal to the difference of two primes. For example: $7, 11, 19, 23, 25, \cdots$. In problem 94 of [1], Professor F.Smarandache asked us to study this sequence. About this problem, it seems that none had studied it before. Let $\mathcal{A}$ denotes the set of the odd sieve numbers. In this paper, we use analytic method to study the mean value properties of this sequence, and give two interesting asymptotic formulae. That is, we shall prove the following:

Theorem 1. For any positive number $x > 1$, we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} n = \frac{x^2}{4} - \frac{x^2}{2\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Theorem 2. For any positive number $x > 1$, we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} \frac{1}{n} = \frac{1}{2}\ln\frac{x}{2} - \ln\ln(x+2) + \frac{1}{2}\gamma - A + B + O\left(\frac{1}{\ln x}\right),$$

where A, B are computable constants, γ is the Euler's constant.

§2. Proof of Theorems

In this section, we shall complete the proof of Theorems. Firstly we prove Theorem 1, let

$$a(n) = \begin{cases} 1, & n \text{ is a prime,} \\ 0, & \text{otherwise,} \end{cases}$$

and note that

$$\pi(x) = \sum_{n \le x} a(n) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

Therefore if we take $f(n) = n$ in Abel's identity, we can get the estimate

$$\begin{aligned}
\sum_{p \le x+2} p &= (x+2)\pi(x+2) - 2\pi(2) - \int_2^{x+2} \pi(t) f'(t) dt \\
&= \frac{(x+2)^2}{\ln(x+2)} + O\left(\frac{(x+2)^2}{\ln^2(x+2)}\right) - \int_2^{x+2} \left(\frac{t}{\ln t} + O\left(\frac{t}{\ln^2 t}\right)\right) dt \\
&= \frac{(x+2)^2}{2\ln(x+2)} + O\left(\frac{(x+2)^2}{\ln^2(x+2)}\right).
\end{aligned}$$

Then from the definition of the odd sieve sequence and the Euler's summation formula, we have

$$\begin{aligned}
\sum_{\substack{n \le x \\ n \in \mathcal{A}}} n &= \sum_{2n-1 \le x} (2n-1) - \sum_{p-2 \le x} (p-2) \\
&= \frac{(x+1)(x+3)}{4} + O(x) - \sum_{p \le x+2} p + 2 \sum_{p \le x+2} 1 \\
&= \frac{(x+1)(x+3)}{4} - \frac{(x+2)^2}{2\ln(x+2)} + O\left(\frac{(x+2)^2}{\ln^2(x+2)}\right) \\
&= \frac{x^2}{4} - \frac{x^2}{2\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).
\end{aligned}$$

This completes the proof of Theorem 1.

Now we prove Theorem 2. From the Euler's summation formula, we have

$$\sum_{n \le x} \frac{1}{n} = \ln x + \gamma + O\left(\frac{1}{x}\right), \tag{1}$$

where γ is the Euler's constant.

Since

$$\sum_{n \le x} \frac{1}{2n(2n-1)} \le \sum_{n=1}^{\infty} \frac{1}{(n-1)^2},$$

we have

$$\sum_{n \le x} \frac{1}{2n(2n-1)} = O(1). \tag{2}$$

Note

$$\sum_{n \le x} \frac{1}{p} = \ln\ln x + A + O\left(\frac{1}{\ln x}\right), \tag{3}$$

where A is a constant.

From the definition of odd sieve and formulae (1), (2) and (3), we can obtain

$$\begin{aligned}
&\sum_{\substack{n\le x\\ n\in\mathcal{A}}}\frac{1}{n}=\sum_{2n-1\le x}\frac{1}{2n-1}-\sum_{p-2\le x}\frac{1}{p}\\
&=\sum_{n\le\frac{x+1}{2}}\left(\frac{1}{2n}+\frac{1}{2n(2n-1)}\right)-\sum_{3\le p\le x+2}\frac{1}{p-2}\\
&=\frac{1}{2}\sum_{n\le\frac{x+1}{2}}\frac{1}{n}-\sum_{p\le x+2}\frac{1}{p}+\sum_{n\le\frac{x+1}{2}}\frac{1}{2n(2n-1)}-\sum_{3\le p\le x+2}\frac{2}{p(p-2)}+\frac{1}{2}\\
&=\frac{1}{2}\ln\frac{x}{2}-\ln\ln(x+2)+\frac{1}{2}\gamma-A+B+O\left(\frac{1}{\ln x}\right),
\end{aligned}$$

where B is a computable constant.

This completes the proof of Theorem 2.

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ON THE K-POWER PART RESIDUE FUNCTION

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Abstract The main purpose of this paper is using the elementary and analytic methods to study the asymptotic properties of the k-power part residue, and give an interesting asymptotic formula for it.

Keywords: k-power part residues function; Asymptotic formula.

§1. Introduction and results

For any positive integer n, the Smarandache k-th power complements $b_k(n)$ is the smallest positive integer such that $nb_k(n)$ is a complete k-th power (see probem 29 of [1].) Similar to the Smarandache k-th power complements, the additive k-th power complements $a_k(n)$ is defined as the smallest nonnegative integer such that $a_k(n)+n$ is a complete k-th power. About this problem, some authors had studied it, and obtained some interesting results. For example, in [4] Xu Z.F. used the elementary method to study the mean value properties of $a_k(n)$ and $d(a_k(n))$. in [5] Yi Y. and Liang F.C. used the analytic method to study the mean value properties of $d(a_2(n))$, and obtained a sharper asymptotic formula for it.

Similarly, we will define the k-power part residue function as following: For any positive integer n, it is clear that there exists a positive integer N such that $N^k \leq n < (N+1)^k$. Let $n = N^k + r$, then $f_k(n) = r$ is called the k-power part residue of n. In this paper, we use the elementary and analytic methods to study the asymptotic properties of this sequence, and obtain two interesting asymptotic formulae for it. That is, we shall prove the following:

Theorem. *For any real number $x > 1$ and any fixed positive integer m and k, we have the asymptotic formula*

$$\sum_{n\leq x}\delta_m(f_k(n)) = \frac{k^2}{2(2k-1)}\prod_{p|m}\frac{p}{p+1}x^{2-\frac{1}{k}} + O\left(x^{2-\frac{2}{k}}\right),$$

where $\prod_{p|m}$ *denotes the product over all prime divisors of* m*, and*

$$\delta_m(n) = \begin{cases} max\{d \in N \quad | \quad d|n, (d,m) = 1\}, & if \quad n \neq 0, \\ 0, & if \quad n = 0. \end{cases}$$

Especially taking $m = 1$, and note that $\delta_1(f_k(n)) = f_k(n)$ we may immediately get the following:

Corollary . *For any real number* $x > 1$ *and any fixed positive integer* k*, we have the asymptotic formula*

$$\sum_{n \leq x} f_k(n) = \frac{k^2}{2(2k-1)} x^{2-\frac{1}{k}} + O\left(x^{2-\frac{2}{k}}\right).$$

§2. Proof of Theorem

In this section, we will complete the proof of Theorem. First we need following

Lemma. *For any real number* $x > 1$ *and positive integer m, we have*

$$\sum_{n \leq x} \delta_m(n) = \frac{x^2}{2} \prod_{p|k} \frac{p}{p+1} + O(x^{\frac{3}{2}+\epsilon}),$$

where ϵ *is any positive number.*

Proof. Let $s = \sigma + it$ be a complex number and $f(s) = \sum\limits_{n=1}^{\infty} \frac{\delta_m(n)}{n^s}$. Note that $\delta_m(n) \ll n$, so it is clear that $f(s)$ is an absolutely convergent series for Re(s)$>$ 2, by the Euler product formula [2] and the definition of $\delta_m(n)$ we get

$$\begin{aligned} f(s) = \sum_{n=1}^{\infty} \frac{\delta_m(n)}{n^s} &= \prod_{p} \left(1 + \frac{\delta_m(p)}{p^s} + \frac{\delta_m(p^2)}{p^{2s}} + \cdots + \frac{\delta_m(p^{2n})}{p^{ns}} + \cdots\right) \\ &= \prod_{p|m} \left(1 + \frac{\delta_m(p)}{p^s} + \frac{\delta_m(p^2)}{p^{2s}} + \cdots + \frac{\delta_m(p^{2n})}{p^{ns}} + \cdots\right) \\ &\quad \times \prod_{p\dagger m} \left(1 + \frac{\delta_m(p)}{p^s} + \frac{\delta_m(p^2)}{p^{2s}} + \cdots + \frac{\delta_m(p^{2n})}{p^{ns}} + \cdots\right) \\ &= \prod_{p|m} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{ns}} + \cdots\right) \\ &\quad \times \prod_{p\dagger m} \left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots + \frac{p^{2n}}{p^{ns}} + \cdots\right) \\ &= \prod_{p|m} \left(\frac{1}{1-\frac{1}{p^s}}\right) \prod_{p\dagger m} \left(\frac{1}{1-\frac{1}{p^{s-1}}}\right) \end{aligned}$$

$$= \zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta-function and $\prod_p$ denotes the product over all primes.

From (1) and Perron's formula [3], we have

$$\sum_{n\le x}\delta_m(n) = \frac{1}{2\pi i}\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT}\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds + O\left(\frac{x^{\frac{5}{2}+\epsilon}}{T}\right), \tag{2}$$

where ϵ is any positive number.

Now we move the integral line in (2) from $s=\frac{5}{2}\pm iT$ to $s=\frac{3}{2}\pm iT$. This time, the function $\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}$ has a simple pole point at $s=2$ with residue

$$\frac{x^2}{2}\prod_{p|m}\frac{p}{p+1}. \tag{3}$$

Hence, we have

$$\frac{1}{2\pi i}\left(\int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT}+\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT}+\int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}\right)\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds$$
$$= \frac{x^2}{2}\prod_{p|m}\frac{p}{p+1}. \tag{4}$$

We can easily get the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT}+\int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT}\right)\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds\right| \ll \frac{x^{\frac{5}{2}+\epsilon}}{T}, \tag{5}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds\right| \ll x^{\frac{3}{2}+\epsilon}. \tag{6}$$

Taking $T=x$, combining (2), (4), (5) and (6) we deduce that

$$\sum_{n\le x}\delta_m(n) = \frac{x^2}{2}\prod_{p|m}\frac{p}{p+1} + O(x^{\frac{3}{2}+\epsilon}). \tag{7}$$

This completes the proof of Lemma.

Now we shall use the above lemma to complete the proof of Theorem. For any real number $x\ge 1$, let M be a fixed positive integer such that

$$M^k \le x < (M+1)^k. \tag{8}$$

Then from (7) and the definition of $f_k(n)$, we have

$$\begin{aligned}
&\sum_{n\le x}\delta_m(f_k(n)) \qquad (9)\\
=&\sum_{t=1}^{M}\sum_{(t-1)^k\le n<t^k}\delta_m(f_k(n))+\sum_{M^k\le n<x}\delta_m(f_k(n))\\
=&\sum_{t=1}^{M-1}\sum_{t^k\le n<(t+1)^k}\delta_m(f_k(n))+\sum_{M^k\le n\le x}\delta_m(f_k(n))\\
=&\sum_{t=1}^{M}\sum_{j=0}^{(t+1)^k-t^k}\delta_m(j)+O\left(\sum_{M^k\le n<(M+1)^k}\delta_m(f_k(n))\right)\\
=&\sum_{t=1}^{M}\left(\frac{\left((t+1)^k-t^k\right)^2}{2}\prod_{p|m}\frac{p}{p+1}+O\left((t+1)^k-t^k\right)^{\frac{3}{2}+\epsilon}\right)+O\left(M^k\right)\\
=&\frac{1}{2}\prod_{p|m}\frac{p}{p+1}\left(\sum_{t=1}^{M}\left((t+1)^k-t^k\right)^2\right)+O\left(\sum_{t=1}^{M}t^{(k-1)(\frac{3}{2}+\epsilon)}\right)\\
=&\frac{k^2}{2}\prod_{p|m}\frac{p}{p+1}\sum_{t=1}^{M}t^{2(k-1)}+O\left(\sum_{t=1}^{M}t^{2k-3}\right)\\
=&\frac{k^2M^{2k-1}}{2(2k-1)}\prod_{p|m}\frac{p}{p+1}+O\left(M^{2k-2}\right). \qquad (10)
\end{aligned}$$

On the other hand, we also have the estimate

$$0\le x-M^k<(M+1)^k-M^k\ll x^{\frac{k-1}{k}}.$$

Now combining (9) and (10) we may immediately obtain the asympotic formula

$$\sum_{n\le x}\delta_m(f_k(n))=\frac{k^2}{2(2k-1)}\prod_{p|m}\frac{p}{p+1}x^{2-\frac{1}{k}}+O\left(x^{2-\frac{2}{k}}\right).$$

This completes the proof of Theorem.

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MEAN VALUE OF THE ADDITIVE ANALOGUE OF SMARANDACHE FUNCTION *

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Abstract For any positive integer n, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined the smallest $m \in N^+$, where $n|m!$. In this paper, we study the mean value properties of the additive analogue of $S(n)$, and give an interesting mean value formula for it.

Keywords: Smarandache function; Additive Analogue; Mean Value formula.

§1. Introduction and results

For any positive integer n, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined the smallest $m \in N^+$, where $n|m!$. In paper [2], Jozsef Sandor defined the following analogue of Smarandache function:

$$S_1(x) = \min\{m \in N : x \leq m!\}, \quad x \in (1, \infty), \tag{1}$$

which is defined on a subset of real numbers. Clearly $S(x) = m$ if $x \in ((m-1)!, m!]$ for $m \geq 2$ (for $m = 1$ it is not defined, as $0! = 1! = 1!$), therefore this function is defined for $x > 1$.

About the arithmetical properties of $S(n)$, many people had studied it before (see reference [3]). But for the mean value problem of $S_1(n)$, it seems that no one have studied it before. The main purpose of this paper is to study the mean value properties of $S_1(n)$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. For any real number $x \geq 2$, we have the mean value formula

$$\sum_{n \leq x} S_1(n) = \frac{x \ln x}{\ln \ln x} + O(x).$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. That is,

*This work is supported by the Zaizhi Doctorate Foundation of Xi'an Jiaotong University.

Lemma. For any fixed positive integers m and n, if $(m-1)! < n \leq m!$, then we have

$$m = \frac{\ln n}{\ln \ln n} + O(1).$$

Proof. From $(m-1)! < n \leq m!$ and taking the logistic computation in the two sides of the inequality, we get

$$\sum_{i=1}^{m-1} \ln i < \ln n \leq \sum_{i=1}^{m} \ln i. \tag{2}$$

Using the Euler's summation formula, then

$$\sum_{i=1}^{m} \ln i = \int_1^m \ln t dt + \int_1^m (t - [t])(\ln t)' dt = m \ln m - m + O(\ln m) \tag{3}$$

and

$$\sum_{i=1}^{m-1} \ln i = \int_1^{m-1} \ln t dt + \int_1^{m-1} (t - [t])(\ln t)' dt = m \ln m - m + O(\ln m). \tag{4}$$

Combining (2), (3) and (4), we can easily deduce that

$$\ln n = m \ln m - m + O(\ln m). \tag{5}$$

So

$$m = \frac{\ln n}{\ln m - 1} + O(1). \tag{6}$$

Similarly, we continue taking the logistic computation in two sides of (5), then we also have

$$\ln m = \ln \ln n + O(\ln \ln m), \tag{7}$$

and

$$\ln \ln m = O(\ln \ln \ln n). \tag{8}$$

Hence,

$$m = \frac{\ln n}{\ln \ln n} + O(1).$$

This completes the proof of Lemma.

Now we use Lemma to complete the proof of Theorem. For any real number $x \geq 2$, by the definition of $s_1(n)$ and Lemma we have

$$\begin{aligned} \sum_{n \leq x} S_1(n) &= \sum_{\substack{n \leq x \\ (m-1)! < n \leq m!}} m \qquad (9) \\ &= \sum_{n \leq x} \left(\frac{\ln n}{\ln \ln n} + O(1) \right) \\ &= \sum_{n \leq x} \frac{\ln n}{\ln \ln n} + O(x). \end{aligned}$$

By the Euler's summation formula, we deduce that

$$\begin{aligned}
&\sum_{n\leq x}\frac{\ln n}{\ln\ln n}\\
=\ &\int_2^x\frac{\ln t}{\ln\ln t}dt+\int_2^x(t-[t])\left(\frac{\ln t}{\ln\ln t}\right)'dt+\frac{\ln x}{\ln\ln x}(x-[x]) \quad (10)\\
=\ &\frac{x\ln x}{\ln\ln x}+O\left(\frac{x}{\ln\ln x}\right).
\end{aligned}$$

So, from (9) and (10) we have

$$\sum_{n\leq x}S_1(n)=\frac{x\ln x}{\ln\ln x}+O(x).$$

This completes the proof of Theorem.

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HYBRID MEAN VALUE ON SOME SMARANDACHE-TYPE MULTIPLICATIVE FUNCTIONS AND THE MANGOLDT FUNCTION*

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Abstract In this paper, we study the hybrid mean value of some Smarandache-type multiplicative functions and the Mangoldt function, and give two asymptotic formulae.

Keywords: Smarandache-type multiplicative functions; Mangoldt function; Hybrid mean value.

§1. Introduction

In [1], Henry Bottomley considered eleven particular families of interrelated multiplicative functions, which are listed in Smarandache's problems.

It might be interesting to discuss the mean value of these functions on $\{p^\alpha\}$, since they are multiplicative. In this paper we study the hybrid mean value of some Smarandache-type multiplicative functions and the Mangoldt function. One is $C_m(n)$, which is defined as the m-th root of largest m-th power dividing n. The other function $J_m(n)$ is denoted as m-th root of smallest m-th power divisible by n. We will give two asymptotic formulae on these two functions. That is, we shall prove the following:

Theorem 1. *For any integer $m \geq 3$ and real number $x \geq 1$, we have*

$$\sum_{n \leq x} \Lambda(n) C_m(n) = x + O\left(\frac{x}{\log x}\right),$$

where $\Lambda(n)$ is the Mangoldt function.

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Theorem 2. *For any integer $m \geq 2$ and real number $x \geq 1$, we have*

$$\sum_{n\leq x}\Lambda(n)J_m(n) = x^2 + O\left(\frac{x^2}{\log x}\right).$$

Using our methods one should be able to get some similar mean value formulae. We are hoping to see more papers.

§2. Proof of the theorems

Now we prove the theorems. Noting that

$$C_m(p^\alpha) = p^k, \quad \text{if } mk \leq \alpha < m(k+1) \tag{1}$$

and

$$C_m(p^\alpha) \leq p^{\frac{\alpha}{m}}, \tag{2}$$

then we have

$$\sum_{n\leq x}\Lambda(n)C_m(n) = \sum_{p^\alpha\leq x}\log pC_m(p^\alpha) = \sum_{p\leq x}\log pC_m(p) + \sum_{\substack{p^\alpha\leq x\\ \alpha\geq 2}}\log pC_m(p^\alpha). \tag{3}$$

Let

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is prime;}\\ 0, & \text{otherwise,}\end{cases}$$

then

$$\sum_{n\leq x}a(n) = \pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

By Abel's identity and (1) we have

$$\begin{aligned}\sum_{p\leq x}\log pC_m(p) &= \sum_{p\leq x}\log p\sum_{n\leq x}a(n)\log n = \pi(x)\log x - \int_2^x\frac{\pi(t)}{t}\mathrm{d}t\\ &= x + O\left(\frac{x}{\log x}\right) + O\left(\int_2^x\frac{1}{\log t}\mathrm{d}t\right) = x + O\left(\frac{x}{\log x}\right).\end{aligned} \tag{4}$$

From (2) we also have

$$\begin{aligned}\sum_{\substack{p^\alpha\leq x\\ \alpha\geq 2}}\log pC_m(p^\alpha) &= \sum_{2\leq\alpha\leq\frac{\log x}{\log 2}}\sum_{p\leq x^{\frac{1}{\alpha}}}\log pC_m(p^\alpha) \leq \sum_{2\leq\alpha\leq\frac{\log x}{\log 2}}\sum_{p\leq x^{\frac{1}{\alpha}}}\log p\cdot p^{\frac{\alpha}{m}}\\ &\ll \sum_{2\leq\alpha\leq\frac{\log x}{\log 2}} x^{\frac{1}{m}+\frac{1}{\alpha}} \ll x^{\frac{1}{m}+\frac{1}{2}}.\end{aligned} \tag{5}$$

Therefore for any integer $m \geq 3$ and real number $x \geq 1$, from (3), (4) and (5) we have

$$\sum_{n\leq x}\Lambda(n)C_m(n) = x + O\left(\frac{x}{\log x}\right).$$

This proves Theorem 1.

On the other hand, noting that

$$J_m(p^\alpha) = p^{k+1}, \quad \text{if } mk < \alpha \leq m(k+1)$$

and

$$J_m(p^\alpha) \leq p^{\frac{\alpha}{m}+1},$$

then using the methods of proving Theorem 1 we can easily get Theorem 2.

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ON A NUMBER SET RELATED TO THE K-FREE NUMBERS

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Abstract Let F_k denotes the set of k-free number. For any positive integers $l \geq 2$, we define a number set $A_{k,l}$ as follows

$$A_{k,l} = \{n : n = m^l + r, m^l \leq n < (m+1)^l, r \in F_k, n \in N\}.$$

In this paper, we study the arithmetical properties of the number set $A_{k,l}$, and give some interesting asymptotic formulae for it.

Keywords: Number set; k-free number; Asymptotic formula.

§1. Introduction

Let $k \geq 2$ be an integer. The k-free numbers set F_k is defined as follows

$$F_k = \{n : \text{if prime } p|n \text{ then } p^k \dagger n, n \in N\}.$$

In problem 31 of [1], Professor F.Smarandache asked us to study the arithmetical properties of the numbers in F_k. About this problem, many authors had studied it, see [2], [3], [4]. For any positive integer n and $l \geq 2$, there exist an integer m such that

$$m^l \leq n \leq (m+1)^l.$$

So we can define the following number set $A_{k,l}$:

$$A_{k,l} = \{n : n = m^l + r, m^l \leq n < (m+1)^l, r \in F_k, n \in N\}.$$

In this paper, we use the elementary methods to study the asymptotic properties of the number of integers in $A_{k,l}$ less than or equal to a fixed real number x, and give some interesting asymptotic formulae. That is, we shall prove the following results:

Theorem 1. Let $k, l \geq 2$ be any integers. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{\substack{n \leq x \\ n \in A_{k,l}}} 1 = \frac{x}{\zeta(k)} + O_{k,l}\left(x^{\frac{1}{l}+\frac{1}{k}-\frac{1}{kl}}\right),$$

where $\zeta(s)$ denotes the Riemann zeta function and $O_{k,l}$ means the big Oh constant related to k, l.

Theorem 2. Assuming the Riemann Hypothesis, there holds

$$\sum_{\substack{n\leq x\\ n\in A_{2,2}}} 1 = \frac{6}{\pi^2}x + O\left(x^{\frac{29}{44}+\epsilon}\right),$$

where ϵ is any fixed positive number.

§2. Two Lemmas

Lemma 1. For any real number $x > 1$ and integer $k \geq 2$, we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in F_k}} 1 = \frac{x}{\zeta(k)} + O\left(x^{\frac{1}{k}}\right).$$

Proof. See reference [5].

Lemma 2. Assuming the Riemann Hypothesis, we have

$$\sum_{\substack{n\leq x\\ n\in F_2}} 1 = \frac{6}{\pi^2}x + O\left(x^{\frac{7}{22}+\epsilon}\right).$$

Proof. See reference [6].

§3. Proof of the theorems

In this section, we shall complete the proofs of the theorems. For any real number $x \geq 1$ and integer $l \geq 2$, there exist a positive integer M such that

$$M^l \leq x < (M+1)^l. \tag{1}$$

So from the definition of the number set $A_{k,l}$ and Lemma 1, we can write

$$\begin{aligned}
\sum_{\substack{n\leq x\\ n\in A_{k,l}}} 1 &= \sum_{t=1}^{M-1}\sum_{\substack{m=1\\ m\in F_k}}^{(t+1)^l-t^l} 1 \sum_{\substack{m\leq x-M^l\\ m\in F_k}} 1 \\
&= \sum_{t=1}^{M-1}\frac{(t+1)^l-t^l}{\zeta(k)} + O\left(\sum_{t=1}^{M-1}\left((t+1)^l-t^l\right)^{\frac{1}{k}}\right) \\
&\quad + \frac{x-M^l}{\zeta(k)} + O\left(\left(x-M^l\right)^{\frac{1}{k}}\right) \\
&= \sum_{t=1}^{M-1}\frac{(t+1)^l-t^l}{\zeta(k)} + \frac{x-M^l}{\zeta(k)} + O_{k,l}\left(M^{1+\frac{l-1}{k}}\right) \\
&= \frac{x}{\zeta(k)} + O_{k,l}\left(M^{1+\frac{l-1}{k}}\right),
\end{aligned} \tag{2}$$

On the other hand, from (1) we have the estimates

$$0 \leq x - M^l < (M+1)^l - M^l \ll x^{\frac{l-1}{l}} \tag{3}$$

Now combining (2) and (3), we have

$$\sum_{\substack{n \leq x \\ n \in A_{k,l}}} 1 = \frac{x}{\zeta(k)} + O_{k,l}\left(x^{\frac{1}{l}+\frac{1}{k}-\frac{1}{kl}}\right).$$

This completes the proof of Theorem 1. From the same argue as proving Theorem 1 and Lemma 2, we can get

$$\sum_{\substack{n \leq x \\ n \in A_{2,2}}} 1 = \quad = \frac{12}{\pi^2}\sum_{t=1}^{M-1} t + O\left(\sum_{t=1}^{M-1} t^{\frac{7}{22}+\epsilon}\right) \tag{4}$$

$$= \frac{6}{\pi^2}M^2 + O\left(M^{\frac{29}{22}+\epsilon}\right) \tag{5}$$

$$= \frac{6}{\pi^2}x + O_{k,l}\left(x^{\frac{29}{44}+\epsilon}\right). \tag{6}$$

This completes the proof of Theorem 2.

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SMARANDACHE PSEUDO– HAPPY NUMBERS

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Happy numbers are defined by Grudman and Teeple [1], Muneer Jebral [2] and C. Asbacher [3] as:

"A natural number n is a Happy Number if the sum of squares of its digits, when added iteratively, terminates to 1." 7 is a happy number because $7^2 \rightarrow 49 \rightarrow 4^2 + 9^2 = 97 \rightarrow 9^2 + 7^2 = 130 \rightarrow 1^2 + 3^2 + 0^2 = 10 \rightarrow 1$ But 5 is not a happy number!

This paper deals with Smarandache Pseudo Happy Number, which similar to above concept, with some change in the definition. And many properties of these numbers are derived.

1.1. Definition

A natural number n is called a Smarandache Pseudo Happy Number(SPHN), if the digits of n^2, when simply added iteratively, terminates to 1; that is, the digital root of n^2 is 1

For, 8 is SPHN, because $8^2 = 64 -> 6 + 4 -- > 10 -> 1$ Incidentally, 7 is a happy number but it is not a SPHN !!

Now, we give a general definition of SPHN: Let $a \in N$, Let $a^2 = \sum a_i 10^i$ Let H: $N -> N$, Let $H(a) = \sum a_i$, H is a many-one function.

If $\sum a_i$, terminates to 1 when added simply and iteratively, then a is a Smarandache Pseudo Happy Number (SPHN)

1.2 The following is the set of SPHN, up to first 100 **only .**

Since they terminate at 1, the set of SPHN is denoted by [1].

$$[1] = \{1, 8, 10, 17, 19, 26, 28, 35, 37, 44, 46, 53, 55, 62, 64, 71, 73, 80, 82, 89, 91, 98, \ldots\ldots\}$$

We say that $H(26) = 1$ because $26 \in [1]$

Note:

(i)In general, 23 of the natural numbers are SPHN.

(ii)The negative numbers $-1, -8, -10, -17, \ldots\ldots$ are also SPHN;
But here, we will restrict to set of naturals only.

1.3. Let $[1] = a_n$
This set of SPHN is generated as: $a_1 = 1$, $a_{2n} = a_{2n-1} + 7$, $a_{2n+1} = a_{2n} + 2$, where $n \in N$

1.4.
As we notice above, 17 and 71 are both SPHN, it is obvious that the number formed by the reversal of digits of a SPHN is also a SPHN. For, the following pairs are SPHN: $(19, 91)$; $(26, 62)$; $(28, 82)$; $(35, 53)$; $(37, 73)$; $(46, 64)$; ... etc. A proof for this result is presented later on.

1.5.
Adding zeros in between or on right hand side of a SPHN do not add to the sum of the digits of the number. Hence new number, by adding zeros, is also a SPHN.
For, 17 is a SPHN. And $107^2 = 11449 \rightarrow 19 \rightarrow 1$. Hence 107 is also a SPHN.
This shows that there is infinite number of SPHN.

1.6.
Let $a_i = ith$ SPHN Then it is easy to prove the following results: (i) $a_i \equiv (mod 9)$.
(ii) $a_i^2 \equiv 1(mod 9)$.
(iii) $a_{2n-1} + a_{2n}$, when iterated, terminates to 9.
(iv) a_i, when iterated, terminates to 1 or 9
(v) $a_i \equiv 1(mod)2$.
(vi) $\| \ a_i$, when iterated, terminates to 1 or 8.
(vii) $a_i \bullet a_j$ is also a SPHN.
(viii) $(a_{2n})^3 + (a_{2n+1})^3$, when iterated ,terminates to 9.
(ix) $1/a_n \rightarrow 0$ as n infinity since an is an increasing sequence.

1.7.
Let $A = 1, 10, 19, 28, \ldots$ $B = 8, 17, 26, 35, \ldots$ Then $AUB = [1]$ The sequences A and B are both arithmetic progressions.

1.8.
When the digits of a SPHN are reversed, the new number is also a SPHN.
Proof. Let a be a natural number. Let $a = b_1 + b_2 \cdot 10$
$a' = b_2 + b_1 \cdot 10$
Then $a^2 = b_1^2 + 2b_1b_2 \cdot 10 + b_2^2 \cdot 100$,
And $a'^2 = b_2^2 + 2b_1b_2 \cdot 10 + b_1^2 \cdot 100$,
And the sum of the digits of

$a^2 = b_1^2 + 2b_1b_2 + b_2^2$
= sum of digits of a'^2
$= (b_1 + b_2)^2$

Hence if the number is reversed, the sum of digits remains same , and then, the new number is also SPHN.

Obviously, all the PHN palindromes are also SPHN.

Corollary *(i). Now, it is sufficient to find the square of the sum of digits of any number to test its SPHN status.*

For example, 13200432175211431501 *is a SPHN, because sum of digits of this* $20 - digit$ *number is* 46*; and* $46'^2 = 2116 \to 10 \to 1$

(ii). We have, $a^2 - a'^2 = 99 \cdot (b_1^2 - b_2^2)$ *This is another formula for finding the PHN status.*

(iii) $1, 6$*, are triangular numbers which are SPHN;*

2.1 Non-SPHN numbers.

What about the other natural numbers which are not SPHN?

We have defined above, if the digits of n^2, when added simply and iteratively], terminates to 1. and that the set of PHN is denoted by [1]

The other numbers, when iteratively added as defined in PHN, terminate at either $4, 7$ or9. Hence the set of numbers belonging to these categories are denoted by [4], [7] or [9] respectively.

Hence we have:

$[4] = 2, 7, 11, 16, 20, 25, 29, 34 \ldots,$
$[7] = 4, 5, 13, 14, 22, 23, 31, 32 \ldots,$
$[9] = 3, 6, 9, 12, 15, 18, 21, 24 \ldots.$

2.2 We note the following:

(i) The set N of natural numbers is partitioned into [1], [4], [7] and [9]; that is, every natural number belongs to either of these sets.

(ii) No number, as added above, terminates to $2, 3, 5, 6$ or 8.

(iii) All multiples of 3 belong to [9] only.

2.3 The above sets are generated as follows: for $n \in N$**,**

(i) Let $[4] = b_n$, then, $b_1 = 2, b_{2n} = b_{2n-1} + 5, b_{2n+1} = b_{2n} + 4,$

(ii) Let $[7] = c_n$, then $c_1 = 3, c_{2n} = c_{2n-1} + 1, c_{2n+1} = c_{2n} + 4,$

(iii) $[9] = 3n.$

2.4 We define the multiplication [1] and [4] as:

$[1] \cdot [4] = a_r.b_r/a_r \in [1], b_r \in [4]$, i.e. the set of products of corresponding elements. The other multiplications of sets are defined similarly. Then $[1] \cdot [1] \subset [1]$, that is, $[1] \cdot [1]$. a subset of [1]

Also, $[1] \cdot [4] \subset [4]$,

$[1] \cdot [7] \subset [7]$,
$[1] \cdot [9] \subset [9]$,
Considering the other products similarly, we have the following table:
[1] [4] [7] [9] .....
[1] [1] [4] [7] [9]
[4] [4] [7] [9] [9]
[7] [7] [1] [4] [9]
[9] [9] [9] [9] [9]
It is obvious from the above table, that $H^n(a) = 1$, if $a \in [1]$

2.5.
(i) Let $X = [1], [4], [7]$ Then , from the above table,$(X, \cdot)$ is an abelian group, under the subset condition, with identity as [1].
ii) Let $Y = [1], [4], [7], [9]$ Then $(Y, \cdot)$ is a monoid, under the subset condition, with identity as [1].
iii) Unfortunately, the addition of these sets, in similar way ,does not yield any definite result.

3.1 Lemma:
The sum of digits of a^3 is equal to cube of sum of digits of a. Proof: We consider a two digit number. Let $a = a_1 + a_2 \cdot 10$
$a^3 = a_1^3 + (3a_1^2 \cdot a_2) \cdot 10 + (3a_1 \cdot a_2^2) \cdot 10^2 + a_2^3.10^3$
sum of digits of
$a^3 = a_1^3 + (3a_1^2 \cdot a_2) + (3a_1 \cdot a_2^2) + a_2^3.$
$= (a_1 + a_2)^3$
$= cubeofsumofdigitsofa.$
Hence we generalize this as: The sum of digits of a n is equal to n th power of sum of digits of a.
Now this result can be used to find the PHN status of a number As:
$(13)^6 \to (1+3)^6 \to 4^6 \to 4096 \to 19 \to 1.$
Therefore $(13)^6 \in [1]$, hence $(13)^6$ is a PHN
Incidentally,
$(13)^k \in [1]$, if k is a multiple of 3
$(13)^k \in [7]$, if $k = 1 + 3i, i = 1, 2, 3, \ldots$
$(13)^k \in [4]$, if $k = 2 + 3i$
Similar results can be obtained for the higher powers of any number.
Also, it can be shown that if $a^m \cdot a^n \in [i]$, then $a^{m+n} \in [i], i = 1, 4, 7, 9.$

3.2 Concatenation of SPHN.
We have, $[1] = 1, 8, 10, 17, 19, 26, 28, 35, 37, \ldots.$ All the SPHN are concatenated one after another and the new number is tested.
(i) We note that:
$1 \in [1]$, $18 \in [9]$,
$1810 \in [1]$, $181017 \in [9]$,

$18101719 \in [1]$, $1810171926 \in [9]$,
$181017192628 \in [1]$, $18101719262835 \in [9]$. etc.
Hence we have, for $a_i \in [1]$,
The Concatenation $a_1 \cdot a_2 \cdot a_3 \ldots a_k \in [1]$, if k is odd, and hence it a SPHN $\in [9]$, if k is even.
(ii)A similar result is also obtained : product $a_{i+1} \cdot a_{i+2} \cdot a_{i+3} \ldots a_{i+k} \in [1]$, if$k$ is even, and it is a SPHN [9], if k is odd.

3.3 Twin Primes.
(i) The first twin primes, up to 100, are: $[5, 7], [11, 13], [17, 19].[29, 31]$, $[41, 43], [59, 61], [71, 73]$.
The sum of each twin prime pair is a multiple of 3 Hence, The sum of each twin prime pair is a member of [9].
(ii) Let the twin primes be$2p - 1, 2p + 1, p \in N$ The product of these twin $primes = 4p^2 - 1 = 36k^2 - 1, forp = 3k$ Now the sum of digits of $36k^2 - 1$, in iteration, is 8 for all k. Hence the product belongs to [1]. Therefore the product of numbers in each twin pair is SPHN .

4.1 Change of base.
Up till now, the base of the numbers was 10.
Now change the $base \geq 2$. Then we note that the status of SPHN changes with the base. Following are some examples of numbers which are already PHN.
$35 = (55)_6 \in [1]$; Hence 35 is SPHN at the base 6 also. (additions with ref. to base 10) $71 = (107)_8 \in [1]$; Hence 71 is SPHN at the base 8.
Similarly, $89 = (118)_9 \in [1]$; Hence 89 is SPHN at the base 9.
However, some numbers, which are not SPHN with base 10, become SPHN with change of base, as: $49 = (100)_7$ is now SPHN; $50 = (62)_8$ is a SPHN

Lemma *Square of any natural number n is SPHN with ref. to n as a base.*

4.2 Product Sequences.
(i) Let S_n be a square product sequence defined as:
$S_n = 1 + s_1 \cdot s_2 \cdot s_3 \ldots s_n$, where $s_n = n^2$
we get, $S = 2, 5, 37, 577, 14401, 51849, 25401601, 1625702401 \ldots$ here, all the elements of this set, except 2 and 5, are SPHN.
(ii) Let C_n be a square product sequence defined as: $C_n = 1 + c_1 \cdot c_2 \cdot c_3 \ldots c_n$, where $c_n = n^3$
we get, $C = 2, 9, 217, 13825, 1728001, 373248001, \ldots$ here, all the elements of this set, except 2 and 9 are SPHN
(iii) Let F_n be a square product sequence defined as:
$F_n = 1 + f_1.f_2.f_3 \ldots f_n$, where $f_n = factorial \qquad n$
we get, $F = 2, 3, 13, 289, 34561, 24883201, 125411328001, \ldots$
here, all the elements of this set, except $2, 3$ and 13, are SPHN.

(iv) Let S be a sequence of continued sequence of natural numbers, as: $Sn = (12345 \ldots n)$

That is $S = 1, 12, 123, 1234, 12345, 123456 \ldots 12345 \ldots n, \ldots$. If $n = 3k + 1$, $k = 0, 1, 2, 3, \ldots$ then S_n is a SPHN. In all other cases, S_n belongs to [9]

(v) All factorial numbers, $(n)!$, belong to [9] because they are the multiples of 3

4.3 Summation.

We have, set $[1] = 1, 8, 10, 17, 19, 26, 28, 35, \ldots$.

This set is partitioned into two sets A and B as $A = 1, 10, 19, 28, 37, \ldots$. Its r^{th} term $a_r = 9r - 8$ and $B = 8, 17, 26, 35, 44, \ldots$, r^{th} term $b_r = 9r - 1$

now, the sum of first $2n$ terms of $A = \sum a_r = 9n(n + 1)/2 - 8n$

Also, the sum of first $2n$ terms of $B = \sum b_r = 9n(n + 1)/2 - n$

Hence sum of first $2n$ terms of $[1] = \sum a_r + \sum b_r = 9n^2$

Surprisingly, sum first of $2n$ terms of $[4] = 9n^2$

Also, sum of first $2n$ terms of $[7] = 9n^2$

But, sum of first $2n$ terms of $[9] = 6n^2 + 3n$.

4.4 Indices.

(i) If $a \in [1]$, then $a^k \in [1]$ for all k.

(ii) $a \in [4]$, then

$a^{3k-1} \in [7]$,

$a^{3k} \in [1]$,

$a^{3k+1} \in [4]$, for all k

(iii) $a \in [7]$, then

$a^{3k-1} \in [4]$,

$a^{3k} \in [1]$,

$a^{3k+1} \in [7]$, for all k

(iv)$a \in [9]$, $a^{3k} \in [9]$, for all k

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A NUMBER THEORETIC FUNCTION AND ITS MEAN VALUE *

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Abstract Let $q \geq 3$ be a fixed positive integer, $e_q(n)$ denotes the largest exponent of power q which divides n. In this paper, we use the elementary method to study the properties of the sequence $e_q(n)$, and give some sharper asymptotic formulas for the mean value $\sum_{n\leq x} e_q^k(n)$.

Keywords: Largest exponent; Asymptotic formula; Mean value.

§1. Introduction

Let $q \geq 3$ be a fixed positive integer, $e_q(n)$ denotes the largest exponent of power q which divides n. It is obvious that $e_q(n) = m$, if $q^m | n$, and $q^{m+1} \dagger n$. In problem 68 of [3], Professor F.Smarandach asked us to study the properties of the sequence $e_q(n)$. About this problem, lv chuan in [2] had given the following result:

If p is a prime, $m \geq 0$ is an integer

$$\sum_{n\leq x} e_p^m(n) = \frac{p-1}{p} a_p(m)x + O\left(\log^{m+1} x\right),$$

where $a_p(m)$ is a computable number.

The author had used the analytic method to consider the special case: p_1 and p_2 are two fixed distinct primes. That is, for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} e_{p_1p_2}(n) = \frac{x}{p_1p_2 - 1} + O\left(x^{1/2+\varepsilon}\right), \tag{1}$$

where ε is any fixed positive number.

In this paper, we use the elementary method to improve the error term of (1), and give some sharper asymptotic formula for the mean value $\sum_{n\leq x} e_q^k(n)$. That is we shall prove the following:

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Theorem 1. *Let $q \geq 3$ be any fixed positive integer, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} e_q(n) = \frac{x}{q-1} + O(\log x).$$

.

Theorem 2. *If $q \geq 3$ is any fixed positive integer, $k \geq 2$ is an integer, then we have the asymptotic formula*

$$\sum_{n \leq x} e_q^k(n) = \frac{q-1}{q} B_q(k) x + O\left(\log^{k+1} x\right),$$

where $B_q(k)$ is given by the recursion formulas: $B_q(0) = \frac{1}{q-1}$,

$$B_q(k) = \frac{1}{q-1}\left(\binom{k}{1} B_q(k-1) + \binom{k}{2} B_q(k-2) + \cdots + \binom{k}{k-1} B_q(1) + B_q(0) + 1\right).$$

Taking $q = p_1 p_2$ in Theorem 1, where p_1, p_2 are two fixed distinct primes, we may immediately obtain the following

Corollary. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} e_{p_1 p_2}(n) = \frac{x}{p_1 p_2 - 1} + O(\log x).$$

§2. Proof of the theorems

In this section, we shall complete the Theorems.

Let $M = [x]$, the greatest integer $\leq x$, S denotes the set of $\{1, 2, 3, \cdots, M\}$. We distribute the integers of S into disjoint sets as follows. For each integer $m \geq 0$, let

$$A(m) = \{n| \quad e_q(n) = m, 1 \leq n \leq M\}.$$

That is, $A(m)$ contains those elements of S which satisfies: $q^m | n$, but $q^{m+1} \nmid n$.

Therefore if $f(m)$ denotes the number of integers in $A(m)$, we have

$$f(m) = \left[\frac{M}{q^m}\right] - \left[\frac{M}{q^{m+1}}\right]$$

So we have

$$\sum_{n \leq x} e_q(n) = \sum_{n \leq M} e_q(n) = \sum_{m=0}^{\infty} m f(m)$$

$$= \sum_{m=1}^{\infty} m\left(\left[\frac{M}{q^m}\right] - \left[\frac{M}{q^{m+1}}\right]\right) = \sum_{m=1}^{\infty} \left[\frac{M}{q^m}\right]$$

$$\begin{aligned}
&= \sum_{m=1}^{\infty}\frac{M}{q^m}+O\left(\sum_{m\le\frac{\log M}{\log q}}1\right)+O\left(\sum_{m>\frac{\log M}{\log q}}\frac{M}{q^m}\right)\\
&= \sum_{m=1}^{\infty}\frac{x}{q^m}+O\left(\sum_{m=1}^{\infty}\frac{1}{q^m}\right)+O(\frac{\log M}{\log q})\\
&= \frac{x}{q-1}+O\left(\log x\right).
\end{aligned}$$

This completes the proof of Theorem1.

Before proving Theorem 2, we consider the series $B_q(k)=\sum\limits_{m=1}^{\infty}\frac{m^k}{q^m}$, it is easy to show that

$B_q(0)=\sum\limits_{m=1}^{\infty}\frac{1}{q^m}=\frac{1}{q-1}$, and $B_q(k)$ satisfies the recursion formula

$$B_q(k)=\frac{1}{q-1}\left(\tbinom{k}{1}B_q(k-1)+\tbinom{k}{2}B_q(k-2)+\cdots+\tbinom{k}{k-1}B_q(1)+B_q(0)+1\right).$$

Now we complete the proof of theorem2, with the same method as above, we have

$$\begin{aligned}
&\sum_{n\le x}e_q^k(n)=\sum_{n\le M}e_q^k(n)=\sum_{m=0}^{\infty}m^kf(m)\\
=\ &\sum_{m=1}^{\infty}m^k\left(\left[\frac{M}{q^m}\right]-\left[\frac{M}{q^{m+1}}\right]\right)\\
=\ &\sum_{m=1}^{\infty}m^k\left(\frac{M}{q^m}-\frac{M}{q^{m+1}}\right)+O\left(\sum_{m\le\frac{\log M}{\log q}}m^k\right)+O\left(\sum_{m>\frac{\log M}{\log q}}\frac{Mm^k}{q^m}\right)\\
=\ &\frac{(q-1)M}{q}\sum_{m=1}^{\infty}\frac{m^k}{q^m}+O(\log^{k+1}M)+O\left(\frac{1}{q^{[\frac{\log M}{\log q}]}}\sum_{u=1}^{\infty}\frac{M(\frac{\log M}{\log q}+u)^k}{q^u}\right)\\
=\ &\frac{(q-1)M}{q}B_q(k)+O(\log^{k+1}M)\\
&+O\left(\left(\frac{\log M}{\log q}\right)^k\sum_{u=1}^{\infty}\frac{1}{q^u}+\tbinom{k}{1}\left(\frac{\log M}{\log q}\right)^{k-1}\sum_{u=1}^{\infty}\frac{u}{q^u}+\cdots+\tbinom{k}{k}\sum_{u=1}^{\infty}\frac{u^k}{q^u}\right)\\
=\ &\frac{(q-1)M}{q}B_q(k)+O(\log^{k+1}M)\\
=\ &\frac{q-1}{q}B_q(k)x+O\left(\log^{k+1}x\right).
\end{aligned}$$

This completes the proof of Theorem2.

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A NEW FUNCTION AND ITS MEAN VALUE*

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Abstract The main purpose of this paper is using the elementary method to study the mean value properties of a new function for n, and give a sharp asymptotic formula for it.

Keywords: Elementary method; Mean value; Asymptotic formula.

§1. Introduction

For any positive integer n, let $Sg(n)$ denotes the smallest square greater than or equal to n. For example, $Sg(1) = 1$, $Sg(2) = 4$, $Sg(3) = 4$, $Sg(4) = 4$, $Sg(5) = 9$, $Sg(6) = 9$, $Sg(7) = 9$, $\cdots$, $Sg(9) = 9$, $Sg(10) = 16\cdots$. In problem 40 of book [1], Professor F. Smarandache asks us to study the properties of the sequence $Sg(n)$. About this problem, we know very little. Let x be any real number, in this paper we will study function $Sk(x) - x$. The problem is very important because it can help us to study the distribution of the square root sequence. In this paper, we generalized this problem for generalization. That is, let $Sk(n)$ denotes the smallest power k greater than or equal to n, and $Gk(n) = Sk(n) - n$. In this paper use the elementary methods to study the mean value properties of $Gk(n)$ for n, and give a sharp asymptotic formula for it. That is, we shall prove the following:

Theorem. Let $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} Gk(n) = \frac{k^2}{2(2k-1)} x^{\frac{2k-1}{k}} + O\left(x^{\frac{2k-2}{k}}\right).$$

Especially, when $k = 2, 3$, we have the following

Corollary 1. Let $x \geq 1$, we have

$$\sum_{n\leq x} (Sg(n) - n) = \frac{2}{3} x^{\frac{3}{2}} + O(x).$$

*This work is supported by the P.S.F.(2005A09) of P.R.China.

Corollary 2. Let $x \geq 1$ and $Sc(n)$ denotes the smallest cube greater than or equal to n, we have

$$\sum_{n\leq x}(Sc(n)-n)=\frac{9}{10}x^{\frac{5}{3}}+O\left(x^{\frac{4}{3}}\right).$$

§2. A Lemma

To complete the proof of the theorem, we need the following:

Lemma. Let x be any real number and $\alpha \geq 0$, we have

$$\sum_{n\leq x} n^{\alpha}=\frac{x^{\alpha+1}}{\alpha+1}+o(x^{\alpha}).$$

Proof (See reference [2]).

§3. Proof of the theorem

In this section, we complete the proof of Theorem. For any real number $x \geq 1$, let M be a fixed positive integer such that

$$M^k \leq x < (M+1)^k.$$

Then from the definition of $Gk(n)$, we have

$$\begin{aligned}\sum_{n\leq x} Gk(n) &= \sum_{t=1}^{M}\sum_{(t-1)^k\leq n<t^k}(t^k-n)+O\left(\sum_{M^k\leq n<x}(M^k-n)\right)\\ &= \sum_{t=1}^{M}\sum_{0\leq u<t^k-(t-1)^k} u+O\left(\sum_{0\leq n<M^k-x} u\right)\\ &= \sum_{t=1}^{M}\left[\frac{k^2t^{2k-2}}{2}+O(t^{2k-3})\right]+O(M^{2k-2})\\ &= \frac{k^2}{2(2k-1)}M^{2k-1}+O(M^{2k-2})\end{aligned}$$

On the other hand, note that the estimates

$$0\leq x-M^k \ll (M+1)^k-M^k \ll M^{k-1} \ll X^{\frac{k-1}{k}}$$

them we have

$$M^{2k-1}=x^{\frac{2k-1}{k}}+O(x^{\frac{2k-2}{k}})$$

and

$$M^{2k-2} \ll x^{\frac{2k-2}{k}}$$

Now combining the above, we have obtain the asymptotic formula

$$\sum_{n\leq x} Gk(n)=\frac{k^2}{2(2k-1)}x^{\frac{2k-1}{k}}+O\left(x^{\frac{2k-2}{k}}\right).$$

This completes the proof of the theorem.

Acknowledgments

The author express his gratitude to his supervisor Professor Zhang Wenpeng for his very helpful and detailed instructions.

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ON THE M-POWER COMPLEMENT NUMBERS

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Abstract The main purpose of this paper is using the elementary method to study the asymptotic properties of the m-power complement numbers, and give an interesting asymptotic formula for it.

§1. Introduction and results

Let $n \geq 2$ is any integer, $a_m(n)$ is called a m-power complement about n if $a_m(n)$ is the smallest integer such that $n \times a_m(n)$ is a perfect m-power. For example $a_m(2) = 2^{m-1}$, $a_m(3) = 3^{m-1}$, $a_m(4) = 2^{m-2}$, $a_m(2^m) = 1, \cdots$. The famous Smarandache function $S(n)$ is defined as following:

$$S(n) = \min\{m : m \in N, n \mid m!\}.$$

For example, $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3, \cdots$. In reference [1], Professor F.Smarandache asked us to study the properties of m-power complement number sequence. About this problem, some authors have studied it before. See [4]. In this paper, we use the elementary method to study the mean value properties of m-power complement number sequence, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. *Let $x \geq 1$ be any real number and $m \geq 2$, then we have the asymptotic formula*

$$\sum_{n \leq x} a_m(S(n)) = \frac{x^m \zeta(m)}{m \ln x} + O\left(\frac{x^m}{\ln^2 x}\right).$$

§2. Proof of the theorem

To complete the proof of the theorem, we need some lemmas.

Lemma 1. *If $p(n) > \sqrt{n}$, then $S(n) = p(n)$.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r} p(n)$; so we have

$$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r} < \sqrt{n}$$

then

$$p_i^{\alpha_i} \mid p(n)!, \ \ i = 1, 2, \cdots, r.$$

So $n \mid p(n)!$, but $p(n) \dagger (p(n)-1)!$, so $S(n) = p(n)$.

This completes the proof of the lemma 1.

Lemma 2. *If $x \geq 1$ be any real number and $m \geq 2$, then we have the two asymptotic formulae:*

$$\sum_{\substack{n \leq x \\ p(n) \leq \sqrt{n}}} S^{m-1}(n) = O\left(x^{\frac{m+1}{2}} \ln^{m-1} x\right);$$

$$\sum_{\substack{n \leq x \\ p(n) > \sqrt{n}}} S^{m-1}(n) = \frac{x^m \zeta(m)}{m \ln x} + O\left(\frac{x^m}{\ln^2 x}\right).$$

Proof. First, from the Euler summation formula [2] we can easily get

$$\begin{aligned} & \sum_{\substack{n \leq x \\ p(n) \leq \sqrt{n}}} S^{m-1}(n) \ll \sum_{n \leq x} (\sqrt{n} \ln n)^{m-1} \\ = & \int_1^x (\sqrt{t} \ln t)^{m-1} dt + \int_1^x (t - [t]) \left((\sqrt{t} \ln t)^{m-1}\right)' dt + (\sqrt{x} \ln x)^{m-1}(x - [x]) \\ = & \frac{m+3}{m+1} x^{\frac{m+1}{2}} \ln^{m-1} x + O\left(x^{\frac{m}{2}} \ln^{m-1} x\right). \end{aligned}$$

And then, we have

$$\begin{aligned} \sum_{\substack{n \leq x \\ p(n) > \sqrt{n}}} S^{m-1}(n) &= \sum_{\substack{np \leq x \\ p > \sqrt{np}}} S^{m-1}(np) = \sum_{\substack{n \leq \sqrt{x} \\ \sqrt{n} < p \leq \frac{x}{n}}} p^{m-1} \\ &= \sum_{n \leq \sqrt{x}} \sum_{\sqrt{n} < p \leq \frac{x}{n}} p^{m-1}. \end{aligned}$$

Let $\pi(x)$ denote the number of the primes up to x. From [3], we have

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

Using Abel's identity [2], we can write

$$\begin{aligned} & \sum_{\sqrt{x} < p \leq \frac{x}{n}} p^{m-1} = \pi(\frac{x}{n})(\frac{x}{n})^{m-1} - \pi(\sqrt{x})(\sqrt{x})^{m-1} - \int_{\sqrt{x}}^{\frac{x}{n}} \pi(t)(t^{m-1})' dt \\ = & \left(\frac{x^m}{n^m(\ln x - \ln n)} + O\left(\frac{x^m}{n^m(\ln x - \ln n)^2}\right)\right) \end{aligned}$$

$$\begin{aligned} & - \left(\frac{2x^{\frac{m}{2}}}{\ln x} + O\left(\frac{4x^{\frac{m}{2}}}{\ln^2 x}\right)\right) - (m-1)\int_{\sqrt{x}}^{\frac{x}{n}} \left(\frac{t^{m-1}}{\ln t} + O\left(\frac{t^{m-1}}{\ln^2 x}\right)\right) dt \\ = & \frac{x^m}{mn^m \ln x} + O\left(\frac{x^m}{n^m \ln^2 x}\right). \end{aligned}$$

According to [2], we know that

$$\sum_{n\le x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}) \quad if\ s > 0, s \neq 1.$$

so we have

$$\sum_{n\le\sqrt{x}} \sum_{\sqrt{n}<p\le\frac{x}{n}} p^{m-1} = \frac{x^m \zeta(m)}{m \ln x} + O\left(\frac{x^m}{\ln^2 x}\right).$$

This completes the proof of the lemma 2.

3 Proof of the Theorem

In this section, we complete the proof of the Theorem. Combining Lemma 1, Lemma 2 and the definition of $a_m(n)$ it is clear that

$$\begin{aligned} \sum_{n\le x} a_m(S(n)) &= \sum_{\substack{n\le x \\ p(n)>\sqrt{n}}} p^{m-1} + O\left(\sum_{\substack{n\le x \\ p(n)\le\sqrt{n}}} (\sqrt{n}\ln n)^{m-1}\right) \\ &= \frac{x^m \zeta(m)}{m \ln x} + O\left(\frac{x^m}{\ln^2 x}\right). \end{aligned}$$

This completes the proof of the Theorem.

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ON THE PRIMITIVE NUMBERS OF POWER *P* AND ITS ASYMPTOTIC PROPERTY *

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Abstract Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer $m \in N^+$, where $p^n|m!$. In this paper, we study the mean value properties of $S_p(n)$, and give an interesting asymptotic formula for it.

Keywords: Smarandache function; Primitive numbers; Asymptotic formula

§1. Introduction and results

Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer such that $S_p(n)!$ is divisible by p^n. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = 9$, $S_3(4) = 9$, $\cdots\cdots$. In problem 49 of book [1], Professor F. Smarandache ask us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Professor Zhang and Liu in [2] have studied it and obtained an interesting asymptotic formula. That is, for any fixed prime p and any positive integer n,

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p} \cdot \ln n\right).$$

In this paper, we will use the elementary method to study the asymptotic properties of $S_p(n)$ in the following form:

$$\frac{1}{p}\sum_{n\leq x} |S_p(n+1) - S_p(n)|,$$

where x be a positive real number, and give an interesting asymptotic formula for it. In fact, we shall prove the following result:

Theorem. *For any real number $x \geq 2$, let p be a prime and n be any positive integer. Then we have the asymptotic formula*

$$\frac{1}{p}\sum_{n\leq x} |S_p(n+1) - S_p(n)| = x\left(1 - \frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right).$$

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§2. Proof of the Theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. That is,

Lemma. *Let p be a prime and n be any positive integer, then we have*

$$|S_p(n+1) - S_p(n)| = \begin{cases} p, & \text{if } p^n \parallel m!; \\ 0, & \text{otherwise}, \end{cases}$$

where $S_p(n) = m$, $p^n \parallel m!$ denotes that $p^n | m!$ and $p^{n+1} \dagger m!$.

Proof. Now we will discuss it in two cases.

(i) Let $S_p(n) = m$, if $p^n \parallel m!$, then we have $p^n | m!$ and $p^{n+1} \dagger m!$. From the definition of $S_p(n)$ we have $p^{n+1} \dagger (m+1)!$, $p^{n+1} \dagger (m+2)!$, $\cdots$, $p^{n+1} \dagger (m+p-1)!$ and $p^{n+1} | (m+p)!$, so $S_p(n+1) = m + p$, then we get

$$|S_p(n+1) - S_p(n)| = p. \tag{1}$$

(ii) Let $S_p(n) = m$, if $p^n | m!$ and $p^{n+1} | m!$, then we have $S_p(n+1) = m$, so

$$|S_p(n+1) - S_p(n)| = 0. \tag{2}$$

Combining (1) and (2), we can easily get

$$|S_p(n+1) - S_p(n)| = \begin{cases} p, & \text{if } p^n \parallel m!; \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof of Lemma.

Now we use above Lemma to complete the proof of Theorem. For any real number $x \geq 2$, by the definition of $S_p(n)$ and Lemma we have

$$\frac{1}{p} \sum_{n \leq x} |S_p(n+1) - S_p(n)| = \frac{1}{p} \sum_{\substack{n \leq x \\ p^n \| m!}} p = \sum_{\substack{n \leq x \\ p^n \| m!}} 1, \tag{3}$$

where $S_p(n) = m$. Note that if $p^n \parallel m!$, then we have (see reference [3], Theorem 1.7.2)

$$\begin{aligned} n &= \sum_{i=1}^{\infty} \left[\frac{m}{p^i} \right] = \sum_{i \leq \log_p m} \left[\frac{m}{p^i} \right] \\ &= m \cdot \sum_{i \leq \log_p m} \frac{1}{p^i} + O\left(\log_p m \right) \\ &= \frac{m}{p-1} + O\left(\frac{\ln m}{\ln p} \right). \end{aligned} \tag{4}$$

From (4), we can deduce that

$$m = (p-1)n + O\left(\frac{p \ln n}{\ln p} \right). \tag{5}$$

So that

$$1 \le m \le (p-1) \cdot x + O\left(\frac{p \ln x}{\ln p}\right), \quad \text{if} \quad 1 \le n \le x.$$

Note that for any fixed positive integer n, if there has one m such that $p^n \parallel m!$, then $p^n \parallel (m+1)!$, $p^n \parallel (m+2)!$, $\cdots$, $p^n \parallel (m+p-1)!$. Hence there have p times of m such that $n = \sum\limits_{i=1}^{\infty} \left[\frac{m}{p^i}\right]$ in the interval $1 \le m \le (p-1) \cdot x + O\left(\frac{p \ln x}{\ln p}\right)$. Then from this and (3), we have

$$\begin{aligned} \frac{1}{p} \sum_{n \le x} |S_p(n+1) - S_p(n)| &= \frac{1}{p} \sum_{\substack{n \le x \\ p^n \| m!}} p = \sum_{\substack{n \le x \\ p^n \| m!}} 1 \\ &= \frac{1}{p}\left((p-1) \cdot x + O\left(\frac{p \ln x}{\ln p}\right)\right) \\ &= x \cdot \left(1 - \frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right). \end{aligned}$$

This completes the proof of Theorem.

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MEAN VALUE OF THE ADDITIVE ANALOGUE OF SMARANDACHE FUNCTION

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Abstract For any positive integer n, let $Sdf(n)$ denotes the Smarandance double factorial function, then $Sdf(n)$ is defined as least positive integer m such that $m!!$ is divisible by n. In this paper, we study the mean value properties of the additive analogue of $Sdf(n)$ and give an interesting mean value formula for it.

Keywords: Smarandance function; Additive Analogue; Mean value formula

§1. Introduction and result

For any positive integer n, let $Sdf(n)$ denotes the Smarandance double factorial function, then $Sdf(n)$ defined the least positive integer n such that $m!!$ is divisible by n, where

$$m!! = \begin{cases} 2 \cdot 4 \cdots m, & \text{if } 2|m; \\ 1 \cdot 3 \cdots m, & \text{if } 2\dagger\text{m}. \end{cases}$$

In reference [2], Professor Jozsef Sandor defined the following analogue of Smarandance double factorial function as:

$$Sdf_1(2x) = \min\{2m \in N : 2x \leq (2m)!!\}, x \in (1, \infty),$$

$$Sdf_1(2x+1) = \min\{2m+1 \in N : (2x+1) \leq (2m+1)!!\}, x \in (1, \infty),$$

which is defined on a subset of real numbers. Clearly $Sdf_1(n) = m$ if $x \in ((m-2)!!, m!!]$ for $m \geq 2$, therefore this function is defined for $x \geq 1$.

About the arithmetical properties of $Sdf(n)$, many people had ever studied it. But for the mean value properties of $Sdf_1(n)$, it seems that no one have studied before. The main purpose of this paper is to study the mean value properties of $Sdf_1(n)$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n \leq x} Sdf_1(n) = \frac{2x \ln x}{\ln \ln x} + O\left(\frac{x(\ln x)(\ln \ln \ln x)}{(\ln \ln x)^2}\right).$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need the following one simple Lemma. That is,

Lemma. *For any fixed positive integer m and n with $(m-2)!! < n \leq m!!$, we have the asymptotic formula*

$$m = \frac{2\ln n}{\ln\ln n} + O\left(\frac{(\ln n)(\ln\ln\ln n)}{(\ln\ln n)^2}\right).$$

Proof. To complete the proof the Lemma, we separate it into two cases:

(I) If $m = 2u$, we have $(2u-2)!! < n \leq (2u)!!$. Taking the logistic computation in the two sides of the inequality, we get

$$(u-1)\ln 2 + \sum_{i=1}^{u-1} \ln i < \ln n \leq u\ln 2 + \sum_{i=1}^{u} \ln i. \tag{1}$$

Then using the Euler's summation formula we have

$$\sum_{i=1}^{u} \ln i = \int_1^u \ln t dt + \int_1^u (t-[t])(\ln t)' dt = u\ln u - u + O(\ln u) \tag{2}$$

and

$$\sum_{i=1}^{u-1} \ln i = \sum_{i=1}^{u} \ln i + O(\ln u) = u\ln u - u + O(\ln u). \tag{3}$$

Combining (1), (2) and (3), we can easily deduce that

$$\ln n = u\ln u + (\ln 2 - 1)u + O(\ln u). \tag{4}$$

So

$$u = \frac{\ln n}{\ln u + (\ln 2 - 1)} + O(1). \tag{5}$$

Similarly, we continue taking the logistic computation in two sides of (5), then we also have

$$\ln u = \ln\ln n + O(\ln\ln u) \tag{6}$$

and

$$\ln\ln u = O(\ln\ln\ln n). \tag{7}$$

Hence, by (5), (6) and (7) we have

$$u = \frac{\ln n}{\ln\ln n} + O\left(\frac{(\ln n)(\ln\ln\ln n)}{(\ln\ln n)^2}\right).$$

This completes the proof of the first case.

(II) If $m = 2u+1$, we have $(2u-1)!! < n \leq (2u+1)!!$. Taking the logistic computation in the two sides of the inequality, we get

$$\sum_{i=1}^{2u} \ln i - (u\ln 2 + \sum_{i=1}^{u} \ln i) < \ln n \leq \sum_{i=1}^{2u+1} \ln i - (u\ln 2 + \sum_{i=1}^{u} \ln i). \quad (8)$$

Then using the Euler's summation formula we have

$$\sum_{i=1}^{2u} \ln i = \int_1^{2u} \ln t dt + \int_1^{2u} (t-[t])(\ln t)' dt = 2u\ln u + 2(\ln 2 - 1)u + O(\ln u) \quad (9)$$

and

$$\sum_{i=1}^{2u+1} \ln i = \sum_{i=1}^{2u} \ln i + O(\ln 2u + 1) = 2u\ln u + 2(\ln 2 - 1)u + O(\ln u). \quad (10)$$

From (2), (3), (8), (9) and (10) we have

$$\ln n = u\ln u + (\ln 2 - 1)u + O(\ln u).$$

Therefore, we may obtain (5).

Using the similar method on the above, we may have

$$u = \frac{\ln n}{\ln\ln n} + O\left(\frac{(\ln n)(\ln\ln\ln n)}{(\ln\ln n)^2}\right).$$

This completes the proof of the second case.

Combining the above two cases, we can easily get

$$m = \frac{2\ln n}{\ln\ln n} + O\left(\frac{(\ln n)(\ln\ln\ln n)}{(\ln\ln n)^2}\right).$$

This completes the proof of Lemma.

Now we use the above Lemma to complete the proof of Theorem. For any real number $x \geq 2$, by the definition of $Sdf_1(n)$ and the above Lemma we have

$$\begin{aligned}\sum_{n\leq x} Sdf_1(n) &= \sum_{\substack{n\leq x \\ (m-2)!!<n\leq m!!}} m \\ &= \sum_{n\leq x}\left(\frac{2\ln n}{\ln\ln n} + O\left(\frac{(\ln n)(\ln\ln\ln n)}{(\ln\ln n)^2}\right)\right) \\ &= 2\sum_{n\leq x}\frac{\ln n}{\ln\ln n} + O\left(\frac{x(\ln x)(\ln\ln\ln x)}{(\ln\ln x)^2}\right). \quad (11)\end{aligned}$$

By the Euler's summation formula, we deduce that

$$\sum_{n\leq x}\frac{\ln n}{\ln\ln n} = \int_2^x \frac{\ln t}{\ln\ln t}dt + \int_2^x (t-[t])(\frac{\ln t}{\ln\ln t})'dt + \frac{\ln x}{\ln\ln x}(x-[x])$$

$$= \frac{x\ln x}{\ln\ln x} + O\left(\frac{x}{\ln\ln x}\right). \tag{12}$$

Therefore, from (11) and (12) we have

$$\sum_{n\leq x} Sdf_1(n) = \frac{2x\ln x}{\ln\ln x} + O\left(\frac{x(\ln x)(\ln\ln\ln x)}{(\ln\ln x)^2}\right).$$

This completes the proof of Theorem.

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ON THE GENERALIZATION OF THE FLOOR OF THE SQUARE ROOT SEQUENCE

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Abstract The floor of the square root sequence is the natural sequence, where each number is repeated $2n+1$ times. In this paper, we use analytic method to study the mean value properties of its generalization, and give an interesting asymptotic formula.

Keywords: the floor of the square root sequence; mean value; asymptotic formula.

§1. Introduction

The floor of the square roots of the natural numbers are:
0,1,1,1,2,2,2,2,2,3,3,3,3,3,3,3,4,4,4,4,4,4,4,4,4, 5,5,5,5,5,5,5,5,5,5,5,6,6,
6,6,6,6,6,6,6,6,6,6,6,6, 7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,8, $\cdots$

This sequence is the natural sequence, where each number is repeated $2n+1$ times. In reference [1], Professor F.Smarandache asked us to study the properties of this sequence. We denote the generalization of the sequence as $b(n)$, in which each number is repeated $kn+1$ times, and express it as $b(n)=[n^{1/k}]$. In reference [2], He Xiaolin and Guo Jinbao studied the mean value of $d(b(n))$, and obtain an asymptotic formula for it. In this paper, as a generalization of [2], we use analytic method to study the mean value properties of $\sigma_\alpha(b(n))$, and give a general asymptotic formula for $\sigma_\alpha(b(n))$. That is, we shall prove the following :

Theorem. For any real number $x>1$ and integer $n\geq 1$, we have

$$\sum_{n\leq x}\sigma_\alpha(b(n))=\begin{cases}\frac{k\zeta(\alpha+1)}{\alpha+k}x^{\frac{\alpha+k}{k}}+O(x^{\frac{\beta+k-1}{k}}), & \text{if } \alpha>0;\\ \frac{1}{k}x\log x+O(x), & \text{if } \alpha=0;\\ \zeta(2)x+0(x^{\frac{k+\varepsilon-1}{k}}), & \text{if } \alpha=-1;\\ \zeta(1-\alpha)x+O(x^{\frac{\delta+k-1}{k}}), & \text{if } \alpha<0 \text{ and } \alpha\neq -1\end{cases}$$

where $\sigma_\alpha(n)=\sum\limits_{d|n}d^\alpha$ be the divisor function, $\zeta(n)$ be the Riemann Zeta function, $\beta=\max\{1,\alpha\},\delta=\max\{0,1+\alpha\}$ and $\varepsilon>0$ be an arbitrary real number.

§2. Proof of the Theorem

In this section, we shall complete the proof of the theorem. Firstly, we need following:

Lemma 1. Let $\alpha > 0$ be a fixed real number. Then for $x > 1$, we have

$$\sum_{n\leq x}\sigma_\alpha(n) = \frac{\zeta(\alpha+1)}{\alpha+1}x^{\alpha+1} + O(x^\beta),$$

and

$$\sum_{n\leq x}\sigma_{-\alpha}(n) = \begin{cases} \zeta(\alpha+1)x + O(x^\delta), & \text{if } \alpha \neq 1; \\ \zeta(2)x + O(\log x), & \text{if } \alpha = 1. \end{cases}$$

where $\beta = \max\{1, \alpha\}$, $\delta = \max\{0, 1-\alpha\}$, $\zeta(n)$ denotes the Riemann zeta-function.

Proof See reference [3].

Lemma 2. Let n be a positive integer, and $b(n) = [n^{1/k}]$, $d(n)$ be the divisor function, then

$$\sum_{n\leq x}\sigma_0(b(n)) = \sum_{n\leq x} d\left([n^{1/k}]\right) = \frac{1}{k}x\log x + O(x).$$

Proof. See reference [2].

Now we use the above Lemmas to complete the proof of Theorem. We separate α into three cases respectively.

Case 1 , when $\alpha > 0$, we have

$$\begin{aligned} &\sum_{n\leq x}\sigma_\alpha(b(n)) = \sum_{n\leq x}\sigma_\alpha([n^{1/k}]) \\ = &\sum_{1^k\leq n<2^k}\sigma_\alpha\left([n^{1/k}]\right) + \sum_{2^k\leq n<3^k}\sigma_\alpha\left([n^{1/k}]\right) + \cdots \\ &+ \sum_{N^k\leq n<(N+1)^k}\sigma_\alpha\left([n^{1/k}]\right) + O(N^\beta) \\ = &\sum_{j\leq N}\left((j+1)^k - j^k\right)\sigma_\alpha(j) + O(N^\beta). \end{aligned}$$

Let $A(n) = \sum\limits_{n\leq x}\sigma_\alpha(j)$ and $f(j) = \sum\limits_{j\leq N}\left((j+1)^k - j^k\right)$, applying Abel's identity and Lemma 1, we have

$$\begin{aligned} &\sum_{n\leq x}\sigma_\alpha(b(n)) \\ = &A(N)f(N) - A(1)f(1) - \int_1^N A(t)f^{'}(t)dt + O(N^\beta) \\ = &\frac{k\zeta(\alpha+1)}{\alpha+1}N^{\alpha+k} - k(k-1)\int_1^N \frac{\zeta(\alpha+1)}{\alpha+k}t^{\alpha+k+1}dt \end{aligned}$$

$$\begin{aligned} & +O(N^{\beta+k-1}) \\ = & \frac{k\zeta(\alpha+1)}{\alpha+k}N^{\alpha+k}+O(N^{\beta+k-1}) \\ = & \frac{k\zeta(\alpha+1)}{\alpha+k}x^{\frac{\alpha+k}{k}}+O(x^{\frac{\beta+k-1}{k}}). \end{aligned}$$

Case 2, when $\alpha=-1$, we have

$$\begin{aligned} & \sum_{n\le x}\sigma_{-1}(b(n))=\sum_{n\le x}\sigma_{-1}([n^{1/k}]) \\ = & \sum_{1^k\le n<2^k}\sigma_{-1}\left([n^{1/k}]\right)+\sum_{2^k\le n<3^k}\sigma_{-1}\left([n^{1/k}]\right)+\cdots \\ & +\sum_{N^k\le n<(N+1)^k}\sigma_{-1}\left([n^{1/k}]\right)+O(N^{\varepsilon}) \\ = & \sum_{j\le N}\left((j+1)^k-j^k\right)\sigma_{-1}(j)+O(N^{\varepsilon}). \end{aligned}$$

Let $A(n)=\sum_{j\le N}\sigma_{-1}(j)$ and $f(j)=\sum_{j\le N}\left((j+1)^k-j^k\right)$, we have

$$\begin{aligned} & \sum_{n\le x}\sigma_{-1}(b(n))=\sum_{n\le x}\sigma_{-1}[n^{1/k}] \\ = & A(N)f(N)-A(1)f(1)-\int_1^N A(t)f'(t)dt+O(N^{\varepsilon}) \\ = & \zeta(2)N^k+O(N^{k+\varepsilon-1}) \\ = & \zeta(2)x+O(x^{\frac{k+\varepsilon-1}{k}}), \end{aligned}$$

where $\varepsilon>0$ be an arbitrary real number.

Case 3, when $\alpha<0$ and $\alpha\neq-1$, we have

$$\begin{aligned} & \sum_{n\le x}\sigma_{\alpha}(b(n))=\sum_{n\le x}\sigma_{\alpha}([n^{1/k}]) \\ = & \sum_{1^k\le n<2^k}\sigma_{\alpha}\left([n^{1/k}]\right)+\sum_{2^k\le n<3^k}\sigma_{\alpha}\left([n^{1/k}]\right)+\cdots \\ & +\sum_{N^k\le n<(N+1)^k}\sigma_{\alpha}\left([n^{1/k}]\right)+O(N^{\delta}) \\ = & \sum_{j\le N}\left((j+1)^k-j^k\right)\sigma_{\alpha}(j)+O(N^{\delta}). \end{aligned}$$

Let $A(n)=\sum_{j\le N}\sigma_{\alpha}(j)$ and $f(j)=\sum_{j\le N}\left((j+1)^k-j^k\right)$, we have

$$\sum_{n\le x}\sigma_{\alpha}(b(n))=\sum_{n\le x}\sigma_{\alpha}([n^{1/k}])$$

$$\begin{aligned} &= A(N)f(N) - A(1)f(1) - \int_1^N A(t)f^{'}(t)dt + O(N^{\delta}) \\ &= k\zeta(1-\alpha)N^k + O(N^{k+\delta-1}) - (k-1)\zeta(1-\alpha)N^k \\ &= \zeta(1-\alpha)N^k + O(N^{k+\delta-1}) \\ &= \zeta(1-\alpha)x + O(x^{\frac{\delta+k-1}{k}}). \end{aligned}$$

Combining the above result, we have

$$\sum_{n\leq x}\sigma_{\alpha}(b(n)) = \begin{cases} \frac{k\zeta(\alpha+1)}{\alpha+k}x^{\frac{\alpha+k}{k}} + O(x^{\frac{\beta+k-1}{k}}), & \text{if } \alpha > 0; \\ \frac{1}{k}x\log x + O(x), & \text{if } \alpha = 0; \\ \zeta(2)x + 0(x^{\frac{k+\varepsilon-1}{k}}), & \text{if } \alpha = -1; \\ \zeta(1-\alpha)x + O(x^{\frac{\delta+k-1}{k}}), & \text{if } \alpha < 0 \text{ and } \alpha \neq -1 \end{cases}$$

This completes the proof of Theorem.

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MEAN VALUE OF A NEW ARITHMETIC FUNCTION

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Abstract The main purpose of this paper is using elementary method to study a new arithmetic function, and give an interesting asymptotic formula for it.

Keywords: Arithmetic function; Mean value; Asymptotic formula

§1. Introduction

For any positive integer n, we have $n = u^k v$, where v is a k-power free number. Let $b_k(n)$ be the k-power free part of n. Let p be any fixed prime, n be any positive integer, $e_p(n)$ denotes the largest exponent of power p. That is, $e_p(n) = m$,if $p^m | n$ and $p^{m+1} \nmid n$. In [1], Professor F.Smarandache asked us to study the properties of these two arithmetic functions. It seems that no one knows the relationship between these two arithmetic functions before. The main purpose of this paper is to study the mean value properties of $e_p(b_k(n))$, and obtain an interesting mean value formula for it. That is, we shall prove the following conclusion:

Theorem. *Let p be a prime, k be any fixed positive integer. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} e_p(b_k(n)) = \left(\frac{p^k - p}{(p^k - 1)(p - 1)} - \frac{k - 1}{p^k - 1} \right) x + O\left(x^{\frac{1}{2}+\epsilon} \right),$$

where ϵ denotes any fixed positive number.

Taking $k = 2$ in the theorem, we may immediately obtain the following

Corollary. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} e_p(b_k(n)) = \frac{1}{p+1}x + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

§2. Proof of the theorem

In this section, we shall use analytic method to complete the proof of the theorem. In fact we know that $e_p(n)$ is not a multiplicative function, but we can use the properties of the Riemann zeta-function to obtain a generating function. For any complex s, if $\mathrm{Re}(s) > 1$, we define the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{e_p(b_k(n))}{n^s}.$$

Let positive integer $n = p^\alpha n_1$, where $(n_1, p) = 1$, then from the definition of $e_p(n)$ and $b_k(n)$, we have:

$$e_p(b_k(n)) = e_p(b_k(p^\alpha n_1)) = e_p(b_k(p^\alpha)).$$

From the above formula and the Euler product formula (See Theorem 11.6 of [3]) we can get

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} \frac{e_p(b_k(n))}{n^s} \\ &= \sum_{\alpha=0}^{\infty} \sum_{\substack{n_1=1 \\ (n_1,p)=1}}^{\infty} \frac{e_p(b_k(p^\alpha))}{p^{\alpha s} n_1^s} \\ &= \zeta(s)(1-\frac{1}{p^s}) \sum_{\alpha=1}^{\infty} \frac{e_p(b_k(p^\alpha)}{p^{\alpha s}}. \end{aligned}$$

Let

$$\begin{aligned} A &= \sum_{\alpha=1}^{\infty} \frac{e_p(b_k(p^\alpha)}{p^{\alpha s}} \\ &= \frac{1}{p^s} + \frac{2}{p^{2s}} + \cdots + \frac{k-1}{p^{(k-1)s}} + \frac{1}{p^{(k+1)s}} + \frac{2}{p^{(k+2)s}} + \cdots + \frac{k-1}{p^{(2k-1)s}} \\ &\quad + \cdots + \frac{1}{p^{(uk+1)s}} + \frac{2}{p^{(uk+2)s}} + \cdots + \frac{k-1}{p^{(uk+k-1)s}} \\ &= \sum_{u=0}^{\infty} \frac{1}{p^{uks}} \sum_{r=1}^{k-1} \frac{r}{p^{rs}} \\ &= \frac{1}{1-\frac{1}{p^{ks}}} \frac{1}{1-\frac{1}{p^s}} \left(\frac{1-\frac{1}{p^{(k-1)s}}}{p^s-1} - \frac{k-1}{p^{ks}} \right). \end{aligned}$$

So we have

$$f(s) = \sum_{n=1}^{\infty} \frac{e_p(b_k(n))}{n^s} = \left(\frac{p^{ks}-p^s}{(p^{ks}-1)(p^s-1)} - \frac{k-1}{p^{ks}-1} \right) \zeta(s).$$

Because the Riemann zeta-function $\zeta(s)$ have a simple pole point at $s=1$ with the residue 1, we know $f(s)\frac{x^s}{s}$ also have a simple pole point at $s=1$ with the residue $\left(\frac{p^k-p}{(p^k-1)(p-1)} - \frac{k-1}{p^k-1}\right) x$. By Perron formula (See [2]), taking $s_0 = 0$, $b = \frac{3}{2}$, $T > 1$, then we have

$$\sum_{n \le x} e_p(b_k(n)) = \frac{1}{2\pi i} \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{x^{\frac{3}{2}}}{T} \right),$$

we move the integral line to Re $s = \frac{1}{2} + \epsilon$, then taking $T = x$, we can get

$$\begin{aligned} \sum_{n \le x} e_p(b_k(n)) &= \left(\frac{p^k-p}{(p^k-1)(p-1)} - \frac{k-1}{p^k-1} \right) x \\ &\quad + \frac{1}{2\pi i} \int_{\frac{1}{2}+\epsilon-iT}^{\frac{1}{2}+\epsilon+iT} f(s) \frac{x^s}{s} ds + O\left(x^{\frac{1}{2}+\epsilon} \right) \\ &= \left(\frac{p^k-p}{(p^k-1)(p-1)} - \frac{k-1}{p^k-1} \right) x \\ &\quad + O\left(\int_{-T}^{T} \left| f(\frac{1}{2}+\epsilon+it) \right| \frac{x^{\frac{1}{2}+\epsilon}}{1+|t|} dt \right) + O\left(x^{\frac{1}{2}+\epsilon} \right) \\ &= \left(\frac{p^k-p}{(p^k-1)(p-1)} - \frac{k-1}{p^k-1} \right) x + O\left(x^{\frac{1}{2}+\epsilon} \right). \end{aligned}$$

This completes the proof of Theorem.

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ON THE NUMBER OF NUMBERS WITH A GIVEN DIGIT SUM

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Abstract We consider the sum of digits function which maps an integer to the sum of it's digits, for example 142 is mapped to $1 + 4 + 2 = 7$. This papers examines the question of how many other integers are mapped to a given digit in the range 1 to 10^z.

§1. Introduction

To begin with, we need a sum of digits function [1]. The code in this paper has been written in Pari/GP [2].

x is used to determine the number of digits of n, and d is used to store the cumulative sum of the digits, which are extracted by considering the last digit, and removing it, and then divide by 10.

To test the function, and to see the values to $n = 100$:
for $(n = 1, 100)$,print1$(", "sd(n))$

$1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 3,$
$4, 5, 6, 7, 8, 9, 10, 11, 12, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 5, 6, 7, 8, 9, 10, 11, 12,$
$13, 14, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 8, 9,$
$10, 11, 12, 13, 14, 15, 16, 17, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19$

This is $A007953$ at the Online Encyclopaedia of Integer Sequences [3].

The function has obvious patterns. We will only look at digit sums to 9, as the theory for higher digit sums is too complex for this paper.

§2. Partition Theory.

The sum of digits function is closely related to the theory of partitions and compositions.

A partition of n is a sum $d_1 + d_2 + \ldots + d_k$, for some k less than or equal to n such that the sum is n, and the d's are ordered from largest to smallest.

For example, the partitions of 4 are $4, 3+1, 2+2, 2+1+1$ and $1+1+1+1$.

A composition is the same, except for the order of the d's doesn't matter, so the compositions of 4 are as with the partitions, but also including $1+3, 1+2+1$ and $1 + 1 + 2$.

With the sum of digits we need a new concept, as we are dealing with compositions that can be 'stretched' using zeroes in-between the digits.

We also need to add the constraint that any d can be no larger than 9.

The picture is further complicated by the fact that as we consider larger integers, we have more stretched compositions associated with any given composition.

For example, if we consider the composition $1+3$, this is directly associated with 13.

In three digits, we may have 103 and 130. In four digits we can have $1003, 1030$ and 1300.

Easy enough, but try and calculate the number of stretched compositions of 121 over eight digits!

§3. Length of Compositions

To do this, we consider a composition of n, and we define the number of individual digits in a sum as its length. $1 + 2 + 1$ has length 3, etc...

We know that there are 2^{n-1} compositions of n, but determination of the length of each composition is more complicated.

The following tables examine this up to $n = 4$:

The sum of the lengths is $1, 3, 8, 20$, which is [4], and has a simple formula : $(n + 2) * 2^{n-1}$.

The number of compositions with a given length is given by $C(n - 1, k)$, where k is the $length - 1$.

Here is the table for n=5:

Note that the formula quoted in this section are only applicable to the full version of compositions, and only hold to $n = 9$ in this paper.

§4. The function $\boldsymbol{NND(y, X)}$

In order to perform our calculations, we need to first determine an X which is our upper bound, and we do not include X. We then define a function $Number_Of_Numbers_With_Digit_Sum_d(X)$, or $NND(y, X)$ for short.

We count the number of digits of X, and this is the maximum length that we need to consider.

For simplicities sake, we will only consider X to be a power of 10.

If $X = 10^z$, then $NND(1, X) = z$, and as the maximum digit sum is $9 * (z - 1), NND(9 * (z - 1)) = 1$.

Next we consider $NND(2, X)$ through to $NND(9, X)$, as these obey the formulas given in the previous section.

$NND(2, X)$

$NND(2, X)$ asks for the number of integers with a digit sum of 2 on the range 1 to 10^z.

This requires either an opening digit of 2, and the rest zeroes, or an opening digit of 1, some zeroes, another 1, and some more zeroes, possibly zero zeroes in each case.

There are z examples of the first case, for example if $z = 3$ we have $2, 20$ and 200.

The second case we consider the stretched partitions $11, 101, 1001, 10001,$ etc..., as stems for the numbers we are interested in.

If $z = 3$, then the number of each stem present is $2[11, 110]$ and $1[101]$.

If $z = 4$, then we have $3[11, 110, 1100]$ plus $2[101, 1010]$ plus $1[1001]$.

Therefore $NND(2, X) = z(z + 1)/2$.

$NND(3, X)$

We have 4 cases:

I. 3

II. 21

III. 12

IV. 111

Case I : Easy - contributes z

Case II and III : contributes $(z - 1)z/2$ once each - total $z(z - 1)$

Case IV : We have two gaps (a stem must end with a 1), and this creates k possibilities for each stretched composition into $k + 2$ digits;

Stems: $111, [1011, 1101], [10011, 10101, 11001], [100011, 100101, 101001, 110001], \ldots$

To gain some idea of how this pans out, we create a table:

These are the tetrahedral numbers [5] given by $(z - 2)(z - 1)(z)/6$

So we have

$$z + z^2 - z + z^3/6 - z^2/2 + z/3 = z^3/6 + z^2/2 + z/3$$
$$= (z^3 + 3z^2 + 2z)/6 = (z)(z + 1)(z + 2)/6$$

This predicts $2 * 3 * \frac{4}{6} = 4$ entries less than 100, and these are $3, 12, 21, 30$. The 10 entries less than 1000 include these 4 and also $102, 111, 120, 201, 210$ and 300.

$NND(4, X)$ **to** $NND(9, X)$

From the previous results, we might reasonably expect that for y between 0 and 9 we have:

$$NND(y,x) = \frac{\prod_{j=0}^{y-1} z+j}{y!}$$

Before we attempt to prove this, we should test our hypothesis:

First construct a table of z values to say $z = 6$ and the predicted value of $NND(y, X)$:

Define a function $PNND(y, z)$ as in the equation above:

$PNND(y, z) = 1/y! * prod(j = 0, y - 1, z + j)$

And run through y and z to create the table above.

To physically determine the table use:

$DNND(y, z) = local(c); c = 0; for(i = 1, 10^z - 1, if(sd(i) == y, c + +)); c$

which produces the desired output (8 minutes at $3Ghz$).

§5.Proof

We can see how this might be true due to the nature of the table. Each row (or column) is the partial sums of the previous row (or column).

This leads us to consider that the answer for a digit sum is constructible from the previous digit sum if we read the candidates in a different way.

For example, the $y = 1$ row gives $[1], [1, 10], [1, 10, 100], etc \ldots$

Therefore for $y = 2$, consider the grouping:

[1],

[[1], [1,10]],

[[1], [1,10], [1,10,100], etc...

Can we interpret this as $[2], [2, 11, 20], [2, 11, 20, 101, 110, 200], etc \ldots$?

And can we find a mapping between the values at $y = 3$, namely $[3]$, $[3, 12, 21, 30]$, $[3, 12, 21, 30, 102, 111, 120, 201, 210, 300]$ and the groups formed from $y = 2$ under our hypothesis 2?

And can we find a method of extending this through to $n = 9$?

A logical map involves adding 1 to each element, and then extending elements in group with zeroes depending on the relative position of the group inside the main group.

So looking at the first case, $[1]$ becomes $[2]$.

$[1]$ becomes $[20]$ as it is a 1 digit value required to become a 2 digit value. $[1, 10]$ becomes $[2, 11]$

Next, $[1]$ becomes $[200]$, $[1, 10]$ becomes $[20, 110]$ and $[1, 10, 100]$ becomes $[2, 11, 101]$.

This works for 3 as well, but we need to justify the argument up to the map between $y = 8$ and $y = 9$.

And we can do this by first observing that the number of trailing zeroes makes each group unique, and within each group each new element is obviously unique.

$NND(10+, X)$

With y greater than 9, the pattern stops, as for example with $NND(10, X)$ we cannot have 10 as a composition.

The formula predicts $1, 11, 66, 286, 1001, 3003$ and $DNND(10, z)$ states that the actual number of numbers with a digit sum 10 is $0, 9, 63, 282, 996, 2997$

So $NND(10, X) = predicted - z$, due to the missing composition.

With $NND(11, X)$, we have neither 11 or $10+1$ or $1+10$ as compositions.

Predicted is $1, 12, 78, 364, 1365, 4368$ Actual is $0, 8, 69, 348, 1340, 4332$

Which leads us to believe $NND(11, X) = predicted - z^2$

With $NND(12, X)$;

Predicted is $1, 13, 91, 455, 1820, 6188$ Actual is $0, 7, 73, 415, 1745, 6062$

The difference here is the pentagonal pyramidal numbers [6], so

$NND(12, X) = predicted - z^2(z + 1)/2$

There seems to be a pattern, so let's continue for a while:

NND(13,X) Predicted : $1, 14, 105, 560, 2380, 8568$ Actual : $0, 6, 75, 480, 2205, 8232$

$NND(13, X) = predicted - z^2(z + 1)(z + 2)/6$ [7]

$NND(14, X)$ Predicted : $1, 15, 120, 680, 3060, 11628$

Actual :$0, 5, 75, 540, 2710, 10872$

$NND(14, X) = predicted - (z + 1)^2(z + 2)(z + 3)(z + 4)/24$ [8]

The pattern is reasonably obvious, but unstable - we cannot say when, if ever, a jump to $(n + 2)^2$ as a start occurs.

If we put the data we have into a table, and incorporate the previous table for $y = 0$ to 9, we might spot a pattern:

And we do have a $pattern - each$ column is symmetric!, i.e. after $y = 9$, column 2 is itself reversed, after $y = 13$, column 3 is itself reversed, and (we guess) so on. The switch happens at $y = 9z/2$, which is the maximum sum of digits $(i.e. 99 \ldots 99)$ divided by 2.

The symmetry can be seen by considering that if a number n has a digit sum d, then $(10^z - 1) - n$ has a digit sum $9z - d$.

And we have another $pattern - after\ y = 9$, a column continues to be the partial sums of the previous column, however the $y - 10'th$ entry is subtracted.

For example, consider the $z = 4$ column. $(9, 4) = 220$, and $(10, 4) = (9, 4) + (10, 3) - (0, 3) = 220 + 63 - 1 = 282$. $(11, 4) = (10, 4) + (11, 3) - (1, 3) = 282 + 69 - 3 = 348$.

In general $(y, z) = (y-1, z)+(y, z-1)-(y-10, z-1)$, where $(y, z) = 0$ if y is less than 0.

This becomes obvious if we see that the stretched compositions not present in y greater than 9 are equal in number to the number of stretched compositions of $y - 10$.

Summary

We prove that $NND(y, X)$ for $X = 10^z$ and y less than 10 is given by product$(j = 0, y - 1, z + j)/y!$.

We show that $NND(y, X)$ with y greater than 9 behaves reasonably predictably.

We prove that $NND(y, X) = NND(9z - y, X)$

We prove that $(y, z) = (y - 1, z) + (y, z - 1) - (y - 10, z - 1)$

Open problems

1. Find a formula for $NND(y, X)$ valid for all y with $X = 10^z$
2. Find a formula for $NND(y, X)$ valid for y less than 10 with a general X
3. Find a formula for $NND(y, X)$ valid for all y with a general X

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ON THE MEAN VALUE OF THE SMARANDACHE DOUBLE FACTORIAL FUNCTION

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Abstract For any positive integer n, the Smarandache double factorial function $Sdf(n)$ is defined as the least positive integer m such that $m!!$ is divisible by n. In this paper, we study the mean value properties of $Sdf(n)$, and give an interesting mean value formula for it.

Keywords: F.Smarandache problem; Smarandache function; Mean Value.

§1. Introduction and results

For any positive integer n, the Smarandache double factorial function $Sdf(n)$ is defined as the least positive integer m such that $m!!$ is divisible by n, where

$$m!! = \begin{cases} 2 \cdot 4 \cdots m, & \text{if } 2|m; \\ 1 \cdot 3 \cdots m, & \text{if } 2\dagger \text{m}. \end{cases}$$

About the arithmetical properties of $Sdf(n)$, many people had studied it before (see reference [2]). The main purpose of this paper is to study the mean value properties of $Sdf(n)$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n \leq x} Sdf(n) = \frac{7\pi^2}{24} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need the following two simple Lemmas.

Lemma 1. *if $2 \dagger n$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the factorization of n, where $p_1, p_2, \cdots, p_k$ are distinct odd primes and $\alpha_1, \alpha_2, \cdots, \alpha_k$ are positive integers, then*

$$Sdf(n) = \max(Sdf(p_1^{\alpha_1}), Sdf(p_2^{\alpha_2}), \cdots, Sdf(p_k^{\alpha_k}))$$

Proof. Let $m_i = Sdf(p_i^{\alpha_i})$ for $i = 1, 2, \cdots, k$. Then we get $2 \dagger m_i (i = 1, 2, \cdots, k)$ and

$$p_i^{\alpha_i} | (m_i)!!, i = 1, 2, \cdots, k.$$

Let $m = \max(m_1, m_2, \cdots, m_k)$. Then we have

$$(m_i)!! | m!!, i = 1, 2, \cdots, k.$$

Thus we get

$$p_i^{\alpha_i} | m!!, i = 1, 2, \cdots, k.$$

Notice that $p_1, p_2, \cdots, p_k$ are distinct odd primes. We have

$$gcd(p_i^{\alpha_i}, p_j^{\alpha_j}) = 1, 1 \leq i < j \leq k.$$

Therefore, we obtain $n | m!!$. It implies that

$$Sdf(n) \leq m.$$

On the other hand, by the definition of m, if $Sdf(n) < m$, then there exists a prime power $p_j^{\alpha_j} (1 \leq j \leq k)$ such that

$$p_j^{\alpha_j} | Sdf(n)!!.$$

We get $n | Sdf(n)!!$, a contradiction. Therefore, we obtain $Sdf(n) = m$.

This proves Lemma 1.

Lemma 2. *For positive integer n($2 \dagger n$), let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n and $P(n) = \max\limits_{1 \leq i \leq k} \{p_i\}$. if there exists $P(n)$ satisfied with $P(n) > \sqrt{n}$, then we have the identity*

$$Sdf(n) = P(n).$$

Proof. First we let $Sdf(n) = m$, then m is the smallest positive integer such that $n | m!!$. Now we will prove that $m = P(n)$. We assume $P(n) = p_0$. From the definition of $P(n)$ and lemma 1, we know that $Sdf(n) = \max(p_0, (2\alpha_i - 1)p_i)$. Therefore we get

(I) If $\alpha_i = 1$, then $Sdf(n) = p_0 \geq n^{\frac{1}{2}} \geq (2\alpha_i - 1)p_i$;

(II) If $\alpha_i \geq 2$, then $Sdf(n) = p_0 > 2 \ln n n^{\frac{1}{4}} > (2\alpha_i - 1)p_i$.

Combining (I)-(II), we can easily obtain

$$Sdf(n) = P(n)$$

This proves Lemma 2.

Now we use the above Lemmas to complete the proof of Theorem. First we separate the summation in the Theorem into two parts.

$$\sum_{n \leq x} Sdf(n) = \sum_{u \leq \frac{x-1}{2}} Sdf(2u+1) + \sum_{u \leq \frac{x}{2}} Sdf(2u), \tag{1}$$

For the first part. we let the sets $\mathcal{A}$ and $\mathcal{B}$ as following:

$$\mathcal{A} = \{2u+1 | 2u+1 \leq x, P(2u+1) \leq \sqrt{2u+1}\}$$

and

$$\mathcal{B} = \{2u+1 | 2u+1 \leq x, P(2u+1) > \sqrt{2u+1}\}.$$

Using the Euler summation formula, we get

$$\sum_{2u+1 \in \mathcal{A}} Sdf(2u+1) \ll \sum_{2u+1 \leq x} \sqrt{2u+1}\ln(2u+1) \ll x^{\frac{3}{2}} \ln x. \quad (2)$$

Similarly, from the Abel's identity we also get

$$\begin{aligned}
&\sum_{2u+1 \in \mathcal{B}} Sdf(2u+1) \\
= &\sum_{\substack{2u+1 \leq x \\ P(2u+1) > \sqrt{2u+1}}} P(2u+1) \\
= &\sum_{1 \leq 2l+1 \leq \sqrt{x}} \sum_{2l+1 \leq p \leq \frac{x}{2l+1}} p + O\left(\sum_{2l+1 \leq \sqrt{x}} \sum_{\sqrt{2l+1} \leq p \leq \frac{x}{2l+1}} \sqrt{x}\right) \\
= &\sum_{1 \leq 2l+1 \leq \sqrt{x}} \left(\frac{x}{2l+1}\pi(\frac{x}{2l+1}) - (2l+1)\pi(2l+1) - \int_{\sqrt{x}}^{\frac{x}{2l+1}} \pi(s)ds\right) \\
&+O\left(x^{\frac{3}{2}} \ln x\right), \quad (3)
\end{aligned}$$

where $\pi(x)$ denotes all the numbers of prime which is not exceeding x.

For $\pi(x)$, we have

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$$

and

$$\begin{aligned}
&\sum_{1 \leq 2l+1 \leq \sqrt{x}} \left(\frac{x}{2l+1}\pi(\frac{x}{2l+1}) - (2l+1)\pi(2l+1) - \int_{\sqrt{x}}^{\frac{x}{2l+1}} \pi(s)ds\right) \\
&= \sum_{1 \leq 2l+1 \leq \sqrt{x}} \left(\frac{1}{2}\frac{x^2}{(2l+1)^2 \ln \frac{x}{(2l+1)}} - \frac{1}{2}\frac{(2l+1)^2}{\ln(2l+1)}\right. \\
&\quad +O\left(\frac{x^2}{(2l+1)^2 \ln^2 \frac{x}{(2l+1)}}\right) + O\left(\frac{(2l+1)^2}{\ln^2(2l+1)}\right) \\
&\quad \left. +O\left(\frac{x^2}{(2l+1)^2 \ln^2 \frac{x}{(2l+1)}} - \frac{(2l+1)^2}{\ln^2(2l+1)}\right)\right). \quad (4)
\end{aligned}$$

Hence

$$\sum_{1\le 2l+1\le\sqrt{x}} \frac{x^2}{(2l+1)^2 \ln\frac{x}{2l+1}} = \sum_{0\le l\le\frac{\sqrt{x}-1}{2}} \frac{x^2}{(2l+1)^2 \ln\frac{x}{2l+1}}$$

$$= \sum_{0\le l\le\frac{\ln x-1}{2}} \frac{x^2}{(2l+1)^2 \ln x} + O\left(\sum_{\frac{\ln x-1}{2}\le l\le\frac{\sqrt{x}-1}{2}} \frac{x^2\ln(2l+1)}{(2l+1)^2\ln^2 x}\right)$$

$$= \frac{\pi^2}{8}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \tag{5}$$

Combining (2), (3),(4) and (5) we obtain

$$\sum_{u\le\frac{x-1}{2}} Sdf(2u+1) = \frac{\pi^2}{8}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \tag{6}$$

For the second part, we notice that $2u = 2^\alpha n_1$ where α, n_1 are positive integers with $2\dagger n_1$, let $S(2u) = \min\{m \mid 2u|m!\}$, from the definition of $Sdf(2u)$, we have

$$\sum_{2u\le x} Sdf(2u) = \sum_{\substack{2^\alpha n_1\le x\\ 2^\alpha > n_1}} Sdf(2^\alpha n_1) \ll \sum_{\alpha\le\frac{\ln x}{\ln 2}} \sqrt{x} \ll \sqrt{x}\ln x, \tag{7}$$

and

$$\sum_{2u\le x} Sdf(2u) = 2\sum_{2u\le x} S(2u) + O(\sqrt{x}\ln x) = \frac{\pi^2}{6}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \tag{8}$$

Combining (7) and (8) we obtain

$$\sum_{u\le\frac{x}{2}} Sdf(2u) = \frac{\pi^2}{6}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \tag{9}$$

From (1), (6) and (9) we obtain the asymptotic formula

$$\sum_{n\le x} Sdf(n) = \frac{7\pi^2}{24}\frac{x^2}{\ln x} + \left(\frac{x^2}{\ln^2 x}\right).$$

This completes the proof of Theorem.

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ON THE M-POWER FREE PART OF AN INTEGER

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Abstract The main purpose of this paper is using the elementary method to study the mean value properties of a new arithmetical function involving the m-power free part of an integer, and give an interesting asymptotic formula for it.

Keywords: Arithmetical function; Mean value; Asymptotic formula

§1. Introduction

For any positive integer n, it is clear that we can assume $n = u^m v$, where v is a m-power free number. Let $b_m(n) = v$ be the m-power free part of n. For example, $b_3(8) = 1$, $b_3(24) = 3$, $b_2(12) = 3$, $\cdots\cdots$. Now for any positive integer $k > 1$, we define another function $\delta_k(n)$ as following:

$$\delta_k(n) = \max\{d : d \mid n, (d, k) = 1\}.$$

From the definition of $\delta_k(n)$, we can prove that $\delta_k(n)$ is also a completely multiplicative function. In reference [1], Professor F.Smarandache asked us to study the properties of the sequence $\{b_m(n)\}$. It seems that no one knows the relations between sequence $\{b_m(n)\}$ and the arithmetical function $\delta_k(n)$ before. The main purpose of this paper is to study the mean value properties of $\delta_k(b_m(n))$, and obtain an interesting mean value formula for it. That is, we shall prove the following conclusion:

Theorem. *Let m and k be any fixed positive integer. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} \delta_k(b_m(n)) = \frac{x^2}{2} \frac{\zeta(2m)}{\zeta(m)} \prod_{p|k} \frac{p^m + 1}{p^{m-1}(p+1)} + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where ϵ denotes any fixed positive number, $\zeta(s)$ is the Riemann zeta-function, and $\prod_{p|k}$ denotes the product over all different prime divisors of k.

Taking $m = 2$ in this Theorem, we may immediately obtain the following:

Corollary. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \delta_k(b_2(n)) = \frac{\pi^2}{30}x^2 \prod_{p|k} \frac{p^2+1}{p(p+1)} + O(x^{\frac{3}{2}+\varepsilon}).$$

§2. Proof of the Theorem

In this section, we shall use the analytic method to complete the proof of the theorem. In fact, we know that $b_m(n)$ is a completely multiplicative function, so we can use the properties of the Riemann zeta-function to obtain a generating function. For any complex s, if $\mathrm{Re}(s) > 2$, we define the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{\delta_k(b_m(n))}{n^s}.$$

If positive integer $n = p^\alpha$, then from the definition of $\delta_k(n)$ and $b_m(n)$ we have:

$$\delta_k(b_m(n)) = \delta_k(b_m(p^\alpha)) = 1, \qquad if \qquad p|k,$$

and

$$\delta_k(b_m(n)) = \delta_k(b_m(p^\alpha)) = p^\beta, \qquad if \alpha \equiv \beta mod m, 0 \leq \beta < m \qquad and \qquad p\dagger k.$$

From the above formula and the Euler product formula (See Theorem 11.6 of [3]) we can get

$$\begin{aligned}
f(s) &= \prod_p \left(1 + \frac{\delta_k(b_m(p))}{p^s} + \frac{\delta_k(b_m(p^2))}{p^{2s}} + \frac{\delta_k(b_m(p^3))}{p^{3s}} + \cdots\right) \\
&= \prod_{p|k} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{(m-1)s}} + \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}} + \cdots\right) \\
&\quad \times \prod_{p\dagger k} \left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots + \frac{p^{m-1}}{p^{(m-1)s}} + \frac{1}{p^{ms}} + \frac{p}{p^{(m+1)s}} + \cdots\right) \\
&= \prod_{p|k} \frac{1}{1-\frac{1}{p^s}} \prod_{p\dagger k} \left[\left(1 + \frac{p}{p^s} + \ldots + \frac{p^{m-1}}{p^{(m-1)s}}\right)\left(1 + \frac{1}{p^{ms}} + \frac{1}{p^{2ms}} + \cdots\right)\right] \\
&= \prod_{p|k} \frac{1}{1-\frac{1}{p^s}} \prod_{p\dagger k} \frac{1}{1-\frac{1}{p^{ms}}} \prod_{p\dagger k} \left(1 + \frac{p}{p^s} + \ldots + \frac{p^{m-1}}{p^{(m-1)s}}\right)
\end{aligned}$$

$$\begin{aligned} &= \prod_{p|k}\frac{1}{1-\frac{1}{p^s}}\prod_{p\dagger k}\frac{1-\frac{1}{p^{m(s-1)}}}{1-\frac{1}{p^{s-1}}}\times\frac{1}{1-\frac{1}{p^{ms}}} \\ &= \frac{\zeta(s-1)\zeta(ms)}{\zeta(ms-m)}\prod_{p|k}\frac{(1-\frac{1}{p^{s-1}})(1-\frac{1}{p^{ms}})}{(1-\frac{1}{p^{m(s-1)}})(1-\frac{1}{p^s})}. \end{aligned}$$

Because the Riemann zeta-function $\zeta(s)$ have a simple pole point at $s=1$ with the residue 1, we know that $f(s)\frac{x^s}{s}$ also have a simple pole point at $s=2$ with the residue $\frac{\zeta(2m)}{\zeta(m)}\prod_{p|k}\frac{p^m+1}{p^{m-1}(p+1)}\frac{x^2}{2}$. By Perron formula (See [2]), taking $s_0=0$, $b=3$, $T>1$, then we have

$$\sum_{n\le x}\delta_k(b_m(n)) \quad = \tfrac{1}{2\pi i}\int_{3-iT}^{3+iT} f(s)\tfrac{x^s}{s}ds + O\left(\tfrac{x^{3+\epsilon}}{T}\right).$$

Now we move the integral line to Re $s=\frac{3}{2}+\epsilon$, then taking $T=x^{\frac{3}{2}}$, we can get

$$\begin{aligned} &\sum_{n\le x}\delta_k(b_m(n)) \\ &= \frac{\zeta(2m)}{\zeta(m)}\prod_{p|k}\frac{p^m+1}{p^{m-1}(p+1)}\frac{x^2}{2} + \frac{1}{2\pi i}\int_{\frac{3}{2}+\epsilon-iT}^{\frac{3}{2}+\epsilon+iT} f(s)\frac{x^s}{s}ds + O\left(x^{\frac{3}{2}+\epsilon}\right) \\ &= \frac{\zeta(2m)}{\zeta(m)}\prod_{p|k}\frac{p^m+1}{p^{m-1}(p+1)}\frac{x^2}{2} + O\left(\int_{-T}^{T}\left|f(\frac{3}{2}+\epsilon+it)\right|\frac{x^{\frac{3}{2}+\epsilon}}{1+|t|}dt\right) \\ &\quad +O\left(x^{\frac{3}{2}+\epsilon}\right) \\ &= \frac{\zeta(2m)}{\zeta(m)}\prod_{p|k}\frac{p^m+1}{p^{m-1}(p+1)}\frac{x^2}{2} + O\left(x^{\frac{3}{2}+\epsilon}\right). \end{aligned}$$

This completes the proof of Theorem.

Note that $\zeta(2)=\frac{\pi^2}{6}$ and $\zeta(4)=\frac{\pi^4}{90}$, taking $m=2$ in the theorem, we may immediately obtain the Corollary.

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ON THE MEAN VALUE OF THE $SCBF$ FUNCTION

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Abstract The main purpose of this paper is using the elementary method to study the asymptotic properties of the SCBF function on simple numbers, and give an interesting asymptotic formula for it.

Keywords: SCBF function; Mean value; Asymptotic formula.

§1. Introduction

In reference [1], the Smarandache Sum of Composites Between Factors function $SCBF(n)$ is defined as: The sum of composite numbers between the smallest prime factor of n and the largest prime factor of n. For example, $SCBF(14)$=10, since $2\times7 = 14$ and the sum of the composites between 2 and 7 is: $4 + 6 = 10$. In reference [2]: A number n is called simple number if the product of its proper divisors is less than or equal to n. Let A denotes set of all simple numbers. That is, $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, \cdots\}$.

According to reference [1], Jason Earls has studied the arithmetical properties of $SCBF(n)$ and proved that $SCBF(n)$ is not a multiplicative function. For example, $SCBF(14 \times 15) = 10$ and $SCBF(14) \times SCBF(15) = 40$. He also got that if i and j are positive integers then $SCBF(2^i \times 5^j) = 4$, $SCBF(2^i \times 7^j) = 10$, etc. In this paper, we use the elementary method to study the mean value properties of $SCBF(n)$ on simple numbers, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. Let $x \geq 1$, A denotes the set of all simple numbers. Then we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in A}} SCBF(n) = B\frac{x^3}{\ln x} + O\left(\frac{x^3}{\ln^2 x}\right),$$

where $B = \frac{1}{3}\sum\limits_{p}\frac{1}{p^3}$ is a constant, $\sum\limits_{p}$ denotes the summation over all primes.

§2. Some Lemmas

To complete the proof of the theorem, we need the following lemmas:

Lemma 1. For any prime p and positive integer k, we have the asymptotic formula

$$SCBF(p^k) = 0.$$

Proof. (See reference [1]).

Lemma 2. Let $n \in A$, then we have $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$ four case, where p, q denote the distinct primes.

Proof. First let n be a positive integer, $p_d(n)$ is the product of all positive divisors of n, that is, $p_d(n) = \prod_{d|n} d$. $q_d(n)$ is the product of all positive divisors of n but n. That is, $q_d(n) = \prod_{d|n,d<n} d$. Then from the definition of $p_d(n)$ we know that

$$p_d(n) = \prod_{d|n} d = \prod_{d|n} \frac{n}{d}.$$

So from this formula we have

$$p_d^2(n) = \prod_{d|n} d \times \prod_{d|n} \frac{n}{d} = \prod_{d|n} n = n^{d(n)}.$$

where $d(n) = \sum_{d|n} 1$. Then we may immediately get $p_d(n) = n^{\frac{d(n)}{2}}$ and

$$q_d(n) = \prod_{d|n,d<n} d = \frac{\prod_{d|n} d}{n} = n^{\frac{d(n)}{2}-1}.$$

By the definition of the simple numbers, we get $n^{\frac{d(n)}{2}-1} \leq n$. Therefore, we have

$$d(n) \leq 4.$$

This inequality holds only for $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$ four cases. This completes the proof of Lemma 2.

Lemma 3. For any distinct prime p and q, we have the asymptotic formula

$$SCBF(pq) = \frac{q^2}{2}\left(1 - \frac{1}{\ln q}\right) - \frac{p^2}{2}\left(1 - \frac{1}{\ln p}\right) + O\left(\frac{q^2}{\ln^2 q}\right).$$

Proof. From the definition of $SCBF(n)$, we have

$$SCBF(pq) = \sum_{p<n<q} n - \sum_{p<q_1<q} q_1,$$

where q_1 is a prime. Using the Abel's Identity [3] and note that the asymptotic formula

$$\sum_{n\leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$$

we can get

$$\begin{aligned} SCBF(pq) &= \sum_{p<n<q} n - \sum_{p<q_1<q} q_1 \\ &= \sum_{p<n\le q-1} n - \sum_{p<q_1\le q-1} q_1 \\ &= \sum_{n\le q-1} n - \sum_{n\le p} n - \sum_{p<q_1\le q-1} q_1 \\ &= \frac{(q-1)^2}{2} - \frac{(p-1)^2}{2} + O(q) - (q-1)\pi(q-1) + p\pi(p) \\ &\quad + \int_p^{q-1} \pi(t)dt \\ &= \frac{q^2}{2} - \frac{q^2}{2\ln q} - \frac{p^2}{2} + \frac{p^2}{2\ln p} + O\left(\frac{q^2}{\ln^2 q}\right). \end{aligned}$$

This completes the proof of Lemma 3.

Lemma 4. For real number $x \ge 1$, we have the asymptotic formula

$$\sum_{pq\le x} SCBF(pq) = B\frac{x^3}{\ln x} + O\left(\frac{x^3}{\ln^2 x}\right),$$

where p and q are two distinct primes, $B = \frac{1}{3}\sum\limits_p \frac{1}{p^3}$ is a constant, and $\sum\limits_p$ denotes the summation over all primes.

Proof. From the definition of $SCBF(n)$ and Lemma 1, Lemma 3, we get

$$\begin{aligned} \sum_{pq\le x} SCBF(pq) &= 2\sum_{pq\le x, p<q} SCBF(pq) - \sum_{p^2\le x} SCBF(p^2) \\ &= 2\sum_{p\le\sqrt{x}} \sum_{p<q\le\frac{x}{p}} SCBF(pq) \\ &= \sum_{p\le\sqrt{x}} \sum_{p<q\le\frac{x}{p}} \left(q^2 - \frac{q^2}{\ln q} - p^2 + \frac{p^2}{\ln p} + O\left(\frac{q^2}{\ln^2 q}\right)\right). \end{aligned}$$

Noting that $\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$, using Abel's Identity [3] we get

$$\begin{aligned} \sum_{p<q\le\frac{x}{p}} q^2 &= \pi(\frac{x}{p})\frac{x^2}{p^2} - \pi(p)p^2 - 2\int_p^{\frac{x}{p}} \pi(t)tdt \\ &= \frac{x^3}{3p^3\ln\frac{x}{p}} - \frac{p^3}{3\ln p} + O\left(\frac{x^3}{p^3\ln^2\frac{x}{p}}\right) \end{aligned}$$

and

$$\begin{aligned}\sum_{p<q\leq\frac{x}{p}}\frac{q^2}{\ln q} &= A(\frac{x}{p})f(\frac{x}{p}) - A(p)f(p) - \int_p^{\frac{x}{p}} A(t)f(t)'dt \\ &= \frac{x^3}{3p^3\ln^2\frac{x}{p}} - \frac{p^3}{3\ln^2 p} - \frac{p^3}{9\ln^3 p} + O\left(\frac{x^3}{p^3\ln^3\frac{x}{p}}\right),\end{aligned}$$

where $A(\frac{x}{p}) = \sum_{p<q\leq\frac{x}{p}} q^2$, $f(x) = \frac{1}{\ln x}$. From reference [3], we know that

$$\sum_{p\leq x}\frac{1}{p} = \ln\ln x + C + O\left(\frac{1}{\ln x}\right),$$

where C is a computable constant. And then we also get

$$\sum_{p\leq\sqrt{x}} p = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$$

and

$$\sum_{p\leq\sqrt{x}} p^3 = \frac{x^2}{2\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Using the same method, we obtain

$$\sum_{p\leq\sqrt{x}}\frac{p}{\ln p} = \frac{2x}{\ln^2 x} + O\left(\frac{x}{\ln^3 x}\right)$$

and

$$\sum_{p\leq\sqrt{x}}\frac{p^3}{\ln p} = \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right).$$

Noting that $\frac{1}{1-\frac{\ln p}{\ln x}} = 1 + \frac{\ln p}{\ln x} + \frac{\ln^2 p}{\ln^2 x} + \cdots + \frac{\ln^m p}{\ln^m x} + \cdots$, then we get the following two formulae:

$$\begin{aligned}&\sum_{p\leq\sqrt{x}}\sum_{p<q\leq\frac{x}{p}} q^2 \\ =\ &\sum_{p\leq\sqrt{x}}\left(\frac{x^3}{3p^3\ln\frac{x}{p}} - \frac{p^3}{3\ln p} + O\left(\frac{x^3}{p^3\ln^2\frac{x}{p}}\right)\right) \\ =\ &\frac{x^3}{3\ln x}\sum_{p\leq\sqrt{x}}\frac{1}{p^3}\left(1 + \frac{\ln p}{\ln x} + \frac{\ln^2 p}{\ln^2 x} + \cdots\right) \\ &-\frac{1}{3}\sum_{p\leq\sqrt{x}}\frac{p^3}{\ln p} + O\left(\frac{x^3}{\ln^2 x}\sum_{p\leq\sqrt{x}}\frac{1}{p^3}\left(1 + 2\frac{\ln p}{\ln x} + 3\frac{\ln^2 p}{\ln^2 x} + \cdots\right)\right) \\ =\ &C_1\frac{x^3}{\ln x} + O\left(\frac{x^3}{\ln^2 x}\right);\end{aligned}$$

$$\begin{aligned}
&\sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}\frac{q^2}{\ln q}\\
=\;&\sum_{p\le\sqrt{x}}\left(\frac{x^3}{3p^3\ln^2\frac{x}{p}}-\frac{p^3}{3\ln^2 p}-\frac{p^3}{9\ln^3 p}+O\left(\frac{x^3}{p^3\ln^3\frac{x}{p}}\right)\right)\\
=\;&\frac{x^3}{3\ln^2 x}\sum_{p\le\sqrt{x}}\frac{1}{p^3}\left(1+2\frac{\ln p}{\ln x}+3\frac{\ln^2 p}{\ln^2 x}+\cdots\right)-\frac{1}{3}\sum_{p\le\sqrt{x}}\frac{p^3}{\ln^2 p}\\
&-\frac{1}{9}\sum_{p\le\sqrt{x}}\frac{p^3}{\ln^3 p}+O\left(\sum_{p\le\sqrt{x}}\frac{x^3}{p^3\ln^3\frac{x}{p}}\right)\\
=\;&C_2\frac{x^3}{\ln^2 x}+O\left(\frac{x^2}{\ln^2 x}\right),
\end{aligned}$$

where $C_1=C_2=\frac{1}{3}\sum_p\frac{1}{p^3}$.

So we have

$$\begin{aligned}
&2\sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}SCBF(pq)\\
=\;&\sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}\left(q^2-\frac{q^2}{\ln q}-p^2+\frac{p^2}{\ln p}+O\left(\frac{q^2}{\ln^2 q}\right)\right)\\
=\;&\sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}q^2-\sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}\frac{q^2}{\ln q}-\sum_{p\le\sqrt{x}}p^2\sum_{p<q\le\frac{x}{p}}1\\
&+\sum_{p\le\sqrt{x}}\frac{p^2}{\ln p}\sum_{p<q\le\frac{x}{p}}1+O\left(\sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}\frac{q^2}{\ln^2 q}\right)\\
=\;&B\frac{x^3}{\ln x}+O\left(\frac{x^3}{\ln^2 x}\right),
\end{aligned}$$

where $B=\frac{1}{3}\sum_p\frac{1}{p^3}$. This proves Lemma 4.

§3. Proof of the theorem

In this section, we complete the proof of Theorem. According to the definition of simple numbers and Lemma 2, we have

$$\sum_{\substack{n\le x\\ n\in A}}SCBF(n)$$

$$= \sum_{p\leq x} SCBF(p) + \sum_{p^2\leq x} SCBF(p^2) + \sum_{p^3\leq x} SCBF(p^3) + \sum_{pq\leq x} SCBF(pq).$$

And then, using Lemma 1 and Lemma 4 we obtain

$$\begin{aligned}\sum_{\substack{n\leq x\\ n\in A}} SCBF(n) &= \sum_{pq\leq x} SCBF(pq)\\ &= B\frac{x^3}{\ln x} + O\left(\frac{x^3}{\ln^2 x}\right).\end{aligned}$$

This completes the proof of Theorem.

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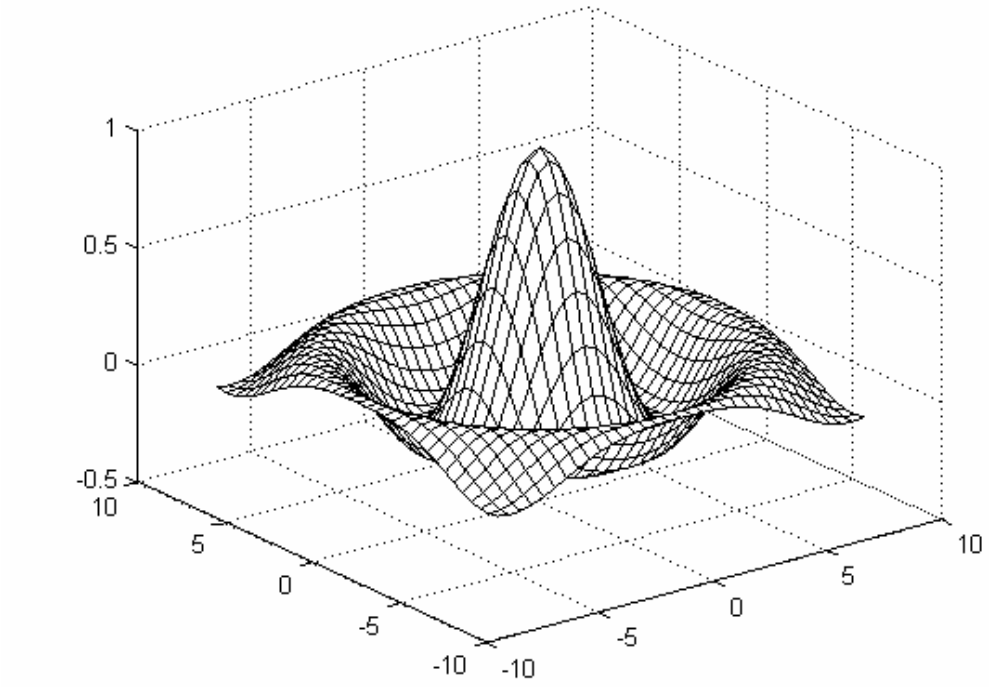

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