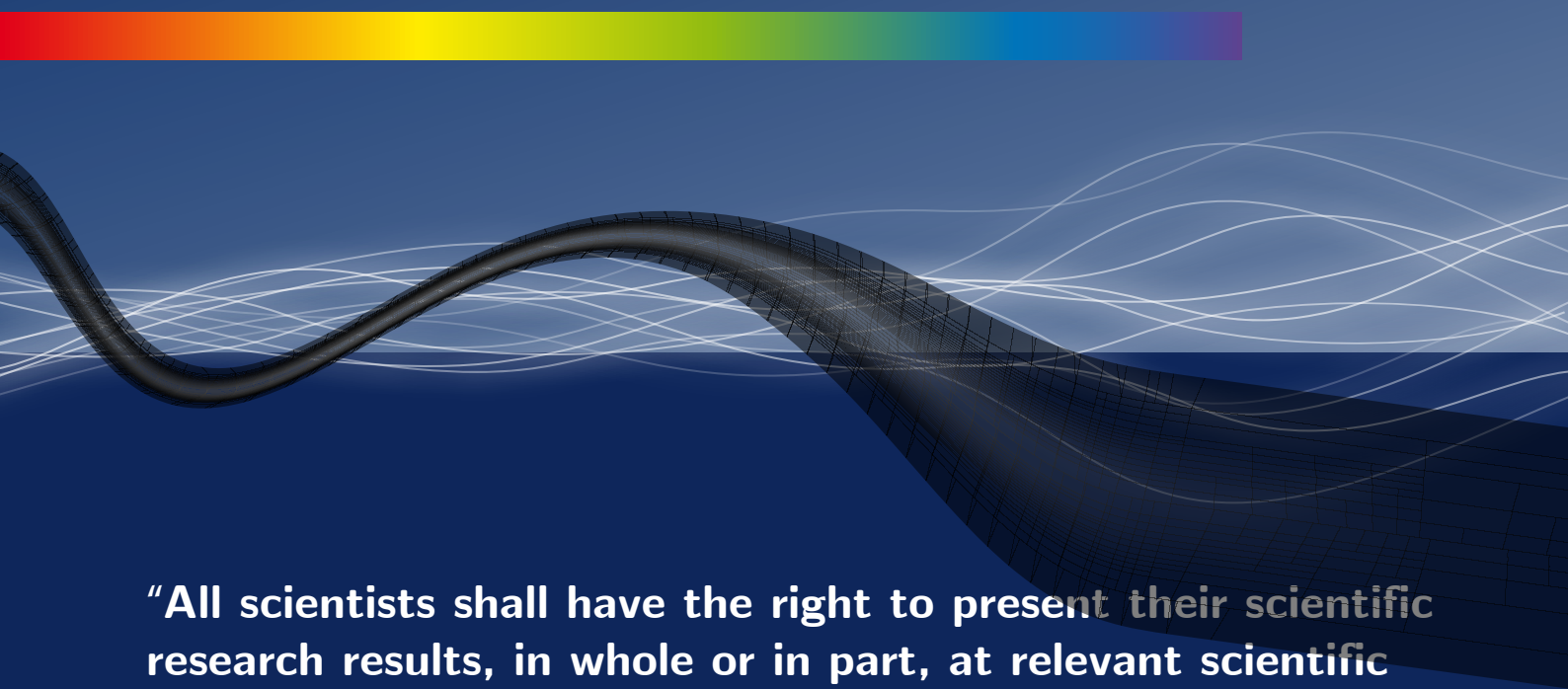


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including Related Themes from Mathematics

PROGRESS IN PHYSICS



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Editorial Board

Pierre Millette
millette@ptep-online.com
Andreas Ries
ries@ptep-online.com
Florentin Smarandache
fsmarandache@gmail.com
Ebenezer Chifu
chifu@ptep-online.com

Postal Address

Department of Mathematics and Science,
University of New Mexico,
705 Gurley Ave., Gallup, NM 87301, USA

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Physical Observables in General Relativity and the Zelmanov Chronometric Invariants

Dmitri Rabounski and Larissa Borissova

Puschino, Moscow Region, Russia

E-mail: rabounski@yahoo.com, lborissova@yahoo.com

Chronomic invariants are mathematically determined as the projections of four-dimensional tensorial quantities onto the three-dimensional spatial section and the line of time belonging to a real particular observer. Such projections are physical observables to the observer; it is these quantities that are measurable in his real laboratory and depend on the physical and geometric properties of his local physical space. In other words, chronometric invariants are physical observable quantities in the space-time of General Relativity. Chronometric invariants and the mathematical apparatus for their calculation were introduced in 1944 by Abraham L. Zelmanov. In this article, we have collected everything (or almost everything) that we know about chronometric invariants to provide a convenient and most detailed reference to this mathematical apparatus originally scattered throughout many publications.

Physical observables were mathematically determined and introduced into General Relativity in 1941–1944 by Abraham L. Zelmanov (1913–1987), who called them chronometrically invariant quantities or, in brief, *chronometric invariants*. Zelmanov first presented his mathematical apparatus for calculating physical observables in 1944, in the form of his PhD thesis [1]. Later, in 1956–1957, he published a brief review of his theory in two journal articles [2, 3], of which his 1957 presentation is the most useful and complete. A more detailed account of Zelmanov's mathematical apparatus can be found in the respective chapters of our three research monographs [4–6] and in one of our recent journal publications [7].

Chronometrically invariant quantities are determined as the projections of four-dimensional tensorial quantities onto the three-dimensional spatial section and the line of time in the real physical reference frame belonging to a particular observer. Such quantities depend on the physical and geometric properties of his local physical space (his physical reference space) and can be measured in his laboratory. In other words, chronometric invariants are physical observable quantities in the space-time of General Relativity.

For this reason and since we have always sought to obtain a theoretical result that can be registered in laboratory measurements, we used Zelmanov's mathematical apparatus in our research studies. The chronological list of our publications in English and French, wherein we used chronometric invariants, is given in the end of this article.

Unfortunately, it just so happened that after Zelmanov's death in 1987, we remain the only ones in the world who professionally master this mathematical apparatus and apply it in scientific research. In addition, Zelmanov's mathematical apparatus was fragmentarily scattered throughout the aforementioned publications. Some of them pretended to be more or less complete, but were also limited due to the omission of some important parts (not relevant to the specific problem).

For this reason, and also because the problem of physical observables in General Relativity is of great importance for

experiment, Pierre A. Millette, Editor of *Progress in Physics*, prompted us to write a compendium containing “everything we know about chronometric invariants and would like to say”. Such an article, despite the obvious repetitions with the previous ones, would contain the entire mathematical apparatus of chronometric invariants, which is very convenient for ourselves and our future followers.

We are grateful to Pierre A. Millette for his proposal and will implement it here in this article.

Usually, when doing a research study on General Relativity, we present all equations and their terms in the general covariant (four-dimensional) form. This form has its own advantage as well as a substantial drawback. The advantage is the invariance of general covariant equations and their terms in all transitions from one reference frame to another. The drawback is that they do not show actual three-dimensional quantities, which can be measured in experiments by a real observer in his real physical laboratory. In other words, general covariant equations do not give us *physical observable quantities*, but only an intermediate theoretical result, which is not applicable in practice. Therefore, in order to obtain a theoretical result applicable in practice, we need to formulate our equations in terms of physical observables — the quantities that are experimentally measurable and depend on the physical and geometric properties of the physical local reference space belonging to a real particular observer.

Meanwhile, to determine physical observable quantities in the space-time of General Relativity is not a trivial problem. For instance, a four-dimensional vector, i.e., a contravariant tensor of the 1st rank, has just 4 components: 1 time component and 3 spatial components. In this case, we can heuristically assume that its three spatial components form a three-dimensional observable vector, while its time component is the observable potential of the vector field (which, generally speaking, does not prove that these quantities can actually be observed). A tensor of the 2nd rank, e.g., a rota-

tion or deformation tensor, has 16 components: 1 time component, 9 spatial components and 6 mixed (time-spatial) components. Are the mixed components physical observables? This is another question that seemingly has no definite answer. Tensors of higher ranks have even more components. For instance, the Riemann-Christoffel curvature tensor is a tensor of the 4th rank. It has 256 components. In such a case the problem of the heuristic recognition of physically observable components becomes far more complicated, or even impossible. Besides that, there is an obstacle related to the recognition of observable components of covariant tensors (in which indices occupy the lower position) and of mixed type tensors, which have both lower and upper indices.

Therefore, the most reasonable way out of the labyrinth of heuristic guesses is to create a strict mathematical theory that allows us to calculate observable components for any tensor quantity. As mentioned in the beginning of this article, such a complete mathematical theory was created in 1941–1944 by Zelmanov. His theory was called the mathematical apparatus of physical observable quantities in General Relativity, or, in brief, the *theory of chronometric invariants*.

It should be noted that in the 1930's and 1950's, independently from Zelmanov, some other researchers tried to give a mathematical definition to physical observable quantities in the space-time of General Relativity. In 1939, L. D. Landau and E. M. Lifshitz in their famous *The Classical Theory of Fields* [8] introduced observable time and observable three-dimensional interval similar to Zelmanov's definitions. But, Landau and Lifshitz limited themselves only to this particular case and they did not arrive at general mathematical methods to calculate physical observable quantities in the four-dimensional space-time. In the 1950's, the idea of presenting physical observables in the form of the projections of four-dimensional tensorial quantities onto the three-dimensional spatial section and the time line belonging to an observer was also voiced by the Italian mathematician Carlo Cattaneo [9–12]. Cattaneo highly appreciated Zelmanov's theory of chronometric invariants, and referred to it in his last publication [12]. Nevertheless, when evaluating the scientific contribution of Cattaneo, we must take two facts into account. Firstly, his research was done only in 1958, i.e. 14 years later than Zelmanov. And secondly, his result was very far from a complete theory: he limited himself to general considerations on this problem and did not take into account the physical and geometric observable properties of the local physical space belonging to an observer (as Zelmanov did). Therefore, the projections of four-dimensional tensor quantities considered by Cattaneo do not depend on the observable properties of the observer's reference space and cannot be considered physical observables.

We therefore call physical observable quantities in the space-time of General Relativity the *Zelmanov chronometric invariants* in order to fix this term and Zelmanov's priority in the history of science.

It is also necessary to understand that Zelmanov's mathematical apparatus of chronometric invariants is not just one of many other mathematical techniques used in the General Theory of Relativity, which require an experimental verification of their applicability in practice. The Zelmanov chronometric invariants are physical observables by definition, and there is no other mathematical technique to determine physical observables in General Relativity. In this sense, the mathematical apparatus of chronometric invariants does not require experimental verification, since all quantities that we register in experiments and astronomical observations are chronometric invariants by definition. This fact should always be taken into account, when a researcher seeks to obtain a theoretical result that can be verified in a laboratory experiment or astronomical observations.

Below we present the mathematical apparatus of Zelmanov's chronometric invariants in its entirety, based on his original publications, our personal conversations with him, as well as our own works. So, let us begin.

In order to recognize which of the components of a four-dimensional quantity are physical observables, we consider a physical frame of reference belonging to a real observer, which includes a three-dimensional *coordinate grid* spanned over his *reference body* (a real physical body near him, such as the planet Earth for an Earth-bound observer), at each point of which a *real physical clock* is installed. His reference body, like any other real physical body, has a gravitational field, can rotate and deform, thereby making the local reference space of the observer inhomogeneous and anisotropic. In fact, the reference body and its reference space can be considered as a set of the real physical standards to which the observer compares the results of his measurements. Mathematically, this means that the physical observable quantities registered by an observer are the projections of four-dimensional quantities onto the three-dimensional space (coordinate grid) and the time line of his reference body.

From a geometric point of view, the three-dimensional space of an observer is a *three-dimensional spatial section* drawn in space-time at the time coordinate $x^0 = ct = \text{const}$ determined by the moment of observation t . In fact, at any point in space-time, a local spatial section (local space) can be drawn orthogonally to the line of time. If there exists an enveloping curve to such local spatial sections (local three-dimensional spaces) in space-time, these local spatial sections create a global spatial section, everywhere orthogonal to the lines of time that "pierce" it. Such a space is known as a *holonomic space*. If there is not an enveloping curve for such local spaces, then there are only spatial sections locally orthogonal to the lines of time: such a space is *non-holonomic*.

Assume that an observer is at rest with respect to his physical references (his reference body). The reference frame of such an observer always accompanies his reference body in any of its displacements, so such a system is called an *accompanying reference frame*. Any coordinate grid that is at

rest with respect to its reference body is connected to another coordinate grid through the transformation

$$\left. \begin{aligned} \tilde{x}^0 &= \tilde{x}^0(x^0, x^1, x^2, x^3) \\ \tilde{x}^i &= \tilde{x}^i(x^1, x^2, x^3), \quad \frac{\partial \tilde{x}^i}{\partial x^0} = 0 \end{aligned} \right\},$$

where the latter equation means that spatial coordinates in the tilde-marked grid are independent of time in the non-tilde coordinate grid, which is the same as setting a coordinate grid of fixed time lines $x^i = \text{const}$ at any point of the grid. Transformation of spatial coordinates is nothing but only transition from one coordinate grid to another within the same spatial section. Transformation of time means changing the whole set of clocks, so this is transition to another spatial section (another three-dimensional reference space). In practice, this means replacing one reference body and all its physical references with another one that has its own physical references. But when using different physical references, the observer will obtain different results of measurement (other observable quantities). Therefore, all physical observable quantities in the reference frame accompanying an observer must be invariant with respect to transformations of time throughout his entire three-dimensional spatial section $x^i = \text{const}$. In other words, such quantities must have the property of *chronometric invariance*. That is, all physical observable quantities in the reference frame accompanying an observer are “chronometrically” invariant quantities or, in brief, *chronometric invariants*.

Since the aforementioned transformations of time determine a set of fixed time lines “piercing” the observer’s three-dimensional spatial section, chronometric invariants (physical observable quantities) are all those quantities that are invariant with respect to these transformations.

In practice, in order to obtain physical observable quantities in the physical reference frame that accompanies a real observer, we need to calculate chronometrically invariant projections of four-dimensional quantities onto the spatial section and the time line of the observer’s physical reference body, and then formulate the projections with chronometrically invariant (physically observable) properties of his local physical reference space.

Therefore, Zelmanov had introduced projection operators that completely characterize the reference space of a particular observer.

The operator of projection onto the time line of an observer is the unit vector of the observer’s four-dimensional velocity b^α with respect to his reference body

$$b^\alpha = \frac{dx^\alpha}{ds},$$

which is tangential to his four-dimensional (space-time) trajectory at each of its points. Because any individual reference frame is characterized by its own tangential unit vector b^α ,

Zelmanov referred to the b^α as the *monad vector*. It is easy to see that since the vector b^α is tangential to the observer’s four-dimensional trajectory at each of its points, this vector has unit length

$$b_\alpha b^\alpha = g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = \frac{g_{\alpha\beta} dx^\alpha dx^\beta}{ds^2} = +1.$$

The operator of projection onto the three-dimensional reference space of the observer (which is an instant spatial section of space-time at the moment of observation) is a four-dimensional symmetric tensor $h_{\alpha\beta}$ having the form

$$h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta,$$

$$h^{\alpha\beta} = -g^{\alpha\beta} + b^\alpha b^\beta,$$

$$h^\beta_\alpha = -g^\beta_\alpha + b_\alpha b^\beta.$$

It is easy to see that the vector b^α and the tensor $h_{\alpha\beta}$ have all the necessary properties characteristic of projection operators, namely — the properties

$$b_\alpha b^\alpha = +1, \quad h^\beta_\alpha b^\alpha = 0,$$

where the second property follows from the fact that the vector b^α and the tensor $h_{\alpha\beta}$ are orthogonal to each other in space-time: mathematically this means that their common contraction is zero

$$h_{\alpha\beta} b^\alpha = -g_{\alpha\beta} b^\alpha + b_\alpha b^\alpha b_\beta = 0,$$

$$h^{\alpha\beta} b_\alpha = -g^{\alpha\beta} b_\alpha + b^\beta b_\alpha b^\alpha = 0,$$

$$h^\alpha_\beta b^\alpha = -g^\alpha_\beta b^\alpha + b_\beta b^\alpha b_\alpha = 0,$$

$$h^\beta_\alpha b^\alpha = -g^\beta_\alpha b^\alpha + b^\beta b_\alpha b^\alpha = 0.$$

In the reference frame accompanying the observer, his three-dimensional velocity with respect to his reference body is zero, which means that $b^i = 0$. As a result, the components of the b^α in the accompanying reference frame are

$$b^0 = \frac{1}{\sqrt{g_{00}}}, \quad b_0 = g_{0\alpha} b^\alpha = \sqrt{g_{00}},$$

$$b^i = 0, \quad b_i = g_{i\alpha} b^\alpha = \frac{g_{i0}}{\sqrt{g_{00}}}.$$

Therefore, the components of the projection operator $h_{\alpha\beta}$ in the accompanying reference frame ($b^i = 0$) are

$$h_{00} = 0, \quad h^{00} = -g^{00} + \frac{1}{g_{00}}, \quad h^0_0 = 0,$$

$$h_{0i} = 0, \quad h^{0i} = -g^{0i}, \quad h^i_0 = \delta^i_0 = 0,$$

$$h_{i0} = 0, \quad h^{i0} = -g^{i0}, \quad h^0_i = \frac{g_{i0}}{g_{00}},$$

$$h_{ik} = -g_{ik} + \frac{g_{0i} g_{0k}}{g_{00}}, \quad h^{ik} = -g^{ik}, \quad h^i_k = -g^i_k = \delta^i_k.$$

The projection of a tensor onto the time line of an observer is the result of its contraction with the monad vector b^α of his reference frame.

The projection of a tensor onto the three-dimensional spatial section of the observer (his three-dimensional reference space) is the result of its contraction with the tensor $h_{\alpha\beta}$ of his reference frame.

Despite the fact that such projections of a tensor of the 1st rank (a vector) are chronometric invariants, i.e., physical observables, not all such projections (contractions) of higher rank tensors have the property of chronometric invariance. To solve this problem, Zelmanov developed a general mathematical method for calculating chronometrically invariant (physically observable) projections of any four-dimensional general covariant tensor and formulated it as a theorem. We refer to it as *Zelmanov's theorem*.

ZELMANOV'S THEOREM: Let there be a four-dimensional tensor $Q_{\alpha\beta\ldots\sigma}^{\mu\nu\ldots\rho}$ of the r -th rank, where $Q_{00\ldots0}^{ik\ldots p}$ is the three-dimensional part of $Q_{00\ldots0}^{\mu\nu\ldots\rho}$, in which all upper indices are non-zero, and all m lower indices are zeroes. Then,

$$T^{ik\ldots p} = (g_{00})^{-\frac{m}{2}} Q_{00\ldots0}^{ik\ldots p}$$

is a chronometrically invariant three-dimensional contravariant tensor of the $(r - m)$ -th rank. This means that the chr.inv.-tensor $T^{ik\ldots p}$ is the result of m -fold projection of the initial tensor $Q_{\alpha\beta\ldots\sigma}^{\mu\nu\ldots\rho}$ onto the time line by the indices $\alpha, \beta, \ldots, \sigma$ and onto the spatial section by $r - m$ indices μ, ν, \ldots, ρ .

According to this theorem, the chronometrically invariant (physically observable) projections of a four-dimensional vector Q^α are the quantities

$$b^\alpha Q_\alpha = \frac{Q_0}{\sqrt{g_{00}}}, \quad h_\alpha^i Q^\alpha = q^i,$$

while the chr.inv.-projections of a symmetric tensor of the 2nd rank $Q^{\alpha\beta}$ are the quantities

$$b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}, \quad h^{\alpha\beta} b^\beta Q_{\alpha\beta} = \frac{Q_0^i}{\sqrt{g_{00}}}, \quad h_\alpha^i h_\beta^j Q^{\alpha\beta} = Q^{ij},$$

where, in the case of an antisymmetric tensor of the 2nd rank, the first chr.inv.-projection is zero, because $Q_{00} = Q^{00} = 0$ for any antisymmetric 2nd rank tensor.

The chr.inv.-projections of a four-dimensional coordinate interval dx^α are the physically observable time interval

$$d\tau = \sqrt{g_{00}} dt + \frac{g_{0i}}{c\sqrt{g_{00}}} dx^i,$$

and the interval of the physically observable coordinates dx^i , which are the same as the regular spatial coordinates. Thus, the three-dimensional chr.inv.-vector

$$v^i = \frac{dx^i}{d\tau}, \quad v_i v^i = h_{ik} v^i v^k = v^2$$

is the physically observable velocity of a particle, which is different from the particle's coordinate velocity

$$u^i = \frac{dx^i}{dt}.$$

At isotropic trajectories (trajectories of light), the v^i transforms into the three-dimensional chr.inv.-vector of the physically observable velocity of light

$$c^i = v^i = \frac{dx^i}{d\tau}, \quad c_i c^i = h_{ik} c^i c^k = c^2.$$

When we project the fundamental metric tensor $g_{\alpha\beta}$ onto the three-dimensional spatial section of an observer (which is his three-dimensional reference space)

$$h_\alpha^i h_\beta^j g_{\alpha\beta} = g^{ij} = -h^{ik}, \quad h_i^\alpha h_k^\beta g_{\alpha\beta} = g_{ik} - b_i b_k = -h_{ik},$$

we see that the three-dimensional part of the projection operator $h_{\alpha\beta}$, i.e., the three-dimensional tensor h_{ik} , the components of which have the form

$$h_{ik} = -g_{ik} + b_i b_k, \quad h^{ik} = -g^{ik}, \quad h_k^i = -g_k^i = \delta_k^i,$$

is the *chr.inv.-metric tensor* or, in other words, the metric tensor physically observed in the reference frame accompanying the observer.

The chr.inv.-metric tensor h_{ik} has all properties of the fundamental metric tensor $g_{\alpha\beta}$ throughout the observer's three-dimensional spatial section (his three-dimensional reference space), i.e., it satisfies the condition

$$h_\alpha^i h_k^\alpha = \delta_k^i - b_k b^i = \delta_k^i, \quad \delta_k^i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where δ_k^i is the unit three-dimensional tensor. The tensor δ_k^i is the three-dimensional part of the four-dimensional unit tensor δ_β^α , which can be used to lift and lower indices in four-dimensional quantities. For this reason, the chr.inv.-metric tensor h_{ik} can lift and lower indices in chronometrically invariant quantities.

Using $g_{\alpha\beta}$ from $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$, we obtain the four-dimensional interval $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ in the form

$$ds^2 = b_\alpha b_\beta dx^\alpha dx^\beta - h_{\alpha\beta} dx^\alpha dx^\beta$$

expressed with the projection operators b^α and $h_{\alpha\beta}$. Because $b_\alpha dx^\alpha = c d\tau$, the first term of the above formula transforms into $b_\alpha b_\beta dx^\alpha dx^\beta = c^2 d\tau^2$. The second term of this formula, $h_{\alpha\beta} dx^\alpha dx^\beta = d\sigma^2$, in the reference frame accompanying the observer is the square of the three-dimensional physically observable interval

$$d\sigma^2 = h_{ik} dx^i dx^k,$$

since $h_{\alpha\beta}$ has all properties of the fundamental metric tensor $g_{\alpha\beta}$ in the accompanying reference frame.

As a result, the four-dimensional interval written in terms of physically observable chr.inv.-quantities has the form

$$ds^2 = c^2 d\tau^2 - d\sigma^2.$$

Obviously, the physical observables (chr.inv.-projections of four-dimensional quantities) registered by an observer depend on the physical and geometric observable properties of the observer's local space (his physical reference space), with which, therefore, all chr.inv.-quantities and equations must be expressed. Therefore, Zelmanov deduced the basic observable properties of the reference space accompanying an observer and introduced them into the theory.

Two main physical observable properties of the accompanying reference space can be obtained using the chr.inv.-derivation operators with respect to time and the spatial coordinates. The mentioned chr.inv.-derivation operators introduced by Zelmanov have the form

$$\frac{* \partial}{\partial t} = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t}, \quad \frac{* \partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}} \frac{\partial}{\partial x^0}$$

and are non-commutative, so the difference between the 2nd derivatives is not zero

$$\begin{aligned} \frac{* \partial^2}{\partial x^i \partial t} - \frac{* \partial^2}{\partial t \partial x^i} &= \frac{1}{c^2} F_i \frac{* \partial}{\partial t}, \\ \frac{* \partial^2}{\partial x^i \partial x^k} - \frac{* \partial^2}{\partial x^k \partial x^i} &= \frac{2}{c^2} A_{ik} \frac{* \partial}{\partial t}. \end{aligned}$$

Here, A_{ik} is the three-dimensional antisymmetric chr.inv.-tensor of the angular velocity with which the reference space of the observer rotates

$$A_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right) + \frac{1}{2c^2} (F_i v_k - F_k v_i),$$

where v_i is the linear velocity of this rotation

$$\begin{aligned} v_i &= -c \frac{g_{0i}}{\sqrt{g_{00}}}, & v^i &= -c g^{0i} \sqrt{g_{00}}, \\ v_i &= h_{ik} v^k, & v^2 &= v_k v^k = h_{ik} v^i v^k. \end{aligned}$$

In addition, the v_i gives detailed formulae for the physically observable time interval $d\tau$ and the chr.inv.-metric tensor h_{ik} , which are

$$d\tau = \sqrt{g_{00}} dt - \frac{1}{c^2} v_i dx^i, \quad h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k.$$

The quantity F_i is the three-dimensional chr.inv.-vector of the gravitational inertial force

$$F_i = \frac{1}{\sqrt{g_{00}}} \left(\frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t} \right) = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t} \right),$$

where

$$w = c^2 (1 - \sqrt{g_{00}})$$

is the gravitational potential, the origin of which is the gravitational field of the observer's reference body. In the framework of quasi-Newtonian approximation, i.e., in a weak gravitational field at velocities much lower than the velocity of light and in the absence of rotation of the space, the F_i transforms into the non-relativistic gravitational force

$$F_i = \frac{\partial w}{\partial x^i}.$$

It should be noted that the quantities w and v_i do not have the property of chronometric invariance, despite the fact that $v_i = h_{ik} v^k$ is obtained as for a chr.inv.-quantity, through lowering the upper index by the chr.inv.-metric tensor h_{ik} . On the other hand, the vector of the gravitational inertial force F_i and the tensor of the angular velocity of rotation of the observer's space, A_{ik} , built using them, are chr.inv.-quantities.

The chr.inv.-quantities F_i and A_{ik} are related to each other by two identities, which we call the *Zelmanov identities*

$$\frac{* \partial A_{ik}}{\partial t} + \frac{1}{2} \left(\frac{* \partial F_k}{\partial x^i} - \frac{* \partial F_i}{\partial x^k} \right) = 0,$$

$$\frac{* \partial A_{km}}{\partial x^i} + \frac{* \partial A_{mi}}{\partial x^k} + \frac{* \partial A_{ik}}{\partial x^m} + \frac{1}{2} (F_i A_{km} + F_k A_{mi} + F_m A_{ik}) = 0.$$

In addition to rotation and the presence of a gravitational field, the real reference body of an observer can deform. In this case, the observer's reference space with its coordinate grid deforms accordingly, which must be taken into account in experiments. Mathematically, this factor manifests itself in the non-stationarity of the physically observable chr.inv.-metric h_{ik} of the observer's space and must be taken into account in the physically observable chr.inv.-quantities registered by him. For this reason, Zelmanov had introduced the three-dimensional symmetric chr.inv.-tensor D_{ik} characterizing the rate of deformations of the observer's space

$$\begin{aligned} D_{ik} &= \frac{1}{2} \frac{* \partial h_{ik}}{\partial t}, & D^{ik} &= -\frac{1}{2} \frac{* \partial h^{ik}}{\partial t}, \\ D &= h^{ik} D_{ik} = D^n_n = \frac{* \partial \ln \sqrt{h}}{\partial t}, & h &= \det ||h_{ik}||. \end{aligned}$$

Zelmanov had also introduced a theorem linking the holonomy of space-time to the tensor of the angular velocity of rotation of the observer's three-dimensional space.

ZELMANOV'S THEOREM ON THE HOLONOMY OF SPACE-TIME: The identical equality to zero of the tensor A_{ik} in a four-dimensional region of space-time is the necessary and sufficient condition for the orthogonality of the spatial sections to the time lines everywhere in this region.

In other words, $A_{ik} \neq 0$ in a non-holonomic space-time region, and $A_{ik} = 0$ in a holonomic one. Naturally, if the three-dimensional spatial sections are everywhere orthogonal to the time lines (in such a case the space-time region is holonomic), all the quantities g_{0i} are equal to zero. Since $g_{0i} = 0$, we have

$v_i = 0$ and $A_{ik} = 0$ too. Therefore, we also refer to the tensor A_{ik} as the *space non-holonomy tensor*.

The space-time of Special Relativity (Minkowski space) in the Galilean reference frame, as well as some cases of the space-time in General Relativity, do not rotate ($A_{ik} = 0$). These are examples of holonomic spaces: time lines are orthogonal to spatial sections in them. Rotating spaces ($A_{ik} \neq 0$) are non-holonomic; time lines are non-orthogonal to three-dimensional spatial sections in such spaces.

To understand why the rotation of a three-dimensional spatial section of space-time makes this spatial section non-orthogonal to the time lines “piercing” it, consider a *locally geodesic reference frame*. Within the infinitesimal vicinity of any point in such a reference frame, the fundamental metric tensor has the form

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{1}{2} \left(\frac{\partial^2 \tilde{g}_{\mu\nu}}{\partial \tilde{x}^\rho \partial \tilde{x}^\sigma} \right) (\tilde{x}^\rho - x^\rho) (\tilde{x}^\sigma - x^\sigma) + \dots,$$

which means that the numerical values of its components in the infinitesimal vicinity of any point differ from those at this point itself only in the 2nd order terms and the higher other terms, which can be neglected. Therefore, at any point in a locally geodesic reference frame, the fundamental metric tensor (within the 2nd order terms withheld) is constant, while the 1st derivatives of the metric tensor, i.e., the Christoffel symbols, are zeroes.

It is obvious that in any Riemannian space within the infinitesimal vicinity of any point of the space a locally geodesic reference frame can be set up. As a result, at any point belonging to the locally geodesic reference frame, a flat space can be set up tangential to the Riemannian space so that the locally geodesic reference frame in the Riemannian space is a globally geodesic frame in the tangential flat space. Since the fundamental metric tensor is constant in the flat space, there in the infinitesimal vicinity of any point in the Riemannian space the quantities $\tilde{g}_{\mu\nu}$ converge to those of the tensor $g_{\mu\nu}$ in the tangential flat space. This means that, in the tangential flat space, we can set up a system of the basis vectors $\tilde{e}_{(a)}$ tangential to the curved coordinate lines of the Riemannian space. Because the coordinate lines of a Riemannian space are curved (in a general case), and, in the case where the space is non-holonomic, are not even orthogonal to each other, the lengths of the basis vectors are sometimes substantially different from unit length.

Consider the world-vector $d\vec{r}$ of an infinitesimal displacement from such a point, i.e., $d\vec{r} = \{dx^0, dx^1, dx^2, dx^3\}$. Then $d\vec{r} = \tilde{e}_{(a)} dx^a$, where its components $e_{(a)}$ are

$$\begin{aligned} \tilde{e}_{(0)} &= \{e_{(0)}^0, 0, 0, 0\}, & \tilde{e}_{(1)} &= \{0, e_{(1)}^1, 0, 0\}, \\ \tilde{e}_{(2)} &= \{0, 0, e_{(2)}^2, 0\}, & \tilde{e}_{(3)} &= \{0, 0, 0, e_{(3)}^3\}. \end{aligned}$$

The scalar product of the vector $d\vec{r}$ with itself is equal to $d\vec{r}d\vec{r} = ds^2$. On the other hand, it is $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. Thus,

we obtain the general formula

$$g_{\alpha\beta} = \tilde{e}_{(a)} \tilde{e}_{(b)} = e_{(a)} e_{(b)} \cos(x^\alpha; x^\beta).$$

According to this formula we have

$$g_{00} = e_{(0)}^2,$$

while, on the other hand, $\sqrt{g_{00}} = 1 - \frac{w}{c^2}$. Hence, the length $e_{(0)}$ of the time basis vector $\tilde{e}_{(0)}$ tangential to the time line $x^0 = ct$ is expressed with the gravitational potential w as

$$e_{(0)} = \sqrt{g_{00}} = 1 - \frac{w}{c^2}.$$

The stronger the gravitational potential w , the smaller $e_{(0)}$ is than 1. In the case of gravitational collapse ($w = c^2$), the length of the time basis vector $\tilde{e}_{(0)}$ becomes zero: $e_{(0)} = 0$.

Thus, according to the above general formula, the component g_{0i} is expressed as

$$g_{0i} = e_{(0)} e_{(i)} \cos(x^0; x^i),$$

while, according to the definition of v_i , we have

$$g_{0i} = -\frac{1}{c} v_i \left(1 - \frac{w}{c^2} \right) = -\frac{1}{c} v_i e_{(0)},$$

whence we obtain the formula for v_i , which takes into account the angle of inclination of the time lines to the three-dimensional spatial section of space-time, i.e.

$$v_i = -c e_{(i)} \cos(x^0; x^i).$$

In addition, since the above general formula gives

$$g_{ik} = e_{(i)} e_{(k)} \cos(x^i; x^k),$$

and according to the definition of the chr.inv.-metric tensor h_{ik} (page 7), we obtain the formula for h_{ik} , which also takes into account the angle of inclination of the time lines to the three-dimensional spatial section

$$h_{ik} = e_{(i)} e_{(k)} [\cos(x^0; x^i) \cos(x^0; x^k) - \cos(x^i; x^k)].$$

From the above formula for v_i , we see that from a geometric point of view, the linear velocity v_i with which the three-dimensional reference space of an observer rotates is the projection (scalar product) of the time basis vector $\tilde{e}_{(0)}$ of his reference space onto the spatial basis vectors $\tilde{e}_{(i)}$, multiplied by the velocity of light. If the spatial sections of a space (space-time) are everywhere orthogonal to the time lines thereby giving the space holonomy, then $\cos(x^0; x^i) = 0$ and, hence, $v_i = 0$. In a non-holonomic space, the spatial sections are not orthogonal to the lines of time: $\cos(x^0; x^i) \neq 0$.

Generally $|\cos(x^0; x^i)| \leq 1$, hence the linear velocity v_i with which the three-dimensional reference space of an observer rotates cannot exceed the velocity of light.

If somewhere the conditions $F_i = 0$ and $A_{ik} = 0$ are met in common, there the conditions $g_{00} = 1$ and $g_{0i} = 0$ are present as well (the conditions $g_{00} = 1$ and $g_{0i} = 0$ can be satisfied through the transformation of time). In such a region, according to the definition of $d\tau$ (page 6), we have $d\tau = dt$: so, the difference between the coordinate time t and the physically observable time τ disappears in the absence of gravitational fields and rotation of space. In other words, according to the theory of chronometric invariants, the difference between the coordinate time t and the physically observable time τ comes from both gravitation and rotation attributed to the local reference space of the observer (in fact — from his reference body, which is a real physical body near him, for example, the planet Earth for an Earth-bound observer), or from each of the mentioned two factors separately.

On the other hand, it is doubtful to find such a region of the Universe where gravitational fields or rotation of the background space are clearly absent. Therefore, in practice the physically observable time τ differs from the coordinate time t . This means that the real space of our Universe is non-holonomic: it rotates and is filled with gravitational fields, while a holonomic space, free from rotation and gravity, can only be a local approximation to it.

The condition of holonomy of a space (space-time) is directly linked to the problem of integrability of time in it. In a non-holonomic space, the formula for the physically observable time interval $d\tau$ has no integrating multiplier, i.e., it cannot be transformed to the form

$$d\tau = A dt,$$

where the multiplier A depends on only t and x^i . In this case the formula for $d\tau$ (page 6) has a non-zero second term depending on the coordinate interval dx^i and g_{0i} . On the contrary, in a holonomic space, we have $A_{ik} = 0$, so $g_{0i} = 0$. In this case, the second term of the formula for $d\tau$ is zero, while the first term is the coordinate time interval dt with an integrating multiplier

$$A = \sqrt{g_{00}} = f(x^0, x^i),$$

so we can write the integral

$$d\tau = \int \sqrt{g_{00}} dt.$$

Hence time is integrable in a holonomic space ($A_{ik} = 0$), while it cannot be integrated if the space is non-holonomic ($A_{ik} \neq 0$). In the case where time is integrable, i.e., in a holonomic space, we can synchronize the clocks installed at two distantly located points by moving a control clock along the path between these two points. In the case where time cannot be integrated (in a non-holonomic space), synchronization of clocks in two distant points is impossible in principle: the larger is the distance between these two points, the more is the deviation of time on these clocks.

The space of our planet Earth, is non-holonomic due to the daily rotation of it around the Earth's axis. Hence, two clocks installed at different points on the surface of the Earth should manifest a deviation between the intervals of time registered on each of them. The larger is the distance between these clocks, the larger is the deviation of the physically observable time expected to be registered on them. This effect was surely verified by the well-known Hafele-Keating experiment performed in October 1971 by Joseph C. Hafele together with Richard E. Keating [13–15] and then successfully repeated by the UK's National Measurement Laboratory commonly with the BBC on its 25th anniversary in 2005 [16]. This experiment concerned with displacing standard atomic clocks by a jet airplane around the terrestrial globe, where rotation of the Earth's space sensibly changed the measured time. During the flight along the Earth's rotation, the local space of an observer on board of the airplane had more rotation than the space of another observer who stayed fixed on the airfield. During the flight against the Earth's rotation it was vice versa. The atomic clocks on board the airplane showed a significant deviation of the observed time depending on the velocity of rotation of the observer's space.

Since synchronization of clocks at various points on the Earth's surface is the highly important task of metrology, marine navigation, aviation, and orbital space flights, corrections for desynchronization were introduced in early times in the form of tables of empirically obtained corrections that take the Earth's rotation into account. Now, thanks to the theory of chronometric invariants, we know the origin of the corrections and therefore can calculate them on the basis of General Relativity.

With Zelmanov's definitions of chr.inv.-quantities above, we can not only calculate the physically observable chr.inv.-projections of any four-dimensional general covariant quantity or equation of theoretical physics, but also express them in terms of the physically observable chr.inv.-properties F^i , A_{ik} , and D_{ik} characteristic of the local reference space of a particular observer.

The Christoffel symbols (coherence coefficients of space) appear in the absolute derivatives, the equations of motion, and somewhere else in the equations of theoretical physics. The Christoffel symbols are not tensors [17]. Nevertheless, they can be expressed in terms of physical observable quantities. Following the analogy with the regular Christoffel symbols of the 2nd rank $\Gamma_{\mu\nu}^\alpha$ and the regular Christoffel symbols of the 1st rank $\Gamma_{\mu\nu,\sigma}$

$$\Gamma_{\mu\nu}^\alpha = g^{\alpha\sigma} \Gamma_{\mu\nu,\sigma} = \frac{1}{2} g^{\alpha\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right),$$

Zelmanov had introduced the chr.inv.-Christoffel symbols of the 2nd rank and 1st rank

$$\Delta_{jk}^i = h^{im} \Delta_{jk,m} = \frac{1}{2} h^{im} \left(\frac{\partial h_{jm}}{\partial x^k} + \frac{\partial h_{km}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^m} \right),$$

where the only difference is that the chr.inv.-Christoffel symbols use the chr.inv.-metric tensor h_{ik} instead of the fundamental metric tensor $g_{\alpha\beta}$.

It is not a problem to find out how the regular Christoffel symbols are expressed in terms of the physically observable chr.inv.-properties characteristic of the reference space of an observer. Expressing the components of $g^{\alpha\beta}$ and then the 1st derivatives of $g_{\alpha\beta}$ with F_i , A_{ik} , D_{ik} , w , and v_i , after some algebra we obtain

$$\begin{aligned}\Gamma_{00,0} &= -\frac{1}{c^3} \left(1 - \frac{w}{c^2}\right) \frac{\partial w}{\partial t}, \\ \Gamma_{00,i} &= \frac{1}{c^2} \left(1 - \frac{w}{c^2}\right)^2 F_i + \frac{1}{c^4} v_i \frac{\partial w}{\partial t}, \\ \Gamma_{0i,0} &= -\frac{1}{c^2} \left(1 - \frac{w}{c^2}\right) \frac{\partial w}{\partial x^i}, \\ \Gamma_{0i,j} &= -\frac{1}{c} \left(1 - \frac{w}{c^2}\right) \left(D_{ij} + A_{ij} + \frac{1}{c^2} F_j v_i\right) + \frac{1}{c^3} v_j \frac{\partial w}{\partial x^i}, \\ \Gamma_{ij,0} &= \frac{1}{c} \left(1 - \frac{w}{c^2}\right) \left[D_{ij} - \frac{1}{2} \left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right) + \frac{1}{2c^2} (F_i v_j + F_j v_i)\right], \\ \Gamma_{ij,k} &= -\Delta_{ij,k} + \frac{1}{c^2} \left[v_i A_{jk} + v_j A_{ik} + \frac{1}{2} v_k \left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right) - \right. \\ &\quad \left. - \frac{1}{2c^2} v_k (F_i v_j + F_j v_i)\right] + \frac{1}{c^4} F_k v_i v_j, \\ \Gamma_{00}^0 &= -\frac{1}{c^3} \left[\frac{1}{1 - \frac{w}{c^2}} \frac{\partial w}{\partial t} + \left(1 - \frac{w}{c^2}\right) v_k F^k\right], \\ \Gamma_{00}^k &= -\frac{1}{c^2} \left(1 - \frac{w}{c^2}\right)^2 F^k, \\ \Gamma_{0i}^0 &= \frac{1}{c^2} \left[-\frac{1}{1 - \frac{w}{c^2}} \frac{\partial w}{\partial x^i} + v_k \left(D_i^k + A_{i \cdot}^k + \frac{1}{c^2} v_i F^k\right)\right], \\ \Gamma_{0i}^k &= \frac{1}{c} \left(1 - \frac{w}{c^2}\right) \left(D_i^k + A_{i \cdot}^k + \frac{1}{c^2} v_i F^k\right), \\ \Gamma_{ij}^0 &= -\frac{1}{c \left(1 - \frac{w}{c^2}\right)} \left\{-D_{ij} + \frac{1}{c^2} v_n \times \right. \\ &\quad \left. \times \left[v_j (D_i^n + A_{i \cdot}^n) + v_i (D_j^n + A_{j \cdot}^n) + \frac{1}{c^2} v_i v_j F^n\right] + \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right) - \frac{1}{2c^2} (F_i v_j + F_j v_i) - \Delta_{ij}^n v_n\right\}, \\ \Gamma_{ij}^k &= \Delta_{ij}^k - \frac{1}{c^2} \left[v_i (D_j^k + A_{j \cdot}^k) + v_j (D_i^k + A_{i \cdot}^k) + \frac{1}{c^2} v_i v_j F^k\right].\end{aligned}$$

Respectively, some components of the regular Christoffel symbols are linked to the chr.inv.-properties of the observer's

space by the following relations

$$\begin{aligned}D_k^i + A_{k \cdot}^i &= \frac{c}{\sqrt{g_{00}}} \left(\Gamma_{0k}^i - \frac{g_{0k} \Gamma_{00}^i}{g_{00}}\right), \\ F^k &= -\frac{c^2 \Gamma_{00}^k}{g_{00}}, \\ g^{i\alpha} g^{k\beta} \Gamma_{\alpha\beta}^m &= h^{iq} h^{ks} \Delta_{qs}^m.\end{aligned}$$

By analogy with the respective absolute derivatives, Zelmanov had also introduced the chr.inv.-derivatives

$$\begin{aligned}{}^* \nabla_i Q_k &= \frac{{}^* \partial Q_k}{\partial x^i} - \Delta_{ik}^l Q_l, \\ {}^* \nabla_i Q^k &= \frac{{}^* \partial Q^k}{\partial x^i} + \Delta_{il}^k Q^l, \\ {}^* \nabla_i Q_{jk} &= \frac{{}^* \partial Q_{jk}}{\partial x^i} - \Delta_{ij}^l Q_{lk} - \Delta_{ik}^l Q_{jl}, \\ {}^* \nabla_i Q_j^k &= \frac{{}^* \partial Q_j^k}{\partial x^i} - \Delta_{ij}^l Q_l^k + \Delta_{il}^k Q_j^l, \\ {}^* \nabla_i Q^{jk} &= \frac{{}^* \partial Q^{jk}}{\partial x^i} + \Delta_{il}^j Q^{lk} + \Delta_{il}^k Q^{jl}, \\ {}^* \nabla_i Q^i &= \frac{{}^* \partial Q^i}{\partial x^i} + \Delta_{ji}^j Q^i, \quad \Delta_{ji}^j = \frac{{}^* \partial \ln \sqrt{h}}{\partial x^i}, \\ {}^* \nabla_i Q^{ji} &= \frac{{}^* \partial Q^{ji}}{\partial x^i} + \Delta_{il}^j Q^{il} + \Delta_{il}^i Q^{ji}, \quad \Delta_{li}^i = \frac{{}^* \partial \ln \sqrt{h}}{\partial x^i}.\end{aligned}$$

In particular, they show the following properties of the chr.inv.-metric tensor h_{ik}

$${}^* \nabla_i h_{jk} = 0, \quad {}^* \nabla_i h_j^k = 0, \quad {}^* \nabla_i h^{jk} = 0.$$

Next we give an account of tensor calculus in terms of physical observables (chronometric invariants).

Assume that there is a space (not necessarily metric) in which there is an arbitrary reference frame $\{x^\alpha\}$. Let this space contain an object G determined by n functions f_n of the x^α coordinates. Let us know the transformation rule to calculate these n functions in any other reference frame $\{\tilde{x}^\alpha\}$ in this space. If the n functions f_n and also the transformation rule have been given in the space, then G is a *geometric object*, which in the system $\{x^\alpha\}$ has axial components $f_n(x^\alpha)$, while in any other system $\{\tilde{x}^\alpha\}$ it has components $\tilde{f}_n(\tilde{x}^\alpha)$.

Assume that a tensor object (tensor) of zero rank is any geometric object φ , transformable according to the rule

$$\tilde{\varphi} = \varphi \frac{\partial x^\alpha}{\partial \tilde{x}^\alpha},$$

where the index takes numbers of all coordinate axes one-by-one (this notation is also known as *by-component notation* or *tensor notation*). Any tensor of zero rank has a single component and is called *scalar*.

From a geometric point of view, any scalar is a point to which a certain number is attributed. Therefore, a scalar field is a set of points of the space, which have a common property. For instance, a point mass is a scalar, while a distributed mass (a gas, for instance) makes up a scalar field.

It should be noted that the algebraic notations for a tensor and a tensor field are the same. The field of a tensor in a space is represented as the tensor in a given point of the space, but its presence in other points everywhere in this region of the space is assumed.

Contravariant tensors of the 1st rank A^α are geometric objects with components, transformable according to the rule

$$\tilde{A}^\alpha = A^\mu \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}.$$

From a geometric point of view, such an object is an n -dimensional vector. For instance, the vector of an elementary displacement dx^α is a contravariant tensor of the 1st rank.

Contravariant tensors of the 2nd rank $A^{\alpha\beta}$ are geometric objects transformable according to the rule

$$\tilde{A}^{\alpha\beta} = A^{\mu\nu} \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu}.$$

From a geometric point of view, such an object is the area (parallelogram) formed by two vectors. For this reason, contravariant tensors of the 2nd rank are also called *bivectors*.

So forth, contravariant tensors of higher ranks are formulated as the following geometric objects

$$\tilde{A}^{\alpha\dots\sigma} = A^{\mu\dots\tau} \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \dots \frac{\partial \tilde{x}^\sigma}{\partial x^\tau}.$$

A vector field or a higher rank tensor field are space distributions of the respective tensor quantities. For instance, because a mechanical strength characterizes both its own magnitude and direction, its distribution in a physical body can be presented by a vector field.

Covariant tensors of the 1st rank A_α are geometric objects, transformable according to the rule

$$\tilde{A}_\alpha = A_\mu \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}.$$

Thus, the gradient of a scalar field φ , i.e., the quantity

$$A_\alpha = \frac{\partial \varphi}{\partial x^\alpha},$$

is a covariant tensor of the 1st rank. This is because for a regular invariant we have $\tilde{\varphi} = \varphi$, then

$$\frac{\partial \tilde{\varphi}}{\partial \tilde{x}^\alpha} = \frac{\partial \tilde{\varphi}}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} = \frac{\partial \varphi}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}.$$

Covariant tensors of the 2nd rank $A_{\alpha\beta}$ are geometric objects with the transformation rule

$$\tilde{A}_{\alpha\beta} = A_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta},$$

hence, covariant tensors of higher ranks are formulated as

$$\tilde{A}_{\alpha\dots\sigma} = A_{\mu\dots\tau} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \dots \frac{\partial x^\tau}{\partial \tilde{x}^\sigma}.$$

Mixed tensors are tensors of the 2nd rank or of higher ranks with both upper and lower indices. For instance, any mixed symmetric tensor A^α_β is a geometric object, transformable according to the rule

$$\tilde{A}^\alpha_\beta = A^\mu_\nu \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial \tilde{x}^\beta}.$$

Tensor objects exist both in metric and non-metric spaces. In non-metric spaces, as it is known, the distance between any two points cannot be measured. This is in contrast to metric spaces. In the theories of space-time-matter, such as the General Theory of Relativity and its extensions, metric spaces are taken under consideration. This is because the core of such theories is the measurement of time intervals and spatial lengths, that is nonsense in a non-metric space.

Any tensor has a^n components, where a is its dimension and n is the rank. For instance, a four-dimensional tensor of zero rank has 1 component, a tensor of the 1st rank has 4 components, a tensor of the 2nd rank has 16 components, a tensor of the 4th rank (for example, the Riemann-Christoffel curvature tensor) has 256 components, and so on.

Indices in a geometric object, marking its axial components, are found not in tensors only, but in other geometric objects as well. For this reason, if we encounter a quantity in component notation, it is not necessarily a tensor quantity.

In practice, to know whether a given object is a tensor or not, we need to know a formula for this object in a reference frame and to transform it to any other reference frame. For instance, consider the classic question: are Christoffel's symbols (i.e., the coherence coefficients of space) tensors? To answer this question, we need to calculate the Christoffel symbols in a tilde-marked reference frame

$$\tilde{\Gamma}^\alpha_{\mu\nu} = \tilde{g}^{\alpha\sigma} \tilde{\Gamma}_{\mu\nu,\sigma}, \quad \tilde{\Gamma}_{\mu\nu,\sigma} = \frac{1}{2} \left(\frac{\partial \tilde{g}_{\mu\sigma}}{\partial \tilde{x}^\nu} + \frac{\partial \tilde{g}_{\nu\sigma}}{\partial \tilde{x}^\mu} - \frac{\partial \tilde{g}_{\mu\nu}}{\partial \tilde{x}^\sigma} \right)$$

proceeding from the general formula of them in a non-marked reference frame.

First, we calculate the terms in the brackets. The fundamental metric tensor like any other covariant tensor of the 2nd rank, is transformable to the tilde-marked reference frame according to the following rule

$$\tilde{g}_{\mu\sigma} = g_{\varepsilon\tau} \frac{\partial x^\varepsilon}{\partial \tilde{x}^\mu} \frac{\partial x^\tau}{\partial \tilde{x}^\sigma}.$$

Because the quantity $g_{\varepsilon\tau}$ depends on the non-tilde-marked coordinates, its derivative with respect to the tilde-marked coordinates (which are functions of the non-tilded ones) is calculated according to the rule

$$\frac{\partial g_{\varepsilon\tau}}{\partial \tilde{x}^\nu} = \frac{\partial g_{\varepsilon\tau}}{\partial x^\rho} \frac{\partial x^\rho}{\partial \tilde{x}^\nu},$$

and thus the first term in the brackets, taking the rule of transformation of the fundamental metric tensor into account, takes the form

$$\frac{\partial \tilde{g}_{\mu\sigma}}{\partial \tilde{x}^\nu} = \frac{\partial g_{\varepsilon\tau}}{\partial x^\rho} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} \frac{\partial x^\varepsilon}{\partial \tilde{x}^\mu} \frac{\partial x^\tau}{\partial \tilde{x}^\sigma} + g_{\varepsilon\tau} \left(\frac{\partial x^\tau}{\partial \tilde{x}^\sigma} \frac{\partial^2 x^\varepsilon}{\partial \tilde{x}^\nu \partial \tilde{x}^\mu} + \frac{\partial x^\varepsilon}{\partial \tilde{x}^\mu} \frac{\partial^2 x^\tau}{\partial \tilde{x}^\nu \partial \tilde{x}^\sigma} \right).$$

Calculating the rest of the terms of the tilde-marked Christoffel symbols and transposing their free indices we obtain

$$\tilde{\Gamma}_{\mu\nu,\sigma} = \Gamma_{\varepsilon\rho,\tau} \frac{\partial x^\varepsilon}{\partial \tilde{x}^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} \frac{\partial x^\tau}{\partial \tilde{x}^\sigma} + g_{\varepsilon\tau} \frac{\partial x^\tau}{\partial \tilde{x}^\sigma} \frac{\partial^2 x^\varepsilon}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu},$$

$$\tilde{\Gamma}_{\mu\nu}^\alpha = \Gamma_{\varepsilon\rho}^\gamma \frac{\partial \tilde{x}^\alpha}{\partial x^\gamma} \frac{\partial x^\varepsilon}{\partial \tilde{x}^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} + \frac{\partial \tilde{x}^\alpha}{\partial x^\gamma} \frac{\partial^2 x^\gamma}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}.$$

We see that the Christoffel symbols are not transformed in the same way as tensors, hence they are not tensors.

Tensors can be represented as matrices. But in practice, such a form can only be possible for tensors of the 1st or 2nd rank (one-row and flat matrices, respectively). For instance, the tensor of an elementary four-dimensional displacement can be represented in the form of a one-row matrix

$$dx^\alpha = (dx^0, dx^1, dx^2, dx^3),$$

the four-dimensional fundamental metric tensor can be represented in the form of a flat matrix

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix},$$

and tensors of the 3rd rank are three-dimensional matrices. Representing tensors of higher ranks as matrices is problematic and not visual.

Now let us turn to tensor algebra, the branch of tensor calculus that focuses on algebraic operations with tensors.

Only same-type tensors of the same rank with indices in the same position can be added or subtracted. Adding up two n -rank tensors of the same type gives a new tensor of the same type and rank, the components of which are the sums of the corresponding components of the added tensors. For instance

$$A^\alpha + B^\alpha = D^\alpha, \quad A_\beta^\alpha + B_\beta^\alpha = D_\beta^\alpha.$$

Multiplication is allowed not only for tensors of the same type, but also for any tensors of any rank. External multiplication of a tensor of the n -rank by a tensor of the m -rank gives a new tensor of the $(n+m)$ -rank

$$A_{\alpha\beta} B_\gamma = D_{\alpha\beta\gamma}, \quad A_\alpha B^{\beta\gamma} = D_\alpha^{\beta\gamma}.$$

Contraction is the multiplication of tensors of the same rank when some of their indices are the same. Contraction of tensors across all indices yields a scalar

$$A_\alpha B^\alpha = C, \quad A_{\alpha\beta}^\gamma B_\gamma^{\alpha\beta} = D.$$

Often the multiplication of tensors entails the contraction of some of their indices. Such multiplication is known as inner multiplication, which means that some indices become contracted when the tensors are multiplied. Below is an example of internal multiplication

$$A_{\alpha\sigma} B^\sigma = D_\alpha, \quad A_{\alpha\sigma}^\gamma B_\gamma^{\beta\sigma} = D_\alpha^\beta.$$

Using internal multiplication of geometric objects we can determine whether they are tensors or not. This is the so-called *fraction theorem*.

FRACTION THEOREM: If $B^{\sigma\beta}$ is a tensor and its internal multiplication with a geometric object $A(\alpha, \sigma)$ is a tensor $D(\alpha, \beta)$, i.e., $A(\alpha, \sigma) B^{\sigma\beta} = D(\alpha, \beta)$, then this object $A(\alpha, \sigma)$ is also a tensor.

According to this theorem, if internal multiplication of an object $A_{\alpha\sigma}$ with a tensor $B^{\sigma\beta}$ gives another tensor D_α^β

$$A_{\alpha\sigma} B^{\sigma\beta} = D_\alpha^\beta,$$

then this object $A_{\alpha\sigma}$ is a tensor. Or, if internal multiplication of an object A_σ^α and a tensor $B^{\sigma\beta}$ gives a tensor $D^{\alpha\beta}$

$$A_\sigma^\alpha B^{\sigma\beta} = D^{\alpha\beta},$$

then the object A_σ^α is a tensor.

The geometric properties of any metric space are determined by its fundamental metric tensor, which can lift and lower the indices in the objects of this metric space. In Riemannian spaces, the space metric has a square form, which is $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ and is known also as the *Riemannian metric*, so the fundamental metric tensor of a Riemannian space is a tensor of the 2nd rank $g_{\alpha\beta}$. The mixed fundamental metric tensor g_α^β is equal to the unit tensor $g_\alpha^\beta = g_{\alpha\sigma} g^{\sigma\beta} = \delta_\alpha^\beta$. The diagonal components of the unit tensor are units, while its rest (non-diagonal) components are zeroes. Using the unit tensor we can replace the indices in four-dimensional quantities

$$\delta_\alpha^\beta A_\beta = A_\alpha, \quad \delta_\mu^\nu \delta_\rho^\sigma A^{\mu\rho} = A^{\nu\sigma}.$$

Contracting any tensor of the 2nd rank with the fundamental metric tensor $g_{\alpha\beta}$ yields a scalar known as the tensor spur or its trace

$$g^{\alpha\beta} A_{\alpha\beta} = A_\sigma^\sigma = A.$$

For example, the spur of the fundamental metric tensor in a four-dimensional pseudo-Riemannian space is 4

$$g_{\alpha\beta} g^{\alpha\beta} = g_\sigma^\sigma = g_0^0 + g_1^1 + g_2^2 + g_3^3 = \delta_0^0 + \delta_1^1 + \delta_2^2 + \delta_3^3 = 4.$$

As mentioned on page 6, the chr.inv.-metric tensor h_{ik} has all properties of the fundamental metric tensor $g_{\alpha\beta}$ throughout the observer's three-dimensional spatial section (his three-dimensional reference space). Therefore, h_{ik} can lower, lift and replace indices in chr.inv.-quantities. Accordingly, the spur (trace) of any three-dimensional chr.inv.-tensor is obtained by contracting it with h_{ik} . For instance, the spur (trace)

of the tensor of the rate of deformations of the observer's space, D_{ik} , is the chr.inv.-scalar

$$D = D_m^m = h^{ik} D_{ik},$$

the physical sense of which is the rate of relative expansion or contraction of the elementary volume of the observer's reference space.

The scalar product of two vectors A^α and B^α (tensors of the 1st rank) in a four-dimensional pseudo-Riemannian space is formulated as

$$g_{\alpha\beta} A^\alpha B^\beta = A_\alpha B^\alpha = A_0 B^0 + A_i B^i.$$

Scalar product is the result of contraction, because the multiplication of vectors contracts all their indices. Therefore, the scalar product of two vectors (tensors of the 1st rank) is always a scalar (tensor of zero rank). If both the vectors are the same, their scalar product

$$g_{\alpha\beta} A^\alpha A^\beta = A_\alpha A^\alpha = A_0 A^0 + A_i A^i$$

is the square of the given vector A^α , the length of which is expressed as

$$A = |A^\alpha| = \sqrt{g_{\alpha\beta} A^\alpha A^\beta}.$$

The four-dimensional pseudo-Riemannian space, which is the space-time of General Relativity, by its definition has the sign-alternating metric, i.e., the fundamental metric tensor has the sign-alternating signature (+---) or (-+++). In this case, the lengths of four-dimensional vectors can be real, imaginary or zero. Vectors with non-zero (real or imaginary) lengths are known as *non-isotropic vectors*; they are tangential to non-isotropic trajectories. Vectors with zero length are known as *isotropic vectors*; they are tangential to isotropic trajectories (trajectories of light-like particles).

In the three-dimensional Euclidean space, the scalar product of two vectors is a scalar quantity, the numerical value of which is equal to the product of their lengths and the cosine of the angle between them

$$A_i B^i = |A^i| |B^i| \cos(A^i; B^i).$$

From the above formula it follows that the scalar product of two vectors is zero, if the vectors are orthogonal to each other. In other words, from a geometric point of view, the scalar product of two vectors is the projection of one vector onto the other. If the vectors are the same, then the vector is projected onto itself, so the result of this projection is the square of its length.

Theoretically, at each point of any Riemannian space, a tangential flat space can be set, the basis vectors of which are tangential to the basis vectors of the Riemannian space at this point. Then, the metric of the tangential flat space is also the metric of the Riemannian space at this point. Therefore, the

above formula is also true, if we consider the angle between the three-dimensional coordinate lines and the time lines in the space thereby replacing the Roman (three-dimensional spatial) indices with the Greek (four-dimensional) ones.

Denote the chr.inv.-projections of arbitrary vectors A^α and B^α onto the time line and the three-dimensional spatial section of an observer as follows

$$a = \frac{A_0}{\sqrt{g_{00}}}, \quad a^i = A^i, \\ b = \frac{B_0}{\sqrt{g_{00}}}, \quad b^i = B^i,$$

then their remaining components have the form

$$A^0 = \frac{a + \frac{1}{c} v_i a^i}{1 - \frac{w}{c^2}}, \quad A_i = -a_i - \frac{a}{c} v_i, \\ B^0 = \frac{b + \frac{1}{c} v_i b^i}{1 - \frac{w}{c^2}}, \quad B_i = -b_i - \frac{b}{c} v_i.$$

Substituting the chr.inv.-projections of the vectors A^α and B^α into the formulae for $A_\alpha B^\alpha$ and $A_\alpha A^\alpha$, we obtain

$$A_\alpha B^\alpha = ab - a_i b^i = ab - h_{ik} a^i b^k, \\ A_\alpha A^\alpha = a^2 - a_i a^i = a^2 - h_{ik} a^i a^k.$$

From here, we see that the square of the length of any vector is the difference between the squares of the lengths of its time and spatial chr.inv.-projections. If both these projections are the same, then the vector's length is zero, so the vector is isotropic. Hence, any isotropic vector equally belongs to the time line and the spatial section. The equality of its time projection to its spatial projection also means that this vector is orthogonal to itself. If its time projection is "longer" than its spatial projection, then this vector is real. If the spatial projection is "longer", then this vector is imaginary.

The latter can be illustrated by the square of the length of the space-time interval

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = dx_\alpha dx^\alpha = dx_0 dx^0 + dx_i dx^i,$$

which in terms of chr.inv.-quantities has the form

$$ds^2 = c^2 d\tau^2 - dx_i dx^i = c^2 d\tau^2 - h_{ik} dx^i dx^k = c^2 d\tau^2 - d\sigma^2.$$

Its length ds can be real, imaginary or zero, depending on whether ds is time-like $c^2 d\tau^2 > d\sigma^2$, which is the case along sublight-speed real trajectories, space-like $c^2 d\tau^2 < d\sigma^2$, which is the case of imaginary superluminal-speed trajectories, or isotropic $c^2 d\tau^2 = d\sigma^2$, which is the case of light-like (isotropic) trajectories, respectively.

The vector product of two vectors A^α and B^α is a tensor of the 2nd rank $V^{\alpha\beta}$, obtained from their external multiplication according to the rule

$$V^{\alpha\beta} = [A^\alpha; B^\beta] = \frac{1}{2} (A^\alpha B^\beta - A^\beta B^\alpha) = \frac{1}{2} \begin{vmatrix} A^\alpha & A^\beta \\ B^\alpha & B^\beta \end{vmatrix}.$$

As it is easy to see, in this case the order in which the vectors are multiplied matters, i.e., the order in which we write down the tensor indices is important. For this reason, the tensors obtained as vector products are called *antisymmetric tensors*. In an antisymmetric tensor we have $V^{\alpha\beta} = -V^{\beta\alpha}$, where its indices being moved “reserve” their places as dots, $g_{\alpha\sigma} V^{\sigma\beta} = V_{\alpha}^{\cdot\beta}$, thereby showing the place from where the specific index was moved. In symmetric tensors there is no need to “reserve” places for moved indices, because the order in which they appear does not matter. For example, the fundamental metric tensor is symmetric $g_{\alpha\beta} = g_{\beta\alpha}$, and the Riemann-Christoffel tensor of the curvature of space $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ is symmetric with respect to transposition over a pair of its indices and antisymmetric within each pair of the indices. It is obvious that only tensors of the 2nd rank or higher ranks can be symmetric or antisymmetric.

All diagonal components of any antisymmetric tensor by its definition are zeroes. For instance, in an antisymmetric tensor of the 2nd rank we have

$$V^{\alpha\alpha} = [A^{\alpha}; B^{\alpha}] = \frac{1}{2} (A^{\alpha} B^{\alpha} - A^{\alpha} B^{\alpha}) = 0.$$

In the three-dimensional Euclidean space, the numerical value of the vector product of two vectors is defined as the area of the parallelogram formed by them and is equal to the product of their moduli multiplied by the sine of the angle between them

$$V^{ik} = |A^i| |B^k| \sin(A^i; B^k).$$

This means that the vector product of two vectors, i.e., any antisymmetric tensor of the 2nd rank, is a pad oriented in space according to the directions of the vectors forming it.

The contraction of an antisymmetric tensor $V_{\alpha\beta}$ with any symmetric tensor $A^{\alpha\beta} = A^{\alpha} A^{\beta}$ is zero. Naturally, since $V_{\alpha\alpha} = 0$ and $V_{\alpha\beta} = -V_{\beta\alpha}$ we have

$$V_{\alpha\beta} A^{\alpha} A^{\beta} = V_{00} A^0 A^0 + V_{0i} A^0 A^i + V_{i0} A^i A^0 + V_{ik} A^i A^k = 0.$$

According to the theory of chronometric invariants, an antisymmetric tensor of the 2nd rank $V^{\alpha\beta}$ has the following chr.inv.-projections

$$\begin{aligned} \frac{V_{00}}{g_{00}} &= 0, \\ \frac{V_{0\cdot}^i}{\sqrt{g_{00}}} &= -\frac{V_{\cdot 0}^i}{\sqrt{g_{00}}} = \frac{1}{2} (ab^i - ba^i), \\ V^{ik} &= \frac{1}{2} (a^i b^k - a^k b^i), \end{aligned}$$

which are expressed here with the chr.inv.-projections of its forming (multiplied) vectors A^{α} and B^{α} : here a and b are the chr.inv.-projections of the multiplied vectors A^{α} and B^{α} onto the time line of the observer, and a^i and b^i are their chr.inv.-projections onto the observer's spatial section (which is his three-dimensional reference space).

The first chr.inv.-projection of the antisymmetric tensor $V^{\alpha\beta}$ is zero, since in any antisymmetric tensor all its diagonal components are zeroes. The third physically observable chr.inv.-quantity V^{ik} is the projection of the tensor $V^{\alpha\beta}$ onto the observer's spatial section. It is analogous to a vector product in the three-dimensional space. The second chr.inv.-quantity of the above is the space-time (mixed) projection of $V^{\alpha\beta}$. It has no equivalent among the components of a regular three-dimensional vector product.

The square of an antisymmetric tensor of the 2nd rank $V^{\alpha\beta}$, formulated with the chr.inv.-projections of its forming vectors A^{α} and B^{α} , is calculated as

$$\begin{aligned} V_{\alpha\beta} V^{\alpha\beta} &= \frac{1}{2} (a_i a^i b_k b^k - a_i b^i a_k b^k) + \\ &+ ab a_i b^i - \frac{1}{2} (a^2 b_i b^i - b^2 a_i a^i). \end{aligned}$$

The asymmetry of tensor fields is determined by reference antisymmetric tensors. Such references in the Galilean reference frame* are Levi-Civita's tensors: for four-dimensional quantities this is the four-dimensional completely antisymmetric unit tensor $e^{\alpha\beta\mu\nu}$, while for three-dimensional quantities this is the three-dimensional completely antisymmetric unit tensor e^{ikm} . The components of the Levi-Civita tensors, which have all indices different, are either +1 or -1 depending on the number of transpositions of their indices. All the remaining components, i.e., those having at least two coinciding indices, are zeroes. Moreover, with the space signature (+---) we are using, all non-zero contravariant components of the Levi-Civita tensors have the opposite sign to their corresponding covariant components†. For instance, in the Minkowski space we have

$$\begin{aligned} g_{\alpha\sigma} g_{\beta\rho} g_{\mu\tau} g_{\nu\gamma} e^{\sigma\rho\tau\gamma} &= g_{00} g_{11} g_{22} g_{33} e^{0123} = -e^{0123}, \\ g_{i\alpha} g_{k\beta} g_{m\gamma} e^{\alpha\beta\gamma} &= g_{11} g_{22} g_{33} e^{123} = -e^{123}, \end{aligned}$$

since $g_{00} = 1$ and $g_{11} = g_{22} = g_{33} = -1$ with the space signature (+---) we are using. In this case, the components of the tensor $e^{\alpha\beta\mu\nu}$ are

$$\begin{aligned} e^{0123} &= +1, \quad e^{1023} = -1, \quad e^{1203} = +1, \quad e^{1230} = -1, \\ e^{0123} &= -1, \quad e_{1023} = +1, \quad e_{1203} = -1, \quad e_{1230} = +1, \end{aligned}$$

and the components of the tensor e^{ikm} are

$$\begin{aligned} e^{123} &= +1, \quad e^{213} = -1, \quad e^{231} = +1, \\ e_{123} &= -1, \quad e_{213} = +1, \quad e_{231} = -1. \end{aligned}$$

*A Galilean reference frame is one that does not rotate, is not subject to deformation, and falls freely in the space-time of Special Relativity (Minkowski space). The time lines in the Galilean reference frame are linear, as are the three-dimensional coordinate axes.

†If the space signature is (++++), then what has been said is true only for the four-dimensional Levi-Civita tensor $e^{\alpha\beta\mu\nu}$. The components of the three-dimensional Levi-Civita tensor e^{ikm} will have the same sign as well as the corresponding components of the e_{ikm} tensor.

In general, the tensor $e^{\alpha\beta\mu\nu}$ is related to the tensor e^{ikm} as follows $e^{0ikm} = e^{ikm}$. Because we have an arbitrary choice for the sign of the first component, we can choose $e^{0123} = -1$ and $e^{123} = -1$. Then the remaining components of e^{ikm} will change respectively.

Multiplying the four-dimensional antisymmetric unit tensor $e^{\alpha\beta\mu\nu}$ by itself we obtain a regular tensor of the 8th rank with the non-zero components determined by the matrix

$$e^{\alpha\beta\mu\nu} e_{\sigma\tau\rho\gamma} = - \begin{pmatrix} \delta_{\sigma}^{\alpha} & \delta_{\tau}^{\alpha} & \delta_{\rho}^{\alpha} & \delta_{\gamma}^{\alpha} \\ \delta_{\sigma}^{\beta} & \delta_{\tau}^{\beta} & \delta_{\rho}^{\beta} & \delta_{\gamma}^{\beta} \\ \delta_{\sigma}^{\mu} & \delta_{\tau}^{\mu} & \delta_{\rho}^{\mu} & \delta_{\gamma}^{\mu} \\ \delta_{\sigma}^{\nu} & \delta_{\tau}^{\nu} & \delta_{\rho}^{\nu} & \delta_{\gamma}^{\nu} \end{pmatrix}.$$

The remaining properties of the tensor $e^{\alpha\beta\mu\nu}$ are deduced from the previous by means of contraction of their indices

$$e^{\alpha\beta\mu\nu} e_{\sigma\tau\rho\nu} = - \begin{pmatrix} \delta_{\sigma}^{\alpha} & \delta_{\tau}^{\alpha} & \delta_{\rho}^{\alpha} \\ \delta_{\sigma}^{\beta} & \delta_{\tau}^{\beta} & \delta_{\rho}^{\beta} \\ \delta_{\sigma}^{\mu} & \delta_{\tau}^{\mu} & \delta_{\rho}^{\mu} \end{pmatrix},$$

$$e^{\alpha\beta\mu\nu} e_{\sigma\tau\mu\nu} = -2 \begin{pmatrix} \delta_{\sigma}^{\alpha} & \delta_{\tau}^{\alpha} \\ \delta_{\sigma}^{\beta} & \delta_{\tau}^{\beta} \end{pmatrix} = -2 (\delta_{\sigma}^{\alpha} \delta_{\tau}^{\beta} - \delta_{\sigma}^{\beta} \delta_{\tau}^{\alpha}),$$

$$e^{\alpha\beta\mu\nu} e_{\sigma\beta\mu\nu} = -6 \delta_{\sigma}^{\alpha}, \quad e^{\alpha\beta\mu\nu} e_{\alpha\beta\mu\nu} = -6 \delta_{\alpha}^{\alpha} = -24.$$

Multiplying the three-dimensional antisymmetric unit tensor e^{ikm} by itself we obtain a regular tensor of the 6th rank

$$e^{ikm} e_{rst} = \begin{pmatrix} \delta_r^i & \delta_s^i & \delta_t^i \\ \delta_r^k & \delta_s^k & \delta_t^k \\ \delta_r^m & \delta_s^m & \delta_t^m \end{pmatrix}.$$

The remaining properties of the tensor e^{ikm} are

$$e^{ikm} e_{rsm} = - \begin{pmatrix} \delta_r^i & \delta_s^i \\ \delta_r^k & \delta_s^k \end{pmatrix} = \delta_s^i \delta_r^k - \delta_r^i \delta_s^k,$$

$$e^{ikm} e_{rkm} = 2 \delta_r^i, \quad e^{ikm} e_{ikm} = 2 \delta_i^i = 6.$$

The completely antisymmetric unit tensor determines for a tensor object its corresponding *pseudotensor*, marked with asterisk. For instance, any four-dimensional scalar, vector and tensors of the 2nd, 3rd, and 4th ranks have corresponding four-dimensional pseudotensors of the following ranks

$$V^{*\alpha\beta\mu\nu} = e^{\alpha\beta\mu\nu} V, \quad V^{*\alpha\beta\mu} = e^{\alpha\beta\mu\nu} V_{\nu}, \quad V^{*\alpha\beta} = \frac{1}{2} e^{\alpha\beta\mu\nu} V_{\mu\nu},$$

$$V^{*\alpha} = \frac{1}{6} e^{\alpha\beta\mu\nu} V_{\beta\mu\nu}, \quad V^* = \frac{1}{24} e^{\alpha\beta\mu\nu} V_{\alpha\beta\mu\nu},$$

where pseudotensors of the 1st rank, such as $V^{*\alpha}$, are called *pseudovectors*, while pseudotensors of zero rank, such as V^* , are called *pseudoscalars*. Any tensor and its corresponding pseudotensor are known as *dual* to each other to emphasize

their common genesis. So, three-dimensional antisymmetric tensors have their corresponding three-dimensional pseudotensors

$$V^{*ikm} = e^{ikm} V, \quad V^{*ik} = e^{ikm} V_m,$$

$$V^{*i} = \frac{1}{2} e^{ikm} V_{km}, \quad V^* = \frac{1}{6} e^{ikm} V_{ikm}.$$

Pseudotensors are called such because, in contrast to regular tensors, they do not change when reflected with respect to one of the coordinate axes. For instance, when reflected with respect to the abscissa axis $x^1 = -\tilde{x}^1$, $x^2 = \tilde{x}^2$, $x^3 = \tilde{x}^3$, the reflected component of an antisymmetric tensor V_{ik} , orthogonal to x^1 , is $\tilde{V}_{23} = -V_{23}$, while the dual component of the pseudovector V^{*i} retains the original sign unchanged

$$V^{*1} = \frac{1}{2} e^{1km} V_{km} = \frac{1}{2} (e^{123} V_{23} + e^{132} V_{32}) = V_{23},$$

$$\tilde{V}^{*1} = \frac{1}{2} \tilde{e}^{1km} \tilde{V}_{km} = \frac{1}{2} (e^{123} \tilde{V}_{23} + e^{132} \tilde{V}_{32}) = V_{23}.$$

Since any four-dimensional antisymmetric tensor of the 2nd rank and its dual pseudotensor are of the same rank, their contraction yields a pseudoscalar, which is

$$V_{\alpha\beta} V^{*\alpha\beta} = V_{\alpha\beta} e^{\alpha\beta\mu\nu} V_{\mu\nu} = e^{\alpha\beta\mu\nu} B_{\alpha\beta\mu\nu} = B^*.$$

The square of a pseudotensor $V^{*\alpha\beta}$ and a pseudovector V^{*i} , expressed with their dual tensors, are

$$V_{\alpha\beta} V^{*\alpha\beta} = e_{\alpha\beta\mu\nu} V^{\mu\nu} e^{\alpha\beta\rho\sigma} V_{\rho\sigma} = -24 V_{\mu\nu} V^{\mu\nu},$$

$$V_{*i} V^{*i} = e_{ikm} V^{km} e^{ipq} V_{pq} = 6 V_{km} V^{km}.$$

We cannot set a Galilean reference frame in an inhomogeneous and anisotropic pseudo-Riemannian space. In such a general space, the antisymmetry references of tensor fields depend on the inhomogeneity and anisotropy of the space itself, which are determined by the fundamental metric tensor, and a reference antisymmetric tensor is the four-dimensional completely antisymmetric discriminant tensor

$$E^{\alpha\beta\mu\nu} = \frac{e^{\alpha\beta\mu\nu}}{\sqrt{-g}}, \quad E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu} \sqrt{-g}.$$

The proof is the following. Transformation of the four-dimensional completely antisymmetric unit tensor $e_{\alpha\beta\mu\nu}$ from a Galilean (non-tilde-marked) reference frame into an arbitrary (tilde-marked) reference frame is

$$\tilde{e}_{\alpha\beta\mu\nu} = \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\alpha}} \frac{\partial x^{\gamma}}{\partial \tilde{x}^{\beta}} \frac{\partial x^{\varepsilon}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\tau}}{\partial \tilde{x}^{\nu}} e_{\sigma\gamma\varepsilon\tau} = J e_{\alpha\beta\mu\nu},$$

where

$$J = \det \left\| \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\sigma}} \right\| = \det \begin{pmatrix} \frac{\partial x^0}{\partial \tilde{x}^0} & \frac{\partial x^0}{\partial \tilde{x}^1} & \frac{\partial x^0}{\partial \tilde{x}^2} & \frac{\partial x^0}{\partial \tilde{x}^3} \\ \frac{\partial x^1}{\partial \tilde{x}^0} & \frac{\partial x^1}{\partial \tilde{x}^1} & \frac{\partial x^1}{\partial \tilde{x}^2} & \frac{\partial x^1}{\partial \tilde{x}^3} \\ \frac{\partial x^2}{\partial \tilde{x}^0} & \frac{\partial x^2}{\partial \tilde{x}^1} & \frac{\partial x^2}{\partial \tilde{x}^2} & \frac{\partial x^2}{\partial \tilde{x}^3} \\ \frac{\partial x^3}{\partial \tilde{x}^0} & \frac{\partial x^3}{\partial \tilde{x}^1} & \frac{\partial x^3}{\partial \tilde{x}^2} & \frac{\partial x^3}{\partial \tilde{x}^3} \end{pmatrix}$$

is the determinant of Jacobi's matrix known also as the Jacobian of the transformation. Because the fundamental metric tensor $g_{\alpha\beta}$ is transformable according to the rule

$$\tilde{g}_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta}$$

and since its determinant in the tilde-marked frame is

$$\tilde{g} = \det \left\| g_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} \right\| = J^2 g,$$

then, in the Galilean (non-tilde-marked) reference frame,

$$g = \det \|g_{\alpha\beta}\| = \det \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -1,$$

and, hence, $J^2 = -\tilde{g}^2$. Denoting $\tilde{e}_{\alpha\beta\mu\nu}$ in an arbitrary reference frame as $E_{\alpha\beta\mu\nu}$ and writing down the metric tensor in a regular non-tilde-marked form, we obtain

$$E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu} \sqrt{-g}$$

as expected at the very beginning, which was to be proved. In the same way, we obtain the transformation rule

$$E^{\alpha\beta\mu\nu} = \frac{e^{\alpha\beta\mu\nu}}{\sqrt{-g}}$$

for the components $E^{\alpha\beta\mu\nu}$, because for them

$$g = \tilde{g} \tilde{J}^2, \quad \tilde{J} = \det \left\| \frac{\partial \tilde{x}^\alpha}{\partial x^\sigma} \right\|.$$

The discriminant tensor $E^{\alpha\beta\mu\nu}$ is not a physical observable quantity. For this reason, Zelmanov had introduced the four-dimensional discriminant tensor $\varepsilon^{\alpha\beta\gamma}$

$$\varepsilon^{\alpha\beta\gamma} = h_\mu^\alpha h_\nu^\beta h_\rho^\gamma b_\sigma E^{\sigma\mu\nu\rho} = b_\sigma E^{\sigma\alpha\beta\gamma},$$

$$\varepsilon_{\alpha\beta\gamma} = h_\alpha^\mu h_\beta^\nu h_\gamma^\rho b^\sigma E_{\sigma\mu\nu\rho} = b^\sigma E_{\sigma\alpha\beta\gamma},$$

which in the accompanying reference frame of an observer ($b^i = 0$) and taking into account that $\sqrt{-g} = \sqrt{h} \sqrt{g_{00}}$ according to the theory of chronometric invariants transforms into the three-dimensional chr.inv.-discriminant tensor ε^{ikm}

$$\varepsilon^{ikm} = b_0 E^{0ikm} = \sqrt{g_{00}} E^{0ikm} = \frac{e^{ikm}}{\sqrt{h}},$$

$$\varepsilon_{ikm} = b^0 E_{0ikm} = \frac{E_{0ikm}}{\sqrt{g_{00}}} = e_{ikm} \sqrt{h},$$

for which, as is easy to obtain, we have

$$\begin{aligned} {}^*\nabla_l \varepsilon_{ijk} &= 0, & {}^*\nabla_l \varepsilon^{ijk} &= 0, \\ \frac{\partial \varepsilon_{ijk}}{\partial t} &= \varepsilon_{ijk} D, & \frac{\partial \varepsilon^{ijk}}{\partial t} &= -\varepsilon^{ijk} D, \end{aligned}$$

where D is the spur (trace) of the chr.inv.-tensor D_{ik} characterizing the rate of deformations of the observer's space

$$D = h^{ik} D_{ik} = D_n^n = \frac{{}^*\partial \ln \sqrt{h}}{\partial t}, \quad h = \det \|h_{ik}\|.$$

The three-dimensional chr.inv.-discriminant tensor ε^{ikm} is the physical observable reference of the asymmetry of tensor fields in the observer's reference space. Using the ε^{ikm} , we can transform antisymmetric chr.inv.-tensors into the corresponding chr.inv.-pseudotensors.

For example, for the chr.inv.-tensor A_{ik} of the angular velocity of rotation of the observer's space, we have the chr.inv.-pseudovector Ω^{*i} of this rotation

$$\Omega^{*i} = \frac{1}{2} \varepsilon^{ikm} A_{km}, \quad \Omega_{*i} = \frac{1}{2} \varepsilon_{imn} A^{mn}, \quad A^{ik} = \varepsilon^{mik} \Omega_{*m},$$

$$\varepsilon^{ipq} \Omega_{*i} = \frac{1}{2} \varepsilon^{ipq} \varepsilon_{imn} A^{mn} = \frac{1}{2} (\delta_m^p \delta_n^q - \delta_m^q \delta_n^p) A^{mn} = A^{pq}.$$

With the chr.inv.-pseudovector Ω^{*i} the Zelmanov identities (page 7) connecting the chr.inv.-quantities F_i and A_{ik} take the form, respectively,

$$\frac{2}{\sqrt{h}} \frac{{}^*\partial}{\partial t} (\sqrt{h} \Omega^{*i}) + \varepsilon^{ijk} {}^*\nabla_j F_k = 0,$$

$${}^*\nabla_k \Omega^{*k} + \frac{1}{c^2} F_k \Omega^{*k} = 0.$$

Next we consider the absolute differential and absolute directional derivative.

In geometry, a *differential* of a function is its variation between two infinitely close points with the coordinates x^α and $x^\alpha + dx^\alpha$. Respectively, the *absolute differential* in an n -dimensional space represents the change of an n -dimensional quantity between two infinitely close points in this space. For continuous functions, which we commonly deal with in practice, their variations between infinitely close points are infinitesimal. But in order to determine an infinitesimal variation of a tensor quantity, we cannot use a simple "difference" between its numerical values at the neighbouring points x^α and $x^\alpha + dx^\alpha$, because tensor algebra does not determine it. This ratio can only be determined using the rules for transforming tensors from one reference frame to another. As a consequence, differential operators and the results of their application to tensors must be tensors.

For instance, the absolute differential of a tensor quantity is a tensor of the same rank as the original tensor itself. The absolute differential of a scalar φ is the scalar

$$D\varphi = \frac{\partial \varphi}{\partial x^\alpha} dx^\alpha,$$

which in the accompanying reference frame of an observer ($b^i = 0$) takes the form

$$D\varphi = \frac{\partial \varphi}{\partial t} d\tau + \frac{\partial \varphi}{\partial x^i} dx^i,$$

where, apart from the three-dimensional observable differential (second term), there is an additional term that takes into account the dependence of the absolute differential $D\varphi$ on the physically observable time interval $d\tau$.

The absolute differential of a contravariant vector A^α is formulated with the absolute derivation operator ∇ (nabla) and has the following form

$$\begin{aligned} DA^\alpha &= \nabla_\sigma A^\alpha dx^\sigma = \frac{\partial A^\alpha}{\partial x^\sigma} dx^\sigma + \Gamma_{\mu\sigma}^\alpha A^\mu dx^\sigma = \\ &= dA^\alpha + \Gamma_{\mu\sigma}^\alpha A^\mu dx^\sigma, \end{aligned}$$

where $\nabla_\sigma A^\alpha$ is the absolute derivative of A^α with respect to x^σ , and d stands for regular differentials

$$\begin{aligned} \nabla_\sigma A^\alpha &= \frac{\partial A^\alpha}{\partial x^\sigma} + \Gamma_{\mu\sigma}^\alpha A^\mu, \\ d &= \frac{\partial}{\partial x^\alpha} dx^\alpha. \end{aligned}$$

Formulating the absolute differential with physical observable quantities is equivalent to projecting its general covariant form onto the time line and the spatial section in the accompanying reference frame of an observer. According to the theory of chronometric invariants, the physically observable chr.inv.-projections of the absolute differential of a vector A^α are the quantities

$$T = b_\alpha DA^\alpha = \frac{g_{0\alpha} DA^\alpha}{\sqrt{g_{00}}}, \quad B^i = h_\alpha^i DA^\alpha.$$

Denoting the chr.inv.-projections of the vector A^α as

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}, \quad q^i = A^i,$$

we calculate its remaining components, which, when expressed in terms of the φ and q^i take the form

$$A_0 = \varphi \left(1 - \frac{w}{c^2}\right), \quad A^0 = \frac{\varphi + \frac{1}{c} v_i q^i}{1 - \frac{w}{c^2}}, \quad A_i = -q_i - \frac{\varphi}{c} v_i.$$

Taking the chr.inv.-formula for the regular differential

$$d = \frac{\partial}{\partial t} d\tau + \frac{\partial}{\partial x^i} dx^i$$

into account, we substitute them and also the regular Christoffel symbols expressed in terms of chr.inv.-quantities (see page 10) into the T and B^i . As a result we obtain the chr.inv.-projections of the absolute differential of the vector A^α in the final chr.inv.-form

$$\begin{aligned} T &= b_\alpha DA^\alpha = d\varphi + \frac{1}{c} (-F_i q^i d\tau + D_{ik} q^i dx^k), \\ B^i &= h_\sigma^i DA^\sigma = dq^i + \left(\frac{\varphi}{c} dx^k + q^k d\tau\right) (D_k^i + A_{k.}^i) - \\ &\quad - \frac{\varphi}{c} F^i d\tau + \Delta_{mk}^i q^m dx^k. \end{aligned}$$

The *directional derivative* of a function is its change with respect to the elementary displacement along the given direction. The *absolute directional derivative* in an n -dimensional space is the change of an n -dimensional quantity with respect to an elementary n -dimensional interval along the given direction in the space.

For instance, the absolute derivative of a scalar function φ to a direction along a curve $x^\alpha = x^\alpha(\rho)$, where ρ is a non-zero monotone parameter along this curve, expresses the rate at which this function φ changes

$$\frac{D\varphi}{d\rho} = \frac{d\varphi}{d\rho},$$

which in the accompanying reference frame of an observer is

$$\frac{D\varphi}{d\rho} = \frac{\partial\varphi}{\partial t} \frac{d\tau}{d\rho} + \frac{\partial\varphi}{\partial x^i} \frac{dx^i}{d\rho}.$$

The absolute derivative of a vector A^α to the given direction tangential to a curve $x^\alpha = x^\alpha(\rho)$ is

$$\frac{DA^\alpha}{d\rho} = \nabla_\sigma A^\alpha \frac{dx^\sigma}{d\rho} = \frac{dA^\alpha}{d\rho} + \Gamma_{\mu\sigma}^\alpha A^\mu \frac{dx^\sigma}{d\rho},$$

and its chr.inv.-projections are

$$\begin{aligned} b_\alpha \frac{DA^\alpha}{d\rho} &= \frac{d\varphi}{d\rho} + \frac{1}{c} \left(-F_i q^i \frac{d\tau}{d\rho} + D_{ik} q^i \frac{dx^k}{d\rho} \right), \\ h_\sigma^i \frac{DA^\sigma}{d\rho} &= \frac{dq^i}{d\rho} + \left(\frac{\varphi}{c} \frac{dx^k}{d\rho} + q^k \frac{d\tau}{d\rho} \right) (D_k^i + A_{k.}^i) - \\ &\quad - \frac{\varphi}{c} F^i \frac{d\tau}{d\rho} + \Delta_{mk}^i q^m \frac{dx^k}{d\rho}. \end{aligned}$$

The equations of motion of a particle are based on the absolute directional derivative of the particle's world vector. For this reason, the above chr.inv.-projections are the "generic" chr.inv.-equations of motion.

The *divergence* of a tensor field is its "change" along a coordinate axis. Respectively, the *absolute divergence* of an n -dimensional tensor field is its divergence in an n -dimensional space. The divergence of a tensor field is the result of contraction of the field tensor with the absolute derivation operator ∇ . The divergence of a vector field A^α is the scalar

$$\nabla_\sigma A^\sigma = \frac{\partial A^\sigma}{\partial x^\sigma} + \Gamma_{\sigma\mu}^\sigma A^\mu,$$

and the divergence of a field of a 2nd rank tensor, say the tensor $F^{\alpha\beta}$, is the vector

$$\nabla_\sigma F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \Gamma_{\sigma\mu}^\sigma F^{\alpha\mu} + \Gamma_{\sigma\mu}^\alpha F^{\sigma\mu},$$

where, as it can be proved, $\Gamma_{\sigma\mu}^\sigma$ is

$$\Gamma_{\sigma\mu}^\sigma = \frac{\partial \ln \sqrt{-g}}{\partial x^\mu}.$$

To prove this, we use the definition of the regular Christoffel symbols (see page 9), which, when re-written with the above indices has the form

$$\Gamma_{\sigma\mu}^{\sigma} = g^{\sigma\rho} \Gamma_{\mu\sigma,\rho} = \frac{1}{2} g^{\sigma\rho} \left(\frac{\partial g_{\mu\rho}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\rho}} \right),$$

where, since σ and ρ are free indices here, they can change their sites. As a result, after contracting with the tensor $g^{\sigma\rho}$ the first and the last terms of the above formula for $\Gamma_{\sigma\mu}^{\sigma}$ cancel each other, so the formula for $\Gamma_{\sigma\mu}^{\sigma}$ simplifies as

$$\Gamma_{\sigma\mu}^{\sigma} = \frac{1}{2} g^{\sigma\rho} \frac{\partial g_{\sigma\rho}}{\partial x^{\mu}}.$$

The quantities $g^{\sigma\rho}$ are the components of a tensor reciprocal to the tensor $g_{\sigma\rho}$. For this reason, each component of the matrix $g^{\sigma\rho}$ is formulated as

$$g^{\sigma\rho} = \frac{a^{\sigma\rho}}{g}, \quad g = \det \|g_{\sigma\rho}\|,$$

where $a^{\sigma\rho}$ is the algebraic co-factor of the matrix element with indices $\sigma\rho$, equal to $(-1)^{\sigma+\rho}$, multiplied by the determinant of the matrix obtained by crossing the row and the column with numbers σ and ρ out of the matrix $g_{\sigma\rho}$. As a result, we obtain $a^{\sigma\rho} = g g^{\sigma\rho}$.

Because the determinant of the fundamental metric tensor by definition is formulated as

$$g = \det \|g_{\sigma\rho}\| = \sum_{\alpha_0 \dots \alpha_3} (-1)^{N(\alpha_0 \dots \alpha_3)} g_{0(\alpha_0)} g_{1(\alpha_1)} g_{2(\alpha_2)} g_{3(\alpha_3)},$$

then the quantity dg is $dg = a^{\sigma\rho} dg_{\sigma\rho} = g g^{\sigma\rho} dg_{\sigma\rho}$, or

$$\frac{dg}{g} = g^{\sigma\rho} dg_{\sigma\rho}.$$

Integrating the left hand side gives $\ln(-g)$, because the g is negative while logarithm is determined for only positive functions. Then, we have $d \ln(-g) = \frac{dg}{g}$. Taking into account that $\sqrt{-g} = \frac{1}{2} \ln(-g)$, we obtain

$$d \ln \sqrt{-g} = \frac{1}{2} g^{\sigma\rho} dg_{\sigma\rho},$$

so the above $\Gamma_{\sigma\mu}^{\sigma}$ takes the form

$$\Gamma_{\sigma\mu}^{\sigma} = \frac{1}{2} g^{\sigma\rho} \frac{\partial g_{\sigma\rho}}{\partial x^{\mu}} = \frac{\partial \ln \sqrt{-g}}{\partial x^{\mu}},$$

which was to be proved.

The divergence of a vector field A^{α} is a scalar quantity. Hence $\nabla_{\sigma} A^{\sigma}$ cannot be projected onto a time line and a spatial section. But this is enough to express $\nabla_{\sigma} A^{\sigma}$ with the chr.inv.-projections of the A^{α} and the physically observable properties of the observer's reference space. Besides that, the regular derivation operators must be replaced with the chr.inv.-derivation operators.

Assuming the above notation φ and q^i for the chr.inv.-projections of the vector A^{α} , we express the remaining components of the A^{α} with them. Then, substituting the regular derivation operators expressed with the chr.inv.-derivation operators (marked by asterisk, see their definition on page 7)

$$\frac{\partial}{\partial t} = \sqrt{g_{00}} \frac{* \partial}{\partial t}, \quad \sqrt{g_{00}} = 1 - \frac{w}{c^2},$$

$$\frac{\partial}{\partial x^i} = \frac{* \partial}{\partial x^i} - \frac{1}{c^2} v_i \frac{* \partial}{\partial t},$$

into the general formula for $\nabla_{\sigma} A^{\sigma}$ (page 17) and taking into account that $\sqrt{-g} = \sqrt{h} \sqrt{g_{00}}$, after some algebra we obtain the $\nabla_{\sigma} A^{\sigma}$ in the extended chr.inv.-form

$$\nabla_{\sigma} A^{\sigma} = \frac{1}{c} \left(\frac{* \partial \varphi}{\partial t} + \varphi D \right) + \frac{* \partial q^i}{\partial x^i} + q^i \frac{* \partial \ln \sqrt{h}}{\partial x^i} - \frac{1}{c^2} F_i q^i.$$

In the third term of this formula, the quantity

$$\frac{* \partial \ln \sqrt{h}}{\partial x^i} = \Delta_{ji}^j$$

stands for the chr.inv.-Christoffel symbols Δ_{ji}^k contracted by two indices. Therefore, by analogy with the definition of the absolute divergence of a four-dimensional vector field A^{α} (see page 17), Zelmanov called the quantity

$$* \nabla_i q^i = \frac{* \partial q^i}{\partial x^i} + q^i \frac{* \partial \ln \sqrt{h}}{\partial x^i} = \frac{* \partial q^i}{\partial x^i} + q^i \Delta_{ji}^j$$

the *chr.inv.-divergence* of a three-dimensional chr.inv.-vector field q^i . Thus the $\nabla_{\sigma} A^{\sigma}$ takes the final chr.inv.-form

$$\nabla_{\sigma} A^{\sigma} = \frac{1}{c} \left(\frac{* \partial \varphi}{\partial t} + \varphi D \right) + * \nabla_i q^i - \frac{1}{c^2} F_i q^i.$$

The first term of this formula has no equivalent. It is made up of two parts. The first part is the observable change in time of the time projection φ of the vector A^{α} . The second part φD , since the spur (trace) $D = h^{ik} D_{ik}$ of the chr.inv.-tensor D_{ik} is the observable rate of relative expansion or compression of an elementary volume of the observer's space, is the observable change of the elementary volume of the three-dimensional observable vector field q^i in time.

The difference between the last two terms of this formula, which make up the chr.inv.-quantity

$$* \widetilde{\nabla}_i q^i = * \nabla_i q^i - \frac{1}{c^2} F_i q^i,$$

Zelmanov called the *physical chr.inv.-divergence*, because the chr.inv.-quantity $* \widetilde{\nabla}_i q^i$ takes into account the fact that, in a real physical space, the flow of time is different on the opposite walls of an elementary volume.

Generally speaking, when calculating the divergence of a field we consider an elementary volume of the space, so we calculate the difference between the amounts of a "substance"

which flows in and out of the volume over an elementary time interval. The gravitational inertial force F^i results in a different flow of time at different points: the beginnings as well as the ends of the time intervals measured on the opposite walls of a volume will not coincide, which makes these time intervals inapplicable for comparison. Synchronization of clocks on the opposite walls of the volume will give the true result: the measured time intervals will be different. That is, the physical chr.inv.-divergence ${}^*\tilde{\nabla}_i q^i$ is a physical observable in the observer's three-dimensional reference space, which is analogous to a regular divergence.

Next we deduce the chr.inv.-projections of the absolute divergence $\nabla_\sigma F^{\sigma\alpha}$ of an antisymmetric tensor $F^{\alpha\beta} = -F^{\beta\alpha}$

$$\begin{aligned}\nabla_\sigma F^{\sigma\alpha} &= \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \Gamma_{\sigma\mu}^\sigma F^{\alpha\mu} + \Gamma_{\sigma\mu}^\alpha F^{\sigma\mu} = \\ &= \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \frac{\partial \ln \sqrt{-g}}{\partial x^\mu} F^{\alpha\mu},\end{aligned}$$

we need to obtain Maxwell's equations in chr.inv.-form. Here in this formula, the third term $\Gamma_{\sigma\mu}^\alpha F^{\sigma\mu}$ is zero, because contracting the Christoffel symbols $\Gamma_{\sigma\mu}^\alpha$ (they are symmetric by their lower indices) with the antisymmetric tensor $F^{\sigma\mu}$ gives zero as in the case of any symmetric and antisymmetric geometric objects.

The quantity $\nabla_\sigma F^{\sigma\alpha}$ is a four-dimensional vector, therefore its chr.inv.-projections are

$$T = b_\alpha \nabla_\sigma F^{\sigma\alpha}, \quad B^i = h_\alpha^i \nabla_\sigma F^{\sigma\alpha} = \nabla_\sigma F^{\sigma i}.$$

Denoting the chr.inv.-projections of the tensor $F^{\alpha\beta}$ as

$$E^i = \frac{F_{0\cdot}^i}{\sqrt{g_{00}}}, \quad H^{ik} = F^{ik},$$

we obtain the remaining non-zero components of the $F^{\alpha\beta}$ expressed with its chr.inv.-projections

$$\begin{aligned}F_{0\cdot}^0 &= \frac{1}{c} v_k E^k, \\ F_{k\cdot}^0 &= \frac{1}{\sqrt{g_{00}}} \left(E_i - \frac{1}{c} v_n H_{k\cdot}^n - \frac{1}{c^2} v_k v_n E^n \right), \\ F^{0i} &= \frac{E^i - \frac{1}{c} v_k H^{ik}}{\sqrt{g_{00}}}, \\ F_{0i} &= -\sqrt{g_{00}} E_i, \\ F_{i\cdot}^k &= -H_{i\cdot}^k - \frac{1}{c} v_i E^k, \\ F_{ik} &= H_{ik} + \frac{1}{c} (v_i E_k - v_k E_i),\end{aligned}$$

and also the square of the tensor $F^{\alpha\beta}$ in the form as well expressed with its chr.inv.-projections

$$F_{\alpha\beta} F^{\alpha\beta} = H_{ik} H^{ik} - 2 E_i E^i.$$

Substituting these formulae into the above general formulae for T and B^i and then replacing the regular derivation operators with the chr.inv.-derivation operators, after some algebra we obtain the formulae for the chr.inv.-projections T and B^i of the absolute divergence $\nabla_\sigma F^{\sigma\alpha}$ of the antisymmetric tensor $F^{\alpha\beta} = -F^{\beta\alpha}$ in detail

$$\begin{aligned}T &= \frac{\nabla_\sigma F_{0\cdot}^\sigma}{\sqrt{g_{00}}} = \frac{{}^*\partial E^i}{\partial x^i} + E^i \frac{{}^*\partial \ln \sqrt{h}}{\partial x^i} - \frac{1}{c} H^{ik} A_{ik}, \\ B^i &= \nabla_\sigma F^{\sigma i} = \frac{{}^*\partial H^{ik}}{\partial x^k} + H^{ik} \frac{{}^*\partial \ln \sqrt{h}}{\partial x^k} - \frac{1}{c^2} F_k H^{ik} - \\ &\quad - \frac{1}{c} \left(\frac{{}^*\partial E^i}{\partial t} + D E^i \right).\end{aligned}$$

Taking into account that

$$\frac{{}^*\partial E^i}{\partial x^i} + E^i \frac{{}^*\partial \ln \sqrt{h}}{\partial x^i} = {}^*\nabla_i E^i$$

is the chr.inv.-divergence of the vector E^i , and also that

$$\begin{aligned}\frac{{}^*\partial H^{ik}}{\partial x^k} + H^{ik} \frac{{}^*\partial \ln \sqrt{h}}{\partial x^k} - \frac{1}{c^2} F_k H^{ik} &= \\ &= {}^*\nabla_k H^{ik} - \frac{1}{c^2} F_k H^{ik} = {}^*\tilde{\nabla}_k H^{ik}\end{aligned}$$

is the physical chr.inv.-divergence of the tensor H^{ik} , we arrive at the final formulae for chr.inv.-projections of the absolute divergence $\nabla_\sigma F^{\sigma\alpha}$ of the antisymmetric tensor $F^{\alpha\beta}$

$$\begin{aligned}T &= {}^*\nabla_i E^i - \frac{1}{c} H^{ik} A_{ik}, \\ B^i &= {}^*\tilde{\nabla}_k H^{ik} - \frac{1}{c} \left(\frac{{}^*\partial E^i}{\partial t} + D E^i \right).\end{aligned}$$

Calculate the chr.inv.-projections of the absolute divergence $\nabla_\sigma F^{*\sigma\alpha}$ of the pseudotensor $F^{*\alpha\beta}$ dual to the antisymmetric tensor $F^{\alpha\beta}$. For such a dual pseudotensor we have

$$F^{*\alpha\beta} = \frac{1}{2} E^{\alpha\beta\mu\nu} F_{\mu\nu}, \quad F_{*\alpha\beta} = \frac{1}{2} E_{\alpha\beta\mu\nu} F^{\mu\nu}.$$

Denoting its chr.inv.-projections as

$$H^{*i} = \frac{F_{0\cdot}^{*i}}{\sqrt{g_{00}}}, \quad E^{*ik} = F^{*ik},$$

we see that the obvious relations $H^{*i} \sim H^{ik}$ and $E^{*ik} \sim E^i$ exist between the chr.inv.-projections of the antisymmetric tensor $F^{\alpha\beta}$ and the pseudotensor $F^{*\alpha\beta}$, which are due to the duality of these tensors to each other.

As a result of these relations, given that

$$\frac{F_{0\cdot}^{*i}}{\sqrt{g_{00}}} = \frac{1}{2} \varepsilon^{ipq} H_{pq}, \quad F^{*ik} = -\varepsilon^{ikp} E_p,$$

the remaining components of the pseudotensor $F^{*\alpha\beta}$, formulated with the chr.inv.-projections of its dual tensor $F^{\alpha\beta}$ have

the following form

$$\begin{aligned}
F_{0\cdot}^{*\cdot 0} &= \frac{1}{2c} v_k \varepsilon^{kpq} \left[H_{pq} + \frac{1}{c} (v_p E_q - v_q E_p) \right], \\
F_{i\cdot}^{*\cdot 0} &= \frac{1}{2\sqrt{g_{00}}} \left[\varepsilon_{i\cdot}^{pq} H_{pq} + \frac{1}{c} \varepsilon_{i\cdot}^{pq} (v_p E_q - v_q E_p) - \right. \\
&\quad \left. - \frac{1}{c^2} \varepsilon^{kpq} v_i v_k H_{pq} - \frac{1}{c^3} \varepsilon^{kpq} v_i v_k (v_p E_q - v_q E_p) \right], \\
F^{*0i} &= \frac{1}{2\sqrt{g_{00}}} \varepsilon^{ipq} \left[H_{pq} + \frac{1}{c} (v_p E_q - v_q E_p) \right], \\
F_{*0i} &= \frac{1}{2} \sqrt{g_{00}} \varepsilon_{ipq} H^{pq}, \\
F_{i\cdot}^{*k} &= \varepsilon_{i\cdot}^{kp} E_p - \frac{1}{2c} v_i \varepsilon^{kpq} H_{pq} - \frac{1}{c^2} v_i v_m \varepsilon^{mkp} E_p, \\
F_{*ik} &= \varepsilon_{ikp} \left(E^p - \frac{1}{c} v_q H^{pq} \right),
\end{aligned}$$

while the square of the pseudotensor $F^{*\alpha\beta}$ has the form

$$F_{*\alpha\beta} F^{*\alpha\beta} = \varepsilon^{ipq} (E_p H_{iq} - E_i H_{pq}).$$

With the above components, after some algebra we obtain the chr.inv.-projections of the absolute divergence $\nabla_\sigma F^{*\sigma\alpha}$ of the dual pseudotensor $F^{*\alpha\beta}$ in detail

$$\begin{aligned}
\frac{\nabla_\sigma F_{0\cdot}^{*\cdot\sigma}}{\sqrt{g_{00}}} &= \frac{* \partial H^{*i}}{\partial x^i} + H^{*i} \frac{* \partial \ln \sqrt{h}}{\partial x^i} - \frac{1}{c} E^{*ik} A_{ik}, \\
\nabla_\sigma F^{*\sigma i} &= \frac{* \partial E^{*ik}}{\partial x^i} + E^{*ik} \frac{* \partial \ln \sqrt{h}}{\partial x^k} - \frac{1}{c^2} F_k E^{*ik} - \\
&\quad - \frac{1}{c} \left(\frac{* \partial H^{*i}}{\partial t} + D H^{*i} \right),
\end{aligned}$$

then, using the formulae for the chr.inv.-divergence $*\nabla_i H^{*i}$ and the physical chr.inv.-divergence $*\widetilde{\nabla}_k E^{*ik}$, we arrive at the final formulae for chr.inv.-projections of the absolute divergence $\nabla_\sigma F^{*\sigma\alpha}$ of the dual pseudotensor $F^{*\alpha\beta}$

$$\begin{aligned}
\frac{\nabla_\sigma F_{0\cdot}^{*\cdot\sigma}}{\sqrt{g_{00}}} &= *\nabla_i H^{*i} - \frac{1}{c} E^{*ik} A_{ik}, \\
\nabla_\sigma F^{*\sigma i} &= *\widetilde{\nabla}_k E^{*ik} - \frac{1}{c} \left(\frac{* \partial H^{*i}}{\partial t} + D H^{*i} \right).
\end{aligned}$$

Apart from the absolute divergence of vectors, antisymmetric tensors and pseudotensors of the 2nd rank, we need to deduce the chr.inv.-projections of the absolute divergence of a symmetric tensor of the 2nd rank (we need them to obtain the conservation law in chr.inv.-form).

Just as Zelmanov did, we denote the chr.inv.-projections of a symmetric tensor $T^{\alpha\beta}$ as

$$\frac{T_{00}}{g_{00}} = \rho, \quad \frac{T_{0i}}{\sqrt{g_{00}}} = K^i, \quad T^{ik} = N^{ik},$$

whence, following the same algebra as above, we obtain the chr.inv.-projections of the absolute divergence $\nabla_\sigma T^{\sigma\alpha}$ of the symmetric tensor $T^{\alpha\beta}$ in detail

$$\begin{aligned}
\frac{\nabla_\sigma T_0^\sigma}{\sqrt{g_{00}}} &= \frac{* \partial \rho}{\partial t} + \rho D + D_{ik} N^{ik} + c^* \nabla_i K^i - \frac{2}{c} F_i K^i, \\
\nabla_\sigma T^{\sigma i} &= c \frac{* \partial K^i}{\partial t} + c D K^i + 2c (D_k^i + A_k^i) K^k + \\
&\quad + c^2 * \nabla_k N^{ik} - F_k N^{ik} - \rho F^i.
\end{aligned}$$

In addition to the inner (scalar) product of a tensor with the absolute differentiation operator ∇ , which is the absolute divergence of this tensor field, there may also be a difference between the covariant derivatives of the field. This quantity is known as the *curl* of the field, because from a geometric point of view it is the vortex (rotation) of the field itself. The *absolute curl* is the curl of an n -dimensional tensor field in an n -dimensional space.

The curl of an arbitrary four-dimensional vector field A^α is a covariant antisymmetric tensor of the 2nd rank*

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu},$$

where $\nabla_\mu A_\nu$ is the absolute derivative of the A_α with respect to the coordinate x^μ

$$\nabla_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu} - \Gamma_{\nu\mu}^\sigma A_\sigma.$$

The curl contracted with the four-dimensional absolutely antisymmetric discriminant tensor $E^{\alpha\beta\mu\nu}$ is the pseudotensor

$$F^{*\alpha\beta} = E^{\alpha\beta\mu\nu} (\nabla_\mu A_\nu - \nabla_\nu A_\mu) = E^{\alpha\beta\mu\nu} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right).$$

In electrodynamics, the electromagnetic field tensor $F_{\mu\nu}$ (Maxwell's tensor) is the curl of the four-dimensional electromagnetic field potential A^α . Therefore, we need the formulae for the chr.inv.-projections of the four-dimensional curl $F_{\mu\nu}$ and its dual pseudotensor $F^{*\alpha\beta}$ expressed in terms of the chr.inv.-projections of the four-dimensional vector potential A^α that forms them.

After the same algebra as above, we obtain the chr.inv.-projections of the absolute curl $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ expressed in terms of the chr.inv.-projections φ and q^i of the vector A^α forming this curl

$$\begin{aligned}
\frac{F_{0\cdot}^{*i}}{\sqrt{g_{00}}} &= \frac{g^{i\alpha} F_{0\alpha}}{\sqrt{g_{00}}} = h^{ik} \left(\frac{* \partial \varphi}{\partial x^k} + \frac{1}{c} \frac{* \partial q_k}{\partial t} \right) - \frac{\varphi}{c^2} F^i, \\
F^{ik} &= g^{i\alpha} g^{k\beta} F_{\alpha\beta} = h^{im} h^{kn} \left(\frac{* \partial q_m}{\partial x^n} - \frac{* \partial q_n}{\partial x^m} \right) - \frac{2\varphi}{c} A^{ik}.
\end{aligned}$$

*Strictly speaking, a real geometric curl is not a tensor, but its dual pseudotensor. This is because the invariance with respect to reflection is necessary for any rotation. See §98 in the very good textbook *Riemannsche Geometrie und Tensoranalysis* [17] written by Peter Raschewski (1907–1983), the well-known expert in Riemannian geometry.

The remaining components of the curl $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ with taking into account that $F_{00} = F^{00} = 0$ just like for any antisymmetric tensor have the form

$$\begin{aligned} F_{0i} &= \left(1 - \frac{w}{c^2}\right) \left(\frac{\varphi}{c^2} F_i - \frac{* \partial \varphi}{\partial x^i} - \frac{1}{c} \frac{* \partial q_i}{\partial t} \right), \\ F_{ik} &= \frac{* \partial q_i}{\partial x^k} - \frac{* \partial q_k}{\partial x^i} + \frac{\varphi}{c} \left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i} \right) + \\ &\quad + \frac{1}{c} \left(v_i \frac{* \partial \varphi}{\partial x^k} - v_k \frac{* \partial \varphi}{\partial x^i} \right) + \frac{1}{c^2} \left(v_i \frac{* \partial q_k}{\partial t} - v_k \frac{* \partial q_i}{\partial t} \right), \\ F_{0\cdot}^{\cdot 0} &= -\frac{\varphi}{c^3} v_k F^k + \frac{1}{c} v^k \left(\frac{* \partial \varphi}{\partial x^k} + \frac{1}{c} \frac{* \partial q_k}{\partial t} \right), \\ F_{k\cdot}^{\cdot 0} &= -\frac{1}{\sqrt{g_{00}}} \left[\frac{\varphi}{c^2} F_k - \frac{* \partial \varphi}{\partial x^k} - \frac{1}{c} \frac{* \partial q_k}{\partial t} + \right. \\ &\quad + \frac{2\varphi}{c^2} v^m A_{mk} + \frac{1}{c^2} v_k v^m \left(\frac{* \partial \varphi}{\partial x^m} + \frac{1}{c} \frac{* \partial q_m}{\partial t} \right) - \\ &\quad \left. - \frac{1}{c} v^m \left(\frac{* \partial q_m}{\partial x^k} - \frac{* \partial q_k}{\partial x^m} \right) - \frac{\varphi}{c^4} v_k v_m F^m \right], \\ F_{k\cdot}^{\cdot i} &= h^{im} \left(\frac{* \partial q_m}{\partial x^k} - \frac{* \partial q_k}{\partial x^m} \right) - \frac{1}{c} h^{im} v_k \frac{* \partial \varphi}{\partial x^m} - \\ &\quad - \frac{1}{c^2} h^{im} v_k \frac{* \partial q_m}{\partial t} + \frac{\varphi}{c^3} v_k F^i + \frac{2\varphi}{c} A_{k\cdot}^{\cdot i}, \\ F^{0k} &= \frac{1}{\sqrt{g_{00}}} \left[h^{km} \left(\frac{* \partial \varphi}{\partial x^m} + \frac{1}{c} \frac{* \partial q_m}{\partial t} \right) - \frac{\varphi}{c^2} F^k + \right. \\ &\quad \left. + \frac{1}{c} v^n h^{mk} \left(\frac{* \partial q_n}{\partial x^m} - \frac{* \partial q_m}{\partial x^n} \right) - \frac{2\varphi}{c^2} v_m A^{mk} \right]. \end{aligned}$$

Respectively, the chr.inv.-projections of the dual pseudo-tensor $F^{*\alpha\beta}$ of the curl $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ have the form

$$\begin{aligned} \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} &= \frac{g_{0\alpha} F^{*\alpha i}}{\sqrt{g_{00}}} = \varepsilon^{ikm} \left[\frac{1}{2} \left(\frac{* \partial q_k}{\partial x^m} - \frac{* \partial q_m}{\partial x^k} \right) - \frac{\varphi}{c} A_{km} \right], \\ F^{*ik} &= \varepsilon^{ikm} \left(\frac{\varphi}{c^2} F_m - \frac{* \partial \varphi}{\partial x^m} - \frac{1}{c} \frac{* \partial q_m}{\partial t} \right), \end{aligned}$$

where $F_{0\cdot}^{*\cdot i} = g_{0\alpha} F^{*\alpha i} = g_{0\alpha} E^{\alpha i \mu \nu} F_{\mu \nu}$ is calculated using the above components of the curl $F_{\mu \nu}$.

Laplace's operator known also as Laplacian is the three-dimensional derivation operator

$$\Delta = \nabla \nabla = \nabla^2 = -g^{ik} \nabla_i \nabla_k.$$

The four-dimensional generalization of Laplace's operator in a pseudo-Riemannian space is *d'Alembert's operator* known also as *d'Alembertian*

$$\square = g^{\alpha\beta} \nabla_\alpha \nabla_\beta.$$

Let us apply d'Alembert's operator to a scalar field and a vector field in the four-dimensional pseudo-Riemannian space (the space-time of General Relativity), and then express the calculation results in chr.inv.-form.

First we apply d'Alembert's operator to a scalar field φ

$$\square \varphi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \varphi = g^{\alpha\beta} \frac{\partial \varphi}{\partial x^\alpha} \left(\frac{\partial \varphi}{\partial x^\beta} \right) = g^{\alpha\beta} \frac{\partial^2 \varphi}{\partial x^\alpha \partial x^\beta},$$

because in this case the calculation is much simpler: the absolute derivative of a scalar, $\nabla_\alpha \varphi$, does not contain the Christoffel symbols, so it becomes the regular derivative.

We express the components of the fundamental metric tensor in terms of chronometric invariants. For g^{ik} we have $g^{ik} = -h^{ik}$ (see page 5). The components g^{0i} are obtained from the formula for the linear velocity of rotation of the observer's space $v^i = -c g^{0i} \sqrt{g_{00}}$ (see page 7)

$$g^{0i} = -\frac{1}{c \sqrt{g_{00}}} v^i.$$

The component g^{00} is obtained from the main property of the fundamental metric tensor $g_{\alpha\sigma} g^{\beta\sigma} = g_\alpha^\beta$. Setting up $\alpha = \beta = 0$ in the mentioned property, we obtain

$$g_{0\sigma} g^{0\sigma} = g_{00} g^{00} + g_{0i} g^{0i} = \delta_0^0 = 1,$$

whence, taking into account that

$$g_{00} = \left(1 - \frac{w}{c^2}\right)^2, \quad g_{0i} = -\frac{1}{c} v_i \left(1 - \frac{w}{c^2}\right),$$

we obtain the formula

$$g^{00} = \frac{1}{\left(1 - \frac{w}{c^2}\right)^2} \left(1 - \frac{1}{c^2} v_i v^i\right), \quad v_i v^i = h_{ik} v^i v^k = v^2.$$

Substituting the obtained formulae for g^{00} , g^{0i} and g^{ik} into the above general formula for $\square \varphi$ and then replacing the regular derivation operators with the chr.inv.-derivation operators, we obtain the d'Alembertian of the scalar field φ in chr.inv.-form

$$\square \varphi = \frac{1}{c^2} \frac{* \partial^2 \varphi}{\partial t^2} - h^{ik} \frac{* \partial^2 \varphi}{\partial x^i \partial x^k} = * \square \varphi,$$

where $* \square$ is the chr.inv.-d'Alembert operator, and $* \Delta$ is the chr.inv.-Laplace operator

$$* \square = \frac{1}{c^2} \frac{* \partial^2}{\partial t^2} - h^{ik} \frac{* \partial^2}{\partial x^i \partial x^k} = \frac{1}{c^2} \frac{* \partial^2}{\partial t^2} - * \Delta,$$

$$* \Delta = h^{ik} \frac{* \partial^2}{\partial x^i \partial x^k} = -g^{ik} * \nabla_i * \nabla_k.$$

Now, we apply d'Alembert's operator to an arbitrary four-dimensional vector field A^α

$$\square A^\alpha = g^{\mu\nu} \nabla_\mu \nabla_\nu A^\alpha.$$

Because $\square A^\alpha$ is a four-dimensional vector, the chr.inv.-projections of it are

$$T = b_\sigma \square A^\sigma = b_\sigma g^{\mu\nu} \nabla_\mu \nabla_\nu A^\sigma,$$

$$B^i = h_\sigma^i \square A^\sigma = h_\sigma^i g^{\mu\nu} \nabla_\mu \nabla_\nu A^\sigma.$$

It should be noted that the derivation of the d'Alembertian of a vector field in a Riemannian space is not a trivial task. This is because in this case, the Christoffel symbols are not zeroes and, therefore, the formulae for the chr.inv.-projections of the second derivatives take many pages*.

So, after some difficult algebra we had obtained formulae for the chr.inv.-projections of the d'Alembertian of the vector field A^α in the four-dimensional pseudo-Riemannian space. They have the following form†

$$T = {}^*\square\varphi - \frac{1}{c^3} \frac{\partial}{\partial t} (F_k q^k) - \frac{1}{c^3} F_i \frac{\partial q^i}{\partial t} + \frac{1}{c^2} F^i \frac{\partial \varphi}{\partial x^i} +$$

$$+ h^{ik} \Delta_{ik}^m \frac{\partial \varphi}{\partial x^m} - h^{ik} \frac{1}{c} \frac{\partial}{\partial x^i} [(D_{kn} + A_{kn}) q^n] + \frac{D}{c^2} \frac{\partial \varphi}{\partial t} -$$

$$- \frac{1}{c} D_m^k \frac{\partial q^m}{\partial x^k} + \frac{2}{c^3} A_{ik} F^i q^k + \frac{\varphi}{c^4} F_i F^i - \frac{\varphi}{c^2} D_{mk} D^{mk} -$$

$$- \frac{D}{c^3} F_m q^m - \frac{1}{c} \Delta_{kn}^m D_m^k q^n + \frac{1}{c} h^{ik} \Delta_{ik}^m (D_{mn} + A_{mn}) q^n,$$

$$B^i = {}^*\square A^i + \frac{1}{c^2} \frac{\partial}{\partial t} [(D_k^i + A_k^i) q^k] + \frac{D}{c^2} \frac{\partial q^i}{\partial t} +$$

$$+ \frac{1}{c^2} (D_k^i + A_k^i) \frac{\partial q^k}{\partial t} - \frac{1}{c^3} \frac{\partial}{\partial t} (\varphi F^i) - \frac{1}{c^3} F^i \frac{\partial \varphi}{\partial t} +$$

$$+ \frac{1}{c^2} F^k \frac{\partial q^i}{\partial x^k} - \frac{1}{c} (D^{mi} + A^{mi}) \frac{\partial \varphi}{\partial x^m} + \frac{1}{c^4} q^k F_k F^i +$$

$$+ \frac{1}{c^2} \Delta_{km}^i q^m F^k - \frac{\varphi}{c^3} D F^i + \frac{D}{c^2} (D_n^i + A_n^i) q^n -$$

$$- h^{km} \left\{ \frac{\partial}{\partial x^k} (\Delta_{mn}^i q^n) + \frac{1}{c} \frac{\partial}{\partial x^k} [\varphi (D_m^i + A_m^i)] + \right.$$

$$+ (\Delta_{kn}^i \Delta_{mp}^n - \Delta_{km}^n \Delta_{np}^i) q^p + \frac{\varphi}{c} [\Delta_{kn}^i (D_m^n + A_m^n) -$$

$$\left. - \Delta_{km}^n (D_n^i + A_n^i)] + \Delta_{kn}^i \frac{\partial q^n}{\partial x^m} - \Delta_{km}^n \frac{\partial q^i}{\partial x^n} \right\},$$

where ${}^*\square\varphi$ and ${}^*\square q^i$ are the result of applying the chr.inv.-d'Alembert operator to the quantities $\varphi = \frac{A_0}{\sqrt{g_{00}}}$ and $q^i = A^i$,

*This is one of the reasons why applications of the theory of electromagnetic fields are calculated in the Galilean reference frame in the Minkowski space (the space-time of Special Relativity), where the Christoffel symbols are zeroes. General covariant notation hardly allows unambiguous interpretation of calculation results, unless they are formulated with physical observable quantities (chronometric invariants) or demoted to a simple specific case like that in the Minkowski space, for instance.

†The above chr.inv.-projections of the d'Alembertian of a vector field in the four-dimensional pseudo-Riemannian space were deduced not by Zelmanov, but by one of us, L. Borisova, in the 1980s.

which are chr.inv.-projections (physically observable components) of the vector A^α

$${}^*\square\varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - h^{ik} \frac{\partial^2 \varphi}{\partial x^i \partial x^k},$$

$${}^*\square q^i = \frac{1}{c^2} \frac{\partial^2 q^i}{\partial t^2} - h^{km} \frac{\partial^2 q^i}{\partial x^k \partial x^m}.$$

The main criterion for correct calculations in such a complicated case as here is Zelmanov's rule of the chronometric invariance: "Correct calculations make all terms in the final equations chronometrically invariant quantities. That is to say, the final equations consist of the chr.inv.-quantities, their chr.inv.-derivatives, and also the chr.inv.-properties of the observer's reference space. If at least one error was made in the calculations, then some terms of the final equations will not be chronometric invariants."

In the Galilean reference frame in the Minkowski space (the space-time of Special Relativity), Laplace's and d'Alembert's operators take the simplified form

$$\Delta = \frac{\partial^2}{\partial x^1 \partial x^1} + \frac{\partial^2}{\partial x^2 \partial x^2} + \frac{\partial^2}{\partial x^3 \partial x^3},$$

$$\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^1 \partial x^1} - \frac{\partial^2}{\partial x^2 \partial x^2} - \frac{\partial^2}{\partial x^3 \partial x^3} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta.$$

D'Alembert's operator applied to a tensor field and equated to zero or not zero, gives the *d'Alembert equations* for this field. From a physical point of view, these are the equations of propagation of waves of the field. If the d'Alembertian of a field is not zero, these are the equations of propagation of the waves enforced by the sources that induce this field; they are called the *d'Alembert equations with sources*. For instance, the sources of electromagnetic fields are electric charges and currents. If the d'Alembertian of a field is zero, then these are the equations of propagation of waves in the field not related to any sources. If the space-time region under consideration, in addition to the tensor field, is filled with another medium, then the d'Alembert equations gain an additional term characterizing this medium (this term can be found using the equations which determine the medium).

These are the basics of tensor calculus expressed in terms of chronometric invariants.

Next we present formulae for the most common equations used in General Relativity, in the form expressed in terms of physical observables (chronometric invariants).

First, consider the equations of motion of a particle. A particle under the influence of gravitation only falls freely and thus travels along the shortest (*geodesic*) line. Such motion is called *free* or *geodesic motion*. If an additional non-gravitational force also acts on the particle, then the force deviates this particle from its geodesic trajectory, and the motion becomes *non-geodesic*.

From a geometric point of view, motion of a particle in the four-dimensional pseudo-Riemannian space (space-time)

is parallel transport of the four-dimensional vector Q^α , which is tangential to the particle's trajectory at any of its points and completely characterizes this particle. Therefore, the equations of motion of a particle actually determine the parallel transport of the particle's vector Q^α along the particle's four-dimensional trajectory and they are the equations of the absolute derivative of this vector with respect to a parameter ρ , which is non-zero along the trajectory

$$\frac{DQ^\alpha}{d\rho} = \frac{dQ^\alpha}{d\rho} + \Gamma_{\mu\nu}^\alpha Q^\mu \frac{dx^\nu}{d\rho},$$

where $DQ^\alpha = dQ^\alpha + \Gamma_{\mu\nu}^\alpha Q^\mu dx^\nu$ is the absolute differential of the transported vector Q^α (i.e., its absolute increment) along the trajectory.

If a particle travels along a geodesic trajectory (free motion), then the particle's characteristic vector is transported in Levi-Civita's sense: the square of the transported vector remains unchanged $Q_\alpha Q^\alpha = \text{const}$ along the trajectory, while the absolute derivative of the transported vector is zero and such equations are called the *equations of free motion*.

A mass-bearing particle (such particles travel along non-isotropic space-time trajectories) is characterized by its own four-dimensional momentum vector

$$P^\alpha = m_0 \frac{dx^\alpha}{ds}, \quad P_\alpha P^\alpha = m_0^2 = \text{const},$$

where m_0 is the particle's rest-mass. Respectively, the equations of motion of a free mass-bearing particle are

$$\frac{dP^\alpha}{ds} + \Gamma_{\mu\nu}^\alpha P^\mu \frac{dx^\nu}{ds} = 0.$$

A massless light-like particle (such particles travel along isotropic space-time trajectories) is characterized by its own four-dimensional wave vector

$$K^\alpha = \frac{\omega}{c} \frac{dx^\alpha}{d\sigma}, \quad K_\alpha K^\alpha = 0,$$

where ω is the characteristic frequency of the massless particle, and $d\sigma = h_{ik} dx^i dx^k$ is the three-dimensional chr.inv.-interval, which, since $ds^2 = c^2 d\tau^2 - d\sigma^2 = 0$ along isotropic trajectories, is invariant along them. Respectively, the equations of motion of a free massless (light-like) particle are

$$\frac{dK^\alpha}{d\sigma} + \Gamma_{\mu\nu}^\alpha K^\mu \frac{dx^\nu}{d\sigma} = 0.$$

The projections of the above four-dimensional equations of motion onto the time line and the three-dimensional spatial section of an observer are, respectively, the chr.inv.-equations of motion of a free mass-bearing particle

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k = 0,$$

$$\frac{d(mv^i)}{d\tau} + 2m(D_k^i + A_{k.}^i) v^k - mF^i + m\Delta_{nk}^i v^n v^k = 0,$$

and the chr.inv.-equations of motion of a free massless (light-like) particle

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_i c^i + \frac{\omega}{c^2} D_{ik} c^i c^k = 0,$$

$$\frac{d(\omega c^i)}{d\tau} + 2\omega(D_k^i + A_{k.}^i) c^k - \omega F^i + \omega\Delta_{nk}^i c^n c^k = 0,$$

where m is the relativistic mass of the travelling mass-bearing particle, ω is the characteristic frequency of the massless particle, $d\tau$ is the physically observable time interval, and v^i is the chr.inv.-vector of the physically observable velocity of the mass-bearing particle. Along isotropic trajectories (trajectories of light) the v^i transforms into the chr.inv.-vector of the physically observable velocity of light, the square of which is $c_i c^i = h_{ik} c^i c^k = c^2$ (see page 6).

If a particle travels along a non-geodesic trajectory, then $Q_\alpha Q^\alpha \neq \text{const}$, and the absolute derivative of the transported vector Q^α is equal to a force Φ^α that deviates the particle from a geodesic line. Such equations are called the *equations of non-geodesic motion* [5]. In this case, the right hand side of the above chr.inv.-equations of motion is different from zero and contains the respective chr.inv.-projections of the deviating force Φ^α .

The chr.inv.-equations of motion show how the observed motion of particles depends on the physically observable gravitational inertial force F^i , rotation A_{ik} , deformation D_{ik} and inhomogeneity (the coherence coefficients Δ_{kn}^i) of the observer's reference space.

Let us now turn to the basics of electrodynamics in the four-dimensional pseudo-Riemannian space.

The electromagnetic field tensor $F^{\mu\nu}$ is determined as the curl $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ of the four-dimensional electromagnetic field potential A^α . Following the terminology of electrodynamics, we call the chr.inv.-projections of the A^α (page 17) the chr.inv.-scalar potential φ and the chr.inv.-vector potential q^i of the electromagnetic field

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}, \quad q^i = A^i,$$

and the chr.inv.-projections of the electromagnetic field tensor $F^{\mu\nu}$ (page 20) — the chr.inv.-electric strength E^i and the chr.inv.-magnetic strength H^{ik} of the field

$$E^i = \frac{F_{0.}^i}{\sqrt{g_{00}}} = \frac{g^{i\alpha} F_{0\alpha}}{\sqrt{g_{00}}} = h^{ik} \left(\frac{\partial \varphi}{\partial x^k} + \frac{1}{c} \frac{\partial q_k}{\partial t} \right) - \frac{\varphi}{c^2} F^i,$$

$$H^{ik} = F^{ik} = g^{i\alpha} g^{k\beta} F_{\alpha\beta} = h^{im} h^{kn} \left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m} \right) - \frac{2\varphi}{c} A^{ik},$$

where their covariant (lower-index) versions are

$$E_i = h_{ik} E^k = \frac{\partial \varphi}{\partial x^i} + \frac{1}{c} \frac{\partial q_i}{\partial t} - \frac{\varphi}{c^2} F_i,$$

$$H_{ik} = h_{im} h_{kn} H^{mn} = \frac{\partial q_i}{\partial x^k} - \frac{\partial q_k}{\partial x^i} - \frac{2\varphi}{c} A_{ik},$$

and the mixed components $H_k^m = -H_k^m$ are obtained from H^{ik} using the metric chr.inv.-tensor h_{ik} , i.e., $H_k^m = h_{ki} H^{im}$.

Respectively, the electromagnetic field pseudotensor $F^{*\alpha\beta}$ dual to the field tensor, i.e., $F^{*\alpha\beta} = \frac{1}{2} E^{\alpha\beta\mu\nu} F_{\mu\nu}$, has the following chr.inv.-projections

$$H^{*i} = \frac{F^{*i}}{\sqrt{g_{00}}} = \frac{1}{2} \varepsilon^{imn} \left(\frac{* \partial q_m}{\partial x^n} - \frac{* \partial q_n}{\partial x^m} - \frac{2\varphi}{c} A_{mn} \right) = \frac{1}{2} \varepsilon^{imn} H_{mn},$$

$$E^{*ik} = F^{*ik} = \varepsilon^{ikn} \left(\frac{\varphi}{c^2} F_n - \frac{* \partial \varphi}{\partial x^n} - \frac{1}{c} \frac{* \partial q_n}{\partial t} \right) = -\varepsilon^{ikn} E_n,$$

which we call the chr.inv.-magnetic strength pseudovector H^{*i} and the chr.inv.-electric strength pseudotensor E^{*ik} . It is obvious that the quantities H^{*i} and H_{mn} are dually conjugate, and the quantities E^{*ik} and E_m are also dually conjugate.

The above formulae show that the observed electric and magnetic strengths of the electromagnetic field depend on the physically observable gravitational inertial force F^i and rotation A_{ik} of the observer's reference space.

So forth, the electromagnetic field invariants

$$J_1 = F_{\mu\nu} F^{\mu\nu} = H_{ik} H^{ik} - 2 E_i E^i = -2(E_i E^i - H_{*i} H^{*i}),$$

$$J_2 = F_{\mu\nu} F^{*\mu\nu} = \varepsilon^{imn} (E_m H_{in} - E_i H_{nm}) = -4 E_i H^{*i},$$

the first of which is a scalar, and the second is a pseudoscalar, have the following detailed chr.inv.-formulation

$$J_1 = 2 \left[h^{im} h^{kn} \left(\frac{* \partial q_i}{\partial x^k} - \frac{* \partial q_k}{\partial x^i} \right) \frac{* \partial q_m}{\partial x^n} - h^{ik} \frac{* \partial \varphi}{\partial x^i} \frac{* \partial \varphi}{\partial x^k} - \frac{2}{c} h^{ik} \frac{* \partial \varphi}{\partial x^i} \frac{* \partial q_k}{\partial t} - \frac{1}{c^2} h^{ik} \frac{* \partial q_i}{\partial t} \frac{* \partial q_k}{\partial t} + \frac{8\varphi}{c^2} \Omega_{*i} \Omega^{*i} - \frac{2\varphi}{c} \varepsilon^{imn} \Omega_{*m} \frac{* \partial q_i}{\partial x^n} + \frac{2\varphi}{c^2} \frac{* \partial \varphi}{\partial x^i} F^i + \frac{2\varphi}{c^3} \frac{* \partial q_i}{\partial t} F^i - \frac{\varphi}{c^4} F_i F^i \right],$$

$$J_2 = \frac{1}{2} \left[\varepsilon^{imn} \left(\frac{* \partial q_m}{\partial x^n} - \frac{* \partial q_n}{\partial x^m} \right) - \frac{4\varphi}{c} \Omega^{*i} \right] \times \left(\frac{* \partial \varphi}{\partial x^i} + \frac{1}{c} \frac{* \partial q_i}{\partial t} - \frac{\varphi}{c^2} F_i \right).$$

Mathematically, any electromagnetic field in the four-dimensional pseudo-Riemannian space is completely characterized by a system of 10 equations in 10 unknowns. First, this system includes Maxwell's equations

$$\nabla_\sigma F^{\mu\sigma} = \frac{4\pi}{c} j^\mu, \quad \nabla_\sigma F^{*\mu\sigma} = 0,$$

the chr.inv.-projections of which give two groups of equations, which we call the chr.inv.-Maxwell equations* and which

*The chr.inv.-Maxwell equations were first deduced in the late 1960s independently by Nikolai Pavlov and José del Prado (unpublished). Zelmanov asked these students to do it as homework. These equations are deduced on the basis of the chr.inv.-projections of the absolute divergence of a 2nd rank antisymmetric tensor (page 19), as well as the chr.inv.-projections of the absolute divergence of its dual pseudotensor (page 20).

have the following form

$$\left. \begin{aligned} * \nabla_i E^i - \frac{1}{c} H^{ik} A_{ik} &= 4\pi\rho \\ * \nabla_k H^{ik} - \frac{1}{c^2} F_k H^{ik} - \frac{1}{c} \left(\frac{* \partial E^i}{\partial t} + D E^i \right) &= \frac{4\pi}{c} j^i \end{aligned} \right\} \text{I},$$

$$\left. \begin{aligned} * \nabla_i H^{*i} - \frac{1}{c} E^{*ik} A_{ik} &= 0 \\ * \nabla_k E^{*ik} - \frac{1}{c^2} F_k E^{*ik} - \frac{1}{c} \left(\frac{* \partial H^{*i}}{\partial t} + D H^{*i} \right) &= 0 \end{aligned} \right\} \text{II},$$

or, in another notation

$$\left. \begin{aligned} * \nabla_i E^i - \frac{2}{c} \Omega_{*m} H^{*m} &= 4\pi\rho \\ \varepsilon^{ikm} * \widetilde{\nabla}_k (H_{*m} \sqrt{h}) - \frac{1}{c} \frac{* \partial}{\partial t} (E^i \sqrt{h}) &= \frac{4\pi}{c} j^i \sqrt{h} \end{aligned} \right\} \text{I},$$

$$\left. \begin{aligned} * \nabla_i H^{*i} + \frac{2}{c} \Omega_{*m} E^m &= 0 \\ \varepsilon^{ikm} * \widetilde{\nabla}_k (E_m \sqrt{h}) + \frac{1}{c} \frac{* \partial}{\partial t} (H^{*i} \sqrt{h}) &= 0 \end{aligned} \right\} \text{II}.$$

These are 8 equations in 10 unknowns, which are 3 components of the chr.inv.-electric strengths E^i , 3 components of the chr.inv.-magnetic strength H^{*i} , 1 component of the electric charge density ρ and 3 components of the chr.inv.-current density vector j^i . The latter two, known as the electromagnetic field sources, are the chr.inv.-projections

$$\rho = \frac{1}{c} b^\alpha j_\alpha = \frac{1}{c} \frac{j_0}{\sqrt{g_{00}}}, \quad j^i = h_\alpha^i j^\alpha$$

of the four-dimensional current vector j^α of the electromagnetic field (also known as the shift current).

The first equation of Group I is the Biot-Savart law, the second is Gauss' theorem, both in chr.inv.-notation. The first and second equations of Group II represent a chr.inv.-notation of Faraday's law of electromagnetic induction and the conditions for the absence of magnetic charges, respectively.

In particular, the 1st equation in Group II shows that, if the observer's reference space does not rotate, then $* \nabla_i H^{*i} = 0$ (the magnetic field is homogeneous), while the electric field is not, $* \nabla_i E^i = 4\pi\rho$ (the 1st equation in Group I). Therefore, a "magnetic charge", if it really exists, is directly connected with the rotation of space itself.

The 9th equation of the equation system mentioned above is Lorentz' condition

$$\nabla_\sigma A^\sigma = 0,$$

which is the conservation condition for the four-dimensional electromagnetic field potential A^α . The 10th equation that makes this system definite (the number of equations in this system must be the same as the number of unknowns), is the

law of conservation of electric charge (known also as the continuity equation)

$$\nabla_\sigma j^\sigma = 0,$$

which is the mathematical notation of the fact that electric charge cannot be destroyed, but merely redistributed between the charged bodies in contact.

Using the chr.inv.-formula for the divergence of an arbitrary vector field (see page 18), we obtain the Lorentz condition and the continuity condition in chr.inv.-form

$$\begin{aligned} \frac{1}{c} \frac{\partial \varphi}{\partial t} + \frac{\varphi}{c} D + {}^* \nabla_i q^i - \frac{1}{c^2} F_i q^i &= 0, \\ \frac{\partial \rho}{\partial t} + \rho D + {}^* \nabla_i j^i - \frac{1}{c^2} F_i j^i &= 0, \end{aligned}$$

or, replacing the regular chr.inv.-divergence with the physical chr.inv.-divergence (see page 18), we finally have

$$\begin{aligned} \frac{1}{c} \frac{\partial \varphi}{\partial t} + \frac{\varphi}{c} D + {}^* \widetilde{\nabla}_i q^i &= 0, \\ \frac{\partial \rho}{\partial t} + \rho D + {}^* \widetilde{\nabla}_i j^i &= 0. \end{aligned}$$

With the above chr.inv.-Lorentz condition and the chr.inv.-continuity equation, the mentioned system of 10 equations that completely characterizes any electromagnetic field in the four-dimensional pseudo-Riemannian space is complete.

Now consider the energy-momentum tensor of an electromagnetic field. It has the form

$$T^{\mu\nu} = \frac{1}{4\pi} \left(-F^{\mu\sigma} F^\nu{}_\sigma + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right).$$

This tensor is symmetric: $T^{\mu\nu} = T^{\nu\mu}$. For this reason, its chr.inv.-projections are calculated as for any symmetric tensor of the 2nd rank (see page 6)

$$q = \frac{T_{00}}{g_{00}}, \quad J^i = \frac{c T_0^i}{\sqrt{g_{00}}}, \quad U^{ik} = c^2 T^{ik}$$

and have the following form

$$\begin{aligned} q &= \frac{E^2 + H^{*2}}{8\pi}, \\ J^i &= \frac{c}{4\pi} \varepsilon^{ikm} E_k H_{*m}, \\ U^{ik} &= q c^2 h^{ik} - \frac{c^2}{4\pi} (E^i E^k + H^{*i} H^{*k}), \end{aligned}$$

where $E^2 = h_{ik} E^i E^k$ and $H^{*2} = h_{ik} H^{*i} H^{*k}$. These projections have the following physical sense: the scalar q is the physically observable energy density of the electromagnetic field, J^i is the physically observable density of the field momentum (the chr.inv.-Poynting vector), and U^{ik} is the physically

observable density of the field momentum flux (the chr.inv.-stress tensor).

Any electrically charged particle travelling in an electromagnetic field deviates from a geodesic trajectory due to the Lorentz force acting on its electric charge e from the electromagnetic field. The Lorentz force in the four-dimensional pseudo-Riemannian space has the form

$$\Phi^\alpha = \frac{e}{c} F^\alpha{}_\sigma U^\sigma, \quad U^\alpha = \frac{dx^\alpha}{ds},$$

where U^α is the four-dimensional velocity of the charged particle. Respectively, the four-dimensional equations of motion of a charged particle in an electromagnetic field (determined by the electromagnetic field tensor $F_{\alpha\beta}$) have the form

$$\frac{dP^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} P^\mu U^\nu = \frac{e}{c^2} F^\alpha{}_\beta U^\beta,$$

and their chr.inv.-projections

$$\begin{aligned} \frac{dE}{d\tau} - m F_i v^i + m D_{ik} v^i v^k &= -e E_i v^i, \\ \frac{d(m v^i)}{d\tau} - m F^i + 2m (D_k^i + A_k^i) v^k + m \Delta_{nk}^i v^n v^k &= \\ &= -e \left(E^i + \frac{1}{c} \varepsilon^{ikm} v_k H_{*m} \right) \end{aligned}$$

are the chr.inv.-equations of motion of the charged particle. Here, $E = mc^2$ is the relativistic energy of the particle, so the first (scalar) equation is the theorem of live forces represented in chr.inv.-form.

The above chr.inv.-equations of motion show how the observed motion of charged particles is affected by the physically observable gravitational inertial force F^i , rotation A_{ik} , deformation D_{ik} and inhomogeneity Δ_{kn}^i of the observer's reference space.

Zelmanov had also introduced the chr.inv.-curvature tensor. It is deduced similarly to the Riemann-Christoffel tensor from the non-commutativity of the 2nd chr.inv.-derivatives of an arbitrary vector

$${}^* \nabla_i {}^* \nabla_k Q_l - {}^* \nabla_k {}^* \nabla_i Q_l = \frac{2A_{ik}}{c^2} \frac{\partial Q_l}{\partial t} + H_{lki}^{...j} Q_j,$$

where the 4th rank chr.inv.-tensor

$$H_{lki}^{...j} = \frac{\partial \Delta_{il}^j}{\partial x^k} - \frac{\partial \Delta_{kl}^j}{\partial x^i} + \Delta_{il}^m \Delta_{km}^j - \Delta_{kl}^m \Delta_{im}^j$$

is the basis for the chr.inv.-curvature tensor C_{lkij} , which has all properties of the Riemann-Christoffel tensor in the observer's three-dimensional spatial section, and its contraction gives the observable chr.inv.-scalar curvature C

$$\begin{aligned} C_{lkij} &= \frac{1}{4} (H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}), \\ C_{lk} &= C_{lki}^{...i}, \quad C = h^{lk} C_{lk}, \end{aligned}$$

where

$$H_{lkij} = C_{lkij} + \frac{1}{2} (2A_{ki} D_{jl} + A_{ij} D_{kl} + A_{jk} D_{il} + A_{kl} D_{ij} + A_{li} D_{jk}),$$

$$H_{lk} = C_{lk} + \frac{1}{2} (A_{kj} D_l^j + A_{lj} D_k^j + A_{kl} D),$$

$$H = h^{lk} H_{lk} = C.$$

The above formulae show that the observed curvature of a space depends on not only the gravitational inertial force acting in the local reference space of the observer, but also the rotation and deformation of his reference space, and, therefore, does not vanish in the absence of the gravitational field. If the space does not rotate, then we have $H_{lkij} = C_{lkij}$. This is as well true for H_{lk} and C_{lk} . In this particular case, the tensor $C_{lk} = h^{ij} C_{ilkj}$ has the form

$$C_{lk} = \frac{* \partial}{\partial x^k} \left(\frac{* \partial \ln \sqrt{h}}{\partial x^l} \right) - \frac{* \partial \Delta_{kl}^i}{\partial x^i} + \Delta_{il}^m \Delta_{km}^i - \Delta_{kl}^m \frac{* \partial \ln \sqrt{h}}{\partial x^m}.$$

Zelmanov had also deduced chr.inv.-projections for the Riemann-Christoffel curvature tensor

$$R_{jkl}^i = \frac{\partial \Gamma_{lj}^i}{\partial x^k} - \frac{\partial \Gamma_{kj}^i}{\partial x^l} + \Gamma_{kp}^i \Gamma_{lj}^p - \Gamma_{lp}^i \Gamma_{kj}^p.$$

The Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$ is symmetric with respect to transposition over a pair of its indices and antisymmetric within each pair of the indices. Therefore, it has three chr.inv.-projections as follows

$$X^{ik} = -c^2 \frac{R_{0 \cdot 0}^{i \cdot k}}{g_{00}}, \quad Y^{ijk} = -c \frac{R_{0 \dots}^{i j k}}{\sqrt{g_{00}}}, \quad Z^{ijkl} = c^2 R^{ijkl}.$$

Substituting the necessary components of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$ into these formulae and then lowering the indices, Zelmanov had obtained the chr.inv.-projections of the Riemann-Christoffel tensor in the form

$$X_{ij} = \frac{* \partial D_{ij}}{\partial t} - (D_i^l + A_i^l)(D_{jl} + A_{jl}) + (* \nabla_i F_j + * \nabla_j F_i) - \frac{1}{c^2} F_i F_j,$$

$$Y_{ijk} = * \nabla_i (D_{jk} + A_{jk}) - * \nabla_j (D_{ik} + A_{ik}) + \frac{2}{c^2} A_{ij} F_k,$$

$$Z_{iklj} = D_{ik} D_{lj} - D_{il} D_{kj} + A_{ik} A_{lj} - A_{il} A_{kj} + 2A_{ij} A_{kl} - c^2 C_{iklj},$$

where we have $Y_{(ijk)} = Y_{ijk} + Y_{jki} + Y_{kij} = 0$, as in the Riemann-Christoffel tensor. Contraction of the observable spatial projection Z_{iklj} step-by-step as $Z_{il} = h^{kj} Z_{iklj}$ and $Z = h^{il} Z_{il}$ gives

$$Z_{il} = D_{ik} D_l^k - D_{il} D + A_{ik} A_l^k + 2A_{ik} A_l^k - c^2 C_{il},$$

$$Z = h^{il} Z_{il} = D_{ik} D^{ik} - D^2 - A_{ik} A^{ik} - c^2 C.$$

Using the above, Zelmanov was able to deduce chr.inv.-projections for Einstein's field equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\kappa T_{\alpha\beta} + \lambda g_{\alpha\beta},$$

where he used $\kappa = \frac{8\pi G}{c^2}$ instead of $\kappa = \frac{8\pi G}{c^4}$ as used by Landau and Lifshitz in their *The Classical Theory of Fields* [8]. To understand the reason, consider the chr.inv.-projections of the energy-momentum tensor $T_{\alpha\beta}$ of a distributed matter, which are calculated according to the rule

$$\varrho = \frac{T_{00}}{g_{00}}, \quad J^i = \frac{c T_0^i}{\sqrt{g_{00}}}, \quad U^{ik} = c^2 T^{ik}$$

as for any symmetric tensor of the 2nd rank (see page 6). The scalar ϱ is the physically observable mass density of the distributed matter, J^i is its physically observable momentum density, and U^{ik} is its physically observable momentum flux density (stress-tensor). Ricci's tensor $R_{\alpha\beta}$ has the dimension $[\text{cm}^{-2}]$. This means that the scalar chr.inv.-projection of the field equations, $\frac{G_{00}}{g_{00}} = -\frac{\kappa T_{00}}{g_{00}} + \lambda$, as well as $\frac{\kappa T_{00}}{g_{00}} = \frac{8\pi G \varrho}{c^2}$ have the same dimension $[\text{cm}^{-2}]$. Hence, the energy-momentum tensor has the dimension of mass density $[\text{gram}/\text{cm}^3]$. Therefore, if we used $\kappa = \frac{8\pi G}{c^4}$ on the right hand side of the field equations, then we would not use the energy-momentum tensor $T_{\alpha\beta}$ itself, but $c^2 T_{\alpha\beta}$ as Landau and Lifshitz did.

Taking all the above into account, Zelmanov had obtained the chr.inv.-projections of Einstein's field equations. They are called the chr.inv.-Einstein equations and have the form

$$\frac{* \partial D}{\partial t} + D_{jl} D^{jl} + A_{jl} A^{lj} + * \nabla_j F^j - \frac{1}{c^2} F_j F^j = -\frac{\kappa}{2} (\varrho c^2 + U) + \lambda c^2,$$

$$* \nabla_j (h^{ij} D - D^{ij} - A^{ij}) + \frac{2}{c^2} F_j A^{ij} = \kappa J^i,$$

$$\begin{aligned} \frac{* \partial D_{ik}}{\partial t} - (D_{ij} + A_{ij})(D_k^j + A_k^j) + D D_{ik} + 3A_{ij} A_k^j - \\ - \frac{1}{c^2} F_i F_k + \frac{1}{2} (* \nabla_i F_k + * \nabla_k F_i) - c^2 C_{ik} = \\ = \frac{\kappa}{2} (\varrho c^2 h_{ik} + 2U_{ik} - U h_{ik}) + \lambda c^2 h_{ik}. \end{aligned}$$

In addition, the energy-momentum tensor $T_{\alpha\beta}$ of the distributed matter must satisfy the conservation law

$$\nabla_\sigma T^{\sigma\alpha} = 0,$$

the chr.inv.-projections of which are calculated as for the absolute divergence of any symmetric tensor of the 2nd rank (see page 20), and are chr.inv.-conservation law equations

$$\frac{* \partial \varrho}{\partial t} + D \varrho + \frac{1}{c^2} D_{ij} U^{ij} + * \widetilde{\nabla}_i J^i - \frac{1}{c^2} F_i J^i = 0,$$

$$\frac{* \partial J^k}{\partial t} + D J^k + 2(D_i^k + A_i^k) J^i + * \widetilde{\nabla}_i U^{ik} - \varrho F^k = 0.$$

So, we have presented here Zelmanov's mathematical apparatus of chronometric invariants, which are physical observables in General Relativity. This mathematical apparatus is given here in its entirety and in the form it was introduced by Zelmanov in 1944 (except for the chr.inv.-Maxwell equations, the chr.inv.-d'Alembert and chr.inv.-Laplace operators, which were deduced later). The above description of this mathematical apparatus contains all its foundations and definitions, tensor calculus in terms of chronometric invariants, as well as the most common equations used in General Relativity, which are also expressed in terms of chronometric invariants. All this is collected here in one article, which is very convenient. Even if we have missed some details, these details are not essential for understanding and working with this mathematical apparatus.

Zelmanov's mathematical apparatus was applied to many problems of General Relativity. In general, Zelmanov always said that he liked creating "mathematical tools" more than applying them. Nevertheless, his contribution to relativistic cosmology, as well as his calculation of the main effects of General Relativity and the basics of electrodynamics in terms of chronometric invariants, are significant. We also made a contribution: the list of our works, published in English and French, can be found just after the References*.

We recommend the present article to all those readers who would like to work independently in the field of General Relativity using the mathematical apparatus of chronometric invariants. Good luck!

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*It is necessary to say a word about the authorship of those articles in this list, which were published before 1991. It was the dark time of the communist dictatorship, when the personal contribution of a researcher, especially a woman, was neglected. Therefore, when L. Borisova submitted an article for publication through her superiors (because there was no other way to submit at that time), she could often find their names added to the submission. She was allowed to publish her articles only under her own name only after great troubles and a scandal. As one of the superiors publicly stated: "Science is a man's business. We will not allow this 'Einstein in a skirt' to be present in science." Even Zelmanov, who took custody of her from her student years, could not do anything against this suppression and lawlessness. As a result, those persons whose names can be found as "co-authors" in some of her publications before 1991 had nothing to do with her research: they, having an administrative power, simply added their names to her submissions. Their names must therefore be excluded from those published articles and forgotten (despite the fact that we mentioned them in the bibliography for this article). Fortunately, this dark era of our lives ended in 1991 after the collapse of the USSR and everything connected with it. All the mathematical problems that we considered in our works (from our student years to the present day) were posed and solved only by us, individually or together, but without any assistance or advice of a "supervisor" or another person.

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Applying Chronometric Invariants

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Fission with a Difference

G. S. Burra

Adj. Professor, University of Udine, 201, Via delle Scienze 33100, Udine, Italy. E-mail: gsburra7748@gmail.com

Invoking a model of an elementary particle as a collection of ultrarelativistic transient particles, we show that it is possible to recover the energy of the particle by bombarding it with monochromatic high-energy radiation.

Introduction

We consider the possibility of using an alternative route to releasing fission energy. This is prompted by some recent developments by the team of scientists Cruz-Chu *et al* [1], which leads to the technological development of practically monochromatic radiation in the X-ray region.

Let us start from a relativistic point of view, and the Lorentz transformation,

$$x = \gamma(x' - vt), \quad \gamma = (1 - v^2/c^2)^{-1/2}. \quad (1)$$

Indeed it is known that for a collection of relativistic particles, the various mass centres form a two-dimensional disc perpendicular to the angular momentum vector \vec{L} and with radius [3]

$$r = \frac{L}{mc}. \quad (2)$$

Further if the system has positive energies, then it must have an extension greater than r , while at distances of the order of r , we begin to encounter negative energies.

If we consider the system to be a particle of spin or angular momentum $L = \hbar/2$, then (2) gives $r = \hbar/2mc$. That is, we are in the Compton wavelength region. Another interesting feature which is the two dimensionality of the disc of mass centres.

On the other hand it is known that (cf. [4]), if a Dirac particle is represented by a Gaussian packet, then we begin to encounter negative energies precisely at the same Compton wavelength as above. Thus a particle can indeed be treated as a spherical shell of relativistic transient sub-constituents or “particlets”. Indeed, this is an alternative description of Dirac’s zitterbewegung or rapid oscillation.

The above picture is also reminiscent of Dirac’s shell or membrane model of the electron [5–7].

Outside this Compton region we have the usual space (or space time) of physics. But as we approach the Compton wavelength region we encounter a region where the space axis becomes as it were a complex plane. This has been described at length by the author, in terms of the Feschbach formalism [8] which leads to the double Weiner process. Consider the following system [9]

$$\begin{aligned} i\hbar \frac{\partial \phi}{\partial t} &= \frac{1}{2m} \left(\frac{\hbar}{i\nabla} - \frac{e\mathbf{A}}{c} \right)^2 (\phi + \chi) + (e\phi + mc^2)\phi \\ i\hbar \frac{\partial \phi}{\partial t} &= -\frac{1}{2m} \left(\frac{\hbar}{i\nabla} - \frac{e\mathbf{A}}{c} \right)^2 (\phi + \chi) + (e\phi - mc^2)\phi. \end{aligned} \quad (3)$$

The merit of this formalism is that it enables us to give a particle interpretation to the usual wave-formalism (see [8] for further details.) However the advantage of the Feschbach Villars formalism is that we can now work with an ostensible particle interpretation.

In any case, we encounter the Compton scale again and again. Wigner [10] pointed out its remarkable universality.

From the above it is apparent that if an elementary particle in the above characterisation is bombarded with very high frequency radiation of the order of the Compton frequency such a particle would break up and yield its energy. What happens in this case is that the Bell curve becomes so compressed that it will be like a straight line or spike, almost (see [12, 13]). This sharp spike would break up the elementary particle releasing its mass as energy.

It is well known in Quantum Mechanics that what may be called monochromatic waves are an idealization. This is in the sense that we have in general a wave packet made up of several frequencies [2]. But suppose we can single out a pure or nearly pure frequency? This is a technological problem. Let us start with the Schrodinger equation [2]:

$$\frac{d^2\psi}{dx^2} + \frac{p^2}{\hbar^2}\psi = 0$$

where

$$p = \sqrt{2m[E - V(x)]}.$$

This leads to

$$\phi(x) \exp\left(\pm \frac{i}{\hbar} \int^x p(x) dx\right) \quad (4)$$

where $\phi(x)$ is the solution of the free equation, and we already have a wave packet over different values of p or effectively frequencies. However, if we have a wave function like $\psi' = e^{ikx-pt}$, such a wave would be an extreme idealization and at the same time would be monochromatic. Can we achieve this, is the question. There has been recently some progress in this direction thanks to the experiment of Cruz-Chu and co-workers [1] who have been able to conduct an experiment where single particle X-ray diffraction patterns could be analysed thanks to a machine learning algorithm.

Remarks

What happens in this case is, the Bell curve becomes so compressed that it will be like a straight line. This sharp spike

could break up the elementary particle releasing its mass as energy. Fortunately, in recent years there has been some progress in this direction [11–13]. Furthermore, it may be pointed out that a pure monochromatic signal would be useful in communications as well. This is because, effectively the bandwidth would increase [14]. Finally, we observe that, if we can break up quarkonium particles, we can extract even greater energy. There is one way of doing this: we know that with $g = 2$ factor, there is a sort of precession and, if we could radiate with resonant frequencies, the particle would break up. This could be a technological problem.

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Avoiding Negative Energies in Quantum Mechanics

G. G. Nyambuya

National University of Science and Technology, Faculty of Applied Sciences – Department of Applied Physics,
Fundamental Theoretical and Astrophysics Group, P. O. Box 939, Ascot, Bulawayo, Republic of Zimbabwe.
E-mail: physicist.ggn@gmail.com

Quantum mechanical observables are naturally assumed to be real. Herein, we depart from this traditional and seemingly natural assumption whereby we consider a Quantum Mechanics (QM) whose operators have corresponding complex eigenvalues. The motivation for this is that complex eigenvalues lead us directly to positive definite energy solutions, hence mass. The resulting QM is able to qualitatively explain in a coherent manner some physical phenomenon that are currently inexplicable from a QM whose operators have corresponding real eigenvalues – e.g. one is now able to explain the instability of particles, their localization, the observed matter-antimatter asymmetry and the supposed variation of fundamental natural constants amongst others. In addition to this, there is the difficulty in Dirac's interpretation of negative energy states appearing in his theory. While Dirac's negative energy problem is not considered a problem anymore, we provide an alternative way out of this problem. We propose that eigenvalues corresponding to quantum mechanical operators associated with physical observables ought to be complex. From this seemingly simple hypothesis, we demonstrate that negative energy states leading to negative mass can be avoided altogether.

I cannot imagine a reasonable Unified Theory containing an explicit number which the whim of the Creator could just as easily have chosen differently ... Numbers arbitrarily chosen by God do not exist. Their alleged existence relies on our incomplete understanding [of the Laws of Nature and how God designed and fashioned the Universe].

Albert Einstein (1879-1955)

1 Introduction

Looking back – thus far, one can most confidently and safely say that the time period of the first thirty years of the twentieth century was perhaps a special time in the intellectual discourse of humanity with this period being a period of the greatest intellectual leaps in all the history of human thought and intellectual endeavour. For to date, these great intellectual leaps have found no equal. Perhaps, apart from CERN's famous 4th of July 2012 announcement that a strong signal mimicking a *Higgs-like* particle has been detected in the LHC data, it appears as though real progress in Physics has hit a serious brick wall. In all probability, it ought to be said that there has not been any real noteworthy and new exciting discoveries this century as those witnessed at the beginning of the twentieth century, especially on the *frontiers of fundamental theoretical Physics*.

Take for example: in 1905, Germany's youthful 26 years' old third class Swiss patent clerk Albert Einstein (1879-1955) [1] discovered the Special Theory of Relativity (STR), and shortly thereafter, in the period 1923-4, France's aristocrat and physicist Louis Victor Pierre Raymond de Broglie (1892-1987) [2–5] opened *Pandora's Box* with his wave-particle duality hypothesis, Germany's great physicist Werner Karl

Heisenberg (1901-1976) [6] theoretically argued his uncertainty principle into existence and Austria's own theoretical physicist Erwin Rudolf Josef Alexander Schrödinger (1887-1961) [7, 8] discovered the key wave equation of Quantum Mechanics (QM) which now bears his name, *etc.*

Once QM was inceptioned in the mid-1920s, no sooner was it realised that there was a need to unite these two theories which stand to this day as a major part of the twin pillars of modern physics – i.e. the STR and QM. At the time of these great discoveries and revolutionary paradigm shifts, nobody yet knew how to make the two theories consistent with each other. In 1928 while QM was still in its nascence, the then little-known British preeminent Paul Adrien Maurice Dirac (1902-1984), who ranks as one of the greatest fundamental theoretical physicists of his time, then only 26 years' old, succeeded where others found it difficult. Dirac [9, 10] successfully unified Einstein [1]'s STR and de Broglie [2–5], Heisenberg [6] and Schrödinger [7, 8]'s QM.

Dirac [9, 10]'s unified theory was an unprecedented success, except for one detail: a quantum system could have either positive or negative energy. How can something have negative energy? For example, according to Einstein [1]'s mass-energy equivalence, the mass (m) of a particle is related to its energy (E) by the relation $m = E/c_0^2$ (where: $c_0 = 2.99792458 \times 10^8 \text{ m s}^{-1}$ is the speed of light in *vacuo*), such that negative energy entails negative mass. For all we know, the measure of the resistance to any change of the state of motion of a given substance is a measure of its mass. Further, mass was and is understood as a measure of the quantity of matter in a substance. From this understanding, what does negative mass mean?

According to Newton's first law of motion, since a posi-

tive mass quantum system has the property that it has the tendency to preserve its current state of motion in such a manner that it resists all efforts to change this state of motion, does it then mean that a negative mass quantum system will have the exact opposite properties, that is, have the property that it has the tendency not to preserve the particle's current state of motion in such a manner that it does not resist any efforts to change the particle's current state of motion but only engenders it? Such are some of the plausible questions that have puzzled those that have attempted to comprehend what negative mass might actually be or mean. What will happen when positive energy-mass matter comes in contact with negative energy-mass matter? Will they nullify upon contact? These are just some of the pertinent questions out of many plausible ones that come to mind.

Be that as it may, Dirac was an extraordinary brilliant man who sought beauty in his work. He did not think of the negative energy quantum systems implied by his equation in ordinary terms, but thought of them mathematically and quantum mechanically. The negative energy solutions first appeared in the Klein [11] and Gordon [12] theory (KG-theory) on whose shoulders the Dirac's theory stands. In order to get rid of these negative energies, some notable figures of the time suggested that these negative energy solutions must be discarded with the simple remark that "these solutions have no correspondence with physical and natural reality". To that, Dirac [9] replied:

One gets over the difficulty on the classical theory by arbitrarily excluding those solutions that have a negative E . One cannot do this in the QM, since in general a perturbation will cause transitions from states with E -positive to states with E -negative.

So, it would strongly appear that negative energy states were here to stay – at least in theory. They needed a satisfying physical explanation.

While Dirac's theory was met with both enthusiasm and scepticism (e.g. by physicists Werner Heisenberg, Wolfgang Ernst Pauli (1900-1958), Ernst Pascual Jordan (1902-1980), George Gamow (1904-1968), amongst others), the enthusiasm was on the latent power wielded by the equation, e.g. the equation solved the difficult contemporary problem of spin; and scepticism was with respect to the negative energy solutions. Against this scepticism, Dirac [13] further proposed that the *vacuo* was an unobservable infinite sea of negative energy states, such that all negative energy states were filled! This invisible sea of negative energy states became known as the Dirac Sea.

According to Pauli [14]'s *Exclusion Principle* (PEP) that forbids more than two fermions to be in the same quantum state, a Universe in which there exists a Dirac Sea would forbid the transitions of positive energy quantum states to transit into negative energy states thereby resulting in a Universe that has stable positive energy states. Transitions from states with E -positive to states with E -negative are forbidden because the

E -positive state once in the E -negative state is going to be in the same quantum state as the E -negative state thus violating the PEP, hence, forbidden by Nature. In this way, the Dirac [9, 10] theory was safe.

To further clarify Dirac's theory, the preeminent American physicist Richard Phillips Feynman (1924-1987) proposed that the negative energy states be interpreted as antiparticles: they move backwards in time such that, in a Universe where time moves in a forward direction, these quantum states would appear as positive energy states. This is the current *de facto* interpretation of antiparticles. Other than the negative energy problem, Dirac [9, 10]'s equation exhibits a perfect symmetry and this property of the equation has no correspondence nor bearing with physical and natural reality as we know it. Often, the theory has had to be patched [15, for example] in order to measure up to physical and natural reality. These patches often propose that the combined Charge (C) and Space or Parity (P) reversal symmetry (CP violation) must explain the apparent matter-antimatter asymmetry [16]. While CP violation has been observed [17–22, for example], it is yet to be verified by experiment as the mechanism responsible for the observed matter-antimatter asymmetry.

We must hasten to say that, while this work will touch on other subjects that we had not intended to cover, the original and sole aim of this work is twofold:

1. To demonstrate that Dirac's negative energy solutions can be eliminated altogether by resorting to particles endowed with Complex Energy and Momentum (CEM) wherein under this new proposal, the energy and momentum of the quantum system of concern is measured as the magnitude of these complex quantities.
2. To show that the resultant energy from the resulting complex energy and momentum does solve without any need for exogenous ideas, the matter-antimatter asymmetry problem that the Dirac theory has so far failed to solve and possibly the recent issue to do with the plausible variation of *Fundamental Natural Constants* (FNCs).

To achieve our desired objective, we adopt the working hypothesis, that in general, all quantum mechanical observables such as the energy and momentum of quantum mechanical systems can take complex values ($z = x + iy$) such that the resultant observable that we measure in the laboratory is the magnitude of this complex quantity in question (i.e. $|z| = \sqrt{x^2 + y^2} \geq 0$). This assumption is all that we shall require in our exploration. As a result, we shall formulate a new basis for the further development of a QM that allows for observables to take complex values and from thereon, proceed to show that the theory resulting from the CEM hypothesis not only provides a plausible and perdurable solution to Dirac's problem of negative energies, but that, it also provides a plausible solution to the matter-antimatter asymmetry problem which the bare Dirac theory is unable to solve by its own.

In closing this introductory section, we give the synopsis of the reminder of this paper. That is: in §2, we dis-

cuss the idea of complex quantum mechanical observables: we discuss how this idea may provide a perdurable solution to Dirac's negative energy problem. In §3, we write down the usual Dirac equation and thereafter proceed to incorporate into its structure the CEM hypothesis. In §4, we apply the idea of complex quantum mechanical observables to the notion of the variation of FNCs. In §5, we work out the symmetries of the new CEM-Dirac equation, and lastly, in §6 and §7, we give a general discussion and the conclusion drawn thereof, respectively.

2 Complex energy and momentum

The negative probabilities manifesting in the KG-theory are a result of the fact that the emergent quantum probability (P_Q) expression in this theory is directly proportional to the energy (E) of the quantum system in question – i.e. $P_Q \propto E$, the consequential meaning of which is that, for negative energy quantum systems, the corresponding quantum probability will be negative. From this very fact $P_Q \propto E$, Dirac hatched the idea that these negative energies appearing in the quantum probability of the KG-theory could be removed if a theory linear in the temporal and spatial derivatives were possible because a linear system of equations will always have one solution, a quadratic two, a cubic three, a quad four, etc.

Further on his effort to eliminate these meaningless negative probabilities, Dirac hoped that with his linear solution, he might also eliminate the negative energy solutions as well. Because of the pivotal constraint that he imposed on his theory, namely that when his equation is *squared* it must yield the quantum mechanical wave equation of the KG-theory, this directly translates to the fact that the energy solutions of Dirac's quantum systems would exactly be as those obtained in the KG-theory, thus leading back to the same problem of negative energies faced by the KG-theory. The only way to eliminate these supposedly *nagging* negative energy solutions would be to build a theory from an energy-momentum equation that only admits positive definite energy solutions from the outset. This is the approach that we take here. We make use of a property of complex numbers – namely that the magnitude of a complex number is always a positive definite quantity.

To that end, we postulate that every physical observable ($O \in \mathbb{C}$) shall be considered to have two parts to it, namely: the real part ($O_R \in \mathbb{R}$), and the imaginary part ($O_I \in \mathbb{R}$), that is to say:

$$O = O_R + i O_I. \quad (1)$$

The subscripts R and I in (1) are used to label the real and imaginary parts of the complex physical quantity in question. For example, if the energy of a quantum system were complex, then $E = E_R + i E_I$, where E_R and E_I are the real and imaginary parts of the energy respectively. The imaginary part of the energy may lead to the possibility of naturally explaining the phenomenon of particle decay. In the case of

momentum, $\vec{p} = \vec{p}_R + i \vec{p}_I$, where \vec{p}_R and \vec{p}_I are the real and imaginary parts of the momentum respectively. Likewise, the imaginary part of the momentum may very well lead one to be able to naturally explain why particles are localised. These are interesting issues that can be tackled in a separate paper in the future.

Once the energy and momentum are complex physical variables, the rest-mass m_0 cannot be spared – i.e. $m_0 = m_R + i m_I$, where $(m_R, m_I) \in \mathbb{R}$. In summary:

$$E = E_R + i E_I, \quad (2a)$$

$$\vec{p} = \vec{p}_R + i \vec{p}_I, \quad (2b)$$

$$m_0 = m_R + i m_I. \quad (2c)$$

What (2) implies is that the four momentum p_μ , will have two parts to it – with one part that is associated with the real part and the other with the imaginary part, i.e.

$$\begin{aligned} p_\mu &= \left(\vec{p}, \frac{i E}{c_0^2} \right) \\ &= \left(\vec{p}_R, \frac{i E_R}{c_0^2} \right) + i \left(\vec{p}_I, \frac{i E_I}{c_0^2} \right) \\ &= p_\mu^R + i p_\mu^I, \end{aligned} \quad (3)$$

where:

$$\vec{p}_R = p_1^R \vec{i} + p_2^R \vec{j} + p_3^R \vec{k}, \quad (4a)$$

$$\vec{p}_I = p_1^I \vec{i} + p_2^I \vec{j} + p_3^I \vec{k}. \quad (4b)$$

For p^μ , we will have $p^\mu = (\vec{p}, i E/c_0^2)^* = (\vec{p}^*, -i E^*/c_0^2)$, so that the relativistic invariant quantity $p^\mu p_\mu$ is now such that $p^\mu p_\mu = m_0^* m_0 c_0^2$, i.e.

$$|E|^2 - |\vec{p}|^2 c_0^2 = |m_0|^2 c_0^4, \quad (5)$$

where:

$$|E| = \sqrt{E^* E} = \sqrt{E_R^2 + E_I^2} \geq 0, \quad (6a)$$

$$|\vec{p}| = \sqrt{\vec{p}^* \cdot \vec{p}} = \sqrt{|\vec{p}_R|^2 + |\vec{p}_I|^2} \geq 0, \quad (6b)$$

$$|m_0| = \sqrt{m_0^* m_0} = \sqrt{m_R^2 + m_I^2} \geq 0, \quad (6c)$$

hence, when written in full, (5) is given by:

$$(E_R^2 + E_I^2) - (|\vec{p}_R|^2 + |\vec{p}_I|^2) c_0^2 = (m_R^2 + m_I^2) c_0^4. \quad (7)$$

While the energy and momentum of the quantum system are complex, what we measure as the energy, momentum and the rest-mass of the quantum system are the magnitudes of these complex quantities. These magnitudes can only take positive values. So from (5), we will have:

$$|E| = m c_0^2 = \sqrt{|\vec{p}|^2 c_0^2 + |m_0|^2 c_0^4} \geq 0. \quad (8)$$

In this way, we find a clever and clear mathematical fix to Dirac [13]'s long-standing issue of negative mass and energies as these are now positive definite (i.e. $|E| \geq 0$; $m = |E|/c_0^2 \geq 0$) as we would naturally expect. As a disclaimer, we must say that we are not saying that this is the scheme which *Nature* has chosen in order to solve this problem, but that this is a plausible solution which can be taken seriously. In the next section, we will show how this idea of complex observables can be applied to the supposed problem of temporal and spatial variation of *Fundamental Constants of Nature* (FNCs).

3 CEM-Dirac equation

What kind of a Dirac equation does one get from the CEM hypothesis? Before we can answer this important question, we write down, for completeness purposes, the usual Dirac equation that assumes real-valued physical observables. Written in Dirac [23]'s *Bra-Ket notation*, the Dirac equation is given by:

$$[i\hbar\gamma^\mu\partial_\mu - m_0c_0]|\psi\rangle = 0, \quad (9)$$

where:

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (10)$$

is a four-component wavefunction which can further be written as a composition of two spinors, the left-hand $|\psi_L\rangle$ and the right-hand $|\psi_R\rangle$ spinors respectively, i.e.:

$$|\psi\rangle = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad (11a)$$

$$|\psi_L\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}, \quad (11b)$$

$$|\psi_R\rangle = \begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix}, \quad (11c)$$

and

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \text{ and } \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (12)$$

are the 4×4 Dirac gamma matrices with I_2 and 0, being the 2×2 identity and null matrices respectively. Throughout this paper, the Greek indices will be understood to mean $\mu, \nu, \dots = 0, 1, 2, 3$; and lower case English alphabet indices: $i, j, k, \dots = 1, 2, 3$. The matrices σ^j are the three 2×2 Pauli [24] matrices and are given by:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (13a)$$

$$\sigma^2 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (13b)$$

$$\sigma^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (13c)$$

The Dirac equation admits free particle solutions of the form $\psi = ue^{-iS/\hbar}$, where $u = u(E, \vec{p})$, is a 4×1 component object and $S = p_\mu x^\mu = \vec{p} \cdot \vec{r} - Et \in \mathbb{R}$ is the phase of the quantum system in question. The Quantum Probability Amplitude (QPA) ρ of such a quantum system is such that $\rho = u^\dagger u$, and this QPA has no temporal nor spatial dependence. As we shall soon find out, for the CEM version of the Dirac equation, things are very different.

The phase of the CEM-Dirac quantum system is such that $S = S_R + iS_I$, where $S_R = p_\mu^R x^\mu \in \mathbb{R}$ and $S_I = p_\mu^I x^\mu \in \mathbb{R}$ are the real and imaginary parts of the phase of the quantum system in question. Another major difference is that the rest-mass will be a complex quantity as opposed to it being real as is the case with the original Dirac equation. As will be demonstrated in §5, this complex rest-mass leads to a Lorentz [25–27] invariant C , CP , CT , and CPT -violating equation. The QPA of a CEM-Dirac quantum system is such that $\rho = u^\dagger ue^{S_I/\hbar}$, and unlike the QPA of the normal Dirac particle, the QPA of a CEM-Dirac quantum system does have an explicit temporal and spatial dependence. It is this explicit temporal and spatial dependence that we strongly believe will lead to an explanation of why particles decay and why they appear to be localized. Like we said (in the text above), we are not in the present going to investigate this issue, but shall leave it for a future paper. This we have done so that we keep our focus on the paramount issue at hand.

In closing this section, we must say that what we have presented herein is what we have coined the *CEM-Dirac equation*. While the CEM-Dirac equation and the usual Dirac equation are identical in their symbols – i.e. the way we write these two equations down, the main difference between them is that the energy, momentum and rest-mass of the CEM-Dirac equation are complex physical variables while in the usual Dirac equation these are real physical variables. The real part of the energy and momentum (E_R, \vec{p}_R) can perhaps be understood as the four-momentum of the quantum system that we measure in the laboratory while the imaginary part (E_I, \vec{p}_I), can be understood as the energy responsible for the decay and localization of the particle in question. Once more, we shall reiterate that these are issues for a separate paper. In the next section, we shall apply the *CEM-hypothesis* to the contemporary issue of the supposed variation of FNCs.

4 Variation of fundamental physical constants

In this section, we show that the supposed variation of fundamental physical constants such as the dimensionless Fine Structure Constant (FSC) (or the Sommerfeld [28] constant) α_0 can be explained from the idea just laid down above – i.e. the idea of CEM eigenvalues. The FSC is given by:

$$\alpha_0 = \frac{e^2}{4\pi\epsilon_0\hbar c_0}. \quad (14)$$

Present measurements give $1/\alpha_0 = 137.035999084(21)$ (CODATA, 2018). If the FSC is varying, it could be any one,

or a combination, of the constituents making up this dimensionless quantity, namely e , ε_0 , \hbar , c_0 , or any one of the combination of these supposed constants.

The idea that fundamental constants may vary during the course of the Universe's evolution was first considered by the preeminent British physicists Edward Arthur Milne (1896–1950) and Dirac [29]. Independently, Milne [30] and Dirac [29] considered cosmological models which incorporated a time-variable gravitational constant G , thus setting the ball rolling for the serious theoretical consideration of FNCs. In the intervening years 1938 to about 1999, the idea that FNCs may vary over cosmological times had no backing from experimental philosophy, and because of this, the idea was considered as purely nothing more than an academic pursuit, speculation and curiosity with no bearing whatsoever to do with physical and natural reality. With Web *et al.* [31]'s ground breaking work, this position has since changed as further and strong evidence from observational experience suggesting a plausible variation of the supposedly sacrosanct constants of Nature has been put forward for serious consideration [32–36, for example]. The question is: *Is there a fundamental basis for this variation?* We think that the QM of complex eigenvalues might have something to say about this.

Without any doubt whatsoever, FNCs (e.g. e , ε_0 , \hbar , c_0 , etc) are observables since they cannot only be measured in the laboratory, but are intimately, intrinsically and inherently associated with quantum systems. With that having been said, it is clear that if a physical observable such as an FNC is a true constant of Nature, i.e. having no spatial nor temporal variation, then its total (and not partial) time derivative must vanish identically – i.e. $d\langle O \rangle / dt \equiv 0$. The total (and not partial) time derivative operator is given by:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}. \quad (15)$$

Applying this to the expectation value $\langle O \rangle = \langle \Psi | \hat{T} | \Psi \rangle$ of an arbitrary observable O , one gets:

$$i\hbar \frac{d\langle O \rangle}{dt} = \langle \Psi | [\hat{T}^\dagger, \hat{H}] | \Psi \rangle + \vec{v} \cdot \langle \Psi | [\hat{T}^\dagger, \vec{P}] | \Psi \rangle, \quad (16)$$

where:

$$[\hat{T}^\dagger, \hat{H}] = \hat{T}^\dagger \hat{H} - \hat{H}^\dagger \hat{T}, \quad (17a)$$

$$[\hat{T}^\dagger, \vec{P}] = \hat{T}^\dagger \vec{P} - \vec{P}^\dagger \hat{T}, \quad (17b)$$

and $\vec{P} = -i\hbar\vec{\nabla}$ is the quantum mechanical momentum operator and \vec{v} is the velocity of the quantum system under consideration. We must say that it is more appropriate to think of this velocity:

$$\vec{v} = \frac{\hbar}{m} \text{Im} \left(\frac{\Psi^\dagger \vec{\nabla} \Psi}{\Psi^\dagger \Psi} \right), \quad (18)$$

as the Bohmian [37–39] velocity* field of the quantum system in question and the possible justification for this has been provided in [40].

What (16) is telling us, is that if an observable is a true constant, that is, it does not vary neither with time nor space, then the operator corresponding to this observable must commute with both the Hamiltonian and the momentum operator – i.e. $[\hat{T}^\dagger, \hat{H}] = 0$, and $[\hat{T}^\dagger, \vec{P}] = 0$. Against the seemingly sacrosanct dictates of our current understanding of QM, the condition $[\hat{T}^\dagger, \hat{H}] = 0$ is here found not to be sufficient to guarantee that the observable O will be a truly conserved quantity and constant quantity throughout all of space and time. If for some reason we have that $[\hat{T}^\dagger, \hat{H}] \neq 0$, and $[\hat{T}^\dagger, \vec{P}] \neq 0$, then for an observable to be a true constant, the spatial variation ought to be compensated by the temporal variation and *vice-versa*, and this will be in accordance with (16) under the setting $d\langle O \rangle / dt = 0$.

At this point, in order for us to proceed, we need to evaluate (16) in terms of *observable* quantities, i.e. we need to compute $\langle \Psi | [\hat{T}^\dagger, \hat{H}] | \Psi \rangle$ and $\vec{v} \cdot \langle \Psi | [\hat{T}^\dagger, \vec{P}] | \Psi \rangle$. To that end, we know that:

$$\begin{aligned} \left| \hat{T} \frac{\partial \Psi}{\partial t} \right| &= \frac{1}{i\hbar} \left| \hat{T} i\hbar \frac{\partial \Psi}{\partial t} \right|, \\ &= \frac{1}{i\hbar} \left| \hat{T} E \Psi \right|, \\ &= -\frac{1}{i\hbar} E \left| \hat{T} \Psi \right|, \end{aligned} \quad (19)$$

and that:

$$\begin{aligned} \left| \hat{T} \frac{\partial \Psi}{\partial t} \right| &= -\frac{1}{i\hbar} \left| \hat{T} i\hbar \frac{\partial \Psi}{\partial t} \right|, \\ &= -\frac{1}{i\hbar} \left| \hat{T} E \Psi \right|, \\ &= -\frac{1}{i\hbar} E^* \left| \hat{T} \Psi \right|. \end{aligned} \quad (20)$$

Multiplying (19) from the left by $\langle \Psi |$ and (20) from the right by $|\Psi\rangle$ respectively, and thereafter adding the resulting equations, we will have:

$$\begin{aligned} \langle \Psi | [\hat{T}^\dagger, \hat{H}] | \Psi \rangle &= (E - E^*) \langle O \rangle, \\ &= 2i E_I \langle O \rangle, \\ &= i\hbar \frac{\partial \langle O \rangle}{\partial t}, \end{aligned} \quad (21)$$

hence:

$$\langle \Psi | [\hat{T}^\dagger, \hat{H}] | \Psi \rangle = -2i E_I \langle O \rangle. \quad (22)$$

*Im() is an operator which extracts the imaginary part of a complex quantity – i.e. if $z = x + iy$, then $\text{Im}(z) = y$.

Further, we know that:

$$\begin{aligned} |\widehat{\mathcal{T}}\vec{\nabla}\Psi\rangle &= -\frac{1}{i\hbar} |\widehat{\mathcal{T}}(-i\hbar)\vec{\nabla}\Psi\rangle, \\ &= -\frac{1}{i\hbar} |\widehat{\mathcal{T}}\vec{P}\Psi\rangle, \\ &= -\frac{1}{i\hbar} \vec{p} |\widehat{\mathcal{T}}\Psi\rangle, \end{aligned} \quad (23)$$

and:

$$\begin{aligned} \langle\widehat{\mathcal{T}}\vec{\nabla}\Psi| &= \frac{1}{i\hbar} \langle\widehat{\mathcal{T}}(-i\hbar)\vec{\nabla}\Psi|, \\ &= \frac{1}{i\hbar} \langle\widehat{\mathcal{T}}\vec{P}\Psi|, \\ &= \frac{1}{i\hbar} \vec{p}^* \langle\widehat{\mathcal{T}}\Psi|. \end{aligned} \quad (24)$$

Likewise, multiplying (23) from the left by $\langle\P|$ and (24) from the right by $|\Psi\rangle$ respectively, and thereafter adding the resulting equations, we will have:

$$\begin{aligned} \left\langle\P\left|\left[\widehat{\mathcal{T}}, \vec{P}^\dagger\right]\right|\Psi\right\rangle &= -(\vec{p} - \vec{p}^*)\langle O\rangle \\ &= -2i\vec{p}_I\langle O\rangle \\ &= -i\hbar\vec{\nabla}\langle O\rangle, \end{aligned} \quad (25)$$

hence:

$$\vec{v} \cdot \left\langle\P\left|\left[\widehat{\mathcal{T}}, \vec{P}^\dagger\right]\right|\Psi\right\rangle = -2i\vec{v} \cdot \vec{p}_I\langle O\rangle. \quad (26)$$

Now, inserting (22) and (26) into (16), we obtain:

$$i\hbar \frac{d\langle O\rangle}{dt} = -2i\hbar(E_I - \vec{v} \cdot \vec{p}_I)\langle O\rangle. \quad (27)$$

From this, it follows that a system will have all of its observables being constants *if-and-only-if*:

$$E_I - \vec{v} \cdot \vec{p}_I = 0. \quad (28)$$

In passing – out of curiosity, we need to point out an indelible fact of experience namely that (28) has a seductive and irresistible semblance with Bartoli [41, 42] and Maxwell [43]’s energy-momentum dispersion relation for Light – i.e. $E - c_0p = 0$. If any, what connection can one make of this (28) with the nature of the photon? At present, we can only exhibit our curiosity: that is, we shall leave it here and slate it for exploration in future papers.

Now, applying the above ideas to the case of the variation of the FSC and assuming the present Standard Big Bang Cosmology Model [44–46] which assumes co-moving coordinates [47–50], it would appear that this FSC variation ought to be temporal in nature, as logic dictates that it cannot be spatial since co-moving coordinates imply $\vec{v} \equiv 0$. This directly implies that those patches of the sky exhibiting different FSC-values ought to be of different ages! If the temporal homogeneity and isotropy of the Universe is to be preserved,

then the only way to explain the variation of the FSC across the night-sky is to drop the assumption of co-moving coordinates! We are not going to say anything further on this matter of the variation of the FSC and complex observables, as this is something that requires a dedicated piece of work of its own. All that we wanted, we have achieved, and this has been to demonstrate the latent power in the seemingly alien idea of complex quantum mechanical observables that we have here suggested. We shall now move to the next section, where we shall consider the symmetries of the CEM-Dirac equation.

5 Symmetries of the CEM-Dirac equation

Now, if the electromagnetic coupled CEM-Dirac equation $[i\hbar\gamma^\mu\mathcal{D}_\mu - m_0c_0]|\psi\rangle = 0$, with $m_0 \in \mathbb{C}$, is to be symmetric,

$$\text{i.e. } q \mapsto -q \Rightarrow [i\hbar\gamma^\mu\mathcal{D}_\mu^* - m_0c_0]|\psi\rangle = 0$$

under charge conjugation, then we need to show that there exists a set of mathematically legal operations that take this new charge conjugated equation $[i\hbar\gamma^\mu\mathcal{D}_\mu^* - m_0c_0]|\psi\rangle = 0$, back to the original CEM-Dirac equation – i.e. an equation without the **-operation* on the covariant derivative \mathcal{D}_μ . If we can find these legal mathematical operations, it would mean that the CEM-Dirac equation applies equally to particles as to antiparticles – hence, it is symmetric with respect matter and antimatter. On the contrary, if we fail to find the said legal mathematical operations, it invariably means that the CEM-Dirac equation is not symmetric under charge conjugation.

To that end, let us start our attempt by removing the **-operator* on the covariant derivative \mathcal{D}_μ in the equation $[i\hbar\gamma^\mu\mathcal{D}_\mu^* - m_0c_0]|\psi\rangle = 0$. We will do this by taking the complex conjugate throughout this equation. So doing, we obtain $[i\hbar\gamma^{\mu*}\mathcal{D}_\mu + m_0^*c_0]|\psi^*\rangle = 0$, and because $\gamma^5\gamma^0\gamma^{\mu*} = -\gamma^\mu\gamma^5\gamma^0$, we can, in this resulting equation, remove the complex conjugate operator acting on $\gamma^{\mu*}$ and this we can achieve by multiplying throughout the resultant equation by $\gamma^5\gamma^0$ and then making use of the fact that $\gamma^5\gamma^0\gamma^{\mu*} = -\gamma^\mu\gamma^5\gamma^0$. So doing, we will have:

$$[i\hbar\gamma^\mu\mathcal{D}_\mu - m_0^*c_0]|\psi_c\rangle = 0, \quad (29)$$

where $|\psi_c\rangle = \gamma^5\gamma^0|\psi^*\rangle$ is the wavefunction of the corresponding antiparticle. Clearly, if we have that $m_I \propto q$, or $m_I \propto q^n$, where $(n \in \mathbb{O}) = 3, 5, 7, \text{ etc}$, this would mean that the transformation $q \mapsto -q$, would also lead to:

$$m_I \mapsto -m_I \Rightarrow m_0^* \mapsto m_0, \quad (30)$$

and in this way, (29) would simultaneously transform to:

$$[i\hbar\gamma^\mu\mathcal{D}_\mu - m_0c_0]|\psi_c\rangle = 0, \quad (31)$$

thus making this CEM-Dirac equation (whose rest-mass $(m_0 \in \mathbb{C})$ is a complex quantity) symmetric under charge conjugation. Less for the fact that the wavefunction $|\psi\rangle$ has been replaced by the new wavefunction $|\psi_c\rangle$, (31) is the same

CEM-Dirac equation applicable to the particle counterpart. Since $|\psi_c\rangle$ represents the antiparticle, the original Dirac equation is said to be symmetric under charge conjugation. In Dirac [9, 10]'s original theory, m_0 is real, the meaning of which is that $m_I \equiv 0$, hence making this original Dirac [9, 10] equation symmetric under charge conjugation. In the new setting of the CEM-Dirac equation, if m_I is not related to the electrical charge of the particle as suggested in (29), then the CEM-Dirac equation (with $m_0 \in \mathbb{C}$) is going to be asymmetric with respect to charge conjugation. As the reader can verify for themselves, not only is the CEM-Dirac equation going to violate C symmetry, but also CP , CT , and CPT symmetries as well. The only preserved symmetries are the P , T and PT symmetries.

6 Discussion

We have shown herein that the issue to do with negative energies can be solved by way of making a proper choice of the energy and momentum eigenvalues of the energy and momentum operators, respectively. These eigenvalues need to be complex as opposed to them being real as is the case in the present formulation of QM. Once the energy and momentum eigenvalues are complex, the measurable values become the magnitude of the corresponding eigenvalues and these magnitudes are positive definite! In this way, the issues surrounding these negative energies vanish forthwith. What remains is whether or not this is the scheme which *Nature* has chosen in order to go round this problem. We are of the strong opinion that this may very well be the scheme *Nature* has chosen.

This issue of negative energies has similarities with negative probabilities. As already said in the main text, prior to the discovery of his equation, Dirac had hoped that the negative probabilities occurring in the KG-theory, if solved, would also solve, in his new anticipated theory, the issue of the negative energies as well. We now know that Dirac was wrong as his new anticipated theory, which has positive definite probabilities, also has these negative energies. We did show in [40] that the emergence of these negative probabilities in the KG-theory is a result of an improper choice of the quantum mechanical probability current density in the KG-theory. In the same vein, the emergence of negative energies in both the Dirac and the KG-theory is a result of an improper choice of the energy and momentum eigenvalues – they need to be complex as suggested therein.

While we have not explored the richness of the hypothesis of complex energy and momentum eigenvalues, we need to mention the latent power in this new way of thinking, namely that one may very well be able to explain the variation of FNCs using this idea. Apart from this, it should be possible, using the complex part of the energy and momentum, to explain why particles decay, as well as the localization of particles into a finite region of space. What we had wanted here is to show that Dirac's negative energies can be done away with,

once and for all!

7 Conclusion

The following conclusion is drawn on the *proviso* that the hypothesis of complex energy-momentum is acceptable:

1. The complex energy-momentum hypothesis when applied to both the Klein-Gordon and the Dirac theory, does solve the issue of negative energies. This problem ceases to exist as the energy of all particles now is positive definite.
2. Quantum mechanics as currently understood and constituted where *all* quantum mechanical operators are required to be hermitian so that the corresponding eigenvalues are real-valued, may have to be modified or reconsidered.
3. The long-standing issue of the asymmetry in the matter-antimatter constitution of the Universe can be explained by the C , CP , CT and CPT violation that arises from the complex energy-momentum hypothesis when applied to the Dirac equation.

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Novel Insights into Nonlocal Gravity

Elio B. Porcelli, Victo S. Filho

H4D Scientific Research Laboratory, 1100 B15 Av. Sargento Geraldo Santana, CEP: 04674-225, São Paulo, SP, Brazil.

E-mail: elioporcelli@h4dscientific.com

Our various experiments, analyses and theoretical models to describe anomalous phenomena related to so many diverse physical systems like superconductors, capacitors and others led us to consolidate the idea that all existing particles are in a preexisting state of quantum entanglement. Such a generality involving weight reduction of the devices leads us to inevitably infer a direct relationship between such a state and gravity, considering it as a nonlocal force. In this work, we intend to explore this issue of generality and then propose, on the basis of such a theoretical framework of generalized quantum entanglement state, to investigate in this alternative way other issues such as the small order of magnitude of the gravitational force in relation to other known local forces, as the electromagnetic one, explain why the gravitational force is attractive in the Universe and why particles and bodies are limited to the speed of light in the vacuum even interacting through instantaneous interactions. We also explore the issue of quantum interference in neutron experiments as being induced by nonlocal gravity.

1 Introduction

One of most important topics of research in physics relates to the nature of the gravitational field, mainly considering that the quantum mechanics cannot describe the physics in the macroscopic and even astronomical scale, in which the gravity force is the prominent one. In Quantum Field Theory, it is well known that the fundamental interactions of Nature in the nuclear and atomic scale are possible through gauge bosons. In the nuclear medium, gluons are the gauge bosons of the weak and strong interactions. Further, in our macroscopic world, the electromagnetic forces are dominant and the interactions between bodies are mediated by photons. However, despite the proposal of the graviton as the gauge boson for gravity, till now, no evidence of its existence has been found, so that it becomes hard to obtain a theoretical framework that encloses all the interactions in Nature and as consequence a unified theory of fields, although a series of alternative theories [1–4], interpretations [3, 4] and unification theories have been proposed in literature [5, 6].

Although such investigations are very hard to be successful, many physicists have tried alternative theoretical explanations for understanding the nature of gravity and beyond. As examples, we can cite

- the Emergent Gravity theory [7];
- the possibility of a fractal physical space-time [8];
- the existence of the coupling between it and electromagnetism or the hypothesis that considers gravity as derived from the electromagnetic interaction [9];
- the idea from which relevant information on the emergence of space is hidden at the quark / hadron level, by following the line of thought from which space is an attribute of matter [10], so that quantum properties of matter or the discretization of mass induces us to

believe in some form of quantization of space, with intrinsic consequences to gravity.

In this context, it is natural to suppose that quantum mechanisms could really be responsible for generating the gravitational force. The possibility that the collapse of the wave function in quantum mechanics is not merely a mathematical formalism but a real physical effect and ultimately connected to classical gravity has been discussed a long time ago since the proposal of the Diósi-Penrose model (DP) [11–14]. The idea was first conceived by Diósi in the study of the influence of gravitational fluctuations on quantum systems. Next, Penrose reported an estimation for the collapse time of a superposition due to gravitational effects that was the same found by the precise dynamical equation given by Diósi, based on the idea of a noise-based dynamical reduction effect. Such a topic has been still explored up to recently [15].

Another relevant idea on the local action of gravity refers to the inclusion of quantum fluctuations effect, which is a nonlocal component in the description of cosmological physical systems. For instance, in [16] such a point is analyzed by assuming that a mass scale is dynamically generated in the infrared regime, giving rise to nonlocal terms in the quantum effective action of gravity. Hence, the associated nonlocal gravity models are analyzed in many conceptual aspects as causality, degrees of freedom and their cosmological consequences. In a recent work [17], we have an overview on many aspects of nonlocal gravity cosmology.

On the basis of such previous ideas, we think that the hypothesis of generalized quantum entanglements (GQE) that we have developed in some previous works [18–22] could be a candidate for understanding some aspects and properties related to gravity, mainly considering the recent report of the existence of a type of quantum force [23]. In addition, in another work [24], it was asserted that it would be

possible to infer entanglement gravitational generation by using an atom interferometer [24]. The basic idea consists in the hypothesis that if we suppose gravitational perturbations as being quantized into gravitons, then the resulting graviton interactions should lead to an entangling interaction between massive bodies. However, the authors proposal of an experimental test – introducing the concept of interactive quantum information sensing – was not robust as reported and an erratum was published [25] with basis on the calculations showed in [26]. Basically, in [26], the authors showed by means of an explicit example that an interaction between a harmonic oscillator and a two-level test mass mediated by a local operation and classical communication channel produces a signature that in [24] was claimed to be exclusively for transmitting quantum information. Although the result was not really highly robust, in [25] they suggest methods to overcome the weakness in the proposed experiment. So, with basis on [23] and [25], we see that the subject is really intriguing and motivating in the sense of investigating and deepening the possibility, consequences and implications of associating generalized states of quantum entanglement between microscopic particles and the gravitational force. Although preserving quantum entanglements effects over macroscopic scale [27] is very difficult due to physical interactions, in many specific situations, as for instance in cases of physical systems subjected to high magnetic fields (in which the spins of all the component particles point in the same direction), effects of such a quantum state can be experimentally verified [28–30]. In addition, it is also proposed in [27] a physically robust quantum entanglement process that indicates the persistence of such states to classical scales.

From the general lines that we here exposed, it is our proposal in this work to discuss a generality of ideas that corroborate for this line of thought and research, in order to motivate investigations on the area. In more specific terms, we intend to describe at least four relevant lines of work concerning such aspects. One of them refers to a work in which the expansion of the electromagnetic field in a power series indicates that relevant terms of the series are equal to a purely gravitational term. Further we discuss how such an approach can be improved so that quantum entanglements among all the particles in the system can induce the force of gravity. In another study, we discuss some relations between the gravity and quantum mechanisms, that is, that the speed of light in the vacuum is a limit to the matter as a consequence of its origin (quantum fluctuations of the vacuum) and the interferences on beams of neutrons by gravity. At last, we discuss the description of the nonlocal gravity proposal by adopting the GQE hypothesis.

2 Generality of gravity

In the early 1990s, one of the authors of this manuscript began studies in order to investigate possible effects of quantum

entanglements in the macroscopic environment, starting from the premise that all existing particles in the Universe are in a preexisting state of maximum quantum entanglement, considering that at their origin (Big Bang) they were all in causal contact in a very small volume and associating such a collective state with gravity and inertia [18]. At first, one could imagine that this would contradict the concept of quantum monogamy [31], in which is reported that the concept is related to the idea that an entangled state cannot be shared with many parties, that is, the more parties, the less entanglement occurs among them. However, in reality, such a property is valid considering that two particles are entangled with each other, but not with a third or others, so that when the entanglement spreads the state of maximum entanglement is no longer possible. In the model that we consider, all particles are already entangled with each other since the beginning of the Universe.

The generality of both the gravity and the preexisting collective quantum state that governs all particles is one of the main factors that they can be somehow related to each other. Here generality means that the interaction involves all existing particles. For instance, electromagnetism involves only the charged particles and does not present such a characteristic.

Using the quantum mechanics formalism [18], it was verified that the dynamics of particles can be governed by non-local potentials in addition to the local ones and that, therefore, there is a possibility that gravitational potentials are also nonlocal due to other existing evidences described in this paper. A recent work [19] showed in an experiment that there was a correlation between the polarization of electric dipoles and photons (without local interaction between them) via discrete observables and indicated the possible preexistence of generalized quantum entanglements (we will call it GQE or the GQE model from now on). Penrose [32] reported that the evolution of states indicated by the Schrödinger equation inevitably makes all particles entangled and other studies [33] have also indicated that quantum entanglements can exist even in particles that never coexisted, considering entanglement chains. A very interesting work that we will analyze further here by Buniy and Hsu [34] indicated that everything in the Universe is maximally entangled despite not associating this property with gravity. The main argument is that particles had causal contact at the beginning of the Universe and with its expansion, the vast majority of current entanglements occur between particles that are beyond the causal horizon and that must be uniformly distributed in thermodynamic equilibrium (as evidenced by cosmic radiation). Such entanglements cannot even be removed by local interactions. One of the consequences of degrees of freedom being beyond the causal horizon is that two particles or two groups composed of a few particles, called X and Y, chosen at random, are not likely directly entangled with each other. This is because in this condition, the vast majority of degrees of freedom are not

contained between the two subsystems X and Y, but outside them (causal horizon that involves both). Therefore, for this reason the two subsystems share a negligible entanglement with each other.

In order to formalize the analysis, one can describe the system by the equation

$$\rho_{XY} = \sum_i \omega_i \rho_X^i \otimes \rho_Y^i, \quad (1)$$

which shows the density matrix that describes the entanglement between the two subsystems, each of them described by its individual density matrix.

Both of the subsystems are casually connected and separated. Now if the subsystems X (small subsystem) and Y (rest of the Universe with large amount of particles) are separated by the causal horizon (space-like separated) so that the vast majority of degrees of freedom are contained between both, we have that the density matrix describes the entanglement between both subsystems as described in equation

$$\rho_{XY} = \sum_i \omega_i |\phi_{XY}^i\rangle\langle\phi_{XY}^i|, \quad (2)$$

in which the term ϕ_{XY}^i represents a pure state and ω_i are probabilities.

In the situation formalized by (2), unlike the previous situation formalized by (1), the two systems X and Y share a high degree of entanglement. Fig. 1 summarizes the two situations analyzed here about entanglement across causal horizons. On the left side we have the X and Y subsystems with few particles surrounded by the causal horizon (crosshatched background) where mutual entanglement is negligible. On the right side we have the subsystem X with few particles surrounded by the causal horizon (crosshatched background) smaller than the subsystem Y which contains the rest of the Universe (a myriad of particles) where mutual entanglement is immense.

It is notable that the degree of entanglement between such subsystems depends directly on their dimensionality in the correspondent Hilbert space with respect to the dimensionality of the causal horizon and that these grow exponentially according to the number of degrees of freedom they have, that is, with the amount of particles that constitute them and their possible states. This indicates that it is possible to make local manipulations of a myriad of particles so that the effects of entanglements between subsystems become detectable, that is, for local systems to pass from the condition indicated by (1) and diagram on the left side of Fig. 1 to the one indicated by (2) and the right side of Fig. 1.

That's what we actually performed in our previous experiments [20, 22, 35, 36]. In various experiments we polarized a myriad of electric dipoles inside a dielectric under intense electric fields, magnetized a myriad of magnetic dipoles inside solenoids under intense magnetic fields, placed a myriad of electric and magnetic dipoles in collective precession

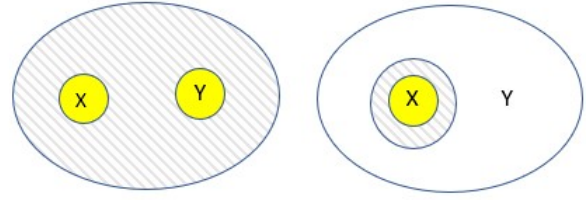


Fig. 1: Scheme of entanglement across horizons. At left, we see the systems X and Y with negligible amount of entanglement between them because their small areas (extremely small quantity of particles) compared to the area of the causal horizon (crosshatched background). At right, we see the opposite, the big amount of mutual entanglement between the systems X and Y, considering the big value of the sum of their areas (myriad of particles) compared to the small area of the causal horizon (crosshatched background).

and mobilized a myriad of charge carriers within conductors, superconductors and semiconductors. Considering only the nonlocal interaction between two separate simple dipoles and no local interaction via known forces as shown in Fig. 2, we have that the action of a local potential (magnetic or electric) in one of the dipoles affects the other dipole of the pair that is in the environment. The state of the pair of dipoles can be represented as being typically entangled [28] as represented by equations

$$|\Psi_1\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \quad (3)$$

and

$$|\Psi_2\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad (4)$$

in which the state zero means orientation along the field and the state one means orientation against the field. These kets represent entangled states of a pair of dipoles in which one of them is oriented by a local field.

The nonlocal connection between the dipoles explains the supposedly anomalous forces in the form of weight variation that are measured in devices where high local potentials are applied and also in forces that such devices induce at a distance. Such inductions cannot be blocked as we have seen in our experiments [20, 22] because in fact there are no isolated systems and this is precisely one of the main properties of gravity.

We deal in our experiments with intense local potentials that have driven myriads of particles, but immense amounts of particles in the Universe are affected by local bound potentials of very weak magnitude so that nonlocal net effects are extremely weak, but we will show later in this work that these may explain the weakness of gravity through the other known forces.

A gravitational-like interaction was detected in our experiments in [20], where a shielded capacitor via Faraday cage

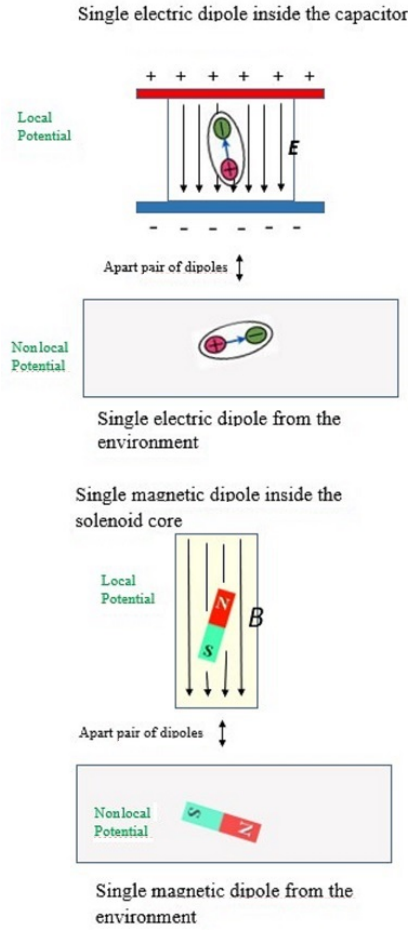


Fig. 2: Simplified analysis of the nonlocal interaction dipole-dipole.

enclosed inside a box was subject to a high voltage applied via shielded and insulated wires. Its weight variation cannot be explained via ionic winds and local interactions such as electrostatic, magnetic or acoustic ones.

The adoption of macroscopic observables as witnesses of entanglement of systems composed of many particles such as the electrical susceptibility χ_e and the magnetic susceptibility χ_m provides the necessary tools that can be applied in theoretical formalisms that explain the experimental results [21, 29, 30] considering the complexity involving quantum systems composed of myriads of particles. According to [29], the entanglement witness is shown as being more general (in the sense that it is not only valid for special materials), associating some macroscopic observables such as magnetic susceptibility χ_m with spin entanglement between individual constituents of a solid. It was proposed in [29] a macroscopic quantum complementary relation basically between magnetization M , representing local properties, and magnetic susceptibility χ_m , representing nonlocal properties. By defining for a system of N spins of an arbitrary spin length s in a lattice

the quantities:

$$G_l = 1 - \frac{kT\bar{\chi}}{N_s} \quad (5)$$

and

$$G_{nl} = \frac{\langle \vec{M} \rangle^2}{N^2 s^2}, \quad (6)$$

in which M is the modulus of the magnetization vector, k is the Boltzmann constant, T the temperature and the susceptibility of the system is defined as

$$\bar{\chi} = \chi_x + \chi_y + \chi_z, \quad (7)$$

hence, it was shown in [29] that one has:

$$G_l + G_{nl} \leq 1. \quad (8)$$

Such quantities have specific meanings, that is, G_{nl} represents the quantum correlations between the spins in the solid (nonlocal properties) and G_l represents the local properties of individual spins. The hypothesis of preexisting GQE indicates that there are no isolated systems as mentioned before, thus the magnetic core and the environment around it are both part of the same system where the inequality (8) can be considered accordingly. In other words, if one quantity increases then the corresponding counterpart quantity has to decrease. If G_l increases and G_{nl} decreases in the magnetic core, G_{nl} decreases and G_l increases in the environment and vice-versa. This is the same framework described before involving a simple system with two entangled magnetic dipoles. If we increase the intensity of a magnetic field (G_l) applied in one then the nonlocal forces (G_{nl}) must increase in the other.

Then, calculating the intensity of the nonlocal force F generated by a series of magnetic spins (dipoles) within the core of solenoid subjected to intense magnetic fields through classical quantities such as the magnetic susceptibility of the solenoid material (macroscopic observable) and the magnetization M can be defined by

$$F = \frac{1}{16\pi^2} \frac{SBI}{\theta}. \quad (9)$$

The product SBI represents the summation of the energy eigenstates of the internal spins (dipoles) of the solenoid, in which S is the area of the solenoid, B is the internal density of magnetic flux and I is the electric current that flows through the wires of the solenoid and which generates the magnetic field applied. The number in the denominator comes from the Planck constant \hbar squared existing in the Hamiltonian of the spin system and θ is the radius of the cylindrical solenoid. Such an equation corroborated the experimental results of experiments with magnetic solenoids.

Another experiment replicating Galileo's experiment in the Tower of Pisa and referring to spins or magnetic dipoles is related to locked magnets [37]. The experiments showed that in free fall two strong magnets in repulsion coupled fall more

slowly than equivalent ordinary (demagnetized) objects and two strong attractive magnets coupled to each other fall faster than demagnetized equivalent objects. In the GQE model, the phenomenon of difference of gravitational acceleration can be explained in a consistent way [37]. The theoretical predictions using such a formalism are corroborated by the results of the experiments and following the same criterion of using macroscopic quantum observables (considered as classical quantities) representing the entanglement of a myriad of particles with the environment, so that it was possible to build models that explain other systems such as dipole electrical charges in dielectrics and electrical charges flowing inside conductors, semiconductors and superconductors. It is known that light beams are deflected and distorted (gravitational lensing) by gravitational fields of massive bodies and considering such phenomena we performed experiments with isolated piezoelectric materials that also showed deformed laser beams effects [35]. This indicates another possible association of the GQE model with gravity.

Another experiment [38, 39] demonstrating force induction at a distance with gravitational characteristics was carried out involving the mechanical displacement of masses of different materials in a pendulum caused by collimated impulses produced by a superconductor subjected to high voltages. An explanation via the Theory of Relativity was proposed in [40]. Our GQE model [21] can also explain that effect in a consistent way with the experimental data, especially the relationship of applied energy and the mechanical energy of oscillation of the pendulum. All this argument possibly demonstrates the generality of both gravity and GQE and that both have a very close relationship with each other. Further we conclude that there is an intrinsic connection between these physical entities that affect all bodies and particles (fermions and bosons), regardless of their constitution.

3 The order of magnitude of gravity

According to GQE, all the particles transfer variations of moment indistinctly among them, considering as basic hypothesis that they are quantically entangled and subjected to known nonlocal forces. Let us assume valid GQE hypothesis and investigate if quantum mechanisms like that can explain the gravity force. The question to be answered is: *But how to explain that the gravity force can be originated from such momenta exchange if it is extremely weak and the magnitudes of the local forces are much higher?* For instance, the gravitational force is 10^{-36} times smaller than the electrostatic force at the same distance [9].

In general, all the local forces such as the electromagnetic one, weak nuclear and strong nuclear forces are attractive and repulsive [41], but what explains the fact that only the gravitational force is attractive? In [9], a very interesting study was reported by Assis, indicating that two neutral electric dipoles where negative charges oscillate with small angular velocity

around an equilibrium position can attract each other through an average net electrostatic force that falls off as the inverse square of the distance between them and whose magnitude are compatible with that of the gravitational force. Besides he also showed that the same behavior is valid for groups of N dipoles; in other words, he showed that in that theoretical framework gravitation can be derived from electromagnetism. To reach this result, Assis used calculations based on Weber's generalized potential energy shown by equation

$$U = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{r_{12}} \left[1 - \alpha \left(\frac{\dot{r}_{12}}{c} \right)^2 - \beta \left(\frac{\dot{r}_{12}}{c} \right)^4 - \gamma \left(\frac{\dot{r}_{12}}{c} \right)^6 - \dots \right], \quad (10)$$

considering dipole oscillations in the three x , y and z directions. Eq. (10) indicates the dependence of the potential energy between the dipoles in terms of the power series in parentheses, in which r_{12} is the average distance between the two particles of the dipole and α , β and γ are the parameters that indicate the magnitude of those power series terms. As known, q_1 and q_2 are the oscillating negative charges of the dipoles, ϵ_0 is the vacuum permittivity and c is the speed of light. The dot in the distance r_{12} is the notation used for the time variation of the distance between the two charges of the dipole.

The force between the dipoles is attractive and given by

$$\vec{F}_{12} = -\hat{r}_{12} \frac{dU}{dr_{12}}. \quad (11)$$

Eq. (11) indicates the force between two dipoles 1 and 2 apart by the distance r_{12} and its attractive feature.

Surprisingly, the calculations resulted in a cancellation of the most significant terms of the series so that the potential energy – and, therefore, also the force – started to decay according to c^{-4} reproducing Newtonian gravitation as being the fourth-order of the electromagnetic effect.

Considering the values of the charges equal to the electron charge, that is, $q_1 = q_2 = e$, and making $A_1 \sim 10^{-10}$ m, which is the typical size of the atom or molecule where the electrons are vibrating around the positive nucleus; and also considering equal the angular velocities of the oscillating charges of the dipoles $\omega_1 = \omega_2 = \omega$ and the coefficient $\beta = 1/8$, from (10) Assis interestingly simulated Newtonian gravitation with electromagnetism demonstrating an interesting relationship between electromagnetic parameters, as shown in the left term of equation:

$$\frac{7}{18} \frac{1}{8} \frac{e^2}{4\pi\epsilon_0} \frac{A_1^4 - \omega^4}{c^4} = GM^2, \quad (12)$$

as gravitational parameters shown in the term on the right, where G is the usual gravitational constant and M is the mass of the neutron or the mass of the hydrogen atom.

Assis also indicated other issues in his model, such as the relationship of inertia with gravity derived from electromagnetism that we will address in future works. In addition, he

also indicated possible limitations such as the fact that the calculations do not include relativistic corrections, as proposed by Phipps [42] for Weber's generalized potential energy. The model here investigated can explain the orders of magnitude of the gravitational interaction, its decay with the distance between the bodies and also its attractiveness characteristic through electromagnetic interactions between neutral dipoles, more specifically between the charged particles that compose them such as electrons. Despite the apparent success of this model, if we are supposed to isolate such dipoles through electromagnetic shields (Faraday cages) we could suppress them and it is known that in principle there is no gravitational shielding. In other words, there is an apparent paradox if we adopt such a model for gravity. Another issue is that electromagnetic interactions between electrically charged particles such as electrons and protons that make up neutral dipoles were considered to derive the gravitational interaction, but it is known that neutral particles such as photons and neutrinos (it is assumed that the latter can contribute most of the mass of the Universe) undergo the action of gravity. So, these arguments lead us to conclude that a very important feature must be added to the model studied here in order to derive gravity from local forces, which is to consider the GQE hypothesis. Thus, a potential energy such as Weber's generalized one given by (10) can be considered as a local potential energy V in the Hamiltonian \hat{H} of a multiparticle system represented by (18) that we discuss in more detail in section 5. Such a local potential can provide the interaction with particles external to the system through nonlocal forces that we can consider to be gravitational.

Our previous work [43–45] has corroborated that such nonlocal forces can indeed be induced and measured externally when, for example, strong electric fields are applied locally to a myriad of atomic and molecular dipoles contained in dielectrics even though they are shielded by a Faraday cage. In the model here discussed and represented by (10), we assume local potentials between dipoles. Assis indicated in [9] that terms lower than fourth-order are cancelled and are preponderant in the Universe groups of particles that interact with each other or particles that interact with themselves (for example, when a neutrino splits into two virtual particles and then the virtual particles become a neutrino again). This phenomenon occurs via electromagnetism but also via other known local forces.

To validate such a model, it is necessary to use a quantum approach as done with London dispersion forces [46] and also a relativistic approach.

4 The quantum origin of the speed of light

Quantum mechanics successfully demonstrates that particle dynamics have a dual nature where the mutual transfer of momentum is governed both by local interactions mediated by force-carrying particles such as the photon in the case of

electromagnetism and by nonlocal interactions arising from entangled quantum states between particles. In the first case, the interaction speed follows a finite upper limit and in the second case the interaction is instantaneous. Another fundamental feature of the theory of relativity is that the speed of light is independent of any source or reference, although a proposal to challenge such physical property has already been done in the literature [47]. Knowing the origin of the finite limiting velocity of local interactions, more specifically the speed of light, is critical because both special relativity and general relativity are built on this fundamental characteristic. A very interesting work by Urban and his collaborators [48] proposes to derive the speed of light from quantum mechanics. The model in question treats the velocity of the real photon as being instantaneous as well as with nonlocal interactions, but its propagation through the quantum vacuum occurs in leaps like those of a frog. It “jumps” instantaneously between the pairs of virtual particles of ephemeral duration being absorbed and re-emitted in a chain. The delay inherent in the absorption and re-emission process is what determines the finite propagation velocity c of the photon. Electromagnetic properties such as permeability μ_0 and permittivity ϵ_0 of the vacuum are determined by the creation and destruction of ephemeral particles such as electrons and positrons in addition to other fermions, and therefore both statistically determine on average the speed of light in the vacuum, given that

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}, \quad (13)$$

allowing some small variation to be confirmed experimentally. Einstein [49, 50] showed that the total mass m of a body is the measure of its energy content E (mass-energy equivalence) according to the relation $E = mc^2$, where c is the speed of light in vacuum. In order to reach this conclusion, he considered that the body yielded or absorbed energy in the form of electromagnetic radiation and that such an action caused its mass to change. So the exchange of force carriers like photons that are constrained in their propagation speed due to their interactions with the ubiquitous quantum vacuum that surrounds all bodies and particles appears to be a crucial factor for both special and general relativity, both experimentally validated.

A pioneering experiment among the various later experiments that validated such theories was the Hafele-Keating one [51] in the 1970s, which compared the measurements of time between precision atomic clocks inside an airplane that spent a certain time at high altitude and speed synchronized with others that were at rest on land. After the plane landed, the measurements of the clocks were compared and showed to be in notable disagreement with each other, obtaining an accuracy above 98% in relation to the theoretical predictions [52, 53] given that the clocks on board suffered an advance in the proper time due to the fact that they were at a higher altitude, that is, subject to a weaker gravitational

potential energy than at the surface, corroborating the relationship between proper time $d\tau_1$ (plane) and dt (surface) indicated in (14) derived from general relativity (Schwarzschild metric):

$$d\tau_1 = \sqrt{1 - \frac{2m}{R}} dt. \quad (14)$$

The clocks were also delayed due to the effect of special relativity with the plane's speed v , corroborating the relationship between the proper time $d\tau_2$ (plane) and dt (plane at rest on the ground) indicated in (15):

$$d\tau_2 = \sqrt{1 - \frac{v^2}{c^2}} dt. \quad (15)$$

In (14), R indicates the distance from the planet's center of mass (altitude) and m represents the relativistic mass of the planet according to the relationship $m = GM/c^2$, where G is the usual Newtonian gravitational constant and M is the planet's rest mass. The theory of relativity also predicts other effects [52, 53] such as the advanced perihelion of the planets, the deflection of light by the gravitational field and the spectral displacement of gravitational origin, all of which are experimentally proven. According to GQE theory, all particles are quantum entangled and, therefore, interact not locally instantaneously, but due to the fact that they also interact via local interactions mediated by force carriers limited to the velocity c taking into account their delayed propagation, through the ubiquitous quantum vacuum, the predictions of both general and special relativity support under certain conditions part of phenomena such as those mentioned above that Newtonian physics cannot explain.

The understanding of such "complementarity" in the co-existence of nonlocal and local interactions is analogous to what exists in the corpuscular and wave aspects because they seem contradictory, but are actually complementary according to quantum mechanics. Therefore, it is essential to study certain aspects of contradictory phenomena in relation to general relativity and special relativity, as indicated by van Flantern [54]. For example, according to him, the photons emitted by the Sun arrive at planet Earth 8.3 minutes after being emitted, time in which the Sun moves 20 arcs of a second in relation to the terrestrial reference. This causes the classic optical aberration studied by Bradley in 1728 to occur [54]. If such a phenomenon of aberration occurred with gravity, there would be a slight radial decrease in the intensity of the force so that the radial distance of the Earth's orbit would increase by 150 million kilometers every 1200 years, which in fact does not occur. Such an effect occurs with the radiation emitted by the Sun absorbed by dust particles where a transverse component affects their orbits (Poynting-Robertson effect). It is clear in this example the need to understand the complementary nature in which the gravitational interaction is instantaneous as proposed by the GQE theory and that the optical aberration in the radiation emitted by the Sun occurs

due to the photons having a finite propagation speed like the other force carriers. More examples are given by van Flantern such as the fact that gravity and light do not act in parallel directions, anomalies that occur during solar eclipses, *etc.* Other works such as [55] tried to answer why there is no aberration of gravity via General Relativity Theory without superluminal propagation of gravity assuming an approximation that the Sun and Earth have a mutual uniform motion. On the other hand, other works report possible superluminal different phenomena such as superluminal photonics tunneling [56] and superluminal X-rays [57]. Regarding the gravity waves with approximately the speed of light that were supposedly detected by the huge laser beam interferometers of LIGO-VIRGO collaboration, it must be necessary to investigate deeply as the authors [58] are proceeding in order to explain if the deviation of the beams was produced by other physical effects such as propagating vacuum fluctuations caused by huge cosmological events or by another local events. There is another experiment that supposedly evidences the nonlocal nature of gravity in [39], in which is reported the generation of a supposed gravitational-type interaction using superconductors under high-voltage discharges carried out, where impulses of up to 70 ns were induced at a distance at an apparently superluminal velocity (supposedly) 64 times the speed of light within the limitations of the equipment. The study of light interaction with gravity impulses and measurements of the speed of gravity impulses were also reported in [59].

Based on these promising analyses, the authors intend to continue the studies to deepen the understanding of the mentioned complementarity (local and nonlocal) of the interactions as well as to carry out new experiments involving the measurement of the velocity of the gravitational interaction. The authors also intend to publish another work containing important topics related to the association between inertia and gravity, the Mach principle and the principle of equivalence between gravitational mass and inertial mass.

5 Quantum interference via gravitation

The theory of Entropic Gravity or Emergent Gravity [7] proposes that gravity is not a fundamental interaction based on Quantum Field Theory, and therefore is not mediated by particles called force carriers such as gravitons. This characteristic is analogous to the GQE theory, which also proposes that gravity is not mediated by force carriers, but is the result of the transfer of momentum at a distance between particles that undergo the action of fundamental or canonical potential energies at their locations, considering that they are all in a pre-existing state of generalized quantum entanglement. The theory of Entropic Gravity has had a lot of opposition [60–62], but in this work we want to describe one of the oppositions [63, 64] that emphasizes that such a theory is not consistent with the result of the pioneering experiment of gravitational

induction of neutron phase shift [65]. In the aforementioned experiment that uses interferometry, a beam of neutrons with coherent quantum wave functions is split and separated into two beams that pass through different paths and are then recombined again to form an interference figure. The diagram in Fig. 3 shows one of the beams passing through positions A, B and D on a trajectory with higher altitude with respect to the Z axis (vertical) and another beam passing through positions A, C and D on a trajectory with lower altitude. Point D indicates the interference region where the two beams recombine.

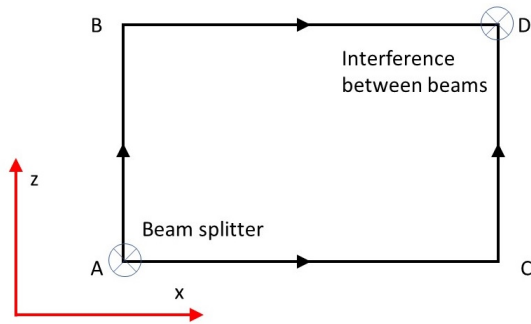


Fig. 3: Neutron interferometry experiment where a phase difference between the ABD and ACD beams was detected due to being subjected to different gravitational potentials [65].

The analysis of the interference figure clearly indicated that there was a phase difference depending on the neutron trajectory, related to the fact that the gravitational potential energy has a lower magnitude in the higher altitude trajectory (BD section) and higher magnitude in the lower altitude trajectory (CD section). The neutron momentum is therefore different on the trajectory ABD ($p_1 = \hbar k_{BD}$), with respect to the momentum of the trajectory ACD ($p_2 = \hbar k_{CD}$). The Schrödinger equation [66, 67]

$$-\frac{\hbar^2}{2m_n} \frac{\partial^2 \Psi}{\partial z^2} + m_n g z \Psi = i\hbar \frac{\partial \Psi}{\partial t} \quad \text{for } z > 0 \quad (16)$$

indicates dependence of the neutron dynamics on the gravitational potential energy represented by the term $m_n g z \Psi$ (the other term to the left of the equality represents the kinetic energy). The two terms to the left of the equality represent the standard Hamiltonian operator \hat{H} .

This Schrödinger equation (16) represents the dynamics of the neutron in the physical system described. The gravitational potential energy used in the formalism is that of classical physics, that is, it depends on the neutron mass m_n , the gravitational constant g and the altitude z (Z axis) according to the term $m_n g z \Psi$. The predicted phase difference theoretically calculated according to the standard Hamiltonian \hat{H} is consistent with the phase difference obtained experimentally.

But according to [63], in the case of entropic gravity, the interference pattern is destroyed because the gravitational interaction is not fundamental or canonical for this theory, that is, it behaves like a typical chaotic thermodynamic interaction so that the wave function of the neutron loses its coherence. In the theory of entropic gravity, the gradient of variation of the gravitational field with respect to altitude (Z axis) is a statistical average of the thermal fluctuations involving a myriad of microstates. The last two terms in the equation

$$\hat{H} = \frac{\hat{p}^2}{2m} + V_{grav}(r) = -\frac{\partial^2}{\partial r^2} - 4\pi m \frac{\partial}{\partial r} - 4\pi^2 m^2 + mgr. \quad (17)$$

indicate deviation from the standard Hamiltonian and demonstrate that entropic gravity does not explain the experimentally measured phase difference. Eq. (17) shows the Hamiltonian operator according to the entropic gravity theory, where m is the neutron mass and r is the altitude value (Z axis). In the experiment [65], according to the interpretation of the nonlocal gravitational interaction through the GQE theory, we can define the relevant Hamiltonian operator of the neutron-Earth system via equation

$$\hat{H} = -\frac{\hbar^2}{2} \left(\sum_{M=1}^N \frac{1}{M} \frac{\partial^2}{\partial z^2} + \frac{1}{m_n} \frac{\partial^2}{\partial z^2} \right) + \hat{V}_M. \quad (18)$$

which determines its dynamics in order to change the phase of its wave function coherently.

Eq. (18) shows the “standard” Hamiltonian operator of the nonlocal gravitational interaction of the neutron-Earth system according to GQE theory. The neutron moves on a trajectory with a certain kinetic energy (term on the right in parentheses in (18)) and in a state of quantum entanglement with the myriad of particles that make up the planet Earth whose total kinetic energy is represented in the sum of the term from the left in parentheses in (18). As the neutron is subject only to gravitational interaction with the planet Earth (its electromagnetic interaction is negligible because it has practically zero electric charge), the potential energy \hat{V}_M (fundamental or canonical) inherent to the particles of the planet Earth is represented in (18).

The analysis performed here allows us to suppose that the phase difference theoretically predicted via GQE theory and calculated according to the standard Hamiltonian \hat{H} shown in (18) is consistent with the phase difference obtained experimentally in the experiment in [65]. The potential energy inherent to the particles of the planet Earth of (18) can be equivalent to the classical gravitational potential energy of (16) with its main gravitational characteristics such as decay according to the inverse of the distance (height), attractiveness of the force and order of magnitude, for example, if it corresponds to potential energy like Weber’s generalized one, as shown earlier via (10). The fact that nonlocal gravity within the framework of GQE theory seems to be consistent with gravitational induction of quantum interference, considering

that the nonlocal potential \hat{V}_M is canonical or fundamental such as the local potential, is very interesting and encourages further studies for a more detailed understanding.

6 Conclusions

In this work, we outlined a discussion of the state of the art of the research on some gravitational phenomena and theories of gravity. Specifically, we discuss how the GQE hypothesis can be associated with gravity, explain all aspects of such an interaction as its very weak magnitude, the nonlocal effects of gravity, the limit of the light velocity as consequence of the quantum vacuum, the validity of the nonlocal gravity and its description by means of GQE and many other interesting theoretical issues concerning the gravitational interaction.

We assert that some previous experiments indicated that GQE is consistent with some gravitational effects reported in the literature, as the weight reduction of capacitors, the gravitational shield generated by superconductors or the change in the value of the gravity acceleration of two magnets locked with each other in free fall.

It is worth to mention that in a next work we also intend to analyze a lot of relevant topics that deserve a more profound study, as the earlier mentioned complementarity (local and nonlocal) of the interactions and the association between inertia and gravity, Mach's principle and the principle of equivalence between gravitational mass and inertial mass. In addition, we also intend to explore other themes not discussed here as the dark matter or MOND, in order to investigate more profoundly the consistency of GQE for explaining all such phenomena.

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Space and Gravity

Anatoly V. Belyakov

Tver, Russia. E-mail: belyakov.lih@gmail.com

Based on the mechanistic interpretation of J. Wheeler's geometrodynamics, where space has the properties of an ideal fluid surface, it was found that the ratio of the forces acting in the surface wave transverse component to the forces acting in its longitudinal component is equal to the ratio of the electric forces to the gravitational ones. The surface of a finite thickness is the original material entity, the fractalization of which leads to forming material bodies and the tension of the above surface, which is manifested as an attraction between the bodies. The speed of light has been determined and the gravitational constant calculation formula has been obtained. It is concluded that a radiating cell of the surface wave generates a radiation having the wavelength corresponding to the background radiation maximum.

1 Introduction and main provisions

Geometrodynamics, introduced by the famous scientist John A. Wheeler, who died in 2008, does not seem to be approved by modern physicists, since it requires the presence of some medium (ether). According to Wheeler's concept, charged microparticles are singular points on a non-unitary coherent connected two-dimensional surface of our world, connected by a "wormhole", a vortex tube or a current force line of the drain-source type in an additional dimension, forming a closed contour [1]. But "wormholes", if they are not considered purely mathematical constructions, in their physical embodiment can only be the vortex formations based on the surface (or phase boundary) of some substance that has the properties of an ideal fluid.

The presence of contours (vortex tubes) is also postulated, for example, in [2], where the vacuum structure is considered as a network of one-dimensional flow tubes (knotted/linked flux tubes), and it is claimed it is such a network that provides the spatial three-dimensionality of the Universe. At the same time this network, infinitely densely filled with such vortex formations, forms a continuous surface (the possibility of this was proved in the 19th century by J. Peano [3]). This surface, in turn, as it becomes more complex, can form three-dimensional material objects, which are, in fact, highly fractalized, up to the parameters of the microworld, *surfaces* with a fractional dimension. An undeformed (non-fractalized) surface is equivalent to the empty space, and bodies when driving in such a continuous medium does not feel any resistance up to the speed of light, i.e., until surface waves forms, and *for any observer the vacuum medium remains undetectable*. Recall that even when moving in a real liquid body, an observer does not feel a resistance up to the speed when a surface wave is formed (for water, the speed is 0.3...0.5 m/sec).

As for a completely entire three-dimensional body, it does not have an internal structure, does not carry any information about its structure (except for its own mass), and such bodies do not really exist. The fact that all objects are fractalized

surfaces is especially well manifested in the organic world: under the surface of outer shells there are the surfaces of organs, vessels, then – the surfaces of their fibers, then – the surfaces of cells, etc.

The Wheeler model's closest analogy on the scale of our world would be the ideal fluid surface, the vortex formations arising in it and corresponding interactions between them. In the mechanistic interpretation of Wheeler's idea, the charge reflects a measure of the medium nonequilibrium and is proportional to its momentum along the vortical current tube contour, spin, respectively, is proportional to its angular momentum relative to the contour longitudinal axis, and magnetic interaction between the conductors is similar to the forces acting between the current tubes. In this model, a point or a line is considered to be physical objects with certain dimensions, where the electron volume with mass m_e and radius r_e can be taken as a medium unit element. A free charged particle in such a scheme is represented as part of an open contour or a unipolar vortex resting on the surface of our world and directed along the "extra" dimension, where the particle charge and spin are determined by the "hidden mass" dynamics [4].

In such a model, the electric constant becomes the density per unit of the vortex tube length

$$\varepsilon_0 = \frac{m_e}{r_e} = 3.23 \times 10^{-16} \text{ kg/m}, \quad (1)$$

and the reciprocal of the magnetic constant is the centrifugal force

$$\frac{1}{\mu_0} = c^2 \varepsilon_0 = 29.06 \text{ N}, \quad (2)$$

arising from the rotation of the vortex tube element with mass m_e , at the speed of light c along the radius r_e ; it is also equivalent to the force acting between two elementary charges at this radius.

The paper [4] defines the vortex thread parameters: its mass M , circumferential velocity v , radius r , and length l for

an arbitrary p^+e^- -contour in dimensionless units of m_e , r_e and c :

$$M = l = (an)^2 \quad (3)$$

$$v = \frac{c_0^{1/3}}{(an)^2}, \quad (4)$$

$$r = \frac{c_0^{2/3}}{(an)^4}, \quad (5)$$

where n is the main quantum number, a is the reciprocal of the fine structure constant, c_0 is the dimensionless speed of light, $c/[m/sec]$.

This approach has justified itself in determining the electron charge and radiation constants numerical value and other parameters both for the microworld [4, 5, 6, 7] and for cosmological objects [8]. The mechanistic interpretation of geometrodynamics does not introduce any additional entities, but, on the contrary, reduces them. So, the Coulomb is excluded from the set of dimensions and is replaced by the electron limiting momentum, which radically simplifies all the dimensions associated with electromagnetism [9].

2 On the surface wave parameters

The speed of light is one of the few fundamental quantities not derived in theory. However, as established in [10], it turns out the propagation of light to be similar to wave's moving on the liquid surface and *has the maximum equal to the speed of light*, which is determined from the well-known equation

$$v^2 = \frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\rho\lambda}, \quad (6)$$

where g is the acceleration, λ is the wavelength, σ is the surface tension (force relative to the perimeter), ρ is the specific density. The first term reflects the gravity effect on the wave speed, the second – the surface tension effect.

When this equation was solved, a radiating cell (toroid) was considered, in which the medium circulates along the contour with radius $R = a^2 n^2 r_e$ and rotates helically about the toroid longitudinal axis, creating z structurally ordered units (waves or photons) with centrifugal acceleration $g = zv^2/R$. The surface tension was defined as $(1/\mu_0)/R$, and the specific density as $m_p m_e / R^3$, where m_p is the relative proton mass in units of m_e .

The solution obtained, can be considered as a special case of the wave velocity maximum at $n = 4.23$ and does not depend on the parameters n and z . However, unlike a liquid where the surface wave velocity has a minimum, and, actually, their capillary and gravitational waves velocity depends on the surface tension and the basin depth, there is some natural mechanism for electromagnetic waves ensuring of their speed from the wavelength independence.

Let us express the wavelength from (6). Assuming $v = c$, after transformations we get (a plus radical formula is accepted)

$$\lambda = \frac{\pi r_e a^6 n^6}{c_0^{2/3}} \left[1 + \left(1 - \frac{4c_0^{2/3}}{a^2 n^2 m_p} \right)^{1/2} \right]. \quad (7)$$

The critical value $n = 0.227$ corresponds to the wavelength minimum value

$$\lambda_{min} = \frac{\pi r_e a^6 n^6}{c_0^{2/3}} = 1.81 \times 10^{-11} \text{ m}, \quad (8)$$

which is already gamma radiation. Thus, for $n < 0.227$, either a more accurate equation is required, or the radiation already completely loses its longitudinal component and becomes the capillary waves analogue; anyway, it is known gamma rays to behave like particles at $\lambda < 10^{-10}$ m.

It should be borne in mind that, as applied to (7), the parameter n determines the radiating cell physical size, i.e., the circular trajectory size, along which particles move under the surface in the liquid medium under the action of gravitational forces (in contrast to the proton-electron system main quantum number, which characterizes the atom excited state). Note that at $n = 4.23$, when $v = c$, from (7) follows $\lambda = 1.52 \times 10^{-3}$ m, which corresponds to the *background microwave radiation maximum*. Thus, it turns out the optimal radiative cell at which the wave speed is compared with the speed of light to be the cosmic background radiation natural source or, at least, be its longitudinal component.

As shown in [5], the proton-electron contour parameters are determined from the condition of the equality of the magnetic repulsive forces and gravitational attractive forces, which in the Coulombless form has the form

$$\frac{z_{g1} z_{g2} \gamma m_e^2}{r^2} = \frac{z_{e1} z_{e2} \mu_0 m_e^2 c^2 l}{2\pi r \times [\text{sec}^2]}, \quad (9)$$

where z_{g1} , z_{g2} , z_{e1} , z_{e2} are the gravitational masses and charges in the masses and charges of an electron, γ is the gravitational constant.

The largest contour size is possible when the entire proton mass, corrected for the Weinberg projection angle Θ , is involved in the circulation contour. Then at $z_{g2} = m_p / \cos \Theta$ for unit charges after transformations we obtain the geometric mean from (9) in units of r_e

$$l_k = (lr)^{1/2} = \left(\frac{m_p}{\cos \Theta} \right)^{1/2} (2\pi \gamma \rho_e)^{1/2} \times [\text{sec}], \quad (10)$$

where ρ_e is the specific electron density m_e / r_e^3 , $\Theta \approx 28.70^\circ$ is the Weinberg angle.

The l_k parameter is compound. Taking into account (3), (5) and (10), the values of l and r in units of r_e have the form:

$$l = \frac{c_0^{2/3}}{l_k^2}, \quad (11)$$

$$r = \frac{l_k^4}{c_0^{2/3}}. \quad (12)$$

Taking the parameter r as the major axis of the contour, bearing in mind (3) and (12), from (10) the limiting quantum number is determined

$$n_{max} = 2\pi \frac{m_p}{\cos \Theta} \frac{\gamma \rho_e \times [\text{sec}^2]}{ac_0^{1/3}}. \quad (13)$$

It follows from (13) that $n_{max} \approx 390$, then the largest circulation contour size (the surface wave depth) is $390^2 a^2 r_e$, and the recombination wavelength $\lambda_{max} = n^2 / R_\infty = 0.0139$ m, where R_∞ is the Rydberg constant. This result is consistent with the fact that there are no excited hydrogen radio lines at $n > 301$ even in open space [11]; recombination radio lines with more n were detected only in absorption and not from hydrogen, but from the hydrogen-like atoms [12].

As for the parameter n as applied to the radiating cell, even at λ_{max} , as it follows from formula (7), $n \approx 6$. That is, the hydrogen atom radiating cell size (this cell is, as it were, the analogue of an antenna) at any possible wavelength does not go beyond the VI-th period atoms size (the atoms containing electrons in the seventh shell are unstable). The location of an electron at a greater distance from the nucleus is his excited and short-term state.

The longitudinal waves length, apparently, will be determined by the same equation (6) and, if it is limited not by $n = 6$, but by $n_{max} = 390$, then their length can be very large. Perhaps, in some range, electromagnetic waves also have a longitudinal component, since there are studies indicating the existence of longitudinal electromagnetic waves [13].

3 Determination of the gravitational constant

Let us consider an extremely simplified scheme of a single radiating cell, when a medium with an arbitrary mass m circulates along the toroid contour with a radius R , and at the same time it also rotates in the spiral about the toroid longitudinal axis in a radius r . The surface wave is known to have longitudinal and transverse components. Let the circulation along R occur under the action of gravitational forces with acceleration v^2/R , and the spiral rotation along r occur under the action of surface tension forces (capillary forces) with acceleration v^2/r . Considering the surface wave components separately, it is logical to correlate these components with gravitational waves and electromagnetic waves. One can say the electromagnetic oscillations to form, as it were, a small "ripple" on the surface of gravitational waves. Such ripples – a real physical analogue – are easy to observe on the water surface over its ordinary disturbance.

So, it is possible to draw up the single ratio of electric forces to gravitational ones, or, bearing in mind the equality of masses, the ratio of accelerations, which will be the largest under extreme conditions, i.e., it should be equal to

the value $c^2 r_e / \gamma m_e$, where the gravitational constant should be considered unknown. For the transverse component, the highest velocity $v = c$, and the vortex thread smallest size r is the circumscribed circle size around three Planck dimensions r_h , which from geometric constructions is equal to

$$r = r_h \left(1 + \frac{2}{3^{1/2}} \right), \quad (14)$$

where

$$r_h = \left(\frac{\hbar \gamma}{c^3} \right)^{1/2}, \quad (15)$$

and

$$\hbar = am_e r_e c, \quad (16)$$

since it has been established the quantity r_h to have a physical meaning and be the neutrino vortex thread minimum size [14].

For the longitudinal component, bearing in mind (4), the lowest speed

$$v = \frac{c_0^{1/3} c}{(an_{max})^2} \quad (17)$$

and, bearing in mind (3), the largest radius (contour length)

$$R = (an_{max})^2 r_e. \quad (18)$$

Thus, for the largest ratio of accelerations, taking into account the above and bearing in mind (17) and (18), one should write

$$\frac{(an_{max})^6 r_e}{c_0^{2/3} r} = \frac{c^2 r_e}{\gamma m_e}. \quad (19)$$

As a result, bearing in mind (14), (15), (16) and separating the parameter γ , after transformations from (19) we obtain:

$$\gamma = \left(1 + \frac{2}{3^{1/2}} \right)^{2/13} a^{1/13} c_0^{16/39} \left(\frac{\cos \Theta}{2\pi m_p} \right)^{12/13} \times \\ \times w_e \left(\frac{c}{r_e} \right)^{2/13} \times [\text{sec}^{-24/13}], \quad (20)$$

where w_e is the electron specific volume, equal to r_e^3 / m_e , $m_p = 1836.2$.

This equation is exact, but the Weinberg angle Θ (parameter in the electroweak interaction theory) is determined experimentally and lies within $28.13^\circ \dots 28.75^\circ$, i.e., $\cos \Theta = 0.882 \dots 0.877$. However, it can be calculated as the projection angle [4], and in this case

$$\cos \Theta = \frac{c_0^{1/6}}{(2\pi a)^{1/2}} = 0.882, \quad (21)$$

and also as the radius to circumference reduced ratio [5]

$$\cos \Theta = \left(\frac{1}{2\pi} \right)^{1/14} = 0.877, \quad (22)$$

and in other ways, which gives the same results. It can be assumed that the currently observed experimental data inconsistency when determining the gravitational constant is associated not so much with the Weinberg angle's uncertainty, and how much with the projection uncertainty of "hidden" parameters on the selected direction of our world, i.e., with the position uncertainty of the medium velocity vector relative to this selected direction [4].

Interestingly, the product of geometry-related parameters $(1 + 2/3^{1/2})^{2/13}(\cos \Theta)^{12/13}$ is very close to 1, which is apparently not accidental. Indeed, the fundamental constants formulas, such as r_h , \hbar , r_e , also do not contain geometric coefficients. Formula (20) can be written, as a result, by making transformations and replacing c by $c_0 \times [\text{m/sec}]$ in a more compact form

$$\gamma = a^{1/13} c_0^{22/39} (2\pi m_p)^{-12/13} m_e^{-1} r_e^{37/13} \times [\text{m}^{2/13} \text{sec}^{-2}], \quad (23)$$

which gives the value γ with a negligible error (here $m_p = 1836.2$).

It should be noted that the parameter r_e can be excluded from the formula for γ , since by definition it is determined by the electron mass, speed of light, dielectric constant and charge values, the latter, in turn, being determined through the electron mass, speed of light and Weinberg' angle [4].

4 Conclusion

The connection of gravity with the proton-electron contour size confirms the existence of limiting sizes, both in the microcosm and in space. So the vortex thread smallest size (the neutrino size) turned out to be equal to the Planck value \hbar [14], and the hydrogen atom largest size, still capable of radiation, corresponds to $n = 390$; this is confirmed by the fact that even in space, no excited hydrogen radio lines with $n > 301$ have been found [11]. The ratio of these limiting values makes it possible to calculate the gravitational constant.

Determining the speed of light and the gravitational constant magnitude based on the proposed physical model proves space to have the ideal fluid surface properties, where the speed of light is the surface wave speed. It has been established the ratio of forces acting in the transverse component of this wave (capillary waves) to the forces in its longitudinal component (gravitational waves) to be equal to the electric forces to gravitational forces ratio.

Thus, the original material entity is the finite thickness surface, whose deformation (fractalization) leads to the formation of material objects. Since the surface wave longitudinal component is essentially an elastic medium, in which tension-compression forces are possible, the gravity forces are the forces of attraction between material bodies, arising due to tension of this surface during its fractalization (deformation, thickening, condensation) in these bodies formation process. This is consistent with the conclusions drawn by Pierre A. Millette in his elastodynamics based on the analysis

of space-time deformation in terms of continuum mechanics [15, 16]. That is, the very *existence of material bodies is the cause of their mutual attraction*, and electromagnetic waves contain the longitudinal component, which is the gravitational waves conductor.

It has also been established that the surface wave optimal radiating cell generates radiation with the length of 1.52×10^{-3} m, which corresponds to the background radiation maximum and, therefore, may be its natural cause.

And so, on the basis of the mechanistic interpretation of Wheeler's geometrodynamics, which, not being essentially physical and mathematical, but rather physical and logical, have determined the gravitational constant value (as well as the speed of light, electron charge, neutrino mass, etc., which was stated in the relevant articles). This model's possibility to obtain and predict results not achieved by mathematized methods proves the macroanalogies underlying its full compliance with the corresponding physical natural laws, which indicates the need for further development of this model at a higher level.

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Fermion Mass Derivations: I. Neutrino Masses via the Linear Superposition of the 2T, 2O, and 2I Discrete Symmetry Binary Subgroups of SU(2)

Franklin Potter

Sciencegems.com, 8642 Marvale Drive, Huntington Beach, CA 92646 USA. E-mail: frank11hb@yahoo.com

We derive neutrino masses from discrete symmetry binary subgroups of SU(2), 2T for the electron family, 2O for the muon family, and 2I for the tau family, acting collectively to generate the PMNS mixing angles. Using the modulus τ near $\omega = \exp(2\pi i/3)$ in the domain of SU(2) converts the PMNS matrix into the 24th root of unity and produces a factor of 3^{11} to predict neutrino masses: $m_1 = 0.3$ meV, $m_2 = 8.9$ meV, $m_3 = 50.7$ meV.

1 Introduction

One of the most challenging fundamental problems in particle physics is to calculate the mass values of the leptons and quarks. We tackle this problem within the framework of the Standard Model by considering the three specific discrete symmetry binary subgroups of SU(2) that we have established previously [1,2]. The three lepton families represent the binary tetrahedral group 2T for the electron family, the binary octahedral group 2O for the muon family, and the binary icosahedral group 2I for the tau family. The mass values for the quark families will be derived via an identical approach in a separate article.

After a brief review of some of the limitations of the Standard Model, we explain some of the consequences of the discrete symmetry binary subgroups of SU(2), including how we utilized their generators to derive the correct mixing angles for the lepton PMNS[‡] mixing matrix. These subgroups have a domain in the upper half of the complex plane and we use their modulus τ for fractional linear transformations near its symmetry point $\tau_0 = \omega = \exp(2\pi i/3)$ in our procedure to predict the lepton mass values. Note that the modular subgroups of SL(2,Z) used to calculate lepton masses via many parameters [3,4] are isomorphic to our subgroups of SU(2).

We find that by treating the three lepton families equivalently leads to the circulant matrix method used to derive [5,6] the 1982 Koide formula [7] that accurately predicted the mass value of the tau lepton. We move the value of τ slightly away from ω , thereby introducing CP symmetry breaking, to convert our PMNS mixing matrix into the 24th root of unity, from which we calculate neutrino mass values by using the factor of 3^{11} difference from the charged-lepton mass values.

Finally, we examine how the unique invariant N for each binary subgroup can be used to derive the lepton mass values from geometry. According to F. Klein [8] in 1884, each of the three binary subgroups has an invariant N inversely related to $j(\tau)$ of elliptic modular functions, the N being: 1 for 2T, 108 for 2O, and 1728 for 2I, integer values that have a similar hierarchy to the 0.511 MeV, 105.66 MeV, and 1776.82 MeV

mass values for the charged leptons!

2 SM limitations

The Standard Model (SM) of leptons and quarks has been an extremely successful effective field theory [9–12] for combining the unified electroweak interaction with the nuclear color interaction since its formulation in the 1970s. Its fundamental particles represent quantum fields, with the SM probably being an approximation to an underlying theory.

The physical world is artificially partitioned into a (3+1)-D spacetime and an internal symmetry space at each point in spacetime. The known fundamental particle quantum states are defined in the internal symmetry space, but the number of dimensions of the internal symmetry space has yet to be established.




The two particle quantum states for each lepton family and for each quark family represent the continuous symmetry group SU(2), i.e. the $\pm 1/2$ weak isospin states which are also called the up and down flavor states. Of the 3 known lepton families, the electron family (ν_e, e^-), the muon family (ν_μ, μ^-), and the tau family (ν_τ, τ^-), the more massive muon and tau charged leptons are known to not be excited higher mass states of the electron. Likewise, the two known quark families beyond the first quark family are not simply higher mass states of the first quark family.

The SM as presently understood cannot predict the number of lepton families nor the number of quark families. However, the weak interaction Z^0 boson decays suggest that there are exactly the 3 lepton families [13] if there are only neutrinos with mass values below about 90 GeV, which appears to be the case. In addition, there is a cosmological limit of 15 total fundamental leptons plus quarks. There being 12 known fundamental leptons plus quarks, at least one more family of two particles is possible. [14, 15]

Lepton mixing occurs [16–18] when one neutrino type or charged-lepton can change into another on the journey from source to detector. This behavior is in direct conflict with the SM expectation for massless neutrinos. However, most conserved quantities still hold true, such as electric charge conservation with the electromagnetic interaction being equiva-

[‡]Pontecorvo-Maki-Nakagawa-Sakata

Table 1: Lepton Family Group Assignments.

Family	Group	Order	3-D	Mass (MeV)
ν_e				< 0.001
e^-	2T	24		0.511
ν_μ				< 0.001
μ^-	2O	48		105.7
ν_τ				< 0.001
τ^-	2I	120		1776.8

lent for all electrically charged particles as well as the weak interaction being identical for each of the lepton and quark family particles, a property called weak universality. Further tests challenging this weak interaction lepton flavor universality (LFU) continue to be carried out at many different experiments worldwide.

3 Lepton mixing

In order to better understand the physical behavior of the SM particle states, in the 1990s we introduced [1] specific different discrete symmetry binary subgroups of SU(2) in R^3 for each family of leptons and in R^4 for each family of quarks. This approach has gained in importance in the past decade as other approaches have become less likely or eliminated. The discrete symmetry binary subgroups for the lepton families are the assignments listed in Table 1 along with their 3-D representations as the Platonic solids at the Planck scale. The justification for these specific binary subgroup assignments includes the correct mixing angles for the lepton PMNS matrix that relates the wave functions for the SM flavor states to their mass states.

One major consequence of having the fundamental particles represent specific discrete symmetry binary subgroups is that the lepton and quark SM weak isospin states are not the same as the mass states, in agreement with experimental results. Otherwise, in the traditional SM view with the lepton and quark families representing the continuous symmetry group SU(2), there is no known reason for the mass states to be different from the SM weak isospin states and this difference is simply attributed to a mismatch between the weak isospin states and the mass states!

We proposed [2] that the reason for the difference between the SM weak isospin flavor states and the mass states depends upon the continuous symmetry requirement of quantum field

Table 2: Quaternion Generators. $\phi = (1 + \sqrt{5})/2$

Fam.	Grp.	U3 Generator	Factor	Ang./2
ν_e, e^-	SU(2) 2T	k $-\frac{1}{2}i - \frac{1}{2}j + \frac{1}{2}k$	-0.2642	52.66°
ν_μ, μ^-	2O	$-\frac{1}{2}i - \frac{1}{\sqrt{2}}j + \frac{1}{2}k$	+0.8012	18.38°
ν_τ, τ^-	2I	$-\frac{1}{2}i - \frac{\phi}{2}j + \frac{1}{2\phi}k$	-0.5370	61.24°

Table 3: Comparison to NuFit 5.2 values for neutrino observables.

	bfp $\pm 1\sigma$	3σ range	predicted
$\sin^2 \theta_{12}$	$0.303^{+0.012}_{-0.012}$	$0.270 \rightarrow 0.341$	0.3172
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \rightarrow 35.74$	34.29°
$\sin^2 \theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.408 \rightarrow 0.603$	0.4627
$\theta_{23}/^\circ$	$42.2^{+1.1}_{-0.9}$	$39.7 \rightarrow 51.0$	42.85°
$\sin^2 \theta_{13}$	$0.0222^{+0.0006}_{-0.0006}$	$0.0205 \rightarrow 0.0240$	0.0223
$\theta_{13}/^\circ$	$8.58^{+0.12}_{-0.12}$	$8.23 \rightarrow 8.91$	8.56°

theory (QFT) because the fields are required to be continuous. Having our specific discrete symmetry binary subgroups define their weak isospin states within the framework of the SM violates this QFT continuous symmetry requirement. Therefore, to eliminate this violation, we determined that a linear superposition of the binary subgroup generators was needed so that acting collectively the three discrete symmetry binary subgroups could mimic the continuous symmetry group SU(2).

This linear superposition is achieved separately for the lepton families and for the quark families. The quaternion generators for each of the three lepton binary subgroups are the same for the first two generators, i.e. the quaternions U1 = i and U2 = j of SU(2), but the third generators, the U3's, which should each be k, are different for each subgroup and are listed in Table 2. The normalized contributing factors to the linear superposition for each lepton family binary subgroup are listed in column four of Table 2 as well as their half-angle contributions whose differences determine the PMNS mixing angles.

The absolute values of our predicted mixing angles for the lepton PMNS mixing matrix are listed in Table 3, showing that they agree with the empirically determined ranges of values. Note that we predict the θ_{23} angle of 42.85° to be in the first quadrant, in agreement with some of the empirical values but in contrast to other results that suggest the second quadrant.

The PMNS matrix for the lepton families is the product

of the charged-lepton and the neutrino matrices

$$U_{PMNS} = U_e^\dagger U_\nu. \quad (1)$$

If the charged-lepton states do not mix, or their mixing is minimal, U_e is diagonal, an assumption that is discussed in a later section, then the PMNS mixing matrix represents neutrino mixing only. Therefore, the PMNS matrix relates the neutrino mass states (ν_1, ν_2, ν_3) to the SM weak isospin states $(\nu_e, \nu_\mu, \nu_\tau)$ as

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \quad (2)$$

Keeping a phase factor δ for CP violation consideration, our PMNS matrix in the standard popular 3x3 formulation is

$$\begin{pmatrix} 0.817 & 0.557 & -0.1491e^{-i\delta} \\ -0.413 - 0.084e^{i\delta} & 0.606 - 0.057e^{i\delta} & -0.669 \\ -0.383 + 0.051e^{i\delta} & 0.558 + 0.062e^{i\delta} & 0.725 \end{pmatrix}. \quad (3)$$

Therefore, we have established the very important result that each lepton family represents its own specific discrete symmetry binary subgroup of SU(2) because our assigned groups lead directly to correct predictions of the mixing angles for the PMNS mixing matrix. And we know that the origin of this mixing is the QFT requirement for continuous symmetry behavior. So the discrete symmetries of the lepton families mix collectively via a linear superposition to mimic the continuous symmetry group SU(2).

Our binary subgroups of SU(2) have their fundamental domain \mathcal{D} in the upper half of the complex plane between $-1/2$ and $+1/2$ as shown in Fig. 1 with three symmetric points $\tau_{sym} = i\infty, i$, and $\omega = \exp(2\pi i/3)$. Although no value of the modulus τ preserves the full symmetry of SU(2) (or its isomorphic modular group $SL(2, \mathbb{Z})$), at the three τ_{sym} values, specific \mathbb{Z}_N symmetries are preserved, with $N = 2, 3$, or 4. When τ lies on the border of \mathcal{D} , CP symmetry is preserved [3, 4], but small deviations expressed by $|\tau - \tau_{sym}|$ lead to CP symmetry being broken and hierarchical mass patterns emerging according to the sequence $(1, \epsilon, \epsilon^2)$, where $\epsilon \ll 1$. See Appendix A for the details which were introduced in a modular group analysis.

4 Circulant matrix approach

We know from the collective action dictated by the continuous symmetry constraint of QFT that perhaps the three lepton families should be treated as equals, a symmetry that suggests they obey the group U(3). If we assume U(3) symmetry for this equal treatment, we can utilize its expression as a 3x3 circulant matrix [5, 6], from which the famous Koide formula [7] has been derived.

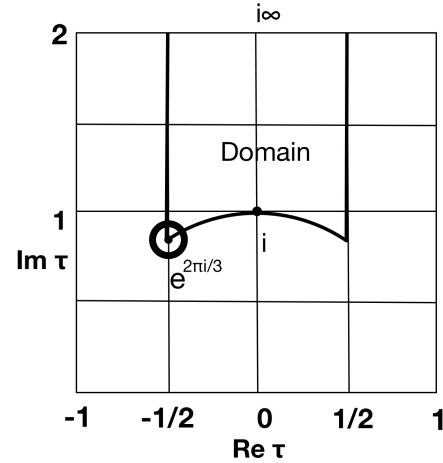


Fig. 1: The fundamental domain of our three SU(2) subgroups 2T, 2O, and 2I (and of the modular group $\Gamma = SL(2, \mathbb{Z})$) with its three symmetric points $\tau_{sym} = i\infty, i, \omega$, where $\omega = \exp(2\pi i/3) = -0.5 + 0.866i$, with a small ring of acceptable values around ω .

We now paraphrase a 2006 article by C. Brannen [5], which shows how to use this type of equality to derive the Koide formula for the charged-lepton mass values from a circulant matrix and then proceeds to derive the mathematical relations that lead to the prediction of reasonable neutrino mass values in the meV energy range.

The 3x3 1-circulant matrix

$$G(A, B, C) = \begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix}, \quad (4)$$

where A, B and C are complex constants, has eigenvectors of the form

$$|n\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \exp(+2n\pi i/3) \\ \exp(-2n\pi i/3) \end{pmatrix}, \quad (5)$$

with $n = 1, 2, 3$. By requiring the eigenvalues λ_n to be real, the circulant matrix can be rewritten in the form

$$G(\mu, \eta, \beta) = \mu \begin{pmatrix} 1 & \eta \exp(+i\beta) & \eta \exp(-i\beta) \\ \eta \exp(-i\beta) & 1 & \eta \exp(+i\beta) \\ \eta \exp(+i\beta) & \eta \exp(-i\beta) & 1 \end{pmatrix}, \quad (6)$$

with η assumed to be non-negative. The η and β are pure numbers, whereas μ will scale with the eigenvalues given by

$$G(\mu, \eta, \beta) |n\rangle = \lambda_n |n\rangle = \mu (1 + 2\eta \cos(\beta + 2n\pi/3)) |n\rangle. \quad (7)$$

From the traces of G and G^2 one derives the eigenvalue relationships

$$\lambda_1 + \lambda_2 + \lambda_3 = 3\mu \quad (8)$$

and

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 3\mu^2 (1 + 2\eta^2). \quad (9)$$

From here one obtains the Koide formula by setting $\eta^2 = 0.5$:

$$\frac{(\lambda_1 + \lambda_2 + \lambda_3)^2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} = \frac{3}{2}. \quad (10)$$

By setting the eigenvalues $\lambda_i = \sqrt{m_i}$, the 1982 formula proposed by Koide for the masses of the charged leptons is:

$$\frac{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2}{m_e + m_\mu + m_\tau} = \frac{3}{2}. \quad (11)$$

Using the known mass values of the electron and the muon, the mass value of the tau was predicted [7] to be in agreement with future experimental results to better than two decimal places!

Consequently, from knowing the masses of the charged leptons, one determines [5] that

$$\begin{aligned} \mu_1 &= 17716.13(109) \text{ eV}^{0.5} \\ \eta_1^2 &= 0.500003(23) \\ \beta_1 &= 0.2222220(19) \end{aligned} \quad (12)$$

where the subscript 1 has been added to distinguish these parameters from the future neutrino parameters. Notice that β_1 is essentially $2/9$ and perhaps could be related to the phase $\phi = -2\pi/9$ of the scalar potential in the modular group approach introduced in the Appendix.

5 Lepton mass hierarchy

Before there was any evidence of tau neutrino mixing with the electron neutrino, the tribimaximal matrix with its zero value in the (1,3) position was thought by researchers to be the PMNS matrix that best represented the neutrino data:

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (13)$$

Of course, we will substitute our PMNS matrix for this approximate matrix, but first we shall continue to follow the original article [5] in order to reveal its amazing result.

Left-multiplying this tribimaximal matrix by a matrix of the circulant eigenvectors achieves a simple product with the value of $\tau = \omega$, i.e. the lower left corner at $\tau_0 = \exp(2\pi i/3)$ in the domain region:

$$\alpha = \omega = e^{2\pi i/3} = -0.5 + 0.866i : \quad (14)$$

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \alpha & \alpha^* \\ 1 & 1 & 1 \\ 1 & \alpha^* & \alpha \end{pmatrix} \begin{pmatrix} 0.8165 & 0.5773 & 0 \\ -0.4082 & 0.5773 & -0.7071 \\ -0.4082 & 0.5773 & 0.7071 \end{pmatrix} = \quad (15)$$

$$= \begin{pmatrix} 0.7071 & 0 & -0.7071i \\ 0 & 1 & 0 \\ 0.7071 & 0 & 0.7071i \end{pmatrix}.$$

This resulting matrix is the 24th root of unity! That is, its 24th power is the unit matrix.

Note there exists many mathematical relationships from here which we could list, such as relationships to the expansions of the j-invariant $j(\tau)$, the eta function, etc., which involve 24th powers or 24th roots, but we do not need these mathematical functions to derive the neutrino mass values. However, these functions would be needed for expressing the wave functions of the particles.

Continuing onward, we know that the true PMNS mixing matrix is not the tribimaximal matrix but our PMNS matrix determined by our binary subgroups. We can achieve the same result, i.e. the 24th root of unity matrix, by using a value of τ slightly different from ω . After trying several different values, using this value of τ :

$$\alpha = \tau = -0.496 + 0.877i, \quad (16)$$

to multiply the values in our PMNS matrix leads to

$$\begin{aligned} &\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \alpha & \alpha^* \\ 1 & 1 & 1 \\ 1 & \alpha^* & \alpha \end{pmatrix} \begin{pmatrix} 0.817 & 0.557 & -0.149 \\ -0.4213 & 0.6084 & -0.669 \\ -0.3936 & -0.5654 & 0.7248 \end{pmatrix} = \\ &= \begin{pmatrix} 0.7014 - 0.0731i & 0.021 & 0.021 - 0.7059i \\ 0.008 & 0.927 & 0.116 \\ 0.7014 - 0.0731i & 0.021 & 0.0707 + 0.7075i \end{pmatrix}. \end{aligned} \quad (17)$$

The result is within 1% of the 24th root of unity when using our PMNS mixing matrix and this value of α . A slight adjustment in the α value could make the fit closer.

As shown in the Appendix, the modular subgroup approach agrees that this value of α is a universal fit for the SU(2) subgroups or their equivalent modular subgroups.

Therefore, we will consider our α to be close enough and continue with this approach in order to establish the relationship between the charged-lepton states and the neutrino states as well as to determine the neutrino mass values.

Following the procedure, we define the mass operator M associated with the eigenvalue $\lambda_i = \sqrt{m_i}$ to take left-handed states to right-handed states and vice-versa:

$$\begin{aligned} M |R\rangle &= |L\rangle \\ M |L\rangle &= |R\rangle. \end{aligned} \quad (18)$$

In general, M^2 picks up a Berry-Panchartnam or topological phase to become complex upon returning to the original state, so we can express

$$M^2 |L\rangle = p^2 \exp(2ik) |L\rangle. \quad (19)$$

Note that if $\kappa = 2\pi/24 = \pi/12$, then the state $|L\rangle$ is brought back to a multiple of $|L\rangle$ by

$$M^{24}|L\rangle = p^{24}|L\rangle. \quad (20)$$

Therefore, if M^2 operates on the left-handed electron as

$$M^2|e_L\rangle = p^2|e_L\rangle, \quad (21)$$

then we would have

$$M^{24}|\nu_L\rangle = p^{24}|\nu_L\rangle, \quad (22)$$

meaning that the masses of the two particle states in the lepton family differ by a factor of p^{22} .

The mass scale factors μ_1 for charged leptons and μ_0 for neutrinos are therefore related by

$$\mu_0^2 = \mu_1^2/3^{22} = 0.100^2, \quad (23)$$

where the factor of three comes from the square of the normalization factor $1/\sqrt{3}$ for the three eigenvectors in the matrix multiplying the PMNS matrix above. Likewise, there is a phase difference

$$\beta_1 - \beta_0 = -\frac{\pi}{12}. \quad (24)$$

Neutrino mass predictions using $\mu_1/\mu_0 = 3^{11}$ and the phase difference $\beta_1 - \beta_0 = -\pi/12$ in the eigenvalue $G(\mu, \eta, \beta)$ results in these reasonable predicted neutrino mass values:

$$\begin{aligned} m_1 &= 0.3 \text{ meV} \\ m_2 &= 8.9 \text{ meV} \\ m_3 &= 50.7 \text{ meV}, \end{aligned} \quad (25)$$

assuming still that $\eta^2 = 0.5$. Although these predicted mass values fit the neutrino values estimated from experimental results, we will need to wait for confirmation from ongoing and future experiments.

However, do these values produce the 3/2 value in the Koide formula? Not with the original version, but they do agree if we utilize the valid alternative version in which the square root of the lowest mass neutrino m_1 is preceded by a negative sign [5,6].

Our results for the leptons being 3-D objects with the discrete symmetries of the binary subgroups 2T, 2O, 2I of SU(2) not only predict reasonable neutrino mass values but also predict the normal mass hierarchy NH of the neutrino mass states as $m_1 < m_2 < m_3$. However, the present experimental data also allows for an inverse hierarchy IH as $m_3 < m_1 < m_2$.

In addition, we have exactly 3 lepton families, in agreement with the Z^0 decay results, but there continues to be speculation about an additional lepton, such as a sterile neutrino. And, our approach treats the 3 lepton families as symmetrical contributors as eigenvectors of a 1-circulant matrix, whereas all other analyses place the charged leptons and the neutrinos into irreducible representations of a subgroup, usually in a 3 or 3' irreducible representation. Which method Nature has chosen will be determined by experiments in the near future.

6 Review of the derivation

Our sequence of steps to neutrino mass predictions were:

1. We first established that the three discrete symmetry 3-D binary subgroups 2T, 2O, 2I of SU(2) are represented by the three lepton families and are 3-D objects instead of point particles at the Planck scale. These are the correct groups because when they collectively mimic the continuous group SU(2) to satisfy the requirement of QFT they produce the correct mixing angles for the PMNS mixing matrix.
2. By treating the lepton families as equals in symmetry group U(3), the famous Koide formula is derived via a 3x3 1-circulant matrix, revealing that the important mass quantity is the square root of the mass values $\lambda = \sqrt{m}$ instead of the mass value itself. Three parameters μ_1, η_1 , and β_1 for calculating the mass eigenvalues of the charged leptons were determined.
3. With the value $\alpha = \tau = -0.496 + 0.877i$ in the domain of SU(2), we derived the 24th root of the unity matrix by multiplying our PMNS matrix by the appropriate 1-circulant eigenvector matrix. This specific value of α agreed with the findings of the modular group approach that uses subgroups of SL(2,Z), i.e. that the value applies equally to our three subgroups of SU(2).
4. By inserting the Berry-Panchartnam phase factor when returning a left-handed lepton state back to its original state for the mass-squared operator M^2 , there resulted a factor of p^{22} difference between the charged-lepton states and the neutrino states as well as a phase difference of $\pi/12$.
5. Finally, using the factor of 3^{11} that connected the neutrino mass values to the charged-lepton mass values for the parameter ratio μ_1/μ_0 , with the eigenvalue expression $G(\mu, \eta, \beta)$ we predicted reasonable neutrino mass values in the meV range in NH: $m_1 = 0.3 \text{ meV}, m_2 = 8.9 \text{ meV}, m_3 = 50.7 \text{ meV}$.

In the next section, our goal is to relate the above results to the invariants $N = 1, 108, 1728$ of the three lepton family binary subgroups 2T, 2O, 2I respectively. Therefore, we should be able to understand how the lepton family mass values originate from their 3-D geometric properties.

7 Invariant theory connection

Invariant theory connects the elliptic modular function $j(\tau)$ to invariants of our specific discrete symmetry binary subgroups. Each invariant N is related by

$$j(\tau) = \frac{W_1}{NW_2}, \quad (26)$$

where W_1 is expressed in two complex variables for the vertices and W_2 for the face centers of the polyhedrons [8] for

the binary groups 2T, 2O, and 2I, with $N = 1, 108$, and 1728 , respectively.

These invariants are similar to the charged-lepton mass values in MeV, i.e. 0.511 , 105.66 , and 1776.82 , but they have no energy units, so we would naturally consider their ratios instead. However, the question remains, why is there a change from the original geometrical values N that are invariant under all fractional linear transformations to the experimentally determined universal values for the charged lepton masses?

One possible answer could be related to the change of the value of τ from $\omega = \exp(2\pi i/3) = -0.5 + 0.866i$ to the nearby value, $\alpha = \tau_0 = -0.496 + 0.877i$ in the domain. However, we realize that we have simply changed the question without providing the reason for the change.

However, recall that the lepton PMNS mixing matrix

$$U_{PMNS} = U_e^\dagger U_\nu, \quad (27)$$

relates the wave functions, so we can speculate that there could be a slight mixing among the charged-lepton states, particularly among the electron and the muon states. Experiments are being planned specifically to check for this mixing possibility.

If we want the tentative geometrical state mass values suggested by the SM binary group N values to become the measured mass state values, one would have a mass matrix very close to being the unitary matrix but containing some small off-diagonal terms. Such a mass matrix might look like

$$\begin{pmatrix} 1 & -0.0274 & 0 \\ 0.0274 & 1 & -0.0033 \\ 0 & 0.0558 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1} \\ \sqrt{108} \\ \sqrt{1728} \end{pmatrix} \quad (28)$$

$$= \begin{pmatrix} \sqrt{0.511} \\ \sqrt{105.66} \\ \sqrt{1776.82} \end{pmatrix},$$

in which we have used the square root of the mass values as determined by the Koide relationship. That is, the slight mixing among the charged-lepton wave functions could be carried over to a mass matrix relating our N values to the measured mass values. Of course, mass ratios would be preferred. But we are still left with determining an energy scale for these mass values.

8 Conclusions

We have been able to calculate the mass values of the neutrinos by following a series of steps beginning with the correct identification of the discrete symmetry binary subgroups of $SU(2)$, which are equivalent to subgroups of the modular group $SL(2, \mathbb{Z})$. The three lepton families represent 2T, 2O, and 2I, and we derived their PMNS mixing matrix for

their wave functions from their quaternion generators in order to agree with a continuous symmetry constraint dictated by quantum field theory (QFT).

Assuming that these binary subgroups together act as a $U(3)$ symmetry, the famous Koide formula follows directly via a 1-circulant matrix approach that also relates the PMNS matrix to the 24th root of unity matrix by using a modulus τ value slightly different from the symmetry point value $\omega = \exp(2\pi i/3) = -0.5 + 0.866i$ in the fundamental domain of $SU(2)$ and its isomorphic modular group $SL(2, \mathbb{Z})$. That is, we set $\tau = -0.496 + 0.877i$. This method then produced a factor of 3^{11} difference in the mass values of the charged leptons and the neutrinos, which led directly to the predicted neutrino mass values being $m_1 = 0.3$ meV, $m_2 = 8.9$ meV, $m_3 = 50.7$ meV.

Although we assumed that the charged-lepton mixing matrix was diagonal, the invariants $N = 1, 108$, and 1728 from geometry and invariant theory for the electron family, muon family, and tau family binary subgroups, respectively, indicated that there is a slight mixing of the charged leptons also. We suggested a matrix that has unit values on the diagonal but also has a few very small off-diagonal terms to relate the N values to the actual charged lepton universal mass values 0.511 MeV, 105.66 MeV, and 1776.82 MeV. Of course, the mass scale would still remain to be determined.

In a future article, i.e. part II, we determine the origin of the quark mass values. We will establish that a similar approach succeeds for modulus τ values near to the other symmetric point $\tau = i$ within the fundamental domain. In the quark case, we predict 4 quark families, (u,d), (c,s), (t,b), and (t' , b'), which represent [1, 2] the discrete symmetry binary subgroups [333], [433], [343], and [533], respectively, in R^4 . QFT dictates a continuous symmetry group behavior, so the linear superposition of their generators to mimic $SU(2)$ produces the CKM4 mixing matrix with CKM[‡] submatrix values.

The quark mass values fit a four term Koide formula separately for the up and the down states, and a 4x4 circulant matrix defines eigenvectors. The predicted t' quark should have a mass value of about 3 TeV, a mass value large enough to gain a factor of about 10^{13} multiplying the present Jarlskog constant, thereby providing a value large enough to help explain the baryon asymmetry of the Universe [BAU] in terms of CP violation [19].

Appendix: Modular group

A brief look into what researchers in the past decade have achieved using subgroups of the modular group $SL(2, \mathbb{Z})$ in order to calculate neutrino mass values will demonstrate some agreement with our results. We therefore provide a summary of their research by paraphrasing a recent article [3, 4] to illustrate how our bottoms-up approach from the binary sub-

[‡]Cabibbo-Kobayashi-Maskawa

groups of SU(2) can relate to the top-down calculations using modular groups related to superstring theory. The modulus τ of $SL(2, \mathbb{Z})$ is the single field quantity associated with the fermion particle states.

Our three discrete symmetry binary subgroups 2T, 2O, and 2I of SU(2) for the lepton families are isomorphic to these modular double cover subgroups:

$$2T = \Gamma'_3, \quad 2O = \Gamma'_4, \quad 2I = \Gamma'_5. \quad (29)$$

Therefore, their modular mathematical properties apply to our discrete symmetry binary subgroups of SU(2) as well.

Lepton flavor models based upon the modular symmetry group $\Gamma = SL(2, \mathbb{Z})$ utilize its subgroups $\Gamma'_N = SL(2, \mathbb{Z}_N)$, such as the double covers $\Gamma'_2 = S'_3$, $\Gamma'_3 = A'_4$, $\Gamma'_4 = S'_4$, $\Gamma'_5 = A'_5$ of the permutation groups S_3 , A_4 , S_4 , A_5 . With significant fine-tuning and a number of coupling constants, the mass hierarchies of the leptons can be reproduced in terms of a small parameter when the three lepton families are assigned to an irreducible representation of a modular subgroup, such as $\Gamma'_3 = A'_4$.

The modular group's fundamental domain \mathcal{D} shown in Fig. 1 has the three symmetric points $\tau_{sym} = i\infty, i$, and $\omega = \exp(2\pi i/3)$ with its three τ_{sym} values preserving specific \mathbb{Z}_N symmetries, i.e. those with $N = 2, 3$, or 4 . When τ lies on the border, CP symmetry is preserved, but small deviations lead to CP symmetry being broken and hierarchical mass patterns emerging according to the sequence $(1, \epsilon, \epsilon^2)$.

This recent research has revealed that the lepton data suggests a value of τ near the cusp $\tau_0 = \omega = -0.5 + 0.866i$, with the best fit being

$$\tau = -0.496 + 0.877i \quad (30)$$

with a viable region being a small ring of values around the cusp ω , as shown in Fig. 1. The result is universal, meaning that its value is independent of which modular subgroup is being considered.

The research defined a scalar potential V_m near $\tau_0 = \omega$ that depends upon an integer parameter m and a phase angle ϕ , with a minimum in the scalar potential at

$$\frac{0.0145}{m + 0.0025}. \quad (31)$$

If the phase angle is included, the minimum occurs at

$$\phi_{min} = \frac{-2\pi}{9} \quad (32)$$

independent of m , producing for $m = 2$ the result

$$\frac{0.0145}{2 + 0.0025} \exp\left(\frac{-2\pi i}{9}\right) \leftrightarrow \tau_{min} = -0.492 + 0.875i. \quad (33)$$

The scalar potential V_m has a deep trench from ω upward from ω in the first quadrant direction that depends upon the quantity

$$[j(\tau) - 1728]^{m/2} \quad (34)$$

where $j(\tau)$ is the j -invariant of elliptic modular functions.

Therefore the modular group approach has revealed some very important results, particularly telling us that there seems to be no dependence upon which modular subgroup Γ'_N is being used as the modular subgroup for lepton flavor symmetry! Whence, the above results apply to all the modular subgroups equally or, equivalently, to our specific binary subgroups of SU(2) for the lepton families.

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A Derivation of Planck's Constant from the Principles of Electrodynamics

Yuri Heymann

3 rue Chandieu, 1202 Geneva, Switzerland. E-mail: y.heyman@yahoo.com

A formula for Planck's constant is derived from the Bohr model and Larmor formula, leading to its expression as a function of the proton-to-electron mass ratio, the elementary charge of an electron, and variables as the speed of light and vacuum permittivity. While Planck's constant obtained from its theoretical formula deviates from the Committee on Data of the International Science Council (CODATA) value by a tiny epsilon due to modelling assumptions or geometrical aspects, 98.6% of this deviation is explained by the relativistic effect of electron mass and the mass gap due to the binding energy of electron. As such the relative error of Planck's constant adjusted for the aforementioned factors remains about 22.2 parts per million.

1 Introduction

The Planck's constant known as quantity h , is a fundamental constant in physics of importance in quantum mechanics, statistical mechanics, electronics and metrology. The constant h appears in Max Planck's work on black-body radiation and its spectrum [11–13], a collaborative effort on Kirchhoff's law. In 1905, Einstein publishes the photoelectric effect for the measurement of quantized energies of photons $E = h\nu$, where ν is the frequency of electromagnetic waves [3]. The photoelectric experiment is conducted inside a vacuum chamber exposed to light at different frequencies, causing electrons to be ejected from a metal plate. Einstein's photoelectric relation expresses the kinetic energy of ejected electrons by the relation $eV = h\nu - w$, where w is the work function of the metal, representing the energy level that electromagnetic waves must exceed to eject electrons from the plate. Early photoelectric experiments by Hughes [14] and Richardson and Compton [4], yield estimates of h/e with uncertainties of about 10%. As Millikan refined the experiment, he obtained a value of $h = 6.57 \times 10^{-34}$ J s [9].

The Kibble balance, formerly called a watt balance, is a metrological instrument to measure the weight of a tiny object very precisely by the electric current and voltage powering the balance. This instrument, developed in 1975 by Bryan Kibble, is used to measure Planck's constant on the basis of the Josephson and quantum Hall effect. The Josephson effect, is described by the set of equations $I(t) = I_c \sin(\varphi(t))$ and

$$\frac{\partial \varphi}{\partial t} = \frac{2e}{\hbar} V(t),$$

where $V(t)$ and $I(t)$ are the voltage and current flowing through the Josephson junction, I_c the critical current, \hbar the reduced Planck's constant, and e the elementary charge. A Josephson junction is a superconducting tunnel junction made of a thin film of a few micrometers separating superconducting wires [5, 6], whereas the Hall effect is produced by a current flowing through a conductor exposed to a magnetic field perpendicular to the current. The method exploits discretised

jumps in the resistivity computed as

$$R = \frac{V_{Hall}}{I_{ch}} = \frac{h}{e^2 \nu},$$

where V_{Hall} is the Hall voltage and I_{ch} the channel current, e the elementary charge, and h Planck's constant. The divisor ν can be an integer $\nu = 1, 2, 3, \dots$ or a fractional number $\nu = 1/3, 2/5, 3/7, \dots$ producing jumps as the density of electrons varies. An example of such quantization are Landau levels representing discretised energies as a proposed solution to the Schrödinger's equation [7].

A Planck's constant of $h = 6.62607034(12) \times 10^{-34}$ J s was obtained in recent work by a team of researchers using a watt balance to demonstrate its capability [15]. The joule balance is an enhanced watt balance where dynamic measurements are replaced by a static measurement for convenience purpose. The performance of the joule balance was demonstrated by measuring Planck's constant, $h = 6.626104(59) \times 10^{-34}$ J s with an 8.9 ppm uncertainty [18]. A detailed view of the historical development of Planck's constant measurements is provided in Reiner [16].

In the present work, a formula for Planck's constant was obtained from the Bohr model and Larmor formula, see Section 2. The coupling between both models into a single expression for quantity h involves a membrane representation of the electron as a surface covering the Bohr sphere, where the flux of energies radiating across the membrane is determined by the mass of the proton.

Table 1: Fundamental constants from Committee on Data for Science and Technology (CODATA), 2014 [10].

Constant	Symbol	Value	Unc. u [*]
Planck constant	h	$6.626070040(81) \times 10^{-34}$ J s	8.7×10^{-8}
Electron mass	m_e	$9.10938356(11) \times 10^{-31}$ kg	8.8×10^{-8}
Proton mass	m_p	$1.672621898(21) \times 10^{-27}$ kg	8.9×10^{-8}
Elementary charge	q, e	$1.602176620(89) \times 10^{-19}$ C	4.4×10^{-8}
Vacuum permittivity	ϵ_0	$8.8541878128(13) \times 10^{-12}$ F / m	–
Speed of light	c	299 792 458 m/s	–

SI units, Intern. Committee for Weights and Measures.

* u, means relative standard uncertainty, source [17].

Planck's constant predicted by the present model and its deviation from CODATA (see Table 1) are provided in Section 3. The attribution of errors by modeling assumptions is described at the end. Of the deviation, 98.6% is explained by the relativistic effect of electron mass and mass gap due to the binding energy of an electron in its orbital, which is a fairly promising result.

2 Method

2.1 Larmor formula

The Larmor formula expresses the power radiated by a non-relativistic charged particle as a result of acceleration [8]. The Larmor formula in its current form appears in more recent works, see the Bremsstrahlung effect and the study of electromagnetic radiation emitted in cyclotrons. The electromagnetic wave as a bimodal function is often represented as a tuple of two undulatory waves moving in the same direction, where functions in Hilbert space \mathcal{L}^2 are orthogonal by the inner product. The magnetic cardioid or lemniscate are geometric representations, involving the interaction between an electron and an electric field. These basics of currents and electromagnetism are useful wave representation of the electron. The magnetic field, commonly denoted by the letter B , is represented by an E-field in the current context. Such an E-field is denoted as E_θ , where θ is the angle between the radial electric field E_r and the orientation of E_θ itself.

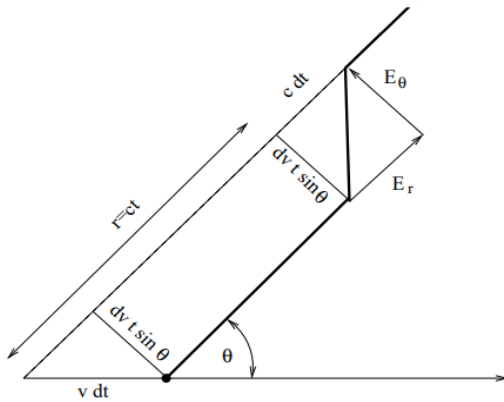


Fig. 1: E-field in the region of an electromagnetic pulse in polar coordinates.

As we suppose the E-field is proportional to the inverse of the wave frequency, where the ratio of wave frequencies is equal to the ratio of velocities, we have $\frac{E_\theta}{E_r} = \frac{v_r}{v_\theta}$. By the Pythagorean theorem, we get:

$$\frac{E_\theta}{E_r} = \frac{\Delta v t \sin(\theta)}{c \Delta t}, \quad (1)$$

where E_θ and E_r are the tangential and radial components of the E-field respectively, c the speed of light, and t the time of

a pulse Δt . From equality $v/t = dv/dt$, we get $v\Delta t = t\Delta v$. By definition, t is the time to accelerate a charged particle q from rest to velocity v .

The radial component of an E-field as in Coulomb's law, is expressed as follows:

$$E_r = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2}, \quad (2)$$

where r is the radius, q the charge of the particle and ϵ_0 the vacuum permittivity.

Given the acceleration term $a = \frac{\Delta v}{\Delta t}$ and joint relation $r = ct$, (1) and (2) lead to:

$$E_\theta = \frac{q a}{4\pi\epsilon_0 c^2 r} \sin(\theta). \quad (3)$$

By Poynting's theorem, i.e. $S = c\epsilon_0 E^2$, the flux is expressed as:

$$S = \frac{1}{16\pi^2 c^3 \epsilon_0 r^2} q^2 a^2 \sin^2 \theta. \quad (4)$$

The angular element in spherical coordinates is

$$d\Omega = r^2 \sin \theta d\theta d\varphi,$$

leading to the below expression for the power radiated by an electron:

$$P = \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} S r^2 \sin \theta d\theta d\varphi. \quad (5)$$

As

$$\int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \sin^3 \theta d\theta d\varphi = \frac{8\pi}{3},$$

we obtain:

$$P = \frac{8\pi}{3} \frac{q^2 a^2}{16\pi^2 c^3 \epsilon_0}, \quad (6)$$

which is the Larmor formula for the power radiated by a particle of charge q under acceleration a , in say Watt per squared steradians where the variables in the argument are expressed in the International System of Units (SI).

2.2 Thomson cross section to Planck formula

Considering an E-field where the field lines are collinear and pulsed in the direction orthogonal to the electron orbital, the energy flux over a cross section σ_e transverse to the power inflow, is given by:

$$P_{in} = c\epsilon_0 E_r^2 \sigma_e, \quad (7)$$

where the energy flux is the speed of light times the energy density as given by Poynting's theorem.

The power radiated by a ground state electron revolving around a nucleus is given by the Larmor formula, which can be expressed as follows:

$$P_{out} = \frac{8\pi}{3} \frac{q^2 (q E_r / m_e)^2}{16\pi^2 c^3 \epsilon_0}. \quad (8)$$

As $P_{in} = P_{out}$, (7) and (8) lead to the well-known Thomson cross section for a free electron in its orbital:

$$\sigma_e = \frac{8\pi}{3} \left(\frac{q^2}{4\pi\epsilon_0 m_e c^2} \right)^2. \quad (9)$$

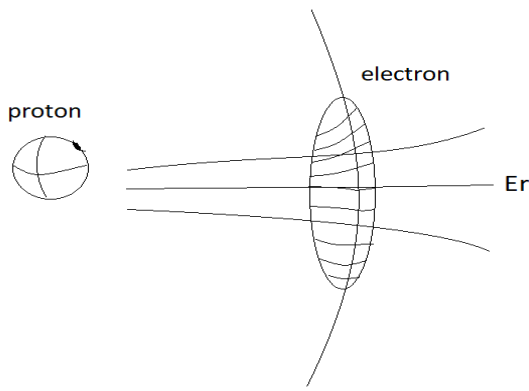


Fig. 2: Membrane representation of the electron where the electron is represented as a surface covering the Bohr sphere. A more natural shape for the atom of hydrogen would be a Horn Torus, or “apple shape” having field lines connecting its poles. The flux of energy crossing the membrane is determined by the mass of the proton as used in the scaling of the Thomson cross section.

By the squared-mass scaling rule, we multiply (9) by $(m_p/m_e)^2$, a scaling of the Thomson cross section to the Bohr sphere, yielding:

$$\sigma_0 = \frac{8\pi}{3} \left(\frac{q^2 m_p}{4\pi\epsilon_0 m_e^2 c^2} \right)^2. \quad (10)$$

The scaled Bohr radius, expressed as

$$r_1 = \frac{\epsilon_0 h^2}{4\pi^2 m_e q^2},$$

is a non-standard Bohr radius of electron orbital obtained by rescaling in a way that E in Poynting's theorem $S = c\epsilon_0 E^2$ is the standard wave of an electric field, for consistency with the Thomson cross section. By the scaled Bohr radius, the surface of the Bohr sphere $4\pi r_1^2$ is expressed as follows:

$$\sigma_s = \frac{\epsilon_0^2 h^4}{4\pi^3 m_e^2 e^4}. \quad (11)$$

The standard Bohr radius

$$r_0 = \frac{\epsilon_0 h^2}{\pi m_e e^2}$$

representing the radius of an electron orbital in the Bohr model [1, 2], is based on the electron identity $n \frac{h}{2\pi r} = m_e v$, where h is a quantity defined as the product of electron momentum

by one circumference of the ring, n the number of electrons, m_e the mass of an electron, and v its velocity.

As the electron from the Thomson cross section rescaled by the squared-mass scaling rule covers the whole surface of the Bohr sphere, we can match σ_0 with σ_s , i.e. (10) and (11). As such the Bohr sphere stands as a membrane of the electron, as seen in Fig. 2. As a result, the one circumference momentum of the electron, also known as the Planck's constant, is expressed as follows:

$$h = \frac{e^2}{c \epsilon_0} \sqrt{\pi \sqrt{\frac{2}{3}} \frac{m_p}{m_e}}, \quad (12)$$

where e is the elementary charge of an electron, m_e the mass of an electron, m_p the mass of a proton, ϵ_0 the vacuum permittivity, and c the speed of light.

3 Results

The Planck's constant computed from (12) with values in Table 1, yields $h = 6.6368 \times 10^{-34}$ J s, deviating from its CODATA value by 1.62 parts per thousand. Of this deviation, 87.4% is explained by the non-relativistic approximation of electron mass, 11.2% by the binding energy of the electron orbital, and 1.37% remains unexplained (see Fig. 3).

By introducing the relativistic mass of the electron $m_{el} = \frac{m_e}{\sqrt{1-(v_e/c)^2}}$ into (12), with the electron velocity

$$v_e = \frac{e}{\sqrt{4\pi\epsilon_0 r_0 m_e}}$$

resulting from the equilibrium between centripetal and Coulomb's force, where e is the elementary charge of the electron, r_0 the standard Bohr radius, m_e the mass of an electron, and ϵ_0 the vacuum permittivity, leads to the new value $h = 6.6247 \times 10^{-34}$ J s.

The binding energy of the electron in its orbital, as given by the potential energy using the rescaled Bohr radius r_e ,

Planck's constant deviation from CODATA

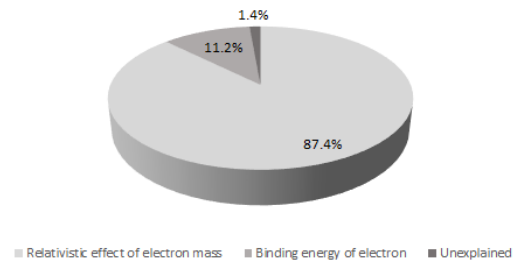


Fig. 3: Attribution of Planck's constant deviation from its CODATA value. Of a relative error of 1.62 parts per thousand, 87.4% is explained by the non-relativistic approximation of electron mass, 11.2% by the binding energy of the electron orbital, and 1.37% remains unexplained.

gives $K = \frac{1}{2} \frac{e^2}{4\pi\epsilon_0 r_e}$. By subtracting the mass gap $\Delta m_e = K/c^2 \simeq 3.059 \times 10^{-34}$ kg from the mass of the electron and applying relativistic adjustment (multiplying electron mass by the inverse of the Lorentz-FitzGerald contraction), yields a new Planck's value $h = 6.62592 \times 10^{-34}$ J s, of an accuracy of about 22.2 parts per million with respect to actual measurements (as explained by modelling assumptions or geometrical aspects, e.g. shape of atom departing from a perfect sphere).

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Zitterbewegung and the Non-Holonomy of Pseudo-Riemannian Spacetime

Pierre A. Millette

E-mail: pierre.millette@uottawa.ca, Ottawa, Canada

In this paper, we explore the connection between *zitterbewegung* for free particles, and the work of Rabounski and Borissova on Zelmanov's chronometric invariant formulation of General Relativity to calculate space and time physical observables [2, 6]. In the chr.inv.-analysis, the spin of a particle interacts with the space non-holonomy field of pseudo-Riemannian spacetime. From this, the particle gains an additional momentum which imparts a non-geodesic component to the particle's motion. The solution of the particle with spin chr.inv.-equation of motion is a spiral that can be visualized as being wound on a pulsating cylinder. Free electron oscillations occur at a frequency equal to the double angular velocity of the space rotation Ω , with fluctuations of the particle position on the order of its reduced Compton wavelength. We thus show that *zitterbewegung* is a direct manifestation of general relativistic space and time physical observables at the elementary particle level.

1 Introduction

In this paper, we explore the connection between *zitterbewegung*, German for “jittery” or “trembling motion”, as calculated for Dirac free particles [1], and the work of Rabounski and Borissova on Zelmanov's chronometric invariant formulation of General Relativity to calculate space and time physical observables [2, 6]. We will show that *zitterbewegung* is a direct manifestation of general relativistic space and time physical observables at the elementary particle level.

2 Zitterbewegung

Zitterbewegung was first recognized by Breit [7] and further analyzed and the name coined by Schrödinger [8, 9]. This solution is obtained in the Heisenberg representation equation of motion for the velocity operator α of the Dirac equation for a free particle

$$H_0 = \alpha \cdot \mathbf{p} + \beta m, \quad (1)$$

where m and \mathbf{p} are the mass and momentum of the free particle respectively, and the α and β matrices are used instead of the γ^μ ($\beta = \gamma_0$ and $\alpha_i = \gamma_0 \gamma_i$) [1, 10].

The space operator solution in the Heisenberg representation $\mathbf{x}(t)$ (i.e. $\alpha = \dot{\mathbf{x}}$) is then given by

$$\mathbf{x}(t) = \mathbf{x}(0) + \frac{\mathbf{p} c^2}{H_0} t + \left(\alpha(0) - \frac{\mathbf{p} c}{H_0} \right) \frac{i\hbar c}{2H_0} \exp(-2iH_0 t/\hbar), \quad (2)$$

where the first two terms on the right hand side of (2) correspond to the classical equation of motion trajectory of the particle, with the third term corresponding to a rapid oscillatory motion (*zitterbewegung*) about the classical trajectory.

The angular frequency of these oscillations is of order $2mc^2/\hbar \sim 2 \times 10^{21} \text{ s}^{-1}$ and their amplitude of order $\hbar/mc \equiv \lambda_C$, corresponding to fluctuations of the particle position on the order of its reduced Compton wavelength. Schrödinger found that the *zitterbewegung* results from the interference

between positive and negative-energy state amplitudes. Consequently, there has been a tendency to dismiss *zitterbewegung*, as its expectation value vanishes for wave-packets consisting entirely of positive-energy or negative-energy waves. In addition, it has not been observed experimentally due to its high-frequency, low amplitude motion, although indirect evidence of its presence has been suggested in numerous areas by some investigators [11–14]. One is reminded of the situation with Brownian motion, where it has not been observed directly, but evidence of its presence is now unquestionably accepted.

However, *zitterbewegung* has been investigated by many researchers, and identified in many areas. Indeed, there is other evidence that points to the reality of *zitterbewegung*. For example, the Darwin term which provides a small correction in the fine-structure of the energy level of *s*-orbitals of the hydrogen atom can be shown to result from *zitterbewegung* [15]. In the 1990s, David Hestenes revived *zitterbewegung* as a physical process when he recast it in terms of his Geometric Algebra [16–19]. Since then, much work has been done on modelling and detecting *zitterbewegung* — see for example [20–25] among many others.

3 Physical observables in General Relativity

Many practitioners of General Relativity do not realize that the theory is based on a 4-dimensional pseudo-Riemannian representation of spacetime and that the calculations they perform give results in that particular spacetime description. The pseudo-Riemannian characterization refers to the three space and one time dimensions, described by a metric with signature $(+---)$ or $(-+++)$, which uniquely results in space-like and time-like intervals. To properly understand the results obtained, the 4-dimensional calculations in general covariant form must be projected onto the observer's 3+1 space and time dimensions separately as space and time physical

observables.

This requires developing a mathematical theory to enable the calculation of observable components for any tensor. This work started in the 1930s — Landau and Lifshitz introduced the observable time interval and the observable three-dimensional interval in their classic *The Classical Theory of Fields* [26, §84]. Zelmanov, starting in 1941, developed such a comprehensive theory over many decades — it is known as the *theory of chronometric invariants* [2, 3]. The most complete description of the mathematical apparatus of physically observable quantities in General Relativity is given in the recent review article by Rabounski and Borissova [4]. It provides an up-to-date compendium of the results obtained by Zelmanov and the authors over the past decades, and allows for the calculation of the physical observable components of any tensor.

The basic approach consists in projecting a general covariant 4-dimensional tensor onto an observer's physical object frame of reference (e.g. the Earth's surface), consisting of a three-dimensional coordinate grid with “real” physical clocks (a spatial section $x_0 = ct = \text{constant}$, orthogonal to the observer's physical time line at time of observation t), known as the observer's *accompanying reference frame*.

The projection operator onto an observer's time line is the unit vector of the observer's four-dimensional velocity b^α with respect to his physical object frame of reference, which is tangential at each point of the observer's four-dimensional trajectory

$$b^\alpha = \frac{dx^\alpha}{ds}. \quad (3)$$

The projection of a tensor onto an observer's time line is given by its contraction with the vector b^α of his reference frame. In an observer's accompanying reference frame, his three-dimensional velocity with respect to his reference object is zero, $b^i = 0$, and its components are given by

$$b^0 = \frac{1}{\sqrt{g_{00}}}, \quad b_0 = g_{0\alpha}b^\alpha = \sqrt{g_{00}}, \quad b_i = g_{i\alpha}b^\alpha = \frac{g_{i0}}{\sqrt{g_{00}}}.$$

The projection operator onto an observer's three-dimensional spatial section is the four-dimensional symmetric tensor

$$h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta, \quad h^{\alpha\beta} = -g^{\alpha\beta} + b^\alpha b^\beta. \quad (4)$$

The projection of a tensor onto an observer's three-dimensional spatial section is given by its contraction with the tensor $h_{\alpha\beta}$ of his reference frame.

The observer's physical object reference frame has a gravitational field that can be rotated and deformed, and hence, the observer's local reference space can be inhomogeneous and anisotropic. If there is a spatial section everywhere orthogonal to the time lines, then the space is an *holonomic space*. If only spatial sections locally orthogonal to the time lines exist, then the space is a *non-holonomic space*.

Any coordinate grid that is at rest with respect to its reference physical object can be transformed to another coordinate grid through standard coordinate transformations, within the same spatial section. However, time transformations imply a change of spatial section (i.e. new clocks), and hence a change in the measurements of observable quantities. This requires that physical observable quantities in an observer's reference frame must be invariant with respect to time transformations throughout his three-dimensional spatial section $x^i = \text{constant}$, so these must be *chronometric invariant quantities*, and are named *chr.inv.-quantities* for short. Thus Zelmanov developed a general mathematical method to calculate physically observable chr.inv.-projections of any four-dimensional general covariant tensor (see [4] for details).

Accordingly, Zelmanov introduced chr.inv.-derivative operators with respect to time and the spatial coordinates given by

$$\frac{* \partial}{\partial t} = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t}, \quad \frac{* \partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}} \frac{\partial}{\partial x^0}, \quad (5)$$

where g_{00} and g_{0i} are components of the metric tensor $g_{\mu\nu}$, and the superscripted symbol $*$ indicates a chr.inv.-partial derivative. These are non-commutative: the order in which their second derivatives are taken gives different results, and their difference is not zero.

From these, three tensors can be defined:

1. A_{ik} : three-dimensional antisymmetric chr.inv.-tensor of the angular velocity with which the reference space of the observer rotates.
2. F_i : three-dimensional chr.inv.-vector of the gravitational inertial force.
3. D_{ik} : three-dimensional symmetric chr.inv.-tensor characterizing the rate of deformation of the observer's space.

Specifically, these tensors are explicitly given by:

$$A_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right) + \frac{1}{2c^2} (F_i v_k - F_k v_i), \quad (6)$$

where v_i is the tangential (linear) velocity of the rotation and c is the speed of light *in vacuo*,

$$F_i = \frac{1}{\sqrt{g_{00}}} \left(\frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t} \right) = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t} \right), \quad (7)$$

where $w = c^2 (1 - \sqrt{g_{00}})$ is the gravitational potential, originating from the gravitational field of the observer's reference object,

$$D_{ik} = \frac{1}{2} \frac{* \partial h_{ik}}{\partial t}, \quad D = \frac{* \partial \ln \sqrt{h}}{\partial t}, \quad h = \det \|h_{ik}\|, \quad (8)$$

where h_{ik} is the physically observable chr.inv.-metric of the observer's space, $D = h^{ik} D_{ik} = D_m^m$, the trace of the tensor of

the space deformation rate, is the relative dilatation rate of an elementary volume of the observer's space.

In addition, the tensor A_{ik} is further identified as the *space non-holonomy tensor*, which Zelmanov defined in the following theorem:

Zelmanov's theorem on the holonomy of space-time:

The identical equality to zero of the tensor A_{ik} in a four-dimensional region of space-time is the necessary and sufficient condition for the orthogonality of the spatial sections to the time lines everywhere in this region.

In other words, $A_{ik} \neq 0$ in a non-holonomic space-time region, and $A_{ik} = 0$ in a holonomic one. [4, p. 7]

Rotating spaces ($A_{ik} \neq 0$) are non-holonomic, as three-dimensional spatial sections are non-orthogonal to time lines in rotating spaces.

This section has covered the basics of Zelmanov's chronometric invariants theory to generate physically observable quantities in General Relativity by projecting general covariant 4-dimensional tensors onto an observer's physical object frame of reference to obtain physically observable chr.inv.-projections. The reader is encouraged to consult the recent compendium article of Rabounski and Borissova [4] for a deeper complete coverage of the chr.inv.-theory.

4 Geodesic motion of particles in pseudo-Riemannian spacetime

We first apply this formalism to the equations of motion of a particle. The motion of a particle under the influence of gravitation is characterized as freely falling along a geodesic (shortest-distance) line, known as free or geodesic motion. Under the action of additional non-gravitational forces, the particle deviates from its geodesic trajectory, and its motion is known as non-geodesic.

In a four-dimensional pseudo-Riemannian spacetime, the motion of a particle is geometrically determined by the parallel transport of the four-dimensional vector Q^α tangential to the points along the particle's four-dimensional trajectory, given by [6, see p. 9]

$$\frac{DQ^\alpha}{ds} = \frac{dQ^\alpha}{ds} + \Gamma_{\mu\nu}^\alpha Q^\mu \frac{dx^\nu}{ds}, \quad Q_\alpha Q^\alpha = \text{constant}, \quad (9)$$

where DQ^α is the absolute differential of the transported vector Q^α along the trajectory, dQ^α is the differential of the vector and $\Gamma_{\mu\nu}^\alpha$ is the Christoffel symbol of the second kind.

For a particle of rest mass m_0 and four-dimensional momentum vector P^α given by [6, see p. 12]

$$P^\alpha = m_0 \frac{dx^\alpha}{ds}, \quad P_\alpha P^\alpha = m_0^2 = \text{constant}, \quad (10)$$

the equation of motion of the free particle is given by

$$\frac{dP^\alpha}{ds} + \Gamma_{\mu\nu}^\alpha P^\mu \frac{dx^\nu}{ds} = 0. \quad (11)$$

For a massless particle of four-dimensional wave vector K^α given by

$$K^\alpha = \frac{\omega}{c} \frac{dx^\alpha}{d\sigma}, \quad K_\alpha K^\alpha = 0, \quad (12)$$

where ω is the characteristic frequency of the massless particle and $d\sigma = h_{ik} dx^i dx^k$ is the three-dimensional chr.inv.-interval, the equation of motion of the free massless particle is given by

$$\frac{dK^\alpha}{d\sigma} + \Gamma_{\mu\nu}^\alpha K^\mu \frac{dx^\nu}{d\sigma} = 0. \quad (13)$$

The projection of the four-dimensional equation of motion (11) onto the time line and the spatial section of an observer for a free particle is then given respectively by [4, see p. 23]

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k = 0, \quad (14)$$

$$\frac{d(mv^i)}{d\tau} + 2m(D_k^i + A_k^i) v^k - mF^i + m\Delta_{nk}^i v^n v^k = 0,$$

where m is the relativistic mass of the particle, $d\tau$ is the physically observable time interval, v^i is the chr.inv.-vector of the physically observable velocity of the particle and Δ_{nk}^i is the chr.inv.-Christoffel symbol of the second kind, while the equivalent chr.inv.-equations of motion for a free massless particle are given by

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_i c^i + \frac{\omega}{c^2} D_{ik} c^i c^k = 0, \quad (15)$$

$$\frac{d(\omega c^i)}{d\tau} + 2\omega(D_k^i + A_k^i) c^k - \omega F^i + \omega\Delta_{nk}^i c^n c^k = 0,$$

where c^i is the chr.inv.-vector of the physically observable velocity of light, with $c^i c_i = c^2$.

In the case where $Q_\alpha Q^\alpha \neq \text{constant}$, the trajectory of the particle is non-geodesic and the absolute derivative of the transported vector $\frac{DQ^\alpha}{ds} = \Phi^\alpha$, which is a force that deviates the particle from a geodesic trajectory. The right hand side of (14) and (15) are set equal to the chr.inv.-projections of the deviating force Φ^α instead of 0. These are called the equations of non-geodesic motion.

5 Fields and charged spin particles in pseudo-Riemannian spaces

The previous section §4 has covered the necessary background on the calculation of equations of motion in the theory of chronometric invariants to permit their generalization to charged particles with spin. In their book *Fields, Vacuum and the Mirror Universe: Fields and particles in the space-time of General Relativity*, Rabounski and Borissova apply the chronometric invariants formalism to the analysis of fields and charged particles with spin [6, see Chapters 3 & 4].

Chapter 3 provides the chronometrically invariant theory of electrodynamics in a pseudo-Riemannian space. It takes

into account the impact on the electromagnetic field of the physically observable chr.inv.-properties of the reference space, specifically the gravitational inertial force (i.e. acceleration) F_i , the space non-holonomy tensor of space rotation A_{ik} , and the rate of deformation of space tensor D_{ik} . This theory will not be covered here as it is beyond the scope of this paper.

Chapter 4 covers the chronometrically invariant theory of particles with spin in a pseudo-Riemannian space. It is based on the premise that spin is a fundamental property of matter, such as mass and charge. The analysis will show that the field of the space non-holonomy from the spatial rotation of the space A_{ik} interacts with the particle's spin and imparts it an additional momentum. From this will be derived the equations of motion of a particle with an internal rotation momentum (i.e. spin).

5.1 Spin particle equation of motion

Based on these considerations, the four-dimensional dynamic vector Q^α for the parallel transport equations is assumed to be given by [6, see pp. 155]

$$Q^\alpha = P^\alpha + S^\alpha, \quad (16)$$

where P^α is given by (10) and S^α is the spin momentum which the particle gains from its internal momentum resulting from the spin, thus making the motion of the particle non-geodesic.

To deduce the spin momentum vector S^α , we start from the known properties of the spin of elementary particles. Their numerical value is given by $\pm n\hbar$, where \hbar is the reduced Planck constant which has units of angular momentum, and $n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, with the \pm sign indicating right-wise or left-wise internal rotation of the spin particle respectively. This suggests that the spin vector would be an antisymmetric tensor of the 2nd rank, similar to a tensor of angular momentum.

From Bohr's second postulate on the length of an electron orbit in an atom and the experimental finding that an electron has an internal magnetic moment proportional to its internal rotation spin momentum, Rabounski and Borissova make an argument to define a four-dimensional antisymmetric 2nd rank angular momentum-like tensor, which they call the Planck tensor and write as $\hbar^{\alpha\beta}$, given by [6, see pp. 155–156]

$$[r^j; p^k] = \frac{1}{2} (r^j p^k - r^k p^j) = k\hbar^{jk} \quad (17)$$

for some constant k , to characterize the spin of a particle in four-dimensional pseudo-Riemannian space.

The diagonal and space-time components of the Planck tensor are zero, while the non-diagonal spatial components are $\pm\hbar$, based on the spatial direction of the spin and the right- or left-handedness of the reference frame. Note that the antisymmetric Planck tensor \hbar^{ik} is not to be confused with

the symmetric physically observable chr.inv.-metric of the observer's space tensor h^{ik} .

This represents a general mathematical approach that requires no assumption on the internal structure of a particle's spin. Instead, it is based on a fundamental quantum space rotation. We have already encountered an antisymmetric rotation of space chr.inv.-tensor A_{ik} in §3, given by (6). In the absence of gravitational fields, the tensor of angular velocity A_{ik} is given by

$$A_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right), \quad (18)$$

which can be more specifically denoted as $A_{\alpha\beta} = \Omega_{\alpha\beta}$, with components

$$\Omega_{00} = 0 \quad \Omega_{0i} = -\Omega_{i0} = 0 \quad \Omega_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right). \quad (19)$$

The quantum principle of wave-particle duality results in a particle's energy being given by $E = mc^2 = \hbar\omega$ where ω is the characteristic frequency of the particle with relativistic mass m . Rabounski and Borissova suggest a generalization of that equation into the geometric tensor relation $mc^2 = \hbar^{\alpha\beta}\omega_{\alpha\beta}$.

The additional momentum S^α in (16) gained by a particle from its spin can be determined from the *action* S of a particle with spin. The action to displace a spin particle generated by the interaction of its spin with the space non-holonomy field $A_{\alpha\beta}$ is given by [6, see pp. 162]

$$S = \alpha(S) \int_a^b \hbar^{\alpha\beta} A_{\alpha\beta} ds = \frac{n}{c} \int_a^b \hbar^{\alpha\beta} A_{\alpha\beta} ds, \quad (20)$$

where $\alpha(S)$ is a scalar constant characteristic of the particle in the spin interaction. One then obtains [6, see pp. 164]

$$S^\alpha = \frac{1}{c^2} n \hbar^{\mu\nu} A_{\mu\nu} \frac{dx^\alpha}{ds}, \quad (21)$$

such that the dynamic vector Q^α that characterizes the motion of the spin particle is given by

$$Q^\alpha = P^\alpha + S^\alpha = m_0 \frac{dx^\alpha}{ds} + \frac{1}{c^2} n \hbar^{\mu\nu} A_{\mu\nu} \frac{dx^\alpha}{ds}, \quad (22)$$

where P^α is given by (10).

The equations of motion of a spin particle are obtained from the parallel transport equations of Q^α given by (22) along the trajectory of the particle

$$\frac{d}{ds} (P^\alpha + S^\alpha) + \Gamma_{\mu\nu}^\alpha (P^\mu + S^\mu) \frac{dx^\nu}{ds} = 0, \quad (23)$$

where $Q_\alpha Q^\alpha = \text{constant}$. The chr.inv.-equations of a particle

with mass and spin is given by [6, see pp. 170]

$$\begin{aligned} \frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k &= \\ &= -\frac{1}{c^2} \frac{d\eta}{d\tau} + \frac{\eta}{c^4} F_i v^i - \frac{\eta}{c^4} D_{ik} v^i v^k, \\ \frac{d(mv^i)}{d\tau} + 2m(D_k^i + A_k^i)v^k - mF^i + m\Delta_{nk}^i v^n v^k &= \\ &= -\frac{1}{c^2} \frac{d(\eta v^i)}{d\tau} - \frac{2\eta}{c^2} (D_k^i + A_k^i)v^k + \frac{\eta}{c^2} F^i - \frac{\eta}{c^2} \Delta_{nk}^i v^n v^k, \end{aligned} \quad (24)$$

where η is given by

$$\eta = \frac{n \hbar^{\mu\nu} A_{\mu\nu}}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (25)$$

The left hand side of equations (24) is the same as that of equations (14), and represents the geodesic part of a spinless particle's motion. However, while the right hand side of equations (14) are equal to zero, in the case of a particle with spin, the right hand side of equations (24) are non-zero, and thus represent the non-geodesic component of the motion of a particle with spin. That, is the component that gives rise to zitterbewegung, while the left hand side represents the classical geodesic trajectory of the particle.

Allowing for the weak gravitational interaction, compared to others, by setting $w \rightarrow 0$ in (7) and $D = 0$ in (8) [6, p. 176], results in the elimination of the F_i and D_{ik} terms, and a simplification of (24). The kinematic equations of motion (24) become

$$\frac{dv^i}{d\tau} + 2A_k^i v^k + \Delta_{nk}^i v^n v^k = 0. \quad (26)$$

Assuming that the space rotates with a constant angular velocity Ω around the x^3 -axis (z -axis), from (18) and (19) and the linear velocity of rotation of the space given by $v_i = \Omega_{ik} x^k$, then the space non-holonomy tensor A_{ik} has only two non-zero components,

$$A_{12} = -A_{21} = -\Omega, \quad (27)$$

and the chr.inv.-vector equations of motion become

$$\frac{dv^1}{d\tau} + 2\Omega v^2 = 0, \quad \frac{dv^2}{d\tau} - 2\Omega v^1 = 0, \quad \frac{dv^3}{d\tau} = 0, \quad (28)$$

where the superscripts are numerical vector indices.

Solving the equations of motion, we obtain the solutions [6, p. 179–183]

$$v^1 = v_{(0)}^1 \cos(2\Omega\tau), \quad v^2 = v_{(0)}^2 \sin(2\Omega\tau), \quad v^3 = v_{(0)}^3, \quad (29)$$

where the $v_{(0)}^i$ represent the initial values of v^i . Integrating (29) with respect to $d\tau$, we obtain the particle's trajectory dis-

placements

$$\begin{aligned} x^1 &= x_{(0)}^1 + \frac{v_{(0)}^1}{2\Omega} \sin(2\Omega\tau) \\ x^2 &= x_{(0)}^2 + \frac{v_{(0)}^1}{2\Omega} - \frac{v_{(0)}^1}{2\Omega} \cos(2\Omega\tau) \\ x^3 &= x_{(0)}^3 + v_{(0)}^3 \tau, \end{aligned} \quad (30)$$

where the $x_{(0)}^i$ represent the initial values of x^i .

Setting the initial displacement of the particle to be zero, $x_{(0)}^1 = x_{(0)}^2 = x_{(0)}^3 = 0$, (30) can be simplified as

$$\begin{aligned} x^1 &= x = a \sin(2\Omega\tau) \\ x^2 &= y = a [1 - \cos(2\Omega\tau)] \\ x^3 &= z = b\tau, \end{aligned} \quad (31)$$

where $a = \frac{v_{(0)}^1}{2\Omega}$ and $b = v_{(0)}^3$. From this, we can move from the τ parametric representation to the coordinate representation of the solution to determine the shape of the three-dimensional trajectory covered by the particle. We obtain [6, p. 184]

$$x^2 + y^2 = 2a^2 [1 - \cos(2\Omega\tau)] = 4a^2 \sin^2(\Omega\tau), \quad (32)$$

where $\tau = z/b$, which is similar to a spiral line equation $x^2 + y^2 = a^2$, $z = b\tau$. The particle has a constant velocity $b = v_{(0)}^3$ along the axis of the spiral, with the radius of the particle's trajectory oscillating with a frequency Ω in the range 0 to $2a = v_{(0)}^1/\Omega$ at distances $z = \frac{\pi k b}{2\Omega}$, for $k = 0, 1, 2, 3, \dots$. The spiral can be visualized as being wound on a pulsating cylinder.

5.2 Charged spin particle in an electromagnetic field

For a charged spin particle in an electromagnetic field, the four-dimensional dynamic vector Q^α for the parallel transport equations takes the form [6, p. 186]

$$Q^\alpha = P^\alpha + \frac{e}{c^2} A^\alpha + S^\alpha, \quad (33)$$

where e is the electric charge and A^α is the electromagnetic field potential. There is thus an additional momentum gained by the particle from the interaction of its charge with the electromagnetic field. The chr.inv.-scalar equation of motion of a charged spin particle in an electromagnetic field is then given by [6, p. 204]

$$\frac{d}{d\tau} \left(m + \frac{\eta}{c^2} \right) = -\frac{e}{c^2} E_i v^i, \quad (34)$$

where E_i is the i^{th} component of the electric field. Then for particles with mass,

$$m_0 c^2 = -n \hbar^{mn} A_{mn} \quad (35)$$

where again \hbar^{mn} is the Planck tensor and A_{mn} is the rotation of space chr.inv.-tensor. The right hand side of this equation (without the negative sign) characterizes the interaction energy of the particle's spin with the space non-holonomy field, i.e. the "spin energy". Rabounski and Borissova refer to (35) as the *law of quantization of the masses of elementary particles*:

The rest-energy of any mass-bearing spin particle is equal to the energy of its spin interaction with the space non-holonomy field, taken with the opposite sign. [6, p. 205]

From (35), it can be shown that for any elementary particle with mass, the following relationship exists between its rest-mass m_0 and the angular velocity of the space rotation Ω [6, p. 207]:

$$\Omega = \frac{m_0 c^2}{2n\hbar}. \quad (36)$$

5.3 The Compton wavelength and zitterbewegung

The wavelength corresponding to the frequency of the space rotation Ω given by (36) can be calculated by assuming that the wave of the space non-holonomy propagates at the speed of light c [6, p. 209]:

$$\lambda_\Omega = \frac{c}{\Omega} = 2n \frac{\hbar}{m_0 c}. \quad (37)$$

For an electron, with $n = \frac{1}{2}$, (37) becomes

$$\lambda_C = \frac{\hbar}{m_0 c}, \quad (38)$$

i.e. the wavelength of the space non-holonomy rotation Ω is equal to the reduced Compton wavelength of the electron.

This confirms that (31) and (32) are the candidate equations to describe zitterbewegung: free electron oscillations occur at a frequency equal to the double angular velocity of the space rotation Ω given by (31), with fluctuations of the particle position on the order of its reduced Compton wavelength given by (38) while following a trajectory described by a pulsating spiral equation of motion.

6 Discussion and conclusion

In this paper, we have explored the connection between *zitterbewegung* for free particles, and the work of Rabounski and Borissova on Zelmanov's chronometric invariant formulation of General Relativity to calculate space and time physical observables [2, 6]. They introduced a four-dimensional antisymmetric tensor of the 2nd rank they called the Planck tensor to characterize the spin of an elementary particle. In the chr.inv.-analysis, the spin of a particle interacts with the space non-holonomy field of pseudo-Riemannian spacetime.

From this, the particle gains an additional momentum which imparts a non-geodesic component to the particle's

motion. The solution of the particle with spin chr.inv.-equation of motion is a spiral that can be visualized as being wound on a pulsating cylinder. It has a constant velocity $b = v_{(0)}^3$ along the x^3 -axis of the spiral, with the radius of the particle's trajectory oscillating with a frequency Ω in the range 0 to $2a = v_{(0)}^1/\Omega$ at distances $z = \frac{\pi k b}{2\Omega}$, for $k = 0, 1, 2, 3, \dots$. The wavelength of the space non-holonomy rotation Ω is equal to the reduced Compton wavelength of the electron.

Free electron oscillations occur at a frequency equal to the double angular velocity of the space rotation Ω , with fluctuations of the particle position on the order of its reduced Compton wavelength. Thus, we have shown that within the chr.inv.-equation of motion of particles with spin derived in Rabounski and Borissova's work [6], zitterbewegung is a direct manifestation of general relativistic space and time physical observables at the elementary particle level.

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Reduction of Matter in the Universe to Protons and Electrons via the Lie-isotopic Branch of Hadronic Mechanics

Ruggero Maria Santilli

The Institute for Basic Research, 35246 U. S. 19N, Suite 215, Palm Harbor, FL 34684, USA.
E-mail: research@i-b-r.org

Matter was originally conceived as bound states of the permanently stable protons and electrons because stars initiate their lives as sole aggregates of Hydrogen atoms, and must synthesize neutrons from protons and electrons as a necessary condition to produce light via nuclear fusions. In oblivion to the Einstein-Podolsky-Rosen argument that *quantum mechanics is not a complete theory*, said conception was abandoned despite its plausibility because of the unverified assumption that the exact validity of Heisenberg's uncertainty principle for point-like particles in vacuum was equally valid for extended protons and neutrons under strong nuclear forces, resulting in the assumption that electrons cannot remain within a nuclear structure. In this paper, we review and update: the insufficiencies of quantum mechanics in nuclear physics; the completion of quantum mechanics into the axiom-preserving, Lie-isotopic branch of *hadronic mechanics* for the invariant representation of *extended* protons and neutrons under potential and contact/non-potential interactions; the exact hadronic representation of *all* characteristics of the neutron in its synthesis from the proton and the electron at the non-relativistic and relativistic levels; the completions of Bell's inequalities with ensuing *iso-deterministic principle* for strong interactions. We then present the apparent resolution of the historical objections against the reduction of all stable matter in the universe to protons and electrons and point out a number of open problems whose treatment is beyond the capabilities of quantum mechanics, such as: the cosmological implications of the missing energy in the neutron synthesis, the prediction of negatively charged pseudo-protons, and the possible recycling of radioactive nuclear waste by nuclear power plants via their stimulated decay.

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Acknowledgments

References

1 Introduction

1.1 Historical notes

As it is well known to historians (see, e.g. [1] [2]), nuclei were originally conceived to be bound states of protons and electrons because stars initiate their lives as aggregates of Hydrogen atoms and they must synthesize neutrons from protons

and electrons as a necessary condition to initiate the production of light via nuclear fusions.

The above original conception of the nuclear structure was abandoned in oblivion of the Einstein-Podolsky-Rosen (EPR) argument that *Quantum mechanics is not a complete theory* [3] (see also the recent verifications [4]–[8]), under the experimentally unverified *assumption* that the validity of Heisenberg's uncertainty principle for *point-like* particles in vacuum was also valid for the *extended* protons and neutrons under strong nuclear forces, resulting in the *assumption* that electrons cannot remain within the dense nuclear structure on various grounds, such as:

1.1) The inability for the electron to remain within a nucleus [1]. By recalling the value of the electron mass $m_e = 0.511 \text{ MeV} = 9.1 \times 10^{-31} \text{ kg}$ and the nuclear radius $R = 10^{-14} \text{ m}$, *Heisenberg's uncertainty principle* [9]

$$\Delta r \Delta p = \frac{1}{2} |\langle \psi | [r, p] | \psi \rangle| \geq \frac{1}{2} \hbar = 5.26548578 \times 10^{-34} \text{ J Hz}^{-1}, \quad (1)$$

would imply the electron to have the superluminal velocity

$$v \geq \frac{\hbar}{\Delta r \times m_e} = 5.79 \times 10^{10} \text{ m/s}. \quad (2)$$

1.2) Under the validity of principle (1), an electron would have the linear momentum uncertainty [10]

$$\Delta p = 1.05 \times 10^{20} \text{ kg m/s}, \quad (3)$$

with corresponding energy

$$E = 19.5 \text{ MeV}, \quad (4)$$

contrary to the evidence that electrons emitted in Beta decays have a maximum energy of 3 MeV.

1.3) The excessive value for nuclear standards of the magnetic moment of the electron [7]. In fact, expressed in nuclear magnetron μ_N , the magnetic moment of the electron has the value

$$\begin{aligned} \mu_e^{spin} &= -9.284764 \times 10^{-24} \text{ J/T} \\ &= -9.284764 \times 10^{-24} \times 1.9798907610^{26} \mu_N \\ &= -928.4784 \times 1.979890 \mu_N = 1838.2851 \mu_N, \end{aligned} \quad (5)$$

which is 961 times the magnetic moment of the neutron $\mu_n = -1.91304 \mu_N$.

In this paper we show that, thanks to the availability of new mathematics for the time-invariant representation of extended protons and neutrons under strong nuclear forces, and the related completion of quantum into hadronic mechanics, Heisenberg's uncertainty principle for point-like particles in vacuum is replaced by a progressive validity of Einstein's determinism for extended protons and neutron under strong nuclear forces [3]–[8], with ensuing resolution of the historical objections against the reduction of matter to protons and electrons.

1.2 Insufficiencies of quantum mechanics in nuclear physics

By using well known nuclear experimental data [11]–[18], we recall the following, century-old, generally ignored insufficiencies of quantum mechanics in nuclear physics:

Quantum mechanical insufficiency I: Inability to represent the synthesis of the neutron from a proton and an electron in the core of stars [19]. Notwithstanding the extremely big (for particle standards) *attractive* Coulomb force of about 230 Newtons between the (negatively charged) electron and the (positively charged) proton,

$$\begin{aligned} F &= -\frac{e^2}{r^2} = \\ &= -(8.99 \times 10^9) \frac{(1.60 \times 10^{-19})^2}{(10^{-15})^2} = -230 \text{ N}, \end{aligned} \quad (6)$$

quantum mechanics allows no quantitative representation of the fundamental synthesis of the neutron in the core of stars. This insufficiency was first identified by R. M. Santilli in the 1978 Harvard's Lyman Laboratory of Physics [20] (see also the subsequent 1979 paper from Harvard's Department of Mathematics [22] on grounds that the mass/rest energy of the neutron is 0.782 MeV *bigger* than the sum of the masses/ rest energies of the proton and of the electron

$$\begin{aligned} E_p &= 938.272 \text{ MeV}, \quad E_e = 0.511 \text{ MeV}, \\ E_n &= 939.565 \text{ MeV}, \\ \Delta E &= E_n - (E_p + E_e) = 0.782 \text{ MeV} > 0, \end{aligned} \quad (7)$$

by therefore requiring a *positive binding energy* and resulting in a *rest energy excess* for which the Schrödinger equation admits no physically meaningful solutions (for a two-body bound state). A similar case occurs for the Dirac equation, which after achieving an exact relativistic representation of the bound state of a proton and the electron at large mutual distances in the Hydrogen atom, the Dirac equation fails to provide any quantitative representation of the bound state of the same particles at nuclear mutual distances.

By no means the neutron synthesis is an isolated case because as we shall see in Sect. 4.1, the representation of *unstable* leptons, mesons and baryons as generalized bound states of particles and antiparticles generally produced free in their spontaneous decays, permits the numerically exact representation of *all* their characteristics, including the mechanism of their spontaneous decays, which has been impossible to date via quantum mechanics.

Quantum mechanical insufficiency II: Inability to achieve a numerically exact representation of nuclear magnetic moments. In fact, under the use of the tabulated values of the magnetic moments of the proton and of the neutron in vacuum [12]

$$\mu_p = +2.79285 \mu_N, \quad \mu_n = -1.91304 \mu_N, \quad (8)$$

quantum mechanics (qm) predicts that the magnetic moment of the Deuteron is given by

$$\mu_D^{qm} = (2.79285 - 1.91304)\mu_N = 0.87981\mu_N, \quad (9)$$

while the experimentally measured value is given by

$$\mu_D^{ex} = 0.85647\mu_N, \quad (10)$$

resulting in the *deviation* of the quantum mechanical prediction from the experimental value of about 3%, with embarrassing deviations for heavier nuclei such as the zirconium.

Quantum mechanical insufficiency III: Inability to achieve a consistent representation of nuclear spins. According to quantum mechanics, the only stable bound state of two particles with spin 1/2, such as the proton and the neutron, is the *singlet coupling*. Consequently, quantum mechanics predicts that the Deuteron D has the structure

$$D = (p_{\uparrow}, n_{\downarrow})_{qm}, \quad (11)$$

for which the total angular momentum is null, $J_D = 0$, contrary to the experimental value of the spin of the Deuteron $J_D = 1$. As a result of this insufficiency, quantum mechanics represents the spin of the Deuteron via such a *collection of orbital contributions to have the value $L_D = 1$* (see, e.g. [21]) in clear disagreement with experimental evidence for which *the spin $S_D = 1$ has been measured for the Deuteron in its true ground state*, i.e. the state for which $L_D \equiv 0$.

Quantum mechanical insufficiency IV: Inability to represent the nuclear stability despite the natural instability of the neutron. As it is well known, the neutron is naturally unstable with spontaneous decay following 887.7 s [17], at which point nuclei should disintegrate evidently due to the excessive number of positive charges. In view of the inability to represent the neutron synthesis from the proton and the electron, quantum mechanics does not allow a meaningful treatment of the mechanism according to which neutrons become stable when members of a nuclear structure.

Quantum mechanical insufficiency V: Inability to represent the nuclear stability despite strongly repulsive protonic Coulomb forces. As it is well known [11], nuclei contain a number of positively charged protons indicated with the atomic number Z , thus experiencing a *repulsive* Coulomb force of type (6) which is so big to overcome known nuclear forces.

Needless to say, the above insufficiencies also apply to relativistic quantum mechanics, as well as to related space time symmetries and relativities.

1.3 Rudiments of isotopic theories

The indicated insufficiencies of quantum mechanical methods, space time symmetries and relativities for the representation of the synthesis of the neutron from the Hydrogen are primarily due to the *local* character of quantum mechanical

methods [3], here referred to the sole dependence of the wave function $\psi(r)$, the potential $V(r)$, and the differential calculus, on a finite number of isolated points r in empty space, as it is the case, e.g. for the linear momentum

$$p\psi(r) = -i\partial_r\psi(r), \quad (12)$$

of the Schrödinger equation

$$\left[\sum_{k=1,2,\dots,A} \frac{1}{2m_k} p_k p_k + V(r) \right] \psi(r) = E\psi(r). \quad (13)$$

Such an approximation of nature has been effective for *atomic structures* due to the large mutual distances between the constituents which allow particles to be approximated as the Newtonian *massive points*. However, the indicated local character of quantum mechanics is excessively approximated for nuclear structures since, according to clear experimental measurements [16]–[18], protons and neutron are *extended charge distributions*, and nuclear volumes are generally *smaller* than the sum of the volumes of their protons and neutrons.

Consequently, *nuclei are generally composed by extended protons and neutrons in condition of partial mutual penetration*, resulting in the expectation that nuclear forces comprise conventional, action-at-a-distance, linear, local and potential interactions (herein called *Hamiltonian interactions*), plus contact, thus zero-range, non-linear, non-local and non-potential interactions (herein called *non-Hamiltonian interactions*).

By noting that a point-like electron cannot possibly be bonded to a point-like proton, we expect that the neutron synthesis requires the representation of the charge distribution of the proton and of the electron wave packet as being extended, with ensuing Hamiltonian and non-Hamiltonian interactions at mutual distances smaller than their size.

Since at the time of the initiation of the studies herein reported (late 1970's), mathematical and physical theories for the time invariant representation of extended particles did not exist, they had to be constructed. In this paper, we adopt *isotopic methods* comprising:

- 1) The *Lie-isotopic mathematics*, or *iso-mathematics* for short.
- 2) The *Lie-isotopic branch of hadronic mechanics*, or *iso-mechanics* for short.
- 3) The non-relativistic and relativistic *iso-symmetries and iso-relativities*.

The above isotopic methods were proposed by R. M. Santilli (when at Harvard University under DOE support) in the 1978 Springer-Verlag monographs [23, 24] and they do achieve the needed time invariant representation of extended particles and/or their wave packets, with consequential Hamiltonian and non-Hamiltonian interactions.

As it is well known, the mathematics of quantum mechanics is based on the universal, enveloping, associative algebra $\xi\{A, B, \dots; A \times B, I\}$ of operators A, B, \dots on a linear space \mathcal{H}

with conventional associative product and related (multiplicative) unit

$$AB = A \times B, \quad (14)$$

$$I : IA = AI \equiv A \quad \forall A \in \xi,$$

which envelope allows a rigorous treatment of Lie's theory via algebra L isomorphic to the antisymmetric sub-algebra $L \approx \xi^-$ with the familiar Lie product $[A, B] = AB - BA$, and ensuing mechanics, symmetries and relativities.

Santilli's iso-mathematics is based on the *axiom-preserving*, thus isotopic lifting of the enveloping algebra $\xi\{A, B, \dots; A \times B, I\}$ into the universal enveloping *iso-associative algebra* $\hat{\xi}\{\hat{A}, \hat{B}, \dots; A \hat{\times} B, \hat{I}\}$ of iso-operators \hat{A}, \hat{B}, \dots on an iso-linear iso-space $\hat{\mathcal{H}}$ with *iso-product* introduced in the 1978 Harvard's paper [20], extended in the 1979 paper [22] and systematically studied in Sect. 5.2, p. 154 on of [24])

$$\hat{A} \hat{\times} \hat{B} = \hat{A} \times \hat{I} \times \hat{B}, \quad (15)$$

and related *iso-unit*

$$\hat{I} = 1/\hat{I} : \hat{I} \hat{\times} \hat{A} = \hat{A} \hat{\times} \hat{I} \equiv \hat{A} \quad \forall \hat{A} \in \hat{\mathcal{H}}. \quad (16)$$

Under the condition that, for consistency, iso-product (15) is applied to the *totality* of the products of the new mathematics, including numbers, functions, operators, etc., the associativity-preserving lifting $\xi \rightarrow \hat{\xi}$ allowed in 1978:

1) The foundations of iso-mathematics, including the Lie-Isotopic theory (nowadays called the *Lie-Santilli iso-theory*) consisting of the step by step isotopic lifting of Lie's theory, including Lie algebras, Lie groups and the transformation theory, with generic N -dimensional iso-algebra \hat{L} of Hermitian operators X_k , $k = 1, \dots, N$ and *iso-commutation rules* (Eq. (38c), p 170 of [42])

$$[X_i \hat{\times} X_j] = X_i \hat{\times} X_j - X_j \hat{\times} X_i =$$

$$= X_i \times \hat{I} \times X_j - X_j \times \hat{I} \times X_i = C_{ij}^k X_k. \quad (17)$$

After leaving Harvard University, Santilli completed the above studies with the 1994 construction of the new *iso-number theory* [89] with iso-unit (16), the 1996 construction of the new *iso-differential calculus* [50] defined for *volumes*, rather than points, and other advances.

2) The foundations of iso-mechanics comprising the *Schrödinger-Santilli iso-equation* (Eq. (14), p. 259 of [24])

$$\hat{H} \hat{\times} |\hat{\psi}\rangle = \left[\sum_{k=1,2,\dots,A} \frac{1}{2m_k} \hat{p}_k \hat{\times} \hat{p}_k + V(r) \right] \hat{\times} |\hat{\psi}\rangle =$$

$$= \hat{E} \hat{\times} |\hat{\psi}\rangle = (E \times \hat{I}) \times \hat{I} \times |\hat{\psi}\rangle = E \times |\hat{\psi}\rangle,$$

and the *Heisenberg-Santilli iso-equation* (Eq. (16), p. 153 of [24]) in its infinitesimal and finite form

$$i \frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}] = \hat{A} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A}, \quad (19)$$

$$A(t) = e^{\hat{H}\hat{I}t} \times A(0) \times e^{-i\hat{H}\hat{I}t},$$

thus requiring *two* quantities for the characterization of nuclear structures, the conventional Hamiltonian $H > 0$ for the representation of linear, local and potential interactions, and the isotopic element $\hat{I} > 0$ for the representation of the extended character of particles and their non-linear, non-local and non-potential interactions.

3) The iso-Galilean symmetry and relativity (Chapter 6, p. 199 on of [24]).

Following the above foundations, hadronic mechanics has been studied by various scholars (see monographs [25]–[34] and papers quoted therein) at about thirty workshops and various international conferences (see representative proceedings [35]–[40], comprehensive presentations [41]–[43]) (see also the summary of the various branches of hadronic mechanics [49], the overviews [45]–[49], and the recent summaries [46]–[48]).

Nowadays, hadronic mechanics has various branches of increasing complexity for the description of particles with increasingly complex physical conditions [49].

The above mathematical and theoretical studies, combined with experimental verifications [43], allowed the identification of the following explicit form of the *isotopic element* (15) and *iso-unit* (16) for a two-body hadronic system [44]

$$\hat{I} = \Pi_{\alpha=1,2} \text{Diag.} \left(\frac{1}{n_{1,\alpha}^2}, \frac{1}{n_{2,\alpha}^2}, \frac{1}{n_{3,\alpha}^2}, \frac{1}{n_{4,\alpha}^2} \right) \times e^{-\Gamma} \ll 1,$$

$$\hat{I} = 1/\hat{I}$$

$$= \Pi_{\alpha=1,2} \text{Diag.} (n_{1,\alpha}^2, n_{2,\alpha}^2, n_{3,\alpha}^2, n_{4,\alpha}^2) \times e^{+\Gamma} \gg 1, \quad (20)$$

$$\Gamma(r, p, a, E, d, \pi, \tau, \psi, \dots) > 0, \quad n_{\mu,\alpha} > 0,$$

$$\mu = 1, 2, 3, 4, \quad \alpha = 1, 2,$$

where \hat{I} is solely restricted by the condition of being positive-definite, but otherwise possess an unrestricted functional dependence (hereon tacitly assumed) on coordinates r , momenta p , accelerations a , energy E , density d , pressure π , temperature τ , wave functions ψ , and any other needed local variable:

1) The representation of the dimension and shape of the individual nucleons is done via semi-axes $n_{k,\alpha}^2$, $k = 1, 2, 3$ (with n_3 parallel to the spin) and normalization for the vacuum $n_{k,\alpha}^2 = 1$.

2) The representation of the density is done via the characteristic quantity $n_{4,\alpha}^2$ per individual nucleons with normalization for the vacuum $n_{4,\alpha}^2 = 1$.

3) The representation of the non-Hamiltonian interactions between extended nucleons which is achieved by the exponential term $e^{-\Gamma}$.

On pedagogical grounds, it should be indicated that any given quantum mechanical model with point-like nucleons and sole Hamiltonian interactions can be uniquely and unambiguously completed into the covering hadronic model for

extended nucleons with Hamiltonian and non-Hamiltonian interactions via the simple *non-unitary transformation* (first proposed in Eq. (11), p. 249 of [24])

$$U \times U^\dagger = \hat{I} = 1/\hat{T} > 0, \quad (21)$$

provided that, to avoid insidious inconsistencies, it is applied to the totality of the quantum formalism with no exception known to this author. In fact, under transformation (8), the conventional associative product of quantum operators A, B is mapped into the *iso-product* of *iso-operators*

$$U \times (A \times B) \times U^\dagger = \hat{A} \hat{\times} \hat{B} = \hat{A} \times \hat{T} \times \hat{B}, \quad (22)$$

$$\hat{T} = (U U^\dagger)^{-1}, \quad \hat{A} = U \times A \times U^\dagger, \quad \hat{B} = U \times B \times U^\dagger,$$

and the same holds for all aspects of iso-mechanics as we shall see in detail in Section 3.

Finally, it is important to indicate from these initial notes that *the representation of the dimensions of particles and their non-Hamiltonian interactions via hadronic mechanics is invariant over time*, of course, not under the *unitary* time evolution of Heisenberg's equations, but under the *iso-unitary* time evolution of the Heisenberg-Santilli iso-equation,

$$U = \hat{U} \hat{T}^{1/2}, \quad (23)$$

$$\hat{U} \hat{\times} \hat{U}^\dagger = \hat{U}^\dagger \hat{\times} \hat{U} = \hat{I},$$

under which the iso-unit and the isotopic element of hadronic mechanics are *numerically invariant* [51]

$$\hat{U} \hat{\times} \hat{I} \hat{\times} \hat{U}^\dagger \equiv \hat{I}, \quad (24)$$

$$\hat{U} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{U}^\dagger = \hat{A}' \times \hat{T}' \times \hat{B}', \quad \hat{T}' \equiv \hat{T}.$$

By using a language accessible to the general physics audience, in Section 2 we review half a century of mathematical, theoretical, experimental and industrial studies in the *non-relativistic* synthesis of the neutron from the proton and the electron.

In Section 3, we report the relativistic studies in the synthesis of the neutron with particular reference to the space time iso-symmetries and iso-relativities necessary for their derivation.

In Section 4, we show that all objections against electrons being part of the nuclear structure are resolved by the recent EPR verifications [4]–[8] and more particularly, by the progressive validity of the *iso-deterministic principle under strong interactions* which occurs in the structure of hadrons, nuclei and stars and the full achievement of Einstein's determinism at the limit of the Schwarzschild horizon.

An initial understanding of this paper can be reached via a knowledge of reviews [46]–[48], with the understanding that a technical knowledge of this paper can solely be reached via a technical knowledge of hadronic mechanics according to the general presentations [41]–[43].

2 Non-relativistic representation of the neutron synthesis from the Hydrogen atom

In this section, we shall outline and update one century of studies on the synthesis of the neutron from the Hydrogen atom in the core of stars as well as in laboratory. Needless to say, we can only outline the main aspects of such a vast topic and provide the references for detailed studies.

2.1 Historical notes

As recalled in Sect. 1.1, stars initiate their lives as an aggregate of Hydrogen that grows by accretion during travel in interstellar spaces. At the moment when the temperature in the core of the aggregate reaches a value of the order of 10 MK, E. Rutherford [19] suggested in 1920 that the Hydrogen atom is “compressed” into a new neutral particle which he called the *neutron*,

$$e^- + p^+ \rightarrow n. \quad (25)$$

The existence of the neutron was experimentally established in 1932 by J. Chadwick [52]. In 1933, W. Pauli [53] pointed out that synthesis (11) violates the conservation of angular momentum. Therefore, E. Fermi [54] submitted in 1935 the hypothesis that the synthesis of the neutron occurs with the joint *emission* of a neutral and massless particle ν with spin 1/2 which he called the *neutrino* (meaning “little neutron” in Italian)

$$e^- + p^+ \rightarrow n + \nu. \quad (26)$$

Subsequent tests (see the recent review [17]) established that *the neutron is naturally unstable* with a mean life of $\tau = 877$ s and spontaneous decay

$$n \rightarrow e^- + p^+ + \bar{\nu}, \quad (27)$$

where $\bar{\nu}$ is the *antineutrino*.

Predictably, the synthesis of the neutron from the Hydrogen attracted attention soon following the Chadwick confirmation. According to the historical account [55], Ernest J. Sternglass conducted in 1951 the first test for the laboratory synthesis of the neutron from Hydrogen, followed by tests in 1952 by E. Trounson and others, although none of these initial tests were reported in published papers in view of the incompatibility of the neutron synthesis with quantum mechanics (Insufficiency I) and for other reasons.

To the author's best knowledge, the first published tests on the laboratory synthesis of the neutron from Hydrogen were done in the 1960's by the Italian priest-physicist, Don Carlo Borghi and his associates [56]. In essence the experimentalists constructed a cylindrical metal chamber (called *klystron*) filled up with the Hydrogen gas (at a fraction of 1 bar pressure) kept the gas mostly ionized via an electric arc with about 500 V and 10 mA. Additionally, the gas was traversed by microwaves with the frequency of 10^{-10} s^{-1} . The experimentalists then placed in the exterior of the Klystron

various materials suitable to be activated when exposed to a neutron flux (such as gold or silver). Following exposures over several weeks, the experimentalists reported clear and reproducible nuclear transmutations that can only be due to a neutral hadron emitted from the Klystron. Due to insufficient evidence on neutron emission, the experimentalists conjectured that the detected nuclear transmutations were due to a new neutral particle with the mass of the neutron but spin different than $1/2$ that they called the *neutroid*.

2.2 Santilli's studies on the neutron synthesis

In view of its fundamental character for all quantitative sciences, R. M. Santilli has conducted over the past five decades mathematical, theoretical, experimental and industrial research on the synthesis of the neutron from the Hydrogen atom in the core of stars, as well as in laboratory (see the mathematical studies [20,23,24,41,50] [57]–[67], the physical studies [42] [68]–[73], the experimental studies [43, 74, 80], and the independent studies [25]–[34] [81]–[85]).

These studies were initiated in the late 1970's at Harvard University under DOE support with the inapplicability of quantum mechanics for the neutron synthesis [20] (Quantum insufficiency I) followed by the proposal to construct *hadronic mechanics* in monographs [23,24].

By far the biggest difficulty of the above studies has been the representation of the spin of the neutron $S_n = 1/2$ from two particles each having spin $1/2$, as originally conceived by Rutherford [19]. This problem stimulated the construction of the Lie-Santilli iso-theory (see Sect. 4.4, p. 173 on of [24] and independent work [26]), followed by systematic studies on the isotopies of spacetime symmetry [57]–[67], with particular reference to the isotopies $\widehat{SO}(3)$ and $\widehat{SU}(2)$ of the angular momentum and spin symmetries at the classical and operator levels [57]–[60] and then passing to the isotopies of spacetime symmetries [61]–[67].

As a result of these preparatory studies, Santilli was able to achieve a numerically exact and time invariant representation of *all* characteristics of the neutron at the non-relativistic level in the 1990 paper [68], and at the relativistic level in the 1995 paper [72], with additional studies available in monograph [73].

Following, and only following, the achievement of a consistent representation of the neutron synthesis via the Lie-isotopic branch of hadronic mechanics, Santilli initiated in 2007 experimental tests on the laboratory synthesis of the neutron from Hydrogen [74]–[80]. According to these experiments, the neutron synthesis from Hydrogen can be generated by *hadronic reactors* consisting of a metal vessel containing in their interior a commercially available Hydrogen gas at pressure and a pair of submerged carbon electrodes powered by a specially designed (patent pending) DC source with a gap controllable from the outside. During operations (Fig. 1), the DC arc is continuously connected and disconnected be-

cause of the consumption of the carbon electrodes. During its activation (left of Fig. 2), the special form of the DC arc ionizes the Hydrogen gas by creating a plasma mostly composed by protons and electrons in its cylindrical surroundings, while during its deactivation (right of Fig. 2), the specially designed DC electric arc compresses the ionized gas from all radial directions toward its symmetry axis.

Interested readers should be aware that commercially available DC electric arcs between carbon electrodes submerged within a Hydrogen gas may synthesize neutroids (Fig. 2) and other unstable hadronic bound states under their big Coulomb attraction, but they are not designed to *compress electrons inside the proton* according to Rutherford's original conception [19].

Experiments [74]–[80] have confirmed: 1) The production of Don Borghi's neutroids (Fig. 2) for DC power of the order of 5 kw, gas pressure of 5 psi and electrode gap of 2 mm. 2) The production of neutrons (Fig. 3) for DC power with at least 50 kw, gas pressure from 10 psi on and electrode gap of at least 5 mm. In particular, the synthesis of neutroids (Fig. 2) resulted to be an unavoidable step prior to the synthesis of the neutron (Fig. 3).

Following, and only following sufficient experimental evidence on the laboratory synthesis of the neutron from a Hydrogen gas, Santilli founded in 2012 the U.S. publicly traded company *Thunder Energies Corporation* (now the privately held Hadronic Technologies Corporation www.hadronictechnologies.com) for the production and sale of a thermal neutron source (see Sect. 4.1).

2.3 Non-relativistic representation of the neutron synthesis

This study was initiated by Santilli with his 1978 Harvard University memoir [20], continued in various works [68]–[73] thanks to the collaboration by various scholars, and reviewed in the 2021 paper [48].

These studies have been conducted under the assumption [20] that the angular momentum of the electron compressed inside the proton is *constrained* to be equal to the spin of the proton as a necessary condition to prevent *extreme resistive forces* caused by the motion of its *extended* wave packet against the *dense* medium in the interior of the proton.

More particularly, when compressed inside the dense proton, the electron e is mutated into a new particle called the *eleton* in Sect. 5.1 of [20] and indicated with the symbol ϵ^- to distinguish it from the electron and the elementary charge e , but recently called the *iso-electron*

$$\hat{\epsilon}^- = U\epsilon^-U^\dagger, \quad (28)$$

because characterized by the complex lifting of the elementary charge (identified in Sect. 3 as an open problem) generated by the isotopic completion $\hat{G}(3.1)$ of the Galilean symmetry [87, 88] (see monograph [25] for an extensive inde-

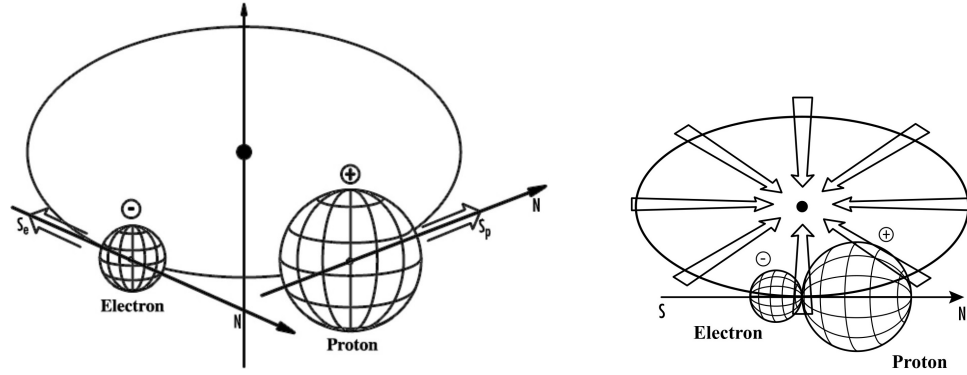


Fig. 1: In this figure, we illustrate the mechanism used by hadronic reactors for the synthesis of neutroids and neutrons via a specially designed (patent pending) DC electric arc between Carbon electrodes submerged within a Hydrogen gas. The mechanism comprises the ionization of Hydrogen atoms into electrons and protons by the activation of the arc (left view) and the compression of the electron within the proton by the de-activation of the arc (right view).

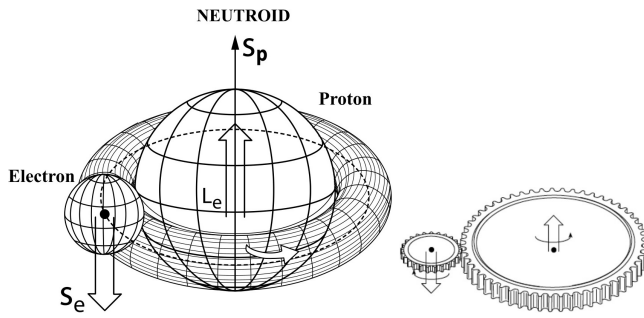


Fig. 2: In the left of this figure, we illustrate the predicted structure of the neutroid in its ground state as a hadronic bond of electrons and protons under their very big Coulomb attraction, Eq. (6), in singlet couplings with null eigenvalues of the angular momentum and of the spin. In the right view, we present a conceptual gear equivalent of the left view to illustrate the reason for the half life of neutroids as being about 10% that of neutrons, i.e. of about 8 s.

pendent study), with corresponding relativistic extension characterized by the isotopy $\widetilde{SO}(3.1)$ of the Lorentz symmetry $SO(3.1)$ [61] and the isotopy $\widetilde{P}(3.1)$ of the spinorial covering of the Poincaré symmetry $\mathcal{P}(3.1)$ characterizing the 20th century notion of *particle*.

By comparison, the proton is assumed in first approximation to be un-mutated, $\hat{p}^+ = p^+$ since the iso-electron is about 1800-times lighter than the proton.

The above assumptions imply the following *structure model of the neutron* as a bound state of a proton p^+ and an iso-electron \hat{e}^- according to hadronic mechanics (hm)

$$n = (\hat{e}_\downarrow^{spin}, \hat{e}_\uparrow^{orb}, p_\uparrow^{spin})_{hm}, \quad (29)$$

under the Coulomb attraction in the macroscopic value of 230 Newton, Eq. (6).

It should be stressed that, in view of the extremely big value of Coulomb attraction (6), the numeric value of the

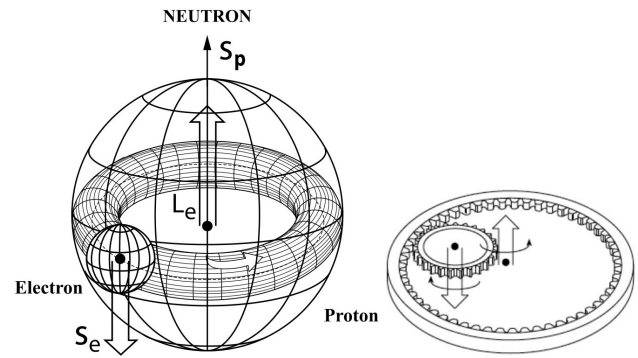


Fig. 3: In the left view, we illustrate the compression of the neutroid of Fig. 2 via the mechanism of Fig. 1, resulting in a constrained hadronic angular momentum of the electron within the dense medium inside the proton that, to avoid extreme resistive forces, has to be equal to the proton spin with ensuing total angular momentum 1/2. In the right of this figure, we provide a conceptual rendering of the left view via coupled gears to illustrate the rather large half life of the neutron of 887 s.

mean life of the neutron according to model (29) can be subject to scientific debates, but not its existence.

2.3.1 Representation of the neutron mass, mean life and charge radius

Let us recall the well known essential elements of the non-relativistic, quantum mechanical representation of the Hydrogen atom as a bound state of a proton p and an electron e , which are given by:

1) The geometric representation on the Euclidean space $E(r, \delta, I)$ with relative coordinate $r = r_p - r_e$, metric $\delta = \text{Diag.}(1, 1, 1)$, unit $I = \text{Diag.}(1, 1, 1)$, and invariant

$$r^2 = r^i \times \delta_{ij} \times r^j = r_1^2 + r_2^2 + r_3^2. \quad (30)$$

2) The operator representation on the *Hilbert space* \mathcal{H} over the field of complex numbers C with states $|\psi(r)\rangle$, normalization

$$\langle\psi(r)|\times|\psi(r)\rangle = I, \quad (31)$$

and expectation value of a Hermitean operator A

$$\langle A \rangle = \langle\psi(r)|\times A \times |\psi(r)\rangle. \quad (32)$$

3) The *Schrödinger representation*, comprising the linear momentum

$$p \times |\psi(r)\rangle = -i \times \hbar \times \partial_r |\psi(r)\rangle, \quad (33)$$

the eigenvalue equation

$$\begin{aligned} H(r, p) \times |\psi(r)\rangle &= E_H \times |\psi(r)\rangle \\ &= \left[\sum_{k=1,2,3} \frac{1}{2m} \times p_k \times p_k - \frac{e^2}{r} \right] \times |\psi(r)\rangle, \end{aligned} \quad (34)$$

where m is the reduced mass

$$m = \frac{m_e \times m_p}{m_e + m_p}, \quad (35)$$

and the canonical commutation rules

$$\begin{aligned} [r^i, p_j] \times |\psi(r)\rangle &= (r^i \times p_j - p_j \times r^i) \times |\psi(r)\rangle = \\ &= -i \times \hbar \times \delta_j^i \times |\psi(r)\rangle, \end{aligned} \quad (36)$$

$$[r^i, r^j] \times |\psi(r)\rangle = [p_i, p_j] \times |\psi(r)\rangle = -0.$$

As it is well known, the above formulation characterizes the Hydrogen atom binding energy

$$E_H = 13.6 \text{ eV}, \quad (37)$$

stability, Bohr's radius and all other features.

According to studies first done in the 1990 paper [68], completed in the 1995 monograph [73] and updated in Sect. 2 of the 2021 memoir [47], the non-relativistic hadronic treatment of structure model (29) is given by the following, step-by-step, *non-unitary* transformation of the quantum treatment of the Hydrogen atom

$$U \times U^\dagger = \hat{I} = 1/\hat{T} > 0, \quad (38)$$

where, for the non-relativistic treatment, we assume iso-unity (20) with value for the density $n_{4,k} = 1$, $k = 1, 2$ whose treatment is done at the relativistic level (Sect. 3.4).

In fact, the geometric treatment of model (29) is done in the *iso-Euclidean iso-space* $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ [41, 61] over the iso-real iso-field $\hat{R}(\hat{n}, \hat{\times}, \hat{I})$ [41, 89] (see also monograph [29]) with iso-unit (20), iso-coordinates $\hat{r} = UrU^\dagger = r\hat{I}$, iso-metric $\hat{\delta} = \hat{T} \times \delta$ and *iso-invariant*

$$\begin{aligned} \hat{r}^2 &= Ur^2U^\dagger = U(r^i \times \delta_{ij} r^j)U^\dagger = \\ &= (Ur^iU^\dagger)(UU^\dagger)^{-1}[(U\delta_{ij}U^\dagger)(UU^\dagger)^{-1}](Ur^jU^\dagger) = \\ &= \hat{r}^i \hat{\times} \delta_{ij} \hat{\times} \hat{r}^j = \left(\frac{r_1^2}{n_1^2} + \frac{r_2^2}{n_1^2} + \frac{r_3^2}{n_1^2} \right) \hat{I}, \end{aligned} \quad (39)$$

where the exponential term of iso-unit (20) has been embedded in the characteristic n -quantities, and one should note the final multiplication by \hat{I} which is necessary for the iso-invariant to be an *iso-scalar*, that is an element of $\hat{R}(\hat{n}, \hat{I})$.

The operator treatment of structure model (29) is done in the *Hilbert-Myung-Santilli isospace* [90] over the iso-field of iso-complex iso-numbers \hat{C} [89] with *iso-states*

$$|\hat{\psi}(\hat{r})\rangle = U(|\psi(r)\rangle)U^\dagger, \quad (40)$$

iso-normalization

$$\langle\hat{\psi}(\hat{r})|\hat{\times}|\hat{\psi}(\hat{r})\rangle = \langle\hat{\psi}(\hat{r})|\times\hat{T}\times|\hat{\psi}(\hat{r})\rangle = \hat{T}, \quad (41)$$

and *iso-expectation values* of an iso-operator

$$\begin{aligned} \hat{\langle A \rangle} &= \langle\hat{\psi}(\hat{r})|\hat{\times}\hat{A}\hat{\times}|\hat{\psi}(\hat{r})\rangle = \\ &= \langle\hat{\psi}(\hat{r})|\times\hat{T}\times\hat{A}\times\hat{T}\times|\hat{\psi}(\hat{r})\rangle. \end{aligned} \quad (42)$$

The reader should note that iso-normalization (41) is characterized by the isotopic element \hat{T} (rather than the iso-unit \hat{I}) for consistency because \hat{T} can be a constant as a particular case, but also because from normalization (31), we expect

$$[|\hat{\psi}(\hat{r})\rangle]^\dagger |\hat{\psi}(\hat{r})\rangle = \langle\hat{\psi}(\hat{r})|\times|\hat{\psi}(\hat{r})\rangle = I. \quad (43)$$

Iso-Schrödinger iso-representation (see Chapter 5, p. 182 of [42] for a detailed treatment). It should be indicated that despite considerable efforts reviewed earlier, by the early 1990's the hadronic form of the Schrödinger equation was still unknown due to the inapplicability of the Newton-Leibnitz differential calculus in general and in particular, the inapplicability for hadronic mechanics of the conventional form (33) of the quantum mechanical linear momentum, with ensuing inability to compute the iso-Hamiltonian.

The axiomatic origin of this impasse was the incompatibility between the sole applicability of the differential calculus to *isolated points* r compared to isotopic methods which are entirely devoted to the representation of *volumes* via iso-unit $\hat{I} = \hat{I}(r, \dots)$, Eq. (20), iso-coordinates $\hat{r} = r\hat{I}(r, \dots)$ and iso-functions $\hat{f}(\hat{r}) = [f(r\hat{I})]\hat{I}$.

This impasse left R. M. Santilli with no other option than that of generalizing the Newton-Leibnitz differential calculus from its sole applicability to isolated points r to volumes \hat{r} . This generalization was first achieved in the 1994 paper submitted for the 1996 memoir [50] (see the 1995 general study [41, 42] and systematic independent works from 2014 on [33, 34]) via the introduction of the infinite class of *iso-differentials of an iso-coordinate* $\{\hat{d}\hat{r}\}$ on $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ on $\hat{\mathcal{R}}$ solely restricted to admit the conventional differential dr for the particular case $\hat{I} = 1$

$$\{\hat{d}\hat{r}\}_{\hat{I}=1} = dr, \quad (44)$$

with selected solution (Eq. (1.27), p. 20 of [50])

$$\hat{d}\hat{r} = \hat{T} d[r\hat{I}(r, \dots)] = dr + r\hat{T} d\hat{I}(r, \dots), \quad (45)$$

consequential *iso-derivative*

$$\frac{\hat{\partial} \hat{f}(\hat{r})}{\hat{\partial} \hat{r}} = \hat{I} \times \frac{\partial \hat{f}(\hat{r})}{\partial \hat{r}}, \quad (46)$$

and finally, the needed expression for the *iso-linear iso-momentum* of hadronic mechanics, first achieved in Sect. 2.5, p. 52 of [50] and Eq. (3.1.10), p. 82 of [42]

$$\hat{p} \hat{\times} |\hat{\psi}(\hat{r})\rangle = -\hat{i} \hat{\times} \hat{h} \hat{\times} \hat{\partial}_{\hat{r}} |\hat{\psi}(\hat{r})\rangle = -\hat{i} \hat{I} \hat{\partial}_{\hat{r}} |\hat{\psi}(\hat{r})\rangle. \quad (47)$$

By using non-unitary transformations of the type

$$\begin{aligned} U \left[\sum_{k=1,2,3} \frac{1}{2m} p_k p_k - \frac{e^2}{r} \right] |\psi(r)\rangle U^\dagger &= \\ = \left[\sum_{k=1,2,3} \frac{1}{2m} (U p_k U^\dagger) (U U^\dagger)^{-1} (U p_k U^\dagger) - \right. \\ \left. - (U \frac{e^2}{r} U^\dagger) \right] (U U^\dagger)^{-1} (U |\psi(r)\rangle U^\dagger) &= \\ = U [E |\psi(r)\rangle] U^\dagger = E [U |\psi(r)\rangle U^\dagger] = E |\hat{\psi}(\hat{r})\rangle, \\ U \left(\frac{e^2}{r} \right) U^\dagger = \frac{e^2}{\hat{r}} = \frac{\hat{I}^2 e^2}{\hat{I} \hat{r}} = \frac{\hat{e}^2}{\hat{r}}. \end{aligned} \quad (48)$$

Schrödinger's equation (34) for the Hydrogen atom on \mathcal{H} over C is mapped into the *iso-Schrödinger equation for the neutron on iso-space $\hat{\mathcal{H}}$ over the iso-field \hat{C}*

$$\begin{aligned} \hat{H}(\hat{r}, \hat{p}) \hat{\times} |\hat{\psi}(\hat{r})\rangle &= \\ = \left[\sum_{k=1,2,3} \frac{\hbar^2}{2m} \hat{p}_k \hat{\times} \hat{p}_k - \frac{\hat{e}^2}{\hat{r}} \right] \hat{\times} |\hat{\psi}(\hat{r})\rangle &= E_n |\hat{\psi}(\hat{r})\rangle, \end{aligned} \quad (49)$$

and the canonical commutation rules (36) are mapped into the *iso-canonical iso-commutation rules*

$$\begin{aligned} [\hat{r}^i, \hat{p}_j] \hat{\times} |\hat{\psi}(\hat{r})\rangle &= (\hat{r}^i \hat{T} \hat{p}_j - \hat{p}_j \hat{T} \hat{r}^i) \hat{T} |\hat{\psi}(\hat{r})\rangle = \\ = -\hat{i} \hat{\times} \hat{h} \hat{\times} \delta_j^i \hat{\times} |\hat{\psi}(\hat{r})\rangle &= -\hat{i} \hbar \delta_j^i |\hat{\psi}(\hat{r})\rangle, \\ [\hat{r}^i, \hat{r}^j] \hat{\times} |\hat{\psi}(\hat{r})\rangle &= [\hat{p}_i, \hat{p}_j] \hat{\times} |\hat{\psi}(\hat{r})\rangle = 0. \end{aligned} \quad (50)$$

As one can see, (49) is formally equivalent to (34) and therefore, it can be solved on the iso-space over the iso-field, yielding the following value of the *neutron binding energy* similar to that for the positronium [14]

$$E_n \approx 7 \text{ eV}, \quad (51)$$

by therefore confirming the expectation, from the high centripetal force of the iso-electrons compressed inside the proton, that the *neutron is a quasi-free hadronic bound state of an (iso-)proton and an iso-electron*.

To identify the impact of the non-Hamiltonian interactions in the neutron structure model (29), it is necessary to assume an explicit realization of the isotopic element and

iso-unit of (20). We here assume the original realization of Sect. 5.1, p. 827 on of the 1978 memoir [20], merely reformulated according to iso-mathematics and iso-mechanics with the simplifying assumptions $n_{\mu,\alpha} = 1$, $\mu = 1, 2, 3, 4$, $k = 1, 2$,

$$\begin{aligned} \hat{I} = 1/\hat{T} = U U^\dagger &= e^{+\frac{V_h(r)}{V_c(r)}} \approx 1 + \frac{V_h(r)}{V_c(r)} \gg 1, \\ \hat{T} = (U U^\dagger)^{-1} &= e^{-\frac{V_h(r)}{V_c(r)}} \approx 1 - \frac{V_h(r)}{V_c(r)} \ll 1, \end{aligned} \quad (52)$$

where $V_h(r)$ is the *Hulten potential* first adopted in Eq. (5.1.6) p. 833 of [20]

$$V_h(r) = K \frac{e^{-br}}{1 - e^{-br}}, \quad (53)$$

with

$$b = R^{-1} \approx 10^{-13} \text{ cm}, \quad (54)$$

and $V_c(r)$ is the conventional Coulomb potential

$$V_c(r) = \frac{e^2}{r}. \quad (55)$$

We consider now the *projection* of iso-equation (49) into the conventional Euclidean and Hilbert spaces. By using isotopic element (52), the needed projection can be written

$$\begin{aligned} \left[\sum_{1,2,3} \frac{\hbar}{2m} (-i \hat{I} \partial_r) (-i \hat{I} \partial_r) - (U V_c(r) U^\dagger) \right] |\psi(r)\rangle &= \\ = E_n |\psi(r)\rangle, \end{aligned} \quad (56)$$

where, in first approximation,

$$U V_c(r) U^\dagger = V_c(r) \hat{I} \approx V_c(r) + V_h(r). \quad (57)$$

But the Hulten potential behaves at very short distances like the Coulomb potential (Eq. (5.1.5) p. 936 of [20]) by therefore absorbing the latter with a mere re-definition K' of the constant K . Consequently, (56) can be reduced in one space dimension to

$$\left[\frac{1}{2m} (-i \hat{I} \partial_r) (-i \hat{I} \partial_r) + K' \frac{e^{-br}}{1 - e^{-br}} \right] |\psi(r)\rangle = E |\psi(r)\rangle, \quad (58)$$

whose radial form

$$\left[\frac{1}{r^2} \left(\frac{d}{dr} r^2 \frac{d}{dr} \right) + \bar{m} K' \frac{e^{-br}}{1 - e^{-br}} \right] = 0, \quad (59)$$

has been studied in great details in Sect. 5.1, p. 827 on of the 1978 memoir [20], including its full analytic solution with boundary conditions.

By adding the isotopy of the non-relativistic quantum mechanical mean life, yielding the expression (Eq. (5.1.13), p. 835 of [20])

$$\tau^{-1} = 2\pi \lambda^2 |\hat{\psi}(0)|^2 \frac{\alpha^2 E_{\hat{e}}}{\hbar}, \quad (60)$$

we reach the *hadronic equations for the mass mean life and charge radius of the neutron according to model (29)* (Eq. (5.1.14), p. 836 of [20])

$$\left[\frac{1}{r^2} \left(\frac{d}{dr} r^2 \frac{d}{dr} \right) + \bar{m} \left(E + K' \frac{e^{-br}}{1 - e^{-br}} \right) \right] \psi(r) = 0,$$

$$E_n^{tot} = E_p + E_{\tilde{e}} - E_n - E = 939.565 \text{ MeV},$$

$$\tau^{-1} = 2\pi\lambda^2 |\hat{\psi}(0)|^2 \frac{\alpha^2 E_{\tilde{e}}}{\hbar} = 877 \text{ s},$$

$$R = b^{-1} - 10^{-13} \text{ cm},$$
(61)

where \bar{m} is the *iso-renormalized reduced mass* (Eq. (5.1.7), p. 833 of [20]), and the last three equations are subsidiary constraints on the first equation.

The analytic solution of the above equations was reduced to the solution of the following two algebraic equations on the parameters k_1 and k_2 (Eq. (5.1.32), p. 840 of [20])

$$E_n^{tot} = \frac{2\hbar ck_1}{b} [k_1 - (k_2 - 1)^2] = 939.37 \text{ MeV},$$

$$\tau = \frac{48 \times (137)^2}{4\pi bc} \frac{k_1}{(k_2 - 1)^3} = 877 \text{ s},$$
(62)

with numeric solutions (Eq. (2.20), p. 521 of [68])

$$k_1 = 0.34, \quad k_2 = 1 + 0.81 \times 10^{-8}. \quad (63)$$

The energy spectrum results to be the typical *finite* spectrum of the Hulthen potential (for $k_2 = 1$)

$$E = \frac{1}{4R^2\bar{m}} \left(\frac{1}{n} - n \right)^2, \quad n = 1, 2, 3, \dots \quad (64)$$

whose sole consistent solution occurs for $n = 1$, as a result of which *the sole possible value for the binding energy E caused by the Hulthen potential is null*

$$E = \frac{1}{4R^2\bar{m}} \left(\frac{1}{n} - n \right)^2 = 0, \quad n = 1, \quad (65)$$

because, as expected, *all the excited states of neutron structure (29) are the various states of the Hydrogen atom*. Alternatively, the null value of the binding energy E is expected from the fact that *contact, zero-range interactions have no potential energy by central assumption*.

2.3.2 Representation of the neutron spin

The central assumption of hadronic model (29) requires that, to avoid extreme resistive forces, the hadronic angular momentum of the iso-electron \tilde{e}^- be equal to the spin of the proton, thus having value $\hat{L}_{3,\tilde{e}} = 1/2$. The study of this assumption was initiated in the 1984 papers on the isotopies of the rotational symmetry [57, 58] and continued in the 1990 paper [68] via the iso-trigonometric iso-functions (see p. 304 on



Fig. 4: In this figure, we illustrate some of the hadronic reactors used for the synthesis of the neutron from Hydrogen (see [80] for a complete presentation).

of [41]), under the use of the Lie-Santilli iso-algebra $\widehat{SO}(3)$. Regrettably, we cannot review these studies to avoid an excessive length.

We here present, apparently for the first time, the non-relativistic representation of the hadronic angular momentum $\hat{L}_3 = 1/2$ under the assumptions that *the orbit of the extended iso-electron within the dense proton is a perfect circle perpendicular to the proton symmetry axis with radius $R = 10^{-13}$ cm*. In fact, deviations from the above assumptions imply instabilities generally preventing a representation of the significant (for particle standards) neutron mean life of 887 s, under which assumptions the acting iso-symmetry is the two-dimensional Lie-Santilli iso-group $\widehat{SO}(2)$ [57, 58] (see also Sect. 6.4, p. 233 on of [42]).

Consider the conventional $O(2)$ symmetry which is classically formulated on the two-dimensional Euclidean space $E(\mathbb{Z}, \delta, I)$, and quantum mechanically treated on a Hilbert space \mathcal{H} over \mathbb{C} . By continuing the construction of hadronic models via a non-unitary transformation of quantum models of the preceding section, we map the entire classical and quantum mechanical formulation of $O(2)$ under the nonunitary transformation

$$UU^\dagger = \hat{I} = 1/\hat{T} = \text{Diag.}(n_1^2, n_2^2) =$$

$$= \text{Diag.}(b_1^{-2}, b_2^{-2}), \quad b_k = 1/n_k > 0, \quad k = 1, 2, \quad (66)$$

and represent the orthogonality condition via Bohm's hidden variable [91]

$$\frac{1}{n_1} = \frac{1}{n_2} = b_1 = b_2 = \lambda > 0. \quad (67)$$

Therefore, the iso-representation occurs in the two-dimensional iso-Euclidean iso-space $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ over the isofield \hat{C} with iso-coordinates

$$\hat{r} = r\hat{I} = \{x, y\}\lambda^2 I_{2 \times 2}, \quad (68)$$

iso-metric

$$\hat{\delta} = \hat{T}\delta = \lambda^2 \delta, \quad (69)$$

iso-invariant

$$\hat{r}^2 = \lambda^2 r^2, \quad (70)$$

and iso-trigonometric representation (Appendix 5C, p. 300 on of [41])

$$\begin{aligned} x &= r\lambda^{-1} \cos \hat{\phi}, \quad y = r\lambda^{-1} \sin \hat{\phi}, \\ \hat{\phi} &= T_\phi \phi = n_1 n_2 \phi = \lambda^{-2} \phi, \quad \hat{T}_\phi = b_1 - 1b_2^{-2} = \lambda^{-2}. \end{aligned} \quad (71)$$

The iso-unitary and iso-irreducible iso-representations of $\widehat{\text{SO}}(2)$ is defined on the iso-space \mathcal{H} [90] \hat{C} with iso-states $|\hat{\psi}(\hat{r})\rangle$, iso-normalization (41), iso-generator $\hat{R}(\hat{\phi})$ and related iso-eigenvalues

$$\begin{aligned} \hat{R}(\hat{\phi})\hat{\times}|\hat{\psi}\rangle &= \hat{e}^{iM\hat{\phi}}\hat{\times}|\hat{\psi}\rangle = (e^{i\hat{M}\hat{\phi}})\hat{I}_\psi\hat{\times}|\hat{\psi}\rangle = (e^{i\lambda^2 M\hat{\phi}})|\hat{\psi}\rangle, \\ \hat{M} &= b_1 b_2 M = \frac{1}{n_1 n_2} M, \\ \lambda^2 &= b_1 b_2 = \frac{1}{n_1} \frac{1}{n_2}, \end{aligned} \quad (72)$$

(where \hat{e} is, this time, the iso-exponentiation in the ϕ -plane) with Lie-Santilli iso-group laws

$$\begin{aligned} \hat{R}(\hat{\phi})\hat{\times}\hat{R}(\hat{\phi}')\hat{\times}|\hat{\psi}\rangle &= \hat{R}(\hat{\phi}')\hat{\times}\hat{R}(\hat{\phi})\hat{\times}|\hat{\psi}\rangle = \hat{R}(\hat{\phi} + \hat{\phi}')\hat{\times}|\hat{\psi}\rangle, \\ \hat{R}(\hat{\phi})\hat{\times}\hat{R}(-\hat{\phi})\hat{\times}|\hat{\psi}\rangle &= \hat{R}(0)\hat{\times}|\hat{\psi}\rangle = |\hat{\psi}\rangle. \end{aligned} \quad (73)$$

The iso-eigenvalue of the hadronic angular momentum \hat{L} is given by

$$\hat{L}\hat{\times}|\hat{\psi}\rangle = \hat{M}|\hat{\psi}\rangle = \lambda^2 M|\hat{\psi}\rangle. \quad (74)$$

But isotopies preserve original numeric values. Therefore,

$$\hat{M} = \lambda^2 M = 0, 1, 2, 3, \dots \quad (75)$$

Consequently, the angular momentum measured by the experimentalist in our space is given by

$$M = \frac{\hat{M}}{\lambda^2}, \quad (76)$$

and can represent the *constrained* angular momentum of the electron inside the proton for the value

$$M = \frac{\hat{M}}{\lambda^2} = \frac{1}{2}, \quad (77)$$

resulting in the *numeric value of Bohm's hidden variable*, here presented for the first time,

$$\lambda = \sqrt{b_1 b_2} = \sqrt{\frac{1}{n_1 n_2}} = \sqrt{2} = 1.4142, \quad (78)$$

which should be compared with essentially the double value of Bohm's hidden variable for the representation of the Deuteron spin [7, 8].

The total spin of the neutron is then given by

$$S_n = s_p + s_{\hat{e}} + L_{\hat{e}} = \frac{1}{2} - \frac{1}{2} + \frac{1}{2}. \quad (79)$$

Hence, according to hadronic structure model (29), *the spin of the neutron coincides with the spin of the proton* as expected. Alternatively, we can say that *the total angular momentum of the iso-electron compressed inside the proton is identically null*, with intriguing applications, e.g. for the exact representation of nuclear spins to be studied in a separate work.

For brevity, we leave to the interested reader the representation of the spin $S = 1/2$ of the iso-electron via the iso-symmetry $\widehat{\text{SO}}(2)$, which can be derived from the above treatment with the $\widehat{\text{SO}}(2)$ symmetry.

The spin of the neutroid according to Fig. 2 is characterized by the following value for the hadronic angular momentum of the iso-electron

$$M = \frac{\hat{M}}{\lambda} = 0, \quad \hat{M} = 0, \quad \lambda > 0. \quad (80)$$

Consequently, *the spin of the neutroid according to Fig. 2 is predicted to be zero*, by therefore explaining the reason for their lack of detection via commercially available neutron detectors.

2.3.3 Representation of the neutron magnetic moment

The anomalous magnetic moment of the neutron according to model (29) has been first represented in the 1990 original paper [68] via the following *three* contributions

$$\mu_n = \mu_p + \mu_{\hat{e}}^{spin} + \mu_{\hat{e}}^{orb}. \quad (81)$$

The biggest difficulty for the above representation is that the magnetic moment of the electron $\nu_{\hat{e}}^{spin}$, Eq. (5), is so big for nuclear standard to prevent a quantum mechanical model of the neutron synthesis as well as to prevent that electrons can be members of nuclear structures (Section 1). These insufficiencies are here resolved, apparently for the first time via the magnetic moment of the *orbital* motion of the iso-electron $\mu_{\hat{e}}^{orb}$ which is *opposite* that of the iso-electron (Fig. 3) and its value is predicted to be [68]

$$\mu_{\hat{e}}^{orb} = 1833.580 \mu_N. \quad (82)$$

By recalling the known values of the magnetic moments of the proton and the neutron [12] $\mu_p = 2.792\mu_N$, $\mu_n = -1.913\mu_N$, we reach in this way the *numerically exact and time invariant representation of the anomalous magnetic moment of the neutron* [68]

$$\mu_n = \mu_p + \mu_e^{spin} + \mu_e^{orb} = 2.792\mu_N - 1838.285\mu_N + 1833.5801\mu_N = -1.913\mu_N \quad (83)$$

It should be noted that the assumption of the above orbital contribution of the iso-electron not only allows a representation of the *numeric value* of the anomalous magnetic moment of the neutron, but also of its *negative value*.

3 Relativistic representation of the neutron synthesis from the Hydrogen atom

Recall that the relativistic treatment of the Hydrogen atom is based on the rotational symmetry $SO(3)$, the spin symmetry $SU(2)$, the Lorentz symmetry $SO(3,1)$, the Poincaré symmetry $P(3,1) = SO(3,1) \times \mathcal{T}(3,1)$, the spinorial covering of the Poincaré symmetry $\mathcal{P}(3,1) = SL(2, \mathbb{C}) \times \mathcal{T}(3,1)$ and related special relativity.

Immediately following the construction in 1983 of the isotopies of the various branches of Lie's theory (Sect. 5.2 on p. 154 of [24]), Santilli constructed the isotopies of the above symmetries and relativities on iso-spaces over iso-fields as a condition to achieve a relativistic representation of the neutron synthesis from the proton and electron, and prove its compatibility with the non-relativistic treatment [57]–[67], including:

- 1) The rotational iso-symmetry $\widehat{SO}(3)$ [57]–[59].
- 2) The spin iso-symmetry $\widehat{SU}(2)$ [60].
- 3) The Lorentz iso-symmetry $\widehat{SO}(3,1)$ [61, 62].
- 4) The Poincaré iso-symmetry $\widehat{P}(3,1) = \widehat{SO}(3,1) \times \widehat{\mathcal{T}}(3,1)$ [63, 64].
- 5) The spinorial covering of the Poincaré iso-symmetry $\widehat{\mathcal{P}}(3,1) = \widehat{SL}(2, \mathbb{C}) \times \widehat{\mathcal{T}}(3,1)$ [66, 67].

The use of the above iso-symmetries then allowed Santilli to construct the unique and unambiguous isotopies of special relativity for the description of extended particles and electromagnetic waves propagating within a physical medium, known under the name of *special iso-relativity*, or *iso-relativity* for short, which was first presented in the 1983 Nuovo Cimento paper [61] for the classical part and in the adjoining paper [62] for the operator counterpart, and subsequently treated in the 1991 monographs [92, 93] with 1996 update [41]–[43] and in 2021 overview [44] (see also the review in monograph [25] from Santilli's lecture notes at the ICTP, Trieste, Italy, monographs [28, 32], and papers quoted therein).

Note that the above extended scientific journey was necessary for the *time invariant representation of the size and density of extended particles* without which experimental verifications cannot be consistently formulated.

Note also that *iso-symmetries and iso-relativities coincide at the abstract level with conventional symmetries and relativities*. Therefore, the representation of the dynamics within physical media solely occur in their *projection* on conventional spaces over conventional fields.

Therefore, *the same symmetries and relativities represent, at the abstract level, both the Hydrogen atom and the neutron*. All differences between the two bound states of a proton and an electron solely occur in their *realizations*.

3.1 The main open problem for particle fusions

As indicated in Sect. 5, p. 819 on of the 1978 Harvard University memoir [20], hadronic mechanics was proposed and constructed not only for a more accurate representation of *nuclear fusions*, but also for the representation of *particle fusions* (also called synthesis), beginning with the fusion of the proton and the electron into the neutron. Additionally, the 1978 memoir [20] proposed isotopic methods for the representation of the structure of *unstable* particles as hadronic bound states of lighter particles and antiparticles generally produced free in their spontaneous decays.

While the quantum mechanical point-like abstraction of particles and nuclei has provided a first approximation of nuclear fusions, quantum mechanics is inapplicable for the representation of particle fusions (Sect. 1.2) due to the *mass excess/rest energy excess*, namely, the mass of the synthesized particle is bigger than the sum of the masses of the constituents as it is clearly the case for the neutron synthesis, (7) while by comparison, nuclear fusions cause the well known *mass defect/energy defect*.

Following the identification of the open problem of the neutron synthesis, in Sect. 5.1, p. 827 on of [20], Santilli achieved the first known representation of *all* characteristics of the π^0 meson as the hadronic bound state (i.e. the fusion) of a mutated electron, then called *eletron* ϵ^- (more recently called *iso-electron*) and a mutated positron ϵ^+ ,

$$\pi^0 = (\epsilon_1^-, \epsilon_1^+)_{hm} . \quad (84)$$

This proposal was based in the following experimental evidence: 1) The extremely big Coulomb attraction (6) between the $\epsilon^- - \epsilon^+$ constituents. 2) The spontaneous decay of the π^0 into an electron and a positron

$$\pi^0 \rightarrow e^- + e^+, \quad 7.5 \times 10^{-8}, \quad (85)$$

via a process interpreted as a *hadronic tunnel effect of the constituents*. 3) The π^0 primary decay which is evidently due to electron-positron annihilation

$$\pi^0 \rightarrow \gamma + \gamma, \quad 98.5\%, \quad (86)$$

which decay allowed the first known identification of the *mechanism* triggering the spontaneous decay of the π^0 and the *exact* representation of its mean life $\tau = 0.828 \times 10^{-16}$ s. The

extension of the model to all remaining mesons was also proposed in the same section 5.1 of [20].

Another important aim of Sect. 5.1 of [20] was to show that quantum mechanics is completely inapplicable for any structure model of the π^0 as a bound state of lighter constituents due to the *rest energy excess* similar to that for the neutron (7) which, for the case of model (84) is given by

$$\pi^0 = (\hat{e}_\uparrow^-, \hat{e}_\downarrow^+)_{hm}, \quad \Delta E = -133.954 \text{ MeV}. \quad (87)$$

In the author's view, the indicated inapplicability of quantum mechanics for the *structure* of particles, jointly with the unavailability at the time of a suitable covering method, explains (and justifies) the sole studies of particles in the 20th century via *classification* methods, such as *mass spectra* that as such, has never produced a structure equation for any particle.

In the subsequent Sect 5.2, p. 849 on of [20] (see also the recent confirmations [48, 98, 99]), Santilli confirmed the results of Sect. 5.1 by reaching the first known representation of *all* characteristics of the μ^\pm leptons via the hadronic structure model (i.e. particle fusion)

$$\mu_\uparrow^\pm = (\epsilon_\uparrow^-, \epsilon_\uparrow^\pm, \epsilon_\downarrow^\pm)_{hm}, \quad (88)$$

on the experimental ground that the μ^\pm leptons decay spontaneously into the indicated constituents via a hadronic tunnel effect

$$\mu^\pm \rightarrow e^- + e^\pm + e^+, \quad 1.0 \times 10^{-12}, \quad (89)$$

while the electron-positron pair annihilation explains the spontaneous character of the decay and its mean life, which annihilation is experimentally confirmed by the muon decay

$$\mu^\pm \rightarrow e^\pm + 2\gamma, \quad 7.2 \times 10^{-11}. \quad (90)$$

Santilli concluded Sect. 5.2, p. 849 on of [20] by indicating the complete inapplicability of quantum mechanics for any structure model of the leptons with lighter constituents due to the rest energy excess

$$\mu_\uparrow^\pm = (\epsilon_\uparrow^-, \epsilon_\uparrow^\pm, \epsilon_\downarrow^\pm)_{hm} \rightarrow \Delta E = -104.636 \text{ MeV}. \quad (91)$$

The extension of the model to the remaining (unstable) leptons was also proposed in the same section 5.2 of [20].

The use of hadronic mechanics under the same principles (the physical constituents of unstable particles are produced free in the spontaneous decays) allowed similar structure models of unstable baryons, such as the model for the Λ^0 [48, 100]

$$\Lambda_\uparrow^0 = (\hat{p}_\uparrow^+, \hat{\pi}^-)_{hm}, \quad (92)$$

(where the “hat” indicates isotopic mutation due to total mutual immersion) based on the primary spontaneous decay

$$\Lambda_\uparrow^0 \rightarrow p_\uparrow^+ + \pi^-, \quad 20 - 30\%, \quad (93)$$

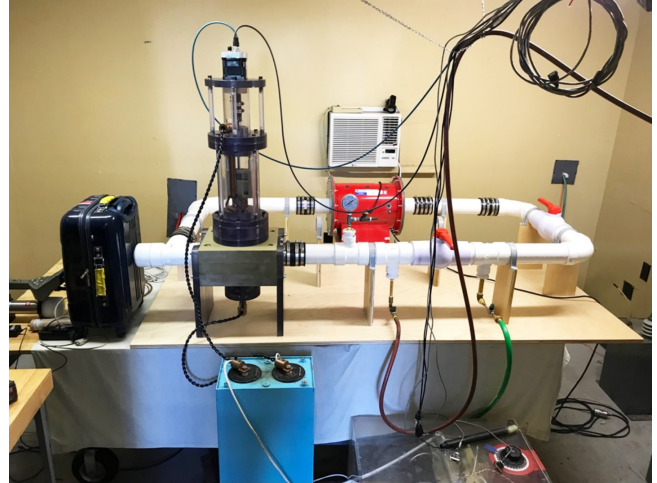


Fig. 5: In this picture, we illustrate the Directional Neutron Source (DNS) produced and sold by Thunder Energies Corporation (now Hadronic Technologies Corporation) generating a flux of thermal neutrons in the desired direction and intensity. The DNS is suggested for the detection of fissile material that may be hidden in baggages, and other applications.

with rest energy excess

$$\Lambda_\uparrow^0 = (\hat{p}_\uparrow^+, \hat{\pi}^-)_{hm} \quad \Delta E = -37.812 \text{ MeV}. \quad (94)$$

The extension of the above hadronic structure model to the remaining (unstable) baryons is left to the interested reader.

The compatibility of the above *structure* models of unstable particles with their known *classification* was shown to be possible via the *iso-units* of the representations that turned out to be different for different particles (see Fig. 4 for mesons and Fig. 12 for baryons of [48]).

Evidently, the indicated excess energies are *physically* acquired by the constituents. The open problem to be addressed in this section is that we are currently unable to calculate the kinetic energy of an extended particle moving within a dense hadronic medium. Consequently, in this section we shall review and upgrade the isotopic methods used for the *geometric* representation of the excess energy for the neutron, in a form extendable to all other particle fusions.

3.2 Iso-Minkowskian iso-spaces

As it is well known, the Minkowski space in 3+1-dimensions provides a geometric representation of the *homogeneity and isotropy of empty space*. By contrast, the primary function of the *Minkowski iso-space* in 3+1-dimensions (first proposed in the 1983 paper [61] and called *Minkowski-Santilli iso-space*) is to provide a geometric representation of the *inhomogeneity and anisotropy of physical media*.

Let $M(x, \eta, I)$ be the conventional Minkowski space over the reals \mathcal{R} with space time coordinates, metric, unit and in-

variant

$$\begin{aligned} x &= \{x^\mu\} = \{x^1, x^2, x^3, x^4 = ct, \}, \\ \eta &= \text{Diag.}(1, 1, 1, -1), \quad I = \text{Diag.}(1, 1, 1, 1), \\ \mu, \nu &= 1, 2, 3, 4, \end{aligned} \quad (95)$$

and invariant

$$x^2 = x^\mu \eta_{\mu,\nu} x^\nu = x_1^2 + x_2^2 + x_3^2 - t^2 c^2. \quad (96)$$

Relativistic isotopic methods, including most importantly the Lie-Santilli iso-theory [24] (see also independent studies [25, 26] and review [47]), are uniquely and unambiguously characterized by the conventional Minkowski space $M(x, \eta, I)$ and the infinite family of positive-definite isotopic elements which, for the case of iso-relativities, are assumed to have the simplified form of the general expression (20)

$$\begin{aligned} T &= 1/\hat{I} = \text{Diag.}\left(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}, \frac{1}{n_4^2}\right) = \\ &= \text{Diag.}(b_1^2, b_2^2, b_3^2, b_4^2), \quad n_\mu > 0, \quad b_\mu > 0, \end{aligned} \quad (97)$$

where we have indicated the characteristic quantities $b_\mu = 1/n_\mu$ mostly used in the early literature in the field, and the exponent of isotopic element (20) is embedded in the characteristic quantities to be factored out whenever needed.

Relativistic methods are then formulated on the infinite family of *iso-Minkowski iso-spaces* $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I})$ over the iso-real iso-field $\hat{\mathcal{R}}$ with iso-unit $\hat{I} = 1/\hat{T}$ [89], *iso-coordinates*

$$\hat{x} = x\hat{I} = \left(\frac{x_1}{n_1}, \frac{x_2}{n_2}, \frac{x_3}{n_3}, \frac{x_4}{n_4} = t^2 \frac{c}{n_4}\right), \quad (98)$$

iso-metric

$$\hat{\Omega} = \hat{\eta}\hat{I} = (\hat{T}\eta)\hat{I}, \quad (99)$$

where one should note the final multiplication by \hat{I} as a necessary consistency condition for the *iso-metric to be an iso-matrix* (namely, a matrix whose elements are iso-numbers) [41]), and *iso-invariant*

$$\begin{aligned} \hat{x}^2 &= \hat{x}^\mu \hat{\Omega}_{\mu,\nu} \hat{x}^\nu = \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 - t^2 \hat{c}^2 = \\ \hat{x}^\mu \hat{T} \hat{\Omega}_{\mu,\nu} \hat{T} \hat{x}^\nu &= x^\mu \hat{\eta}_{\mu,\nu} x^\nu = \\ \frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - t^2 \frac{c^2}{n_4^2}, \end{aligned} \quad (100)$$

illustrating the *identity* at the abstract level between the Minkowski invariant (85) and its iso-Minkowskian image in the first line of invariant (88), all differences occurring in the projection of the latter in the conventional Minkowski space.

It should be noted that, as it was the case for isotopic element (20), the iso-metric has the Minkowskian topological structure $(+, +, +, -)$ but an unrestricted functional dependence on local (space time) coordinates x , momenta p , acceleration a , energy E , density d , pressure π , temperature τ ,

wave function ψ , and any other needed local variable,

$$\hat{\eta}_{\mu\nu} = \hat{\eta}_{\mu\nu}(x, p, a, E, d, \pi, \tau, \psi, \dots). \quad (101)$$

Consequently, the *iso-Minkowskian geometry* with iso-invariant (89) (first introduced in the 1996 paper [50] on the iso-differential calculus and treated in more details in the 1998 paper [67]) is the most general possible geometry with a symmetric invariant in $(3 + 1)$ -dimensions, thus including in particular the Minkowskian, Riemannian, Finslerian and other geometries (see Sect. 3.8 for details).

3.3 The Fundamental theorem on iso-symmetries

The following theorem was first presented in the 1983 paper [61] and upgraded in Section 4.6, page 169 on of [41] as well as in other publications.

3.3.1. FUNDAMENTAL THEOREM ON ISO-SYMMETRIES: Let G be an N -dimensional Lie symmetry of a K -dimensional space $S(x, m, F)$ with coordinates x and metric m over a numeric field F ,

$$G: x' = a(w)x, \quad y' = a(w)y, \quad x, y \in S, \quad w \in F, \quad (102)$$

leaving invariant the interval

$$(x' - y')^\dagger m(x' - y') \equiv (x - y)^\dagger m(x - y) \quad (103)$$

with main property

$$\begin{aligned} (x' - y')^\dagger a^\dagger(w)ma(w)(x - y) &\equiv (x - y)^\dagger m(x - y), \\ a^\dagger(w)ma(w) &\equiv m, \quad \forall x, y \in S. \end{aligned} \quad (104)$$

Then, all infinitely possible iso-symmetries \hat{G} on iso-spaces $\hat{S}(\hat{x}, \hat{M}, \hat{F})$, where $\hat{M} = \hat{m}\hat{I} = (\hat{T}_i^k m_{kj})\hat{I}$ over iso-fields \hat{F} with iso-unit $\hat{I} = 1/\hat{T}$

$$\hat{G}: \hat{x}' = \hat{A}(\hat{w})\hat{x}\hat{I} = (\hat{A}\hat{I})\hat{T}\hat{x} = \hat{A}\hat{x}, \quad (105)$$

$$\hat{y}' = \hat{A}(\hat{w})\hat{y}\hat{I} = (\hat{A}\hat{I})\hat{T}\hat{y} = \hat{A}\hat{y}, \quad \forall \hat{x}, \hat{y} \in \hat{S}, \quad \hat{w} \in \hat{F},$$

leave invariant the iso-interval

$$\begin{aligned} (\hat{x}' - \hat{y}')^\dagger \hat{\Omega}^\dagger(\hat{w})\hat{\Omega}(\hat{w})\hat{M}\hat{\Omega}(\hat{w})\hat{\Omega}(\hat{x} - \hat{y}) &\equiv \\ &\equiv (\hat{x} - \hat{y})^\dagger \hat{\Omega}^\dagger(\hat{w})\hat{\Omega}(\hat{w})\hat{M}\hat{\Omega}(\hat{w})\hat{\Omega}(\hat{x} - \hat{y}), \end{aligned} \quad (106)$$

with main property

$$\hat{A}^\dagger(\hat{w})\hat{\Omega}^\dagger(\hat{w})\hat{M}\hat{\Omega}(\hat{w})\hat{A}(\hat{w}) \equiv \hat{M}, \quad \forall \hat{x}, \hat{y} \in \hat{S}, \quad \hat{w} \in \hat{F}, \quad (107)$$

and all so-constructed iso-symmetries \hat{G} are isomorphic to the original symmetry G .

The verification of the above theorem by all space time iso-symmetries [57]–[67] is an instructive exercise by the interested reader. Note that all iso-symmetries are uniquely and unambiguously characterized by the original symmetry and the infinite class of possible isotopic elements $\hat{T} > 0$.

We finally note that the *iso-exponentiation* [41]

$$\hat{e}^{X_k w_k} = (e^{X_k \hat{T} w_k})\hat{I} = \hat{I}(e^{w_k \hat{T} X_k}), \quad (108)$$

allows the explicit construction of iso-transformations (105).

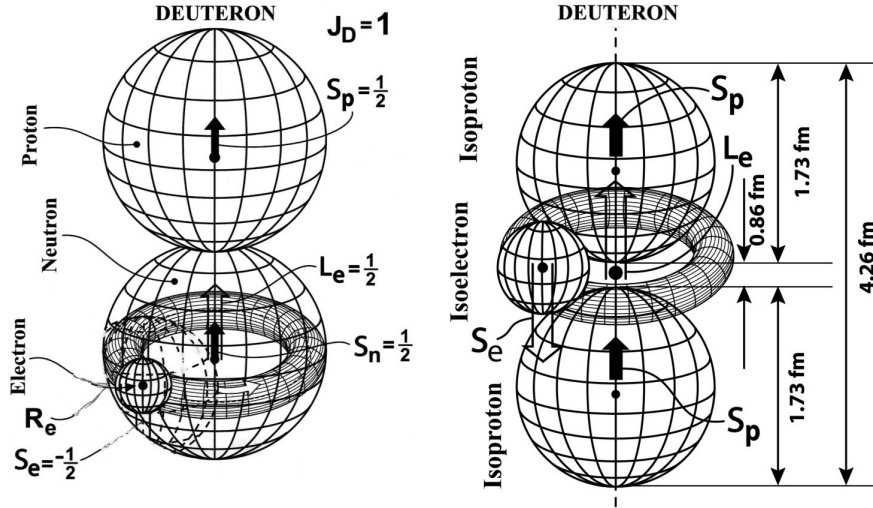


Fig. 6: In the left view, we illustrate the new axial triplet coupling of a proton and a neutron that has achieved the first known representation of the spin $S_D = 1$ of the Deuteron in its true ground state, that with null orbital contributions (Insufficiency II of Sect. 1.2) [44, 102]. In the right view, we illustrate the decoupling of the iso-electron from the neutron to achieve the first known representation of the stability of the Deuteron despite the natural instability of the neutron (Insufficiency IV of Sect. 1.2). Note that said stability is possible if and only if the proton and the electron are the actual physical constituents of the neutron.

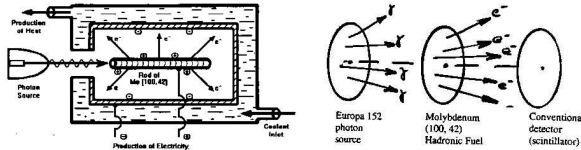


Fig. 7: In this figure, we illustrate the stimulated nuclear transmutations (174) which are predicted to be triggered by irradiation with resonating photons γ_r with energy $E_r = E_{\hat{r}} = 1,293 \text{ MeV}$ [115] which has been tentatively verified in [117]. In this figure, we reproduce the original drawing of paper [115] showing (from the left) a beam of resonating photons irradiating a cylinder of Mo(100, 42, 0) which emits electrons easily trapped by a metal casing with the production of a clean DC electric current of nuclear origin, plus clean heat triggered by said metal screen absorbing electrons with 0.782 MeV kinetic energy. In the right of this figure, we reproduce the original figure of paper [115] illustrating the simplicity as well as the low cost of experimental verifications consisting of the purchase of a small sample of the commercially available radioisotope Europium-152 (emitting photons with E_r) and of the pure isotope Mo(100, 42, 0), which samples are placed next to each other. In the event of confirmation of the emission of electrons from the Mo(100, 42, 0) sample, or of traces of Ru(100, 44, 0) in the originally pure sample of Mo(100, 42, 0), the production of clean nuclear energies via stimulated neutron decays would be confirmed.

3.4 Lorentz iso-symmetries

As it is well known, the Lorentz symmetry characterizes the propagation of point-like particles and electromagnetic waves in the homogeneous and isotropic vacuum represented by the Minkowskian space. The six generators of the connected component of the Lorentz algebra $so(3,1)$ are given by the (Hermitean) generators of rotations J_k , $k = 1, 2, 3$ and the Lorentz boosts M_k on the Hilbert space \mathcal{H} over the field of complex numbers calC with commutation rules

$$\begin{aligned} [J_i, J_j] &= -\epsilon_{ij}^k J_k, \\ [M_i, M_j] &= c^2 \epsilon_{ij}^k J_k, \quad [J_i, M_j] = -\epsilon_{ij}^k M_k. \end{aligned} \quad (109)$$

The exponentiation of the above commutation rules according to Lie's theorems yields the celebrated *Lorentz transformations* on the (3,4)-Minkowski space $M(x, \eta, I)$ according to Theorem 3.2.1

$$x'^3 = \gamma(x^3 - vt), \quad x'^4 = \gamma\left(t - \frac{vx^3}{c^2}\right), \quad (110)$$

where

$$\beta = \frac{v^2}{c^2}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad (111)$$

whose historical role has been *the invariance of the speed of light c in vacuum expressed in line element (84)*.

The infinite family of *Lorentz iso-symmetries* $\widehat{SO}(3,1)$, first introduced in the 1983 Nuovo Cimento paper [61] following the preparatory papers on the *rotational and spin iso-symmetries* [57]–[60] (whose knowledge is here assumed to

prevent a prohibitive length), are defined on the Iso-Minkowski iso-spaces $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I})$.

The Lie-Santilli iso-algebra $\widehat{\text{SO}}(3.1)$ is characterized by six iso-generators defined on the Hilbert-Myung-Santilli iso-space $\hat{\mathcal{H}}$ [90] over the iso-complex iso-field $\hat{\mathbb{C}}$ [89] (Eq. (54), p. 44 of [58])

$$\hat{J}_k = J_k \hat{I}, \quad \hat{M}_k = M_k \hat{I}, \quad (112)$$

(where J_k, M_k are the conventional 4-dimensional matrix generators of $\text{SO}(3.1)$) with explicit expressions (Eq. (10), p. 550 of [61])

$$\hat{J}_1 = n_2 n_3 J_1, \quad \hat{J}_2 = n_1 n_3 J_2, \quad \hat{J}_3 = n_1 n_2 J_3, \quad (113)$$

$$\hat{M}_k = n_k M_k,$$

with iso-commutation rules

$$[\hat{J}_i, \hat{J}_j] = \hat{J}_i \hat{\times} \hat{J}_j - \hat{J}_j \hat{\times} \hat{J}_i = -\epsilon_{ij}^k \hat{J}_k, \quad (114)$$

$$[\hat{M}_i, \hat{M}_j] = c^2 \epsilon_{ij}^k \hat{J}_k, \quad [\hat{J}_i, \hat{M}_j] = -\epsilon_{ij}^k \hat{M}_k,$$

with related Casimir iso-invariant (Eq. (13), p.551 of [61])

$$\hat{C}_1 = \hat{J}^2 - \frac{1}{c} \hat{M}^2 = -3\hat{I}, \quad (115)$$

$$\hat{C}_2 = \hat{J} \hat{\times} \hat{M} = \hat{J}_k \hat{I}^{kk} \hat{M}_k = 0.$$

The realization of the Lorentz-Santilli iso-group via iso-exponents (108) (Eq. (11), p. 550 of [61]) yields the *Lorentz-Santilli iso-transformations* in the (3, 4) plane (see [42] for the general case) for motion of an extended particle with speed v along the x^3 -axis *under the initial assumption that its density has unit value*, $n_4 = 1$ (Eq. (15), p. 551 of [61])

$$x'^3 = \hat{\gamma}(x^3 - vt), \quad (116)$$

$$t' = \hat{\gamma}\left(t - \frac{vb_3^2 x^3}{c^2}\right) = \hat{\gamma}\left(t - \frac{vx^3}{n_3^2 c^2}\right),$$

where

$$\hat{\beta} = \frac{v^2 b_3^2}{c^2}, \quad \hat{\gamma} = \frac{1}{\sqrt{(1 - \hat{\beta}^2)}}. \quad (117)$$

By reinstating generic values of the density $n_4 \neq 1$, and by noting that

$$\hat{\beta} \frac{n_3}{n_4} = \frac{v_3/n_3}{c/n_4} \frac{n_3}{n_4} = \frac{v_3}{c}, \quad (118)$$

$$\hat{\beta} \frac{n_4}{n_3} = \frac{v_3/n_3}{c/n_4} \frac{n_4}{n_3} = \frac{v_3}{c} \frac{n_4^2}{n_3^2},$$

iso-transforms (116) acquire the symmetrized form [42, 44, 93]

$$x'^1 = x^1, \quad x'^2 = x^2, \quad (119)$$

$$x'^3 = \hat{\gamma}\left(x^3 - \hat{\beta} \frac{n_3}{n_4} x^4\right) = \hat{\gamma}\left(x^3 - \hat{\beta} \frac{b_4}{b_3} x^4\right),$$

$$x'^4 = \hat{\gamma}\left(x^4 - \hat{\beta} \frac{n_4}{n_3} x^3\right) = \hat{\gamma}\left(x^4 - \hat{\beta} \frac{b_3}{b_4} x^3\right),$$

where

$$\hat{\beta} = \frac{v_3}{c} \frac{n_4}{n_3}, \quad \hat{\gamma} = \frac{1}{\sqrt{1 - \hat{\beta}^2}}, \quad (120)$$

which achieved in 1983: 1) The invariance of the iso-Minkowskian line element (100). 2) The first known invariant description of extended, thus deformable particles or extended wave packets with semi-axes n_1^2, n_2^2, n_3^2 propagating within a physical medium with density n_4 . 3) The first known *invariance of the local speed of light propagating within transparent physical media*, called *iso-light*

$$C = \frac{c}{n_4} \leq c, \quad (121)$$

which, as we shall see, is generally *smaller* than c ($n_4 > 1$) for media of low density (such as Earth's atmosphere) and *bigger* than c ($n_4 < 1$) for media of high density (such as hadrons).

The following comments are in order:

3.4.1. On historical grounds, let us recall that Lorentz first attempted the invariance of the local speed of light of his times, $C = c/n_4$, but had to restrict his study to the invariance of the constant speed c , due to unsolvable technical difficulties caused by the fact that Lie's theory is notoriously a *linear* problem, while the invariance of local speeds (121) is a highly *non-linear* problem. Hence, the Lie-Santilli iso-theory was constructed with a non-linear structure precisely for the solution of the historical Lorentz problem.

3.4.2. As it is well known, Albert Einstein justly received the 1921 Nobel Prize in Physics for the *quantized absorption*, and not for the *quantized propagation* of light. For the evident intent of preserving special relativity, light propagating within physical media is generally reduced to photons scattering among the molecules of the medium, thus traveling in vacuum with speed c rather than the Lorentz speed $C = c/n_4$. While the quantized *absorption* of light is a historical reality, the quantized *propagation* of light reduced to photons is disproved by experimental evidence, e.g. because it has to occur for all visible frequencies, thus including the reduction to photons of infrared light (see Inconsistencies 1 to 7 in Sect. 8.4.4, p. 134 of [44]). As one can see, the only way known to this author to resolve these inconsistencies is that *visible light propagating within transparent physical media is a "wave" with local speed (121)*, as well known since Lorentz's time.

3.4.3. It was generally believed in the 20th century that the Lorentz symmetry is broken for locally varying speed of light within physical media. In reality, *the axioms of the Lorentz symmetry are fully preserved under isotopies* in view of the evident isomorphism $\text{SO}(3.1) \approx \widehat{\text{SO}}(3.1)$.

3.4.4. It was also believed in the 20th century that *the axioms of the Lorentz symmetry do not allow speeds bigger than c* , with ensuing academic opposition for the initiation studies on interstellar travel, and other problems implying speeds

$C > c$. In 1982, Santilli [95] pointed out that *strong interactions may accelerate particles faster than the speed of light in vacuum* under the admission that strong interactions have a contact non-potential component between the extended protons and neutrons. In fact, the acceleration of point-particles via potential energy up to c notoriously requires infinite energy. By contrast, non-potential interactions can accelerate particles without any use of potential energy and, in any case, special relativity is inapplicable under non-potential interactions.

3.4.5. In 1997, Santilli [96] (see the 2016 update [97]) showed that the following simple transformation of the Minkowski coordinates

$$x^\mu \rightarrow \tilde{x}^\mu = \frac{x^\mu}{n_\mu}, \quad (122)$$

maps the conventional Minkowski invariant (96) with maximal speed c into iso-invariant (100) for which the local speed of light (121) is arbitrary.

To conclude, in view of the isomorphism $\widehat{\text{SO}}(3.1) \approx \text{SO}(3.1)$, we can state that *the abstract axioms of the Lorentz symmetry do indeed predict arbitrary speeds of light*.

3.5 Poincaré iso-symmetries

Consider the conventional Poincaré symmetry on the Minkowskian space $M(x, \eta, I)$ over a field F , as the semi-direct product of the Lorentz symmetry $\text{SO}(3.1)$ and the translations in space time $\mathcal{T}(3.1)$,

$$\text{P}(3.1) = \text{SO}(3.1) \times \mathcal{T}(3.1), \quad (123)$$

with generators

$$J_{\mu\nu} = \{J_k, M_k\}, \quad P_\mu \quad \mu, \nu = 1, 2, 3, 4, \quad k = 1, 2, 3, \quad (124)$$

commutation rules

$$\begin{aligned} [J_{\mu\nu}, J_{\alpha\beta}] &= i(\eta_{\nu\alpha}J_{\beta\mu} - \eta_{\mu\alpha}J_{\beta\nu} - \eta_{\nu\beta}J_{\alpha\mu} + \eta_{\mu\beta}J_{\alpha\nu}), \\ [J_{\mu\nu}, P_\alpha] &= i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu), \quad [P_\mu, P_\nu] = 0, \end{aligned} \quad (125)$$

and Casimir invariants

$$\begin{aligned} C_1 &= I, \\ C_2 &= P^2 = P_\mu P^\mu, \quad (\eta_{\mu\nu} P^\mu P^\nu), \\ C_3 &= W^2 = W_\mu W^\mu, \quad W_\mu = \epsilon_{\mu\alpha\beta\rho} J^{\alpha\beta} P^\rho. \end{aligned} \quad (126)$$

The infinite family of *Poincaré iso-symmetries*, first presented by Santilli in 1993 at the Department of Physics of Moscow State University [63, 64], also called the *Poincaré-Santilli iso-symmetries*

$$\hat{\text{P}}(3.1) = \widehat{\text{SO}}(3.1) \hat{\times} \hat{\mathcal{T}}(3.1), \quad (127)$$

is defined on iso-Minkowski iso-spaces $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I})$ (where $\hat{\Omega} = \hat{\eta}\hat{I}$) over iso-real iso-field $\hat{\mathcal{R}}$ with iso-generators from definition (110)

$$\{\hat{J}_{\mu\nu}\} = \{\hat{J}_k, \hat{M}_k\}, \quad \hat{P}_\mu, \quad (128)$$

and iso-commutation rules [63, 64]

$$\begin{aligned} [\hat{J}_{\mu\nu}, \hat{J}_{\alpha\beta}] &= i(\hat{\eta}_{\nu\alpha}\hat{J}_{\beta\mu} - \hat{\eta}_{\mu\alpha}\hat{J}_{\beta\nu} - \hat{\eta}_{\nu\beta}\hat{J}_{\alpha\mu} + \hat{\eta}_{\mu\beta}\hat{J}_{\alpha\nu}), \\ [\hat{J}_{\mu\nu}, \hat{P}_\alpha] &= i(\hat{\eta}_{\mu\alpha}\hat{P}_\nu - \hat{\eta}_{\nu\alpha}\hat{P}_\mu), \\ [\hat{P}_\mu, \hat{P}_\nu] &= 0, \end{aligned} \quad (129)$$

where one should note the appearance of the *structure functions* with the functional dependence (98), i.e. $\hat{\eta}(x, p, a, E, d, \pi, \tau, \psi, \dots)$, rather than the traditional structure constants. Consequently, the Poincaré-Santilli iso-symmetry is *irregular*, namely, it cannot be obtained from the original symmetry via non-unitary transforms, as it is the case for regular Lie-Santilli iso-algebra [101].

The use of iso-commutation rules (128) yields the *Casimir-iso-invariants* [63, 64]

$$\begin{aligned} \hat{C}_1 &= \hat{I} > 0, \\ \hat{C}_2 &= \hat{P}^2 = \hat{P}_\mu \hat{\times} \hat{P}^\mu = (\hat{\eta}^{\mu\nu} P_\mu P_\nu) \hat{I} = \\ &= \left(\sum_{k=1,2,3} \frac{1}{n_k^2} \hat{P}_k^2 - \frac{c^2}{n_4^2} \hat{P}_4^2 \right) \hat{I}, \\ \hat{C}_3 &= \hat{W}^2 = \hat{W}_\mu \hat{\times} \hat{W}^\mu, \quad \hat{W} = W \hat{I}, \\ \hat{W}_\mu &= \hat{\epsilon}_{\mu\alpha\beta\rho} \hat{\times} J^{\alpha\beta} \hat{\times} \hat{P}^\rho, \end{aligned} \quad (130)$$

and they are at the foundation of classical and operator *relativistic iso-mechanics* with deep implications for structure models of particles, nuclei and stars.

Note that *all possible Poincaré-Santilli iso-symmetries are isomorphic to the conventional Poincaré symmetry*. However, the conventional Poincaré symmetry is *linear* in view of the commutativity of the linear momenta $[P_\mu, P_\nu] = 0$, while the *Poincaré-Santilli iso-symmetry is iso-linear* because the property $[\hat{P}_\mu, \hat{P}_\nu] = 0$ holds on iso-spaces over iso-fields, but its projection into conventional spaces over conventional fields is not, in null, $[\hat{P}_\mu, \hat{P}_\nu] \neq 0$, with ensuing non-linearity of the theory. Consequently, the iso-translations $\hat{\mathcal{T}}(3.1)$ are generally nonlinear.

3.6 Dirac iso-equations

As it is well known, the Dirac equation achieved a justly historical role for the relativistic representation of the electron of the Hydrogen atom under the external field of the proton. The infinite family of *Dirac iso-equations*, first introduced in the 1995 papers [65, 66] and called the *Dirac-Santilli iso-equations* have been constructed for the relativistic representation of the iso-electron of the neutron while considering the proton as external.

The Dirac equation is generally obtained via the linearization of the second order Casimir invariant of the Poincaré symmetry (125). The Dirac-Santilli iso-equations are then best obtained via the linearization of the second order iso-Casimir invariant (130).

The carrier iso-spaces of the Dirac-Santilli iso-equations are given by the iso-product of the real-valued, *orbital* (or) iso-Minkowskian iso-spaces and of the complex-valued, *spin* (sp) iso-Euclidean iso-space

$$\hat{M}^{tot} = \hat{M}(\hat{x}, \hat{\Omega}^{or}, \hat{I}^{or}) \hat{\times} \hat{E}(\hat{z}, \hat{\Delta}^{sp}, \hat{I}^{sp}), \quad (131)$$

with orbital specifications

$$\begin{aligned} \hat{x} &= x \hat{I}^{or}, \quad \hat{\Omega}^{or} = \hat{\eta}^{or} \hat{I}^{or} = (\hat{T}^{or} \hat{\eta}) \hat{I}^{or}, \\ \hat{I}^{or} &= \text{Diag.}(n_1^2, n_2^2, n_3^2, n_4^2) = 1/\hat{T}^{or} > 0, \\ \hat{x}^{\hat{2}} &= \hat{x}^{\mu} \hat{\times}^{or} \hat{\Omega}_{\mu\nu}^{or} \hat{\times}^{or} \hat{x}^{\nu} = \left(\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - \frac{x_4^2}{n_4^2} \right)^{or}, \end{aligned} \quad (132)$$

and spin specifications

$$\begin{aligned} \hat{z} &= (z_1, z_2) \hat{I}^{sp}, \quad \hat{\Delta}^{sp} = \hat{\delta}^{sp} \hat{I}^{sp} = (\hat{T}^{sp} \hat{\delta}) \hat{I}^{sp}, \\ \hat{I}^{sp} &= \text{Diag.}(\lambda^{-1}, \lambda) = 1/\hat{T}^{sp} > 0, \quad \text{Det.} \hat{I}^{sp} = 1, \\ \hat{z}^{\hat{2}} &= \hat{z}^i \hat{\times}^{sp} \hat{\Delta}_{ij}^{sp} \hat{\times}^{sp} \hat{z}^j = (\lambda z_1^2 + \lambda^{-1} z_2^2)^{sp}, \end{aligned} \quad (133)$$

where λ is Bohm's "hidden variable" [91].

Let us recall the explicit form of the iso-linear four-momentum on a Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ over an iso-complex iso-field $\hat{\mathcal{C}}$

$$\hat{p}_{\mu} \hat{\times}^{or} |\hat{b}\rangle = -\hat{i}^{or} \hat{\times}^{or} \hat{\partial}_{\mu}^{or} |\hat{b}\rangle = -i \hat{I}^{or} \partial_{\mu} |\hat{b}\rangle. \quad (134)$$

By using the iso-mass of iso-particles and the iso-speed of iso-light

$$\hat{m} = m \hat{I}^{tot}, \quad \hat{C} = C \hat{I}^{tot} = \frac{c}{n_4} \hat{I}^{tot}, \quad (135)$$

we have the iso-linearization of the second order iso-Casimir invariant (130) acting on an iso-basis $|\hat{b}\rangle$ (see Eq. (6.1), page 189, [66])

$$\begin{aligned} &(\hat{\Omega}^{\mu\nu} \hat{\times}^{tot} \hat{P}_{\mu} \hat{\times}^{tot} \hat{P}_{\nu} - \hat{m}^{\hat{2}} \hat{\times}^{tot} \hat{C}^{\hat{2}}) \hat{\times}^{tot} |\hat{b}\rangle = \\ &= (\hat{\Omega}^{\mu\nu} \hat{\times}^{tot} \hat{I}_{\mu} \hat{\times}^{tot} \hat{P}_{\nu} + i^{tot} \hat{\times}^{tot} \hat{m} \hat{\times} \hat{C}) \hat{\times}^{tot} \\ &\hat{\times}^{tot} (\hat{\Omega}^{\mu\nu} \hat{\times}^{tot} \hat{I}_{\mu} \hat{\times}^{tot} \hat{P}_{\nu} - i^{tot} \hat{\times}^{tot} \hat{m} \hat{\times} \hat{C}) \hat{\times}^{tot} |\hat{b}\rangle, \end{aligned} \quad (136)$$

which holds if and only if the following conditions are verified

$$\begin{aligned} \hat{I}_{\mu} &= \hat{\gamma}_{\mu} \hat{I}^{or}, \\ \{\hat{\gamma}_{\mu}, \hat{\gamma}_{\nu}\}^{or} &= \hat{\gamma}_{\mu} \hat{\times}^{or} \hat{\gamma}_{\nu} + \hat{\gamma}_{\nu} \hat{\times}^{or} \hat{\gamma}_{\mu} = 2\hat{\eta}_{\mu\nu}, \end{aligned} \quad (137)$$

with realization given by the *Dirac-Santilli iso-gamma matrices*

$$\begin{aligned} \hat{\gamma}_k &= \frac{1}{n_k} \begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \\ \hat{\gamma}_4 &= \frac{i}{n_4} \begin{pmatrix} I_{2 \times 2} & 0 \\ 0 & -I_{2 \times 2} \end{pmatrix}, \end{aligned} \quad (138)$$

where $\hat{\sigma}_k$ are the *Pauli-Santilli iso-matrices* first proposed in Eq. (6.8.20), p. 248, [42]

$$\begin{aligned} \hat{\sigma}_1 &= \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix}, \\ \hat{\sigma}_3 &= \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \end{aligned} \quad (139)$$

with Lie-Santilli iso-commutation rules

$$\begin{aligned} [\hat{\sigma}_i, \hat{\sigma}_j] &= \hat{\sigma}_i \hat{\times} \hat{\sigma}_j - \hat{\sigma}_j \hat{\times} \hat{\sigma}_i = \\ &= \hat{\sigma}_i \hat{T} \hat{\sigma}_j - \hat{\sigma}_j \hat{T} \hat{\sigma}_i = i2\epsilon_{ijk} \hat{\sigma}_k, \end{aligned} \quad (140)$$

with iso-eigenvalues on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$

$$\begin{aligned} \hat{S}_k &= \frac{1}{2} \hat{\times} \hat{\sigma}_k = \frac{1}{2} \hat{\sigma}_k, \\ \hat{\sigma}_3 \hat{\times} |\hat{b}\rangle &= \hat{\sigma}_3 \hat{T} |\hat{b}\rangle = \pm |\hat{b}\rangle, \\ \hat{\sigma}^{\hat{2}} \hat{\times} |\hat{b}\rangle &= (\hat{\sigma}_1 \hat{T} \hat{\sigma}_1 + \hat{\sigma}_2 \hat{T} \hat{\sigma}_2 + \hat{\sigma}_3 \hat{T} \hat{\sigma}_3) \hat{T} |\hat{b}\rangle = 3 |\hat{b}\rangle, \end{aligned} \quad (141)$$

clearly showing the representation of the spin 1/2 of the considered iso-particle.

The Dirac-Santilli iso-equations can then be written

$$\begin{aligned} &(\hat{\Omega}^{\mu\nu} \hat{\times}^{or} \hat{I}_{\mu} \hat{\times}^{or} \hat{P}_{\nu} + i \hat{\times}^{or} \hat{m} \hat{\times} \hat{C}) \hat{\times}^{or} |\hat{b}\rangle = \\ &= (\hat{\eta}^{\mu\nu} \hat{\gamma}_{\mu} \hat{\times}^{or} \hat{P}_{\nu} + i \hat{m} \hat{\times} \hat{C}) \hat{\times}^{or} |\hat{b}\rangle = 0, \end{aligned} \quad (142)$$

which will be used in Sect. 3.9 for the relativistic representation of the neutron structure.

To avoid insidious, because unfounded inconsistencies in applications, the reader should keep in mind that the iso-metric $\Omega^{\mu\nu}$ for iso-momenta is the *contra-variant version* of the iso-metric $\Omega^{\mu\nu}$ for iso-coordinates.

3.7 Iso-spinorial Poincaré iso-symmetries

In view of the spin 1/2 of the electron, the space time symmetry for the relativistic treatment of the Hydrogen atom is given by the spinorial covering of the Poincaré symmetry

$$\mathcal{P}(3.1) = \mathcal{SL}(\in, \mathcal{C}) \times \mathcal{T}(\in, \infty), \quad (143)$$

with realization of the generators in terms of the Dirac gamma matrices

$$\begin{aligned} \mathcal{SL}(2, \mathcal{C}) : \quad S_k &= \frac{1}{2} \gamma_k \times \Gamma_4, \quad R_k = \frac{1}{2} \epsilon_i^k \gamma_i \times \gamma_j, \\ \mathcal{T}(3.1) : \quad P_{\mu}, \end{aligned} \quad (144)$$

which verify commutation rules (136).

Similarly, the *iso-spinorial coverings of the Poincaré iso-symmetries*, first presented in the 1995 paper [66] is given by

$$\hat{\mathcal{P}}(3.1) = \hat{\mathcal{S}}\hat{\mathcal{L}}(\hat{2}, \hat{C}) \hat{\times} \hat{\mathcal{T}}(3.1), \quad (145)$$

and admits the realization of the iso-generators in terms of the Dirac-Santilli iso-gamma iso-matrices $\hat{\Gamma}_\mu = \hat{\gamma} \hat{I}^{or}$

$$\begin{aligned} \hat{\mathcal{S}}\hat{\mathcal{L}}(\hat{2}, \hat{C}) : \quad \hat{S}_k &= \frac{1}{2} \hat{\Gamma}_k \hat{\times} \hat{\Gamma}_4, \quad \hat{R}_k = \frac{1}{2} \epsilon_{ijk} \hat{\Gamma}_i \hat{\times} \hat{\Gamma}_j, \\ \hat{\mathcal{T}}(3.1) : \quad \hat{P}_\mu, \end{aligned} \quad (146)$$

which verify iso-commutation rules (128).

We also have the *rotational iso-sub-symmetries*

$$\hat{O}(3) : L_k = \epsilon_{kj}^i r_j p_i, \quad [L_i, L_j] = \epsilon_{ij}^k n_k^2 L_k, \quad (147)$$

with iso-eigenvalues

$$\begin{aligned} \hat{L}^2 \hat{\times} |\hat{b}\rangle &= (\hat{L}_1 \hat{\times} \hat{L}_1 + \hat{L}_2 \hat{\times} \hat{L}_2 + \hat{L}_3 \hat{\times} \hat{L}_3) \hat{\times} |\hat{b}\rangle = \\ &= (n_1^2 n_2^2 + n_2^2 n_3^2 + n_3^2 n_1^2) |\hat{b}\rangle, \\ \hat{L}_3 \hat{\times} |\hat{b}\rangle &= n_1 n_2 |\hat{b}\rangle, \end{aligned} \quad (148)$$

and the *spinorial iso-sub-symmetries*

$$\begin{aligned} \hat{S}\hat{U}(2) : \quad \hat{S}_k &= \frac{1}{2} \epsilon_{ij}^{kj} \hat{\gamma}_i \hat{\times} \hat{\gamma}_j, \\ [\hat{S}_i, \hat{S}_j] &= \frac{1}{n_k} \hat{S}_k \text{ (no sum)}, \end{aligned} \quad (149)$$

with iso-eigenvalues

$$\begin{aligned} \hat{S}^2 &= (\hat{S}_1 \hat{\times} \hat{S}_1 + \hat{S}_2 \hat{\times} \hat{S}_2 + \hat{S}_3 \hat{\times} \hat{S}_3) \hat{\times} |\hat{b}\rangle = \\ &= \frac{1}{4} \left(\frac{1}{n_1^2 n_2^2} + \frac{1}{n_2^2 n_3^2} + \frac{1}{n_3^2 n_1^2} \right) |\hat{b}\rangle \\ \hat{S}_3 \hat{\times} |\hat{b}\rangle &= \frac{1}{2} \frac{1}{n_1 n_2} |\hat{b}\rangle, \end{aligned} \quad (150)$$

that will be used in Sect. 3.9 for the identification of the main characteristics of the iso-electron in the neutron structure.

3.8 Special iso-relativities

Special Relativity (SR) has achieved a justly historical role for the characterization of *time reversal invariant, thus stable systems of point-like particles and electromagnetic waves propagating in the homogeneous and isotropic vacuum*, where the restriction to time reversal invariance follows from the dependence of Minkowski's invariant (96) on t^2 .

SR is only approximately valid for the characterization of time reversal invariant, thus stable systems of *extended* particles (such as the proton in a nucleus) because extended

particles imply features outside the representational capabilities of the *mathematics* underlying SR, such as the existence of contact, thus zero-range interactions without potential, the *mass/energy excess* of particle fusions (Sect. 3.1), the generally inhomogeneous and anisotropic character of the medium, and other problems.

In the author's view, SR is inapplicable (rather than violated) for an axiomatically consistent representation of *irreversible processes, such as nuclear fusions* for various axiomatic and physical reasons, including the possible violation of causality (e.g. the admission of solutions in which the effect precedes the cause) [102].

Special isotopic (i.e. axiom-preserving) relativity, or *Special Iso-Relativity* (SIR) for short, has been introduced by R. M. Santilli in the 1983 *Nuovo Cimento* papers [61, 62] for their classical and operator formulations, respectively, and then studied in various works [92]–[97] (see also reviews [25, 28, 32]) for the characterization of *time reversal invariant systems of extended, thus deformable and dense particles under conditions of mutual penetration, as occurring in stable nuclei, under the most general known, linear, local and potential interactions represented by a Hamiltonian H and the most general possible non-linear, non-local and non-potential interactions represented by the isotopic element \hat{T} of (20)*. In this paper, we use SIR with *constant n -characteristic quantities* for the representation of the neutron synthesis even though the neutron is *unstable* (when isolated), yet it decays into the original constituents (27), as a result of which the neutron synthesis can be assumed to be *reversible over time*.

The formulation of SIR used in this paper is *not* recommended for the treatment of nuclear fusions (because of the possible violation of causality indicated earlier) in favor of the *Lie-admissible relativity* studied in [24, 42] with an *irreversible axiomatic structure* [103, 104].

The correct classical formulation of SIR should be done on iso-Minkowskian iso-spaces $\hat{M}(\hat{x}, \hat{\Omega}, \hat{t})$ over iso-fields $\hat{\mathcal{R}}$, while the operator formulation should be done on Hilbert-Myung-Santilli iso-spaces $\hat{\mathcal{H}}$ over iso-complex iso-fields $\hat{\mathcal{C}}$. At the abstract level, SR and SIR coincide by conception and construction. Therefore, by continuing to follow [66] we present below the *projection* of SIR iso-axioms in the conventional Minkowski space $M(x, \eta, t)$ over the field \mathcal{R} . Said iso-axioms are then uniquely and unambiguously characterized by the iso-symmetries reviewed in preceding sections, and are expressed below for the k -direction, e.g. that of the third space component,

ISO-AXIOM I: *The speed of light within (transparent) physical media is given by the locally varying speed:*

$$C = \frac{c}{n_4} \lesseqgtr c. \quad (151)$$

ISO-AXIOM II: *The maximal causal speed within physical*

media is given by:

$$V_{max,K} = c \frac{n_k}{n_4}. \quad (152)$$

ISO-AXIOM III: The addition of speeds within physical media follows the isotopic law:

$$V_{tot} = \frac{\frac{v_{1,k}}{n_k} + \frac{v_{2,k}}{n_k}}{1 + \frac{v_{1,v_2}}{c^2} \frac{n_4^2}{n_k^2}}. \quad (153)$$

ISO-AXIOM IV: The iso-dilation of time, the iso-contraction of lengths, the iso-variation of mass with speed and the mass-energy iso-equivalence (iso-renormalization) within physical media follow the isotopic laws:

$$t'_k = \hat{\gamma}_k t, \quad (154)$$

$$\ell'_k = \hat{\gamma}_k^{-1} \ell, \quad (155)$$

$$m'_k = \hat{\gamma}_k m, \quad (156)$$

$$\hat{E}_k = m V_{max,k}^2 = m_k c^2 \frac{n_k^2}{n_4^2}. \quad (157)$$

ISO-AXIOM V: The frequency shift within physical media follows the isotopic law (for null aberration)

$$\omega_{exp} = \frac{\omega_{sou}}{\hat{\gamma} [1 - \hat{\beta} \cos(\hat{\alpha})]}. \quad (158)$$

To avoid a prohibitive length, in regard to the experimental verifications of Iso-Axioms I–V in classical physics, particle physics, nuclear physics, astrophysics and other fields, we suggest the interested reader to inspect the 1995 [43] and the 2021 upgrade [44].

The following comments are now in order:

3.8.1. Note that the maximal causal speed in SIR is no longer given by the speed of light, and it is given instead by value (152), because *physical media are generally opaque to light*, thus requiring the broader geometric notion $v_{max,k}$ derivable from the expression in (3, 4)-space coordinates

$$\frac{dx_k^2}{n_k^2} - dt^2 \frac{c^2}{n_4^2} = 0. \quad (159)$$

3.8.2. By recalling that we are dealing with inhomogeneous and anisotropic physical media, the reader should be aware that the numeric values of Iso-Axioms (151)–(158) generally vary with the variation of the k -direction.

3.8.3. The sole known geometric representation of the excess mass/excess rest energy of the neutron synthesis, as well as of particle fusions at large (Sect. 3.1), will be done in the next section with Iso-Axiom (157).

3.8.4. When the isotopic element \hat{T} , and therefore, the n -characteristic quantities, solely depend on space time coordinates $\hat{T} = \hat{T}(x)$, $n_\mu = n_\mu(x)$, iso-Minkowskian intervals (00)

coincide with Riemannian intervals [67], and characterize the *Exterior General Iso-Relativity* (EGIR) for the formulation of Einstein's field equations under the universal Poincaré-Santilli *iso-symmetry* $\hat{\mathcal{P}}(3.1)$ [64] (rather than the known covariance), including the representation of the Schwartzschild metric with the isotopic element (for brevity, see Sect. 8.5, p. 155 on of [44])

$$\hat{T}_{kk} = \frac{\delta_{kk}}{(1 - \frac{2M}{r})}, \quad \hat{T}_{44} = 1 - \frac{2M}{r}, \quad (160)$$

with the apparent resolution of the century-old problematic aspects of general relativity [105].

3.8.5. When the isotopic element \hat{T} has the general functional dependence (101), Iso-Axioms I–V characterize the *Interior General Iso-Relativity* (IGIR) which is intended to study the *origin* (rather than the sole description) of the gravitational field, that expectedly occurs in the nuclear structure (see Santilli's paper [106] from his stay at MIT in 1974–1977), thus including the structure of the neutron (see Sect. 8.6, p. 161 of [44]).

3.9 Relativistic representation of the neutron synthesis

Recall that, under the invariance of the spinorial covering of the Poincaré symmetry $\mathcal{P}(3.1) = \text{SL}(2, \mathbb{C}) \times \mathcal{T}(3.1)$ (which is needed for the spin $S = 1/2$ of the electron), the Dirac equation has provided an exact and time invariant relativistic representation of the *point-like* electron under the *external* field of the proton in the structure of the Hydrogen atom.

The Dirac-Santilli iso-equation (142) has been constructed to attempt the exact and time invariant representation of the *extended* wave packet of the iso-electron within the *extended* proton in the structure of the neutron according to Fig. 3, thus requiring its characterization via the isotopies of the spinorial covering of the Poincaré symmetry $\hat{\mathcal{P}}(3.1) = \widehat{\text{SL}}(2, \hat{\mathbb{C}}) \hat{\times} \hat{\mathcal{T}}(3.1)$, first introduced in the 1995 paper [66] jointly with the Dirac-Santilli iso-equation and the first relativistic representation of the neutron synthesis.

For consistency, the neutron structure model (29) requires that the hadronic angular momentum of the iso-electron \hat{L}_3 be equal to the proton spin \hat{S}_3 , thus requiring that

$$\hat{L}_3 = \hat{S}_3, \quad \hat{L}^2 = \hat{S}^2. \quad (161)$$

From (148) and (150) of the iso-spinorial iso-symmetry $\hat{\mathcal{P}}(3.1)$, we therefore obtain the following two conditions on the characteristic quantities for the basic isotopic element (97) expressed in the symbols $b_\mu = 1/n_\mu$ of [66]

$$\begin{aligned} b_1^{-1} b_2^{-1} &= \frac{1}{2} b_1 b_2, \\ b_1^{-2} b_2^{-2} + b_2^{-2} b_3^{-2} + b_3^{-2} b_1^{-2} &= \\ \frac{1}{4} (b_1^2 b_2^{-2} + b_2^2 b_3^{-2} + b_3^2 b_1^{-2}), \end{aligned} \quad (162)$$

with numeric value confirming the expected spheroidal shape of the neutron (Eqs. (7.2), (7.3), p. 192 of [66])

$$\begin{aligned} b_1^2 &= b_2^2 = \frac{1}{n_1^2} = \frac{1}{n_2^2} = \sqrt{2} = 1.415, \\ b_1 &= b_2 = \frac{1}{n_1} = \frac{1}{n_2} = 1.189. \end{aligned} \quad (163)$$

Consequently, the above relativistic representation is in remarkable axiomatic and numerical agreement with the corresponding non-relativistic value (Sect. 2.3.2) via Bohm's hidden variable (78),

$$\lambda = \sqrt{b_1 b_2} = b = \sqrt{\frac{1}{n_1} \frac{1}{n_2}} = \frac{1}{n} \sqrt{2} = 1.189, \quad (164)$$

by therefore establishing the *compatibility between the non-relativistic and the relativistic structure models (29) of the neutron*.

The value of the third semi-axis $1/b_3^2 = n_3^2$ of the iso-electron can be found by assuming the preservation of the volume V of the original sphere with semi-axes $n_k^2 = 1$, $k = 1, 2, 3$ plus values (162) for the first two semi-axes

$$\begin{aligned} V &= \frac{4}{3} \pi (n_1^2)^2 n_3^2 = 4.192 \left(\frac{1}{1.415} \right)^2 n_3^2 = 4.19, \\ n_3^2 &= \frac{4.19}{2.087} = 2.007, \end{aligned} \quad (165)$$

resulting in the values

$$n_1^2 = n_2^2 = 0.707, \quad n_3^2 = 2.007, \quad (166)$$

suggesting that the spheroidal shape of the iso-electron is *prolate* (because $n_3^2 > n_1^2 = n_2^2$).

The representation of the excess energy $\Delta E = 0.782$ MeV in the neutron synthesis from the proton and the electron (7), is done via Iso-Axiom (157), requiring a numeric value of $n_4^2 = 1/b_4^2$ which in this case, represents the density of the *proton*, since the charge of the iso-electron has no dimension.

From Iso-Axiom (157) we obtain the iso-renormalized rest energy of the neutron

$$\begin{aligned} \tilde{E}_n &= m_e C^2 = \\ &= m_e c^2 \frac{b_3^2}{b_4^2} = m_e c^2 \frac{n_3^2}{n_4^2} = 939.565 \text{ MeV}, \end{aligned} \quad (167)$$

from which

$$\frac{b_4^2}{b_3^2} = \frac{n_3^2}{n_4^2} = \frac{1.293}{0.511} = 2.530. \quad (168)$$

From values (166) we then obtain the numeric value of the density n_4^2 which is needed for the iso-renormalization of the mass/rest energy of the iso-electron here presented apparently for the first time

$$n_4^2 = \frac{1}{b_4^2} = \frac{n_3^2}{2.530} = 0.793, \quad n_4 = \frac{1}{b_4} = 0.891, \quad (169)$$

which is compatible with the density $n_4^2 = \frac{1}{b_4^2} = 0.429$ of the fireball of the proton-antiproton annihilation of the Bose-Einstein correlation [107,108], see Eq. (10.27), p.127 of [107] (see also [108]).

Intriguingly, taken in *prima facie*, the above data suggest that *the proton is about 50% denser than the proton-antiproton fireball of the Bose-Einstein correlation*.

4 Applications of the neutron synthesis

In this section, we briefly indicate some of the applications of the synthesis/fusion of the proton and the electron into the neutron with related references.

4.1 Detection of smuggled fissile material

Recall that fissile material, such as Uranium-233, Uranium-235 and Plutonium-239, are *metals* that, as such, cannot be distinguished from ordinary metals via all scanning equipment currently available at airports and ports. Thanks to the studies reported in this paper, the U.S. publicly traded company *Thunder Energies Corporation*, (now the private *Hadronic Technologies Co*) did develop, produce and sell a scanner permitting a clear detection of fissile material via the irradiation of baggages with the *Directional Neutron Source* (DNS) of Fig. 5 which produces on demand from a commercially available Hydrogen gas *a beam of thermal neutrons* ($E < 100$ eV) *in the desired direction and intensity*, resulting in a shower of easily detectable radiation from the disintegration of a few fissile nuclei [68]–[80].

It should be noted that various neutron sources are commercially available but they all produce *high energy neutrons* that, as such, are not recommendable for use in public places because of the risk of triggering a chain reaction which is absent for irradiation of fissile material with a controlled small beam of thermal neutrons.

4.2 Representation of nuclear stability

It appears that hadronic mechanics has permitted a quantitative solution of the problem of nuclear instability despite the neutron natural instability (Insufficiency IV of Sect. 1.2) via the *decoupling of the permanently stable electron from the neutron when members of a nuclear structure* (Fig. 6), which was first presented in Appendix C.1 and Fig. 13, p. 152 of [102]. Note that the indicated decoupling introduces a new, very strong, Coulomb *attraction* in the Deuteron structure between the iso-electron and the proton pair. Note also that *the indicated nuclear stability is possible if and only if the proton and the electron are the actual physical constituents of the neutron*.

The resolution of Insufficiency V of Sect. 1.2 (on the nuclear stability despite the very big, repulsive, protonic, Coulomb force) requires separate future studies on the *structure* of the elementary iso-charge.

4.3 Representation of the gravitational stability of the Sun

The Sun releases into light the energy of [109]

$$\Delta E_{out}^{Sun} = 2.3 \times 10^{38} \text{ MeV/s}, \quad (170)$$

which corresponds to about 4.3×10^6 t/s. Since, in a Gregorian year, there are 10^7 seconds, the loss of mass by the Sun per year ΔM_{year}^{Sun} due to light emission is given by

$$\Delta M_{year}^{Sun} = 10^{23} \text{ metric tons per year}. \quad (171)$$

The above loss of mass by the Sun is of such a magnitude to cause a change of planetary orbits that should be detectable by contemporary, sufficiently sensitive instruments in astrophysical laboratories contrary to centuries of measurements on the stability of planetary orbits, i.e. the stability of the Sun's gravitational field.

For these and other reasons, Santilli [110] proposed in 2007 the hypothesis that *the missing energy in the neutron synthesis is provided by the ether conceived as a universal substratum with extremely big energy density*, and that the energy of 0.782 MeV is transferred from the ether to the neutron by a massless, chargeless and spinless *longitudinal impulse* called *etherino* (denoted with the letter *a* from the Latin *aether*) in the *left hand side* of the neutron synthesis

$$\hat{e}^- + a + \hat{p}^+ \rightarrow n. \quad (172)$$

In fact, a medium size star such as our Sun synthesizes about 10^{40} neutrinos per second [13], that requires the total energy of about

$$\Delta E_n^{star} = 7.8 \times 10^{39} \text{ MeV per second}. \quad (173)$$

The etherino hypothesis [110] was formulated on grounds that the energy needed for the neutron synthesis by the Sun (173), is essentially equal to the Sun's loss of energy into light (170). Consequently, the assumption that the missing energy for the neutron synthesis is provided by the ether as a universal substratum permits a quantitative representation of the stability of the Sun's gravitational field.

In any case, the missing energy of 0.782 MeV cannot be provided by the relative kinetic energy between the proton and the electron, because at that value, the $e - p$ cross section is essentially null, thus prohibiting any synthesis. Similarly, said missing energy cannot be provided by the Sun because the total missing energy (173) is so big that the Sun would cool down and never produce light.

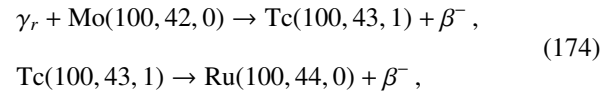
Note that the indicated representation of the gravitational stability of the Sun implies a *return to the continuous creation of matter in the universe* [112], with intriguing implications, e.g. for a realistic representation of the energy released in supernova explosions. Note finally that experiments on the

predictions of the neutrino hypothesis [113] may be numerically representable via the corresponding *predictions* of the etherino hypothesis.

Note also that the indicated gravitational stability of the Sun requires the acceptance of the ether as a universal substratum for the structure and propagation of truly “elementary” particles and electromagnetic waves without any real conflict with special relativity due to our evident inability to reach a reference frame at rest with the ether (for the absence of the “etheral wind” under the indicated conditions, see the 1956 paper [114] and Chapter 3 of [32]).

4.4 Stimulated decay of the neutron

The hypothesis that the neutron is a hadronic bound state of a proton and an electron implies the *possible stimulated decay of one or more neutrons when members of selected nuclear structures* via irradiation with resonating photons γ_r with energy equal to the total energy of the iso-electron $E_r = E_{\hat{e}} = 1,293 \text{ MeV}$. Intriguing, said stimulated decay implies the production of nuclear energy without the emission of harmful radiation and without the release of radioactive waste, e.g. as occurring in the stimulated decay [115]



which has been tentatively verified by the experimental team [117] (see also [83]). Regrettably, no physics laboratory contacted by the author has shown interest to date in dismissing or confirming Tsagas' results via the repetition of the very simple and inexpensive measurements of transmutation (174) (see Fig. 7 for details).

4.5 The pseudo-proton hypothesis

The synthesis of the neutron via Rutherford's “compression” of an electron within the dense proton, implies the synthesis, in statistical smaller amounts of *negatively charged, strongly interacting particles* preliminarily confirmed by tests [118], such as: the *protoid* \tilde{p}_1^- with spin 0, mass essentially that of the neutron and mean-life predicted to be of about 7 s, and the *pseudo-proton* \tilde{p}_2^- with spin 1/2, mass equal to that of the neutron and mean-life of the order of 5 s, both representable with the synthesis/fusion $\tilde{p}^- = (\hat{e}^-, n)_{hm}$.

Note that, being negatively charged and strongly interacting, *protoids and pseudo-protons are attracted by nuclei* with new nuclear transmutations here expressed for *N* protoids

$$N\tilde{p}_1^- + N(Z, A, J) \rightarrow \tilde{N}(Z - N, A + N, J), \quad (175)$$

having an evident significance for possible new forms of nuclear energies and recycling of nuclear waste.

4.6 Recycling of nuclear waste

Due to known public opposition, it appears that the sole possible recycling of radioactive nuclear waste should be done by the nuclear power plants themselves via their stimulated decay. Unfortunately, the latter recycling is prohibited by quantum mechanics with ensuing mainstream academic opposition against its study. Interested readers may be interested to know that hadronic mechanics predicts a number of mechanisms for the recycling of radioactive nuclear waste via their stimulated decay triggered by irradiation with thermal neutrons (Sect. 4.1 and Fig. 5), pseudo-protons (Sect. 4.5) as well as other means, and ensuing production of new nuclear energy (see, for brevity, Sect. 8.2.10-II, p. 111 of [44] and [115, 116]).

4.7 Resolution of the Coulomb barrier for nuclear fusion

In the author's view, the most important environmental implication of the synthesis/fusion of the proton and the electron into the neutron is the consequential synthesis/fusion, under the extremely strong Coulomb attraction (6), of at least a pair negatively charged electrons generally coupled in singlet and a positively charged Deuteron into a new negatively charged nucleus $\tilde{D}(-1, 2, 1)$, called *pseudo-Deuteron* with sufficiently long mean life (of the order of $\tau = 1$ s) to be attracted by a natural, positively charged Deuteron, resulting in a new nuclear fusion, called *HyperFusion*, without the historical Coulomb barrier that has prevented the achievement to date of new clean nuclear energies (see [102] for brevity).

5 Reduction of matter to protons and electrons

During his graduate studies at the University of Torino, Italy, in the mid 1960's, after learning that stars initiate their lives as aggregates of Hydrogen atoms, R. M. Santilli accepted the historical hypothesis [1,2] that matter is composed of the permanently stable protons and electrons, and could not accept the various opposing arguments [10] on grounds that Heisenberg's uncertainty principle has been experimentally verified solely for *point-like particles* (i.e. as the electron) in vacuum under electromagnetic/Hamiltonian interactions. In line with the 1935 legacy by A. Einstein, B. Podolsky and N. Rosen that *quantum mechanics is not a complete theory* [3], Santilli argued that the same principle should not be applied to the *extended protons and neutrons under strong nuclear interactions* without due scrutiny.

Subsequently, Santilli learned from experimental measurements [16, 18] that nuclear volumes are generally *smaller* than the sum of the volumes of the constituents, thus implying that the hyper-dense protons and neutrons are in conditions of partial mutual penetration in a nuclear structure. In turn, this implies the expectation that strong nuclear forces have a contact-zero range, non-linear, non-local and non-potential, thus non-Hamiltonian component under which *Heisenberg's*

uncertainty principle cannot be consistently formulated, let alone tested.

In the late 1970's, when he was at Harvard University under DOE support, Santilli proposed the foundation of the EPR completion of quantum into *hadronic mechanics* for the invariant representation of extended nucleons under Hamiltonian and non-Hamiltonian interactions [23, 24]. He then initiated in 1981 studies [4] on the *completion of Heisenberg's uncertainties for strong interactions* via generalized uncertainties of the type (Eq. (2.18), p. 654 of [4])

$$\Delta r \times \Delta p \approx \frac{1}{2} \hbar F(r, p, \psi, \dots), \quad F > 0, \quad (176)$$

and conducted systematic mathematical, theoretical, experimental and industrial studies (reported in the preceding sections) on the synthesis/fusion of a proton and an electron into the neutron.

Santilli became aware in the early 1990's that mathematical and physical theories can be completed into a form representing the astrophysical evidence that the neutron, and therefore all matter in the universe, is a collection of suitable bound states of the permanently stable protons and electrons. Final studies in the field are reported in this section on the explicit form of the uncertainty principle which is applicable under the most general possible, Hamiltonian and non-Hamiltonian strong nuclear forces.

In 1964, J. S. Bell published the theorem below under the assumption of quantum mechanics according to its Copenhagen interpretation, thus including Heisenberg's uncertainty principle, the representation of the spin 1/2 of particles via the SU(2)-invariant Pauli matrices, and other assumptions:

THEOREM 5.1 [119]: *A system of two point-like particles verifying the SU(2) Lie symmetry does not admit a classical counterpart.*

The theorem was proved by showing that a certain expression D^{Bell} (whose numeric value depends on the relative conditions of the two particles) is always *smaller* than the corresponding classical value D^{Clas} ,

$$D^{Bell} < D^{Clas}, \quad (177)$$

for all possible values of D^{Bell} .

The importance of the Theorem 5.1 for the identification of the ultimate constituents of matter is that of *strengthening* the general acceptance of Heisenberg's uncertainty principle for all possible conditions existing in the universe, thus leading to the unverified *assumption* that electrons cannot be members of a nuclear structure (Sect. 1.2).

Following the achievement of maturity of the iso-mathematical and iso-mechanical branches of hadronic mechanics [50, 89], and following the formulation of the $\widehat{SU}(2)$ -iso-invariant Pauli-Santilli iso-matrices reviewed in Sect. 3.6, Santilli proved in 1998 the following:

THEOREM 5.2 [5]: *A system of two extended particles verifying the $\widehat{\text{SU}}(2)$ Lie-Santilli iso-symmetry does admit a classical counterpart.*

The theorem was first proved on grounds that contact, zero-range, non-potential interactions are outside the class of unitary equivalence of quantum mechanics, while being fully representable via a non-unitary transformation of quantum mechanical models (21). Consequently, there always exists a non-unitary transformation $UU^\dagger = \hat{I}$ of Bell's quantity D^{Bell} such to verify the equality

$$D^{hm} = U(D^{Bell})U^\dagger \equiv D^{Clas}. \quad (178)$$

Additionally, Santilli conducted a step-by-step isotopic lifting of Bell's proof of Theorem 5.1 via the Pauli-Santilli iso-matrices (139) resulting in the equality (Eq. (5.8), p. 189 of [5])

$$D^{hm} = \frac{1}{2} (\lambda_1 \lambda_2^{-1} + \lambda_1^{-1} \lambda_2) D^{Bell} \equiv D^{class}, \quad (179)$$

which is always verified by particular values of Bohm's hidden variables [91] λ_1 and λ_2 . [5] also provided specific examples of identity (178) in terms of the iso-Minkowski iso-spaces over iso-fields.

Finally, by combining the results of [4] and [5], in 2019 Santilli proved the following:

THEOREM 5.3 [6]: *The iso-standard iso-deviations for iso-coordinates Δr and iso-momenta Δp , as well as their product, progressively approach Einstein's determinism for extended particles in the interior of hadrons, nuclei and stars, and achieve the full determinism at the limit of Schwarzschild's singularity (ss).*

The theorem was proved by showing that the invariance under the Lie-Santilli iso-symmetry $\widehat{\text{SU}}(2)$ implies the following property known as *iso-deterministic principle* derived via iso-commutation rules (50) and iso-normalization (41) (see for details Lemma 3.7, p. 34 of review [47] and its Corollary 3.7.1 on the ensuing removal of divergencies here ignored for brevity)

$$\begin{aligned} \Delta r \Delta p &\approx \frac{1}{2} \langle \hat{\psi}(\hat{r}) | \hat{\times} [\hat{r}, \hat{p}] \hat{\times} | \hat{\psi}(\hat{r}) \rangle = \\ &= \frac{1}{2} \langle \hat{\psi}(\hat{r}) | \hat{T} [\hat{r}, \hat{p}] \hat{T} | \hat{\psi}(\hat{r}) \rangle = \frac{1}{2} \hbar \hat{T} \ll 1, \\ \{\Delta r \Delta p\}_{ss} &= 0. \end{aligned} \quad (180)$$

Theorem 5.3 then holds in view of the fact that the isotopic element has always values smaller than $\hat{T} \ll 1$, from the fitting of all experimental data dealing with hadronic media [43], and the value of the isotopic element is null for gravitational collapse (160) $\hat{T}_{ss} = 0$.

It is easy to see that Theorems 5.2 and 5.3 resolve Objection 1.1 and 1.2 against electrons being members of a nuclear

structure. In fact, under iso-principle (180), electrons would have the *sub-luminal speed*

$$v \geq \frac{\hbar}{\Delta r \times m_e} = 5.79 \hat{T} 10^{10} \text{ m/s}, \quad \hat{T} \ll 1. \quad (181)$$

Similarly, the linear momentum uncertainty would have the value

$$\Delta p = 1.05 \hat{T} 10^{20} \text{ kg m/s}, \quad \hat{T} \ll 1, \quad (182)$$

as a result of which the energy of the electrons can be the expected value $E_e = 1.293 \text{ MeV}$, thus being much less than the 18.5 MeV predicted via Heisenberg's uncertainty principle (4). The understanding is that the final numerical values of the isotopic element for the neutron require additional studies as well as experimental measurements. Objection 1.3 has been resolved in Section 2.3.3 by showing that excessive value (5) of the magnetic moment of the electron for nuclear standards is counterbalanced by the magnetic moment of the constrained angular momentum within the proton structure.

In conclusion, rather than adapting experimental evidence to a preferred theory, it appears that mathematical and physical methods can indeed be completed to verify the evidence that the permanently stable proton and electron are the constituents of the neutron, with ensuing reduction of all matter in the universe to protons and electron in conditions of increasing complexity.

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LETTERS TO PROGRESS IN PHYSICS**Calculation of Outgoing Longwave Radiation
in the Absence of Surface Radiation of the Earth**

Y. C. Zhong

ERICHEN Consulting, Queensland, Australia. E-mail: drzhong88@yahoo.com

Based on the observed equilibrium at the surface of the earth, it is argued that almost no infrared radiation would be emitted by the surface of the earth that is in physical contact with the nearest isothermic air layer. By assuming the outgoing longwave radiation is the cumulative upward thermal radiation by the air, an analytic formula with four dependent observables is proposed which is used for the first time to calculate the effective air emissivities at different lapse rates in the troposphere. Given the observed global mean outgoing longwave radiation 239 W m^{-2} and the stable tropospheric lapse rate 6.5 K km^{-1} , the calculated effective air emissivity near the surface is 0.135, in agreement with early experimental observations.

1 Introduction

It has been recently shown that the earth is capable of self-regulating outgoing infrared radiation without changing the long-term global mean surface temperature [1]. In line with this study, it becomes clear that the radiation cooling at the surface seems unrealistically overestimated. Since 1896, it has been assumed that the surface of the earth emits infrared at radiation flux close to 390 W m^{-2} , similar to a blackbody at its thermal equilibrium temperature 288 K in vacuum, based on a model atmosphere that is physically separated from the surface [2,3]. Nevertheless, it could be argued that the widely used assumption cannot be justified in the presence of the isothermic gaseous atmosphere that is physically attached to the surface. At such a thermodynamic equilibrium, the net energy transfer between the condensed-matter surface and the nearest layer of air should be negligible if not zero. This implies that the surface infrared radiation should be absent as far as the long-term global climate stability is concerned, which is supported by recent experimental measurements that the proportion of the non-radiative heat and mass transfer at the sea level is close to 99.6% [4]. In light of this argument, an analytical formula is introduced to directly calculate the outgoing longwave radiation (OLR) in the absence of the surface infrared radiation as reported in this Letter.

2 Formulation

In the absence of the atmosphere, the thermal temperature of vacuum space is close to 4 K. Under this condition, the terrestrial infrared radiation intensity can be described by the Stefan-Boltzmann law,

$$I = \sigma T_S^4. \quad (1)$$

where σ is the Stefan-Boltzmann constant, T_S is the thermal equilibrium temperature of the condensed-matter surface that is approximated as a blackbody. However, (1) becomes in-

valid as the temperature gradient should be zero at the surface in the presence of the gaseous atmosphere. Thus, it is reasonable to assume that the OLR is merely the cumulative thermal radiation by the atmosphere from different isothermic layers. Further, it is assumed that the effective air emissivity ϵ is scaled by the air density, viz.

$$\epsilon = \epsilon_0 \frac{\rho}{\rho_0}. \quad (2)$$

where ρ is the air density with its value at the surface $\rho_0 = 1.225 \text{ kg m}^{-3}$, respectively; ϵ_0 is the atmospheric emissivity measured near the surface. To be specific, the vertical air density distribution in this study is written as

$$\rho = \rho_0 \exp(-0.135z). \quad (3)$$

where z is the altitude in km. The assumption (2) is consistent with the fact that the air thermal radiation must vanish in the absence of air molecules in the atmosphere. By approximating each thin atmospheric layer as isothermic with its local thermal equilibrium temperature, the OLR in W m^{-2} observable at the top of the atmosphere can be formulated in terms of the Stefan-Boltzmann law by the following integral

$$\text{OLR} = \int_0^\infty \epsilon \sigma T_a^4 dz. \quad (4)$$

where T_a denotes the atmospheric temperature at different altitudes.

3 Calculation

To proceed further, the troposphere and the stratosphere from the ground to altitude 85 km are divided into four parts whose vertical temperature distributions can be approximated as a step-wise linear function based on the International Standard Atmosphere [5]. Substituting (2) and (3) into (4) and integrating in each of the four parts yields

$$\text{OLR} = \epsilon_0 \sigma (A + B + C + D), \quad (5)$$

where

$$A = \int_0^a (T_s - Lz)^4 \exp(-0.135z) dz \quad (6)$$

$$B = \int_a^{20} (210)^4 \exp(-0.135z) dz \quad (7)$$

$$C = \int_{20}^{50} (164 + 2.3z)^4 \exp(-0.135z) dz \quad (8)$$

$$D = \int_{50}^{85} (389 - 2.2z)^4 \exp(-0.135z) dz, \quad (9)$$

where L denotes the lapse rate in the troposphere, the altitude a is dependent of L . Notice that $T_a = T_s$ at the surface in (6). It is apparent that the OLR is determined by two variables, the lapse rate and the effective air emissivity close to the surface when the surface temperature is fixed. It is found that the integration is nearly a constant above 85 km, as the air density exponentially decreases with the altitude. For the lapse rate 6.5 K km^{-1} , the calculated effective air emissivity near the surface is 0.135. The range of the calculated effective air emissivity, 0.12 to 0.16, for the lapse rates between 4 K km^{-1} and 10.5 K km^{-1} is consistent with some early observed atmospheric emissivities [6].

Using the observed long-term global mean OLR value, 239 W m^{-2} , the explicit dependence of the effective emissivity on the lapse rate can be fitted with a linear function with $R^2 = 0.996$,

$$\epsilon_0 = 0.0065L + 0.091. \quad (10)$$

By way of extrapolation, it is predicted that the effective emissivity of the atmosphere near the surface is 0.091 as the troposphere becomes isothermic. Besides, when the effective air emissivity and the lapse rate are fixed at 0.135 and 6.5 K km^{-1} , respectively, it is found that the calculated OLR also linearly depends on the surface temperature with $R^2 = 0.999$, viz.

$$\text{OLR} = 3.24 T_s - 695.49, \quad (11)$$

which gives the gradient

$$\frac{d(\text{OLR})}{dT_s} = 3.24 \text{ W m}^{-2} \text{ K}^{-1}. \quad (12)$$

4 Discussion and conclusion

To explore the implications of the zero surface radiation hypothesis, the outgoing thermal radiation by the air is formulated and quantitatively calculated in the absence of the surface infrared radiation. Based on the calculation, it appears that long-term global climate stability might be simply explained in relation to the tropospheric lapse rate, adjustable by changing the water vapor in the troposphere, that provides a natural mechanism to control the OLR for the earth to re-emit the absorbed solar radiation back to outer space while

keeping the global mean surface temperature constant. Further, it is revealed that the four coupled variables, namely OLR, effective air emissivity, the tropospheric lapse rate, and the surface temperature, are *linearly* dependent on each other, as shown in (10) and (11). So far, the linear dependence of the monthly mean OLR on the sea surface temperature (SST) has been observed on several locations [7], but the theoretical interpretations in terms of water vapor feedback and speculated emergent properties seem complicated and confined to the cloud-free observations [8]. By way of contrast, (11) is simply deduced from the hypothesis that the surface radiation is zero.

Without invoking the greenhouse effect, it seems the current global energy balance can be quantitatively explained, i.e. the solar shortwave radiation at the surface, 161 W m^{-2} , is completely transferred into the atmosphere by means of convection and conduction and then is thermally radiated by the atmosphere into outer space, together with the shortwave absorption by the atmosphere at 78 W m^{-2} , which makes the OLR at the top of the atmosphere equal to

$$161 + 78 = 239 \text{ W m}^{-2}$$

as observed [3]. Further experimental observations both in lab and in space are necessary for further evaluating this proposed description with fundamental implications for understanding the long-term global climate stability.

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Natural Metrology in Physics of Numerical Relations

Hartmut Müller

Rome, Italy

E-mail: hm@interscalar.com

The paper introduces the natural electron metrology that is based on the electron mass, the speed of light in a vacuum, and the Planck constant. Since the units of the electron metrology are natural, their application gives physical meaning to the numerical properties of the readings and allows to identify and predict physical effects caused by numerical relations. In this paper, the electron metrology is applied to real systems of coupled periodic processes, in particular to the solar system and exoplanetary systems. It is shown that the application of the electron metrology allows to define numerical conditions for lasting stability and to identify evolutionary trends.

Introduction

In physics, measurement is the source of data that allows to develop and verify theoretical models of reality. The result of a measurement is the ratio of physical quantities where one of them is the reference quantity called unit of measurement. Obviously, the value of this ratio depends on the chosen unit of measurement. Moreover, any change of the unit of measurement changes also the numerical properties of the value. For example, a 20 cm microwave and a 7.874... inch microwave both have the same wavelength. However, 20 is integer, but 7.874... is not. Thus, an arbitrarily chosen unit of measurement results in random values of the measured ratios. In this case, also the numerical properties of the measured values are random, and their physical interpretation has no sense. This is why in theoretical physics numerical ratios usually remain outside the realm of interest.

The situation changes fundamentally, if we choose natural units of measurement, for instance, a natural frequency of a real periodical process. In this case, all the harmonics have rational values. Thus, the use of natural units gives physical meaning to the numerical properties of the readings. Now the numerical properties of the measured frequencies provide information about whether they are harmonics or not.

Indeed, the history of metrology shows a clear trend to natural units of measurement. For instance, the current SI definition [1] of a second is based on the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom. One second takes 9,192,631,770 periods of this radiation. However, the number of periods is arbitrarily chosen. Therefore, in the current definition, one second is not a natural unit of measurement, although it is based on the frequency of a natural subatomic process. Also the current SI unit meter is not a natural unit as it is based on the current definition of a second.

The current SI definition of the kilogram is based on the fixed numerical value of the Planck constant, expressed in units of meter and second. Therefore, one kilogram is not a natural unit of measurement. Consequently, all secondary units of measurement based on kilogram, meter and second

cannot be considered natural. Therefore, the current SI is not a system of natural units.

The concept of natural units was first introduced in 1874, when George Stoney [2], noting that electric charge is quantized, derived units of length, time, and mass, now named Stoney units in his honor. Stoney chose his units so that the Newtonian gravitational constant, the speed of light in a vacuum, and the electron charge would be numerically equal to 1. In 1899, Max Planck proposed a system of units that is based on the quantum of action. Planck underlined the universality of the new system, writing [3]: ... it is possible to set up units for length, mass, time and temperature, which are independent of special bodies or substances, necessarily retaining their meaning for all times and for all civilizations, including extraterrestrial and non-human ones, which can be called natural units of measure. Planck derived units for length, time, mass, and temperature from the Newtonian gravitational constant, the speed of light, the quantum of action, and the Boltzmann constant.

Regrettably, using Newton's gravitational constant G increases not only the uncertainty of the Planck system, but also its dependence on theoretical assumptions. The constancy of G is only postulated, its value is measured in laboratory scale only, and there is no guaranty of its universality in astronomical scales, because the mass of a planet, planetoid or moon cannot be measured without using G .

In [4] we proposed a system of natural units that is based on the electron mass, the speed of light in a vacuum, the Planck constant, and the Boltzmann constant. The only difference to the Planck system is that we use the electron mass instead of G . However, this difference seems to be significant enough to give physical meaning to the numerical properties of the readings.

In [5] we have shown that in electron units, the masses of elementary particles including the proton have numerical values that approximate integer and reciprocal integer powers of Euler's transcendental number $e = 2.71828...$

As we have shown in [6], the orbital and rotational periods of the planets, planetoids and large moons of the solar

system have numerical values that approximate integer powers of Euler's number, if expressed in electron units (table 1). This we have shown also for 1430 exoplanets. Furthermore, the gravitational parameters of the Sun and the planets of the solar system, if expressed in electron units, approximate integer powers of Euler's number [7].

The electron mass is actually the key component in the natural metrology that we propose in this paper. The electron mass defines an absolute reference value, and the Planck constant in combination with the speed of light are interdimensional converters that allow to derive absolute spatial and temporal reference values, which are the Compton wavelength of the electron, and its natural frequency. The Boltzmann constant allows to derive the electron black body temperature as additional natural unit.

The electron is not a rare substance since it is ubiquitous in the universe. The uniqueness of the electron stems from its elementarity and exceptional stability, with an estimated lifetime of over 10^{28} years. In fact, stability and high precision are fundamental requirements for units of measurement. The electron mass is given with an accuracy of 10^{-10} , as shown in table 1. Since the speed of light and the Planck constant are fixed, the accuracy of the electron metrology depends only on the accuracy of the electron mass.

In the following we will show that the application of the electron metrology gives physical meaning to the numerical properties of the readings and allows to identify and predict physical effects caused by numerical relations. For reasons of clarity, in this paper we deal with periodical processes.

Theoretical Approach

The starting point of our approach is frequency as obligatory characteristic of a periodical process. As the result of a measurement is always a *ratio* of physical quantities, one can measure only *ratios* of frequencies. This ratio is always a real number. Being a real value, this ratio can approximate an integer, rational, irrational algebraic or transcendental number. In [8] we have shown that the difference between rational, irrational algebraic and transcendental numbers is not only a mathematical task, but it is also an essential aspect of stability in systems of coupled periodical processes. For instance, integer frequency ratios, in particular fractions of small integers, make possible parametric resonance that can destabilize such a system [9, 10]. This is why asteroids cannot maintain orbits that are unstable because of their resonance with Jupiter [11]. These orbits form the Kirkwood gaps that are areas in the asteroid belt where asteroids are absent.

According to this idea, irrational frequency ratios should not cause destabilizing parametric resonance, because irrational numbers cannot be represented as a ratio of integers. However, algebraic irrational numbers, being real roots of algebraic equations, can be converted to rational numbers by multiplication. For example, $\sqrt{2} = 1.41421\dots$ cannot be

ELECTRON UNITS	DEFINITION	VALUE
Electron rest energy	$E = m/c^2$	0.51099895000(15) MeV
Angular frequency	$\omega = E/\hbar$	$7.76344 \cdot 10^{20}$ Hz
Oscillation period	$\tau = 1/\omega$	$1.28809 \cdot 10^{-21}$ s
Compton wavelength	$\lambda = c/\omega$	$3.86159 \cdot 10^{-13}$ m

Table 1: Basic units of the electron metrology. The units are calculated from the measured electron rest energy. The speed of light c in a vacuum, and the Planck constant \hbar are fixed. Data from Particle Data Group [12].

come a frequency scaling factor in real systems of coupled periodical processes, because $\sqrt{2} \cdot \sqrt{2} = 2$ creates the conditions for the occurrence of parametric resonance. Thus, only transcendental ratios can prevent parametric resonance, because they cannot be converted to rational or integer numbers by multiplication. Actually, it is transcendental numbers that define the preferred frequency ratios which allow to avoid destabilizing parametric resonance [13]. In this way, transcendental frequency ratios sustain the lasting stability of coupled periodical processes.

Among all transcendental numbers, Euler's number $e = 2.71828\dots$ is unique, because its real power function e^x coincides with its own derivatives. In the consequence, Euler's number allows avoiding parametric resonance between any coupled periodical processes including their derivatives.

Because of this unique property of Euler's number, we expect that periodical processes in real systems prefer frequency ratios close to Euler's number and its roots. For rational exponents, the natural exponential function is always transcendental [14]. The natural logarithms of those frequency ratios are therefore close to integer or reciprocal integer values, which are attractors of transcendental numbers of the type e^x , as we have shown in [13]. With reference to the evolution of a planetary system and its stability, we may therefore expect that the ratio of any two orbital periods should finally approximate an integer or reciprocal integer power of Euler's number [15].

The electron shares its exceptional stability with the proton with an estimated lifetime of over 10^{29} years [12]. Within our approach, the stability of the proton results from the numerical properties of the proton-to-electron ratio that approximates the 7th power of Euler's number and its square root [7]. In this way, the metric properties of the proton can be derived from the metric properties of the electron theoretically.

The eigenfrequencies and harmonics of the proton and the electron are natural frequencies of any type of matter, also of the accreted matter of a planet. Conventional models of the solar system do not take into account this aspect, which lies at the core of our numeric physical approach to the electron metrology. Given the enormous number of protons and electrons that form a planet, eigenresonance must be avoided in

the long term. This affects any periodical process including orbital and rotational motion. This is why the planets in the solar system and in hundreds of exoplanetary systems have orbital periods that approximate integer and rational powers of Euler's number relative to the natural oscillation periods of the proton and the electron, as shown in my paper [6].

In the following, we discuss exemplary applications of the electron metrology to the analysis of orbital and rotational periods in the solar system.

Exemplary Applications

Kepler's laws of planetary motion do not explain why the planets of the solar system have the orbital periods 87.969, 224.701, 365.256, 686.971 days, and 11.862, 29.457, 84.02, 164.8, 247.94 years, because there are infinitely many pairs of orbital periods and distances that fulfill Kepler's laws. Einstein's field equations do not reduce the theoretical variety of possible orbits, but increases it even more.

However, if we express the orbital periods in electron units, we can realize that they approximate integer powers and roots of Euler's number, and in this way, they avoid destabilizing parametric resonance. This requirement reduces dramatically the number of possible orbits.

For instance, if we express Jupiter's orbital period in years (11.862), in days (4332.59) or in seconds ($3.74343 \cdot 10^8$), there is no way to verify whether this value is special or not. If we express Jupiter's orbital period in oscillation periods of the electron, we can realize that it is indeed very special, because it approximates the 66^{th} power of Euler's number:

$$\ln\left(\frac{T_O(\text{Jupiter})}{2\pi \cdot \tau_e}\right) = \ln\left(\frac{3.74343 \cdot 10^8 \text{ s}}{2\pi \cdot 1.28809 \cdot 10^{-21} \text{ s}}\right) = 66.00$$

The same is valid for the orbital period 686.98 days ($5.93551 \cdot 10^7$ seconds) of the planet Mars that equals the 66^{th} power of Euler's number multiplied by the *angular* oscillation period of the electron:

$$\ln\left(\frac{T_O(\text{Mars})}{\tau_e}\right) = \ln\left(\frac{5.93551 \cdot 10^7 \text{ s}}{1.28809 \cdot 10^{-21} \text{ s}}\right) = 66.00$$

Consequently, the Jupiter-to-Mars orbital period ratio is 2π :

$$T_O(\text{Jupiter}) = 2\pi \cdot T_O(\text{Mars})$$

This transcendental ratio allows Mars to avoid parametric orbital resonance with Jupiter. Approaching an integer power of Euler's number relative to the electron's natural period of oscillation prevents both Jupiter's and Mars' periodic orbital motion from provoking electron based eigenresonance. Since the proton-to-electron ratio approximates an integer power of Euler's number and its square root, both planets avoid also proton based eigenresonance.

In [16] we have shown that integer and rational powers of $e = 2.71828 \dots$ and $\pi = 3.14159 \dots$ form two complementary fractal scalar fields of transcendental attractors – the *Euler field* and the *Archimedes field*.

The rotational periods of planets and planetoids of the solar system approximate integer powers of Euler's number and its square root relative to the angular oscillation period of the electron. Since the proton-to-electron ratio approximates the 7^{th} power of Euler's number and its square root, the rotational periods approximate integer powers of Euler's number relative to the angular oscillation period of the proton, as we have shown in [16].

For instance, the current sidereal rotational period of the Earth equals 23 h, 56 min and 4.1 s, or 86164.1 s. In general, the duration of the sidereal day should increase, because it is believed that the rotation of the Earth is slowing down. Indeed, if we express the sidereal rotational period of the Earth in electron units, we can realize that it must increase in order to reach the 59^{th} power of Euler's number and its square root:

$$\ln\left(\frac{T_R(\text{Earth})}{\tau_e}\right) = \ln\left(\frac{86164.1 \text{ s}}{1.28809 \cdot 10^{-21} \text{ s}}\right) = 59.47$$

However, our numeric physical approach suggests that the rotation of the Earth will slow down only until the sidereal day reaches a duration of 24 hours, 47 minutes and 1 second, or 89221 s that corresponds with the Euler-attractor:

$$\tau_e \cdot e^{59} \cdot \sqrt{e} = 89221 \text{ s}$$

When the sidereal period of rotation has reached that Euler-attractor, the rotation of the Earth should be stabilized, and should not slow down more. By the way, the sidereal rotational period of the planet Mars 24 hours, 37 minutes and 22.7 seconds, or 88642.7 s is much closer to that attractor:

$$\ln\left(\frac{T_R(\text{Mars})}{\tau_e}\right) = \ln\left(\frac{88642.7 \text{ s}}{1.28809 \cdot 10^{-21} \text{ s}}\right) = 59.49$$

Probably, smaller bodies with faster rotation can reach numerical attractors faster than larger bodies. The sidereal rotational period 9.07417 h = 32667 s of the planetoid Ceres, for example, has already reached an Euler-attractor:

$$\ln\left(\frac{T_R(\text{Ceres})}{\tau_e}\right) = \ln\left(\frac{32667 \text{ s}}{1.28809 \cdot 10^{-21} \text{ s}}\right) = 58.50$$

In general, every prime, irrational or transcendental number generates a unique fundamental fractal field of its own integer and rational powers that causes physical effects which are typical for that number.

For instance, integer and rational powers of 2 and 3 generate two different fractal scalar fields – the fundamental binary and the fundamental ternary fields, which are the strongest providers of parametric resonance.

On the contrary, the golden ratio $\phi = (\sqrt{5} + 1)/2 = 1.618 \dots$ makes difficult its rational approximation, since its continued fraction does not contain large denominators. So, the fundamental field of its integer and rational powers should be a perfect inhibitor of resonance amplification. This is why

the Venus-to-Earth orbital period ratio approximates $1/\phi$, as already shown by Butusov [17] in 1978.

In [16] we have proposed to name this field after Hippasus of Metapontum who was an ancient Greek philosopher and early follower of Pythagoras, and is widely credited with the discovery of the existence of irrational numbers, and the first proof of the irrationality of the golden ratio.

Although the golden ratio is irrational, it is a Pisot number, so its powers are getting closer and closer to whole numbers, for example, $\phi^{10} = 122.99\dots$. This is why the Hippasus field can inhibit resonance within small frequency ranges only. Hence, in systems with many coupled periodic processes, the Hippasus field can produce two opposing effects: over small frequency ranges, the Hippasus field can inhibit parametric resonance, but over large frequency ranges, it provides the long-period appearance of resonance amplification. Euler's number is not a Pisot number, so that the Euler field permits coupled periodic processes to avoid parametric resonance also over very large frequency ranges. As we have shown in [6, 7], typical examples are the orbital and rotational periods of planets and planetoids.

Conclusion

The use of natural units of measure gives physical meaning to the numerical properties of the readings and allows the study of physical effects caused by their numerical relations.

In the case of frequency ratios, the readings are real numbers that can approximate integer, rational, irrational algebraic or transcendental values.

In application to real systems of coupled periodic processes, transcendental numerical relations can avoid destabilizing parametric resonance and provide lasting stability.

In units of the electron metrology (table 1), the orbital and rotational periods of large bodies of the solar system approximate integer powers of Euler's number and its roots multiplied by the natural oscillation period of the electron. This we have verified [6] also for 1430 exoplanets.

The perihelion and aphelion of a planetary orbit, if expressed in units of the electron metrology, give the lower and upper approximations of integer powers of Euler's number, as we have shown in [7]. As a consequence, the gravitational parameters of the Sun and its planets, if expressed in electron units, approximate integer powers of Euler's number.

The maxima in the frequency distribution of the number of stars as function of the distance between them, expressed in electron units, correspond with integer powers of Euler's number and its roots. In [18] we have shown this for 18336 interstellar distances in the solar neighborhood.

All these findings allow us to interpret the approximation of integer powers of Euler's number and its roots as general evolutionary trend.

In this context, also the current temperature 2.726 K of the cosmic microwave background radiation (CMBR) does

not appear as to be accidental. In [8] we have shown that this temperature, if expressed in electron units, approximates an integer power of Euler's number. Consequently, it is very unlikely that the temperature of the CMBR will still decrease. This conclusion contradicts the big bang model of a cooling down universe. However, a resonating with protons and electrons fulfilling the entire cosmic space microwave radiation could probably impede the formation of molecules essential for life. By obeying the Euler field, the CMBR allows life to arise. From this point of view, the Euler field can be seen as a promoter of life on a cosmic scale.

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