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On Generalized Difference Rough Ideal Statistical Convergence in Neutrosophic Normed Spaces

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Abstract. This article's main goal is to provide and investigate a novel statistical convergence generalisation for generalized difference sequences in Neutrosophic Normed Spaces (*NNS*) called rough ideal statistical convergence. The collection of approximate optimal statistical limit points and cluster points is defined, and their algebraic and topological features are examined. Additionally, we included the relationship for generalized difference sequences in *NNS* between rough I-statistical cluster points and rough I-statistical limit points.

Keywords: Neutrosophic normed space; ideal statistical convergence; rough ideal statistical convergence; difference sequence

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1. **Introduction**

Numerous fields of analysis and number theory have found use for statistical convergence, which has been thoroughly researched to comprehend the convergence behaviour of various kinds of sequences and series. It enables mathematicians to investigate convergence in more versatile and general contexts, which may produce fresh ideas and discoveries in a variety of mathematical fields. Zygmund first introduced statistical convergence in his 1935 monograph, which was first published in Warsaw [52]. Steinhaus [50] and Fast [20] developed the idea of statistical convergence, and then later reintroduced by Schoenberg [42]. The idea of natural density serves as its foundation.

The Natural density, denoted by $\delta(K_p)$ of the set $K_p = \{p \in \mathbb{N} : p \leq n\} \subseteq \mathbb{N}$, is defined as $\delta(K_p) = \lim_{p\to\infty} \frac{|K_p|}{n}$, where $|K_p|$ denotes the cardinality of the set K_p. If, for each $\epsilon > 0$, $\delta(\{p \in \mathbb{N} : ||y_p - y_0|| \geq \epsilon\}) = 0$, then a sequence $y = \{y_p\}$ in $\mathbb{R}$ is considered as statistically convergent to $y_0 \in \mathbb{R}$.

For sequences on finite-dimensional normed linear spaces, Phu [38] first proposed rough convergence in 2001. After that, numerous authors were inspired to work on this type of convergence on various sequence-spaces, including those for double sequences [32, 33], triple sequences [14], lacunary sequences [26], ideals [33,36] etc. Despite this, it has been established in a variety of spaces, see [2, 3, 9, 14] etc. It was later expanded to infinite-dimensional normed linear spaces [39]. In 2008, Aytar [5] also worked on same and introduced new generalized convergence named rough statistical convergence.

In 2000, Kostryko *et al.* [30] proposed the concept of ideal convergence(I-convergence) by generalizing the statistical convergence with the helps of ideals. For more details we refer [6–8,29,31]. With the help of ideals, In 2011, a new generalisation named rough ideal statistical convergence in normed spaces was defined by Das, Savas and Ghosal [13]. They studied its fundamental properties.

Initially, Kizmaz [28] proposed the concept of difference sequence spaces as $\mathcal{Z}(\Delta) = \{y = (y_p) : (\Delta y_p) \in \mathcal{Z}\}$ for $\mathcal{Z} = l_\infty$(spaces of all bounded sequences), C(convergent sequences), C_0(null sequences), where $\Delta y = (\Delta y_p) = (y_p - y_{p+1})$. In particular, $l(\Delta), C(\Delta)$ and $C_0(\Delta)$ are also Banach spaces, relative to a norm induced by $||y_p||_\Delta = |y_1| + \sup_k |\Delta y_p|$ and the generalized difference sequence spaces was introduced as (see [18]): $\mathcal{Z}(\Delta^m y_p) = \{y = (y_p) : \Delta^m y_p \in \mathcal{Z}\}$ where $\Delta^m y = (\Delta^m y_p) = (\Delta^{m-1} y_p - \Delta^{m-1} y_{p+1})$ so that $\Delta^m y_p = \sum_{r=0}^{m} (-1)^r \binom{m}{r} y_{k+r}$.

Various characteristics and properties of difference sequences have been explored by researchers can be found in [17–19, 21, 35]. Demir and Gümüş [15] investigated rough convergence through difference sequences on finite dimensional normed space. In 2022, Gümüş [16] defined rough statistical convergence for (Δy_p) sequences and established some properties for the collection of approximate optimal statistical limit points. Karabacak and Or [22] also studied these generalized convergence in normed linear spaces.

In order to examine the ambigious qualitative or quantitative data, Zadeh [51] invented an extension of classical sets known as fuzzy sets. Atannassov [4] prsented a new generalization named intuitionistic fuzzy sets(I.F.s). Smarandache [44] suggested Neutrosophic sets, a new variant of I.F.s. Additionally, Neutrosophic soft linear spaces [11] and Neutrosophic metric spaces [25] are also defined using this idea. Not only these, further extensions related to super soft hyper set, Revolutionary topologies, Neutrosophic topologies, Neutrosophic algebra, Neutrosophic SuperHyperStructure and Neutrosophic numbers can be seen in [10, 45–49]. In addition to establishing a number of sequential concepts in these spaces, including convergence, Cauchy, and convexity, Bera and Mahapatra [12] introduced the term neutrosophic norm. These places are employed in situations where complicated, ambiguous, and indeterminate information must be handled at the same time. There are several applications in decision-making, control theory, and other domains where imprecision and uncertainty exist.

Neutrosophic fuzzy normed spaces are a relatively advanced and specialized topic in mathematics and are primarily used in areas that require the modeling of complex and uncertain data. More recently, statistical convergence and its features in these spaces have been expanded by Kirişci and Şimşek [25].

Notable advancements in the area of rough convergence in past few years served as inspiration for the current research work. Very recently, Kişi and Yildil [27], added the theory of rough statistical convergence via difference operator in *NNS* and Kaur and Meenakshi also [24] worked on generalized ideal convergence in same space, which are the motivation for this research paper. The significance of this work is to investigate the theory of rough I-statistical convergence(rough I-st-convergence) for generalized difference sequences in the setup of *NNS* including the algebraic and topological characteristics for the set of rough I-st-limit points and rough I-st-cluster points for generalised difference sequences. Along with investigating some features, we also established the relationship between limit points set and cluster points set. The main objective is to find the feasibility of new notion convergence with classical one convergence. Also to identify whether this convergence i.e. rough I-st-converegnce for generalized difference sequences in *NNS*, serves the theory of convergence in *NNS*.
In the next section, the basic definitions are included which are necessary for the development of this work. Section 3 and 4 includes the main results of the paper. In section 3, rough I-st-convergence for generalized difference sequences in *NNS* has been introduced. The algebraic and topological characteristics for the set of rough I-st-limit points for generalized difference sequences like convexity, closedness etc. are proved. In section 4, rough I-st-cluster points for generalized difference sequences in *NNS* have been defined and the relationship between limit points and cluster points has been explored.

2. Preliminaries

we begin this section with fundamental definitions utilised for perceptual progression, which are as follows:

Definition 2.1. [34] A t-norm is a continuous correspondance $\circledast : [0,1]\times[0,1] \to [0,1]$, which is associative, commutative and having identity 1 and $f \circledast g \leq j \circledast k$ for each whenever $f \leq j$ and $g \leq k$ for each f, g, j and $k \in [0,1]$.

Definition 2.2. [34] A t-conorm is a continuous correspondance $\diamond : [0,1] \times [0,1] \to [0,1]$, which is associative, commutative and having identity 0 and $f \diamond g \leq j \diamond k$ whenever $f \leq j$ and $g \leq k$ for each f, g, j and $k \in [0,1]$.

In veiw of these notations, Kirişci and Şimşek [25], recently defined *NNS* and worked on statistical convergence in same spaces.

Definition 2.3. [25] Consider $N = \{p, \psi(p), \eta(p), \sigma(p) : p \in \mathcal{V}\}$ be a normed space such that $N : \mathcal{V} \times \mathbb{R}^+ \to [0, 1]$, $\mathcal{V}$ be a vector space and $\circledast$, $\diamond$ are continuous t-norm and continuous t-conorm respectively. Then $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is named as Neutrosophic Normed Spaces (*NNS*) if the following axioms hold:

For each $p, q \in \mathcal{V}$ and $\xi, \lambda > 0$ and for every $\alpha \neq 0$ we have

(1) $0 \leq \psi(p, \xi), \eta(p, \xi), \sigma(p, \xi) \leq 1$ for every $\xi \in \mathbb{R}^+$;
(2) $\psi(p, \xi) + \eta(p, \xi) + \sigma(p, \xi) \leq 3$ for $\xi \in \mathbb{R}^+$;
(3) $\psi(p, \xi) = 1, \eta(p, \xi) = 0$ and $\sigma(p, \xi) = 0$ for $\xi > 0$ iff $p = 0$;
(4) $\psi(p, \xi) = 0, \eta(p, \xi) = 1$ and $\sigma(p, \xi) = 1$ for $\xi \leq 0$;
(5) $\psi(\alpha p, \xi) = \psi(p, \frac{\xi}{|\alpha|}), \eta(\alpha p, \xi) = \eta(p, \frac{\xi}{|\alpha|})$ and $\sigma(\alpha p, \xi) = \sigma(p, \frac{\xi}{|\alpha|})$;
(6) $\psi(p, \lambda) \circledast \psi(p, \xi) \leq \psi(p + q, \lambda + \xi)$;
(7) $\psi(p, \circledast)$ is continuous non-decreasing function;
(8) $\eta(p, \lambda) \diamond \eta(q, \xi) \geq \eta(p + q, \lambda + \xi)$;
(9) $\eta(p, \diamond)$ is continuous non-decreasing function;
(10) $\sigma(p, \lambda) \diamond \sigma(q, \xi) \geq \sigma(p + q, \lambda + \xi)$;
(11) $\sigma(p, \diamond)$ is continuous non-decreasing function;
(12) $\lim_{\xi\to\infty} \psi(p, \xi) = 1, \lim_{\xi\to\infty} \eta(p, \xi) = 0$ and $\lim_{\xi\to\infty} \sigma(p, \xi) = 0$.

Here, $N(\psi, \eta, \sigma)$ is the Neutrosophic norm on $\mathcal{V}$.

Definition 2.4. [25] A sequence $\{y_p\}$ in *NNS* $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is called *statistically convergent* to $\kappa \in \mathbb{Y}$ w.r.t to norms (ψ, η, σ) if for $\epsilon > 0$ and $\lambda \in (0, 1)$

$$\lim_{n\to\infty} \frac{1}{n} |\{p \in \mathbb{N} : \psi(y_p - \kappa, \epsilon) \leq 1 - \lambda \text{ or } \eta(y_p - \kappa, \epsilon) \geq \lambda \text{ or } \sigma(y_p - \kappa, \epsilon) \geq \lambda\}| = 0.$$

equivalently $\delta(\mathbb{A}(\epsilon, \lambda)) = 0$ where
$\mathbb{A}(\epsilon, \lambda) = \{p \in \mathbb{N} : \psi(y_p - \kappa, \epsilon) \leq 1 - \lambda \text{ or } \eta(y_p - \kappa, \epsilon) \geq \lambda \text{ or } \sigma(y_p - \kappa, \epsilon) \geq \lambda\}$.

In 2023, Kişi and Yildil [27], defined rough statistical convergence via difference operator in *NNS* as follow:

Definition 2.5. [27] A sequence $\varDelta y = \{\varDelta y_p\}$ in *NNS* $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is called *rough statistically convergent* to $\kappa \in \mathbb{Y}$ w.r.t. norms (ψ, η, σ) for some $r > 0$ if for each $\epsilon > 0$ and $\lambda \in (0, 1)$

$$\lim_{n\to\infty} \frac{1}{n} |\{p \leq n : \psi(\varDelta y_p - \kappa; r + \epsilon) \leq 1 - \lambda \text{ or } \eta(\varDelta y_p - \kappa; r + \epsilon) \geq \lambda \text{ or } \sigma(\varDelta y_p - \kappa; r + \epsilon) \geq \lambda\}| = 0, .$$

or

$$\delta(\{p \leq n : \psi(\varDelta y_p - \kappa; r + \epsilon) \leq 1 - \lambda \text{ or } \eta(\varDelta y_p - \kappa; r + \epsilon) \geq \lambda \text{ or } \sigma(\varDelta y_p - \kappa; r + \epsilon) \geq \lambda\}) = 0, .$$

It is denoted by $\varDelta y_p \xrightarrow{r-st_{(\psi,\eta,\sigma)}} \kappa$ or $r - st_{(\psi,\eta,\sigma)} - \lim_{p\to\infty} \varDelta y_p = \kappa$.
Let $st_{(\psi,\eta,\sigma)} - LIM^r_{y_p}$ represents the the set of all rough st-limit points of the differenc sequence $\varDelta y = \{\varDelta y_p\}$.

3. Rough ideal statistical convergence for generalized difference sequences in *NNS*

In this section, we introduce the notion of Rough I-Statistical Convergence for generalized difference sequences in *NNS* and then examined some results using generalized difference sequence. Throughout the article I is an admissible ideal and $(\Delta^m y_p) = \Delta^{m-1} y_p - \Delta^{m-1} y_{p+1}$, where $m \in \mathbb{N}$, be the generalized difference sequence.

Definition 3.1. Let $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ be *NNS*, $\Delta^m y = (\Delta^m y_p)$ where $m \in \mathbb{N}$, a generalized difference sequence is considered as *rough Δ^m-statistical convergent* to $\xi \in \mathbb{Y}$ w.r.t. neutrosophic norm (ψ, η, σ) for $r \geq 0$ if for every $\epsilon > 0$ and $\lambda \in (0,1)$

$$\lim_{n\to\infty} \frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p - \xi; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda\}| = 0,$$

or

$$\delta(\{p \leq n : \psi(\Delta^m y_p - \xi; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda\}) = 0.$$

It is denoted by $\Delta^m y_p \xrightarrow{r-st_{(\psi,\eta,\sigma)}} \xi$ or $r - st_{(\psi,\eta,\sigma)} - \lim_{p\to\infty} \Delta^m y_p = \xi$.
Let $st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ represnts the collection of all rough st-limit points of $(\Delta^m y_p)$.

Remark 3.2. For $r = 0$, the notion rough Δ^m-st-convergence is equivalent to the Δ^m- st-convergence for $(\Delta^m y_p)$ in *NNS*.

Remark 3.3. For $m = 1$, the notion rough Δ^m-statistical convergence agrees with rough Δ-st-convergence studied in [27].

The $r - st_{(\psi,\eta,\sigma)}$-limit of a generalized difference sequence may not be unique in *NNS*. So, consider the set of rough st-limit points of $(\Delta^m y_p)$ as $st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \left[\xi : \Delta^m y_p \xrightarrow{r-st_{(\psi,\eta,\sigma)}} \xi\right]$. Note that the sequence $(\Delta^m y_p)$ is $r_{(\psi,\eta,\sigma)}$-convergent if $LIM^{r(\psi,\eta,\sigma)}_{\Delta^m y_p} \neq \phi$, where

$$LIM^{r(\psi,\eta,\sigma)}_{\Delta^m y_p} = \left[\xi^* \in \mathbb{Y} : \Delta^m y_p \xrightarrow{r_{(\psi,\eta,\sigma)}} \xi^*\right].$$

Example 3.4. Let $(\mathbb{Y}, \|.\|)$ be real normed space. For every $q > 0$ and for all $\Delta^m y = (\Delta^m y_p) \in \mathbb{Y}$, define $\psi(\Delta^m y_p, q) = \frac{q}{q+||\Delta^m y_p||}$, $\eta(\Delta^m y_p, q) = \frac{\Delta^m y_p}{q+||\Delta^m y_p||}$ and $\sigma(\Delta^m y_p, q) = \frac{\Delta^m y_p}{q+||\Delta^m y_p||}$. Then $\mathbb{Y} = (V, N, \circledast, \diamond)$ is *NNS*. Consider a sequence $(\Delta^m y_p)_{m\in\mathbb{N}}$ such that

$$\Delta^m y_p = \begin{cases} (-1)^p, & \text{if } p \neq n^2 \\ p, & \text{if } p = n^2 \ . \end{cases}$$

Then $\Delta^m y_p = (-1, 2, 3, 1, 5, 6, 7, 8, -1.........)$ and Clearly for every $\epsilon > 0$ and $\lambda \in (0,1)$

$$\lim_{n\to\infty} \frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p - \xi; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda\}| = 0.$$

Also,

$$st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \begin{cases} \phi & r < 1 \\ [1-r, r-1] & \text{otherwise.} \end{cases}$$

For unbounded sequences, $LIM^{r(\psi,\eta,\sigma)}_{\Delta^m y_p} = \phi$, But $st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \neq \phi$. That is the sequence might be rough st-convergent. Above example shows that a generalized sequence can be rough st-convergent but not rough convergent.

Definition 3.5. Let $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ be *NNS*, $\Delta^m y = (\Delta^m y_p)$, be a generalized difference sequence in $\mathbb{Y}$ is considered to be *I-Δ^m-statistically convergent* to $\xi \in \mathbb{Y}$ w.r.t. neutrosophic norm (ψ, η, σ) if for every $\epsilon > 0$ and $\lambda \in (0,1)$

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \xi; \epsilon) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p - \xi; \epsilon) \geq \lambda \text{ or}$$
$$\sigma(\Delta^m y_p - \xi; r + \epsilon) \geq \lambda\}| \geq \delta\} \in I,$$

It is denoted by $\Delta^m y_p \xrightarrow{I-st_{(\psi,\eta,\sigma)}} \xi$. Let $I - st_{(\psi,\eta,\sigma)} - LIM_{\Delta^m y_p}$ represnts the collection of all I- statistical limit points of $(\Delta^m y_p)$.

Next, we added rough I-st-convergence for generalized differenece sequences in *NNS*.

Definition 3.6. Let $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ be *NNS*, $\Delta^m y = (\Delta^m y_p)$, be a generalized difference sequence in $\mathbb{Y}$ is considered to be *rough I-Δ^m-statistically convergent* to $\xi \in \mathbb{Y}$ w.r.t. neutrosophic norm (ψ, η, σ) for some $r \geq 0$ if for every $\epsilon > 0$ and $\lambda \in (0,1)$

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \xi; r + \epsilon) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p - \xi; r + \epsilon) \geq \lambda$$
$$\text{or } \sigma(\Delta^m y_p - \xi; r + \epsilon) \geq \lambda\}| \geq \delta\} \in I,$$

It is denoted by $\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} \xi$. Let $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ represents the collection of all rough I-st-limit points of $(\Delta^m y_p)$.

Remark 3.7. For $r = 0$, the notion *rough I-Δ^m-st-convergence* agrees with the *I-Δ^m-st-convergence* in *NNS*.

Example 3.8. Let $(\mathbb{Y}, \|.\|)$ be any real normed space. For every $q > 0$ and for all $\Delta^m y = (\Delta^m y_p) \in \mathbb{Y}$, define $\psi(\Delta^m y_p, q) = \frac{q}{q+||\Delta^m y_p||}$, $\eta(\Delta^m y_p, q) = \frac{\Delta^m y_p}{q+||\Delta^m y_p||}$ and $\sigma(\Delta^m y_p, q) = \frac{\Delta^m y_p}{q+||\Delta^m y_p||}$. Then $\mathbb{Y} = (V, N, \circledast, \diamond)$ is *NNS*. If we take $I_d = \{A \in \mathbb{N} \text{ with } \delta(A) = 0\}$ then for this admissible ideal, define $(\Delta^m y_p)_{m \in \mathbb{N}}$ such that

$$\Delta^m y_p = \begin{cases} p, & \text{if } p \in A \\ (-1)^p, & p \notin A. \end{cases}$$

Then

$$I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \begin{cases} \phi & r < 1 \\ [1-r, r-1] & \text{otherwise.} \end{cases}$$

Hence $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \neq \phi$.

Definition 3.9. A sequence $\Delta^m y = (\Delta^m y_p)$ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is $I - \Delta^m$-st bounded if $\exists\ G > 0$, for $\epsilon > 0$ and $0 < \lambda < 1$ such that

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p; G) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p; G) \geq \lambda \text{ or } \sigma(\Delta^m y_p; G) \geq \lambda\}| \geq \delta\} \in I$$

We found the following results for generalised difference sequences using the aforementioned definitions in *NNS*.

Theorem 3.10. *Let $\mathbb{Y} = (V, N, \circledast, \diamond)$ be NNS. A sequence $(\Delta^m y_p)$ in $\mathbb{Y}$ is I-Δ^m-st-bounded iff $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \neq \phi$ for some $r > 0$.*

Proof. Firstly suppose the sequence $(\Delta^m y_p)$ is I-Δ^m-st-bounded in NNS $\mathbb{Y} = (V, N, \circledast, \diamond)$. Then for each $\epsilon > 0$, $\lambda \in (0, 1)$ and some $r > 0$, $\exists G > 0$ such that

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p; G) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p; G) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \xi; r + \epsilon) \geq \lambda\}| \geq \delta\} \in I.$$

Since I is admissible ideal , therefore $M = \mathbb{N} \setminus H$ is a non-empty set, where

$$H = \left\{n \in \mathbb{N} : \frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p; G) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p; G) \geq \lambda \text{ or } \sigma(\Delta^m y_p; G) \geq \lambda\}| \geq \delta\right\}.$$

Choose $p \in\ M$, then

$$\frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p; G) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p; G) \geq \lambda \text{ or } \sigma(\Delta^m y_p; G) \geq \lambda\}|\ < \delta$$

$$\Longrightarrow\ \frac{1}{n} |\{p \leq n : \psi(\Delta^m y_p; G) > 1 - \lambda \text{ and } \eta(\Delta^m y_p; G) < \lambda \text{ or } \sigma(\Delta^m y_p; G) \geq \lambda\}|\ \geq 1-\delta. \quad (1)$$

Let $\mathbb{K} = \{p \leq n : \psi(\Delta^m y_p; G) > 1 - \lambda \text{ and } \eta(\Delta^m y_p; G) < \lambda \text{ and } \sigma(\Delta^m y_p; G) < \lambda\}$.
Also for $p \in \mathbb{K}$,

$$\begin{aligned}\psi(\Delta^m y_p; r + G) &\geq \min\{\psi(0, r), \psi(\Delta^m y_p, G)\}\\ &= \min\{1, \psi(\Delta^m y_p; G)\}\\ &> 1 - \lambda\end{aligned}$$

$$\begin{aligned}\eta(\Delta^m y_p; r + G) &\leq \max\{\eta(0, r), \eta(\Delta^m y_p, G)\}\\ &= \max\{0, \eta(\Delta^m y_p; G)\}\\ &< \lambda\end{aligned}$$

$$\begin{aligned}\sigma(\Delta^m y_p; r + G) &\leq \max\{\sigma(0, r), \sigma(\Delta^m y_p, G)\}\\ &= \max\{0, \sigma(\Delta^m y_p; G)\}\\ &< \lambda\end{aligned}$$

Thus $\mathbb{K} \subset \{p \leq n : \psi(\Delta^m y_p; r+G) > 1-\lambda, \eta(\Delta^m y_p; r+G) < \lambda, \sigma(\Delta^m y_p; r+G) < \lambda\}$.
Using (1), we have

$$1-\delta \leq \frac{|\mathbb{K}|}{n} \leq \frac{1}{n}\left|\{p \leq n : \psi(\Delta^m y_p; r+G) > 1-\lambda, \eta(\Delta^m y_p; r+G) < \lambda, \sigma(\Delta^m y_p; r+G) < \lambda\}\right|.$$

Therefore,

$$\frac{1}{n}\left|\{p \leq n : \psi(\Delta^m y_p; r+G) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p; r+G) \geq \lambda \text{ or } \sigma(\Delta^m y_p; r+G) < \lambda\}\right| < 1-(1-\delta) < \delta.$$

Then for $n \in \mathbb{N}$,

$$\frac{1}{n}|\{p \leq n : \psi(y_p; r+G) \leq 1-\lambda \text{ or } \eta(y_p; r+G) \geq \lambda \text{ or } \sigma(\Delta^m y_p; r+G) \geq \lambda\}| \geq \delta \subset H \in I.$$

Hence $0 \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y}$. Therefore, $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y} \neq \phi$.
Conversely; for some $r > 0$, let $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \neq \phi$. Then $\exists$ some $\omega \in \mathbb{Y}$ such that $\omega \in I - st_{(\psi,\eta)} - LIM^r_{\Delta^m y_p}$.
For every $\epsilon > 0$ and $0 < \lambda < 1$, we have

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \omega; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \omega; r+\epsilon) \geq \lambda \text{ or}$$

,

$$\sigma(\Delta^m y_p - \omega; r+\epsilon) < \lambda\}| \geq \delta\} \in I$$

It follows that nearly all $\Delta^m y_p$'s are encircled in a ball with centre ω in *NNS*, This suggests $(\Delta^m y_p)$ is *I-Δ^m-statistically bounded* in *NNS*. □

Next, we will show that the algebraic characterization also hold for rough I-st-convergent sequences for generalized difference sequences in *NNS*.

Theorem 3.11. *Let $(\Delta^m y_p)$ and $(\Delta^m y_q)$ be two generalized difference sequences in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ with I as admissible ideal and $r \geq 0$, then the following results holds:*
(i) if $\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} L$ and $\beta \in \mathbb{R}$, then $\beta\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} \beta L$.
(ii) if $\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} L_1$ and $\Delta^m y_q \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} L_2$ then $(\Delta^m y_p + \Delta^m y_q) \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} (L_1 + L_2)$

Proof. (i) If $\beta = 0$ then the result is obvious. So assume $\beta \neq 0$. As $\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} L$ then for given $\lambda > 0$ and $r \geq 0$,
Consider $G = \{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - L; r+\epsilon) \leq 1-\lambda$ or $\eta(\Delta^m y_p - L; r+\epsilon) \geq \lambda$ or $\sigma(\Delta^m y_p - L; r+\epsilon) < \lambda\}| \geq \delta\} \in I$. As I is admissible ideal, therefore take $M = \mathbb{N} \setminus G$ as an non-empty set. Choose $m \in M$, then

$$\frac{1}{n}\left|\{p \leq n : \psi(\Delta^m y_p - L; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - L; r+\epsilon) \geq \lambda \text{ or } \sigma(\Delta^m y_p - L; r+\epsilon) < \lambda\}\right| < \delta$$

$$\implies \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - L; r+\epsilon) > 1-\lambda \text{ or } \eta(\Delta^m y_p - L; r+\epsilon) < \lambda$$
$$\text{or } \sigma(\Delta^m y_p - L; r+\epsilon) < \lambda\}| \geq 1-\delta.$$
$$\implies \frac{1}{n}|\mathbb{K}| \geq 1-\delta. \tag{2}$$

Where

$$\mathbb{K} = \{p \in \mathbb{N} : \psi(\Delta^m y_p - L; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - L; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - L; r+\epsilon) < \lambda\}.$$

It suffices to demonstrate that for each $\lambda > 0$ and $r \geq 0$;

$$\mathbb{K} \subset \{m \in \mathbb{N} : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) > 1-\lambda, \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda, \sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda\}.$$

Let $k \in \mathbb{K}$, then $\psi(\Delta^m y_k - L; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_k - L; r+\epsilon) < \lambda$ and $\sigma(\Delta^m y_p - L; r+\epsilon) < \lambda$.
So;

$$\begin{aligned}
\psi(\beta\Delta^m y_p - \beta L; r+\epsilon) &= \psi\left(\Delta^m y_p - L, \frac{r+\epsilon}{|\beta|}\right) \\
&\geq \min\left\{\psi(\Delta^m y_p - L, r+\epsilon), \psi\left(0, \frac{r+\epsilon}{|\beta|} - (r+\epsilon)\right)\right\} \\
&\geq \min\{\psi(\Delta^m y_p - L, r+\epsilon), 1\} \\
&= \psi(\Delta^m y_p - L, r+\epsilon) > 1-\lambda, \\
\eta(\beta\Delta^m y_p - \beta L; r+\epsilon) &= \eta\left(\Delta^m y_p - L, \frac{r+\epsilon}{|\beta|}\right) \\
&\leq \max\left\{\eta(\Delta^m y_p - L, r+\epsilon), \eta\left(0, \frac{r+\epsilon}{|\beta|} - (r+\epsilon)\right)\right\} \\
&\leq \max\{\eta(\Delta^m y_p - L, r+\epsilon), 0\} \\
&= \sigma(\Delta^m y_p - L, r+\epsilon) < \lambda. \\
\text{and } \sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) &= \sigma\left(\Delta^m y_p - L, \frac{r+\epsilon}{|\beta|}\right) \\
&\leq \max\left\{\sigma(\Delta^m y_p - L, r+\epsilon), \sigma\left(0, \frac{r+\epsilon}{|\beta|} - (r+\epsilon)\right)\right\} \\
&\leq \max\{\sigma(\Delta^m y_p - L, r+\epsilon), 0\} \\
&= \sigma(\Delta^m y_p - L, r+\epsilon) < \lambda.
\end{aligned}$$

which gives;

$$\mathbb{K} \subset \{m \in \mathbb{N} : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) > 1-\lambda, \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda, \sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda\}.$$

Using (2), we have

$$1-\delta \leq \frac{|\mathbb{K}|}{n} \leq \frac{1}{n}|\{p \leq n : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) > 1-\lambda, \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda,$$
$$\sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda\}|.$$

Therefore,

$$\frac{1}{n}|\{p \leq n : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) \geq \lambda,$$
$$\text{or } \sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) \geq \lambda\}| < 1-(1-\delta) < \delta.$$

Then

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) \geq \lambda,$$
$$\text{or } \sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) \geq \lambda\}| \geq \delta\} \subset \mathbb{G} \in I.$$

which shows that $\beta\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} \beta L$.

(ii) In the similar manner, we can prove (ii) part. So, we are omitting its proof. □

In next result, we will show the set $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ is closed.

Theorem 3.12. *The set $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ of a generalized difference sequence $(\Delta^m y_p)$ is a closed set in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$.*

Proof. If $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \phi$ then the result is obvious as $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y}$ is either empty set or singleton set.

Let $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \neq \phi$.

Let $\Delta^m x = (\Delta^m x_p)$ be a convergent sequence in $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ which converges to $x_0 \in \mathbb{Y}$.

For $\lambda \in (0,1)$ and $\epsilon > 0 \quad \exists \ m_0 \in \mathbb{N}$ such that

$$\psi\left(\Delta^m x_p - x_0; \frac{\epsilon}{2}\right) > 1-\lambda, \eta\left(\Delta^m x_p - x_0; \frac{\epsilon}{2}\right) < \lambda, \sigma(\Delta^m x_p - x_0; \frac{\epsilon}{2}) < \lambda \text{ for all } \ p \geq m_0.$$

Let us take $\Delta^m x_{m_1} \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ with $m_1 > m_0$ such that

$$\mathbb{A} = \{p \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \geq \lambda,$$
$$\text{or } \sigma(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \geq \lambda\}| \geq \delta\} \in I$$

Take $G = \mathbb{N} \setminus \mathbb{A}$ is a non-empty set. Choose $n \in \ G$, then

$$\frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \geq \lambda$$
$$\text{or } \sigma(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \geq \lambda\}| < \delta$$
$$\Rightarrow \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) < \lambda,$$
$$\sigma(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) < \lambda\}| \geq 1-\delta.$$
$$\text{Let } \mathbb{B}_n = \{p \leq n : \psi(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) < \lambda,$$
$$\sigma(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) < \lambda\}.$$

Then for $j \in \mathbb{B}_n$, $j \geq m_0$, we have

$$\begin{aligned}
\psi(\Delta^m y_j - x_0; r+\epsilon) &\geq \min \left\{\psi\left(\Delta^m y_j - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right), \psi\left(\Delta^m x_{m_1} - x_0; \frac{\epsilon}{2}\right)\right\} \\
&> 1-\lambda, \\
\eta(\Delta^m y_j - x_0; r+\epsilon) &\leq \max \left\{\eta\left(\Delta^m y_j - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right), \eta\left(\Delta^m x_{m_1} - x_0; \frac{\epsilon}{2}\right)\right\} \\
&< \lambda \\
\sigma(\Delta^m y_j - x_0; r+\epsilon) &\leq \max \left\{\sigma\left(\Delta^m y_j - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right), \sigma\left(\Delta^m x_{m_1} - x_0; \frac{\epsilon}{2}\right)\right\} \\
&< \lambda.
\end{aligned}$$

Therefore;

$$j \in \{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}.$$

Hence

$$\mathbb{B}_n \subset \{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}$$

which implies

$$1-\delta \leq \frac{|\mathbb{B}_n|}{n} \leq \frac{1}{n}|\{p \leq n : \psi\left(\Delta^m y_p - x_0; r+\epsilon\right) > 1-\lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}|.$$

Therefore,

$$\frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - x_0; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - x_0; r+\epsilon) \geq \lambda$$
$$\text{or } \sigma(\Delta^m y_p - x_0; r+\epsilon) \geq \lambda\}| < 1-(1-\delta) = \delta.$$

Then

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - x_0; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - x_0; r+\epsilon) \geq \lambda \text{ or}$$
$$\sigma(\Delta^m y_p - x_0; r+\epsilon) \geq \lambda\}| \geq \delta\} \subset \mathbb{A} \in I.$$

implies $x_0 \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ in $(\mathbb{Y}, \psi, \eta)$. □

The convexity of the set $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y}$ is demonstrated in the following result.

Theorem 3.13. *The set $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ of a generalized difference sequence in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is a convex set for some non-negative number r.*

Proof. Let $\varphi_1, \varphi_2 \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$. For convexity we have to show that

$$(1-\omega)\varphi_1 + \omega\varphi_2 \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y}$$

for any real number $\omega \in (0,1)$.

Since $\varphi_1, \varphi_2 \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$, then $\exists \;\; p \in \mathbb{N}$ for each $\epsilon > 0$ and $\lambda \in (0,1)$ such that

$$\mathbb{A}_0 = \{p \in \mathbb{N} : \psi\left(\Delta^m y_p - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right) \leq 1-\lambda \text{ or } \eta\left(\Delta^m y_p - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right) \geq \lambda$$
$$\text{or } \sigma\left(\Delta^m y_p - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right) \geq \lambda\},$$

and

$$\mathbb{A}_1 = \{p \in \mathbb{N} : \psi\left(\Delta^m y_p - \varphi_2; \frac{r+\epsilon}{2\omega}\right) \leq 1-\lambda \text{ or } \eta\left(\Delta^m y_p - \varphi_2; \frac{r+\epsilon}{2\omega}\right) \geq \lambda$$
$$\text{or } \sigma\left(\Delta^m y_p - \varphi_2; \frac{r+\epsilon}{2\omega}\right) \geq \lambda\}.$$

For $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{p \leq n : p \in \mathbb{A}_0 \cup \mathbb{A}_1\}\right| \geq \delta\right\} \in I,$$

Now choose $\delta_1 \in (0,1)$ such that $(1-\delta_1) \in (0,\delta)$ Let

$$\mathbb{A} = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{p \leq n : p \in \mathbb{A}_0 \cup \mathbb{A}_1\}\right| \geq \delta_1\right\} \in I,$$

Now for $n \notin \mathbb{A}$, $\frac{1}{n}\left|\{p \leq n : p \in \mathbb{A}_0 \cup \mathbb{A}_1\}\right| < 1-\delta_1$ or $\frac{1}{n}\left|\{p \leq n : p \notin \mathbb{A}_0 \cup \mathbb{A}_1\}\right| \geq 1-(1-\delta_1) = \delta_1$ This implies $\{p \leq n : m \notin \mathbb{A}_0 \cup \mathbb{A}_1\} \neq \phi$

Let $m_0 \in (\mathbb{A}_0 \cup \mathbb{A}_1)^c = \mathbb{A}_0^c \cap \mathbb{A}_1^c$

Then

$$\begin{aligned}
\psi(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) &= \psi[(1-\omega)(\Delta^m y_{m_0} - \varphi_1) + \omega(\Delta^m y_{m_0} - \varphi_2); r+\epsilon] \\
&\geq \min\left\{\psi\left((1-\omega)(\Delta^m y_{m_0} - \varphi_1); \frac{r+\epsilon}{2}\right), \psi\left(\omega(\Delta^m y_{m_0} - \varphi_2); \frac{r+\epsilon}{2}\right)\right\} \\
&= \min\left\{\psi\left(\Delta^m y_{m_0} - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right), \psi\left(\Delta^m y_{m_0} - \varphi_2; \frac{r+\epsilon}{2\omega}\right)\right\} \\
&> 1-\lambda,
\end{aligned}$$

$$\begin{aligned}
\eta(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) &= \eta[(1-\omega)(\Delta^m y_{m_0} - \varphi_1) + \omega(\Delta^m y_{m_0} - \varphi_2); r+\epsilon] \\
&\leq \max\left\{\eta\left((1-\omega)(\Delta^m y_{m_0} - \varphi_1); \frac{r+\epsilon}{2}\right), \eta\left(\omega(\Delta^m y_{m_0} - \varphi_2); \frac{r+\epsilon}{2}\right)\right\} \\
&= \max\left\{\eta\left(\Delta^m y_{m_0} - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right), \eta\left(\Delta^m y_{m_0} - \varphi_2; \frac{r+\epsilon}{2\omega}\right)\right\} \\
&< \lambda,
\end{aligned}$$

$$\begin{aligned}
\sigma(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) &= \sigma[(1-\omega)(\Delta^m y_{m_0} - \varphi_1) + \omega(\Delta^m y_{m_0} - \varphi_2); r+\epsilon] \\
&\leq \max\left\{\sigma\left((1-\omega)(\Delta^m y_{m_0} - \varphi_1); \frac{r+\epsilon}{2}\right), \sigma\left(\omega(\Delta^m y_{m_0} - \varphi_2); \frac{r+\epsilon}{2}\right)\right\} \\
&= \max\left\{\sigma\left(\Delta^m y_{m_0} - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right), \sigma\left(\Delta^m y_{m_0} - \varphi_2; \frac{r+\epsilon}{2\omega}\right)\right\} \\
&< \lambda.
\end{aligned}$$

This implies $\mathbb{A}_0^c \cap \mathbb{A}_1^c \subset \mathbb{B}^c$ where

$$\mathbb{B} = \{p \in \mathbb{N} : \psi(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) \leq 1-\lambda \text{ or } \eta\,(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) \geq \lambda$$
$$\text{or } \sigma\,(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) \geq \lambda\}.$$

So for $n \notin \mathbb{A}$,

$$\delta_1 \leq \frac{1}{n}\,|\{p \leq n : p \notin \mathbb{A}_0 \cup \mathbb{A}_1\}| \leq \frac{1}{n}\,|\{p \leq n : p \notin \mathbb{B}\}|$$

or

$$\frac{1}{n}\,|\{p \leq n : p \in \mathbb{B}\}| < 1 - \delta_1 < \delta$$

Thus $\mathbb{A}^c \subset \left\{n \in \mathbb{N} : \frac{1}{n}\,|p \leq n : p \in \mathbb{B}| < \delta\right\}$. Since $\mathbb{A}^c \in \mathbb{F}(I)$, So, $\left\{n : \frac{1}{n}\,|p \leq n : p \in \mathbb{B}| < \delta\right\} \in \mathbb{F}(I)$, which implies
$\left\{n : \frac{1}{n}\,|p \leq n : p \in \mathbb{B}| \geq \delta\right\} \in I$. This implies that $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ is a convex set. □

Theorem 3.14. *A generalized difference sequence $\Delta^m y = (\Delta^m y_p)$ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is rough-I-Δ^m-statistically convergent to $\rho \in \mathbb{Y}$ w.r.t. the norm (ψ,η,σ) for some $r > 0$ if there exists a sequence $\Delta^m z = (\Delta^m z_p)$ in $\mathbb{Y}$ such that $I - st_{(\psi,\eta,\sigma)} - LIM_{\Delta^m z_p} = \rho$ in $\mathbb{Y}$ and for each $\lambda \in (0,1)$ we have $\psi(\Delta^m y_p - \Delta^m z_p; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m z_p; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \Delta^m z_p; r+\epsilon) < \lambda$ for all $p \in \mathbb{N}$.*

Proof. Since $\Delta^m z = (\Delta^m z_p)$ be a generalized difference sequence in $\mathbb{Y}$, which is *I-Δ^m-statistically convergent* to $\rho \in \mathbb{Y}$ and $\psi(\Delta^m y_p - \Delta^m z_p; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m z_p; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \Delta^m z_p; r+\epsilon) < \lambda$ for all $p \in \mathbb{N}$ and $\lambda \in (0,1)$.
Then by definition, for any $\epsilon, \delta > 0$ and $\lambda \in (0,1)$ the set

$$\mathbb{M} = \{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m z_p - \rho; \epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m z_p - \rho; \epsilon) \geq \lambda$$
$$\text{or } \sigma(\Delta^m z_p - \rho; \epsilon) \geq \lambda\}| \geq \delta\} \in I.$$

Define

$$\mathbb{A}_1 = \{p \in \mathbb{N} : \psi(\Delta^m z_p - \rho; \epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m z_p - \rho; \epsilon) \geq \lambda \text{ or } \sigma(\Delta^m z_p - \rho; \epsilon) \geq \lambda\}$$

$$\mathbb{A}_2 = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \Delta^m z_p; r) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \Delta^m z_p; r) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \Delta^m z_p; r) \geq \lambda\}.$$

For $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\,|\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} \geq \delta\right\} \in I,$$

Now choose $\delta_1 \in (0,1)$ such that $(1-\delta_1) \in (0,\delta)$ and let

$$\mathbb{A} = \left\{n : \frac{1}{n}\,|\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} \geq \delta_1\right\} \in I,$$

Now for $n \notin \mathbb{A}$

$$\frac{1}{n}\,|\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} < 1 - \delta_1$$

$$\frac{1}{n}\left|\{p \leq n : p \notin \mathbb{A}_1 \cup \mathbb{A}_2|\} \geq 1-(1-\delta_1) = \delta_1$$

This implies $\{p \leq n : p \notin \mathbb{A}_1 \cup \mathbb{A}_2\} \neq \phi$

Let $p \in (\mathbb{A}_1 \cup \mathbb{A}_2)^c = \mathbb{A}_1^c \cap \mathbb{A}_2^c$

Then

$$\begin{aligned}
\psi\left(\Delta^m y_p - \rho; r+\epsilon\right) &\geq \ \min\ \{\psi(\Delta^m y_p - \Delta^m z_p; r), \psi(\Delta^m z_p - \rho; \epsilon)\} \\
&> 1-\lambda \\
\eta\left(\Delta^m y_p - \rho; r+\epsilon\right) &\leq \ \max\ \{\eta(\Delta^m y_p - \Delta^m z_p; r), \eta(\Delta^m z_p - \rho; \epsilon)\} \\
&< \lambda \\
\sigma\left(\Delta^m y_p - \rho; r+\epsilon\right) &\leq \ \max\ \{\sigma(\Delta^m y_p - \Delta^m z_p; r), \sigma(\Delta^m z_p - \rho; \epsilon)\} \\
&< \lambda.
\end{aligned}$$

Which gives $\mathbb{A}_1^c \cap \mathbb{A}_2^c \subset \mathbb{B}^c$, where

$\mathbb{B} = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \rho; r+\epsilon) \leq 1-\lambda$ or $\eta(\Delta^m y_p - \rho; r+\epsilon) \geq \lambda$ or $\sigma(\Delta^m y_p - \rho; r+\epsilon) \geq \lambda\}$.

So for $n \notin \mathbb{A}$,

$$\delta_1 \leq \frac{1}{n}|\{p \leq n : p \notin \mathbb{A}_1 \cup \mathbb{A}_2\}| \leq \frac{1}{n}|\{p \leq n : p \notin \mathbb{B}\}|$$

or

$$\frac{1}{n}|\{p \leq n : p \in \mathbb{B}\}| < 1-\delta_1 < \delta$$

Thus $\mathbb{A}^c \subset \{n : \frac{1}{n}|p \leq n : p \in \mathbb{B}| < \delta\}$. Since $\mathbb{A}^c \in \mathbb{F}(I)$, So, $\left\{n : \frac{1}{n}\left|p \leq n : p \in \mathbb{B}\right| < \delta\right\} \in \mathbb{F}(I)$, which implies $\left\{n : \frac{1}{n}\left|p \leq n : p \in \mathbb{B}\right| \geq \delta\right\} \in I$.

Hence, $\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} \rho$ in *NNS* $\mathbb{Y} = (V, N, \circledast, \diamond)$. □

Theorem 3.15. *Let $\Delta^m y = (\Delta^m y_p)$ be a generalized differenece sequence in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ then the existence of two elements $\alpha_1, \alpha_2 \in I-st_{(\psi,\eta,\sigma)}-LIM^r_{\Delta^m y}$ is not possible for $r > 0$ and $\lambda \in (0,1)$ such that $\psi(\alpha_1-\alpha_2; cr) \leq 1-\lambda$ or $\eta(\alpha_1-\alpha_2; cr) \geq \lambda$ or $\sigma(\alpha_1-\alpha_2; cr) \geq \lambda$ for $c > 2$.*

Proof. Let us assume the existence of two elements $\alpha_1, \alpha_2 \in I-st_{(\psi,\eta,\sigma)}-LIM^r_{\Delta^m y}$ is possible such that

$$\psi(\alpha_1-\alpha_2; cr) \leq 1-\lambda \text{ or } \eta(\alpha_1-\alpha_2; cr) \geq \lambda \text{ or } \sigma(\alpha_1-\alpha_2; cr) \geq \lambda \text{ for } c > 2. \tag{3}$$

Then for each $\epsilon > 0$ and $\lambda \in (0,1)$. Define,

$$\begin{aligned}
\mathbb{A}_1 = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \alpha_1; r+\frac{\epsilon}{2}) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \alpha_1; r+\frac{\epsilon}{2}) \geq \lambda \\
\text{or } \sigma(\Delta^m y_p - \alpha_1; r+\frac{\epsilon}{2}) \geq \lambda\}
\end{aligned}$$

$$\mathbb{A}_2 = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}) \geq \lambda$$
$$\text{or } \sigma(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}) \geq \lambda\}.$$

Then

$$\frac{1}{n}|\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2\}| \leq \frac{1}{n}|\{p \leq n : p \in \mathbb{A}_1\}| + \frac{1}{n}|\{p \leq n : p \in \mathbb{A}_2\}|$$

So, by the property of I-*convergence*, we have

$$I-\lim_{n\to\infty} \frac{1}{n} |\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2\}| \leq I-\lim_{n\to\infty} \frac{1}{n} |\{p \leq n : p \in \mathbb{A}_1\}| + I-\lim_{n\to\infty} \frac{1}{n} |\{p \leq n : p \in \mathbb{A}_2\}| = 0$$

Thus

$$\left\{n : \frac{1}{n} |\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} \geq \delta\right\} \in I, \text{for all } \ \delta > 0$$

Now choose $0 < \delta_1 = 1/2 < 1$ such that $(1 - \delta_1) \in (0, \delta)$

Let

$$\mathbb{K} = \left\{n : \frac{1}{n} |\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} \geq \delta_1\right\} \in I,$$

Now for $n \notin \mathbb{K}$

$$\frac{1}{n} |\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} < 1 - \delta_1 = 1/2$$
$$\frac{1}{n} |\{p \leq n : p \notin \mathbb{A}_1 \cup \mathbb{A}_2|\} \geq 1 - (1 - \delta_1) = 1/2$$

This implies $\{p \leq n : p \notin \mathbb{A}_1 \cup \mathbb{A}_2\} \neq \phi$. Then for $p \in \mathbb{A}_1^c \cap \mathbb{A}_2^c$ we have

$$\begin{aligned}
\psi(\alpha_1 - \alpha_2; 2r + \epsilon) &\geq \min \left\{\psi\left(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}\right), \psi\left(\Delta^m y_p - \alpha_1; r + \frac{\epsilon}{2}\right)\right\} \\
&> 1 - \lambda, \\
\eta(\alpha_1 - \alpha_2; 2r + \epsilon) &\leq \max \left\{\eta\left(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}\right), \eta\left(\Delta^m y_p - \alpha_1; r + \frac{\epsilon}{2}\right)\right\} \\
&< \lambda \\
\sigma(\alpha_1 - \alpha_2; 2r + \epsilon) &\leq \max \left\{\sigma\left(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}\right), \sigma\left(\Delta^m y_p - \alpha_1; r + \frac{\epsilon}{2}\right)\right\} \\
&< \lambda.
\end{aligned}$$

Hence,

$$\psi(\alpha_1 - \alpha_2; 2r + \epsilon) > 1 - \lambda, \eta(\alpha_1 - \alpha_2; 2r + \epsilon) < \lambda, \sigma(\alpha_1 - \alpha_2; 2r + \epsilon) < \lambda. \tag{4}$$

Then from (4) we have

$$\psi(\alpha_1 - \alpha_2; cr) > 1 - \lambda, \eta(\alpha_1 - \alpha_2; cr) < \lambda, \sigma(\alpha_1 - \alpha_2; cr) < \lambda \text{ for } c > 2.$$

It contradicts (3). So, existence of two elements is not possible such that $\psi(\alpha_1 - \alpha_2; cr) \leq 1 - \lambda$ or $\eta(\alpha_1 - \alpha_2; cr) \geq \lambda$ or $\sigma(\alpha_1 - \alpha_2; cr) \geq \lambda$ for $c > 2$. □

4. Rough ideal statistical cluster points for generalized difference sequences in *NNS*

Definition 4.1. Let $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ be *NNS*. Then $\gamma \in \mathbb{Y}$ is called *rough I-Δ^m-statistical cluster point* of the sequence $\Delta^m y = (\Delta^m y_p)$ in $\mathbb{Y}$ w.r.t. norm (ψ, η, σ) for some $r > 0$ if for every $\epsilon > 0$ and $\lambda \in (0, 1)$

$$\delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \gamma; r + \epsilon) > 1 - \lambda, \eta(\Delta^m y_p - \gamma; r + \epsilon) < \lambda, \eta(\Delta^m y_p - \gamma; r + \epsilon) < \lambda\}) \neq 0$$

where $\delta_I(A) = I - \lim\limits_{n\to\infty} \frac{1}{n} |\{p \leq n : p \in A\}|$ if exists. Here γ is known as *r-I-Δ^m-st-cluster point* of a sequence $(\Delta^m y_p)$.
Let $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ indicates the collection of all *r-I-Δ^m-st-cluster points* w.r.t. the norm (ψ, η, σ) of a sequence $(\Delta^m y_p)$ in *NNS* $\mathbb{Y} = (V, N, \circledast, \diamond)$. If $r = 0$ then the notion stands for only *I-Δ^m-st-cluster point* in *NNS* $\mathbb{Y} = (V, N, \circledast, \diamond)$. Symbolically; $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = \Gamma^{I}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.

In the next result, we have derived the closedness of the set $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ of generalized difference sequence $(\Delta^m y_p)$ in $\mathbb{Y}$.

Theorem 4.2. *The set $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ of generalized difference sequence $\Delta^m y = (\Delta^m y_p)$ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is closed for some $r > 0$.*

Proof. If $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = \phi$, then the result is obvious. So nothing to prove.
Let us suppose $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \neq \phi$. Consider $\Delta^m x = (\Delta^m x_p)$ be a generalized difference sequence such that

$$(\Delta^m x) \subseteq \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \text{ and } \Delta^m x_p \xrightarrow{(\psi,\eta)} x_0.$$

To prove closedness, we will prove $x_0 \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.
As $\Delta^m x_p \xrightarrow{(\psi,\eta,\sigma)} x_0$, so for $\lambda \in (0, 1)$ and $\epsilon > 0$, $\exists$ $p_\epsilon \in \mathbb{N}$ such that
$\psi(\Delta^m x_p - x_0; \frac{\epsilon}{2}) > 1 - \lambda, \eta(\Delta^m x_p - x_0; \frac{\epsilon}{2}) < \lambda, \sigma(\Delta^m x_p - x_0; \frac{\epsilon}{2}) < \lambda$ for $p \geq p_\epsilon$.
Choose some $p_0 \in \mathbb{N}$ such that $p_0 \geq p_\epsilon$. Then we have

$$\psi(\Delta^m x_{p_0} - x_0; \frac{\epsilon}{2}) > 1 - \lambda, \eta(\Delta^m x_{p_0} - x_0; \frac{\epsilon}{2}) < \lambda, \sigma(\Delta^m x_{p_0} - x_0; \frac{\epsilon}{2}) < \lambda.$$

Again as $\Delta^m x = (\Delta^m x_p) \subseteq \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$, we have $\Delta^m x_{p_0} \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.

$$\implies \delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - c; r + \frac{\epsilon}{2}) > 1 - \lambda, \eta(\Delta^m y_p - \Delta^m x_{p_0}); r + \frac{\epsilon}{2}) < \lambda,$$
$$\sigma(\Delta^m y_p - \Delta^m x_{p_0}); r + \frac{\epsilon}{2}) < \lambda\}) \neq 0. \tag{5}$$

Consider

$$\mathbb{G} = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) > 1 - \lambda, \eta(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) < \lambda, \sigma(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) < \lambda\}$$

Choose $j \in \mathbb{G}$, then we have $\psi(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) > 1 - \lambda, \eta(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) < \lambda, \sigma(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) < \lambda$.
Now,

$$\begin{aligned}\psi(\Delta^m y_j - x_0; r+\epsilon) &\geq \min \left\{\psi\left(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right), \psi\left(\Delta^m x_{p_0} - x_0; r + \frac{\epsilon}{2}\right)\right\}\\ &> 1-\lambda,\\ \eta(\Delta^m y_j - x_0; r+\epsilon) &\leq \max \left\{\eta\left(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right), \eta\left(\Delta^m x_{p_0} - y_0; r + \frac{\epsilon}{2}\right)\right\}\\ &< \lambda\\ \sigma(\Delta^m y_j - x_0; r+\epsilon) &\leq \max \left\{\sigma\left(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right), \sigma\left(\Delta^m x_{p_0} - y_0; r + \frac{\epsilon}{2}\right)\right\}\\ &< \lambda.\end{aligned}$$

Thus

$$j \in \{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda, (\Delta^m y_p - x_0; r+\epsilon) < \lambda\}.$$

Hence

$$\{p \in \mathbb{N} : \psi(\Delta^m y_p - \Delta^m x_{p_0}; r+\frac{\epsilon}{2}) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m x_{p_0}; r+\frac{\epsilon}{2}) < \lambda, \sigma(\Delta^m y_p - \Delta^m x_{p_0}; r+\frac{\epsilon}{2}) < \lambda\}$$

$$\subseteq \{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda, \eta(y_m - x_0; r+\epsilon) < \lambda, \sigma(y_m - x_0; r+\epsilon) < \lambda\}.$$

On the other side, we have the inequality:

$$\begin{aligned}&\delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) < \lambda,\\ &\qquad \sigma(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) < \lambda\})\\ &\leq \delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda,\\ &\qquad \sigma(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}).\end{aligned} \tag{6}$$

Using (5), we conclude that

$$\delta_I\left(\{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}\right) \neq 0,$$

as the set on L.H.S.of (6) possesses natural density more than zero.
So, $x_0 \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$. □

Theorem 4.3. *Let $\Gamma^{I}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ be the collection of all I-Δ^m-st-cluster points of $\Delta^m y = (\Delta^m y_p)$ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$. Then for any arbitrary $\nu \in \Gamma^{I}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$, $\lambda \in (0,1)$ and $r \geq 0$, we have $\psi(\zeta - \nu; r) > 1-\lambda, \eta(\zeta - \nu; r) < \lambda, \sigma(\zeta - \nu; r) < \lambda$ for all $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.*

Proof. Since $\nu \in \Gamma^{I}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ then for $\lambda \in (0,1)$ and $\epsilon > 0$,

$$\delta_I\left(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \nu; \epsilon) > 1-\lambda, \eta(\Delta^m y_p - \nu; \epsilon) < \lambda, \sigma(\Delta^m y_p - \nu; \epsilon) < \lambda\}\right) \neq 0. \tag{7}$$

Now it is sufficient to show that if any $\zeta \in \mathbb{Y}$ satisfying $\psi(\zeta - \nu; \epsilon) > 1-\lambda, \eta(\zeta-\nu;\epsilon) < \lambda, \eta(\zeta - \nu; \epsilon) < \lambda$ then $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.
Suppose $j \in \{p \in \mathbb{N} : \psi(\Delta^m y_p - \nu; \epsilon) > 1-\lambda, \eta(\Delta^m y_p - \nu; \epsilon) < \lambda, \sigma(\Delta^m y_p - \nu; \epsilon) < \lambda\}$ then $\psi(\Delta^m y_j - \nu; \epsilon) > 1-\lambda, \eta(\Delta^m y_j - \nu; \epsilon) < \lambda, \sigma(\Delta^m y_j - \nu; \epsilon) < \lambda$.
On the other side,

$$\begin{aligned}
\psi\left(\Delta^m y_j - \zeta; r+\epsilon\right) &\geq \ \min\ \left\{\psi\left(\Delta^m y_j - \nu; \epsilon\right), \psi\left(\zeta - \nu; r\right)\right\} \\
&> 1-\lambda, \\
\eta\left(\Delta^m y_j - \zeta; r+\epsilon\right) &\leq \ \max\ \left\{\eta\left(\Delta^m y_j - \nu; \epsilon\right), \eta\left(\zeta - \nu; r\right)\right\} \\
&< \lambda \\
\sigma\left(\Delta^m y_j - \zeta; r+\epsilon\right) &\leq \ \max\ \left\{\sigma\left(\Delta^m y_j - \nu; \epsilon\right), \sigma\left(\zeta - \nu; r\right)\right\} \\
&< \lambda
\end{aligned}$$

So, we have $\psi\left(\Delta^m y_j - \zeta; r+\epsilon\right) > 1-\lambda, \eta\left(\Delta^m y_j - \zeta; r+\epsilon\right) < \lambda, \sigma\left(\Delta^m y_j - \zeta; r+\epsilon\right) < \lambda$.
Thus
$j \in \{p \in \mathbb{N} : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}$
which gives the inclusion

$$\begin{aligned}
&\{p \in \mathbb{N} : \psi(\Delta^m y_p - \nu; \epsilon) > 1-\lambda, \eta(\Delta^m y_p - \nu; \epsilon) < \lambda, \sigma(\Delta^m y_p - \nu; \epsilon) < \lambda\} \\
\subseteq &\left\{p \in \mathbb{N} : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\delta_I\left(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \nu; \epsilon) > 1-\lambda, \eta(\Delta^m y_p - \nu; \epsilon) < \lambda, \sigma(\Delta^m y_p - \nu; \epsilon) < \lambda\}\right) \\
\leq &\,\delta_I\left(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}\right).
\end{aligned}$$

Therefore, from (7),

$$\delta_I\left(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}\right) \neq 0.$$

Hence $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$. □

Theorem 4.4. *Let $\Delta^m y = (\Delta^m y_p)$ be a generalized difference sequence in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$. and $\overline{B(\rho,\lambda,r)} = \{\Delta^m y \in \mathbb{Y} : \psi(\Delta^m y - \rho; r) \geq 1-\lambda, \eta(\Delta^m y - \rho; r) \leq \lambda, \sigma(\Delta^m y - \rho; r) \leq \lambda\}$, denotes the closure of the open ball $B(\rho,\lambda,r) = \{\Delta^m y \in \mathbb{Y} : \psi(\Delta^m y - \rho; r) > 1-\lambda, \eta(\Delta^m y - \rho; r) < \lambda, \sigma(\Delta^m y - \rho; r) < \lambda\}$ for some $r > 0$ and $0 < \lambda < 1$ with $\rho \in \mathbb{Y}$ then $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = \bigcup_{\rho \in \Gamma^{I}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$.*

Proof. Let $\zeta \in \bigcup_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$ then there exists some $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ for $r > 0,$ and $0 < \lambda < 1$ such that $\psi(\rho-\zeta;r) > 1-\lambda, \eta(\rho-\zeta;r) < \lambda, \sigma(\rho-\zeta;r) < \lambda$.
As $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ then there exists a set

$$\mathbb{M} = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \rho; \epsilon) > 1-\lambda, \eta(\Delta^m y_p - \rho; \epsilon) < \lambda, \sigma(\Delta^m y_p - \rho; \epsilon) < \lambda\}$$

with $\delta_I(\mathbb{M}) \neq 0$. For $p \in \mathbb{M}$,

$$\begin{aligned}
\psi\left(\Delta^m y_p - \zeta; r+\epsilon\right) &\geq \min\left\{\psi\left(\Delta^m y_p - \rho; \epsilon\right), \psi\left(\rho - \zeta; r\right)\right\} \\
&> 1-\lambda, \\
\eta\left(\Delta^m y_p - \zeta; r+\epsilon\right) &\leq \max\left\{\eta\left(\Delta^m y_p - \rho; \epsilon\right), \eta\left(\rho - \zeta; r\right)\right\} \\
&< \lambda \\
\sigma\left(\Delta^m y_p - \zeta; r+\epsilon\right) &\leq \max\left\{\sigma\left(\Delta^m y_p - \rho; \epsilon\right), \sigma\left(\rho - \zeta; r\right)\right\} \\
&< \lambda.
\end{aligned}$$

This implies that
$\delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}) \neq 0.$
Hence $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$. So, $\bigcup_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} \subseteq \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.
Conversely, Take $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ if possible let $\zeta \notin \bigcup_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$ i.e. $\zeta \notin \overline{B(\rho,\lambda,r)}$ for all $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.
Then for all $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$, we have $\psi(\zeta-\rho;r) \leq 1-\lambda$ or $\eta(\zeta-\rho;r) \geq \lambda$ or $\sigma(\zeta-\rho;r) \geq \lambda$. According to theorem (4.3) for any arbitrary $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$, we have $\psi(\zeta-\rho;r) > 1-\lambda, \eta(\zeta-\rho;r) < \lambda, \sigma(\zeta-\rho;r) < \lambda$ which is contradiction to our supposition. Hence $\zeta \in \bigcup_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$. Hence $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \bigcup_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$. This completes the proof. □

Theorem 4.5. *Let $\Delta^m y = (\Delta^m y_p)$ be a generalized difference sequence in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$, Then for $\lambda \in (0,1)$ and $r > 0$,*
(i) If $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ then $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \subseteq \overline{B(\rho,\lambda,r)}$.
(ii) $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} = \left\{\xi \in \mathbb{Y} : \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B(\xi,\lambda,r)}\right\}$.

Proof. Let $\xi \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ and $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$
For $\epsilon > 0$ and $\lambda \in (0,1)$,

Consider

$$\mathbb{G} = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \xi; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \xi; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \xi; r+\epsilon) < \lambda\}$$

and

$$\mathbb{H} = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \rho; \epsilon) > 1-\lambda, \eta(\Delta^m y_p - \rho; \epsilon) < \lambda, \sigma(\Delta^m y_p - \rho; \epsilon) < \lambda\}$$

with $\delta_I(\mathbb{G}^c) = 0$ and $\delta_I(\mathbb{H}) \neq 0$ respectively. Now for $p \in \mathbb{G} \cap \mathbb{H}$,

$$\begin{aligned}
\psi(\xi - \rho; r) &\geq \min \{\psi(\Delta^m y_p - \rho; \epsilon), \psi(\Delta^m y_p - \xi; r+\epsilon)\} \\
&> 1-\lambda, \\
\eta(\xi - \rho; r) &\leq \max \{\eta(\Delta^m y_p - \rho; \epsilon), \eta(\Delta^m y_p - \xi; r+\epsilon)\} \\
&< \lambda, \\
\sigma(\xi - \rho; r) &\leq \max \{\sigma(\Delta^m y_p - \rho; \epsilon), \sigma(\Delta^m y_p - \xi; r+\epsilon)\} \\
&< \lambda.
\end{aligned}$$

Thus $\xi \in \overline{B(\rho, \lambda, r)}$. Hence $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \subseteq \overline{B(\rho, \lambda, r)}$.
(ii) It follows from (i) part that $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \subseteq \bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho, \lambda, r)}$

Take $y \in \bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho, \lambda, r)}$ then
$\psi(y-\rho; r) \geq 1-\lambda, \eta(y-\rho; r) \leq \lambda, \sigma(y-\rho; r) \leq \lambda$ for all $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$. This implies that $\Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B(y, \lambda, r)}$ i.e. $\bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho, \lambda, r)} \subseteq \{\xi \in \mathbb{Y} : \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B(\xi, \lambda, r)}\}$.
Now assume $y \notin I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$, then for $\lambda \in (0,1)$ and $\epsilon > 0$, we have

$$\delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - y; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - y; r+\epsilon) \geq \lambda \text{ or } \sigma(\Delta^m y_p - y; r+\epsilon) \geq \lambda\}) \neq 0.$$

It means there exists some I-statistical cluster point ρ for the sequence $\Delta^m y = (\Delta^m y_p)$ with $\psi(y-\rho; r+\epsilon) \leq 1-\lambda$ or $\eta(y-\rho; r+\epsilon) \geq \lambda$ or $\sigma(y-\rho; r+\epsilon) \geq \lambda$.
Thus $\Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \nsubseteq \overline{B(y, \lambda, r)}$ and $y \notin \{\xi \in \mathbb{Y} : \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B(\xi, \lambda, r)}\}$
Hence $\{\xi \in \mathbb{Y} : \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B(\xi, \lambda, r)}\} \subseteq I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$
And $\bigcap_{\rho \in \Gamma^I_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho, \lambda, r)} \subseteq I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$.
So, $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho, \lambda, r)} = \{\xi \in \mathbb{Y} : \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B(\xi, \lambda, r)}\}$. □

Theorem 4.6. *Let $\Delta^m y = (\Delta^m y_p)$ is ideal statistically convergent to ρ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$, then $\overline{B(\rho, \lambda, r)} = I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$.*

Proof. Since $(\Delta^m y_p)$ is ideal statistically convergent to ρ w.r.t.the norms (ψ,η) i.e. $(\Delta^m y_p) \xrightarrow{I-st(\psi,\eta,\sigma)} \rho$, then by definition

$$\mathbb{A} = \left\{ n : \frac{1}{n} \left| \{ p \leq n : \psi(\Delta^m y_p - \rho; \epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \rho; \epsilon) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \rho; \epsilon) \geq \lambda \} \right| > \delta \right\} \in I.$$

Take $G = \mathbb{N} \setminus \mathbb{A}$, as non-empty set then for $p \in G^c$,

$$\frac{1}{n} \left| \{ p \leq n : \psi(\Delta^m y_p - \rho; \epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \rho; \epsilon) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \rho; \epsilon) \geq \lambda \} \right| < \delta$$

$$\Rightarrow \frac{1}{n} \left| \{ p \leq n : \psi(\Delta^m y_p - \rho; \epsilon) > 1-\lambda, \eta(\Delta^m y_p - \rho; \epsilon) < \lambda, \sigma(\Delta^m y_p - \rho; \epsilon) < \lambda \} \right| \geq 1-\delta.$$

Put $\mathbb{B}_n = \{ p \leq n : \psi\left(\Delta^m y_p - \rho; \epsilon\right) > 1-\lambda, \eta\left(\Delta^m y_p - \rho; \epsilon\right) < \lambda, \sigma\left(\Delta^m y_p - \rho; \epsilon\right) < \lambda \}$ for $j \geq m$.

Now for $j \in \mathbb{B}_n$, we have $\psi\left(\Delta^m y_j - \rho; \epsilon\right) > 1-\lambda, \eta\left(\Delta^m y_j - \rho; \epsilon\right) < \lambda, \sigma\left(\Delta^m y_j - \rho; \epsilon\right) < \lambda$.

Let $y \in \overline{B\left(\rho, \lambda, r\right)}$. We will prove $y \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$

$$\begin{aligned}
\psi(\Delta^m y_j - y; r+\epsilon) &\geq \min\left\{ \psi(\Delta^m y_j - \rho, \epsilon), \psi(y-\rho, r) \right\} \\
&> 1-\lambda \\
\eta(\Delta^m y_j - y; r+\epsilon) &\leq \max\left\{ \eta(\Delta^m y_j - \rho, \epsilon), \eta(y-\rho, r) \right\} \\
&< \lambda \\
\sigma(\Delta^m y_j - y; r+\epsilon) &\leq \max\left\{ \sigma(\Delta^m y_j - \rho, \epsilon), \sigma(y-\rho, r) \right\} \\
&< \lambda.
\end{aligned}$$

Hence $\mathbb{B}_n \subset \{ p \in \mathbb{N} : \psi(\Delta^m y_p - y; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - y; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - y; r+\epsilon) < \lambda \}$, which gives:

$1-\delta \leq \frac{|\mathbb{B}_n|}{n} \leq \frac{1}{n} |\{ p \leq n : \psi(\Delta^m y_p - y; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - y; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - y; r+\epsilon) < \lambda \}|$.

Therefore,

$$\begin{aligned}
&\frac{1}{n} |\{ p \leq n : \psi(\Delta^m y_p - y; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - y; r+\epsilon) \geq \lambda \\
&\qquad \text{or } \sigma(\Delta^m y_p - y; r+\epsilon) \geq \lambda \}| < 1-(1-\delta) = \delta.
\end{aligned}$$

Then

$$\begin{aligned}
&\big\{ n \in \mathbb{N} : \frac{1}{n} |\{ p \leq n : \psi(\Delta^m y_p - y; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - y; r+\epsilon) \geq \lambda \\
&\qquad \text{or } \sigma(\Delta^m y_p - y; r+\epsilon) \geq \lambda \}| \geq \delta \big\} \subset \mathbb{A} \in I.
\end{aligned}$$

i.e. $y \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ in NNS $\mathbb{Y} = (V, N, \circledast, \diamond)$.

Hence $\overline{B\left(\rho, \lambda, r\right)} \subseteq I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$. Also $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \subseteq \overline{B\left(\rho, \lambda, r\right)}$

Hence, $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \overline{B\left(\rho, \lambda, r\right)}$. □

Theorem 4.7. *Let $\Delta^m y = (\Delta^m y_p)$ be a generalized difference sequence in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$, which is ideal statistically convergent to ξ then $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$.*

Proof. Firstly, Assume $y_p \xrightarrow{I-st_{(\psi,\eta,\sigma)}} \xi$, which gives $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = \{\xi\}$. Then for $r > 0$ and $\lambda \in (0,1)$ by Theorem (4.4) , we have $\Gamma^{r(I)}_{st_{(\psi,\eta,\sigma)}}(\Delta^m y_p) = \overline{B(\xi,\lambda,r)}$. Also from Theorem (4.6), $\overline{B(\xi,\lambda,r)} = I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$.
Hence $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$.
Conversely, Assume $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$, then by Theorem (4.4) and (4.5)(ii),

$$\bigcap_{\xi \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} = \bigcup_{\xi \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$$

This is possible only if either $\Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = \phi$ or $\Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ is a singlton set. Then $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} = \overline{B(\xi,\lambda,r)}$ for some $\xi \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$. Also, By Theorem (4.4), $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \xi$. □

Conclusions

The present work is more generalized than rough statistical convergence for difference sequences on *NNS* defined by Kişi and Yildil [27]. For this type of convergence various properties like statistical boundness, algebraic properties, closedness, convexity and relationship between limit points and cluster points have been obtained. Further this type of convergence can be investigated for double sequences, triple sequences in various sapces like Iintuitionistic fuzzy normed spaces, probablistic norm spaces or neutrosophic normed spaces.

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