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Approach to Multi-Criteria Decision-Making in a Neutrosophic Picture Hyper-Soft Set Environment using Generalized Neutrosophic TOPSIS

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Abstract: Given the complexity of today's world, we might need to work with numbers requiring multi-attribute functions, such as those having positive, neutral, and negative membership and those having truth, indeterminacy, and falsity membership. Adding these numbers together to get a single real number is the most important factor. In situations like this, decision-makers have more difficult choices and are unable to apply the single attribute function of the soft set theory. To address this constraint, the hyper-soft set theory with multi-attribute functions is introduced. We combine the notion of neutrosophic hypersoft set with picture fuzzy hypersoft set to form a single theory of neutrosophic picture hypersoft set in this study. We introduce the notions of correlation coefficient and weighted correlation coefficient and demonstrate its fundamental properties for neutrosophic picture hypersoft set. Then, we introduce the notions of a weighted average operator and a weighted geometric operator for neutrosophic picture hypersoft set by using the various aggregation operators with a suitable example. Making decisions based on several factors and choosing the best option is multi-criteria decision-making or MCDM. When ranking and choosing options based on a distance metric, one essential and useful strategy is the Technique of order preference by similarity to an ideal solution (TOPSIS). We demonstrate the accuracy of the fuzzy TOPSIS methodology by extending it to neutrosophic fuzzy TOPSIS and using neutrosophic picture hypersoft set theory to describe the MCDM problem in this study. We provide a generalized neutrosophic TOPSIS approach to demonstrate correlation coefficients and the effectiveness of this approach with an appropriate example. Finally, we offer a comparison to prior studies to demonstrate the viability of the proposed approach.

Keywords: Picture fuzzy set, soft set, Hyper-soft set, Neutrosophic set.

1. Introduction

The subject of how to express worry in mathematical modelling has nevertheless received a lot of attention. Different techniques have been suggested and endorsed by numerous researchers from across the world to reduce ambiguity [1]. To cope with uncertain, ambiguous, and indefinite things, Molodtsov [2] has presented Soft Sets (SSs) as a practical statistical technique. By Maji et al. [3], SSs have been enlarged to include the idea of fuzzy soft sets (FSSs) and detailed the various operations and attributes of SSs [4]. Maji's method of SSs has been modified by Ali et al. [5], who also used its features to create a few new operations. The idea of using soft matrices in operations has been introduced by Cagman and Enginoglu [6], who also have described their characteristics. They have also devised a decision-making (DM) strategy to tackle issues associated with uncertainty. Smarandache [7] has initially established a neutrosophic set (NS). A single-valued NS has been first used for the CC of IFS by Wang et al. [8]. NS has been extended into neutrosophic soft set (NSS) by Maji [9]. With certain operations and attributes, Broumi [10] has created a general NSS and applied it to DM problems. Cuong and Kreinovich [11] have extended the concepts of FS and IFS to form a picture fuzzy set (PFS). None of the above research can help with the problems when different characteristics have associated sub-attributes.

1.1. Literature Review

Zadeh [12] 1965 initially introduced the concept of fuzzy sets (FSs) to address issues that include uncertainty and anxiety. Atanassov [13] has extended FSs into an intuitionistic fuzzy set (IFS). However, IFS's theory is not able to address the issue of inconsistent data. To cope with uncertain, ambiguous, and indefinite things, Molodtsov [2] has presented soft sets (SSs) as a practical statistical technique. The introduction of the neutrosophic set (NS) concept, featuring the assignment of truth, indeterminacy, and falsity grades to individual set elements, has been made by Smarandache [7]. Maji et al. [3] have formulated fuzzy soft sets (FSSs) by integrating the principles of SSs and FSs. To address the limitations of NS, the introduction of the single-valued neutrosophic set (SVNS) with constrained membership values has been undertaken by Wang et al. [8]. Maji [9] has extended NS to a neutrosophic soft set (NSS). Cuong and Kreinovich [11] have further developed the concept of FS and IFS into a picture fuzzy set (PFS). Yang et al. [14] have proposed the combination of PFS and SS, resulting in a picture fuzzy soft set (PFSS). Hypersoft sets (HSSs) have been introduced by Smarandache [15] as a solution to the limitations of SSs. Jaber et al. [16] presented a generalized picture fuzzy soft set (GPFSS) by merging the concepts of PFSS and PFS in a hybrid mode to improve DM accuracy. Saqlain et al. [17] have built the aggregation operators for a neutrosophic hypersoft set (NHSS) and modified a DM approach for NHSS. Yolcu et al. [18, 19] have extended the notion of HS by introducing both a fuzzy hypersoft set (FHSS) and an intuitionistic fuzzy hypersoft set (IFHSS). Chinnadurai and Bobin [20] have extended the concept of HSS and have proposed a picture fuzzy hypersoft set (PFHSS) theory.

A summary of the conceptual framework alongside the preceding structures is presented in Table 1 below:

Table 1: Conceptual Framework Summary

Sr. No.	Proposed Structure	Corresponding Authors	Year of Publication	Key Findings
1	Fuzzy Set (FS)	Zadeh	1965	Every element in the universal set is assigned a membership value within the range of 0 to 1.
2	Intuitionistic Fuzzy Set (IFS)	Atanassov	1986	It signifies the degree to which an element of a set either belongs or does not belong.
3	Soft Set (SS)	Molodtsov	1999	It handles uncertainty in a parametric way.
4	Neutrosophic Set (NS)	Smarandache	1999	Every element within the set is assessed based on its levels of truth, indeterminacy, and falsity.
5	Fuzzy Soft Set (FSS)	Maji et al.	2001	Every power set within the universal set has been assigned fuzzy values.
6	Single-valued Neutrosophic Set (SVNS)	Wang et al.	2010	It imposes specific limitations on membership values to address the limitations encountered in NS.

7	Neutrosophic Soft Set (NSS)	Maji	2013	It combines SVNS with SS.
8	Picture Fuzzy Set (PFS)	Cuong et al.	2013	An extension of FS and IFS.
9	Picture Fuzzy Soft Set (PFSS)	Yang et al.	2015	It combines PFS with SS and addresses the inconsistent problem of data.
10	Hyper-soft Set (HSS)	Smarandache	2018	It imposes specific constraints on membership values to address the limitations encountered in SS.
11	Generalized Picture Fuzzy Soft Set (GPFSS)	Jaber et al.	2019	Integrating PFSS and PFS in a hybrid mode involves incorporating additional data into the PFS output to enhance decision-making accuracy.
12	Neutrosophic Hyper-Soft Set (NHSS)	Saqlain et al.	2020	It combines NS with HSS.
13	Fuzzy Hyper-soft Set (FHSS)	Yolcu et al.	2021	A fuzzy membership degree is assigned to each element in the power set.
14	Intuitionistic Fuzzy Hyper-Soft Set (IFHSS)	Yolcu et al.	2021	It combines IFS with HSS.
15	Picture Fuzzy Hyper-Soft Set (PFHSS)	Chinnadurai and Bobin	2021	Combination of PFSS and FHSS.

1.2. Motivation

The introduction of the concept of a simplified neutrosophic set (SNS) with various aggregation operators (AOs) was carried out by Ye [21] who introduced a DM approach based on the suggested AOs. Karaaslan [22] has defined Possibility NS and provided an NSS selection procedure to resolve those and-product-based uncertain issues. To deal with uncertainty, a function is changed into a multi-attribute function, Smarandache [15] has expanded the SSs to a hypersoft set (HSS) to overcome these formulations. To convert the fuzzily formatted neutrosophic number into the crisp form, Saqlain et al. [23] have modified a DM approach for the neutrosophic hypersoft set (NHSS). Chinnadurai and Bobin [20] have extended the concept of HSS and have proposed a picture fuzzy hypersoft set (PFHSS) theory. Zulqarnain et al. [24] have expanded on the idea of NHSS and provided specific methods for NHSS. Saqlain et al. [25] have expanded NHSS's concept and offered an interval-valued neutrosophic hypersoft set (IVNHSS). An intuitionistic fuzzy hypersoft set (IFHSS) is being developed by Zulqarnain et al. [26] who established an approach also by generating a CC to overcome DM challenges. Rahman et al. [27] have introduced the neutrosophic parameterized hypersoft set (NPHS) in DM problems. The HSS theory's foundational ideas have been researched by Saeed et al. [28]. Rahman et al. [29] have designed a model of NPHS under the environment of FS, IFS and NS. Ihsan et al. [30] have expanded a soft expert set to a hypersoft expert set to address DM issues. The IVNHSS's basic operations have been researched by Rana Muhammad Zulqarnain et al. [31]. Numerous additional studies have been performed in a neutrosophic environment, and their implications for daily life are discussed [32-54].

Today's practical applications can't be handled by a single attribute function like IFS or SS [55-57]. Improved versions of SSs and new varieties of soft sets have been made available by Smarandache [58]. He has also established a novel interpretation of the super hyper soft set [59] and added a fuzzy extension to it [60]. Several additional studies have been conducted within a neutrosophic context, and various researchers delve into the implications these studies have for everyday life [61-65]. Applying the multi-attribute function in HSS, an extension of SS, allows it to get around this restriction. Additionally, HSS may be used for any DM issues without imposing any constraints on the characteristics used by the decision-makers (DMs). Data collection for DMs may be accomplished without information loss by merging HSS with other hybrid fuzzy structures. Positive, neutral, and negative evaluations all depend on one another in the picture fuzzy set (PFS), and their aggregate cannot be larger than one. The main reason for choosing the Hypersoft Set (HSS) is because a soft set's circumstances cannot manage scenarios where attributes are more than one and further divided. Therefore, it is important to create a new strategy to address these. Taking into account both the positive and negative aspects of each option, decision-making techniques aid specialists in choosing a suitable one. Drawing inspiration from the work of Chinnadurai and Bobin [20] who have combined the structures of PFS and HSS to create a theory named a "picture fuzzy hypersoft set" (PFHSS). Thus, this study's main goal is to combine the structure of PFHSS with NS to formulate a new theory called neutrosophic picture hypersoft set (NPHSS). This study is limited to its theory and any related advancement. We illustrate the merits of the proposed theory using the instances.

The paper is structured in the following manner: Section 2: This Section covers the key terminologies and concepts that have been used for this study. Section 3: The notion of NPHSS is established with a suitable example. Section 4: The notion of correlation-coefficient for NPHSS is introduced and its main characteristics are established. Section 5: The notion of weighted correlation-coefficient for NPHSS is presented and its main characteristics are established. Section 6: The notions of a weighted average operator as well as a weighted geometric operator for NPHSS by using the various AOs are presented with suitable examples. Section 7: An algorithm for addressing MCDM problems using NPHSS through the TOPSIS method and its applications in DM problems are highlighted in this Section. Section 8: This Section discusses the comparison study of the suggested notion. Section 9: The study presents the prospects of the proposed measure in another environment.

2. Preliminaries

In this Section, we introduce fundamental definitions pertinent to the subject, and we will consistently adhere to the specified notations unless explicitly stated otherwise.

Let us assume that U be the universal set and u ∈ U, P_U be the power set of U.

Definition 2.1. A set $\mathcal{F} = \{(u, \mathcal{M}(u)) : u \in U\}$ is known as a fuzzy set (FS) [12], where $\mathcal{M}(u): U \to [0,1]$ be the membership's degree of u over U.

Definition 2.2. A set $I = \{(u, \mathcal{M}(u), N(u)) : u \in U\}$ is known as an intuitionistic fuzzy set (IFS) [13], where $\mathcal{M}(u): U \to [0,1]$ be the membership's degree of u and $N(u): U \to [0,1]$ be the non-membership degree of u such that $\forall\, u \in U$, and $0 \leq \mathcal{M}(u) + N(u) \leq 1$, where $\mathcal{H}(u) = 1 - \mathcal{M}(u) - N(u)$ be the hesitancy's degree of u.

Definition 2.3. A set $\wp = \{(u, \mathcal{M}_p(u), \mathcal{M}_N(u), \mathcal{M}_n(u)) : u \in U\}$ is known as a picture fuzzy set (PFS) [11], where $\mathcal{M}_p(u), \mathcal{M}_N(u), \mathcal{M}_n(u): U \to [0,1]$ be the values of positive membership, neutral membership and negative membership of u respectively, such that $\forall\, u \in U$, and $0 \leq \mathcal{M}_p(u) + \mathcal{M}_N(u) + \mathcal{M}_n(u) \leq 1$, where $\mathcal{R}(u) = 1 - (\mathcal{M}_p(u) + \mathcal{M}_N(u) + \mathcal{M}_n(u))$ be the refusal membership's degree of u.

Definition 2.4. An ordered pair of the form (M_S, S_p) is known as a soft set (SS) [2], if $M_S: S_p \to P_U$, S_p be a parameters' set and $S \subseteq S_p$. Here, M_S is known as an approximate function of SS such that $M_S(x) = \emptyset \text{ if } x \notin S$ where $M_S(x)$ is known as an x-approximate value set which is made up of the parameter's associated objects $x \in S_p$.

Definition 2.5. An ordered pair of the form ($\widetilde{M_S}, S_p$) defines a fuzzy soft set (FSS) [3], if $\widetilde{M_S}: S_p \to C_F(U)$, $C_F(U)$ be a set of all fuzzy subsets of U, S_p be a set of parameters and $S \subseteq S_p$. Here, $\widetilde{M_S}(x) = \emptyset \text{ if } x \notin S$ where $\emptyset$ be a null FS.

Definition 2.6. A set $NS = \{(u, T_{NS}(u), I_{NS}(u), F_{NS}(u)) : u \in U\}$ is called a neutrosophic set (NS) [8], which T_{NS} stands for truth-membership's function, I_{NS} stands for indeterminacy-membership's function, and F_{NS} stands for falsity-membership's function such that $T_{NS}(u), I_{NS}(u)$ and $F_{NS}(u) \in [0,1]$ and $0 \leq T_{NS}(u) + I_{NS}(u) + F_{NS}(u) \leq 3\ \forall\, u \in U$. The simplest way to express an element in NS is to use a single-valued neutrosophic number (SVNN) as $u = \langle T_u, I_u, F_u \rangle \forall\, u \in U$.

Definition 2.7. An ordered pair of the form (N, S) defines a neutrosophic soft set (NSS) [9, 10], if $N: S \to C_N(U)$, $C_N(U)$ be a collection of all the neutrosophic sets of U, S_p be a parameters' set and $S \subseteq S_p$.

Definition 2.8. If $(\alpha_1, \alpha_2, \dots, \alpha_i)$, be i distinct attributes, for all $i \geq 1$, with the corresponding attribute values are $(S_1, S_2, \dots, S_i)$ respectively, such that $\alpha_m \cap \alpha_n = \emptyset$ for $m \neq n$ where $m, n \in \{1, 2 \dots, i\}$. Then, an ordered pair $(H, S_1 \times S_2 \times \dots \times S_i)$ where $H: S_1 \times S_2 \times \dots \times S_i \to P_U$ is said to be a hyper-soft set (HSS) [15] in U.

Definition 2.9. If $(\alpha_1, \alpha_2, \dots, \alpha_i)$, be i distinct attributes, for all $i \geq 1$, with the corresponding attribute values are respectively $(S_1, S_2, \dots, S_i)$, such that $\alpha_m \cap \alpha_n = \emptyset$ for $m \neq n$ where $m, n \in \{1, 2 \dots, i\}$. Then, an ordered pair (N, H) is defined as a neutrosophic hyper-soft set (NHSS) [23] if there a relation exists $S_1 \times S_2 \times \dots \times S_i = H$ where $N: H \to N^U, H = S_1 \times S_2 \times \dots \times S_i$ and N^U be a set of all the neutrosophic subsets in U and $N(S_1 \times S_2 \times \dots \times S_i) = \{(u, T_{NS}(u), I_{NS}(u), F_{NS}(u)) : u \in U\}$ where T_{NS} stands for truth-membership function, I_{NS} stands for indeterminacy-membership function, and F_{NS} stands for falsity-membership function such that $T_{NS}(u), I_{NS}(u)$ and $F_{NS}(u) \in [0,1]$ and $0 \leq T_{NS}(u) + I_{NS}(u) + F_{NS}(u) \leq 3\ \forall\, u \in U$.

Remark: For two neutrosophic hypersoft subsets $(NH)_1$ and $(NH)_2$ in U, $(NH)_1$ is considered to be a neutrosophic hypersoft subset of $(NH)_2$ if the followings are satisfied:

$T_{NS}(NH)_1 \leq T_{NS}(NH)_2$, $I_{NS}(NH)_1 \leq I_{NS}(NH)_2$, $F_{NS}(NH)_2 \geq F_{NS}(NH)_2$.

Definition 2.10. An ordered pair (P, H) defines a picture fuzzy hyper-soft set (PFHSS) [20] where $P: H \to C_P, H = S_1 \times S_2 \times \dots \times S_i$ and C_P be a set of the picture fuzzy subsets in U.

Then, PFHSS represents as follows: $(P,H) = \{(p,h): p \in H = S_1 \times S_2 \times \ldots \times S_i, h \in C_P, C_P \in [0,1]\}$.

Here, $h = \{(u, \mathcal{M}_p(u), \mathcal{M}_N(u), \mathcal{M}_n(u)) : u \in U\}$ and $\mathcal{M}_p(u), \mathcal{M}_N(u), \mathcal{M}_n(u): U \to [0,1]$ be the positive membership, neutral membership and negative membership values respectively, such that $\forall\, u \in U$, and $0 \leq \mathcal{M}_p(u) + \mathcal{M}_N(u) + \mathcal{M}_n(u) \leq 1$, where $\mathcal{R}(u) = 1 - (\mathcal{M}_p(u) + \mathcal{M}_N(u) + \mathcal{M}_n(u))$ be the refusal membership's degree of u.

3. Proposed Neutrosophic Picture Hypersoft Set (NPHSS):

The notion of NPHSS is developed in this Section and goes through its fundamental characteristics of correlation coefficient (CC) as well as weighted CC.

Definition 3.1. Assume that U be the universal set, $(NH)^U$ be a set of the neutrosophic hypersoft subsets and C_P be a set of the picture fuzzy subsets. Let $(\alpha_1, \alpha_2, \ldots, \alpha_i)$ be i distinct attributes, for all $i \geq 1$, with the corresponding attribute values $(S_1, S_2, \ldots, S_i)$ respectively, such that $\alpha_m \cap \alpha_n = \emptyset$ for $m \neq n$ and $m, n \in \{1, 2 \ldots, i\}$ and $S_1 \times S_2 \times \ldots \times S_i = H$. Then, an ordered pair (P^N, H) is called as a neutrosophic picture hyper-soft set (NPHSS) in U such that $P^N: S_1 \times S_2 \times \ldots \times S_i \to (NH)^U \times C_P$.

By using the Definition 2.9 and Definition 2.10, it is represented as follows:

$P_H^N(u) = \{(u_j, \mathcal{M}_p(P^N)_{S_i}(u_j), \mathcal{M}_N(P^N)_{S_i}(u_j), \mathcal{M}_n(P^N)_{S_i}(u_j))\, \forall u_j \in U\}$ such that $0 \leq \mathcal{M}_p(P^N)_{S_i}(u_j) + \mathcal{M}_N(P^N)_{S_i}(u_j) + \mathcal{M}_n(P^N)_{S_i}(u_j) \leq 1$ where $\mathcal{M}_p(P^N), \mathcal{M}_N(P^N), \mathcal{M}_n: U \to [0,1]$ be positive membership, neutral membership and negative membership, respectively.

Remark: For two neutrosophic picture hypersoft subsets, $(P^N)_1$ and $(P^N)_2$ in U, $(P^N)_1$ is considered to be a neutrosophic picture hypersoft subset of $(P^N)_2$ if the followings are satisfied:

$\mathcal{M}_p(P^N)_1 \leq \mathcal{M}_p(P^N)_2$, $\mathcal{M}_N(P^N)_1 \leq \mathcal{M}_N(P^N)_2$, and $\mathcal{M}_n(P^N)_1 \geq \mathcal{M}_n(P^N)_2$.

Here, $\mathcal{M}_p(P^N), \mathcal{M}_N(P^N), \mathcal{M}_n(P^N): U \to [0,1]$ be the positive membership, neutral membership and negative membership, respectively, such that $\forall\, u \in U$, and $0 \leq \mathcal{M}_p(P^N)(u) + \mathcal{M}_N(P^N)(u) + \mathcal{M}_n(P^N)(u) \leq 1$, where $\mathcal{R}(u) = 1 - (\mathcal{M}_p(P^N)(u) + \mathcal{M}_N(P^N)(u) + \mathcal{M}_n(P^N)(u))$ be the refusal membership's degree of u.

The representation of the extended NPHSS model is shown in Figure 1, as outlined below:

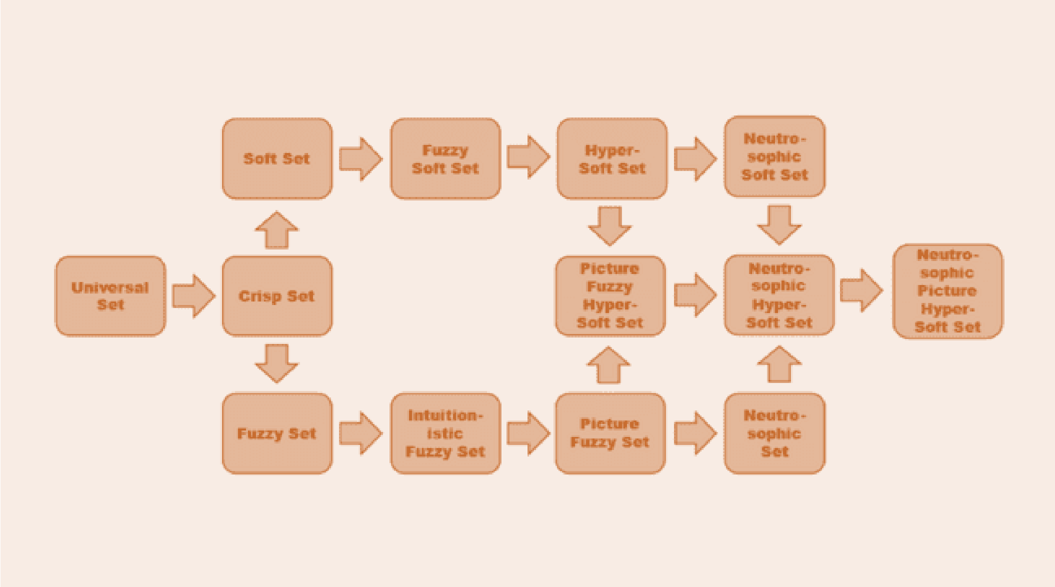


Figure 1: Flow diagram representing the NPHSS Model.

Example 3.1.1. Assuming $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ constitute a team of faculties from three departments that are responsible for evaluating the representative of the department. Let(α_1, α_2): (Criteria Analysis) be three distinct attributes, with the corresponding multi-valued sub-attributes respectively represented as follows:

$$\alpha_1 = \text{representative skills} = \begin{Bmatrix} \mathbb{Q}_{11} = \text{leadership skills}, \\ \mathbb{Q}_{12} = \text{management skills}, \\ \mathbb{Q}_{13} = \text{interpersonal skills} \end{Bmatrix} \text{ and } \alpha_2 = \text{representative experience} = \begin{Bmatrix} \mathbb{Q}_{21} = \text{research experience}, \\ \mathbb{Q}_{22} = \text{teaching experience}, \\ \mathbb{Q}_{23} = \text{industry experience} \end{Bmatrix}$$

Then, $H = \alpha_1 \times \alpha_2 = \{\mathbb{Q}_{11}, \mathbb{Q}_{12}, \mathbb{Q}_{13}\} \times \{\mathbb{Q}_{21}, \mathbb{Q}_{22}, \mathbb{Q}_{23}\} = \begin{Bmatrix} \{\mathbb{Q}_{11}, \mathbb{Q}_{21}, \mathbb{Q}_{22}, \mathbb{Q}_{23}\}, \\ \{\mathbb{Q}_{12}, \mathbb{Q}_{21}, \mathbb{Q}_{22}, \mathbb{Q}_{23}\}, \\ \{\mathbb{Q}_{13}, \mathbb{Q}_{21}, \mathbb{Q}_{22}, \mathbb{Q}_{23}\} \end{Bmatrix} = \{\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3\}.$

An NPHSS (P^N, H) is a set of subsets of $\mathcal{F}$ which is introduced by the faculties for a departmental representative (DR) as given below in Table 2.

Table 2: Qualities of a DR in NPHSS

$\mathcal{F}$	$\mathbb{Q}_1$	$\mathbb{Q}_2$	$\mathbb{Q}_3$
$\mathcal{F}_1$	⟨0.04,0.05,0.02⟩	⟨0.01,0.04,0.02⟩	⟨0.03,0.06,0.02⟩
$\mathcal{F}_2$	⟨0.02,0.04,0.05⟩	⟨0.02,0.01,0.04⟩	⟨0.02,0.03,0.06⟩
$\mathcal{F}_3$	⟨0.05,0.02,0.04⟩	⟨0.04,0.02,0.01⟩	⟨0.06,0.02,0.03⟩

4. Proposed Correlation Coefficient (CC) for NPHSS:

This Section introduces the notion of CC and its fundamental features for NPHSSs.

Let $(P^N{}_1, H_1) = \left\{\left(u_j, \mathcal{M}_p(P^N{}_1)_{S_i}(u_j), \mathcal{M}_N(P^N{}_1)_{S_i}(u_j), \mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right) : u_j \in U\right\}$ and $(P^N{}_2, H_2) = \left\{\left(u_j, \mathcal{M}_p(P^N{}_2)_{S_i}(u_j), \mathcal{M}_N(P^N{}_2)_{S_i}(u_j), \mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right) : u_j \in U\right\}$ be two NPHSSs over U.

Definition 4.1. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs, its neutrosophic informational energies are defined respectively as follows:

$$\mathrm{E}(P^N{}_1, H_1) = \sum_{i=1}^{k}\sum_{j=1}^{l}\left[\begin{array}{c}\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2 + \left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2 + \\ \left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\end{array}\right] \tag{1}$$

$$\mathrm{E}(P^N{}_2, H_2) = \sum_{i=1}^{k}\sum_{j=1}^{l}\left[\begin{array}{c}\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2 + \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2 + \\ \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\end{array}\right] \tag{2}$$

Here, $1 \le i \le k$ and $1 \le j \le l$ where $k, l \in \mathbb{N}$ (natural numbers).

Definition 4.2. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs, a correlation measure between $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ is defined as:

$$\mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right) * \left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)\right] + \\ \left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right) * \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)\right] + \\ \left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right) * \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)\right]\end{array}\right\} \tag{3}$$

Proposition 4.2.1. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs. Then,

(i) $$\mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_1, H_1)] = \mathrm{E}(P^N{}_1, H_1); \tag{4}$$

(ii) $$\mathrm{M}^c[(P^N{}_2, H_2), (P^N{}_2, H_2)] = \mathrm{E}(P^N{}_2, H_2); \tag{5}$$

Proof: By using the Equations (1), (2) and (3), the results of Equations (4) and (5) are obvious.

Definition 4.3. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs, the CC between $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ is defined as:

$$\mathrm{C}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \frac{\mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)]}{\sqrt{\mathrm{E}(P^N{}_1, H_1)}\sqrt{\mathrm{E}(P^N{}_2, H_2)}} \tag{6}$$

Proposition 4.3.1. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs. Then,

a) $$0 \le \mathrm{C}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)] \le 1; \tag{7}$$

b) $\mathrm{C}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \mathrm{C}^c[(P^N{}_2,H_2),(P^N{}_1,H_1)];$ (8)

c) If $(P^N{}_1,H_1) = (P^N{}_2,H_2)$, then $\mathrm{C}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$ (9)

Proof: a) It is obvious from the Equation (6) that,

$$\mathrm{C}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \geq 0 \tag{10}$$

Now, it is required to prove that:

$$\mathrm{C}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \leq 1. \tag{11}$$

From the Equation (3), we have,

$$\mathrm{M}^c = \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right) * \left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)\right] + \\ \left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right) * \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)\right] + \\ \left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right) * \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)\right]\end{array}\right\}.$$

$$\mathrm{M}^c = \sum_{i=1}^{k}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_1)\right) * \left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_1)\right)\right] + \left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_1)\right) * \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_1)\right)\right] + \\ \left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_1)\right) * \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_1)\right)\right] + \left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_2)\right) * \left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_2)\right)\right] + \\ \left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_2)\right) * \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_2)\right)\right] + \left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_2)\right) * \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_2)\right)\right] + \\ \dots + \left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_l)\right) * \left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_l)\right)\right] + \left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_l)\right) * \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_l)\right)\right] + \\ \left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_l)\right) * \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_l)\right)\right]\end{array}\right\} \tag{12}$$

By using Cauchy-Schwarz inequality in Equation (12), we have,

$$\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]^2 \leq \sum_{i=1}^{k}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_1)\right)^2 + \left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_2)\right)^2 + \dots + \left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_l)\right)^2\right] + \\ \left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_1)\right)^2 + \left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_2)\right)^2 + \dots + \left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_l)\right)^2\right] + \\ \left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_1)\right)^2 + \left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_2)\right)^2 + \dots + \left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_l)\right)^2\right]\end{array}\right\} \times$$

$$\sum_{i=1}^{k}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_1)\right)^2 + \left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_2)\right)^2 + \dots + \left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_l)\right)^2\right] + \\ \left[\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_1)\right)^2 + \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_2)\right)^2 + \dots + \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_l)\right)^2\right] + \\ \left[\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_1)\right)^2 + \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_2)\right)^2 + \dots + \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_l)\right)^2\right]\end{array}\right\}.$$

$$\Rightarrow \mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]^2 \leq \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\begin{array}{c}\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2 + \\ \left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2 + \\ \left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\end{array}\right\} \times \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\begin{array}{c}\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2 + \\ \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2 + \\ \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\end{array}\right\}. \tag{13}$$

By using the Equations (1) and (2) in Equation (13), we have,

$$\mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)]^2 \leq \mathrm{E}(P^N{}_1, H_1) \times \mathrm{E}(P^N{}_2, H_2).$$

$$\Rightarrow \mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)] \leq \sqrt{\mathrm{E}(P^N{}_1, H_1)} \times \sqrt{\mathrm{E}(P^N{}_2, H_2)}.$$

$$\Rightarrow \frac{\mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)]}{\sqrt{\mathrm{E}(P^N{}_1, H_1)}\sqrt{\mathrm{E}(P^N{}_2, H_2)}} \leq 1. \tag{14}$$

By using the Equation (6) in Equation (14), we have,

$$\mathrm{C}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)] \leq 1.$$

Hence, the stated result of Equation (11) is established mathematically.

Now, on combining the Equations (10) and (11), it is well proved that:

$$0 \leq \mathrm{C}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)] \leq 1.$$

b) It is obvious from the Equation (6) that,

$\mathrm{C}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \mathrm{C}^c[(P^N{}_2, H_2), (P^N{}_1, H_1)].$ Hence, proof of Equation (8) is simple.

c) Now, it is required to prove that: $\mathrm{C}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)] = 1.$

Proof: If $(P^N{}_1, H_1) = (P^N{}_2, H_2)$, then,

From the Equation (6), we have,

$$\mathrm{C}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \frac{\mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)]}{\sqrt{\mathrm{E}(P^N{}_1, H_1)}\sqrt{\mathrm{E}(P^N{}_2, H_2)}}.$$

Now, using the Equations (1), (2) and (3) in Equation (6), we have,

$$\mathrm{C}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \frac{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2 + \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2 + \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]}{\sqrt{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2 + \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2 + \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]} \times \sqrt{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2 + \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2 + \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]}}$$

$$\Rightarrow \mathrm{C}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)] = 1.$$

Hence, the stated result of Equation (9) is established mathematically.

Definition 4.4. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs. The CC between $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ is defined as:

$$\widetilde{\mathrm{C}^c}[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \frac{\mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)]}{\max\{\mathrm{E}(P^N{}_1, H_1), \mathrm{E}(P^N{}_2, H_2)\}}. \tag{15}$$

By using the Equations (1), (2) and (3) in Equation (15), we have,

$$\widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \frac{\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)\right]\\+\left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)\right]\\+\left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)\right]\end{array}\right\}}{\max\left\{\begin{array}{l}\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\right],\\\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]\end{array}\right\}} \tag{16}$$

Proposition 4.4.1. Let $(P^N{}_1,H_1)$ and $(P^N{}_2,H_2)$ be two NPHSSs. Then,

1) $0 \le \widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le 1;$ (17)

2) $\widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \widetilde{C^c}[(P^N{}_2,H_2),(P^N{}_1,H_1)];$ (18)

3) If $(P^N{}_1,H_1) = (P^N{}_2,H_2)$, then $\widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$ (19)

Proof: 1) It is obvious from the Equations (15) and (16) that,

$$\widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \ge 0. \tag{20}$$

Now, it is required to prove that:

$$\widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le 1. \tag{21}$$

By using Cauchy-Schwarz inequality in Equation (12), we have,

$$M^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le \sqrt{\begin{array}{l}\sum_{i=1}^{k}\left\{\begin{array}{l}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_l)\right)^2\right]+\\\left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_l)\right)^2\right]+\\\left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_l)\right)^2\right]\end{array}\right\}\times\\\sum_{i=1}^{k}\left\{\begin{array}{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_l)\right)^2\right]+\\\left[\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_l)\right)^2\right]+\\\left[\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_l)\right)^2\right]\end{array}\right\}\end{array}}.$$

$$\Rightarrow M^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le \sqrt{\begin{array}{l}\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\right\}\times\\\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right\}\end{array}}.$$

$$\le \sqrt{\left[max\left\{\begin{array}{l}\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\right\}\times\\\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right\}\end{array}\right\}\right]^2}.$$

$$\Rightarrow \mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] = max\left\{\begin{array}{c}\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\right\}\times \\ \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right\}\end{array}\right\} \quad (22)$$

By using the Equations (1) and (2) in Equation (22), we have,

$$\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \leq \max\{\mathrm{E}(P^N{}_1,H_1),\mathrm{E}(P^N{}_2,H_2)\}.$$

$$\Rightarrow \frac{\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]}{\max\{\mathrm{E}(P^N{}_1,H_1),\mathrm{E}(P^N{}_2,H_2)\}} \leq 1. \quad (23)$$

Now, using the Equation (15) in Equation (23), we have,

$$\widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \leq 1.$$

Hence, the stated result of Equation (21) is established mathematically.

Now, on combining the Equations (20) and (21), it is well proved that:

$$0 \leq \widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \leq 1.$$

2) It is clearly stated from the Equations (15) and (16) that,

$$\widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \widetilde{\mathrm{C}^c}[(P^N{}_2,H_2),(P^N{}_1,H_1)]. \ldots\ldots$$

3) Now, it is required to prove that: $\widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$

Proof: If $(P^N{}_1,H_1) = (P^N{}_2,H_2)$, then,

From the Equation (15), we have, $\widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \frac{\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]}{\sqrt{\mathrm{E}(P^N{}_1,H_1)}\sqrt{\mathrm{E}(P^N{}_2,H_2)}}.$

By using the Equations (1), (2) and (3) in Equation (15), we have,

$$\widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \frac{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]}{\begin{array}{c}\sqrt{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]}\times \\ \sqrt{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]}\end{array}}$$

$$\Rightarrow \widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$$

Hence, the stated result of Equation (19) is established mathematically.

5. Proposed Weighted Correlation Coefficient (WCC) for NPHSS:

This Section introduces the notion of WCC and its fundamental features for NPHSSs. Decision-makers can assign various weights to each of the alternatives with the help of WCC.

Let us consider $\vartheta = \{\vartheta_1, \vartheta_2, \dots, \vartheta_v\}$ and $\omega = \{\omega_1, \omega_2, \dots, \omega_w\}$ be the weights of the alternatives and experts, respectively, such that: $\sum_{x=1}^{v} \vartheta_x = 1, \sum_{y=1}^{w} \omega_y = 1 \ \forall\ \vartheta_x, \omega_y > 0$.

Definition 5.1. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs. The WCC between $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ is defined as:

$$W^C[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \frac{\mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)]}{\sqrt{\mathrm{E}(P^N{}_1, H_1)}\sqrt{\mathrm{E}(P^N{}_2, H_2)}}. \tag{24}$$

By using the Equations (1), (2) and (3) in Equation (24), we have,

$$W^C[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \frac{\sum_{x=1}^{v} \vartheta_x \left\{ \sum_{y=1}^{w} \omega_y \left[\begin{array}{c} \left(\left(\mathcal{M}_p (P^N{}_1)_{S_i}(u_j)\right) * \left(\mathcal{M}_p (P^N{}_2)_{S_i}(u_j)\right) \right) + \\ \left(\left(\mathcal{M}_N (P^N{}_1)_{S_i}(u_j)\right) * \left(\mathcal{M}_N (P^N{}_2)_{S_i}(u_j)\right) \right) \\ + \left(\left(\mathcal{M}_n (P^N{}_1)_{S_i}(u_j)\right) * \left(\mathcal{M}_n (P^N{}_2)_{S_i}(u_j)\right) \right) \end{array} \right] \right\}}{\sqrt{\sum_{x=1}^{v} \vartheta_x \left\{ \sum_{y=1}^{w} \omega_y \left[\left(\begin{array}{c} \left(\mathcal{M}_p (P^N{}_1)_{S_i}(u_j)\right)^2 + \left(\mathcal{M}_N (P^N{}_1)_{S_i}(u_j)\right)^2 \\ + \left(\mathcal{M}_n (P^N{}_1)_{S_i}(u_j)\right)^2 \end{array} \right) \right] \right\}} \times \sqrt{\sum_{x=1}^{v} \vartheta_x \left\{ \sum_{y=1}^{w} \omega_y \left[\left(\begin{array}{c} \left(\mathcal{M}_p (P^N{}_2)_{S_i}(u_j)\right)^2 + \left(\mathcal{M}_N (P^N{}_2)_{S_i}(u_j)\right)^2 \\ + \left(\mathcal{M}_n (P^N{}_2)_{S_i}(u_j)\right)^2 \end{array} \right) \right] \right\}}} \tag{25}$$

Remark: If $\vartheta = \left\{\frac{1}{v}, \frac{1}{v}, \dots, \frac{1}{v}\right\}$ and $\omega = \left\{\frac{1}{w}, \frac{1}{w}, \dots, \frac{1}{w}\right\}$, then the WCC given in Equation (24) reduces to the CC as given in Equation (6).

Proposition 5.1.1. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs. Then,

1) $0 \le W^C[(P^N{}_1, H_1), (P^N{}_2, H_2)] \le 1;$ (26)

2) $W^C[(P^N{}_1, H_1), (P^N{}_2, H_2)] = W^C[(P^N{}_2, H_2), (P^N{}_1, H_1)];$ (27)

3) If $(P^N{}_1, H_1) = (P^N{}_2, H_2)$, then $W^C[(P^N{}_1, H_1), (P^N{}_2, H_2)] = 1.$ (28)

Proof: It is as similar to Proposition 4.3.1.

Definition 5.2. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs. The WCC between $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ is defined as:

$$\widetilde{W^C}[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \frac{\mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)]}{\max\{\mathrm{E}(P^N{}_1, H_1), \mathrm{E}(P^N{}_2, H_2)\}}. \tag{29}$$

By using the Equations (1), (2) and (3) in Equation (29), we have,

$$\widetilde{W^C}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \frac{\sum_{x=1}^{v}\vartheta_x\left\{\sum_{y=1}^{w}\omega_y\left[\begin{array}{c}\left(\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)\right)+\\\left(\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)\right)+\\\left(\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)\right)\end{array}\right]\right\}}{\max\left\{\begin{array}{c}\sum_{x=1}^{v}\vartheta_x\left\{\sum_{y=1}^{w}\omega_y\left[\left(\begin{array}{c}\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2\\+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\end{array}\right)\right]\right\},\\\sum_{x=1}^{v}\vartheta_x\left\{\sum_{y=1}^{w}\omega_y\left[\left(\begin{array}{c}\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2\\+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\end{array}\right)\right]\right\}\end{array}\right\}}. \quad (30)$$

Remark: If $\vartheta = \left\{\frac{1}{v},\frac{1}{v},\dots,\frac{1}{v}\right\}$ and $\omega = \left\{\frac{1}{w},\frac{1}{w},\dots,\frac{1}{w}\right\}$, then the WCC given in Equation (29) reduces to the CC as given in Equation (15).

Proposition 5.2.1. Let $(P^N{}_1,H_1)$ and $(P^N{}_2,H_2)$ be two NPHSSs. Then,

1) $0 \le \widetilde{W^C}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le 1;$ (31)

2) $\widetilde{W^C}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \widetilde{W^C}[(P^N{}_2,H_2),(P^N{}_1,H_1)];$ (32)

3) If $(P^N{}_1,H_1) = (P^N{}_2,H_2)$, then $\widetilde{W^C}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$ (33)

Proof: It is similar to Proposition 4.4.1.

6. Proposed Aggregation Operators for NPHSS:

The notions of a weighted average operator as well as a weighted geometric operator are presented in this Section for an NPHSS by using the operational laws as follows:

Let us consider η to be a set of neutrosophic picture hypersoft numbers (NPHSNs).

6.1. The NPHSS's operational laws:

Definition 6.1.1. Let $P^N{}_{11} = \left(\mathcal{M}_{p_{11}},\mathcal{M}_{N_{11}},\mathcal{M}_{n_{11}}\right)$ and $P^N{}_{12} = \left(\mathcal{M}_{p_{12}},\mathcal{M}_{N_{12}},\mathcal{M}_{n_{12}}\right)$ be two NPHSSs and ρ be a positive integer. Then,

i. $P^N{}_{11}\oplus P^N{}_{12} = \left\langle \begin{array}{c}\mathcal{M}_{p_{11}}+\mathcal{M}_{p_{12}}-\mathcal{M}_{p_{11}}\mathcal{M}_{p_{12}},\\\mathcal{M}_{N_{11}}+\mathcal{M}_{N_{12}}-\mathcal{M}_{N_{11}}\mathcal{M}_{N_{12}},\mathcal{M}_{n_{11}}\mathcal{M}_{n_{12}}\end{array}\right\rangle;$ (34)

ii. $P^N{}_{11}\otimes P^N{}_{12} = \left\langle \begin{array}{c}\mathcal{M}_{p_{11}}\mathcal{M}_{p_{12}},\mathcal{M}_{N_{11}}\mathcal{M}_{N_{12}},\\\mathcal{M}_{n_{11}}+\mathcal{M}_{n_{12}}-\mathcal{M}_{n_{11}}\mathcal{M}_{n_{12}}\end{array}\right\rangle;$ (35)

iii. $\rho\,(P^N{}_{11}) = \left\langle\left[1-\left(1-\mathcal{M}_{p_{11}}\right)^{\rho},1-\left(1-\mathcal{M}_{N_{11}}\right)^{\rho},\left(\mathcal{M}_{n_{11}}\right)^{\rho}\right]\right\rangle;$ (36)

iv. $(P^N{}_{11})^{\rho} = \left\langle\left[\left(\mathcal{M}_{p_{11}}\right)^{\rho},\left(\mathcal{M}_{N_{11}}\right)^{\rho},1-\left(1-\mathcal{M}_{n_{11}}\right)^{\rho}\right]\right\rangle.$ (37)

6.2. Weighted Average Operator (WAO) for NPHSS:

Definition 6.2.1. If $\vartheta = \{\vartheta_1, \vartheta_2, \dots, \vartheta_v\}$ and $\omega = \{\omega_1, \omega_2, \dots, \omega_w\}$ be the weights of the alternatives and experts, respectively, such that $\sum_{x=1}^{v} \vartheta_x = 1, \sum_{y=1}^{w} \omega_y = 1\ \forall$ $\vartheta_x, \omega_y > 0$ and $P^N{}_{yx} = \left(\mathcal{M}_{p_{yx}}, \mathcal{M}_{N_{yx}}, \mathcal{M}_{n_{yx}}\right)$ be an NPHSN, where $x = \{1,2,\dots,v\}$; $y = \{1,2,\dots,w\}$.

If $\phi: \eta^n \to \eta$, then, WAO for NPHSS (ϕ) is defined as follows:

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{wv}) = \oplus_{x=1}^{v} \vartheta_x \left(\oplus_{y=1}^{w} \omega_y P^N{}_{yx}\right). \tag{38}$$

Theorem 6.2.2. Let $P^N{}_{yx} = \left(\mathcal{M}_{p_{yx}}, \mathcal{M}_{N_{yx}}, \mathcal{M}_{n_{yx}}\right)$ be an NPHSN, where $x = \{1,2,\dots,v\}$; $y = \{1,2,\dots,w\}$.

Then, an aggregated value of WAO (ϕ) is also an NPHSN which is given as:

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{wv}) = \left\langle \begin{matrix} 1 - \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1 - \mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1 - \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1 - \mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \right\rangle. \tag{39}$$

Proof: Suppose that $w = 1$, then $\omega_1 = 1$ in Equation (38), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{1v}) = \oplus_{x=1}^{v} \vartheta_x P^N{}_{1x}. \tag{40}$$

By using Equation (36) in Equation (40), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{1v}) = \left\langle \begin{matrix} 1 - \prod_{x=1}^{v}\left[\prod_{y=1}^{1}\left(1 - \mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1 - \prod_{x=1}^{v}\left[\prod_{y=1}^{1}\left(1 - \mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ \prod_{x=1}^{v}\left[\prod_{y=1}^{1}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \right\rangle. \tag{41}$$

Suppose that $v = 1$, then $\vartheta_1 = 1$ in Equation (38), we have,

$$\phi(P^N{}_{11}, P^N{}_{21}, \dots, P^N{}_{w1}) = \oplus_{y=1}^{w} \omega_y P^N{}_{y1}. \tag{42}$$

By using Equation (36) in Equation (42), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{1v}) = \langle \begin{matrix} 1 - \prod_{x=1}^{1}\left[\prod_{y=1}^{w}\left(1 - \mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1 - \prod_{x=1}^{1}\left[\prod_{y=1}^{w}\left(1 - \mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ \prod_{x=1}^{1}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \rangle. \tag{43}$$

Thus, the above results of the Equations (41) and (43) are valid for $w = 1$ and $v = 1$.

Now, assuming $v = \alpha + 1$ and $w = \beta$ in Equation (38), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{\beta(\alpha+1)}) = \oplus_{x=1}^{\alpha+1}\vartheta_x(\oplus_{y=1}^{\beta}\omega_y P^N{}_{yx}). \tag{44}$$

By using the Equation (36) in Equation (44), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{\beta(\alpha+1)}) = \langle \begin{matrix} 1 - \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(1 - \mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1 - \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(1 - \mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \rangle. \tag{45}$$

Similarly, by assuming $v = \alpha$ and $w = \beta + 1$ in Equation (38), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{\alpha(\beta+1)}) = \oplus_{x=1}^{\alpha}\vartheta_x(\oplus_{y=1}^{\beta+1}\omega_y P^N{}_{yx}). \tag{46}$$

By using the Equation (36) in Equation (46), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{\alpha(\beta+1)}) = \langle \begin{matrix} 1 \prod_{x=1}^{\alpha}\left[\prod_{y=1}^{\beta+1}\left(1 - \mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1 - \prod_{x=1}^{\alpha}\left[\prod_{y=1}^{\beta+1}\left(1 - \mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ \prod_{x=1}^{\alpha}\left[\prod_{y=1}^{\beta+1}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \rangle. \tag{47}$$

Now, by assuming $v = \alpha + 1$ and $w = \beta + 1$ in Equation (38), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{(\alpha+1)(\beta+1)}) = \oplus_{x=1}^{\alpha+1}\vartheta_x(\oplus_{y=1}^{\beta+1}\omega_y P^N{}_{yx}). \tag{48}$$

$$\Rightarrow \phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{(\alpha+1)(\beta+1)}) = \oplus_{x=1}^{\alpha+1}\vartheta_x(\oplus_{y=1}^{\beta}\omega_y P^N{}_{yx})\oplus_{x=1}^{\alpha+1}\vartheta_x(\omega_\beta P^N{}_{x(\beta+1)}). \tag{49}$$

Using the Equation (36) in Equation (49), we have,

$$\phi(P^N{}_{11},P^N{}_{12},\dots,P^N{}_{(\alpha+1)(\beta+1)}) = \left\langle \begin{matrix} 1-\prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(1-\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \oplus 1-\prod_{x=1}^{\alpha+1}\left[\left(1-\mathcal{M}_{p_{x(\beta+1)}}\right)^{\omega_{\beta+1}}\right]^{\vartheta_x}, \\ 1-\prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(1-\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \oplus 1-\prod_{x=1}^{\alpha+1}\left[\left(1-\mathcal{M}_{N_{x(\beta+1)}}\right)^{\omega_{\beta+1}}\right]^{\vartheta_x}, \\ \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \oplus \prod_{x=1}^{\alpha+1}\left[\left(\mathcal{M}_{n_{x(\beta+1)}}\right)^{\omega_{\beta+1}}\right]^{\vartheta_x} \end{matrix} \right\rangle. \tag{50}$$

Using the Equation (36) in Equation (50), we have,

$$\phi(P^N{}_{11},P^N{}_{12},\dots,P^N{}_{(\alpha+1)(\beta+1)}) = \left\langle \begin{matrix} 1-\prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta+1}\left(1-\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1-\prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta+1}\left(1-\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta+1}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \right\rangle. \tag{51}$$

Thus, the above results of the Equations (50) and (51) are valid for $v = \alpha + 1$ and $w = \beta + 1$, where α, β be any positive integer.

As a result, by using the induction approach, the above results are valid $\forall\ \alpha, \beta \geq 1$.

Now, from the Definition 3.1., we have,

$$0 \leq \mathcal{M}_{p_{yx}} + \mathcal{M}_{N_{yx}} + \mathcal{M}_{n_{yx}} \leq 1. \tag{52}$$

Using the Equation (36) in Equation (52), we have,

$$\Leftrightarrow 1-\prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1-\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} + 1-\prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1-\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} + \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \leq 1. \tag{53}$$

As a result of Equation (53), we can easily say that an aggregated value of WAO (ϕ) is also an NPHSN which is clearly stated that the result of Equation (39) is well proved.

Example 6.2.3. Consider the above Example 3.1.1. Suppose $\omega_y = \{0.20, 0.30, 0.50\}$ and $\vartheta_x = \{0.30, 0.34, 0.36\}$ are the weights assigned to faculties and attributes, respectively.

Using the Equation (39), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{33}) = \langle \begin{matrix} 1 - \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1 - \mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1 - \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1 - \mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \rangle$$

$$= \langle 0.31, 0.30, 0.10 \rangle.$$

6.3. Weighted Geometric Operator (WGO) for NPHSS:

Definition 6.3.1. If $\vartheta = \{\vartheta_1, \vartheta_2, \dots, \vartheta_v\}$ and $\omega = \{\omega_1, \omega_2, \dots, \omega_w\}$ be the weights of the alternatives and experts, respectively, such that $\sum_{x=1}^{v}\vartheta_x = 1, \sum_{y=1}^{w}\omega_y = 1\ \forall\ \vartheta_x, \omega_y > 0$ and $P^N{}_{yx} = \left(\mathcal{M}_{p_{yx}}, \mathcal{M}_{N_{yx}}, \mathcal{M}_{n_{yx}}\right)$ be a NPHSN, where $x = \{1,2, \dots, v\}; y = \{1,2, \dots, w\}$.

If $\psi: \eta^n \to \eta$, then, WGO for NPHSS (ψ) is defined as follows:

$$\psi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{wv}) = \otimes_{x=1}^{v}\left[\otimes_{y=1}^{w}\left(P^N{}_{yx}\right)^{\omega_y}\right]^{\vartheta_x}. \quad (54)$$

Theorem 6.3.2. Let $P^N{}_{yx} = \left(\mathcal{M}_{p_{yx}}, \mathcal{M}_{N_{yx}}, \mathcal{M}_{n_{yx}}\right)$ be an NPHSN, where $x = \{1,2, \dots, v\}$; $y = \{1,2, \dots, w\}$.

Then, an aggregated value of WGO (ψ) is also an NPHSN which is given as:

$$\psi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{wv}) = \langle \begin{matrix} \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ 1 - \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1 - \mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \rangle. \quad (55)$$

Proof: It is similar to the proof of Theorem 6.2.2.

Example 6.3.3. Consider the above Example 3.1.1. Suppose $\omega_y = \{0.20, 0.30, 0.50\}$ and $\vartheta_x = \{0.30, 0.34, 0.36\}$ are the weights assigned to faculties and attributes, respectively.
Then, by using the Equation (55) the value of WGO is obtained as follows:

$$\psi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{wv})$$

$$= \langle \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1 - \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1 - \mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \rangle = \langle 0.34, 0.35, 0.30 \rangle.$$

7. Solving MCDM Problems using NPHSS through the TOPSIS method

TOPSIS approach aids in determining the optimum alternative based on the least and greatest distances between the neutrosophic picture positive ideal solution (NPPIS) and the neutrosophic picture negative ideal solution (NPNIS). Additionally, this approach provides accurate results for calculating closeness coefficients when used with CC rather than similarity measures. To demonstrate the NPHSS TOPSIS technique based on CC, we give an algorithm and a case study.

7.1. Algorithm for addressing MCDM Problems using NPHSS through the TOPSIS:

Suppose that $D = \{D_1, D_2, \dots, D_d\}$ be a collection of the departments of the management-studies cluster. The task is to choose the optimal department through faculty evaluation analysis. Let $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_h\}$ be a collection of students of the departments responsible for assessing faculty members in their respective departments. The evaluation involves assigning weights (student's weightage) $\omega = \{\omega_1, \omega_2, \dots, \omega_w\}$, such that $\sum_{y=1}^{w} \omega_y = 1 \ \forall \ \omega_y > 0$.
Let $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_h\}$ be a collection of students of the departments who are responsible for evaluating the faculties of their respective department with the weights (student's weightage) $\omega = \{\omega_1, \omega_2, \dots, \omega_w\}$, such that $\sum_{y=1}^{w} \omega_y = 1 \ \forall \ \omega_y > 0$. If $H = S_1 \times S_2 \times \dots \times S_i$ be a collection of multi-valued sub-attributes along with their respective weights (faculty's weightage) $\vartheta = \{\vartheta_1, \vartheta_2, \dots, \vartheta_v\}$, such that $\sum_{x=1}^{v} \vartheta_x = 1, \forall \ \vartheta_x > 0$. The faculty evaluation analysis is done by the sub-attributes with multiple values $S_j (j = 1,2, \dots, i)$ which is presented in the NPHSS format. It is represented as $P^N{}_{xy} = \langle \mathcal{M}_{p_{xy}}, \mathcal{M}_{N_{xy}}, \mathcal{M}_{n_{xy}} \rangle$ such that $0 \le \mathcal{M}_{p_{xy}} + \mathcal{M}_{N_{xy}} + \mathcal{M}_{n_{xy}} \le 1 \ \forall \ x, y$.

Step 1: Generate a matrix in the NPHSS format for each multi-valued sub-attribute, as:

$$[D_d, H]_{m \times n} = [D_d]_{m \times n} \tag{56}$$

$$= \begin{array}{c c} & \begin{array}{ccc} S_1 & \dots & S_i \end{array} \\ \begin{array}{c} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_h \end{array} & \begin{bmatrix} \langle \mathcal{M}_{p_{11}}, \mathcal{M}_{N_{11}}, \mathcal{M}_{n_{11}} \rangle & \dots & \langle \mathcal{M}_{p_{1n}}, \mathcal{M}_{N_{1n}}, \mathcal{M}_{n_{1n}} \rangle \\ \vdots & \vdots & \vdots \\ \langle \mathcal{M}_{p_{m1}}, \mathcal{M}_{N_{m1}}, \mathcal{M}_{n_{m1}} \rangle & \dots & \langle \mathcal{M}_{p_{mn}}, \mathcal{M}_{N_{mn}}, \mathcal{M}_{n_{mn}} \rangle \end{bmatrix} \end{array}.$$

Step 2: The process of obtaining the decision matrix, including weights assigned to each multi-valued sub-attribute, is outlined as follows:

$$[\overline{D_d}]_{m \times n} = \left\langle \begin{array}{c} 1 - \prod_{x=1}^{v} \left[\prod_{y=1}^{w} \left(1 - \mathcal{M}_{p_{xy}}\right)^{\omega_y} \right]^{\vartheta_x}, 1 - \prod_{x=1}^{v} \left[\prod_{y=1}^{w} \left(1 - \mathcal{M}_{N_{xy}}\right)^{\omega_y} \right]^{\vartheta_x}, \\ \prod_{x=1}^{v} \left[\prod_{y=1}^{w} \left(\mathcal{M}_{n_{xy}}\right)^{\omega_y} \right]^{\vartheta_x} \end{array} \right\rangle. \tag{57}$$

$$[\overline{D_d}]_{m \times n} = \langle \overline{\mathcal{M}_{p_{xy}}}, \overline{\mathcal{M}_{N_{xy}}}, \overline{\mathcal{M}_{n_{xy}}} \rangle \tag{58}$$

Step 3: The NPPIS and NPNIS for weighted NPHSS can be determined respectively, as:

$$\overline{D}^+ = \langle \overline{\mathcal{M}_p}^+, \overline{\mathcal{M}_N}^+, \overline{\mathcal{M}_n}^+ \rangle_{m \times n} = \langle \overline{\mathcal{M}_p}^{(\Delta_{xy})}, \overline{\mathcal{M}_N}^{(\nabla_{xy})}, \overline{\mathcal{M}_n}^{(\nabla_{xy})} \rangle ; \tag{59}$$

$$\overline{D}^{-} = \langle \overline{\mathcal{M}_p}^{-}, \overline{\mathcal{M}_N}^{-}, \overline{\mathcal{M}_n}^{-} \rangle_{m\times n} = \langle \overline{\mathcal{M}_p}^{(\nabla_{xy})}, \overline{\mathcal{M}_N}^{(\nabla_{xy})}, \overline{\mathcal{M}_n}^{(\Delta_{xy})} \rangle. \quad (60)$$

Here, $\Delta_{xy} = \arg max_d \{\varphi_{xy}^d\}$ and $\nabla_{xy} = \arg min_d \{\varphi_{xy}^d\}$.

Step 4: The CC for each alternative from NPPIS and NPNIS can be determined respectively, as:

$$\mathbb{C}_d^{+} = \mathrm{C}^c(\overline{D_d}, \overline{D}^{+}) = \frac{\mathrm{M}^c(\overline{D_d}, \overline{D}^{+})}{\sqrt{\mathrm{E}(\overline{D_d})}\sqrt{\mathrm{E}(\overline{D}^{+})}}; \quad (61)$$

$$\mathbb{C}_d^{-} = \mathrm{C}^c(\overline{D_d}, \overline{D}^{-}) = \frac{\mathrm{M}^c(\overline{D_d}, \overline{D}^{-})}{\sqrt{\mathrm{E}(\overline{D_d})}\sqrt{\mathrm{E}(\overline{D}^{-})}}. \quad (62)$$

Step 5: The closeness coefficient of the ideal solution for NPHSS can be calculated as:

$$\mathscr{G}_d = \frac{1-\mathbb{C}_d^{-}}{2-\mathbb{C}_d^{+}-\mathbb{C}_d^{-}}. \quad (63)$$

Step 6: The ordering of the alternatives can be established by organizing the $\mathscr{G}_d$ values in descending order, with the highest value indicating the best alternative.

The graphical representation of the proposed method is illustrated below in the Figure 2:

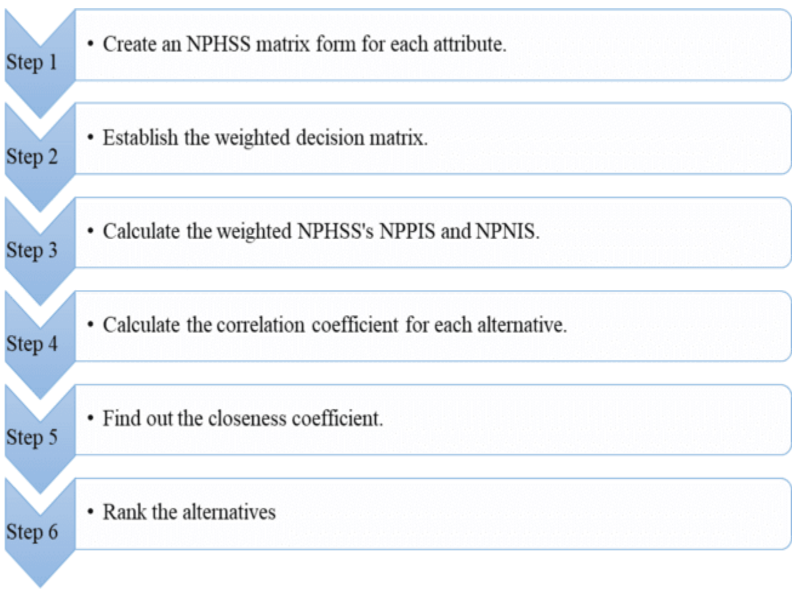


Figure 2: Flow diagram outlining the proposed TOPSIS approach.

5.2. Application in decision-making problem based on TOPSIS Approach for NPHSS:

5.2.1. A Case Study on Faculty Evaluation Analysis: Let us suppose that $D = \{D_1, D_2, D_3, D_4, D_5\}$ be a set of five departments (USB-Commerce; USB-MBA; USB-BBA; MBA-AIT and UITHM - University Institute of Tourism and Hospitality Management) within the Chandigarh University management studies cluster from which we have to select the best department based on faculty evaluation analysis. Let (α_1, α_2): (Criteria Analysis) be two distinct attributes, whose corresponding multi-valued sub-attributes are respectively represented as follows:

Faculty Evaluation Analysis from the feedback of the head of the department: $\alpha_1 = \{S_1, S_2, S_3, S_4\}$

$S_1 =$ Adherence to core values while dealing with students & colleagues =

$$\left\{\begin{matrix} S_{11} = \text{treats every student fairly}, \\ S_{12} = \text{tries relentlessly to motivate and inspire students}, \\ S_{13} = \text{respects divergent views and welcomes disagreement}, \\ S_{14} = \text{Vounteers to address exigencies and solve problems.} \end{matrix}\right\}$$

$S_2 =$ Quality of teaching & student engagement =

$$\left\{\begin{matrix} S_{21} = \text{Expert at delivering highly engaging classroom or online sessions}, \\ S_{22} = \text{Most of students rate him/her as a highly effective instructor}, \\ S_{23} = \text{Expert at using a variety of pedagogical tools like lectures}, \\ \text{multimedia presentations}, \\ \text{simulations, roleplays, bespoke activities, etc.}, \\ S_{24} = \text{Students consider the teacher} - \text{friendly and easy to talk to} \end{matrix}\right\}.$$

$S_3 =$ Quality of course or curriculum content and assessment writing =

$$\left\{\begin{matrix} S_{31} = \text{Delivers high} - \text{quality syllabi or course contents}, \\ S_{32} = \text{ Expert at preparing competency} - \text{based assessments to} \\ \text{demonstrate achievement of outcomes} \end{matrix}\right\}.$$

$S_4 =$ Interpersonal; collaborative and research skills =

$$\left\{\begin{matrix} S_{41} = \text{Is always available for consultation/ help to other faculty members}, \\ S_{42} = \text{Participates in administrative tasks as when required}, \\ S_{43} = \text{Achieves at least 2 Scopus indexed (or higher value) publications/ year}, \\ S_{44} = \text{Is an active member of a Research Cluster} \end{matrix}\right\}.$$

Faculty Evaluation Analysis from the feedback of students of the department: $\alpha_2 = \{S_5, S_6\} =$

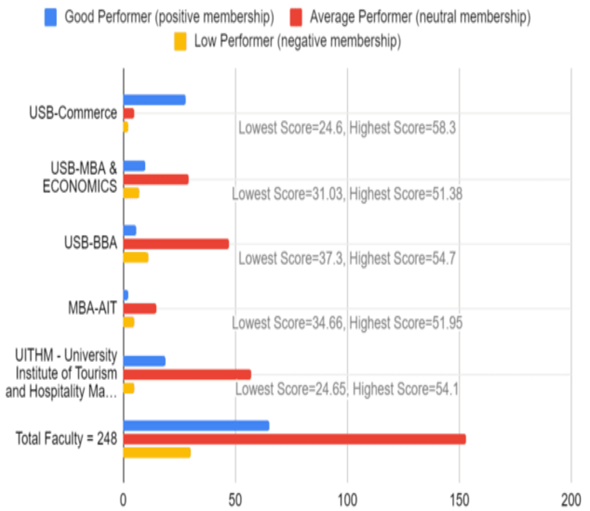
$$\left\{\begin{matrix} S_5 = \text{Student's mid} - \text{semester test score and a number of students' responses for} \\ \text{faculty feedback about the MST exam}, \\ S6 = \text{Student's end} - \text{to} - \text{semester exam score and the number of students' responses} \\ \text{for faculty feedback about end semester exam} \end{matrix}\right\}.$$

Then, $H = \alpha_1 \times \alpha_2 = \{S_1, S_2, S_3, S_4\} \times \{S_5, S_6\} = \{S_1, S_2, S_3, S_4, S_5\} \times \{S_1, S_2, S_3, S_4, S_6\} = \{\tilde{S}_1 \times \tilde{S}_2\}$ = (head's feedback and student's feedback about the faculties of their respective department) be a collection of multi-valued sub-attributes along with their respective weights (faculty's weightage):

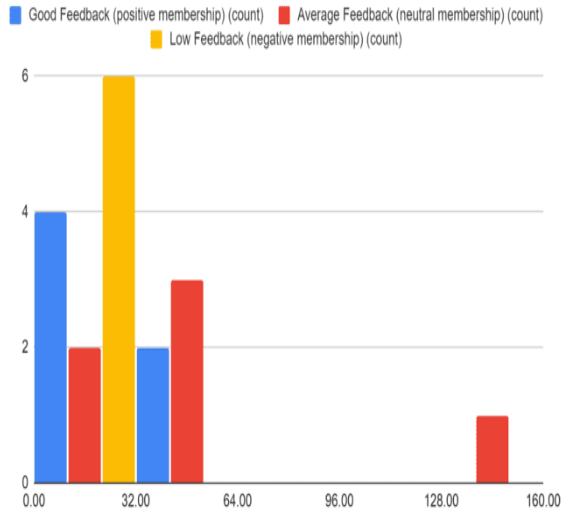
$\vartheta = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5\} = \{0.18, 0.25, 0.26, 0.15, 0.16\}$, $\sum_{x=1}^{5} \vartheta_x = 1, \forall\ \vartheta_x > 0$.

Let $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5\}$ be a set of students of the departments who are responsible for evaluating the faculties of their respective departments with the weights (student's weightage): $\omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} = \{0.14, 0.19, 0.26, 0.09, 0.32\}$, $\sum_{y=1}^{w} \omega_y = 1\ \forall\ \omega_y > 0$.

Based on a faculty evaluation analysis, the goal is to identify the best department within the Chandigarh University management studies cluster. The graphical illustration of faculty evaluation analysis from the received is shown as follows in Figure 3:



(a)



(b)

Figure 3: (a) Faculty performance assessment through the department head's feedback; (b) Faculty performance assessment through the department student's feedback.

Step 1: Create the matrices D_1, D_2, D_3, D_4 and D_5 in the form of NPHSS for each multi-valued sub-attribute by using Equation (56), as shown in Tables 3, 4, 5, 6 and 7, respectively:

Table 3: Showing the values for D_1.

D_1	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	$\langle 0.11, 0.02, 0.01 \rangle$	$\langle 0.01, 0.03, 0.001 \rangle$
$\mathcal{H}_2$	$\langle 0.15, 0.03, 0.01 \rangle$	$\langle 0.01, 0.03, 0.002 \rangle$
$\mathcal{H}_3$	$\langle 0.21, 0.04, 0.02 \rangle$	$\langle 0.02, 0.05, 0.002 \rangle$
$\mathcal{H}_4$	$\langle 0.07, 0.01, 0.01 \rangle$	$\langle 0.01, 0.02, 0.001 \rangle$
$\mathcal{H}_5$	$\langle 0.26, 0.04, 0.02 \rangle$	$\langle 0.02, 0.06, 0.002 \rangle$

Table 4: Showing the values for D_2.

D_2	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	$\langle 0.03, 0.09, 0.02 \rangle$	$\langle 0.02, 0.03, 0.14 \rangle$
$\mathcal{H}_2$	$\langle 0.04, 0.12, 0.03 \rangle$	$\langle 0.02, 0.04, 0.19 \rangle$
$\mathcal{H}_3$	$\langle 0.06, 0.16, 0.04 \rangle$	$\langle 0.03, 0.06, 0.26 \rangle$
$\mathcal{H}_4$	$\langle 0.02, 0.06, 0.01 \rangle$	$\langle 0.01, 0.02, 0.09 \rangle$
$\mathcal{H}_5$	$\langle 0.07, 0.20, 0.05 \rangle$	$\langle 0.04, 0.07, 0.32 \rangle$

Table 5: Showing the values for D_3.

D_3	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	$\langle 0.01, 0.10, 0.02 \rangle$	$\langle 0.02, 0.04, 0.001 \rangle$
$\mathcal{H}_2$	$\langle 0.02, 0.14, 0.03 \rangle$	$\langle 0.02, 0.05, 0.002 \rangle$
$\mathcal{H}_3$	$\langle 0.02, 0.19, 0.04 \rangle$	$\langle 0.03, 0.07, 0.003 \rangle$
$\mathcal{H}_4$	$\langle 0.01, 0.07, 0.02 \rangle$	$\langle 0.01, 0.03, 0.001 \rangle$
$\mathcal{H}_5$	$\langle 0.03, 0.23, 0.05 \rangle$	$\langle 0.04, 0.09, 0.003 \rangle$

Table 6: Showing the values for D_4.

D_4	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	$\langle 0.01, 0.10, 0.03 \rangle$	$\langle 0.01, 0.01, 0.003 \rangle$
$\mathcal{H}_2$	$\langle 0.02, 0.13, 0.04 \rangle$	$\langle 0.02, 0.02, 0.004 \rangle$
$\mathcal{H}_3$	$\langle 0.02, 0.18, 0.06 \rangle$	$\langle 0.02, 0.03, 0.005 \rangle$
$\mathcal{H}_4$	$\langle 0.01, 0.06, 0.02 \rangle$	$\langle 0.01, 0.01, 0.002 \rangle$
$\mathcal{H}_5$	$\langle 0.03, 0.22, 0.07 \rangle$	$\langle 0.03, 0.03, 0.006 \rangle$

Table 7: Showing the values for D_5.

D_5	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	$\langle 0.03, 0.10, 0.01 \rangle$	$\langle 0.08, 0.03, 0.004 \rangle$
$\mathcal{H}_2$	$\langle 0.04, 0.13, 0.01 \rangle$	$\langle 0.11, 0.04, 0.01 \rangle$

$\mathcal{H}_3$	⟨0.06,0.18,0.02⟩	⟨0.15,0.05,0.01⟩
$\mathcal{H}_4$	⟨0.02,0.06,0.01⟩	⟨0.05,0.02,0.002⟩
$\mathcal{H}_5$	⟨0.07,0.22,0.02⟩	⟨0.19,0.07,0.01⟩

Step 2: Obtain the weighted matrices $\overline{D_1}, \overline{D_2}, \overline{D_3}, \overline{D_4}$ and $\overline{D_5}$ in the NPHSS format for each multi-valued sub-attribute by using the Equation (57), as shown in Tables 8, 9, 10, 11 and 12, respectively:

Table 8: Showing the weighted values $\overline{D_1}$.

$\overline{D_1}$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0029,0.0005,0.890⟩	⟨0.003,0.008,0.840⟩
$\mathcal{H}_2$	⟨0.0077,0.0014,0.804⟩	⟨0.0005,0.0014,0.744⟩
$\mathcal{H}_3$	⟨0.0158,0.0028,0.768⟩	⟨0.0014,0.0035,0.657⟩
$\mathcal{H}_4$	⟨0.0010,0.0001,0.940⟩	⟨0.0001,0.0003,0.911⟩
$\mathcal{H}_5$	⟨0.0153,0.0021,0.818⟩	⟨0.0010,0.0032,0.727⟩

Table 9: Showing the weighted values $\overline{D_2}$.

$\overline{D_2}$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0008,0.0024,0.906⟩	⟨0.0005,0.0008,0.952⟩
$\mathcal{H}_2$	⟨0.0019,0.0061,0.847⟩	⟨0.0010,0.0019,0.924⟩
$\mathcal{H}_3$	⟨0.0042,0.0117,0.804⟩	⟨0.0021,0.0042,0.913⟩
$\mathcal{H}_4$	⟨0.0003,0.0008,0.940⟩	⟨0.0001,0.0003,0.968⟩
$\mathcal{H}_5$	⟨0.0037,0.0114,0.858⟩	⟨0.0021,0.0037,0.943⟩

Table 10: Showing the weighted values $\overline{D_3}$.

$\overline{D_3}$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0003,0.0027,0.906⟩	⟨0.0005,0.0010,0.840⟩
$\mathcal{H}_2$	⟨0.0010,0.0071,0.847⟩	⟨0.0010,0.0024,0.744⟩
$\mathcal{H}_3$	⟨0.0014,0.0141,0.804⟩	⟨0.0021,0.0049,0.675⟩
$\mathcal{H}_4$	⟨0.0001,0.0010,0.949⟩	⟨0.0001,0.0004,0.911⟩
$\mathcal{H}_5$	⟨0.0016,0.0133,0.858⟩	⟨0.0021,0.0048,0.743⟩

Table 11: Showing the weighted values $\overline{D_4}$.

$\overline{D_4}$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0003,0.0027,0.915⟩	⟨0.0003,0.003,0.864⟩
$\mathcal{H}_2$	⟨0.0010,0.0066,0.858⟩	⟨0.0010,0.0010,0.769⟩
$\mathcal{H}_3$	⟨0.0014,0.0133,0.827⟩	⟨0.0014,0.0021,0.699⟩
$\mathcal{H}_4$	⟨0.0001,0.0008,0.949⟩	⟨0.0001,0.0001,0.920⟩
$\mathcal{H}_5$	⟨0.0016,0.0126,0.873⟩	⟨0.0016,0.0016,0.770⟩

Table 12: Showing the weighted values $\overline{D_5}$.

$\overline{D_5}$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0008,0.0027,0.890⟩	⟨0.0021,0.0008,0.870⟩
$\mathcal{H}_2$	⟨0.0019,0.0066,0.804⟩	⟨0.0055,0.0019,0.804⟩
$\mathcal{H}_3$	⟨0.0042,0.0133,0.768⟩	⟨0.0109,0.0035,0.732⟩
$\mathcal{H}_4$	⟨0.0003,0.0008,0.940⟩	⟨0.0007,0.0003,0.920⟩
$\mathcal{H}_5$	⟨0.0037,0.0126,0.818⟩	⟨0.0107,0.0037,0.790⟩

Step 3: Calculate the NPPIS and NPNIS by using the weighted NPHSS matrices for D_1, D_2, D_3, D_4 and D_5 by using the Equations (59) and (60), as shown in the Tables 13 and 14, respectively:

Table 13: Showing the values of NPPIS ($\overline{D}^+$)

$\overline{D}^+$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0029,0.0005,0.890⟩	⟨0.003,0.0008,0.840⟩
$\mathcal{H}_2$	⟨0.0077,0.0014,0.804⟩	⟨0.0055,0.0010,0.744⟩

Table 14: Showing the values of NPNIS ($\overline{D}^-$)

$\overline{D}^-$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0003,0.0005,0.915⟩	⟨0.0003,0.0008,0.952⟩
$\mathcal{H}_2$	⟨0.0010,0.0014,0.858⟩	⟨0.0005,0.0010,0.924⟩

$\mathcal{H}_3$	$\langle 0.0158,0.0028,0.768\rangle\langle 0.0109,0.0021,0.657\rangle$	$\mathcal{H}_3$	$\langle 0.0014,0.0028,0.827\rangle\langle 0.0014,0.0021,0.913\rangle$
$\mathcal{H}_4$	$\langle 0.0010,0.0001,0.940\rangle\langle 0.0007,0.0001,0.911\rangle$	$\mathcal{H}_4$	$\langle 0.0001,0.0001,0.949\rangle\langle 0.0001,0.0001,0.968\rangle$
$\mathcal{H}_5$	$\langle 0.0153,0.0021,0.818\rangle\langle 0.0107,0.0016,0.727\rangle$	$\mathcal{H}_5$	$\langle 0.0016,0.0021,0.873\rangle\langle 0.0010,0.0016,0.943\rangle$

Step 4: Calculate the CC by using the values of NPPIS and NPNIS D_1, D_2, D_3, D_4 and D_5 by using the Equations (61) and (62), as shown in the Table 15:

Table 15: Showing the values of CC ($\mathbb{C}_d{}^+$ and $\mathbb{C}_d{}^-$) for D_1, D_2, D_3, D_4 and D_5

	$\mathbb{C}_d{}^+$	$\mathbb{C}_d{}^-$
D_1	0.9999	0.9951
D_2	0.9946	0.9999
D_3	0.9997	0.9948
D_4	0.9999	0.9957
D_5	0.9993	0.9980

Step 5: The closeness coefficient of the ideal solution for NPHSS is determined by using the Equation (63), as shown in Table 16:

Table 16: Showing the closeness coefficient ($\mathcal{G}_d$) values for D_1, D_2, D_3, D_4 and D_5

D_1	$\mathcal{G}_{d_1}$	0.98
D_2	$\mathcal{G}_{d_2}$	0.018
D_3	$\mathcal{G}_{d_3}$	0.945
D_4	$\mathcal{G}_{d_4}$	0.977
D_5	$\mathcal{G}_{d_5}$	0.741

Step 6: The rank of the alternatives can be determined by arranging the $\mathcal{G}_d$ values in descending order: $\mathcal{G}_{d_1} > \mathcal{G}_{d_4} > \mathcal{G}_{d_3} > \mathcal{G}_{d_5} > \mathcal{G}_{d_2} \Rightarrow D_1 > D_4 > D_3 > D_5 > D_2$.

Hence, based on a faculty evaluation analysis, D_1 (USB-Commerce) is the optimal department within the Chandigarh University management studies cluster.

8. The Analysis of Comparison:

This Section presents a comparative analysis of the proposed approach against previous methods, as depicted in Table 17:

Table 17: Comparing the suggested approach with the prior methods.

Authors	Methods	Observations
Das et al. [66]	Neutrosophic Fuzzy Set	(i) Use of a single set of parameters with intuitive fuzzy values. (ii) An intuitionistic fuzzy set's subset is the approximate function.
Khalil et al. [67]	Single-Valued	(i) Use of a single set of parameters with

	Neutrosophic Fuzzy Soft Set	neutrosophic fuzzy values. (ii) A subset of the universal set is the approximate function.
Muhammad et al. [68]	Single-Valued Neutrosophic Fuzzy Hyper-Soft Set	(i) Applying intuitionistic fuzzy values to a single set of parameters that has been separated into distinct attribute-valued sets. (ii) A neutrosophic set's subset is the approximate function.
Farooq and Saqlain [69]	Neutrosophic Hyper-Soft Set	(i) A subset of the neutrosophic set can be considered as an approximate function. (ii) Truth membership, indeterminacy membership, and falsity membership values are all mutually independent.
Chinnadurai and Bobin [20]	Picture Fuzzy Hyper-Soft Set	(i) A portion of the picture fuzzy set can be viewed as an approximate function. (ii) Relationships exist among positive membership grades, neutral membership grades, and negative membership grades. (iii) Have used multi-attribute functions.
Bobin et al. [70]	Interval-valued Picture Fuzzy Hyper-Soft	(i) Interdependencies in grades exist between positive, neutral, and negative memberships from both lower and upper ends, respectively. (ii) Have used multi-attribute functions.
Proposed method	Neutrosophic Picture Hyper-Soft Set	(i) A subset of the neutrosophic hypersoft set is an approximate function. (ii) Dependent grades between positive, neutral, and negative membership. (iii) Used multi-valued sub-attributes.

9. Conclusions:

Engaging in MCDM involves considering various factors to select the optimal choice. A valuable strategy for ranking options using a distance metric is the technique known as TOPSIS. This research focuses on the theoretical aspects of TOPSIS and any related advancements or applications, addressing the decision-making problem by assessing and rating options. Within this study, we introduce the NPHSS concept, expanding on the NHSS and PFHSS to establish the NPHSS framework with a relevant example. We illustrate the NPHSS model's generalization through a flow chart and present the CC and WCC notions, proving their fundamental properties for NPHSS. Additionally, we have introduced the WAO and WGO notions for NPHSS using various aggregation operators along with suitable examples. The research delves into a TOPSIS-based MCDM issue, demonstrating its effectiveness through a case study on faculty evaluation analysis aimed at selecting the best department from the management studies cluster. Comparative examples highlight the proposed method's efficiency in contrast to existing approaches. The research highlights the reliability of NPHSS as a decision-making tool in uncertain scenarios. There are intentions to expand the model into an interval-valued NPHSS as well as explore its applications in various fields in the future.

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