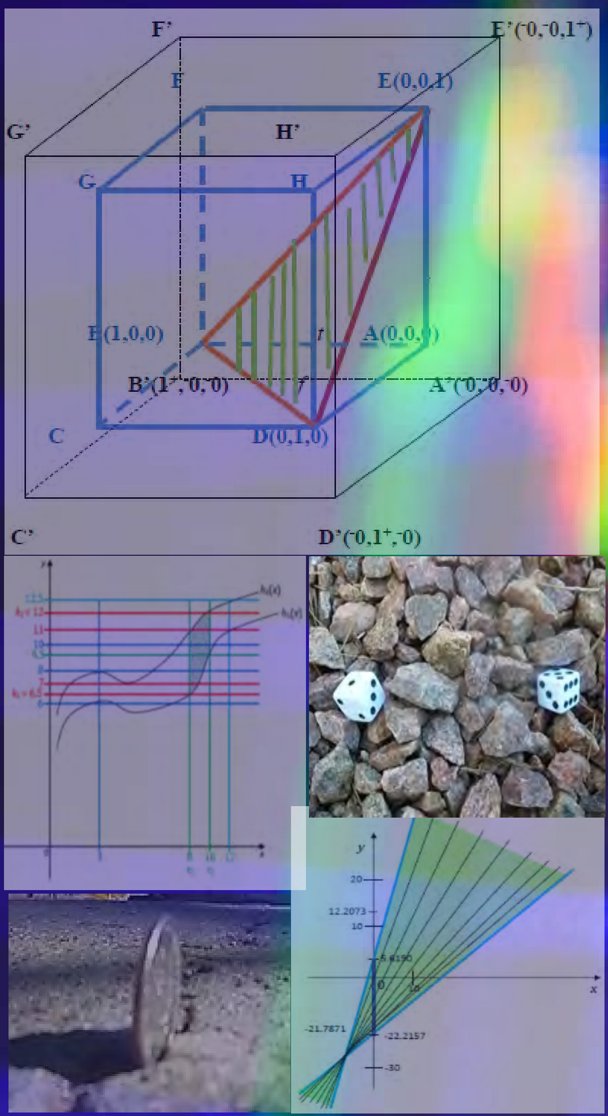
Volume 67,2024

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering



<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
Editors-in-Chief

ISSN 2331-6055 (Print)
ISSN 2331-608X (Online)

Neutrosophic Science
International Association (NSIA)

ISSN 2331-6055 (print) *ISSN 2331-608X (online)*

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

University of New Mexico

ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

NSS

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

Copyright Notice

Information for Authors and Subscribers

"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

All submissions should be designed in MS Word format using our template file:
http://fs.unm.edu/NSS/NSS-paper-template.doc.

A variety of scientific books in many languages can be downloaded freely from the Digital Library of Science:
http://fs.unm.edu/ScienceLibrary.htm.

To submit a paper, mail the file to the Editor-in-Chief. To order printed issues, contact the Editor-in-Chief. This journal is non-commercial, academic edition. It is printed from private donations.

Information about the neutrosophics you get from the UNM website:
http://fs.unm.edu/neutrosophy.htm. The
home page of the journal is accessed on
http://fs.unm.edu/NSS.

ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

NSS

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

**** NSS has been accepted by SCOPUS. Starting with Vol. 19, 2018, the NSS articles are indexed in Scopus.**

NSS ABSTRACTED/INDEXED IN

SCOPUS,
Google Scholar,
Google Plus,
Google Books,
EBSCO,
Cengage Thompson Gale (USA),
Cengage Learning (USA),
ProQuest (USA),
Amazon Kindle (USA),
University Grants Commission (UGC) - India,
DOAJ (Sweden),
International Society for Research Activity (ISRA),
Scientific Index Services (SIS),
Academic Research Index (ResearchBib),
Index Copernicus (European Union),
CNKI (Tongfang Knowledge Network Technology Co.,
Beijing, China),
Baidu Scholar (China),
Redalyc - Universidad Autonoma del Estado de Mexico (IberoAmerica),
Publons,
Scimago, etc.

Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

.

ISSN 2331-6055 (print) NSS *ISSN 2331-608X (online)*

University of New Mexico

Contents

Neutrosophic Sets and Systems, Vol. 67, 2024
University of New Mexico

The indefinite symbolic plithogenic trigonometric integrals

Yaser Ahmad Alhasan[1,*], Suliman Sheen[2], Raja Abdullah Abdulfatah[3] and Mohammed Mustafa Ahmed[4]

[1,*]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, KSA; y.alhasan@psau.edu.sa
[4]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.; S.almleh@psau.edu.sa
[3]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.; r.abdulfatah@psau.edu.sa
[4]Department of mathematics, Faculty of science, Islamic University of Madinah, Madinah 42351, Saudi Arabia; mohamemedm086@gmail.com

*Corresponding author: y.alhasan@psau.edu.sa

Abstract: This paper discussed the indefinite plithogenic trigonometric integrals, where we presented the integrating products of plithogenic trigonometric function, also studying the plithogenic trigonometric identities, which facilitated finding the integral of the associated formulas. In addition to a set of exercises that clarify each idea.

Keywords: trigonometric integrals; plithogenic trigonometric function; plithogenic trigonometric identities.

---

1. Introduction and Preliminaries

To The genesis, origination, formation, development, and evolution of new entities through dynamics of contradictory and/or neutral and/or noncontradictory multiple old entities is known as plithogenic. Plithogeny advocates for the integration of theories from several fields.

We use numerous "knowledges" from domains like soft sciences, hard sciences, arts and literature theories, etc. as "entities" in this study, this is what Smarandache introduced, as he presented a study on plithogeny, plithogenic set, logic, probability, and statistics [2], in addition to presenting introduction to the symbolic plithogenic algebraic structures (revisited), through which he discussed several ideas, including mathematical operations on plithogenic numbers [1]. Also, an overview of plithogenic set and symbolic plithogenic algebraic structures was discussed by him [3]. It is thought that the symbolic n-plithogenic sets are a good place to start when developing algebraic extensions for other classical structures including rings, vector spaces, modules, and equations [4-5-6-7].

Paper is divided into four parts. Provides an introduction in the first portion, which includes a review of Plithogenic science. A few definitions of a Plithogenic and operations with plithogenic numbers are covered in the second section. The third section defined plithogenic functions. The paper's conclusion is provided in the fourth section.

Alhasan also presented several papers on calculus, in which he discussed neutrosophic definite and indefinite integrals. He also presented the most important applications of definite integrals in neutrosophic logic [8-9]. Alhasan, Smarandache and Abdulfatah presented the indefinite symbolic plithogenic integrals [10].

Division of Symbolic Plithogenic Numbers [1]

Let consider two symbolic plithogenic numbers as below:

$$PN_r = a_0 + a_1P_1 + a_2P_2 + \cdots + a_rP_r$$

$$PN_s = b_0 + b_1P_1 + b_2P_2 + \cdots + b_sP_s$$

$$\frac{PN_r}{PN_s} = \begin{cases} none, one \quad many & r \geq s \\ \emptyset & r < s \end{cases}$$

This paper covered a number of topics; the indefinite plithogenic trigonometric integrals were covered in the main discussion section after the introduction and preliminary information were presented in the first section. The paper's conclusion is provided in the final section.

Main Discussion

We will consider C the constant of the symbolic plithogenic integral defined as:
$C = a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n$, where: $a_0, a_1, a_2, \ldots, a_n$ are real numbers.

Integrating products of symbolic plithogenic trigonometric function:

I. $\int PN_r\, sin^m(PN_s x)\, cos^n(PN_s x) \quad dx,$ where m and n are positive integers.

To find this integral, we can distinguish the following two cases:

- Case n is odd:
  - split of $cos(PN_s x)$
  - apply $cos^2(PN_s x) = 1 - sin^2(PN_s x)$
  - we substitution $u = sin(PN_s x)$
- Case m is odd:
  - split of $sin(PN_s x)$
  - apply $sin^2(PN_s x) = 1 - cos^2(PN_s x)$
  - we substitution $u = cos(PN_s x)$

Example 1

Find: $\int (P_3 + 1)\, sin^2(P_2 + 2)x\, cos^3(P_2 + 2)x\, dx$

Solution:

$$\int (P_3 + 2)\, sin^2(P_3 + 1)x\, cos^3(P_3 + 1)x\, dx = \int (P_3 + 2)\, sin^2(P_3 + 1)x\, cos^2(P_3 + 1)x \quad cos(P_3 + 1)x\, dx$$

$$= \int (P_3 + 2)\, sin^2(P_3 + 1)x\, (1 - sin^2(P_3 + 1)x) \quad cos(P_3 + 1)x\, dx$$

$$= \int [(P_3 + 2)\, sin^2(P_3 + 1)x - (P_3 + 2)\, sin^4(P_3 + 1)x]\, cos(P_3 + 1)x\, dx$$

By substitution:

$$u = sin(P_3 + 1)x \qquad \Rightarrow \qquad du = (P_3 + 1)\, cos(P_3 + 1)x \quad dx$$

$$\Rightarrow \qquad = \int \left[\left(\frac{P_3 + 2}{P_3 + 1}\right)u^2 - \left(\frac{P_3 + 2}{P_3 + 1}\right)u^4\right] du$$

$$=\left(\frac{P_3+2}{P_3+1}\right)\left(\frac{u^3}{3}-\frac{u^5}{5}\right)+C=\left(2-\frac{1}{2}P_3\right)\left(\frac{sin^3(P_3+1)x}{3}-\frac{sin^5(P_3+1)x}{5}\right)+C$$

where:

$C=a_0+a_1P_1+a_2P_2+a_3P_3$; $a_0,\ a_1,\ a_2,\ a_3$ are real numbers.

and:

$$\frac{P_3+2}{P_3+1}=x_0+x_1P_1+x_2P_2+x_3P_3$$

$$P_3+2=(P_2+1)(x_0+x_1P_1+x_2P_2+x_3P_3)$$

$$P_3+2=x_0P_3+x_1P_3+x_2P_3+x_3P_3+x_0+x_1P_1+x_2P_2+x_3P_3$$

$P_3+2=x_0+x_1P_1+x_2P_2+(x_0+x_1+x_2+2x_3)P_3$, then:

$x_0=2\ ,x_1=0\ ,x_2=0\ ,x_3=-\frac{1}{2}$

hence: $\frac{P_3+2}{P_3+1}=2-\frac{1}{2}P_3$

<u>let's check the answer:</u>

$$\frac{d}{dx}\left[\left(2-\frac{1}{2}P_3\right)\left(\frac{sin^3(P_3+1)x}{3}-\frac{sin^5(P_3+1)x}{5}\right)+C\right]$$

$$=\left(2-\frac{1}{2}P_3\right)\left(\frac{3(P_3+1)sin^2(P_3+1)x.cos(P_3+1)x}{3}-\frac{5(P_3+1)sin^4(P_3+1)x.cos(P_3+1)x}{5}\right)$$

$$=\left(2-\frac{1}{2}P_3\right)(P_3+1)sin^2(P_3+1)x(1-sin^2(P_3+1)x)\,cos(P_3+1)$$

$$=(P_3+2)sin^2(P_3+1)x(cos^2(P_3+1)x)\,cos(P_3+1)$$

$=(P_3+2)sin^2(P_3+1)xcos^3(P_3+1)x$ (The same integral function)

II. $\int tan^m(PN_sx)\,sec^n(PN_sx)\,dx$, where m and n are positive integers.

To find this integral, we can distinguish the following cases:

- Case n is even:
  - split of $sec^2(PN_sx)$
  - apply $sec^2(PN_sx)=1+tan^2(PN_sx)$
  - we substitution $u=tan(PN_sx)$
- Case m is odd:
  - split of $sec(PN_sx)$ $tan(PN_sx)$
  - apply $tan^2(PN_sx)=sec^2(PN_sx)-1$

- we substitution $u = sec(PN_s x)$

➢ Case m even and n odd:

- apply $tan^2(PN_s x) = sec^2(PN_s x) - 1$
- we substitution $u = sec(PN_s x)$ or $u = tan(PN_s x)$, depending on the case.

Example 2

Find: $\int(-3P_7 + 4P_6)\, tan^2(P_6 x) \quad sec^4(P_6 x)x \quad dx$

Solution:

$n = 4$ (even)

$$\int(-3P_7 + 4P_6)\, tan^2(P_6 x) \quad sec^4(P_6 x)\, dx = \int(-3P_7 + 4P_6)\, tan^2(P_6 x) \quad sec^2(P_6 x)\, sec^2(P_6 x)\, dx$$

$$= \int(-3P_7 + 4P_6)(tan^2(P_6 x) + tan^4(P_6 x))\, sec^2(P_6 x) \quad dx$$

by substitution:

$$u = tan(P_6 x) \qquad \Rightarrow \qquad du = P_6\, sec^2(P_6 x) \quad dx$$

$$\Rightarrow \int(-3P_7 + 4P_6)(tan^2(P_6 x) + tan^4(P_6 x))\, sec^2(P_6 x) \quad dx$$

$$= \frac{-3P_7 + 4P_6}{P_6}\int(u^2 + u^4)\, du$$

$$= (x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 + x_6P_6 - 3P_7)\left(\frac{u^3}{3} + \frac{u^5}{5}\right) + C$$

$$= (x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 + x_6P_6 - 3P_7)\left(\frac{tan^3(P_6 x)}{3} + \frac{tan^5(P_6 x)}{5}\right) + C$$

where:
$C = a_0 + a_1P_1 + a_2P_2 + a_3P_3 + x_4P_4 + a_5P_5 + a_6P_6 + a_7P_7$; $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$ are real numbers.

and:

$$\frac{-3P_7 + 4P_6}{P_6} = x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 + x_6P_6 + x_7P_7$$

$$-3P_7 + 4P_6 = (P_6 x)(x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 + x_6P_6 + x_7P_7)$$

$$-3P_7 + 4P_6 = x_0P_6 + x_1P_6 + x_2P_6 + x_3P_6 + x_4P_6 + x_5P_6 + x_6P_6 + x_7P_7$$

$$-3P_7 + 4P_6 = (x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6)P_6 + x_7P_7$$

then:

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 4 \;\; and \quad x_7 = -3$$

hence:

$\frac{-3P_7+4P_6}{P_6} = x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 + x_6P_6 - 3P_7$

where: $x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 4$

Example 3

Find:

$$\int (P_4 - 3)\, tan^3(P_1 + 1)x\, sec^3(P_1 + 1)x\, dx$$

Solution:

$m = 3$ (odd)

$$\int (P_4 - 3)\, tan^3(P_1 + 1)x\, sec^3(P_1 + 1)x\, dx$$
$$= \int (P_4 - 3)\, tan^2(P_1 + 1)x \quad sec^2(P_1 + 1)x \quad sec(P_1 + 1)x \quad tan(P_1 + 1)x\, dx$$

$$= \int (P_4 - 3)(sec^4(P_1 + 1)x - sec^2(P_1 + 1)x)\, sec(P_1 + 1)x \quad tan(P_1 + 1)x\, dx$$

by substitution:

$$u = sec(P_1 + 1)x \qquad \Rightarrow \qquad du = (P_1 + 1)\, sec(P_1 + 1)x \quad tan(P_1 + 1)x \quad dx$$

$$\Rightarrow \int (P_4 - 3)(sec^4(P_1 + 1)x - sec^2(P_1 + 1)x) \quad sec(P_1 + 1)x \quad tan(P_1 + 1)x\, dx$$

$$= \frac{P_4 - 3}{P_1 + 1} \int (u^4 + u^2)\, du$$

$$= \left(-3 + \frac{3}{2}P_1 + \frac{1}{2}P_4\right)\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$$

$$= \left(-3 + \frac{3}{2}P_1 + \frac{1}{2}P_4\right)\left(\frac{sec^5(P_1 + 1)x}{5} - \frac{sec^3(P_1 + 1)x}{3}\right) + C$$

where:
$C = a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4$; a_0, a_1, a_2, a_3, a_4 are real numbers.

and:

$\frac{P_4 - 3}{P_1 + 1} = x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4$

$$P_4 - 3 = (P_1 + 1)(x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4)$$

$$P_4 - 3 = x_0P_1 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4$$

$$P_4 - 3 = x_0 + (x_0 + 2x_1)P_1 + 2x_2P_2 + 2x_3P_3 + 2x_4P_4$$

then:

$$x_0 = -3 \ \ , x_1 = \frac{3}{2} \ \ , x_2 = 0 \ \ , x_3 = 0 \ \ and \quad x_4 = \frac{1}{2}$$

hence:

$\frac{P_4-3}{P_1+1} = -3 + \frac{3}{2}P_1 + \frac{1}{2}P_4$

III. $\int cot^m(PN_sx)\, csc^n(PN_sx)\, dx$ *, where* m *and* n *are positive integers.*

To find this integral, we can distinguish the following cases:

- Case n is even:
  - split of $csc^2(PN_sx)$
  - apply $csc^2(PN_sx) = 1 + cot^2(PN_sx)$
  - we substitution $u = cot(PN_sx)$
- Case m is odd:
  - split of $csc(PN_sx)$ $cot(PN_sx)$
  - apply $cot^2(PN_sx) = csc^2(PN_sx) - 1$
  - we substitution $u = csc(PN_sx)$
- Case m even and n odd:
  - apply $cot^2(PN_sx) = csc^2(PN_sx) - 1$
  - we substitution $u = csc(PN_sx)$ or $u = cot(PN_sx)$, depending on the case.

Example 6

Find: $\int 2P_9\sqrt{cot(P_8x)} \quad csc^4(P_8x)\, dx$

Solution:

$n = 4$ (even)

$$\int 2P_9\sqrt{cot(P_8x)} \quad csc^4(P_8x)\, dx = \int 2P_9\, cot^{1/2}(P_8x) \quad csc^2(P_8x)\, csc^2(P_8x)\, dx$$

$$= \int 2P_9\left(cot^{1/2}(P_8x) + cot^{3/2}(P_8x)\right) csc^2(P_8x) \quad dx$$

by substitution:

$$u = cot(P_8x) \qquad \Rightarrow \qquad du = -P_8\, csc^2(P_8x) \quad dx$$

$$\Rightarrow \int 2P_9\left(cot^{\frac{1}{2}}(P_8x) + cot^{\frac{3}{2}}(P_8x)\right) csc^2(P_8x) \quad dx$$

$$= -\frac{2P_9}{P_8}\int \left(u^{1/2} + u^{3/2}\right) du$$

$$= -(x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 + x_6P_6 + x_7P_7 + x_8P_8 + 2P_9)\left(\frac{2\ u^{\frac{3}{2}}}{3} + \frac{2\ u^{\frac{5}{2}}}{5}\right) + C$$

$$= (x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 + x_6P_6 + x_7P_7 + x_8P_8 + 2P_9)\left(-\frac{2\ \cot^{\frac{3}{2}}(P_8x)}{3} - \frac{2\ \cot^{5/2}(P_8x)}{5}\right) + C$$

where:

$C = a_0 + a_1P_1 + a_2P_2 + a_3P_3 + x_4P_4 + a_5P_5 + a_6P_6 + a_7P_7 + a_8P_8 + a_9P_9$

; $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9$ are real numbers.

and:

$$\frac{2P_9}{P_8} = x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 + x_6P_6 + x_7P_7 + x_8P_8 + x_9P_9$$

$$2P_9 = (P_8)(x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 + x_6P_6 + x_7P_7 + x_8P_8 + x_9P_9)$$

$$2P_9 = x_0P_8 + x_1P_8 + x_2P_8 + x_3P_8 + x_4P_8 + x_5P_8 + x_6P_8 + x_7P_8 + x_8P_8 + x_9P_9$$

$$2P_9 = (x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8)P_8 + x_9P_9$$

then:

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 0 \quad and \quad x_9 = 2$$

hence:

$$\frac{2P_9}{P_8} = x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 + x_6P_6 + x_7P_7 + x_8P_8 + 2P_9$$

where: $x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 0$

Plithogenic trigonometric identities:

1) $\sin(PN_rx)\cos(PN_sx) = \frac{1}{2}\ [\sin(PN_r + PN_s)x + \sin(PN_r - PN_s)x]$

2) $\cos(PN_rx)\sin(PN_sx) = \frac{1}{2}\ [\sin(PN_r + PN_s)x - \sin(PN_r - PN_s)x]$

3) $\cos(PN_rx)\cos(PN_sx) = \frac{1}{2}\ [\cos(PN_r + PN_s)x + \cos(PN_r - PN_s)x]$

4) $\sin(PN_rx)\sin(PN_sx) = \frac{-1}{2}\ [\cos(PN_r + PN_s)x - \cos(PN_r - PN_s)x]$

Example 7

Find:

$$1)\int P_3 sin(P_2+3)x\,cos(2P_1-1)\,x\ \ dx=\int \frac{1}{2}P_3\ \ [sin(P_2+2P_1+2)x+sin(P_2-2P_1+4)x]\,dx$$

$$=\frac{1}{2}\left[-\left(\frac{P_3}{P_2+2P_1+2}\right)cos(P_2+2P_1+2)x-\left(\frac{P_3}{P_2-2P_1+4}\right)cos(P_2-2P_1+4)x\right]+C$$

$$=\frac{1}{2}\left[-\frac{1}{5}P_3 cos(P_2+2P_1+2)x-\frac{1}{3}P_3\,cos(P_2-2P_1+4)x\right]+C$$

$$=-\frac{1}{10}P_3 cos(P_2+2P_1+2)x-\frac{1}{6}P_3\,cos(P_2-2P_1+4)x+C$$

where:
$C=a_0+a_1P_1+a_2P_2+a_3P_3$; a_0,a_1,a_2,a_3 are real numbers.

and:

$$\frac{P_3}{P_2+2P_1+2}=x_0+x_1P_1+x_2P_2+x_3P_3$$

$$P_3=(P_2+2P_1+2)(x_0+x_1P_1+x_2P_2+x_3P_3+x_0+x_1P_1+x_2P_2+x_3P_3+x_0+x_1P_1+x_2P_2+x_3P_3)$$

$$P_3=2x_0+2x_1P_1+2x_2P_2+2x_3P_3+2x_0P_1+2x_1P_1+2x_2P_2+2x_3P_3+x_0P_2+x_1P_2+x_2P_2+x_3P_3$$

$$P_3=2x_0+(2x_0+4x_1)P_1+(x_0+x_1+5x_2)P_2+5x_3P_3$$

then:

$$x_0=0\ ,\ x_1=0\ ,\ x_2=0\ ,\ x_3=\tfrac{1}{5}$$

hence:

$$\frac{P_3}{P_2+2P_1+3}=\frac{1}{5}P_3$$

and we have:

$$\frac{P_3}{P_2-2P_1+4}=x_0+x_1P_1+x_2P_2+x_3P_3$$

$$P_3=(P_2-2P_1+4)(x_0+x_1P_1+x_2P_2+x_3P_3+x_0+x_1P_1+x_2P_2+x_3P_3+x_0+x_1P_1+x_2P_2+x_3P_3)$$

$$P_3=4x_0+4x_1P_1+4x_2P_2+4x_3P_3-2x_0P_1-2x_1P_1-2x_2P_2-2x_3P_3+x_0P_2+x_1P_2+x_2P_2+x_3P_3$$

$$P_3=4x_0+(-2x_0+2x_1)P_1+(x_0+x_1+3x_2)P_2+3x_3P_3$$

then:

$$x_0=0\ ,\ x_1=0\ ,\ x_2=0\ ,\ x_3=\tfrac{1}{3}$$

hence:

$$\frac{P_3}{P_2-2P_1+4}=\frac{1}{3}P_3$$

$$2)\int P_2 cos(P_2x)\,cos(2P_2x)\ \ dx=\int\frac{1}{2}P_2[cos(3P_2x)+cos(P_2x)]\ \ dx$$

$$= \frac{1}{2}[(x_0 + x_1P_1 + x_2P_2)sin(3P_2x) + (x_0 + x_1P_1 + x_2P_2)sin(P_2x)] + C$$

where:
$C = a_0 + a_1P_1 + a_2P_2$; a_0, a_1, a_2 are real numbers.

and:

$$\frac{P_2}{3P_2} = x_0 + x_1P_1 + x_2P_2$$

$$P_2 = (3P_2)(x_0 + x_1P_1 + x_2P_2)$$

$$P_2 = 3x_0P_2 + 3x_1P_2 + 3x_2P_2$$

$$P_2 = (3x_0 + 3x_1 + 3x_2)P_2$$

then:

$$x_0 + x_1 + x_2 = \frac{1}{3}$$

hence:

$\frac{P_2}{3P_2} = x_0 + x_1P_1 + x_2P_2$, where: $x_0 + x_1 + x_2 = \frac{1}{3}$

and we have:

$$\frac{P_2}{P_2} = \acute{x}_0 + \acute{x}_1P_1 + \acute{x}_2P_2$$

$$P_2 = (P_2)(\acute{x}_0 + \acute{x}_1P_1 + \acute{x}_2P_2)$$

$$P_2 = \acute{x}_0P_2 + \acute{x}_1P_2 + \acute{x}_2P_2$$

$$P_3 = (\acute{x}_0 + \acute{x}_1 + \acute{x}_2)P_2$$

then:

$$\acute{x}_0 + \acute{x}_1 + \acute{x}_2 = 1$$

hence:

$\frac{P_2}{P_2} = \acute{x}_0 + \acute{x}_1P_1 + \acute{x}_2P_2$, where: $\acute{x}_0 + \acute{x}_1 + \acute{x}_2 = 1$

3) $\int P_4sin(P_7 + 3)x\, sin(P_6 - 3)\,x \quad dx = \int \frac{-1}{2}P_4[cos(P_7 + P_6 + 6)x - \cos(P_7 - P_6)x] \quad dx$

$$= dose \quad not \quad exist$$

because:

$$\frac{P_4}{P_7 + P_6 + 6} = (dose \quad not \quad exist)$$

$$and \quad \frac{P_4}{P_7 - P_6} = (dose \quad not \quad exist)$$

5. Conclusions

In this paper, we studied the indefinite plithogenic trigonometric integrals by presenting integration methods specific to the plithogenic field; we concluded that we could obtain an integration that does not exist in the plithogenic field, as the division process is not possible according to the concept presented by Smarandache.

Acknowledgments "This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2024/R/1445)".

References

[1] Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)". Neutrosophic Sets and Systems, Volume 53, PP: 653-665, 2023

[2] Smarandache, F., "Plithogeny, Plithogenic Set, Logic, Probability, and Statistics", 2017.

[3] Smarandache, F., "An Overview of Plithogenic Set and Symbolic Plithogenic Algebraic Structures", Journal of Fuzzy Extension and Applications, Volume 4, pp. 48-55, 2023.

[4] Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.

[5] Rawashdeh, A., "An Introduction To The Symbolic 3-plithogenic Number Theory", Neoma Journal Of Mathematics and Computer Science, 2023.

[6] Abobala, M., and Allouf, A., " On A Novel Security Scheme for The Encryption and Decryption Of 2×2 Fuzzy Matrices with Rational Entries Based on The Algebra of Neutrosophic Integers and El-Gamal Crypto-System", Neutrosophic Sets and Systems, vol.54, 2023.

[7] M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, “Introduction to Symbolic 2-Plithogenic Probability Theory,” Galoitica: Journal of Mathematical Structures and Applications, vol. 7, no. 1, 2023.

[8] Alhasan,Y., "The neutrosophic integrals and integration methods", Neutrosophic Sets and Systems, Volume 43, pp. 290-301, 2021.

[9] Alhasan,Y., "The definite neutrosophic integrals and its applications", Neutrosophic Sets and Systems, Volume 49, pp. 277-293, 2022.

[10] Alhasan,Y., Smarandache,F., Abdulfatah, R., "The indefinite symbolic plithogenic integrals", Neutrosophic Sets and Systems, Volume 60, pp. 29-38, 2023.

Received: Feb 1, 2024. Accepted: April 25, 2024

Solving the Minimum Spanning Tree Problem Under Interval-Valued Fermatean Neutrosophic Domain

Souhail Dhouib 1 , Vidhya K 2 and Said Broumi 3,4,5 *, Mohamed Talea

[1] Laboratory OLID, Higher Institute of Industrial Management, University of Sfax, Tunisia.; souh.dhou@gmail.com
[2] SRM University of Science and Technology, Tamil Nadu, India; kvidhya0911@gmail.com
[3] STIE Team, Regional Center for the Professions of Education and Training (C.R.M.E.F), Casablanca-Settat, Morocco
[4] Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, Casablanca, Morocco; broumisaid78@gmail.com
[5] Department of Mathematics Saveetha School of Engineering, SIMATS Thandalam, Chennai – 602105 Tamilnadu, India

* Correspondence: broumisaid78@gmail.com;

Abstract: In classical graph theory, the minimal spanning tree (MST) is a subgraph that lacks cycles and efficiently connects every vertex by utilizing edges with the minimum weights. The computation of a minimum spanning tree for a graph has been a pervasive problem over time. However, in practical scenarios, uncertainty often arises in the form of fuzzy edge weights, leading to the emergence of the Fuzzy Minimum Spanning Tree (FMST). This specialized approach is adept at managing the inherent uncertainty present in edge weights within a fuzzy graph, a situation commonly encountered in real-world applications. This study introduces the initial optimization approach for the Minimum Spanning Tree Problem within the context of interval-valued fermatean neutrosophic domain. The proposed solution involves the adaptation of the Dhouib-Matrix-MSTP (DM-MSTP) method, an innovative technique designed for optimal resolution. The DM-MSTP method operates by employing a column-row navigation strategy through the adjacency matrix. To the best of our knowledge, instances of this specific problem have not been addressed previously. To address this gap, a case study is generated, providing a comprehensive application of the novel DM-MSTP method with detailed insights into its functionality and efficacy.

Keywords: Minimum Spanning Tree Problem; Fermatean Neutrosophic Domain; Dhouib-Matrix-MSTP; Artificial Intelligence; Operations Research; Combinatorial Optimization

1. Introduction

The Minimum Spanning Tree (MST) is a fundamental concept in graph theory, serving as a crucial optimization problem with widespread applications in various fields. At its core, the MST seeks to identify the most efficient way to connect a set of vertices within a graph by utilizing edges with minimal total weights. This problem has garnered significant attention due to its relevance in operational research, communication systems, transportation networks, logistics, supply chain management, image processing, wireless telecommunication, and cluster analysis. Consider a connected graph G=(V,E) where V represents the set of vertices and E is the set of edges. In the context of this graph, a tree denoted as T is considered a spanning tree if it serves as a subgraph of G and encompasses all the nodes present in G. Referred to as a maximal tree subgraph, T stands out among other trees within G as the most extensive, containing the maximum number of arcs.

Conventional spanning trees operate under the assumption of precise and deterministic edge weights, which may not align with the realities of many real-world situations characterized by ambiguous or imprecise information. The introduction of fuzzy logic offers a valuable means to address uncertainty, providing a more accurate model for scenarios where the nature of relationships between nodes is not fully understood or easily quantifiable. The necessity for a Fuzzy Spanning Tree (FST) arises as a response to the limitations of traditional spanning tree models in effectively handling the inherent uncertainties found in practical, real-world scenarios. Recognizing that real-world systems often grapple with imprecise, uncertain, or incomplete information, the adoption of fuzzy logic in spanning tree models emerges as a solution. This approach enhances the models' ability to capture and represent inherent uncertainties, resulting in solutions that are more robust and adaptable across various application domains. Zadeh [34] introduced the concept of fuzzy sets to address ambiguity and uncertainty by employing a degree of membership function. Subsequently, Atanassov [35] proposed intuitionistic fuzzy sets (IFS), capable of managing both membership and non-membership functions, offering enhanced flexibility in dealing with ambiguity.

Building upon these ideas, Smarandache [36] developed the concept of neutrosophic sets (NS), which includes membership, indeterminacy, and non-membership functions. NS gained considerable attention, leading to numerous studies [37]-[41]. In the domain of minimum spanning trees (MST), Agnes et al. [29] formulated a Fuzzy Clustering model, while Oscan et al. [30] addressed uncertain cost and demand parameters with a capacitated fuzzy MST approach. Mohanta et al. [31] proposed an algorithm for intuitionistic fuzzy MST, and Mandal et al. [32] explored robust MST in cancer detection using intuitionistic fuzzy graphs. For neutrosophic fuzzy graphs, Dey et al. [33] implemented an algorithm for MST.

Senapati et al. [42] first introduced Fermatean fuzzy sets as an extension of Pythagorean fuzzy sets to overcome their inherent limitations. Following this, Jeevaraj [43] further developed the concept by proposing interval-valued Fermatean fuzzy sets. Subsequently, Palani et al. [44] elaborated on the topic by presenting a decision-making methodology within the framework of Pythagorean interval-valued Fermatean fuzzy sets. Expanding on this research, Ruan et al. [45] discussed a multi-criteria decision-making approach tailored for interval-valued Fermatean neutrosophic fuzzy sets.

Despite these advancements, there has been a notable absence of matrix-based methods for MST in the literature. This gap in existing methods has motivated our work, leading us to extend the concept to the Dhouib-Matrix-MSTP (DM-MSTP) method for interval-valued fermatean fuzzy graphs.

This paper is structured as follows: Section 2 introduces the fundamental concepts of IFS, NFS, FFS, and IVFNS. Section 3 outlines the Dhouib Matrix MST method for IVFNN. Section 4 provides numerical examples, and Section 5 concludes the paper.

2. Preliminaries

Definition 1. The Fermatean fuzzy Set (FFS) $\tilde{F}$ in the universal set X is defined by $\tilde{F} = \{\langle x, \mu_{\tilde{F}}(x), \nu_{\tilde{F}}(x)\rangle : x \in X\}$ where the membership function $\mu_{\tilde{F}}(x)$: X → [0, 1] and the non-membership function $\nu_{\tilde{F}}(x)$: X → [0, 1] satisfy the condition $[\mu_{\tilde{F}}(x)]^3 + [\nu_{\tilde{F}}(x)]^3 \leq 1$ is said to be the degree of hesitation of x to $\tilde{F}$.

Definition 2. Let X be the universe of discourse. Then $N = \{\langle x, T_N(x), I_N(x), F_N(x)\rangle : x \in X\}$ is defined as Neutrosophic Fuzzy Set (NFS), where the truth-membership function is represented as $T_N(x): X \to [0,1]$ an indeterminacy-membership function $I_N(x): X \to [0,1]$ and the falsity membership function $F_N(x): X \to [0,1]$ which satisifes the conditions $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3, \forall x \in X$.

Definition 3. A neutrosophic fuzzy set ℓ in the universe X is the form of $\ell = \{\langle u, T_\ell(u), I_\ell(u), F_\ell(u)\rangle : u \in \ell\}$ represents the degree of truth, indeterminacy and falisty-membership of ℓ respectively. The mapping $T_\ell(u): \ell \to [0,1]$, $I_\ell(u): \ell \to [0,1]$, $F_\ell(u): \ell \to [0,1]$ and $0 \leq T_\ell(u)^3 + I_\ell(u)^3 + F_\ell(u)^3 \leq 2$. Here $\ell = (T_\ell, I_\ell, F_\ell)$ is denoted as fermatean neutrosophic number (FNN).

Definition 4. [28] An interval-valued fermatean neutrosophic set (IVFNS) A on the universe of discourse X is defined as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}$, where $T_A(x) = [T^-_A(x), T^+_A(x)]$, $I_A(x) = [I^-_A(x), I^+_A(x)]$, $F_A(x) = [F^-_A(x), F^+_A(x)]$ represents the truth-membership degree, indeterminacy-membership degree and falsity-membership degree, respectively. Hence the mapping $T_A(x): X \to [0,1]$, $I_A(x): X \to [0,1]$, $F_A(x): X \to [0,1]$ and $0 \leq (T_A(x))^3 + (F_A(x))^3 \leq 1$ and $0 \leq (I_A(x))^3 \leq 1$ $0 \leq (T_A(x))^3 + (F_A(x))^3 + (I_A(x))^3 \leq 2$ means $0 \leq (T_A(x))^3 + (F_A(x))^3 + (I_A(x))^3 \leq 2 \ \forall \ x \in X$

Definition 5. [28] Let $A = \langle [T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U] \rangle$ then the score function $S(x)$ is defined as:

$$S(x) = \frac{\left(T_A^L(x)\right)^3 + \left(T_A^U(x)\right)^3 + \left(I_A^L(x)\right)^3 + \left(I_A^U(x)\right)^3 + \left(F_A^L(x)\right)^3 + \left(F_A^U(x)\right)^3}{2}.$$

3. The Dhouib-Matrix-MSTP method

In a very recent work, the first author invented a new optimal method entitled Dhouib-Matrix-MSTP (DM-MSTP) to optimally solve the Minimum Spanning Tree Problem. The general structure of DM-MSTP is depicted in Figure 1 where two lists (the MST-Path and the Min-Columns) are added to the Adjacency-Matrix.

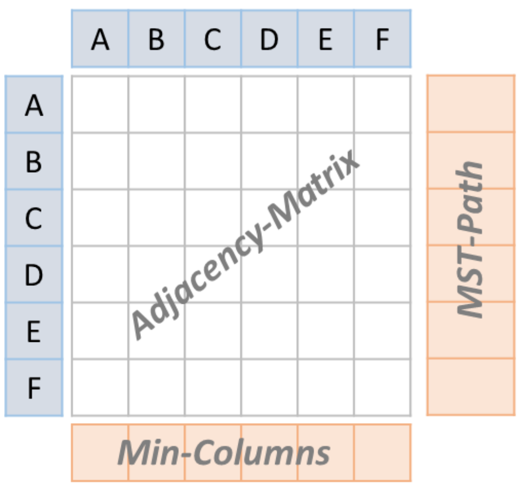


Figure 1. The general structure of DM-MSTP

Indeed, MST-Path gathers the generated components of the spanning tree, Min-Columns is used to drive the research process and Adjacency-Matrix represents the distance between all the vertices.

DM-MSTP is composed of four steps (see Figure 2) and for a more clarification a detailed example will be presented in section 4.

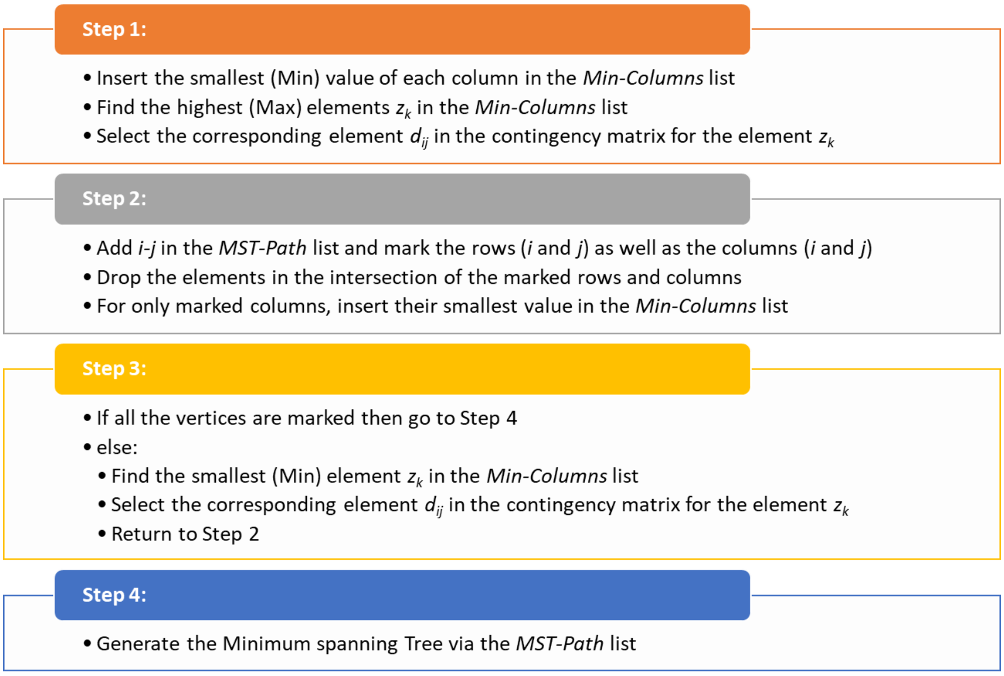


Figure 2. The general structure of DM-MSTP

DM-MSTP is designed under the general concept of Dhouib-Matrix where several optimization methods are developed such as the DM-ALL-SPP to create the shortest path between all-pairs of vertices in (Dhouib, 2024b) and the DM-SPP to solve single-pair, single-source and single-destination Shortest Path Problems (Dhouib, 2023a; Dhouib, 2023b; Dhouib, 2024c; Dhouib, 2024d). in addition, two heuristics (DM-AP1 and DM-AP2) are developed for the Assignment Problems in (Dhouib, 2022a; Dhouib, 2022b; Dhouib, 2023c; Dhouib and Sutikno, 2023) and the DM-TP1 is invented for the Transportation Problems in (Dhouib, 2021a; Dhouib, 2021b). Moreover, two other methods (DM-TSP1 and DM-TSP2) are designed for the Travelling Salesman Problems in (Dhouib, 2021c; Dhouib, 2021d; Dhouib, 2022c; Dhouib et al., 2021; Dhouib et al., 2023). Besides, three innovative metaheuristics are developed: DM4 (Dhouib, 2024e; Dhouib, 2022d; Dhouib and Pezer, 2022; Dhouib and Pezer, 2023; Dhouib, 2023d; Dhouib et al, 2024), DM3 (Dhouib, 2021e; Dhouib and Zouari, 2023a; Dhouib and Zouari, 2023b) and FtN (Dhouib, 2022e).

4. Numerical examples

In this section, the DM-MSTP is simulated on an undirected graph under interval-valued Fermatean neutrosophic domains where the objective is to create the shortest minimum spanning tree. The interval-valued Fermatean neutrosophic number $A = \langle [T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U] \rangle$ is converted to a crisp number using Equation 1 originally developed by (Broumi et al., 2023):

$$S(x) = \frac{\left(T_A^L(x)\right)^3 + \left(T_A^U(x)\right)^3 + \left(I_A^L(x)\right)^3 + \left(I_A^U(x)\right)^3 + \left(F_A^L(x)\right)^3 + \left(F_A^U(x)\right)^3}{2} \qquad (1)$$

Consider the graph in Figure 3, with six vertices and nines edges.

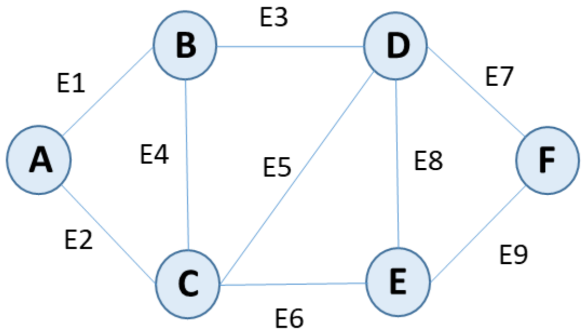


Figure 3. Undirected graph with six vertices and nine edges

The values of the nine edges are represented as interval-valued Fermatean neutrosophic numbers and are given in Table 1.

Table 1. The edge interval-valued Fermatean neutrosophic values

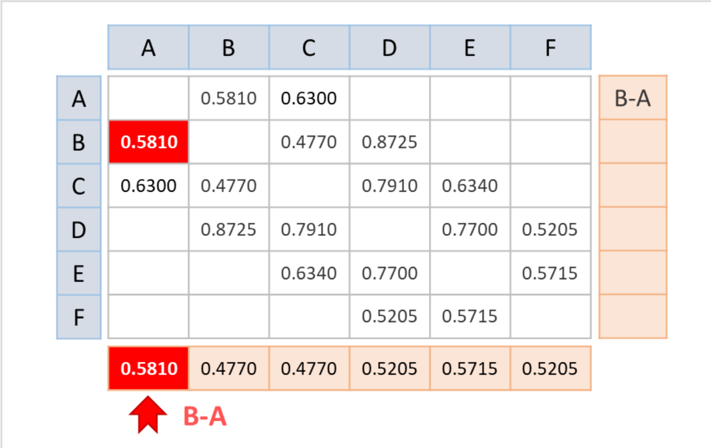
Edge name	Neutrosophic length	Crisp length
E1	⟨[0.4,0.9], [0.5,0.6], [0.1,0.3]⟩	0.5810
E2	⟨[0.5,0.9], [0.4,0.6], [0.1,0.5]⟩	0.6300
E3	⟨[0.2,0.9], [0.3,0.5], [0.1,0.4]⟩	0.4770
E4	⟨[0.2,0.6], [0.4,0.6], [0.8,0.9]⟩	0.8725
E5	⟨[0.1,0.6], [0.5,0.8], [0.6,0.8]⟩	0.7910
E6	⟨[0.2,0.5], [0.4,0.7], [0.6,0.8]⟩	0.6340
E7	⟨[0.1,0.7], [0.5,0.7], [0.6,0.8]⟩	0.7700
E8	⟨[0.3,0.9], [0.3,0.5], [0.2,0.5]⟩	0.5205
E9	⟨[0.5,0.9], [0.2,0.4], [0.1,0.6]⟩	0.5715

The first step is to convert the interval-valued Fermatean neutrosophic set to a crisp one using the score function developed in Equation 1. Figure 4, illustrates the crisp matrix.

$$\begin{bmatrix} \infty & 0.5810 & 0.6300 & \infty & \infty & \infty \\ 0.5810 & \infty & 0.4770 & 0.8725 & \infty & \infty \\ 0.6300 & 0.4770 & \infty & 0.7910 & 0.6340 & \infty \\ \infty & 0.8725 & 0.7910 & \infty & 0.7700 & 0.5205 \\ \infty & \infty & 0.6340 & 0.7700 & \infty & 0.5715 \\ \infty & \infty & \infty & 0.5205 & 0.5715 & \infty \end{bmatrix}$$

Figure 4. The crisp adjacency matrix

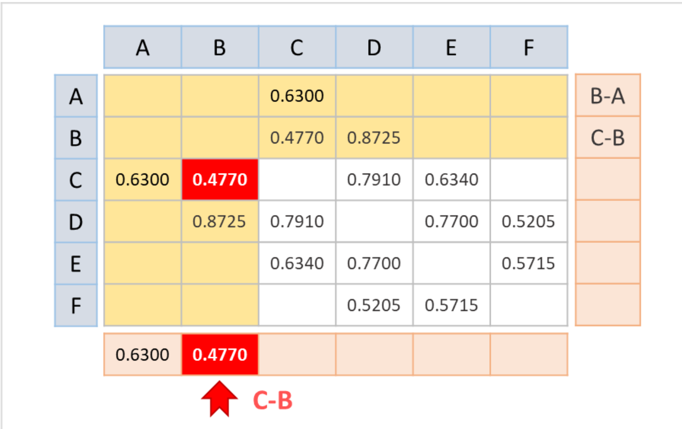
DM-MSTP starts by inserting the smallest elements of each column in Min-Columns and selecting the biggest value (0.5810) at column 1. The corresponding element (dBA) in column 1 is selected to indicate that vertices B and A are connected and 'B-A' is archived in MSTP-Path (see Figure 5).



	A	B	C	D	E	F	
A		0.5810	0.6300				B-A
B	0.5810		0.4770	0.8725			
C	0.6300	0.4770		0.7910	0.6340		
D		0.8725	0.7910		0.7700	0.5205	
E			0.6340	0.7700		0.5715	
F				0.5205	0.5715		
	0.5810	0.4770	0.4770	0.5205	0.5715	0.5205	

Figure 5. The vertices B and A are connected

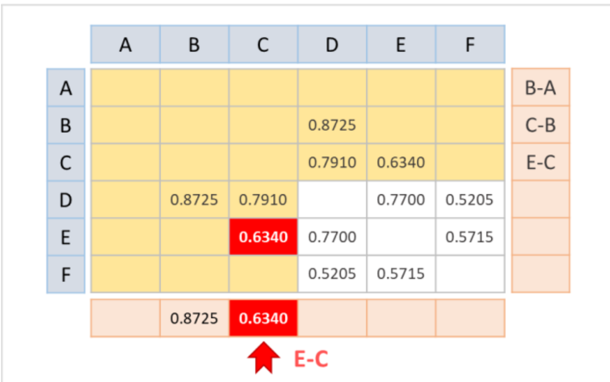
Besides, the rows and columns (B and A) are selected (see elements with the yellow color in Figure 6) and the values of elements in the intersection are dropped. Next, Min-Columns is initiated with the smallest element of only the selected column (with the yellow color), the smallest value is selected (0.4770) and its corresponding element (dCB) is designated. Thus, vertex C is connected to vertex B and 'C-B' is archived in MSTP-Path.



	A	B	C	D	E	F	
A			0.6300				B-A
B			0.4770	0.8725			C-B
C	0.6300	0.4770		0.7910	0.6340		
D		0.8725	0.7910		0.7700	0.5205	
E			0.6340	0.7700		0.5715	
F				0.5205	0.5715		
	0.6300	0.4770					

Figure 6. The vertices C and B are connected

Next, row and column C are selected and the value of the elements in the intersection of A, B and C are dropped (see Figure 7). Also, the smallest elements for each selected column are inserted in Min-Columns, their smallest value (0.6340) is selected and its corresponding element (dEC) is identified. Then, vertex E is connected to vertex C and 'E-C' is added to MSTP-Path.



	A	B	C	D	E	F	
A							B-A
B				0.8725			C-B
C				0.7910	0.6340		E-C
D		0.8725	0.7910		0.7700	0.5205	
E			0.6340	0.7700		0.5715	
F				0.5205	0.5715		
		0.8725	0.6340				

Figure 7. The vertices E and C are connected

Following, row and column E are selected and the value of the elements in the intersection are discarded (see Figure 8). Similarly, Min-Columns is initiated, the smallest value (0.5715) is selected and its corresponding element (dFE) is identified. Then, vertex F is connected to vertex E and 'F-E' is added to MSTP-Path.

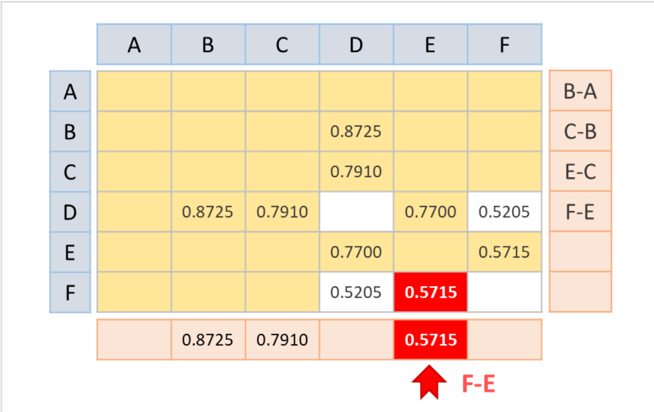


Figure 8. The vertices F and E are connected

Subsequent, row and column E are selected and the value of the elements in the intersection are discarded (see Figure 9). Similarly, the Min-Columns is initiated, the smallest value (0.5205) is selected and its correspondent element is (dDF) identified. Then, vertex DE is connected to vertex F and 'D-F' is added to MSTP-Path.

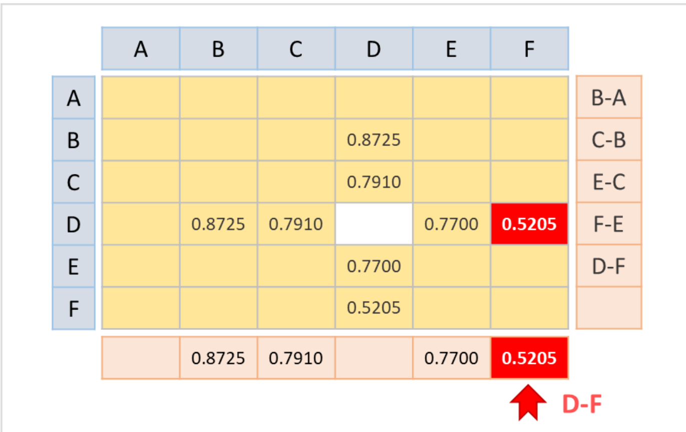


Figure 9. The vertices D and F are connected

Finally, row and column D are selected and the value of the elements in the intersection are discarded. After the above steps, the adjacency matrix is empty and the minimum spanning tree can be generated from MSTP-Path with a total cost of 0.5810+0.4770+0.6340+0.5715+0.5205 = 2.784. Clearly, from Figure 10 you can see the proposed solution generated by DM-SPP.

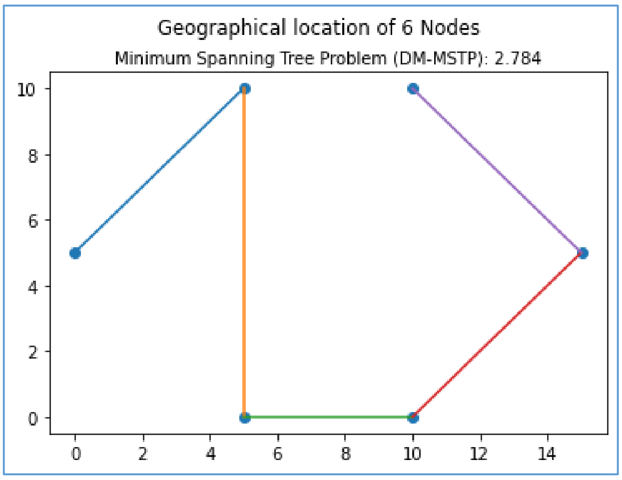


Figure 10. The minimum spanning tree generated by DM-MSTP

5. Conclusions

This paper introduces enhancements to the Dhouib-Matrix-MSTP (DM-MSTP) method to optimize the Minimum Spanning Tree Problem within an interval-valued Fermatean neutrosophic framework. The DM-MSTP method, known for its efficiency, incorporates two key lists: the Path-Memory list, which stores spanning tree edges, and the Min-Column list, used to determine the next column to activate. Additionally, the paper includes a detailed case study illustrating the application of the method. Future research will explore the application of DM-MSTP to other variants of the Minimum Spanning Tree Problem, such as the capacitated version, and its extension to other neutrosophic domains.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Dhouib S (2024a) Innovative Method to Solve the Minimum Spanning Tree Problem: The Dhouib-Matrix-MSTP (DM-MSTP). Results in Control and Optimization, 14:100359. https://doi.org/10.1016/j.rico.2023.100359.
2. Dhouib S (2024b) An Original Optimal All-Pairs Shortest Path Method: The Dhouib-Matrix-ALL-SPP (DM-ALL-SPP). Data Science and Management In press. https://doi.org/10.1016/j.dsm.2024.01.005.
3. Dhouib S (2024c) Intelligent Path Planning for Cognitive Mobile Robot Based on Dhouib-Matrix-SPP Method. Cognitive Robotics, Vol. 4:62-73, https://doi.org/10.1016/j.cogr.2024.02.001.
4. Dhouib S (2024d) Faster than Dijkstra and A* Methods for the Mobile Robot Path Planning Problem Using Four Movement Directions: The Dhouib-Matrix-SPP-4. Advances in Transdisciplinary Engineering, Mechatronics and Automation Technology 46:284-290. https://doi.org/10.3233/ATDE231116.
5. Dhouib S (2023a) An optimal method for the Shortest Path Problem: The Dhouib-Matrix-SPP (DM-SPP). Results in Control and Optimization 12:100269. https://doi.org/10.1016/j.rico.2023.100269.
6. Dhouib S (2023b) Shortest Path Planning via the Rapid Dhouib-Matrix-SPP (DM-SPP) Method for the Autonomous Mobile Robot. Results in Control and Optimization 13:100299. https://doi.org/10.1016/j.rico.2023.100299.
7. Dhouib S (2022a) An Intelligent Assignment Problem Using Novel Heuristic: The Dhouib-Matrix-AP1 (DM-AP1): Novel Method for Assignment Problem. International Journal of Intelligent Systems and Applications in Engineering 10(1):135-141.
8. Dhouib S. (2022b) Novel Optimization Method for Unbalanced Assignment Problems with Multiple Jobs: The Dhouib-Matrix-AP2. Intelligent Systems with Applications 17:200179. https://doi.org/10.1016/j.iswa.2023.200179.

9. Dhouib S (2023c) Unravelling the Assignment Problem under Intuitionistic Triangular Fuzzy Environment by the Novel Heuristic Dhouib-Matrix-AP1. Yugoslav Journal of Operations Research 33(3):467-480. https://doi.org/10.2298/YJOR220915005D.
10. Dhouib S, Sutikno T (2023) Solving the trapezoidal fuzzy assignment problem using the novel Dhouib-Matrix-AP1 heuristic. Bulletin of Electrical Engineering and Informatics 12(2):950-957. https://doi.org/10.11591/eei.v12i2.4855.
11. Dhouib S (2021a) Solving the Single-Valued Trapezoidal Neutrosophic Transportation Problems through the Novel Dhouib-Matrix-TP1 Heuristic. Mathematical Problems in Engineering 2021:3945808. https://doi.org/10.1155/2021/3945808.
12. Dhouib S (2021b) Solving the Trapezoidal Fuzzy Transportation Problems Via New Heuristic the Dhouib-Matrix-TP1. International Journal of Operations Research and Information Systems 12(4):1-16. https://doi.org/10.4018/IJORIS.294119.
13. Dhouib S (2021c) Neutrosophic Triangular Fuzzy Travelling Salesman Problem Based on Dhouib-Matrix-TSP1 Heuristic. International Journal of Computer and Information Technology 10(5):180-183. https://doi.org/10.24203/ijcit.v10i5.154.
14. Dhouib S (2021d) Optimization of Travelling Salesman Problem on Single Valued Triangular Neutrosophic Number using Dhouib-Matrix-TSP1 Heuristic. International Journal of Engineering 34(12):2642-2647. https://doi.org/ 10.5829/IJE.2021.34.12C.09.
15. Dhouib S (2022c) Novel Heuristic for New Pentagonal Neutrosophic Travelling Salesman Problem. Neutrosophic Sets and Systems 51:344-359. https://doi.org/10.5281/zenodo.7135315.
16. Dhouib S, Broumi S, Lathamaheswari M (2021) Single Valued Trapezoidal Neutrosophic Travelling Salesman Problem with Novel Greedy Method: The Dhouib-Matrix-TSP1 (DM-TSP1). International Journal of Neutrosophic Science, 17, 2, 144-157. https://doi.org/10.54216/IJNS.170205.
17. Dhouib S., Zouari A., Dhouib S. and Chabchoub H. (2023) “Integrating the Artificial Bee Colony Metaheuristic with Dhouib-Matrix-TSP1 heuristic for Holes Drilling Problems”, Journal of Industrial and Production Engineering, Vol. 40, No. 3, pp. 177-187, DOI: 10.1080/21681015.2022.2158499.
18. Dhouib S (2024e) Multi-Start Constructive Heuristic through Descriptive Statistical Metrics: The Dhouib-Matrix-4 (DM4) Metaheuristic. International Journal of Operational Research, In press. https://doi.org/10.1504/IJOR.2021.10045069.
19. Dhouib S., Kharrat A., Loukil T. and Chabchoub H. (2024) "Enhancing the Dhouib-Matrix-4 Metaheuristic to Generate the Pareto Non-Dominated Set Solutions for Multi-objective Travelling Salesman Problem: The DM4-PMO Method". Results in Control and Optimization, In press, DOI. https://doi.org/10.1016/j.rico.2024.100402.
20. Dhouib S (2022d) Finding the Shortest Holes Drilling Path in Printed Circuit Board via the Dhouib-Matrix-4 Technique. Advances in Transdisciplinary Engineering, Mechatronics and Automation Technology 33:396-401. https://doi.org/10.3233/ATDE221192.
21. Dhouib S, Pezer D (2022) A Novel Metaheuristic Approach for Drilling Process Planning Optimization: Dhouib-Matrix-4 (DM4). International Journal of Artificial Intelligence 20(2):80-92.
22. Dhouib S, Pezer D (2023) Increasing the Performance of Computer Numerical Control Machine via the Dhouib-Matrix-4 Metaheuristic. Inteligencia Artificial 26(71):142-152. https://doi.org/10.4114/intartif.vol26iss71pp142-152.
23. Dhouib S (2023d) Hierarchical Coverage Repair Policies Optimization by Dhouib-Matrix-4 (DM4) Metaheuristic for Wireless Sensor Networks using Mobile Robot. International Journal of Engineering 36(12):2153-2160. https://doi.org/10.5829/ije.2023.36.12c.03
24. Dhouib S (2021e) Novel Metaheuristic Based on Iterated Constructive Stochastic Heuristic: Dhouib-Matrix-3 (DM3). Applied Computational Intelligence and Soft Computing 2021:7761993. https://doi.org/10.1155/2021/7761993.
25. Dhouib S, Zouari A (2023a) Adaptive Iterated Stochastic Metaheuristic to Optimize Holes Drilling Path in Food Industry: The Adaptive-Dhouib-Matrix-3 (A-DM3). Engineering Applications of Artificial Intelligence 120:105898. https://doi.org/10.1016/j.engappai.2023.105898.
26. Dhouib S, Zouari A (2023b) Optimizing the Non-Productive Time of Robotic Arm for Drilling Circular Holes Network Patterns via the Dhouib-Matrix-3 Metaheuristic. International Journal of Mechatronics and Manufacturing Systems 16(1). https://doi.org/10.1504/IJMMS.2023.10054319.
27. Dhouib S (2022e) Holes Drilling Route Optimization in Printed Circuit Board Using Far-to-Near Metaheuristic. International Journal of Strategic Engineering 5(1):1-12. https://doi.org/10.4018/IJoSE.301568.
28. Broumi, Said, et al. "Interval-valued fermatean neutrosophic graphs." Collected Papers. Volume XIII: On various scientific topics (2022): 496.

29. Vathy-Fogarassy, A., Balázs Feil, and János Abonyi. "Minimal spanning tree based fuzzy clustering." Proc World Acad Sci Eng Tech 8 (2005): 7-12.
30. Öncan, Temel. "Design of capacitated minimum spanning tree with uncertain cost and demand parameters." Information Sciences 177.20 (2007): 4354-4367.
31. Mohanta, Kartick, et al. "An algorithmic approach for finding minimum spanning tree in a intuitionistic fuzzy graph." Proceedings of 32nd International Conference on. Vol. 63. 2019.
32. Nithya, S., S. Bhuvaneswari, and S. Senthil. "Robust minimal spanning tree using intuitionistic fuzzy c-means clustering algorithm for breast cancer detection." Am. J. Neural Netw. Appl 5.1 (2019).
33. Dey, Arindam, et al. "A new algorithm for finding minimum spanning trees with undirected neutrosophic graphs." Granular Computing 4 (2019): 63-69.
34. L. A. Zadeh, “Information and control,” Fuzzy sets, vol. 8, no. 3, pp. 338–353, 1965.
35. K. T. Atanassov, “Intuitionistic fuzzy sets,” Fuzzy sets and Systems, vol. 20, no. 1, pp. 87–96, 1986
36. Smarandache, Florentin. (1999). A unifying field in Logics: Neutrosophic Logic. Philosophy. American Research Press. 1-141.
37. S. Dey and G. C. Ray, “Properties of redefined neutrosophic composite relation,” Neutrosophic Systems with Applications, vol. 7, pp. 1–12, 2023.
38. S. Broumi, A. Bakali, M. Talea, F. Smarandache, and L. Vladareanu, “Computation of shortest path prblem in a network with sv-trapezoidal neutrosophic numbers,” in Advanced Mechatronic Systems (ICAMechS), 2016 International Conference on. IEEE, 2016, pp. 417–422
39. Broumi, Said, et al. (2022). Interval-valued fermatean neutrosophic graphs. Collected Papers. Volume XIII: On various scientific topics: 496.
40. Broumi, Said, et al. (2019). Shortest path problem in fuzzy, intuitionistic fuzzy and neutrosophic environment: an overview. Complex and Intelligent Systems. 5, 371-378.
41. Broumi, S., S. krishna Prabha, & Vakkas Ulucay. (2023). Interval-Valued Fermatean Neutrosophic Shortest Path Problem via Score Function. Neutrosophic Systems With Applications, 11, 1-10.
42. Senapati, T., Yager, R.R. Fermatean fuzzy sets. J Ambient Intell Human Comput 11, 663–674 (2020). https://doi.org/10.1007/s12652-019-01377-0
43. Jeevaraj, S. "Ordering of interval-valued Fermatean fuzzy sets and its applications." Expert Systems with Applications 185 (2021): 115613.
44. Palanikumar, M., et al. "Multi-criteria group decision making method in Pythagorean interval-valued neutrosophic fuzzy soft soft using VIKOR approach." International Journal of Neutrosophic Science 22.1 (2023): 104-113.
45. Ruan, Chuanyang, and Xiangjing Chen. "Probabilistic Interval-Valued Fermatean Hesitant Fuzzy Set and Its Application to Multi-Attribute Decision Making." Axioms 12.10 (2023): 979.

Received: Feb 5, 2024. Accepted: April 26, 2024

Neutrosophic Sets and Systems, Vol. 67, 2024
University of New Mexico

TripleMAsk Spatial Linear Filter and Neutrosophic Entropy for Video Denoising, Face Detection and Recognition in Forensic Crime Analysis Using Deep Learning

Anita Sigamani [1], Dr. Prema Selvaraj [2]

[1] Department of Computer Science, Bhusanayana Mukundadas Sreenivasaiah College for Women, Bengaluru, Karnataka, India; anita@bmscw.edu.in

[2] Department of Computer applications, Arulmigu Arthanareeswarar Arts and Science college, Thiruchengodu, Tamilnadu, India; s.prema@ksrcas.edu.in

* Correspondence: anita@bmscw.edu.in; Tel.: +91 8197675637

Abstract: Forensic Science is the application of Scientific methods to resolve crime and legal issues. It involves various disciplines, such as Computer Science, Biology, Chemistry and Anthropology. Forensic scientists examine and analyze evidence from crime scenes, such as fingerprints, DNA, blood, or weapons. Digital proof is one of the forms of forensic evidence. It provide real time eye witness of the incident. Video recordings enable investigators to find out what exactly has transpired. Investigators use video evidence as a source for witness statements, and it aids in the search for the missing person or suspect. Video evidence is also used to testify in court and help with investigations and prosecutions. Failure of forensic science results in wrong judgement convicting innocent people and escaping criminals [1]. For most crimes high quality video recordings are often not available. video quality issues such as blurry, speckled, pixelated and low-resolution videos captured at low light are a real challenge in forensic analysis. To address such issues in this research a hybrid model using set of filters including triplemask spatial linear filter, median filter and bilateral filters are used. For denoising images, a novel image filter using sliding window convolution is proposed. For image sharpening a triplemask spatial linear filter is proposed. Triplemask spatial linear filter is created by cascading a series of filters. Identity, shift and fraction-based approach is used in mask processing. For image smoothing and to preserve the edges bilateral filter is used [2]. The performance of convolution operation is compared with distinct convolution, shift rotational convolution and scipy convolution. To handle uncertainty, imprecision, and ambiguity in real-world image data in a precise manner neutrosophic science is used in image analysis. By the generated neutrosophic set of the given input image ambiguous regions in the image are detected. Feature selection is made by calculating the entropy of different image regions. From the generated neutrosophic set entropy the degree of uncertainty, within the input image is quantified. The intensity distributions are measured using entropy values. In feature selection regions with highest and lowest entropy values containing face images are selected, visualized and processed to further aid in forensic analysis in detecting the culprits. Neutrosophic AHP is used for prioritizing criteria based on face detection and indeterminancy. Face detection is performed using single shot detector framework with a resnet base network, trained using caffe deep learning framework. face recognition process is performed using dlibs [9] state-of-the-art face recognition model built with deep learning [10]. Face recognition in this research distance-based similarity measure using neutrosophic sets is performed. These measures are used in conjunction with facenet[59] face recognition algorithm to improve the robustness and accuracy over traditional methods. The model has an accuracy of 99.38% on the labelled faces in the wild benchmark.

Keywords: Forensic; video-preprocessing; neutrosophic-sets; face-detection; face-recognition, deep learning.

1. Introduction

Forensic investigations depend on the quality and clarity of video evidence. However, several challenges are associated with video resolution in forensic applications such as low-Quality Source Footage, Digital Zoom Artifacts, Compression Artifacts, Limited Field of View, Frame Rate Issues, Interlacing, Challenges in Enhancement, Authentication and Tampering, Low-Light Conditions and Integration of Multiple Sources. To address these challenges, forensic experts often employ various techniques and tools, such as image and video enhancement software, upscaling algorithms, and specialized video analysis software. Additionally, advances in surveillance technology, including high-resolution cameras and improved compression algorithms, are gradually improving the quality of video evidence available for forensic purposes. In this paper, to address Low-Quality Source Footage and Low-Light Condition issues in Forensic video analysis Deep learning techniques and neutrosophic set theory are used. In studying the related works based on the selected research problem bibliometrics analysis is performed. In this paper vos-viewer tool is used to construct and visualize bibliometric networks based on the research problem. Dimensions database is used to study the related work based on the selected problem statement. The search analysis is made with the Co-authorship and authors relation in Full counting mode. The search query “crime AND Videoevidence AND forensics” is used. The query response displays only 9 publications from 2014 to 2023 in forensics, crime, and video evidence. The search analysis was performed with the threshold criteria: Minimum number of documents of an author: 1. Minimum number of citations of an author: 20. Out of 14 authors 6 meet the threshold.

Table 1. Author documents and citations

S.No.	Author	Documents	Citations	Total Link Strength
1	**Green, Sarah L**	1	24	2
2	**Mine, Becky**	1	23	2
3	**Niche, Robert A.**	1	24	2
4	**Powell, Martine B.**	1	27	1
5	**Wade, Kimberely A.**	1	24	2
6	**Westera, Nina J.**	1	27	1

After verifying the details of 6 selected authors a total-strength of Co-Authorship is calculated. The authors with the maximum link strength are selected. Some of the 6 items in the network are connected to each other. The network represents the Author and Co-Author relation between Kimberley A, Wade, Sarah. L, Green, Robert. A, Nash. The work carried by the authors is taken for reference in understanding how tampered video evidences induce false eye-witness and misleads in judgement.

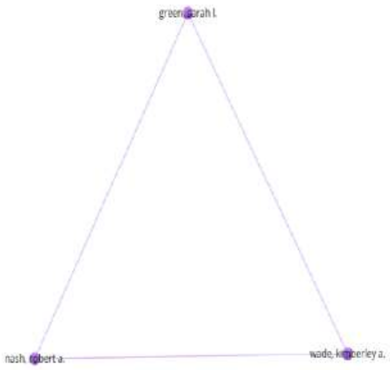


Figure 1. Bibliometric networks

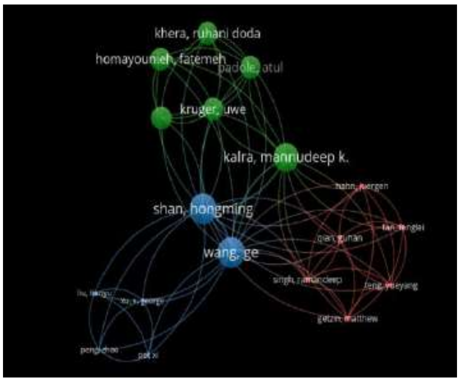


Figure 2. Network Visualization

The authors have proposed cognitive mechanisms in overcoming such defects in video analysis [21][22]. Performing Analysis with less amount of digital evidence in crime and violation prosecutions is a persistent problem. In Prosecutors' perceptions authors describe on improving the quality of evidence [23]. In "lost in the detail" paper authors discuss about prosecutors' perceptions of the utility of video recorded police interviews as rape complainant evidence. Strategy for improving police interviews in dealing with rape complaint cases to collect better evidences is discussed [24]. In amateur censoring visual information is described [25].

A neutrosophic set is a mathematical theory introduced by Florentin Smarandache in late 1990s as an extension of fuzzy sets [33]. Neutrosophic sets are used to represent and handle uncertainty, indeterminacy, and imprecision in a more comprehensive manner. A neutrosophic set consists of three components: the truth-membership function (T(x)), the indeterminacy-membership function (I(x)), and the falsity-membership function (F(x)). The sum of these membership values for any element x in the neutrosophic set should satisfy the inequality: $T(x) + I(x) + F(x) \leq 3$.

Neutrosophic science is an interdisciplinary field which collaborates with a diverse set of branches in scientific, engineering, computer science, mathematics, philosophy, and much more. Neutrosophic science has a wide range of applications including artificial intelligence, computer vision and pattern recognition [34,36], medical image processing, Segmentation [35] and Object Extraction, Video Analysis, Content-Based Image Retrieval, Image and Video Compression, Human-Computer Interaction, decision making, information fusion, philosophical implications and inter-disciplinary research. Recent researches using Neutrosophic Set in varying applications include the following: This study aims to analyse the cardiovascular disease risks by a new distance-based similarity measure motivated from intuitionistic fuzzy set theory[40]. They concluded that the existing measures failed in some circumstances, while the proposed measures classify them more appropriate and precisely. In this article the shortest route issue is formulated using an interval set of values in a Fermatean neutrosophic setting[39]. In this study the authors aim for an Optimal solution for Agricultural Land Use using an Efficient Neutrosophic Linear Programming Method[41]. This study highlights Innovations of Industry 5.0 Technologies Toward Smart Supply Chain Partners using AHP and single value neutrosophic sets (SVNSs)[42]. In this study Heart Disease Prediction under Machine Learning and Association Rules under Neutrosophic Environment is performed[43]. In image processing the input image is converted to digital data and it is processed to find more accurate information. In video processing, the input video is decomposed into a set of frames. Each frame is converted to an array containing details of hue, saturation and intensity of pixels. Pixel operations such as log, inverse log, inverse intensity, log, and gamma corrections [37] are performed on individual pixels to correct dead pixels, for denoising, smoothing and image quality enhancement.

2. Video Evidence based Forensic Analysis and its challenges

In the process of finding the real accused, recorded video evidences are used in Forensic analysis. For forensic analysis based on video evidences accurate visual information is essential for proper investigation and to make right judgment. A good quality video aids in proper investigation and clear analysis of crime cases. Certain video quality issues significantly hampers forensic analysis in following scenarios:

2.1. Surveillance video footage analysis

Low standard surveillance video footages affected by pixelation, blurriness and poor lighting, becomes an obstacle in identifying the individuals, vehicles and other objects of interest.

2.2. Crime scene remaking

Video footages of crime scenes are used as base reference for remaking the scene and to derive conclusions from visual information. Bad quality videos lead to ambiguity in finding certain minute details.

2.3. Facial detection and recognition

One of the applications of facial recognition technology is to detect and recognize faces. From poor quality videos affected with noise or blur it is difficult to accurately match faces to known individuals.

2.4. Vehicle license plate Identification

In cases of accidents video footages involving vehicle details are used in analysis. Poor video quality obscures readability of plates, making it challenging to identify vehicles involved in incidents.

2.5. Digital Forensics and Authentication

Preserving the integrity and quality of the original video footage is vital for authentication and creating a chain of custody.

2.6. Video enhancement and evidence analysis

Videos collected certain times requires enhancement to reveal hidden details. Bad quality video footages hinder the effectiveness of enhancement techniques leading to inaccurate interpretations.

2.7. Expert witness testimony

Forensic experts often depend on video evidences to support their testimony in court. Poor video quality weakens the reliability of expert opinions and deteriorate the influential impact of visual evidence on the judges.

2.8. Documenting human rights violations

In cases involving human rights abuses and criminal activities, video recordings are used as vital evidence. If the quality of such recordings is not to the standard, it can hamper the efforts in documentation and impeach criminals.

2.9. Accident Reconstruction

Video footage of accidents captured by traffic cameras and surveillance cameras, is important for remaking events and determining accountability. Poor video quality obscures important details for accurate reconstruction.

3. Proposed Methodology

Spatial Linear filter is chosen over spatial non-linear filter because of its computational efficiency, simplicity and interpretability, linearity preserving nature, preserves structural information, imposes smoothness and denoising, and compatible with convolutional neural network.

The computational complexity of linear filters is based only on the size of the filter/kernel which has reduced computations. Spatial non-linear filters involve complex sorting and iterative processes which makes computationally inefficient. Spatial linear filters are known for its simplicity and interpretability. Linear filters are easy to understand and analyze. To detect faces in videos involving various linear transformations and interpret the image using neutrosophic set mathematical operations linear filter is used. Spatial linear filter is chosen for preserving the structural details of the image. Linear filters are selected to impose smoothness over the output image by denoising it. Linear filters are compatible with convolutional neural networks. To this end the convolution operations are directly imposed at convolutional layers for face detection.

3.1. TripleMask Spatial Linear filter

Convolution operation is performed on the selected input image. Convolution operation on an image is a process of adding each element of the image to its local neighbors, weighted by a small matrix called the kernel [26]. The kernel acts as a filter that modifies the output image according to some desired effect, such as blurring, sharpening, edge detection, etc. [27]. The properties of convolution, such as commutativity, associativity and distributivity, still hold for image convolution [28]. Convolution properties are as follows:

1. It is commutative: $a * b = b * a$
2. It is distributive with respect to addition: $a * (b1 + b2) = a * b1 + a * b$
3. It is linear: $t\,(a * b) = (t \cdot a) * b$
4. It is separable if the kernel is separable: $a * (bi \otimes bj) = (a * bi) * bj$

A Triple Spatial Linear filter is proposed for image sharpening. The filter is created by cascading a series of filters. Identity, Shift and Fraction based approach is used in selecting the Mask. A 64x64 size image is taken as input. The input image is processed by applying spatial linear filter. Spatial linear filter is created by choosing an identity kernel and a fractional kernel. To increase the strength of kernel values by using the properties of convolution, the identity kernel elements are multiplied by an integer. The resultant product kernel is then subtracted with fractional kernel. By doing this we have obtained a simple sharpening filter. For Image smoothing and to preserve the edges bilateral filter is used [2].

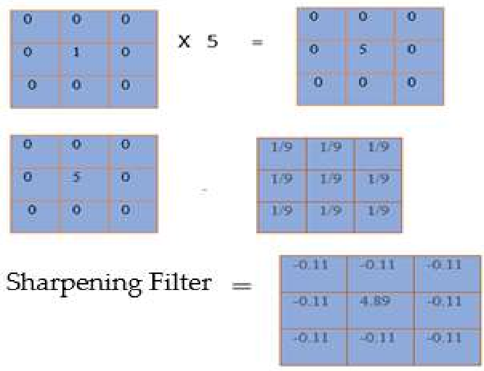


Figure 3: Sharpening Filter

Triple Spatial Linear filter procedure:

Convert input image as array

Arrayimage = asarray(image)

Declare identity kernel of size m x n

//assign kernel elements

Identity kernel = [[0] * n for i in range (m)]

Display identity kernel

Declare fractionalkernel of size m x n

//assign kernel elements

Fractionalkernel[m][n] = [[1/9] * n for i in Range (m)]

Display fractionalkernel

Set constant = k

Multiple identity kernel elements by constant k

Product identity kernel[i][j] = identity kernel * k

Obtain new sharpeningfilter

Sharpeningfilter[i][j] = productidentitykernel[i][j] - fractionalkernel[m][n]

Perform convolution operation

Rotate Sharpeningfilter by 180 degree

//apply rotatedsharpeningfilter on input image

Sharpenedimage=filter2d(arrayimage, ddepth = -1, kernel= rotatedsharpeningfilter)

Display sharpenedimage

(a) (b) **(c)** **(d)**

Figure 4. Input image processed using sharpening filter and bilateral filter is listed as: (a) Original image and sharpen image; (b) Cropped original; (c) sharpened image; (d) Image smoothing and edge preserving.

3.2. Advantages of using proposed TripleMask spatial linear filter

3.2.1. Improved feature representation

The convolution operation using identity filter on image preserves the original feature, the shift operation on the image introduces translational invariance and fractional mask/kernel operation enables precise fine-grained feature extraction for precise localization to detect and segment faces in a video and image. By combining these three kernels/filters we obtain more comprehensive image feature representations.

3.2.2. Enhanced robustness to basic geometric transformations

Shift and fractional kernel operations on input image makes the working model more robust to translation, rotation, scaling and distortion.

3.2.3. Reduction in overfitting

By combining three filters identity, shift and fraction kernels the model is made to learn diverse representations of data to reduce overfitting when trained with limited data.

3.2.4. Improved level of performance

By using the combined filter approach the model can leverage the strengths of each kernel operation to result in improved performance in face detection and segmentation.

3.2.5. Adaptability

The kernel operation is refined using distributive, commutative and linear convolution properties to fit to the model architecture based on the input dataset.

3.3. Convolution Operations, Impact on Video Quality Enhancement

Different convolution operations excel in various scenarios based on their characteristics and the requirements of the task at hand. Here we write few insights on each convolution operation, their influence to video quality enhancement and the scenarios where they excel:

3.3.1. Mean Filter:

Scenario: Mean filtering is operative for reducing Gaussian noise in images or videos. It is mainly useful in scenarios where the noise is uniform and can be modelled as Additive-white gaussian noise.

Contribution to Video Quality Enhancement: By smoothing noise, the mean filter helps improve the visual quality of videos, making them appear cleaner and less grainy.

3.3.2. Gaussian Filter:

Scenario: Gaussian filtering is appropriate for noise reduction while preserving edges and fine details in the image or video. It is commonly used when a more natural and visually pleasing result is looked-for compared to traditional mean filtering.

Contribution to Video Quality Enhancement: Gaussian filtering effectively reduces noise while maintaining the sharpness of edges and textures, resulting in visually attractive videos with improved clarity.

3.3.3. Sobel and Prewitt Operators:

Scenario: Sobel and Prewitt operators are commonly used detecting edges in images and videos. They are effective in detecting regions of rapid intensity changes, and object-boundaries.

Contribution to Video Quality Enhancement: These operators are used in enhancing the perceptual quality of videos by highlighting the edges. They advance in the delineation of objects and structures within the video frames, resulting in much sharper and well-defined visual content.

3.3.4. Laplacian Filter:

Scenario: The Laplacian filter is useful for edge detection and image sharpening. It highlights regions of rapid intensity changes and enhances the overall contrast in the image or video.

Contribution to Video Quality Enhancement: By highlighting edges and fine details the Laplacian filter enhances the visual quality of videos by making them appear much sharper and more detailed.

3.3.5. Gabor Filter:

Scenario: Gabor filters are commonly used for texture analysis and feature extraction in images and videos. They are effective in capturing both spatial and frequency information by making them suitable for tasks such as texture classification and segmentation.

Contribution to Video Quality Enhancement: Gabor filters is used to improve the perceptual quality of videos by capturing and preserving texture details. They enhance the richness and depth of textures by leading to more visually pleasing videos.

3.3.6. Wiener Filter:

Scenario: The Wiener filter is an adaptive filter used for noise reduction in images and videos. It estimates the power spectrum of the noise and the signal to compute an optimal filter that minimizes the mean square error.

Contribution to Video Quality Enhancement: By adaptively reducing noise while preserving image details the Wiener filter significantly enhances the visual quality of videos. It ensures that noise reduction is optimized for the specific characteristics of the noise and the underlying signal by resulting in clear and high-quality video content.

3.4. Performance Comparison of convolution operation

The performance of convolution operation is compared with a distinct convolution, a shift rotational convolution and scipy convolution. The performance is measured using various metrics in terms of CPU times: user time and sys time, Wall time and Total time. Scipy convolution operation comparatively is efficient in which user time- the amount of CPU time taken outside of the kernel and the sys time- the amount of time taken inside of the kernel is very least amount measuring only to 2 μs and 1e+03 ns where as traditional Convolution operation takes user time: 208 ms and sys time: 50.1 ms. Comparing Sliding window convolution with Distinct and traditional convolution the CPU times its measures only 10% of 305 ms to 14% of 208 ms and Wall time measures 11% of 311 ms to 12% of 294 ms.

3.4.1. Traditional Convolution

$$Z(i, j) = w * f(i, j) = \sum_{di=-a}^{a} \sum_{dj=-b}^{b} w(di, dj)\, f(i-di)(j-dj) \quad (1)$$

Z(i,j) is the filtered image, f(i,j) is the original image, w is the filter kernel. Where $-a \le di \le a$ and $-b \le dj \le b$.

3.4.1.1. Image smoothening

Performing convolution operation on an input image with, selected kernel of Ones is shown as follows:

Kernel= Ones ((size, size)) / (size ^2) (2)

size=3, kernel equation (2) generates a kernel as follows: [[1. 1. 1.], [1. 1. 1.], [1. 1. 1.]]

(a) (b) (c)

Figure 5. Traditional Convolution operation for Image smoothening with kernel in equation 2 using mean kernel smoothening produces results with varying size listed as: (a) Original image; (b) 3x3 kernel; (c) 20x20 kernel. (CPU times: user 208 ms, sys: 50.1 ms, total: 258 ms, Wall time: 294 ms)

3.4.2. Distinct Convolution

$$Z(i, j) = M * f(i, j) = \sum_{di,\, dj} f(i+di, j+dj) * M[di, dj] \quad (3)$$

Z(i,j) is the filtered image, f(i,j) is the original image, M is the filter kernel. Where -a≤di≤a and -b≤dj≤b. Performing convolution operation on an input image with, selected kernel of Ones is shown as follows:

Kernel = ones((21,3))/(21/3) (4)

Figure 6. Distinct convolution operation (CPU times: user 305 ms, sys: 0 ns, total: 305 ms, Wall time: 311 ms)

3.4.3. Sliding window convolution

A sliding window which highlights the image pixels to be operated is covered by the window. Image values are not copied but viewed. It is faster with the sliding window since there is no loop. The creation of the rolling window does not take much time because no data is copied, it is just a view on the original matrix. Performing sliding window convolution operation on an input image with selected kernel ones using equation (4) is shown as follows:

Figure 7. Sliding window convolution operation

CPU times: user 31.5 ms, sys: 6.34 ms, total: 37.9 ms, Wall time: 36.9 ms

3.4.4. Scipy convolution

Scipy convolution operation is performed using functions linear and non-linear filtering, binary morphology, B-spline interpolation [29].

Figure 8. Scipy convolution operation

CPU times: user 2 µs, sys: 1e+03 ns, total: 3 µs, Wall time: 5.96 µs

Table 2. Performance Comparison of convolution operation

S.no	Operation	CPU times		Wall time	Total CPU times
		user	sys		
1	Traditional Convolution	208 ms	50.1 ms	294 ms	258 ms
2	Distinct Convolution	50.1 ms	0 ns	311 ms	305 ms

3	Sliding Window Convolution	305 ms	6.34 ms	36.9 ms	37.9 ms
4	Scipy Convolution	2 µs	1e+03 ns	5.96 µs	3 µs

Scipy convolution operation comparatively is efficient in which (user time) the amount of CPU time taken outside of the kernel and (sys time) the amount of time taken inside of the kernel is very least amount measuring only to 2 µs and 1e+03 ns where as Traditional Convolution operation takes user time: 208 ms and sys time: 50.1 ms. Comparing Sliding window convolution with Distinct and traditional convolution the CPU times its measures only 10% of 305 ms to 14% of 208 ms and Wall time measures 11% of 311 ms to 12% of 294 ms.

3.5. Image Denoising

To denoise the input image sliding Window Convolution operation is used. Creating a new novel filter by using sliding window convolution to denoise images is shown below. For each sliding window sort the pixels and take the mean(r) of the pixel. When r=1 the filter functions similar to median filter. If r=3 then novel filter function is invoked. An image affected with impulse noise is taken as input. Its is processed using novel filter. The output is compared with median filter. The obtained result is 99% similar to that of median filter.

(a) (b) (c)

Figure 9. Denoising input image using sliding Window Convolution operation produces results as listed (a) Original image; (b) Median filter; (c) Novel filter.

3.6. Neutrosophic Science in Forensic Analysis: Addressing Uncertainty and Ambiguity, Image Data

Neutrosophic science has several advantages in addressing uncertainty and ambiguity in image data over forensic analysis:

1. *Handling uncertainty in Video Evidence Interpretation*: Forensic analysis often deals with imperfect or incomplete evidence leading to uncertainty in interpretation. Neutrosophic logic provides a formal framework to represent and reason about uncertainty in image data. By assigning truth-membership, indeterminacy membership, and falsehood membership degrees to pixel values or image features. Neutrosophic science allows forensic analysts to model and quantify uncertainty more effectively.
2. *Robustness to Noise and Distortion:* Images obtained as evidence in forensic investigations may be subjected to noise, compression artifacts, or other forms of distortion. Neutrosophic image processing techniques, such as neutrosophic filtering and denoising, are designed to handle such imperfections while preserving important details. By incorporating neutrosophic set theory into image processing algorithms, forensic

analysts can enhance the quality of image evidence and reduce the impact of noise and distortion on analysis results.

3. *Accurate Boundary Detection and Segmentation:* Neutrosophic edge detection and segmentation methods are robust to ambiguity in object boundaries, making them well suited for forensic applications where precise delineation of objects or regions of interest is crucial. By considering indeterminacy membership degrees in addition to truth membership degrees, Neutrosophic segmentation algorithms can accurately segment forensic images even in cases where object boundaries are unclear or ambiguous.
4. *Effective Fusion of Multimodal Data:* Forensic investigations often involve the analysis of multiple types of evidence, including images, videos, and other sensor data. Neutrosophic image fusion techniques enable the integration of information from diverse sources while accounting for uncertainties inherent in each modality. By combining neutrosophic set theory with fusion algorithms, forensic analysts can extract more comprehensive and reliable information from multimodal data thus enhancing the overall accuracy and completeness of forensic analyses.
5. *Transparent Representation of Evidence Confidence:* Neutrosophic science provides a transparent way to represent the confidence level or reliability of evidence in forensic analyses. By explicitly quantifying truth membership, indeterminacy membership, and falsehood membership degrees. Neutrosophic logic allows forensic analysts to convey the degree of uncertainty associated with different aspects of image data. This transparency fosters better communication and interpretation of forensic findings.

4. Preliminaries

This section introduced some preliminary notions which will be applied in the final analysis.

4.1. Single Valued Neutrosophic Set

A neutrosophic set which can be used in real scientific and engineering applications is known as Single valued neutrosophic set (SVNS).

Definition 4.1.1 [46]. Let X be a space of points (objects) with a generic element in X denoted by x. A single valued neutrosophic set A in X is characterized by a truth membership function, $TA(x)$, an indeterminacy membership function, $IA(x)$, and a falsity membership function $FA(x)$. Here $TA\ (x)$, $IA\ (x)$, $FA\ (x)$ are real subsets of [0,1].

$$A = \{\langle x,\ TA\ (x), IA\ (x),\ FA\ (x)\rangle \mid x \in X\}$$

4.2. Distance-based Similarity Measure of Neutrosophic set

Definition 4.2.1 [47]. Normalized Hamming distance measure $dNS\ NH(A,\ B)$ operator between neutrosophic set A and B is defined as follows:

$$d_{NS}^{NH}(A,B) = \frac{1}{3n}\sum_{i=1}^{n}(|T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)|)$$

Definition 4.2.2 [47]. Normalized Euclidean distance measure $dNS\ NE(A,\ B)$ operator between neutrosophic set A and B is defined as follows:

$$d_{NS}^{NE}(A,B) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}((T_A(x_i) - T_B(x_i))^2 + (I_A(x_i) - I_B(x_i))^2 + (F_A(x_i) - F_B(x_i))^2)}$$

Definition 4.2.3 [47]. Let A, B be two neutrosophic sets in X. The similarity measure between the neutrosophic sets A and B can be evaluate from distance measures, as follows:

$$S_N(A,\ B) = 1 - d_{NS}(A,\ B)$$

where $d_{NS}(A,\ B)$ is represent the distance measure between neutrosophic set A and B for all $xi \in X$.

Proposition 4.2.3.1: The distance measures for neutrosophic set $d_{NS}(A,\ B)$and similarity measure for neutrosophic set $S_N(A,\ B)$ satisfies the following properties:

$0 \leq d_{NS}(A,\ B) \leq 1$; $0 \leq S_N(A,\ B) \leq 1$;

$d_{NS}\ (A,\ B) = 0$ if and only if A = B; $S_N\ (A,\ B) = 1$ if and only if for $A = B$

$d_{NS}\ (A,\ B) = d_{NS}\ (B,\ A)$; $S_N\ (A,\ B) = S_N\ (B,\ A)$;

$d_{NS}\ (A,\ C) \leq d_{NS}\ (A,\ B)$ and $d_{NS}\ A,\ C) \leq d_{NS}\ (B,\ C)$ if C is neutrosophic set in X and $A \subseteq B \subseteq C$;

$S_N\ (A,\ C) \leq S_N\ (A,\ B)$ and $S_N\ (A,\ C) \leq S_N\ (B,\ C)$ if C is neutrosophic set in X and $A \subseteq B \subseteq C$. [44,45]

5. Represent Image In Neutrosophic Set

Representing an image in neutrosophic set theory involves characterizing the uncertainty and indeterminacy present in the image's pixel values. Neutrosophic set theory is an extension of fuzzy set theory that introduces a third membership function, representing the degree of indeterminacy. First the input image is read and preprocessed to convert to a format suitable for applying neutrosophic science techniques. The image is loaded using Keras with parameters colormode rgb and interpolation nearest. The input image is converted to a tensor using tensorflow. The shape of the tensor is determined using tensorflow. The input image has a tensor shape of 194 X 259 with 3 colour channels [194, 259,3], with its data elements as int32 type. The created tensor is converted to a NumPy array.

Figure 10. Input image

[[13. 8. 14.]	[6. 6. 6.]	[[3. 0. 4.]	[6. 6. 6.]	[[1. 0. 4.]	[6. 6. 6.]
[11. 6. 12.]	[6. 6. 6.]	[1. 0. 4.]	[6. 6. 6.]	[0. 0. 4.]	[6. 6. 6.]
[4. 0. 5.]	[6. 6. 6.]]	[3. 0. 4.]	[6. 6. 6.]]	[1. 0. 4.]	[
	...		...		6. 6. 6.]]

Figure 1l. Image to Array

Generate the Neutrosophic Set Elements for the image_array created above. Representing an image as a neutrosophic set in Python involves assigning each pixel in the image degrees of membership in the T (true), I (indeterminate), and F (false) subsets as per the definition in 4.1.1 [46]. For representing in neutrosophic set domain the degrees of membership for each pixel is defined using two membership rules:

Rule 1: If intensity > 128, assign high T-membership, low I-membership, and low F-membership. In first rule T-membership value is normalized to [0, 1], having T_membership = (pixel - 128) / 127.0, I_membership = 0.2 and F_membership = 0.1

Rule 2: If intensity <= 128, assign high F-membership, low I-membership, and low T-membership. In second rule F-membership value is normalized to [0,1], having T_membership = 0.1, I_membership = 0.2, F_membership = (128 - pixel) / 127.0.

The image is converted to gray image and the intensity of each pixel is read from pixel coordinates x,y. It is observed that pixels with 0 intensity are ambiguous and unclear.

Pixel at (0, 175) has intensity: 0
Pixel at (1, 175) has intensity: 0
Pixel at (2, 175) has intensity: 0
Pixel at (3, 175) has intensity: 0
Pixel at (4, 175) has intensity: 0
Pixel at (5, 175) has intensity: 0
Pixel at (6, 175) has intensity: 0
Pixel at (7, 175) has intensity: 0

Figure 12. Ambiguous and unclear

[0.08661418, 0.11811024, 0.33070865]
[0.00000000, 0.00000000, 0.22047244]
[0.18897638, 0.17322835, 0.33858266]
[0.24409449, 0.24409449, 0.43307087]

Figure 13. (F,I,T)-membership for pixels in figure 12

The pixels displayed in figure:12 consists of regions which are difficult to interpret. Recognizing such regions are crucial for image understanding and decision-making. For these pixels neutrosophic set contains high F-membership, low I-membership, and low T-membership.

Pixel at (134, 181) has intensity: 128	
Pixel at (135, 181) has intensity: 107	[0.42519686, 0.26771653, 0.33858266],
Pixel at (136, 181) has intensity: 143	[0.44094488, 0.28346458, 0.35433072],
Pixel at (137, 181) has intensity: 164	[0.40157488, 0.24409449, 0.31496063],
Pixel at (138, 181) has intensity: 190	[0.40944883, 0.25196858, 0.32283464],
Pixel at (139, 181) has intensity: 207	[0.44881899, 0.31496063, 0.37795275],
Pixel at (140, 181) has intensity: 211	[0.41732284, 0.28346458, 0.34645677],
Pixel at (141, 181) has intensity: 210	[0.47244096, 0.33858266, 0.40157489],
Pixel at (142, 181) has intensity: 211	[0.62992126, 0.52755904, 0.58267724]

Figure 14. Pixels with intensity > 128: unambiguous and clear **Figure 17.** High T-membership, low I-membership, and low F-membership for pixels in figure 14

5.1. Pixel-wise Neutrosophic Entropy Calculation

Neutrosophic-entropy has wide significance in image processing [38]. Neutrosophic set entropy is a measure of the uncertainty. To calculate entropy for input image, the input_arr equivalaent for the input image is converted to float type value using the operation input_arr.astype(float)/255.0. Calculate neutrosophic entropy for each pixel having membership > 0, using the following equation:

entropy -= membership * np.log(membership + epsilon) (5)

Where, epsilon = 1e-10 and pixel_memberships = [0.3, 0.4, 0.3].

[[[0.15173366 0.10860618 0.15933681]	[[0.05226649 0. 0.06517599]
[0.13559628 0.08822363 0.14382856]	[0.02173045 0. 0.06517599]
[0.06517599 0. 0.07709462]	[0.05226649 0. 0.06517599]
...	...
[0.08822363 0.08822363 0.08822363]	[0.08822363 0.08822363 0.08822363]
[0.08822363 0.08822363 0.08822363]	[0.08822363 0.08822363 0.08822363]
[0.08822363 0.08822363 0.08822363]]	[0.08822363 0.08822363 0.08822363]]

Figure 12. Neutrosophic entropy of input image in figure 10

It is observed for those pixels having intensity values with equal probability, the entropy is maximum, and for pixels having low intensity the entropy values are low indicating nonuniform intensity distribution. Pixel at (10,60) has intensity: 2 resulting with low entropy ((10, 60), 0.8812909).

5.2. Identification of Crucial Image Regions from Calculated Entropy Values

The input image shape is read and stored as "height, width = gray_image.shape", for each pixel coordinates x,y in for y in range(height) and for x in range(width), intensity is found as "f'Pixel at ({x}, {y}) has intensity: {intensity}'". The input image is read in rgb color mode with 'nearest' interpolation, convert to input image to array and degrees of membership for each pixel is defined, for each pixel in row: If (pixel intensity > 128).any() then t_membership = (pixel - 128) / 127.0, normalize T-membership to [0, 1], i_membership = 0.2, f_membership = 0.1. Else: t_membership = 0.1, i_membership = 0.2, f_membership = (128 - pixel) / 127.0, normalize F-membership to [0, 1]. Append the degrees of membership for each pixel, row_set.append((t_membership, i_membership, f_membership)).

Thresholding is applied to identify indeterminate regions. An indeterminacy_filter is defined to work the input image with threshold =100. Indeterminacy_filter(image, threshold=100). _, binary_image = cv2.threshold(image, threshold, 255, cv2.THRESH_BINARY). Then the binary image is inverted to get indeterminate regions. Indeterminate_regions = cv2.bitwise_not(binary_image).

Neutrosophic_entropy is calculated for pixel_memberships >0, initialy entropy is set to be 0.0 and a small constant epsilon, to avoid taking the logarithm of zero is used, where epsilon = 1e-10. for membership in pixel_memberships, if membership > 0 then entropy is calculated as entropy -= membership * np.log(membership + epsilon). Example degrees of membership for a pixel (T, I, F) as pixel_memberships = [0.3, 0.4, 0.3], Neutrosophic entropy for the pixel is calculated as Neutrosophic Entropy: 1.0888999750452237. Calculating the neutrosophic entropy for the entire input image, entropy = neutrosophic_entropy(image_float), resulting as: Neutrosophic Entropy: [[[0.15173366 0.10860618 0.15933681], [0.13559628 0.08822363 0.14382856], [0.06517599 0. 0.07709462] ... [0.08822363 0.08822363 0.08822363], [0.08822363 0.08822363 0.08822363].

It is observed for those pixels having intensity values with equal probability, the entropy is maximum, and for pixels having low intensity the entropy values are low indicating nonuniform intensity distribution. Pixel at (10,60) has intensity: 2 resulting with low entropy ((10, 60), 0.8812909).

From the calculated entropy values, it is observed, when pixels having low intensity its entropy values are also low. Such pixels with low entropy values has nonuniform intensity distribution in the input image. Such finding helps in finding crucial image regions.

5.3. Entropy a Suitable Measure for Feature Selection in Forensic Face Feature Detection

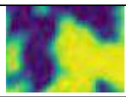
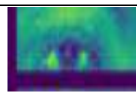
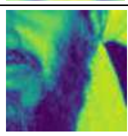
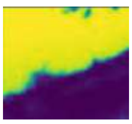
Entropy is a suitable measure for feature selection in this context due to several reasons. In handling uncertainty in image data, such as noise, occlusion, and variations in illumination and pose. Faces in images may exhibit varying degrees of uncertainty due to factors like partial occlusion or changes in lighting conditions. Neutrosophic entropy considers truth-membership, indeterminacy-membership, and falsehood-membership degrees, allowing it to capture uncertainty in feature selection more comprehensively compared to traditional entropy measures. Neutrosophic Entropy is robustness to Ambiguity in feature selection. Faces in images may appear differently due to variations in facial expressions, age, gender, and ethnicity, leading to ambiguity in feature representation. Neutrosophic entropy considers indeterminacy-membership degrees, which reflect the degree of ambiguity associated with feature selection. Neutrosophic entropy can integrate information from multiple modalities for feature selection. In face detection, different modalities such as color, texture, and shape may contain complementary information about facial features. Neutrosophic entropy allows for the fusion of information from these modalities while considering their respective truth-membership and indeterminacy-membership degrees, enabling more comprehensive feature selection.

The functionality of feature selection process using entropy for forensic face feature detection is described below. Load the image in grayscale, a function is defined to calculate entropy as follows: def calculate_entropy(image_patch): calculate histogram, hist = cv2.calcHist([image_patch], [0], None, [256], [0, 256]), normalize histogram, hist = hist / hist.sum(), calculate entropy for image patches, entropy = -np.sum(hist * np.log2(hist + np.finfo(float).eps)). Define the size of the image regions or patches, patch_size = 64, Iterate over the image and calculate entropy for each patch as follows:

```
for y in range(0, image.shape[0] - patch_size + 1, patch_size):
    for x in range(0, image.shape[1] - patch_size + 1, patch_size):
        patch = image[y:y+patch_size, x:x+patch_size]
        entropy = calculate_entropy(patch)
        patch_coordinates.append((x, y))
        patch_entropies.append(entropy)
```

Sort patches by entropy, sorted_patches = sorted(zip(patch_coordinates, patch_entropies), key=lambda x: x[1]), Select the patches with the highest entropy (or lowest, based on your needs), num_selected_patches = 10 # Adjust this value as needed, selected_patches = sorted_patches[-num_selected_patches:], Visualize or process the selected patches as needed, for (x, y), entropy in selected_patches: patch = image[y:y+patch_size, x:x+patch_size], cv2.rectangle(image, (x, y), (x+patch_size, y+patch_size), (0, 255, 0), 2) by drawing a rectangle around selected patches.

Table 3. Observation from calculated entropy values

Input	Parameters	High Entropy Features and values			Low Entropy Features and values		
Image	Degree of uncertainty	Greater uncertainty	((40, 30), 4.8966913)		More certainty	((130, 50), 5.8985624)	
Image	Feature selection	Important, variable information	((130, 70), 5.72637)		More constant color	((220, 110), -0.0)	
Image	Information content	More diverse, unpredictable information	((70, 80), 4.902795)		Constant, predictable information	((130, 50), 5.8985624),	

The entropy values calculated for patch coordinates for the figure: highlighted in red, has negative entropy values, with constant color, represents a kind of artifact or anomaly present in the image patch due to sensor noise or image corruption. The entropy values highlighted in blue, has high entropy values for the selected coordinates, signifies greater uncertainty in the image. The entropy values highlighted in light blue has low entropy values, representing more certainty facial feature an eyebrow. The entropy values highlighted in light green has high entropy values, representing more important, variable information such as mouth, mustache, part of nose, beard and ear tip.

5.3.1. Feature Selection based on Entropy values

Feature selection in image processing using neutrosophic sets is an approach that combines neutrosophic set theory with traditional feature selection techniques [39] to extract relevant information from images. First the input image is loaded. The function to calculate entropy is defined. the size of the image region is defined. To store region coordinates and entropy values a list is declared and initialized. The entropy is computed for each region of interest by iterating over the image. The entropy is calculated using equation (6). Histogram for the image_regions are calculated and normalized. The regions are sorted based on entropy values in ascending and descending order. The sorted regions are printed in the format of (region_coordinates, region_entropies). Upon selected regions a rectangle box is drawn to highlight the region.

$$\text{Entropy} = \text{sum}(\text{histogram} * \log2(\text{histogram}) \quad (5)$$

[((200, 100), -0.0), ((220, 110), -0.0), ((230, 110), -0.0), ((240, 140), -0.0), ((200, 100), -0.0), ((220, 110), -0.0), ((230, 110), -0.0), ((240, 140), -0.0), ((240, 50), 0.22194068), ((240, 50), 0.22194068), ((210, 110), 0.5293609), ((210, 110), 0.5293609), ((200, 110), 0.72192806),((110, 50), 6.121209), ((110, 80), 6.121209), ((200, 80), 6.1412086), ((200, 80), 6.1412086), ((130, 80), 6.1536603), ((130, 80), 6.1536603), ((140, 50), 6.156307), ((140, 50), 6.156307), ((170, 70), 6.161209), ((170, 70), 6.161209), ((210, 80), 6.2163067), ((210, 80), 6.2163067), ((10, 100), 6.308758), ((10, 100), 6.308758), ((170, 80), 6.3763065), ((170, 80), 6.3763065)]

Figure 13. Patch_coordinates and patch_entropies generated for input image in figure 14.

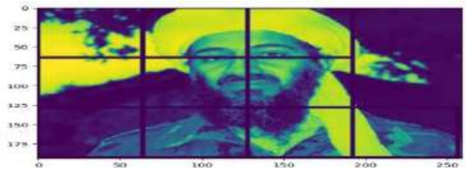
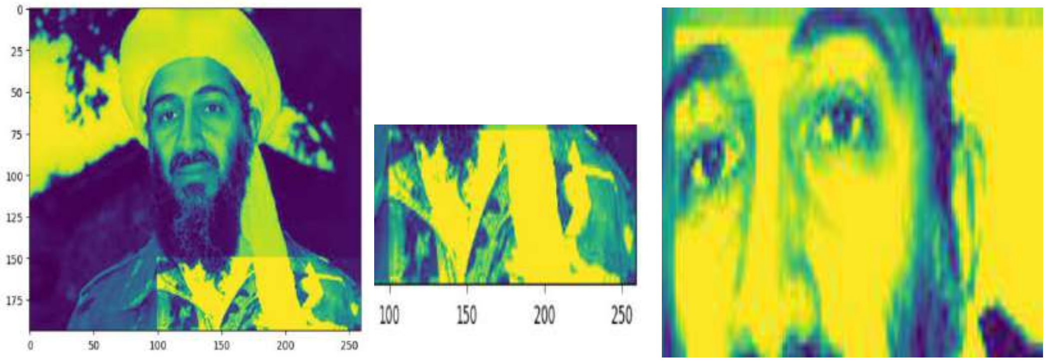


Figure 14. Patch_coordinates for input image

5.3.2. Region-Of-Interest (Roi) Enhancement Using Image Entropy

Region-of-Interest (ROI) enhancement involves applying image enhancement techniques to a specific region within an image. In this paper ROI enhancement is performed using OpenCV library. To enhance the region of interest selected first the image is loaded. The the coordinates (x, y) of the region to be enhanced is defined with its height and width factors as following: x, y, width, height = 100, 150, 200, 200. Using array slicing the selected region of interest is extracted. Then the enhancement technique is applied to the ROI. To enhance the ROI in this work colour contrast is selected. The colour contrast is increased using convertScaleAbs property of cv2.



(a) (b) (c)

Figure 15. Region-of-Interest enhancement using image entropy produces result as listed: (a) Increased color contrast for the region of interest: x, y, width, height = 100, 150, 200, 200; (b) Extracted region of interest using arrayslicing; (c) Color contrast increased using convertScaleAbs for the region of interest coordinates x, y, width, height = 100, 50, 200, 200

6. Distance-Based Similarity Measure For Face Recognition Using Neutrosophic Sets

In this paper for finding similarity between faces for face recognition in forensics investigation we use distance-based similarity measures using neutrosophic sets. Neutrosophic sets handle three types of indeterminacy: truth, indeterminacy, and falsity. Three types of distance-based similarity measures are used for face recognition using neutrosophic sets, such as Neutrosophic Euclidean Distance, Neutrosophic Cosine Similarity, Neutrosophic Manhattan Distance. The computations of these distances are given below.

6.1. Neutrosophic Euclidean Distance

Given two neutrosophic feature vectors A= (at,ai,af) and B= (bt,bi,bf). where at,ai,af and bt,bi,bf represent the truth, indeterminacy, and falsity degrees of feature A, and similarly for B, the neutrosophic Euclidean distance is computed as:

$$d(a,b) = \sqrt{(a_t - b_t)^2 + (a_i - b_i)^2 + \left(a_f - b_f\right)^2}$$

6.2. Neutrosophic Cosine Similarity

Given two neutrosophic feature vectors A and B, the neutrosophic cosine similarity is compu333ted as:

$$cos(a,b) = \frac{a_t \cdot b_t + a_i \cdot b_i + a_f \cdot b_f}{\sqrt{a_t^2 + a_i^2 + a_f^2} \cdot \sqrt{b_t^2 + b_i^2 + b_f^2}}$$

6.3. Neutrosophic Manhattan Distance

Given two neutrosophic feature vectors A and B, the neutrosophic Manhattan distance is computed as:

$$d(a,b) = |a_t - b_t| + \quad |a_i - b_i| + |a_f - b_f|$$

6.4. Example:

Suppose we have the following neutrosophic feature vectors: A=(0.8,0.1,0.1), B=(0.6,0.2,0.2) , Here, A has a truth degree of 0.8, an indeterminacy degree of 0.1, and a falsity degree of 0.1. Similarly, B has a truth degree of 0.6, an indeterminacy degree of 0.2, and a falsity degree of 0.2.

Calculate the neutrosophic Euclidean distance between A and B using equation given in 6.1.

$$d(a,b) = \sqrt{(0.8 - 0.6)^2 + (0.1 - 0.2)^2 + (0.1 - 0.2)^2}$$

$$d(A,B) = \sqrt{0.06}$$

d(A,B) ≈ 0.24494897427831783

Calculate the neutrosophic cosine similarity between A and B using equation given in 6.2.

$$cos(a,b) = \frac{0.8 \cdot 0.6 + 0.1 \cdot 0.2 + 0.1 \cdot 0.2}{\sqrt{0.64 + 0.1 + 0.1}.\sqrt{0.36 + 0.4 + 0.4}}$$

$$cos(a,b) = \frac{0.52}{\sqrt{0.66}.\sqrt{0.44}}$$

$$cos(a,b) = \frac{0.52}{\sqrt{0.2904}}$$

$$cos(a,b) \approx \frac{0.52}{0.5393544708}$$

cos(a,b) ≈ 0.964019

Calculate the neutrosophic Manhattan Distance between A and B using equation given in 6.3.

$$d(a,b) = |0.8 - 0.6| + \quad |0.1 - 0.2| + |0.1 - 0.2|$$

$$d(a,b) = |0.2| + \quad |0.1| + |0.1|$$

$$d(a,b) = 0.4$$

These measures are used in conjunction with Facenet[59] face recognition algorithm to improve the robustness and accuracy of face recognition systems over traditional methods. Traditional methods rely on handcrafted features or geometric features. However, facial expressions are highly complex with delicate patterns in facial muscles and configurations. Facenet a Deep learning-based approache in conjuction with Distance-based Similarity Measure USING NEUTROSOPHIC SETS can capture the intricate details and variations in facial expressions more effectively. The convolutional neural network based Facenet with neutrosophic Distance-based Similarity Measure trained on a dataset of facial images is capable of learning hierarchical representations related to facial expressions, such as wrinkles around the eyes.

Figure. 15. Captured wrinkles around the eyes.

7. Neutrosophic AHP for prioritizing criteria

Analytic hierarchy process (AHP) is utilized to evaluate and prioritize various factors related to the problem statement of this research. Neutrosophic AHP[43] is used in prioritizing critereas for Video enhancement. Priority Trade-offs are set for criteria's like "Detect Faces, Deocclude and Recognize Faces. Priority values are selected based on equal, moderate and unequal importance.

Table. 4. Criteria Priorities

	Detect Faces	Deocclude	Recognize Faces	Priorities
Detect Faces	1	2	2	0.49
Deocclude	0.5	1	2	0.312
Recognize Faces	0.5	0.5	1	0.198

Table.5. Metrics

Option Name	Priorities
Known	0.5
Unknown	0.5

The consistency of the pairwise comparisons are made using consistency Ratio (CR). According To Thomas L.Saaty if the CR exceeds a predefined threshold (e.g., 0.1), the comparisons need to be revised to improve consistency. We obtained a Consistency Ratio calculated as 0.046. Pairwise Comparisons of Options for Criteria is done using the function below:

Multi-Criteria Utility Function = 0.49 * [Detect Faces] + 0.31 * [Deocclude] + 0.2 * [Recognize Faces]

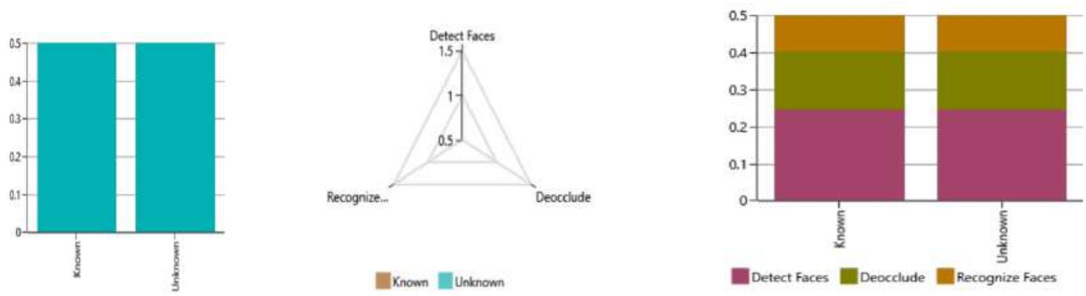


Figure. 16. a). Priorities b). Attributes c). Weighted Attributes

[Criterion : Recognize Faces]→[Option : Known]→[Value]

[Pair Comparison Against]→[Option : Unknown]

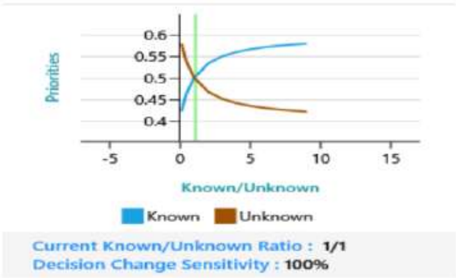


Figure. 17. Sensitive Variables

7.1. Indeterminacy Inherent Pixel Analysis Using Neutrosophic Analytic Hierarchy

Applying analytic hierarchy process (AHP) in neutrosophic sets to pixel analysis in our research aids in assessing and prioritizing various factors related to pixel-level characteristics and video frame processing tasks by taking into account uncertainties and indeterminacies in data. The objective is to optimize pixel analysis tasks such as Video enhancement, and denoise.

We begin by identifying two criteria's relevant to pixel analysis, such as video quality and noise.

The criteria is organized in a hierarchical structure, with the goal at the top. Each criterion is paired with its corresponding parent criterion. Using the neutrosophic pairwise comparison results, the priority weights for each criterion is calculated. The consistency of neutrosophic pairwise comparisons are assessed using indeterminacy, and falsity metrics. We then aggregate the neutrosophic priority weights of criteria using to determine the overall ranking and prioritize pixel analysis tasks. Weight calculation using approximate Eigenvector and Priority Calculation using Weighted Sum: Weighted Sum = ∑(Weight * Attribute). Sensitivity analysis is performed to evaluate the pairwise comparisons considering the indeterminacy inherent in the decision-making process following results were obtained.

Table. 6. Priority Tradeoffs

	video quality	noise	Priorities
video quality	1	9	0.9
noise	0.111	1	0.1

Table. 7. Comparisons for Criteria 'video quality'

video quality 0.9	Indeterminancy	falsity	Priorities
Indeterminancy	1	2	0.667
falsity	0.5	1	0.333

Table. 8. Pairwise for Criteria 'video quality'

noise 0.1	Indeterminancy	falsity	Priorities
Indeterminancy	1	1	0.5
falsity	1	1	0.5

Table. 9. Metrics

Option Name	Priorties
Indeterminancy	0.65
falsity	0.35

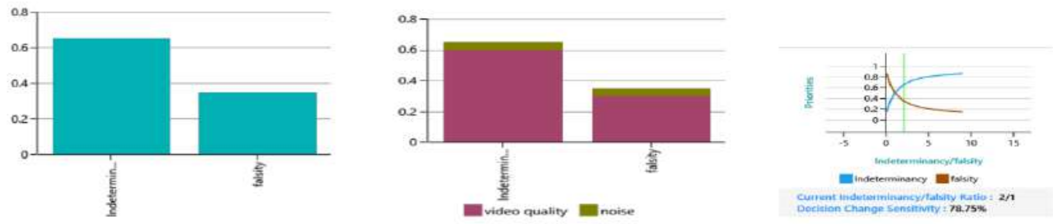


Figure.18. a). Priorities b). Weighted Attributes c). Sensitivity Variables

Multi-Criteria Utility Function = 0.9 * [video quality] + 0.1 * [noise] , Sensitive Variables (1): Criterion : video quality]→[Option : Indeterminancy]→[Value][Pair Comparison Against]→[Option : falsity]

8. Forensic Face Detection and Recognition Using Deep Learning

Face detection and recognition techniques are used in image and video analysis. They are widely used for making the crime investigation process easier. Existing face detection methods include, Face detection with Haar cascades which is extremely fast but prone to false-positives and in general less accurate than deep learning-based face detectors. Face detection with dlib (HOG and CNN) which is more accurate than Haar cascades but computationally more expensive. Dlib's CNN face detector is the most accurate of the bunch but cannot run in real-time without a GPU. Multi-task Cascaded Convolutional Networks (MTCNNs), which is very accurate deep learning-based face detector. MTCNNs is easily compatible with both Keras and TensorFlow.

8.1. Single Shot Detector

Single shot detector(SSD) is an object detection algorithm, it can detect multiple objects from a given image or a video stream in one shot. SSDs detect real time objects in video surveillance systems, selfdriving cars and Forensics. SSDs use a single pretrained network(CNN) to detect objects. The SSD functions upon ResNet base network. SSDs are better than R-CNN object detection algorithm which uses multiple networks and various stages to perform object detection. To this end, SSD is selected to detect objects in this work.

The CNN based ResNet model is chosen as the base network and SSD is made to work on top of it. The ResNets last layer which is a classification layer is truncated and convolutional feature layers are added. Convolutional predictors are used for detection. The model is compared with other object detection methods such as cf Overfeat[49] and YOLO[50] allows only single scale feature maps.

SSDs default boxes are similar to the anchor boxes used in Faster R-CNN [48]. The matching each ground truth box to the default box is made with the best jaccard overlap MultiBox [53]). Unlike MultiBox, in SSD default boxes are matched to any ground truth with jaccard overlap higher than a threshold (0.5) by simplifying the learning problem. The model loss is a weighted sum between localization loss, Smooth L1 [52] and confidence loss, Softmax.

The SSD training objective is derived from the MultiBox objective [53,54] but is extended to handle multiple object categories.

$$L(x,c,l,g) = \frac{1}{N}(L_{conf}(x,c) + \alpha L_{loc}(x,l,g))$$

If N = 0, the loss is set to 0. The localization loss is a Smooth L1 loss [51] between the predicted box (l) and the ground truth box (g) parameters. Similar to Faster R-CNN [48], we regress to offsets for the center (cx, cy) of the default bounding box (d) and for its width (w) and height (h). The confidence loss over multiple classes confidences(c).

$$L_{loc}(x,l,g) = \sum_{i \in Pos}^{N} \sum_{m \in \{cx,cy,w,h\}} x_{ij}^{k} \text{smooth}_{\text{L1}}(l_i^m - \hat{g}_j^m)$$

$$\hat{g}_j^{cx} = (g_j^{cx} - d_i^{cx})/d_i^w \qquad \hat{g}_j^{cy} = (g_j^{cy} - d_i^{cy})/d_i^h$$

$$\hat{g}_j^{w} = \log\left(\frac{g_j^w}{d_i^w}\right) \qquad \hat{g}_j^{h} = \log\left(\frac{g_j^h}{d_i^h}\right)$$

The confidence loss is the softmax loss over multiple classes confidences (c).

$$L_{conf}(x,c) = -\sum_{i \in Pos}^{N} x_{ij}^{p} log(\hat{c}_i^p) - \sum_{i \in Neg} log(\hat{c}_i^0) \quad \text{where} \quad \hat{c}_i^p = \frac{\exp(c_i^p)}{\sum_p \exp(c_i^p)}$$

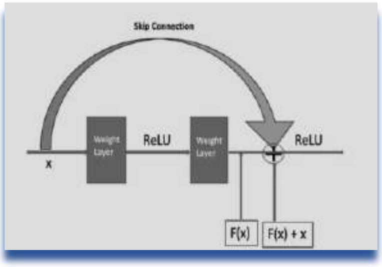
and the weight term α is set to 1 by cross validation.

The scale of the default boxes for each feature map is computed as: sk = smin + smax − smin m − 1 (k − 1), k ∈ [1, m] (49) where smin is 0.2 and smax is 0.9, meaning the lowest layer has a scale of 0.2 and the highest layer has a scale of 0.9, and all layers in between are regularly spaced.

$$s_k = s_{\text{min}} + \frac{s_{\text{max}} - s_{\text{min}}}{m-1}(k-1), \quad k \in [1,m]$$

8.2. Residual Neural Network

Deep networks often result in Gradients that vanishes as it is back-propagated to previous layers, repeated multiplication may make the Gradient infinitely small. ResNet[55] uses the concept of Residual blocks that include Skip connections.



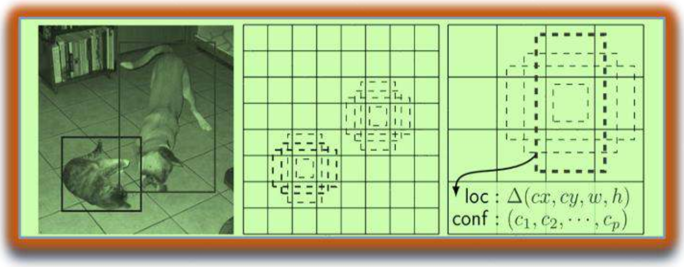


Figure:19. a). Residual Block **b).** Feature Map

Figure. 19.b) Image with Ground-Truth boxes, 8 × 8 feature map and 4 × 4 feature map [51])

8.3. ResNet-34

ResNets[55] can easily be optimized and can gain increased accuracy by relative depth increment. The model is trained using FaceScrub and VGGFace2 datasets and tested with labeled faces in the wild (LFW) data set. The aligned faces are passed to the ResNet model and it represent faces 128 dimensional vector.

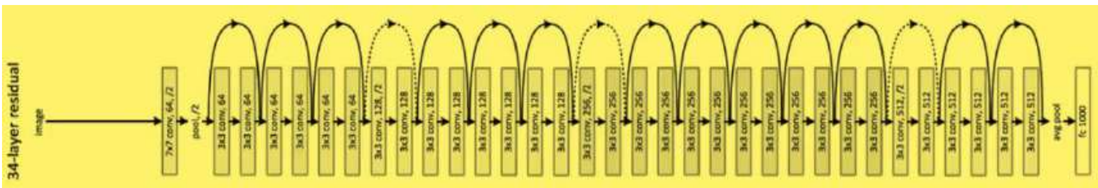


Figure. 20. ResNet Architecture

It consists of four residual blocks with three, four and six respectively. 64, 128, 256 and 512 Channels are used. Except for the first block, each block starts with a 3x3 kernel of stride of 2. Caffe deep learning framework is known for speed and its modularity[30]. Additionally, the network combines predictions from multiple feature maps with different resolutions to naturally handle objects of various sizes [31].

8.4. Implementation of Face detection Model

In this study for face detection, single shot detector framework with a ResNet base network is used which is trained using caffe deep learning framework. An image captured using webcam is displayed below. For detecting face from a captured image, a pre-trained face detection model built using ResNet base network, trained using caffe deep learning framework is used. The pre-trained face detection model, consists of ResNet base network definition and learned weights from caffemodel.

Initially the captured image is passed through the trained Res-Net base network for making detections and predictions. A loop construct is used to loop over the detections and to draw boxes around the detected faces. The confidence probability associated with the predictions are extracted. The minimum confidence threshold is selected to be 0.5. All the weak detections are filtered to ensure the `confidence` is greater than the minimum confidence threshold 0.5. If confidence is greater than minimum confidence threshold then the (x, y)-coordinates of the bounding box for the object is computed. The bounding box of the face along with the associated probability with (x, y) coordinates are drawn. The model being trained on different captured images has detected face accurately at 99.88% similarity.

(a) (b) (c)

Figure 21. Face detection using single shot detector framework with ResNet base network produces results as listed: (a) Captured image; (b) Face detected with 98.84% accuracy; (c) Face detected with 99.88% accuracy

8.5. Face Recognition Using Deep Learning

A facial recognition system[1] is a technology potentially capable of matching a human face from a digital image or a video frame against a database of faces. This can be done through an eyewitness or from digitally stored pictures. Facial recognition software uses specific points on an image, compares those points to the same points of images in a database and finds similarity percentage to recognize faces.

Face_recognition library is used in this paper for implementing face recognition module built using dlib. Dlib is a modern C++ toolkit containing machine learning algorithms and tools for creating complex software in C++ to solve real world problems. It is used in a wide range of applications including robotics, embedded devices, and large high performance computing environments [32]. To run Face_recognition module efficiently AMD Ryzen RTX 3050 GPU is used. The module operates on a facedataset which consists of both known and unknown face images. First an input image is loaded into the model, and the model generates 128 face encodings for the loaded image. Faces which are similar will have same encodings and for faces which are different the encodings will differ. The length of the encodings is verified. If the length of encodings is greater than one then the model has detected more than one face from the input image. If the length of the encoding is equal to zero then no faces are detected from the input video and hence the file is ignored. If the length of encodings is neither 0 nor greater than one then appends the base image name to known_names array and appends the image encodings to know_face_encodings array. First an unknown image is loaded to test the module for face recognition task.

8.5.1. Dlib - Face Recognition Library

Dlibs face recognition pipeline consists of 4 common stages: detect, align, represent and verify. Dlib is mainly inspired from a ResNet-34 model[55]. Davis E. King in his work modified the regular ResNet structure and dropped some layers and re-build a neural network consisting of 29 convolution layers. Accepts 150x150x3 size input and represent 128 dimensional vectors. Dlib is capable of finding 68 facial landmark points on the face including eyes, eye-brows, mouth, lips, jaw, chin, and nose. Face detection does not have to be applied for rectangle areas as Haar cascade does.

Using Dlib we are able, to do out-of-the-box face recognition as well. Sixty eight landmarks of dlib is shown below.

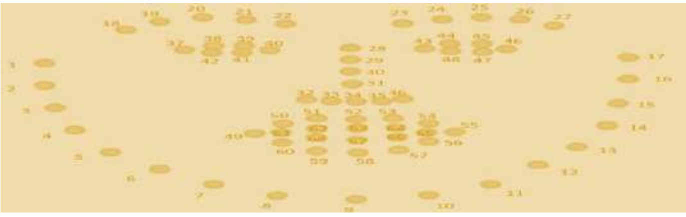


Figure. 22. Sixty Eight different landmarks of a face

8.5.2. Face Recognition implementation

The loaded image is preprocess to resize the image if its size is greater than 1600. The image is resized using thumbnail() resizing filter which takes a resampling filter called lanczos kernel as its parameter. It is the normalized sinc function sinc(x), windowed (multiplied) by the Lanczos window, or sin c window, which is the central lobe of a horizontally stretched sin c function sin c(x/a) for $-a \leq x \leq a$. After resizing, the image is converted into array of values. Next for unknown input image generate encodings. Calculate distance between known face encodings and unknown face encodings. The tolerance is set to 0.5. Then all images whose distances are less than or equal to tolerance are extracted. The details of image name and distance from all extracted images are retrieved. The image with less distance is considered to be more similar to the search image. The deep learning module has successfully detected and recognized faces. Face detection and recognition of Radullan Sahiron and Ahmad abousamra is successful by the face_recognition module as shown in figure 12.

Figure 23. Deep learning Face_recognition module: Detected and recognised image of Ahmad Abousamra with 98.84% accuracy.

Dlib finds representations in dlib.vector type, to find the Euclidean distance using tuned threshold between images we can convert it to numpy in order to find the similar faces with threshold value 0.5 which results in 99.88% confidence score on LFW data set. On the other hand, human beings hardly have 97.53% score on same dataset. Dlibs face recognition model competes with the other state-of-the-art face recognition models and human beings as well.

9. Advantages of Amd Ryzen Rtx 3050 Gpu in Face Detection and Recognition Task

The AMD Ryzen RTX 3050 GPU offers several advantages for face detection and recognition tasks, especially when paired with implementations like SSD (Single Shot Multibox Detector), ResNet (Residual Neural Network), and dlib. The proposed model with NSS, SSD, ResNet, and dlib implementations with GPU acclerations is capable of performing complex convolutions and extract facial features, by passing through deep ResNets, resulting in faster inference times. NVIDIA's optimized Tensor Cores, present in the RTX 3050 GPU, accelerates deep learning workloads which makes suitable for face detection and recognition. The RTX 3050 GPU offers improved performance over previous generations, with higher clock speeds, more CUDA cores, and enhanced memory bandwidth. This translates to faster inference speeds and better overall performance for face detection and recognition tasks. The deep learning frameworks, TensorFlow and PyTorch, have CUDA support. CUDA, is developed by NVIDIA for GPU-accelerated computing. By implementing the work in tensorflow we are able to leverage the GPU's processing power which results in enhanced performance of SSD, ResNet, and dlib implementations on the RTX 3050 GPU. RTX 3050 GPU with its VRAM (Video Random Access Memory) feature with its adequate memory capacity supports us to efficiently run face detection and recognition model with smaller datasets.The RTX 3050 GPU's scalability depends on the specific application requirements and dataset sizes. The GPU handles real-time inference for face detection and recognition tasks efficiently working with smaller to moderate-sized datasets. NVIDIAs RTX 3050 GPU combined with AMDs RDNA architecture, results in better energy efficiency compared to previous generations. The GPU consumes less energy which makes the model suitable to be deployed in systems with power consumption concern. The AMD Ryzen RTX 3050 GPU offers a promising solution for face detection and recognition tasks, particularly when used with SSD, ResNet, and dlib implementations. Its GPU acceleration, optimized tensor cores, increased performance, CUDA support, memory capacity, scalability, and energy efficiency make it well-applicable for small to medium datasets and potentially for forensics real-time applications.

10. Key Contributions

A Novel Filter is using Sliding Window Convolution is proposed for Image Denoising. Sliding window is also called as Rolling Window. Using Sliding window concept we first highlight the pixels to be operated by the kernel. Convolution is faster with sliding window since there is no loop. The creation of the rolling window does not take much time because no data is copied, it is just a view on the original matrix. An image affected with impulse noise is taken as input. For each sliding window first the pixels are sorted and the Mean(r) pixel is calculated. When r=1 the filter functions similar to median filter. If r=3 then novel filter function is invoked. The output is compared with median filter. The obtained result as in figure 11 has 99% similarity to that of median filter. For image and video frame sharpening a TripleMask Spatial Linear filter is proposed. TripleMask Spatial Linear filter is created by cascading a series of filters. Identity, Shift and Fraction based approach is used in Mask processing. For Image smoothing and to preserve the edges Bilateral filter is used [2]. The results obtained are shown in Figure.4. The performance of convolution operation is compared as in table.3 in terms of CPU time with distinct convolution, shift rotational convolution, sliding window convolution and scipy convolution. We address the Video quality issues issues involving shaky or unstable camera movements which leads to blurry footage. In our work we use a Hybrid model for video deblurring using Deep Video Deblurring for Hand-held Cameras[56], DEblurGAN[57] and Spatio-Temporal Transformer Networks for Video Deblurring[58] and proposed TripleMask spatial linear filter. Through the combined approach and leveraging the power of deep learning our model is capable to surpass traditional methods by performing a data-driven mapping from blurry to sharp frames, to learn complex motion patterns, to generate high-quality deblurred results, to obtain superior deblurring performance, mainly for dynamic scenes. Results produced by the combined approach using set of filters including TripleMask spatial linear filter, Median Filter and Bilateral filters are shown below. Sharpening the video frames at an amount of 0.50, the model enables to see clearly the objects in the frame. Deblurring the frame to see very minute details at an amount of 0.78. By applying median filter by setting mean filter threshold to 50, edge threshold 50, motion threshold 50 we get a more clear visual appearance. We obtain temporal softening with radius 5, luma threshold 4 and croma threshold 8 which causes a smoothing effect and removes fluctuations in pixel values over time. This helps reduce noise and improve the overall quality of the video. By turning right or left and changing the colors of the object it enables to visualize in different directions to further recognize the object. It smooths the video frame by preserving edges by considering both the spatial distance and the intensity difference between neighboring pixels. Median Filter: threshold:50, Edge threshold:50, Motion threshold:50. Temporal Soften: radius 10, luma threshold 10 and croma threshold 20 .

Figure. **24**. a). Sharpen scale:1.00 b). Blur scale:1.00 c). Median Filter

Figure. **25**. a). Temporal Soften b). TripleMask Spatial linear Soften

10.1. Video Frame processing and analysis using Histogram

A histogram is a graphical representation that displays the distribution of data. It consists of a series of vertical bars, where each bar represents the frequency or count of data values falling within a particular range or interval. Histograms are commonly used in statistics, data analysis, and image processing to visualize the distribution of numerical data. In our work we use histogram for video processing to represent the distribution of pixel intensities within a frame. Each pixel in a frame has an intensity value, which can range from 0 (black) to 255 (white) in an 8-bit grayscale image. The histogram displays the frequency of occurrence of each intensity value within the image.

Histogram representations are used to perform analysis and understand the image in terms of its colour, contrast, and brightness. Each pixel in an image has red, green, and blue colour components, and the red channel histogram shows the frequency of different red intensity levels present in the image. Green channel histogram shows the frequency of different green intensity levels present in the image. Blue channel histogram shows the frequency of different blue intensity levels present in the image. Histogram representations are used in this research work to assess color balance, identify color casts, adjust pixel intensity levels to obtaing correct color balance and to achieve more accurate color reproduction.

Figure. **26**. Original Frame

Figure. **27**. Low:120, Gamma: 1.76 High: 144 and blue histogram

Figure. **28**. Low:166, Gamma: 3.16 High: 223 and Green histogram

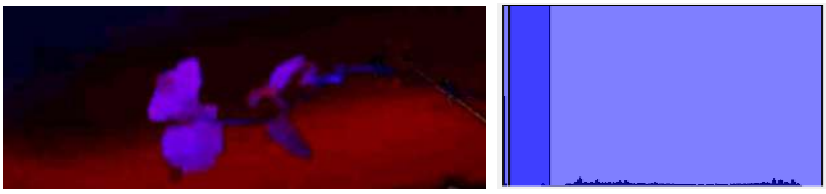


Figure. **29.** Low:38, Gamma: 1.16 High: 6 and Red histogram

10.2. Traditional entropy measures

Traditional entropy measures come from information theory and statistical mechanics and are used to quantify the uncertainty or disorder in a system. The most common traditional entropy measures include:

10.2.1 Shannon Entropy (Information Entropy): Introduced by Claude Shannon in his seminal work on information theory, Shannon entropy quantifies the uncertainty associated with a random variable. It is defined as the average amount of information produced by a random variable. Mathematically, for a discrete random variable X with probability mass function p(x), Shannon entropy H(X) is given by: $H(X) = -\Sigma\ p(x) * \log2(p(x))$

10.2.2 Boltzmann Entropy: In statistical mechanics, Boltzmann entropy is a measure of the microscopic disorder or randomness of a system. It's defined as: $S = kB * \ln(W)$

Where S is the entropy, kB is the Boltzmann constant, and W is the number of microstates corresponding to a given macrostate.

10.2.3. Gibbs Entropy: In thermodynamics, Gibbs entropy is a measure of the disorder in a system at equilibrium.

It's defined as: $S = -k * \Sigma\, pi * \ln(pi)$, Where S is the entropy, k is the Boltzmann constant, and pi is the probability of the system being in the i-th microstate.

10.2.4 Renyi Entropy: It is a generalization of Shannon entropy and Boltzmann entropy, which introduces a parameter α. When α approaches 1, Rényi entropy converges to Shannon entropy.

It's defined as: $H\alpha(X) = 1/(1-\alpha) * \log2(\Sigma\, p(x)^\wedge\alpha)$

Where p(x) is the probability mass function of the random variable X.

10.3. *Selection Of Neutrosophic Entropy in Feature Selection Instead of Traditional Entropy Measures*:

Neutrosophic entropy considers truth-membership, indeterminacy-membership, and falsehood-membership degrees, allowing it to capture uncertainty in feature selection more comprehensively compared to traditional entropy measures. For performing Image/Video frame analysis and for making feature selection in this work Neutrosophic Entropy is used. For forensic face feature detection Neutrosophic Entropy values are used. The input image is loaded in grayscale. Image patches are created. For each image patch Histogram values are created using calcHist method of opencv.

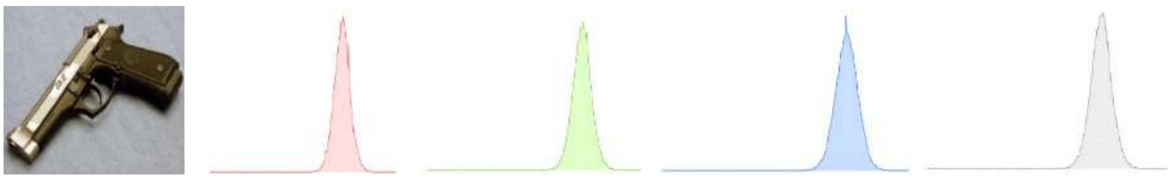


Figure. 30. **Image, Histogram: Red, Green, Blue and Luminosity**

The calculated histogram is normalized using equation, x=x/x.sum(). Then entropy values are calculated for each patch based on its histogram created using the following calculation.entropy = (hist * np.log2(hist + np.finfo(float).eps)). The patches with the highest entropy and lowest entropy is selected, visualizes or processed to detect face features.

Histogram analysis is often used for automatic or manual determination of threshold values for image segmentation. In this paper thresholding is applied to identify indeterminate regions. An indeterminacy_filter is defined to work the input image with threshold =100. Then the binary image is inverted to get indeterminate regions. Neutrosophic_entropy is calculated even for checking individual pixel_memberships >0, initialy entropy is set to be 0.0 and a small constant epsilon, to avoid taking the logarithm of zero is used, where epsilon = 1e-10. for membership in pixel_memberships, if membership > 0 then entropy is calculated as entropy -= membership * np.log(membership + epsilon). It is observed for those pixels having intensity values with equal probability, the entropy is maximum, and for pixels having low intensity the entropy values are low indicating nonuniform intensity distribution. Pixel at (10,60) has intensity: 2 resulting with low entropy ((10, 60), 0.8812909). From the calculated entropy values, it is observed, when pixels having low intensity its entropy values are also low. Such pixels with low entropy values has nonuniform intensity distribution in the input image. Such finding helps in finding crucial image regions. By using distance-based similarity measures using neutrosophic sets in face recognition we are able to find uncertainty and imprecision inherent in facial feature representations. Region-of-Interest enhancement is performed using image entropy. The produced result shows Increased color contrast for the region of interest: x, y, width, height = 100, 150, 200, 200, Extracted region of interest using arrayslicing, Color contrast increased using convertScaleAbs for the region of interest coordinates x, y, width, height = 100, 50, 200, 200 in figure 15. Face detection using single shot detector framework with ResNet base network has produced results as shown in figure. 22, Captured image, Face detected with 98.84% accuracy and Face detected with 99.88% accuracy. Deep learning Face_recognition module result is shown in figure .23 , Detected and recognised image of Ahmad Abousamra with 98.84% accuracy.

11. Future Research Directions

Neutrosophic sets can be employed to delineate image regions or objects based on their degrees of truth, indeterminacy, and falsity. As a challenging task neutrosophic set can be used in uncertain boundaries of images in the process of image segmentation. Neutrosophic entropy value can provide information about how much information is contained in the neutrosophic set representation of the image. Neutrosophic set entropy can be used in image segmentation and clustering oriented problems. Researchers can apply neutrosophic set for detecting faces from a video. Neutrosophic methods can be employed in risk assessment and management, especially when dealing with complex and uncertain risks in fields like insurance and project management. Neutrosophic logic can be integrated into AI and ML models to handle uncertain or contradictory data. Neutrosophic approaches can be used to assess and manage environmental data that is often uncertain or incomplete. This can be particularly used in climate modeling and ecological studies. In our work we have only used pre-recorded videos for processing and to detect faces and recognize them from a set of manually collected videos using AMD Ryzen RTX 3050 GPU with limited memory capacity. But in future the proposed framework can be extended to work with real time surveillance systems to detect faces and recognize for forensic investigations. Also high end GPUs can be used instead of AMD Ryzen RTX 3050 GPU to make the system compatible to work with dynamic large scale datasets.

12. Conclusions

Video quality issues such as blurry, speckled, pixelated and low-resolution videos captured at low light are a real challenge in forensic analysis. Such issues are addressed in this research using a set of algorithms and techniques. For denoising images, a novel image filter using sliding window convolution is proposed and used. For image sharpening a TripleMask Spatial Linear filter is proposed and applied. TripleMask Spatial Linear filter is created by cascading a series of filters. Identity, Shift and Fraction based approach is used in Mask processing. For Image smoothing and to preserve the edges Bilateral filter is used [2]. The performance of convolution operation is compared with distinct convolution, shift rotational convolution and scipy convolution. To handle uncertainty, imprecision, and ambiguity in real-world image data in a precise manner neutrosophic science is used in image analysis. By the generated Neutrosophic set of the given input image ambiguous regions in the image are detected. Feature selection is made by calculating the entropy of different image regions. From the generated Neutrosophic set entropy the degree of uncertainty, within the input image is quantified. The intensity distributions are measured using entropy values. In Feature selection regions with highest and lowest entropy values containing face images are selected, visualized and processed to further aid in forensic analysis in detecting the culprits. Face detection is performed using single shot detector framework with a ResNet base network, trained using caffe deep learning framework. Face recognition process is performed using dlibs [9] state-of-the-art face recognition model built with deep learning [10]. The model was successfully implemented using AMD Ryzen RTX 3050 GPU. The model has an accuracy of 99.38% on the Labeled Faces in the Wild benchmark.

Funding: "This research received no external funding"

References

1. K. Sita Manikyam, Forensic Science and its Limitations in Rape and Murder Cases in India, March 2023, *Journal of Forensic Science and Medicine* 9(1):91-97
2. Sylvain Paris; Pierre Kornprobst; Jack Tumblin; Frédo Durand, Bilateral Filtering: Theory and Applications, now, 2009.
3. N. Dalal and B. Triggs, "Histograms of oriented gradients for human detection," San Diego, CA, USA, 2005, pp. 886-893 *vol. 1*, doi: 10.1109/CVPR.2005.177.
4. Kazemi, Vahid ,One Millisecond Face Alignment with an Ensemble of Regression Trees, 2014/06/01
5. Hassan, Rondik, Deep Learning Convolutional Neural Network for Face Recognition: A Review, 2021/01/27
6. Triantafyllidou, Danai & Tefas, Anastasios, 3560-3565. 10.1109/ICPR.2016.7900186.
7. FaceNet, https://doi.org/10.48550/arXiv.1503.03832
8. Brandon amos,Bartosz Ludwiczuk, Mahadev Satyanarayanan, OpenFace: A general-purpose face recognition library with mobile applications, 1 Jan 2016
9. Davis E. King. Dlib-ml: A Machine Learning Toolkit.
10. Adam Geitgey, Face Recognition, Copyright 2017, Adam Geitgey Revision 2e2dccea.
11. Geitgey, *A.,Deep Learning: Face Recognition, BOOK*, 2018.

12. Mark M. Pollitt, Handbook of Digital and Multimedia Forensic Evidence, 2008, ISBN : 978-1-58829-782-2
13. Forensic science services and the criminal justice system, published Friday, 12 November, 2021
14. The Guardian, Forensic Science Failures putting jusice at risk, February 2020
15. Graeme Horsman, Digital evidence and the crime scene, Science & Justice, Volume 61, Issue 6, 2021, Pages 761-770, ISSN 1355-0306.
16. Video Evidence – A Law Enforcement Guide to Resources and Best Practices , March 2014.
17. Abdul Rehman Javed, Zunera Jalil, Wisha Zehra, Thippa Reddy Gadekallu, Doug Young Suh, Md. Jalil Piran, A survey on digital video forensics, *Volume 106*, 2021, 104456, ISSN 0952-1976.
18. Sowmya K.N., H.R. Chennamma, A Survey On Video Forgery Detection, arXiv:1503.00843
19. J. Xiao et al.: Video-Based Evidence Analysis and Extraction in Digital Forensic Investigation
20. Kim, J., Lee, D. & Park, N. Multimedia Tools Appl 79, 23461–23481 (2020). https://doi.org/10.1007/s11042-020-09016-z
21. Kimberley A. Wade, Sarah L. Green, Robert A. Nash, Can fabricated evidence induce false eyewitness testimony? Applied Cognitive Psychology, 24(7), 899-908 - August
22. 2009K. A. Wade, S. L. Green, & R. A. Nash, from the archive: ‘can fabricated evidence induce false eyewitness testimony?’ by (2010), 24, 899–908 with commentary
23. JNina J. Westera - Griffith University, Martine B. Powell - Deakin University, Prosecutors’ perceptions of how to improve the quality of evidence in domestic violence cases Policing & Society, 27(2), 157-172 - June 2015, https://doi.org/10.1080/10439463.2015.1039002
24. Nina J Westera - Griffith University, Lost in the detail: Prosecutors' perceptions of the utility of video recorded police interviews as rape complainant evidence, Journal of Criminology, 50(2), 252-268 - December 2015, https://doi.org/10.1177/0004865815620705
25. Annamaria Motrescu-Mayes, Susan Aasman, Amateur Media and Participatory Cultures, Film, Video, and Digital Media, January 2019, https://doi.org/10.4324/9781315396149
26. https://en.wikipedia.org/wiki/Kernel_%28image_processing%29
27. https://web.pdx.edu/~jduh/courses/Archive/geog481w07/Students/Ludwig_ImageConvolution.pdf
28. https://www.cs.cornell.edu/courses/cs1114/2013sp/sections/S06_convolution.pdf
29. https://docs.scipy.org/doc/scipy/tutorial/ndimage.html
30. Yangqing Jia, Evan Shelhamer, Jeff Donahue, Sergey Karayev, Jonathan Long, Ross Girshick, Sergio Guadarrama, Trevor Darrell, Caffe: Convolutional Architecture For Fast Feature Embedding, https://doi.org/10.48550/arXiv.1408.5093
31. Wei liu, Dragomir anguelov, Dumitru erhan, Christian szegedy, Scott reed, Cheng-yang fu, A, SSD: Single Shot MultiBox Detector, https://docs.scipy.org/doc/scipy/tutorial/ndimage.html
32. Davis E. King. Dlib-ml: A Machine Learning Toolkit, Journal of Machine Learning Research, 2009, *volume 10*, pages 1755-1758.
33. Haitham A. El-Ghareeb: Novel Open Source Python Neutrosophic Package, *Neutrosophic Sets and Systems, vol. 25*, 2019, pp. 136-160, http://fs.unm.edu/NSS/NovelOpenSourcePython.pdf.

34. M. R. Faraji and Xiaojun Qi, "An effective neutrosophic set-based preprocessing method for face recognition," 2013 IEEE International Conference on Multimedia and Expo Workshops (ICMEW), San Jose, CA, USA, 2013, pp. 1-4, doi: 10.1109/ICMEW.2013.6618251.

35. A neutrosophic approach to image segmentation based on watershed method, *Signal Processing, Volume 90*, Issue 5, 2010, Pages 1510-1517, ISSN 0165- 684.

36. Mohammad Reza Faraji and Xiaojun Qi, An Effective Neutrosophic Set-Based Preprocessing Method For Face Recognition, Http://fs.unm.edu/NeutrosophicPreprocessing.pdf

37. A. Salama, Florentin Smarandache and Mohamed Eisa, Introduction to Image Processing via Neutrosophic Techniques, Neutrosophic Sets and Systems, Vol. 5, 2014.

38. Bolón-Canedo, V., Remeseiro, B. Feature selection in image analysis: a survey. Artif Intell Rev 53, 2905–2931 (2020). https://doi.org/10.1007/s10462-019-09750-3

39. Broumi, S., S. krishna Prabha, & Vakkas Uluçay. (2023). *Neutrosophic Systems With Applications*, *11*, 1–10. https://doi.org/10.61356/j.nswa.2023.83

40. Norzieha Mustapha1 , Suriana Alias2*, Roliza Md Yasin3 , Ilyani Abdullah4 and Said Broumi, cardiovascular diseases risk analysis using distance-based similarity measure of Neutrosophic Sets

41. Jdid, M., & Smarandache, F. (2023). Optimal agricultural land use, using Neutrosophic Linear Programming Method. *Neutrosophic Systems With Applications*, *10*, 53–59. https://doi.org/10.61356/j.nswa.2023.76

42. Mohamed, M., Karam M. Sallam, & Ali Wagdy Mohamed. (2023). Innovations of Industry 5.0 Technologies in smart supply chain partners. *Neutrosophic Systems With Applications*, *10*, 1–11. https://doi.org/10.61356/j.nswa.2023.74

43. Ahmed A. El-Douh, SongFeng Lu, Ahmed Abdelhafeez, Ahmed M. Ali, & Alber S. Aziz. (2023). Heart disease prediction under machine learning and association rules under Neutrosophic Environment. *Neutrosophic Systems with Applications*, *10*, 35–52. https://doi.org/10.61356/j.nswa.2023.75

44. Vakkas, U., Kılı¸c, Sahin, M., Deniz, H. Distance-based similarity measure for refined neutrosophic sets and its application in Medical Diagnosis, Matematika, 2019, 35(1), 83–96.

45. Guleria, A., Srivastava, S., Bajaj. R.,K. Application in Decision-making Models, Neutrosophic Sets and Systems, 2019, 29, 101–120

46. Broumi, S.; Smarandache, F. Several Similarity Measures of Neutrosophic Sets, Neutrosophic Sets And Systems, 2013, 1(1), 54–62.

47. Majumdar, P.; Samanta, S. K. On Similarity and Entropy of Neutrosophic Sets, Journal of Intelligent and Fuzzy Systems, 2013, I, 1–13.

48. Ren, S., He, K., Girshick, R., Sun, J.: Faster R-CNN: Towards real-time object detection with region proposal networks. In: NIPS. (2015)

49. Sermanet, P. Eigen, D. Zhang, X. Mathieu, M. Fergus, R. LeCun, Y. Overfeat, Integrated recognition, localization and detection using convolutional networks. In: ICLR. (2014)

50. Redmon, J., Divvala, S., Girshick, R., Farhadi, A.: You only look once: Unified, real-time object detection. In: CVPR. (2016)

51. Wei Liu, Dragomir Anguelov , Dumitru Erhan , Christian Szegedy , Scott Reed , Cheng-Yang Fu, Alexander C. Berg, SSD: Single Shot MultiBox Detector

52. Girshick, R.: Fast R-CNN. In: ICCV. (2015)
53. Erhan, D., Szegedy, C., Toshev, A., Anguelov, D.: Scalable object detection using deep neural networks. In: CVPR. (2014)
54. Szegedy, C., Reed, S., Erhan, D., Anguelov, D.: Scalable, high-quality object detection. arXiv preprint arXiv:1412.1441 v3 (2015)
55. [1512.03385] Deep Residual Learning for Image Recognition - arXiv.org, DOI: https://doi.org/10.48550/arXiv.1512.03385
56. Seungjun Nah Tae Hyun Kim Kyoung Mu Lee, Deep Multi-scale Convolutional Neural Network for Dynamic Scene Deblurring Department of ECE, ASRI, Seoul National University, CVPR 2017
57. Orest Kupyn, DeblurGAN: Blind Motion Deblurring Using Conditional Adversarial Networks, 2018
58. Jinshan Pan et al., Deep Discriminative Spatial and Temporal Network for Efficient Video Deblurring Nanjing University of Science and Technology.
59. Schroff, Florian, FaceNet: Unified Embedding for Face Recognition and Clustering, DOI: 10.1109, 2015, 7298682

Received: Feb 2, 2024. Accepted: April 26, 2024

Single-Valued Pentapartitioned Neutrosophic Soft Set

Bimal Shil[1], Suman Das[2], Rakhal Das[3], and Surapati Pramanik[4,*]

[1]Department of Statistics, Tripura University, Agartala, 799022, Tripura, India.
Email: bimalshil738@gmail.com
[2]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.
Email: sumandas18842@gmail.com, dr.suman1995@yahoo.com

[3]Department of Mathematics, The ICFAI University, Kamalghat, Agartala, 799210, Tripura, India.
Email: rakhaldas95@gmail.com

[4]Department of Mathematics, Nandalal Ghosh B.T. College, Narayanpur, 743126, West Bengal, India.

Email: surapati.math@gmail.com
*** Correspondence:** surapati.math@gmail.com

Abstract:

Soft set (SS) and neutrosophic set (NS) are important mathematical concepts to deal with uncertainty. NS was further extended to pentapartitioned neutrosophic set (PNS) to deal with uncertainty comprehensively. In order to expand the concept of single-valued pentapartitioned neutrosophic set (SVPNS) and SS, the paper aims to introduce the concept of single-valued pentapartitioned neutrosophic soft set (SVPNS-Set) by adding the ideas of SS and SVPNS together. Furthermore, based on SVPNS-Set, several definitions, examples, properties, propositions and theorems have been established.

Keywords: Single-Valued Neutrosophic Set; SVPN-Set; SVPNS-Set.

1. Introduction:

In 1999, Molodtsov [1] grounded the idea of SS theory. Afterwards, Ali et al. [2] presented some new operations on SS. In 2002, Maji et al. [3] proposed a decision making strategy under the SS environment. SS theory was further studied by Maji et al. [4] in 2003. In 1998, Smarandache [5] introduced the notion of NS by combining the notions of Fuzzy Set (FS) [6] and Intuitionistic FS (IFS) [7]. Afterwards, the concept of bipolar NS was introduced by Deli et al. [8] in 2015. Biswas et al. [9] established the multi-attribute decision making (MADM) strategy using entropy and Grey Relational Analysis (GRA) in the context of NS theory. Later on, Biswas et al. [10] presented the MADM strategy dealing with unknown weight information in the NS environment. Pramanik et al. [11] presented the cross-entropy based MADM strategy in the NS setting. Later on, Maji [12] presented the idea of Single-Valued Neutrosophic Soft Set (SVNS-Set) in the year 2013 by combining the notions of NS and SS. Thereafter, Maji [13] further studied SVNS-Set in 2013. Many researchers around the globe proposed MADM strategies such as TOPSIS [14], GRA [15, 16, 17], etc. Later on, Karaaslan [18] presented two algorithms for group decision making in SVNS-Set environment. Das et al. [19] presented a group decision making strategy based on neutrosophic soft matrix (NSM) and relative weights of experts applying the notion of SVNS-Set. Jha et al. [20] applied SVNS-Set to present a stock tending analysis. Pramanik et al. [21] proposed the TOPSIS based MADM strategy under the single-valued neutrosophic soft expert set environment. Subsequently, the neutrosophic bipolar vague soft set was introduced by Mukherjee and Das [22], who also suggested a MADM strategy for this setting. In 2017, Bera and Mahapatra [23] grounded the idea of topology on SVNS-Sets. Later on, Bera and Mahapatra [24] further studied the neutrosophic soft topological space. Thereafter, the notion of separation axioms via neutrosophic soft topological space was introduced by Aras et al. [25] in 2019. Neurosophic soft compactness via neutrosophic simply soft open set was introduced by Das and Pramanik [26] in 2020. Mehmood et al. [27] grounded the concept of neutrosophic soft α-open set via neutrosophic soft topological space. Smarandache [28] expanded the concept of the SS in 2018 to include the hyper SS and plithogenic hyper SS.

NS and multi-valued neutrosophic refined logic [30] were expanded in 2020 by Mallick and Pramanik [29] by including contradiction, ignorance, and unknown components in order to handle indeterminacy and uncertainty thoroughly. This gave rise to the PNS. Das et al. [31] proposed the GRA based MADM strategy under the PNS setting. Das et a1. [32] proposed the tangent similarity measure based MADM strategy in the PNS setting. Pramanik [33] presented the ARAS strategy in the PNS environment. Later on, Majumder et al. [34] established a MADM strategy based on the hyperbolic tangent similarity measure to determine the most significant environmental risks during the COVID-19 pandemic. Das and Tripathy [36] introduced the concept of topology on PNSs. Afterwards, Das et al. [35] grounded the idea of Q-algebra on PNSs, and introduced pentapartitioned neutrosophic Q-algebra and pentapartitioned neutrosophic Q-Ideal.

Research gap: Several studies of PNS, NS, single-valued NS [37], SVNS-Set and their applications have been depicted in [38]. However, no studies have been reported on the combination of PNS and soft set to deal with uncertainty. This research is important as the combination of PNS and soft set is more powerful in dealing with real problems with uncertainty over the existing NS.

Motivation: Having realizing the advantage and to address the research gap, we initiate to combine the PNS and SS which we call the notions of SVPNS-Set.

This article's primary goal is to define the concept of SVPNS-Set and outline its various properties.

This article's remaining portion is structured as follows:

Section-2 goes over the fundamental definitions and characteristics of NS, SVNS-Set and SVPNS. The concept of SVPNS-Set and its characteristics are introduced in Section-3. In section-4, we finally wrap up the paper by outlining some potential areas for further research.

2. Some Preliminary Results:

This section includes some fundamental definitions and findings on SVNS-Set and SVPN-Set that are pertinent to the article's main findings.

Definition 2.1.[15] Suppose that P be a collection of parameters. Assume that NS(Û) be the family of all NSs defined over a fixed set Û. Then, for any $S\subseteq P$, a pair (N, S) is referred to as a SVNS-Set over Û, where N is a function from S to NS(Û).

An SVNS-Set (N, S) is defined as follows:

$$(N, S)=\{(b, \{(l, Ý_{N(b)}(l), Ī_{N(b)}(l), Ŕ_{N(b)}(l)): l\in Û\}): b\in P, l\in Û\},$$

where $Ý_{N(b)}(l)$, $Ī_{N(b)}(l)$, $Ŕ_{N(b)}(l)$ are the degrees of truth membership function, indeterminacy membership function and false membership function of each $c\in$Û with respect to the parameter $b\in P$.

Example 2.1. Suppose that Û={$\wp*_1$, $\wp*_2$, $\wp*_3$} be a set of three mobiles, and S={b_1(looks), b_2(RAM), b_3(cost)} be a family of parameters with respect to which the nature of mobile will be described. Assume that $K(b_1)$ = {($\wp*_1$, 0.6, 0.5, 0.5), ($\wp*_2$, 0.3, 0.8, 0.5), ($\wp*_3$, 0.5, 0.3, 0.4)}, $K(b_2)$ = {($\wp*_1$, 0.7, 0.4, 0.6), ($\wp*_2$, 0.6, 0.5, 0.4), ($\wp*_3$, 0.7, 0.3, 0.3)}, $K(b_3)$ = {($\wp*_1$, 0.8, 0.5, 0.4), ($\wp*_2$, 0.7, 0.8, 0.5), ($\wp*_3$, 0.5, 0.3, 0.6)} be three single-valued NSs over Û. Then, (K, S) = {(b_1, $K(b_1)$), (b_2, $K(b_2)$), (b_3, $K(b_3)$)} is an SVNS-Set over Û with respect to the set S.

Definition 2.2.[15] The complement (K^c, S) of an SVNS-Set (K, S) is defined as follows:

$$(K^l, S) = \{(b, \{(l, 1\text{-}Ý_{K(b)}(l), 1\text{-}Ī_{K(b)}(l), 1\text{-}Ŕ_{K(b)}(l)): l\in Û\}): b\in S\}.$$

Definition 2.3.[15] Assume that (K_1, S) and (K_2, S) are any two SVNS-Sets over Û. Then, (K_1, S) is referred to as a SVNS-Set of (K_2, S) if and only if $Ý_{K_1(§)}(\wp*) \le Ý_{K_2(§)}(\wp*)$, $Ī_{K_1(§)}(\wp*) \ge Ī_{K_2(§)}(\wp*)$ and $Ŕ_{K_1(§)}(\wp*) \ge Ŕ_{K_2(§)}(\wp*)$, for all $§\in S$ and $\wp*\in$Û. One may write, (K_1, S) $\subseteq$ (K_2, S). Then, (K_2, S) is referred to as single-valued neutrosophic soft super-set of (K_1, S).

Definition 2.4.[15] Suppose that (K_1, S) and (K_2, S) are any two SVNS-Sets over Û. Then, union of (K_1, S) and (K_2, S) is denoted by (K, S), where $K=K_1\cup K_2$ is defined as follows:

$$(K, S) = \{(§, \{(l, Ý_{K(§)}(l), Ī_{K(§)}(l), Ŕ_{K(§)}(l)): l\in Û\}): §\in S\},$$

where $Ý_{K(§)}(l) = \max\{Ý_{K_1(§)}(l), Ý_{K_2(§)}(l)\}$, $Ī_{K(§)}(l) = \min\{Ī_{K_1(§)}(l), Ī_{K_2(§)}(l)\}$ and $Ŕ_{K(§)}(l) = \min\{Ŕ_{K_1(§)}(l), Ŕ_{K_2(§)}(l)\}$.

Definition 2.5.[15] Suppose that (K_1, S) and (K_2, S) are any two SVNS-Sets over Û. Then, intersection of (K_1, S) and (K_2, S) is denoted by (K, S), where $K=K_1\cap K_2$ is defined as follows:

$$(K, S) = \{(§, \{(l, Ý_{K(§)}(l), Ī_{K(§)}(l), Ŕ_{K(§)}(l)): l\in Û\}): §\in S\},$$

where $Ý_{K(§)}(l) = \min\{Ý_{K_1(§)}(l), Ý_{K_2(§)}(l)\}$, $Ī_{K(§)}(l) = \max\{Ī_{K_1(§)}(l), Ī_{K_2(§)}(l)\}$, and $Ŕ_{K(§)}(l) = \max\{Ŕ_{K_1(§)}(l), Ŕ_{K_2(§)}(l)\}$.

Definition 2.6.[15] An SVNS-Set (N, S) over a fixed set Û is referred to as a null SVNS-Set if $Ý_{N(g)}(l)=0$, $Ī_{N(g)}(l)=1$, $Ŕ_{N(g)}(l)=1$, $\forall l\in$Û with respect to the parameter $g\in S$. The null SVNS-Set may be denoted by $0_{(N, S)}$.

Definition 2.7.[15] An SVNS-S (N, S) over a fixed set Û is called an absolute SVNS-Set if $Ý_{N(g)}(l)=1$, $Ī_{N(g)}(l)=0$, $Ŕ_{N(g)}(l)=0$, $\forall l\in$Û with respect to the parameter $g\in S$. The absolute SVNS-Set may be denoted by $1_{(N, S)}$. Clearly, $1^c{}_{(N, S)} = 0_{(N, S)}$ and $0^c{}_{(N, S)} = 1_{(N, S)}$.

Definition 2.8.[29] A SVPNS D over a fixed set Û is defined as follows:

$$D = \{(\eth, Ý_D(\eth), Ć_D(\eth), Ź_D(\eth), Ú_D(\eth), Ŕ_D(\eth)) : \eth \in Û\},$$

where $Ý_D$: Û→[0, 1], $Ć_D$: Û→[0, 1], $Ź_D$: Û→[0, 1], $Ú_D$: Û→[0, 1], $Ŕ_D$: Û→[0, 1] are respectively referred to as the truth membership function, contradiction membership function, ignorance membership function, unknown membership function and false membership function, such that

$$0 \le Ý_D(\eth) + Ć_D(\eth) + Ź_D(\eth) + Ú_D(\eth) + Ŕ_D(\eth) \le 5, \text{ for all } \eth \in Û.$$

Definition 2.9.[29] Assume that $X = \{(\eth, Ý_X(\eth), Ć_X(\eth), Ź_X(\eth), Ú_X(\eth), Ŕ_X(\eth)) : \eth \in Û\}$ and $Y = \{(\eth, Ý_Y(\eth), Ć_Y(\eth), Ź_Y(\eth), Ú_Y(\eth), Ŕ_Y(\eth)) : \eth \in Û\}$ are two SVPNSs over Û. Then, the following results hold:

(*i*) $X \subseteq Y$ if and only if $Ý_X(\eth) \le Ý_Y(\eth)$, $Ć_X(\eth) \le Ć_Y(\eth)$, $Ź_X(\eth) \ge Ź_Y(\eth)$, $Ú_X(\eth) \ge Ú_Y(\eth)$, $Ŕ_X(\eth) \ge Ŕ_Y(\eth)$, for all $\eth \in Û$.

(*ii*) $X \cup Y = \{(\eth, \max\{Ý_X(\eth), Ý_Y(\eth)\}, \max\{Ć_X(\eth), Ć_Y(\eth)\}, \min\{Ź_X(\eth), Ź_Y(\eth)\}, \min\{Ú_X(\eth), Ú_Y(\eth)\}, \min\{Ŕ_X(\eth), Ŕ_Y(\eth)\}) : \eth \in Û\}$.

(*iii*) $X \cap Y = \{(\eth, \min\{Ý_X(\eth), Ý_Y(\eth)\}, \min\{Ć_X(\eth), Ć_Y(\eth)\}, \max\{Ź_X(\eth), Ź_Y(\eth)\}, \max\{Ú_X(\eth), Ú_Y(\eth)\}, \max\{Ŕ_X(\eth), Ŕ_Y(\eth)\}) : \eth \in Û\}$.

(*iv*) $X^c = \{(\eth, Ý_X(\eth), Ć_X(\eth), 1-Ź_X(\eth), Ú_X(\eth), Ŕ_X(\eth)) : \eth \in Û\}$.

3. Single-Valued Pentapartitioned Neutrosophic Soft Set:

In this section, we procure the notion SVPNS-Sets and study some operations on them. Then, we formulate some results on SVPNS-Sets.

Definition 3.1. Let Û is a non-empty fixed set and Q be a collection of parameters. Suppose that, SVPN-Set(Û) denotes the set of all SVPN-Sets defined over Û. Then, for any $S \subseteq Q$, a pair (P_N, S) is referred to as an SVPNS-Set over Û, where P_N is a mapping from S to SVPN-Set(Û).

An SVPNS-Set (P_N, S) is defined as follows:

$$(P_N, S) = \{(h, \{(l, Ý_{P\_N(h)}(l), Ć_{P\_N(h)}(l), Ź_{P\_N(h)}(l), Ú_{P\_N(h)}(l), Ŕ_{P\_N(h)}(l)) : l \in Û\}) : h \in Q, l \in Û\},$$

where $Ý_{P\_N(h)}(l)$, $Ć_{P\_N(h)}(l)$, $Ź_{P\_N(h)}(l)$, $Ú_{P\_N(h)}(l)$, and $Ŕ_{P\_N(h)}(l)$ are the truth, contradiction, ignorance, unknown, and falsity membership values of each u with respect to the parameter $h \in Q$.

Example 3.1. Suppose that Û=$\{t_1, t_2, t_3, t_4\}$ is a fixed set consisting of four different colleges, and Q= $\{h_1$(grade), h_2(infrastructure), h_3(semester fee), h_4(placement), h_5(laboratory)$\}$ is a set of parameters corresponding to different colleges. Let $P\_N(h_1)$=$\{(t_1$,0.9,0.6,0.4,0.2,0.2), (t_2,0.8,0.4,0.5, 0.3,0.3,0.4)$\}$, $P\_N(h_2)$=$\{(t_1$,0.61,0.35,0.22,0.4,0.21), (t_2,0.55,0.4,0.18,0.24,0.32), $P\_N(h_3)$=$\{(t_1$,0.92,0.45, 0.56, 0.41,0.32), (t_2,0.75,0.3,0.2,0.41,0.55), $P\_N(h_4)$=$\{(t_1$,0.82,0.5,0.4,0.6,0.2), (t_2,0.65,0.47,0.6,0.2,0.1), $P\_N(h_5)$= $\{(t_1$,0.82,0.25,0.54,0.55,0.23), (t_2,0.77,0.6,0.57,0.8,0.9). Then, (P_N, Q) = $\{(h_1, P\_N(h_1)), (h_2, P\_N(h_2)), (h_3, P\_N(h_3)), (h_4, P\_N(h_4)), (h_5, P\_N(h_5))\}$ is an SVPNS-Set over Û with respect to the set Q.

Definition 3.2. The complement of an SVPNS-Set (P_N, Q) is denoted by $(P_N, Q)^c = (P_N{}^c, Q)$ and is defined by $(P_N{}^c, Q) = \{(h, \{(l, 1-Ý_{P\_N(h)}(l), 1-Ć_{P\_N(h)}(l), 1-Ź_{P\_N(h)}(l), 1-Ú_{P\_N(h)}(l), 1-Ŕ_{P\_N(h)}(l)) : l \in Û\}) : h \in Q\}$.

Definition 3.3. Suppose that (S_1, Q) and (S_2, Q) are any two SVPNS-Sets over Û. Then, (S_1, Q) is referred to as a single-valued pentapartitioned neutrosophic soft sub-set of (S_2, Q) if and only if $Ý_{S_1(§)}(\ell) \leq Ý_{S_2(§)}(\ell)$, $Ć_{S_1(§)}(\ell) \geq Ć_{S_2(§)}(\ell)$, $Ź_{S_1(§)}(\ell) \geq Ź_{S_2(§)}(\ell)$, $Ú_{S_1(§)}(\ell) \geq Ú_{S_2(§)}(\ell)$, and $Ŕ_{S_1(§)}(\ell) \geq Ŕ_{S_2(§)}(\ell)$, ∀ §∈$Q$ and ℓ∈Û. We write (S_1, Q) ⊆ (S_2, Q). Then, (S_2, Q) is referred to as a single-valued pentapartitioned neutrosophic soft super-set of (S_1, Q).

Definition 3.4. Suppose that (S_1, Q) and (S_2, Q) be any two SVPNS-Sets over a fixed set Û. Then, intersection of (S_1, Q) and (S_2, Q) is denoted by (S, Q), where $S=S_1 \cap S_2$ is defined as follows:

$$(S, Q)=\{(§, \{(k, Ý_{S(§)}(\wp*), Ć_{S(§)}(\wp*), Ź_{S(§)}(\wp*), Ú_{S(§)}(\wp*), Ŕ_{S(§)}(\wp*)): \wp*\in Û\}): §\in Q\},$$

where $Ý_{S(§)}(\wp*) = \min\{Ý_{S_1(§)}(\wp*), Ý_{S_2(§)}(\wp*)\}$, $Ć_{S(§)}(\wp*) = \max\{Ć_{S_1(§)}(\wp*), Ć_{S_2(§)}(\wp*)\}$, $Ź_{S(§)}(\wp*) = \max\{Ź_{S_1(§)}(\wp*), Ź_{S_2(§)}(\wp*)\}$, $Ú_{S(§)}(\wp*) = \max\{Ú_{S_1(§)}(\wp*), Ú_{S_2(§)}(\wp*)\}$ and $Ŕ_{S(§)}(\wp*) = \max\{Ŕ_{S_1(§)}(\wp*), Ŕ_{S_2(§)}(\wp*)\}$.

Definition 3.5. Suppose that (S_1, Q) and (S_2, Q) are any two SVPNS-Sets over a fixed set Û. Then, union of (S_1, Q) and (S_2, Q) is denoted by (S, Q), where $S = S_1 \cup S_2$ is defined as follows:

$$(S, Q) = \{(§, \{(\wp*, Ý_{S(§)}(\wp*), Ć_{S(§)}(\wp*), Ź_{S(§)}(\wp*), Ú_{S(§)}(\wp*), Ŕ_{S(§)}(\wp*)): \wp*\in Û\}): §\in Q\},$$

where $Ý_{S(§)}(\wp*) = \max\{Ý_{S_1(§)}(\wp*), Ý_{S_2(§)}(\wp*)\}$, $Ć_{S(§)}(\wp*) = \min\{Ć_{S_1(§)}(\wp*), Ć_{S_2(§)}(\wp*)\}$, $Ź_{S(§)}(\wp*) = \min\{Ź_{S_1(§)}(\wp*), Ź_{S_2(§)}(\wp*)\}$, $Ú_{S(§)}(\wp*) = \min\{Ú_{S_1(§)}(\wp*), Ú_{S_1(§)}(\wp*)\}$ and $Ŕ_{S(§)}(\wp*) = \min\{Ŕ_{S_1(§)}(\wp*), Ŕ_{S_1(§)}(\wp*)\}$.

Definition 3.6. An SVPNS-Set (S, Q) over a fixed set Û is called a null SVPNS-Set if and only if $Ý_{S(§)}(\wp*) = 0$, $Ć_{S(§)}(\wp*) = 1$, $Ź_{S(§)}(\wp*) = 1$, $Ú_{S(§)}(\wp*) = 1$ and $Ŕ_{S(§)}(\wp*) = 1$, ∀$\wp*$∈Û with respect to the parameter §∈Q. The null SVPNS-Set is denoted by $0_{(S, Q)}$.

Definition 3.7. An SVPNS-Set (S, Q) over a fixed set Û is referred to as an absolute SVPNS-Set if and only if $Ý_{S(§)}(\wp*) = 1$, $Ć_{S(§)}(\wp*) = 0$, $Ź_{S(§)}(\wp*) = 0$, $Ú_{S(§)}(\wp*) = 0$ and $Ŕ_{S(§)}(\wp*) = 0$, ∀$\wp*$∈Û with respect to the parameter § ∈Q. The absolute SVPNS-Set is denoted by $1_{(S, Q)}$.

Clearly, $1^c_{(S, Q)} = 0_{(S, Q)}$ and $0^c_{(S, Q)} = 1_{(S, Q)}$.

Theorem 3.1. Suppose that (K, A) and (L, A) are two SVPNS-Sets over the same universe Û. Then, the following results hold:

(*i*) (K, A) ∪ (K, A) = (K, A) & (K, A) ∩ (K, A) = (K, A);

(*ii*) (K, A) ∪ (L, A) = (L, A) ∪ (K, A) & (K, A) ∩ (L, A) = (L, A) ∩ (K, A);

(*iii*) (K, A) ∪ $0_{(K, A)}$ = (K, A) & (K, A) ∩ $0_{(K, A)}$ = $0_{(K, A)}$;

(*iv*) (K, A) ∪ $1_{(K, A)}$ = $1_{(K, A)}$ & (K, A) ∩ $1_{(K, A)}$ = (K, A);

(*v*) $[(K, A)^c]^c = (K, A)$.

Proof. (*i*) Assume that $(K, A) = \{(§, \{(\wp*, Ý_{K(§)}(\wp*), Ć_{K(§)}(\wp*), Ź_{K(§)}(\wp*), Ú_{K(§)}(\wp*), Ŕ_{K(§)}(\wp*)):\wp*\in Û\}): §\in C\}$ is an SVPNS-Set over a universe Û.

Now, (K, A) ∪ (K, A)

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\} \cup \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$.

$= \{(\S, \{(\wp*, \max\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{K(\S)}(\wp*)\}, \min\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*)\}, \min\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*)\}, \min\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*)\}, \min\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$.

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$.

$= (K, A)$.

Therefore, $(K, A) \cup (K, A) = (K, A)$.

Now, $(K, A) \cap (K, A)$

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\} \cap \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$.

$= \{(\S, \{(\wp*, \min\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{K(\S)}(\wp*)\}, \max\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*)\}, \max\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*)\}, \max\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*)\}, \max\{\acute{R}_{K(\Omega)}(\wp*), \acute{R}_{K(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$.

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$.

$= (K, A)$.

Therefore, $(K, A) \cap (K, A) = (K, A)$.

(*ii*) Assume that $(K, C)=\{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$ and $(L, C)= \{(\S, \{(\wp*, \acute{Y}_{L(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$ is any two SVPNS-Sets over the same universe $\hat{U}$.

Now, $(K, A) \cup (L, A)$

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\} \cup \{(\S, \{(\wp*, \acute{Y}_{L(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$.

$= \{(\S, \{(\wp*, \max\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{L(\S)}(\wp*)\}, \min\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*)\}, \min\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*)\}, \min\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*)\}, \min\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$.

$= \{(\S, \{(\wp*, \max\{\acute{Y}_{L(\S)}(\wp*), \acute{Y}_{K(\S)}(\wp*)\}, \min\{\acute{C}_{L(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*)\}, \min\{\acute{Z}_{L(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*)\}, \min\{\acute{U}_{L(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*)\}, \min\{\acute{R}_{L(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$.

$= (L, A) \cup (K, A)$.

Therefore, $(K, A) \cup (L, A) = (L, A) \cup (K, A)$.

Further, we have

$(K, A) \cap (L, A)$

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\} \cap \{(\S, \{(\wp*, \acute{Y}_{L(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$.

$= \{(\S, \{(\wp*, \min\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{L(\S)}(\wp*)\}, \max\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*)\}, \max\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*)\}, \max\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*)\}, \max\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$.

= {(§, {(℘∗, min{$Ý_{L(§)}$ (℘∗), $Ý_{K(§)}$ (℘∗)}, max{$Ć_{L(§)}$ (℘∗), $Ć_{K(§)}$ (℘∗)}, max{$Ź_{L(§)}$ (℘∗), $Ź_{K(§)}$ (℘∗)}, max{$Ú_{L(§)}$(℘∗), $Ú_{K(§)}$(℘∗)}, max{$Ŕ_{L(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)}):℘∗∈Û}): §∈*C*}.

= (*L*, *A*) ∩ (*K*, *A*).

Therefore, (*K*, *A*) ∩ (*L*, *A*) = (*L*, *A*) ∩ (*K*, *A*).

(*iii*) Assume that (*K*, *A*) = {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈*C*} is an SVPNS-Set over a universe Û.

Now, (*K*, *A*) ∪ $0_{(K, A)}$

= {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈*C*} ∪ {(§, {(℘∗, 0, 1, 1, 1, 1):℘∗∈Û}): §∈*C*}

= {(§, {(℘∗, max{$Ý_{K(§)}$(℘∗), 0}, min{$Ć_{K(§)}$(℘∗), 1}, min{$Ź_{K(§)}$(℘∗), 1}, min{$Ú_{K(§)}$(℘∗), 1}, min{$Ŕ_{K(§)}$(℘∗), 1}):℘∗∈Û}): §∈*C*}.

= {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈*C*}.

= (*K*, *A*).

Therefore, (*K*, *A*) ∪ $0_{(K, A)}$ = (*K*, *A*).

Further, we have

(*K*, *A*) ∩ $0_{(K, A)}$

= {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈*C*} ∩ {(§, {(℘∗, 0, 1, 1, 1, 1):℘∗∈Û}): §∈*C*}

= {(§, {(℘∗, min{$Ý_{K(§)}$(℘∗), 0}, max{$Ć_{K(§)}$(℘∗), 1}, max{$Ź_{K(§)}$(℘∗), 1}, max{$Ú_{K(§)}$(℘∗), 1}, max{$Ŕ_{K(§)}$(℘∗), 1}):℘∗∈Û}): §∈*C*}.

= {(§, {(℘∗, 0, 1, 1, 1, 1):℘∗∈Û}): §∈*C*}.

= $0_{(K, A)}$.

Therefore, (*K*, *A*) ∩ $0_{(K, A)}$ = $0_{(K, A)}$.

(*iv*) Let (*K*, *A*) = {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈*C*} be an SVPNS-Set over a universe Û.

Now, (*K*, *A*) ∪ $1_{(K, A)}$

= {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈*C*} ∪ {(§, {(℘∗, 1, 0, 0, 0, 0):℘∗∈Û}): §∈*C*}.

= {(§, {(℘∗, max{$Ý_{K(§)}$(℘∗), 1}, min{$Ć_{K(§)}$(℘∗), 0}, min{$Ź_{K(§)}$(℘∗), 0}, min{$Ú_{K(§)}$(℘∗), 0}, min{$Ŕ_{K(§)}$(℘∗), 0}):℘∗∈Û}): §∈*C*}.

= {(§, {(℘∗, 1, 0, 0, 0, 0):℘∗∈Û}): §∈*C*}.

= $1_{(K, A)}$.

Therefore, (*K*, *A*) ∪ $1_{(K, A)}$ = $1_{(K, A)}$.

Further, we have

$(K, A) \cap 1_{(K, A)}$

= {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈C} ∩ {(§, {(℘∗, 1, 0, 0, 0, 0):℘∗∈Û}): §∈C}.

= {(§, {(℘∗, min{$Ý_{K(§)}$(℘∗), 1}, max{$Ć_{K(§)}$(℘∗), 0}, max{$Ź_{K(§)}$(℘∗), 0}, max{$Ú_{K(§)}$(℘∗), 0}, max{$Ŕ_{K(§)}$(℘∗), 0}):℘∗∈Û}): §∈C}.

= {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈C}.

= (K, A).

Therefore, $(K, A) \cap 1_{(K, A)} = (K, A)$.

(v) Assume that (K, A) = {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈C} is an SVPNS-Set over a universe Û. Then, $(K, A)^c$ = {(§, {(℘∗, 1-$Ý_{K(§)}$(℘∗), 1-$Ć_{K(§)}$(℘∗), 1-$Ź_{K(§)}$(℘∗), 1-$Ú_{K(§)}$(℘∗), 1-$Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈C}.

Now, we have

$((K, A)^c)^c$

= {(§, {(℘∗, 1-(1-$Ý_{K(§)}$(℘∗)), 1-(1-$Ć_{K(§)}$(℘∗)), 1-(1-$Ź_{K(§)}$(℘∗)), 1-(1-$Ú_{K(§)}$(℘∗)), 1-(1-$Ŕ_{K(§)}$(℘∗))):℘∗∈Û}): §∈C}

= {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈C}

= (K, C)

Therefore, $((K, A)^c)^c = (K, A)$.

Theorem 3.2. Suppose that (K, A), (L, A) and (M, A) are three SVPNS-Sets over the same universe Û. Then, the following results hold:

(i) $(K, A) \cup [(L, A) \cup (M, A)] = [(K, A) \cup (L, A)] \cup (M, A)$.

(ii) $(K, A) \cap [(L, A) \cap (M, A)] = [(K, A) \cap (L, A)] \cap (M, A)$.

(iii) $(K, A) \cup [(L, A) \cap (M, A)] = [(K, A) \cup (L, A)] \cap [(K, A) \cup (M, A)]$.

(iv) $(K, A) \cap [(L, A) \cup (M, A)] = [(K, A) \cap (L, A)] \cup [(K, A) \cap (M, A)]$.

Proof. (i) Suppose that (K, A), (L, A) and (M, A) are any three SVPNS-Sets over the same universe Û. Now, we have

$(K, A) \cup [(L, A) \cup (M, A)]$

= {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈C} ∪ [{(§, {(℘∗, $Ý_{L(§)}$(℘∗), $Ć_{L(§)}$(℘∗), $Ź_{L(§)}$(℘∗), $Ú_{L(§)}$(℘∗), $Ŕ_{L(§)}$(℘∗)):℘∗∈Û}): §∈C} ∪ {(§, {(℘∗, $Ý_{M(§)}$(℘∗), $Ć_{M(§)}$(℘∗), $Ź_{M(§)}$(℘∗), $Ú_{M(§)}$(℘∗), $Ŕ_{M(§)}$(℘∗)):℘∗∈Û}): §∈C}]

= {(§, {(℘∗, $Ý_{K(§)}$(℘∗), $Ć_{K(§)}$(℘∗), $Ź_{K(§)}$(℘∗), $Ú_{K(§)}$(℘∗), $Ŕ_{K(§)}$(℘∗)):℘∗∈Û}): §∈C} ∪ {(§, {(℘∗, max {$Ý_{L(§)}$(℘∗), $Ý_{M(§)}$(℘∗)}, min {$Ć_{L(§)}$(℘∗), $Ć_{M(§)}$(℘∗)}, min {$Ź_{L(§)}$(℘∗), $Ź_{M(§)}$(℘∗)}, min {$Ú_{L(§)}$(℘∗), $Ú_{M(§)}$(℘∗)}, min {$Ŕ_{L(§)}$(℘∗), $Ŕ_{M(§)}$(℘∗)}):℘∗∈Û}): §∈C}

= {(§, {($\wp*$, max {$Ý_{K(§)}(\wp*)$, $Ý_{L(§)}(\wp*)$, $Ý_{M(§)}(\wp*)$}, min {$Ć_{K(§)}(\wp*)$, $Ć_{L(§)}(\wp*)$, $Ć_{M(§)}(\wp*)$}, min {$Ź_{K(§)}(\wp*)$, $Ź_{L(§)}(\wp*)$, $Ź_{M(§)}(\wp*)$}, min {$Ú_{K(§)}(\wp*)$, $Ú_{L(§)}(\wp*)$, $Ú_{M(§)}(\wp*)$}, min {$Ŕ_{K(§)}(\wp*)$, $Ŕ_{L(§)}(\wp*)$, $Ŕ_{M(§)}(\wp*)$}):$\wp*$∈Û}): §∈*C*}

= {(§, {($\wp*$, max {$Ý_{K(§)}(\wp*)$, $Ý_{L(§)}(\wp*)$}, min {$Ć_{K(§)}(\wp*)$, $Ć_{L(§)}(\wp*)$}, min {$Ź_{K(§)}(\wp*)$, $Ź_{L(§)}(\wp*)$}, min {$Ú_{K(§)}(\wp*)$, $Ú_{L(§)}(\wp*)$}, min {$Ŕ_{K(§)}(\wp*)$, $Ŕ_{L(§)}(\wp*)$}):$\wp*$∈Û}): §∈*C*} ∪ {(§, {($\wp*$, $Ý_{M(§)}(\wp*)$, $Ć_{M(§)}(\wp*)$, $Ź_{M(§)}(\wp*)$, $Ú_{M(§)}(\wp*)$, $Ŕ_{M(§)}(\wp*)$):$\wp*$∈Û}): §∈*C*}

= [(*K*, *C*) ∪ (*L*, *C*)] ∪ (*M*, *C*)

Therefore, (*K*, *A*) ∪ [(*L*, *A*) ∪ (*M*, *A*)] = [(*K*, *A*) ∪ (*L*, *A*)] ∪ (*M*, *A*).

(*ii*) Assume that (*K*, *A*), (*L*, *A*) and (*M*, *A*) are any three SVPNS-Sets over the same universe Û.

Now, we have

(*K*, *A*) ∩ [(*L*, *A*) ∩ (*M*, *A*)]

= {(§, {($\wp*$, $Ý_{K(§)}(\wp*)$, $Ć_{K(§)}(\wp*)$, $Ź_{K(§)}(\wp*)$, $Ú_{K(§)}(\wp*)$, $Ŕ_{K(§)}(\wp*)$):$\wp*$∈Û}): §∈*C*} ∩ [{(§, {($\wp*$, $Ý_{L(§)}(\wp*)$, $Ć_{L(§)}(\wp*)$, $Ź_{L(§)}(\wp*)$, $Ú_{L(§)}(\wp*)$, $Ŕ_{L(§)}(\wp*)$):$\wp*$∈Û}): §∈*C*} ∩ {(§, {($\wp*$, $Ý_{M(§)}(\wp*)$, $Ć_{M(§)}(\wp*)$, $Ź_{M(§)}(\wp*)$, $Ú_{M(§)}(\wp*)$, $Ŕ_{M(§)}(\wp*)$):$\wp*$∈Û}): §∈*C*}]

= {(§, {($\wp*$, $Ý_{K(§)}(\wp*)$, $Ć_{K(§)}(\wp*)$, $Ź_{K(§)}(\wp*)$, $Ú_{K(§)}(\wp*)$, $Ŕ_{K(§)}(\wp*)$):$\wp*$∈Û}): §∈*C*} ∩ {(§, {($\wp*$, min {$Ý_{L(§)}(\wp*)$, $Ý_{M(§)}(\wp*)$}, max {$Ć_{L(§)}(\wp*)$, $Ć_{M(§)}(\wp*)$}, max {$Ź_{L(§)}(\wp*)$, $Ź_{M(§)}(\wp*)$}, max {$Ú_{L(§)}(\wp*)$, $Ú_{M(§)}(\wp*)$}, max {$Ŕ_{L(§)}(\wp*)$, $Ŕ_{M(§)}(\wp*)$}):$\wp*$∈Û}): §∈*C*}

= {(§, {($\wp*$, min {$Ý_{K(§)}(\wp*)$, $Ý_{L(§)}(\wp*)$, $Ý_{M(§)}(\wp*)$}, max {$Ć_{K(§)}(\wp*)$, $Ć_{L(§)}(\wp*)$, $Ć_{M(§)}(\wp*)$}, max {$Ź_{K(§)}(\wp*)$, $Ź_{L(§)}(\wp*)$, $Ź_{M(§)}(\wp*)$}, max {$Ú_{K(§)}(\wp*)$, $Ú_{L(§)}(\wp*)$, $Ú_{M(§)}(\wp*)$}, max {$Ŕ_{K(§)}(\wp*)$, $Ŕ_{L(§)}(\wp*)$, $Ŕ_{M(§)}(\wp*)$}):$\wp*$∈Û}): §∈*C*}

= {(§, {($\wp*$, min {$Ý_{K(§)}(\wp*)$, $Ý_{L(§)}(\wp*)$}, max {$Ć_{K(§)}(\wp*)$, $Ć_{L(§)}(\wp*)$}, max {$Ź_{K(§)}(\wp*)$, $Ź_{L(§)}(\wp*)$}, max {$Ú_{K(§)}(\wp*)$, $Ú_{L(§)}(\wp*)$}, max {$Ŕ_{K(§)}(\wp*)$, $Ŕ_{L(§)}(\wp*)$}):$\wp*$∈Û}): §∈*C*} ∩ {(§, {($\wp*$, $Ý_{M(§)}(\wp*)$, $Ć_{M(§)}(\wp*)$, $Ź_{M(§)}(\wp*)$, $Ú_{M(§)}(\wp*)$, $Ŕ_{M(§)}(\wp*)$):$\wp*$∈Û}): §∈*C*}

= [(*K*, *A*) ∩ (*L*, *A*)] ∩ (*M*, *A*)

Therefore, (*K*, *A*) ∩ [(*L*, *A*) ∩ (*M*, *A*)] = [(*K*, *A*) ∩ (*L*, *A*)] ∩ (*M*, *A*).

(*iii*) Suppose that (*K*, *A*), (*L*, *A*) and (*M*, *A*) are any three SVPNS-Sets over the same universe Û.

Now, we have

(*K*, *A*) ∪ [(*L*, *A*) ∩ (*M*, *A*)]

= {(§, {($\wp*$, $Ý_{K(§)}(\wp*)$, $Ć_{K(§)}(\wp*)$, $Ź_{K(§)}(\wp*)$, $Ú_{K(§)}(\wp*)$, $Ŕ_{K(§)}(\wp*)$):$\wp*$∈Û}): §∈*C*} ∪ [{(§, {($\wp*$, $Ý_{L(§)}(\wp*)$, $Ć_{L(§)}(\wp*)$, $Ź_{L(§)}(\wp*)$, $Ú_{L(§)}(\wp*)$, $Ŕ_{L(§)}(\wp*)$):$\wp*$∈Û}): §∈*C*} ∩ {(§, {($\wp*$, $Ý_{M(§)}(\wp*)$, $Ć_{M(§)}(\wp*)$, $Ź_{M(§)}(\wp*)$, $Ú_{M(§)}(\wp*)$, $Ŕ_{M(§)}(\wp*)$):$\wp*$∈Û}): §∈*C*}]

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\} \cup \{(\S, \{(\wp*, \min\{\acute{Y}_{L(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}, \max\{\acute{C}_{L(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}, \max\{\acute{Z}_{L(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}, \max\{\acute{U}_{L(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}, \max\{\acute{R}_{L(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$

$= \{(\S, \{(\wp*, \max\{\acute{Y}_{K(\S)}(\wp*), \min\{\acute{Y}_{L(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}\}, \min\{\acute{C}_{K(\S)}(\wp*), \max\{\acute{C}_{L(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}\}, \min\{\acute{Z}_{K(\S)}(\wp*), \max\{\acute{Z}_{L(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}\}, \min\{\acute{U}_{K(\S)}(\wp*), \max\{\acute{U}_{L(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}\}, \min\{\acute{R}_{K(\S)}(\wp*), \max\{\acute{R}_{L(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}\}):\wp*\in\hat{U}\}): \S\in C\}$

Further, we have

$[(K, A) \cup (L, A)]$

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\} \cup \{(\S, \{(\wp*, \acute{Y}_{L(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$

$= \{(\S, \{(\wp*, \max\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{L(\S)}(\wp*)\}, \min\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*)\}, \min\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*)\}, \min\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*)\}, \min\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$,

and $[(K, A) \cup (M, A)]$

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\} \cup \{(\S, \{(\wp*, \acute{Y}_{M(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$

$= \{(\S, \{(\wp*, \max\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}, \min\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}, \min\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}, \min\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}, \min\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$.

Now, $[(K, A) \cup (L, A)] \cap [(K, A) \cup (M, A)]$

$= \{(\S, \{(\wp*, \max\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{L(\S)}(\wp*)\}, \min\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*)\}, \min\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*)\}, \min\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*)\}, \min\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\} \cap \{(\S, \{(\wp*, \max\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}, \min\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}, \min\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}, \min\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}, \min\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$

$= \{(\S, \{(\wp*, \min\{\max\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{L(\S)}(\wp*)\}, \max\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}\}, \max\{\min\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*)\}, \min\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}\}, \max\{\min\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*)\}, \min\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}\}, \max\{\min\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*)\}, \min\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}\}, \max\{\min\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)\}, \min\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}\}):\wp*\in\hat{U}\}): \S\in C\}$

$= \{(\S, \{(\wp*, \max\{\acute{Y}_{K(\S)}(\wp*), \min\{\acute{Y}_{L(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}\}, \min\{\acute{C}_{K(\S)}(\wp*), \max\{\acute{C}_{L(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}\}, \min\{\acute{Z}_{K(\S)}(\wp*), \max\{\acute{Z}_{L(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}\}, \min\{\acute{U}_{K(\S)}(\wp*), \max\{\acute{U}_{L(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}\}, \min\{\acute{R}_{K(\S)}(\wp*), \max\{\acute{R}_{L(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}\}):\wp*\in\hat{U}\}): \S\in C\}$

$= (K, A) \cup [(L, A) \cap (M, A)]$.

Therefore, $(K, A) \cup [(L, A) \cap (M, A)] = [(K, A) \cup (L, A)] \cap [(K, A) \cup (M, A)]$.

(*iv*) Assume that (K, A), (L, A) and (M, A) are any three SVPNS-Sets over the same universe Û.

Now, we have

$(K, A) \cap [(L, A) \cup (M, A)]$

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*))\wp*\in\hat{U}\}): \S\in C\} \cap [\{(\S, \{(\wp*, \acute{Y}_{L(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\} \cup \{(\S, \{(\wp*, \acute{Y}_{M(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}]$

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\} \cap \{(\S, \{(\wp*, \max\{\acute{Y}_{L(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}, \min\{\acute{C}_{L(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}, \min\{\acute{Z}_{L(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}, \min\{\acute{U}_{L(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}, \min\{\acute{R}_{L(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$

$= \{(\S, \{(\wp*, \min\{\acute{Y}_{K(\S)}(\wp*), \max\{\acute{Y}_{L(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}\}, \max\{\acute{C}_{K(\S)}(\wp*), \min\{\acute{C}_{L(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}\}, \max\{\acute{Z}_{K(\S)}(\wp*), \min\{\acute{Z}_{L(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}\}, \max\{\acute{U}_{K(\S)}(\wp*), \min\{\acute{U}_{L(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}\}, \max\{\acute{R}_{K(\S)}(\wp*), \min\{\acute{R}_{L(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$

Further, we have

$[(K, A) \cap (L, A)]$

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\} \cap \{(\S, \{(\wp*, \acute{Y}_{L(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$

$= \{(\S, \{(\wp*, \min\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{L(\S)}(\wp*)\}, \max\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*)\}, \max\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*)\}, \max\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*)\}, \max\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$,

and $[(K, A) \cap (M, A)]$

$= \{(\S, \{(\wp*, \acute{Y}_{K(\S)}(\wp*), \acute{C}_{K(\S)}(\wp*), \acute{Z}_{K(\S)}(\wp*), \acute{U}_{K(\S)}(\wp*), \acute{R}_{K(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\} \cap \{(\S, \{(\wp*, \acute{Y}_{M(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)):\wp*\in\hat{U}\}): \S\in C\}$

$= \{(\S, \{(\wp*, \min\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}, \max\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}, \max\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}, \max\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}, \max\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$.

Now, $[(K, A) \cap (L, A)] \cup [(K, A) \cap (M, A)]$

$= \{(\S, \{(\wp*, \min\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{L(\S)}(\wp*)\}, \max\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*)\}, \max\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*)\}, \max\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*)\}, \max\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\} \cup \{(\S, \{(\wp*, \min\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}, \max\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}, \max\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}, \max\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}, \max\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$

$= \{(\S, \{(\wp*, \max\{\min\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{L(\S)}(\wp*)\}, \min\{\acute{Y}_{K(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}\}, \min\{\max\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{L(\S)}(\wp*)\}, \max\{\acute{C}_{K(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}\}, \min\{\max\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{L(\S)}(\wp*)\}, \max\{\acute{Z}_{K(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}\}, \min\{\max\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{L(\S)}(\wp*)\}, \max\{\acute{U}_{K(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}\}, \min\{\max\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{L(\S)}(\wp*)\}, \max\{\acute{R}_{K(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}\}\}):\wp*\in\hat{U}\}): \S\in C\}$

$= \{(\S, \{(\wp*, \min\{\acute{Y}_{K(\S)}(\wp*), \max\{\acute{Y}_{L(\S)}(\wp*), \acute{Y}_{M(\S)}(\wp*)\}\}, \max\{\acute{C}_{K(\S)}(\wp*), \min\{\acute{C}_{L(\S)}(\wp*), \acute{C}_{M(\S)}(\wp*)\}\}, \max\{\acute{Z}_{K(\S)}(\wp*), \min\{\acute{Z}_{L(\S)}(\wp*), \acute{Z}_{M(\S)}(\wp*)\}\}, \max\{\acute{U}_{K(\S)}(\wp*), \min\{\acute{U}_{L(\S)}(\wp*), \acute{U}_{M(\S)}(\wp*)\}\}, \max\{\acute{R}_{K(\S)}(\wp*), \min\{\acute{R}_{L(\S)}(\wp*), \acute{R}_{M(\S)}(\wp*)\}):\wp*\in\hat{U}\}): \S\in C\}$

$= (K, C) \cap [(L, C) \cup (M, C)]$.

Therefore, $(K, A) \cap [(L, A) \cup (M, A)] = [(K, A) \cap (L, A)] \cup [(K, A) \cap (M, A)]$.

Definition 3.8. Assume that (K, Y) and (L, Z) are any two SVPNS-Sets over the same universe Û. Then, the operation 'AND' or 'Meet' is defined as follows:

$$(K, Y) \wedge (L, Z) = (P, Y \times Z),$$

where the corresponding truth, contradiction, ignorance, unknown and false membership values of (P, $Y\times Z$) are measured by $Ý_{P(u,\, v)}(\wp*) = \min \{Ý_{K(u)}(\wp*), Ý_{L(v)}(\wp*)\}$, $Ć_{P(u,\, v)}(\wp*) = \min \{Ć_{K(u)}(\wp*), Ć_{L(v)}(\wp*))$, $Ź_{P(u,\, v)}(\wp*) = \frac{(Ź_{K(u)}(\wp*)+Ź_{L(v)}(\wp*))}{2}$, $Ú_{P(u,\, v)}(\wp*) = \frac{(Ú_{K(u)}(\wp*)+\ Ú_{L(v)}(\wp*))}{2}$, $Ŕ_{P(u,\, v)}(\wp*) = \max \{Ŕ_{K(u)}(\wp*), Ŕ_{L(v)}(\wp*)\}$, for all $u \in K$, $v \in L$, $\wp* \in$ Û.

Example 3.2. Assume that (K, E) and (L, F) are two SVPNS-Sets over the common universe Û.

The tabulated representation of SVPNS-Set (K, E) is given as follows:

Û	Grade	Infrastructures	Faculties
t_1	[0.88,0.45,0.66,0.2,0.1]	[0.73,0.2,0.88,0.44,0.64]	[0.89,0.82,0.45,0.25,0.75]
t_2	[0.85,0.27,0.58,0.26,0.72]	[0.65,0.14,0.52,0.36,0.52]	[0.78,0.98,0.65,0.43,0.64]
t_3	[0.76,0.44,0.77,0.65,0.25]	[0.41,0.25,0.69,0.57,0.74]	[0.98,0.65,0.55,0.35,0.39]
t_4	[0.82,0.65,0.14,0.86,0.37]	[0.66,0.65,0.14,.34,0.46]	[0.78,0.35,0.48,0.65,0.66]

The tabulated representation of SVPNS-Set (L, F) is given as follows:

Û	Semester Fee	Faculties	Students facilities
t_1	[0.85,0.5,0.47,0.65,0.2]	[0.88,0.54,0.47,0.58,0.22]	[0.46,0.48,0.35,0.14,0.36]
t_2	[0.77,0.52,0.19,0.77,0.25]	[0.65,0.42,0.75,0.65,0.69]	[0.78,0.65,0.87,0.69,0.49]
t_3	[0.73,0.2,0.88,0.44,0.64]	[0.95,0.25,0.85,0.45,0.19]	[0.98,0.36,0.97,0.54,0.63]
t_4	[0.69,0.68,0.61,0.25,0.96]	[0.58,0.45,0.64,0.85,0.47]	[0.87,0.69,0.55,0.45,0.47]

The tabulated representation of SVPNS-Set (K, E) $\wedge$ (L, F) is as follows:

Û	(Grade, Semester Fee)	(Grade, Faculties)	(Grade, Students facilities)
t_1	[0.85,0.45,0.565,0.425,0.2]	[0.88,0.45,0.565,0.39,0.22]	[0.46,0.45,0.505,0.17,0.36]
t_2	[0.77,0.27,0.385,0.515,0.72]	[0.65,0.42,0.76,0.65,0.72]	[0.78,0.27,0.725,0.475,0.72]
t_3	[0.73,0.2,0.825,0.545,0.64]	[0.76,0.25,0.81,0.55,0.25]	[0.76,0.47,0.87,0.595,0.63]
t_4	[0.69,0.65,0.375,0.555,0.96]	[0.58,0.45,0.39,0.855,0.47]	[0.82,0.65,0.345,0.655,0.47]

Û	(Infrastructures, Semester Fee)	(Infrastructures, Faculties)	(Infrastructures, Students facilities)

t_1	[0.73,0.2,0.675,0.545,0.64]	[0.73,0.2,0.675,0.51,0.64]	[0.46,0.2,0.615,0.29,0.64]
t_2	[0.65,0.14,0.355,0.565,0.52]	[0.65,0.14,0.635,0.505,0.69]	[0.65,0.14,0.695,0.525,0.52]
t_3	[0.41,0.2,0.785,0.505,0.74]	[0.41,0.25,0.77,0.51,0.74]	[0.41,0.25,0.83,0.555,0.74]
t_4	[0.66,0.65,0.375,0.295,0.96]	[0.58,0.45,0.39,0.595,0.47]	[0.66,0.65,0.345,0.395,0.47]

Û	(Faculties, Semester Fee)	(Faculties, Faculties)	(Faculties, Students facilities)
t_1	[0.85,0.5,0.46,0.45,0.75]	[0.88,0.54,0.46,0.415,0.75]	[0.46,0.48,0.4,0.195,0.75]
t_2	[0.77,0.52,0.42,0.6,0.64]	[0.65,0.42,0.7,0.54,0.69]	[0.78,0.65,0.76,0.56,0.64]
t_3	[0.73,0.2,0.715,0.395,0.64]	[0.95,0.25,0.7,0.4,0.39]	[0.98,0.36,0.76,0.445,0.63]
t_4	[0.69,0.35,0.545,0.45,0.96]	[0.58,0.35,0.56,0.75,0.66]	[0.78,0.35,0.515,0.55,0.66]

Definition 3.9. Let us consider two SVPNS-Sets (K, Y) and (L, Z) over the same universe Û. Then, the operation 'OR' or 'Join' operation is defined by

$$(K, Y) \vee (L, Z) = (\S, Y \times Z),$$

where the corresponding truth membership value, contradiction membership value, ignorance membership value, unknown membership value and false membership value of ($\S$, $Y\times Z$) are measured by $\acute{Y}_{Q(u,v)}(\wp*) = \max\{\acute{Y}_{K(u)}(\wp*), \acute{Y}_{L(v)}(\wp*)\}$, $\acute{C}_{Q(u,v)}(\wp*) = \max\{\acute{C}_{K(u)}(\wp*), \acute{C}_{L(v)}(\wp*)\}$, $\acute{Z}_{Q(u,v)}(\wp*) = \frac{(\acute{Z}_{K(u)}(\wp*)+\ \acute{Z}_{L(v)}(\wp*))}{2}$, $\acute{U}_{Q(u,v)}(\wp*) = \frac{(\acute{U}_{K(u)}(\wp*)+\ \acute{U}_{L(v)}(\wp*))}{2}$, $\acute{R}_{Q(u,v)}(\wp*) = \min\{\acute{R}_{K(u)}(\wp*), \acute{R}_{L(v)}(\wp*)\}$, for all $u\in K$, $v\in L$, $\wp*\in$Û.

Example 3.3. Assume that (K, E) and (L, F) are two SVPNS-Sets over the same universe Û as shown in Example 3.2. Then, the tabulated representation of 'OR' or 'Join' operation between (K, E) and (L, F) is as follows:

Û	(Grade, Semester Fee)	(Grade, Faculties)	(Grade, Students facilities)
t_1	[0.88,0.5,0.565,0.425,0.1]	[0.88,0.54,0.565,0.39,0.1]	[0.88,0.48,0.505,0.17,0.1]
t_2	[0.85,0.52,0.385,0.515,0.25]	[0.85,0.42,0.665,0.455,0.69]	[0.85,0.65,0.725,0.475,0.49]
t_3	[0.76,0.44,0.825,0.545,0.25]	[0.95,0.44,0.81,0.55,0.19]	[0.98,0.44,0.87,0.595,0.25]
t_4	[0.82,0.68,0.375,0.555,0.37]	[0.82,0.65,0.39,0.855,0.37]	[0.87,0.69,0.345,0.655,0.37]

Û	(Infrastructures, Semester Fee)	(Infrastructures, Faculties)	(Infrastructures, Students facilities)
t_1	[0.87,0.69,0.715,0.445,0.47]	[0.88,0.54,0.675,0.51,0.22]	[0.73,0.48,0.615,0.29,0.36]
t_2	[0.77,0.52,0.355,0.565,0.25]	[0.65,0.42,0.635,0.505,0.52]	[0.78,0.65,0.695,0.525,0.49]

t_3	[0.73,0.25,0.785,0.505,0.64]	[0.95,0.25,0.77,0.51,0.19]	[0.98,0.36,0.83,0.555,0.63]
t_4	[0.69,0.68,0.375,0.295,0.46]	[0.66,0.65,0.39,0.595,0.46]	[0.87,0.69,0.345,0.395,0.46]

Û	(Faculties, Semester Fee)	(Faculties, Faculties)	(Faculties, Students facilities)
t_1	[0.89,0.82,0.46,0.45,0.2]	[0.89,0.82,0.46,0.415,0.22]	[0.89,0.82,0.4,0.195,0.36]
t_2	[0.78,0.98,0.42,0.6,0.25]	[0.78,0.98,0.7,0.54,0.64]	[0.78,0.98,0.76,0.56,0.49]
t_3	[0.98,0.65,0.715,0.395,0.39]	[0.98,0.65,0.7,0.4,0.19]	[0.98,0.65,0.76,0.445,0.39]
t_4	[0.78,0.68,0.545,0.45,0.66]	[0.78,0.45,0.56,0.75,0.47]	[0.87,0.69,0.515,0.55,0.47]

Theorem 3.3. Let us consider two SVPNS-Sets (K, Y) and (L, Z) defined over the same universe Û. Then, the following results hold:

(*i*) $[(K, Y) \vee (L, Z)]^c = (K, Y)^c \wedge (L, Z)^c$;

(*ii*) $[(K, Y) \wedge (L, Z)]^c = (K, Y)^c \vee (L, Z)^c$.

Proof. (*i*) Assume that $(K, Y) = \{(y, \{(\wp*, Ý_{K(y)}(\wp*), Ć_{K(y)}(\wp*), Ź_{K(y)}(\wp*), Ú_{K(y)}(\wp*), Ŕ_{K(y)}(\wp*)):\wp*\in Û\}): y\in Y\}$ and $(L, Z) = \{(y, \{(\wp*, Ý_{L(y)}(\wp*), Ć_{L(y)}(\wp*), Ź_{L(y)}(\wp*), Ú_{L(y)}(\wp*), Ŕ_{L(y)}(\wp*)):\wp*\in Û\}): y\in Z\}$ are any two SVPNS-Sets over the same universe Û. Suppose that $(§, Y\times Z) = (K, Y) \vee (L, Z)$, where $Q(u, v) = \{(\wp*,$ $\max\{Ý_{K(u)}(\wp*), Ý_{L(v)}(\wp*)), \max\{Ć_{K(u)}(\wp*), Ć_{L(v)}(\wp*)\}, \frac{(Ź_{K(u)}(\wp*)+ Ź_{L(v)}(\wp*))}{2}, \frac{(Ú_{K(u)}(\wp*)+ Ú_{L(v)}(\wp*))}{2}, \min\{Ŕ_{K(u)}(\wp*), Ŕ_{L(v)}(\wp*)\}) : \wp*\in Û, u\in K, v\in L\}$.

We have, $[(K, Y) \vee (L, Z)]^c$

$= \{(\wp*, \min\{1-Ý_{K(u)}(\wp*), 1-Ý_{L(v)}(\wp*)\}, \min\{1-Ć_{K(u)}(\wp*), 1-Ć_{L(v)}(\wp*)\}, \frac{(1-Ź_{K(u)}(\wp*))+(1-Ź_{L(v)}(\wp*))}{2},$ $\frac{(1-Ú_{K(u)}(\wp*))+(1- Ú_{L(v)}(\wp*))}{2}, \max\{1-Ŕ_{K(u)}(\wp*), 1-Ŕ_{L(v)}(\wp*)\}): \wp*\in Û, u\in K, v\in L\}$.

Now, $(K, Y)^c \wedge (L, Z)^c$

$= \{(\wp*, Ý_{K(y)}(\wp*), Ć_{K(y)}(\wp*), Ź_{K(y)}(\wp*), Ú_{K(y)}(\wp*), Ŕ_{K(y)}(\wp*)): \wp*\in Û, u\in K, v\in L)\}^c \wedge \{(\wp*, Ý_{L(y)}(\wp*), Ć_{L(y)}(\wp*), Ź_{L(y)}(\wp*), Ú_{L(y)}(\wp*), Ŕ_{L(y)}(\wp*)): \wp*\in Û, u\in K, v\in L\}^c$.

$= \{(\wp*, 1-Ý_{K(y)}(\wp*), 1-Ć_{K(y)}(\wp*), 1-Ź_{K(y)}(\wp*), 1-Ú_{K(y)}(\wp*), 1-Ŕ_{K(y)}(\wp*)): \wp*\in Û, u\in K, v\in L\} \wedge \{(\wp*, 1-Ý_{L(y)}(\wp*), 1-Ć_{L(y)}(\wp*), 1-Ź_{L(y)}(\wp*), 1-Ú_{L(y)}(\wp*), 1- Ŕ_{L(y)}(\wp*)): \wp*\in Û, u\in K, v\in L\}$.

$= \{(\wp*, \min\{1-Ý_{K(u)}(\wp*), 1-Ý_{L(v)}(\wp*)\}, \min\{1-Ć_{K(u)}(\wp*), 1-Ć_{L(v)}(\wp*)\}, \frac{(1-Ź_{K(u)}(\wp*))+(1- Ź_{L(v)}(\wp*))}{2},$ $\frac{(1-Ú_{K(u)}(\wp*))+(1- Ú_{L(v)}(\wp*))}{2}, \max\{1-Ŕ_{K(u)}(\wp*), 1-Ŕ_{L(v)}(\wp*)\}): \wp*\in Û, u\in K, v\in L\}$.

Therefore, $[(K, Y) \vee (L, Z)]^c = (K, Y)^c \wedge (L, Z)^c$.

(*ii*) Assume that $(K, Y) = \{(\wp*, \acute{Y}_{K(y)}(\wp*), \acute{C}_{K(y)}(\wp*), \acute{Z}_{K(y)}(\wp*), \acute{U}_{K(y)}(\wp*), \acute{R}_{K(y)}(\wp*)): \wp*\in\hat{U}\}$ and $(L, Z) = \{(\wp*, \acute{Y}_{L(y)}(\wp*), \acute{C}_{L(y)}(\wp*), \acute{Z}_{L(y)}(\wp*), \acute{U}_{L(y)}(\wp*), \acute{R}_{L(y)}(\wp*)): \wp*\in\hat{U}\}$ be any two SVPNS-Sets over the same universe $\hat{U}$. Suppose that $(P, Y\times Z) = [(K, Y) \wedge (L, Z)]$, where $P(u, v) = \{(\wp*, \min\{\acute{Y}_{K(u)}(\wp*), \acute{Y}_{L(v)}(\wp*)\}, \min\{\acute{C}_{K(u)}(\wp*), \acute{C}_{L(v)}(\wp*)\}, \frac{(\acute{Z}_{K(u)}(\wp*)+\ \acute{Z}_{L(v)}(\wp*))}{2}, \frac{(\acute{U}_{K(u)}(\wp*)+\ \acute{U}_{L(v)}(\wp*))}{2}, \max\{\acute{R}_{K(u)}(\wp*), \acute{R}_{L(v)}(\wp*)): \wp*\in\hat{U}, u\in K, v\in L\}$.

Now, $[(K, Y) \wedge (L, Z)]^c$

$= \{(\wp*, \min\{1\text{-}\acute{Y}_{K(u)}(\wp*), 1\text{-}\acute{Y}_{L(v)}(\wp*)\}, \min\{1\text{-}\acute{C}_{K(u)}(\wp*), 1\text{-}\acute{C}_{L(v)}(\wp*)\}, \frac{(1-\acute{Z}_{K(u)}(\wp*))+(1-\ \acute{Z}_{L(v)}(\wp*))}{2}, \frac{(1-\acute{U}_{K(u)}(\wp*))+(1-\ \acute{U}_{L(v)}(\wp*))}{2}, \max\{1\text{-}\acute{R}_{K(u)}(\wp*), 1\text{-}\acute{R}_{L(v)}(\wp*)\}): \wp*\in\hat{U}, u\in K, v\in L\}^c$.

$= \{(\wp*, \max\{1\text{-}\acute{Y}_{K(u)}(\wp*), 1\text{-}\acute{Y}_{L(v)}(\wp*)\}, \max\{1\text{-}\acute{C}_{K(u)}(\wp*), 1\text{-}\acute{C}_{L(v)}(\wp*)\}, \frac{(1-\acute{Z}_{K(u)}(\wp*))+(1-\ \acute{Z}_{L(v)}(\wp*))}{2}, \frac{(1-\acute{U}_{K(u)}(\wp*))+(1-\ \acute{U}_{L(v)}(\wp*))}{2}, \min\{1\text{-}\acute{R}_{K(u)}(\wp*), 1\text{-}\acute{R}_{L(v)}(\wp*)\}): \wp*\in\hat{U}, u\in K, v\in L\}$.

Now, $(K, Y)^c \vee (L, Z)^c$

$= \{(\wp*, \acute{Y}_{K(y)}(\wp*), \acute{C}_{K(y)}(\wp*), \acute{Z}_{K(y)}(\wp*), \acute{U}_{K(y)}(\wp*), \acute{R}_{K(y)}(\wp*)): \wp*\in\hat{U}, u\in K, v\in L\}^c \vee \{(\wp*, \acute{Y}_{L(y)}(\wp*), \acute{C}_{L(y)}(\wp*), \acute{Z}_{L(y)}(\wp*), \acute{U}_{L(y)}(\wp*), \acute{R}_{L(y)}(\wp*)): \wp*\in\hat{U}, u\in K, v\in L\}^c$.

$= \{(\wp*, 1\text{-}\acute{Y}_{K(y)}(\wp*), 1\text{-}\acute{C}_{K(y)}(\wp*), 1\text{-}\acute{Z}_{K(y)}(\wp*), 1\text{-}\acute{U}_{K(y)}(\wp*), 1\text{-}\acute{R}_{K(y)}(\wp*)): \wp*\in\hat{U}, u\in K, v\in L\} \vee \{(\wp*, 1\text{-}\acute{Y}_{L(y)}(\wp*), 1\text{-}\acute{C}_{L(y)}(\wp*), 1\text{-}\acute{Z}_{L(y)}(\wp*), 1\text{-}\acute{U}_{L(y)}(\wp*), 1\text{-}\acute{R}_{L(y)}(\wp*)): \wp*\in\hat{U}, u\in K, v\in L\}$.

$= \{(\wp*, \max\{1\text{-}\acute{Y}_{K(u)}(\wp*), 1\text{-}\acute{Y}_{L(v)}(\wp*)\}, \max\{1\text{-}\acute{C}_{K(u)}(\wp*), 1\text{-}\acute{C}_{L(v)}(\wp*)\}, \frac{(1-\acute{Z}_{K(u)}(\wp*))+(1-\ \acute{Z}_{L(v)}(\wp*))}{2}, \frac{(1-\acute{U}_{K(u)}(\wp*))+(1-\ \acute{U}_{L(v)}(\wp*))}{2}, \min\{1\text{-}\acute{R}_{K(u)}(\wp*), 1\text{-}\acute{R}_{L(v)}(\wp*)\}): \wp*\in\hat{U}, u\in K, v\in L\}$.

Therefore, $[(K, Y) \wedge (L, Z)]^c = (K, Y)^c \vee (L, Z)^c$.

4. Conclusions:

In this article, we have extended the notion of single-valued neutrosophic soft set, and grounded the idea of SVPNS-Set. By introducing the notion of SVPNS-Set, we have formulated some results on them. It is hoped that, researchers of different branches of science can done many new investigations based on these notion of SVPNS-Set.

Conflict of Interest: The authors declare that they have no conflict of interest.

Authors Contribution: Each author contributed equally to the formation of this article.

References:

1. Molodtsov, D. (1999). Soft set theory-First results. Computer & Mathematics with Applications, 37, 19-31.
2. Ali, M.I., Feng, F., Liu, X., Min, W.K., & Shabir, M. (2009). On some new operations in soft set theory. Computer & Mathematics with Applications, 57(9), 1547-1553.
3. Maji, P.K., Biswas, R., & Roy, A.R. (2002). An application of soft sets in a decision making problem. Computer & Mathematics with Applications, 44, 1077-1083.
4. Maji, P.K., Biswas, R., & Roy, A.R. (2003). Soft set theory. Computer & Mathematics with Applications, 45, 555-562.
5. Smarandache, F. (1998). A unifying field in logics, neutrosophy: neutrosophic probability, set and logic. Rehoboth, American Research Press.
6. Zadeh, L.A. (1965). Fuzzy sets. Information and Control, 8(3), 338-353.
7. Atanassov, K. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20, 87-96.
8. Deli, I., Ali, M., & Smarandache, F. (2015). Bipolar neutrosophic sets and their applications based on multicriteria decision making problems. International Conference on Advanced Mechatronic Systems(ICAMechS), Beijing, 2015, 249-254, doi: 10.1109/ICAMechS.2015.7287068.
9. Biswas, P., Pramanik, S., & Giri, B.C. (2014). Entropy based grey relational analysis method for multi-attribute decision making under single valued neutrosophic assessments. Neutrosophic Sets and Systems, 2, 102-110.
10. Biswas, P, Pramanik, S. & Giri, B.C. (2014). A new methodology for neutrosophic multi-attribute decision-making with unknown weight information. Neutrosophic Sets and Systems, 3, 42-50.
11. Pramanik, S., Dalapati, S., Alam, S., Smarandache, S., & Roy, T.K. (2018). NS-cross entropy based MAGDM under single valued neutrosophic set environment. Information, 9(2), 37.
12. Maji, P.K. (2012). A neutrosophic soft set approach to a decision making problem. Annals of Fuzzy Mathematics and Informatics, 3(2), 313-319.
13. Maji, P.K. (2013). Neutrosophic soft set. Annals of Fuzzy Mathematics and Informatics, 5(1), 157-168.
14. Dey, P.P., Pramanik, S., & Giri, B.C. (2015). Generalized neutrosophic soft multi-attribute group decision making based on TOPSIS. Critical Review, 11, 41-55.
15. Dey, P.P., Pramanik, S., & Giri, B.C. (2016). Neutrosophic soft multi-attribute group decision making based on grey relational analysis method. Journal of New Results in Science, 10, 25-37.
16. Dey, P.P., Pramanik, S., & Giri, B.C. (2016). Neutrosophic soft multi-attribute decision making based on grey relational projection method. Neutrosophic Sets and Systems, 11, 98-106.

17. Das, S., Kumar, S., Kar, S., & Pal, T. (2019). Group decision making using neutrosophic soft matrix: An algorithmic approach. Journal of King Saud University-Computer and Information Sciences, 31(4), 459-468.
18. Karaaslan, F. (2015). Neutrosophic soft sets with applications in decision making. International Journal of Information Science and Intelligent System, 4(2), 1-20.
19. Jha, S., Kumar, R., Son, L.H., Chatterjee, J.M., Khari, M., Yadav, N., & Smarandache, F. (2019). Neutrosophic soft set decision making for stock trending analysis. Evolving Systems, 10, 621-627.
20. Pramanik, S., & Dalapati, S. (2016). GRA based multi criteria decision making in generalized neutrosophic soft set environment. Global Journal of Engineering Science and Research Management, 3(5), 153-169.
21. Pramanik, S., Dey, P.P., & Giri, B.C. (2015). TOPSIS for single valued neutrosophic soft expert set based multi-attribute decision making problems. Neutrosophic Sets and Systems, 10, 88-95.
22. Mukherjee, A., & Das, R. (2020). Neutrosophic bipolar vague soft set and its application to decision making problems. Neutrosophic Sets and Systems, 32, 410-424.
23. Bera, T., & Mahapatra, N.K. (2017). Introduction to neutrosophic soft topological space. OPSEARCH, 54, 841-867.
24. Bera, T., & Mahapatra, N.K. (2018). On neutrosophic soft topological space. Neutrosophic Sets and Systems, 19, 3-15.
25. Aras, Ç.G., Ozturk, T.Y., & Bayramov, S. (2019). Separation axioms on neutrosophic soft topological spaces. Turkish Journal of Mathematics, 43, 498-510.
26. Das, S., & Pramanik, S. (2020). Neutrosophic simply soft open set in neutrosophic soft topological space. Neutrosophic Sets and Systems, 38, 235-243.
27. Mehmood, A., Nadeem, F., Park, C., Gorge, N., Humaira, K., Khan, M.R., & Abbas, N. (2020). Neutrosophic soft α-open set in neutrosophic soft topological spaces. Journal of Algorithms and Computation, 52(1), 37-66.
28. Smarandache, F. (2018). Extension of soft set to hypersoft set, and then to plithogenic hypersoft set, Neutrosophic Sets and Systems, 22, 168-170.
29. Mallick, R., & Pramanik, S. (2020). Pentapartitioned neutrosophic set and its properties. Neutrosophic Sets and Systems, 36, 184-192.
30. Das, S., Shil, B., & Pramanik, S. (2021). SVPNS-MADM strategy based on GRA in SVPNS Environment. *Neutrosophic Sets and Systems*, 47, 50-65.
31. Das, S., Shil, B., & Tripathy, B.C. (2021). Tangent similarity measure based MADM-strategy under SVPNS-environment. Neutrosophic Sets and Systems, 43, 93-104.

32. Pramanik, S. (2023). SVPNN-ARAS strategy for MCGDM under pentapartitioned neutrosophic number environment. *Serbian Journal of Management*, *18*(2), 405-420. doi: 10.5937/sjm18-44545.
33. Majumder, P., Paul, A., & Pramanik, S. (2023). Single-valued pentapartitioned neutrosophic weighted hyperbolic tangent similarity measure to determine the most significant environmental risks during the COVID-19 pandemic. Neutrosophic Sets and Systems, 57, 57-75.
34. Das, S., Das, R., Granados, C., & Mukherjee, A. (2021). Pentapartitioned neutrosophic Q-Ideals of Q-algebra. Neutrosophic Sets and Systems, 41, 52-63.
35. Das, S., & Tripathy, B.C. (2021). Pentapartitioned neutrosophic topological space. *Neutrosophic Sets and Systems*, 45, 121-132.

Received: Feb 5, 2024. Accepted: April 26, 2024

Neutrosophic Sets and Systems, Vol. 67, 2024
University of New Mexico

Approach to Multi-Criteria Decision-Making in a Neutrosophic Picture Hyper-Soft Set Environment using Generalized Neutrosophic TOPSIS

Manpreet Kaur[1*] and Akanksha Singh[2]

[1] Department of Mathematics, Chandigarh University, Gharuan, Mohali, India; manpreet.e9900@cumail.in
[2] Department of Mathematics, Chandigarh University, Gharuan, Mohali, India; akanksha.e10462@cumail.in

* Correspondence: manpreettk24@gmail.com

Abstract: Given the complexity of today's world, we might need to work with numbers requiring multi-attribute functions, such as those having positive, neutral, and negative membership and those having truth, indeterminacy, and falsity membership. Adding these numbers together to get a single real number is the most important factor. In situations like this, decision-makers have more difficult choices and are unable to apply the single attribute function of the soft set theory. To address this constraint, the hyper-soft set theory with multi-attribute functions is introduced. We combine the notion of neutrosophic hypersoft set with picture fuzzy hypersoft set to form a single theory of neutrosophic picture hypersoft set in this study. We introduce the notions of correlation coefficient and weighted correlation coefficient and demonstrate its fundamental properties for neutrosophic picture hypersoft set. Then, we introduce the notions of a weighted average operator and a weighted geometric operator for neutrosophic picture hypersoft set by using the various aggregation operators with a suitable example. Making decisions based on several factors and choosing the best option is multi-criteria decision-making or MCDM. When ranking and choosing options based on a distance metric, one essential and useful strategy is the Technique of order preference by similarity to an ideal solution (TOPSIS). We demonstrate the accuracy of the fuzzy TOPSIS methodology by extending it to neutrosophic fuzzy TOPSIS and using neutrosophic picture hypersoft set theory to describe the MCDM problem in this study. We provide a generalized neutrosophic TOPSIS approach to demonstrate correlation coefficients and the effectiveness of this approach with an appropriate example. Finally, we offer a comparison to prior studies to demonstrate the viability of the proposed approach.

Keywords: Picture fuzzy set, soft set, Hyper-soft set, Neutrosophic set.

---

1. Introduction

The subject of how to express worry in mathematical modelling has nevertheless received a lot of attention. Different techniques have been suggested and endorsed by numerous researchers from across the world to reduce ambiguity [1]. To cope with uncertain, ambiguous, and indefinite things, Molodtsov [2] has presented Soft Sets (SSs) as a practical statistical technique. By Maji et al. [3], SSs have been enlarged to include the idea of fuzzy soft sets (FSSs) and detailed the various operations and attributes of SSs [4]. Maji's method of SSs has been modified by Ali et al. [5], who also used its features to create a few new operations. The idea of using soft matrices in operations has been introduced by Cagman and Enginoglu [6], who also have described their characteristics. They have also devised a decision-making (DM) strategy to tackle issues associated with uncertainty. Smarandache [7] has initially established a neutrosophic set (NS). A single-valued NS has been first used for the CC of IFS by Wang et al. [8]. NS has been extended into neutrosophic soft set (NSS) by Maji [9]. With certain operations and attributes, Broumi [10] has created a general NSS and applied it to DM problems. Cuong and Kreinovich [11] have extended the concepts of FS and IFS to form a picture fuzzy set (PFS). None of the above research can help with the problems when different characteristics have associated sub-attributes.

1.1. Literature Review

Zadeh [12] 1965 initially introduced the concept of fuzzy sets (FSs) to address issues that include uncertainty and anxiety. Atanassov [13] has extended FSs into an intuitionistic fuzzy set (IFS). However, IFS's theory is not able to address the issue of inconsistent data. To cope with uncertain, ambiguous, and indefinite things, Molodtsov [2] has presented soft sets (SSs) as a practical statistical technique. The introduction of the neutrosophic set (NS) concept, featuring the assignment of truth, indeterminacy, and falsity grades to individual set elements, has been made by Smarandache [7]. Maji et al. [3] have formulated fuzzy soft sets (FSSs) by integrating the principles of SSs and FSs. To address the limitations of NS, the introduction of the single-valued neutrosophic set (SVNS) with constrained membership values has been undertaken by Wang et al. [8]. Maji [9] has extended NS to a neutrosophic soft set (NSS). Cuong and Kreinovich [11] have further developed the concept of FS and IFS into a picture fuzzy set (PFS). Yang et al. [14] have proposed the combination of PFS and SS, resulting in a picture fuzzy soft set (PFSS). Hypersoft sets (HSSs) have been introduced by Smarandache [15] as a solution to the limitations of SSs. Jaber et al. [16] presented a generalized picture fuzzy soft set (GPFSS) by merging the concepts of PFSS and PFS in a hybrid mode to improve DM accuracy. Saqlain et al. [17] have built the aggregation operators for a neutrosophic hypersoft set (NHSS) and modified a DM approach for NHSS. Yolcu et al. [18, 19] have extended the notion of HS by introducing both a fuzzy hypersoft set (FHSS) and an intuitionistic fuzzy hypersoft set (IFHSS). Chinnadurai and Bobin [20] have extended the concept of HSS and have proposed a picture fuzzy hypersoft set (PFHSS) theory.

A summary of the conceptual framework alongside the preceding structures is presented in Table 1 below:

Table 1: Conceptual Framework Summary

Sr. No.	Proposed Structure	Corresponding Authors	Year of Publication	Key Findings
1	Fuzzy Set (FS)	Zadeh	1965	Every element in the universal set is assigned a membership value within the range of 0 to 1.
2	Intuitionistic Fuzzy Set (IFS)	Atanassov	1986	It signifies the degree to which an element of a set either belongs or does not belong.
3	Soft Set (SS)	Molodtsov	1999	It handles uncertainty in a parametric way.
4	Neutrosophic Set (NS)	Smarandache	1999	Every element within the set is assessed based on its levels of truth, indeterminacy, and falsity.
5	Fuzzy Soft Set (FSS)	Maji et al.	2001	Every power set within the universal set has been assigned fuzzy values.
6	Single-valued Neutrosophic Set (SVNS)	Wang et al.	2010	It imposes specific limitations on membership values to address the limitations encountered in NS.

7	Neutrosophic Soft Set (NSS)	Maji	2013	It combines SVNS with SS.
8	Picture Fuzzy Set (PFS)	Cuong et al.	2013	An extension of FS and IFS.
9	Picture Fuzzy Soft Set (PFSS)	Yang et al.	2015	It combines PFS with SS and addresses the inconsistent problem of data.
10	Hyper-soft Set (HSS)	Smarandache	2018	It imposes specific constraints on membership values to address the limitations encountered in SS.
11	Generalized Picture Fuzzy Soft Set (GPFSS)	Jaber et al.	2019	Integrating PFSS and PFS in a hybrid mode involves incorporating additional data into the PFS output to enhance decision-making accuracy.
12	Neutrosophic Hyper-Soft Set (NHSS)	Saqlain et al.	2020	It combines NS with HSS.
13	Fuzzy Hyper-soft Set (FHSS)	Yolcu et al.	2021	A fuzzy membership degree is assigned to each element in the power set.
14	Intuitionistic Fuzzy Hyper-Soft Set (IFHSS)	Yolcu et al.	2021	It combines IFS with HSS.
15	Picture Fuzzy Hyper-Soft Set (PFHSS)	Chinnadurai and Bobin	2021	Combination of PFSS and FHSS.

1.2. Motivation

The introduction of the concept of a simplified neutrosophic set (SNS) with various aggregation operators (AOs) was carried out by Ye [21] who introduced a DM approach based on the suggested AOs. Karaaslan [22] has defined Possibility NS and provided an NSS selection procedure to resolve those and-product-based uncertain issues. To deal with uncertainty, a function is changed into a multi-attribute function, Smarandache [15] has expanded the SSs to a hypersoft set (HSS) to overcome these formulations. To convert the fuzzily formatted neutrosophic number into the crisp form, Saqlain et al. [23] have modified a DM approach for the neutrosophic hypersoft set (NHSS). Chinnadurai and Bobin [20] have extended the concept of HSS and have proposed a picture fuzzy hypersoft set (PFHSS) theory. Zulqarnain et al. [24] have expanded on the idea of NHSS and provided specific methods for NHSS. Saqlain et al. [25] have expanded NHSS's concept and offered an interval-valued neutrosophic hypersoft set (IVNHSS). An intuitionistic fuzzy hypersoft set (IFHSS) is being developed by Zulqarnain et al. [26] who established an approach also by generating a CC to overcome DM challenges. Rahman et al. [27] have introduced the neutrosophic parameterized hypersoft set (NPHS) in DM problems. The HSS theory's foundational ideas have been researched by Saeed et al. [28]. Rahman et al. [29] have designed a model of NPHS under the environment of FS, IFS and NS. Ihsan et al. [30] have expanded a soft expert set to a hypersoft expert set to address DM issues. The IVNHSS's basic operations have been researched by Rana Muhammad Zulqarnain et al. [31]. Numerous additional studies have been performed in a neutrosophic environment, and their implications for daily life are discussed [32-54].

Today's practical applications can't be handled by a single attribute function like IFS or SS [55-57]. Improved versions of SSs and new varieties of soft sets have been made available by Smarandache [58]. He has also established a novel interpretation of the super hyper soft set [59] and added a fuzzy extension to it [60]. Several additional studies have been conducted within a neutrosophic context, and various researchers delve into the implications these studies have for everyday life [61-65]. Applying the multi-attribute function in HSS, an extension of SS, allows it to get around this restriction. Additionally, HSS may be used for any DM issues without imposing any constraints on the characteristics used by the decision-makers (DMs). Data collection for DMs may be accomplished without information loss by merging HSS with other hybrid fuzzy structures. Positive, neutral, and negative evaluations all depend on one another in the picture fuzzy set (PFS), and their aggregate cannot be larger than one. The main reason for choosing the Hypersoft Set (HSS) is because a soft set's circumstances cannot manage scenarios where attributes are more than one and further divided. Therefore, it is important to create a new strategy to address these. Taking into account both the positive and negative aspects of each option, decision-making techniques aid specialists in choosing a suitable one. Drawing inspiration from the work of Chinnadurai and Bobin [20] who have combined the structures of PFS and HSS to create a theory named a "picture fuzzy hypersoft set" (PFHSS). Thus, this study’s main goal is to combine the structure of PFHSS with NS to formulate a new theory called neutrosophic picture hypersoft set (NPHSS). This study is limited to its theory and any related advancement. We illustrate the merits of the proposed theory using the instances.

The paper is structured in the following manner: Section 2: This Section covers the key terminologies and concepts that have been used for this study. Section 3: The notion of NPHSS is established with a suitable example. Section 4: The notion of correlation-coefficient for NPHSS is introduced and its main characteristics are established. Section 5: The notion of weighted correlation-coefficient for NPHSS is presented and its main characteristics are established. Section 6: The notions of a weighted average operator as well as a weighted geometric operator for NPHSS by using the various AOs are presented with suitable examples. Section 7: An algorithm for addressing MCDM problems using NPHSS through the TOPSIS method and its applications in DM problems are highlighted in this Section. Section 8: This Section discusses the comparison study of the suggested notion. Section 9: The study presents the prospects of the proposed measure in another environment.

2. Preliminaries

In this Section, we introduce fundamental definitions pertinent to the subject, and we will consistently adhere to the specified notations unless explicitly stated otherwise.

Let us assume that U be the universal set and u ∈ U, P_U be the power set of U.

Definition 2.1. A set $\mathcal{F} = \{(u, \mathcal{M}(u)) : u \in U\}$ is known as a fuzzy set (FS) [12], where $\mathcal{M}(u): U \to [0,1]$ be the membership's degree of u over U.

Definition 2.2. A set $I = \{(u, \mathcal{M}(u), N(u)) : u \in U\}$ is known as an intuitionistic fuzzy set (IFS) [13], where $\mathcal{M}(u): U \to [0,1]$ be the membership's degree of u and $N(u): U \to [0,1]$ be the non-membership degree of u such that $\forall\, u \in U$, and $0 \leq \mathcal{M}(u) + N(u) \leq 1$, where $\mathcal{H}(u) = 1 - \mathcal{M}(u) - N(u)$ be the hesitancy's degree of u.

Definition 2.3. A set $\wp = \{(u, \mathcal{M}_p(u), \mathcal{M}_N(u), \mathcal{M}_n(u)) : u \in U\}$ is known as a picture fuzzy set (PFS) [11], where $\mathcal{M}_p(u), \mathcal{M}_N(u), \mathcal{M}_n(u): U \to [0,1]$ be the values of positive membership, neutral membership and negative membership of u respectively, such that $\forall\, u \in U$, and $0 \leq \mathcal{M}_p(u) + \mathcal{M}_N(u) + \mathcal{M}_n(u) \leq 1$, where $\mathcal{R}(u) = 1 - (\mathcal{M}_p(u) + \mathcal{M}_N(u) + \mathcal{M}_n(u))$ be the refusal membership's degree of u.

Definition 2.4. An ordered pair of the form (M_S, S_p) is known as a soft set (SS) [2], if $M_S: S_p \to P_U$, S_p be a parameters' set and $S \subseteq S_p$. Here, M_S is known as an approximate function of SS such that $M_S(x) = \emptyset \text{ if } x \notin S$ where $M_S(x)$ is known as an x-approximate value set which is made up of the parameter's associated objects $x \in S_p$.

Definition 2.5. An ordered pair of the form ($\widetilde{M_S}, S_p$) defines a fuzzy soft set (FSS) [3], if $\widetilde{M_S}: S_p \to C_F(U)$, $C_F(U)$ be a set of all fuzzy subsets of U, S_p be a set of parameters and $S \subseteq S_p$. Here, $\widetilde{M_S}(x) = \emptyset \text{ if } x \notin S$ where $\emptyset$ be a null FS.

Definition 2.6. A set $NS = \{(u, T_{NS}(u), I_{NS}(u), F_{NS}(u)) : u \in U\}$ is called a neutrosophic set (NS) [8], which T_{NS} stands for truth-membership's function, I_{NS} stands for indeterminacy-membership's function, and F_{NS} stands for falsity-membership's function such that $T_{NS}(u), I_{NS}(u)$ and $F_{NS}(u) \in [0,1]$ and $0 \leq T_{NS}(u) + I_{NS}(u) + F_{NS}(u) \leq 3\ \forall\, u \in U$. The simplest way to express an element in NS is to use a single-valued neutrosophic number (SVNN) as $u = \langle T_u, I_u, F_u \rangle \forall\, u \in U$.

Definition 2.7. An ordered pair of the form (N, S) defines a neutrosophic soft set (NSS) [9, 10], if $N: S \to C_N(U)$, $C_N(U)$ be a collection of all the neutrosophic sets of U, S_p be a parameters' set and $S \subseteq S_p$.

Definition 2.8. If $(\alpha_1, \alpha_2, \ldots, \alpha_i)$, be i distinct attributes, for all $i \geq 1$, with the corresponding attribute values are $(S_1, S_2, \ldots, S_i)$ respectively, such that $\alpha_m \cap \alpha_n = \emptyset$ for $m \neq n$ where $m, n \in \{1, 2 \ldots, i\}$. Then, an ordered pair $(H, S_1 \times S_2 \times \ldots \times S_i)$ where $H: S_1 \times S_2 \times \ldots \times S_i \to P_U$ is said to be a hyper-soft set (HSS) [15] in U.

Definition 2.9. If $(\alpha_1, \alpha_2, \ldots, \alpha_i)$, be i distinct attributes, for all $i \geq 1$, with the corresponding attribute values are respectively $(S_1, S_2, \ldots, S_i)$, such that $\alpha_m \cap \alpha_n = \emptyset$ for $m \neq n$ where $m, n \in \{1, 2 \ldots, i\}$. Then, an ordered pair (N, H) is defined as a neutrosophic hyper-soft set (NHSS) [23] if there a relation exists $S_1 \times S_2 \times \ldots \times S_i = H$ where $N: H \to N^U, H = S_1 \times S_2 \times \ldots \times S_i$ and N^U be a set of all the neutrosophic subsets in U and $N(S_1 \times S_2 \times \ldots \times S_i) = \{(u, T_{NS}(u), I_{NS}(u), F_{NS}(u)) : u \in U\}$ where T_{NS} stands for truth-membership function, I_{NS} stands for indeterminacy-membership function, and F_{NS} stands for falsity-membership function such that $T_{NS}(u), I_{NS}(u)$ and $F_{NS}(u) \in [0,1]$ and $0 \leq T_{NS}(u) + I_{NS}(u) + F_{NS}(u) \leq 3 \; \forall\, u \in U$.

Remark: For two neutrosophic hypersoft subsets $(NH)_1$ and $(NH)_2$ in U, $(NH)_1$ is considered to be a neutrosophic hypersoft subset of $(NH)_2$ if the followings are satisfied:

$T_{NS}(NH)_1 \leq T_{NS}(NH)_2$, $I_{NS}(NH)_1 \leq I_{NS}(NH)_2$, $F_{NS}(NH)_2 \geq F_{NS}(NH)_2$.

Definition 2.10. An ordered pair (P, H) defines a picture fuzzy hyper-soft set (PFHSS) [20] where $P: H \to C_P, H = S_1 \times S_2 \times \ldots \times S_i$ and C_P be a set of the picture fuzzy subsets in U.

Then, PFHSS represents as follows: $(P,H) = \{(p,h): p \in H = S_1 \times S_2 \times \ldots \times S_i, h \in C_P, C_P \in [0,1]\}$.

Here, $h = \{(u, \mathcal{M}_p(u), \mathcal{M}_N(u), \mathcal{M}_n(u)) : u \in U\}$ and $\mathcal{M}_p(u), \mathcal{M}_N(u), \mathcal{M}_n(u): U \to [0,1]$ be the positive membership, neutral membership and negative membership values respectively, such that $\forall\, u \in U$, and $0 \leq \mathcal{M}_p(u) + \mathcal{M}_N(u) + \mathcal{M}_n(u) \leq 1$, where $\mathcal{R}(u) = 1 - (\mathcal{M}_p(u) + \mathcal{M}_N(u) + \mathcal{M}_n(u))$ be the refusal membership's degree of u.

3. Proposed Neutrosophic Picture Hypersoft Set (NPHSS):

The notion of NPHSS is developed in this Section and goes through its fundamental characteristics of correlation coefficient (CC) as well as weighted CC.

Definition 3.1. Assume that U be the universal set, $(NH)^U$ be a set of the neutrosophic hypersoft subsets and C_P be a set of the picture fuzzy subsets. Let $(\alpha_1, \alpha_2, \ldots, \alpha_i)$ be i distinct attributes, for all $i \geq 1$, with the corresponding attribute values $(S_1, S_2, \ldots, S_i)$ respectively, such that $\alpha_m \cap \alpha_n = \emptyset$ for $m \neq n$ and $m, n \in \{1,2 \ldots, i\}$ and $S_1 \times S_2 \times \ldots \times S_i = H$. Then, an ordered pair (P^N, H) is called as a neutrosophic picture hyper-soft set (NPHSS) in U such that $P^N: S_1 \times S_2 \times \ldots \times S_i \to (NH)^U \times C_P$.

By using the Definition 2.9 and Definition 2.10, it is represented as follows:

$P_H^N(u) = \{(u_j, \mathcal{M}_p(P^N)_{S_i}(u_j), \mathcal{M}_N(P^N)_{S_i}(u_j), \mathcal{M}_n(P^N)_{S_i}(u_j)) \forall u_j \in U\}$ such that $0 \leq \mathcal{M}_p(P^N)_{S_i}(u_j) + \mathcal{M}_N(P^N)_{S_i}(u_j) + \mathcal{M}_n(P^N)_{S_i}(u_j) \leq 1$ where $\mathcal{M}_p(P^N), \mathcal{M}_N(P^N), \mathcal{M}_n: U \to [0,1]$ be positive membership, neutral membership and negative membership, respectively.

Remark: For two neutrosophic picture hypersoft subsets, $(P^N)_1$ and $(P^N)_2$ in U, $(P^N)_1$ is considered to be a neutrosophic picture hypersoft subset of $(P^N)_2$ if the followings are satisfied:

$\mathcal{M}_p(P^N)_1 \leq \mathcal{M}_p(P^N)_2$, $\mathcal{M}_N(P^N)_1 \leq \mathcal{M}_N(P^N)_2$, and $\mathcal{M}_n(P^N)_1 \geq \mathcal{M}_n(P^N)_2$.

Here, $\mathcal{M}_p(P^N), \mathcal{M}_N(P^N), \mathcal{M}_n(P^N): U \to [0,1]$ be the positive membership, neutral membership and negative membership, respectively, such that $\forall\, u \in U$, and $0 \leq \mathcal{M}_p(P^N)(u) + \mathcal{M}_N(P^N)(u) + \mathcal{M}_n(P^N)(u) \leq 1$, where $\mathcal{R}(u) = 1 - (\mathcal{M}_p(P^N)(u) + \mathcal{M}_N(P^N)(u) + \mathcal{M}_n(P^N)(u))$ be the refusal membership's degree of u.

The representation of the extended NPHSS model is shown in Figure 1, as outlined below:

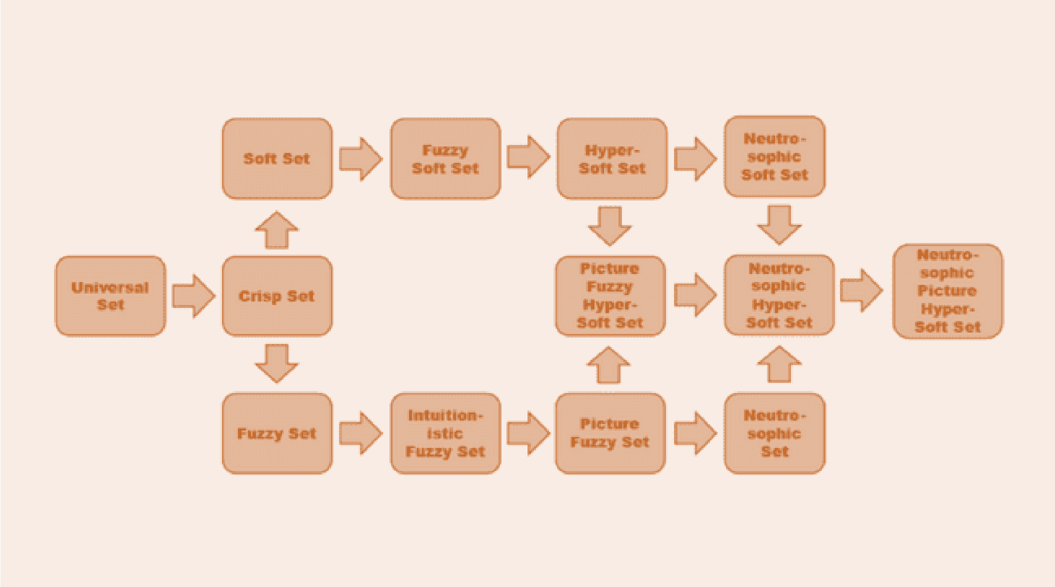


Figure 1: Flow diagram representing the NPHSS Model.

Example 3.1.1. Assuming $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ constitute a team of faculties from three departments that are responsible for evaluating the representative of the department. Let(α_1, α_2): (Criteria Analysis) be three distinct attributes, with the corresponding multi-valued sub-attributes respectively represented as follows:

$$\alpha_1 = \text{representative skills} = \begin{Bmatrix} \mathbb{Q}_{11} = \text{leadership skills}, \\ \mathbb{Q}_{12} = \text{management skills}, \\ \mathbb{Q}_{13} = \text{interpersonal skills} \end{Bmatrix} \text{and} \alpha_2 = \text{representative experience} = \begin{Bmatrix} \mathbb{Q}_{21} = \text{research experience}, \\ \mathbb{Q}_{22} = \text{teaching experience}, \\ \mathbb{Q}_{23} = \text{industry experience} \end{Bmatrix}$$

Then, $H = \alpha_1 \times \alpha_2 = \{\mathbb{Q}_{11}, \mathbb{Q}_{12}, \mathbb{Q}_{13}\} \times \{\mathbb{Q}_{21}, \mathbb{Q}_{22}, \mathbb{Q}_{23}\} = \begin{Bmatrix} \{\mathbb{Q}_{11}, \mathbb{Q}_{21}, \mathbb{Q}_{22}, \mathbb{Q}_{23}\}, \\ \{\mathbb{Q}_{12}, \mathbb{Q}_{21}, \mathbb{Q}_{22}, \mathbb{Q}_{23}\}, \\ \{\mathbb{Q}_{13}, \mathbb{Q}_{21}, \mathbb{Q}_{22}, \mathbb{Q}_{23}\} \end{Bmatrix} = \{\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3\}.$

An NPHSS (P^N, H) is a set of subsets of $\mathcal{F}$ which is introduced by the faculties for a departmental representative (DR) as given below in Table 2.

Table 2: Qualities of a DR in NPHSS

$\mathcal{F}$	$\mathbb{Q}_1$	$\mathbb{Q}_2$	$\mathbb{Q}_3$
$\mathcal{F}_1$	⟨0.04,0.05,0.02⟩	⟨0.01,0.04,0.02⟩	⟨0.03,0.06,0.02⟩
$\mathcal{F}_2$	⟨0.02,0.04,0.05⟩	⟨0.02,0.01,0.04⟩	⟨0.02,0.03,0.06⟩
$\mathcal{F}_3$	⟨0.05,0.02,0.04⟩	⟨0.04,0.02,0.01⟩	⟨0.06,0.02,0.03⟩

4. Proposed Correlation Coefficient (CC) for NPHSS:

This Section introduces the notion of CC and its fundamental features for NPHSSs.

Let $({P^N}_1,H_1)=\left\{\left(u_j,\mathcal{M}_p({P^N}_1)_{S_i}(u_j),\mathcal{M}_N({P^N}_1)_{S_i}(u_j),\mathcal{M}_n({P^N}_1)_{S_i}(u_j)\right):u_j\in U\right\}$ and $({P^N}_2,H_2)=\left\{\left(u_j,\mathcal{M}_p({P^N}_2)_{S_i}(u_j),\mathcal{M}_N({P^N}_2)_{S_i}(u_j),\mathcal{M}_n({P^N}_2)_{S_i}(u_j)\right):u_j\in U\right\}$ be two NPHSSs over U.

Definition 4.1. Let $({P^N}_1,H_1)$ and $({P^N}_2,H_2)$ be two NPHSSs, its neutrosophic informational energies are defined respectively as follows:

$$\mathrm{E}({P^N}_1,H_1)=\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p({P^N}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N({P^N}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n({P^N}_1)_{S_i}(u_j)\right)^2\right] \quad (1)$$

$$\mathrm{E}({P^N}_2,H_2)=\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p({P^N}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N({P^N}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n({P^N}_2)_{S_i}(u_j)\right)^2\right] \quad (2)$$

Here, $1\le i\le k$ and $1\le j\le l$ where $k,l\in\mathbb{N}$ (natural numbers).

Definition 4.2. Let $({P^N}_1,H_1)$ and $({P^N}_2,H_2)$ be two NPHSSs, a correlation measure between $({P^N}_1,H_1)$ and $({P^N}_2,H_2)$ is defined as:

$$\mathrm{M}^c[({P^N}_1,H_1),({P^N}_2,H_2)]=\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p({P^N}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_p({P^N}_2)_{S_i}(u_j)\right)\right]+\\ \left[\left(\mathcal{M}_N({P^N}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_N({P^N}_2)_{S_i}(u_j)\right)\right]+\\ \left[\left(\mathcal{M}_n({P^N}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_n({P^N}_2)_{S_i}(u_j)\right)\right]\end{array}\right\} \quad (3)$$

Proposition 4.2.1. Let $({P^N}_1,H_1)$ and $({P^N}_2,H_2)$ be two NPHSSs. Then,

(i) $\mathrm{M}^c[({P^N}_1,H_1),({P^N}_1,H_1)]=\mathrm{E}({P^N}_1,H_1);$ (4)

(ii) $\mathrm{M}^c[({P^N}_2,H_2),({P^N}_2,H_2)]=\mathrm{E}({P^N}_2,H_2);$ (5)

Proof: By using the Equations (1), (2) and (3), the results of Equations (4) and (5) are obvious.

Definition 4.3. Let $({P^N}_1,H_1)$ and $({P^N}_2,H_2)$ be two NPHSSs, the CC between $({P^N}_1,H_1)$ and $({P^N}_2,H_2)$ is defined as:

$$\mathrm{C}^c[({P^N}_1,H_1),({P^N}_2,H_2)]=\frac{\mathrm{M}^c[({P^N}_1,H_1),({P^N}_2,H_2)]}{\sqrt{\mathrm{E}({P^N}_1,H_1)}\sqrt{\mathrm{E}({P^N}_2,H_2)}} \quad (6)$$

Proposition 4.3.1. Let $({P^N}_1,H_1)$ and $({P^N}_2,H_2)$ be two NPHSSs. Then,

a) $0\le\mathrm{C}^c[({P^N}_1,H_1),({P^N}_2,H_2)]\le 1;$ (7)

b) $C^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] = C^c[(P^N{}_2,H_2),(P^N{}_1,H_1)];$ (8)

c) If $(P^N{}_1,H_1) = (P^N{}_2,H_2)$, then $C^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$ (9)

Proof: a) It is obvious from the Equation (6) that,

$$C^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \geq 0 \tag{10}$$

Now, it is required to prove that:

$$C^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \leq 1. \tag{11}$$

From the Equation (3), we have,

$$M^c = \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)\right]+\\ \left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)\right]+\\ \left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)\right]\end{array}\right\}.$$

$$M^c = \sum_{i=1}^{k}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_1)\right)*\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_1)\right)\right]+\left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_1)\right)*\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_1)\right)\right]+\\ \left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_1)\right)*\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_1)\right)\right]+\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_2)\right)*\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_2)\right)\right]+\\ \left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_2)\right)*\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_2)\right)\right]+\left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_2)\right)*\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_2)\right)\right]+\\ \dots+\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_l)\right)*\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_l)\right)\right]+\left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_l)\right)*\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_l)\right)\right]+\\ \left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_l)\right)*\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_l)\right)\right]\end{array}\right\} \tag{12}$$

By using Cauchy-Schwarz inequality in Equation (12), we have,

$$M^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]^2 \leq \sum_{i=1}^{k}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_2)\right)^2+\dots+\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_l)\right)^2\right]+\\ \left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_2)\right)^2+\dots+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_l)\right)^2\right]+\\ \left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_2)\right)^2+\dots+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_l)\right)^2\right]\end{array}\right\}\times$$

$$\sum_{i=1}^{k}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_2)\right)^2+\dots+\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_l)\right)^2\right]+\\ \left[\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_2)\right)^2+\dots+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_l)\right)^2\right]+\\ \left[\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_2)\right)^2+\dots+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_l)\right)^2\right]\end{array}\right\}.$$

$$\Rightarrow M^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]^2 \leq \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\begin{array}{c}\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2+\\ \left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2+\\ \left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\end{array}\right\}\times\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\begin{array}{c}\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\\ \left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\\ \left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\end{array}\right\}. \tag{13}$$

By using the Equations (1) and (2) in Equation (13), we have,

$$\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]^2 \le \mathrm{E}(P^N{}_1,H_1)\times \mathrm{E}(P^N{}_2,H_2).$$

$$\Rightarrow \mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le \sqrt{\mathrm{E}(P^N{}_1,H_1)}\times\sqrt{\mathrm{E}(P^N{}_2,H_2)}.$$

$$\Rightarrow \frac{\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]}{\sqrt{\mathrm{E}(P^N{}_1,H_1)}\sqrt{\mathrm{E}(P^N{}_2,H_2)}} \le 1. \tag{14}$$

By using the Equation (6) in Equation (14), we have,

$$\mathrm{C}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le 1.$$

Hence, the stated result of Equation (11) is established mathematically.
Now, on combining the Equations (10) and (11), it is well proved that:

$$0 \le \mathrm{C}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le 1.$$

b) It is obvious from the Equation (6) that,

$\mathrm{C}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \mathrm{C}^c[(P^N{}_2,H_2),(P^N{}_1,H_1)].$ Hence, proof of Equation (8) is simple.

c) Now, it is required to prove that: $\mathrm{C}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$

Proof: If $(P^N{}_1,H_1) = (P^N{}_2,H_2)$, then,

From the Equation (6), we have,

$$\mathrm{C}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \frac{\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]}{\sqrt{\mathrm{E}(P^N{}_1,H_1)}\sqrt{\mathrm{E}(P^N{}_2,H_2)}}.$$

Now, using the Equations (1), (2) and (3) in Equation (6), we have,

$$\mathrm{C}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \frac{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]}{\sqrt{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]}\times\sqrt{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]}}$$

$$\Rightarrow \mathrm{C}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$$

Hence, the stated result of Equation (9) is established mathematically.

Definition 4.4. Let $(P^N{}_1,H_1)$ and $(P^N{}_2,H_2)$ be two NPHSSs. The CC between $(P^N{}_1,H_1)$ and $(P^N{}_2,H_2)$ is defined as:

$$\widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \frac{\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]}{\max\{\mathrm{E}(P^N{}_1,H_1),\mathrm{E}(P^N{}_2,H_2)\}}. \tag{15}$$

By using the Equations (1), (2) and (3) in Equation (15), we have,

$$\widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \frac{\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)\right] \\ +\left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)\right] \\ +\left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)\right]\end{array}\right\}}{\max\left\{\begin{array}{c}\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\right], \\ \sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]\end{array}\right\}} \tag{16}$$

Proposition 4.4.1. Let $(P^N{}_1,H_1)$ and $(P^N{}_2,H_2)$ be two NPHSSs. Then,

1) $0 \le \widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le 1;$ (17)

2) $\widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \widetilde{C^c}[(P^N{}_2,H_2),(P^N{}_1,H_1)];$ (18)

3) If $(P^N{}_1,H_1) = (P^N{}_2,H_2)$, then $\widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$ (19)

Proof: 1) It is obvious from the Equations (15) and (16) that,

$$\widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \ge 0. \tag{20}$$

Now, it is required to prove that:

$$\widetilde{C^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le 1. \tag{21}$$

By using Cauchy-Schwarz inequality in Equation (12), we have,

$$\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le \sqrt{\begin{array}{c}\sum_{i=1}^{k}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_l)\right)^2\right]+ \\ \left[\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_l)\right)^2\right]+ \\ \left[\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_l)\right)^2\right]\end{array}\right\}\times \\ \sum_{i=1}^{k}\left\{\begin{array}{c}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_l)\right)^2\right]+ \\ \left[\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_l)\right)^2\right]+ \\ \left[\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_1)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_2)\right)^2+\cdots+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_l)\right)^2\right]\end{array}\right\}\end{array}}.$$

$$\Rightarrow \mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le \sqrt{\begin{array}{c}\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\right\}\times \\ \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right\}\end{array}}.$$

$$\le \sqrt{\left[max\left\{\begin{array}{c}\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\right\}\times \\ \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right\}\end{array}\right\}\right]^2}.$$

$$\Rightarrow \mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] = max\left\{\begin{array}{l}\sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\right\}\times \\ \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right\}\end{array}\right\} \quad (22)$$

By using the Equations (1) and (2) in Equation (22), we have,

$$\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le \max\{\mathrm{E}(P^N{}_1,H_1),\mathrm{E}(P^N{}_2,H_2)\}.$$

$$\Rightarrow \frac{\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]}{\max\{\mathrm{E}(P^N{}_1,H_1),\mathrm{E}(P^N{}_2,H_2)\}} \le 1. \quad (23)$$

Now, using the Equation (15) in Equation (23), we have,

$$\widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le 1.$$

Hence, the stated result of Equation (21) is established mathematically.

Now, on combining the Equations (20) and (21), it is well proved that:

$$0 \le \widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le 1.$$

2) It is clearly stated from the Equations (15) and (16) that,

$$\widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \widetilde{\mathrm{C}^c}[(P^N{}_2,H_2),(P^N{}_1,H_1)]. \ldots\ldots$$

3) Now, it is required to prove that: $\widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$

$Proof$: If $(P^N{}_1,H_1) = (P^N{}_2,H_2)$, then,

From the Equation (15), we have, $\widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \frac{\mathrm{M}^c[(P^N{}_1,H_1),(P^N{}_2,H_2)]}{\sqrt{\mathrm{E}(P^N{}_1,H_1)}\sqrt{\mathrm{E}(P^N{}_2,H_2)}}.$

By using the Equations (1), (2) and (3) in Equation (15), we have,

$$\widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \frac{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]}{\sqrt{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]}\times\sqrt{\sum_{i=1}^{k}\sum_{j=1}^{l}\left[\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\right]}}$$

$$\Rightarrow \widetilde{\mathrm{C}^c}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$$

Hence, the stated result of Equation (19) is established mathematically.

5. Proposed Weighted Correlation Coefficient (WCC) for NPHSS:

This Section introduces the notion of WCC and its fundamental features for NPHSSs. Decision-makers can assign various weights to each of the alternatives with the help of WCC.

Let us consider $\vartheta = \{\vartheta_1, \vartheta_2, \dots, \vartheta_v\}$ and $\omega = \{\omega_1, \omega_2, \dots, \omega_w\}$ be the weights of the alternatives and experts, respectively, such that: $\sum_{x=1}^{v} \vartheta_x = 1, \sum_{y=1}^{w} \omega_y = 1 \; \forall \; \vartheta_x, \omega_y > 0$.

Definition 5.1. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs. The WCC between $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ is defined as:

$$W^C[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \frac{\mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)]}{\sqrt{\mathrm{E}(P^N{}_1, H_1)}\sqrt{\mathrm{E}(P^N{}_2, H_2)}}. \tag{24}$$

By using the Equations (1), (2) and (3) in Equation (24), we have,

$$W^C[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \frac{\sum_{x=1}^{v} \vartheta_x \left\{ \sum_{y=1}^{w} \omega_y \begin{bmatrix} \left(\left(\mathcal{M}_p (P^N{}_1)_{S_i}(u_j) \right) * \left(\mathcal{M}_p (P^N{}_2)_{S_i}(u_j) \right) \right) + \\ \left(\left(\mathcal{M}_N (P^N{}_1)_{S_i}(u_j) \right) * \left(\mathcal{M}_N (P^N{}_2)_{S_i}(u_j) \right) \right) \\ + \left(\left(\mathcal{M}_n (P^N{}_1)_{S_i}(u_j) \right) * \left(\mathcal{M}_n (P^N{}_2)_{S_i}(u_j) \right) \right) \end{bmatrix} \right\}}{\sqrt{\sum_{x=1}^{v} \vartheta_x \left\{ \sum_{y=1}^{w} \omega_y \left[\begin{pmatrix} \left(\mathcal{M}_p (P^N{}_1)_{S_i}(u_j) \right)^2 + \left(\mathcal{M}_N (P^N{}_1)_{S_i}(u_j) \right)^2 \\ + \left(\mathcal{M}_n (P^N{}_1)_{S_i}(u_j) \right)^2 \end{pmatrix} \right] \right\}} \times \sqrt{\sum_{x=1}^{v} \vartheta_x \left\{ \sum_{y=1}^{w} \omega_y \left[\begin{pmatrix} \left(\mathcal{M}_p (P^N{}_2)_{S_i}(u_j) \right)^2 + \left(\mathcal{M}_N (P^N{}_2)_{S_i}(u_j) \right)^2 \\ + \left(\mathcal{M}_n (P^N{}_2)_{S_i}(u_j) \right)^2 \end{pmatrix} \right] \right\}}} \tag{25}$$

Remark: If $\vartheta = \left\{ \frac{1}{v}, \frac{1}{v}, \dots, \frac{1}{v} \right\}$ and $\omega = \left\{ \frac{1}{w}, \frac{1}{w}, \dots, \frac{1}{w} \right\}$, then the WCC given in Equation (24) reduces to the CC as given in Equation (6).

Proposition 5.1.1. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs. Then,

1) $0 \le W^C[(P^N{}_1, H_1), (P^N{}_2, H_2)] \le 1;$ (26)

2) $W^C[(P^N{}_1, H_1), (P^N{}_2, H_2)] = W^C[(P^N{}_2, H_2), (P^N{}_1, H_1)];$ (27)

3) If $(P^N{}_1, H_1) = (P^N{}_2, H_2)$, then $W^C[(P^N{}_1, H_1), (P^N{}_2, H_2)] = 1.$ (28)

Proof: It is as similar to Proposition 4.3.1.

Definition 5.2. Let $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ be two NPHSSs. The WCC between $(P^N{}_1, H_1)$ and $(P^N{}_2, H_2)$ is defined as:

$$\widetilde{W^C}[(P^N{}_1, H_1), (P^N{}_2, H_2)] = \frac{\mathrm{M}^c[(P^N{}_1, H_1), (P^N{}_2, H_2)]}{\max\{\mathrm{E}(P^N{}_1, H_1), \mathrm{E}(P^N{}_2, H_2)\}}. \tag{29}$$

By using the Equations (1), (2) and (3) in Equation (29), we have,

$$\widetilde{W^C}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \frac{\sum_{x=1}^{v}\vartheta_x\left\{\sum_{y=1}^{w}\omega_y\left[\begin{array}{c}\left(\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)\right)+\\ \left(\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)\right)+\\ \left(\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)*\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)\right)\end{array}\right]\right\}}{\max\left\{\begin{array}{c}\sum_{x=1}^{v}\vartheta_x\left\{\sum_{y=1}^{w}\omega_y\left[\left(\begin{array}{c}\left(\mathcal{M}_p(P^N{}_1)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_1)_{S_i}(u_j)\right)^2\\+\left(\mathcal{M}_n(P^N{}_1)_{S_i}(u_j)\right)^2\end{array}\right)\right]\right\},\\ \sum_{x=1}^{v}\vartheta_x\left\{\sum_{y=1}^{w}\omega_y\left[\left(\begin{array}{c}\left(\mathcal{M}_p(P^N{}_2)_{S_i}(u_j)\right)^2+\left(\mathcal{M}_N(P^N{}_2)_{S_i}(u_j)\right)^2\\+\left(\mathcal{M}_n(P^N{}_2)_{S_i}(u_j)\right)^2\end{array}\right)\right]\right\}\end{array}\right\}}. \quad (30)$$

Remark: If $\vartheta = \left\{\frac{1}{v},\frac{1}{v},\dots,\frac{1}{v}\right\}$ and $\omega = \left\{\frac{1}{w},\frac{1}{w},\dots,\frac{1}{w}\right\}$, then the WCC given in Equation (29) reduces to the CC as given in Equation (15).

Proposition 5.2.1. Let $(P^N{}_1,H_1)$ and $(P^N{}_2,H_2)$ be two NPHSSs. Then,

1) $0 \le \widetilde{W^C}[(P^N{}_1,H_1),(P^N{}_2,H_2)] \le 1;$ (31)

2) $\widetilde{W^C}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = \widetilde{W^C}[(P^N{}_2,H_2),(P^N{}_1,H_1)];$ (32)

3) If $(P^N{}_1,H_1) = (P^N{}_2,H_2)$, then $\widetilde{W^C}[(P^N{}_1,H_1),(P^N{}_2,H_2)] = 1.$ (33)

Proof: It is similar to Proposition 4.4.1.

6. Proposed Aggregation Operators for NPHSS:

The notions of a weighted average operator as well as a weighted geometric operator are presented in this Section for an NPHSS by using the operational laws as follows:

Let us consider η to be a set of neutrosophic picture hypersoft numbers (NPHSNs).

6.1. The NPHSS's operational laws:

Definition 6.1.1. Let $P^N{}_{11} = \left(\mathcal{M}_{p_{11}},\mathcal{M}_{N_{11}},\mathcal{M}_{n_{11}}\right)$ and $P^N{}_{12} = \left(\mathcal{M}_{p_{12}},\mathcal{M}_{N_{12}},\mathcal{M}_{n_{12}}\right)$ be two NPHSSs and ρ be a positive integer. Then,

i. $P^N{}_{11} \oplus P^N{}_{12} = \left\langle \begin{array}{c}\mathcal{M}_{p_{11}}+\mathcal{M}_{p_{12}}-\mathcal{M}_{p_{11}}\mathcal{M}_{p_{12}},\\ \mathcal{M}_{N_{11}}+\mathcal{M}_{N_{12}}-\mathcal{M}_{N_{11}}\mathcal{M}_{N_{12}},\mathcal{M}_{n_{11}}\mathcal{M}_{n_{12}}\end{array}\right\rangle;$ (34)

ii. $P^N{}_{11} \otimes P^N{}_{12} = \left\langle \begin{array}{c}\mathcal{M}_{p_{11}}\mathcal{M}_{p_{12}},\mathcal{M}_{N_{11}}\mathcal{M}_{N_{12}},\\ \mathcal{M}_{n_{11}}+\mathcal{M}_{n_{12}}-\mathcal{M}_{n_{11}}\mathcal{M}_{n_{12}}\end{array}\right\rangle;$ (35)

iii. $\rho\,(P^N{}_{11}) = \left\langle\left[1-\left(1-\mathcal{M}_{p_{11}}\right)^\rho, 1-\left(1-\mathcal{M}_{N_{11}}\right)^\rho, \left(\mathcal{M}_{n_{11}}\right)^\rho\right]\right\rangle;$ (36)

iv. $(P^N{}_{11})^\rho = \left\langle\left[\left(\mathcal{M}_{p_{11}}\right)^\rho, \left(\mathcal{M}_{N_{11}}\right)^\rho, 1-\left(1-\mathcal{M}_{n_{11}}\right)^\rho\right]\right\rangle.$ (37)

6.2. Weighted Average Operator (WAO) for NPHSS:

Definition 6.2.1. If $\vartheta = \{\vartheta_1, \vartheta_2, \dots, \vartheta_v\}$ and $\omega = \{\omega_1, \omega_2, \dots, \omega_w\}$ be the weights of the alternatives and experts, respectively, such that $\sum_{x=1}^{v} \vartheta_x = 1, \sum_{y=1}^{w} \omega_y = 1 \ \forall \quad \vartheta_x, \omega_y > 0$ and $P^N{}_{yx} = \left(\mathcal{M}_{p_{yx}}, \mathcal{M}_{N_{yx}}, \mathcal{M}_{n_{yx}}\right)$ be an NPHSN, where $x = \{1,2,\dots,v\}$; $y = \{1,2,\dots,w\}$.

If $\phi: \eta^n \to \eta$, then, WAO for NPHSS (ϕ) is defined as follows:

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{wv}) = \oplus_{x=1}^{v} \vartheta_x \left(\oplus_{y=1}^{w} \omega_y P^N{}_{yx}\right). \tag{38}$$

Theorem 6.2.2. Let $P^N{}_{yx} = \left(\mathcal{M}_{p_{yx}}, \mathcal{M}_{N_{yx}}, \mathcal{M}_{n_{yx}}\right)$ be an NPHSN, where $x = \{1,2,\dots,v\}$; $y = \{1,2,\dots,w\}$.

Then, an aggregated value of WAO (ϕ) is also an NPHSN which is given as:

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{wv}) = \left\langle \begin{matrix} 1 - \prod_{x=1}^{v} \left[\prod_{y=1}^{w} \left(1 - \mathcal{M}_{p_{yx}}\right)^{\omega_y} \right]^{\vartheta_x}, 1 - \prod_{x=1}^{v} \left[\prod_{y=1}^{w} \left(1 - \mathcal{M}_{N_{yx}}\right)^{\omega_y} \right]^{\vartheta_x}, \\ \prod_{x=1}^{v} \left[\prod_{y=1}^{w} \left(\mathcal{M}_{n_{yx}}\right)^{\omega_y} \right]^{\vartheta_x} \end{matrix} \right\rangle. \tag{39}$$

Proof: Suppose that $w = 1$, then $\omega_1 = 1$ in Equation (38), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{1v}) = \oplus_{x=1}^{v} \vartheta_x P^N{}_{1x}. \tag{40}$$

By using Equation (36) in Equation (40), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{1v}) = \left\langle \begin{matrix} 1 - \prod_{x=1}^{v} \left[\prod_{y=1}^{1} \left(1 - \mathcal{M}_{p_{yx}}\right)^{\omega_y} \right]^{\vartheta_x}, 1 - \prod_{x=1}^{v} \left[\prod_{y=1}^{1} \left(1 - \mathcal{M}_{N_{yx}}\right)^{\omega_y} \right]^{\vartheta_x}, \\ \prod_{x=1}^{v} \left[\prod_{y=1}^{1} \left(\mathcal{M}_{n_{yx}}\right)^{\omega_y} \right]^{\vartheta_x} \end{matrix} \right\rangle. \tag{41}$$

Suppose that $v = 1$, then $\vartheta_1 = 1$ in Equation (38), we have,

$$\phi(P^N{}_{11}, P^N{}_{21}, \dots, P^N{}_{w1}) = \oplus_{y=1}^{w} \omega_y P^N{}_{y1}. \tag{42}$$

By using Equation (36) in Equation (42), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{1v}) = \left\langle \begin{matrix} 1-\prod_{x=1}^{1}\left[\prod_{y=1}^{w}\left(1-\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1-\prod_{x=1}^{1}\left[\prod_{y=1}^{w}\left(1-\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ \prod_{x=1}^{1}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \right\rangle. \tag{43}$$

Thus, the above results of the Equations (41) and (43) are valid for $w = 1$ and $v = 1$.

Now, assuming $v = \alpha + 1$ and $w = \beta$ in Equation (38), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{\beta(\alpha+1)}) = \oplus_{x=1}^{\alpha+1}\vartheta_x\left(\oplus_{y=1}^{\beta}\omega_y P^N{}_{yx}\right). \tag{44}$$

By using the Equation (36) in Equation (44), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{\beta(\alpha+1)}) = \left\langle \begin{matrix} 1-\prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(1-\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1-\prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(1-\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \right\rangle. \tag{45}$$

Similarly, by assuming $v = \alpha$ and $w = \beta + 1$ in Equation (38), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{\alpha(\beta+1)}) = \oplus_{x=1}^{\alpha}\vartheta_x\left(\oplus_{y=1}^{\beta+1}\omega_y P^N{}_{yx}\right). \tag{46}$$

By using the Equation (36) in Equation (46), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{\alpha(\beta+1)}) = \left\langle \begin{matrix} 1\prod_{x=1}^{\alpha}\left[\prod_{y=1}^{\beta+1}\left(1-\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1-\prod_{x=1}^{\alpha}\left[\prod_{y=1}^{\beta+1}\left(1-\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ \prod_{x=1}^{\alpha}\left[\prod_{y=1}^{\beta+1}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \right\rangle. \tag{47}$$

Now, by assuming $v = \alpha + 1$ and $w = \beta + 1$ in Equation (38), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{(\alpha+1)(\beta+1)}) = \oplus_{x=1}^{\alpha+1}\vartheta_x\left(\oplus_{y=1}^{\beta+1}\omega_y P^N{}_{yx}\right). \tag{48}$$

$$\Rightarrow \phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{(\alpha+1)(\beta+1)}) = \oplus_{x=1}^{\alpha+1}\vartheta_x\left(\oplus_{y=1}^{\beta}\omega_y P^N{}_{yx}\right)\oplus_{x=1}^{\alpha+1}\vartheta_x\left(\omega_\beta P^N{}_{x(\beta+1)}\right). \tag{49}$$

Using the Equation (36) in Equation (49), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{(\alpha+1)(\beta+1)}) = \langle 1 - \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(1-\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \oplus 1 - \prod_{x=1}^{\alpha+1}\left[\left(1-\mathcal{M}_{p_{x(\beta+1)}}\right)^{\omega_{\beta+1}}\right]^{\vartheta_x}, 1 - \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(1-\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \oplus 1 - \prod_{x=1}^{\alpha+1}\left[\left(1-\mathcal{M}_{N_{x(\beta+1)}}\right)^{\omega_{\beta+1}}\right]^{\vartheta_x}, \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \oplus \prod_{x=1}^{\alpha+1}\left[\left(\mathcal{M}_{n_{x(\beta+1)}}\right)^{\omega_{\beta+1}}\right]^{\vartheta_x} \rangle. \quad (50)$$

Using the Equation (36) in Equation (50), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{(\alpha+1)(\beta+1)}) = \langle 1 - \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta+1}\left(1-\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1 - \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta+1}\left(1-\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \prod_{x=1}^{\alpha+1}\left[\prod_{y=1}^{\beta+1}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \rangle. \quad (51)$$

Thus, the above results of the Equations (50) and (51) are valid for $v = \alpha + 1$ and $w = \beta + 1$, where α, β be any positive integer.

As a result, by using the induction approach, the above results are valid $\forall\ \alpha, \beta \geq 1$.

Now, from the Definition 3.1., we have,

$$0 \leq \mathcal{M}_{p_{yx}} + \mathcal{M}_{N_{yx}} + \mathcal{M}_{n_{yx}} \leq 1. \quad (52)$$

Using the Equation (36) in Equation (52), we have,

$$\Leftrightarrow 1 - \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1-\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} + 1 - \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1-\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} + \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \leq 1. \quad (53)$$

As a result of Equation (53), we can easily say that an aggregated value of WAO (ϕ) is also an NPHSN which is clearly stated that the result of Equation (39) is well proved.

Example 6.2.3. Consider the above Example 3.1.1. Suppose $\omega_y = \{0.20, 0.30, 0.50\}$ and $\vartheta_x = \{0.30, 0.34, 0.36\}$ are the weights assigned to faculties and attributes, respectively.

Using the Equation (39), we have,

$$\phi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{33}) = \langle \begin{matrix} 1-\prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1-\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1-\prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1-\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \rangle$$

$$= \langle 0.31, 0.30, 0.10 \rangle.$$

6.3. Weighted Geometric Operator (WGO) for NPHSS:

Definition 6.3.1. If $\vartheta = \{\vartheta_1, \vartheta_2, \dots, \vartheta_v\}$ and $\omega = \{\omega_1, \omega_2, \dots, \omega_w\}$ be the weights of the alternatives and experts, respectively, such that $\sum_{x=1}^{v}\vartheta_x = 1, \sum_{y=1}^{w}\omega_y = 1\ \forall\ \vartheta_x, \omega_y > 0$ and $P^N{}_{yx} = \left(\mathcal{M}_{p_{yx}}, \mathcal{M}_{N_{yx}}, \mathcal{M}_{n_{yx}}\right)$ be a NPHSN, where $x = \{1,2,\dots,v\}$; $y = \{1,2,\dots,w\}$.

If $\psi: \eta^n \to \eta$, then, WGO for NPHSS (ψ) is defined as follows:

$$\psi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{wv}) = \otimes_{x=1}^{v}\left[\otimes_{y=1}^{w}\left(P^N{}_{yx}\right)^{\omega_y}\right]^{\vartheta_x}. \quad (54)$$

Theorem 6.3.2. Let $P^N{}_{yx} = \left(\mathcal{M}_{p_{yx}}, \mathcal{M}_{N_{yx}}, \mathcal{M}_{n_{yx}}\right)$ be an NPHSN, where $x = \{1,2,\dots,v\}$; $y = \{1,2,\dots,w\}$.

Then, an aggregated value of WGO (ψ) is also an NPHSN which is given as:

$$\psi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{wv}) = \langle \begin{matrix} \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \\ 1-\prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1-\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \end{matrix} \rangle. \quad (55)$$

Proof: It is similar to the proof of Theorem 6.2.2.

Example 6.3.3. Consider the above Example 3.1.1. Suppose $\omega_y = \{0.20, 0.30, 0.50\}$ and $\vartheta_x = \{0.30, 0.34, 0.36\}$ are the weights assigned to faculties and attributes, respectively.
Then, by using the Equation (55) the value of WGO is obtained as follows:

$$\psi(P^N{}_{11}, P^N{}_{12}, \dots, P^N{}_{wv})$$

$$= \langle \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{p_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, \prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(\mathcal{M}_{N_{yx}}\right)^{\omega_y}\right]^{\vartheta_x}, 1-\prod_{x=1}^{v}\left[\prod_{y=1}^{w}\left(1-\mathcal{M}_{n_{yx}}\right)^{\omega_y}\right]^{\vartheta_x} \rangle = \langle 0.34, 0.35, 0.30 \rangle.$$

7. Solving MCDM Problems using NPHSS through the TOPSIS method

TOPSIS approach aids in determining the optimum alternative based on the least and greatest distances between the neutrosophic picture positive ideal solution (NPPIS) and the neutrosophic picture negative ideal solution (NPNIS). Additionally, this approach provides accurate results for calculating closeness coefficients when used with CC rather than similarity measures. To demonstrate the NPHSS TOPSIS technique based on CC, we give an algorithm and a case study.

7.1. Algorithm for addressing MCDM Problems using NPHSS through the TOPSIS:

Suppose that $D = \{D_1, D_2, \ldots, D_d\}$ be a collection of the departments of the management-studies cluster. The task is to choose the optimal department through faculty evaluation analysis. Let $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_h\}$ be a collection of students of the departments responsible for assessing faculty members in their respective departments. The evaluation involves assigning weights (student's weightage) $\omega = \{\omega_1, \omega_2, \ldots, \omega_w\}$, such that $\sum_{y=1}^{w} \omega_y = 1 \; \forall \; \omega_y > 0$.

Let $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_h\}$ be a collection of students of the departments who are responsible for evaluating the faculties of their respective department with the weights (student's weightage) $\omega = \{\omega_1, \omega_2, \ldots, \omega_w\}$, such that $\sum_{y=1}^{w} \omega_y = 1 \; \forall \; \omega_y > 0$. If $H = S_1 \times S_2 \times \ldots \times S_i$ be a collection of multi-valued sub-attributes along with their respective weights (faculty's weightage) $\vartheta = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_v\}$, such that $\sum_{x=1}^{v} \vartheta_x = 1, \forall \; \vartheta_x > 0$. The faculty evaluation analysis is done by the sub-attributes with multiple values $S_j (j = 1, 2, \ldots, i)$ which is presented in the NPHSS format. It is represented as $P^N{}_{xy} = \langle \mathcal{M}_{p_{xy}}, \mathcal{M}_{N_{xy}}, \mathcal{M}_{n_{xy}} \rangle$ such that $0 \le \mathcal{M}_{p_{xy}} + \mathcal{M}_{N_{xy}} + \mathcal{M}_{n_{xy}} \le 1 \; \forall \; x, y$.

Step 1: Generate a matrix in the NPHSS format for each multi-valued sub-attribute, as:

$$[D_d, H]_{m \times n} = [D_d]_{m \times n} \tag{56}$$

$$= \begin{matrix} \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_h \end{matrix} \begin{matrix} \begin{matrix} S_1 & \cdots & S_i \end{matrix} \\ \begin{bmatrix} \langle \mathcal{M}_{p_{11}}, \mathcal{M}_{N_{11}}, \mathcal{M}_{n_{11}} \rangle & \cdots & \langle \mathcal{M}_{p_{1n}}, \mathcal{M}_{N_{1n}}, \mathcal{M}_{n_{1n}} \rangle \\ \vdots & \vdots & \vdots \\ \langle \mathcal{M}_{p_{m1}}, \mathcal{M}_{N_{m1}}, \mathcal{M}_{n_{m1}} \rangle & \cdots & \langle \mathcal{M}_{p_{mn}}, \mathcal{M}_{N_{mn}}, \mathcal{M}_{n_{mn}} \rangle \end{bmatrix} \end{matrix}.$$

Step 2: The process of obtaining the decision matrix, including weights assigned to each multi-valued sub-attribute, is outlined as follows:

$$[\overline{D_d}]_{m \times n} = \left\langle \begin{matrix} 1 - \prod_{x=1}^{v} \left[\prod_{y=1}^{w} \left(1 - \mathcal{M}_{p_{xy}}\right)^{\omega_y} \right]^{\vartheta_x}, 1 - \prod_{x=1}^{v} \left[\prod_{y=1}^{w} \left(1 - \mathcal{M}_{N_{xy}}\right)^{\omega_y} \right]^{\vartheta_x}, \\ \prod_{x=1}^{v} \left[\prod_{y=1}^{w} \left(\mathcal{M}_{n_{xy}}\right)^{\omega_y} \right]^{\vartheta_x} \end{matrix} \right\rangle. \tag{57}$$

$$[\overline{D_d}]_{m \times n} = \langle \overline{\mathcal{M}_{p_{xy}}}, \overline{\mathcal{M}_{N_{xy}}}, \overline{\mathcal{M}_{n_{xy}}} \rangle \tag{58}$$

Step 3: The NPPIS and NPNIS for weighted NPHSS can be determined respectively, as:

$$\overline{D}^{+} = \langle \overline{\mathcal{M}_p}^{+}, \overline{\mathcal{M}_N}^{+}, \overline{\mathcal{M}_n}^{+} \rangle_{m \times n} = \langle \overline{\mathcal{M}_p}^{(\Delta_{xy})}, \overline{\mathcal{M}_N}^{(\nabla_{xy})}, \overline{\mathcal{M}_n}^{(\nabla_{xy})} \rangle; \tag{59}$$

$$\overline{D}^{-} = \langle \overline{\mathcal{M}_p}^{-}, \overline{\mathcal{M}_N}^{-}, \overline{\mathcal{M}_n}^{-} \rangle_{m\times n} = \langle \overline{\mathcal{M}_p}^{(\nabla_{xy})}, \overline{\mathcal{M}_N}^{(\nabla_{xy})}, \overline{\mathcal{M}_n}^{(\Delta_{xy})} \rangle. \quad (60)$$

Here, $\Delta_{xy} = \arg max_d \{\varphi_{xy}^d\}$ and $\nabla_{xy} = \arg min_d \{\varphi_{xy}^d\}$.

Step 4: The CC for each alternative from NPPIS and NPNIS can be determined respectively, as:

$$\mathbb{C}_d^{\,+} = \mathrm{C}^c(\overline{D_d}, \overline{D}^{+}) = \frac{\mathrm{M}^c(\overline{D_d}, \overline{D}^{+})}{\sqrt{\mathrm{E}(\overline{D_d})}\sqrt{\mathrm{E}(\overline{D}^{+})}}; \quad (61)$$

$$\mathbb{C}_d^{\,-} = \mathrm{C}^c(\overline{D_d}, \overline{D}^{-}) = \frac{\mathrm{M}^c(\overline{D_d}, \overline{D}^{-})}{\sqrt{\mathrm{E}(\overline{D_d})}\sqrt{\mathrm{E}(\overline{D}^{-})}}. \quad (62)$$

Step 5: The closeness coefficient of the ideal solution for NPHSS can be calculated as:

$$\mathcal{G}_d = \frac{1 - \mathbb{C}_d^{\,-}}{2 - \mathbb{C}_d^{\,+} - \mathbb{C}_d^{\,-}}. \quad (63)$$

Step 6: The ordering of the alternatives can be established by organizing the $\mathcal{G}_d$ values in descending order, with the highest value indicating the best alternative.

The graphical representation of the proposed method is illustrated below in the Figure 2:

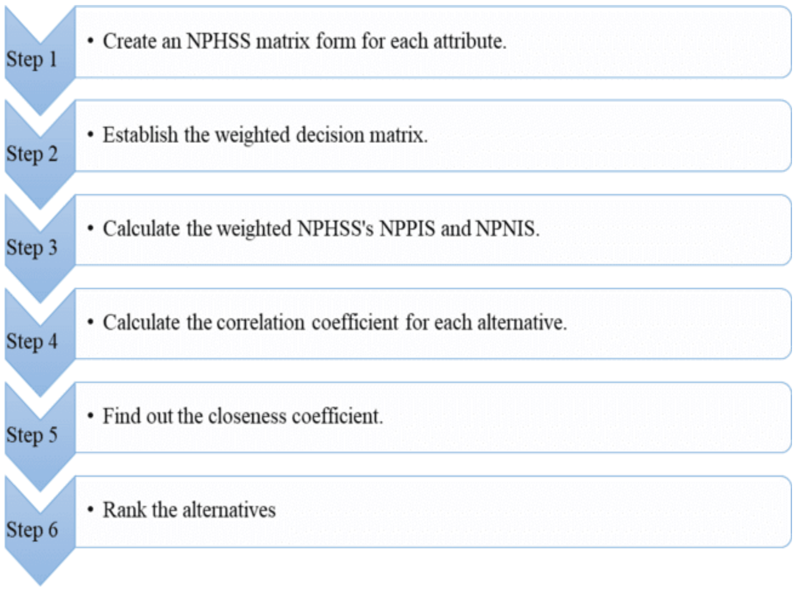


Figure 2: Flow diagram outlining the proposed TOPSIS approach.

5.2. Application in decision-making problem based on TOPSIS Approach for NPHSS:

5.2.1. A Case Study on Faculty Evaluation Analysis: Let us suppose that $D = \{D_1, D_2, D_3, D_4, D_5\}$ be a set of five departments (USB-Commerce; USB-MBA; USB-BBA; MBA-AIT and UITHM - University Institute of Tourism and Hospitality Management) within the Chandigarh University management studies cluster from which we have to select the best department based on faculty evaluation analysis. Let (α_1, α_2): (Criteria Analysis) be two distinct attributes, whose corresponding multi-valued sub-attributes are respectively represented as follows:

Faculty Evaluation Analysis from the feedback of the head of the department: $\alpha_1 = \{S_1, S_2, S_3, S_4\}$

$S_1 =$ Adherence to core values while dealing with students & colleagues =

$$\left\{\begin{matrix} S_{11} = \text{treats every student fairly}, \\ S_{12} = \text{tries relentlessly to motivate and inspire students}, \\ S_{13} = \text{respects divergent views and welcomes disagreement}, \\ S_{14} = \text{Vounteers to address exigencies and solve problems.} \end{matrix}\right\}$$

S_2= Quality of teaching & student engagement =

$$\left\{\begin{matrix} S_{21} = \text{Expert at delivering highly engaging classroom or online sessions}, \\ S_{22} = \text{Most of students rate him/her as a highly effective instructor}, \\ S_{23} = \text{Expert at using a variety of pedagogical tools like lectures}, \\ \text{multimedia presentations}, \\ \text{simulations, roleplays, bespoke activities, etc.}, \\ S_{24} = \text{Students consider the teacher} - \text{friendly and easy to talk to} \end{matrix}\right\}.$$

S_3= Quality of course or curriculum content and assessment writing =

$$\left\{\begin{matrix} S_{31} = \text{Delivers high} - \text{quality syllabi or course contents}, \\ S_{32} = \text{ Expert at preparing competency} - \text{based assessments to} \\ \text{demonstrate achievement of outcomes} \end{matrix}\right\}.$$

S_4=Interpersonal; collaborative and research skills =

$$\left\{\begin{matrix} S_{41} = \text{Is always available for consultation/ help to other faculty members}, \\ S_{42} = \text{Participates in administrative tasks as when required}, \\ S_{43} = \text{Achieves at least 2 Scopus indexed (or higher value) publications/ year}, \\ S_{44} = \text{Is an active member of a Research Cluster} \end{matrix}\right\}.$$

Faculty Evaluation Analysis from the feedback of students of the department: $\alpha_2 = \{S_5, S_6\} =$

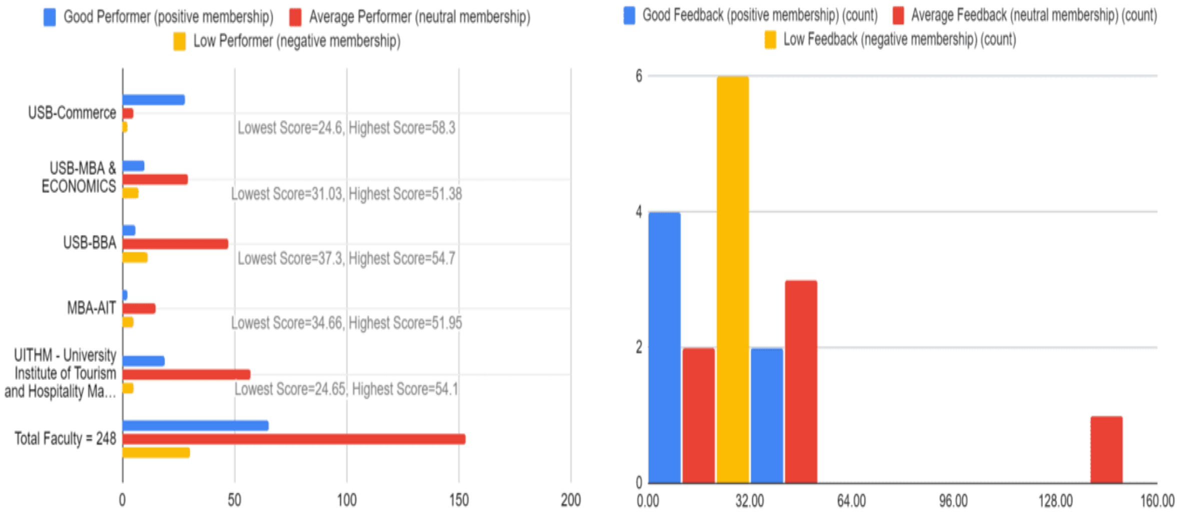
$$\left\{\begin{matrix} S_5 = \text{Student's mid} - \text{semester test score and a number of students' responses for} \\ \text{faculty feedback about the MST exam}, \\ S6 = \text{Student's end} - \text{to} - \text{semester exam score and the number of students' responses} \\ \text{for faculty feedback about end semester exam} \end{matrix}\right\}.$$

Then, $H = \alpha_1 \times \alpha_2 = \{S_1, S_2, S_3, S_4\} \times \{S_5, S_6\} = \{S_1, S_2, S_3, S_4, S_5\} \times \{S_1, S_2, S_3, S_4, S_6\} = \{\tilde{S}_1 \times \tilde{S}_2\}$ = (head's feedback and student's feedback about the faculties of their respective department) be a collection of multi-valued sub-attributes along with their respective weights (faculty's weightage):

$\vartheta = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5\} = \{0.18, 0.25, 0.26, 0.15, 0.16\}, \sum_{x=1}^{5} \vartheta_x = 1, \forall\, \vartheta_x > 0$.

Let $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5\}$ be a set of students of the departments who are responsible for evaluating the faculties of their respective departments with the weights (student's weightage): $\omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} = \{0.14, 0.19, 0.26, 0.09, 0.32\}, \sum_{y=1}^{w} \omega_y = 1\ \forall\, \omega_y > 0$.

Based on a faculty evaluation analysis, the goal is to identify the best department within the Chandigarh University management studies cluster. The graphical illustration of faculty evaluation analysis from the received is shown as follows in Figure 3:



(a) (b)

Figure 3: (a) Faculty performance assessment through the department head's feedback; (b) Faculty performance assessment through the department student's feedback.

Step 1: Create the matrices D_1, D_2, D_3, D_4 and D_5 in the form of NPHSS for each multi-valued sub-attribute by using Equation (56), as shown in Tables 3, 4, 5, 6 and 7, respectively:

Table 3: Showing the values for D_1.

D_1	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	$\langle 0.11, 0.02, 0.01 \rangle$	$\langle 0.01, 0.03, 0.001 \rangle$
$\mathcal{H}_2$	$\langle 0.15, 0.03, 0.01 \rangle$	$\langle 0.01, 0.03, 0.002 \rangle$
$\mathcal{H}_3$	$\langle 0.21, 0.04, 0.02 \rangle$	$\langle 0.02, 0.05, 0.002 \rangle$
$\mathcal{H}_4$	$\langle 0.07, 0.01, 0.01 \rangle$	$\langle 0.01, 0.02, 0.001 \rangle$
$\mathcal{H}_5$	$\langle 0.26, 0.04, 0.02 \rangle$	$\langle 0.02, 0.06, 0.002 \rangle$

Table 4: Showing the values for D_2.

D_2	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	$\langle 0.03, 0.09, 0.02 \rangle$	$\langle 0.02, 0.03, 0.14 \rangle$
$\mathcal{H}_2$	$\langle 0.04, 0.12, 0.03 \rangle$	$\langle 0.02, 0.04, 0.19 \rangle$
$\mathcal{H}_3$	$\langle 0.06, 0.16, 0.04 \rangle$	$\langle 0.03, 0.06, 0.26 \rangle$
$\mathcal{H}_4$	$\langle 0.02, 0.06, 0.01 \rangle$	$\langle 0.01, 0.02, 0.09 \rangle$
$\mathcal{H}_5$	$\langle 0.07, 0.20, 0.05 \rangle$	$\langle 0.04, 0.07, 0.32 \rangle$

Table 5: Showing the values for D_3.

D_3	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	$\langle 0.01, 0.10, 0.02 \rangle$	$\langle 0.02, 0.04, 0.001 \rangle$
$\mathcal{H}_2$	$\langle 0.02, 0.14, 0.03 \rangle$	$\langle 0.02, 0.05, 0.002 \rangle$
$\mathcal{H}_3$	$\langle 0.02, 0.19, 0.04 \rangle$	$\langle 0.03, 0.07, 0.003 \rangle$
$\mathcal{H}_4$	$\langle 0.01, 0.07, 0.02 \rangle$	$\langle 0.01, 0.03, 0.001 \rangle$
$\mathcal{H}_5$	$\langle 0.03, 0.23, 0.05 \rangle$	$\langle 0.04, 0.09, 0.003 \rangle$

Table 6: Showing the values for D_4.

D_4	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	$\langle 0.01, 0.10, 0.03 \rangle$	$\langle 0.01, 0.01, 0.003 \rangle$
$\mathcal{H}_2$	$\langle 0.02, 0.13, 0.04 \rangle$	$\langle 0.02, 0.02, 0.004 \rangle$
$\mathcal{H}_3$	$\langle 0.02, 0.18, 0.06 \rangle$	$\langle 0.02, 0.03, 0.005 \rangle$
$\mathcal{H}_4$	$\langle 0.01, 0.06, 0.02 \rangle$	$\langle 0.01, 0.01, 0.002 \rangle$
$\mathcal{H}_5$	$\langle 0.03, 0.22, 0.07 \rangle$	$\langle 0.03, 0.03, 0.006 \rangle$

Table 7: Showing the values for D_5.

D_5	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	$\langle 0.03, 0.10, 0.01 \rangle$	$\langle 0.08, 0.03, 0.004 \rangle$
$\mathcal{H}_2$	$\langle 0.04, 0.13, 0.01 \rangle$	$\langle 0.11, 0.04, 0.01 \rangle$

$\mathcal{H}_3$	⟨0.06,0.18,0.02⟩	⟨0.15,0.05,0.01⟩
$\mathcal{H}_4$	⟨0.02,0.06,0.01⟩	⟨0.05,0.02,0.002⟩
$\mathcal{H}_5$	⟨0.07,0.22,0.02⟩	⟨0.19,0.07,0.01⟩

Step 2: Obtain the weighted matrices $\overline{D_1}, \overline{D_2}, \overline{D_3}, \overline{D_4}$ and $\overline{D_5}$ in the NPHSS format for each multi-valued sub-attribute by using the Equation (57), as shown in Tables 8, 9, 10, 11 and 12, respectively:

Table 8: Showing the weighted values $\overline{D_1}$.

$\overline{D_1}$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0029,0.0005,0.890⟩	⟨0.003,0.008,0.840⟩
$\mathcal{H}_2$	⟨0.0077,0.0014,0.804⟩	⟨0.0005,0.0014,0.744⟩
$\mathcal{H}_3$	⟨0.0158,0.0028,0.768⟩	⟨0.0014,0.0035,0.657⟩
$\mathcal{H}_4$	⟨0.0010,0.0001,0.940⟩	⟨0.0001,0.0003,0.911⟩
$\mathcal{H}_5$	⟨0.0153,0.0021,0.818⟩	⟨0.0010,0.0032,0.727⟩

Table 9: Showing the weighted values $\overline{D_2}$.

$\overline{D_2}$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0008,0.0024,0.906⟩	⟨0.0005,0.0008,0.952⟩
$\mathcal{H}_2$	⟨0.0019,0.0061,0.847⟩	⟨0.0010,0.0019,0.924⟩
$\mathcal{H}_3$	⟨0.0042,0.0117,0.804⟩	⟨0.0021,0.0042,0.913⟩
$\mathcal{H}_4$	⟨0.0003,0.0008,0.940⟩	⟨0.0001,0.0003,0.968⟩
$\mathcal{H}_5$	⟨0.0037,0.0114,0.858⟩	⟨0.0021,0.0037,0.943⟩

Table 10: Showing the weighted values $\overline{D_3}$.

$\overline{D_3}$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0003,0.0027,0.906⟩	⟨0.0005,0.0010,0.840⟩
$\mathcal{H}_2$	⟨0.0010,0.0071,0.847⟩	⟨0.0010,0.0024,0.744⟩
$\mathcal{H}_3$	⟨0.0014,0.0141,0.804⟩	⟨0.0021,0.0049,0.675⟩
$\mathcal{H}_4$	⟨0.0001,0.0010,0.949⟩	⟨0.0001,0.0004,0.911⟩
$\mathcal{H}_5$	⟨0.0016,0.0133,0.858⟩	⟨0.0021,0.0048,0.743⟩

Table 11: Showing the weighted values $\overline{D_4}$.

$\overline{D_4}$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0003,0.0027,0.915⟩	⟨0.0003,0.003,0.864⟩
$\mathcal{H}_2$	⟨0.0010,0.0066,0.858⟩	⟨0.0010,0.0010,0.769⟩
$\mathcal{H}_3$	⟨0.0014,0.0133,0.827⟩	⟨0.0014,0.0021,0.699⟩
$\mathcal{H}_4$	⟨0.0001,0.0008,0.949⟩	⟨0.0001,0.0001,0.920⟩
$\mathcal{H}_5$	⟨0.0016,0.0126,0.873⟩	⟨0.0016,0.0016,0.770⟩

Table 12: Showing the weighted values $\overline{D_5}$.

$\overline{D_5}$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0008,0.0027,0.890⟩	⟨0.0021,0.0008,0.870⟩
$\mathcal{H}_2$	⟨0.0019,0.0066,0.804⟩	⟨0.0055,0.0019,0.804⟩
$\mathcal{H}_3$	⟨0.0042,0.0133,0.768⟩	⟨0.0109,0.0035,0.732⟩
$\mathcal{H}_4$	⟨0.0003,0.0008,0.940⟩	⟨0.0007,0.0003,0.920⟩
$\mathcal{H}_5$	⟨0.0037,0.0126,0.818⟩	⟨0.0107,0.0037,0.790⟩

Step 3: Calculate the NPPIS and NPNIS by using the weighted NPHSS matrices for D_1, D_2, D_3, D_4 and D_5 by using the Equations (59) and (60), as shown in the Tables 13 and 14, respectively:

Table 13: Showing the values of NPPIS ($\overline{D}^+$)

$\overline{D}^+$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0029,0.0005,0.890⟩	⟨0.003,0.0008,0.840⟩
$\mathcal{H}_2$	⟨0.0077,0.0014,0.804⟩	⟨0.0055,0.0010,0.744⟩

Table 14: Showing the values of NPNIS ($\overline{D}^-$)

$\overline{D}^-$	$\tilde{S}_1$	$\tilde{S}_2$
$\mathcal{H}_1$	⟨0.0003,0.0005,0.915⟩	⟨0.0003,0.0008,0.952⟩
$\mathcal{H}_2$	⟨0.0010,0.0014,0.858⟩	⟨0.0005,0.0010,0.924⟩

$\mathcal{H}_3$	$\langle 0.0158,0.0028,0.768\rangle\langle 0.0109,0.0021,0.657\rangle$		$\mathcal{H}_3$	$\langle 0.0014,0.0028,0.827\rangle\langle 0.0014,0.0021,0.913\rangle$
$\mathcal{H}_4$	$\langle 0.0010,0.0001,0.940\rangle\langle 0.0007,0.0001,0.911\rangle$		$\mathcal{H}_4$	$\langle 0.0001,0.0001,0.949\rangle\langle 0.0001,0.0001,0.968\rangle$
$\mathcal{H}_5$	$\langle 0.0153,0.0021,0.818\rangle\langle 0.0107,0.0016,0.727\rangle$		$\mathcal{H}_5$	$\langle 0.0016,0.0021,0.873\rangle\langle 0.0010,0.0016,0.943\rangle$

Step 4: Calculate the CC by using the values of NPPIS and NPNIS D_1, D_2, D_3, D_4 and D_5 by using the Equations (61) and (62), as shown in the Table 15:

Table 15: Showing the values of CC ($\mathbb{C}_d{}^+$ and $\mathbb{C}_d{}^-$) for D_1, D_2, D_3, D_4 and D_5

	$\mathbb{C}_d{}^+$	$\mathbb{C}_d{}^-$
D_1	0.9999	0.9951
D_2	0.9946	0.9999
D_3	0.9997	0.9948
D_4	0.9999	0.9957
D_5	0.9993	0.9980

Step 5: The closeness coefficient of the ideal solution for NPHSS is determined by using the Equation (63), as shown in Table 16:

Table 16: Showing the closeness coefficient ($\mathcal{G}_d$) values for D_1, D_2, D_3, D_4 and D_5

D_1	$\mathcal{G}_{d_1}$	0.98
D_2	$\mathcal{G}_{d_2}$	0.018
D_3	$\mathcal{G}_{d_3}$	0.945
D_4	$\mathcal{G}_{d_4}$	0.977
D_5	$\mathcal{G}_{d_5}$	0.741

Step 6: The rank of the alternatives can be determined by arranging the $\mathcal{G}_d$ values in descending order: $\mathcal{G}_{d_1} > \mathcal{G}_{d_4} > \mathcal{G}_{d_3} > \mathcal{G}_{d_5} > \mathcal{G}_{d_2} \Rightarrow D_1 > D_4 > D_3 > D_5 > D_2$.

Hence, based on a faculty evaluation analysis, D_1 (USB-Commerce) is the optimal department within the Chandigarh University management studies cluster.

8. The Analysis of Comparison:

This Section presents a comparative analysis of the proposed approach against previous methods, as depicted in Table 17:

Table 17: Comparing the suggested approach with the prior methods.

Authors	Methods	Observations
Das et al. [66]	Neutrosophic Fuzzy Set	(i) Use of a single set of parameters with intuitive fuzzy values. (ii) An intuitionistic fuzzy set's subset is the approximate function.
Khalil et al. [67]	Single-Valued	(i) Use of a single set of parameters with

	Neutrosophic Fuzzy Soft Set	neutrosophic fuzzy values. (ii) A subset of the universal set is the approximate function.
Muhammad et al. [68]	Single-Valued Neutrosophic Fuzzy Hyper-Soft Set	(i) Applying intuitionistic fuzzy values to a single set of parameters that has been separated into distinct attribute-valued sets. (ii) A neutrosophic set's subset is the approximate function.
Farooq and Saqlain [69]	Neutrosophic Hyper-Soft Set	(i) A subset of the neutrosophic set can be considered as an approximate function. (ii) Truth membership, indeterminacy membership, and falsity membership values are all mutually independent.
Chinnadurai and Bobin [20]	Picture Fuzzy Hyper-Soft Set	(i) A portion of the picture fuzzy set can be viewed as an approximate function. (ii) Relationships exist among positive membership grades, neutral membership grades, and negative membership grades. (iii) Have used multi-attribute functions.
Bobin et al. [70]	Interval-valued Picture Fuzzy Hyper-Soft	(i) Interdependencies in grades exist between positive, neutral, and negative memberships from both lower and upper ends, respectively. (ii) Have used multi-attribute functions.
Proposed method	Neutrosophic Picture Hyper-Soft Set	(i) A subset of the neutrosophic hypersoft set is an approximate function. (ii) Dependent grades between positive, neutral, and negative membership. (iii) Used multi-valued sub-attributes.

9. Conclusions:

Engaging in MCDM involves considering various factors to select the optimal choice. A valuable strategy for ranking options using a distance metric is the technique known as TOPSIS. This research focuses on the theoretical aspects of TOPSIS and any related advancements or applications, addressing the decision-making problem by assessing and rating options. Within this study, we introduce the NPHSS concept, expanding on the NHSS and PFHSS to establish the NPHSS framework with a relevant example. We illustrate the NPHSS model's generalization through a flow chart and present the CC and WCC notions, proving their fundamental properties for NPHSS. Additionally, we have introduced the WAO and WGO notions for NPHSS using various aggregation operators along with suitable examples. The research delves into a TOPSIS-based MCDM issue, demonstrating its effectiveness through a case study on faculty evaluation analysis aimed at selecting the best department from the management studies cluster. Comparative examples highlight the proposed method's efficiency in contrast to existing approaches. The research highlights the reliability of NPHSS as a decision-making tool in uncertain scenarios. There are intentions to expand the model into an interval-valued NPHSS as well as explore its applications in various fields in the future.

Acknowledgement: Kaur and Singh would like to acknowledge the thanks to Management Cluster of Chandigarh University as the data taken for evaluation in Section…. is real and uncontroversial.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] Kaur, M., & Buttar, G. S. (2019). A brief review of different measures of entropy. *Int. J. Emerg. Technol.*, *10*(2), 31-38.

[2] Molodtsov, D. (1999). Soft set theory—first results. *Computers & mathematics with applications*, *37*(4-5), 19-31.

[3] Maji, P. K., Biswas, R. K., & Roy, A. (2001). Fuzzy soft sets.

[4] Maji, P. K., Biswas, R., & Roy, A. R. (2003). Soft set theory. *Computers & mathematics with applications*, *45*(4-5), 555-562.

[5] Ali, M. I., Feng, F., Liu, X., Min, W. K., & Shabir, M. (2009). On some new operations in soft set theory. *Computers & Mathematics with Applications*, *57*(9), 1547-1553.

[6] Çağman, N., & Enginoğlu, S. (2010). Soft matrix theory and its decision making. *Computers & Mathematics with Applications*, *59*(10), 3308-3314.

[7] Smarandache, F. (1999). A unifying field in logic. neutrosophy: Neutrosophic probability, set and logic.

[8] Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). Single-valued neutrosophic sets. *Infinite study*, *12*.

[9] Maji, P. K. (2013). *Neutrosophic soft set*. Infinite Study.

[10] Broumi, S. (2013). *Generalized neutrosophic soft set*. Infinite Study.

[11] Cuong, B. C., & Kreinovich, V. (2013, December). Picture fuzzy sets: a new concept for computational intelligence problems. In *2013 Third World Congress on Information and Communication Technologies (WICT 2013)* (pp. 1-6). IEEE.

[12] Zadeh, L. A., Fuzzy sets. Inf. Control, 1965, 8, 338-353. 3.

[13] Atanassov, K., Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 1986, 20, 87-96.

[14] Yang, Y., Liang, C., Ji, S., Liu, T. Adjustable soft discernibility matrix based on picture fuzzy soft sets and its application in decision making. *J. Int. Fuzzy Syst.* 2015, 29, 1711-1722.

[15] Smarandache, F. (2018). Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, *22*(1), 168-170.

[16] Khan. J.M., Kumam, P., Ashraf. S., Kumam, W., Generalized Picture Fuzzy Soft Sets and Their Application in Decision Support Systems, *Symmetry*. 2019.

[17] Saqlain, M., Moin, S., Jafar, M. N., Saeed, M., & Smarandache, F. (2020). *Aggregate operators of neutrosophic hypersoft set*. Infinite Study.

[18] Yolcu, A., Ozturk, T.Y. Fuzzy hypersoft sets and its application to decision making. *Pons Publishing House: Brussels, Belgium*. 2021, 50-64. 20.

[19] Yolcu, A., Smarandache, F., Ozturk, T.Y. Intuitionistic fuzzy hypersoft sets. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* 2021, 70, 443-455.

[20] Chinnadurai, V., & Bobin, A. (2021). PICTURE FUZZY HYPERSOFT TOPSIS METHOD BASED ON CORRELATION COEFFICIENT. *Journal of Hyperstructures*, *10*(2).

[21] Ye, J. (2014). A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *Journal of Intelligent & Fuzzy Systems*, *26*(5), 2459-2466.

[22] Karaaslan, F. (2017). Possibility of neutrosophic soft sets and PNS-decision-making method. *Applied Soft Computing*, *54*, 403-414.

[23] Saqlain, M., Saeed, M., Ahmad, M. R., & Smarandache, F. (2019). *Generalization of TOPSIS for Neutrosophic Hypersoft set using Accuracy Function and its Application*. Infinite Study.

[24] Zulqarnain, R. M., Xin, X. L., Saqlain, M., & Smarandache, F. (2020). Generalized aggregate operators on neutrosophic hypersoft set. *Neutrosophic Sets and Systems*, *36*(1), 271-281.

[25] Saqlain, M., & Xin, X. L. (2020). *Interval-valued, m-polar and m-polar interval-valued neutrosophic hypersoft sets*. Infinite Study.

[26] Zulqarnain, R. M., Xin, X. L., & Saeed, M. (2020). Extension of TOPSIS method under intuitionistic fuzzy hypersoft environment based on correlation coefficient and aggregation operators to solve decision-making problem. *AIMS mathematics*, *6*(3), 2732-2755.

[27] Rahman, A. U., Saeed, M., & Dhital, A. (2021). *Decision-making application based on neutrosophic parameterized hypersoft set theory*. Infinite Study.

[28] Saeed, M., Rahman, A. U., Ahsan, M., & Smarandache, F. (2021). An inclusive study on fundamentals of a hypersoft set. *Theory and Application of Hypersoft Set*, *1*, 1-23.

[29] Rahman, A. U., Saeed, M., Alodhaibi, S. S., & Khalifa, H. A. E. W. (2021). Decision Making Algorithmic Approaches Based on Parameterization of Neutrosophic Set under Hypersoft Set Environment with Fuzzy, Intuitionistic Fuzzy and Neutrosophic Settings. *CMES-Computer Modeling in Engineering & Sciences*, *128*(2).

[30] Ihsan, M., Rahman, A. U., & Saeed, M. (2021). Hypersoft expert set with application in decision-making for the recruitment process. *Neutrosophic Sets and Systems*, *42*(1), 12.

[31] Zulqarnain, R. M., Xin, X. L., Saqlain, M., Saeed, M., Smarandache, F., & Ahamad, M. I. (2021). Some fundamental operations on interval-valued neutrosophic hypersoft set with their properties. *Neutrosophic sets and systems*, *40*, 134-148.

[32] Bobin, A., Thangaraja, P., Prabu, E., & Chinnadurai, V. (2022). Interval-valued picture fuzzy hypersoft TOPSIS method based on the correlation coefficient. *Journal of Mathematics and Computer Science*, *27*(2), 142-163.

[33] Bobin, A., & Chinnadurai, V. (2022). Interval-valued intuitionistic neutrosophic hypersoft TOPSIS method based on correlation coefficient. *Neutrosophic Sets and Systems*, *51*(1), 38.

[34] Chinnadurai, V., Bobin, A., & Cokilavany, D. (2022). Simplified intuitionistic neutrosophic hypersoft TOPSIS method based on the correlation coefficient. *Neutrosophic Sets and Systems*, *51*(1), 37.

[35] Chinnadurai, V., & Bobin, A. (2021). Interval-valued intuitionistic neutrosophic soft set and its application on diagnosing psychiatric disorder by using a similarity measure. *Neutrosophic Sets and Systems*, *41*, 215-245.

[36] Chinnadurai, V., & Bobin, A. (2021). *Simplified intuitionistic neutrosophic soft set and its application in diagnosing psychological disorder by using a similarity measure*. Infinite Study.

[37] Singh, A., Kumar, A., & Appadoo, S. S. (2019). A novel method for solving the fully neutrosophic linear programming problems: Suggested modifications. *Journal of intelligent & fuzzy systems*, *37*(1), 885-895.

[38] Singh, A., Kumar, A., & Appadoo, S. S. (2017). Modified approach for optimization of real-life transportation problem in neutrosophic environment. *Mathematical Problems in Engineering*, *2017*.

[39] Singh, A., & Bhat, S. A. (2021). A novel score and accuracy function for neutrosophic sets and their real-world applications to the multi-criteria decision-making process. *Neutrosophic Sets and Systems*, *41*, 168-197.

[40] Singh, A., Kumar, A., & Appadoo, S. S. (2018). Mehar ranking method for comparing connection numbers

and its application in decision making. *Journal of Intelligent & Fuzzy Systems*, *35*(5), 5523-5528.
[41] Singh, A. (2018). Modified method for solving non-linear programming for multi-criteria decision-making problems under interval neutrosophic set environment. Infinite Study.
[42] Singh, A. (2022, November). Modified Expressions to Evaluate the Correlation Coefficient Between Two Dual Hesitant Fuzzy Soft Sets and Their Application in Decision-Making. In *2022 International Conference on Electrical, Computer, Communications and Mechatronics Engineering (ICECCME)* (pp. 1-9). IEEE.
[43] Bhat, S. A., Singh, A., & Qudaimi, A. A. (2021). A new Pythagorean fuzzy analytic hierarchy process based on interval-valued Pythagorean fuzzy numbers. *Fuzzy Optimization and Modeling Journal*, *2*(4), 38-51.
[44] Singh, A., & Singh, N. (2020). A note on "A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multi-criteria decision-making problem". *International Journal of Research in Engineering, Science and Management*, *3*(5), 1235-1237.
[45] Singh, A. (2021, October). Modified non-linear programming methodology for multi-attribute decision-making problems with interval-valued intuitionistic fuzzy soft sets information. In *2021 2nd Global Conference for Advancement in Technology (GCAT)* (pp. 1-9). IEEE.
[46] Singh, A. (2022, December). A Novel Shortest Path Problem Using Dijkstra Algorithm in Interval-Valued Neutrosophic Environment. In *2022 International Conference on Smart Generation Computing, Communication and Networking (SMART GENCON)* (pp. 1-6). IEEE.
[47] Singh, A. (2022, November). Modified Expressions to Evaluate the Correlation Coefficient Between Two Dual Hesitant Fuzzy Soft Sets and Their Application in Decision-Making. In *2022 International Conference on Electrical, Computer, Communications and Mechatronics Engineering (ICECCME)* (pp. 1-9). IEEE.
[48] Singh, A., & Bhat, S. (2021). A novel score function and accuracy function for real-life multi-criteria decision-making problems under a neutrosophic environment. *Neutrosophic Sets and Systems*, *41*, 168-197.
[49] Dalkılıç, O., & Demirtaş, N. (2023). A novel perspective for Q-neutrosophic soft relations and their application in decision making. *Artificial Intelligence Review*, *56*(2), 1493-1513.
[50] Al-Sharqi, F., Ahmad, A. G., & Al-Quran, A. (2023). Fuzzy parameterized-interval complex neutrosophic soft sets and their applications under uncertainty. *Journal of Intelligent & Fuzzy Systems*, (Preprint), 1-25.
[51] Al-Hijjawi, S., & Alkhazaleh, S. (2023). Possibility Neutrosophic Hypersoft Set (PNHSS). *Neutrosophic Sets and Systems*, *53*(1), 7.
[52] Al-Sharqi, F., Al-Qudah, Y., & Alotaibi, N. (2023). Decision-making techniques based on similarity measures of possibility neutrosophic soft expert sets. *Neutrosophic Sets and Systems*, *55*(1), 22.
[53] Karataş, E., & Ozturk, T. Y. (2023). An application method for the use of neutrosophic soft mappings in decision-making for the diagnosis of COVID-19 and other lung diseases. *Process Integration and Optimization for Sustainability*, *7*(3), 545-558.
[54] Mahmood, A., Abbas, M., & Murtaza, G. (2023). Multi-valued multi-polar Neutrosophic Sets with an application in Multi-Criteria Decision-Making. *Neutrosophic Sets and Systems*, *53*(1), 32.
[55] Kaur, M., & Singh, A. (2022, October). A Novel Divergence and Fuzzy Divergence Measure in an Uncertain Environment for Multi-Criteria Decision-Making Problems. In *2022 IEEE 3rd Global Conference for Advancement in Technology (GCAT)* (pp. 1-7). IEEE.
[56] Kaur, M., Singh, A., & Buttar, G. S. (2023, September). Multi-criteria decision-making approach by fuzzy TOPSIS using an intuitionistic fuzzy entropy measure. In *AIP Conference Proceedings* (Vol. 2735, No. 1). AIP Publishing.
[57] Kaur, M., & Buttar, G. S. (2022, October). A novel measure of intuitionistic fuzzy entropy in multi-criteria

decision-making problems using TOPSIS. In *AIP Conference Proceedings* (Vol. 2555, No. 1). AIP Publishing.

[58] Smarandache, F. (2023). *New Types of Soft Sets: HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set*. Infinite Study.

[59] Smarandache, F. (2023). Foundation of the SuperHyperSoft Set and the Fuzzy Extension SuperHyperSoft Set: A New Vision. *Neutrosophic Systems with Applications*, *11*, 48-51.

[60] Smarandache, F. (2024). Foundation of SuperHyperStructure & Neutrosophic SuperHyperStructure. *Neutrosophic Sets and Systems*, *63*(1), 21.

[61] Al-shami, T. M., Alcantud, J. C. R., & Mhemdi, A. (2023). New generalization of fuzzy soft sets:(a, b)-Fuzzy soft sets. *Aims Math*, *8*(2), 2995-3025.

[62] Rahman, A. U., Saeed, M., & Smarandache, F. (2022). A theoretical and analytical approach to the conceptual framework of convexity cum concavity on fuzzy hypersoft sets with some generalized properties. *Soft Computing*, *26*(9), 4123-4139.

[63] Smarandache, F. (2023). Decision Making Based on Valued Fuzzy Superhypergraphs.

[64] Sudha, S., Martin, N., & Smarandache, F. (2023). State of Art of Plithogeny Multi-Criteria Decision-Making Methods. *Neutrosophic Sets and Systems*, *56*(1), 27.

[65] Elrawy, A., Smarandache, F., & Temraz, A. A. (2024). Investigation of a neutrosophic group. *Journal of Intelligent & Fuzzy Systems*, *46*(1), 2273-2280.

[66] Das, S., Roy, B. K., Kar, M. B., Kar, S., & Pamučar, D. (2020). Neutrosophic fuzzy set and its application in decision making. *Journal of Ambient Intelligence and Humanized Computing*, *11*, 5017-5029.

[67] Khalil, A. M., Cao, D., Azzam, A., Smarandache, F., & Alharbi, W. R. (2020). Combination of the single-valued neutrosophic fuzzy set and the soft set with applications in decision-making. *Symmetry*, *12*(8), 1361.

[68] Saeed, M., & Rahman, A. U. (2021). *Optimal supplier selection via decision-making algorithmic technique based on single-valued neutrosophic fuzzy hypersoft set*. Infinite Study.

[69] Farooq, M. U., & Saqlain, M. (2021). *The Application of Neutrosophic Hypersoft Set TOPSIS (NHSS-TOPSIS) in the Selection of Carbon Nano Tube Based Field Effective Transistors CNTFETs*. Infinite Study.

[70] Bobin, A., Thangaraja, P., Prabu, E., & Chinnadurai, V. (2022). Interval-valued picture fuzzy hypersoft TOPSIS method based on the correlation coefficient. *Journal of Mathematics and Computer Science*, *27*(2), 142-163.

Received: Feb 8, 2024. Accepted: April 28, 2024

Neutrosophic Sets and Systems, Vol. 67, 2024
University of New Mexico

Single-Valued Quadripartitioned Neutrosophic *d*-Ideal of *d*-Algebra

Bimal Shil[1], Rakhal Das[2], Carlos Granados[3], Suman Das[4,*], and Binod Chandra Tripathy[5]
[1]Department of Statistics, Tripura University, Agartala, 799022, Tripura, India.
[2]Department of Mathematics, ICFAI University, Agartala, 799210, Tripura, India.
[3]Universidad del Atlantico, Barranquilla, Colombia.
[4,5]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.
E-mail: [1]bimalshil738@gmail.com, [2]rakhaldas95@gmail.com, [3]carlosgranadosortiz@outlook.es, [4]sumandas18842@gmail.com, dr.suman1995.yahoo.com, and [5]tripathybc@gmail.com, tripathybc@yahoo.com

***Correspondence:** sumandas18842@gmail.com

Abstract: The conception of single-valued quadripartitioned neutrosophic *d*-ideal (SVQN-*d*-I) of single-valued quadripartitioned neutrosophic *d*-algebra (SVQN-*d*-A) as an expansion of neutrosophic *d*-Ideal and neutrosophic *d*-Algebra has been attempted to be introduced in this article. Additionally, we identify various characteristics of them. Additionally, SVQN-*d*-I and SVQN-*d*-A examples have been provided.

Keywords: NS; SVNS; SVQNS; *d*-Algebra; *d*-Ideal; SVQN-*d*-I; SVQN-*d*-A.

1. Introduction: In the year 1996, Imai & Iseki [34] grounded the notion of BCI-Algebra as an extension of BCK-Algebra [35], and studied several properties of them. Neggers & Kim [41] later extended the framework of BCK-Algebra by incorporating the concept of d-Algebra (*d*-A). Neggers et al. [40] utilised the principle of ideal theory to *d*-A in 1999 and proposed the thought of *d*-Ideal (*d*-I) of *d*-A. Abdullah and Hasan [1] grounded the notion of semi *d*-I of *d*-A in 2013. In order to convey the membership of an expression in mathematics, Zadeh [48] originally suggested the term fuzzy set (FS) in 1965. Later, by extending the ideas of FS theory, Atanassov [3] devised the intuitionistic FS (IFS) concept. The concept of fuzzy *d*-I of *d*-A was first proposed by Jun et al. [37] in 2000. Subsequently, Jun et al. [36] presented the intuitionistic fuzzy *d*-A and utilized the concept of *d*-A on IFS. The idea of intuitionistic fuzzy *d*-I of *d*-A was developed by Hasan [29]. Hasan [30] further studied the concept of semi *d*-I of *d*-A in the context of IFS theory. Afterwards, Hasan and Saqban [33] defined the concept of doubt intuitionistic fuzzy semi *d*-I of *d*-A in 2020. Later, Hasan [31] also presented the idea of intuitionistic fuzzy *d*-Filter in 2020. In 2021, Hasan [32] developed the idea of direct product of intuitionistic fuzzy topological *d*-A (IF-*d*-A). Smarandache [45] introduced the concept of the neutrosophic set (NS) as a logical development of IFS theory. Later, as a modification of NS, Wang et al. [47] invented the idea of single-valued NS (SVNS) in 2010. Till now, many mathematicians around the globe gives their contribution [2, 5-9, 11-28, 38-39, 42-44, 46] in the area of NS and its extensions. Following that, in 2021, Das and Hasan [10] presented the idea of neutrosophic d-I of neutrosophic d-A. In 2016, Chatterjee et al. [4] improved upon the basic idea of NS and proposed the notion of a single-valued quadripartitioned neutrosophic set (SVQNS). Subsequently, single-valued quadripartitioned neutrosophic topological space has been investigated by Das et al. [6].

We obtain the concept of SVQN-d-I of SVQN-d-A in this article as a generalisation of neutrosophic d-I and neutrosophic d-A. Additionally, we identify various characteristics of them. Furthermore, we provide a few examples that demonstrate SVQN-d-I and SVQN-d-A.

Research gap: There hasn't been any research on SVQN-d-A or SVQN-d-I published in the most recent publications.

Motivation: In order to close the investigation gap, we describe the principles of SVQN-d-A and SVQN-d-I and provide some first findings.

The following sections make up the remaining portion of our paper:

The definitions and preliminary information on d-A, d-I, fuzzy d-A and fuzzy d-I are reviewed in section 2. The idea of SVQN-d-A and SVQN-d-I are introduced in section 3, along with some propositions, theorems and other information regarding SVQN-d-I of d-A. The conclusion of the work we have done for this article is covered in section 4.

2. Some Relevant Results:

We offer some current definitions and findings in this section that are highly helpful in developing the article's primary findings.

Let us consider a universal set Z, and 0 be a constant in it. Consider a binary operation ' * ' on Z. Then, the pair (Z, *) is referred to as a d-A [41] if the following axiom holds:

(i) $y * y = 0$, for all $y \in Z$;

(ii) $0 * y = 0$, for all $y \in Z$;

(iii) $y * u = 0$ and $u * y = 0 \Rightarrow y = u$, for all $y, u \in Z$.

We will consider $y \leq u$ if and only if $y * u = 0$.

Let us consider a d-A (Z, *). Then, (Z, *) is referred [41] to as

(i) bounded d-A if $\exists$ an element $r \in Z$ such that $i * r = 0$, $\forall i \in Z$, i.e. $i \leq r$, $\forall i \in Z$.

(ii) commutative d-A iff $i * (i * r) = r * (r * i)$, $\forall i, r \in Z$.

Let us consider a d-A (Z, *) with binary operator ' * '. Then, S ($\subseteq Z$) is said to be a d-sub-A [41] of (Z, *) iff $\tilde{\eta}, \tilde{\alpha} \in S \Rightarrow \tilde{\eta} * \tilde{\alpha} \in S$.

Let ' * ' be a binary operator on a d-A (W, *). Then, Z ($\subseteq W$) is referred to as a [41] d-I if the following holds:

(i) $\tilde{\alpha} * \tilde{\eta} \in Z, \tilde{\eta} \in Z \Rightarrow \tilde{\eta} \in Z$;

(ii) $\tilde{\alpha} \in Z, \tilde{\eta} \in W \Rightarrow \tilde{\alpha} * \tilde{\eta} \in Z$.

Assume that (W, *) be a d-A with binary operator ' * '. Suppose that $Z = \{(i, T_Z(i)) : i \in W\}$ be a FS over W. Then, Z is referred to as a [37] fuzzy d-A (F-d-A) iff $T_Z(i * r) \geq \min\{T_Z(i), T_Z(r)\}$, $\forall i, r \in W$.

Suppose that $\tilde{\tilde{A}} = \{(\tilde{\tilde{o}}, T_{\tilde{\tilde{A}}}(\tilde{\tilde{o}})) : \tilde{\tilde{o}} \in W\}$ be a FS defined over a *d*-A *W* satisfying the following conditions:

(*i*) $T_{\tilde{\tilde{A}}}(\tilde{\tilde{o}}) \geq \min \{T_{\tilde{\tilde{A}}}(\tilde{\tilde{o}} * \tilde{\eta}), T_{\tilde{\tilde{A}}}(\tilde{\eta})\}$;

(*ii*) $T_{\tilde{\tilde{A}}}(\tilde{\tilde{o}} * \tilde{\eta}) \geq T_{\tilde{\tilde{A}}}(\tilde{\tilde{o}}), \forall \tilde{\tilde{o}}, \tilde{\eta} \in W$.

Then, $\tilde{\tilde{A}}$ is referred to as a fuzzy *d*-I [37] (F-*d*-I).

The notion of SVQNS was grounded by Chatterjee et al. [4] as follows:

An SVQNS *P* over an universal set *W* is defined as follows:

$\tilde{\tilde{A}} = \{(\tilde{\eta}, T_{\tilde{\tilde{A}}}(\tilde{\eta}), C_{\tilde{\tilde{A}}}(\tilde{\eta}), U_{\tilde{\tilde{A}}}(\tilde{\eta}), F_{\tilde{\tilde{A}}}(\tilde{\eta})) : \tilde{\eta} \in W\}$.

Here, $T_{\tilde{\tilde{A}}}(\tilde{\eta})$, $C_{\tilde{\tilde{A}}}(\tilde{\eta})$, $U_{\tilde{\tilde{A}}}(\tilde{\eta})$ and $F_{\tilde{\tilde{A}}}(\tilde{\eta})$ ($\in[0, 1]$) denotes the degree of truth, contradiction, unknown and falsity membership value of each $\tilde{\eta}\in W$ respectively. So, $0 \leq T_{\tilde{\tilde{A}}}(\tilde{\eta}) + C_{\tilde{\tilde{A}}}(\tilde{\eta}) + U_{\tilde{\tilde{A}}}(\tilde{\eta}) + F_{\tilde{\tilde{A}}}(\tilde{\eta}) \leq 4, \forall \tilde{\eta}\in W$.

Assume that $\tilde{\tilde{A}} = \{(\hat{a}, T_{\tilde{\tilde{A}}}(\hat{a}), C_{\tilde{\tilde{A}}}(\hat{a}), U_{\tilde{\tilde{A}}}(\hat{a}), F_{\tilde{\tilde{A}}}(\hat{a})) : \hat{a} \in W\}$ and $\ddot{E} = \{(\hat{a}, T_{\ddot{E}}(\hat{a}), C_{\ddot{E}}(\hat{a}), U_{\ddot{E}}(\hat{a}), F_{\ddot{E}}(\hat{a})) : \hat{a} \in W\}$ be two SVQNSs over a fixed set *W*. Then,

(*i*) $\tilde{\tilde{A}} \subseteq \ddot{E}$ iff $T_{\tilde{\tilde{A}}}(\hat{a}) \leq T_{\ddot{E}}(\hat{a})$, $C_{\tilde{\tilde{A}}}(\hat{a}) \leq C_{\ddot{E}}(\hat{a})$, $U_{\tilde{\tilde{A}}}(\hat{a}) \geq U_{\ddot{E}}(\hat{a})$, $F_{\tilde{\tilde{A}}}(\hat{a}) \geq F_{\ddot{E}}(\hat{a}), \forall \hat{a} \in W$.

(*ii*) $\tilde{\tilde{A}} \cap \ddot{E} = \{(\hat{a}, \min \{T_{\tilde{\tilde{A}}}(\hat{a}), T_{\ddot{E}}(\hat{a})\}, \min \{C_{\tilde{\tilde{A}}}(\hat{a}), C_{\ddot{E}}(\hat{a})\}, \max \{U_{\tilde{\tilde{A}}}(\hat{a}), U_{\ddot{E}}(\hat{a})\}, \max \{F_{\tilde{\tilde{A}}}(\hat{a}), F_{\ddot{E}}(\hat{a})\}) : \hat{a} \in W\}$.

(*iii*) $\tilde{\tilde{A}} \cup \ddot{E} = \{(\hat{a}, \max \{T_{\tilde{\tilde{A}}}(\hat{a}), T_{\ddot{E}}(\hat{a})\}, \max \{C_{\tilde{\tilde{A}}}(\hat{a}), C_{\ddot{E}}(\hat{a})\}, \min \{U_{\tilde{\tilde{A}}}(\hat{a}), U_{\ddot{E}}(\hat{a})\}, \min \{F_{\tilde{\tilde{A}}}(\hat{a}), F_{\ddot{E}}(\hat{a})\}) : \hat{a} \in W\}$.

(*iv*) $\tilde{\tilde{A}}^c = \{(\hat{a}, F_{\tilde{\tilde{A}}}(\hat{a}), U_{\tilde{\tilde{A}}}(\hat{a}), C_{\tilde{\tilde{A}}}(\hat{a}), T_{\tilde{\tilde{A}}}(\hat{a})) : \hat{a} \in W\}$ and $\ddot{E}^c = \{(\hat{a}, F_{\ddot{E}}(\hat{a}), U_{\ddot{E}}(\hat{a}), C_{\ddot{E}}(\hat{a}), T_{\ddot{E}}(\hat{a})) : \hat{a} \in W\}$.

3. Single-Valued Quadripartitioned Neutrosophic *d*-Ideal:

We define the concept of single-valued quadripartitioned neutrosophic *d*-I of *d*-A in this section and provide a number of intriguing results regarding it.

Definition 3.1. Assume that $\hat{O} = \{(i, T_{\hat{O}}(i), C_{\hat{O}}(i), U_{\hat{O}}(i), F_{\hat{O}}(i)): i\in Z\}$ be an SVQNS defined over a *d*-A *Z*, which satisfies the following conditions:

i $T_{\hat{O}}(i * r) \geq \min \{T_{\hat{O}}(i), T_{\hat{O}}(r)\}$, for all $i, r \in Z$;

ii $C_{\hat{O}}(i * r) \geq \min \{C_{\hat{O}}(i), C_{\hat{O}}(r)\}$, for all $i, r \in Z$;

iii $U_{\hat{O}}(i * r) \leq \max \{U_{\hat{O}}(i), U_{\hat{O}}(r)\}$, for all $i, r \in Z$;

iv $F_{\hat{O}}(i * r) \leq \max \{F_{\hat{O}}(i), F_{\hat{O}}(r)\}$, for all $i, r \in Z$.

Then, the SVQNS Ô is referred to as an SVQN-*d*-A of *d*-A *Z*.

Theorem 3.1. For any SVQN-*d*-A $\hat{O} = \{(\tilde{\eta}, T_{\hat{O}}(\tilde{\eta}), C_{\hat{O}}(\tilde{\eta}), U_{\hat{O}}(\tilde{\eta}), F_{\hat{O}}(\tilde{\eta})): \tilde{\eta}\in Z\}$ of a *d*-A (*Z*, *),

i $T_{\hat{O}}(0) \geq T_{\hat{O}}(\tilde{\eta}), \forall \tilde{\eta} \in Z$;

ii $C_{\hat{O}}(0) \geq C_{\hat{O}}(\tilde{\eta}), \forall \tilde{\eta} \in Z$;

iii $U_{\hat{O}}(0) \leq U_{\hat{O}}(\tilde{\eta}), \forall \tilde{\eta} \in Z$;

iv $F_{\hat{O}}(0) \leq F_{\hat{O}}(\tilde{\eta}), \forall \tilde{\eta} \in Z$.

Proof. Suppose that $\hat{O} = \{(\tilde{\eta}, T_{\hat{O}}(\tilde{\eta}), C_{\hat{O}}(\tilde{\eta}), U_{\hat{O}}(\tilde{\eta}), F_{\hat{O}}(\tilde{\eta})): \tilde{\eta}\in Z\}$ be an SVQN-*d*-A of a *d*-A (*Z*, *). Assume that $\tilde{\eta} \in Z$. Then, by Definition 2.1 and Definition 3.1, we have

i $T_{\hat{O}}(0) = T_{\hat{O}}(\tilde{\eta} * \tilde{\eta}) \geq \min \{T_{\hat{O}}(\tilde{\eta}), T_{\hat{O}}(\tilde{\eta})\} = T_{\hat{O}}(\tilde{\eta}), \forall \tilde{\eta} \in Z$;

ii $C_{\hat{O}}(0) = C_{\hat{O}}(\tilde{\eta} * \tilde{\eta}) \geq \min \{C_{\hat{O}}(\tilde{\eta}), C_{\hat{O}}(\tilde{\eta})\} = C_{\hat{O}}(\tilde{\eta}), \forall \tilde{\eta} \in Z$;

iii $U_{\hat{O}}(0) = U_{\hat{O}}(\tilde{\eta} * \tilde{\eta}) \leq \max \{U_{\hat{O}}(\tilde{\eta}), U_{\hat{O}}(\tilde{\eta})\} = U_{\hat{O}}(\tilde{\eta}), \forall \tilde{\eta} \in Z$;

iv $F_{\hat{O}}(0) = F_{\hat{O}}(\tilde{\eta} * \tilde{\eta}) \leq \max\{F_{\hat{O}}(\tilde{\eta}), F_{\hat{O}}(\tilde{\eta})\} = F_{\hat{O}}(\tilde{\eta}), \forall \tilde{\eta} \in Z$.

Theorem 3.2. Let $\{\Omega_k : k \in \Delta\}$ be a collection of SVQN-*d*-As of Z. Then, their intersection $\bigcap_{k\in\Delta}\Omega_k$ is also an SVQN-*d*-A of Z.

Proof. Suppose that $\{\Omega_k : k \in \Delta\}$ be a family of SVQN-*d*-As of Z. We have, $\bigcap_{k\in\Delta}\Omega_k = \{(\hat{e}, \wedge T_{\Omega_k}(\hat{e}), \wedge C_{\Omega_k}(\hat{e}), \vee U_{\Omega_k}(\hat{e}), \vee F_{\Omega_k}(\hat{e})) : \hat{e} \in Z\}$. Suppose that $\hat{e}, \hat{a} \in Z$. Then, we have

i $\wedge T_{\Omega_k}(\hat{e} * \hat{a}) \geq \wedge \min\{T_{\Omega_k}(\hat{e}), T_{\Omega_k}(\hat{a})\} = \min\{\wedge T_{\Omega_k}(\hat{e}), \wedge T_{\Omega_k}(\hat{a})\}$
$\Rightarrow \wedge T_{\Omega_k}(\hat{e} * \hat{a}) \geq \min\{\wedge T_{\Omega_k}(\hat{e}), \wedge T_{\Omega_k}(\hat{a})\}$;

ii $\wedge C_{\Omega_k}(\hat{e} * \hat{a}) \geq \wedge \min\{C_{\Omega_k}(\hat{e}), C_{\Omega_k}(\hat{a})\} = \min\{\wedge C_{\Omega_k}(\hat{e}), \wedge C_{\Omega_k}(\hat{a})\}$
$\Rightarrow \wedge C_{\Omega_k}(\hat{e} * \hat{a}) \geq \min\{\wedge C_{\Omega_k}(\hat{e}), \wedge C_{\Omega_k}(\hat{a})\}$;

iii $\vee U_{\Omega_k}(\hat{e} * \hat{a}) \leq \vee \max\{U_{\Omega_k}(\hat{e}), U_{\Omega_k}(\hat{a})\} = \max\{\vee U_{\Omega_k}(\hat{e}), \wedge U_{\Omega_k}(\hat{a})\}$
$\Rightarrow \vee U_{\Omega_k}(\hat{e} * \hat{a}) \leq \max\{\vee U_{\Omega_k}(\hat{e}), \wedge U_{\Omega_k}(\hat{a})\}$;

iv $\vee F_{\Omega_k}(\hat{e} * \hat{a}) \leq \vee \max\{F_{\Omega_k}(\hat{e}), F_{\Omega_k}(\hat{a})\} = \max\{\vee F_{\Omega_k}(\hat{e}), \wedge F_{\Omega_k}(\hat{a})\}$
$\Rightarrow \vee F_{\Omega_k}(\hat{e} * \hat{a}) \leq \max\{\vee F_{\Omega_k}(\hat{e}), \wedge F_{\Omega_k}(\hat{a})\}$;

Hence, $\bigcap_{k\in\Delta}\Omega_k$ is an SVQN-*d*-A of Z.

Theorem 3.3. Assume that $\hat{O} = \{(b, T_{\hat{O}}(b), C_{\hat{O}}(b), U_{\hat{O}}(b), F_{\hat{O}}(b)) : b \in Z\}$ be an SVQN-*d*-A of a *d*-A Z. Then, the sets $Z_T = \{b \in Z: T_{\hat{O}}(b) = T_{\hat{O}}(0)\}$, $Z_C = \{b \in Z: C_{\hat{O}}(b) = C_{\hat{O}}(0)\}$, $Z_U = \{b \in Z: U_{\hat{O}}(b) = U_{\hat{O}}(0)\}$ and $Z_F = \{b \in Z: F_{\hat{O}}(b) = F_{\hat{O}}(0)\}$ are *d*-Sub-As of Z.

Proof. Suppose that $\hat{O} = \{(b, T_{\hat{O}}(b), C_{\hat{O}}(b), U_{\hat{O}}(b), F_{\hat{O}}(b)) : b \in Z\}$ be an SVQN-*d*-A of a *d*-A $(Z, *)$. Given $Z_T = \{b \in Z : T_{\hat{O}}(b) = T_{\hat{O}}(0)\}$, $Z_C = \{b \in Z : C_{\hat{O}}(b) = C_{\hat{O}}(0)\}$, $Z_U = \{b \in Z : U_{\hat{O}}(b) = U_{\hat{O}}(0)\}$, and $Z_F = \{b \in Z : F_{\hat{O}}(b) = F_{\hat{O}}(0)\}$.

Let $b, r \in Z_T$. Therefore, $T_{\hat{O}}(b)=T_{\hat{O}}(0)$, $T_{\hat{O}}(r)=T_{\hat{O}}(0)$. By Definition 3.1, $T_{\hat{O}}(b * r) \geq \min\{T_{\hat{O}}(b), T_{\hat{O}}(r)\} = \min\{T_{\hat{O}}(0), T_{\hat{O}}(0)\} = T_{\hat{O}}(0)$. This implies, $T_{\hat{O}}(b * r) \geq T_{\hat{O}}(0)$, for all $b, r \in Z_T$. Now, by Theorem 3.1, we have $T_{\hat{O}}(0) \geq T_{\hat{O}}(b * r)$. Therefore, $T_{\hat{O}}(b * r) = T_{\hat{O}}(0)$. This implies, $b * r \in Z_T$. Hence, $b, r \in Z_T \Rightarrow b * r \in Z_T$. Therefore, the set $Z_T = \{b \in Z : T_{\hat{O}}(b)=T_{\hat{O}}(0)\}$ is a *d*-Sub-A of Z.

Assume that $b, r \in Z_C$. Therefore, $C_{\hat{O}}(b)=C_{\hat{O}}(0)$, $C_{\hat{O}}(r)=C_{\hat{O}}(0)$. By using the Definition 3.1, we have $C_{\hat{O}}(b * r) \geq \min\{C_{\hat{O}}(b), C_{\hat{O}}(r)\} = \min\{C_{\hat{O}}(0), C_{\hat{O}}(0)\} = C_{\hat{O}}(0)$. This shows that, $C_{\hat{O}}(b * r) \geq C_{\hat{O}}(0)$. By Theorem 3.1, $C_{\hat{O}}(0) \geq C_{\hat{O}}(b * r)$. Therefore, $C_{\hat{O}}(b * r) = C_{\hat{O}}(0)$, which implies $b * r \in Z_C$. Hence, $b, r \in Z_C \Rightarrow b * r \in Z_C$. Therefore, the set $Z_C = \{b \in Z: C_{\hat{O}}(b)=C_{\hat{O}}(0)\}$ is a *d*-Sub-A of Z.

Let $b, r \in Z_U$. Therefore, $U_{\hat{O}}(b)=U_{\hat{O}}(0)$, $U_{\hat{O}}(r)=U_{\hat{O}}(0)$. By using the Definition 3.1, we have $U_{\hat{O}}(b * r) \leq \max\{U_{\hat{O}}(b), U_{\hat{O}}(r)\} = \max\{U_{\hat{O}}(0), U_{\hat{O}}(0)\} = U_{\hat{O}}(0)$. Therefore, $U_{\hat{O}}(b * r) \leq U_{\hat{O}}(0)$. By Theorem 3.1, $U_{\hat{O}}(0) \leq U_{\hat{O}}(b * r)$. Hence, $U_{\hat{O}}(b * r) = U_{\hat{O}}(0)$. This shows that, $b * r \in Z_U$. Therefore, $b * r \in Z_U$ whenever $b, r \in Z_U$. Hence, the set $Z_U = \{b \in Z: U_{\hat{O}}(b)= U_{\hat{O}}(0)\}$ is a *d*-Sub-A of Z.

Let $b, r \in Z_F$. Therefore, $F_{\hat{O}}(b)=F_{\hat{O}}(0)$, $F_{\hat{O}}(r)=F_{\hat{O}}(0)$. Then, by using Definition 3.1, we have $F_{\hat{O}}(b * r) \leq \max\{F_{\hat{O}}(b), F_{\hat{O}}(r)\} = \max\{F_{\hat{O}}(0), F_{\hat{O}}(0)\} = F_{\hat{O}}(0)$. Therefore, $F_{\hat{O}}(b * r) \leq F_{\hat{O}}(0)$. By Theorem 3.1, $F_{\hat{O}}(0) \leq F_{\hat{O}}(b * r)$. Hence, $F_{\hat{O}}(b * r)=F_{\hat{O}}(0)$. This shows that, $b * r \in Z_F$. Therefore, $b * r \in Z_F$ whenever $b, r \in Z_F$. Hence, the set $Z_F = \{b \in Z : F_{\hat{O}}(b)=F_{\hat{O}}(0)\}$ is a *d*-Sub-A of Z.

Definition 3.2. Suppose that $\hat{O} = \{(\hat{a}, T_{\hat{O}}(\hat{a}), C_{\hat{O}}(\hat{a}), U_{\hat{O}}(\hat{a}), F_{\hat{O}}(\hat{a})): \hat{a} \in Z\}$ be an SVQNS over a *d*-A Z. Then, the T-level α-cut, C-level α-cut, U-level α-cut, F-level α-cut of Ω are defined as follows:

i $Z(T_{\hat{O}}, \alpha) = \{\hat{a} \in Z: T_{\hat{O}}(\hat{a}) \geq \alpha\}$;

ii $Z(C_{\hat{O}}, \alpha) = \{\hat{a} \in Z: C_{\hat{O}}(\hat{a}) \geq \alpha\}$;

iii $Z(U_{\hat{O}}, \alpha) = \{\hat{a} \in Z: U_{\hat{O}}(\hat{a}) \leq \alpha\}$;

iv $Z(F_{\hat{O}}, \alpha) = \{\hat{a} \in Z: F_{\hat{O}}(\hat{a}) \leq \alpha\}$.

Theorem 3.4. If $\Omega = \{(\hat{e}, T_\Omega(\hat{e}), C_\Omega(\hat{e}), U_\Omega(\hat{e}), F_\Omega(\hat{e})): \hat{e} \in Z\}$ be an SVQN-*d*-A of an *d*-A *Z*, then, the *T*-level α-cut, *C*-level α-cut, *U*-level α-cut and *F*-level α-cut of Ω are *d*-sub-As of *Z*, for any $\alpha \in [0, 1]$.

Proof. Suppose that $\Omega = \{(\hat{e}, T_\Omega(\hat{e}), C_\Omega(\hat{e}), U_\Omega(\hat{e}), F_\Omega(\hat{e})): \hat{e} \in Z\}$ be an SVQN-*d*-A of a *d*-A *Z*. Then, the *T*-level α-cut of Ω is $Z(T_\Omega, \alpha) = \{\hat{e} \in Z: T_\Omega(\hat{e}) \geq \alpha\}$, *C*-level α-cut of Ω is $Z(C_\Omega, \alpha) = \{\hat{e} \in Z: C_\Omega(\hat{e}) \geq \alpha\}$, *U*-level α-cut of Ω is $Z(U_\Omega, \alpha) = \{\hat{e} \in Z: U_\Omega(\hat{e}) \leq \alpha\}$ and *F*-level α-cut of Ω is $Z(F_\Omega, \alpha) = \{\hat{e} \in Z: F_\Omega(\hat{e}) \leq \alpha\}$.

Let $\hat{e}, \tilde{\eta} \in Z(T_\Omega, \alpha)$. So, $T_\Omega(\hat{e}) \geq \alpha$, $T_\Omega(\tilde{\eta}) \geq \alpha$. Now, we have $T_\Omega(\hat{e} * \tilde{\eta}) \geq \min\{T_\Omega(q), T_\Omega(\tilde{\eta})\} \geq \min\{\alpha, \alpha\} \geq \alpha$. This implies, $\hat{e} * \tilde{\eta} \in Z(T_\Omega, \alpha)$. Therefore, $\hat{e} * \tilde{\eta} \in Z(T_\Omega, \alpha)$, whenever $\hat{e}, \tilde{\eta} \in Z(T_\Omega, \alpha)$. Hence, *T*-level α-cut of Ω i.e., $Z(T_\Omega, \alpha)$ is a *d*-Sub-A of *Z*.

Let $\hat{e}, \tilde{\eta} \in Z(C_\Omega, \alpha)$. So, $C_\Omega(\hat{e}) \geq \alpha$, $C_\Omega(\tilde{\eta}) \geq \alpha$. Now, we have $C_\Omega(\hat{e} * \tilde{\eta}) \geq \min\{C_\Omega(\hat{e}), C_\Omega(\tilde{\eta})\} \geq \min\{\alpha, \alpha\} \geq \alpha$. This implies, $\hat{e} * \tilde{\eta} \in Z(C_\Omega, \alpha)$. Therefore, $\hat{e} * \tilde{\eta} \in Z(C_\Omega, \alpha)$, whenever $\hat{e}, \tilde{\eta} \in Z(C_\Omega, \alpha)$. Hence, *C*-level α-cut of Ω i.e., $Z(C_\Omega, \alpha)$ is a *d*-Sub-A of *Z*.

Let $\hat{e}, \tilde{\eta} \in Z(U_\Omega,\alpha)$. So, $U_\Omega(\hat{e}) \leq \alpha$, $U_\Omega(\tilde{\eta}) \leq \alpha$. Now, we have $U_\Omega(\hat{e} * \tilde{\eta}) \leq \max\{U_\Omega(\hat{e}), U_\Omega(\tilde{\eta})\} \leq \max\{\alpha, \alpha\} \leq \alpha$. This implies, $\hat{e} * \tilde{\eta} \in Z(U_\Omega, \alpha)$. Therefore, $\hat{e} * \tilde{\eta} \in Z(U_\Omega, \alpha)$, whenever $\hat{e}, \tilde{\eta} \in Z(U_\Omega, \alpha)$. Hence, *U*-level α-cut of Ω i.e., $Z(U_\Omega, \alpha)$ is a *d*-Sub-A of *Z*.

Let $\hat{e}, \tilde{\eta} \in Z(F_\Omega,\alpha)$. So, $F_\Omega(\hat{e}) \leq \alpha$, $F_\Omega(\tilde{\eta}) \leq \alpha$. Now, we have $F_\Omega(\hat{e} * \tilde{\eta}) \leq \max\{F_\Omega(\hat{e}), F_\Omega(\tilde{\eta})\} \leq \max\{\alpha, \alpha\} \leq \alpha$. This implies, $\hat{e} * \tilde{\eta} \in Z(F_\Omega, \alpha)$. Therefore, $\hat{e} * \tilde{\eta} \in Z(F_\Omega, \alpha)$, whenever $\hat{e}, \tilde{\eta} \in Z(F_\Omega, \alpha)$. Hence, *F*-level α-cut of Ω i.e., $Z(F_\Omega, \alpha)$ is a *d*-Sub-A of *Z*.

Definition 3.3. Suppose that $\tilde{\hat{O}} = \{(\tilde{\eta}, T_{\check{o}}(\tilde{\eta}), C_{\check{o}}(\tilde{\eta}), U_{\check{o}}(\tilde{\eta}), F_{\check{o}}(\tilde{\eta})): \tilde{\eta} \in W\}$ be an SVQNS of a *d*-A *W*. Then, $\tilde{\hat{O}}$ is referred to as SVQN-*d*-I if the following conditions hold:

i $T_{\check{o}}(\tilde{\eta}) \geq \min\{T_{\check{o}}(\tilde{\eta} * \tilde{\alpha}), T_{\check{o}}(\tilde{\alpha})\}$ & $T_{\check{o}}(\tilde{\eta} * \tilde{\alpha}) \geq T_{\check{o}}(\tilde{\eta})$, $\forall\, \tilde{\eta}, \tilde{\alpha} \in \hat{O}$;

ii $C_{\check{o}}(\tilde{\eta}) \geq \min\{C_{\check{o}}(\tilde{\eta} * \tilde{\alpha}), C_{\check{o}}(\tilde{\alpha})\}$ & $C_{\check{o}}(\tilde{\eta} * \tilde{\alpha}) \geq C_{\check{o}}(\tilde{\eta})$, $\forall\, \tilde{\eta}, \tilde{\alpha} \in \hat{O}$;

iii $U_{\check{o}}(\tilde{\eta}) \leq \max\{U_{\check{o}}(\tilde{\eta} * \tilde{\alpha}), U_{\check{o}}(\tilde{\alpha})\}$ & $U_{\check{o}}(\tilde{\eta} * \tilde{\alpha}) \geq U_{\check{o}}(\tilde{\eta})$, $\forall\, \tilde{\eta}, \tilde{\alpha} \in \hat{O}$;

iv $F_{\check{o}}(\tilde{\eta}) \leq \max\{F_{\check{o}}(\tilde{\eta} * \tilde{\alpha}), F_{\check{o}}(\tilde{\alpha})\}$ & $F_{\check{o}}(\tilde{\eta} * \tilde{\alpha}) \geq F_{\check{o}}(\tilde{\eta})$, $\forall\, \tilde{\eta}, \tilde{\alpha} \in \hat{O}$.

Theorem 3.5. Let us consider a *d*-A (*W*, *) with a binary operator ‘ * ’. Suppose that $\tilde{\hat{O}} = \{(\hat{e}, T_{\check{o}}(\hat{e}), C_{\check{o}}(\hat{e}), I_{\check{o}}(\hat{e}), F_{\check{o}}(\hat{e})) : \hat{e} \in W\}$ be an SVQN-*d*-I of *W*. Then, $T_{\check{o}}(0) \geq T_{\check{o}}(\hat{e})$, $C_{\check{o}}(0) \geq C_{\check{o}}(\hat{e})$, $U_{\check{o}}(0) \leq U_{\check{o}}(\hat{e})$, $F_{\check{o}}(0) \leq F_{\check{o}}(\hat{e})$, $\forall \hat{e} \in W$.

Proof. Let $\tilde{\hat{O}} = \{(\hat{e}, T_{\check{o}}(\hat{e}), C_{\check{o}}(\hat{e}), U_{\check{o}}(\hat{e}), F_{\check{o}}(\hat{e})) : \hat{e} \in W\}$ be an SVQN-*d*-I of *W*. Now,
since $T_{\check{o}}(\hat{e} * \hat{e}) \geq T_{\check{o}}(\hat{e})$, so $T_{\check{o}}(0) \geq T_{\check{o}}(\hat{e})$;
since $C_{\check{o}}(\hat{e} * \hat{e}) \geq C_{\check{o}}(\hat{e})$, so $C_{\check{o}}(0) \geq C_{\check{o}}(\hat{e})$;
since $U_{\check{o}}(\hat{e} * \hat{e}) \leq U_{\check{o}}(\hat{e})$, so $U_{\check{o}}(0) \leq U_{\check{o}}(\hat{e})$;
since $F_{\check{o}}(\hat{e} * \hat{e}) \leq F_{\check{o}}(\hat{e})$, so $F_{\check{o}}(0) \leq F_{\check{o}}(\hat{e})$.

Theorem 3.6. Suppose that (*W*, *) be a *d*-A with a binary operator ‘ * ’. Assume that $\hat{O} = \{(\hat{e}, T_{\hat{o}}(\hat{e}), C_{\hat{o}}(\hat{e}), U_{\hat{o}}(\hat{e}), F_{\hat{o}}(\hat{e})) : \hat{e} \in W\}$ be an SVQN-*d*-I of *W*. If $\hat{e} * \hat{a} \leq s$, then $T_{\hat{o}}(\hat{e}) \geq \min\{T_{\hat{o}}(\hat{a}), T_{\hat{o}}(s)\}$, $C_{\hat{o}}(\hat{e}) \geq \min\{C_{\hat{o}}(\hat{a}), C_{\hat{o}}(s)\}$, $U_{\hat{o}}(\hat{e}) \leq \max\{U_{\hat{o}}(\hat{a}), U_{\hat{o}}(s)\}$ and $F_{\hat{o}}(\hat{e}) \leq \max\{F_{\hat{o}}(\hat{a}), F_{\hat{o}}(s)\}$.

Proof. Assume that $\hat{O} = \{(\hat{e}, T_{\hat{o}}(\hat{e}), C_{\hat{o}}(\hat{e}), U_{\hat{o}}(\hat{e}), F_{\hat{o}}(\hat{e})) : \hat{e} \in W\}$ be an SVQN-*d*-Ideal of *W*. Let us consider three elements $\hat{e}, \hat{a}, \tilde{\eta}$ ($\in W$) such that $\hat{e} * \hat{a} \leq \tilde{\eta}$. By Definition 2.1, we have $(\hat{e} * \hat{a}) * \tilde{\eta} = 0$.

Now, we have

i. $T_{\hat{o}}(\hat{e}) \geq \min\{T_{\hat{o}}(\hat{e} * \hat{a}), T_{\hat{o}}(\hat{a})\} \geq \min\{\min\{T_{\hat{o}}((\hat{e} * \hat{a}) * \tilde{\eta}), T_{\hat{o}}(\tilde{\eta})\}, T_{\hat{o}}(\hat{a})\} = \min\{\min\{T_{\hat{o}}(0), T_{\hat{o}}(\tilde{\eta})\}, T_{\hat{o}}(\hat{a})\} \geq \min\{T_{\hat{o}}(\tilde{\eta}), T_{\hat{o}}(\hat{a})\}$.

Therefore, $T\hat{o}(\hat{e}) \geq \min\{T\hat{o}(\hat{a}), T\hat{o}(\tilde{\eta})\}$.

ii. $C\hat{o}(\hat{e}) \geq \min\{C\hat{o}(\hat{e} * \hat{a}), C\hat{o}(\hat{a})\} \geq \min\{\min\{C\hat{o}((\hat{e} * \hat{a}) * \tilde{\eta}), C\hat{o}(\tilde{\eta})\}, C\hat{o}(\hat{a})\} = \min\{\min\{C\hat{o}(0), C\hat{o}(\tilde{\eta})\}, C\hat{o}(\hat{a})\} \geq \min\{C\hat{o}(\tilde{\eta}), C\hat{o}(\hat{a})\}$.

Therefore, $C\hat{o}(\hat{e}) \geq \min\{C\hat{o}(\hat{a}), C\hat{o}(\tilde{\eta})\}$.

iii. $U\hat{o}(\hat{e}) \leq \max\{U\hat{o}(\hat{e} * \hat{a}), U\hat{o}(\hat{a})\} \leq \max\{\max\{U\hat{o}((\hat{e} * \hat{a}) * \tilde{\eta}), U\hat{o}(\tilde{\eta})\}, U\hat{o}(\hat{a})\} = \max\{\max\{U\hat{o}(0), U\hat{o}(\tilde{\eta})\}, U\hat{o}(\hat{a})\} \leq \max\{U\hat{o}(\tilde{\eta}), U\hat{o}(\hat{a})\}$.

Therefore, $U\hat{o}(\hat{e}) \leq \max\{U\hat{o}(\hat{a}), U\hat{o}(\tilde{\eta})\}$.

iv. $F\hat{o}(\hat{e}) \leq \max\{F\hat{o}(\hat{e} * \hat{a}), F\hat{o}(\hat{a})\} \leq \max\{\max\{F\hat{o}((\hat{e} * \hat{a}) * \tilde{\eta}), F\hat{o}(\tilde{\eta})\}, F\hat{o}(\hat{a})\} = \max\{\max\{F\hat{o}(0), F\hat{o}(\tilde{\eta})\}, F\hat{o}(\hat{a})\} \leq \max\{F\hat{o}(\tilde{\eta}), F\hat{o}(\hat{a})\}$.

Therefore, $F\hat{o}(\hat{e}) \leq \max\{F\hat{o}(\hat{a}), F\hat{o}(\tilde{\eta})\}$.

Theorem 3.7. Let us consider an SVQN-*d*-I $\tilde{\hat{O}} = \{(\hat{e}, T\check{o}(\hat{e}), C\check{o}(\hat{e}), U\check{o}(\hat{e}), F\check{o}(\hat{e})) : \hat{e}\in W\}$ of W, and let $\hat{e}$, $\hat{a}\in W$. If $\hat{e} \leq \hat{a}$, then $T\check{o}(\hat{e}) \geq T\check{o}(\hat{a})$, $C\check{o}(\hat{e}) \geq C\check{o}(\hat{a})$, $U\check{o}(\hat{e}) \leq U\check{o}(\hat{a})$ and $F\check{o}(\hat{e}) \leq F\check{o}(\hat{a})$.

Proof. Let $\tilde{\hat{O}} = \{(\hat{e}, T\check{o}(\hat{e}), C\check{o}(\hat{e}), U\check{o}(\hat{e}), F\check{o}(\hat{e})) : \hat{e}\in W\}$ be an SVQN-*d*-I of W, and let $\hat{e}, \hat{a}\in W$ such that $\hat{e} \leq \hat{a}$. By Definition 2.1, we have $\hat{e} * \hat{a} = 0$.

Now, we have

$T\check{o}(\hat{e}) \geq \min\{T\check{o}(\hat{e} * \hat{a}), T\check{o}(\hat{a})\} = \min\{T\check{o}(0), T\check{o}(\hat{a})\}, T\check{o}(\hat{a})\} = T\check{o}(\hat{a})$.

$\Rightarrow T\check{o}(\hat{e}) \geq T\check{o}(\hat{a})$.

$C\check{o}(\hat{e}) \geq \min\{C\check{o}(\hat{e} * \hat{a}), C\check{o}(\hat{a})\} = \min\{C\check{o}(0), C\check{o}(\hat{a})\}, C\check{o}(\hat{a})\} = C\check{o}(\hat{a})$.

$\Rightarrow C\check{o}(\hat{e}) \geq C\check{o}(\hat{a})$.

$U\check{o}(\hat{e}) \leq \max\{U\check{o}(\hat{e} * \hat{a}), U\check{o}(\hat{a})\} = \max\{U\check{o}(0), U\check{o}(\hat{a})\}, U\check{o}(\hat{a})\} = U\check{o}(\hat{a})$.

$\Rightarrow U\check{o}(\hat{e}) \leq U\check{o}(\hat{a})$.

and $F\check{o}(\hat{e}) \leq \max\{F\check{o}(\hat{e} * \hat{a}), F\check{o}(\hat{a})\} = \max\{F\check{o}(0), F\check{o}(\hat{a})\}, F\check{o}(\hat{a})\} = F\check{o}(\hat{a})$.

$\Rightarrow F\check{o}(\hat{e}) \leq F\check{o}(\hat{a})$.

Theorem 3.8. If $\{\hat{O}_j : j\in\Delta\}$ be a collection of SVQN-*d*-Is of a *d*-A W, then $\cap_{j\in\Delta}\hat{O}_j$ is also an SVQN-*d*-I of W.

Proof. Suppose that $\{\hat{O}_j : j\in\Delta\}$ be a collection of SVQN-*d*-Is of a *d*-A W. Then, we have $\cap_{j\in\Delta}\hat{O}_j = \{(s, \wedge T_{\hat{O}_j}(s), \wedge C_{\hat{O}_j}(s), \vee U_{\hat{O}_j}(s), \vee F_{\hat{O}_j}(s)) : s\in W\}$.

Now, we have

$\wedge T_{\hat{O}_j}(s) \geq \wedge\{\min\{T_{\hat{O}_j}(s * r), T_{\hat{O}_j}(r)\}\} \geq \min\{\wedge T_{\hat{O}_j}(s * r), \wedge T_{\hat{O}_j}(r)\}$;

$\wedge C_{\hat{O}_j}(s) \geq \wedge\{\min\{C_{\hat{O}_j}(s * r), C_{\hat{O}_j}(r)\}\} \geq \min\{\wedge C_{\hat{O}_j}(s * r), \wedge C_{\hat{O}_j}(r)\}$;

$\vee U_{\hat{O}_j}(s) \leq \vee\{\max\{U_{\hat{O}_j}(s * r), U_{\hat{O}_j}(r)\}\} \leq \max\{\vee U_{\hat{O}_j}(s * r), \vee U_{\hat{O}_j}(r)\}$;

and $\vee F_{\hat{O}_j}(s) \leq \vee\{\max\{F_{\hat{O}_j}(s * r), F_{\hat{O}_j}(r)\}\} \leq \max\{\vee F_{\hat{O}_j}(s * r), \vee F_{\hat{O}_j}(r)\}$.

Since $T_{\hat{O}_j}(s * r) \geq T_{\hat{O}_j}(s)$, $C_{\hat{O}_j}(s * r) \geq C_{\hat{O}_j}(s)$, $U_{\hat{O}_j}(s * r) \leq U_{\hat{O}_j}(s)$, $F_{\hat{O}_j}(s * r) \leq F_{\hat{O}_j}(s)$, $\forall j\in\Delta$, so $\wedge T_{\hat{O}_j}(s * r) \geq \wedge T_{\hat{O}_j}(s)$, $\wedge C_{\hat{O}_j}(s * r) \geq \wedge C_{\hat{O}_j}(s)$, $\vee U_{\hat{O}_j}(s * r) \leq \vee U_{\hat{O}_j}(s)$, $\vee F_{\hat{O}_j}(s * r) \leq \vee F_{\hat{O}_j}(s)$. Hence, $\cap_{j\in\Delta}\hat{O}_j = \{(s, \wedge T_{\hat{O}_j}(s), \wedge C_{\hat{O}_j}(s), \vee U_{\hat{O}_j}(s), \vee F_{\hat{O}_j}(s)) : s\in W\}$ is an SVQN-*d*-I of W.

Theorem 3.9. Suppose that W be a *d*-A, and ' * ' be a binary operator defined on it. Suppose that $\tilde{\hat{O}} = \{(\tilde{\alpha}, T\check{o}(\tilde{\alpha}), C\check{o}(\tilde{\alpha}), U\check{o}(\tilde{\alpha}), F\check{o}(\tilde{\alpha})) : \hat{e}\in W\}$ be an SVQN-*d*-I of W. Then, the FSs $\{(\hat{e}, T\check{o}(\hat{e})) : \hat{e}\in W\}$, $\{(\hat{e}, C\check{o}(\hat{e})) : \hat{e}\in W\}$, $\{(\hat{e}, 1\text{-}U\check{o}(\hat{e})) : \hat{e}\in W\}$ and $\{(\hat{e}, 1\text{-}F\check{o}(\hat{e})) : \hat{e}\in W\}$ are the F-*d*-Is of W.

Proof. Assume that $\tilde{\hat{O}}$= {($\hat{e}$, $T\check{o}(\hat{e})$, $C\check{o}(\hat{e})$, $U\check{o}(\hat{e})$, $F\check{o}(\hat{e})$) : $\hat{e}\in W$} be an SVQN-*d*-I of W. Therefore, $T\check{o}(\hat{e}) \geq \min \{T\check{o}(\hat{e} * \tilde{\eta}), T\check{o}(\tilde{\eta})\}$; $T\check{o}(\hat{e} * \tilde{\eta}) \geq T\check{o}(\hat{e})$; $C\check{o}(\hat{e}) \geq \min \{C\check{o}(\hat{e} * \tilde{\eta}), C\check{o}(\tilde{\eta})\}$; $C\check{o}(\hat{e} * \tilde{\eta}) \geq C\check{o}(\hat{e})$; $U\check{o}(\hat{e}) \leq \max \{U\check{o}(\hat{e} * \tilde{\eta}), U\check{o}(\tilde{\eta})\}$; $U\check{o}(\hat{e} * \tilde{\eta}) \leq U\check{o}(\hat{e})$; $F\check{o}(\hat{e}) \leq \max \{F\check{o}(\hat{e} * \tilde{\eta}), F\check{o}(\tilde{\eta})\}$; $F\check{o}(\hat{e} * \tilde{\eta}) \leq F\check{o}(\hat{e})$, $\forall \hat{e}, \tilde{\eta} \in W$.
Since $T\check{o}(\hat{e}) \geq \min \{T\check{o}(\hat{e} * \tilde{\eta}), T\check{o}(\tilde{\eta})\}$ and $T\check{o}(\hat{e} * \tilde{\eta}) \geq T\check{o}(\hat{e})$, $\forall \hat{e}, \tilde{\eta}\in W$, so {($\hat{e}$, $T\check{o}(\hat{e})$) : $\hat{e}\in W$} is a F-*d*-I of W.
Similarly, it is very easy to shown that, the FS {($\hat{e}$, $C\check{o}(\hat{e})$) : $\hat{e}\in W$} is also a F-*d*-I of W.
Further, since $U\check{o}(\hat{e}) \leq \max \{U\check{o}(\hat{e} * \tilde{\eta}), U\check{o}(\tilde{\eta})\}$ and $U\check{o}(\hat{e} * \tilde{\eta}) \leq U\check{o}(\hat{e})$, $\forall \hat{e}, \tilde{\eta}\in W$, so $1\text{-}U\check{o}(\hat{e}) \geq \min \{1\text{-}U\check{o}(\hat{e} * \tilde{\eta}), 1\text{-}U\check{o}(\tilde{\eta})\}$, $1\text{-}U\check{o}(\hat{e} * \tilde{\eta}) \geq 1\text{-}U\check{o}(\hat{e})$. Hence, the FS {($\hat{e}$, $1\text{-}U\check{o}(\hat{e})$) : $\hat{e}\in W$} is a F-*d*-I of W.
Similarly, it can be easily shown that, the FS {($\hat{e}$, $1\text{-}F\check{o}(\hat{e})$) : $\hat{e}\in W$} is also a F-*d*-I of W.

Theorem 3.10. If $\tilde{\hat{O}}$ = {($\hat{e}$, $T\check{o}(\hat{e})$, $C\check{o}(\hat{e})$, $U\check{o}(\hat{e})$, $F\check{o}(\hat{e})$) : $\hat{e}\in W$} be an SVQN-*d*-I of a *d*-A W, then the sets (i) $T\check{o}(W)$ = {$\hat{e}\in W$: $T\check{o}(\hat{e})=T\check{o}(0)$}, (ii) $C\check{o}(W)$ = {$\hat{e}\in W$: $C\check{o}(\hat{e})=C\check{o}(0)$}, (iii) $U\check{o}(W)$ = {$\hat{e}\in W$: $U\check{o}(\hat{e})=U\check{o}(0)$}, and (iv) $F\check{o}(W)$ = {$\hat{e}\in W$: $F\check{o}(\hat{e})=F\check{o}(0)$} are *d*-Is of W.

Proof. Assume that $\tilde{\hat{O}}$ = {($\hat{e}$, $T\check{o}(\hat{e})$, $C\check{o}(\hat{e})$, $U\check{o}(\hat{e})$, $F\check{o}(\hat{e})$) : $\hat{e}\in W$} be an SVQN-*d*-I of W.

(i) Suppose that $\hat{e} * \tilde{\alpha} \in T\check{o}(W)$ and $\tilde{\alpha} \in T\check{o}(W)$. So, $T\check{o}(\hat{e} * \tilde{\alpha})=T\check{o}(0)$, and $T\check{o}(\tilde{\alpha})=T\check{o}(0)$. Since $\tilde{\hat{O}}$ is an SVQN-*d*-I of W, so we have $T\check{o}(\hat{e}) \geq \min \{T\check{o}(\hat{e} * \tilde{\alpha}), T\check{o}(\tilde{\alpha})\} = \min \{T\check{o}(0), T\check{o}(0)\} = T\check{o}(0)$, which implies $T\check{o}(\hat{e}) \geq T\check{o}(0)$. We have, $T\check{o}(0) \geq T\check{o}(\hat{e})$ by using Theorem 3.1. Hence, $T\check{o}(\hat{e})=T\check{o}(0)$. This implies, $\hat{e}\in T\check{o}(W)$. Therefore, $\hat{e} * \tilde{\alpha}\in T\check{o}(W)$ and $\tilde{\alpha} \in T\check{o}(W)$ implies $\hat{e}\in T\check{o}(W)$.
Again, let $\hat{e} \in T\check{o}(W)$ and $\tilde{\alpha} \in W$. Therefore, $T\check{o}(\hat{e})=T\check{o}(0)$. Since $\tilde{\hat{O}}$ is an SVQN-*d*-I of W, so $T\check{o}(\hat{e} * \tilde{\alpha}) \geq T\check{o}(\hat{e}) = T\check{o}(0)$. Therefore, $T\check{o}(\hat{e} * \tilde{\alpha}) \geq T\check{o}(0)$. We have, $T\check{o}(0) \geq T\check{o}(\hat{e} * \tilde{\alpha})$ by using Theorem 3.1. This implies, $T\check{o}(\hat{e} * \tilde{\alpha}) = T\check{o}(0)$ i.e., $\hat{e} * \tilde{\alpha} \in T\check{o}(W)$. Hence, $\hat{e} * \tilde{\alpha} \in T\check{o}(W)$, whenever $\hat{e}\in T\check{o}(W)$ and $\tilde{\alpha}\in W$. Therefore, the set $T\check{o}(W)$ = {$\hat{e}\in W$: $T\check{o}(\hat{e})=T\check{o}(0)$} is a *d*-I of W.

Similarly, it can be easily verified that, the sets (ii) $C\check{o}(W)$ = {$\hat{e}\in W$: $C\check{o}(\hat{e})=C\check{o}(0)$}, (iii) $U\check{o}(W)$ = {$\hat{e}\in W$: $U\check{o}(\hat{e})=U\check{o}(0)$} and (iv) $F\check{o}(W)$ = {$\hat{e}\in W$: $F\check{o}(\hat{e})=F\check{o}(0)$} are the *d*-Is of W.

4. Conclusions:

In this paper, the concepts of SVQN-*d*-I of *d*-A are introduced. Also, we have looked into a number of SVQN-*d*-Is of *d*-A properties and relations. Furthermore, a number of intriguing findings on SVQN-*d*-I of *d*-A have been developed into theorems, remarks, and corollaries. It is hoped that numerous new studies such as single-valued quadripartitioned neutrosophic semi-*d*-I, doubt single-valued quadripartitioned neutrosophic semi-*d*-I, and single-valued quadripartitioned neutrosophic semi-*d*-Filter of *d*-A can be conducted in the future using the concept of SVQN-*d*-I as a foundation.

References:

1. Abdullah, H.K., & Hasan, A.K. (2013). Semi *d*-ideal in *d*-algebra. *journal of kerbala university*, 11(3), 192-197.
2. Alhazaymeh, K., Gulistan, M., Khan, M., & Kadry, S. (2019). Neutrosophic Cubic Einstein Hybrid Geometric Aggregation Operators with Application in Prioritization Using Multiple Attribute Decision-Making Method. *Mathematics*, 7(4), 346-362. doi:10.3390/math7040346
3. Atanassov, K.T. (1986). Intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 35, 87-96.

4. Chatterjee, R., Majumdar, P., & Samanta, S. K. (2016). On some similarity measures and entropy on quadripartitioned single-valued neutrosophic sets. *Journal of Intelligent & Fuuzzy Systems*, 30 (4), 2475-2485.
5. Das, S. (2021). Neutrosophic Supra Simply Open Set and Neutrosophic Supra Simply Compact Space. *Neutrosophic Sets and Systems*, 43, 105-113.
6. Das, S., Das, R., & Granados, C. (2021). Topology on Quadripartitioned Neutrosophic Sets. *Neutrosophic Sets and Systems*, 45, 54-61.
7. Das, S., Das, R., Granados, C., & Mukherjee, A. (2021). Pentapartitioned Neutrosophic *Q*-Ideals of *Q*-Algebra. *Neutrosophic Sets and Systems*, 41, 52-63.
8. Das, S., Das, R., & Tripathy, B.C. (2020). Multi criteria group decision making model using single-valued neutrosophic set. *LogForum*, 16 (3), 421-429.
9. Das, S., Das, R., & Tripathy, B.C. (In Press). Topology on Rough Pentapartitioned Neutrosophic Set. *Iraqi Journal of Science*.
10. Das, S., & Hasan, A.K. (2021). Neutrosophic *d*-Ideal of Neutrosophic *d*-Algebra. *Neutrosophic Sets and Systems*, 46, 246-253.
11. Das, S., Shil, B., & Pramanik, S. (2021). SVPNS-MADM strategy based on GRA in SVPNS Environment. *Neutrosophic Sets and Systems*, 47, 50-65.
12. Das, S., Shil, B., & Tripathy, B. C. (2021). Tangent Similarity Measure Based MADM-Strategy under SVPNS-Environment. *Neutrosophic Sets and Systems*, 43, 93-104.
13. Das, R., Smarandache, F. & Tripathy, B. C. (2020). Neutrosophic fuzzy matrices and some algebraic operation. *Neutrosophic Sets and Systems*, 32, 401-409.
14. Das, R. & Tripathy B.C. (2020). Neutrosophic multiset topological space. *Neutrosophic Sets and Systems*, 35, 142-152.
15. Das, S., & Tripathy, B.C. (2020). Pairwise neutrosophic-*b*-open set in neutrosophic bitopological spaces. *Neutrosophic Sets and Systems*, 38, 135-144.
16. Das, S., & Tripathy, B.C. (2021). Neutrosophic simply *b*-open set in neutrosophic topological spaces. *Iraqi Journal of Science*, 62(12), 4830-4838. DOI: 10.24996/ijs.2021.62.12.21
17. Das, S., & Tripathy, B.C. (2021). Pentapartitioned Neutrosophic Topological Space. *Neutrosophic Sets and Systems*, 45, 121-132.
18. Granados, C., Das, A. K., & Das, S. (2024). New Tauberian Theorems for Ceasaro Summable Triple Sequences of Fuzzy Numbers. *Kragujevac Journal of Mathematics*, 48(5), 713-722.
19. Gulistan, M., Ali, M., Azhar, M., Raho, S., & Kadry, S. (2019). Novel Neutrosophic Cubic Graphs Structures with Application in Decision Making Problems. *IEEE Access*, 1-9. doi:10.1109/ACCESS.2019.2925040.
20. Gulistan, M., Elmoasry, A., & Yaqoob, N. (2021). N-Version of the Neutrosophic Cubic Set; Application in the Negative Influences of Internet. *The Journal of Supercomputing*, 77, 11410-11431.
21. Gulistan, M., & Khan, S. (2019). Extentions of neutrosophic cubic sets via complex fuzzy sets with application. *Complex & Intelligent Systems*, 6, 309-320. doi:10.1007/s40747-019-00120-8
22. Gulistan, M., Mohammad, M., Karaaslan, F., Kadry, S., Khan, S., & Wahab, H.A. (2019). Neutrosophic cubic Heronian mean operators with applications in multiple attribute group

decision-making using cosine similarity functions. *International Journal of Distributed Sensor Networks*, 15(9), 1-21. doi:10.1177/1550147719877613

23. Gulistan, M., Nawaz, S., & Hassan, N. (2018). Neutrosophic Triplet Non-Associative Semihypergroups with Application. *Symmetry*, 10(11), 613-624. doi:10.3390/sym10110613
24. Gulistan, M., Rashid, S., Jun, Y.B., Kadry, S., & Khan, S. (2019). N -Cubic sets and aggregation operators. *Journal of Intelligent & Fuzzy Systems*, 37(4), 1-15. doi:10.3233/JIFS-182595
25. Gulistan, M., Yaqoob, N., Elmoasry, A., & Alebraheem, J. (2020). Complex bipolar fuzzy sets: An application in a transport's company. *Journal of Intelligent & Fuzzy Systems*, 40(2), 1-17. doi:10.3233/JIFS-200234
26. Gulistan, M. Yaqoob, N., Rashid, Z., Smarandache, F., & Wahab, H. (2018). A Study on Neutrosophic Cubic Graphs with Real Life Applications in Industries. *Symmetry*, 10(6), 203-225. doi:10.3390/sym10060203
27. Gulistan, M., Yaqoob, N., Nawaz, S., & Azhar, M. (2019). A study of (α, β)-complex fuzzy hyperideals in non-associative hyperrings. *Journal of Intelligent & Fuzzy Systems*, 36, 6025-6036. Doi: 10.3233/JIFS-181829.
28. Gulistan, M., Yaqoob, N., Vougiouklis, T., & Abdul Wahab, H. (2018). Extensions of cubic ideals in weak left almost semihypergroups. *Journal of Intelligent & Fuzzy Systems*, 34(6), 4161-4172. doi:10.3233/JIFS-171744
29. Hasan, A.K. (2017). On Intuitionistic fuzzy *d*-ideal of *d*-algebra. *Journal University of Kerbala*, 15(1), 161-169.
30. Hasan, A.K. (2017). Intuitionistic fuzzy semi d-ideal of d-algebra. *Journal of Iraqi Al-Khwarizmi society*, 1(1), 85-91.
31. Hasan, A.K. (2020). Intuitionistic fuzzy *d*-filter of *d*-algebra. *Journal of mechanics of continua and mathematical sciences*, 15(6), 360-370.
32. Hasan, A.K. (2021). On the Direct Product of Intuitionistic Fuzzy Topological *d*-algebra. *Iraqi Journal of Science*, 62(7), 2343-2350.
33. Hasan, A.K., & Saqban, M.A. (2020). On doubt Intuitionistic fuzzy semi *d*-ideal of *d*-algebra. *IOP Conference Series: Materials Science and Engineering*, 928(4), 042028.
34. Iami, Y., & Iseki, K. (1966). On Axiom System of Propositional Calculi XIV. *Proceedings of the Japan Academy*, 42, 19-20.
35. Iseki, K. (1966). An algebra Relation with Propositional Calculus. *Proceedings of the Japan Academy*, 42, 26-29.
36. Jun, Y.B., Kim, H.S, & Yoo, D.S. (e-2006). Intuitionistic fuzzy *d*-algebra. *Scientiae Mathematicae Japonicae Online*, 1289-1297.
37. Jun, Y.B., Neggers, J., & Kim, H.S. (2000). Fuzzy *d*-ideals of *d*-algebras. *Journal of Fuzzy Mathematics*, 8(1), 123-130.
38. Majumder, P., Das, S., Das, R., & Tripathy, B.C. (2021). Identification of the Most Significant Risk Factor of COVID-19 in Economy Using Cosine Similarity Measure under SVPNS-Environment. *Neutrosophic Sets and Systems*, 46, 112-127.
39. Mukherjee, A., & Das, R. (2020). Neutrosophic Bipolar Vague Soft Set and Its Application to Decision Making Problems. *Neutrosophic Sets and Systems*, 32, 410-424.

40. Neggers, J., Jun, Y.B., & Kim, H.S. (1999). On *d*-ideals in *d*-algebras. *Mathematica Slovaca*, 49(3), 243-251.
41. Neggers, J., & Kim, H.S. (1999). On *d*-algebra. *Mathematica Slovaca*, 49(1), 19-26.
42. Saha, P., Majumder, P., Das, S., Das, P.K., & Tripathy, B.C. (2022). Single-Valued Pentapartitioned Neutrosophic Dice Similarity Measure and Its Application in the Selection of Suitable Metal Oxide Nano-Additive for Biodiesel Blend on Environmental Aspect. *Neutrosophic Sets and Systems*, 48, 154-171.
43. Shil, B., Das, S., Das, R., & Pramanik, S. (In Press). Single Valued Pentapartitioned Neutrosophic Soft Set. *Neutrosophic Sets and Systems*.
44. Shil, B., Das, R., Granados, C., Das, S., & Chowdhury, B.D. (In Press). Hyperbolic Cosine Similarity Measure Based MADM-Strategy under the SVNS Environment. *Neutrosophic Sets and Systems*.
45. Smarandache, F. (1998). A unifying field in logics, neutrosophy: neutrosophic probability, set and logic. *Rehoboth: American Research Press*.
46. Tripathy, B. C., & Das, S. (2021). Pairwise Neutrosophic *b*-Continuous Function in Neutrosophic Bitopological Spaces. *Neutrosophic Sets and Systems*, 43, 82-92.
47. Wang, H., Smarandache, F., Zhang, Y.Q., & Sunderraman, R. (2010). Single valued neutrosophic sets. *Multispace and Multistructure*, 4, 410-413.
48. Zadeh, L.A. (1965). Fuzzy set. *Information And Control*, 8, 338-353.

Received: Feb 7, 2024. Accepted: April 29, 2024

Neutrosophic Sets and Systems, Vol. 67, 2024
University of New Mexico

Fuzzy Metric Spaces Of The Two-Fold Fuzzy Algebra

Hazim M. Wali Al-Tameemi

The Osol Aldeen Private University College, Iraq

Drhazim8@oue.edu.iq

Abstract:

This paper is dedicated to defining and studying for the first time the concept of fuzzy metric spaces based on two-fold fuzzy algebras, where the elementary properties of this new concept will be studied and presented by many theorems and related examples that explain the validity of our work. Also, many different types of open and closed balls will be discussed, as well as the relationships between these metric substructures.

Keywords: fuzzy metric space, two-fold algebra, open ball, closed ball, torus

Introduction and basic concepts

The applications of neutrosophic sets and fuzzy sets are very wide and open research areas. In the literature, we can find many neutrosophic and fuzzy algebraic structures with deep connection with applied mathematics and number theory [5-11]. The concept of two-fold algebra was presented by Smarandache in [4], where many suggestions for the algebraic structure related to this algebra were defined and presented. This new idea has been used in [1] to study the two-fold algebra based on the standard fuzzy number theoretical system [3].

In [2], Hatip et.al. proposed the two-fold vector space and two-fold algebraic module based on fuzzy mappings, where they have studied the elementary properties of these new generalizations with many interesting examples.

This work is motivated by the modern idea of two-fold algebra, and metric spaces, where we can combine those to different structures in one algebraic structure called two-fold fuzzy metric space. On the other hand, we concentrate on deriving the essential properties and substructures of this new concept.

First, we recall some basic definitions:

Main discussion

Definition:

Let $\mathbb{R}$ be the real field, we define the two-fold fuzzy real algebra as follows:

$\mathbb{R}_{[0,1]} = \{x_a; x \in \mathbb{R}. a \in [0.1]\}$ and $a = \mu(y)$; $y \in \mathbb{R}$ and μ: $\mathbb{R} \to [0,1]$

Definition:

We define the following operations on $\mathbb{R}_{[0.1]}$:

$*: \mathbb{R}_{[0,1]} \times \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$ such that:

$x_{\mu(y)} * z_{\mu(t)} = (x + z)_{\mu(ty)}$

$\circ: \mathbb{R}_{[0,1]} \times \mathbb{R}_{[[0,1]} \to \mathbb{R}_{[0,1]}$ such that:

$x_{\mu(y)} \circ z_{\mu(t)} = (x \cdot z)_{\mu(yt)}$

Theorem 1:

Let $\mathbb{R}_{[0.1]}$ be the two-fold fuzzy real algebra, then:

1] (*,○) are commutative.

2] (*,○) are associative.

3] (*,○) have identities.

4] (*,○) are anti- inverse in general.

Example:

Take μ: $\mathbb{R} \to [0,1]$; $\mu(x) = \begin{cases} |x| & ; \quad 0 < |x| < 1 \\ \frac{1}{|x|} & ; \quad |x| \geq 1 \\ 0 & ; \quad x = 0 \end{cases}$

Consider $x_{\mu(y)} = 5_{\mu(6)}. z_{\mu(t)} = \left(\frac{1}{2}\right)_{\mu(\frac{1}{3})}$ $\in \mathbb{R}_{[0,1]}$, then:

$x_{\mu(y)} * z_{\mu(t)} = \left(5 + \frac{1}{2}\right)_{\mu\left(\frac{6}{3}\right)} = \left(\frac{11}{2}\right)_{\mu(2)} = \left(\frac{11}{2}\right)_{\frac{1}{2}}.$

$x_{\mu(y)} \circ z_{\mu(t)} = \left(5 \cdot \frac{1}{2}\right)_{\mu\left(\frac{6}{3}\right)} = \left(\frac{5}{2}\right)_{\frac{1}{2}}.$

Definition:

Let $\mathbb{R}_{[0.1]}$ be the two fold fuzzy real algebra, with: μ: $\mathbb{R} \to [0.1]$, then we say that $x_{\mu(y)} \geq z_{\mu(t)}$ if and only if: $\begin{cases} x \geq z \\ \mu(y) \geq \mu(t) \end{cases}$.

Also, $x_{\mu(y)} \geq 0$ if and only if: $\begin{cases} x \geq 0 \\ \mu(y) \geq 0 \end{cases}$

Example:

For μ: $\mathbb{R} \to [0.1]$; $\mu(x) = \begin{cases} x^2 & -1 \leq x \leq 1 \\ \frac{1}{|x|} & |x| > 1 \end{cases}$, and for:

$x_{\mu(y)} = 4_{\mu(\frac{1}{2})}$. $z_{\mu(t)} = (5)_{\mu(4)}$, we can see:

$\begin{cases} x = 4 & \leq z = 5 \\ \mu\left(\frac{1}{2}\right) = \frac{1}{4} & \leq \quad \mu(4) = \frac{1}{4} \end{cases}$. hence $x_{\mu(y)} \leq z_{\mu(t)}$.

Remark:

If $x_{\mu(y)} = z_{\mu(t)}$. then $\begin{cases} x = z \\ \mu(y) = \mu(t) \end{cases}$

Theorem2:

Consider the relation (≤) defined previonsly over $\mathbb{R}_{[0.1]}$, then:

1] $x_{\mu(y)} \leq x_{\mu(y)}$ $for \ \ all \ \ x.y \ \ \in \mathbb{R}$.

2] If $x_{\mu(y)} \leq z_{\mu(t)}$ and $z_{\mu(t)} \leq x_{\mu(y)}$. then $x_{\mu(y)} = z_{\mu(t)}$ for all $x.y.z.t \in \mathbb{R}$.

3] If $x_{\mu(y)} \leq z_{\mu(t)}$ and $z_{\mu(t)} \leq N_{\mu(s)}$. then $x_{\mu(y)} \leq N_{\mu(s)}$ for all $x.y.z.N.t.s \in \mathbb{R}$

Remark:

Theorem (2) means that (≤) is a partial order relation on $\mathbb{R}_{[0.1]}$.

Definition:

Let U, V be two non- empty sets, with:

$d = \mathrm{U} \times \mathrm{U} \to \mathbb{R}^+$. μ: $\mathrm{V} \times \mathrm{V} \to [0,1]$ such that:

(d) is a metric on U, (μ) is a fuzzy metric on V.

We define the corresponding twofold algebra fuzzy metric as:

$\Delta = \begin{cases} x \\ y \end{cases}$; $x \in U$. $y \in V$, with: d_μ: $\Delta \times \Delta \to \mathbb{R}^+_{[0.1]}$

Such that:

$d_\mu(x_y . z_t) = [d(x.z)]_{\mu(y.t)}$ the mapping (d_μ) is called the twofold algebra fuzzy metric.

Theorem 3:

Let $(\Delta. d_\mu)$ be two fold algebra fuzzy metric space defined above, then:

1] $d_\mu(x_a.x_a) = o_\circ$ and $d_\mu(x_a.y_b) \geq o_\circ$

2] $d_\mu(x_a.y_b) = d_\mu(y_b.x_a)$.

3] $d_\mu(x_a.z_c) \leq d_\mu(x_a.y_b) + d_\mu(y_b.z_c)$ for all $x.y.z \in U .a.b.c \in V$.

Example:

Take $U = \mathbb{R}$ with $d = \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+ ; d(x.y) = \{|x - y|$

And $V = \mathbb{R}$ with $\mu = \mathbb{R}\times\mathbb{R} \to [0.1] ; \mu(a.b) = \begin{cases} \frac{1}{2} ; & a \neq b \\ 0 ; & a = b \end{cases}$

We have: $\Delta = \{x_a ; x.a \in \mathbb{R}\}$. For example:

$x_a = 3_5 .y_b = 4_6 . d_\mu(x_a.y_b) = (|3-4|)_{\mu(5.6)} = 1_{\frac{1}{2}}$.

Definition:

Let $B_d(x.r) = \{y \in U ; d(x.y) < r \}$ be an open ball in U, with $x \in U$ as a center and $r \in \mathbb{R}^+$ as a radius.

$\overline{B_d}(x.r) = \{y \in U ; d(x.y) \leq r \}$ be the corresponding closed ball, and $T_d(x.r) = \{y \in U ; d(x.y) = r \}$ be the corresponding torus.

Also, let $B_\mu(a.t) = \{b \in V ; \mu(a.b) < t \}$ be an open ball in V, with $a \in V$ as a center and $t \in [0.1]$ as a radius.

$\overline{B_\mu}(a.t) = \{b \in V ; \mu(a.b) \leq t \}$ be the corresponding closed ball, and $T_\mu(a.t) = \{b \in V ; \mu(a.b) = t \}$ be the corresponding torus.

We define the following different types of balls in the twofold algebra fuzzy metric spaces:

1] $\Delta_{B_\mu}^{B_d} = \{x_a \in \Delta ; x \in B_d . a \in B_\mu\}$.

2] $\Delta_{\overline{B_\mu}}^{B_d} = \{x_a \in \Delta ; x \in B_d . a \in \overline{B_\mu}\}$.

3] $\Delta_{T_\mu}^{B_d} = \{x_a \in \Delta ; x \in B_d . a \in T_\mu\}$.

4] $\Delta_{B_\mu}^{\overline{B_d}} = \{x_a \in \Delta ; x \in \overline{B_d} . a \in B_\mu\}$.

5] $\Delta_{\overline{B_\mu}}^{\overline{B_d}} = \{x_a \in \Delta ; x \in \overline{B_d} . a \in \overline{B_\mu}\}$.

6] $\Delta_{T_\mu}^{\overline{B_d}} = \{x_a \in \Delta ; x \in \overline{B_d} . a \in T_\mu\}$.

7] $\Delta_{B_\mu}^{T_d} = \{x_a \in \Delta ; x \in T_d . a \in B_\mu\}$.

8] $\Delta_{B_\mu}^{T_d}= \{x_a \in \Delta \ ; x \in T_d. \quad a \in \overline{B_\mu}\}$.

9] $\Delta_{T_\mu}^{T_d}= \{x_a \in \Delta \ ; x \in T_d. \quad a \in T_\mu\}$.

Remark:

$(\Delta. d_\mu)$ has 9 different types of balls.

Theorem4:

Consider $B_d(x.r) = \{y \in U \ ; d(x.y) < r \} \subset U. \overline{B_d}(x.r)$ and $T_d(x.r)$.

Consider $B_\mu(a.t) = \{b \in V \ ; \mu(a.b) < t \} \subset V. \overline{B_\mu}(a.t)$. and $T_\mu(a.t)$.

Then we have:

1] $\Delta_{B_\mu}^{B_d} \subseteq \Delta_{B_\mu}^{\overline{B_d}} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$

2] $\Delta_{B_\mu}^{B_d} \subseteq \Delta_{\overline{B_\mu}}^{B_d} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$

3] $\Delta_{T_\mu}^{T_d} \subseteq \Delta_{T_\mu}^{\overline{B_d}} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$

4] $\Delta_{T_\mu}^{T_d} \subseteq \Delta_{\overline{B_\mu}}^{T_d} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$

5] $\Delta_{B_\mu}^{T_d} \subseteq \Delta_{\overline{B_\mu}}^{T_d}$

6] $\Delta_{T_\mu}^{B_d} \subseteq \Delta_{T_\mu}^{\overline{B_d}}$

Example:

Consider $U = \mathbb{R} . V = \mathbb{R}$, with $d = \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ . \mu = \mathbb{R} \times \mathbb{R} \to [0.1]$

Such that: $d(x.y) = |x - y|$ and $\mu(a.b) =$

$$\begin{cases} 0 & ; \ a = b \\ \frac{1}{2} & ; \ |a|.|b| \geq 1 \\ \frac{1}{4} & ; \ |a| < 1 \ \ or \ \ |b| < 1 \end{cases}$$

We have for $= 3 \ . r = 1 \ . a = 2 \ . t = \frac{1}{3}$:

$B_d(x.r) = \{y \in \mathbb{R} \ ; |y - 3| < 1 \} = \{y \in \mathbb{R} \ ; 2 < y < 4\}$.

$\overline{B_d}(x.r) = \{y \in \mathbb{R} \ ; 2 \leq y \leq 4\}. T_d(x.r) = \{2.4\}$.

Also, $B_\mu(a.t) = \left\{b \in \mathbb{R} \ ; \quad \mu(a.b) < \frac{1}{3} \right\} = \{b \in \mathbb{R} \ ; |b| < 1 \} \cup \{2\}$.

$\overline{B_\mu}(a.t) = \{b \in \mathbb{R} \ ; \quad |b| < 1 \} \cup \{2\} = B_\mu(a.t)$.

$T_{\mu}(a.t) = \left\{ b \in \mathbb{R} \quad ; \quad \mu(a.b) = \frac{1}{3} \right\} = \emptyset.$

Proof of theorem (1):

1] $x_{\mu(y)} * z_{\mu(t)} = (x+z)_{\mu(yt)} = (z+x)_{\mu(ty)} = z_{\mu(t)} * x_{\mu(y)}$.

Also, $x_{\mu(y)} \circ z_{\mu(t)} = (x \cdot z)_{\mu(yt)} = (z \cdot x)_{\mu(ty)} = z_{\mu(t)} \circ x_{\mu(y)}$.

2] $x_{\mu(a)} * \left(y_{\mu(b)} * z_{\mu(c)}\right) = x_{\mu(a)} * (y+z)_{\mu(bc)} = (x+y+z)_{\mu(abc)} = (x+y)_{\mu(ab)} * z_{\mu(c)} = \left(x_{\mu(a)} * y_{\mu(b)}\right) * z_{\mu(c)}$.

Also, $x_{\mu(a)} \circ \left(y_{\mu(b)} \circ z_{\mu(c)}\right) = x_{\mu(a)} \circ (yz)_{\mu(bc)} = (xyz)_{\mu(abc)} = (xy)_{\mu(ab)} \circ z_{\mu(c)} = \left(x_{\mu(a)} \circ y_{\mu(b)}\right) \circ z_{\mu(c)}$.

3] if $x_{\mu(a)} \circ y_{\mu(b)} = x_{\mu(a)}$ .then $\begin{cases} xy = x \\ \mu(ab) = \mu(a) \end{cases}$ for all $x.a \in \mathbb{R}$.

So that: $y = 1. b = 1.$ and the identity of $(\circ)$ is $1_{\mu(1)}$.

If $x_{\mu(a)} * y_{\mu(b)} = x_{\mu(a)}$ .then $\begin{cases} x + y = x \\ \mu(ab) = \mu(a) \end{cases}$ for all $x.a \in \mathbb{R}$.

So that: $\begin{cases} y = 0 \\ b = 1 \end{cases}$. and the identity of $(*)$ is $\circ_1$.

4] if $x_{\mu(a)} * y_{\mu(b)} = \circ_1$ .then $\begin{cases} \mu(ab) = 1 \\ x + y = 0 \end{cases}$.

This implies that (*) is anti- inverse in general, that is because finding (b) for each (a) such that $\mu(ab) = 1$ is depended on μ.

For (○), it can be proved by the same.

Proof of theorem (2):

1] since $\begin{cases} x \leq x \\ \mu(y) \leq \mu(y) \end{cases}$ .then $x_{\mu(y)} \leq x_{\mu(y)}$.

2] $x_{\mu(y)} \leq z_{\mu(t)}$ implies that $\begin{cases} x \leq z \\ \mu(y) \leq \mu(t) \end{cases}$.

$z_{\mu(t)} \leq x_{\mu(y)}$ implies that $\begin{cases} z \leq x \\ \mu(t) \leq \mu(y) \end{cases}$ .thus $\begin{cases} x = z \\ \mu(t) = \mu(y) \end{cases}$ .and $x_{\mu(y)} = z_{\mu(t)}$.

3] Assume that $x_{\mu(y)} \leq z_{\mu(t)}$ *and* $z_{\mu(t)} \leq N_{\mu(s)}$ .then $\begin{cases} x \leq z \leq N \\ \mu(y) \leq \mu(t) \leq \mu(s) \end{cases}$ .thus $x_{\mu(y)} \leq N_{\mu(s)}$.

Proof of theorem (3):

1] $d_{\mu}(x_a . x_a) = [d(x.x)]_{\mu(a.a)} = o_{\circ}$.

$d_{\mu}(x_a . y_b) = [d(x.y)]_{\mu(a.b)}$ on the other hand, we have:

$\begin{cases} d(x.y) \geq 0 \\ \mu(a.b) \geq 0 \end{cases}$. hence $d_\mu(x_a.y_b) \geq o_\circ$.

2] $d_\mu(x_a.y_b) = [d(x.y)]_{\mu(a.b)} = [d(y.x)]_{\mu(b.a)} = d_\mu(y_b.x_a)$.

3] We have: $\begin{cases} d(x.z) \leq d(x.y) + d(y.z) \\ \mu(a.c) \leq \mu(a.b) + \mu(b.c) \end{cases}$

Thus: $[d(x.z)]_{\mu(a.c)} \leq [d(x.y)]_{\mu(a.b)} + [d(y.z)]_{\mu(b.c)}$, hence:

$d_\mu(x_a.z_c) \leq d_\mu(x_a.y_b) + d_\mu(y_b.z_c)$.

Proof of theorem (4):

1] Let $y_b \in \Delta_{B_\mu}^{B_d}$ $.then$: $\begin{cases} y \in B_d & \subseteq \overline{B_d} \\ b \in B_\mu & \subseteq \overline{B_\mu} \end{cases}$

Thus $\Delta_{B_\mu}^{B_d} \subseteq \Delta_{B_\mu}^{\overline{B_d}} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$.

2] It can be proved by a similar argument of $\boxed{1}$.

3] Let $y_b \in \Delta_{T_\mu}^{T_d}$ $.then$: $\begin{cases} y \in T_d & \subseteq \overline{B_d} \\ b \in T_\mu & \subseteq \overline{B_\mu} \end{cases}$

Thus: $\Delta_{T_\mu}^{T_d} \subseteq \Delta_{T_\mu}^{\overline{B_d}} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$.

4] It can be proved by a similar way.

5] Let $y_b \in \Delta_{B_\mu}^{T_d}$ $.then$: $\begin{cases} y \in T_d & \subseteq T_d \\ b \in B_\mu & \subseteq \overline{B_\mu} \end{cases}$

Thus $\Delta_{B_\mu}^{T_d} \subseteq \Delta_{\overline{B_\mu}}^{T_d}$.

6] It can be proved by a similar argument of $\boxed{5}$.

Definition:

Consider $B_d(x.r).\overline{B_d}(x.r).B_\mu(a.t).\overline{B_\mu}(a.t)$, and:

$\sim$ $B_d(x.r) = \{y \in U \ ; \quad d(x.y) \geq r \}$.

$\sim$ $\overline{B_d}(x.r) = \{y \in U \ ; \quad d(x.y) > r \}$.

$\sim$ $B_\mu(a.t) = \{b \in V \ ; \quad \mu(a.b) \geq t \}$.

$\sim$ $\overline{B_\mu}(a.t) = \{b \in V \ ; \quad \mu(a.b) > t \}$.

We define:

1] $\Delta_{B_\mu}^{\sim \ B_d} = \{x_a \in \Delta \ ; \quad x \in \sim B_d \ . a \in B_\mu\}$.

2] $\Delta_{\sim \ B_\mu}^{\sim \ B_d} = \{x_a \in \Delta \ ; \quad x \in \sim B_d \ . a \in \sim B_\mu\}$.

3] $\Delta^{\sim\ \overline{B_d}}_{B_\mu} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ .a \in B_\mu\}$.

4] $\Delta^{\sim\ \overline{B_d}}_{\sim\ B_\mu} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ .a \in \sim B_\mu\}$.

5] $\Delta^{\sim\ B_d}_{\overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in \sim B_d \ .a \in \overline{B_\mu}\}$.

6] $\Delta^{\sim\ B_d}_{\sim\ \overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in \sim B_d \ .a \in \sim \overline{B_\mu}\}$.

7] $\Delta^{\sim\ \overline{B_d}}_{B_\mu} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ .a \in B_\mu\}$.

8] $\Delta^{\sim\ \overline{B_d}}_{\sim\ B_\mu} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ .a \in \sim B_\mu\}$.

9] $\Delta^{\sim\ \overline{B_d}}_{\overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ .a \in \overline{B_\mu}\}$.

10] $\Delta^{\sim\ \overline{B_d}}_{\sim\ \overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ .a \in \sim \overline{B_\mu}\}$.

11] $\Delta^{B_d}_{\sim B_\mu} = \{x_a \in \Delta \ ; \quad x \in B_d \ .a \in \sim B_\mu\}$.

12] $\Delta^{B_d}_{\sim \overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in B_d \ .a \in \sim \overline{B_\mu}\}$.

13] $\Delta^{\overline{B_d}}_{\sim B_\mu} = \{x_a \in \Delta \ ; \quad x \in \overline{B_d} \ .a \in \sim B_\mu\}$.

14] $\Delta^{\overline{B_d}}_{\sim \overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in \overline{B_d} \ .a \in \sim \overline{B_\mu}\}$.

Theorem (5):

1] $\Delta^{\sim\ B_d}_{B_\mu} \cap \Delta^{\sim\ B_d}_{\sim\ B_\mu} = \emptyset \ .\Delta^{\sim\ B_d}_{\overline{B_\mu}} \cap \Delta^{\sim\ B_d}_{\sim\ \overline{B_\mu}} = \emptyset$.

2] $\Delta^{B_d}_{\sim B_\mu} \cap \Delta^{\sim\ B_d}_{\sim\ B_\mu} = \emptyset \ .\Delta^{B_d}_{\sim \overline{B_\mu}} \cap \Delta^{\sim\ B_d}_{\sim\ \overline{B_\mu}} = \emptyset$.

3] $\Delta^{\overline{B_d}}_{\sim B_\mu} \cap \Delta^{\sim\ \overline{B_d}}_{\sim\ B_\mu} = \emptyset \ .\Delta^{\overline{B_d}}_{\sim \overline{B_\mu}} \cap \Delta^{\sim\ \overline{B_d}}_{\sim\ \overline{B_\mu}} = \emptyset$.

4] $\Delta^{\overline{B_d}}_{B_\mu} \cap \Delta^{\overline{B_d}}_{\sim B_\mu} = \emptyset \ .\Delta^{\overline{B_d}}_{\overline{B_\mu}} \cap \Delta^{\overline{B_d}}_{\sim \overline{B_\mu}} = \emptyset$.

5] $\Delta^{\sim\ B_d}_{B_\mu} \subseteq \Delta^{\sim\ B_d}_{\overline{B_\mu}} \quad .\Delta^{\sim\ \overline{B_d}}_{B_\mu} \subseteq \Delta^{\sim\ \overline{B_d}}_{\overline{B_\mu}}$.

6] $\Delta^{B_d}_{\sim B_\mu} \subseteq \Delta^{\overline{B_d}}_{\sim B_\mu} \quad .\Delta^{B_d}_{\sim B_\mu} \subseteq \Delta^{\overline{B_d}}_{\sim \overline{B_\mu}}$.

7] $\Delta^{\overline{B_d}}_{\sim B_\mu} \cap \Delta^{\sim\ B_d}_{\sim B_\mu} = \Delta^{T_d}_{\sim B_\mu} \ . \quad \Delta^{\overline{B_d}}_{\sim\ \overline{B_\mu}} \cap \Delta^{\sim\ B_d}_{\sim\ \overline{B_\mu}} = \Delta^{T_d}_{\sim\ \overline{B_\mu}}$.

8] $\Delta_{\overline{B_\mu}}^{B_d} \cap \ \Delta_{\sim B_\mu}^{B_d} = \Delta_{T_\mu}^{B_d} \ . \Delta_{\overline{B_\mu}}^{\overline{B_d}} \cap \ \Delta_{\sim B_\mu}^{\overline{B_d}} = \Delta_{T_\mu}^{\overline{B_d}}$.

9] $\Delta_{\sim B_\mu}^{\sim B_d} \cap \ \Delta_{\overline{B_\mu}}^{\overline{B_d}} = \Delta_{T_\mu}^{T_d}$

Proof:

For the proof, we must regard that: $\Delta_y^x \cap \Delta_B^A = \Delta_{y\cap B}^{x\cap A}$ for all $x. A \subseteq U \ . y. B \subseteq V$.

Also, $\Delta_y^x = \emptyset$ if and only if $x = \emptyset \ or \ y = \emptyset$.

According to the definitions, we can write:

$$\begin{cases} B_\mu \cap \sim B_\mu = \overline{B_\mu} \cap \sim \overline{B_\mu} = \emptyset \\ B_d \cap \sim B_d = \overline{B_d} \cap \sim \overline{B_d} = \emptyset \end{cases}$$

Thus 1 , 2, 3, 4 hold directly.

Also, $\begin{cases} \overline{B_d} \cap \sim B_d = T_d \\ \overline{B_\mu} \cap \sim B_\mu = T_\mu \end{cases}$. thus: 7, 8, 9 hold directly.

On the other hand, we have:

$\begin{cases} B_d \subseteq \overline{B_d} \\ B_\mu \subseteq \overline{B_\mu} \end{cases}$,thus 5, 6 hold directly.

Example:

For $U = V = \mathbb{R} \ . d = \mathbb{R} \times \mathbb{R} \ \to \mathbb{R}^+ . \mu = \mathbb{R} \times \mathbb{R} \ \to [0.1]$. with $d(x.y) = |x - y| \ . \mu(a.b) =$

$$\begin{cases} 0 & ; \ a = b \\ \frac{1}{2} & ; \ |a| \text{and} |b| \geq 1 \\ \frac{1}{4} & ; \ |a| < 1 \ \text{ or } \ |b| < 1 \end{cases}$$, we have:

$\sim B_d(3.1) = \{y \in \mathbb{R} \ ; \ |y - 3| \geq 1\} = \{y \in \mathbb{R} \ ; \quad y \geq 4 \ \ or \ \ y \leq 2\}$,

$\sim \overline{B_d}(3.1) = \{y \in \mathbb{R} \ ; \quad y > 4 \quad or \quad y < 2\}$,

$\sim B_\mu \left(2.\frac{1}{3}\right) = \left\{b \in \mathbb{R} \ ; \quad \mu(2.b) \geq \frac{1}{3}\right\} = \{b \in \mathbb{R} \ ; |b| \geq 1 \ \}$,

$\sim \overline{B_\mu} \left(2.\frac{1}{3}\right) = \left\{b \in \mathbb{R} \ ; \quad \mu(2.b) > \frac{1}{3}\right\} = \{b \in \mathbb{R} \ ; |b| \geq 1 \ \} = \sim B_\mu \left(2.\frac{1}{3}\right)$.

So that, we have:

$\Delta_{B_\mu}^{B_d} = \{x_a \in \Delta \ ; \quad 2 < x < 4 \ . \quad |a| < 1 \ \text{ or } \ a = 2\}$

$\Delta_{\overline{B_\mu}}^{B_d} = \Delta_{B_\mu}^{B_d}$

$\Delta_{T_\mu}^{B_d} = \emptyset$

$\Delta_{\sim B_\mu}^{B_d} = \{x_a \in \Delta \;;\quad 2 < x < 4 \;.\quad |a| \geq 1 \}$

$\Delta_{\sim \overline{B_\mu}}^{B_d} = \{x_a \in \Delta \;;\quad 2 < x < 4 \;.\quad |a| \geq 1 \}$

$\Delta_{B_\mu}^{\overline{B_d}} = \{x_a \in \Delta \;;\quad 2 \leq x \leq 4 \;.\quad |a| < 1 \}$

$\Delta_{\overline{B_\mu}}^{\overline{B_d}} = \{x_a \in \Delta \;;\quad 2 \leq x \leq 4 \;.\quad |a| < 1 \}$

$\Delta_{T_\mu}^{\overline{B_d}} = \emptyset$

$\Delta_{\sim B_\mu}^{\overline{B_d}} = \{x_a \in \Delta \;;\quad 2 \leq x \leq 4 \;.\quad |a| \geq 1 \}$

$\Delta_{\sim \overline{B_\mu}}^{\overline{B_d}} = \{x_a \in \Delta \;;\quad 2 \leq x \leq 4 \;.\quad |a| \geq 1 \}$

$\Delta_{B_\mu}^{T_d} = \{x_a \in \Delta \;;\quad x \in \{2.4\} \;.\quad |a| < 1 \quad \text{or} \;\; a = 2\}$

$\Delta_{\overline{B_\mu}}^{T_d} = \{x_a \in \Delta \;;\quad x \in \{2.4\} \;.\quad |a| < 1 \quad \text{or} \;\; a = 2\}$

$\Delta_{T_\mu}^{T_d} = \emptyset$

$\Delta_{\sim B_\mu}^{T_d} = \{x_a \in \Delta \;;\quad x \in \{2.4\} \;.\quad |a| \geq 1 \}$

$\Delta_{\sim \overline{B_\mu}}^{T_d} = \{x_a \in \Delta \;;\quad x \in \{2.4\} \;.\quad |a| \geq 1 \}$.

$\Delta_{B_\mu}^{\sim B_d} = \{x_a \in \Delta \;;\quad x \geq 4 \;\; \text{or} \;\; x \leq 2 \;.\quad |a| < 1 \;\; \text{or} \;\; a = 2\}$

$\Delta_{\overline{B_\mu}}^{\sim B_d} = \{x_a \in \Delta \;;\quad x \geq 4 \;\; \text{or} \;\; x \leq 2 \;.\quad |a| < 1 \;\; \text{or} \;\; a = 2\}$

$\Delta_{T_\mu}^{\sim B_d} = \emptyset$

$\Delta_{\sim B_\mu}^{\sim B_d} = \{x_a \in \Delta \;;\quad x \geq 4 \;\; \text{or} \;\; x \leq 2 \;.\quad |a| \geq 1\}$

$\Delta_{\sim \overline{B_\mu}}^{\sim B_d} = \{x_a \in \Delta \;;\quad x \geq 4 \;\; \text{or} \;\; x \leq 2 \;.\quad |a| \geq 1\}$

$\Delta_{B_\mu}^{\sim \overline{B_d}} = \{x_a \in \Delta \;;\quad x > 4 \;\; \text{or} \;\; x > 2 \;.\quad |a| < 1 \quad or \;\; a = 2\}$

$\Delta_{\overline{B_\mu}}^{\sim \overline{B_d}} = \{x_a \in \Delta \;;\quad x > 4 \;\; \text{or} \;\; x > 2 \;.\quad |a| < 1 \quad \text{or} \;\; a = 2\}$

$\Delta_{T_\mu}^{\sim \overline{B_d}} = \emptyset$

$\Delta_{\sim B_\mu}^{\sim \overline{B_d}} = \{x_a \in \Delta \;;\quad x > 4 \;\; \text{or} \;\; x < 2 \;.\quad |a| \geq 1 \}$

$\Delta_{\sim B_\mu}^{\sim \overline{B_d}} = \{x_a \in \Delta \;; \quad x > 4 \text{ or } x < 2 \quad . \quad |a| \geq 1 \}$.

Conclusion

In this paper, we defined for the first time the concept of fuzzy metric spaces based on two-fold fuzzy algebras, where the elementary properties of this new concept were studied and presented by many theorems and related examples that explain the validity of this work. Also, many different types of open and closed balls were discussed, as well as the relationships between these metric substructures.

References

[1] Mohammad Abobala. (2023). On a Two-Fold Algebra Based on the Standard Fuzzy Number Theoretical System. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 24-29 (**Doi** : https://doi.org/10.54216/JNFS.070202).

[2] Ahmed Hatip, Necati Olgun. (2023). On the Concepts of Two- Fold Fuzzy Vector Spaces and Algebraic Modules. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 46-52 (**Doi** : https://doi.org/10.54216/JNFS.070205).

[3] Mohammad Abobala. (2023). On The Foundations of Fuzzy Number Theory and Fuzzy Diophantine Equations. Galoitica: Journal of Mathematical Structures and Applications, 10 (1), 17-25 (**Doi** : https://doi.org/10.54216/GJMSA.0100102).

[4] Florentine Smarandache. "Neutrosophic Two-Fold Algebra", Plithogenic Logic and Computation, Vol.1, No.1 2024. PP.11-15.

[5] Sankari, H., and Abobala, M., " On The Group of Units Classification In 3-Cyclic and 4-cyclic Refined Rings of Integers And The Proof of Von Shtawzens' Conjectures", International Journal of Neutrosophic Science, 2023.

[6] Abobala, M., " n-Cyclic Refined Neutrosophic Algebraic Systems of Sub-Indeterminacies, An Application to Rings and Modules", International Journal of Neutrosophic Science, 2020.

[7] Alfahal, A., Abobala, M., Alhasan, Y., & Abdulfatah, R. (2023). Generating Weak Fuzzy Complex and Anti Weak Fuzzy Complex Integer Solutions for Pythagoras Diophantine

Equation $\boldsymbol{X}^2 + \boldsymbol{Y}^2 = \boldsymbol{Z}^2$. *International Journal of Neutrosophic Science (IJNS*), Vol. 22, No. 01, PP. 08-14. https://doi.org/10.54216/IJNS.220201

[8] Galarza, F., Flores, M., Rivero, D., & Abobala, M. (2023). On Weak Fuzzy Complex Pythagoras Quadruples. *International Journal of Neutrosophic Science (IJNS*), Vol. 22, No. 02, PP. 108-113. https://doi.org/10.54216/IJNS.220209 .

[9] Mayada Abualhomos, Wael Mahmoud M. Salameh, Malik Bataineh, Mowafaq Omar Al-Qadri, Ayman Alahmade, Abdallah Al-Husban. (2024). An Effective Algorithm for Solving Weak Fuzzy Complex Diophantine Equations in Two Variables. International Journal of Neutrosophic Science, 23 (4), 386-394 (**Doi** : https://doi.org/10.54216/IJNS.230431).

[10] Abobala, M., Sankari, H., & Zeina, M. (2024). On novel security systems based on the 2-cyclic refined integers and the foundations of 2-cyclic refined number theory. *Journal of Fuzzy Extension and Applications*, *5*(1), 69-85. doi: 10.22105/jfea.2024.423818.1321.

[11] Noor Edin Rabeh, Othman Al-Basheer, Sara Sawalmeh, Rozina Ali. (2023). An Algebraic Approach to n-Plithogenic Square Matrices For $\mathbf{18} \leq \boldsymbol{n} \leq \mathbf{19}$. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 08-23 (**Doi** : https://doi.org/10.54216/JNFS.070201).

[12] Rashel Abu Hakmeh, Murhaf Obaidi. (2024). On Some Novel Results About Fuzzy n-Standard Number Theoretical Systems and Fuzzy Pythagoras Triples. Journal of Neutrosophic and Fuzzy Systems, 8 (1), 18-22 (**Doi** : https://doi.org/10.54216/JNFS.080102)

The refined indefinite neutrosophic integral

Yaser Ahmad Alhasan[1,*], Mohammed Mustafa Ahmed[2], Raja Abdullah Abdulfatah[3] and Suliman Sheen[4]

[1,*]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, KSA; y.alhasan@psau.edu.sa
[2]Department of mathematics, Faculty of science, Islamic University of Madinah, Madinah 42351, Saudi Arabia; mohamemedm086@gmail.com
[3]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.; r.abdulfatah@psau.edu.sa
[4]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.; S.almleh@psau.edu.sa

*Corresponding author: y.alhasan@psau.edu.sa

Abstract: This work presented the refined neutrosophic indefinite integral, where the substitution method for calculating integrals in the refined neutrosophic field that contain two part of indeterminacy (I_1, I_2) was presented. We also proved a theorem through which we were able to find most of the integrals for the refined neutrosophic functions.

Keywords: refined neutrosophic indefinite integral; substitution; indeterminacy.

1. Introduction and Preliminaries

To describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction, Smarandache suggested the neutrosophic Logic as an alternative to the current logics. Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R$ or C [1]

Agboola introduced the concept of refined neutrosophic algebraic structures [2]. Also, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that: [3]

$$I_1I_1 = {I_1}^2 = I_1 \quad (1)$$

$$I_2I_2 = {I_2}^2 = I_2 \quad (2)$$

$$I_1I_2 = I_2I_1 = I_1 \quad (3)$$

In addition, there are many papers presenting studies on refined neutrosophic numbers [4-5-6-7-8].
Smarandache discussed neutrosophic indefinite integral (Refined Indeterminacy) [11]
Let $g: \mathbb{R} \to \mathbb{R} \cup \{I_1\} \cup \{I_2\} \cup \{I_3\}$, where I_1, I_2, and I_3 are types of sub indeterminacies,

$$g(x) = 7x - 2I_1 + x^2I_2 + 4x^3I_3$$

then:

$$F(x) = \int [7x - 2I_1 + x^2I_2 + 4x^3I_3]dx$$

$$= \frac{7x^2}{2} - 2xI_1 + \frac{x^3}{3}I_2 + x^4 I_3 + a + bI_1 + cI_2 + dI_3$$

where a and b are real constants.

Alhasan also presented several papers on calculus, in which he discussed neutrosophic definite and indefinite integrals. He also presented the most important applications of definite integrals in neutrosophic logic [9-10].

Integration is important in human life, and one of its most important applications is the calculation of area, size and arc length. In our reality we find things that cannot be precisely defined, and that contain an indeterminacy part. This is the reason for studying neutrosophic integration and methods of its integration in this paper.

This paper dealt with several topics, in the first part of which introduction and preliminaries were presented, and in the main discussion part the refined neutrosophic indefinite integral that contain two part of indeterminacy (I_1, I_2). In the last part, a conclusion to the paper is given.

2. Main Discussion

The refined neutrosophic indefinite integral

Definition 1

Let $f: R(I_1, I_2) \to R(I_1, I_2)$, to evaluate $\int f(x, I_1, I_2)dx$

put: $x = g(u) \quad \Rightarrow \quad dx = \acute{g}(u)du$

by substitution, we get:

$$\int f(x, I_1, I_2)dx = \int f(u)\acute{g}(u)du$$

then we can directly integral it.

Theorem 1

If $\int f(x, I_1, I_2)dx = \varphi(x, I_1, I_2)$, then:

$$\int f\left((\dot{a} + \dot{b}I_1 + \dot{c}I_2)x + r + sI_1 + tI_2\right)dx$$
$$= \left(\frac{1}{\dot{a}} + \left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1 - \left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)\varphi\left((\dot{a} + \dot{b}I_1 + \dot{c}I_2)x + r + sI_1 + tI_2\right) + C$$

where C is an indeterminate real constant (i.e. constant of the form $a + bI_1 + cI_2$, where a, b, c are real numbers, while $I_1, I_2 =$ indeterminacy) and $\dot{a}_2 \neq 0$, $\dot{a}_2 \neq -\dot{c}_2$ and $\dot{a}_2 \neq -\dot{b}_2 - \dot{c}_2$

Proof:

put: $(\dot{a} + \dot{b}I_1 + \dot{c}I_2)x + r + sI_1 + tI_2 = u \quad \Rightarrow \quad (\dot{a} + \dot{b}I_1 + \dot{c}I_2)dx = du$

$$\Rightarrow \quad dx = \frac{1}{\dot{a} + \dot{b}I_1 + \dot{c}I_2}\, du$$

$$\Rightarrow dx = \left(\frac{1}{\dot{a}} + \left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1 - \left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)du$$

$$\int f\left((\dot{a} + \dot{b}I_1 + \dot{c}I_2)x + r + sI_1 + tI_2\right)dx = \int f(u)\left(\frac{1}{\dot{a}} + \left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1 - \left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)du$$

$$=\left(\frac{1}{\dot{a}}+\left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1-\left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)\varphi(u)+C$$

back to the variable x, we get:

$$\int f\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)dx$$
$$=\left(\frac{1}{\dot{a}}+\left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1-\left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)\varphi\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)+C$$

Using the previous theorem, we get on:

1) $\int\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)^n dx$

$$=\left(\frac{1}{\dot{a}}+\left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1-\left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)\frac{\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)^n}{n+1}+C$$

2) $\int\frac{1}{(\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2}dx$

$$=\left(\frac{1}{\dot{a}}+\left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1-\left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)ln\left|(\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right|+C$$

3) $\int e^{(\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2}dx$

$$=\left(\frac{1}{\dot{a}}+\left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1-\left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)e^{(\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2}+C$$

4) $\int\frac{1}{\sqrt{(\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2}}dx$

$$=2\left(\frac{1}{\dot{a}}+\left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1-\left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)\sqrt{(\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2}+C$$

5) $\int cos\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)dx$

$$=\left(\frac{1}{\dot{a}}+\left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1-\left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)sin\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)+C$$

6) $\int sin\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)dx$

$$=-\left(\frac{1}{\dot{a}}+\left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1-\left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)cos\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)+C$$

7) $\int sec^2\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)dx$

$$=\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)tan\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)+C$$

8) $\int csc^2\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)dx$

$$=-\left(\frac{1}{\dot{a}}+\left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1-\left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)cot\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)+C$$

9) $\int sec\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)\ tan\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)dx$

$$=\left(\frac{1}{\dot{a}}+\left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1-\left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)sec\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)+C$$

10) $\int csc\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)\ cot\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)dx$

$$=-\left(\frac{1}{\dot{a}}+\left[\frac{-\dot{a}\dot{b}}{\dot{a}(\dot{a}+\dot{c})(\dot{a}+\dot{b}+\dot{c})}\right]I_1-\left[\frac{\dot{c}}{\dot{a}(\dot{a}+\dot{c})}\right]I_2\right)csc\left((\dot{a}+\dot{b}I_1+\dot{c}I_2)x+r+sI_1+tI_2\right)+C$$

Example 1

1) $\int\left((2-4I_1+3I_2)x+7\right)^8dx=\left(\frac{1}{2}+\left[\frac{8}{(2)(5)(1)}\right]I_1-\left[\frac{3}{(2)(5)}\right]I_2\right)\frac{\left((2-4I_1+3I_2)x+7\right)^9}{9}+C$

$$=\left(\frac{1}{2}+\frac{4}{5}I_1-\frac{3}{10}I_2\right)\frac{\left((2-4I_1+3I_2)x+7\right)^9}{9}+C$$

2) $\int\frac{1}{(7-5I_1+6I_2)x+2I_1+I_2}dx$

$$=\left(\frac{1}{7}+\left[\frac{35}{(7)(13)(8)}\right]I_1-\left[\frac{6}{(7)(13)}\right]I_2\right)ln|(7-5I_1+6I_2)x+2I_1+I_2|+C$$

$$=\left(\frac{1}{7}+\frac{5}{104}I_1-\frac{6}{91}I_2\right)ln|(7-5I_1+6I_2)x+2I_1+I_2|+C$$

3) $\int e^{(1+I_2)x-6I_2}\,dx=\left(1+\left[\frac{0}{(1)(2)(2)}\right]I_1-\left[\frac{1}{(1)(2)}\right]I_2\right)e^{(1+I_2)x-6I_2}+C$

$$=\left(1-\frac{1}{2}I_2\right)e^{(1+I_2)x-6I_2}+C$$

4) $\int cos\left((3+3I_1+9I_2)x+7I_1\right)dx$

$$=\left(\frac{1}{3}+\left[\frac{-9}{(3)(12)(15)}\right]I_1-\left[\frac{9}{(3)(12)}\right]I_2\right)sin\left((3+3I_1+9I_2)x+7I_1\right)+C$$

$$=\left(\frac{1}{3}-\frac{1}{6}I_1-\frac{1}{4}I_2\right)sin\left((3+3I_1+9I_2)x+7I_1\right)+C$$

5) $\int sec^2\big((-4+3I_1+8I_2)x-2I_2\big)\,dx$

$$=\left(\frac{-1}{4}+\left[\frac{12}{(-4)(4)(7)}\right]I_1-\left[\frac{8}{(-4)(4)}\right]I_2\right)tan\big((-4+3I_1+8I_2)x-2I_2\big)+C$$

$$=\left(\frac{-1}{4}-\frac{3}{28}I_1+\frac{1}{2}I_2\right)tan\big((-4+3I_1+8I_2)x-2I_2\big)+C$$

6) $\int csc\big((1-3I_1+I_2)x\big)\ \ cot\big((1-3I_1+I_2)x\big)\,dx$

$$=-\left(1+\left[\frac{3}{(1)(2)(-1)}\right]I_1-\left[\frac{1}{(1)(2)}\right]I_2\right)csc\big((1-3I_1+I_2)x\big)+C$$

$$=-\left(1-\frac{3}{2}I_1-\frac{1}{2}I_2\right)csc\big((1-3I_1+I_2)x\big)+C$$

$$=\left(-1+\frac{3}{2}I_1+\frac{1}{2}I_2\right)csc\big((1-3I_1+I_2)x\big)+C$$

7) $\int \frac{1}{\sqrt{(10-I_1+8I_2)x+9-4I_1}}\,dx$

$$=2\left(\frac{1}{10}+\left[\frac{10}{(10)(18)(17)}\right]I_1-\left[\frac{8}{(10)(18)}\right]I_2\right)\sqrt{(10-I_1+8I_2)x+9-4I_1}+C$$

$$=2\left(\frac{1}{10}+\frac{1}{306}I_1-\frac{2}{45}I_2\right)\sqrt{(10-I_1+8I_2)x+9-4I_1}+C$$

$$=\left(\frac{1}{5}+\frac{2}{306}I_1-\frac{4}{45}I_2\right)\sqrt{(10-I_1+8I_2)x+9-4I_1}+C$$

Theorem 2

Let $f: R(I_1,I_2)\to R(I_1,I_2)$, then:

$$\int\frac{\acute{f}(x,I_1,I_2)}{f(x,I_1,I_2)}dx=ln|f(x,I_1,I_2)|+C$$

Proof:

put: $f(x,I_1,I_2)=u \qquad \Rightarrow \acute{f}(x,I_1,I_2)dx=\ du$

$$\Rightarrow dx=\frac{1}{\acute{f}(x,I_1,I_2)}\ du$$

$$\Rightarrow dx=\frac{1}{\acute{u}}\ du$$

$$\int\frac{\acute{f}(x,I_1,I_2)}{f(x,I_1,I_2)}dx=\int\frac{\acute{u}}{u}\frac{1}{\acute{u}}du=\int\frac{1}{u}du=ln|u|+C$$

back to the $f(x,I_1,I_2)$, we get:

$$\int\frac{\acute{f}(x,I_1,I_2)}{f(x,I_1,I_2)}dx=ln|f(x,I_1,I_2)|+C$$

Example 2

1) $\int\frac{(1-2I_1+3I_2)x^3}{(3-6I_1+9I_2)x^4+4I_1+I_2}dx=\frac{1}{3}ln|(3-6I_1+9I_2)x^4+4I_1+I_2|+C$

2) $\int \frac{(4-I_1+I_2)e^{(4-I_1+I_2)x+5}}{e^{(4-I_1+I_2)x+5}-10I_2}dx = ln\left|e^{(4-I_1+I_2)x+5}-10I_2\right|+C$

3) $\int tan\big((1+I_1+I_2)x+7I_1\big)\,dx = \int \frac{sin\big((1+I_1+I_2)x+7I_1\big)}{cos\big((1+I_1+I_2)x+7I_1\big)}dx$

$$= -\left(1+\left[\frac{-1}{(1)(2)(3)}\right]I_1-\left[\frac{1}{(1)(2)}\right]I_2\right)ln\left|\cos\big((1+I_1+I_2)x+7I_1\big)\right|+C$$

$$= \left(-1+\frac{1}{6}I_1+\frac{1}{2}I_2\right)ln\left|cos\big((1+I_1+I_2)x+7I_1\big)\right|+C$$

4) $\int \frac{1}{1+tan(5-2I_1-2I_2)x}dx = \int \frac{1}{1+\frac{sin(5-2I_1-2I_2)x}{cos(5-2I_1-2I_2)x}}dx$

$$= \frac{1}{2}\int \frac{2\,cos(5-2I_1-2I_2)x}{cos(5-2I_1-2I_2)x+sin(5-2I_1-2I_2)x}dx$$

$$= \frac{1}{2}\int \frac{cos(5-2I_1-2I_2)x+sin(5-2I_1-2I_2)x+cos(5-2I_1-2I_2)x-sin(5-2I_1-2I_2)x}{cos(5-2I_1-2I_2)x+sin(5-2I_1-2I_2)x}dx$$

$$= \frac{1}{2}\int dx+\frac{1}{2}\int \frac{cos(5-2I_1-2I_2)x-sin(5-2I_1-2I_2)\,x}{cos(5-2I_1-2I_2)x+sin(5-2I_1-2I_2)x}dx$$

$$= \frac{1}{2}x+\frac{1}{2}\left(\frac{1}{5}+\left[\frac{10}{(5)(3)(1)}\right]I_1-\left[\frac{-2}{(5)(3)}\right]I_2\right)ln|cos(5-2I_1-2I_2)x+sin(5-2I_1-2I_2)x|+C$$

$$= \frac{1}{2}x+\left(\frac{1}{10}+\frac{1}{3}I_1+\frac{1}{15}I_2\right)ln|cos(5-2I_1-2I_2)x+sin(5-2I_1-2I_2)x|+C$$

Theorem 3

Let $f: R(I_1, I_2) \to R(I_1, I_2)$, then:

$$\int \frac{\acute{f}(x, I_1, I_2)}{\sqrt{f(x, I_1, I_2)}}dx = 2\sqrt{f(x, I_1, I_2)}+C$$

Proof:

put: $f(x, I_1, I_2) = u$ $\Rightarrow \acute{f}(x, I_1, I_2)dx = du$

$$\Rightarrow dx = \frac{1}{\acute{f}(x, I_1, I_2)}\ du$$

$$\Rightarrow dx = \frac{1}{\acute{u}}\ du$$

$$\int \frac{\acute{f}(x, I_1, I_2)}{\sqrt{f(x, I_1, I_2)}}dx = \int \frac{\acute{u}}{\sqrt{u}}\frac{1}{\acute{u}}du = \int \frac{1}{\sqrt{u}}du = 2\sqrt{u}+C$$

back to $f(x, I_1, I_2)$, we get:

$$\int \frac{\acute{f}(x, I_1, I_2)}{\sqrt{f(x, I_1, I_2)}}dx = 2\sqrt{f(x, I_1, I_2)}+C$$

Example 3

1) $\int \frac{-(1+2I_1+2I_2)x+5I_2}{\sqrt{(2+4I_1+4I_2)x^2-10I_2x}}dx = -\sqrt{(2+4I_1+4I_2)x^2-10I_2x}+C$

2) $\int \frac{(5-3I_1+8I_2)x^2}{\sqrt{(5-3I_1+8I_2)x^3-2I_1+7I_2}}dx = \frac{2}{3}\sqrt{(5-3I_1+8I_2)x^3-2I_1+7I_2}+C$

Theorem 4

$f: R(I_1,I_2) \to R(I_1,I_2)$, then:

$$\int [f(x,I_1,I_2)]^n \acute{f}(x,I_1,I_2)\ \ dx = \frac{[f(x,I_1,I_2)]^{n+1}}{n+1}+C$$

Proof:

put: $f(x,I_1,I_2)=u \qquad \Rightarrow \acute{f}(x,I_1,I_2)dx = \ du$

$$\Rightarrow dx = \frac{1}{\acute{f}(x,I_1,I_2)}\ \ du$$

$$\Rightarrow dx = \frac{1}{\acute{u}}\ \ du$$

$$\int [f(x,I_1,I_2)]^n \acute{f}(x,I_1,I_2)\ \ dx = \int u^n \acute{u}\frac{1}{\acute{u}}du = \int u^n du = \frac{u^{n+1}}{n+1}+C$$

back to $f(x,I_1,I_2)$, we get:

$$\int [f(x,I_1,I_2)]^n \acute{f}(x,I_1,I_2)\ \ dx = \frac{[f(x,I_1,I_2)]^{n+1}}{n+1}+C$$

Example 5

1) $\int x^2[(3+2I_1+2I_2)x^3]^{12}dx = \frac{1}{3}\int 3x^2[(3+2I_1+2I_2)x^3]^{12}dx$

$$= \frac{1}{9+6I_1+6I_2}\frac{[(3+2I_1+2I_2)x^3]^{13}}{13}+C$$

$$= \left(\frac{1}{9}+\left[\frac{-54}{(9)(15)(21)}\right]I_1-\left[\frac{6}{(9)(15)}\right]I_2\right)\frac{[(3+2I_1+2I_2)x^3]^{13}}{13}+C$$

$$= \left(\frac{1}{9}-\frac{2}{105}I_1-\frac{2}{45}I_2\right)\frac{[(3+2I_1+2I_2)x^3]^{13}}{13}+C$$

2) $\int \frac{1}{\sqrt{(2+I_1+I_2)x-I_1+2I_2}}\left(\sqrt{(2+I_1+I_2)x-I_1+2I_2}\right)^{14}dx$

$$= 2\left(\frac{1}{2}-\left[\frac{1}{(3)(4)}\right]I_1-\left[\frac{1}{(2)(3)}\right]I_2\right)\frac{\left(\sqrt{(2+I_1+I_2)x-I_1+2I_2}\right)^{15}}{15}+C$$

$$= \left(1 - \frac{1}{6}I_1 - \frac{1}{3}I_2\right)\frac{\left(\sqrt{(2 + I_1 + I_2)x - I_1 + 2I_2}\right)^{15}}{15} + C$$

3. Conclusions

The integral is very important in our life, and is used especially for example in calculating areas whose shape is not familiar. This led us to study the refined neutrosophic indefinite integral that contain two parts of indeterminacy (I_1, I_2). Where the method of integration by substitution are applied to the neutrosophic functions by presenting several theorems through which we were able to apply them to directly find the refined neutrosophic indefinite integral.

Acknowledgments "This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2024/R/1445)".

References

[1] Smarandache, F., " (T,I,F)- Neutrosophic Structures", Neutrosophic Sets and Systems, vol 8, pp. 3-10, 2015.

[2] Agboola,A.A.A. "On Refined Neutrosophic Algebraic Structures", Neutrosophic Sets and Systems, vol 10, pp. 99-101, 2015.

[3] Adeleke, E.O., Agboola, A.A.A.,and Smarandache, F., "Refined Neutrosophic Rings I", International Journal of Neutrosophic Science, Vol. 2(2), pp. 77-81. 2020.

[4] Smarandache, F.,"n-Valued Refined Neutrosophic Logic and Its Applications in Physics", Progress in Physics, USA, vol 4, pp. 143-146, 2013.

[5] Vasantha Kandasamy,W.B; Smarandache,F. "Neutrosophic Rings" Hexis, Phoenix, Arizona, 2006, http://fs.gallup.unm.edu/NeutrosophicRings.pdf

[6] Zeina, M., Abobala, M., "On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, Volume 54, pp. 158-168, 2023.

[7] Agboola, A.A.A.; Akinola, A.D; Oyebola, O.Y., "Neutrosophic Rings I", Int. J. of Math. Comb., vol 4, pp.1-14, 2011.

[8] Celik, M., and Hatip, A., " On The Refined AH-Isometry And Its Applications In Refined Neutrosophic Surfaces", Galoitica Journal Of Mathematical Structures And Applications, 2022.

[9] Alhasan,Y., "The neutrosophic integrals and integration methods", Neutrosophic Sets and Systems, Volume 43, pp. 290-301, 2021.

[10] Alhasan,Y., "The definite neutrosophic integrals and its applications", Neutrosophic Sets and Systems, Volume 49, pp. 277-293, 2022.

[11] Smarandache, F., "Neutrosophic Precalculus and Neutrosophic Calculus", book, 2015.

Received: Feb 10, 2024. Accepted: April 30, 2024

Neutrosophic Sets and Systems, Vol. 67, 2024

University of New Mexico

Critical Path Method & Project Evaluation and Review Technique: A Neutrosophic Review

Navya Pratyusha M[1], Arindam Dey[1], S Broumi[2] A Ranjan Kumar[1,*]

[1]VIT-AP University, Inavolu, Beside AP Secretariat, Amaravati AP, India;

[2]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Morocco.

[*]Correspondence: ranjank.nit52@gmail.com

Abstract. Fuzzy, intuitionistic, and neutrosophic sets are the primary focus of the review investigation under the extension concepts. The significance and use of triangular shape for data representation are investigated. For modelling and expressing ambiguous or complex data, the triangle shape is a useful tool due to its simplicity and computational efficiency. Further study involves the tool of the operations research technique i.e., Network Analysis that helps in implementing and developing using the fuzzy extension principle. Critical Path Method (CPM) & Project Evaluation and Review Technique (PERT) plays a major part in the field of network in various decision-making scenarios using the real-life applications. In this comprehensive study, the insights of understanding extended fuzzy using the CPM/PERT under various applications are reviewed and analyzed for the future advancements in making more accurate and optimum results.

Keywords: Network Analysis; Critical Path Method (CPM); Project Evaluation and Review Technique (PERT); Uncertainty; Extended Fuzzy Principle

1. Introduction

Uncertainty plays a leading role in various fields of modern upgrowth in science and technology, due to ruling world of vague and ambiguity. Followed by which, zadeh introduced the fuzzy concept in 1965, uncertainty theory has risen significantly [1], many researchers from the distinctive domains have worked and executed the uncertainty study in various fields such as science and technology, medical experimental, social media, financial mathematics, ecology etc. Fuzzy sets play a standard role in many technical problems. However, there is an essential issue regarding to relate or use the idea of impreciseness in our computational mathematical modelling. From which a few years later, Chang and Zadeh [2] gave the outline sketch of fuzzy development in the idea of fuzzy sets and formative numbers. Many experts in the subject have proposed various ways to describe it, offered suggestions such as the triangle fuzzy number [3], and shared their views on the best way to deal with uncertainty.

In real-life, environments often express their preferences in more complicated ways, where the expression involves the degree of belief and degree of disbelief towards a certain statement, as if these degrees overlap. In this regard, Krassimir Atanassov came up with the idea of an Intuitionistic fuzzy number (IFN) [4], involving both membership and non-membership belongingness. Following this, the fuzzy triangle IFN shape [5] was created and used in a specific area of mathematics. In furthermore study, Inter-valued IFN [6] study was manifested as it the extension work of IFN. The previous research led to the conclusion that the combined membership and non-membership functions cannot exceed to 1. But in real life, it's not always feasible to stick to the rules and provide under restriction. As an example, if someone says they are 0.8% satisfied the other 0.4% termed as dissatisfied. Therefore, to such situations, IFS theory cannot be handled. To get around this, Yager [7] modified the IFS criteria by squaring the sum of the related sets, corresponding to be Pythagorean fuzzy sets (PFS). The advancement of uncertainty theory, particularly fuzzy set theory and its extensions like IFS and PFS, features the continuous effort to model and interpret the complexity of real-world phenomena more accurately. These advancements reflect on understanding the real-life situations often exhibit degrees of uncertainty that traditional binary logic cannot be captured, as to get a brief view Figure.1 can be addressed for the flowchart of different uncertain parameters.

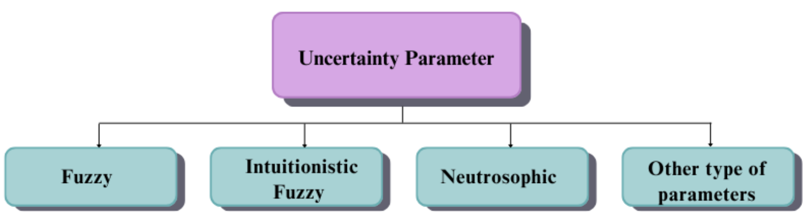


Figure 1. Flowchart for different uncertain parameter

Additionally, multiple researchers have put the theories into action, by creating and improving numerous approaches, and offered several recommendations for using the uncertainty philosophy. The lack of a representation of the unclear parameter leads the literature to address various ideas for classifying these characteristics. Different applications have granted decision-making authority to address such difficulty. Smarandache has created an IFN extension for the previously anticipated setting, termed as the Neutrosophic set (NS) [8]. It determines the truthiness (T), indeterminacy (I), or falsity (F) of each element using membership functions.

Contrarily, NS vary from IFN by sum of truth, indeterminacy, and falsity can be equal or less than equal to 3 on any real number between the interval [0,1]. Considering intuitionistic fuzzy set theories and fuzzy set theories with fuzzy boundaries, where the membership value is always between 0 and 1. Decision making (DM) problems and mathematical modelling are two of its most common modern applications. An increasingly effective tool for tackling complicated problems was created by Wang et.al., [9] via ongoing study; it is the enhanced perception of a single-typed neutrosophic set. In addition, Chakraborty et.al., 10 classified the concept of triangular neutrosophic. Further, the implementation of basic information about uncertain parameters, differing from each other using the concept of uncertainty using some definitions, flowcharts, and diagrams are shown in further sections.

1.1. *Verbal phase related to uncertainty*

In the daily life, researchers often focus on the point of establishing a logical relationship in the state of uncertainty concept in the real-world scenario with the idea of verbal phase for better understanding the concept. In this phenomenon, using the idea mentioned, the construction for such raised questioned can be explained using the example mentioned below:

Example: To maintain a democratic approach while forming the committee, for employing a vote system that incorporates the sentiments, hopes, feeling, ethics and dreams of the group members. An expressed statement describing the amount of support, opposition, or neutrality for the prospective committee member is used by each member to indicate their choice to capture the uncertain and inherent information in decision-making environment was captured by Table 1.

TABLE 1. Verbal phrases for different uncertain parameters

Uncertainty	**Verbal**	**Description**
Crisp Parameter	Support/Oppose	Members expressing the response in the form of binary logic indicating either support as 1 or oppose as 0.
Fuzzy Parameter	Degree of Support	Members expressing the response indicating their fully support or completely Non-support.

Continued on next page

Table 1 – Continued from previous page

Uncertainty	**Verbal**	**Description**
Intuitionistic Fuzzy	Support and Non-Support	Members expressing their support and non-support levels separately, using the level of degrees of uncertainty.
Neutrosophic	Support, Neutral, Non-Support	In NS, the members expressing in three distinct stances, in capturing the complexity of human opinion, including indecisiveness and neutrality.

1.2. *Some basic differences between some uncertain parameters*

Various kinds of sets having expression with different strengths and weaknesses that deals with the real-world challenges, can be seen in the Table 2 and better visualization is presented in Figure 2.

TABLE 2. Differences between uncertain parameters

Various types of sets	**Advantages**	**Limitations**
Crisp Sets	Can correctly decide without any hesitation	Dealing with the information having uncertainty
Fuzzy sets	Can correctly describe and define uncertain	Cannot define the non-membership degree having the uncertain Information
Intuitionistic Fuzzy set	Can define the information having non-membership (NMS) and membership (MS) degree function	The addition of MS and NMS degree obtains above 1

Continued on next page

Table 2 – Continued from previous page

Various types of sets	Advantages	Limitations
Neutrosophic	Dealing with indeterminacy and getting the optimum result	Interval data type cannot be handled

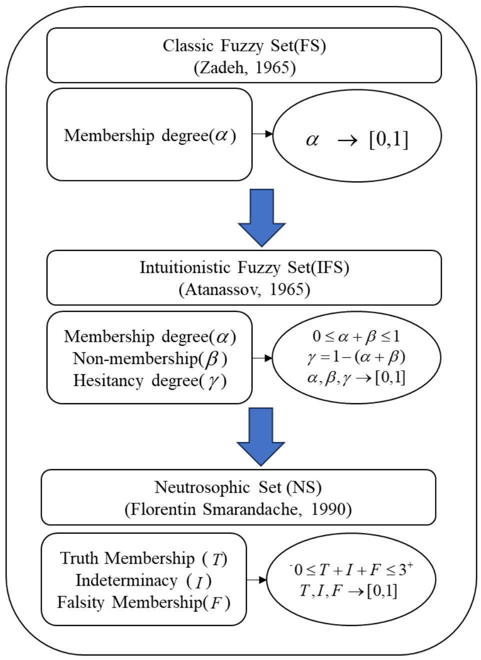
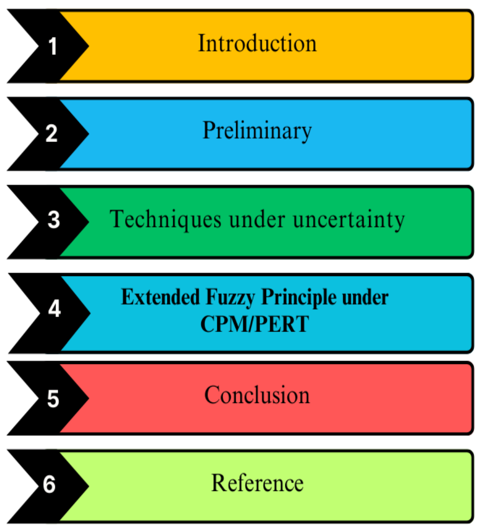


FIGURE 2. The development of NS and the extension of the classical fuzzy sets

Project scheduling strategies like Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT) are the major emphasis of the study. The network analysis fields of project management have benefited greatly from these two well-known approaches because of the helpful information they provide on timeframes, resource allocation, and risk analysis, as they are the integral part of project planning, scheduling, and controlling. To find the best possible time or solution relies heavily on CPM's primary focus on finding the route, which is comprised of interdependent actions [11]. A single time factor is used to anticipate the long route duration using CPM [12]. But in actual projects, these assumptions don't take unexpectedness and variety into consideration. One way to deal with these assumptions is Project Evaluation and Review Technique (PERT), as it recognizes the nature of possible activity periods to be unknown [13]. It considers these unknowns and help in making the project to find more optimality. Project managers may maximize performance by organizing tasks using CPM and taking uncertainties into account using PERT. Management of project hazards, resource allocation, and overall project length may all be improved with the use of these methods. Combining CPM with PERT has changed the game for project management, allowing businesses to make better use of their resources and, in the end, discover better solutions. While most cases display activity times accurately, in deterministic assumptions. But, when dealing with uncertain or imprecise data, to predict and analyze potential outcomes by employing numerical representations of the data is critical. To involve uncertainty and imprecision in project parameters more realistically, the use of triangle shapes delves into CPM/PERT act as the crucial role in task durations and activity periods. Several studies have explored cases where activity times in a project are approximately known and can be suitably represented by sets rather, than precise numbers [14, 15].

1.3. *Structure of the paper*

The article is developed as follows:



2. Mathematical Preliminary

Some preliminary concepts of fuzzy sets and fuzzy numbers are discussed Buckley [16], Dubois and Prade [17], Atanassov [18], Li [19], Subas [20], Kaufmann and Gupta [21], Klir and Yuan [22], and Zadeh [23]. The basic definitions and notations below will be used throughout the paper.

Definition 2.1. The set $\widetilde{f}$ is illustrated as $\widetilde{f} = \left\{ \left(\psi, \mu_{\widetilde{f}}(\psi) \right) : \psi \in f, \mu_{\widetilde{f}}(\psi) \in [0, 1] \right\}$ is often represented by the paired set $\left(\psi, \mu_{\widetilde{f}}(\psi) \right)$, here $\psi \in f$ termed as crisp $\mu_{\widetilde{f}}(\psi) \in [0, 1]$; such that $0 \leq \mu_{\widetilde{f}}(\psi) \leq 1$, $\widetilde{f}$ represents fuzzy set.

Definition 2.2. In the fuzzy set, $\widetilde{f}$ be the fuzzy number on the real line R that satisfies the normality condition and the convexity condition.

Definition 2.3. The triangular fuzzy number $(\widetilde{tfn})$ be the fuzzy number with a linear piece-wise membership function $\mu_{\widetilde{tfn}}(\psi)$ is described as follows:

$$\mu_{\widetilde{tfn}}(\psi) = \begin{cases} \frac{\psi - tfn_1}{tfn_2 - tfn_1}, & tfn_1 \leq \psi \leq tfn_2 \\ 1, & \psi = tfn_2 \\ \frac{tfn_3 - \psi}{tfn_3 - tfn_2}, & tfn_2 \leq \psi \leq tfn_3 \\ 0, & \text{Otherwise.} \end{cases}$$

Further, TFN can be represented as an ordered triplet $\widetilde{tfn} = (tfn_1, tfn_2, tfn_3)$ and below the diagrammatic view of triangle fuzzy number:

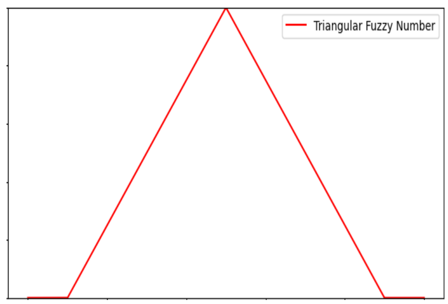


Figure 3:Triangular Fuzzy Number

Definition 2.4. A set $\widetilde{IFS}$, denoted as $\widetilde{IFS} = \{\langle \alpha; [\tau(\alpha), \gamma(\alpha)] \rangle : \alpha \in \psi\}$ may be visually shown as a membership degree in which $\tau(\alpha), \gamma(\alpha) : \psi \to [0,1]$, as truth be $\tau(\alpha)$ and false be $\gamma(\alpha)$ the membership degree. The condition for the set $\tau(\alpha), \gamma(\alpha)$ to satisfy $0 \leq \tau(\alpha) + \gamma(\alpha) \leq 1$.

Definition 2.5. The Triangular Intuitionistic fuzzy number $(\widetilde{TIF})$ is presented in Figure 3 termed as intuitionistic fuzzy set in R with the following membership $\lambda_{\widetilde{TIF}}(\psi)$ and non-membership function $\upsilon_{\widetilde{TIF}}(\psi)$ as follows:

$$\lambda_{\widetilde{tif}}(\psi) = \begin{cases} \frac{\psi - tif_1}{tif_2 - tif_1}, & tif_1 \leq \psi \leq tif_2 \\ \frac{tif_3 - \psi}{tif_3 - tif_2}, & tif_2 \leq \psi \leq tif_3 \\ 0, & \text{Otherwise.} \end{cases} \quad \upsilon_{\widetilde{TIF}}(\psi) = \begin{cases} \frac{tif_1 - \psi}{tif_2 - tif_1{}'}, & tif_1{}' \leq \psi \leq tif_2 \\ \frac{\psi - tif_2}{tif_3{}' - tif_2}, & tif_2 \leq \psi \leq tif_3{}' \\ 1, & \text{Otherwise.} \end{cases}$$

Further, TIF can be represented as $\widetilde{tif} = (tif_1{}', tif_1, tif_2, tif_3, tif_3{}')$, where $tif_1{}' < tif_1 < tif_2 < tif_3 < tif_3{}'$ and $\lambda_{\widetilde{tif}}(\psi), \upsilon_{\widetilde{tif}}(\psi) \leq 0.5$ for $\lambda_{\widetilde{tif}}(\psi) = \upsilon_{\widetilde{tif}}(\psi)\ \forall\ \psi \in$ R.

Definition 2.6. The universal set ψ ,denoted by $(neu)^N$ if it satisfies the condition, is considered a neutrosophic number $(neu)^N = \left\{ \left\langle \alpha; [\lambda_{(neu)^N}(\alpha), \beta_{(neu)^N}(\alpha), \gamma_{(neu)^N}(\alpha)] \right\rangle : \alpha \in \psi \right\}$, where $\lambda_{(neu)^N}(\alpha) : \psi \to [0,1]$,$\beta_{(neu)^N}(\alpha) : \psi \to [0,1]$ and $\gamma_{(neu)^N}(\alpha) : \psi \to [0,1]$ represent membership functions for truth, indeterminacy, and falsity, correspondingly and exhibits the following relation:

$$0 \leq \lambda_{(neu)^N}(\alpha) + \beta_{(neu)^N}(\alpha) + \gamma_{(neu)^N}(\alpha) \leq 3$$

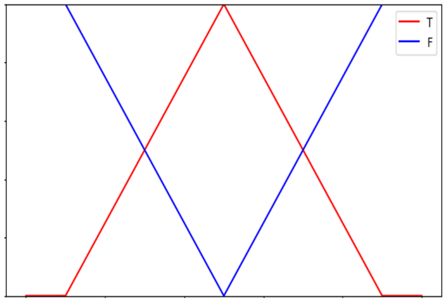


FIGURE 3. Triangular Intuitionistic Fuzzy Number

Definition 2.7. A triangular single valued neutrosophic ($\widetilde{tr_{SVN}}$) represented as $\widetilde{tr_{SVN}} = <((s,t,q);\sigma_{\widetilde{tr_{SVN}}},\varsigma_{\widetilde{tr_{SVN}}},\tau_{\widetilde{tr_{SVN}}})$ is represented in Figure 4, where $\sigma_{\widetilde{tr_{SVN}}},\varsigma_{\widetilde{tr_{SVN}}},\tau_{\widetilde{tr_{SVN}}} \in [0,1]$. Here the truth, indeterminacy, and falsity membership functions are defined as follows:

$$\sigma_{\widetilde{tr_{SVN}}} = \begin{cases} \frac{(\psi-s)\sigma_{\widetilde{tr_{SVN}}}}{t-s}, & \text{s} \leq \psi \leq \text{t}; \\ \frac{(q-\psi)\sigma_{\widetilde{tr_{SVN}}}}{q-t}, & \text{t} \leq \psi \leq \text{q}; \\ 0, & \text{Otherwise.} \end{cases} \quad \varsigma_{\widetilde{tr_{SVN}}} = \begin{cases} \frac{(t-\psi+\varsigma_{\widetilde{tr_{SVN}}}(\psi-s))}{t-s}, & \text{s} \leq \psi \leq \text{t}; \\ \frac{(\psi-t+\varsigma_{\widetilde{tr_{SVN}}}(q-\psi))}{q-t}, & \text{t} \leq \psi \leq \text{q}; \\ 0, & \text{Otherwise.} \end{cases}$$

$$\tau_{\widetilde{tr_{SVN}}} = \begin{cases} \frac{(t-\psi+\tau_{\widetilde{tr_{SVN}}}(\psi-s))}{t-s}, & \text{s} \leq \psi \leq \text{t}; \\ \frac{(\psi-t+\tau_{\widetilde{tr_{SVN}}}(q-\psi))}{q-t}, & \text{t} \leq \psi \leq \text{q}; \\ 0, & \text{Otherwise.} \end{cases}$$

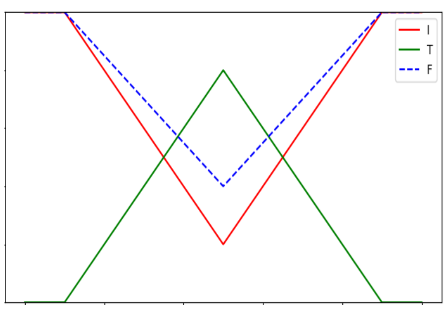


FIGURE 4. Triangular Single-Valued Neutrosophic

3. Impact of various techniques under uncertainty

Decisions in many areas have greatly benefited from the application of operational research methodologies. To solve such complex problems and apply more efficient statistical analysis methods, statistical modelling and optimization techniques come into the field to measure and control uncertainty, necessarily due to inherent unpredictability and variability in real-world conditions shed light on possible consequences and risks, enabling decision-makers to make choices that they choose very informedly. Organizations could make better selections and broaden stronger techniques after they incorporate uncertainty into their decision making. The following Table 3 provides a top-level view of several well-known overall performance appraisal techniques for dealing with uncertainty in numerous industries:

Table 3. Influencing the impact of uncertainty in various techniques

Main Contribution	**Environment**	**Author(s) Information and Year**
To resolve the shortest direction hassle via network problem considering the uncertainty and imprecision	Single valued Triangular Neutrosophic	Broumi et.al., (2019) [24]
To solve the transportation trouble and gain a most useful solution	Triangular Intuitionistic	Pathade et.al., (2019) [25]
A novel ranking feature and demonstration of the utility in the subject of integer programming	Triangular Neutrosophic	Das and Edalatpanah (2020) [26]
Implementing a direct model that facilitates the inclusion in linear programming to allow extra accurate consequences	Triangular Neutrosophic	Edalatpanah (2020) [27]
To endorse and investigate at the software of inventory backorder problem	Triangular Neutrosophic	Mullai and Surya (2020) [28]

Based at the studies above, those strategies have emerged as beneficial systems in the discipline of operations studies incorporating unique environments in their fields for acquiring the most efficient solution. Among the operations observer's strategies, the CPM and PERT have emerged as powerful gear for coping with uncertainty in project making plans and control.

4. Extended Fuzzy principle under CPM/PERT

Fuzzy theory is an extension of traditional fuzzy theory, whose objectives satisfy limitations, solving the robustness of fuzzy discrimination. As the extension under the concept of fuzzy gadgets by parameters and methods adding more accuracy and greater complexity. Fuzzy numbers can express uncertainty through their ability to describe the membership of a set. These membership functions can take many forms, including triangular, trapezoidal, or Gaussian, and can be used to generate imprecision and ambiguity that allows for a more accurate model, because it considers a feasible range of values within a triangular shape. This preferred modeling function is especially valuable in formulating and problem-solving situations where uncertainty plays a dominant role. This advice led to the use of multiple areas, where the addition of CPM and PERT made it difficult to control applications.

To effectively plan and handle challenges in the face of uncertainty, project managers may benefit of combining CPM and PERT with protracted fuzzy theories. It enables accounting for genetic ambiguity and variability in project data, leading to improved decision-making and more reliable project outcomes Furthermore many researchers such as Chakraborty et.al., (2018) [10] explore in-depth analysis on the categories of triangular neutrosophic numbers. Further, Mohamed et.al., (2017) [29] directly establish a connection with critical route and neutrosophic numbers by the use of triangles by their performance on the critical path problem as well as Mohamed et.al., (2017) [30] explore critical path problems in a neutrosophic context, and shed light on the use of neutrosophic numbers analysis in business processes and critical paths. On working with different categories, Vijaya et.al., (2022) 31 provide insights into the critical approach required in fuzzy project networks using neutrosophic fuzzy numbers, establishing a relationship between fuzzy project networks and the use of neutrosophic fuzzy numbers for project schedule analysis is summarized by the use of extended fuzzy theory and triangular fuzzy numbers that presents valuable extensions combining fuzzy logic with CPM/PERT technique that enables to handle the uncertainty in project management, and leads to better project planning, risk assessment, and overall project performance.

Conclusion

In task management, CPM and PERT are widely recognized as an effective tool in planning and scheduling complicated projects. Extending the application of these fuzzy concepts gives big improvements, especially in coping with the uncertainties and ambiguities having inherent information in large-scale projects. Fuzzy CPM/PERT adds fuzzy logic to traditional project management techniques for a more realistic and flexible way to estimate activity time and schedule. This integration accommodates unrealistic and variable real-world projects with fuzzy numbers, rather than crisp criteria. Mostly predictable resulting is more efficient and effective in project execution field, while working with uncertainty. The use of fuzzy CPM/PERT is especially beneficial in large projects, where it helps the project size, reducing the delivery time frame, and the margin of error to be minimal.

References

1. Zadeh,L.A. Fuzzy sets, Inf Control (1965), 8, 338–353.
2. Chang,S.S.L.; Zadeh,L.A. On fuzzy mapping and control, IEEE Transactions on Systems, Man, and Cybernetics(1972), vol.2, pp.30–34.
3. Gorcun,O.F.; Senthil,S.; Küçükönder,H. Evaluation of tanker vehicle selection using a novel hybrid fuzzy mcdm technique. Decision Making: Applications in Management and Engineering 2021, vol.4, pp.140–162.
4. Atanassov,K.T.; Stoeva,S. Intuitionistic fuzzy sets, Fuzzy Sets And Systems(1986), vol.20,87–96.
5. Prasenjit,C. Model for selecting a route for the transport of hazardous materials using a fuzzy logic system, Vojnotehnički glasnik (2021), vol.69, pp.355–390.
6. Atanassov,K.T.; Gargov,G. Interval valued intuitionistic fuzzy sets, Intuitionistic Fuzzy Sets: Theory and Applications (1999), pp.139–177.
7. Yager,R.R. Pythagorean fuzzy subsets, Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Edmonton, AB, Canada, 2013; pp.57-61.
8. Smarandache,aZF. A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth, 1999.
9. Wang,H.; Smarandache,F.; Zhang, Y.; Sunderraman, R. Single valued neutrosophic sets (2010), Infinite study, vol.12.
10. Chakraborty,A.; Mondal,S.P.; Ahmadian,A.; Senu,N.; Alam,S.; Salahshour, S. Different forms of triangular neutrosophic numbers, de-neutrosophication techniques, and their applications (2018), Symmetry, vol.10, pp. 327.
11. Kelley,J.; James,E.; Walker,M.R. Critical-path planning and scheduling, eastern joint IRE-AIEE-ACM computer conference, in Papers presented at the December 1-3, 1959; pp.160-173.
12. Heizer,J.; Render,B. Operation Management's text book, Tenth edition; Publisher:Pearson, 2011.
13. Malcolm,D.G.; Roseboom,J.H.; Clark,C.E.; Fazar,W. Application of a technique for research and development program evaluation (1959), Operations Research, vol.7, pp. 646–669.
14. Chen,C.T. Extensions of the topsis for group decision-making under fuzzy environment (2000), Fuzzy Sets and Systems, vol.114, pp.1–9.
15. Zammori,F.A.; Braglia,M.;Frosolini,M. A fuzzy multi-criteria approach for critical path definition (2009), International Journal of Project Management, vol.27, pp.278–291.
16. Buckley,J.J. Fuzzy hierarchical analysis (1985), Fuzzy Sets And Systems, vol.17, pp.233–247.
17. Dubois,D.; Prade,H. Fuzzy numbers: an overview (1993), Readings in Fuzzy Sets for Intelligent Systems, pp.112–148.
18. Atanassov,K.T. Intuitionistic fuzzy sets, Springer, 1999.
19. Li,D.F. A ratio ranking method of triangular intuitionistic fuzzy numbers and its application to madm problems (2010), Computers & Mathematics with Applications, vol.60, pp. 1557–1570.
20. Subas,Y. Neutrosophic numbers and their application to multi-attribute decision making problems, master's thesis, kilis 7 aralık university, graduate school of natural and applied science (2015), Neutrosophic Sets and Systems, vol.19, pp. 141, 2018.

21. Kaufmann,A.; Gupta,M. Introduction to fuzzy arithmetic: theory and application, Van nonstrand reinhold company inc, New York, 1985.
22. Klir G.J.; Yuan,B. Fuzzy sets and fuzzy logic: theory and applications (1996),Possibility Theory versus Probability Theory, vol. 32, pp. 207–208.
23. Zadeh,L.A. The concept of a linguistic variable and its application to approximate reasoning—i (1975), Information Sciences, vol. 8, pp. 199–249.
24. Broumi,S.; Bakali,A.; Talea,M.; Smarandache,F. Computation of shortest path problem in a network with sv-triangular neutrosophic numbers (2019), Uluslararası Yönetim Bilişim Sistemleri ve Bilgisayar Bilimleri Dergisi, vol.3, pp.41–51.
25. Pathade,P.A.; Ghadle,K.P.; Hamoud,A.A. Optimal solution solved by triangular intuitionistic fuzzy transportation problem, Advances in Intelligent Systems and Computing: Proceedings of ICCET (2019), vol.1025, pp.379-385.
26. Das,S.K.; Edalatpanah,S.A. A new ranking function of triangular neutrosophic number and its application in integer programming (2020), International Journal of Neutrosophic Science, vol.4, pp. 82–92.
27. Edalatpanah,S.A. A direct model for triangular neutrosophic linear programming (2020), International Journal of Neutrosophic Science, vol.1, pp. 19–28.
28. Mullai,M.; Surya,R. Neutrosophic inventory backorder problem using triangular neutrosophic numbers (2020), Neutrosophic Sets and Systems, vol.31, pp. 148–155.
29. Mohamed,M.; Abdel-Basset,M.; Hussien,A.N.; Smarandache, F. Using neutrosophic sets to obtain pert three-times estimates in project management (2017), Infinite Study, vol.1.
30. Mohamed,M.; Abdel-Baset,M.; Smarandache,F.; Zhou,Y. A critical path problem in neutrosophic environment, Infinite Study: El Segundo, CA, USA, 2017.
31. Vijaya,V.; Rajalaxmi,D.; Manikandan,H. Finding critical path in a fuzzy project network using neutrosophic fuzzy number (2022), Advances and Applications in Mathematical Sciences, vol.21, pp. 5743–5753.

Received: Feb 2, 2024. Accepted: April 26, 2024

Harnessing Pliancy Tree Soft Sets in Heart Diseases for Extracting Beneficial Rules of Association Rules

Gawaher S. Hussein [1], Khalid A. Eldrandaly [1] , Abdel Nasser H. Zaied[2] ,Samar L. Elhawy[1,*] , Mona Mohamed[3]

[1]Faculty of Computers and Informatics, Zagazig University, Sharqiyah, Zagazig, 44519, Egypt.Emails: Gawahersoliman@zu.edu.eg, Khalid_Eldrandaly@zu.edu.eg , SLAlhawy@fci.zu.edu.eg.

[2]Department of Information Systems, Misr International University, Egypt; Emails abdelnasser.riad@miuegypt.edu.eg.

[3]Higher Technological Institute, 10th of Ramadan City 44629, Egypt, Email: mona.fouad@hti.edu.eg.

Abstract

Cardiovascular diseases (CVDs)continue to be the primary cause of mortality, accounting for approximately one-third of all fatalities globally. This spawned the proposal of models in several studies. Accordingly, this study contributed to diagnosing heart disease through suggesting soft diagnosis paradigm. Various techniques have been volunteered for serving the suggested paradigm toward achieving its objective. Additionally, this study provided set of contributions. For instance, Tree Soft Technique (TrST) is applied for the first time for forming attributes and sub attributes of patients into nodes and sub-nodes of Tree to obtain relations between it. Even, the study support stakeholders to making accurate decision in mysterious circumstances and in problems with incomplete information through Collaborating the utilized techniques of entropy and Technique for order of preference by similarity to ideal solution (TOPSIS) in this study with Single Value Neutrosophic Sets (SVNSs) forked from neutrosophic uncertainty theory. As well, the relationship between sub-attributes which consider antecedent for obtaining consequent of detecting and diagnosing through collaborating TrST with association rules. Accordingly, we applied four transactions (cases) for obtaining findings of the relations in transactions as listed in Table 13 and Table14.

Keywords:

Cardiovascular diseases (CVDs), Tree Soft Technique (TrST), Single Value Neutrosophic Sets (SVNSs), association rules.

1. Introduction

1.1 Contextual Study

Indeed, according to the World Health Organization (WHO)[1] , cardiovascular diseases (CVDs) are the primary cause of mortality globally, especially in nations with the greatest poverty. The encouragement for this [2] comes from the alarming global statistics on cardiovascular disease, which indicate that by 2030, the total number of deaths per year will surpass 20 million. Overall, the expression of CVDs described in [3] as a variety of conditions that impact the heart and blood arteries. Likewise thought that CVDs or heart diseases are one of the most difficult, deadly, and life-threatening illnesses that affect people worldwide. CVDs in [4] classified into various categories as represented in Fig 1. Also, this Fig illustrates factors causing the disease and its consequences.

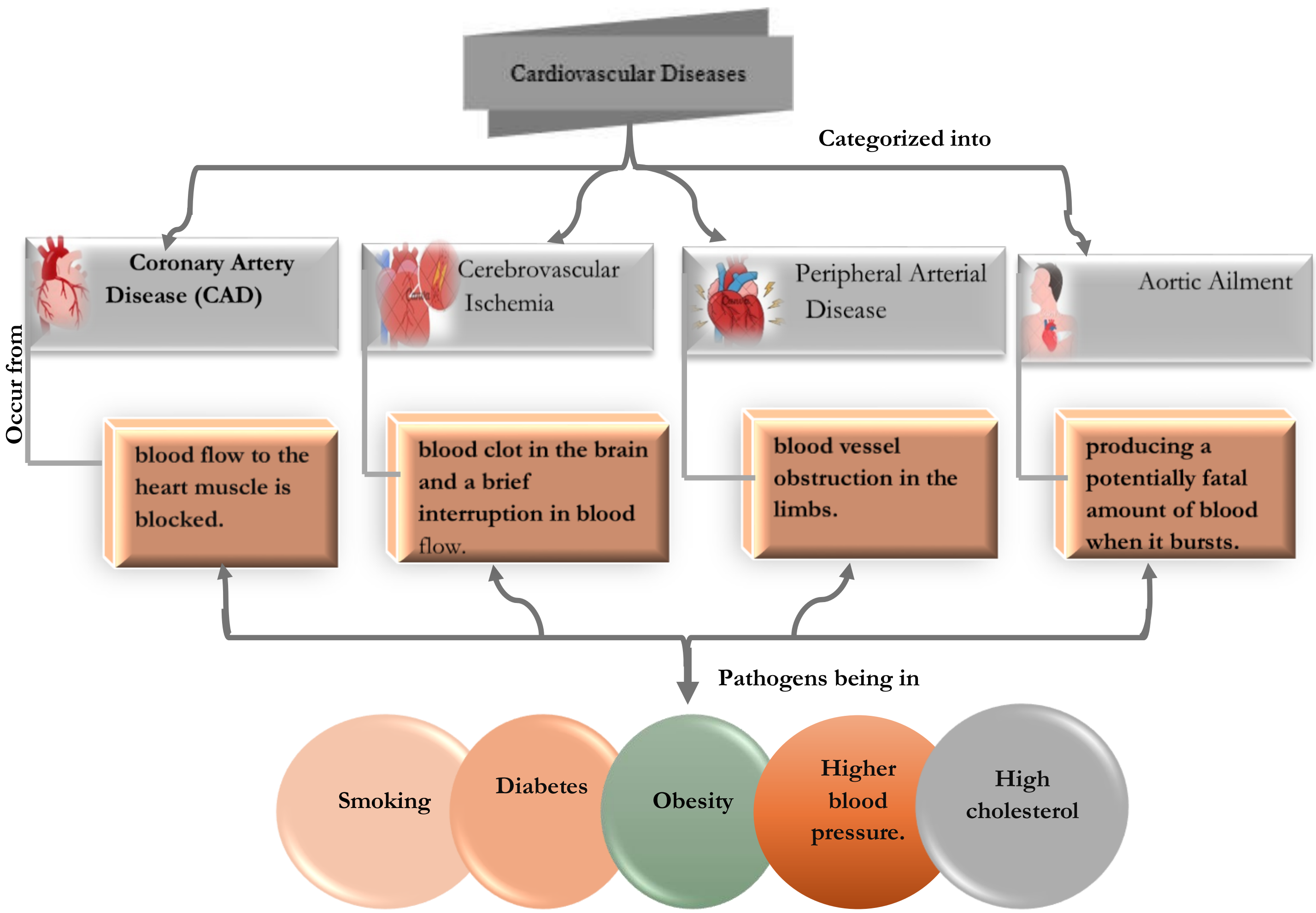


Fig 1. Cardiovascular Diseases Categories

Even though stressing that heart disease is among the most common diseases in humans, [5]also acknowledged that the diagnosis of heart disease can be complicated and delay the right diagnosis decision due to several factors, including symptoms of the disease and the relationship between the disease's pathological and functional manifestations and human organs other than the heart. In this context [6] indicated that improving clinical outcomes and preventing significant adverse cardiac events is possible with early detection of CVDs, which also enables the escalation of guideline-directed medical treatment. Arguably early identification of cardiovascular disorders is imperative. From the perspective of [7], forecast with precision is a challenge, particularly in developing and Asian nations where resources, technology, and peripheral devices are few. In the same context [8] , was revealed that the capacity for controlling CVD is currently inadequate. Individuals lack knowledge about CVDs and possible detrimental practices that contribute to the disease. Another challenge stated in [9] where a wealth of information on heart disease in the healthcare sector needs to be analyzed to aid with decision-making. The mentioned challenges are catalysts for the notion of diagnosing CVDs early to lessen the risk embraced by [10] through using appropriate and precise diagnostic techniques. Moreover, earlier studies tackle the difficulties associated with CVDs and diagnostics by developing a variety of diagnostic methods for the techniques-based prediction of heart disease [11]. For instance [12] where medical practitioners are integrating their experiences as physicians with artificial intelligence (AI) techniques to automate the diagnosing process. Recently, several machine learning (ML)techniques of AI have been constructed to improve the prediction of cardiovascular diseases. Utilizing Extreme Gradient Boosting (XGBoost) classifier [13] to predicate CVDs. On other hand [14] adapted uncertainty theory of Complex intuitionistic fuzzy set (CIFS) for choosing the best method for diagnosing cardiovascular diseases.

1.2 Novelty of Study: Rendered Contributions

Surveys for earlier research that were connected to our study as [15] indicated that doctors frequently make their judgments not on the knowledge-rich material concealed in the database, but rather on their experience and intuition. This results in unintentional prejudices, mistakes, and exorbitant medical expenses, and these issues have an impact on the standard of care given to patients.

The study discussed some of the challenges that retard the process of CVDs diagnosis. As well, this study attempts to avoid these handicaps by constructing an innovative soft diagnostic paradigm. The bedrock of this paradigm is harnessing the various soft computing techniques where neutrosophic theory plays a vital role as vagueness theory. This theory was proposed by Smarandache and can treat dynamic and uncertain environment[16]. As well Tree Soft technique is utilized for forming association between determined attributes which contribute to the diagnosis process. Table 1 shows the contribution of constructing an innovative soft diagnostic paradigm by covering the set of aspects.

Table1. Study Contributions

Aspects	Challenges	Study's Contribution
Theoretically	- From the perspective of [17] CVDs are difficult to diagnose as a Scarcity of diagnostic capabilities, as well as fewer physicians and other healthcare providers. This will affect CVDs patients' optimal prognosis.	- Soft diagnostic paradigm is constructed for diagnosing cases of CVDs to guarantee the accuracy of the diagnosis
Practically	- Due to ambiguity and uncertainty in many complicated health situations as well as limited information regarding the patient's medical status[18], the medical field—including experienced doctors—faces challenges in making accurate diagnoses.	-Set of techniques has been harnessed in this study for constructing soft diagnostic paradigm. -Each technique is responsible for a certain role. i. Tree Soft Technique (TrST) is utilized for forming a patient's attributes into nodes and sub-nodes of a tree. This technique can illustrate the associative rules between determined patients' attributes. ii. Soft computing technique of neutrosophic is utilized for analyzing attributes formed into TrST. This technique known as the uncertainty technique can treat mysterious circumstances.
Credibility	It takes time and experience to identify CVD early and treat patients more successfully, which endangers the patient's life[6].	- To guarantee the paradigm's validity, it is applied to real case studies to guarantee patient safety and life.

2. Methodology of Soft Diagnostic Paradigm

The objective of this section is exhibiting through cover the following points (P_n):

P1: The Preliminaries of techniques that contributed to constructing the proposed paradigm.

P2: The role of each technique is exhibited by presenting the proposed model in the form of steps in the proposed paradigm.

P3: The problem is formed into tree form by adopting TrST. The main attributes related to diagnosis for patients are represented into nodes and its sub-attributes are represented into sub-nodes.

P4: Employing soft computing (SC) technique is neutrosophic which is harnessed in TrST for treating uncertainty problems related to attributes for diagnosing the patients.
As a result, this section is divided into two sub-sections. Each one is responsible for a certain function for serving the previously listed points.

2.1 Preliminaries

Herein, the basic concepts and fundamentals of utilized techniques of the proposed paradigm are clarified.

2.1.1 Tree Soft Technique

The technique of TrST is suggested by Smarandache [19] who is the founder of uncertainty theory is neutrosophic. The objective of TrST is to illustrate the relationship between attributes and sub-attributes of patients and SVDs. Hence, the technique's basic aspects and relationship formed according to [20] as:

- Assum $\mathfrak{H}$ be a universe of discourse, and $\mathcal{H}$ a non-empty subset of $\mathfrak{H}$, with the powerset of $\mathcal{H}$ $P(\mathcal{H})$.
- Suppose ∂ be a set of attributes for main nodes as $\partial = \{ \partial_1, \partial_2, \ldots, \partial_n\}$ where $n \geq 1$ and considering attributes of ∂ resident at the first level.
- Accordingly, sub-attributes of the main attributes are located in the second level as sub-nodes ∂_1 symbolled as $\{ \partial_{1-1}, \partial_{1-2}, \ldots, \partial_{1-n}\}$ also, sub-attributes of ∂_2 expressed as $\{ \partial_{2-1}, \partial_{2-2}, \ldots, \partial_{2-n}\}$.
- Considering ∂ is root and located at level zero, sequentially nodes of level 1, level 2, up to level n are inherent of ∂. Moreover, Then Tree Soft is expressed as F: $P(\text{Tree}(\partial)) \rightarrow P(\mathcal{H})$.

2.1.2 The notion of Tree Soft Technique in Association Rule

various studies as [21] described association rule as a technique deployed for identifying intriguing and recurring patterns in transactional. Therefore, the objective of this technique [22] is discovering recurrent patterns, relationships, correlations, or chains of causality among sets of items in transactional and relational databases where called item sets[23].Suppose that number of transactions [24]expressed as $D = \{T_1, T_2, \ldots T_n\}$ which are including items are symbolled as $I = \{i_1, i_2, \ldots i_n\}$.Generally, association rule consists of two parts are antecedent leads to consequent of the rule. Also, there are important factors that should take not considerations:(i) support ($\wp$)is characterized as proportion of transactions $T_i \in D$ and $\wp \subseteq T_i$.(ii) Let $\wp \rightarrow \vartheta$ and confidence of ($\wp \rightarrow \vartheta$)refers to support $(\wp \cup \vartheta)/\text{support}\,(\wp)$.(iii)moreover, lift indicated to confidence$(\wp \rightarrow \vartheta)/\text{support}\,(\wp)$.

Herein, we will gain from TrST by deploying it in the association rule technique especially, multi-level as a type of association rule [25] to discover relationships between attributes and sub-attributes that are resident in nodes and sub-nodes in form of tree.

Example 1: suppose that the universal set is U consists of elements of houses as U={h_1, h_2,h_3,h_4,h_5,h_6} as in node 0 as in Fig 2.

- Nodes 1,2,4 are main attributes which are inherent of Node 0.
- Node 1 has sub nodes (sub- attributes) are N_{1-1},..N_{1-n}.Also, Node 2 has N_{2-1},..N_{2-n} . Similarly, Node 3 has N_{3-1},..N_{3-n} .
- Considering the mapping between attributes/nodes and powerset of U as: F(Ns) → P(U).
- Table 2 is representing the mapping based on Boolean-valued[26].

$$(F,N)=\begin{cases}\{N_{1-1},N_{2-2},N_{3-2}\}\rightarrow\{h_1,h_5\}\\\{N_{1-2},N_{2-3},N_{3-1}\}\rightarrow\{h_2,h_6\}\\\{N_{1-3},N_{2-1},N_{3-3}\}\rightarrow\{h_{3,}\ h_4\}\end{cases}$$

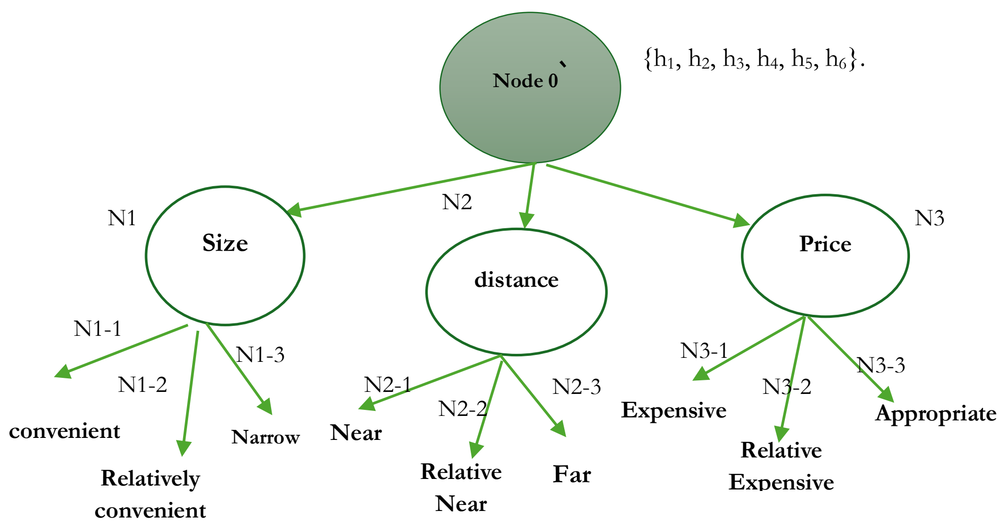


Fig 2. Mapping attributes and sub-attributes in Tree Soft Technique

Table 2. Mapping based on Boolean-valued.

U / Sub-Nodes	N_{1-1}	N_{1-2}	N_{1-3}	N_{2-1}	N_{2-2}	N_{2-3}	N_{3-1}	N_{3-2}	N_{3-3}
h_1	1	0	0	0	1	0	0	1	0
h_2	0	1	0	0	0	1	1	0	0
h_3	0	0	1	1	0	0	0	0	1
h_4	0	0	1	1	0	0	0	0	1

h5	1	0	0	0	1	0	0	1	0
h6	0	1	0	0	0	1	1	0	0

2.2 Methodology of Extracting Knowledge Toward CVDs Diagnosis

Herein, we are achieving the study's objective through constructing soft diagnostic paradigm. Thereby, this section is illustrating the needed steps which contributing to construct paradigm as following:

2.2.1Forming the most influencing heart diseases' attributes based on TrST

- The group of patients as $p = \{p_1, p_2, \ldots p_n\}$ are volunteered in this study to detect CVDs.
- The most influencing heart diseases' attributes are determined for group of patients through utilizing Cleveland heart disease dataset from the UCI repository.
- Forming the main attributes into initial nodes moreover, the sub-attributes are represented into sub-nodes.
- The group of decision makers (DMs) who are related to the medical field is formed to evaluate the patient's medical condition.

2.2.2Weighting TrST's attributes of heart diseases

DMs are evaluating patient's medical condition through analyzing attributes and sub-attributes encoded in TrST which related to heart diseases. The evaluation process is conducted through harnessing Single Value Neutrosophic Sets(SVNSs) as branch of uncertainty technique for evaluating patient's medical condition. SVNSs are leveraged as guide for DMs during evaluation process in hazy situations. The purpose of this phase is generating weights for encoded attributes in TrST. Hence, SVNSs based entropy are utilized for generating weights through following steps:

- Various Neutrosophic decision matrices are constructed based on evaluation of each DM.
- Eq.(1) utilized in constructed neutrosophic decision matrices for transforming these matrices into crisp matrices.

$$\boldsymbol{s}(\partial_{\mathrm{ij}}) = \frac{(2+g-q-\delta)}{3} \tag{1}$$

Where:

g, q, δ refers to truth, false, and indeterminacy respectively.

- Crisp matrices are aggregated into an aggregated decision matrix based on Eq.(2).

$$Q_{ij} = \frac{(\sum_{\mathrm{j}=1}^{\mathrm{N}} \partial_{\mathrm{ij}})}{T} \tag{2}$$

Where:

∂_{ij} refers to value of criterion in matrix, T refers to number of decision makers.

- Eq.(3) employed for normalizing the aggregated matrix

$$D_{ij}=\frac{Q_{ij}}{\sum_{j=1}^{n} Q_{ij}} \quad (3)$$

Where:

$\sum_{j=1}^{n} Q_{ij}$ represents sum of each criterion in aggregated matrix per column

- normalized matrix computes its entropy by Eq. (4).

$$e_{j}=-h\sum_{i=1}^{n} D_{ij} \ln D_{ij} \quad (4)$$

Where:

$$h=\frac{1}{\ln(Ps)} \quad (5)$$

Ps refers to number of alternatives of patients.

- weight vectors are generated through Eq.(6).

$$w_{j}=\frac{1-e_j}{\sum_{j=1}^{n}(1-e_j)} \quad (6)$$

2.2.3 Technique for order of preference by similarity to ideal solution (TOPSIS)

Through collaborating TrST with TOPSIS based on SVNSs, a set of sub-nodes or sub-attributes are inherent in 9 of the main attributes to diagnose the patients. Hence, sub-attributes $(N_{n\text{-}m})=\{N_{1\text{-}3},N_{2\text{-}3},N_{3\text{-}2},N_{4\text{-}2},N_{5\text{-}3},N_{6\text{-}1},N_{7\text{-}2},N_{8\text{-}3},N_{9\text{-}3}\}$ are employed in this step to detect CVDs for patients through implementing the following steps.

- The Neutrosophic decision matrices are constructed and deploying Eqs.(1),(2) to deneutrosphic matrices and aggregated it into an aggregated matrix
- The aggregated decision matrix is normalized according to based on the following Eq.(7).

$$\aleph_{ij}=\frac{x_{ij}}{\sqrt{\sum_{j=1}^{m}(x^2{}_{ij})}} \quad (7)$$

- the weighted decision matrix is generalized through Eq. (8).

$$wz_{ij}=weight_j * \aleph_{ij} \quad (8)$$

- positive ideal solution and negative ideal solution are computing based on Eqs (9,10) respectively.

$$ð^{*}=(wz_1{}^{*},wz_2{}^{*},\ldots.wz_n{}^{*}),wz_j{}^{*}=max_i\{wz_{ij}\} \quad (9)$$

$$\sigma^{-}=(wz_1{}^{-},wz_2{}^{-},\ldots.wz_n{}^{-}),wz_j{}^{-}=min_i\{wz_{ij}\} \quad (10)$$

Where:

$wz_1{}^* \dots wz_n{}^*, wz_1{}^- \dots wz_n{}^-$ are max and min values of weighted normalized criteria per column respectively.

- Eqs (11,12) deployed for computing the distance between the positive ideal solution and negative ideal solution to each patient.

$$d_i{}^* = \sum_{j=1}^{n} d(\ wz_{ij}, wz_j{}^*) \tag{11}$$

$$d_i{}^- = \sum_{j=1}^{n} d(\ wz_{ij}, wz_j{}^-) \tag{12}$$

- The diagnosing for patients is determining based on the values of $\mathrm{CC_i}$ in Eq.(13). After that Eq.(14) deployed to determine heart patients [27].

$$CC_i = \frac{d_i{}^-}{d_i{}^* + d_i{}^-} \tag{13}$$

$$\text{diagnosis} = \begin{cases} \mathrm{CVD} = 0, & \mathrm{CC_i} < 0.5 \\ \mathrm{CVD} = 1, & \mathrm{CC_i} \geq 0.5 \end{cases} \tag{14}$$

2.2.4 Collaboration of TrST -Association rules through applying various cases for extracting the relationships between sub-attributes which resulting in decision of diagnosis

3. Application of Paradigm

In this section, we implement the constructed soft diagnostic paradigm in case study.

3.1 Problem Description

According to Cleveland Heart Disease, which is sourced from the UCI repository, we volunteered attributes and its descriptions in our study. Fig 3 illustrates the utilized attributes encoded into TrST for diagnosis CVDs. The process of diagnosing is conducted by volunteering five patients and evaluating their medical conditions based on nine attributes and 27 sub-attributes in Fig 3.

3.2 SVNSs based Entropy

The attributes' weights are generated through utilizing entropy with the support of SVNSs as following:

- Three Neutrosophic matrices are constructed for each DM based on SVN scale listed in Table 3.
- The constructed matrices transformed to crisp matrices based on Eq.(1) and aggregated based on Eq.(2) into an aggregated decision matrix as in Table 4.
- Eq.(3) utilized in aggregated decision matrix to generate normalized matrix as in Table 5.

- After employing Eq.(4) to compute entropy, the attributes' (nodes') weights are obtained based on Eq.(6) and illustrated in Fig 4. Based on this Fig, attribute 9 is superior to other attributes In contrast to attribute 4.

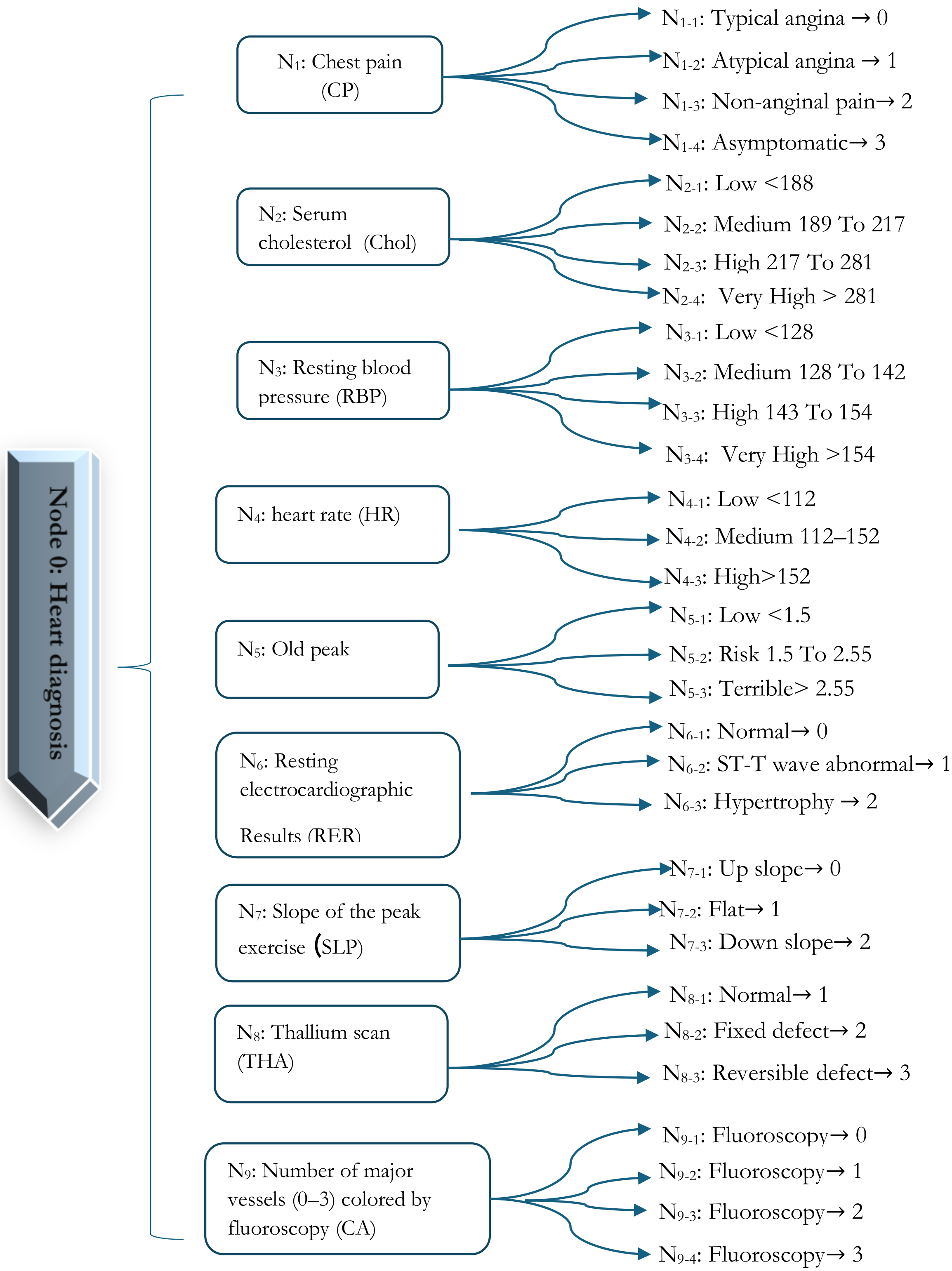


Fig 3. Attributes of Cleveland heart disease encoded into Tree Soft Tree

Table 3. SVN Scale

Synonmy	Acronym	Scale		
		T	I	F
Extremly Weak	EW	0.00	1.00	1.00
Absolutely Weak	AW	0.10	0.90	0.90
Very Weak	VW	0.20	0.85	0.80
Weak	W	0.30	0.75	0.70
Fairly Weak	FW	0.40	0.65	0.60
Fairly	F	0.50	0.50	0.50
Fairly Well	FW	0.60	0.35	0.40
Well	W	0.70	0.25	0.30
Very Well	VW	0.80	0.15	0.20
Absolutely Well	AW	0.90	0.10	0.10
Extremly Well	EW	1.00	0.00	0.00

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9
P_1	0.7467	0.78	0.7067	0.6133	0.7533	0.79	0.58	0.6133	0.46
P_2	0.6533	0.78	0.787	0.54	0.747	0.653	0.54	0.42	0.82
P_3	0.607	0.66	0.507	0.7	0.4267	0.5	0.82	0.607	0.46
P_4	0.38	0.3533	0.46	0.46	0.58	0.75	0.347	0.58	0.313
P_5	0.46	0.42	0.393	0.5067	0.3133	0.813	0.82	0.78	0.3533

Table 4. Aggregated decision Matrix

Table 5. Normalized decision Matrix

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9
P_1	0.262295082	0.2605791	0.24766355	0.21749409	0.2671395	0.224762	0.186695	0.204444	0.191136
P_2	0.229508197	0.2605791	0.27570093	0.191489362	0.2647754	0.186667	0.17382	0.14	0.34072
P_3	0.213114754	0.22049	0.17757009	0.24822695	0.1513002	0.142857	0.263948	0.202222	0.191136
P_4	0.133489461	0.1180401	0.16121495	0.163120567	0.2056738	0.213333	0.111588	0.193333	0.130194
P_5	0.161592506	0.1403118	0.13785047	0.179669031	0.1111111	0.232381	0.263948	0.26	0.146814

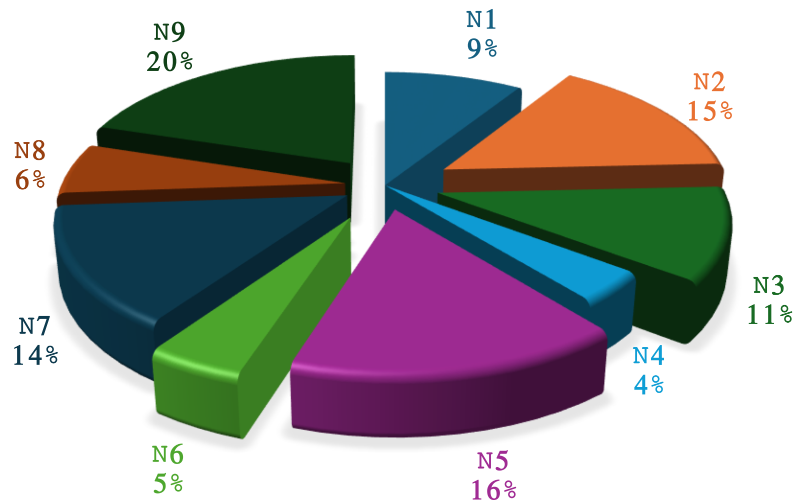


Fig 4 weighs of attributes/nodes encoded into Tree Soft Technique

3.3 Toward Diagnosis: SVNSs based TOPSIS

Herein, the heart patient is detecting based on implementing TOPSIS under authority of SVNSs as in the following steps.

- The aggregated matrix is obtained as in Table 6 after employing Eq.s (1,2) to deneutrosophic the matrices and aggregated it into single matrix.
- The aggregated matrix normalized through Eq.(7) as listed in Table 7.
- -the normalized matrix leveraged to construct weighted decision matrix in Table 8 based on Eq.(8).
- According to Eq.s(11,12) the distance between positive and negative ideal solution is calculated and consider important step for obtaining CC_i as obtained in Table 9.

- According to Values of CC_i in Table 9 which contributed to Eq.(14) for diagnosing the patient condition where P1,P4 classify absence of heart disease whereas P2,P3,P5 classify to presence of heart disease.

Table 6. Aggregated matrix

	N_{1-3}	N_{2-3}	N_{3-2}	N_{4-2}	N_{5-3}	N_{6-1}	N_{7-2}	N_{8-3}	N_{9-3}
P_1	0.845556	0.54	0.52	0.31333	0.53333	0.53333	0.39333	0.72	0.813333
P_2	0.5	0.74667	0.78	0.8333	0.9	0.62	0.667	0.94	0.553333
P_3	0.39333	0.78	0.87333	0.54	0.54	0.81333	0.70667	0.593333	0.54
P_4	0.72	0.5	0.5333	0.52	0.43333	0.526667	0.39333	0.5	0.653333
P_5	0.42667	0.6867	0.84	0.7533	0.43333	0.81333	0.71333	0.7466	0.346667

Table 7. Normalized matrix

	N_{1-3}	N_{2-3}	N_{3-2}	N_{4-2}	N_{5-3}	N_{6-1}	N_{7-2}	N_{8-3}	N_{9-3}
P_1	0.626746859	0.365816197	0.320551	0.22600333	0.401832516	0.354044692	0.296310367	0.449863344	0.605163
P_2	0.370612466	0.505819926	0.4808264	0.60107269	0.678092372	0.411576954	0.502220961	0.587321587	0.411709
P_3	0.291548473	0.528401173	0.5383612	0.38949511	0.406855423	0.539918155	0.532354218	0.370720718	0.401788
P_4	0.533681951	0.338718701	0.3287702	0.37506936	0.32648892	0.349619133	0.296310367	0.3124051	0.486114
P_5	0.316255971	0.465173682	0.5178131	0.54336972	0.32648892	0.539918155	0.537376428	0.466524949	0.257938

Table 8. Weighted decision Matrix

	N_{1-3}	N_{2-3}	N_{3-2}	N_{4-2}	N_{5-3}	N_{6-1}	N_{7-2}	N_{8-3}	N_{9-3}
P_1	0.117162978	0.009118738	0.05824655	0.038864036	0.014126615	0.034807326	0.043939085	0.027528672	0.055381574
P_2	0.069281656	0.012608625	0.087369824	0.103361798	0.023838664	0.040463516	0.074473025	0.035940211	0.037677628
P_3	0.054501569	0.01317151	0.097824333	0.066978445	0.014303198	0.053081171	0.078941406	0.022685665	0.036769734
P_4	0.099765585	0.008443276	0.059740051	0.064497762	0.011477875	0.034372234	0.043939085	0.019117133	0.044486838
P_5	0.059120346	0.011595432	0.09409058	0.093439066	0.011477875	0.053081171	0.079686136	0.028548253	0.023605261

Table 9. Diagnosing the patient based on **CCi**

	d*	**d-**	**Cci**	**Diagnosing**
P_1	0.086412	0.065964	0.432905	0
P_2	0.053804	0.081547	0.602488	1
P_3	0.076572	0.063048	0.52	1
P_4	0.07359	0.052609	0.416872	0
P_5	0.068568	0.078642	0.534218	1

4. Analysis and discussion

Herein, we discuss the proposed paradigm's findings. Thus, this section divides into two sub-sections. Generally speaking, in the constructed paradigm there are set of steps have been conducted for diagnosing the heart conditions for various patients. Hence, the paradigm's notion is based on soft set and uncertainty techniques; for bolstering the constructed paradigm in diagnosing vague patients' cases. Moreover, TrST and neutrosophic are collaborating with entropy and TOPSIS in diagnosing the cases.

4.1 Soft Diagnosis Paradigm's Findings

The constructed paradigm generated group of finding:

- Firstly, implementing SVNSs based entropy in attributes and sub-attributes encoded into Tree Soft which illustrated in Fig 3 to generate attributes' weights indicated that attribute 9 (CA) is superior to other attributes and attribute 4 (HR) is worst as mentioned in Fig 4.
- Secondly, SVNSs based TOPSIS are working in sub-attributes which encoded into Tree Soft to diagnosing five patients over the determined sub-attributes/sub-nodes.
- The sub-attributes are determining and nominating through TrST where $F: N_1 \times N_2 \times N_3 \times N_4 \times N_5 \times N_6 \times N_7 \times N_8 \times N_9 \to P(P)$.Hence, sub-attributes ($N_{n-m}$) ={$N_{1-3}$,$N_{2-3}$,$N_{3-2}$,$N_{4-2}$,$N_{5-3}$,$N_{6-1}$,$N_{7-2}$,$N_{8-3}$,$N_{9-3}$} are employed in SVNSs based TOPSIS doe detecting and diagnosing five patients . The findings indicated that two patients of P1, P4 classify absence of heart disease whereas P2, P3, P5 classify to presence of heart disease as listed in Table 9.

4.2 Association Rule and Soft Paradigm Collaboration Findings

The objective of association rules is to extract and discover the relation between sub-attributes/sub-nodes in Fig 3 as antecedent to diagnose the patient cases as consequent. Thereby, we implemented various cases which consider our problem as transactions for obtaining relation between sun-nodes/sub-attributes which consider items toward diagnosing patients. Thus, D={ T_1,T_2,T_{3-},T_4 } is transactions for items of sub-attributes I={ I_{1-n},I_{2-n},I_{3-n-},I_{4-n},I_{5-n},I_{6-n},I_{7-n-},I_{8-n}, I_{9-n} } .

4.2.1 Case 2 (T_2): Let sub-attributes (N_{n-m}) ={N_{1-2},N_{2-1},N_{3-1},N_{4-1},N_{5-2},N_{6-2},N_{7-3},N_{8-2},N_{9-1}} are employed in this step to detect CVDs.

- According to values of $\mathrm{CC_i}$ in Table 10 which contributed to Eq.(14) for diagnosing the patient condition where P4,P5 classify absence of heart disease whereas P2,P3,P1 classify to presence of heart disease.

Table 10. Diagnosing the patient based on **CCi** in case 2

	d*	**d-**	**Cci**	**Diagnosing**
P_1	0.051868007	0.064332043	0.55	1
P_2	0.053525933	0.059938779	0.53	1
P_3	0.049792455	0.060048848	0.54	1
P_4	0.076682931	0.029550656	0.3	0
P_5	0.059870647	0.05437008	0.4	0

4.2.2 Case 3 (T_3): Let sub-attributes (N_{n-m}) ={N_{1-3},N_{2-1},N_{3-3},N_{4-3},N_{5-1},N_{6-1},N_{7-3},N_{8-1},N_{9-1}} are employed in this step to detect CVDs.

- According to values of CC_i in Table 11 which contributed to Eq.(14) for diagnosing the patient condition where P4,P5 classify absence of heart disease whereas P2,P3,P1 classify to presence of heart disease.

Table 11. Diagnosing the patient based on **CCi** in case 3

	d*	**d-**	**Cci**	**Diagnosing**
P_1	0.127123265	0.111984744	0.5	1
P_2	0.052719105	0.187088011	0.78	1
P_3	0.013877717	0.237815505	0.95	1
P_4	0.237783628	0.008110786	0.033	0
P_5	0.21115882	0.028586828	0.12	0

4.2.3 Case 4 (T_4): Let sub-attributes (N_{n-m}) ={N_{1-1},N_{2-3},N_{3-1},N_{4-3},N_{5-1},N_{6-3},N_{7-1},N_{8-1},N_{9-2}} are employed in this step to detect CVDs.

- According to values of CC_i in Table 12 which contributed to Eq.(14) for diagnosing the patient condition where P4,P5, P1 classify absence of heart disease whereas P2,P3 classify to presence of heart disease

Table 12. Diagnosing the patient based on **CCi** in case 4

	d*	**d-**	**Cci**	**Diagnosing**
P_1	0.92568692	0.372601434	0.287	0
P_2	0.494731035	0.891295757	0.643	1
P_3	0.668434479	0.801959546	0.545	1
P_4	1.064228169	0.08716001	0.0758	0
P_5	0.955166414	0.3815659	0.2855	0

Finally, Table 13 aggregated the set of transactions for set of items for discover and diagnose the patients according to Boolean value[26].Based on transaction ID(TID)in Table 13, support and confidence for items are computing as listed in Table 14.

$$
(\mathrm{F,N}) = \begin{cases} \{\mathrm{N_{1-3}, N_{2-3}, N_{3-2}, N_{4-2}, N_{5-3}, N_{6-1}, N_{7-2}, N_{8-3}, N_{9-3}}\} \rightarrow \{(\mathrm{p_2, p_3, p_5}) = 1\} \\ \{\mathrm{N_{1-2}, N_{2-1}, N_{3-1}, N_{4-1}, N_{5-2}, N_{6-2}, N_{7-3}, N_{8-2}, N_{9-1}}\} \rightarrow \{(\mathrm{p_2, p_3, p_1}) = 1\} \\ \{\mathrm{N_{1-3}, N_{2-1}, N_{3-3}, N_{4-3}, N_{5-1}, N_{6-1}, N_{7-3}, N_{8-1}, N_{9-1}}\} \rightarrow \{(\mathrm{p_2, p_3, p_1}) = 1\} \\ \{\mathrm{N_{1-1}, N_{2-3}, N_{3-1}, N_{4-3}, N_{5-1}, N_{6-3}, N_{7-1}, N_{8-1}, N_{9-2}}\} \rightarrow \{(\mathrm{p_2, p_3}) = 1\} \end{cases}
$$

Table 13. Total Transactions (D) based on items (I_n)

TID	Items (sub-nodes)																									
	$\mathbf{N_{1-1}}$	$\mathbf{N_{1-2}}$	$\mathbf{N_{1-3}}$	$\mathbf{N_{2-1}}$	$\mathbf{N_{2-3}}$	$\mathbf{N_{3-1}}$	$\mathbf{N_{3-2}}$	$\mathbf{N_{3-3}}$	$\mathbf{N_{4-1}}$	$\mathbf{N_{4-2}}$	$\mathbf{N_{4-3}}$	$\mathbf{N_{5-1}}$	$\mathbf{N_{5-2}}$	$\mathbf{N_{5-3}}$	$\mathbf{N_{6-1}}$	$\mathbf{N_{6-2}}$	$\mathbf{N_{6-3}}$	$\mathbf{N_{7-1}}$	$\mathbf{N_{7-2}}$	$\mathbf{N_{7-3}}$	$\mathbf{N_{8-1}}$	$\mathbf{N_{8-2}}$	$\mathbf{N_{8-3}}$	$\mathbf{N_{9-1}}$	$\mathbf{N_{9-2}}$	$\mathbf{N_{9-3}}$
T_1	0	0	1	0	1	0	0	0	0	1	0	0	0	1	1	0	0	0	1	0	0	0	1	0	0	1
T_2	0	1	0	1	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	1	0	1	0	0
T_3	0	0	1	1	0	0	0	1	0	0	1	1	0	0	1	0	0	0	0	1	1	0	0	1	0	0
T_4	1	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	1	1	0	0	1	0	0	0	1	0

Table 14. Support and confidence of items

Items	**Support**	**Confidence**
N_{1-1}	1	¼=25%
N_{1-2}	1	¼=25%
N_{1-3}	2	2/4=50%
N_{2-1}	2	2/4=50%
N_{2-3}	2	2/4=50%
N_{3-1}	1	¼=25%
N_{3-2}	1	¼=25%
N_{3-3}	1	¼=25%
N_{4-1}	1	¼=25%
N_{4-2}	1	¼=25%
N_{4-3}	2	2/4=50%
N_{5-1}	2	2/4=50%
N_{5-2}	1	¼=25%
N_{5-3}	1	¼=25%
N_{6-1}	1	¼=25%
N_{6-2}	1	¼=25%
N_{6-3}	1	¼=25%
N_{7-1}	1	¼=25%
N_{7-2}	1	¼=25%
N_{7-3}	2	2/4=50%
N_{8-1}	2	2/4=50%
N_{8-2}	1	¼=25%
N_{8-3}	1	¼=25%
N_{9-1}	2	2/4=50%
N_{9-2}	1	¼=25%
N_{9-3}	1	¼=25%
$\mathbf{N_{1-3}, N_{2-3}, N_{3-2}, N_{4-2}, N_{5-3}, N_{6-1}, N_{7-2}, N_{8-3}, N_{9-3}}$	1	¼=25%
$\mathbf{N_{1-2}, N_{2-1}, N_{3-1}, N_{4-1}, N_{5-2}, N_{6-2}, N_{7-3}, N_{8-2}, N_{9-1}}$	1	¼=25%
$\mathbf{N_{1-3}, N_{2-1}, N_{3-3}, N_{4-3}, N_{5-1}, N_{6-1}, N_{7-3}, N_{8-1}, N_{9-1}}$	1	¼=25%
$\mathbf{N_{1-1}, N_{2-3}, N_{3-1}, N_{4-3}, N_{5-1}, N_{6-3}, N_{7-1}, N_{8-1}, N_{9-2}}$	1	¼=25%

5. Conclusion

Cardiovascular disease (CVD) is one of the leading causes of death worldwide. It affects not only the heart and blood vessels but also heart failure, blood vessel disorders, stroke, arrhythmia, and myocardial infarction. To intervene with the patient promptly, it is essential to identify the critical risk factors.

Hence, the process of early detecting and diagnosing this disease is important. Scientifically, there are many studies that embraced the process of diagnosing CVDs. Moreover, various techniques and models are suggested for serving this process. Therefore, herein we attempted to treat such problem through constructing soft diagnosis paradigm. For constructing this paradigm, we are harassed various techniques each one responsible for vital role. Firstly,

TrST is employed in this problem for the first time where this technique is based on the notion of soft sets. We leveraged this technique for representing attributes and sub-attributes of patients into Tree soft form. Secondly, SVNSs based entropy are implementing in the formed Tree for analyzing the main nodes of attributes and generating weights for it to showcase the most influenced attribute and the least influenced attribute.

Thirdly, SVNSs based TOPSIS is responsible for detecting the diagnosis for five patients based on the value of CC_i and according to Eq.(14).The findings of soft diagnosis paradigm indicated that P2,P3,P5 are belongs to presence of heart disease otherwise, P1,P4 belongs to absence of heart disease. Fourthly, we exploited the notion of soft set in TrST through collaborating TrST with association rule the relationship between sub-attributes through applying set of transactions (cases) for exhibiting relationships of sub-attributes(items) which considering antecedent that lead to consequent of detecting diagnosis of medical conditions of patients as mentioned in Table 13 and Table 14.

Reference

[1] Stephen, M., & Felix, A. (2023). Fuzzy AHP point factored inference system for detection of cardiovascular disease. *Journal of intelligent & fuzzy systems*, *44*(4), 6655–6684.

[2] Stajić, D., & Đonović, N. (2016). Cardiovascular diseases – risk factors. *Medicinski casopis*, *50*(2), 43–48. DOI:10.5937/mckg50-11761

[3] Gaidai, O., Cao, Y., & Loginov, S. (2023). Global Cardiovascular Diseases Death Rate Prediction. *Current problems in cardiology*, *48*(5), 101622. DOI:10.1016/j.cpcardiol.2023.101622

[4] Swathy, M., & Saruladha, K. (2022). A comparative study of classification and prediction of Cardio-Vascular Diseases (CVD) using Machine Learning and Deep Learning techniques. *ICT express*, *8*(1), 109–116.

[5] Hassan, D., Hussein, H. I., & Hassan, M. M. (2023). Heart disease prediction based on pre-trained deep neural networks combined with principal component analysis. *Biomedical signal processing and control*, *79*, 104019.

[6] Khozeimeh, F., Sharifrazi, D., Izadi, N. H., Joloudari, J. H., Shoeibi, A., Alizadehsani, R., … Islam, S. M. S. (2022). RF-CNN-F: random forest with convolutional neural network features for coronary artery disease diagnosis based on cardiac magnetic resonance. *Scientific reports*, *12*(1), 1–12. DOI:10.1038/s41598-022-15374-5

[7] Rustam, F., Ishaq, A., Munir, K., Almutairi, M., Aslam, N., & Ashraf, I. (2022). Incorporating cnn features for optimizing performance of ensemble classifier for cardiovascular disease prediction. *Diagnostics*, *12*(6), 1474.

[8] Wei, X., Rao, C., Xiao, X., Chen, L., & Goh, M. (2023). Risk assessment of cardiovascular disease based on SOLSSA-CatBoost model. *Expert systems with applications*, *219*(February), 119648. DOI:10.1016/j.eswa.2023.119648

[9] Azmi, J., Arif, M., Nafis, M. T., Alam, M. A., Tanweer, S., & Wang, G. (2022). A systematic review on machine learning approaches for cardiovascular disease prediction using medical big data. *Medical engineering & physics*, *105*, 103825.

[10] Ashraf, M., Salal, Y. K., Abdullaev, S. M., Zaman, M., & Bhut, M. A. (2022). Introduction of feature selection and leading-edge technologies viz. tensorflow, pytorch, and keras: an empirical study to improve prediction accuracy of cardiovascular disease. *International conference on innovative computing and communications: proceedings of icicc 2021, volume 3* (pp. 19–31). Springer.

[11] Ali, L., Niamat, A., Khan, J. A., Golilarz, N. A., Xingzhong, X., Noor, A., … Bukhari, S. A. C. (2019). An optimized stacked support vector machines based expert system for the effective prediction of heart failure. *IEEE access*, *7*, 54007–54014.

[12] Arumugam, K., Naved, M., Shinde, P. P., Leiva-Chauca, O., Huaman-Osorio, A., & Gonzales-Yanac, T. (2023). Multiple disease prediction using Machine learning algorithms. *Materials today: proceedings*, *80*, 3682–3685.

[13] Budholiya, K., Shrivastava, S. K., & Sharma, V. (2022). An optimized XGBoost based diagnostic system for effective prediction of heart disease. *Journal of king saud university-computer and information sciences*, *34*(7), 4514–4523.

[14] Alghazzawi, D., Liaqat, M., Razaq, A., Alolaiyan, H., Shuaib, U., & Liu, J. B. (2023). Selection of Optimal Approach for Cardiovascular Disease Diagnosis under Complex Intuitionistic Fuzzy Dynamic Environment. *Mathematics*, *11*(22). DOI:10.3390/math11224616

[15] Hossain, M. E., Khan, A., Moni, M. A., & Uddin, S. (2019). Use of electronic health data for disease prediction: A comprehensive literature review. *IEEE/acm transactions on computational biology and bioinformatics*, *18*(2), 745–758.

[16] Hussein, G. S., Zaied, A. N. H., & Mohamed, M. (2023). ADM: Appraiser Decision Model for Empowering Industry 5.0-Driven Manufacturers toward Sustainability and Optimization: A Case Study. *Neutrosophic systems with applications*, *11*, 22–30.

[17] Balamurugan, R., Ratheesh, S., & Venila, Y. M. (2022). Classification of heart disease using adaptive Harris hawk optimization-based clustering algorithm and enhanced deep genetic algorithm. *Soft computing*, *26*(5), 2357–2373. DOI:10.1007/s00500-021-06536-0

[18] Mariadoss, S., & Augustin, F. (2023). Enhanced sugeno fuzzy inference system with fuzzy

AHP and coefficient of variation to diagnose cardiovascular disease during pregnancy. *Journal of king saud university-computer and information sciences*, *35*(8), 101659.

[19] Smarandache, F. (2023). New Types of Soft Sets" HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set": An Improved Version. Neutrosophic Systems With Applications, 8, 35-41. https://doi.org/10.61356/j.nswa.2023.41.

[20] AL-baker, S. F., El-henawy, I., & Mohamed, M. (2024). Pairing New Approach of Tree Soft with MCDM Techniques: Toward Advisory an Outstanding Web Service Provider Based on QoS Levels. *Neutrosophic systems with applications*, *14*, 17–29.

[21] Amoozad Mahdiraji, H., Tavana, M., Mahdiani, P., & Abbasi Kamardi, A. A. (2022). A multi-attribute data mining model for rule extraction and service operations benchmarking. *Benchmarking: an international journal*, *29*(2), 456–495.

[22] Buxton, E. K., Vohra, S., Guo, Y., Fogleman, A., & Patel, R. (2019). Pediatric population health analysis of southern and central Illinois region: A cross sectional retrospective study using association rule mining and multiple logistic regression. *Computer methods and programs in biomedicine*, *178*, 145–153.

[23] Shah, J. J., & Gagnani, L. P. (2015). Introduction to Multilevel Association Rule and Its Methods, *3*(10), 531–534.

[24] Sarıyer, G., & Öcal Taşar, C. (2020). Highlighting the rules between diagnosis types and laboratory diagnostic tests for patients of an emergency department: Use of association rule mining. *Health informatics journal*, *26*(2), 1177–1193. DOI:10.1177/1460458219871135

[25] Hameed, N., Shabut, A. M., Ghosh, M. K., & Hossain, M. A. (2020). Multi-class multi-level classification algorithm for skin lesions classification using machine learning techniques. *Expert systems with applications*, *141*, 112961.

[26] Herawan, T., & Deris, M. M. (2011). A soft set approach for association rules mining. *Knowledge-based systems*, *24*(1), 186–195. DOI:10.1016/j.knosys.2010.08.005

[27] Kishore, A. H. N., & Jayanthi, V. E. (2018). Multi Criteria Decision Making Methods to Predict the Prevalence of Coronary Artery Disease. *Journal of medical imaging and health informatics*, *8*(4), 719–726. DOI:10.1166/jmihi.2018.2357

Received: Feb 5, 2024. Accepted: April 28, 2024

An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers

Abdallah Shihadeh[1], Khaled Ahmad Mohammad Matarneh[2], Raed Hatamleh[3], Randa Bashir Yousef Hijazeen[4], Mowafaq Omar Al-Qadri[5], Abdallah Al-Husban[6]

[1]Department of Mathematics, Faculty of Science, The Hashemite University, Zarqa 13133, PO box 330127, Jordan, abdallaha_ka@hu.edu.jo

[2]Faculty of Computer Studies, Arab Open University (AOU), Riyadh, Kingdom of Saudi Arabia, K.matarneh@arabou.edu.sa

[3]Department of Mathematics, Faculty of Science and Information Technology, Jadara University, P.O. Box 733, Irbid 21110, Jordan, raed@jadara.edu.jo

[4]Department of Basic Sciences and informatics, Al-Balqa applied university, Al Karak, Jordan, randa.hijazeen@bau.edu.jo

[5]Department of Mathematics, Jerash University, Jerash, Jordan, m.alqadri@jpu.edu.jo

[6]Department of Mathematics, Faculty of Science and Technology, Irbid National University, P.O. Box: 2600 Irbid, Jordan, dralhosban@inu.edu.jo

Abstract: This paper is dedicated to defining and studying for the first time a two-fold algebra over the neutrosophic real number ring by merging the fuzzy set mapping with the algebraic operations of the neutrosophic real number ring.

Also, we study the elementary algebraic properties of the defined two-fold algebra through its algebraic operations and substructures such as homomorphisms and ideals.

Keywords: two-fold algebraic structure, neutrosophic ring, fuzzy set, two-fold fuzzy neutrosophic real number

Introduction

The concept of fuzzy algebraic structure is considered as a direct application of fuzzy sets and fuzzy mappings [1-2, 4, 6-8], where a fuzzy mapping with truth and falsity value is used to build many algebraic structures.

Also, the concept of neutrosophic set was used by many different authors to generalize classical algebraic structures by using logical conditions instead of algebraic elements [13], where we can see neutrosophic rings, neutrosophic matrices, and neutrosophic mappings [5, 9-12].

Recently, Smarandache in [14] has defined two-fold neutrosophic algebras as novel algebraic structures, and this new concept has been used in [16] to define two-fold fuzzy algebra by combining the standard fuzzy number theoretical system defined in [15], with the concept of two-fold algebraic structure, and many interesting theorems and examples were illustrated about this topic.

On the other hand, Hatip et.al [17], have combined real vector spaces, complex vector spaces, and algebraic modules with a fuzzy well-defined mapping to define and study two-fold fuzzy vector spaces and two-fold fuzzy modules, where they studied many elementary properties of these new structures.

This has prompted us to define and study for the first time a two-fold algebra over the neutrosophic real number ring by merging the fuzzy set mapping with the algebraic operations of the neutrosophic real number ring.

For more details about two-fold structures, and their properties, see [14, 16-17].

Main discussion

Definition:

Let $f:\mathbb{R}\to[0,1]$ with: $\begin{cases} f(0)=0 \\ f(1)=1 \end{cases}$, then f is called a fuzzy mapping.

We use this definition of fuzzy mappings, that is because the property $\begin{matrix} f(0)=0 \\ f(1)=1 \end{matrix}$ is very useful in algebraic structures and operations.

Example:

To understand the concept of fuzzy mapping, we will illustrate two different fuzzy mappings defined on the real field R.

Define: $f,g,h:\mathbb{R}\to[0,1]$ such that:

$$f(x)=\begin{cases} x^2 \ \ if -1<x\le 1 \\ \frac{1}{|x|} \ \ if \ \ x>1 \ \ or \ \ x<-1, \\ 0.4 \ \ if \ \ x=-1 \end{cases} \quad g(x)=\begin{cases} |x^5| \ \ if -1<x\le 1 \\ \frac{1}{|x^5|} \ \ if \ \ x>1 \ \ or \ \ x<-1 \\ 0.6 \ \ if \ \ x=-1 \end{cases}$$

We can see that f and g lie in the closed interval [0,1], with $f(0)=g(0)=0, f(1)=g(1)=1$.

Definition:

Let $\mathbb{R}(I)=\{x+yI \ \ ; x,y\in\mathbb{R} \ \ , I^2=I\}$ be the ring of neutrosophic real numbers , and $f:\mathbb{R}\to[0,1]$ with: $\begin{cases} f(0)=0 \\ f(1)=1 \end{cases}$ be a fuzzy mapping

We define $f_I:\mathbb{R}(I)\to[0,1] : f_I(x+yI)=\max(f(x),f(y))$

And $\mathbb{R}_{f_I}(I)=\{(x+yI)_{f_I(x+yI)} \ \ ; x,y\in\mathbb{R}\}$ is called the two- fold neutrosophic real numbers fuzzy algebra (NRNFA).

Definition:

The algebraic operations on $\mathbb{R}_{f_I}(I)$ are defined as:

*: $\mathbb{R}_{f_I}(I)\times\mathbb{R}_{f_I}(I)\to$ such that: $\mathbb{R}_{f_I}(I)$

○: $\mathbb{R}_{f_I}(I)\times\mathbb{R}_{f_I}(I)\to$ $\mathbb{R}_{f_I}(I)$

$(x+yI)_{f_I(x+yI)} * (z+tI)_{f_I(z+tI)} = [(x+z)+(y+t)I]_{f_I[(x+z)+(y+t)I]}$

$(x+yI)_{f_I(x+yI)} \circ (z+tI)_{f_I(z+tI)} = [\ x\cdot z+I(xt+yz+yt)]_{f_I[\ x\cdot z+I(xt+yz+yt)]}$

Theorem 1:

1] *,○ are commutative operations.

2] *,○ are associative operations.

3] * has $o_\circ$ as an identity, ○ has 1_1 as an identity.

4] ○ is distributive with respect to *.

5] Every $(x+yI)_{f_I(x+yI)}$ has an inverse with respect to (*).

6] $(x+yI)_{f_I(x+yI)}$ has an inverse with respect to (○) if and only if :$x\neq 0$. $x+y\neq 0$

Definition:

Let $P = A_0 + A_1 I$ be an ideal of $\mathbb{R}(\mathrm{I})$, then we define the corresponding two-fold ideal as follows:

$$P_F = \{(x + yI)_{f_I(x+yI)} \in \mathbb{R}_{f_I}(I) \; ; \; x + yI \in P\}$$

Definition:

Let $P = A_0 + A_1 I$ be an AH-ideal of $\mathbb{R}(\mathrm{I})$, then we define the corresponding two-fold AH-ideal as follows:

$$P_F = \{(x + yI)_{f_I(x+yI)} \in \mathbb{R}_{f_I}(I) \; ; \; x \in A_0 \; . y \in A_1\}$$

Theorem 2:

Let $P_F = (A_0 + A_1 I)_F$ be a two-fold ideals of $\mathbb{R}_{f_I}(I)$, then:

$$\begin{cases} B_{f_I(B)} * \; C_{f_I(C)} \in P_F \\ r_{f_I(r)} \circ B_{f_I(B)} \in P_F \end{cases} \quad ; where \;\; B.C \in P_F \;\; .r \in \mathbb{R}(\mathrm{I})$$

Definition:

Let $\varphi: \mathbb{R}(\mathrm{I}) \to \mathbb{R}(\mathrm{I})$ be a ring homomorphism, we define:

$$\varphi_I(B_{f_I(B)}) = (\varphi(B))_{f_I(\varphi(B))} \quad ; \; \varphi_I : \; \mathbb{R}_{f_I}(I) \to \mathbb{R}_{f_I}(I)$$

The mapping φ_I is called two-fold homomorphism.

If φ is an isomorphism, then φ_I is called two-fold isomorphism.

Theorem 3:

Let φ_I be two-fold homomorphism, then:

1] $\varphi_I(o_\circ) = o_\circ \;\; . \;\; \varphi_I(1_1) = 1_1$

2] $\varphi_I(B_{f_I(B)} * \; C_{f_I(C)}) = \varphi_I(B) * \; \varphi_I(C)$

3] $\varphi_I(B \circ \mathrm{C}) = \varphi_I(B) \circ \; \varphi_I(C)$

4] $\varphi_I(-B) = - \; \varphi_I(B)$

5] $\varphi_I\left(\frac{1}{B}\right) = \frac{1}{\varphi_I(B)} \quad ; \quad B \;\; is \;\; invertible.$

6] $k_{er}(\; \varphi_I)$ is an ideal of $\mathbb{R}_{f_I}(I)$.

7] $I_m(\; \varphi_I)$ is a subring of $\mathbb{R}_{f_I}(I)$.

8] If P_F is an ideal of $\mathbb{R}(I)$, then $\varphi_I(P_F)$ is an ideal.

9] If P_F is an AH-ideal of $\mathbb{R}(I)$, then $\varphi_I(P_F)$ is an AH-ideal.

Definition:

Let φ_I: $\mathbb{R}_{f_I}(I) \to \mathbb{R}_{f_I}(I)$. Ψ_I: $\mathbb{R}_{f_I}(I) \to \mathbb{R}_{f_I}(I)$, we define:

$\varphi_I \times \Psi_I$: $\mathbb{R}_{f_I}(I) \to \mathbb{R}_{f_I}(I)$ such that: $\varphi_I \times \Psi_I\big(B_{f_I(B)}\big) = \varphi_I(\Psi_I\big(B_{f_I(B)}\big))$

Theorem 4:

Let $\varphi_I . \Psi_I$: $\mathbb{R}_{f_I}(I) \to \mathbb{R}_{f_I}(I)$ be two-fold homomorphisms, then:

1] $\varphi_I \times \Psi_I$ is two-fold homomorphism.

2] if $\varphi_I . \Psi_I$ are two isomorphisms, then $\varphi_I \times \Psi_I$ is an isomorphism.

Definition:

Let P_F be an ideal of $\mathbb{R}_{f_I}(I)$, then:

$\mathbb{R}_{f_I}(I)/P_F = \{B_{f_I(B)} \circ P_F \quad ; B \in \mathbb{R}(I)\}$ is called the factor of P_F.

Theorem 5:

Let P_F be an ideal of $\mathbb{R}_{f_I}(I)$, then:

$(\mathbb{R}_I(I)/P_F\ ,\ *',\ \circ')$ is a ring, with:

$*': \mathbb{R}_I(I)/P_F \times \mathbb{R}_I(I)/P_F \to \mathbb{R}_I(I)/P_F$

$\circ': \mathbb{R}_I(I)/P_F \times \mathbb{R}_I(I)/P_F \to \mathbb{R}_I(I)/P_F$

$$\begin{cases} \big(B_{f_I(B)} \circ P_F\big) *' \big(C_{f_I(C)} \circ P_F\big) = (B * C) \circ P_F \\ \big(B_{f_I(B)} \circ P_F\big) \circ' \big(C_{f_I(C)} \circ P_F\big) = (B \circ C) \circ P_F \end{cases}$$

Theorem 6:

Let $\mathbb{R}_I(I)/P_F$ be the two- fold factor ring of P_F, then:

If S/P_F is an ideal of $\mathbb{R}_I(I)/P_F$, then S is an ideal of $\mathbb{R}_I(I)$ and contains (P).

Theorem 7:

Let φ_I be two-fold homomorphism, then:

$\mathbb{R}_{f_I}(I)/k_{er}(\ \varphi_I) \cong I_m(\ \varphi_I)$

Proof of theorem 1:

Let $B = b_0 + b_1 I\ . C = c_0 + c_1 I\ . D = d_0 + d_1 I \ \in \mathbb{R}(I)$, then:

1] $B_{f_I(B)} * C_{f_I(C)} = (B+C)_{f_I(B+c)} = (C+B)_{f_I(C+B)} = C_{f_I(C)} * \ B_{f_I(B)}$

$B_{f_I(B)} \circ C_{f_I(C)} = (B \cdot C)_{f_I(Bc)} = (CB)_{f_I(CB)} = C_{f_I(C)} \circ \ B_{f_I(B)}$

2] $B_{f_I(B)} * (C_{f_I(C)} * D_{f_I(D)}) = B_{f_I(B)} * (C+D)_{f_I(C+D)} = (B+C+D)_{f_I(B+C+D)} =$

$(B+C)_{f_I(B+C)} * D_{f_I(D)} = (B_{f_I(B)} * C_{f_I(C)}) * D_{f_I(D)}$

$B_{f_I(B)} \circ (C_{f_I(C)} \circ D_{f_I(D)}) = B_{f_I(B)} \circ (CD)_{f_I(CD)} = (BCD)_{f_I(BCD)} = (BC)_{f_I(BC)} * D_{f_I(D)}$

$= (B_{f_I(B)} \circ C_{f_I(C)}) \circ D_{f_I(D)}$

3] $B_{f_I(B)} * o_\circ = (B + \circ)_{f_I(B+\circ)} = B_{f_I(B)}$

$B_{f_I(B)} \circ 1_1 = (B \cdot 1)_{f_I(B \cdot 1)} = B_{f_I(B)}$

4] $B_{f_I(B)} \circ (C_{f_I(C)} * D_{f_I(D)}) = B_{f_I(B)} \circ (C+D)_{f_I(C+D)} = (BC+BD)_{f_I(BC+BD)} =$

$(BC)_{f_I(BC)} * (BD)_{f_I(BD)} = (B_{f_I(B)} \circ C_{f_I(C)}) * (B_{f_I(B)} \circ D_{f_I(D)})$

5] The inverse of $(x+yI)_{f_I(x+yI)}$ for (*) is: $(-x-yI)_{f_I(-x-yI)}$

6] The inverse of $(x+yI)_{f_I(x+yI)}$ for (○) is: $\left(\frac{1}{x} + I(\frac{1}{x+y} - \frac{1}{x})\right)_{f_I\left(\frac{1}{x}+I(\frac{1}{x+y}-\frac{1}{x})\right)}$

Proof of theorem (2):

Let $.C \quad \in P \quad .r \in \mathbb{R}(I)$, then:

$$\begin{cases} B + C \in P \\ r \cdot B \in P \end{cases}$$

So that: $\begin{cases} (B+C)_{f_I(B+C)} = B_{f_I(B)} * C_{f_I(C)} \in P_F \\ r_{f_I(r)} \circ B_{f_I(B)} = (r \cdot B)_{f_I(r \cdot B)} \quad \in P_F \end{cases}$

Proof of theorem (3):

1] $\begin{cases} \varphi_I(o_\circ) = (\varphi(o))_{f_I(\varphi(o))} = o_\circ \\ \varphi_I(1_1) = (\varphi(1))_{f_I(\varphi(1))} = 1_1 \end{cases}$

2] $\varphi_I(B_{f_I(B)} * C_{f_I(C)}) = \ \varphi_I(B+C)_{f_I(B+c)} = (\varphi(B) + \varphi(C))_{f_I(\varphi(B)+\varphi(C))} =$

$\varphi(B)_{f_I(\varphi(B))} * \varphi(C)_{f_I(\varphi(C))} = \ \varphi_I(\ B_{f_I(B)}) * \ \varphi_I(\ C_{f_I(C)})$

3] $\varphi_I(B_{f_I(B)} \circ C_{f_I(C)}) = \ \varphi_I(BC)_{f_I(Bc)} = (\varphi(B)\varphi(C))_{f_I(\varphi(B)\varphi(C))} = \varphi(B)_{f_I(\varphi(B))} \circ$

$\varphi(C)_{f_I(\varphi(C))} = \varphi \ (\ B_{f_I(B)}) \circ \varphi(\ C_{f_I(C)})$

4] $\varphi_I((-B)_{f_I(-B)}) = \ (-\varphi(B))_{f_I(-\varphi(B))} = \ -\varphi(B)_{f_I(\varphi(B))} = -\varphi_I(B_{f_I(B)})$

5] It can be proved by the same.

6] $k_{er}(\ \varphi_I) = (k_{er}\ \ \varphi)_{f_I} = \ \{B_{f_I(B)}\ \ ;\ \ B \in k_{er}\ \ \varphi\}$, which is an ideal of $\mathbb{R}_{f_I}(I)$, that is because $(k_{er}\ \ \varphi)$ is an ideal of $\mathbb{R}(I)$.

7] $I_m(\ \varphi_I) = (I_m\ \ \varphi)_{f_I} = \ \{B_{f_I(B)}:\ B \in I_m\ \ \varphi\}$, which is a subring of $\mathbb{R}_{f_I}(I)$, that is because $(I_m\ \ \varphi)$ is a subring of $\mathbb{R}(I)$.

8] $\varphi_I(P_F) = (\varphi(P))_{f_I} = \ \left\{(\varphi(B))_{f_I(\varphi(B))}:\ B \in P\right\}$, and it is an ideal of $\mathbb{R}_{f_I}(I)$ because $\varphi(P)$ is an ideal.

9] It can be proved by the same.

Proof of theorem (4):

1] $\varphi_I \times \Psi_I\left(B_{f_I(B)}\right) = \varphi_I(\Psi(B))_{f_I(\Psi(B))} = (\varphi\Psi(\mathrm{B}))_{f_I(\varphi\Psi(\mathrm{B}))}$

Thus $\varphi_I \times \Psi_I\left(B_{f_I(B)} * C_{f_I(C)}\right) = (\varphi\Psi(B + C))_{f_I(\varphi\Psi(B+C))} = (\varphi\Psi(\mathrm{B}) + \varphi\Psi(\mathrm{C}))_{f_I(\varphi\Psi(\mathrm{B})+\varphi\Psi(\mathrm{C}))} = \ [\varphi_I * \Psi_I(B)] \circ [\varphi_I * \Psi_I(C)]$.

2] It holds directly from the definition.

Proof of theorem (5):

$$\left(B_{f_I(B)} \circ P_F\right) *' \left(C_{f_I(C)} \circ P_F\right) = (B * C) \circ P_F = (C * B) \circ P_F$$

$$= \left(C_{f_I(C)} \circ P_F\right) *' \left(B_{f_I(B)} \circ P_F\right)$$

$$\left(B_{f_I(B)} \circ P_F\right) \circ' \left(C_{f_I(C)} \circ P_F\right) = (B \circ C) \circ P_F = (C \circ \mathrm{B}) \circ P_F$$

$$= \left(C_{f_I(C)} \circ P_F\right) \circ' \left(B_{f_I(B)} \circ P_F\right)$$

$$\left(B_{f_I(B)} \circ P_F\right) *' \left[\left(C_{f_I(C)} \circ P_F\right) *' \left(D_{f_I(D)} \circ P_F\right)\right] = (B * C * D)_{f_I(B*C*D)} \circ P_F$$

$$= \left[\left(B_{f_I(B)} \circ P_F\right) *' \left(C_{f_I(C)} \circ P_F\right)\right] *' \left(D_{f_I(D)} \circ P_F\right)$$

$$\left(B_{f_I(B)} \circ P_F\right) \circ' \left[\left(C_{f_I(C)} \circ P_F\right) \circ' \left(D_{f_I(D)} \circ P_F\right)\right] = (B \circ C \circ D)_{f_I(BCD)} \circ P_F$$

$$= \left[\left(B_{f_I(B)} \circ P_F\right) \circ' \left(C_{f_I(C)} \circ P_F\right)\right] \circ' \left(D_{f_I(D)} \circ P_F\right)$$

$$\left(B_{f_I(B)} \circ P_F\right) *' \left(-B_{f_I(B)} \circ P_F\right) = o_\circ \circ P_F.$$

$\left(B_{f_I(B)} \circ P_F\right) \circ' \left[\left(C_{f_I(C)} \circ P_F\right) *' \left(D_{f_I(D)} \circ P_F\right)\right] = \left(B_{f_I(B)} \circ P_F\right) \circ' \left[(C * D)_{f_I(CD)} \circ P_F\right] =$

$\left[\left(B_{f_I(B)} \circ P_F\right) \circ' \left(C_{f_I(C)} \circ P_F\right)\right] *' \left[\left(B_{f_I(B)} \circ P_F\right) \circ' \left(D_{f_I(D)} \circ P_F\right)\right].$

$\left(B_{f_I(B)} \circ P_F\right) *' (o_\circ \circ P_F) = B_{f_I(B)} \circ P_F,$

$\left(B_{f_I(B)} \circ P_F\right) \circ' (1_1 \circ P_F) = B_{f_I(B)} \circ P_F.$

Thus $(\mathbb{R}_I(I)/P_F .*'.\circ')$ is a commutative ring with unity.

Proof of theorem (6):

Assume that S/P_F is an ideal of $\mathbb{R}_I(I)/P_F$, then:

$S \subseteq \mathbb{R}(I)$ is an ideal, with $P \subseteq S$.

Proof of theorem (7):

Since $\mathbb{R}(I)/k_{er}\ (\varphi) \cong I_m\ (\varphi)$, we can write:

$\mathbb{R}_{f_I}(I)/(k_{er}(\varphi))_{f_I} \cong (I_m(\varphi))_{f_I}$, thus:

$\mathbb{R}_{f_I}(I)/k_{er}(\ \varphi_I) \cong I_m(\ \varphi_I)$

Example:

Consider $f\colon \mathbb{R} \to [0.1]$; $\begin{cases} f(0) = 0 \\ f(1) = 1 \\ f(x) = \frac{1}{|x|} & ; |x| > 1 \\ f(x) = |x| & ; 0 < |x| < 1 \end{cases}$

For $B = 3 + 2I \in \mathbb{R}(I)$. $B_{f_I(B)} = (3 + 2I)_{\frac{1}{2}}$.

For $C = 2 + 5I \in \mathbb{R}(I)$. $C_{f_I(C)} = (2 + 5I)_{\frac{1}{2}}$.

$B * C = (5 + 7I)_{\frac{1}{5}}$. $B \circ \mathrm{C} = (6 + 29I)_{\frac{1}{6}}$

$-B = (-3 - 2I)_{\frac{1}{2}}$. $-C = (-2 - 5I)_{\frac{1}{2}}$

Conclusion

In this paper, we have defined and study for the first time a two-fold algebra over neutrosophic real number ring by merging the fuzzy set mapping with the algebraic operations of the neutrosophic real number ring.

Also, we studied the elementary algebraic properties of the defined two-fold algebra through its algebraic operations and substructures such as homomorphisms and ideals.

In the future, we aim to generalize our study to other neutrosophic algebraic structures.

References

[1] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971), 512-517.

[2] P. Sivaramakrishna Das, Fuzzy groups and level subgroups, J. Math. Anal. Appl. 84 (1981), 264-269.

[3] Mohammad Abobala. (2020). n-Cyclic Refined Neutrosophic Algebraic Systems of Sub-Indeterminacies, An Application to Rings and Modules. International Journal of Neutrosophic Science, 12 (2), 81-95 (**Doi** : https://doi.org/10.54216/IJNS.120203)

[4] Malath Fared Alaswad, Rasha Dallah. (2023). Neutrosophic Divisor Point of A Straight Line Segment With A Given Ratio. Pure Mathematics for Theoretical Computer Science, 2 (1), 18-23 (**Doi** : https://doi.org/10.54216/PMTCS.020102)

[5] Rashel Abu Hakmeh, Murhaf Obaidi. (2024). On Some Novel Results About Fuzzy n-Standard Number Theoretical Systems and Fuzzy Pythagoras Triples. Journal of Neutrosophic and Fuzzy Systems, 8 (1), 18-22 (**Doi** : https://doi.org/10.54216/JNFS.080102)

[6] T. M. Anthony, H. Sherwood, A characterization of fuzzy subgroups, Fuzzy Sets and Systems 7 (1982), 297-305.

[7] W. M.Wu, "Normal fuzzy subgroups," Fuzzy Mathematics, vol.1, no. 1, pp. 21–30, 1981.

[8] Palaniappan, N, Naganathan,S and Arjunan, K " A study on Intuitionistic L-Fuzzy Subgroups", Applied Mathematical Sciences, vol. 3 , 2009, no. 53 , 2619-2624.

[9] Noor Edin Rabeh, Othman Al-Basheer, Sara Sawalmeh, Rozina Ali. (2023). An Algebraic Approach to n-Plithogenic Square Matrices For $\mathbf{18} \leq \boldsymbol{n} \leq \mathbf{19}$. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 08-23, (**Doi** : https://doi.org/10.54216/JNFS.070201)

[10] M.A. Ibrahim, A.A.A. Agboola , E.O. Adeleke, S.A. Akinleye. (2020). On Neutrosophic Quadruple Hypervector Spaces. International Journal of Neutrosophic Science, 4 (1), 20-35 (**Doi** : https://doi.org/10.54216/IJNS.040103)

[11] Mohammad Abobala. (2020). Classical Homomorphisms Between n-Refined Neutrosophic Rings. International Journal of Neutrosophic Science, 7 (2), 74-78 (**Doi** : https://doi.org/10.54216/IJNS.070204)

[12] Mohammad Abobala. (2020). Classical Homomorphisms Between Refined Neutrosophic Rings and Neutrosophic Rings. International Journal of Neutrosophic Science, 5 (2), 72-75 (**Doi** : https://doi.org/10.54216/IJNS.050202)

[13] P.K. Sharma , "(α , β) – Cut of Intuitionistic fuzzy Groups" International Mathematical Forum,Vol. 6, 2011 , no. 53 , 2605-2614.

[14] Florentine Smarandache. "Neutrosophic Two-Fold Algebra", Plithogenic Logic and Computation, Vol.1, No.1 2024. PP.11-15.

[15] Mohammad Abobala. "On The Foundations of Fuzzy Number Theory and Fuzzy Diophantine Equations." Galoitica: Journal of Mathematical Structures and Applications, Vol. 10, No. 1, 2023 ,PP. 17-25 (**Doi :** https://doi.org/10.54216/GJMSA.0100102).

[16] Mohammad Abobala. (2023). On a Two-Fold Algebra Based on the Standard Fuzzy Number Theoretical System. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 24-29 (**Doi** : https://doi.org/10.54216/JNFS.070202).

[17] Ahmed Hatip, Necati Olgun. (2023). On the Concepts of Two- Fold Fuzzy Vector Spaces and Algebraic Modules. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 46-52 (**Doi** : https://doi.org/10.54216/JNFS.070205).

Received: Feb 7, 2024. Accepted: April 29, 2024

Theory of Distances in NeutroGeometry

Erick González Caballero [1] and Florentin Smarandache [2]

[1] Member, Asociación Latinoamericana de Ciencias Neutrosóficas, Havana, Cuba. E-mail: erickgc@yandex.com
[2] University of New Mexico, Mathematics, Physics, and Natural Sciences Division 705 Gurley Ave., Gallup, NM 87301, USA, smarand@unm.edu.

Abstract. NeutroGeometry is one of the most recent approaches to geometry. In NeutroGeometry models, the main condition is to satisfy an axiom, definition, property, operator and so on, that is neither entirely true nor entirely false. When one of these concepts is not satisfied at all it is called AntiGeometry. One of the problems that this new theory has had is the scarcity of models. Another open problem is the definition of angle and distance measurements within the framework of NeutroGeometry. This paper aims to introduce a general theory of distance measures in any NeutroGeometry. We also present an algorithm for distance measurement in real-life problems.

Keywords: NeutroGeometry, path, rectifiable path, single-valued neutrosophic set, Taxicab geometry, Chinese checker metric

---

1 Introduction

NeutroGeometries have recently emerged as new proposals by Professor F. Smarandache within the extensive history of geometry. NeutroGeometries are heirs of the neutrosophic vision of the world founded and created by F. Smarandache himself [1, 2]. Neutrosophy is the branch of philosophy that addresses the existence of neutrality, the erroneous, unknown, contradictory, paradoxical, incoherent, inconsistent and neutral, among other concepts of this type [3].

Neutrosophy as a new branch of philosophy has had a favorable impact on some branches of knowledge, such as the emergence of neutrosophic sets as a generalization of fuzzy sets, intuitionistic fuzzy sets and interval-valued fuzzy sets. Also, those logic has led to neutrosophic logic. On the other hand, applications of neutrosophy have appeared in decision-making, digital image processing and statistics, to solve problems in psychology, sociology, economics, physics and mathematics.

NeutroAlgebras appeared before NeutroGeometries, where the degree of indeterminacy is part of some algebraic structure models, besides the degrees of truthfulness and falsehood [4]. However, we must highlight that the study of NeutroGeometries has as a precedent mixed geometries or Smarandachean geometries several decades ago. The definition of a mixed geometry is the acceptance of geometries where at least one axiom is Smarandachely negated in one of the following ways: (a) It is fulfilled by one part of the elements of the structure and is not fulfilled by the rest of them, (b) It is not fulfilled in one way by one part of the elements and is not fulfilled by another part of the elements, but in a different way [5].

That is to say, when an axiom is Smarandachely negated, a geometric structure can exist where the axiom is fulfilled for some elements of the space and not fulfilled for others. For example, a plane that is defined where the fifth Euclidean postulate is satisfied by some straight lines and other lines do not satisfy it. Also, we can have that Euclid's fifth postulate is never fulfilled, but in two different ways: on the one hand, with elements that satisfy the axiom of multiple parallels, and on the other hand with lines and points where there is no parallelism.

This initial historical Smarandache's idea of denying concepts within geometric structure led him to the notion of also including indeterminacy as part of the structure. Thus, NeutroGeometry is understood as the geometric structure where there may be some degree of indeterminacy besides the degrees of truthfulness and falsehood, that is, at least one concept within the structure is partially satisfied by the elements of the structure with some degree of indeterminacy. When one of the concepts, definitions, operations or axioms, among others, is not fulfilled in any case, it is called AntiGeometry. A Smarandache

Geometry is a NeutroGeometry when one axiom is partially false and partially true (and it may also be partially indeterminate), or an AntiGeometry when at least one axiom is totally false.

These more modern geometric approaches recover the primeval emergence of geometry as the branch of mathematics dedicated to the study of real-life physical objects. This first objective had become over time an unreal abstract approximation of physical reality. However, it is obvious that in many cases it is not possible to travel a path through the shortest distance. This is because, in the geometric spaces of the real world, there is indeterminacy, vagueness, uncertainty and so on. This has been Smarandache's main motivation for the creation of NeutroGeometries.

Today, this new theory lacks some points that would make it more solid. Some of them are the scarcity of geometric models where this theory is manifested. Hence, this paper aims to contribute to the solidification of NeutroGeometry introducing a theory of distance within the geometric structures in the neutrosophic framework. However, this is not a simple task. There are multiple circumstances recognized by some authors within the geographical scope, where a region can change in time and space [6]. In the so-called vague regions, one of the elements to consider is geometry [7]. In them, uncertainty is associated with each point, line and region that is studied, where indeterminacy can be due to multiple causes and is not easy to model.

The purpose of vague regions is the use of spatial databases within the geographical scope, such as geographical information systems (GIS), and their application in the modeling of geographical conditions that are uncertain in time and space [8]. They use probabilities, fuzzy sets and supervaluationism to model uncertainty or vagueness [9]. The elements that are components of geography such as forests, plains and mountains cannot be defined exactly in their dimensions, and in everyday life people refer to them vaguely. Neither the relationships between these elements escape the vagueness of natural language for referring to these geographical elements. It is common to use vague terms such as "near", "far" and "small area" rather than precise distance or area measures such as 1 km, 10 km and 3 km^2.

In this paper, however, we propose other objectives, although the motivations are the same. We work directly with geometric properties to solve real problems. Nevertheless, we treat geometric structures beyond the Euclidean terrestrial geometry of GIS. Specifically, we deal with the problem of distances by trajectories in the so-called NeutroGeometries from any geometry applied to any region of the universe.

In classical geometry, the distance between two points is calculated in a segment contained in the geodesic line, where the distance is the shortest one. In NeutroGeometry this is not necessarily true. It depends on the person's knowledge and the obstacles that may be encountered along the way. It is a subjective and cognitive distance, in the sense that it depends on each person - on their knowledge, the historical knowledge about the region and the obstacles that objectively interrupt the path - to determine the shortest path to follow between one point and another. The indeterminacy may be due to the presence of unwanted phenomena such as a conflictive zone, there is a territory dominated by wild animals, there is a river with impassable parts, there are mountains that are difficult to climb and many more examples that we can think of.

It also depends on the orientation; to walk from point A to point B is not always equivalent to walking in the opposite direction. For example, if there is a hill, it is not the same to walk uphill as downhill for the same track. Transiting a river is not the same as navigating with the current or against it.

Also, if we are traveling through a river, it is not efficient to travel a section by boat, then go to land and later continue by boat, only to continue along a straight line. It is better to navigate the river, even if this means deviating at times from the path that coincides with the geodesic line.

They are reasons that motivate this article to propose a general solution to find the shortest distance from one point to another regardless of the geometry in question, whether it is Euclidean or non-Euclidean. This is an important element to consider once a geometric model is established. The other point is the definition of angular measurements, but depending on the geometry in question it is possible to maintain the predefined measurements of angles. However, the problem of finding distances is critical in situations where there are indeterminate or uncertain regions, where the real characteristics of the terrain to be covered may not be known or have degrees of uncertainty.

This paper is divided into section 2 where the preliminary concepts necessary to understand the contribution of this paper to NeutroGeometry theory are explained. Section 3 contains the elements of the theoretical proposal that is the objective of this article. These results are argued with some examples of geometries such as the well-known Taxicab geometry or Chinese checker metric that are cases of the herein proposed theory. Section 4 consists of conclusions.

2. Basic Notions of NeutroGeometry

Smarandache added the prefixes Neutro- and Anti- to certain mathematical structures' names [1, 2, 10-12]. The first of them indicates that the concept presents degrees of truth, falsehood and indeterminacy, and the second one denotes that the concept is not fulfilled in all cases. For example, an Algebra is made up of a set of elements with one or more operations among them that satisfy at least one axiom. NeutroAlgebra satisfies these conditions with a degree of truthfulness, a degree of indeterminacy and a degree of falsity. On the other hand, AntiAlgebra does not satisfy at least one of these conditions at all. So, NeutroAlgebra satisfies its conditions in a triad (t, i, f) of truthfulness, indeterminacy and falseness, where $(t, i, f) \neq (1, 0, 0)$ in at least one of them. All the conditions of classical algebra are satisfied in a triad of $(1, 0, 0)$. AntiAlgebra fulfills that any of its conditions are never satisfied corresponding to the triad $(0, 0, 1)$.

Similarly, NeutroGeometry is made up of elements, generally they are the concepts of "point", "line", "plane", "space" and "hyperspace". There are relationships between them, for example, a point is contained in a line and a line is contained in a plane, and so on. They must satisfy certain axioms. These axioms usually either satisfy or contradict the Euclidean axioms, specifically Euclid's five postulates, with emphasis placed on the fifth postulate.

It is not possible to talk about NeutroGeometry without referring to mixed geometries or Smarandache geometries that emerged several decades ago:

"A Smarandache geometry is a geometry which has at least one Smarandache denied axiom, i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalidated, or only invalidated but in multiple distinct ways and a Smarandache n-manifold is an n-manifold that supports a Smarandache geometry." [5]

However, a mixed geometry only takes into account the negation of the concept in a certain sense. NeutroGeometry may deny some components of classical geometry, but it also takes into account situations where there is indeterminacy due to indistinguishability, error, ignorance, paradoxes, contradictions, vagueness, inconsistency and so on.

A mixed geometry may contain some elements that satisfy Euclid's fifth postulate of parallelism, while other elements do not satisfy it. Moreover, a mixed geometry can contradict the fifth postulate, on the one hand, because more than one parallel to a given line passes through an external point and on the other hand because of the absence of parallelism for other elements.

There are real-life situations that cannot be modeled using a mixed geometry. An example illustrated in Smarandache's writings is a river with unknown areas. Due to the lengths of the river compared to the terrestrial measures, we can utilize a geometric Euclidean model. However, it is unrealistic to consider a straight line for measuring the shortest distance between two points within the river. This is because the straight line that joins both points can pass through an unknown area, so we are not sure whether this area is passable or not. Then this can no longer be the shortest line to travel through both points. It is necessary to deviate in some way along another path avoiding this region or otherwise risk exploring an unknown place, see Figure 1.

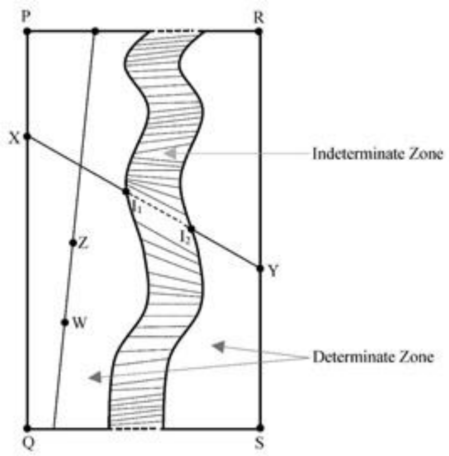


Figure 1: NeutroGeometry example. River with an Indeterminate Zone [1].

In the next section, we propose a theory of distances where there is indeterminacy. Nevertheless, we can consult some references where this topic is discussed, [13-16].

3. Distances in NeutroGeometries

This section contains the details of the proposed theory of distances within the framework of NeutroGeometry.

The first point to consider is that NeutroGeometry must be characterized by the geometry from which it is defined. This can be either Euclidean, hyperbolic, elliptical or mixed geometry. Such a starting geometric structure must have a distance function defined between any two points of the geometric structure. We denote as (G, d_G) this initial geometric space with such a distance. To simplify the results, in this paper we will limit the study to the plane, therefore the geometric structures will only contain points and lines, and the plane is generally defined as a 2-dimensional surface on Euclidean space.

Thus, the two-dimensional coordinate system XOY, where O is the origin together with the distance function forms a metric space. Let us call it (X, d_X) and recall that a metric space is defined from a set of elements and a distance function among them [17].

The other important definition is a *path*, which is a parametric function $p: [0,1] \to X$. That is, it is a function defined on the coordinate system. The paths will be assumed to be rectifiable. In other words, considering the partitions $T = [t_0 = 0, t_1, t_2, \cdots, t_{n-1}, t_n = 1]$ there is always:

$$L(p) = sup_T\left\{\sum_{i=1}^{n} d\left(p(t_{i-1}), p(t_i)\right)\right\} \quad (1)$$

$L(p)$ is called the *length of p* in (G, d_G), where $\sup_T$ is the supremum over the set of finite partitions T. What is more, let us suppose that we are in the presence of a *path metric space* that is, given $x_1, x_2 \in X$, then:

$$d_G(x_1, x_2) = inf\{L(p): p \text{ is a rectifiable path from } x_1 \text{ to } x_2\} \quad (2)$$

Where inf is the abbreviation of infimum.

Let us also consider other definitions of paths. For example, each path is oriented when for all partitions $T = [t_0 = 0, t_1, t_2, \cdots, t_{n-1}, t_n = 1]$ we obtain paths that go from x_1 to x_2, however, we can define $-p(t_i) = p(t_{n-i})$ for obtaining the same path in the opposite direction.

Let us call *subpath* when the path is obtained from p such that for $p(0) = x_1$ and $p(1) = x_2$ the subpath is taken as $q(0) = p(u)$ or $q(1) = p(v)$ for $u > 0$ or $v < 1$. That is, geometrically the subpath is a path contained in the trajectory of the path.

Definition 1. Suppose that besides $L(p)$ we define a *passability function* $\varepsilon(p) \in [0,1]$ for each rectifiable path p. $\varepsilon(p)$ satisfies the following conditions:

i. If $p = \emptyset$ or one of the points $p(t_i)$ is an ideal point, then $\varepsilon(p) = 0$. Let us recall that the ideal points in models of non-Euclidean geometries do not belong to the models, but they are on the border. E.g., the Poincaré Disk model contains the interior points of the unit circle except for the border, thus the circumference of radius 1 is the set of ideal points. So, the distance from one point to an ideal point is infinite.

ii. When $p \equiv x \in X$, that is, the path is formed by a single point, we have $\varepsilon(p) = 1$.

iii. $\forall q$ subpath of p it is fulfilled $\varepsilon(p) \le \varepsilon(q)$.

Definition 1 does not require that $\varepsilon(p) = \varepsilon(-p)$.

As a consequence of Definition 1, we have the following properties:

1. p, q and r are three paths such that $p(1) = q(0)$. Let us denote by $r = p \cup q$ the operation of obtaining the union of the two paths where $r\left(\frac{t}{2}\right) = p(t)$ and $r\left(\frac{t+1}{2}\right) = q(t)$. Therefore, we have $r(0) = p(0)$, $r(1) = q(1)$, $r\left(\frac{1}{2}\right) = p(1) = q(0)$. In this case $\varepsilon(r) \le min(\varepsilon(p), \varepsilon(q))$ according to Definition 1 condition (iii).

2. Let p, q and r be three paths equally oriented, such that for certain intervals $I_1, I_2 \subset I = [0,1]$ $r(I) \equiv p(I_1) \equiv q(I_2)$. This operation is denoted by $r = p \cap q$. So $max(\varepsilon(p), \varepsilon(q)) \le \varepsilon(r)$, according to Definition 1 condition (iii) as well.

Definition 2. Given (G, d_G) is a metric space ([17]) in a geometric structure. The *NeutroGeometric distance* between x_1 and x_2 for $x_1, x_2 \in X$ is defined as:

$$d_{NG}(x_1, x_2) = inf\left\{\frac{L(p)}{\varepsilon(p)}: p \text{ is a rectifiable path from } x_1 \text{ to } x_2\right\} \quad (3)$$

Where $\varepsilon(p)$ is the passability function of p.

Some properties of $d_{NG}(x_1, x_2)$ are the following:

1. Fixing p, when the passability function increases, the distance $d_{NG}(x_1, x_2)$ decreases.

2. When $\forall p\ \varepsilon(p) = 1$ then we have $d_{NG}(x_1, x_2) = d_G(x_1, x_2)$. This happens when there is no indeterminacy or obstacle between x_1 and x_2, which is the hypothesis assumed in classical geometries.

3. In general, $d_{NG}(x_1, x_2)$ is not necessarily symmetrical, thus $d_{NG}(x_1, x_2) \neq d_{NG}(x_2, x_1)$.

4. $\forall x \in X\ d_{NG}(x, x) = 0$, since if p is the path that goes from x to itself then since definition we have $\varepsilon(p) = 1$ and $L(p) = 0$.

5. In some recent models, there are cases where $d_G(x_1, x_2) = \infty$ when an ideal point is contained in the path. Then $d_{NG}(x_1, x_2) = \infty$, since from Equation 3 we have $d_{NG}(x_1, x_2) = \frac{\infty}{0} = \infty \cdot \infty = \infty$.

6. When $\varepsilon(p) = 0$, that is, when there is total certainty of the impassibility along the path p, then we have $\frac{L(p)}{\varepsilon(p)} = \infty$ either for $L(p) < \infty$ or $L(p) = \infty$.

$L(p)$ is defined from the characteristics of geometry (G, d_G). However, for the geometric space (NG, d_{NG}) we must define the passability function $\varepsilon(p)$.

For example, for each path it can be determined $\varepsilon(p)$ as the probability that there is transitibility along the path p. Also $\varepsilon(p)$ can be the function of the possibility of traveling along the path p, where this is a possibility measure developed by Dubois and Prade in the field of fuzzy logic [18, 19]. Equivalently, if a multi-agent system is modeled, $\varepsilon(p)$ can be interpreted as the permission that exists to travel the path in the scope of the deontic logic. $\varepsilon(p)$ can be a truth value in fuzzy logic [20], can be a numerical function obtained from the ordered pair of both membership and non-membership values within the intuitionistic fuzzy logic [21], can be an element of an interval-valued fuzzy set [22, 23] or may be a function of a neutrosophic number [24, 25].

In the latter case when there is a neutrosophic valuation formed by the triple $n = (t, i, f)$ we will need to convert n into a real value using the following formula of the Score function [26]:

$$\mathcal{S}(n) = \frac{2+t-i-f}{3} \quad (4)$$

For the neutrosophic order, defined as $n_1 \preccurlyeq n_2$ if and only if $t_1 \le t_2$, $i_2 \le i_1$ and $f_2 \le f_1$, where $n_1 = (t_1, i_1, f_1)$ and $n_2 = (t_2, i_2, f_2)$; then we have $(0,1,1)$ is the smallest value of n and $(1,0,0)$ is the maximum value. Therefore, the score of $(1,0,0)$ corresponds to $\mathcal{S}\big((1,0,0)\big) = 1$ and the score of $(0,1,1)$ corresponds to $\mathcal{S}\big((0,1,1)\big) = 0$.

When we are using intervals, they can be des-neutrosophied converting them into a real single value using [27]:

$$\lambda([a, b]) = \frac{a+b}{2} \quad (5)$$

Also, we can adapt the calculation of the infimum to intervals, and as a final result we obtain an interval containing indeterminacy that can be converted into the so-called neutrosophic number.

Note that $d_{NG}(x_1, x_2)$ is increasing for the length of the path but is decreasing for the passability function of the path. Therefore, the longer the path is, the greater the value of the distance in NeutroGeometry is, both because the length increases and also because the passability function could decrease.

The theory presented so far is a generalization of already known geometries. For example, Minkowski's Taxicab geometry and the distance function that bears this name [28, 29]:

$$\mathrm{d_T(A, B)} = |\mathrm{x_A} - \mathrm{x_B}| + |\mathrm{y_A} - \mathrm{y_B}| \quad (6)$$

Where (x_A, y_A) are the coordinates of A and (x_B, y_B) are the coordinates of B in the Euclidean Cartesian plane.

This geometry is based on Euclidean geometry (G, d_G), where the points, lines and measurements between angles are the usual ones. Only the distance function between two points is changed to the one represented in Equation 6.

It is assumed that the only valid trajectories are those that go in a straight line in angles of 0, $\frac{\pi}{2}$, π or $\frac{3\pi}{2}$ concerning the x-axis. That is, the only possible paths follow trajectories parallel to one of the two coordinate axes. This geometry models the movement of a car in a city where the blocks are square and have the same dimensions. The taxicab can only turn right or left $\frac{\pi}{2}$ radians when it arrives at the corner of the block, see Figure 2.

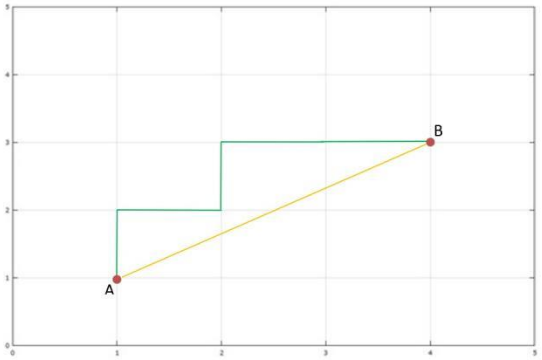


Figure 2: Trajectory from point A to B using paths in the form of broken lines (green lines) in Taxicab geometry. Observe the Euclidean straight line in yellow.

Therefore, paths that contain other types of straight lines are not allowed, thus the passability function is defined by:

$$\varepsilon(p) = \begin{cases} 1, if\ p \text{ is a path containing segments always parallels to one axis} \\ 0, \text{otherwise} \end{cases}$$

So, $\frac{L(p)}{\varepsilon(p)} = \begin{cases} L(p), if\ p \text{ is a path containing segments always parallel to one axis} \\ \infty, \text{otherwise} \end{cases}$

It is easy to check that the previous condition is satisfied only when $d_{NG}(A,B) = d_T(A,B)$. In Figure 2 the path represented with yellow line is therefore not allowed $\frac{L(p)}{\varepsilon(p)} = \infty$. The path of broken lines in green has a length equal to 5 and it can be proven that this is the distance between the two points for this geometry.

This same idea can be extended for other distances inspired by the Taxicab geometry. For example, when we have the following distance:

$$d_C(\mathrm{A},\mathrm{B}) = \max(|\mathrm{x_A} - \mathrm{x_B}|, |\mathrm{y_A} - \mathrm{y_B}|) + (\sqrt{2} - 1)\min(|\mathrm{x_A} - \mathrm{x_B}|, |\mathrm{y_A} - \mathrm{y_B}|) \quad (7)$$

This is known as the *Chinese checker metric*. So, in this geometry it is only allowed movements in straight diagonal lines of $\frac{\pi}{4}$ radians in addition to those already defined in Taxicab geometry [30]. See Figure 3.

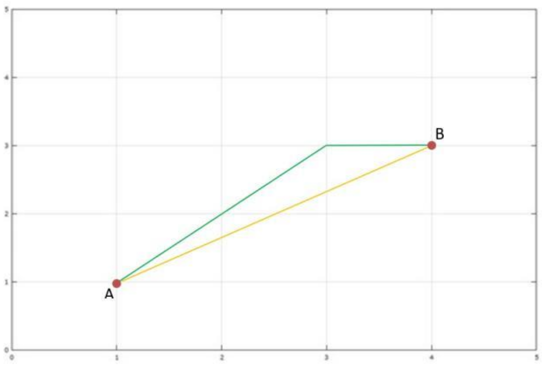


Figure 3: Trajectory from point A to B using paths in the form of lines corresponding to the Chinese checker metric.

In general, it is known that in the cities there are streets with blocks of different geometric shapes, they can be roundabouts, triangular, irregular, as well as square or rectangular, among others. Hence, our theory of distances is more exact, since each of these paths is measurable regardless of the type of shape it presents, see Figure 4.

Figure 4: The streets of a city (in black) have different shapes.

Let us observe that Taxicab or Chinese Checker geometry cannot be applied to measure distances in Figure 4. However, in this city, it is possible to apply Equation 3 because the paths to travel are rectifiable curves in Euclidean geometry as circular and rectilinear paths.

Also, in the case of real traffic, there are additional restrictions. E.g., although ideally we can turn down some streets, this is not always allowed, so we have to comply with traffic laws. These types of real-life situations are modeled thanks to the non-symmetry of the defined distance NG.

Apart from this, an interesting question is why we choose neutrosophy to model the function $\varepsilon(p)$. The neutrosophic theory offers greater possibilities for representing the different types of situations that may arise. In addition, it allows greater accuracy that is achieved at the cost of greater indeterminacy to represent the transitibility from one point to another through a path.

In general, we propose the following operations on the passability functions when there are unions or intersections of paths.

1. $\varepsilon(p \cup q) = min(\varepsilon(p), \varepsilon(q))$,
2. $\varepsilon(p \cap q) = max(\varepsilon(p), \varepsilon(q))$,

A clear limitation is that Equation 3 may be impractical since it requires analyzing all possible paths. In daily life, people make decisions about which path to take according to a good enough decision-making rather than an optimal one. Then, the amount of calculation of possible trajectories is reduced which would be too cumbersome.

The calculation of the path that minimizes the distance between two points in Equation 2 is simplified when we know the type of geometry we are dealing with. So, the path is identified with the geodesic and its equation. For example, in the Euclidean plane, the straight line between the points is the shortest path, and it is also very simple to calculate its algebraic equation. This is not necessarily the case in real life, where we need to know the terrain to get from one point to another throughout the shortest trajectory.

In daily life, human beings use Equation 3 more than Equation 2. Using the theory explained so far is an arduous task to select each path and calculate the shortest distance. In reality, decision-making in daily life is not based on selecting the optimal one, but rather on a "good enough" solution. From the distance defined in Equation 3, it is possible to define an approximate distance based on a finite subset of paths instead of all possible paths.

This leads to the following algorithm shown in Table 1.

Algorithm for deciding on the shortest path

1. Given a region R of the plane, two points A and B are defined as contained in R, and we want to measure the approximate minimum distance to go from A to B.
2. Let $E = \{e_1, e_2, \cdots, e_m\}$ be a group of locals, specialists, geographers, among others who know the terrain to be measured.
3. Γ denotes the subset of possible paths p from A to B, such that it has finite cardinality.
4. Each of the respondents in E is asked based on a scale of 0-10 to rate each $p \in \Gamma$ in terms of:
   (a) Feasibility to go from A to B passing through p.
   (b) Indeterminacy (due to ignorance, indifference, among others) to go from A to B passing through p.
   (c) Impossibility to go from A to B (passing through p).
5. The values of (t, i, f) are obtained in the following way:

$$t = \frac{sum(assessments\ in\ (a))}{10 \cdot m}, i = \frac{sum(assessments\ in\ (b))}{10 \cdot m}, f = \frac{sum(assessments\ in\ (c))}{10 \cdot m}$$

6. It is calculated $\varepsilon(p)$ according to Equation 4 for each of the possible values of (t, i, f). It is also calculated $L'(p)$ as a sufficiently approximate value of the actual length $L(p)$ of the path p. I.e., given $\epsilon > 0$ a prefixed allowed error then $|L'(p) - L(p)| \leq \epsilon$.

7. The approximate distance NG between both points is calculated by Equation 8.

$d'_{NG}(A,B) = inf\left\{\frac{L\prime(p)}{\varepsilon(p)} : p \in \Gamma\right\}$	(8)

Table 1: Algorithm for calculating the shortest approximate distance in NG based on neutrosophic numbers.

In this way, the calculation of a distance is linked to the collective knowledge about the paths to travel, rather than through an inherent single function for each geometry.

Example 1. Suppose we are in a boat, and we want to navigate a river whose bed has irregularities so that the left bank has calm waters and the right bank has turbulent waters in a certain section. However, we wish to go from point A to point B on the right side of the river. Furthermore, the path of the river itself is sinuous, see Figure 5.

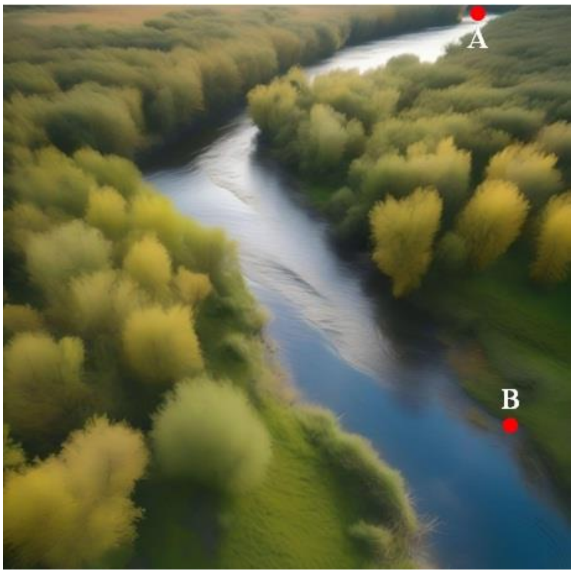


Figure 5: Picture of the river of the example. We need to go from point A to point B. This picture was generated by an AI tool.

That is why the shortest path that passes on the right is impassable and hence $\varepsilon(p_1) = \mathcal{S}((0,1,1)) = 0$. The path that goes through the center has some areas with a percentage of danger, let us say $\varepsilon(p_2) = \mathcal{S}((0.5,0.1,0.4)) = 0.66667$. While the path that goes left and then turns right is the longest one and it is completely safe, therefore $\varepsilon(p_3) = \mathcal{S}((1,0,0)) = 1$, see Figure 6.

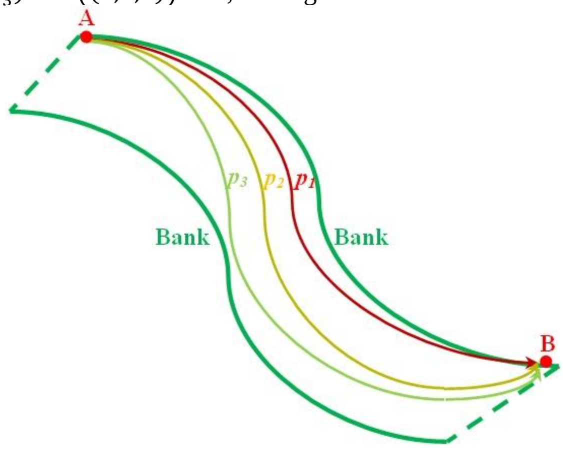


Figure 6: Map of the river showing the three paths, p_1, p_2 and p_3.

Let us suppose that the Euclidean length of each path from A to B is viz., $l(p_1) = 1.5\ Km$, $l(p_2) = 1.9\ Km$ and

$l(p_3) = 2.1\ Km$. To go through the Euclidean straight line, we would have to navigate sections of the river, then go overland in a swampy area, then navigate another section, and so on, which is not practical.

The NG lengths of the three paths in the example are, $l_{NG}(p_1) = \frac{1.5}{0}\ Km = \infty\ Km$, $l_{NG}(p_2) = \frac{1.9}{0.66667}\ Km = 2.85\ Km$, while $l_{NG}(p_3) = \frac{2.1}{1}\ Km = 2.1\ Km$, therefore we prefer the path p_3 to make the crossing, even though it is the longest one according to Euclidean geometry.

That is why the distance NG between A and B is at most 2.1 Km. Note that we have selected a finite number of paths because otherwise, we would have to search for the length of an infinite number of them.

Furthermore, if we have a path p_4 that begins at a point C before A and then continues through p_1, according to our definition of $\varepsilon(\cdot)$ we have $\varepsilon(p_4) \leq \varepsilon(p_1) = 0$, therefore $l_{NG}(p_1) = \infty\ Km$ as well. That is, any path that contains p_1 as a subpath will have infinite length in this geometry.

Another topic of interest is the way that path lengths are calculated. In this paper, we have proposed using the Euclidean arc length, which is a method that can be adapted to other geometries always respecting the definitions of geodesics and lengths, see Figure 7.

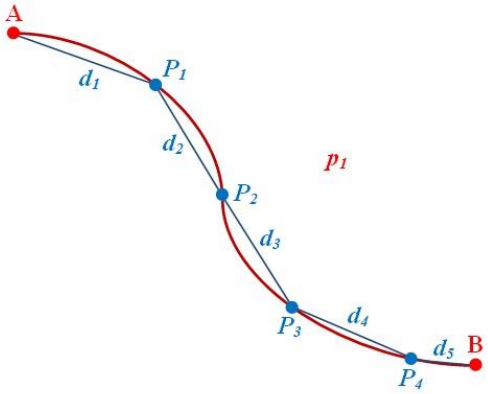


Figure 7: Arc length of the path p_1.

Figure 7 exemplifies the calculus of length of p_1. We select the points $A = P_0, P_1, P_2, P_3, P_4$ and $B = P_5$, all of them are on the curve. Then, we calculate the distances, $d_1 = d_E(A, P_1)$, $d_2 = d_E(P_1, P_2)$, $d_3 = d_E(P_2, P_3)$, $d_4 = d_E(P_3, P_4)$ and $d_5 = d_E(P_4, B)$, where $d_E(\cdot,\cdot)$ is the Euclidean metric. So, the length of p_1 is approximated by $d = \sum_{i=1}^{5} d_i$. For more accuracy, we must use more points on the curve.

In any case, in real life, there are cartographic methods to calculate the length of a path, including satellite spatial vision. Also in certain situations, we can walk along a path at a more or less constant speed and take the transit time from one point to another. Hence, we can approximately calculate the distance (length of the path) by the formula $distance = speed \cdot (time_B - time_A)$.

Conclusion

In this paper, we propose for the first time a theory to measure distance in NeutroGeometry. NeutroGeometry is assumed to be defined from a geometric structure that can be Euclidean or non-Euclidean. Using the distance measure in that base geometric structure, we introduced a distance defined in the NeutroGeometry structure, which depends on a passability function and the length of the path that joins both points. We specify some axioms that the passability function must fulfill. We also verify that the new measure generalizes the cases of classical geometries, as well as other geometries such as Taxicab and Chinese checker metrics. It is more convenient that the passability function depends on neutrosophy and not on other uncertainty models such as fuzzy sets or intuitionistic fuzzy sets models, because a neutrosophic set allows a greater number of possible states of knowledge to be expressed more accurately. Additionally, we propose an algorithm to calculate the approximate distance between two points in the NeutroGeometry plane based on a survey applied to a group of locals, geographers and experts, among others. The theory presented so far, and the proposed algorithm allows us to solve the problem of calculating the distance when there is uncertainty or indeterminacy in a two-dimensional region. These results bring us closer to solving real-life problems than utilizing classical geometries.

References

[1]. Smarandache, F. (2021). NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries. Neutrosophic Sets and Systems, 46, 456-477, http://fs.unm.edu/NSS/NeutroGeometryAntiGeometry31.pdf
[2]. Smarandache, F. (2023). Real Examples of NeutroGeometry & AntiGeometry, 55, 568-575, https://fs.unm.edu/NSS/ExamplesNeutroGeometryAntiGeometry35.pdf
[3]. Latreche, A., Barkat, O., Milles, S., and Ismail, F. (2020). Single valued neutrosophic mappings defined by single valued neutrosophic relations with applications. Neutrosophic Sets and Systems, 1, 203-220.
[4]. Smarandache, F. (2020). NeutroAlgebra is a Generalization of Partial Algebra. International Journal of Neutrosophic Science, 2(1), 8-17.
[5]. Chimienti, S. P. and Bencze, M. (1998). Smarandache paradoxist geometry. Smarandache Notions Journal, 9(1-3), 42.
[6]. Pfoser, D. and Tryfona, N. (2001). Capturing Fuzziness and Uncertainty of Spatiotemporal Objects. In: Advances in Databases and Information Systems: 5th East European Conference, ADBIS 2001 Vilnius, Lithuania, September 25-28, 2001 Proceedings 5 (pp. 112-126). Springer Berlin-Heidelberg.
[7]. Gonzalez-Perez, C., Pereira-Fariña, M., Martín-Rodilla, P., and Tobalina-Pulido, L. (2023). Dealing with Vagueness in Archaeological Discourses. In Discourse and Argumentation in Archaeology: Conceptual and Computational Approaches (pp. 137-157). Cham: Springer International Publishing.
[8]. Jana, S. and Mahanta, J. (2023). Boundary of a fuzzy set and its application in GIS: a review. Artificial Intelligence Review, 56(7), 6477-6507.
[9]. Kulik, L. (2001). A Geometric Theory of Vague Boundaries Based on Supervaluation. In: Spatial Information Theory Foundations of Geographic Information Science, COSIT 2001, Morro Bay, CA, USA (pp. 44-59) Berlin: Springer.
[10]. Singh, P. K. (2021). Anti-geometry and neutrogeometry characterization of non-Euclidean data. Journal of Neutrosophic and Fuzzy Systems, 1(1), 24-33.
[11]. Singh, P. K. (2022). NeutroAlgebra and NeutroGeometry for Dealing the Heteroclinic Patterns. In Theory and Applications of NeutroAlgebras as Generalizations of Classical Algebras (pp. 90-102). IGI Global.
[12]. Granados, C. (2023). A Review of AntiGeometry and NeutroGeometry and Their Application to Real Life. NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World, 1-16; https://www.igi-global.com/book/neutrogeometry-neutroalgebra-superhyperalgebra-today-world/292031 and https://www.igi-global.com/book/neutrogeometry-neutroalgebra-superhyperalgebra-today-world/292031
[13]. González-Caballero, E. (2023). Introduction to the Finite NeutroGeometries: The Mixed Projective-Affine Geometry. In NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World (pp. 52-80). IGI Global.
[14]. González-Caballero, E. (2023). "NeutroGeometry Laboratory": The New Software Dedicated to Finite NeutroGeometries. In NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World (pp. 131-155). IGI Global.
[15]. González-Caballero, E. (2023). A Constructive Introduction to Finite Mixed Projective-Affine-Hyperbolic Planes. In NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World (pp. 156-186). IGI Global.
[16]. Gonzalez-Caballero, E. (2023). Applications of NeutroGeometry and AntiGeometry in Real World. International Journal of Neutrosophic Science, 21(1), 14-33.
[17]. O'Searcoid, M. (2006). Metric spaces. Springer Science & Business Media.
[18]. Dubois, D., Godo, L., and Prade, H. (2023). An elementary belief function logic. Journal of Applied Non-Classical Logics, 33(3-4), 582-605.
[19]. Liu, C., and Martin, R. (2024). Inferential models and possibility measures. In Handbook of Bayesian, Fiducial, and Frequentist Inference (pp. 344-363). Chapman and Hall/CRC.
[20]. Zimmermann, H. J. (2010). Fuzzy set theory. Wiley interdisciplinary reviews: computational statistics, 2(3), 317-332.
[21]. Atanassov, K. T. (2012). On intuitionistic fuzzy sets theory (Vol. 283). Springer.
[22]. Gehrke, M., Walker, C., and Walker, E. (1996). Some comments on interval valued fuzzy sets. structure, 1(2), 751-759.
[23]. Lee, K. M., Lee, K. M., and Cios, K. J. (2001). Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets. In Computing and information technologies: exploring emerging technologies (pp. 433-439).

[24]. Luo, M., Zhang, G., and Wu, L. (2022). A novel distance between single valued neutrosophic sets and its application in pattern recognition. Soft Computing, 26(21), 11129-11137.
[25]. Smarandache, F. (2024) Foundation of Appurtenance and Inclusion Equations for Constructing the Operations of Neutrosophic Numbers Needed in Neutrosophic Statistics. Neutrosophic Systems with Applications, 15, 16-32.
[26]. Smarandache, F. (2020) The Score, Accuracy, and Certainty Functions determine a Total Order on the Set of Neutrosophic Triplets (T,I,F). Neutrosophic Sets and Systems, 38, 1-14.
[27]. Mandal, K. (2020). On Deneutrosophication. Neutrosophic Sets and Systems, 38, 409-423.
[28]. Krause, E.F. (1986) Taxicab Geometry. An Adventure in Non-Euclidean Geometry. New York: Dover Publications.
[29]. Thompson, K. P. (2011). The nature of length, area, and volume in taxicab geometry. International Electronic Journal of Geometry, 4(2), 193-207.
[30]. Chen, G. (1992). Lines and circles in taxicab geometry (Doctoral dissertation, Central Missouri State University).

Received: Feb 7, 2024. Accepted: April 30, 2024

NSS
Neutrosophic Sets and Systems, Vol. 67, 2024
University of New Mexico

A Comprehensive Decision Algorithm for the Analysis of Renewable Energy Source Selection Problem using Pythagorean Neutrosophic Fuzzy Sets

D. Sasirekha[1], P. Senthilkumar [2]
[1]Research Scholar, PG and Research Department of Mathematics, Government Arts and Science College, Kangeyam, Tiruppur, Tamil Nadu, India; sasirekhasaba@gmail.com
[2]Assistant Professor, PG and Research Department of Mathematics, Government Arts and Science College, Kangeyam, Tiruppur, Tamil Nadu, India; rajenranpsk@gmail.com

Abstract. Nowadays, to achieve sustainable development and avoid devastating impacts on the environment, India is making rapid and broad changes to green energy technology. In order to provide long-term energy security with lower emissions, renewable energy sources are essential. It is well known that renewable energy technologies, or RETs, have the capacity to significantly meet the demand for electricity while lowering pollution levels. The nation has set up an ecologically friendly energy route in recent years. In this paper, we apply multi-criteria decision making (MCDM) models to determine the best renewable energy technology for India. We use the MULTIMOORA model to identify the best technology under the pythagorean neutrosophic fuzzy set (PNFS) and the FUCOM to obtain the criteria weights. In this study, we investigate sources of renewable energy using the newly developed idea of the PNFS and recommend the best renewable energy source for India.

Keywords: Neutrosophic fuzzy set, Pythagorean neutrosophic fuzzy set, MULTIMOORA, Renewable energy technology, MCDM.

---

1. Introduction

Renewable energy (RE) is defined as energy generated by natural sources that is restocked at a faster rate than it is ingested. Sunlight and wind are two examples of such frequently restoring sources. Alternative forms of energy abound and are readily accessible to us. On the other hand, fossil fuels (FFs), coal, oil, and gas are resources that are non-renewable which require a large number of years to grow into existence. When FFs are burned to generate energy, they emit harmful greenhouse gases like CO_2. RE produces far fewer pollutants than burning fossil fuels. The transition away from FFs, which presently contribute the largest share of emissions, as well as towards renewable energy is critical for focusing on the issue of global warming. RE sources are now less expensive in numerous nations and create nearly as many employment opportunities as FFs [1]. Today, we primarily use fossil fuels to heat and power our homes, as well as to fuel our automobiles. It is most appropriate to provide for the demands for energy with natural gas, oil, and coal, but these FFs have a finite supply and ecological footprint. We are using them much faster than they are being created. They will eventually deplete. In addition, the US will retire a sizable chunk of its nuclear capacity by 2020 due to safety and waste disposal concerns. It is anticipated that during the next 20 years, the country's energy needs will rise by 33%. That gap can be filled in part by renewable energy. Both decreasing greenhouse gas emissions in the US and greatly enhancing energy security are possible with RE. As the main contributors to CO_2 emissions in the US, the consumption of fossil fuels and energy imports can be mitigated by using renewable energy [2].

India's energy needs are rising in tandem with the country's plans for economic expansion. The development of a country depends critically on a consistent supply of energy in ever-increasing amounts. India ranks fourth in the world after China (26.83%), the United States (14.36%), and the European Union (9.66%) [5], with its contribution to global carbon emissions coming from the World Resource Institute Report 2017 [3,4]. According to the World Energy Council (WEC), the global electrical consumption peak is expected to occur in 2030. In addition to importing pricey fossil fuels, India is one of the world's biggest users of coal [6]. A study by the Centre for Monitoring the Indian Economy [7] states that the nation imported 171 million metric tons (MMTs) of coal in 20132014, 215 MMTs in 20142015, 207 MMTs in 20152016, 195 MMTs in 20162017, and 213 MMTs in 20172018. Consequently, the development of new energy sources for the production of electricity is imperative.

Renewable energy has progressed significantly. Energy is now more productive, accessible, and effective. The majority of families are able to make investments in renewable energy. Consequently, substantial numbers of individuals are unaware of the advantages of energy. RE is now being considered for many residential and commercial properties, such as solar, wind, and geothermal have an advantageous effect on welfare, the financial system, society, and the entire globe. While some people are still suspicious of renewable energy, it is becoming more widely accepted due to its numerous benefits. In this paper, we use MCDM methods to establish the finest alternate energy source for India. The Multi-Objective Optimization on the basis of Ratio Analysis plus full multiplicative form (MULTIMOORA) model is examined to figure out the most sophisticated technology within the PNFS.

2. **Literature Review**

In recent times, the fusion of MCDM frameworks with fuzzy logic has surged in popularity for assessing situations marked by inherent conflicts, owing to their effectiveness in handling the uncertainties and imprecisions commonly encountered in decision-making contexts. To address the increasing level of uncertainties associated with physical problems, various concepts from fuzzy logic have been incorporated. Smarandache [8] introduced the neutrosophic set (NSS) and neutrosophic probability concepts in 1998, alongside a logical framework comprising trueness, neutrality, and falseness. This framework also incorporates the term reflection, enabling its application across diverse study areas. In a fuzzy set, trueness is denoted by H, neutrality by K, and falseness by F, each distinct with $0 \leq H + K + F \leq 3$. Unlike the degree of belongingness and non-belongingness in intuitionistic fuzzy sets, the indeterminacy factor in NFS is independent of the H and F values. A neutrosophic fuzzy number (NFN) can depict unpredictability, falsity, and uncertainty in real-life issues. Recently, Yager [9–11] introduced a pythagorean fuzzy set (PFS) as an alternative evaluation tool for acquiring more relevant data in inaccurate and uncertain scenarios, defined by a combination of membership and non-membership levels fulfill the condition that the square of their sum is less than 1. Zhang and Xu [12] introduced the notion of PFN and an extensive mathematical approach for PFS. Combining these two frameworks offers a comprehensive approach to handling uncertainties in decision-making processes.

Assigning weights to criteria is crucial for evaluating alternatives in decision-making, as it involves considering various factors. These weights can be derived through either an objective or a subjective process. The Full Consistency Method (FUCOM) represents a subjective approach wherein the relative importance of each attribute is estimated through pairwise comparisons [13]. This method involves systematically evaluating the attributes against each other to discern their relative significance, thereby aiding in the establishment of weighted criteria for the decision-making process. Further fuzzy based extension of the method was presented by Pamucar et al. [14] and is been used in solving the green supplier evaluation problem. As a result, several extension of the method has been suggested by various researchers for handling the fuzziness that arose with the criteria weight estimation in sustainable fuel vehicle selection [15], sustainable supplier selection [16], modelling of sustainable mobility plans [17], healthcare waste treatment method selection [18] and so on.

Multi-Objective Optimization on the basis of a Ratio Analysis plus the full MULTIplicative form (MULTIMOORA) is a MCDM model developed as an extension of the Multi-objective optimization by ratio analysis (MOORA). This model is an integrated approach which offers ranking of alternatives based on dominance theory [19]. Balezentis et al. [20] extended the traditional model to the fuzzy numbers and used for evaluating the decision making problems. Stanujkic et al. [21] proposed the neutrosophic extension of the method. Further several extensions of the method using the Pythagorean [22], fermatean [23], intuitionistic linguistic

Table 1. MADM methods used various researchers

Authors	MCDM methodology	Country	Suggested Option
Van Thanh et al. (2022) [41]	Spherical fuzzy AHP-TOPSIS	Vietnam	Solar energy
Goswami et al. (2022) [42]	MEREC-PIV	India	Hydroelectric power plant
Saraswat et al. (2021) [43]	Fuzzy AHP-VIKOR,WSM, PROMETHEE-II	India	Solar energy
Wang et al. (2021) [44]	Grey AHP-WASPAS	Vietnam	Solar energy
Sarkodie et al. (2022) [45]	CRITIC-MOORA, TOPSIS, COPRAS	Ghana	Hydropower
Alkan et al. (2020) [46]	Triangular fuzzy entropy-COPRAS, MULTIMOORA	Turkey	Wind energy

fuzzy numbers [24], picture [25] and spherical fuzzy numbers [26] has been suggested by various researchers for solving diverse decision making problems.

Many researchers are looking into renewable energy technologies and their benefits in various countries. Hussian et al. [27] proposed that wind and solar energy production, as well as economic development, have an impact on environmental quality. Using the fuzzy MCDM model, Jahangiri et al. [28] investigated the best location for capturing wind and solar energy. Aljaghoub et al. [29] proposed MCDM-based solar PV cleaning techniques. Saraswat et al. [30] studied the spatial suitability of solar and wind farm locations in India from economical, technological, and infrastructure-environmental perspectives. Goswami et al. [31] proposed a suitable RE power plant for India based on six key factors using an integrated MCDM model. Li et al. [32] created an innovative structure for evaluating the highest-priority areas of energy from renewable sources growth and implementation in China from the viewpoint of environmental sustainability, consequently improving renewable energy management. Kaur et al. [33] presented an MCDM-based method for selecting the best solar panel for rural electrification. Khalifa et al. [34] examined the inverse capacitated transportation problem in neutrosophic environment. Priyadharshini and Irudayam [35] analyzed the obesity problems in school children using plithogenic single valued fuzzy sets. Mohamed et al. [36] proposed the transition supply chain 4.0 to supply chain 5.0. Jdid and Smarandache [37] examined an efficient optimal solution model for transport models under neutrosophic environment. Khalifa et al. [38] proposed the neutrosophic complex programming using lexicographic order. Alizadeh et al. [39] investigated the role of RE facilities, regulations, and organizational frameworks in facilitating their growth. Kumar and Samuel [40] considered an optimum selection of best RE source with the help of VIKOR method.

Further MCDM methods have been used for evaluating the RE sources suitable for sustainable energy generation. Table 1 presents the MADM methods used and the suggested option.

The reviewed studies primarily aimed to pinpoint the most cost-effective options for Renewable Energy Technologies (RET) in different countries, leveraging the continually advancing Multiple Criteria Decision Making (MCDM) methods that play a pivotal role in the realm of sustainable energy management. Given the diversity of solutions proposed by various methodologies, it becomes imperative to conduct a nation-specific, in-depth analysis to address the problem effectively. The potential synergy of combining the Full Consistency Method (FUCOM) with MULTIMOORA in the context of RET selection remains untapped. Similarly, the capability of Pythagorean Neutrosophic Fuzzy Sets (PNFS) to encapsulate not just the dimensions of truth and falsehood but also of neutrality offers a robust model for tackling complex decision-making scenarios, yet its integration with the FUCOM-MULTIMOORA approach warrants further exploration. This combination could significantly broaden its applicability across different fields and bolster decision-making efficacy. Consequently, this research endeavors to forge a comprehensive framework for selecting the most apt alternative energy sources for India, considering social, economic, and technical criteria. This objective is pursued through the employment of the FUCOM-MULTIMOORA method combined with a Pythagorean neutrosophic fuzzy set, aiming to unearth the optimal solution.

3. Preliminaries

Definition 3.1. [8] Let T be a non-empty set. A NFS L on T is given below:

$$L = \{< a, H_L(a), K_L(a), F_L(a) > | a \in T\} \tag{1}$$

where $H_L(a), K_L(a), F_L(a) \in [0, 1]$, for all $a \in T$, $H_L(a)$ is trueness level, $K_L(a)$ is neutral level and $F_L(a)$ is falseness level. Here, the degree of trueness and falseness are dependent components and neutral level is an independent component.

Definition 3.2. [9, 10] The PFS O is a set over U:

$$O = \{< a, \phi_O(a), \omega_O(a) > | a \in T\} \tag{2}$$

Where $\phi_O(a) : T \to [0, 1]$ and $\omega_T(a) : U \to [0, 1]$ describe the membership and non-membership degree respectively, $a \in O$ on T,

$$0 \leq ((\phi)_O(a))^2 + ((\omega)_O(a))^2 \leq 1 \tag{3}$$

Suppose $((\phi)_O(a))^2 + ((\omega)_O(u))^a \leq 1$ then there is a degree of indeterminacy is defined by $\lambda_O(u) : \sqrt{1 - ((\phi)_O(a))^2 + ((\omega)_O(a))^2}$ and which belongs to 1.

Definition 3.3. [9, 11] A PNFS Q with H and F are dependent components on T is in the form of

$$Q = \{< a, \phi_Q(a), \delta_Q(a), \omega_Q(a) > | a \in T\} \tag{4}$$

Where $\phi_Q(a), \delta_Q(a), \omega_Q(a)$ are belongs to $[0, 1]$ and $0 \leq (\phi_Q(a))^2 + (\delta_Q(a))^2 + (\omega_Q(a))^2 \leq 2$, $\phi_Q(a)$ is trueness level, $\delta_Q(a)$ is neutral level and $\omega_Q(a)$ is falseness level.

Definition 3.4. [8–10] The score function of the PNFS with dependent components N and F are described as:

$$S_Q(a) = (H + (1 - N) + (1 - F)) \tag{5}$$

4. Mathematical Methods

The developed framework consist of evaluating the criteria weights using the FUCOM method and the ranking of the options are provided using the PNFS based MULTIMOORA method. The algorithm of the developed methods are provided in the following sections.

4.1. *Proposed method*

The proposed method employs three techniques: the ratio system (RS), the reference point (RP), and the full multiplicative form (FMF). The algorithm of the proposed model is given below: [47]

4.2. *The RS technique*

The importance of overall i^{th} alternative is:

$$Z_i = z_i^+ - z_i^- \tag{6}$$

Where

$$z_i^+ = \sum_{j \in P} r_{ij} \tag{7}$$

$$z_i^- = \sum_{j \in N} r_{ij} \tag{8}$$

Where z_i^+ and z_i^- represents the addition of normalized performance values (NPV) of the significance. These are obtain based on the positive and negative criteria. Here, P and N denotes the positive and negative criteria respectively; $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., n$. The NPVs are obtained as:

$$r_{ij} = \frac{h_{ij}}{\sum_{i=1}^{m} (h_{ij})^2} \tag{9}$$

Where h_{ij} is the performance value of the i^{th} alternative to the j^{th} criteria. Based on their z_i values, the compared options are ranked descendingly, with the option with the highest z_i value being the best ranked.

4.3. *The RP technique*

The reference point based optimization:

$$Y_i = min_i(max_j W_j \times d(e_j - r_{ij})) \tag{10}$$

The overall performance of the RP technique is denoted by Y_i, and the distance between the RP and the NDM, multiplied by the criteria weights, is denoted by $d(e_j - r_{ij})$. The j^{th} coordinate of the RP is represented by e_j as follows:

$$e_j = max_i r_{ij}; j \in P \tag{11}$$

$$e_j = min_i r_{ij}; j \in N \tag{12}$$

The alternatives under comparison are arranged in order of importance according to their Y_i values; the option with the lowest Y_i value is considered the best.

4.4. *The FMF technique*

The following formula can be used to determine the alternative's overall utility:

$$B_i = \frac{P_i}{N_i} \tag{13}$$

Where P_i represents the combination of the positive criteria's weighted evaluations of performance and N_i represents the combination of the negative criteria's weighted evaluations of performance. The best outcome is indicated by the largest value of B_i, and the compared options are arranged in descending order.

5. **FUCOM method**

Pairwise comparison serves as the foundation for FUCOM, which authenticates outcomes by deviating from maximum consistency (DMC). In comparison to AHP, it minimizes the number of paired criteria comparisons and provides an opportunity to check the results by identifying the transitivity of pairwise criteria comparisons and defining the DMC [48].

The steps of FUCOM is given below: [49]

Step 1: Form the ranking set using the given evaluation attribute $(S_1, S_2, ..., S_n)$ The following are the criteria that are ranked by intended importance as follows:

$$S_{j(1)} > S_{j(2)} > ... > S_{j(b)}, \tag{14}$$

where b represents the criteria order.

Step 2: Comparing every neighboring attribute pair results in the computation of comparative priority are $\Psi_{\frac{(b-1)}{b}}, b = 1, 2, ..., n$. Here, $\Psi_{\frac{(b-1)}{b}}$ represent the criterion value $S_j(b-1)$ relative to criterion $S_j(b-1)$ is expressed by $(b-1), b$. Then, the comparative preferences is

$$\Psi = \{\Psi_{\frac{1}{2}}, \Psi_{\frac{2}{3}}, ..., \Psi_{\frac{(b-1)}{b}}\} \tag{15}$$

Step 3: Compute the final weight values$(w_1, w_2, , w_j)$, it should satisfies two conditions:

- The ratio of weight is equal to the comparative priority $\Psi_{\frac{(b-1)}{b}}$ (from step 2). (i.e),

$$\frac{w_{b-1}}{w_b} = \Psi_{\frac{(b-1)}{b}} \tag{16}$$

- The weight should satisfy the transitivity condition, i.e., $\Psi_{\frac{(b-2)}{b-1}} \otimes \Psi_{\frac{(f-1)}{f}} = \Psi_{\frac{(f-2)}{f}}$. The another condition is

$$\frac{w_{(b-2)}}{w_b} = \Psi_{\frac{(b-2)}{(b-1)}} \otimes \Psi_{\frac{(b-1)}{b}} \tag{17}$$

The construction of a nonlinear constrained programming model is as follows:

$$\begin{aligned} & min \;\; \theta, \\ & s.t \left| \frac{w_{(b-1)}}{w_b} - \Psi_{\frac{(b-1)}{b}} \right| \leq \theta, \left| \frac{w_{(b-2)}}{w_b} - \Psi_{\frac{(b-2)}{b-1}} \otimes \Psi_{\frac{(b-1)}{b}} \right| \leq \Psi, \\ & \sum_{j=1}^{n} w_j = 1, \forall j \\ & w_j \geq 0, \forall j \end{aligned} \tag{18}$$

The ideal weight values for the assessment are given by solving the model, which is $(w_1, w_2, ..., w_n)$.

6. **Application**

An energy source is an indispensable aspect of a country's socioeconomic progress. The explosive growth in the economies of nations that are developing in the past few years has resulted in a rapid rise in energy consumption, which is expected to continue. Research and development of suitable ecological and financial methods will necessitate a projection of future electricity usage. Analogously, an estimate of future electricity usage guides future renewable energy investment decisions. The population size as well as the expansion of a country have an important effect on energy demand. India is the country with the highest annual population growth worldwide, with some of its states having populations as large as several other countries. India's energy consumption is expected to rise at the fastest rate among major nations by 2040. The majority of this demand will come from coal, with renewable energy following closely behind. RES will surpass gas and then oil as the second most important source of domestic power generation by 2020. The demand for renewable energy in India will grow exponentially [5, 6].

Table 2. PNFS linguistic scale

Linguistic term	trueness values	neutral values	falseness values
Extremely high (EH)	0.85	0.10	0.15
moderately high (MH)	0.65	0.30	0.35
Moderate (M)	0.55	0.40	0.45
moderately low (ML)	0.35	0.60	0.65
Extremely low (EL)	0.15	0.80	0.85

Criterion	S_2	S_3	S_4	S_1
Ψ	1	2.4	3.5	4

Developing nations tend to concentrate on modern techniques to generate energy from natural sources, which aids in the search for green energy while managing the energy demand problem. In this paper, we propose the MULIMOORA for determining the best renewable energy technology for India using a PNFS. We selected four types of energy sources based on economic, environmental, social, and technological considerations.

7. Numerical Example

In this part, we use the MULTIMOORA technique to discuss the RET problem under the pythagorean neutrosophic fuzzy set. Based on the chosen criteria, the experts assessed this issue in this case. The renewable energy sources include solar energy (R_1), geothermal energy (R_2), wind energy (R_3), and biomass energy (R_4). Experts analyze the RES, utilizing the suggested way to tackle this issue. A decision matrix derived from the linguistic scale is displayed in table 2.

7.1. *FUCOM weight finding method*

Step 1: The factors are ranked by experts in decreasing order of significance: $S_2 > S_3 > S_4 > S_1$

Step 2: Experts compare the rating criteria in pairs, starting with step 1. The scale [1, 5] serves as its foundation (Table 2). All attributes are listed in order of priority below. Next, we calculate the comparative priorities based on the priorities of attribute as follows: $\Psi_{\frac{2}{3}} = \frac{2.4}{1} = 2.4$; $\Psi_{\frac{3}{4}} = \frac{3.5}{2.4} = 1.458$; $\Psi_{\frac{4}{1}} = \frac{4}{3.5} = 1.142$;

Step 3: Calculate final weights

$\lambda_{\frac{2}{3}} = 2.4$; $\lambda_{\frac{3}{4}} = 1.458$; $\lambda_{\frac{4}{1}} = 1.142$;

$\lambda_{\frac{2}{4}} = 3.499$; $\lambda_{\frac{3}{1}} = 1.665$

Using Eq. (18), we can calculate the weight for coefficient of decision maker, $Min\ \theta$, Subject to

TABLE 3. Decision matrix

	S_1	S_2	S_3	S_4
R_1	1.8	1.6	1.8	1.8
R_2	2.1	1.4	1.8	1.4
R_3	1.8	1.8	1.4	1.2
R_4	1.8	1.9	1.1	1.4

TABLE 4. Normalize decision matrix

	S_1	S_2	S_3	S_4
R_1	0.4788	0.4745	0.5794	0.6138
R_2	0.5586	0.4151	0.5794	0.4774
R_3	0.4788	0.5338	0.4506	0.4092
R_4	0.4788	0.5634	0.3541	0.4774

TABLE 5. The final ranking results for RS

Alternatives	Ranking values	Rank
R_1	1.1889	1
R_2	0.9133	4
R_3	0.9148	3
R_4	0.9161	2

$$|\frac{\lambda_2}{\lambda_3} - 2.4| \leq \theta; |\frac{\lambda_3}{\lambda_4} - 1.458| \leq \theta; |\frac{\lambda_4}{\lambda_1} - 1.142| \leq \theta;$$
$$|\frac{\lambda_2}{\lambda_4} - 3.499| \leq \theta; |\frac{\lambda_3}{\lambda_1} - 1.1665| \leq \theta;$$
$$\sum_{j=1}^{n} w_j = 1 \text{ for every } j$$

After this model is solved, the weight coefficient's ideal values are (0.1281536, 0.5121079, 0.2133819, and 0.1463566), and the DFC of the outcome is $\theta = 0.0004116877$.

7.2. *MULTIMOORA method*

The RS technique

The RS method was used to determine the ranking outcomes as well as the order of the RE technology. After applying Eqs. (7) and (8) to generate the decision matrix displayed in Table 3 using the PNFSs scoring function, we compute the NDM, which is displayed in Table 4. Table 5 presents the final ranking result of the RS technique using Eq. (6).

Table 6. Weighted distance between RP and NDM

	S_1	S_2	S_3	S_4
R_1	0	0.0455	0	0
R_2	-0.0102	0.0759	0	0.0199
R_3	0	0.0151	0.0274	0.0299
R_4	0	0	0.0480	0.0199

Table 7. The final ranking results for RP approach

Alternatives	Ranking values	Rank
R_1	0.0455	2
R_2	0.0759	4
R_3	0.0299	1
R_4	0.0480	3

Table 8. Weighted NDM

	S_1	S_2	S_3	S_4
R_1	0.0613	0.2429	0.1236	0.0898
R_2	0.0715	0.2125	0.1236	0.0698
R_3	0.0613	0.2733	0.0961	0.0598
R_4	0.0613	0.2885	0.0755	0.0698

The RP technique

Table 6 presents the weighted distance between the RP and the NDM, which was calculated by executing the RP process and utilizing Equation (10). Table 7 presents the final ranking results. The reference point is derived using Equations (11) and (12), which are (0.1281536, 0.5121079, 0.2133819, and 0.1463566).

The FMF technique

Table 9 presents the ranking results and order of the RE technology based on the FMF technique, using Eq. (13). To begin with, we acquired the weighted NDM, which is provided in Table 8. Dominance theory is used to produce the final ranking results, which are shown in Table 10.

From this Table 10, R_1- Solar energy is the best RE technology is the most suitable renewable energy sources for India which balancing the energy demand and create a new green energy, such as electricity.

Table 9. The final ranking results for FMF

Alternatives	Ranking values	Rank
R_1	0.0438	1
R_2	0.0251	3
R_3	0.0256	2
R_4	0.0245	4

Table 10. The final ranking results of MULTIMOORA

Alternatives	RA	RP	FMF	Final rank
R_1	1	2	1	1
R_2	4	4	3	4
R_3	3	1	2	2
R_4	2	3	4	3

Table 11. Comparison analysis results

Alternatives	TOPSIS	Rank	VIKOR	Rank	Proposed method
R_1	0.5882	3	0.2200	3	1
R_2	0.3828	4	1	4	4
R_3	0.5996	1	0.0737	1	2
R_4	0.5975	2	0.0959	2	3

8. **Comparative and sensitivity analysis**

This section compares this recommended method's effectiveness with other approaches, such as TOPSIS and VIKOR, for PNFN cases. Sensitivity analysis was specifically developed for the purpose of this study.

8.1. *Comparative analysis*

The efficiency and performance of the suggested model are illustrated in this part by a comparative analysis with other MCDM techniques found in the literature. The VIKOR model and the TOPSIS model are two methods that are currently in use, and they were used to assess the suggested methodology. These MCDM approaches make use of the suggested criterion weights. The ranking order comparison findings are displayed in Table 11. Results from the suggested ranking deviate further from the current TOPSIS and VIKOR approaches. Consequently, when compared to other MCDM models, the suggested method yields more trustworthy findings.

Table 12. Weights in sensitivity analysis

RE	Case 1	Case 2	Case 3
R_1	0.1281536	0.5121079	0.2133819
R_2	0.5121079	0.1281536	0.1463566
R_3	0.2133819	0.1463566	0.1281536
R_4	0.1463566	0.2133819	0.5121079

Table 13. Weights in sensitivity analysis

Alternatives	Case 1	Rank	Case 2	Rank	Case 3	Rank
R_1	0.3198	1	0.00273	1	0.01576	1
R_2	0.0018	3	0.00157	3	0.0092	2
R_3	0.1622	2	0.00159	2	0.00920	3
R_4	-0.3289	4	0.00155	4	0.00891	4

8.2. *Sensitivity analysis*

This approach compares the outcomes of three situations in its sensitivity analysis. Such weight values for the properties are displayed in Table 12. The study's result is Case 1, and the additional results found by applying various attribute weights are Cases 2 and 3. Modifying the attribute weights has an impact on the ranking order, as demonstrated by the sensitivity analysis. The findings of the sensitivity analysis are displayed in Table 13.

9. **Conclusion**

This work provided the MULTIMOORA and FUCOM algorithms in a Pythagorean neutrosophic fuzzy environment. PNFNs are used to represent each alternative's characteristics. The safest and most advantageous RE source in the current environment has been determined to be solar energy technology, which is derived from the suggested strategy for RES problem solving. The findings from this study have the potential to significantly assist policymakers and stakeholders in pinpointing the most suitable Renewable Energy Technology (RET) for India's power sector. By identifying the optimal RET, it enables the formulation of strategic plans aimed at fostering sustainable development within the country. The framework developed through this research distinguishes itself by leveraging expert opinions to ascertain the best RET option, enhancing the decision-making process. Additionally, by broadening the range of criteria considered and adopting objective methods for assigning weights to these criteria, the precision and reliability of the results have been substantially improved. This approach not only enriches the robustness of the decision-making process but also ensures that the selected RET solutions are aligned with Indias unique energy needs and sustainability goals, thereby contributing to a more sustainable and efficient energy future for the country. This process helps produce green energy, which will assist in addressing future problems related to energy demand.

References

1. https://www.un.org/en/climatechange/what-is-renewable-energy.
2. K. Tromly, Renewable energy: An overview. 2001.
3. Canadian environmental sustainability indicators, Global greenhouse gas emissions. Available at http://www.ec.gc.ca/indicateurs-indicators/54C061B5-44F7-4A93-A3EC-5F8B253A7235/GlobalGHGEmissions_EN.pdf. Accessed 27 June.2017.
4. D. Pappas, Energy and Industrial Growth in India: The Next Emissions Superpower? *Energy procedia* **105** (2017) 3656-3662.
5. J. Kumar and M.A. Majid, Renewable energy for sustainable development in India: current status, future prospects, challenges, employment, and investment opportunities, *Energy, Sustainability and Society* **10** (2020) 1-36.
6. World Energy Scenarios Composing energy futures to 2050, World energy Council. (2013) https://www.worldenergy.org/wp-content/uploads/2013/09/World-Energy-Scenarios_Composing-energy-futures-to-2050_Full-report. pdf .Accessed 01 Jan 2017.
7. M. Blondeel, T. Van de Graaf, Toward a global coal mining moratorium? A comparative analysis of coal mining policies in the USA, China, India and Australia. *Climatic Change* **150(1-2)** (2018) 89-101
8. F. Smarandache Neutrosophy/neutrosophic probability, set and logic. *American Research Press, Rehoboth* (1998).
9. R. R. Yager, Pythagorean fuzzy subsets. In Proceedings of the Joint IFSAWorld Congress and NAFIPS Annual Meeting, Edmonton, AB, Canada, June 2428 (2013) 25761.
10. R. R. Yager and A. M. Abbasov, Pythagorean membership grades, complex numbers, and decision making. *International Journal of Intelligent Systems* **28(5)** (2013) 436-452.
11. R. R. Yager, Pythagorean membership grades in multi-criteria decision making. *IEEE Transactions on Fuzzy Systems* **22(4)** (2013) 958-965.
12. X. Zhang and Z. Xu, Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *International Journal of Intelligent Systems*, **29(12)** (2014) 1061-1078.
13. Pamuar, D., Stevi, ., and Sremac, S. (2018). A new model for determining weight coefficients of criteria in mcdm models: Full consistency method (fucom). *Symmetry*, **10(9)**, 393.
14. Pamucar, D., and Ecer, F. (2020). Prioritizing the weights of the evaluation criteria under fuzziness: The fuzzy full consistency methodFUCOM-F. Facta Universitatis, Series: *Mechanical Engineering*, **18(3)**, 419-437.
15. Pamucar, D., Ecer, F., and Deveci, M. (2021). Assessment of alternative fuel vehicles for sustainable road transportation of United States using integrated fuzzy FUCOM and neutrosophic fuzzy MARCOS methodology. *Science of the Total Environment*, **788**, 147763.
16. Durmi, E., Stevi, ., Chatterjee, P., Vasiljevi, M., and Tomaevi, M. (2020). Sustainable supplier selection using combined FUCOMRough SAW model. *Reports in mechanical engineering*, **1(1)**, 34-43.
17. Demir, G., Damjanovi, M., Matovi, B., and Vujadinovi, R. (2022). Toward sustainable urban mobility by using fuzzy-FUCOM and fuzzy-CoCoSo methods: the case of the SUMP podgorica. *Sustainability*, **14(9)**, 4972.

18. Saha, A., Mishra, A. R., Rani, P., Hezam, I. M., and Cavallaro, F. (2022). A q-rung orthopair fuzzy FUCOM double normalization-based multi-aggregation method for healthcare waste treatment method selection. *Sustainability*, *14(7)*, 4171.
19. Hafezalkotob, A., Hafezalkotob, A., Liao, H., and Herrera, F. (2019). An overview of MULTIMOORA for multi-criteria decision-making: Theory, developments, applications, and challenges. *Information Fusion*, **51**, 145-177.
20. Baleentis, T., and Baleentis, A. (2013). A Survey on Development and Applications of the Multi-criteria Decision Making Method MULTIMOORA. *Journal of Multi-Criteria Decision Analysis*, **21(3-4)**, 209222.
21. Stanujkic, D., Zavadskas, E. K., Smarandache, F., Brauers, W. K., and Karabasevic, D. (2017). A neutrosophic extension of the MULTIMOORA method. *Informatica*, **28(1)**, 181-192.
22. Huang, Chao, Mingwei Lin, and Zeshui Xu. "Pythagorean fuzzy MULTIMOORA method based on distance measure and score function: its application in multicriteria decision making process." *Knowledge and Information Systems* **62** (2020): 4373-4406.
23. Rani, P., and Mishra, A. R. (2021). Fermatean fuzzy Einstein aggregation operators-based MULTIMOORA method for electric vehicle charging station selection. *Expert Systems with Applications*, **182**, 115267.
24. Liu, P., Gao, H., and Fujita, H. (2021). The new extension of the MULTIMOORA method for sustainable supplier selection with intuitionistic linguistic rough numbers. *Applied Soft Computing*, **99**, 106893.
25. Tian, C., Peng, J. J., Zhang, Z. Q., Wang, J. Q., and Goh, M. (2022). An extended picture fuzzy MULTIMOORA method based on SchweizerSklar aggregation operators. *Soft Computing*, 1-20.
26. Kutlu Gndodu, F. (2020). A spherical fuzzy extension of MULTIMOORA method. *Journal of Intelligent & Fuzzy Systems*, **38(1)**, 963-978.
27. B. Hussain, S. A. A Naqvi, S. Anwar, and M. Usman, Effect of wind and solar energy production, and economic development on the environmental quality: Is this the solution to climate change? *Gondwana Research*, **119** (2023) 27-44.
28. M. Jahangiri, A. A. Shamsabadi, A. Mostafaeipour, M. Rezaei, Y. Yousefi, and L. M. Pomares Using fuzzy MCDM technique to find the best location in Qatar for exploiting wind and solar energy to generate hydrogen and electricity *International Journal of Hydrogen Energy* **45(27)** (2020) 13862-13875.
29. H. Aljaghoub, F. Abumadi, M. N. AlMallahi, K. Obaideen and A. H. Alami, Solar PV cleaning techniques contribute to Sustainable Development Goals (SDGs) using Multi-criteria decision-making (MCDM): Assessment and review. *International Journal of Thermofluids* **16** (2022) 100233.
30. S. K. Saraswat, A. K. Digalwar, S. S. Yadav and G. Kumar, MCDM and GIS based modelling technique for assessment of solar and wind farm locations in India. *Renewable Energy* **169** (2021) 865-884.
31. S. S. Goswami, S. K. Mohanty and D. K. Behera, Selection of a green renewable energy source in India with the help of MEREC integrated PIV MCDM tool. *Materials today: proceedings* **52** (2022) 1153-1160.
32. T. Li, A. Li and X. Guo, The sustainable development-oriented development and utilization of renewable energy industry - a comprehensive analysis of MCDM methods. *Energy*, **212** (2020) 118694.
33. H. Kaur, S. Gupta and A. Dhingra, Selection of solar panel using entropy TOPSIS technique. *Materials Today: Proceedings* (2023).
34. Khalifa, H. A. E. W., Kumar, P., and Mirjalili, S. (2021). A KKM approach for inverse capacitated transportation problem in neutrosophic environment. Sdhan, **46(3)**, 166.
35. Priyadharshini, S. P., and Irudayam, F. N. (2023). An analysis of obesity in school children during the pandemic COVID-19 using plithogenic single valued fuzzy sets. *Neutrosophic Systems with Applications*, **9**, 24-28.
36. Mohamed, M., Karam M. Sallam, and Ali Wagdy Mohamed. (2023). Transition Supply Chain 4.0 to Supply Chain 5.0: Innovations of Industry 5.0 Technologies Toward Smart Supply Chain Partners. *Neutrosophic Systems With Applications*, **10**, 111.

37. Jdid, M., and Smarandache, F. (2023). An Efficient Optimal Solution Method for Neutrosophic Transport Models: Analysis, Improvements, and Examples. *Infinite Study.*
38. Khalifa, H. A. E. W., Kumar, P., and Smarandache, F. (2020). On optimizing neutrosophic complex programming using lexicographic order. *Infinite Study.*
39. R. Alizadeh, L. Soltanisehat, P. D. Lund and H. Zamanisabzi, Improving renewable energy policy planning and decision-making through a hybrid MCDM method. *Energy Policy* **137** (2020) 111174.
40. Kumar, M., and Samuel, C. (2017). Selection of best renewable energy source by using VIKOR method. *Technology and Economics of Smart Grids and Sustainable Energy*, **2**, 1-10.
41. Van Thanh, N. (2022). Sustainable energy source selection for industrial complex in Vietnam: A Fuzzy MCDM Approach. *IEEE Access*, **10**, 50692-50701.
42. Goswami, S. S., Mohanty, S. K., and Behera, D. K. (2022). Selection of a green renewable energy source in India with the help of MEREC integrated PIV MCDM tool. *Materials today: proceedings*, **52**, 1153-1160.
43. Saraswat, S. K., and Digalwar, A. K. (2021). Evaluation of energy sources based on sustainability factors using integrated fuzzy MCDM approach. *International Journal of Energy Sector Management*, **15(1)**, 246-266.
44. Wang, C. N., Kao, J. C., Wang, Y. H., Nguyen, V. T., Nguyen, V. T., and Husain, S. T. (2021). A multi-criteria decision-making model for the selection of suitable renewable energy sources. *Mathematics*, **9(12)**, 1318.
45. Sarkodie, W. O., Ofosu, E. A., and Ampimah, B. C. (2022). Decision optimization techniques for evaluating renewable energy resources for power generation in Ghana: MCDM approach. *Energy Reports*, **8**, 13504-13513.
46. Alkan, ., and Albayrak, . K. (2020). Ranking of renewable energy sources for regions in Turkey by fuzzy entropy based fuzzy COPRAS and fuzzy MULTIMOORA. *Renewable Energy*, **162**, 712-726.
47. W. K. Brauers, A. Baleentis, T. Baleentis, MULTIMOORA for the EU Member States updated with fuzzy number theory. *Technological and Economic Development of Economy* **17(2)** (2011) 259-290.
48. H. Fazlollahtabar, A. Smailbai and Z. Stevi, FUCOM method in group decision-making: Selection of forklift in a warehouse. *Decision Making: Applications in Management and Engineering* **2(1)** (2019) 49-65.
49. Q. Cao, M. O. Esangbedo, S. Bai and C. O. Esangbedo, Grey SWARA-FUCOM weighting method for contractor selection MCDM problem: A case study of floating solar panel energy system installation. *Energies* **12(13)** (2019) 2481.

Received: Feb 10, 2024. Accepted: April 29, 2024

Neutrosophic Sets and Systems, Vol. 67, 2024
University of New Mexico

Neutrosophic Statistics for Enhanced Time Series Analysis of Unemployment Trends in Ecuador

Stephanie M. Delgado Estrada [1*], Katia del Rocio Ruiz Molina [2], Fernando Enrique Ponce Orellana [3], Jorge Luis Chabusa Vargas [4]

[1] Universidad de Guayaquil, Guayas, Ecuador, Email: stephanie.delgadoe@ug.edu.ec*
[2] Universidad de Guayaquil, Guayas, Ecuador, Email: katia.ruizmo@ug.edu.ec
[3] Universidad de Guayaquil, Guayas, Ecuador, Email: fernando.ponceor@ug.edu.ec
[4] Universidad de Guayaquil, Guayas, Ecuador, Email: jorge.chabusav@ug.edu.ec

Abstract. This study harnesses advanced time series models ARIMA, ETS, and SARIMA, coupled with neutrosophic statistics, to forecast unemployment trends through interval-based predictions. Transforming these predictions into neutrosophic forms enables the quantification of indeterminacy, providing a nuanced interpretation of potential economic scenarios. The integration of neutrosophic statistics enhances the interpretative power and accuracy of these models, offering a deeper insight into the inherent uncertainties of economic forecasting. The approach reveals not only the variabilities and potential outcomes within the unemployment rates but also strengthens the decision-making processes by presenting data that encompass both precision and indeterminacy. This paper underscores the importance of advanced statistical methods in economic predictions, suggesting further exploration into other economic metrics and advocating for a broader application of neutrosophic statistics to enhance the reliability of economic forecasting across diverse contexts.

Keywords: Neutrosophic Statistics, Time Series Forecasting, Economic Forecasting, Uncertainty Quantification.

1 Introduction

Unemployment stands as a crucial economic issue, receiving extensive attention in scholarly discussions. It is defined as a condition where individuals actively seeking employment for more than three months remain unable to secure a job. The unemployment rate quantifies this by expressing the proportion of unemployed individuals relative to the total labor force. This economic issue not only undermines living standards but also contributes to rising crime rates and insecurity, particularly noted in places like Ecuador [1].

Time series data, characterized by the dependency of successive observations, supports the analysis of such phenomena by acknowledging the sequence in which data appears. Time series modeling is widely applied in various fields like sales, meteorology, and inventory management, proving critical in scenarios involving uncertainty about the future. These models are particularly effective in forecasting [2].

Predicting time series data, especially when expressed in intervals rather than specific numbers, allows for more accurate yet uncertain outcomes [3]. This method falls under Neutrosophic Statistics, which incorporates interval data into statistical predictions. Enhancing the precision of these forecasts involves integrating multiple models, a growing and significant research area, especially applied to predicting unemployment rates for the future [4].

Unemployment remains a critical economic issue that garners extensive attention in scholarly discussions, largely due to its profound impact on societal welfare and economic stability. Defined as the condition in which individuals who are actively seeking employment for more than three months remain jobless, the unemployment rate quantifies this phenomenon, reflecting the proportion of the unemployed within the total labor force. The ramifications of high unemployment are severe, undermining living standards and contributing to increased crime rates and insecurity, with notable effects observed in regions like Ecuador [5]. Utilizing time series data, which acknowledges the sequential dependency of observations, enhances the analysis of such phenomena. Time series modeling, widely applied across various domains such as sales, meteorology, and inventory management, proves indispensable in scenarios laden with future uncertainties.

This paper explores the application of predictive models enhanced by neutrosophic statistics to project unemployment rates, offering a novel approach in the realm of economic forecasting by integrating interval-based predictions with a neutrosophic framework to manage and interpret uncertainty more effectively.

2 Preliminaries

2.1 Time series and Neutrosophic Statistics

Time series analysis involves the examination of data points arranged in chronological order. This methodology is crucial for understanding trends, cycles, and seasonal variations inherent in various datasets across time. Time series can be represented as [6]:

$$Y_t = f(t) + \varepsilon_t \quad (1)$$

Where:
Y_t is the is the value of the series at time t,
$f(t)$ represents the deterministic components like trends or seasonal effects,
ε_t is the random error component.
To predict future values of a time series and address the uncertainty of these predictions, interval forecasting is used. Interval predictions provide a range (interval) within which future observations are expected to fall, rather than pinpointing a single value. This can be formulated in the equation editor as [7]:

$$\hat{Y}_{(t+h|t)} = \hat{f}(t+h) \pm z * \hat{\sigma} \quad (2)$$

Where:
$\hat{Y}_{(t+h|t)} =$ is the predicted value at time $t+h$, based on the information up to time t,
$\hat{f}(t+h)$ is the predicted deterministic component,
z is the z-score from the normal distribution corresponding to the desired confidence level,
$\hat{\sigma}$ is the estimated standard deviation of the forecast errors.

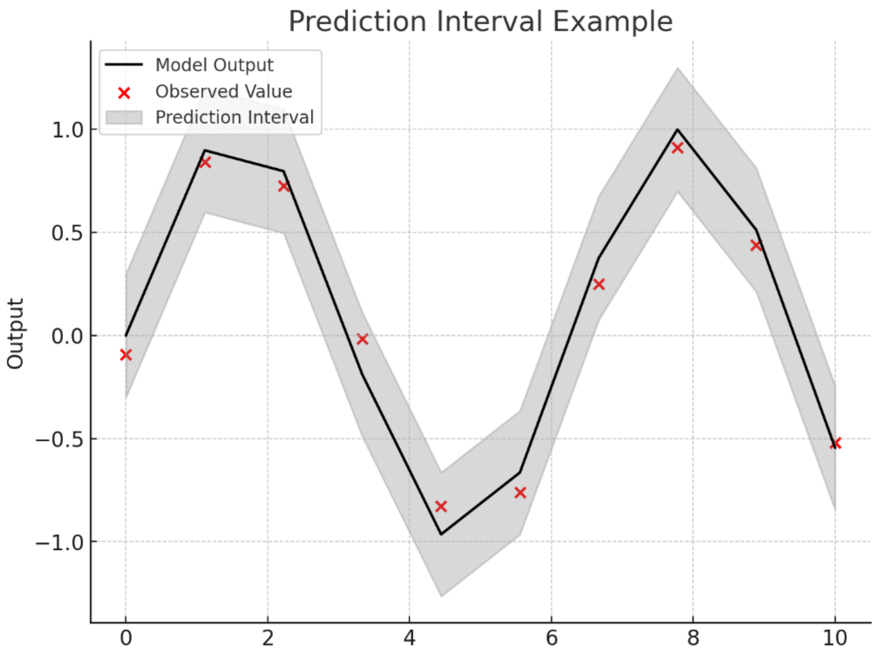


Figure 1. Components of prediction intervals around model outputs [8].

This approach to forecasting accommodates the inherent uncertainty in future predictions and provides a more realistic representation of the expected outcomes, making it invaluable in fields such as economics, finance, and environmental science.
To enhance this model using neutrosophic statistics, which allows for handling data ambiguity and indeterminacy more effectively, the interval can be transformed into a neutrosophic number. This transformation involves expanding the classical interval to include an indeterminacy component, reflecting the uncertainty and imprecision inherent in real-world data. The neutrosophic treatment of the interval is as follows [9]:

$\hat{f}(t+h) - z * \hat{\sigma} \times (\hat{f}(t+h) + z) \cdot I_N$ (3)

Here, I_N represents the indeterminacy factor associated with the prediction, where $I_N \in [I_l, I_u]$, This notation introduces the bounds of indeterminacy.

I_l (lower indeterminacy) and I_u (upper indeterminacy), which define the range of possible deviations due to uncertain elements affecting the forecast.

This refined representation not only aligns with the principles of neutrosophic statistics but also provides a more nuanced and realistic portrayal of the uncertainties inherent in time series forecasting. It makes the prediction interval more robust and informative, particularly useful in scenarios where data ambiguity and fuzziness are prevalent. This approach is instrumental for researchers and practitioners in fields where data quality and precision are variable and often not strictly deterministic.

2 Material and Methods

The analysis presented in this article is based on data sourced from the National Survey of Employment, Unemployment, and Underemployment (ENEMDU), provided by the National Institute of Statistics and Censuses (INEC). This study integrates advanced time series models to enhance the accuracy and understanding of labor trends. Specifically, it combines the established models such as ARIMA[10], the Exponential Smoothing Model (ETS)[11] , and SARIMA (Seasonal ARIMA)[12]. The fusion of these methods is achieved through the use of neutrosophic means, an approach that adeptly handles the uncertainty and indetermination inherent in predictions. This integration not only enhances the robustness of the predictive models but also offers a more profound framework for interpreting the complex dynamics of the labor market[13,14].

The neutrosophic mean, denoted as X_n, is calculated by considering the neutrosophic inclusion $I_N$that belongs to the interval $[I_l, I_u]$. This mean consists of two main elements: X_l, which is the mean of the lower part of the neutrosophic samples, and X_u, which is the mean of the upper part. The respective definitions are:

$$X_l = \frac{\sum_{i=1}^{n_l} x_{il}1}{n_l} \quad (4)$$

$$X_u = \frac{\sum_{i=1}^{n_u} x_{iu}}{n_u} \quad (5)$$

where n_l and $n_u$represent the number of elements in the lower and upper parts of the neutrosophic samples, respectively. Therefore, the neutrosophic mean X_n, is expressed as the sum of X_l and X_u, adjusted by the interval of indetermination I_n:

$$X_N = X_l + X_u I_N;\ I_N \in [I_l, I_u] \quad (6)$$

$I_{l,}$=0, and I_u

$$I_u = \frac{X_u - X_l}{X_u} \quad (7)$$

3 Results

we applied the three methods—ARIMA, ETS, and SARIMA—to the dataset, yielding interval-based predictions. These intervals were then transformed into neutrosophic forms, which effectively capture the measures of indeterminacy associated with each prediction. This transformation allows for a nuanced interpretation of the data, emphasizing the inherent uncertainties within the labor market forecasts. For detailed interval and indeterminacy values, refer to Table 1. This table showcases how each model's predictions are expressed in neutrosophic terms, providing a clear depiction of the range and reliability of the forecasts.

	First Quarter 2024	Second Quarter 2024	Third quarter 2024
ARIMA	[2.97, 4.25]	[2.72, 4.49]	[2.52, 4.69]
ETS	[3.05, 3.72]	[2.91, 3.58]	[2.78, 3.45]
SARIMA	[2.87, 3.30]	[3.16, 3.61]	[2.54, 3.01]
Media	[2.96, 3.76]	[2.93,3.89]	[2.61,3.72
Neutrosophic forms	2.96+3.76I; $I \in$ [0,0.21]	2.93+3.89I; $I \in$ [0,0.247]	2.61+3.72I; $I \in$ [0,0.298]

Table 1: Prediction Intervals and Neutrosophic Forms for Time Series Models.

Table 1 presents the prediction intervals and corresponding neutrosophic forms for three different time series models: ARIMA, ETS, and SARIMA, across three future quarters of 2024. These intervals reflect the range of potential outcomes, illustrating the inherent uncertainty in our predictions. The neutrosophic forms further quantify this uncertainty, highlighting the degree of indeterminacy associated with each interval. This representation allows for a more nuanced understanding of the forecasted data, facilitating more informed decision-making in uncertain conditions.

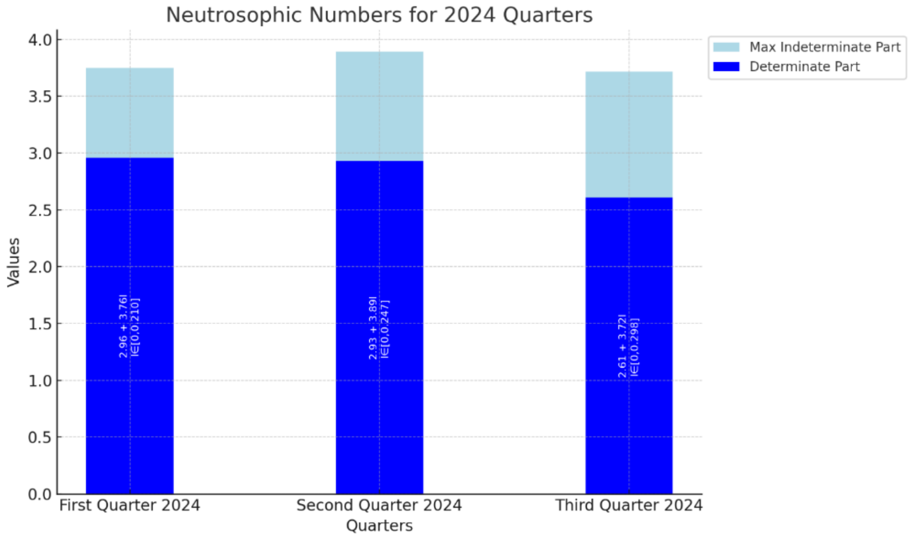


Figure 2. Neutrosophic Number Representation

Figure 2 illustrates the neutrosophic representation of prediction intervals for the first three quarters of 2024, clearly depicting an increasing trend in uncertainty as the year progresses. This rise in indeterminacy, captured through the enlarging segments of the neutrosophic intervals, is crucial for understanding the dynamics at play within the environmental data being analyzed.

The increase in the indeterminate component of the neutrosophic numbers suggests a growing complexity or variability in the underlying data as the year advances. This could be attributed to several factors, including seasonal variations, changes in data collection methodologies, or external economic or environmental impacts that become more pronounced over time.

From a decision-making perspective, this trend underscores the need for adaptive strategies that can accommodate an expanding range of outcomes. It also highlights the importance of continuous monitoring and updating of predictive models to better align with the evolving data landscape, ensuring that decision-making remains robust in the face of increasing uncertainty.

Such observations not only validate the utility of incorporating neutrosophic statistics in the analysis of time series data but also emphasize the critical role of these techniques in enhancing our comprehension of uncertainty in predictive modeling.

Conclusion

This study has demonstrated the effective integration of ARIMA, ETS, and SARIMA models, enhanced by neutrosophic statistics, to forecast unemployment trends through interval-based predictions. By transforming these intervals into neutrosophic forms, we were able to capture and quantify the underlying indeterminacy, providing a more nuanced view of the predictive landscape. This approach not only addresses the uncertainties inherent in economic forecasting but also enhances the interpretative power of the results, offering a more comprehensive understanding of potential future scenarios. The utilization of neutrosophic statistics has proven instrumental in refining the precision of these predictions, allowing for a detailed depiction of uncertainty and variability that traditional models might overlook.
Given the promising results observed in this study, future research could explore the extension of neutrosophic statistics to other areas of economic forecasting, such as inflation rates, GDP growth, or market volatility. Additionally, further refinement of the neutrosophic models to include more dynamic elements of indeterminacy could provide even greater accuracy and reliability in predictions. It would also be beneficial to conduct comparative

studies across different economies to validate the effectiveness of neutrosophic statistics in diverse economic contexts. Such investigations would not only enhance our understanding of neutrosophic methods but also potentially lead to the development of standardized approaches for handling uncertainty in economic time series analysis.

References

[1]. Tulcanaza-Prieto, A. B., Báez Salazar, P. A., & Aguilar-Rodríguez, I. E. (2023). Determinants of Youth Unemployment in Ecuador in 2019. Economies, 11(2), 59.

[2]. Cerqueira, V., Torgo, L., & Mozetič, I. (2020). Evaluating time series forecasting models: An empirical study on performance estimation methods. Machine Learning, 109(11), 1997-2028.

[3]. Cruzaty, L. E. V., Tomalá, M. R., & Gallo, C. M. C. (2020). A neutrosophic statistic method to predict tax time series in Ecuador. Neutrosophic Sets and Systems, 34, 33-39.

[4]. Zhongda, T., Shujiang, L., Yanhong, W., & Yi, S. (2017). A prediction method based on wavelet transform and multiple models fusion for chaotic time series. Chaos, Solitons & Fractals, 98, 158-172.

[5]. Munguía-Cortés, J. A., & Torres-Valdovinos, M. M. (2018). Social representations about employment and unemployment of university students from Quito, Ecuador. Revista Reflexiones, 97(2), 7-22.

[6]. Kočenda, E., & Černý, A. (2015). Elements of time series econometrics: An applied approach. Charles University in Prague, Karolinum Press.

[7]. Chatfield, C. (2001). Prediction intervals for time-series forecasting. Principles of forecasting: A handbook for researchers and practitioners, 475-494.

[8]. Shrestha, D. L., & Solomatine, D. P. (2006). Machine learning approaches for estimation of prediction interval for the model output. Neural networks, 19(2), 225-235.

[9]. Smarandache, F., & Aslam, M. (Eds.). (2023). Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics. Elsevier.

[10]. Khan, S., & Alghulaiakh, H. (2020). ARIMA model for accurate time series stocks forecasting. International Journal of Advanced Computer Science and Applications, 11(7).

[11]. Seong, B. (2020). Smoothing and forecasting mixed-frequency time series with vector exponential smoothing models. Economic Modelling, 91, 463-468.

[12]. Dimri, T., Ahmad, S., & Sharif, M. (2020). Time series analysis of climate variables using seasonal ARIMA approach. Journal of Earth System Science, 129, 1-16.

[13]. Saqlain, M., Garg, H., Kumam, P., & Kumam, W. (2023). Uncertainty and decision-making with multi-polar interval-valued neutrosophic hypersoft set: A distance, similarity measure and machine learning approach. Alexandria Engineering Journal, 84, 323-332.

[14]. Abdullah, W. (2024). A Study of Neutrosophic Sets and Deep Learning Models for Breast Cancer Classification. Multicriteria Algorithms with Applications, 3, 50-59.

Received: Feb 12, 2024. Accepted: April 28, 2024

University of New Mexico

Cubic Spherical Neutrosophic Sets and Selection of Electric Truck Using Cosine Similarity Measure

S. Krishnaprakash[1*] **R. Mariappan**[2], **and Said Broumi**[3]

[1] Department of Mathematics, Sri Krishna College of Engineering and Technology, Coimbatore, Tamilnadu, India - 641 008; mskrishnaprakash@gmail.com

[2]Department of Mathematics, Karpagam College of Engineering, Coimbatore, Tamilnadu, India - 641 032; mariappanbagavathi@gmail.com

[3] Laboratory of Information Processing, Faculty of Science Ben MSik, University Hassan II, Casablanca, Morocco; broumisaid78@gmail.com

[*]Correspondence: mskrishnaprakash@gmail.com

Abstract. The concepts of cubic spherical neutrosophic sets (CSNSs), introduced and investigated by Gomathi et al. [5], offer a geometric representation of collection of neutrosophic sets (NSs), enhancing their ability to capture uncertainty. The formulation characterizes information using points on a sphere with a defined center and radius, providing a more precise depiction of fuzziness inherent in uncertain data. The cubic spherical neutrosophic Archimedean triangular norms(ATN) and conorms (ATCN), expanding the model's capabilities to handle uncertainty. These algebraic operators enable the aggregation and combination of uncertain information, offering a more comprehensive approach to decision-making. The research further presents a method for solving multiple-criteria decision-making problems within the cubic spherical neutrosophic context, leveraging the newly integrated norms and conorms. The algorithm utilizes the cosine similarity measure of cubic spherical neutrosophic sets, exemplified through an application involving the selection of the most effective electric truck. This extended framework provides decision-makers with enhanced tools to navigate complex decision landscapes amidst uncertainty, facilitating more informed and robust choices across diverse domains.

Mathematics Subject Classification: 03E72, 91B06

Keywords: neutrosophic sets; cubic spherical neutrosophic sets; extension of neutrosophic sets; neutrosophic Archimedean triangular norms and conorms.

---

1. Introduction

In 1998, Smarandache [27] early proposed NSs, which are a generalization of fuzzy sets and intuitionistic fuzzy sets. The membership, indeterminacy and non-membership mappings used to describe NSs are such that the sum of their non-negative mapping values is less than three. NSs are frequently utilized in several fields, resulting in decision-making, clustering algorithms, distance measurement, entropy measurement, pattern recognition and medical diagnostics. As a generalization of NSs, numerous sets are Proposed including single-valued NSs [29], interval-valued NSs [28], neutrosophic hesitant FSs [33], bipolar NSs [3], spherical NSs [16], simplified NSs [35], multi-valued NSs [16], and probability multi-valued NSs [21]. In 1954, Menger [15] proposed the first description of TNs, and then Schweizer and Sklar [24,25] revised them to become those that are currently in use. Several investigations [2, 7, 11–13, 36] study the properties of above said norms including continuous, nilpotent, Archimedean, stringent and others.

The area of operations research that is categorized as $MCDM$ focuses on the explicit evaluation of multiple conflicting criteria while making decisions (in daily life and situations like businesses, governments and medicals). Contradictory standards are frequently present while examining possibilities. The cost of the truck is typically among the main criteria and the quality measure is commonly another, simply compared to the cost. When purchasing a truck, we may prioritize cost, towing capacity, loading capability, security and fuel consumption. Typically, the truck with the lowest price also has the maximum towing and loading capabilities. Whenever managing a portfolio, managers want to maximize profits while minimizing risks; yet, the stocks with the highest return potential often have the highest risk of dropping money. Customer happiness and service costs are fundamentally opposing factors in the service sector. People frequently implicitly consider several factors when making daily decisions and they may be satisfied with the results of those judgments if they are solely based on intuition. On the other hand, it's crucial to properly outline the problem when the stakes are high.

Research Gap and Motivation

The study of cubic spherical neutrosophic sets and their associated arithmetic operators present a novel avenue for handling uncertainty and indeterminacy in decision-making processes. However, despite its potential, there remains a notable research gap and several motivating factors for further exploration:

- **Lack of Comprehensive Frameworks:** Existing research on neutrosophic sets and their operators primarily focuses on conventional models, often overlooking the complexities inherent in real-world decision-making scenarios. The introduction of CSNS and its arithmetic operators offers a more comprehensive framework for addressing uncertainty, yet further exploration is needed to fully understand its implications and applicability across diverse domains.

- **Limited Applications and Case Studies:** While the concept of CSNS shows promise, there is a scarcity of practical applications and case studies demonstrating its effectiveness in real-world contexts. The absence of empirical validation hinders the wider adoption and understanding of CSNS-based methodologies, highlighting the need for empirical studies and practical implementations.
- **Potential for Methodological Enhancements:** The development of CSNS-based arithmetic operators opens avenues for further methodological enhancements and refinements. Exploring alternative aggregation techniques, refining parameter estimation methodologies and investigating the scalability of CSNS-based models are areas ripe for exploration and innovation.

The motivation behind this study stems from the need to address the challenges posed by uncertainty and indeterminacy in decision-making processes. Traditional decision-making models often struggle to accommodate the complexities and nuances inherent in real-world scenarios, leading to suboptimal outcomes and missed opportunities. The introduction of CSNS and its associated arithmetic operators offers a promising avenue for overcoming these challenges.

The motivation for studying CSNS lies in its potential to provide a more comprehensive and nuanced representation of uncertain information. By incorporating a spherical representation with a radius r and a triple at its center, CSNS allows decision-makers to capture degrees of membership, indeterminacy and non-participation in a more intuitive and meaningful manner. This in turn, facilitates more informed and robust decision-making processes across various domains.

Contribution

The study makes several significant contributions to the field of decision-making under uncertainty and indeterminacy:

- **Introduction of Cubic Spherical Neutrosophic Sets (CSNS):** The study introduces CSNS as a novel framework for representing uncertainty and indeterminacy in decision-making processes. By extending the concept of Neutrosophic Sets (NS) to include a spherical representation with a radius r and a triple at its center, CSNS offers a more comprehensive and nuanced approach to modeling uncertain information.
- **Development of Arithmetic Operators:** The research proposes two new arithmetic operators specifically tailored for CSNS: Weighted Arithmetic Cubic Spherical Neutrosophic Aggregation Operators and Weighted Geometric Cubic Spherical Neutrosophic Aggregation Operators. These operators address limitations of existing neutrosophic operators and provide more reliable and effective aggregation techniques for handling uncertain data.

- **Methodological Advancement in MCDM:** The study presents an innovative Multiple-Criteria Decision-Making (MCDM) method for selecting the best electric truck based on CSNS and its arithmetic operators. By leveraging CSNS-based aggregation techniques, the proposed method offers a systematic and robust approach to decision-making in complex, uncertain environments.
- **Practical Implications and Future Directions:** The research not only advances theoretical understanding but also holds practical implications for various domains. The introduction of CSNS and its associated arithmetic operators has the potential to enhance decision-making processes in diverse fields such as engineering, finance, healthcare and environmental management. Furthermore, the study opens avenues for future research, including empirical validation, comparative analysis with existing methodologies and exploration of alternative arithmetic operators.

This study presents the concept of algebraic operations between $CSNSs$ using TNs and $TCNs$. Moreover, some weighted aggregation operators that transform input values represented by $CSNVs$ to a single output value using these algebraic operations are proposed. Finally, a cubic spherical CSM depending on the radius is given for evaluating the level of similarity between $CSNVs$. In addition, we propose a technique for converting a set of NVs into a $CSNVs$.

2. **Preliminaries**

Definition 2.1. [8] A mapping $\Gamma : [0,1] \times [0,1] \rightarrow [0,1]$ is a neutrosophic TN if it satisfies the following:
(i) $\Gamma(\alpha, (1,0,0)) = \alpha$ and $\Gamma(\alpha, (0,1,1)) = 0$,
(ii) $\Gamma(\alpha, \beta) = \Gamma(\beta, \alpha)$,
(iii) $\Gamma(\alpha, \Gamma(\beta, \gamma)) = \Gamma(\beta, \Gamma(\alpha, \gamma))$,
(iv) $\Gamma(\alpha, \beta) \leq \Gamma(\alpha', \beta')$ where $\alpha \leq \alpha'$ and $\beta \leq \beta'$,
for all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2) \in [0,1]$.

Definition 2.2. [8] A mapping $\Gamma^* : [0,1] \times [0,1] \rightarrow [0,1]$ is a neutrosophic TCN if it satisfies the following:
(i) $\Gamma^*(\alpha, (0,1,1)) = \alpha$ and $\Gamma^*(\alpha, (1,0,0)) = 0$,
(ii) $\Gamma^*(\alpha, \beta) = \Gamma^*(\beta, \alpha)$,
(iii) $\Gamma^*(\alpha, \Gamma^*(\beta, \gamma)) = \Gamma^*(\beta, \Gamma^*(\alpha, \gamma))$,
(iv) $\Gamma^*(\alpha, \beta) \leq \Gamma^*(\alpha', \beta')$ where $\alpha \leq \alpha'$ and $\beta \leq \beta'$,
for all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2) \in [0,1]$.

Definition 2.3. [9] A monotonically strictly decreasing mapping $* : [0,1] \to [0,\infty)$ defined by $*(1) = 0$ is called an AFG of a t-norm Γ if $\Gamma(\alpha,\beta) = *^{-1}(*(\alpha) + *(\beta))$ for any $(\alpha,\beta) \in [0,1] \times [0,1]$.

Definition 2.4. [9,14] If Γ (resp. Γ^*) is neutrosophic TN (resp. TCN) on $[0,1]$ is said to be dual with respect to $\mathcal{S}$, if $\Gamma(\alpha,\beta) = \mathcal{S}(\Gamma^*(\mathcal{S}(\alpha),\mathcal{S}(\beta)))$ (resp. $\Gamma^*(\alpha,\beta) = \mathcal{S}(\Gamma(\mathcal{S}(\alpha),\mathcal{S}(\beta)))$) for any $\alpha,\beta \in [0,1]$.

Definition 2.5. [8,18] If Γ (resp. Γ^*) is neutrosophic TN (resp. TCN) on $[0,1]$, then the dual t-conorm $\mathcal{T}^*$ is defined as $\mathcal{T}^*(\alpha,\beta) = 1 - \Gamma(1-\alpha, 1-\beta)$, for any $\alpha,\beta \in [0,1]$.
Clearly Γ is an ATN if it is continuous and $\Gamma(\alpha,\alpha) < \alpha$ for all $\alpha \in (0,1)$ and Γ^* is an $ATCN$ if it is continuous $\Gamma^*(\alpha,\alpha) > \alpha$ for any $\alpha \in (0,1)$.

Definition 2.6. [8] A mapping $\mathcal{N} : [0,1] \times [0,1] \to [0,1]$ is a neutrosophic negator, if it satisfies three conditions:
(i) $\mathcal{N}(\alpha,(0,1,1)) = 1$ for any $\alpha \in [0,1]$,
(ii) $\mathcal{N}(\alpha,(1,0,0)) = 0$ for any $\alpha \in [0,1]$,
(iii) $\mathcal{N}(\alpha,\beta) \geq \mathcal{N}(\alpha',\beta')$ where $\alpha \leq \alpha'$ and $\beta \leq \beta'$.

Definition 2.7. [5] A cubic spherical neutrosophic set $(CSNS)$ $\mathbb{C}_{csR}$ in $\mathbb{X}$ is defined by $\mathbb{C}_{csR} = \{< x, \frac{cs\mathbb{T}_C}{x}, \frac{cs\mathbb{I}_C}{x}, \frac{cs\mathbb{F}_C}{x}; csR >: x \in \mathbb{X}\}$, where $cs\mathbb{T}_C, cs\mathbb{I}_C, cs\mathbb{F}_C : \mathbb{X} \to [0,1]$ are mappings satisfies the condition $cs\mathbb{T}_C + cs\mathbb{I}_C + cs\mathbb{F}_C \leq 3$ and $csR > 0$ denote the radius of the sphere centred at the point $(\frac{cs\mathbb{T}_C}{x}, \frac{cs\mathbb{I}_C}{x}, \frac{cs\mathbb{F}_C}{x})$ in the cube.
If a collection $\{< cs\mathbb{T}_{\epsilon,1}, cs\mathbb{I}_{\epsilon,1}, cs\mathbb{F}_{\epsilon,1} >, < cs\mathbb{T}_{\epsilon,2}, cs\mathbb{I}_{\epsilon,2}, cs\mathbb{F}_{\epsilon,2} >, ..., < cs\mathbb{T}_{\epsilon,k_\epsilon}, cs\mathbb{I}_{\epsilon,k_\epsilon}, cs\mathbb{F}_{\epsilon,k_\epsilon} > \}$ of $CSNSs$ is assigned for any x_ϵ in $\mathbb{X}$. Then $\mathbb{U}_{csR} = \{< x_\epsilon, \frac{cs\mathbb{T}_U}{x_\epsilon}, \frac{cs\mathbb{I}_U}{x_\epsilon}, \frac{cs\mathbb{F}_U}{x_\epsilon}; csR_\epsilon >: x_\epsilon \in \mathbb{X}\}$ is a $CSNS$ in $\mathbb{X}$ where $< \frac{cs\mathbb{T}_U}{x_\epsilon}, \frac{cs\mathbb{I}_U}{x_\epsilon}, \frac{cs\mathbb{F}_U}{x_\epsilon} >=< \frac{\sum_{\eta=1}^{k_\epsilon} cs\mathbb{T}_{\epsilon,j}}{k_\epsilon}, \frac{\sum_{\eta=1}^{k_\epsilon} cs\mathbb{I}_{\epsilon,j}}{k_\epsilon}, \frac{\sum_{\eta=1}^{k_\epsilon} cs\mathbb{F}_{\epsilon,j}}{k_\epsilon} >$ and $csR_\epsilon = min\{max_{1\leq j\leq k_\epsilon} \sqrt{(\frac{cs\mathbb{T}_U}{x_\epsilon} - cs\mathbb{T}_{\epsilon,j})^2 + (\frac{cs\mathbb{I}_U}{x_\epsilon} - cs\mathbb{I}_{\epsilon,j})^2 + (\frac{cs\mathbb{F}_U}{x_\epsilon} - cs\mathbb{F}_{\epsilon,j})^2}, 1\}$.

Let $X = \{x, y\}$ and $\lambda_1, \lambda_2 \in NS(X)$ such that

$$\lambda_1 = \{\langle x, 0.88, 0.33, 0.22\rangle, \langle x, 0.77, 0.44, 0.11\rangle, \langle x, 0.55, 0.44, 0.22\rangle, \langle x, 0.66, 0.55, 0.33\rangle\}$$

$$\lambda_2 = \{\langle y, 0.66, 0.22, 0.11\rangle, \langle y, 0.88, 0.11, 0.22\rangle, \langle y, 0.88, 0.33, 0.11\rangle, \langle y, 0.99, 0.44, 0.22\rangle\}.$$

The CSNSs are $\lambda_{(R_1)} = \{\langle x, 0.72, 0.44, 0.22; 0.20\rangle : x \in X\}$ and
$\lambda_{(R_2)} = \{\langle y, 0.85, 0.28, 0.17; 0.22\rangle : y \in X\}$.

Definition 2.8. [5] Let $U_{csR} = \{< x, \frac{cs\mathbb{T}_U}{x}, \frac{cs\mathbb{I}_U}{x}, \frac{cs\mathbb{F}_U}{x}; csR >: x \in \mathbb{X}\}$ and $V_{csS} = \{< x, \frac{cs\mathbb{T}_V}{x}, \frac{cs\mathbb{I}_V}{x}, \frac{cs\mathbb{F}_V}{x}; csS >: x \in \mathbb{X}\}$ be $CSNSs$ in $\mathbb{X}$ and $\dagger \in \{min, max\}$. Then

(1) $U_{csR} \subset V_{csS}$ iff $csR \leq csS$ and $\frac{cs\mathbb{T}_U}{x} \leq \frac{cs\mathbb{T}_V}{x}$, $\frac{cs\mathbb{I}_U}{x} \geq \frac{cs\mathbb{I}_V}{x}$ and $\frac{cs\mathbb{F}_U}{x} \geq \frac{cs\mathbb{F}_V}{x}$ for any $x \in \mathbb{X}$,

(2) $U_{csR} = V_s$ iff $csR = s$ and $\frac{cs\mathbb{T}_U}{x} = \frac{cs\mathbb{T}_V}{x}$, $\frac{cs\mathbb{I}_U}{x} = \frac{cs\mathbb{I}_V}{x}$ and $\frac{cs\mathbb{F}_U}{x} = \frac{cs\mathbb{F}_V}{x}$ for any $x \in \mathbb{X}$,

(3) $U^c_{csR} = \{< x, \frac{cs\mathbb{F}_U}{x}, \frac{cs\mathbb{I}_U}{x}, \frac{cs\mathbb{T}_U}{x}; csR >: x \in \mathbb{X}\}$,

(4) $U_{csR} \cup_\dagger V_s = \{< x, max(\frac{cs\mathbb{T}_U}{x}, \frac{cs\mathbb{T}_V}{x}), min(\frac{cs\mathbb{I}_U}{x}, \frac{cs\mathbb{I}_V}{x}), min(\frac{cs\mathbb{F}_U}{x}, \frac{cs\mathbb{F}_V}{x});$
$\dagger\,(csR, csS) >: x \in \mathbb{X}\}$,

(5) $U_{csR} \cap_\dagger V_{csS} = \{< x, min(\frac{cs\mathbb{T}_U}{x}, \frac{cs\mathbb{T}_V}{x}), max(\frac{cs\mathbb{I}_U}{x}, \frac{cs\mathbb{I}_V}{x}), max(\frac{cs\mathbb{F}_U}{x}, \frac{cs\mathbb{F}_V}{x});$
$\dagger\,(csR, csS) >: x \in \mathbb{X}\}$.

The acronyms used in the current research are listed below.

Table 1. Acronyms

Abbreviation	Description
min	Minimum
max	Maximum
NSs	Neutrosophic Sets
NVs	Neutrosophic Values
$CSNSs$	Cubic Spherical Neutrosophic Sets
$CSNVs$	Cubic Spherical Neutrosophic Values
$MCDM$	Multiple-Criteria Decision-Making
CSM	Cosine Similarity Measure
ATN	Archimedean T-Norm
$ATCN$	Archimedean t-Conorm
AFG	Additive Functional Generator

3. Cubic Spherical Neutrosophic t-norm and t-conorm

Definition 3.1. Let $u =< cs\mathbb{T}_u, cs\mathbb{I}_u, cs\mathbb{F}_u; csR_u >$ and $v =< cs\mathbb{T}_v, cs\mathbb{I}_v, cs\mathbb{F}_u; csR_v >$ be two $CSNVs$ in $\mathbb{X}$ and $\dagger \in \{min, max\}$. The following are some set operations that can be defined between $CSNVs$:

(1) $u \oplus_\dagger v =< cs\mathbb{T}_u + cs\mathbb{T}_v - cs\mathbb{T}_u cs\mathbb{T}_v, cs\mathbb{I}_u cs\mathbb{I}_v, cs\mathbb{F}_u cs\mathbb{F}_v; \dagger(csR_u, csR_v) >$,

(2) $u \otimes_\dagger v =< cs\mathbb{T}_u cs\mathbb{T}_v, cs\mathbb{I}_u + cs\mathbb{I}_v - cs\mathbb{I}_u cs\mathbb{I}_v, cs\mathbb{F}_u + cs\mathbb{F}_v - cs\mathbb{F}_u cs\mathbb{F}_v; \dagger(csR_u, csR_v) >$.

From Definition 3.1, by using TN and TCN, we may extend.
If Γ and Γ^* are dual TN and TCN with respect to the cubic spherical neutrosophic complement $\mathcal{S}$, respectively and $\mathcal{Q}$ is a TN or TCN. Then we can define the following algebraic operations among $CSNVs$.

(1) $u \oplus_Q v =< \Gamma^*(cs\mathbb{T}_u, cs\mathbb{T}_v), \Gamma(cs\mathbb{I}_u, cs\mathbb{I}_v), \Gamma(cs\mathbb{F}_u, cs\mathbb{F}_v); \mathcal{Q}(csR_u, csR_v) >$,

(2) $u \otimes_Q v =< \Gamma(cs\mathbb{T}_u, cs\mathbb{T}_v), \Gamma^*(cs\mathbb{I}_u, cs\mathbb{I}_v), \Gamma^*(cs\mathbb{F}_u, cs\mathbb{F}_v); \mathcal{Q}(csR_u, csR_v) >$.

If $\star : [0,1] \rightarrow [0,\infty)$ is the AFG of a continuous Archimedean ATN and $*(t) = \star(1-t)$ and $\wp : [0,1] \rightarrow [0,\infty)$ is the AFG of a continuous TCN. Then we can define the following algebraic operations among $CSNVs$ and $m > 0$.

(1) $u \oplus_\wp v = < *^{-1}(*(cs\mathbb{T}_u) + *(cs\mathbb{T}_v)), \star^{-1}(\star(cs\mathbb{I}_u) + \star(cs\mathbb{I}_v)),$
$\star^{-1}(\star(cs\mathbb{F}_u) + \star(cs\mathbb{F}_v)); \wp^{-1}(\wp(csR_u) + \wp(csR_v)) >,$

(2) $u \otimes_\wp v = < \star^{-1}(\star(cs\mathbb{T}_u) + \star(cs\mathbb{T}_v)), *^{-1}(*(cs\mathbb{I}_u) + *(cs\mathbb{I}_v)),$
$*^{-1}(*(cs\mathbb{F}_u) + *(cs\mathbb{F}_v)); \wp^{-1}(\wp(csR_u) + \wp(csR_v)) >,$

(3) $m_\wp u = < *^{-1}(m * (cs\mathbb{T}_u)), \star^{-1}(m \star (cs\mathbb{I}_u)), \star^{-1}(m \star (cs\mathbb{F}_u)); \wp^{-1}(m\wp(csR_u)) >,$

(4) $u^{m_\wp} = < \star^{-1}(m \star (cs\mathbb{T}_u)), *^{-1}(m * (cs\mathbb{I}_u)), *^{-1}(m * (cs\mathbb{F}_u)); \wp^{-1}(m\wp(csR_u)) > .$

Definition 3.2. Let $\star, *, \wp, \rho : [0,1] \rightarrow [0,\infty)$ be mappings such that $\star(t) = -log\ t$, $*(t) = -log\ (1-t)$, $\rho(t) = -log\ t$, and $\wp(t) = -log\ (1-t)$ and $m > 0$. Then we can define the following algebraic operations among $CSNVs$.

(1) $u \oplus_\wp v = < cs\mathbb{T}_u + cs\mathbb{T}_v - cs\mathbb{T}_u cs\mathbb{T}_v, cs\mathbb{I}_u cs\mathbb{I}_v, cs\mathbb{F}_u cs\mathbb{F}_v; csR_u csR_v >,$

(2) $u \oplus_\rho v = < cs\mathbb{T}_u + cs\mathbb{T}_v - cs\mathbb{T}_u cs\mathbb{T}_v, cs\mathbb{I}_u cs\mathbb{I}_v, cs\mathbb{F}_u cs\mathbb{F}_v; csR_u + csR_v - csR_u csR_v >,$

(3) $u \otimes_\wp v = < cs\mathbb{T}_u cs\mathbb{T}_v, cs\mathbb{I}_u + cs\mathbb{I}_v - cs\mathbb{I}_u cs\mathbb{I}_v, cs\mathbb{F}_u + cs\mathbb{F}_v - cs\mathbb{F}_u cs\mathbb{F}_v; csR_u csR_v >,$

(4) $u \otimes_\rho v = < cs\mathbb{T}_u cs\mathbb{T}_v, cs\mathbb{I}_u + cs\mathbb{I}_v - cs\mathbb{I}_u cs\mathbb{I}_v, cs\mathbb{F}_u + cs\mathbb{F}_v - cs\mathbb{F}_u cs\mathbb{F}_v;$
$csR_u + csR_v - csR_u csR_v >,$

(5) $m_\wp u = < 1 - (1 - cs\mathbb{T}_u)^m, cs\mathbb{I}_u^m, cs\mathbb{F}_u^m; csR_u^m >,$

(6) $m_\rho u = < 1 - (1 - cs\mathbb{T}_u)^m, cs\mathbb{I}_u^m, cs\mathbb{F}_u^m; 1 - (1 - csR_u)^m >,$

(7) $u^{m_\wp} = < cs\mathbb{T}_u^m, 1 - (1 - cs\mathbb{I}_u)^m, 1 - (1 - cs\mathbb{F}_u)^m; csR_u^m >,$

(8) $u^{m_\rho} = < cs\mathbb{T}_u^m, 1 - (1 - cs\mathbb{I}_u)^m, 1 - (1 - cs\mathbb{F}_u)^m; 1 - (1 - csR_u)^m > .$

Theorem 3.1. *Let* $u =< cs\mathbb{T}_u, cs\mathbb{I}_u, cs\mathbb{F}_u; csR_u >$, $v =< cs\mathbb{T}_v, cs\mathbb{I}_v, cs\mathbb{F}_v; csR_v >$, *and* $w =< cs\mathbb{T}_w, cs\mathbb{I}_w, cs\mathbb{F}_w; csR_w >$ *be* $CSNVs$ *and let* $m, n > 0$. *If* $\star : [0,1] \rightarrow [0,\infty)$ *is the* AFG *of a continuous* ATN *and* $*(t) = \star(1-t)$ *and* $\wp : [0,1] \rightarrow [0,\infty)$ *is the* AFG *of a continuous* ATN *or* $ATCN$. *Then*

(1) $u \oplus_\wp v = v \oplus_\wp u,$

(2) $u \otimes_\wp v = v \otimes_\wp u,$

(3) $(u \oplus_\wp v) \oplus_\wp w = u \oplus_\rho (v \oplus_\wp w),$

(4) $(u \otimes_\wp v) \otimes_\wp w = u \otimes_\wp (v \otimes_\wp w),$

(5) $m_\wp(u \oplus_\wp v) = m_\wp u \oplus_\wp m_\wp v,$

(6) $(m_\wp \oplus n_\wp)u = m_\wp u \oplus_\wp n_\wp u,$

(7) $(u \otimes_\wp v)^{m_\wp} = u^{m_\wp} \otimes_\wp v^{m_\wp},$

(8) $u^{m_\wp} \otimes_\wp u^{n_\wp} = u^{m_\wp + n_\wp}.$

Proof: (1) and (2) are trivial.

$$(3).\ (u \oplus_{\wp} v) \oplus_{\wp} w = < *^{-1}(*(cs\mathbb{T}_u) + *(cs\mathbb{T}_v)), \star^{-1}(\star(cs\mathbb{I}_u) + \star(cs\mathbb{I}_v)), \star^{-1}(\star(cs\mathbb{F}_u) + \star(cs\mathbb{F}_v));$$
$$\wp^{-1}(\wp(csR_u) + \wp(csR_v)) > \oplus_{\wp} < cs\mathbb{T}_w, cs\mathbb{I}_w, cs\mathbb{F}_w; csR_w >$$
$$= < *^{-1}(*(*^{-1}(*(cs\mathbb{T}_u) + *(cs\mathbb{T}_v)) + *(cs\mathbb{T}_w))),$$
$$\star^{-1}\ (\star(\star^{-1}(\star(cs\mathbb{I}_u) + \star(cs\mathbb{I}_v)) + \star(cs\mathbb{I}_w))),$$
$$\star^{-1}\ (\star(\star^{-1}(\star(cs\mathbb{F}_u) + \star(cs\mathbb{F}_v)) + \star(cs\mathbb{F}_w)));$$
$$\wp^{-1}(\wp(\wp^{-1}(\wp(csR_u) + \wp(csR_v)) + \wp(csR_w))) >$$
$$= < *^{-1}(*(cs\mathbb{T}_u) + *(cs\mathbb{T}_v) + *(cs\mathbb{T}_w)), \star^{-1}(\star(cs\mathbb{I}_u) + \star(cs\mathbb{I}_v) + \star(cs\mathbb{I}_w)),$$
$$\star^{-1}\ (\star(cs\mathbb{F}_u) + \star(cs\mathbb{F}_v) + \star(cs\mathbb{F}_w)); \wp^{-1}(\wp(csR_u) + \wp(csR_v) + \wp(csR_w)) >$$
$$= < *^{-1}(*(cs\mathbb{T}_u) + *(*^{-1}(*(cs\mathbb{T}_v) + *(cs\mathbb{T}_w)))),$$
$$\star^{-1}\ (\star(cs\mathbb{I}_u) + \star(\star^{-1}(\star(cs\mathbb{I}_v) + \star(cs\mathbb{I}_w)))),$$
$$\star^{-1}\ (\star(cs\mathbb{F}_u) + \star(\star^{-1}(\star(cs\mathbb{F}_v) + \star(cs\mathbb{F}_w))));$$
$$\wp^{-1}(\wp(csR_u) + \wp(\wp^{-1}(\wp(csR_v) + \wp(csR_w)))) >$$
$$= < cs\mathbb{T}_u, cs\mathbb{I}_u; csR_u > \oplus_{\wp} < *^{-1}(*(cs\mathbb{T}_v) + *(cs\mathbb{T}_w)), \star^{-1}(*(cs\mathbb{I}_v) + \star(cs\mathbb{I}_w)),$$
$$\star^{-1}\ (\star(cs\mathbb{F}_v) + \star(cs\mathbb{F}_w)); \wp^{-1}(\wp(csR_v) + \wp(csR_w)) >$$
$$= u \oplus_{\wp} (v \oplus_{\wp} w).$$

$$(4).\ (u \otimes_{\wp} v) \otimes_{\wp} w = < \star^{-1}(\star(cs\mathbb{T}_u) + \star(cs\mathbb{T}_v)), *^{-1}(*(cs\mathbb{I}_u) + *(cs\mathbb{I}_v)), *^{-1}(*(cs\mathbb{F}_u) + *(cs\mathbb{F}_v));$$
$$\wp^{-1}(\wp(csR_u) + \wp(csR_v)) > \otimes_{\wp} < cs\mathbb{T}_w, cs\mathbb{I}_w, cs\mathbb{F}_w; csR_w >$$
$$= < \star^{-1}(\star(\star^{-1}(\star(cs\mathbb{T}_u) + \star(cs\mathbb{T}_v)) + *(cs\mathbb{T}_w))),$$
$$*^{-1}\ (*(*^{-1}(*(cs\mathbb{I}_u) + *(cs\mathbb{I}_v)) + *(cs\mathbb{I}_w))),$$
$$*^{-1}\ (*(*^{-1}(*(cs\mathbb{F}_u) + *(cs\mathbb{F}_v)) + *(cs\mathbb{F}_w)));$$
$$\wp^{-1}(\wp(\wp^{-1}(\wp(csR_u) + \wp(csR_v)) + \wp(csR_w))) >$$
$$= < \star^{-1}(\star(cs\mathbb{T}_u) + \star(cs\mathbb{T}_v) + \star(cs\mathbb{T}_w)), *^{-1}(*(cs\mathbb{I}_u) + *(cs\mathbb{I}_v) + *(cs\mathbb{I}_w)),$$
$$*^{-1}\ (*(cs\mathbb{F}_u) + *(cs\mathbb{F}_v) + *(cs\mathbb{F}_w)); \wp^{-1}(\wp(csR_u) + \wp(csR_v) + \wp(csR_w)) >$$
$$= < \star^{-1}(\star(cs\mathbb{T}_u) + \star(\star^{-1}(\star(cs\mathbb{T}_v) + \star(cs\mathbb{T}_w)))),$$
$$*^{-1}\ (*(cs\mathbb{I}_u) + *(*^{-1}(*(cs\mathbb{I}_v) + *(cs\mathbb{I}_w)))),$$
$$*^{-1}\ (*(cs\mathbb{F}_u) + *(*^{-1}(*(cs\mathbb{F}_v) + *(cs\mathbb{F}_w))));$$
$$\wp^{-1}(\wp(csR_u) + \wp(\wp^{-1}(\wp(csR_v) + \wp(csR_w)))) >$$

$$
\begin{aligned}
&= < cs\mathbb{T}_u, cs\mathbb{I}_u; csR_u > \otimes_{\wp} < \star^{-1}(\star(cs\mathbb{T}_v) + \star(cs\mathbb{T}_w)),\\
&\quad \ast^{-1}(\ast(cs\mathbb{I}_v) + \ast(cs\mathbb{I}_w)), \ast^{-1}(\ast(cs\mathbb{F}_v) + \ast(cs\mathbb{F}_w));\\
&\quad \wp^{-1}(\wp(csR_v) + \wp(csR_w)) >\\
&= u \otimes_{\wp} (v \otimes_{\wp} w).
\end{aligned}
$$

$$
\begin{aligned}
(5).\ m_{\wp}(u \oplus_{\wp} v) &= m_{\wp} < \ast^{-1}(\ast(cs\mathbb{T}_u) + \ast(cs\mathbb{T}_v)), \star^{-1}(\star(cs\mathbb{I}_u) + \star(cs\mathbb{I}_v)),\\
&\quad \star^{-1}(\star(cs\mathbb{T}_u) + \star(cs\mathbb{T}_v)); \wp^{-1}(\wp(csR_u) + \wp(csR_v)) >\\
&= < \ast^{-1}(m \ast (\ast^{-1}(\ast(cs\mathbb{T}_u) + \ast(cs\mathbb{T}_v)))), \star^{-1}(m \ast (\star^{-1}(\star(cs\mathbb{I}_u) + \star(cs\mathbb{I}_v)))),\\
&\quad \star^{-1}(m \ast (\star^{-1}(\star(cs\mathbb{F}_u) + \star(cs\mathbb{F}_v)))); \wp^{-1}(m \ast (\wp^{-1}(\wp(csR_u) + \wp(csR_v)))) >\\
&= < \ast^{-1}(m \ast (cs\mathbb{T}_u) + m \ast (cs\mathbb{T}_v)), \star^{-1}(m \star (cs\mathbb{I}_u) + m \star (cs\mathbb{I}_v)),\\
&\quad \star^{-1}(m \star (cs\mathbb{F}_u) + m \star (cs\mathbb{F}_v)), \wp^{-1}(m\wp(csR_u) + m\wp(csR_v)) >\\
&= < \ast^{-1}(h = \ast(\ast^{-1}(m \ast (cs\mathbb{T}_u))) + \ast(\ast^{-1}(m \ast (cs\mathbb{T}_v)))),\\
&\quad \star^{-1}(\star(\star^{-1}(m \star (cs\mathbb{I}_u))) + \star(\star^{-1}(m \ast (cs\mathbb{I}_v)))),\\
&\quad \star^{-1}(\star(\star^{-1}(m \star (cs\mathbb{F}_u))) + \star(\star^{-1}(m \ast (cs\mathbb{F}_v))));\\
&\quad \wp^{-1}(\wp(\wp^{-1}(m\wp(csR_u))) + \wp(\wp^{-1}(m\wp(csR_v)))) >\\
&= < \ast^{-1}(\ast(cs\mathbb{T}_{mu}) + \ast(cs\mathbb{T}_{mv})), \star^{-1}(\star(cs\mathbb{I}_{mu}) + \star(cs\mathbb{I}_{mv})),\\
&\quad \star^{-1}(\star(cs\mathbb{F}_{mu}) + \star(cs\mathbb{F}_{mv})); \wp^{-1}(\wp(csR_{mu}) + \wp(csR_{mv})) >\\
&= m_{\wp}u \oplus_{\wp} m_{\wp}v.
\end{aligned}
$$

$$
\begin{aligned}
(6).\ (m_{\wp} + n_{\wp})u &= < \ast^{-1}((m+n) \ast (cs\mathbb{T}_u)), \star^{-1}((m+n) \star (cs\mathbb{I}_u)), \star^{-1}((m+n) \star (cs\mathbb{F}_u))^{-1}\\
&\quad ((m+n)\wp(csR_u)) >\\
&= < \ast^{-1}(m \ast (cs\mathbb{T}_u) + n \ast (cs\mathbb{T}_u)), \star^{-1}(m \star (cs\mathbb{I}_u) + n \star (cs\mathbb{I}_u)),\\
&\quad \star^{-1}(m \star (cs\mathbb{F}_u) + n \star (cs\mathbb{F}_u)); \wp^{-1}(m\wp(csR_u) + n\wp(csR_u)) >\\
&= < \ast^{-1}(\ast(\ast^{-1}(m \ast (cs\mathbb{T}_u))) + \ast(\ast^{-1}(n \ast (cs\mathbb{T}_u)))), \star^{-1}(\star(\star^{-1}(m \star (cs\mathbb{I}_u)))+\\
&\quad \star(\star^{-1}(n \star (cs\mathbb{I}_u)))), \star^{-1}(\star(\star^{-1}(m \star (cs\mathbb{F}_u)))+\\
&\quad \star(\star^{-1}(n \star (cs\mathbb{F}_u)))); \wp^{-1}(\wp(\wp^{-1}(m\wp(csR_u))) + \wp(\wp^{-1}(n\wp(csR_u)))) >\\
&= < \ast^{-1}(\ast(cs\mathbb{T}_{m_{\wp}u}) + (\ast(cs\mathbb{T}_{n_{\wp}u}))), \star^{-1}(\star(cs\mathbb{I}_{m_{\wp}u}) + (\star(cs\mathbb{I}_{n_{\wp}u}))),\\
&\quad \star^{-1}(\star(cs\mathbb{F}_{m_{\wp}u}) + (\star(cs\mathbb{F}_{n_{\wp}u}))); \wp^{-1}(\wp(csR_{m_{\wp}u}) + (\wp(csR_{n_{\wp}u}))) >\\
&= m_{\wp}u \oplus_{\wp} n_{\wp}u.
\end{aligned}
$$

$$
\begin{aligned}
(7).\ (u \otimes_{\wp} v)^{m_{\wp}} &= < \star^{-1}(m \star (cs\mathbb{T}_{u\otimes_{\wp}v})), \ast^{-1}(m \ast (cs\mathbb{I}_{u\otimes_{\wp}v})), \ast^{-1}(m \ast (cs\mathbb{F}_{u\otimes_{\wp}v})); \wp^{-1}(m\wp(csR_{u\otimes_{\wp}v})) >\\
&= < \star^{-1}(m \star (\star^{-1}(\star(cs\mathbb{T}_u)))), \ast^{-1}(m \ast (\ast^{-1}(\ast(cs\mathbb{I}_u)))), \ast^{-1}(m \ast (\ast^{-1}(\ast(cs\mathbb{F}_u))));\\
&\quad \wp^{-1}(m\wp(\wp^{-1}(\wp(csR_u)))) >
\end{aligned}
$$

$$= < \star^{-1}(m \star (cs\mathbb{T}_u) + m \star (cs\mathbb{T}_v)), *^{-1}(m * (cs\mathbb{I}_u) + m * (cs\mathbb{I}_v)),$$
$$*^{-1}\ (m * (cs\mathbb{F}_u) + m * (cs\mathbb{F}_v)); \wp^{-1}(m\wp(csR_u) + m\wp(csR_v)) >$$
$$= < \star^{-1}(\star(\star^{-1}(m \star (cs\mathbb{T}_u))) + \star(\star^{-1}(m \star (cs\mathbb{T}_v)))),$$
$$*^{-1}\ (*(*^{-1}(m * (cs\mathbb{I}_u))) + *(*^{-1}(m * (cs\mathbb{I}_v)))),$$
$$*^{-1}\ (*(*^{-1}(m * (cs\mathbb{F}_u))) + *(*^{-1}(m * (cs\mathbb{F}_v))));$$
$$\wp^{-1}(\wp(\wp^{-1}(m\wp(csR_u))) + \wp(\wp^{-1}(m\wp(csR_v)))) >$$
$$= < \star^{-1}(\star(cs\mathbb{T}_{u^{m_\wp}}) + \star(cs\mathbb{T}_{v^{m_\wp}})), *^{-1}(*(cs\mathbb{I}_{u^{m_\wp}}) + *(cs\mathbb{I}_{v^{m_\wp}})),$$
$$*^{-1}\ (*(cs\mathbb{F}_{u^{m_\wp}}) + *(cs\mathbb{F}_{v^{m_\wp}})); \wp^{-1}(\wp(csR_{u^{m_\wp}}) + \wp(csR_{v^{m_\wp}})) >$$
$$= u^{m_\wp} \otimes v^{m_\wp}.$$

(8). $$u_{csR}^{m_\wp + n_\wp} = < \star^{-1}((m+n) * (cs\mathbb{T}_u)), *^{-1}((m+n) * (cs\mathbb{I}_u)), *^{-1}((m+n) * (cs\mathbb{F}_u));$$
$$\wp^{-1}((m+n)\wp(csR_u)) >$$
$$= < \star^{-1}(m \star (cs\mathbb{T}_u) + n \star (cs\mathbb{T}_u)), *^{-1}(m * (cs\mathbb{I}_u) + n * (cs\mathbb{I}_u)),$$
$$*^{-1}\ (m * (cs\mathbb{F}_u) + n * (cs\mathbb{F}_u)); \wp^{-1}(m\wp(csR_u) + n\wp(csR_u)) >$$
$$= < \star^{-1}(\star(\star^{-1}(m \star (cs\mathbb{T}_u))) + \star(\star^{-1}(n \star (cs\mathbb{T}_u)))), *^{-1}(*(*^{-1}(m * (cs\mathbb{I}_u)))+$$
$$*\ (*^{-1}(n * (cs\mathbb{I}_u)))), *^{-1}(*(*^{-1}(m * (cs\mathbb{F}_u))) + *(*^{-1}(n * (cs\mathbb{F}_u))));$$
$$\wp^{-1}(\wp(\wp^{-1}(m\wp(csR_u))) + \wp(\wp^{-1}(n\wp(csR_u)))) >$$
$$= < \star^{-1}(\star(cs\mathbb{T}_{u^m}) + \star(cs\mathbb{T}_{u^n})), *^{-1}(*(cs\mathbb{I}_{u^m}) + *(cs\mathbb{I}_{u^n})),$$
$$*^{-1}\ (*(cs\mathbb{F}_{u^m}) + *(cs\mathbb{F}_{u^n})); \wp^{-1}(\wp(csR_{u^m}) + \wp(csR_{u^n})) >$$
$$= u_{csR}^{m_\wp} \otimes_\wp u_{csR}^{n_\wp}.$$

4. Weighted Arithmetic Cubic Spherical Neutrosophic Aggregation Operators

Definition 4.1. Consider the collection $\{u_\epsilon =< cs\mathbb{T}_{u_\epsilon}, cs\mathbb{I}_{u_\epsilon}, cs\mathbb{F}_{u_\epsilon}; csR_{u_\epsilon} >: \epsilon = 1, 2, 3, \ldots, k\}$ of $CSNVs$. If $\star : [0, 1] \to [0, \infty)$ is the AFG of a continuous ATN and $*(t) = \star(1 - t)$ and $\wp : [0, 1] \to [0, \infty)$ is the AFG of a continuous ATN or $ATCN$. Then a weighted arithmetic cubic spherical neutrosophic aggregation operator is defined and denoted by $\mathbb{CSNWA}_\wp(u_1, u_2, \ldots, u_k) = (\wp) \bigoplus_{\epsilon=1}^{k} \omega_\epsilon u_\epsilon$, where $0 \leq \omega_\epsilon \leq 1$ for any $\epsilon = 1, 2, 3, \ldots, k$ subject to the condition $\sum_{\epsilon=1}^{k} \omega_\epsilon = 1$.

Theorem 4.1. *Consider* $\{u_\epsilon =< cs\mathbb{T}_{u_\epsilon}, cs\mathbb{I}_{u_\epsilon}, cs\mathbb{F}_{u_\epsilon}; csR_{u_\epsilon} >: \epsilon = 1, 2, 3, \ldots, k\}$ *of* $CSNVs$. *If* $h : [0, 1] \to [0, \infty)$ *is the AFG of a continuous* ATN *and* $*(t) = \star(1-t)$ *and* $\wp : [0, 1] \to [0, \infty)$ *is the AFG of a continuous* ATN *or* $ATCN$. *Then* $\mathbb{CSNWA}_\wp(u_1, u_2, \ldots, u_k) = < *^{-1}(\sum_{\epsilon=1}^{k} \omega_\epsilon * (cs\mathbb{T}_{u_\epsilon}))$, $\star^{-1}(\sum_{\epsilon=1}^{k} \omega_\epsilon \star (cs\mathbb{I}_{u_\epsilon}))$, $\star^{-1}(\sum_{\epsilon=1}^{k} \omega_\epsilon \star (cs\mathbb{F}_{u_\epsilon}))$; $\wp^{-1}(\sum_{\epsilon=1}^{k} \omega_\epsilon \wp(csR_{u_\epsilon})) >$ *where* $0 \leq \omega_\epsilon \leq 1$ *for all* $\epsilon = 1, 2, 3, \ldots, k$ *subject to the condition* $\sum_{\epsilon=1}^{k} \omega_\epsilon = 1$.

Proof. *Clearly* $\mathbb{CSNWA}_{\wp}(u_1, u_2, \ldots, u_k)$ *is a CSNV. The second part can be seen to be true by using mathematical induction. If* $k = 2$, *we have*

$$\begin{aligned}
\mathbb{CSNWA}_{\wp}(u_1, u_2, \ldots, u_k) =& \omega_{1_{\wp}} u_1 \oplus_{\wp} \omega_{2_{\wp}} u_2 \\
=& < *^{-1}(*(cs\mathbb{T}_{\omega_{1_{\wp}}} u_1) + *(cs\mathbb{T}_{\omega_{2_{\wp}}} u_2)), \star^{-1}(\star(cs\mathbb{I}_{\omega_{1_{\wp}}} u_1) + \star(cs\mathbb{I}_{\omega_{2_{\wp}}} u_2)), \\
& \star^{-1}(\star(cs\mathbb{F}_{\omega_{1_{\wp}}} u_1) + \star(cs\mathbb{F}_{\omega_{2_{\wp}}} u_2)); \wp^{-1}(\wp(csR_{\omega_{1_{\wp}}} u_1) + \wp(csR_{\omega_{2_{\wp}}} u_2)) > \\
=& < *^{-1}(*((*^{-1}(\omega_1 h(cs\mathbb{T}_{u_1}))) + *(*^{-1}(\omega_2 * (cs\mathbb{T}_{u_2})))), \\
& \star^{-1}(\star((\star^{-1}(\omega_1 h(cs\mathbb{I}_{u_1}))) + \star(\star^{-1}(\omega_2 * (cs\mathbb{I}_{u_2})))), \star^{-1}(\star((\star^{-1}(\omega_1 * (cs\mathbb{F}_{u_1}))) \\
& + \star(\star^{-1}(\omega_2 * (cs\mathbb{F}_{u_2})))); \wp^{-1}(\wp((\wp^{-1}(\omega_1 h(csR_{u_1}))) + \wp(\wp^{-1}(\omega_1 h(csR_{u_2})))) > \\
=& < *^{-1}(\omega_1 * (cs\mathbb{T}_{u_1}) + \omega_2 h(cs\mathbb{T}_{u_2})), \star^{-1}(\omega_1 \star (cs\mathbb{I}_{u_1}) + \omega_2 \star (cs\mathbb{I}_{u_2})), \\
& \star^{-1}(\omega_1 \star (cs\mathbb{F}_{u_1}) + \omega_2 \star (cs\mathbb{F}_{u_2})); \wp^{-1}(\omega_1 \wp(csR_{u_1}) + \omega_2 \wp(csR_{u_2})) > \\
=& < *^{-1}(\sum_{\eta=1}^{2} \omega_\eta * (cs\mathbb{T}_{u_\eta})), \star^{-1}(\sum_{\eta=1}^{2} \omega_\eta * (cs\mathbb{I}_{u_\eta})), \star^{-1}(\sum_{\eta=1}^{2} \omega_\eta \star (cs\mathbb{F}_{u_\eta})); \\
& \wp^{-1}(\sum_{\eta=1}^{2} \omega_\eta \wp(csR_{u_\eta})) > .
\end{aligned}$$

$$\begin{aligned}
\mathbb{CSNWA}_{\wp}(u_1, u_2, ..., u_{k-1}) =& < *^{-1}(\sum_{\eta=1}^{k-1} \omega_\eta \star (cs\mathbb{T}_{u_\eta})), \star^{-1}(\sum_{\eta=1}^{k-1} \omega_\eta * (cs\mathbb{I}_{u_\eta})), \star^{-1}(\sum_{\eta=1}^{k-1} \omega_\eta \star (cs\mathbb{F}_{u_\eta})); \\
& \wp^{-1}(\sum_{\eta=1}^{k-1} \omega_\eta \wp(csR_{u_\eta})) >
\end{aligned}$$

$$\begin{aligned}
\mathbb{CSNWA}_{\wp}(u_1, u_2, \ldots, u_k) =& \mathbb{CSNWA}_{\wp}(u_1, u_2, \ldots, u_{k-1}) \oplus_{\wp} \omega_{k_{\wp}} u_k \\
=& < *^{-1}(\sum_{\eta=1}^{k-1} \omega_\eta * (cs\mathbb{T}_{u_\eta})), \star^{-1}(\sum_{\eta=1}^{k-1} \omega_\eta \star (cs\mathbb{I}_{u_\eta})), \star^{-1}(\sum_{\eta=1}^{k-1} \omega_\eta \star (cs\mathbb{F}_{u_\eta})); \\
& \wp^{-1}(\sum_{\eta=1}^{k-1} \omega_\eta \wp(csR_{u_\eta})) > \oplus_{\wp} < *^{-1}(*(cs\mathbb{T}_{\omega_{k_{\wp}}} u_1)), \star^{-1}(\star(cs\mathbb{I}_{\omega_{k_{\wp}}} u_1)), \\
& \star^{-1}(\star(cs\mathbb{F}_{\omega_{k_{\wp}}} u_1)); \wp^{-1}(\wp(cs\mathbb{I}_{\omega_{k_{\wp}}} u_1)) > \\
=& < *^{-1}(*((*^{-1}(\sum_{\eta=1}^{k-1} \omega_\eta * (cs\mathbb{T}_{u_\eta}))) + *(*^{-1}(\omega_k * (cs\mathbb{T}_{u_k})))), \\
& \star^{-1}(\star((\star^{-1}(\sum_{\eta=1}^{k-1} \omega_\eta * (cs\mathbb{I}_{u_\eta}))) + \star(\star^{-1}(\omega_k \star (cs\mathbb{I}_{u_k})))), \\
& \star^{-1}(\star((\star^{-1}(\sum_{\eta=1}^{k-1} \omega_\eta * (cs\mathbb{F}_{u_\eta}))) + \star(\star^{-1}(\omega_k \star (cs\mathbb{F}_{u_k}))));
\end{aligned}$$

$$\wp^{-1}(\wp((\wp^{-1}(\sum_{\eta=1}^{k-1}\omega_\eta\wp(csR_{u_\eta}))) + \wp(\wp^{-1}(\omega_k\wp(csR_{u_k})))) >$$

$$= < *^{-1}(\sum_{\eta=1}^{k-1}\omega_\eta * (cs\mathbb{T}_{u_\eta}) + \omega_k * (cs\mathbb{T}_{u_k})), \star^{-1}(\sum_{\eta=1}^{k-1}\omega_\eta \star (cs\mathbb{I}_{u_\eta}) + \omega_k \star (cs\mathbb{I}_{u_k})),$$

$$\star^{-1}(\sum_{\eta=1}^{k-1}\omega_\eta \star (cs\mathbb{F}_{u_\eta}) + \omega_k \star (cs\mathbb{F}_{u_k})); \wp^{-1}(\sum_{\eta=1}^{k-1}\omega_\eta\wp(csR_{u_\eta}) + \omega_k\wp(csR_{u_k}))$$

$$= < *^{-1}(\sum_{\epsilon=1}^{k}\omega_\epsilon * (cs\mathbb{T}_{u_\epsilon})), \star^{-1}(\sum_{\epsilon=1}^{k}\omega_\epsilon \star (cs\mathbb{I}_{u_\epsilon})), \star^{-1}(\sum_{\epsilon=1}^{k}\omega_\epsilon \star (cs\mathbb{F}_{u_\epsilon}));$$

$$\wp^{-1}(\sum_{\epsilon=1}^{k}\omega_\eta\wp(csR_{u_k})) > .$$

This completes the proof.

Definition 4.2. Let $\star, *, \wp, \rho : [0,1] \to [0,\infty)$ be mappings such that $\star(t) = -log\ t$, $*(t) = -log\ (1-t), \wp(t) = -log\ t$ and $\rho(t) = -log\ (1-t)$. The CSNWA aggregating operators listed below can be considered specific cases of definition 4.1.
$\mathbb{CSNWA}_\wp^A(u_1, u_2, \ldots, u_k) =< 1-\prod_{\epsilon=1}^k(1-cs\mathbb{T}_{u_\epsilon})^{\omega_\epsilon}, \prod_{\epsilon=1}^k cs\mathbb{I}_{u_\epsilon}^{\omega_\epsilon}, \prod_{\epsilon=1}^k cs\mathbb{F}_{u_\epsilon}^{\omega_\epsilon}; \prod_{\epsilon=1}^k csR_{u_\epsilon}^{\omega_\epsilon} >$ and $\mathbb{CSNWA}_\rho^A(u_1, u_2, \ldots, u_k) =< 1 - \prod_{\epsilon=1}^k(1 - cs\mathbb{T}_{u_\epsilon})^{\omega_\epsilon}, \prod_{\epsilon=1}^k cs\mathbb{I}_{u_\epsilon}^{\omega_\epsilon}, \prod_{\epsilon=1}^k cs\mathbb{F}_{u_\epsilon}^{\omega_\epsilon}; 1 - \prod_{\epsilon=1}^k(1 - csR_{u_\epsilon})^{\omega_\epsilon} >$.

It can be easily prove that the $\mathbb{CSNWA}$ operator has the following properties.

1. Idempotency property: If all u_η $(\eta = 1, 2, \ldots, k)$ are equal, that is, $u_\eta = u$ for any η, then $\mathbb{CSNWA}_\wp^A(u_1, u_2, \ldots, u_k) = u$.

2. Boundary property: Let $\{u_\epsilon =< cs\mathbb{T}_{u_\epsilon}, cs\mathbb{I}_{u_\epsilon}, cs\mathbb{F}_{u_\epsilon}; csR_{u_\epsilon} >: \epsilon = 1, 2, 3, \ldots, k\}$ be a collection of $CSNVs$ in $\mathbb{X}$, and $u^- = min_\eta u_\eta$, $u^+ = max_\eta u_\eta$. Then $u^- \leq \mathbb{CSNWA}_\wp^A(u_1, u_2, \ldots, u_k) \leq u^+$.

3. Monotonicity property: Let u_η $(\eta = 1, 2, \ldots, k)$ and $u_\eta^{'}$ $(\eta = 1, 2, \ldots, k)$ be two $CSNVs$ in $\mathbb{X}$. If $u_\eta \leq u_\eta^{'}$, then $\mathbb{CSNWA}_\wp^A(u_1, u_2, \ldots, u_k) \leq \mathbb{CSNWA}_\wp^A(u_1^{'}, u_2^{'}, \ldots, u_k^{'})$.

4. Permutation property: Let $\{u_\epsilon =< cs\mathbb{T}_{u_\epsilon}, cs\mathbb{I}_{u_\epsilon}, cs\mathbb{F}_{u_\epsilon}; csR_{u_\epsilon} >: \epsilon = 1, 2, 3, \ldots, k\}$ be a collection of $CSNVs$ in $\mathbb{X}$, Then $\mathbb{CSNWA}_\wp^A(u_1, u_2, \ldots, u_k) = \mathbb{CSNWA}_\wp^A(u_1^{'}, u_2^{'}, \ldots, u_k^{'})$, where $\mathbb{CSNWA}_\wp^A(u_1, u_2, \ldots, u_k)$ is a permutation of $\mathbb{CSNWA}_\wp^A(u_1^{'}, u_2^{'}, \ldots, u_k^{'})$.

5. **Weighted Geometric Cubic Spherical Neutrosophic Aggregation Operators**

Definition 5.1. Consider the collection $\{u_\epsilon =< cs\mathbb{T}_{u_\epsilon}, cs\mathbb{I}_{u_\epsilon}, cs\mathbb{F}_{u_\epsilon}; csR_{u_\epsilon} >: \epsilon = 1, 2, 3, \ldots, k\}$ of $CSNVs$. If $\star : [0,1] \to [0,\infty)$ is the AFG of a continuous ATN and $*(t) = \star(1-t)$ and $\wp : [0,1] \to [0,\infty)$ is the AFG of a continuous ATN or $ATCN$. Then a weighted geometric cubic spherical neutrosophic aggregation operator is defined and denoted by

$\mathbb{CSNWG}_{\wp}(u_1,u_2,\ldots,u_k) = (\wp)\bigotimes_{\epsilon=1}^{k}\omega_\epsilon u_\epsilon$, where $0 \leq \omega_\epsilon \leq 1$ for all $\epsilon = 1,2,3,\ldots,k$ subject to the condition $\sum_{\epsilon=1}^{k}\omega_\epsilon = 1$.

Theorem 5.1. *Consider the collection* $\{u_\epsilon =< cs\mathbb{T}_{u_\epsilon}, cs\mathbb{I}_{u_\epsilon}, cs\mathbb{F}_{u_\epsilon}; csR_{u_\epsilon} >: \epsilon = 1,2,3,\ldots,k\}$ *of* $CSNVs$. *If* $\star : [0,1] \to [0,\infty)$ *is the* AFG *of a continuous* ATN *and* $*(t) = \star(1-t)$ *and* $\wp : [0,1] \to [0,\infty)$ *is the* AFG *of a continuous* ATN *or* $ATCN$. *Then we have*
$\mathbb{CSNWG}_{\wp}(u_1,u_2,\ldots,u_k) =< \star^{-1}(\sum_{\epsilon=1}^{k}\omega_\epsilon f(cs\mathbb{T}_{u_\epsilon}))$,
$*^{-1}(\sum_{\epsilon=1}^{k}\omega_\epsilon * (cs\mathbb{I}_{u_\epsilon})), *^{-1}(\sum_{\epsilon=1}^{k}\omega_\epsilon * (cs\mathbb{F}_{u_\epsilon})); \wp^{-1}(\sum_{\epsilon=1}^{k}\omega_\epsilon\wp(csR_{u_\epsilon})) >$
where $0 \leq \omega_\epsilon \leq 1$ *for any* $\epsilon = 1,2,3,\ldots,k$ *subject to the condition* $\sum_{\epsilon=1}^{k}\omega_\epsilon = 1$.
Proof. *Straightforward to Theorem 4.1.*

Definition 5.2. Let $\star, *, \wp, \rho : [0,1] \to [0,\infty)$ be mappings such that $\star(t) = -log\, t$, $*(t) = -log\,(1-t)$, $\wp(t) = -log\, t$ and $\rho(t) = -log\,(1-t)$. The CSNWA aggregating operators listed below can be considered specific cases of definition 5.1:
$\mathbb{CSNWG}^{a}_{\wp}(u_1,u_2,\ldots,u_k) =$
$< \prod_{\epsilon=1}^{k} cs\mathbb{T}_{u_\epsilon}^{\omega_\epsilon}, 1-\prod_{\epsilon=1}^{k}(1-cs\mathbb{I}_{u_\epsilon})^{\omega_\epsilon}, 1-\prod_{\epsilon=1}^{k}(1-cs\mathbb{F}_{u_\epsilon})^{\omega_\epsilon}; \prod_{\epsilon=1}^{k} csR_{u_\epsilon}^{\omega_\epsilon} >$
$\mathbb{CSNWG}^{a}_{\rho}(u_1,u_2,\ldots,u_k) =$
$< \prod_{\epsilon=1}^{k} cs\mathbb{T}_{u_\epsilon}^{\omega_\epsilon}, 1-\prod_{\epsilon=1}^{k}(1-cs\mathbb{I}_{u_\epsilon})^{\omega_\epsilon}, 1-\prod_{\epsilon=1}^{k}(1-cs\mathbb{F}_{u_\epsilon})^{\omega_\epsilon}; 1-\prod_{\epsilon=1}^{k}(1-csR_{u_\epsilon})^{\omega_\epsilon} >$

6. An Application of Cubic Spherical Neutrosophic Values

For $CSNVs$, we define a similarity measure in this section. Then, in a cubic spherical neutrosophic fuzzy environment, we provide an $MCDM$ technique employing this similarity measure and the suggested aggregation operators. Then, using the suggested approach, we resolve a real-world decision problem from the literature involving picking the optimal solar cell.

6.1. *A similarity Measure for Cubic Spherical Neutrosophic Values*

In an uncertain context, similarity measures are crucial tools for figuring out how similar things are to one another. Due to their ability to handle ambiguity and the fact that they have attracted many research efforts based on similarity measures within neutrosophic research, more and more researchers have begun to explore NSs. The following is the similarity measure for $CSNVs$.

Definition 6.1. If $u =< cs\mathbb{T}_u, cs\mathbb{I}_u, cs\mathbb{F}_u; csR_u >$ and $v =< cs\mathbb{T}_v, cs\mathbb{I}_v, cs\mathbb{F}_u; csR_u >$ be $CSNVs$ in $\mathbb{X}$. Then the cubic spherical cosine similarity measure is defined and denoted by $cs\mathbb{CSM}(u,v) = \frac{cs\mathbb{T}_u cs\mathbb{T}_v + cs\mathbb{I}_u cs\mathbb{I}_v + cs\mathbb{F}_u cs\mathbb{F}_v}{\sqrt{cs\mathbb{T}_u^2+cs\mathbb{I}_u^2+cs\mathbb{F}_u^2}\sqrt{cs\mathbb{T}_v^2+cs\mathbb{I}_v^2+cs\mathbb{F}_v^2}} \times \frac{|csR_u - csR_v|}{\max\{csR_u, csR_v\}}$.

6.2. *A MCDM Method*

In the cubic spherical neutrosophic environment, an $MCDM$ method is suggested in this section. The suggested approach is used to solve an $MCDM$ problem that has been taken from the literature to demonstrate its effectiveness in the next subsection. Following are the steps of the suggested method that we can present:

Step 1: Suppose there are k alternatives that $A = \{A_1, A_2, \ldots, A_k\}$ expert has evaluated in light of a list with j criteria as $C = \{c_1, c_2, \ldots, c_\eta\}$.

Step 2: For each criterion, the expert chooses the weight vector and converts the assessment results of the alternatives into $CSNVs$.

Step 3: If there are any cost criteria based on their values, the complement operation is used.

Step 4: Evaluation findings for each choice that are expressed as "$CSNVs$ are transformed using suggested weighted aggregation operations.

Step 5: The variation in $cs\mathbb{CSM}$ between each alternative's aggregate value and the ideal alternative's positive value $< 1, 0, 0; 1 >$ is determined.

Step 6: The alternative with the greatest similarity value is taken to be the best.

6.3. *Selection of Electric truck using CSNVs*

In recent years, there has been an enormous increase in demand for electric vehicles. As a result of its success in the global and regional markets across many demography, several renowned heavy-duty vehicle manufacturers started investing in developing electric heavy commercial vehicles as a sustainable solution that will replace conventional heavy commercial vehicles for its key advantage of zero air pollution. In India, the demand for electric trucks is constantly growing and the government works on several policies and agendas to meet the increasing demand and to further boost the sales of electric commercial heavy vehicles across all categories. The automobile manufacturers are competing to capture the growing Indian market by launching a range of electric commercial vehicles from electric auto rickshaw that has a payload capacity of a few hundred kilograms to full-scale electric trucks that pull tonnes of load.

At Truck Junction, more than 326 electric commercial vehicles are available. When compared to a conventional commercial vehicle that has a mileage of 8-14 kmpl, an electric vehicle has a range of up to 300km per charge which varies depending on the payload. The typical charging time of an electric vehicle ranges between 4-6 hours. The payload capacity of a typical electric truck ranges between 3.5 tons to 12 tons. When comparing a conventional vehicle with an electric vehicle from the construction point of view, the major upgrade is the replacement of an IC engine with motors. These are BLDC motors which have an efficiency of around 96-98 percent whereas a conventional IC engine can have a maximum efficiency of 36 percent.

Further, the battery packs are mounted under the frame with the motors either placed at center and the wheels are connected via transmission rods or each wheel has a dedicated motor with an integrated gearbox for high torque application. This gives a strategic advantage for an electric truck in terms of distribution of weight evenly across the length of the vehicle and allows more loading space.

Step 1: Consider that the company's engineering, project and purchasing divisions each have three experts ($cs\mathbb{T}_1$, $cs\mathbb{T}_2$ and $cs\mathbb{T}_3$). The set of 6 suppliers, $\mathbb{ET}_1 - \mathbb{ET}_6$, were selected by the three experts from the departments based on seven distinct criteria, including estimated cost ($\mathbb{C}_1$), delivery efficiency ($\mathbb{C}_2$), product flexibility ($\mathbb{C}_3$), reputation and management level ($\mathbb{C}_4$) and eco-design ($\mathbb{C}_5$).

It is essential for the company to ensure that its suppliers care about the environment and follow green guidelines in how they run their business. A supplier is more appealing to a company if they are more environmentally friendly. Furthermore, it may be beneficial to forge a long-term partnership with eco-friendly providers. The alternative ratings on the linguistic scale [LS] used by the decision-makers are shown in Table 2 together with their particular interpretations.

Linguistics Term	Symbolic representation	$< cs\mathbb{T}_U, cs\mathbb{I}_U, cs\mathbb{F}_U > \times 10^{-1}$
No influence	$\emptyset_1$	$< 1, 8, 9 >$
Low influence	$\emptyset_2$	$< 4, 6, 7 >$
Medium influence	$\emptyset_3$	$< 5, 4, 5 >$
High influence	$\emptyset_4$	$< 8, 2, 2 >$
Very high influence	$\emptyset_5$	$< 9, 1, 1 >$

TABLE 2. LS for the calculation of DM's priorities.

Step 2: Expert recommendations over suppliers according to the each criteria is shown in Table 3.

DM's	$\mathbb{T}_1$					$\mathbb{T}_2$					$\mathbb{T}_3$				
$\mathbb{A}s$	$cs\mathbb{C}_1$	$cs\mathbb{C}_2$	$cs\mathbb{C}_3$	$cs\mathbb{C}_4$	$cs\mathbb{C}_5$	$cs\mathbb{C}_1$	$cs\mathbb{C}_2$	$cs\mathbb{C}_3$	$cs\mathbb{C}_4$	$cs\mathbb{C}_5$	$cs\mathbb{C}_1$	$cs\mathbb{C}_2$	$cs\mathbb{C}_3$	$cs\mathbb{C}_4$	$cs\mathbb{C}_5$
$\mathbb{ET}_1$	$\emptyset_4$	$\emptyset_1$	$\emptyset_3$	$\emptyset_2$	$\emptyset_5$	$\emptyset_3$	$\emptyset_2$	$\emptyset_1$	$\emptyset_4$	$\emptyset_3$	$\emptyset_5$	$\emptyset_1$	$\emptyset_5$	$\emptyset_4$	$\emptyset_5$
$\mathbb{ET}_2$	$\emptyset_1$	$\emptyset_4$	$\emptyset_2$	$\emptyset_3$	$\emptyset_1$	$\emptyset_2$	$\emptyset_3$	$\emptyset_4$	$\emptyset_1$	$\emptyset_2$	$\emptyset_4$	$\emptyset_3$	$\emptyset_4$	$\emptyset_3$	$\emptyset_4$
$\mathbb{ET}_3$	$\emptyset_3$	$\emptyset_4$	$\emptyset_5$	$\emptyset_2$	$\emptyset_5$	$\emptyset_5$	$\emptyset_2$	$\emptyset_2$	$\emptyset_3$	$\emptyset_5$	$\emptyset_3$	$\emptyset_4$	$\emptyset_2$	$\emptyset_2$	$\emptyset_3$
$\mathbb{ET}_4$	$\emptyset_1$	$\emptyset_5$	$\emptyset_1$	$\emptyset_3$	$\emptyset_1$	$\emptyset_1$	$\emptyset_3$	$\emptyset_5$	$\emptyset_1$	$\emptyset_1$	$\emptyset_2$	$\emptyset_1$	$\emptyset_3$	$\emptyset_5$	$\emptyset_2$
$\mathbb{ET}_5$	$\emptyset_5$	$\emptyset_3$	$\emptyset_3$	$\emptyset_1$	$\emptyset_2$	$\emptyset_3$	$\emptyset_4$	$\emptyset_4$	$\emptyset_5$	$\emptyset_3$	$\emptyset_3$	$\emptyset_5$	$\emptyset_1$	$\emptyset_1$	$\emptyset_3$
$\mathbb{ET}_6$	$\emptyset_2$	$\emptyset_3$	$\emptyset_1$	$\emptyset_5$	$\emptyset_4$	$\emptyset_1$	$\emptyset_5$	$\emptyset_3$	$\emptyset_2$	$\emptyset_1$	$\emptyset_2$	$\emptyset_4$	$\emptyset_5$	$\emptyset_3$	$\emptyset_2$

TABLE 3. Expert recommendations over suppliers according to the criteria

Step 3: Table 8 in Appendix A, transforms linguistic evaluations into NVs. The main variable pairwise comparison matrix for each decision maker is shown in Table 9. We take the complement of these values since C_1 and C_4 are the cost criteria. Thus, we obtain the cubic spherical neutrosophic group normalized matrix illustrated in Table 4. The NVs in this decision matrix $[DM]$ must be transformed into $CSNVs$. In this approach, The decision matrix in Table 9 is used to determine the greatest radius values.

Sup	$cs\mathbb{C}_1 \times 10^{-1}$	$cs\mathbb{C}_2 \times 10^{-1}$	$cs\mathbb{C}_3 \times 10^{-1}$	$cs\mathbb{C}_4 \times 10^{-1}$	$cs\mathbb{C}_5 \times 10^{-1}$
$\mathbb{ET}_1$	$< 2, 2, 7; 4 >$	$< 2, 7, 8; 3 >$	$< 5, 4, 5; 7 >$	$< 3, 3, 7; 5 >$	$< 8, 2, 2; 4 >$
$\mathbb{ET}_2$	$< 6, 5, 4; 7 >$	$< 6, 3, 4; 3 >$	$< 7, 3, 3; 5 >$	$< 6, 5, 4; 5 >$	$< 4, 5, 6; 7 >$
$\mathbb{ET}_3$	$< 3, 3, 6; 4 >$	$< 7, 3, 3; 5 >$	$< 5, 4, 5; 6 >$	$< 6, 5, 4; 2 >$	$< 8, 2, 2; 4 >$
$\mathbb{ET}_4$	$< 8, 7, 2; 3 >$	$< 5, 4, 5; 7 >$	$< 5, 4, 5; 7 >$	$< 5, 4, 5; 7 >$	$< 2, 7, 8; 3 >$
$\mathbb{ET}_5$	$< 3, 3, 6; 4 >$	$< 7, 2, 2; 4 >$	$< 5, 5, 5; 6 >$	$< 6, 6, 4; 9 >$	$< 5, 5, 5; 2 >$
$\mathbb{ET}_6$	$< 8, 7, 3; 3 >$	$< 7, 2, 2; 4 >$	$< 5, 4, 5; 7 >$	$< 4, 4, 6; 5 >$	$< 4, 5, 6; 7 >$

TABLE 4. The CSN decision matrix

Step 4: Applying the aggregation operations $\mathbb{CSNWA}^A_{\wp}$, $\mathbb{CSNWA}^A_{\rho}$, $\mathbb{CSNWG}^A_{\wp}$ and $\mathbb{CSNWG}^A_{\rho}$ defined via $*(t) = -log\, t, h(t) = -log\,(1-t), \wp(t) = -log\, t$ and $\rho(t) = -log\,(1-t)$, the decision matrix expressed with $CSNVs$ for all decisions is aggregated. Table 5 presents the aggregated cubic spherical neutrosophic decision matrix using $CSNVs$.

Sup	$\mathbb{CSNWA}^A_{\wp} \times 10^{-1}$	$\mathbb{CSNWA}^A_{\rho} \times 10^{-1}$	$\mathbb{CSNWG}^A_{\wp} \times 10^{-1}$	$\mathbb{CSNWG}^A_{\rho} \times 10^{-1}$
$\mathbb{ET}_1$	$< 4, 4, 6; 4 >$	$< 4, 4, 6; 4 >$	$< 3, 5, 7; 4 >$	$< 3, 5, 7; 4 >$
$\mathbb{ET}_2$	$< 6, 4, 4; 5 >$	$< 6, 4, 4; 5 >$	$< 6, 4, 4; 5 >$	$< 6, 4, 4; 5 >$
$\mathbb{ET}_3$	$< 6, 3, 4; 5 >$	$< 6, 3, 4; 5 >$	$< 6, 3, 4; 5 >$	$< 6, 3, 4; 5 >$
$\mathbb{ET}_4$	$< 6, 5, 4; 5 >$	$< 6, 5, 4; 6 >$	$< 5, 6, 5; 5 >$	$< 5, 6, 5; 6 >$
$\mathbb{ET}_5$	$< 6, 3, 4; 4 >$	$< 6, 3, 4; 5 >$	$< 5, 4, 4; 4 >$	$< 5, 4, 4; 5 >$
$\mathbb{ET}_6$	$< 7, 4, 3; 4 >$	$< 7, 4, 3; 5 >$	$< 6, 4, 4; 4 >$	$< 6, 4, 4; 5 >$

TABLE 5. Aggregated values of CSNSs

Step 5: Each aggregated $CSNV$ and positive ideal alternative are compared using the CSM established in Definition 6.1 to determine exactly related or equivalent they are to each other. The evaluation test between alternatives and the ideal positive alternative's results are shown in Table 6.

Step 6: The ranking of electric trucks:

The selection of electric trucks using cubic spherical neutrosophic sets offers a comprehensive framework for decision-making in the procurement process. By aggregating expert recommendations and criteria evaluations into cubic spherical neutrosophic decision matrices, we can

Methods / $cs\mathbb{CSM}$	$cs\mathbb{CSM}$ (A_1, A^+)	$cs\mathbb{CSM}$ (A_2, A^+)	$cs\mathbb{CSM}$ (A_3, A^+)	$cs\mathbb{CSM}$ (A_4, A^+)	$cs\mathbb{CSM}$ (A_5, A^+)	$cs\mathbb{CSM}$ (A_6, A^+)
$\mathbb{CSNWA}^A_\wp$	0.512	0.700	0.786	0.624	0.761	0.788
$\mathbb{CSNWA}^A_\rho$	0.512	0.700	0.786	0.624	0.761	0.788
$\mathbb{CSNWG}^A_\wp$	0.327	0.677	0.714	0.492	0.686	0.732
$\mathbb{CSNWG}^A_\rho$	0.327	0.677	0.714	0.492	0.686	0.732

TABLE 6. Cosine similarity scores

Methods	Ranking	Best EV
$\mathbb{CSNWAAO} - \mathbb{CD}$ [5]	$\mathbb{ET}_6 > \mathbb{ET}_5 > \mathbb{ET}_3 > \mathbb{ET}_2 > \mathbb{ET}_4 > \mathbb{ET}_1$	$\mathbb{ET}_6$
$\mathbb{CSNWGAO} - \mathbb{CD}$ [5]	$\mathbb{ET}_6 > \mathbb{ET}_3 > \mathbb{ET}_5 > \mathbb{ET}_2 > \mathbb{ET}_1 > \mathbb{ET}_4$	$\mathbb{ET}_6$
$\mathbb{CSNWA}^A_\wp$	$\mathbb{ET}_6 > \mathbb{ET}_3 > \mathbb{ET}_5 > \mathbb{ET}_2 > \mathbb{ET}_4 > \mathbb{ET}_1$	$\mathbb{ET}_6$
$\mathbb{CSNWA}^A_\rho$	$\mathbb{ET}_6 > \mathbb{ET}_3 > \mathbb{ET}_5 > \mathbb{ET}_2 > \mathbb{ET}_4 > \mathbb{ET}_1$	$\mathbb{ET}_6$
$\mathbb{CSNWG}^A_\wp$	$\mathbb{ET}_6 > \mathbb{ET}_3 > \mathbb{ET}_5 > \mathbb{ET}_2 > \mathbb{ET}_4 > \mathbb{ET}_1$	$\mathbb{ET}_6$
$\mathbb{CSNWG}^A_\rho$	$\mathbb{ET}_6 > \mathbb{ET}_3 > \mathbb{ET}_5 > \mathbb{ET}_2 > \mathbb{ET}_4 > \mathbb{ET}_1$	$\mathbb{ET}_6$

TABLE 7. Overall ranking of Electric trucks

effectively rank electric trucks based on various criteria. The final Table 7 presents the overall ranking of electric trucks using different aggregation methods, providing valuable insights for decision-makers in selecting the most suitable electric truck for their needs.

Comparison Analysis

We compared the results of proposed methods with the existing CSNS methods and their visualization represents the overall raking of electric trucks are presented : The cubic spherical

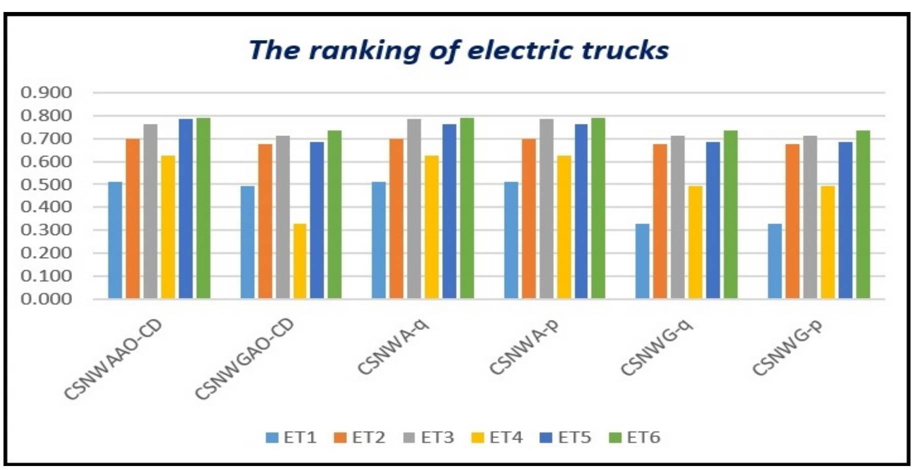


FIGURE 1. Comparison of Ranking of Electric Trucks

neutrosophic set utilizes a spherical framework to portray uncertainty among true, false, and neutral functions, addressing vagueness comprehensively. This method surpasses traditional averaging, providing a collective opinion representation. In Multi-Criteria Decision Making (MCDM), the decision maker's engagement determines criteria weights and preferences; a substantial influence is recommended for sphere representation. Recognizing CSNS limitations is crucial for effective MCDM utilization. Addressing constraints enhances CSNS applicability and reliability in decision-making contexts.

7. Comparison of Cubic Spherical Neutrosophic Sets (CSNS) with Traditional Neutrosophic Sets (NS)

To compare cubic spherical neutrosophic sets (CSNS) with traditional neutrosophic sets (NS), we consider several aspects:

(1) **Representation of Uncertainty:**
Neutrosophic sets and CSNS handle uncertainty through three parameters: truth membership (T), indeterminacy membership (I) and falsity membership (F). However, CSNS extend this representation by adding a fourth parameter, the radius (r), which captures the degree of neutrality or neutrality in the information provided.

(2) **Geometric Interpretation:**
CSNS provide a geometric interpretation of uncertainty in a hypersphere, where the center represents T, I and F and the radius represents the degree of neutrality (r). This geometric interpretation allows for a more intuitive understanding of uncertainty.

(3) **Aggregation Operators:**
While neutrosophic sets have aggregation operators for combining uncertain information, CSNS introduce weighted geometric aggregation operators tailored specifically to handle uncertainty represented in hyperspheres. These operators take into account T, I, F and r, providing a comprehensive way to combine uncertain information.

(4) **Handling Neutrality:**
CSNS explicitly account for neutrality through the radius parameter, which represents the degree to which an element is neither true nor false nor indeterminate. This allows for a more nuanced handling of neutrality compared to traditional neutrosophic sets.

(5) **Archimedean Triangular Norms and Conorms:**
Archimedean triangular norms and conorms are commonly used in fuzzy logic and fuzzy set theory to model conjunction and disjunction operations. CSNATN and CSNATCN extend these operations to CSNS, allowing for the combination of uncertain information in a way that respects the geometric structure of hyperspheres.

8. Conclusion

The main aim of this study is to introduce the concept of a $CSNS$, defined as a sphere with radius r and a triple at its center, representing membership, indeterminacy, and non-participation. $CSNSs$ extend the idea of NSs by depicting these degrees through spheres. Arithmetic operators such as $CSNWA$ and $CSNWG$ are crucial for integrating neutrosophic data. To overcome limitations of existing operators like $NWAO$ and $NWGO$, we propose "weighted arithmetic cubic spherical neutrosophic aggregation operators" and "weighted geometric cubic spherical neutrosophic aggregation operators," offering improved reliability and effectiveness. An $MCDM$ method is developed for selecting the best electric truck based on these operators. Future work aims to enhance other arithmetic operators like Dombi, Hamacher, and Einstein through this framework.

Appendix A

Expert	$\mathbb{A}$	$cs\mathbb{C}_1 \times 10^{-1}$	$cs\mathbb{C}_2 \times 10^{-1}$	$cs\mathbb{C}_3 \times 10^{-1}$	$cs\mathbb{C}_4 \times 10^{-1}$	$cs\mathbb{C}_5 \times 10^{-1}$
$\mathbb{T}_1$	$\mathbb{ET}_1$	$<8,2,2>$	$<1,8,9>$	$<5,4,5>$	$<4,6,7>$	$<9,1,1>$
	$\mathbb{ET}_2$	$<1,8,9>$	$<8,2,2>$	$<4,6,7>$	$<5,4,5>$	$<1,8,9>$
	$\mathbb{ET}_3$	$<5,4,5>$	$<8,2,2>$	$<9,1,1>$	$<4,6,7>$	$<9,1,1>$
	$\mathbb{ET}_4$	$<1,8,9>$	$<9,1,1>$	$<1,8,9>$	$<5,4,5>$	$<1,8,9>$
	$\mathbb{ET}_5$	$<9,1,1>$	$<5,4,5>$	$<5,4,5>$	$<1,8,9>$	$<4,6,7>$
	$\mathbb{ET}_6$	$<4,6,7>$	$<5,4,5>$	$<1,8,9>$	$<9,1,1>$	$<8,2,2>$
$\mathbb{T}_2$	$\mathbb{ET}_1$	$<5,4,5>$	$<4,6,7>$	$<1,8,9>$	$<8,2,2>$	$<5,4,5>$
	$\mathbb{ET}_2$	$<4,6,7>$	$<5,4,5>$	$<8,2,2>$	$<1,8,9>$	$<4,6,7>$
	$\mathbb{ET}_3$	$<9,1,1>$	$<4,6,7>$	$<4,6,7>$	$<5,4,5>$	$<9,1,1>$
	$\mathbb{ET}_4$	$<1,8,9>$	$<5,4,5>$	$<9,1,1>$	$<1,8,9>$	$<1,8,9>$
	$\mathbb{ET}_5$	$<5,4,5>$	$<8,2,2>$	$<8,2,2>$	$<9,1,1>$	$<5,4,5>$
	$\mathbb{ET}_6$	$<1,8,9>$	$<9,1,1>$	$<5,4,5>$	$<4,6,7>$	$<1,8,9>$
$\mathbb{T}_3$	$\mathbb{ET}_1$	$<9,1,1>$	$<1,8,9>$	$<9,1,1>$	$<8,2,2>$	$<9,1,1>$
	$\mathbb{ET}_2$	$<8,2,2>$	$<5,4,5>$	$<8,2,2>$	$<5,4,5>$	$<8,2,2>$
	$\mathbb{ET}_3$	$<5,4,5>$	$<8,2,2>$	$<4,6,7>$	$<4,6,7>$	$<5,4,5>$
	$\mathbb{ET}_4$	$<4,6,7>$	$<1,8,9>$	$<5,4,5>$	$<9,1,1>$	$<4,6,7>$
	$\mathbb{ET}_5$	$<5,4,5>$	$<9,1,1>$	$<1,8,9>$	$<1,8,9>$	$<5,4,5>$
	$\mathbb{ET}_6$	$<4,6,7>$	$<8,2,2>$	$<9,1,1>$	$<5,4,5>$	$<4,6,7>$

TABLE 8. The matrix of pairwise comparisons for the primary DM evaluation

Expert	$\mathbb{A}$	$cs\mathbb{C}_1 \times 10^{-1}$	$cs\mathbb{C}_2 \times 10^{-1}$	$cs\mathbb{C}_3 \times 10^{-1}$	$cs\mathbb{C}_4 \times 10^{-1}$	$cs\mathbb{C}_5 \times 10^{-1}$
$\mathbb{T}_1$	$\mathbb{ET}_1$	$<2,2,8>$	$<1,8,9>$	$<5,4,5>$	$<7,6,4>$	$<9,1,1>$
	$\mathbb{ET}_2$	$<9,8,1>$	$<8,2,2>$	$<4,6,7>$	$<5,4,5>$	$<1,8,9>$
	$\mathbb{ET}_3$	$<5,4,5>$	$<8,2,2>$	$<9,1,1>$	$<7,6,4>$	$<9,1,1>$
	$\mathbb{ET}_4$	$<9,8,1>$	$<9,1,1>$	$<1,8,9>$	$<5,4,5>$	$<1,8,9>$
	$\mathbb{ET}_5$	$<1,1,9>$	$<5,4,5>$	$<5,4,5>$	$<9,8,1>$	$<4,6,7>$
	$\mathbb{ET}_6$	$<7,6,4>$	$<5,4,5>$	$<1,8,9>$	$<1,1,9>$	$<8,2,2>$
$\mathbb{T}_2$	$\mathbb{ET}_1$	$<5,4,5>$	$<4,6,7>$	$<2,2,8>$	$<5,4,5>$	$<5,4,5>$
	$\mathbb{ET}_2$	$<7,6,4>$	$<5,4,5>$	$<8,2,2>$	$<9,8,1>$	$<4,6,7>$
	$\mathbb{ET}_3$	$<1,1,9>$	$<4,6,7>$	$<4,6,7>$	$<5,4,5>$	$<9,1,1>$
	$\mathbb{ET}_4$	$<9,8,1>$	$<5,4,5>$	$<9,1,1>$	$<9,8,1>$	$<1,8,9>$
	$\mathbb{ET}_5$	$<5,4,5>$	$<8,2,2>$	$<8,2,2>$	$<1,1,9>$	$<5,4,5>$
	$\mathbb{ET}_6$	$<9,8,1>$	$<9,1,1>$	$<5,4,5>$	$<7,6,4>$	$<1,8,9>$
$\mathbb{T}_3$	$\mathbb{ET}_1$	$<1,1,9>$	$<1,8,9>$	$<9,1,1>$	$<2,2,8>$	$<9,1,1>$
	$\mathbb{ET}_2$	$<2,2,8>$	$<5,4,5>$	$<8,2,2>$	$<5,4,5>$	$<8,2,2>$
	$\mathbb{ET}_3$	$<5,4,5>$	$<8,2,2>$	$<4,6,7>$	$<7,6,4>$	$<5,4,5>$
	$\mathbb{ET}_4$	$<7,6,4>$	$<1,8,9>$	$<5,4,5>$	$<1,1,9>$	$<4,6,7>$
	$\mathbb{ET}_5$	$<5,4,5>$	$<9,1,1>$	$<1,8,9>$	$<9,8,1>$	$<5,4,5>$
	$\mathbb{ET}_6$	$<7,6,4>$	$<8,2,2>$	$<9,1,1>$	$<5,4,5>$	$<4,6,7>$

Table 9. Normalized decision matrix

References

[1] Ashraf, S.; Abdullah, S.; Smarandache, F.; Amin, N.u. Logarithmic hybrid aggregation operators based on single-valued neutrosophic sets and their applications in decision support systems. *Symmetry* **2019**, 11, 364. https://doi.org/10.3390/sym11030364

[2] Baczynski, W.M.; Jayaram, B. Fuzzy Implications. *Studies in Fuzziness and Soft Computing* **2008**, 231.

[3] Deli, I; Ali, M.; Smarandache, F. Bipolar neutrosophic sets and their applications based on multicriteria decision making problems. Advanced Mechatronic Systems. *(ICAMechs) International Conference* **2015**, 249-254. doi: 10.1109/ICAMechS.2015.7287068

[4] Garg, H.; Nancy. Multi-Criteria Decision-Making Method Based on Prioritized Muirhead Mean Aggregation Operator under Neutrosophic Set Environment. *Symmetry* **2018**, 10, 280. https://doi.org/10.3390/sym10070280

[5] Gomathi, S.; Krishnaprakash, S.; Karpagadevi, M.; Said Broumi. Cubic Spherical Neutrosophic Sets. *International Journal of Neutrosophic Science* **2023**, 21(4), 172-180.

[6] Hamid, M. T.; Riaz, M.; Naeem, K. A study on weighted aggregation operators for q-rung orthopair m-polar fuzzy set with utility to multistage decision analysis, *International Journal of Intelligent Systems* **2022**, 37(9), 6354-6387. https://doi.org/10.1002/int.22847

[7] Hu, B.; Kwong, C. K. On type-2 fuzzy sets and their t-norm operations. *Inf. Sci.* **2014**, 255, 5881. https://doi.org/10.1016/j.ins.2013.07.023

[8] Hu, Q.; Zhang, X. New Similarity Measures of Single-Valued Neutrosophic Multisets Based on the Decomposition Theorem and Its Application in Medical Diagnosis. *Symmetry* **2018**, 10, 466. https://doi.org/10.3390/sym10100466

[9] Klement, E. P.; Mesiar, R.; Pap, E. Triangular norms. Position paper I: basic analytical and algebraic properties, *Fuzzy Sets and Systems* **2004**, 143(1), 5-26. doi:10.1016/j.fss.2003.06.007

[10] Kumam, W.; Naeem, K.; Riaz, M.; Khan, M. J.; Kumam, P. Comparison measures for Pythagorean m-polar fuzzy sets and their applications to robotics and movie recommender system. *AIMS Mathematics* **2023**, 8(5), 10357-10378. doi: 10.3934/math.2023524

[11] Klement, E. P.; Mesiar, R.; Pap, E. Triangular norms. Position paper I: Basic analytical and algebraic properties. *Fuzzy Sets Syst.* **2004**, 143, 526. https://doi.org/10.1016/j.fss.2003.06.007

[12] Li, D. Type-2 triangular norms and their residual operators. *Inf. Sci.* **2015**, 317, 259277. https://doi.org/10.1016/j.ins.2016.01.092

[13] Liu, P. The aggregation operators based on archimedean t-conorm and t-norm for single-valued neutrosophic numbers and their application to decision making. *Int. J. Fuzzy Syst.* **2016**, 18, 849863.

[14] Logical Algebraic, Analytic and Probabilistic Aspects of Triangular Norms. New York, NY, USA: Elsevier; **2005** edited by E.P Klement and R. Mesiar.

[15] Menger, K. Statistical metrics. Natl. *Acad. Sci.* **1942**, 28, 535537. doi: 10.1073/pnas.28.12.535

[16] Molla, M. U.; Giri, B. C. Multi-criteria decision making problem with spherical neutrosophic sets. *OPSEARCH* **2022**. https://doi.org/10.1007/s12597-022-00593-1

[17] Naeem, K.; Divvaz, B. Information measures for MADM under m-polar neutrosophic environment. *Granular Computing* **2023**, 8, 597616. https://doi.org/10.1007/s41066- 022-00340-3

[18] Nguyen H. T; Walker R. A. A First Course in Fuzzy Logic. *Boca Raton, Fla, USA: CRC Press*, **1997**.

[19] Palanikumar, M.; Arulmozhi, K. MCGDM based on TOPSIS and VIKOR using Pythagorean neutrosophic soft with aggregation operators. *Neutrosophic Sets and Systems* **2022**, 51, 1.

[20] Peide Liu; Yanchang Chu; Yanwei Li; Yubao Chen. Some Generalized Neutrosophic Number Hamacher Aggregation Operators and Their Application to Group Decision Making. *International Journal of Fuzzy Systems* **2014**, 16(2), 242-255.

[21] Peng, H g.; Zhang, Hy.; Wang, Jq. Probability multi-valued neutrosophic sets and its application in multi-criteria group decision-making problems. *Neural Comput and Applic* **2018** 30, 563583. https://doi.org/10.1007/s00521-016-2702-0

[22] Rehman, Ateeq Ur; Muhammad Gulistan; Zahid Khan; Fuad S. Al-Duais. A Study of Neutrosophic Cubic Hesitant Fuzzy Hybrid Geometric Aggregation Operators and its Application to Multi Expert Decision Making System. *Neutrosophic Sets and Systems* **2022**, 50, 1. https://digitalrepository.unm.edu/nss_journal/vol50/iss1/5

[23] Riaz, M.; Naeem, K. and Afzal, D. A similarity measure under Pythagorean fuzzy soft environment with applications. *Comp. Appl. Math.* **2020**, 39, 269. https://doi.org/10.1007/s40314-020-01321-5

[24] Schweizer, B.; Sklar, A. Associative mappings and statistical triangle inequalities. *Publ. Math.* **1961**, 8, 169186.

[25] Schweizer, B.; Sklar, A. Probabilistic Metric Spaces. *North-Holl and: New York, NY, USA* **1983**.

[26] Siraj, A; Fatima, T., Afzal, D., Naeem, K. and Karaaslan, F. Pythagorean m-polar Fuzzy Neutrosophic Topology with Applications. *Neutrosophic Sets and Systems* **2022**, 48, 251-290. Retrieved from https://fs.unm.edu/NSS2/index.php/111/article/view/2103

[27] Smarandache, F. Neutrosophy: Neutrosophic Probability. Set and Logic: Analytic Synthesis and Synthetic Analysis. *American Research Press: Santa Fe, NM, USA* **1998**, 2538.

[28] Wang; Haibin; Florentin Smarandache; Yan-Qing Zhang; Rajshekhar Sunderraman. *Interval neutrosophic sets and logic; theory and applications in computing* **2005**. https://digitalrepository.unm.edu/math_fsp/38

[29] Wang, H.; Smarandache, F.; Zhang, Y.; Sunderraman, R. "Single valued neutrosophic sets. Multispace Multistruct. *Neutrosophic Transdiscipl.* **2010**, 4, 410413.

[30] Wang, L.; Zhang, Hy.; Wang, Jq. Frank Choquet Bonferroni Mean Operators of Bipolar Neutrosophic Sets and Their Application to Multi-criteria Decision-Making Problems. *Int. J. Fuzzy Syst* **2018**. 20, 1328. https://doi.org/10.1007/s40815-017-0373-3

[31] Wu, Xh., Wang, Jq.; Peng, Jj. et al. Cross-Entropy and Prioritized Aggregation Operator with Simplified Neutrosophic Sets and Their Application in Multi-Criteria Decision-Making Problems. *Int. J. Fuzzy Syst.* **2016**, 18, 11041116. https://doi.org/10.1007/s40815-016-0180-2

[32] Yingying Zhang; Rui Yong; Jun Ye, Zhen Zhong; Shuang Zheng. Dombi Aggregation Operators of Linguistic Neutrosophic Numbers for Multiple Attribute Group Decision-Making Problems in Landslide Treatment Schemes. *Neutrosophic Sets and Systems* **2022**, 51, 724-743. doi: 10.5281/zenodo.7135407

[33] Ye, J. Single valued neutrosophic minimum spanning tree and its clustering method. *Journal of Intelligent Systems* **2014**, 23(3), 311324.

[34] Ye, Jun and Florentin Smarandache. Similarity measure of refined single-Valued neutrosophic sets and its multicriteria decision making method. *Neutrosophic Sets and Systems* **2016**, 12, 1. https://digitalrepository.unm.edu/nss_journal/vol12/iss1/5

[35] Ye, Jun. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *Journal of Intelligent and Fuzzy Systems* **2014**, 26(5), 2459-2466. doi: 10.3233/IFS-130916

[36] Zhang, B. Notes on type-2 triangular norms and their residual operators. *Inf. Sci.* **2016**, 346, 338350. doi: https://doi.org/10.1016/j.ins.2016.01.092

Received: Feb 5, 2024. / Accepted: April 30, 2024.

NSS Neutrosophic Sets and Systems, Vol. 67, 2024

University of New Mexico

New Similarity measures for Neutrosophic Binary topology using Euclidean distance.

J.Elekiah[1] and G.Sindhu[2]

[1]Research Scholar, Department of Mathematics,Nirmala College for Women, Coimbatore,(TN) India; elekiahmat2020@gmail.com

[2]Associate Professor, Department of Mathematics, Nirmala College for Women, Coimbatore,(TN) India; sindhukannan23@gmail.com

*Correspondence: elekiahmat2020@gmail.com

Abstract. This paper focuses on introducing a new similarity measure for the neutrosophic binary set.Similarity measure are used in multi attribute desicion making problems to find the difference between the alternatives.In this paper a new measure based on Euclidean distance is introduced to find the measure between two binary single valued neutrosophic set.Further its is applied in a multi attribute desicion problem to see the attainability of the proposed measure.

Keywords: Similarity measure, Euclidean distance, Neutrosophic binary set .

1. Introduction

The concept of neutrosophy was mainly used during the problems with uncertinity.Similarity metric is used in multi-attribute decision making problems to measure the difference between the attributes.Majumdar and Samanta [29] proposed a similarity function between Single valued neutrosophic set based on the membership degree.Donghai Liu,Guangyan Liu and Zaiming Liu proposed a new similarity measure based on the similarity measure proposed by Majumdar and Samanta [29] for single valued neutrosophic set.In this paper a new similarity function for a neutrosophic binary set is introduced based on the similarity measure proposed by Donghai Liu,Guangyan

Liu and Zaiming Liu [20] and its been checked with a real life situation. In this paper a sample with both male and female of all age group is taken and their preference for their well being is been analysed using this new similarity measure. The final result is been shown in a pictorial form in five different age category like less than 25 years of age, 26-35 years of age,36-45 years of age, 46-55 years of age and above 56 years of age.

2. Preliminaries

Some basis definition of Similarity measure, Euclidean measure are defined in this section.

Definition 2.1. [22] Let $\tilde{X} = \{\tilde{x}_i, 1 \leq i \leq n\}$ and $\tilde{Y} = \{\tilde{y}_i, 1 \leq i \leq n\}$ be the universal sets. The Neutrosophic binary set $(\mathcal{A}, \mathcal{B}) \subseteq (\tilde{X}, \tilde{Y})$ is given by

$$(\mathcal{A}, \mathcal{B}) = \{< \tilde{X}, (\mu_{\mathcal{A}}(\tilde{x}_i), \sigma_{\mathcal{A}}(\tilde{x}_i), \gamma_{\mathcal{A}}(\tilde{x}_i) >; \tilde{x}_i \in \tilde{X}, \\ < \tilde{Y}, (\mu_{\mathcal{B}}(\tilde{y}_i), \sigma_{\mathcal{B}}(\tilde{y}_i), \gamma_{\mathcal{B}}(\tilde{y}_i) >; \tilde{y}_i \in \tilde{Y}\}$$

where $\mu_{\mathcal{A}}, \sigma_{\mathcal{A}}, \gamma_{\mathcal{A}} \rightarrow [0,1]$; $\mu_{\mathcal{B}}, \sigma_{\mathcal{B}}, \gamma_{\mathcal{B}} \rightarrow [0,1]$ and $0 \leq \mu_{\mathcal{A}}(\tilde{x}) + \sigma_{\mathcal{A}}(\tilde{x}) + \gamma_{\mathcal{A}}(\tilde{x}) \leq 3$; $0 \leq \mu_{\mathcal{B}}(\tilde{y}) + \sigma_{\mathcal{B}}(\tilde{y}) + \gamma_{\mathcal{B}}(\tilde{y}) \leq 3$. The Neutrosophic Binary Set over the universe $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})$ is denoted as $M_N(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})$.

Definition 2.2. [22][Empty set and Universal Set] Let $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ be the universe. Then, (Empty Set) $(0_{\tilde{\mathcal{X}}}, 0_{\tilde{\mathcal{Y}}})$ can be defined as

(0_1) $0_{\tilde{\mathcal{X}}} = \{< \tilde{x}, 0, 0, 1 >: \tilde{x} \in \tilde{\mathcal{X}}\}$, $0_{\tilde{\mathcal{Y}}} = \{< \tilde{y}, 0, 0, 1 >: \tilde{y} \in \tilde{\mathcal{Y}}\}$

(0_2) $0_{\tilde{\mathcal{X}}} = \{< \tilde{x}, 0, 1, 1 >: \tilde{x} \in \tilde{\mathcal{X}}\}$, $0_{\tilde{\mathcal{Y}}} = \{< \tilde{y}, 0, 1, 1 >: \tilde{y} \in \tilde{\mathcal{Y}}\}$

(0_3) $0_{\tilde{\mathcal{X}}} = \{< \tilde{x}, 0, 1, 0 >: \tilde{x} \in \tilde{\mathcal{X}}\}$, $0_{\tilde{\mathcal{Y}}} = \{< \tilde{y}, 0, 1, 0 >: \tilde{y} \in \tilde{\mathcal{Y}}\}$

(0_4) $0_{\tilde{\mathcal{X}}} = \{< \tilde{x}, 0, 0, 1 >: \tilde{x} \in \tilde{\mathcal{X}}\}$, $0_{\tilde{\mathcal{Y}}} = \{< \tilde{y}, 0, 0, 0 >: \tilde{y} \in \tilde{\mathcal{Y}}\}$

(Universal Set) $(1_{\tilde{\mathcal{X}}}, 1_{\tilde{\mathcal{Y}}})$ can be defined as

(1_1) $1_{\tilde{\mathcal{X}}} = \{< \tilde{x}, 1, 0, 0 >: \tilde{x} \in \tilde{\mathcal{X}}\}$, $1_{\tilde{\mathcal{Y}}} = \{< \tilde{y}, 1, 0, 0 >: \tilde{y} \in \tilde{\mathcal{Y}}\}$

(1_2) $1_{\tilde{\mathcal{X}}} = \{< \tilde{x}, 1, 0, 1 >: \tilde{x} \in \tilde{\mathcal{X}}\}$, $1_{\tilde{\mathcal{Y}}} = \{< \tilde{y}, 1, 0, 1 >: \tilde{y} \in \tilde{\mathcal{Y}}\}$

(1_3) $1_{\tilde{\mathcal{X}}} = \{< \tilde{x}, 1, 1, 0 >: \tilde{x} \in \tilde{\mathcal{X}}\}$, $1_{\tilde{\mathcal{Y}}} = \{< \tilde{y}, 1, 1, 0 >: \tilde{y} \in \tilde{\mathcal{Y}}\}$

(1_4) $1_{\tilde{\mathcal{X}}} = \{< \tilde{x}, 1, 1, 1 >: \tilde{x} \in \tilde{\mathcal{X}}\}$, $1_{\tilde{\mathcal{Y}}} = \{< \tilde{y}, 1, 1, 1 >: \tilde{y} \in \tilde{\mathcal{Y}}\}$

Definition 2.3. [22](Complement) Let $(\mathcal{A}, \mathcal{B}) = \{< \mu_{\mathcal{A}}, \sigma_{\mathcal{A}}, \gamma_{\mathcal{A}} >, < \mu_{\mathcal{B}}, \sigma_{\mathcal{B}}, \gamma_{\mathcal{B}} >\}$ be a neutrosophic binary set on $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, \mathcal{M}_{\mathcal{N}})$, then the complement of the set $C(A, B)$ may be defined as

$$
\begin{aligned}
(C_1)\quad C(\mathcal{A},\mathcal{B}) =& \{\tilde{x}, < 1-\mu_{\mathcal{A}}(\tilde{x}), \sigma_{\mathcal{A}}(\tilde{x}), 1-\gamma_{\mathcal{A}}(\tilde{x}) >: \tilde{x}\in\tilde{\mathcal{X}},\\
& < \tilde{y}, 1-\mu_B(\tilde{y}), \sigma_B(\tilde{y}), 1-\gamma_B(\tilde{y}) >: \tilde{y}\in\tilde{\mathcal{Y}}\}\\
(C_2)\quad C(\mathcal{A},\mathcal{B}) =& \{\tilde{x}, < \gamma_{\mathcal{A}}(\tilde{x}), \sigma_{\mathcal{A}}(\tilde{x}), \mu_{\mathcal{A}}(\tilde{x}) >: \tilde{x}\in\tilde{\mathcal{X}},\\
& < \tilde{y}, \gamma_{\mathcal{B}}(\tilde{y}), \sigma_{\mathcal{B}}(\tilde{y}), \mu_{\mathcal{B}}(\tilde{y}) >: \tilde{y}\in\tilde{\mathcal{Y}}\tilde{y}\in\tilde{\mathcal{Y}}\}\\
(C_3)\quad C(\mathcal{A},\mathcal{B}) =& \{\tilde{x}, < \gamma_{\mathcal{A}}(\tilde{x}), 1-\sigma_{\mathcal{A}}(\tilde{x}), \mu_{\mathcal{A}}(\tilde{x}) >: \tilde{x}\in\tilde{\mathcal{X}},\\
& < \tilde{y}, \gamma_{\mathcal{B}}(\tilde{y}), 1-\sigma_{\mathcal{B}}(\tilde{y}), \mu_{\mathcal{B}}(\tilde{y}) >: \tilde{y}\in\tilde{\mathcal{Y}}\}
\end{aligned}
$$

Definition 2.4. [22](Inclusion) Let $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ be two neutrosophic binary sets which is in the form
$(\mathcal{A},\mathcal{B}) = \{< \mu_{\mathcal{A}}, \sigma_{\mathcal{A}}, \gamma_{\mathcal{A}} >, < \mu_{\mathcal{B}}, \sigma_{\mathcal{B}}, \gamma_{\mathcal{B}} >\}$ and
$(\mathcal{C},\mathcal{D}) = \{< \mu_{\mathcal{C}}, \sigma_{\mathcal{C}}, \gamma_{\mathcal{C}} >, < \mu_{\mathcal{D}}, \sigma_{\mathcal{D}}, \gamma_{\mathcal{D}} >\}$.
Then $(\mathcal{A},\mathcal{B}) \subseteq (\mathcal{C},\mathcal{D})$ can be defined as

$$
\begin{aligned}
(\mathcal{A},\mathcal{B}) \subseteq (\mathcal{C},\mathcal{D}) \iff & \mu_{\mathcal{A}}(\tilde{x}) \le \mu_{\mathcal{C}}(\tilde{x}), \sigma_{\mathcal{A}}(\tilde{x}) \le \sigma_{\mathcal{C}}(\tilde{x}), \gamma_{\mathcal{A}}(\tilde{x}) \ge \gamma_{\mathcal{C}}(\tilde{x}) \forall \tilde{x}\in\tilde{\mathcal{X}}\\
& \mu_{\mathcal{B}}(\tilde{y}) \le \mu_{\mathcal{D}}(\tilde{y}), \sigma_{\mathcal{B}}(\tilde{y}) \le \sigma_{\mathcal{D}}(\tilde{y}), \gamma_{\mathcal{B}}(\tilde{y}) \ge \gamma_{\mathcal{D}}(\tilde{y}) \forall \tilde{y}\in\tilde{\mathcal{Y}}
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{A},\mathcal{B}) \subseteq (\mathcal{C},\mathcal{D}) \iff & \mu_{\mathcal{A}}(\tilde{x}) \le \mu_{\mathcal{C}}(\tilde{x}), \sigma_{\mathcal{A}}(\tilde{x}) \ge \sigma_{\mathcal{C}}(\tilde{x}), \gamma_{\mathcal{A}}(\tilde{x}) \ge \gamma_{\mathcal{C}}(\tilde{x}) \forall \tilde{x}\in\tilde{\mathcal{X}}\\
& \mu_{\mathcal{B}}(\tilde{y}) \le \mu_{\mathcal{D}}(\tilde{y}), \sigma_{\mathcal{B}}(\tilde{y}) \ge \sigma_{\mathcal{D}}(\tilde{y}), \gamma_{\mathcal{B}}(\tilde{y}) \ge \gamma_{\mathcal{D}}(\tilde{y}) \forall \tilde{y}\in\tilde{\mathcal{Y}}
\end{aligned}
$$

Definition 2.5. [22][Intersection and Union] Let $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ be two neutrosophic binary sets which is in the form
$(\mathcal{A},\mathcal{B}) = \{< \mu_{\mathcal{A}}, \sigma_{\mathcal{A}}, \gamma_{\mathcal{A}} >, < \mu_{\mathcal{B}}, \sigma_{\mathcal{B}}, \gamma_{\mathcal{B}} >\}$ and
$(\mathcal{C},\mathcal{D}) = \{< \mu_{\mathcal{C}}, \sigma_{\mathcal{C}}, \gamma_{\mathcal{C}} >, < \mu_{\mathcal{D}}, \sigma_{\mathcal{D}}, \gamma_{\mathcal{D}} >\}$.
(1) Intersection: $(\mathcal{A},\mathcal{B}) \cap (\mathcal{C},\mathcal{D})$ can be defined as

$$
\begin{aligned}
(\mathcal{A},\mathcal{B}) \cap (\mathcal{C},\mathcal{D}) = \{ & < \tilde{x}, \mu_{\mathcal{A}}(\tilde{x}) \wedge \mu_{\mathcal{C}}(\tilde{x}), \sigma_{\mathcal{A}}(\tilde{x}) \wedge \sigma_{\mathcal{C}}(\tilde{x}), \gamma_{\mathcal{A}}(\tilde{x}) \vee \gamma_{\mathcal{A}}(\tilde{x}) >\\
& < \tilde{y}, \mu_{\mathcal{A}}(\tilde{y}) \wedge \mu_{\mathcal{C}}(\tilde{y}), \sigma_{\mathcal{A}}(\tilde{y}) \wedge \sigma_{\mathcal{C}}(\tilde{y}), \gamma_{\mathcal{A}}(\tilde{y}) \vee \gamma_{\mathcal{A}}(\tilde{y}) >\}
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{A},\mathcal{B}) \cap (\mathcal{C},\mathcal{D}) = \{ & < \tilde{x}, \mu_{\mathcal{A}}(\tilde{x}) \wedge \mu_{\mathcal{C}}(\tilde{x}), \sigma_{\mathcal{A}}(\tilde{x}) \vee \sigma_{\mathcal{C}}(\tilde{x}), \gamma_{\mathcal{A}}(\tilde{x}) \vee \gamma_{\mathcal{A}}(\tilde{x}) >\\
& < \tilde{y}, \mu_{\mathcal{A}}(\tilde{y}) \wedge \mu_{\mathcal{C}}(\tilde{y}), \sigma_{\mathcal{A}}(\tilde{y}) \vee \sigma_{\mathcal{C}}(\tilde{y}), \gamma_{\mathcal{A}}(\tilde{y}) \vee \gamma_{\mathcal{A}}(\tilde{y}) >\}
\end{aligned}
$$

(2) Union: $(\mathcal{A},\mathcal{B}) \cup (\mathcal{C},\mathcal{D})$ can be defined as

$$
\begin{aligned}
(\mathcal{A},\mathcal{B}) \cup (\mathcal{C},\mathcal{D}) = \{ & < \tilde{x}, \mu_{\mathcal{A}}(\tilde{x}) \vee \mu_{\mathcal{C}}(\tilde{x}), \sigma_{\mathcal{A}}(\tilde{x}) \vee \sigma_{\mathcal{C}}(\tilde{x}), \gamma_{\mathcal{A}}(\tilde{x}) \wedge \gamma_{\mathcal{A}}(\tilde{x}) >\\
& < \tilde{y}, \mu_{\mathcal{A}}(\tilde{y}) \vee \mu_{\mathcal{C}}(\tilde{y}), \sigma_{\mathcal{A}}(\tilde{y}) \vee \sigma_{\mathcal{C}}(\tilde{y}), \gamma_{\mathcal{A}}(\tilde{y}) \wedge \gamma_{\mathcal{A}}(\tilde{y}) >\}
\end{aligned}
$$

$$(\mathcal{A},\mathcal{B})\cap(\mathcal{C},\mathcal{D}) = \{ <\tilde{x},\mu_{\mathcal{A}}(\tilde{x})\vee\mu_{\mathcal{C}}(\tilde{x}),\sigma_{\mathcal{A}}(\tilde{x})\wedge\sigma_{\mathcal{C}}(\tilde{x}),\gamma_{\mathcal{A}}(\tilde{x})\wedge\gamma_{\mathcal{A}}(\tilde{x})> \\ <\tilde{y},\mu_{\mathcal{A}}(\tilde{y})\vee\mu_{\mathcal{C}}(\tilde{y}),\sigma_{\mathcal{A}}(\tilde{y})\wedge\sigma_{\mathcal{C}}(\tilde{y}),\gamma_{\mathcal{A}}(\tilde{y})\wedge\gamma_{\mathcal{A}}(\tilde{y})>\}$$

Definition 2.6. [22] A Neutrosophic binary topology from $\tilde{\mathcal{X}}$ to $\tilde{\mathcal{Y}}$ is a binary structure $M_{\mathcal{N}} \subseteq P(\tilde{\mathcal{X}}) \times P(\tilde{\mathcal{Y}})$ that satisfies the following conditions:

(1) $(0_{\tilde{\mathcal{X}}}, 0_{\tilde{\mathcal{Y}}}) \in M_{\mathcal{N}}$ and $(1_{\tilde{\mathcal{X}}}, 1_{\tilde{\mathcal{Y}}}) \in M_{\mathcal{N}}$.

(2) $(\mathcal{A}_1 \cap \mathcal{A}_2, \mathcal{B}_1 \cap \mathcal{B}_2) \in M_{\mathcal{N}}$ whenever $(\mathcal{A}_1, \mathcal{B}_1) \in M_{\mathcal{N}}$ and $(\mathcal{A}_2, \mathcal{B}_2) \in M_{\mathcal{N}}$.

(3) If $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)_{\alpha\in\Delta}$ is a family of members of $M_{\mathcal{N}}$, then $(\cup_{\alpha\in\Delta}\mathcal{A}_\alpha, \cup_{\alpha\in\Delta}\mathcal{B}_\alpha) \in M_{\mathcal{N}}$.

The triplet $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, M_{\mathcal{N}})$ is called Neutrosophic Binary Topological space. The members of $M_{\mathcal{N}}$ are called the neutrosophic binary open sets and the complement of neutrosophic binary open sets are called the neutrosophic binary closed sets in the neutrosophic binary topological space $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, M_{\mathcal{N}})$.

Definition 2.7. [29] Similarity measure for Single valued neutrosophic set:
Let $X = \{x_1, x_2, ...x_n\}$ be a universal set [15],for any two SVNSs $\mathcal{N}_1 = \{<x_i, T_{N_1}(x_i), I_{N_1}(x_i), F_{N_1}(x_i)>|x_i \in X\}$and $\mathcal{N}_2 = \{<x_i, T_{N_2}(x_i), I_{N_2}(x_i), F_{N_2}(x_i)>|x_i \in X\}$; the similarity measure of SVNSs between $\mathcal{N}_1$ and $\mathcal{N}_2$ is defined as

$$S_{1SVNS}(\mathcal{N}_1,\mathcal{N}_2) = \frac{\sum_{i=1}^{n} min(T_{N_1}(x_i),T_{N_2}(x_i))+min(I_{N_1}(x_i),I_{N_2}(x_i))+min(F_{N_1}(x_i),F_{N_2}(x_i))}{\sum_{i=1}^{n} max(T_{N_1}(x_i),T_{N_2}(x_i))+max(I_{N_1}(x_i),I_{N_2}(x_i))+max(F_{N_1}(x_i),F_{N_2}(x_i))}$$

Definition 2.8. [29] Let $\mathcal{N}_1 = \{<x_i, T_{N_1}(x_i), I_{N_1}(x_i), F_{N_1}(x_i)>|x_i \in X\}$and $\mathcal{N}_2 = \{<x_i, T_{N_2}(x_i), I_{N_2}(x_i), F_{N_2}(x_i)>|x_i \in X\}$; be any two SVNSs in $X = x_1, x_2, ..., x_n$; then, the Euclidean distance between SVNSs $\mathcal{N}_1$ and $\mathcal{N}_2$ is defined as

$$D_{SVNS}(\mathcal{N}_1,\mathcal{N}_2) = \sqrt{\frac{\sum_{i=1}^{n}[(T_{N_1}(x_i)-T_{N_2}(x_i))^2+(I_{N_1}(x_i)-I_{N_2}(x_i))^2+(F_{N_1}(x_i)-F_{N_2}(x_i))^2]}{3n}}$$

Definition 2.9. [20] Similarity measure for Single valued neutrosophic set:
Let $X = \{x_1, x_2, ...x_n\}$ be a universal set for any two SVNSs $\mathcal{N}_1 = \{<x_i, T_{N_1}(x_i), I_{N_1}(x_i), F_{N_1}(x_i)>|x_i \in X\}$and $\mathcal{N}_2 = \{<x_i, T_{N_2}(x_i), I_{N_2}(x_i), F_{N_2}(x_i)>|x_i \in X\}$; the similarity measure of SVNSs between $\mathcal{N}_1$ and $\mathcal{N}_2$ is defined as
$\mathcal{S}^*_{1SVNS}(\mathcal{N}_{B_1},\mathcal{N}_{B_2}) = \frac{1}{2}[S_{1SVNS}(\mathcal{N}_1,\mathcal{N}_2) + 1 - D_{SVNS}(\mathcal{N}_1,\mathcal{N}_2)]$ where $S_{1SVNS}(\mathcal{N}_1,\mathcal{N}_2)$ and $D_{SVNS}(\mathcal{N}_1,\mathcal{N}_2)$ are the similarity measure [29] and the Euclidean distance between SVNSs [29]

3. **Similarity measure for Binary neutrosophic set.**

A new Similarity measure is used to find the measure between two different Binary single valued neutrosophic set which is defined as follows:

Definition 3.1. Let $\tilde{\mathcal{X}} = \{\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n\}; \tilde{\mathcal{Y}} = \{\tilde{y}_1, \tilde{y}_2, .., \tilde{y}_n\}$ be the two universal set of the binary topology and let

$$\mathcal{N}_{B_1} = \{< \tilde{\mathcal{X}}, \mu_{N_{B_1}}(\tilde{x}_i), \sigma_{N_{B_1}}(\tilde{x}_i), \gamma_{N_{B_1}}(\tilde{x}_i) >; \tilde{x}_i \in \tilde{\mathcal{X}}, \\ < \tilde{\mathcal{Y}}, \mu_{N_{B_1}}(\tilde{y}_i), \sigma_{N_{B_1}}(\tilde{y}_i), \gamma_{N_{B_1}}(\tilde{y}_i) >; \tilde{y}_i \in \tilde{\mathcal{Y}}\}$$

and

$$\mathcal{N}_{B_2} = \{< \tilde{\mathcal{X}}, (\mu_{N_{B_2}}(\tilde{x}_i), \sigma_{N_{B_2}}(\tilde{x}_i), \gamma_{N_{B_2}}(\tilde{x}_i) >; \tilde{x}_i \in \tilde{\mathcal{X}}, \\ < \tilde{\mathcal{Y}}, (\mu_{N_{B_2}}(\tilde{y}_i), \sigma_{N_{B_2}}(\tilde{y}_i), \gamma_{N_{B_2}}(\tilde{y}_i) >; \tilde{y}_i \in \tilde{\mathcal{Y}}\}$$

be the two neutrosophic binary sets then the similarity measure between the two neutrosophic binary set is defined as

$$\mathcal{SM}^*_{N_B}(\mathcal{N}_{B_1}, \mathcal{N}_{B_2}) = \frac{1}{2}[\mathcal{SM}_{N_B}(\mathcal{N}_{B_1}, \mathcal{N}_{B_2}) + 1 - \mathcal{D}_{N_B}(\mathcal{N}_{B_1}, \mathcal{N}_{B_2})]$$

where $\mathcal{SM}_{N_B}(\mathcal{N}_{B_1}, \mathcal{N}_{B_2})$ and $\mathcal{D}_{N_B}(\mathcal{N}_{B_1}, \mathcal{N}_{B_2})$ are the Similarity measure and the Euclidean distance respectively and is defined as follows.

Definition 3.2. The similarity measure $\mathcal{SM}_{N_B}(\mathcal{N}_{B_1}, \mathcal{N}_{B_2})$ for the Neutrosophic binary set $\mathcal{N}_{B_1}$ and $\mathcal{N}_{B_2}$ is defined as $\mathcal{SM}_{N_B}(\mathcal{N}_{B_1}, \mathcal{N}_{B_2}) =$

$$\frac{\sum_{i=1}^{n}[\{\mu_{N_{B_1}}(\tilde{x}_i) \wedge \mu_{N_{B_2}}(\tilde{x}_i)\} + \{\sigma_{N_{B_1}}(\tilde{x}_i) \wedge \sigma_{N_{B_2}}(\tilde{x}_i)\} + \{\gamma_{N_{B_1}}(\tilde{x}_i) \wedge \gamma_{N_{B_2}}(\tilde{x}_i)\} + \{\mu_{N_{B_1}}(\tilde{y}_i) \wedge \mu_{N_{B_2}}(\tilde{y}_i)\} + \{\sigma_{N_{B_1}}(\tilde{y}_i) \wedge \sigma_{N_{B_2}}(\tilde{y}_i)\} + \{\gamma_{N_{B_1}}(\tilde{y}_i) \wedge \gamma_{N_{B_2}}(\tilde{y}_i)\}]}{\sum_{i=1}^{n}[\{\mu_{N_{B_1}}(\tilde{x}_i) \vee \mu_{N_{B_2}}(\tilde{x}_i)\} + \{\sigma_{N_{B_1}}(\tilde{x}_i) \vee \sigma_{N_{B_2}}(\tilde{x}_i)\} + \{\gamma_{N_{B_1}}(\tilde{x}_i) \vee \gamma_{N_{B_2}}(\tilde{x}_i)\} + \{\mu_{N_{B_1}}(\tilde{y}_i) \vee \mu_{N_{B_2}}(\tilde{y}_i)\} + \{\sigma_{N_{B_1}}(\tilde{y}_i) \vee \sigma_{N_{B_2}}(\tilde{y}_i)\} + \{\gamma_{N_{B_1}}(\tilde{y}_i) \vee \gamma_{N_{B_2}}(\tilde{y}_i)\}]}$$

Definition 3.3. The Euclidean distance $D_{N_B}(\mathcal{N}_{B_1}, \mathcal{N}_{B_2})$ between the Neutrosophic binary set $\mathcal{N}_{B_1}$ and $\mathcal{N}_{B_2}$ is defined as
$D_{N_B}(\mathcal{N}_{B_1}, \mathcal{N}_{B_2}) =$

$$\sqrt{\frac{\sum_{i=1}^{n}[(\mu_{N_{B_1}}(\tilde{x}_i) - \mu_{N_{B_2}}(\tilde{x}_i))^2 + (\sigma_{N_{B_1}}(\tilde{x}_i) - \sigma_{N_{B_2}}(\tilde{x}_i))^2 + (\gamma_{N_{B_1}}(\tilde{x}_i) - \gamma_{N_{B_2}}(\tilde{x}_i))^2 + (\mu_{N_{B_1}}(\tilde{y}_i) - \mu_{N_{B_2}}(\tilde{y}_i))^2 + (\sigma_{N_{B_1}}(\tilde{y}_i) - \sigma_{N_{B_2}}(\tilde{y}_i))^2 + (\gamma_{N_{B_1}}(\tilde{y}_i) - \gamma_{N_{B_2}}(\tilde{y}_i))^2]}{3n}}$$

Example 3.4. $E_1 = \{a_1, a_2\}$ and $E_2 = \{b_1, b_2\}$ be the universe of the neutrosophic binary topological space $\mathcal{M}_N = \{(0_N(E_1), 0_N(E_2)), (1_N(E_1), 1_N(E_2)), (V_1, W_1), (V_2, W_2)\}$ where $(V_1, W_1) = \{< (0.4, 0.5, 0.5), (0.3, 0.5, 0.6) >, < (0.3, 0.5, 0.5), (0.4, 0.5, 0.7) >\}$ $(V_2, W_2) = \{< (0.3, 0.5, 0.6), (0.2, 0.5, 0.7) >, < (0.2, 0.5, 0.6), (0.3, 0.5, 0.7) >\}$ the similarity measure $\mathcal{SM}_{N_B}\{(V_1, W_1), (V_2, W_2)\} = 0.8833$ and The Euclidean distance

$D_{N_B}\{(V_1,W_1),(V_2,W_2)\} = 0.10798$. The similarity measure between the two neutrosophic binary set $(V_1,W_1),(V_2,W_2)$ is $\mathcal{SM}^*_{N_B}\{(V_1,W_1),(V_2,W_2)\} = \frac{1}{2}[0.8833 + 1 - 0.10798] = 0.88766$.

Theorem 3.5. *The Similarity measure $\mathcal{SM}^*_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2})$ between the neutrosophic Binary set*

$$\mathcal{N}_{B_1} = \{< \tilde{\mathcal{X}}, \mu_{N_{B_1}}(\tilde{x}_i), \sigma_{N_{B_1}}(\tilde{x}_i), \gamma_{N_{B_1}}(\tilde{x}_i) >; \tilde{x}_i \in \tilde{\mathcal{X}}, \\ < \tilde{\mathcal{Y}}, \mu_{N_{B_1}}(\tilde{y}_i), \sigma_{N_{B_1}}(\tilde{y}_i), \gamma_{N_{B_1}}(\tilde{y}_i) >; \tilde{y}_i \in \tilde{\mathcal{Y}}\}$$

$$\mathcal{N}_{B_2} = \{< \tilde{\mathcal{X}}, (\mu_{N_{B_2}}(\tilde{x}_i), \sigma_{N_{B_2}}(\tilde{x}_i), \gamma_{N_{B_2}}(\tilde{x}_i) >; \tilde{x}_i \in \tilde{\mathcal{X}}, \\ < \tilde{\mathcal{Y}}, (\mu_{N_{B_2}}(\tilde{y}_i), \sigma_{N_{B_2}}(\tilde{y}_i), \gamma_{N_{B_2}}(\tilde{y}_i) >; \tilde{y}_i \in \tilde{\mathcal{Y}}\}$$

satisfies the following properties
(i) $0 \leq \mathcal{SM}^*_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2}) \leq 1$
(ii) $\mathcal{SM}^*_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2}) = 1$ *iff* $\mathcal{N}_{B_1} = \mathcal{N}_{B_2}$
(iii) $\mathcal{SM}^*_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2}) = \mathcal{SM}^*_{N_B}(\mathcal{N}_{B_2},\mathcal{N}_{B_1})$

Proof. To prove the above properties it is important to prove that $\mathcal{SM}_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2})$ and $D_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2})$ also satifies the property.
(i)Since the three membership values of the neutrosophic binary set lies between o to 1, the minimum value, the maximum value and their difference also lies between 0 to 1. Hence the value of $\mathcal{SM}_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2})$ and $D_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2})$ also lies between o to 1. hence $\mathcal{SM}^*_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2})$ satisfies (i).
(ii) When two neutrosophic binary sets are equal it is obviouse from the definition of $\mathcal{SM}_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2})$ and $D_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2})$ that $\mathcal{SM}_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2}) = 1$ and $D_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2}) = 0$. Hence $\mathcal{SM}^*_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2}) = \frac{1}{2}[1 + 1 - 0] = 1$
(iii) Since,
$max/min\{\mu_{N_{B_1}}(\tilde{x}_i), \mu_{N_{B_2}}(\tilde{x}_i)\} = max/min\{\mu_{N_{B_2}}(\tilde{x}_i), \mu_{N_{B_1}}(\tilde{x}_i)\}; 1 \leq i \leq n,$
$max/min\{\sigma_{N_{B_1}}(\tilde{x}_i), \sigma_{N_{B_2}}(\tilde{x}_i)\} = max/min\{\sigma_{N_{B_2}}(\tilde{x}_i), \sigma_{N_{B_1}}(\tilde{x}_i)\}; 1 \leq i \leq n,$
$max/min\{\gamma_{N_{B_1}}(\tilde{x}_i), \gamma_{N_{B_2}}(\tilde{x}_i)\} = max/min\{\gamma_{N_{B_2}}(\tilde{x}_i), \gamma_{N_{B_1}}(\tilde{x}_i)\}; 1 \leq i \leq n.$
Similarly,
$max/min\{\mu_{N_{B_1}}(\tilde{y}_i), \mu_{N_{B_2}}(\tilde{y}_i)\} = max/min\{\mu_{N_{B_2}}(\tilde{y}_i), \mu_{N_{B_1}}(\tilde{y}_i)\}; 1 \leq i \leq n,$
$max/min\{\sigma_{N_{B_1}}(\tilde{y}_i), \sigma_{N_{B_2}}(\tilde{y}_i)\} = max/min\{\sigma_{N_{B_2}}(\tilde{y}_i), \sigma_{N_{B_1}}(\tilde{y}_i)\}; 1 \leq i \leq n,$
$max/min\{\gamma_{N_{B_1}}(\tilde{y}_i), \gamma_{N_{B_2}}(\tilde{y}_i)\} = max/min\{\gamma_{N_{B_2}}(\tilde{y}_i), \gamma_{N_{B_1}}(\tilde{y}_i)\}; 1 \leq i \leq n.$
Hence $\mathcal{SM}_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2}) = \mathcal{SM}_{N_B}(\mathcal{N}_{B_2},\mathcal{N}_{B_1})$ and $D_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2}) = D_{N_B}(\mathcal{N}_{B_2},\mathcal{N}_{B_1})$ which implies $\mathcal{SM}^*_{N_B}(\mathcal{N}_{B_1},\mathcal{N}_{B_2}) = \mathcal{SM}^*_{N_B}(\mathcal{N}_{B_2},\mathcal{N}_{B_1})$. □

4. Application of the Proposed Similarity measure

The newly introduced Similarity measure based on set theory for a neutrosophic binary set is now used to find the choice made by the male and female who are categorized by age for their well-being.the data is being collected using a Questionnaire of all age category and segregated into five age groups and its been converted into a neutrosophic data.

4.1. *Methodology*

Let $\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_n$ be the set of male and $\mathfrak{f}_1, \mathfrak{f}_2, ..., \mathfrak{f}_n$ be the set of female; $\mathfrak{H}_{\mathfrak{m}_1}, \mathfrak{H}_{\mathfrak{m}_2}, ..., \mathfrak{H}_{\mathfrak{m}_n}$ be the criteria (well-being) prefered by male and $\mathfrak{H}_{\mathfrak{f}_1}, \mathfrak{H}_{\mathfrak{f}_2}, ..., \mathfrak{H}_{\mathfrak{f}_n}$ be the criteria (well-being) prefered by female ; $\mathfrak{R}_{\mathfrak{m}_1}, \mathfrak{R}_{\mathfrak{m}_2}, ..., \mathfrak{R}_{\mathfrak{m}_n}$ be the alternatives of male individuals and $\mathfrak{R}_{\mathfrak{f}_1}, \mathfrak{R}_{\mathfrak{f}_2}, ..., \mathfrak{R}_{\mathfrak{f}_n}$ be the alternatives of female individual. The ranking of the alternatives is based on the preference made by the individuals against the well-being chosen by them. For a MADM problem, the values associated with the alternatives of male and female individuals can be represented in a decision matrix which is shown in table 1 , table 2.

	$(\mathfrak{R}_{\mathfrak{m}_1}, \mathfrak{R}_{\mathfrak{f}_1})$	$(\mathfrak{R}_{\mathfrak{m}_2}, \mathfrak{R}_{\mathfrak{f}_2})$	$\ldots$	$\ldots$	$(\mathfrak{R}_{\mathfrak{m}_n}, \mathfrak{R}_{\mathfrak{f}_n})$
$(\mathfrak{m}_1, \mathfrak{f}_1)$	ϕ_{11}	ϕ_{12}	$\ldots$	$\ldots$	ϕ_{1n}
$(\mathfrak{m}_2, \mathfrak{f}_2)$	ϕ_{21}	ϕ_{22}	$\ldots$	$\ldots$	ϕ_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\ldots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\ldots$	$\vdots$
$(\mathfrak{m}_n, \mathfrak{f}_n)$	ϕ_{n1}	ϕ_{n2}	$\ldots$	$\ldots$	ϕ_{nn}

TABLE 1. The relation between Individual and attributes

	$(\mathfrak{H}_{\mathfrak{m}_1}, \mathfrak{H}_{\mathfrak{f}_1})$	$(\mathfrak{H}_{\mathfrak{m}_2}, \mathfrak{H}_{\mathfrak{f}_2})$	$\ldots$	$\ldots$	$(\mathfrak{H}_{\mathfrak{m}_n}, \mathfrak{H}_{\mathfrak{f}_n})$
$(\mathfrak{R}_{\mathfrak{m}_1}, \mathfrak{R}_{\mathfrak{f}_1})$	Ψ_{11}	Ψ_{12}	$\ldots$	$\ldots$	Ψ_{1n}
$(\mathfrak{R}_{\mathfrak{m}_2}, \mathfrak{R}_{\mathfrak{f}_2})$	Ψ_{21}	Ψ_{22}	$\ldots$	$\ldots$	Ψ_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\ldots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\ldots$	$\vdots$
$(\mathfrak{R}_{\mathfrak{m}_n}, \mathfrak{R}_{\mathfrak{f}_n})$	Ψ_{n1}	Ψ_{n2}	$\ldots$	$\ldots$	Ψ_{nn}

TABLE 2. The relation between attributes and alternatives

Here ϕ_{ij} and Ψ_{ij} represents the neutrosophic binary sets.

The algorithm for this method is demonstrated below:

Step 1: Deliberate the association between the individuals and the attributes.

	$(\mathfrak{R}_{\mathfrak{m}_1}, \mathfrak{R}_{\mathfrak{f}_1})$	$(\mathfrak{R}_{\mathfrak{m}_2}, \mathfrak{R}_{\mathfrak{f}_2})$	$\ldots\ldots (\mathfrak{R}_{\mathfrak{m}_n}, \mathfrak{R}_{\mathfrak{f}_n})$
$(\mathfrak{m}_1, \mathfrak{f}_1)$	$< \mu_{11}(\tilde{x}_i), \sigma_{11}(\tilde{x}_i), \gamma_{11}(\tilde{x}_i) >, < \mu_{11}(\tilde{y}_i), \sigma_{11}(\tilde{y}_i), \gamma_{11}(\tilde{y}_i) >$	$< \mu_{12}(\tilde{x}_i), \sigma_{12}(\tilde{x}_i), \gamma_{12}(\tilde{x}_i) >, < \mu_{12}(\tilde{y}_i), \sigma_{12}(\tilde{y}_i), \gamma_{12}(\tilde{y}_i) >$	$\ldots\ldots < \mu_{1n}(\tilde{x}_i), \sigma_{1n}(\tilde{x}_i), \gamma_{1n}(\tilde{x}_i) >, < \mu_{1n}(\tilde{y}_i), \sigma_{1n}(\tilde{y}_i), \gamma_{1n}(\tilde{y}_i) >$
$(\mathfrak{m}_2, \mathfrak{f}_2)$	$< \mu_{21}(\tilde{x}_i), \sigma_{21}(\tilde{x}_i), \gamma_{21}(\tilde{x}_i) >, < \mu_{21}(\tilde{y}_i), \sigma_{21}(\tilde{y}_i), \gamma_{21}(\tilde{y}_i) >$	$< \mu_{22}(\tilde{x}_i), \sigma_{22}(\tilde{x}_i), \gamma_{22}(\tilde{x}_i) >, < \mu_{22}(\tilde{y}_i), \sigma_{22}(\tilde{y}_i), \gamma_{22}(\tilde{y}_i) >$	$\ldots\ldots < \mu_{2n}(\tilde{x}_i), \sigma_{2n}(\tilde{x}_i), \gamma_{2n}(\tilde{x}_i) >, < \mu_{2n}(\tilde{y}_i), \sigma_{2n}(\tilde{y}_i), \gamma_{2n}(\tilde{y}_i) >$
$\ldots$	$\ldots$	$\ldots$	$\ldots\ldots\ldots$
$(\mathfrak{m}_n, \mathfrak{f}_n)$	$< \mu_{n1}(\tilde{x}_i), \sigma_{n1}(\tilde{x}_i), \gamma_{n1}(\tilde{x}_i) >, < \mu_{n1}(\tilde{y}_i), \sigma_{n1}(\tilde{y}_i), \gamma_{n1}(\tilde{y}_i) >$	$< \mu_{n2}(\tilde{x}_i), \sigma_{n2}(\tilde{x}_i), \gamma_{n2}(\tilde{x}_i) >, < \mu_{n2}(\tilde{y}_i), \sigma_{n2}(\tilde{y}_i), \gamma_{n2}(\tilde{y}_i) >$	$\ldots\ldots < \mu_{nn}(\tilde{x}_i), \sigma_{nn}(\tilde{x}_i), \gamma_{nn}(\tilde{x}_i) >, < \mu_{nn}(\tilde{y}_i), \sigma_{nn}(\tilde{y}_i), \gamma_{nn}(\tilde{y}_i) >$

Step 2: Deliberate the association between the attributes and the alternatives.

	$(\mathfrak{H}_{\mathfrak{m}_1}, \mathfrak{H}_{\mathfrak{f}_1})$	$(\mathfrak{H}_{\mathfrak{m}_2}, \mathfrak{H}_{\mathfrak{f}_2})$	$\ldots\ldots (\mathfrak{H}_{\mathfrak{m}_n}, \mathfrak{H}_{\mathfrak{f}_n})$
$(\mathfrak{R}_{\mathfrak{m}_1}, \mathfrak{R}_{\mathfrak{f}_1})$	$< \mu_{11}(\tilde{x}_i), \sigma_{11}(\tilde{x}_i), \gamma_{11}(\tilde{x}_i) >, < \mu_{11}(\tilde{y}_i), \sigma_{11}(\tilde{y}_i), \gamma_{11}(\tilde{y}_i) >$	$< \mu_{12}(\tilde{x}_i), \sigma_{12}(\tilde{x}_i), \gamma_{12}(\tilde{x}_i) >, < \mu_{12}(\tilde{y}_i \sigma_{12}(\tilde{y}_i), \gamma_{12}(\tilde{y}_i) >$	$\ldots\ldots < \mu_{1n}(\tilde{x}_i), \sigma_{1n}(\tilde{x}_i), \gamma_{1n}(\tilde{x}_i) >, < \mu_{1n}(\tilde{y}_i), \sigma_{1n}(\tilde{y}_i), \gamma_{1n}(\tilde{y}_i) >$
$\ldots$	$\ldots$	$\ldots$	$\ldots\ldots\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots\ldots\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots\ldots\ldots$
$(\mathfrak{R}_{\mathfrak{m}_n}, \mathfrak{R}_{\mathfrak{f}_n})$	$< \mu_{n1}(\tilde{x}_i), \sigma_{n1}(\tilde{x}_i), \gamma_{n1}(\tilde{x}_i) >, < \mu_{n1}(\tilde{y}_i), \sigma_{n1}(\tilde{y}_i), \gamma_{n1}(\tilde{y}_i) >$	$< \mu_{n2}(\tilde{x}_i), \sigma_{n2}(\tilde{x}_i), \gamma_{n2}(\tilde{x}_i) >, < \mu_{n2}(\tilde{y}_i), \sigma_{n2}(\tilde{y}_i), \gamma_{n2}(\tilde{y}_i) >$	$\ldots\ldots < \mu_{nn}(\tilde{x}_i), \sigma_{nn}(\tilde{x}_i), \gamma_{nn}(\tilde{x}_i) >, < \mu_{nn}(\tilde{y}_i), \sigma_{nn}(\tilde{y}_i), \gamma_{nn}(\tilde{y}_i) >$

TABLE 3

Step: 3 apply the new similarity measure using the formula $\mathcal{SM}^*_{N_B}(\mathcal{N}_{B_1}, \mathcal{N}_{B_2})$ as proposed in the definition 3.1.

Step:4 Ranking of alternatives.

The alternatives are ranked by the decision makers and it is ranked in inclined form of similarity measure $\mathcal{SM}^*_{N_B}(\mathcal{N}_{B_1}, \mathcal{N}_{B_2})$. The highest value of the similarity measure gives the best alternative.

4.2. *Numerical example*

let the two universal set be $\mathfrak{M} = \{\mathfrak{m}_1, ,m_2,m_3,m_4,m_5\}; \mathfrak{F} = \{\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3, \mathfrak{f}_4, \mathfrak{f}_5\}$ where $(\mathfrak{m}_1, \mathfrak{f}_1)$ represent male and female belong to less than or 25 age category;$(\mathfrak{m}_2, \mathfrak{f}_2)$ represent male and female belong to 26-35 age category;$(\mathfrak{m}_3, \mathfrak{f}_3)$ represent male and female belong to 36-45 age category;$(\mathfrak{m}_4, \mathfrak{f}_4)$ represent male and female belong to 46-55 age category;$(\mathfrak{m}_5, \mathfrak{f}_5)$ represent male and female belong to above 56 age category. Let $\mathfrak{H}_{\mathfrak{m}_1}, \mathfrak{H}_{\mathfrak{m}_2}, \mathfrak{H}_{\mathfrak{m}_3}, \mathfrak{H}_{\mathfrak{m}_4}, \mathfrak{H}_{\mathfrak{m}_5}$ and $\mathfrak{H}_{\mathfrak{f}_1}, \mathfrak{H}_{\mathfrak{f}_2}, \mathfrak{H}_{\mathfrak{f}_3}, \mathfrak{H}_{\mathfrak{f}_4}, \mathfrak{H}_{\mathfrak{f}_5}$ be the well-being prefered by male and female of the five age category respectively. Let $\mathfrak{R}_{\mathfrak{m}_1}, \mathfrak{R}_{\mathfrak{m}_2}, \mathfrak{R}_{\mathfrak{m}_3}, \mathfrak{R}_{\mathfrak{m}_4}, \mathfrak{R}_{\mathfrak{m}_5}$ and $\mathfrak{R}_{\mathfrak{f}_1}, \mathfrak{R}_{\mathfrak{f}_2}, \mathfrak{R}_{\mathfrak{f}_3}, \mathfrak{R}_{\mathfrak{f}_4}, \mathfrak{R}_{\mathfrak{f}_5}$ be the Reason prefered by male and female for selecting a particular well-being of the five age category respectively.let $(\mathfrak{H}_{\mathfrak{m}_1}, \mathfrak{H}_{\mathfrak{f}_1})$ represent the set of male and female who prefer yoga /meditation for their well- being who is less than or 25 years of age; $(\mathfrak{H}_{\mathfrak{m}_2}, \mathfrak{H}_{\mathfrak{f}_2})$ represent the set of male and female who prefer Gym/Walking for their well- being who is 26-35 years og age;$(\mathfrak{H}_{\mathfrak{m}_3}, \mathfrak{H}_{\mathfrak{f}_3})$ represent the set of male and female who prefer sports for their well- being who is 36-45 years of age;$(\mathfrak{H}_{\mathfrak{m}_4}, \mathfrak{H}_{\mathfrak{f}_4})$ represent the set of male and female who prefer Travelling for their well- being who is 46-55 years of age;$(\mathfrak{H}_{\mathfrak{m}_5}, \mathfrak{H}_{\mathfrak{f}_5})$ represent the set of male and female who prefer Reading/listening to music for their well- being who is Above 56 years of age.let $(\mathfrak{R}_{\mathfrak{m}_1}, \mathfrak{R}_{\mathfrak{f}_1})$ represent the set of male and female who choose a well-being for Alone time/Self Realization who is less than or 25 years of age;$(\mathfrak{R}_{\mathfrak{m}_2}, \mathfrak{R}_{\mathfrak{f}_2})$ represent the set of male and female who choose a well-being for Relaxation/Calmness who is 26-35 years of age;$(\mathfrak{R}_{\mathfrak{m}_3}, \mathfrak{R}_{\mathfrak{f}_3})$ represent the set of male and female who choose a well-being for Physical Fitness who is 36-45 years of age;$(\mathfrak{R}_{\mathfrak{m}_4}, \mathfrak{R}_{\mathfrak{f}_4})$ represent the set of male and female who choose a well-being for Exploring who is 46-55 years of age;$(\mathfrak{R}_{\mathfrak{m}_5}, \mathfrak{R}_{\mathfrak{f}_5})$ represent the set of male and female who choose a well-being for Mental wellness who is above 56 years of age.
The following table show the relation between male and female of all age category and their Reason for choosing a particular well-being.
The relation between both the male,female individuals and the attributes is represented in the form of neutrosophic binary sets in the table 4.

	$(\mathfrak{R}_{m_1},\mathfrak{R}_{f_1})$	$(\mathfrak{R}_{m_2},\mathfrak{R}_{f_2})$	$(\mathfrak{R}_{m_3},\mathfrak{R}_{f_3})$	$(\mathfrak{R}_{m_4},\mathfrak{R}_{f_4})$	$(\mathfrak{R}_{m_5},\mathfrak{R}_{f_5})$
$(\mathfrak{m}_1,\mathfrak{f}_1)$	$< 0.44, 0.44, 0.55 >, < 0.44, 0.44, 0.55 >$	$< 0.33, 0.33, 0.66 >, < 0.11, 0.55, 0.33 >$	$< 0.11, 0.55, 0.33 >, < 0.22, 0.22, 0.77 >$	$< 0, 0, 1 >, < 0.11, 0, 0.88 >$	$< 0.11, 0.77, 0.22 >, < 0.11, 0.44, 0.44 >$
$(\mathfrak{m}_2,\mathfrak{f}_2)$	$< 0.5, 0.5, 0.5 >, < 0.12, 0, 0.87 >$	$< 0, 0, 1 >, < 0.12, 0.62, 0.25 >$	$< 0.5, 0.5, 0.5 >, < 0.12, 0.37, 0.5 >$	$< 0.5, 0.5, 0.5 >, < 0.25, 0.25, 0.75 >$	$< 1, 1, 0 >, < 0.5, 0.5, 0.5 >$
$(\mathfrak{m}_3,\mathfrak{f}_3)$	$< 0.25, 0.25, 0.75 >, < 0.11, 0.11, 0.88 >$	$< 0.5, 0.5, 0.5 >, < 0.11, 0.33, 0.55 >$	$< 1, 1, 0 >, < 0.11, 0.33, 0.55 >$	$< 0.25, 0.25, 0.75 >, < 0, 0, 1 >$	$< 0.75, 0.75, 0.25 >, < 0.22, 0.44, 0.33 >$
$(\mathfrak{m}_4,\mathfrak{f}_4)$	$< 0.11, 0.22, 0.66 >, < 0.2, 0.2, 0.8 >$	$< 0.33, 0.33, 0.66 >, < 0.2, 0.2, 0.6 >$	$< 0.22, 0.44, 0.33 >, < 0.4, 0, 0.6 >$	$< 0.22, 0.22, 0.77 >, < 0, 0, 1 >$	$< 0.11, 0.44, 0.44 >, < 0.2, 0.2, 0.6 >$
$(\mathfrak{m}_5,\mathfrak{f}_5)$	$< 0.16, 0.16, 0.83 >, < 0.25, 0.25, 0.75 >$	$< 0.16, 0.16, 0.83 >, < 0.25, 0.25, 0.75 >$	$< 0.16, 0.33, 0.5 >, < 0.25, 0.75, 0 >$	$< 0, 0, 1 >, < 0.25, 0.25, 0.75 >$	$< 0.33, 0.33, 0.33 >, < 0.75, 0.75, 0.25 >$

TABLE 4. Realtion between the male,female individuals and Attributes

	$(\mathfrak{H}_{\mathfrak{m}_1},\mathfrak{H}_{\mathfrak{f}_1})$	$(\mathfrak{H}_{\mathfrak{m}_2},\mathfrak{H}_{\mathfrak{f}_2})$	$(\mathfrak{H}_{\mathfrak{m}_3},\mathfrak{H}_{\mathfrak{f}_3})$	$(\mathfrak{H}_{\mathfrak{m}_4},\mathfrak{H}_{\mathfrak{f}_4})$	$(\mathfrak{H}_{\mathfrak{m}_5},\mathfrak{H}_{\mathfrak{f}_5})$
$(\mathfrak{R}_{\mathfrak{m}_1},\mathfrak{R}_{\mathfrak{f}_1})$	$< 0.25, 0.25, 0.75 >, < 0.25, 0.25, 0.75 >$	$< 0.25, 0.25, 0.75 >, < 0.25, 0.25, 0.75 >$	$< 0.25, 0.5, 0.25 >, < 0, 0, 1 >$	$< 0.25, 0.25, 0.75 >, < 0.5, 0.5, 0.5 >$	$< 0.5, 0.5, 0.5 >, < 0.25, 0.75, 0 >$
$(\mathfrak{R}_{\mathfrak{m}_2},\mathfrak{R}_{\mathfrak{f}_2})$	$< 0, 0, 1 >, < 0.33, 0.33, 0.66 >$	$< 0, 0, 1 >, < 0.5, 0.5, 0.5 >$	$< 0, 0, 1 >, < 0, 0, 1 >$	$< 0, 0, 1 >, < 0.16, 0.5, 0.33 >$	$< 0, 0, 1 >, < 0.66, 0.66, 0.33 >$
$(\mathfrak{R}_{\mathfrak{m}_3},\mathfrak{R}_{\mathfrak{f}_3})$	$< 0, 0, 1 >, < 0.25, 0.5, 0.25 >$	$< 0.75, 0.75, 0.25 >, < 0.25, 0, 0.75 >$	$< 0.25, 0.25, 0.75 >, < 0, 0, 1 >$	$< 0.5, 0.5, 0.5 >, < 0.25, 0.25, 0.75 >$	$< 0.75, 0.75, 0.25 >, < 0.5, 0.5, 0.5 >$
$(\mathfrak{R}_{\mathfrak{m}_4},\mathfrak{R}_{\mathfrak{f}_4})$	$< 0.5, 0.5, 0.5 >, < 0, 0, 1 >$	$< 0.5, 0.5, 0.5 >, < 0, 0, 1 >$	$< 0.5, 0.5, 0.5 >, < 0, 0, 1 >$	$< 0.5, 0.5, 0.5 >, < 0, 0, 1 >$	$< 0.5, 0.5, 0 >, < 0, 0, 1 >$
$(\mathfrak{R}_{\mathfrak{m}_5},\mathfrak{R}_{\mathfrak{f}_5})$	$< 0.25, 0.25, 0.5 >, < 1, 1, 0 >$	$< 0, 0, 1 >, < 0, 0, 1 >$	$< 0.5, 0.5, 0.5 >, < 0, 0, 1 >$	$< 0.25, 0.25, 0.75 >, < 0, 0, 1 >$	$< 0.75, 0.75, 0.25 >, < 1, 1, 0 >$

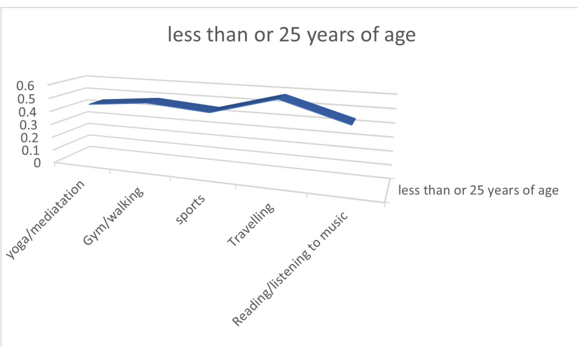
TABLE 5. The relation between the attributes and the alternatives is represented in the form of neutrosophic binary sets

	$(\mathfrak{H}_{\mathfrak{m}_1}, \mathfrak{H}_{\mathfrak{f}_1})$	$(\mathfrak{H}_{\mathfrak{m}_2}, \mathfrak{H}_{\mathfrak{f}_2})$	$(\mathfrak{H}_{\mathfrak{m}_3}, \mathfrak{H}_{\mathfrak{f}_3})$	$(\mathfrak{H}_{\mathfrak{m}_4}, \mathfrak{H}_{\mathfrak{f}_4})$	$(\mathfrak{H}_{\mathfrak{m}_5}, \mathfrak{H}_{\mathfrak{f}_5})$
$(\mathfrak{m}_1, \mathfrak{f}_1)$	0.4429	0.48675	0.45123	0.57821	0.44149
$(\mathfrak{m}_2, \mathfrak{f}_2)$	0.47304	0.4638	0.52207	0.53849	0.55578
$(\mathfrak{m}_3, \mathfrak{f}_3)$	0.41479	0.504125	0.44417	0.49541	0.46420
$(\mathfrak{m}_4, \mathfrak{f}_4)$	0.51916	0.60932	0.57515	0.6248	0.39888
$(\mathfrak{m}_5, \mathfrak{f}_5)$	0.63229	0.40724	0.39026	0.45778	0.4215

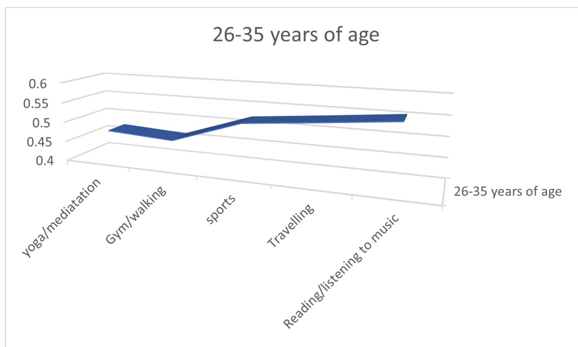
TABLE 6. The computation of Similarity measure between the employees and the investing sectors

Using table1,2,3 table4,5 and 6 is calculated. The highest similarity measure shows the preference of each individuals. From the above table6 it is seen that individuals (both male and female)who belong to age category less than or 25 years of age, 26-35 years of age, 36-45 years of age, 46-55 years of age and above 56 years of age prefer travelling,Reading/ listening to music,gym/walking,travelling and yoga/meditation respectively.

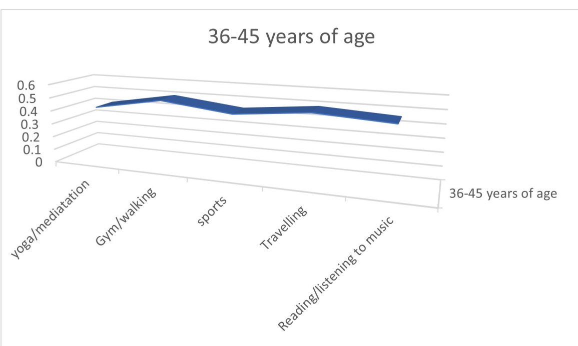
The Graphical representation of this shown in the Figure2



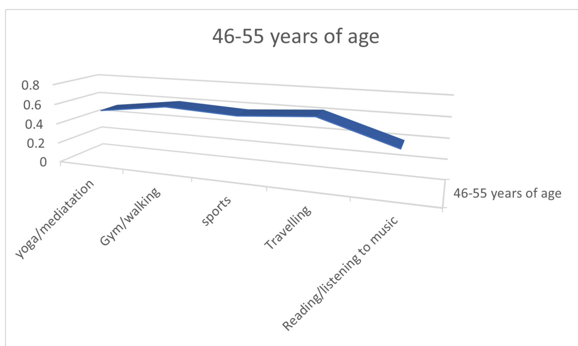
(A) Less than 25 years of age



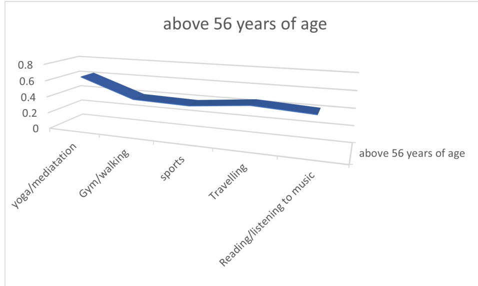
(B) 26-35 years of age



(C) 36-45 years of age



(D) 46-55 years of age



Above 56 years of age

Figure 2. Illustration of the Research

References

[1] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman. Single valued neutrosophic, sets. Multispace and Multistructure, 4(2010), 410–413.

[2] H. Wang, F. Smarandache, Y.Q. Zhang, and R. Sunderraman. Interval neutrosophic sets and logic: Theory and applications in computing, Hexis, Phoenix, AZ, 2005

[3] A. Kharal, A neutrosophic multicriteria decision making method. New Mathematics and Natural Computation, Creighton University, USA, 2013.

[4] S. Ye, and J. Ye. Dice similarity measure between single valued neutrosophic multisets and its application in medical diagnosis, Neutrosophic Sets and Systems, 6(2014). 49-54.

[5] J. Ye. Multicriteria decision-making method using the correlation coefficient under singlevalued neutrosophic environment. International Journal of General Systems, 42(4) (2013), 386-394

[6] Yanqiu Zeng,Haiping Ren ,Tonghua Yang,A Novel Similarity Measure of Single-Valued Neutrosophic Sets Based on Modified Manhattan Distance and Its Applications.

[7] J. Ye. Vector similarity measures of simplified neutrosophic sets and their application in multicriteria decision making. International Journal of Fuzzy Systems, 16(2) (2014), 204- 215

[8] P. Biswas, S. Pramanik, and B. C. Giri. A new methodology for neutrosophic multi-attribute decision making with unknown weight information. Neutrosophic Sets and Systems, 3(2014), 42-52.

[9] S. Pramanik, and S.N. Chackrabarti. A study on problems of construction workers in West Bengal based on neutrosophic cognitive maps. International Journal of Innovative Research in Science, Engineering and Technology 2(11) (2013), 6387-6394.

[10] K. Mondal, and S. Pramanik. A study on problems of Hijras in West Bengal based on neutrosophic cognitive maps. Neutrosophic Sets and Systems, 5(2014), 21-26.

[11] K. Mondal, and S. Pramanik. Multi-criteria group decision making approach for teacher recruitment in higher education under simplified neutrosophic environment, Neutrosophic Sets and Systems, 6(2014), 28-34.

[12] K. Mondal, and S. Pramanik. Neutrosophic decision making model of school choice. Neutrosophic Sets and Systems, 7 (2015), 62-68.

[13] S. Pramanik and T. K. Roy. Neutrosophic game theoretic approach to Indo-Pak conflict over Jammu-Kashmir. Neutrosophic Sets and Systems, 2 (2014), 82-101.

[14] S. Bhattacharya, F. Smarandache, and M. Khoshnevvisan. The Israel- Palestine question-a case for application of neutrosophic game theory.

[15] H. D. Cheng, and Y. Guo. A new neutrosophic approach to image thresholding. New Mathematics and Natural Computation, 4(3) (2008), 291–308.

[16] Y. Guo, and H. D. Cheng. New neutrosophic approach to image segmentation. Pattern Recognition, 42, (2009), 587– 595.

[17] M. Zhang, L. Zhang, and H. D. Cheng. A neutrosophic approach to image segmentation based on watershed method. Signal Processing, 90(5) (2010), 1510-1517.

[18] A. A. Salama and S.A. AL. Blowi. Correlation of neutrosophic data. International Refereed Journal of Engineering and Science, 1(2) (2012), 39-43.

[19] S, Broumi, and F, Smarandache. Several similarity measures of neutrosophic sets. Neutrosophic Sets and Systems, 1 (2013), 54-62.

[20] Donghai Liu,Guangyan Liu and Zaiming Liu,Some Similarity Measures of Neutrosophic Sets Based on the Euclidean Distance and Their Application in Medical Diagnosis,Computational and Mathematical Methods in Medicine Volume 2018, Article ID 7325938, 9 pages.

[21] P. Majumder, and S. K. Samanta. On similarity and entropy of neutrosophic sets. Journal of Intelligent and Fuzzy Systems, 26 (2014), 1245–1252.

[22] S.S.Surekha, J.Elekiah and G.Sindhu, A study on Neutrosophic Binary Topological space, Stochastic Modelling and applications, Vol 26(3), 479-486.

[23] J. Ye, and Q. Zhang. Single valued neutrosophic similarity measures for multiple attribute decision-making Neutrosophic Sets and System, 2(2012), 48-54.

[24] P. Biswas, S. Pramanik, and B.C. Giri. 2015. Cosine similarity measure based multi-attribute decision-making with trapezoidal fuzzy neutrosophic numbers. Neutrosophic Sets and System, 8(2015), 47-58.

[25] J. Ye. Similarity measures between interval neutrosophic sets and their multicriteria decisionmaking method. Journal of Intelligent and Fuzzy Systems, 26, (2014), 165-172.

[26] Rakhal Das, Anjan Mukherjee, and Binod Chandra Tripathycorresponding author , “Application of Neutrosophic Similarity Measures in Covid-19 ,” Ann Data Sci. 2022; 9(1): 55–70.

[27] Chai, J.S., Selvachandran, G., Smarandache, F. et al. New similarity measures for single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems. Complex Intell. Syst. 7, 703–723 (2021)..

[28] H. Wang, F. Smarandache, Y. Zhang et al., Single valued neutrosophic sets, Review of the Air Force Academy, vol. 3, no. 1, pp. 33–39, 2013.

[29] P. Majumdar and S. K. Samanta, On similarity and entropy of neutrosophic sets,Journal of Intelligent and Fuzzy Systems,vol. 26, no. 3, pp. 1245–1252, 2013.

Received: Feb 5, 2024. Accepted: April 28, 2024

Neutrosophic Sets and Systems, Vol. 67, 2024
University of New Mexico

Evaluation Strategies of Leadership Management in Healthcare Systems: An Integrated Type-II Neutrosophic Optimization Approach

Areej Hamoud Alharbi*[1]; Hayam M Jabali[2] and Preethi Rajan[3]
[1,2,3]Management of Department, Applied College, Jazan University, Jazan, KSA
E-mail address: ahamoud2@jazanu.edu.sa; hajabali@jazanu.edu.sa; prajan@jazanu.edu.sa
*Corresponding Author: ahamoud2@jazanu.edu.sa

Abstract: This study presented a decision-making methodology to evaluate leadership management strategies in healthcare systems. This study used the multi-criteria decision-making (MCDM) methodology to deal with conflicting criteria in the decision-making process. The MCDM methodology is integrated with neutrosophic sets to deal with uncertainty and vague information in decision-making. The neutrosophic set has three memberships: truth, indeterminacy, and falsity. The type-2 neutrosophic set is a subset of the neutrosophic set with nine membership degrees. The neutrosophic set integrated with the COPRAS method to rank the alternatives. fourteen criteria and eleven alternatives are used to evaluate leadership management in healthcare systems. The sensitivity analysis was conducted to show the stability of the rank. The sensitivity analysis was performed with 15 cases to change the criteria weights and rank the alternatives under different weights. The results show that the rank of other options is stable in various instances.

Keywords: Neutrosophic Sets; Leadership Management; Management Strategies; Uncertainty; MCDM.

1. Introduction

Leadership theories were not formed in healthcare environments; they were developed in the corporate world and then adapted to the healthcare unit. As a result, the ideas are dynamic and subject to change throughout time. Healthcare organizations are made up of intricate relationships between many different types of experts in various jobs[1], [2].

Healthcare organizations' distinctive structures typically adhere to long-standing customs resistant to change. Effective collaboration is a crucial deficiency in many areas of health care. Thus, it takes competent leadership to implement the changes required for organizations to increase their quality. Some individuals conflate the phrases leadership and management. Peter Drucker claims that although management does things correctly, leadership does the right things[3], [4].

Management places a great emphasis on maintaining the status quo, whereas leadership promotes creativity and change for the organization's future. The healthcare sector is becoming more and more competitive these days due to environmental changes, and leadership is now the cornerstone for encouraging and driving change in the future[5], [6]. There are inner ties within an organization that anybody may access to become a leader. It's merely inspiring others to work tirelessly towards achieving common objectives.

This can strongly emphasize working together inside organizations so that followers and leaders can motivate one another more and value the interdependencies among various stakeholders. A person's behavior while organizing efforts towards a shared goal and adjusting to change has been described as leadership. Many people must develop leadership as a complex collection of behaviors within particular organizational and inter-organizational cultures[7], [8].

Collaboration between diverse groups and healthcare administration is necessary to achieve the long-term objective of lowering sickness and enhancing community health. Managers working in healthcare environments can use cooperative communication techniques that transcend conventional organizational boundaries[9], [10].

Healthcare leadership roles are frequently seen as a highly specialized subset of more general management topics and discussions on management education. Spiritually mature healthcare executives may significantly increase their company's beneficial outcomes by pushing the boundaries, fostering a common vision, and inspiring people to operate traditionally [11], [12].

The multi-criteria decision-making (MCDM) methodology deals with conflict criteria. The COPRAS method is used to rank the alternatives. Zavadskas, Kaklauskas, and Sarka introduced the COmplex PRoportional ASsessment (COPRAS) approach. Using this approach, the maximizing and minimizing of index values are evaluated, and the impact of maximizing and minimizing attribute indexes on evaluating outcomes is considered independently. Several domains use the COPRAS technique, including material selection, investment project selection, and risk assessment. Consequently, the following characteristics are considered for this method: It's a way of compensation; Features stand alone; The process transforms the qualitative attributes into quantitative ones[13], [14].

Zadeh introduced the fuzzy set theory as a generalization of classical sets. Following Zadeh's work, scholars began paying more attention to fuzzy set theory, and the number of academic works on the subject has quickly expanded across several disciplines, including the social sciences, engineering, and economics[15], [16]. A membership function describes a fuzzy set. The fuzzy sets whose degrees of membership are described in language words like low, medium, high, very low, not low, and not high are closely related to the idea of linguistic truth. However, a mapping from discourse universe U to subsets of the interval [0,1] may be applied to a fuzzy set A defined by a membership function containing linguistic variables[17], [18].

Atanassov [19] presented the idea of intuitionistic fuzzy sets as a generalization of fuzzy sets. Membership and non-membership functions are the two functions that define an intuitionistic fuzzy set. A valuable technique for modeling hesitancy-related scenarios is an intuitionistic fuzzy set.

Thus, Smarandache proposed the neutrosophic set theory to address inconsistency-related issues. A neutrosophic set is a generalization of the intuitionistic, classical, fuzzy, paraconsistent, dialetheism, paradoxes, and tautological sets based on neutrosophy[20], [21]. A subfield of philosophy known as neutrosophic examines genesis, characteristics, and use of neutralities. An argument, theory, event, idea, or thing identified as "A" in neurology is compared to its opposite, "Anti-A," and to that which is not A, or "Non-A," as well as to that which is neither "A" nor "Anti-A," or "Neut-A."[22], [23].

The neutrosophic set is defined by three functions, from the universal set to the natural or non-real standard subset. These are known as the independent truth-membership function, indeterminacy membership function, and falsity membership function. While the neutrosophic set theory helps model certain issues, it presents challenges for modeling specific engineering issues[24], [25]. The neutrosophic set was applied in various decision making problems[26]–[31].

1.1 Strategies of Leadership Management

A manager's use of direction, incentive, and influence to assist in the accomplishment of an organizational objective is an example of strategic leadership. A sales manager assembling their team and delivering a motivating speech before a significant sales event is one example of this. The leader guides, inspires, and motivates the sales team to help them reach their target[32], [33].

Leadership is an individual's ability to inspire, persuade, and direct others within an organization toward accomplishing a goal. In business leadership, managers work together to motivate, influence, and guide their subordinates toward a performance goal, such as hitting a sales target and using best practices in directing, motivating, and controlling. The manager's overall leadership qualities become apparent when their team consistently meets the goals that have been established. Strategic leadership is one of the numerous subcategories of leadership styles that fall under the giant leadership umbrella[34], [35].

The capacity of a manager to effectively communicate an organization's strategic vision to its staff members in a way that inspires them to pursue the same goals and underlying assumptions of success is known as strategic leadership. When people start making decisions independently with the organization's goals and vision in mind, it is a sign that strategic management has been successfully implemented in the company. One of the main elements of success in any organization is the manager's ability to apply strategy in the day-to-day management of personnel to assist boost productivity and employee happiness.

In the healthcare industry, strategic management may be implemented at any one of the three organizational levels. Stakeholders, executives, and the board of directors participate in corporate planning. Corporate strategies concentrate on articulating the mission and vision of the organization and achieving its overarching objectives. The management of a healthcare organization is involved in business planning. It describes how managers and staff members may assist in achieving plans and objectives at the corporate level. The interactions between physicians and patients are the main emphasis of functional planning. Production, marketing, research, and service delivery to patients and clients are all included in functional-level strategic planning[36], [37].

1.2 Contributions of this study

- The proposed methodology presents an uncertainty methodology for overcoming uncertainty in evaluation strategies of leadership management healthcare systems by using type-2 neutrosophic numbers.
- The type-2 neutrosophic COPRAS method is used to rank leadership management strategies in healthcare systems.
- The MCDM methodology is used to deal with various conflict criteria in the decision-making the decision-making process.
- Fifteen criteria and eleven alternatives are used to evaluate leadership management strategies in healthcare systems.
- The sensitivity analysis is conducted to show the stability of the rank of alternatives.

1.3 Organization of this study

The rest of this study is organized as follows: Section 2 presents the definitions of type-2 neutrosophic sets. Section 3 presents the proposed methodology and steps of the COPRAS method. Section 4 presents the results of the COPRAS method. Section 5 presents the sensitivity analysis. Section 6 presents the conclusions of this study.

2. Type-II Neutrosophic Sets

This section offers some definitions of type-2 neutrosophic numbers (T2NNs)[38], [39].

Definition 1.

Offer X is a limited universe of discourse and y[0,1] is a group of neutrosophic triangular. A T2NN presented by U defined in X as:

$$U = \{(x, T_U(x), I_U(x), F_U(x) | \, x \in X)\} \quad (1)$$

$$\begin{pmatrix} T_U(x) \to Y[0,1] \\ I_U(x) \to Y[0,1] \\ F_U(x) \to Y[0,1] \end{pmatrix} \quad (2)$$

$$\begin{pmatrix} T_U(x) = \ T_{T_U}(x), T_{I_U}(x), T_{F_U}(x) \\ I_U(x) = \ I_{T_U}(x), I_{I_U}(x), I_{F_U}(x) \\ F_U(x) = \ F_{T_U}(x), F_{I_U}(x), F_{F_U}(x) \end{pmatrix} \quad (3)$$

$$0 \le T_U(x)^3 + I_U(x)^3 + F_U(x)^3 \le 3 \quad (4)$$

$$U = \left(\left(T_{T_U}(x), T_{I_U}(x), T_{F_U}(x)\right), \left(I_{T_U}(x), I_{I_U}(x), I_{F_U}(x)\right), \left(F_{T_U}(x), F_{I_U}(x), F_{F_U}(x)\right)\right) \quad (5)$$

Definition 2.

The summations of two T2NNs can be computed as:

$$U_1 = \left(\left(T_{T_{U_1}}(x), T_{I_{U_1}}(x), T_{F_{U_1}}(x)\right), \left(I_{T_{U_1}}(x), I_{I_{U_1}}(x), I_{F_{U_1}}(x)\right), \left(F_{T_{U_1}}(x), F_{I_{U_1}}(x), F_{F_{U_1}}(x)\right)\right) \text{ and}$$

$$U_2 = \left(\left(T_{T_{U_2}}(x), T_{I_{U_2}}(x), T_{F_{U_2}}(x)\right), \left(I_{T_{U_2}}(x), I_{I_{U_2}}(x), I_{F_{U_2}}(x)\right), \left(F_{T_{U_2}}(x), F_{I_{U_2}}(x), F_{F_{U_2}}(x)\right)\right)$$

$$U_1 \oplus U_2 = \begin{pmatrix} \begin{pmatrix} T_{T_{U_1}}(x) + T_{T_{U_2}}(x) - T_{T_{U_1}}(x) \times T_{T_{U_2}}(x), \\ T_{I_{U_1}}(x) + T_{I_{U_2}}(x) - T_{I_{U_1}}(x) \times T_{I_{U_2}}(x), \\ T_{F_{U_1}}(x) + T_{F_{U_2}}(x) - T_{F_{U_1}}(x) \times T_{F_{U_2}}(x) \end{pmatrix}, \\ \left(I_{T_{U_1}}(x) \times I_{T_{U_2}}(x), I_{I_{U_1}}(x) \times I_{I_{U_2}}(x), I_{F_{U_1}}(x) \times I_{F_{U_2}}(x)\right), \\ \left(F_{T_{U_1}}(x) \times F_{T_{U_2}}(x), F_{I_{U_1}}(x) \times F_{I_{U_2}}(x), F_{F_{U_1}}(x) \times F_{F_{U_2}}(x)\right) \end{pmatrix} \quad (6)$$

Multiplication

$$U_1 \otimes U_2 = \begin{pmatrix} \left(T_{T_{U_1}}(x) \times T_{T_{U_2}}(x), T_{I_{U_1}}(x) \times T_{I_{U_2}}(x), T_{F_{U_1}}(x) \times T_{F_{U_2}}(x)\right), \\ \begin{pmatrix} I_{T_{U_1}}(x) + I_{T_{U_2}}(x) - I_{T_{U_1}}(x) \times I_{T_{U_2}}(x), \\ I_{I_{U_1}}(x) + I_{I_{U_2}}(x) - I_{I_{U_1}}(x) \times I_{I_{U_2}}(x), \\ I_{F_{U_1}}(x) + I_{F_{U_2}}(x) - I_{F_{U_1}}(x) \times I_{F_{U_2}}(x) \end{pmatrix}, \\ \begin{pmatrix} F_{T_{U_1}}(x) + F_{T_{U_2}}(x) - F_{T_{U_1}}(x) \times F_{T_{U_2}}(x), \\ F_{I_{U_1}}(x) + F_{I_{U_2}}(x) - F_{I_{U_1}}(x) \times F_{I_{U_2}}(x), \\ F_{F_{U_1}}(x) + F_{F_{U_2}}(x) - F_{F_{U_1}}(x) \times F_{F_{U_2}}(x) \end{pmatrix} \end{pmatrix} \quad (7)$$

Scaler Multiplication

$$\curlyvee U_1 = \begin{pmatrix} \left(1 - \left(1 - T_{T_{U_1}}(x)\right)^{\curlyvee}, 1 - \left(1 - T_{I_{U_1}}(x)\right)^{\curlyvee}, 1 - \left(1 - T_{F_{U_1}}(x)\right)^{\curlyvee}\right), \\ \left(I_{T_{U_1}}(x)^{\curlyvee}, I_{I_{U_1}}(x)^{\curlyvee}, I_{F_{U_1}}(x)^{\curlyvee}\right), \\ \left(F_{T_{U_1}}(x)^{\curlyvee}, F_{I_{U_1}}(x)^{\curlyvee}, F_{F_{U_1}}(x)^{\curlyvee}\right) \end{pmatrix} \quad (8)$$

Power

$$U_1^{\curlyvee} = \begin{pmatrix} \left(\left(T_{T_{U_1}}(x)^{\curlyvee}, T_{I_{U_1}}(x)^{\curlyvee}, T_{F_{U_1}}(x)^{\curlyvee}\right)\right), \\ \left(1 - \left(1 - I_{T_{U_1}}(x)\right)^{\curlyvee}, 1 - \left(1 - I_{I_{U_1}}(x)\right)^{\curlyvee}, 1 - \left(1 - I_{F_{U_1}}(x)\right)^{\curlyvee}\right), \\ \left(1 - \left(1 - F_{T_{U_1}}(x)\right)^{\curlyvee}, 1 - \left(1 - F_{I_{U_1}}(x)\right)^{\curlyvee}, 1 - \left(1 - F_{F_{U_1}}(x)\right)^{\curlyvee}\right) \end{pmatrix} \quad (9)$$

Definition 3.

The score function can be computed as:

$$S(U) = \frac{1}{12} \begin{pmatrix} 8 + T_{T_U}(x) + 2\left(T_{I_U}(x) + T_{F_U}(x)\right) - \\ \left(I_{T_U}(x) + I_{I_U}(x)\right) + I_{F_U}(x) - \\ \left(F_{T_U}(x) + 2\left(F_{I_U}(x) + F_{F_U}(x)\right)\right) \end{pmatrix} \quad (10)$$

3. Methodology

This methodology presents the COPRAS method to evaluate leadership management in healthcare systems. COPRAS method integrated with type-2 neutrosophic sets[40], [41]. Figure 1 shows the COPRAS method with type-2 neutrosophic sets. The decision matrix is built between criteria and alternatives:

$$Y = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{m1} & \cdots & y_{mn} \end{pmatrix}_{m \times n} ; i = 1,2, \dots, m, \;\; j = 1,2, \dots, n \quad (11)$$

The normalized decision matrix is computed as:

$$y_{ij}^* = \frac{y_{ij}}{\sum_{i=1}^{m} y_{ij}}; \quad j = 1,2, \dots, n \quad (12)$$

The weights of criteria are computed to show the importance of criteria.The weighted normalized decision matrix is computed as:

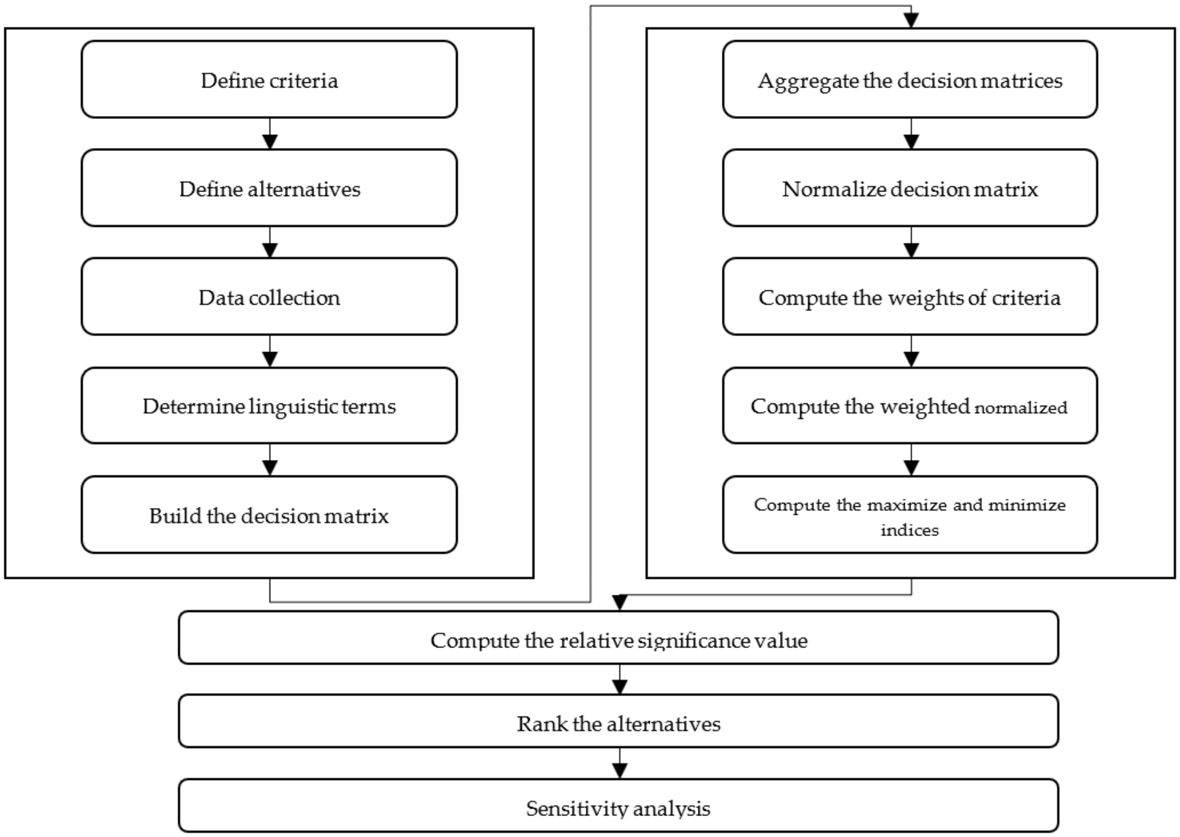


Figure 1. The steps of neutrosophic COPRAS method

$$r_{ij} = y_{ij} * w_j; \quad i = 1,2, \dots, m, \quad j = 1,2, \dots, n \tag{13}$$

Calculate the maximize and minimize indices as:

$$D_{+i} = \sum_{j=1}^{g} r_{ij}\,; \quad i = 1,2, \dots, m \tag{14}$$

$$D_{-i} = \sum_{j=g+1}^{n} r_{ij}\,; \quad i = 1,2, \dots, m \tag{15}$$

Calculate the relative significance value

$$L_i = D_{+i} + \frac{\min_i D_{-i} \sum_{i=1}^{m} D_{-i}}{D_{-i} \sum_{i=1}^{m} \frac{\min_i D_{-i}}{D_{-i}}} \tag{16}$$

$$L_i = D_{+i} + \frac{\sum_{i=1}^{m} D_{-i}}{D_{-i} \sum_{i=1}^{m} \frac{1}{D_{-i}}} \tag{17}$$

4. Results

This section presents the results of the COPRAS method to evaluate the strategies of leadership management in healthcare systems. This study invited three decision-makers to evaluate the criteria and alternatives. The decision-makers used the type-2 neutrosophic numbers as shown in Table 1 [42], [43]. Three decision makers are collected from 14 criteria and 11 alternatives of this study as shown in Figure 2.

Table 1. The linguistic terms of type-2 neutrosophic sets.

Linguistic terms	Type-2 Neutrosophic Numbers
Very Low	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>
Low	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>
Medium Low	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>
Medium	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>
Medium High	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>
High	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>
Very High	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)

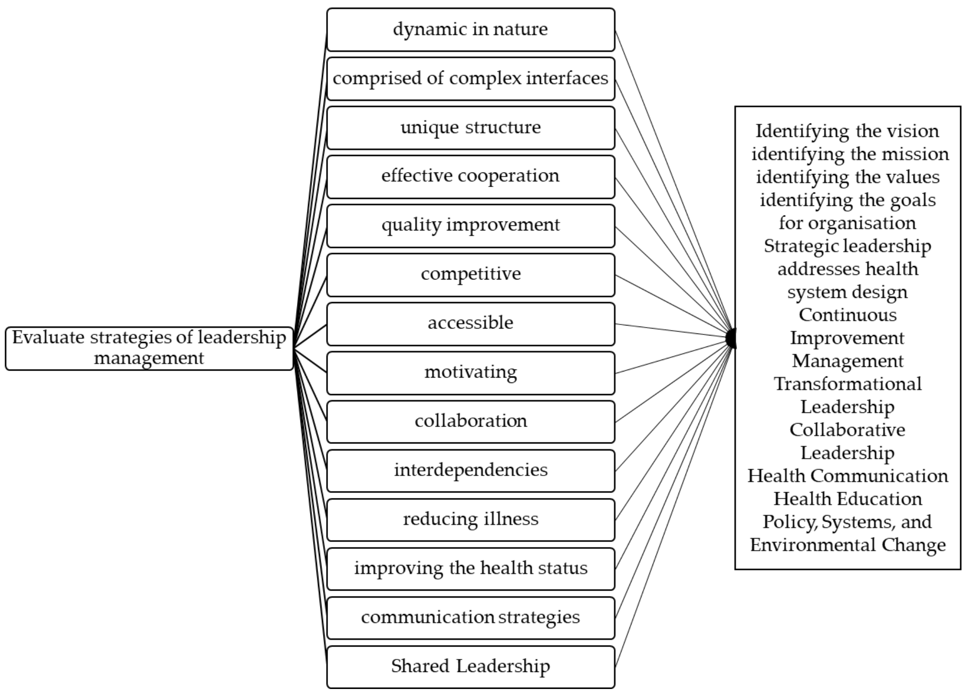


Figure 2. List of criteria and alternatives.

Table 2. Decision matrix by the first decision maker.

	LHC_1	LHC_2	LHC_3	LHC_4	LHC_5	LHC_6	LHC_7	LHC_8	LHC_9	LHC_{10}	LHC_{11}	LHC_{12}	LHC_{13}	LHC_{14}
LHA_1	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>
LHA_2	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>
LHA_3	<(0.20, 0.20, 0.10), (0.65, 0.80,	<(0.60, 0.45, 0.50), (0.20, 0.15,	<(0.60, 0.45, 0.50), (0.20, 0.15,	<(0.70, 0.75, 0.80), (0.15, 0.20,	<(0.95, 0.90, 0.95), (0.10, 0.10,	<(0.95, 0.90, 0.95), (0.10, 0.10,	<(0.60, 0.45, 0.50), (0.20, 0.15,	<(0.70, 0.75, 0.80), (0.15, 0.20,	<(0.60, 0.45, 0.50), (0.20, 0.15,	<(0.35, 0.35, 0.10), (0.50, 0.75,	<(0.40, 0.30, 0.35), (0.50, 0.45,	<(0.50, 0.45, 0.50), (0.40, 0.35,	<(0.50, 0.45, 0.50), (0.40, 0.35,	<(0.50, 0.45, 0.50), (0.40, 0.35,

	0.85), (0.45, 0.80, 0.70)>	0.25), (0.10, 0.25, 0.15)>	0.25), (0.10, 0.25, 0.15)>	0.25), (0.10, 0.15, 0.20)>	0.05), (0.05, 0.05, 0.05)	0.05), (0.05, 0.05, 0.05)	0.25), (0.10, 0.25, 0.15)>	0.25), (0.10, 0.15, 0.20)>	0.25), (0.10, 0.25, 0.15)>	0.80), (0.50, 0.75, 0.65)>	0.60), (0.45, 0.40, 0.60)>	0.50), (0.35, 0.30, 0.45)>	0.50), (0.35, 0.30, 0.45)>	0.50), (0.35, 0.30, 0.45)>
LHA_4	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>
LHA_5	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>
LHA_6	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>
LHA_7	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)
LHA_8	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)
LHA_9	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)
LHA_{10}	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>
LHA_{11}	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>

Table 3. Decision matrix by the second decision maker.

	LHC_1	LHC_2	LHC_3	LHC_4	LHC_5	LHC_6	LHC_7	LHC_8	LHC_9	LHC_{10}	LHC_{11}	LHC_{12}	LHC_{13}	LHC_{14}
LHA_1	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>
LHA_2	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>
LHA_3	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>
LHA_4	<(0.95, 0.90,	<(0.50, 0.45,	<(0.50, 0.45,	<(0.50, 0.45,	<(0.60, 0.45,	<(0.50, 0.45,	<(0.50, 0.45,	<(0.60, 0.45,	<(0.60, 0.45,	<(0.20, 0.20,	<(0.60, 0.45,	<(0.50, 0.45,	<(0.50, 0.45,	<(0.60, 0.45,

	0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>
LHA5	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>
LHA6	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>
LHA7	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)
LHA8	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)
LHA9	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)
LHA10	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>
LHA11	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>

Table 4. Decision matrix by the third decision maker.

	LHC1	**LHC2**	**LHC3**	**LHC4**	**LHC5**	**LHC6**	**LHC7**	**LHC8**	**LHC9**	**LHC10**	**LHC11**	**LHC12**	**LHC13**	**LHC14**
LHA1	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)
LHA2	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>
LHA3	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>
LHA4	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05,	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45,	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45,	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45,	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05,	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50,	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50,	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05,	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50,	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45,	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50,	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50,	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45,	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50,

	0.05, 0.05)	0.80, 0.70)>	0.40, 0.60)>	0.80, 0.70)>	0.05, 0.05)	0.75, 0.65)>	0.75, 0.65)>	0.05, 0.05)	0.75, 0.65)>	0.80, 0.70)>	0.75, 0.65)>	0.75, 0.65)>	0.80, 0.70)>	0.75, 0.65)>
LHA5	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>
LHA6	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)
LHA7	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>
LHA8	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>
LHA9	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>
LHA10	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>
LHA11	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.60, 0.45, 0.50), (0.20, 0.15, 0.25), (0.10, 0.25, 0.15)>	<(0.50, 0.45, 0.50), (0.40, 0.35, 0.50), (0.35, 0.30, 0.45)>	<(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)>	<(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)>	<(0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70)>	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)	<(0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)>	<(0.95, 0.90, 0.95), (0.10, 0.10, 0.05), (0.05, 0.05, 0.05)

Table 5. The normalized decision matrix.

	LHC_1	LHC_2	LHC_3	LHC_4	LHC_5	LHC_6	LHC_7	LHC_8	LHC_9	LHC_{10}	LHC_{11}	LHC_{12}	LHC_{13}	LHC_{14}
LHA_1	0.054986	0.080818	0.081556	0.082318	0.081081	0.098563	0.105107	0.122441	0.130029	0.047704	0.08306	0.07247	0.098423	0.11544
LHA_2	0.04844	0.040647	0.069905	0.076412	0.093615	0.121109	0.09323	0.08229	0.118026	0.146156	0.137025	0.124696	0.107256	0.06828
LHA_3	0.037312	0.133825	0.105078	0.112034	0.114375	0.089785	0.078583	0.11449	0.081129	0.094139	0.071563	0.067409	0.111672	0.071765
LHA_4	0.145974	0.079154	0.078699	0.061462	0.110262	0.080607	0.079968	0.122441	0.084908	0.04339	0.074613	0.07085	0.070032	0.084888
LHA_5	0.145974	0.076539	0.10266	0.070506	0.077752	0.080607	0.086105	0.083085	0.066904	0.08069	0.08306	0.113968	0.080336	0.098011
LHA_6	0.096225	0.075588	0.062431	0.059432	0.066588	0.072825	0.105107	0.080302	0.111136	0.09693	0.109808	0.084818	0.087066	0.124462
LHA_7	0.090334	0.08676	0.107496	0.0921	0.089698	0.11233	0.105107	0.075929	0.098466	0.08881	0.079071	0.098988	0.105152	0.103137
LHA_8	0.076369	0.131448	0.048802	0.107973	0.102233	0.093376	0.109462	0.099384	0.113359	0.126618	0.077428	0.059514	0.102839	0.106623
LHA_9	0.097316	0.121227	0.062871	0.113695	0.131022	0.114924	0.111441	0.109919	0.07335	0.09896	0.052088	0.057895	0.046688	0.106623
LHA_{10}	0.109753	0.121227	0.133436	0.117756	0.03349	0.034118	0.043943	0.044126	0.07335	0.09896	0.097137	0.113968	0.069401	0.051671
LHA_{11}	0.097316	0.052769	0.147065	0.106312	0.099882	0.101756	0.081948	0.065593	0.049344	0.077645	0.135148	0.135425	0.121136	0.0691

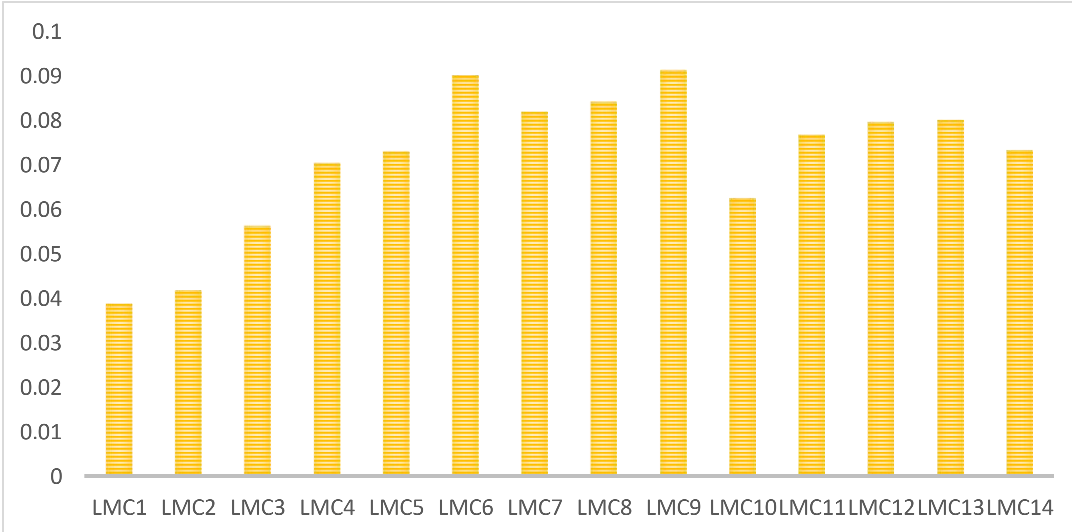


Figure 3. The criteria weights.

Table 6. The weighted normalized decision matrix.

	LHC_1	LHC_2	LHC_3	LHC_4	LHC_5	LHC_6	LHC_7	LHC_8	LHC_9	LHC_{10}	LHC_{11}	LHC_{12}	LHC_{13}	LHC_{14}
LHA_1	0.002132	0.003378	0.004597	0.005791	0.005916	0.008877	0.008615	0.010316	0.01187	0.002981	0.006373	0.005774	0.007874	0.008461
LHA_2	0.001878	0.001699	0.00394	0.005375	0.006831	0.010907	0.007641	0.006933	0.010774	0.009134	0.010514	0.009935	0.00858	0.005004
LHA_3	0.001447	0.005594	0.005922	0.007881	0.008345	0.008086	0.006441	0.009646	0.007406	0.005883	0.005491	0.005371	0.008934	0.00526
LHA_4	0.00566	0.003309	0.004435	0.004324	0.008045	0.00726	0.006554	0.010316	0.007751	0.002712	0.005725	0.005645	0.005603	0.006222
LHA_5	0.00566	0.003199	0.005786	0.00496	0.005673	0.00726	0.007057	0.007	0.006108	0.005043	0.006373	0.00908	0.006427	0.007183
LHA_6	0.003731	0.00316	0.003519	0.004181	0.004859	0.006559	0.008615	0.006766	0.010145	0.006058	0.008425	0.006758	0.006965	0.009122
LHA_7	0.003503	0.003627	0.006058	0.006479	0.006545	0.010117	0.008615	0.006397	0.008989	0.00555	0.006067	0.007887	0.008412	0.007559
LHA_8	0.002961	0.005494	0.00275	0.007596	0.007459	0.00841	0.008972	0.008373	0.010348	0.007913	0.005941	0.004742	0.008227	0.007815
LHA_9	0.003773	0.005067	0.003543	0.007998	0.00956	0.01035	0.009134	0.009261	0.006696	0.006184	0.003997	0.004613	0.003735	0.007815
LHA_{10}	0.004255	0.005067	0.00752	0.008284	0.002444	0.003073	0.003602	0.003718	0.006696	0.006184	0.007453	0.00908	0.005552	0.003787
LHA_{11}	0.003773	0.002206	0.008289	0.007479	0.007288	0.009164	0.006717	0.005526	0.004505	0.004852	0.01037	0.01079	0.009691	0.005064

The decision matrix is built between criteria and alternatives by using Eq. (1) as shown in Tables 2-4. The normalized decision matrix is computed by using Eq. (12) as shown in Table 5. The weights of criteria are computed to show the importance of criteria as shown in Figure 3. The weighted normalized decision matrix is computed by using Eq. (13) as shown in Table 6. Then calculate the maximize and minimize indices by using Eqs. (14 and 15). After that calculate the relative significance value by using Eqs. (16 and 17). Then rank the alternatives as shown in Figure 4.

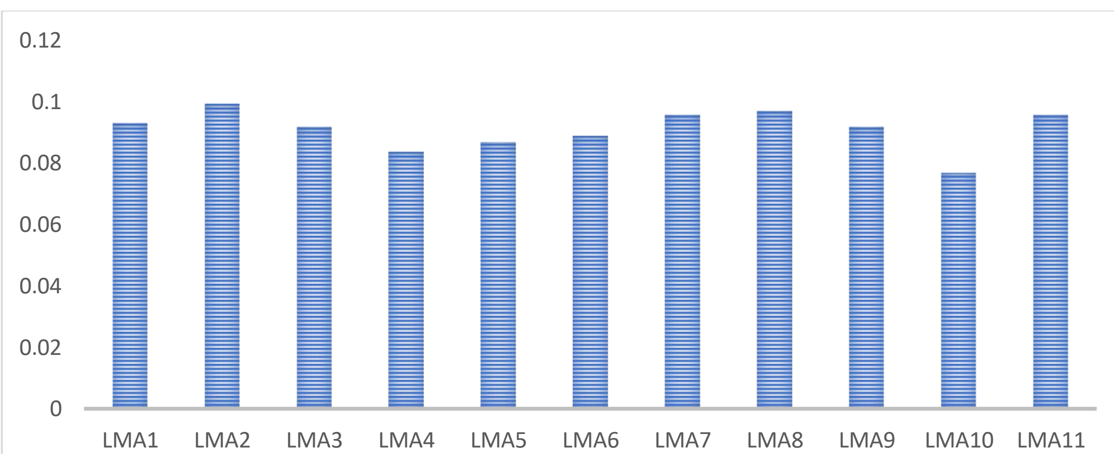


Figure 4. The rank of alternatives.

5. Sensitivity Analysis

This section changes the criteria weights and shows the rank of alternatives to show the stability of the rank. Fifteen cases change the criteria weights with different weights of each criterion. In the first weight, all criteria are given equal weights. In the second weight, the first criterion has a 0.08 weight, and all the other criteria have a weight equal to 0.07142. In the second case, the second criterion has 0.08, and other criteria have weights equal to 0.07142. in the third case, the second criterion has a weight equal to 0.08, and other criteria have weights equal to 0.07142. In the fourth case, the third criterion has weights equal to 0.08, and other criteria have weights equal to 0.07142. Table 7 shows the weights of criteria under sensitivity analysis.

Then, the COPRAS method under a neutrosophic set is applied with different criteria weights. Table 8 shows the rank values of each alternative. Figure 5 shows the rank of other options. The sensitivity analysis results show that the alternatives' rank is stable in different cases.

Table 7. The weights of criteria under sensitivity analysis.

	LHC_1	LHC_2	LHC_3	LHC_4	LHC_5	LHC_6	LHC_7	LHC_8	LHC_9	LHC_{10}	LHC_{11}	LHC_{12}	LHC_{13}	LHC_{14}
LHC1	0.071429	0.08	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769
LHC2	0.071429	0.070769	0.08	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769
LHC3	0.071429	0.070769	0.070769	0.08	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769
LHC4	0.071429	0.070769	0.070769	0.070769	0.08	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769
LHC5	0.071429	0.070769	0.070769	0.070769	0.070769	0.08	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769
LHC6	0.071429	0.070769	0.070769	0.070769	0.070769	0.070769	0.08	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769
LHC7	0.071429	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.08	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769
LHC8	0.071429	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.08	0.070769	0.070769	0.070769	0.070769	0.070769
LHC9	0.071429	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.08	0.070769	0.070769	0.070769	0.070769
LHC10	0.071429	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.08	0.070769	0.070769	0.070769
LHC11	0.071429	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.08	0.070769	0.070769
LHC12	0.071429	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.08	0.070769
LHC13	0.071429	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.08
LHC14	0.071429	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769	0.070769

Table 8. The rank values under sensitivity analysis.

	LHC_1	LHC_2	LHC_3	LHC_4	LHC_5	LHC_6	LHC_7	LHC_8	LHC_9	LHC_{10}	LHC_{11}	LHC_{12}	LHC_{13}	LHC_{14}
LHA_1	0.089571	0.089252	0.08949	0.089497	0.089504	0.089493	0.089654	0.089714	0.089874	0.089945	0.089185	0.089511	0.089413	0.089653
LHA_2	0.094792	0.094364	0.094292	0.094562	0.094622	0.094781	0.095035	0.094778	0.094677	0.095006	0.095266	0.095182	0.095068	0.094907
LHA_3	0.091654	0.091152	0.092043	0.091778	0.091842	0.091864	0.091637	0.091533	0.091865	0.091557	0.091677	0.091469	0.09143	0.091839
LHA_4	0.084803	0.085368	0.084751	0.084747	0.084588	0.085038	0.084765	0.084759	0.085151	0.084804	0.084421	0.084709	0.084675	0.084667
LHA_5	0.089014	0.08954	0.088899	0.08914	0.088843	0.08891	0.088936	0.088987	0.088959	0.08881	0.088937	0.088959	0.089244	0.088934
LHA_6	0.088051	0.088127	0.087936	0.087815	0.087787	0.087853	0.087911	0.088209	0.08798	0.088264	0.088133	0.088252	0.088021	0.088042
LHA_7	0.095241	0.095196	0.095163	0.095355	0.095212	0.09519	0.095399	0.095333	0.095063	0.095271	0.095182	0.095092	0.095276	0.095333
LHA_8	0.096816	0.096627	0.097136	0.096373	0.096919	0.096866	0.096784	0.096933	0.09684	0.096969	0.097091	0.096637	0.096472	0.096872
LHA_9	0.092716	0.092758	0.092979	0.09244	0.092909	0.093069	0.092921	0.092888	0.092874	0.092537	0.092773	0.092341	0.092394	0.092291
LHA_{10}	0.081595	0.081855	0.081961	0.082074	0.081929	0.081151	0.081157	0.081248	0.08125	0.081519	0.081756	0.081739	0.081894	0.081483
LHA_{11}	0.095746	0.09576	0.095349	0.096219	0.095843	0.095784	0.095801	0.095618	0.095467	0.095317	0.095579	0.096109	0.096112	0.09598

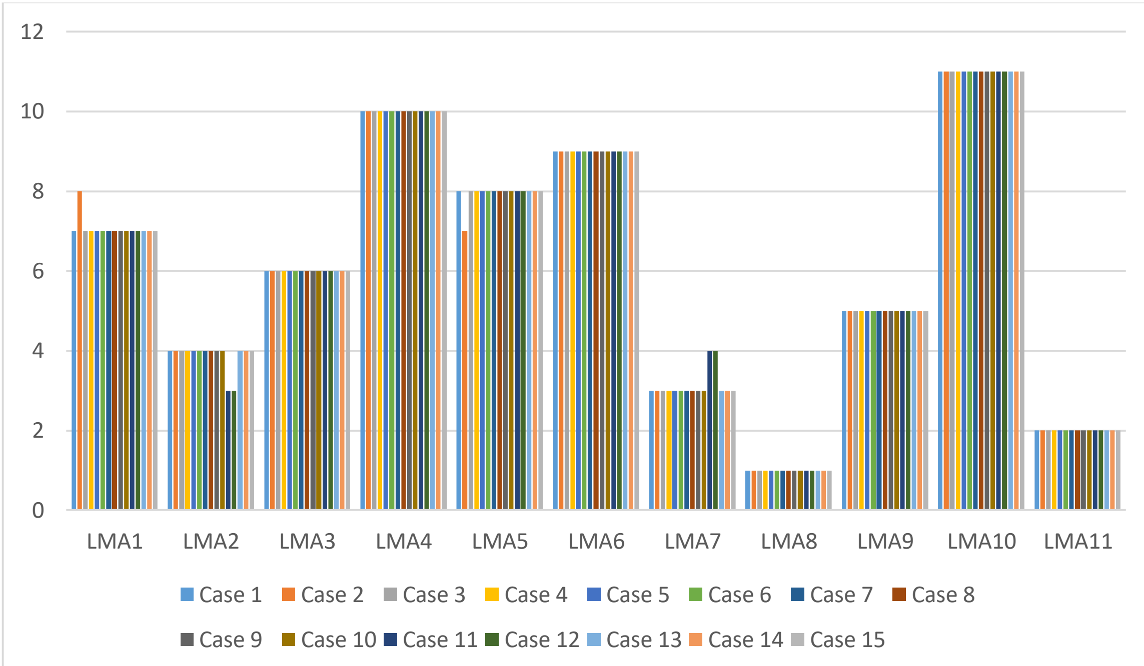


Figure 5. The rank of alternatives under sensitivity analysis.

6. Conclusions

This study proposed a hybrid MCDM methodology to evaluate leadership management strategies in healthcare systems. The MCDM method, such as the COPRAS method, ranks the alternatives. The type-2 neutrosophic sets were used to deal with uncertain information. Three decision-makers are invited in this study to evaluate the criteria and alternatives. This study used 14 criteria and 11 alternatives. Three decision-makers are built on three decision matrices between criteria and alternatives. Decision-makers use linguistic terms to evaluate the requirements and alternatives. Then, the type-2 neutrosophic numbers replace these terms to assess the criteria and alternatives. Then, these decision matrices are aggregated into a single decision matrix. Then, the steps of the COPRAS method are applied to show the rank of other options. The sensitivity analysis was conducted to show the stability of the rank of other possibilities. This study introduced 15 cases of weights. The results show that the rank of other options is stable in different cases. In the future, this proposed methodology can be applied to various decision-making processes. Various MCDM methods, such as VIKOR, TOPSIS, and MABAC, can be used in this study. A comparative analysis can be applied to show the strength of the proposed method.

References

[1] R. D. C. Kumar, "Leadership in healthcare," *Anaesthesia & Intensive Care Medicine*, vol. 14, no. 1, pp. 39–41, 2013.

[2] C. van Diggele, A. Burgess, C. Roberts, and C. Mellis, "Leadership in healthcare education," *BMC medical education*, vol. 20, pp. 1–6, 2020.

[3] R. B. Gunderman, *Leadership in healthcare*. Springer Science & Business Media, 2009.

[4] J. G. Calhoun *et al.*, "Development of an interprofessional competency model for healthcare leadership.," *Journal of healthcare management*, vol. 53, no. 6, 2008.

[5] J.-L. Denis and N. van Gestel, "Medical doctors in healthcare leadership: theoretical and practical challenges," *BMC health services research*, vol. 16, pp. 45–56, 2016.

[6] N. Gopee and J. Galloway, *Leadership and management in healthcare*. Sage, 2017.

[7] Z. Belrhiti, A. N. Giralt, and B. Marchal, "Complex leadership in healthcare: a scoping review," *International journal of health policy and management*, vol. 7, no. 12, p. 1073, 2018.
[8] C. W. Hargett *et al.*, "Developing a model for effective leadership in healthcare: a concept mapping approach," *Journal of healthcare leadership*, pp. 69–78, 2017.
[9] J. Edmonstone, "Healthcare leadership: learning from evaluation," *Leadership in Health Services*, vol. 26, no. 2, pp. 148–158, 2013.
[10] M. E. Stefl, "Common competencies for all healthcare managers: the Healthcare Leadership Alliance model.," *Journal of healthcare management*, vol. 53, no. 6, 2008.
[11] A. S. McAlearney, "Leadership development in healthcare: a qualitative study," *Journal of Organizational Behavior: The International Journal of Industrial, Occupational and Organizational Psychology and Behavior*, vol. 27, no. 7, pp. 967–982, 2006.
[12] R. Chatterjee, R. Suy, Y. Yen, and L. Chhay, "Literature review on leadership in healthcare management," *Journal of social science studies*, vol. 5, no. 1, pp. 38–47, 2018.
[13] N. M. Stefano, N. Casarotto Filho, L. G. L. Vergara, and R. U. G. da Rocha, "COPRAS (Complex Proportional Assessment): state of the art research and its applications," *IEEE Latin America Transactions*, vol. 13, no. 12, pp. 3899–3906, 2015.
[14] A. Alinezhad, J. Khalili, A. Alinezhad, and J. Khalili, "COPRAS method," *New methods and applications in multiple attribute decision making (Madm)*, pp. 87–91, 2019.
[15] L. A. Zadeh, "On fuzzy algorithms," in *fuzzy sets, fuzzy logic, and fuzzy systems: selected papers By Lotfi A Zadeh*, World Scientific, 1996, pp. 127–147.
[16] L. A. Zadeh, "Towards a theory of fuzzy systems," *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A. Zadeh*, pp. 83–104, 1996.
[17] L. A. Zadeh, "Fuzzy logic," in *Granular, Fuzzy, and Soft Computing*, Springer, 2023, pp. 19–49.
[18] L. A. Zadeh, "Fuzzy sets," *Information and control*, vol. 8, no. 3, pp. 338–353, 1965.
[19] K. T. Atanassov and K. T. Atanassov, *Intuitionistic fuzzy sets*. Springer, 1999.
[20] S. Broumi and F. Smarandache, "Several similarity measures of neutrosophic sets," *Infinite Study*, vol. 410, 2013.
[21] W. Al-Omeri and F. Smarandache, "New neutrosophic sets via neutrosophic topological spaces," *Infinite Study*, vol. 2016, pp. 1–15, 2016.
[22] F. Smarandache and S. Pramanik, *New trends in neutrosophic theory and applications*, vol. 1. Infinite Study, 2016.
[23] F. Smarandache, *Neutrosophic Theory and Its Applications, Vol. I: Collected Papers*. Infinite Study, 2014.
[24] F. Smarandache, *Neutrosophic precalculus and neutrosophic calculus: neutrosophic applications*. Infinite Study, 2015.
[25] F. Smarandache *et al.*, "Introduction to neutrosophy and neutrosophic environment," in *Neutrosophic Set in Medical Image Analysis*, Elsevier, 2019, pp. 3–29.
[26] N. Nabeeh, "Assessment and Contrast the Sustainable Growth of Various 1 Road Transport Systems using Intelligent Neutrosophic 2 Multi-Criteria Decision-Making Model," *Sustainable Machine Intelligence Journal*, vol. 2, 2023.
[27] D. M. R. Guerra, O. P. Peña, and J. A. T. Yanez, "Neutrosophic Evaluation of Ethical Factors in Remote Medical Care," *HyperSoft Set Methods in Engineering*, vol. 1, pp. 11–20, 2024.
[28] R. K. Saini, A. Ahirwar, F. Smarandache, and M. Kushwaha, "Multi-Criteria Group Decision-Making with ELECTRE-III Method for Selection of Female Spouse in Pythagorean Neutrosophic Environment," *Plithogenic Logic and Computation*, vol. 1, pp. 36–53, 2024.
[29] M. Jdid, "An Efficient Neutrosophic Method for Generating Random Variates from Erlang Distribution," *Neutrosophic Optimization and Intelligent Systems*, vol. 2, pp. 56–68, 2024.
[30] M. Abouhawwash and M. Jameel, "Evaluation Factors of Solar Power Plants to Reduce Cost Under Neutrosophic Multi-Criteria Decision-Making Model," *Sustainable Machine Intelligence Journal*, vol. 2, 2023.
[31] K. M. Sallam and A. W. Mohamed, "Single Valued Neutrosophic Sets for Assessment Quality of Suppliers under Uncertainty Environment," *Multicriteria Algorithms with Applications*, vol. 1, no. 1, pp. 1–10, 2023.
[32] U. Orukwowu, "Nursing leadership in healthcare: the impact of effective nurse leadership on quality healthcare outcomes," *IPS Interdisciplinary Journal of Social Sciences*, vol. 1, no. 1, pp. 1–6, 2022.
[33] S. Janssens, R. Simon, M. Beckmann, and S. Marshall, "Shared leadership in healthcare action teams: A systematic review," *Journal of Patient Safety*, vol. 17, no. 8, pp. e1441–e1451, 2021.

[34] G. P. Martin and M. Learmonth, “A critical account of the rise and spread of ‘leadership’: the case of UK healthcare,” *Social science & medicine*, vol. 74, no. 3, pp. 281–288, 2012.
[35] B. Alilyyani, C. A. Wong, and G. Cummings, “Antecedents, mediators, and outcomes of authentic leadership in healthcare: A systematic review,” *International journal of nursing studies*, vol. 83, pp. 34–64, 2018.
[36] R. Menaker, “Leadership strategies in healthcare,” *J Med Pract Manage*, vol. 24, no. 6, pp. 339–343, 2009.
[37] P. Weintraub and M. McKee, “Leadership for innovation in healthcare: an exploration,” *International journal of health policy and management*, vol. 8, no. 3, p. 138, 2019.
[38] V. Simic, I. Gokasar, M. Deveci, and A. Karakurt, “An integrated CRITIC and MABAC based type-2 neutrosophic model for public transportation pricing system selection,” *Socio-Economic Planning Sciences*, vol. 80, p. 101157, 2022.
[39] M. Abdel-Basset, M. Saleh, A. Gamal, and F. Smarandache, “An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number,” *Applied Soft Computing*, vol. 77, pp. 438–452, 2019, doi: https://doi.org/10.1016/j.asoc.2019.01.035.
[40] S. Ahmad, F. Ahmad, and M. Sharaf, “Supplier selection problem with type-2 fuzzy parameters: A neutrosophic optimization approach,” *International journal of fuzzy systems*, vol. 23, pp. 755–775, 2021.
[41] A. E. Torkayesh, M. Deveci, S. E. Torkayesh, and E. B. Tirkolaee, “Analyzing failures in adoption of smart technologies for medical waste management systems: a type-2 neutrosophic-based approach,” *Environmental Science and Pollution Research*, pp. 1–14, 2021.
[42] V. Simic, I. Gokasar, M. Deveci, and L. Švadlenka, “Mitigating Climate Change Effects of Urban Transportation Using a Type-2 Neutrosophic MEREC-MARCOS Model,” *IEEE Transactions on Engineering Management*, 2022.
[43] Ş. Özlü and F. Karaaslan, “Hybrid similarity measures of single-valued neutrosophic type-2 fuzzy sets and their application to MCDM based on TOPSIS,” *Soft Computing*, vol. 26, no. 9, pp. 4059–4080, 2022.

Received: Jan 8, 2024. **Accepted**: Apr 30, 2024

Neutrosophic Sets and Systems, Vol. 67, 2024
University of New Mexico

Pythagorean Neutrosophic Triplet Groups

Madad Khan[1], Muhammad Zeeshan [2,] *, Saima Anis[1] and Florentin Smrandache [3]

[1]Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus, Pakistan; madadkhan@cuiatd.edu.pk; saimaanis@cuiatd.edu.pk

[2]Department of Mathematics, COMSATS University Islamabad, Islamabad Campus, Pakistan e-mail@e-mail.com;

FA20-PMT-003@ISBSTUDENT.COMSATS.EDU.PK

[4] Department of Mathematics and Sciences, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu

* Correspondence: FA20-PMT-003@ISBSTUDENT.COMSATS.EDU.PK ; Tel.:+923419205348

Abstract: It is a well-known fact that groups are the only algebraic structures having a single binary operation that is mathematically so perfect that it is impossible to introduce a richer structure within it. The main purpose of this study is to introduce the notion of the Pythagorean neutrosophic triplet (PNT) which is the generalization of neutrosophic triplet (NT). The PNT is an algebraic structure of three ordered pairs that satisfy several properties under the binary operation (B-Operation) "*". Furthermore, we used the PNTs to introduce the novel concept of a Pythagorean neutrosophic triplet group (PNTG). The algebraic structure (AS) of PNTG is different from the neutrosophic triplet group (NTG). We discussed some properties, related results, and particular examples of these novel concepts. We further studied Pythagorean neutro-homomorphism, Pythagorean neutro-isomorphism, etc., for PNTGs. Moreover, we discussed the main distinctions between the neutrosophic triplet group (NTG) and the PNTG.

Keywords: Neutrosophic triplet, Pythagorean neutrosophic triplet, Neutrosophic triplet group, Pythagorean neutrosophic triplet group.

1. Introduction

Neutrosophy is a novel tributary of philosophy associated with the origin, nature, and scope of neutralities. Smarandache [21] defined the notion of neutrosophic set (NS) and neutrosophic logic (NL) in 1995. Each proposition in NL is almost to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. The NS is the extension of fuzzy sets (FSs) [25], classical sets [12], intuitionistic fuzzy sets (IFSs) [6], Pythagorean fuzzy sets (PFSs) [16], and interval-valued fuzzy sets, etc., while as NL is the generalization of fuzzy logic, intuitionistic fuzzy logic, ete. The theory of neutrosophic set [22] is utilized to discuss problems involving imprecision, uncertainty, indeterminacy, incompleteness, inconsistency, and falsity. Smarandache and Kandasamy used neutrosophic theory to investigate neutrosophic ASs in [11], [12], [13] by embedding an indeterminacy "I" into the AS. They combine indeterminacy with the elements of the AS under the binary operation $*$ and say it neutrosophic element (NE), and the novel AS is known as neutrosophic AS. Moreover, they further developed several NASs such as neutrosophic fields, neutrosophic groups, neutrosophic N-groups, neutrosophic bigroups, neutrosophic vector spaces, neutrosophic bisemigroups, neutrosophic semigroups, neutrosophic N-semigroup, neutrosophic groupoids, neutrosophic bigroupoids, neutrosophic biloops, neutrosophic loops, and neutrosophic N-loop, and so on.

In algebraic structures, the groups [8], [9], [24] are so significant that they contribute the role of backbone in practically all the theory of ASs. In the study of algebra, Groups are the most foundational and well-off AS under certain binary operations 🖓. Groups give concrete foundations for many ASs, such as rings, fields, vector spaces, and so on. Many other fields, including physics, chemistry, combinatorics, biology, and others, use groups to explore symmetries and other behavior among their elements. Group action is the most crucial feature of a group. In daily life problems, matrix groups, permutation groups, lie groups, transformation groups, and other forms of groups are widely employed as a mathematical tools. In this regard, generalized groups [8] are very significant. For the first time, Smarandache and Ali [20] gave the notion of neutrosophic triplet (NT). The newly defined notion of NTs is highly dependable on the binary operation $*$. Furthermore, they utilized the concept of NTs to introduce an NTG. Regarding to the structural and foundational properties, the NTG is distinct from the classical group. The NT has a strong structure than Molaei's Generalized Group [15]. Zhang et al. [26] defined a new congruence relation based on commutative NTG. They induced quotient structure by neutrosophic triplet subgroup and proved neutron-homomorphism basic theorem. Zhang et al. [27] gave the notions of neutrosophic triplet subgroups, strong neutrosophic triplet subgroups and weak commutative neutrosophic triplet groups. Further, they established quotient structures on strong neutrosophic triplet subgroups. Hu and Zhang studied the relationships among various neutrosophic extended triplet cyclic associative semihypergroups [10]. The main properties of strong pure neutrosophic extended triplet cyclic associative semihypergroups are obtained. A lot of researchers have been dealing with neutrosophic triplet metric space, neutrosophic triplet vector space, neutrosophic triplet inner product, and neutrosophic triplet normed space in ([17], [18], [19]). The concept of neutrosophic extended triplet was given by by Smarandache in [23]. The neutrosophic extended triplet is the generalization of neutrosophic triplet. Li et al. [14] studied neutrosophic extended triplet group based on neutrosophic quadruple numbers. They proved some significant results with respect to neutrosophic quadruple numbers. Bal et al. [7] defined the neutrosophic image, neutrosophic inverse-image, neutrosophic kernel, and the NET subgroup. The notion of the neutrosophic triplet coset and its relation with the classical coset were discussed. Furthermore, the neutrosophic triplet normal subgroups, and neutrosophic triplet quotient groups were studied. Zhou and Xin proposed the notion of ideals on neutrosophic extended triplet groups and discussed their different properties [29]. The concept of singular neutrosophic extended triplet group was given by Zhang et al. in [28]. They discussed different properties and proved some significant results.

The concept of neutrosophic ring was defined in [11] which is the generalization of classical rings. Agboola et al. [1,2] introduced the notion of refined neutrosophic ring which is the extension of neutrosophic ring. Ahmad et al. [3] solved the imperfect duplets problem in refined neutrosophic rings. Ali et al. [4] proposed the concepts of zero divisor, neutrosophic triplet subring, neutrosophic triplet ideal, nilpotent integral neutrosophic domain, and neutrosophic triplet ring homomorphism. Moreover, they gave the idea of a neutrosophic triplet field.

In this paper, we define the notion of the PNT which is the generalization of NT. We utilize the PNTs to introduce the novel concept of a PNTG. The algebraic structure of PNTG is different from the neutrosophic triplet group (NTG). We discuss some properties, related results, and particular examples of these novel concepts. We further study Pythagorean neutro-homomorphism, Pythagorean neutro-isomorphism, etc., for PNTGs. Moreover, we study the main distinctions between the NTG and the PNTG.

3. Neutrosophic Triplet Groups

In this section, we will recall the notions of NTGs.

Definition [20] Let $\aleph$ be a set together with a B-Operation $*$. Let ℓ be an element of $\aleph$. If there exist a neutral of ℓ characterized by $\overset{+}{\aleph}(\ell)$, and an opposite of ℓ characterized by $\overset{-}{\aleph}(\ell)$, with $\overset{+}{\aleph}(\ell)$ and $\overset{-}{\aleph}(\ell)$ are the elements of $\aleph$, such that:

$$\ell * \overset{+}{\aleph}(\ell) = \overset{+}{\aleph}(\ell) * \ell = \ell,$$

and

$$\ell * \overset{-}{\aleph}(\ell) = \overset{-}{\aleph}(\ell) * \ell = \overset{+}{\aleph}(\ell).$$

Then, $\aleph$ is said to be an NT set.

The elements ℓ, $\overset{+}{\aleph}(\ell)$, and $\overset{-}{\aleph}(\ell)$ are collectively called as NT. The NT set is denoted by $(\ell, \overset{+}{\aleph}(\ell), \overset{-}{\aleph}(\ell))$, where ℓ is the first coordinate of NT, and by $\overset{+}{\aleph}(\ell)$, we mean neutral of ℓ.

Note that $\overset{+}{\aleph}(\ell)$ is not same as the classical algebraic unitary element. For the same element ℓ in N, the neutral "$\overset{+}{\aleph}(\ell)$" and opposites "$\overset{-}{\aleph}(\ell)$" of ℓ are not unique.

Definition [20] Let $(\aleph,*)$ be a NTS. Then, $\aleph$ is said to be a NTG, if the following properties are hold.

(i). If $(\aleph,*)$ is well-defined, i.e. for any ℓ, $\wp \in \aleph$, one has $\ell * \wp \in \aleph$.

(ii). If $(\aleph,*)$ is associative, i.e. $(\ell * \wp) * \hbar = \ell * (\wp * \hbar)$ for all ℓ, $\wp$, $\hbar \in \aleph$.

Definition [20] Let $(\aleph_1,*)$ and $(\aleph_2,\#)$ be two NTGs. Let $F : \aleph_1 \to \aleph_2$ be a mapping. Then, F is called neutro-homomorphism if for all ℓ, $\wp \in \aleph_1$, we have

(i). $F(\ell * \wp) = F(\ell) \# F(\wp)$,

(ii). $F(\overset{+}{\aleph}(\ell)) = \overset{+}{\aleph}(F(\ell))$,

and

(iii). $F(\overset{-}{\aleph}(\ell)) = \overset{-}{\aleph}(F(\ell))$.

4. Pythagorean neutrosophic triplet (PNT)

Definition Let $P_{\aleph}$ be a set together with a B-Operation $*$. Let ℓ_1 and ℓ_2 be any two elements of $P_{\aleph}$. If there exists neutrals of "ℓ_1" and "ℓ_2" known $\overset{+}{\aleph}(\ell_1)$ and $\overset{+}{\aleph}(\ell_2)$, are not same as the classical algebraic unitary elements, and an opposite of "ℓ_1" and "ℓ_2" called $\overset{-}{\aleph}(\ell_1)$ and $\overset{-}{\aleph}(\ell_2)$, with $\overset{+}{\aleph}(\ell_1)$, $\overset{+}{\aleph}(\ell_2)$, $\overset{-}{\aleph}(\ell_1)$, and $\overset{-}{\aleph}(\ell_2)$ are elements of $P_{\aleph}$, such that:

$$\ell_1 * \overset{+}{\aleph}(\ell_1) = \overset{+}{\aleph}(\ell_1) * \ell_1 = \ell_1,\ \ell_1 * \overset{-}{\aleph}(\ell_1) = \overset{-}{\aleph}(\ell_1) * \ell_1 = \overset{+}{\aleph}(\ell_1),$$
$$\ell_2 * \overset{+}{\aleph}(\ell_2) = \overset{+}{\aleph}(\ell_2) * \ell_2 = \ell_2,\ \ell_2 * \overset{-}{\aleph}(\ell_2) = \overset{-}{\aleph}(\ell_2) * \ell_2 = \overset{+}{\aleph}(\ell_2),$$

and

$$\ell_1 * \ell_2 = \overset{+}{\aleph}(\ell_1).$$

Then, $P_{\aleph}$ is said to be a PNT set.

Definition The second component of the PNT is $(\wp_1, \wp_2)$, represented as $\overset{+}{\aleph}(\cdot)$, and $\overset{+}{\aleph}(\bullet)$, if there exist other elements (ℓ_1, ℓ_2), $(\hbar_1, \hbar_2)$ in PNT such that $\ell_1 * \wp_1 = \wp_1 * \ell_1 = \ell_1$, $\ell_1 * \hbar_1 = \hbar_1 * \ell_1 = \wp_1$, and $\ell_2 * \wp_2 = \wp_2 * \ell_2 = \ell_2$, $\ell_2 * \hbar_2 = \hbar_2 * \ell_2 = \wp_2$, and $\ell_1 * \ell_2 = \overset{+}{\aleph}(\ell_1)$. The formed PNT is $[(\ell_1, \ell_2), (\wp_1, \wp_2), (\hbar_1, \hbar_2)]$.

Definition The element $(\hbar_1, \hbar_2)$ is the third component, represented as $\overset{-}{\aleph}(\cdot)$, and $\overset{-}{\aleph}(\bullet)$, of a PNT, if there exist other elements (ℓ_1, ℓ_2), $(\wp_1, \wp_2)$ in PNT such that $\ell_1 * \wp_1 = \wp_1 * \ell_1 = \ell_1$, $\ell_1 * \hbar_1 = \hbar_1 * \ell_1 = \wp_1$, and $\ell_2 * \wp_2 = \wp_2 * \ell_2 = \ell_2$, $\ell_2 * \hbar_2 = \hbar_2 * \ell_2 = \wp_2$, and $\ell_1 * \ell_2 = \overset{+}{\aleph}(\ell_1)$. The formed PNT is $[(\ell_1, \ell_2), (\wp_1, \wp_2), (\hbar_1, \hbar_2)]$.

Example Consider $(Z_{10}, *)$, where "*" is defined as $\ell * \wp = 3\ell\wp(\text{mod}10)$ and $Z_{10} = \{0, 1, 2, 3, ..., 9\}$ then,

*	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	3	6	9	2	5	8	1	4	7
2	0	6	2	8	4	0	6	2	8	4
3	0	9	8	7	6	5	4	3	2	1
4	0	2	4	6	8	0	2	4	6	8
5	0	5	0	5	0	5	0	5	0	5
6	0	8	6	4	2	0	8	6	4	2
7	0	1	2	3	4	5	6	7	8	9
8	0	4	8	2	6	0	4	8	2	6
9	0	7	4	1	8	5	2	9	6	3

1 and 9 produce an PNT because $\overset{+}{\aleph}(1) = 7,$ as $1 * 7 = 21 \equiv 1(\text{mod}10).$ Also $\overset{-}{\aleph}(1) = 9,$ as $1 * 9 = 27 \equiv 7(\text{mod}10).$ Now $\overset{+}{\aleph}(9) = 7,$ as $9 * 7 = 189 \equiv 9(\text{mod}10).$ Also $\overset{-}{\aleph}(9) = 1,$ as $9 * 1 = 27 \equiv 7(\text{mod}10).$ Thus, $[(1,9),(7,7),(9,1)]$ is a PNT because $1 * 9 = 27 \equiv 7(\text{mod}10) = \overset{+}{\aleph}(1).$

Similarly, $[(4,6),(2,2),(6,4)]$ is a PNT. But 2 and 4 does not give rise to a PNT. Since $[(2,4),(2,2),(2,6)]$ does not implies that $2 * 4 = \overset{+}{\aleph}(2),$ as $2 * 4 = 24 \equiv 4(\text{mod}10) \neq \overset{+}{\aleph}(2).$ The trivial PNT is denoted by $[(0,0),(0,0),(0,0)],$ as $\overset{+}{\aleph}(0) = 0$, $\overset{-}{\aleph}(0) = 0$ and $0 * 0 = 0 = \overset{-}{\aleph}(0).$

Theorem If $[(\ell, \wp), (\overset{+}{\aleph}(\ell), \overset{+}{\aleph}(\wp)), (\overset{-}{\aleph}(\ell), \overset{-}{\aleph}(\wp))]$ form a PNT, then $[(\overset{-}{\aleph}(\ell), \overset{-}{\aleph}(\wp)), (\overset{+}{\aleph}(\ell), \overset{+}{\aleph}(\wp)), (\ell, \wp)]$ also form a Pythagorean neutrosophic.

Proof Of course $\overset{-}{\aleph}(\ell) * \ell = \overset{+}{\aleph}(\ell)$ and $\overset{-}{\aleph}(\wp) * \wp = \overset{+}{\aleph}(\wp).$ We have to prove that $\overset{-}{\aleph}(\ell) * \overset{+}{\aleph}(\ell) = \overset{-}{\aleph}(\ell)$ and $\overset{-}{\aleph}(\wp) * \overset{+}{\aleph}(\wp) = \overset{-}{\aleph}(\wp).$ Multiply by ℓ and $\wp$ to the L.H.S, we have

$$\overset{-}{\aleph}(\ell) * \overset{+}{\aleph}(\ell) = \overset{-}{\aleph}(\ell) \text{ and } \overset{-}{\aleph}(\wp) * \overset{+}{\aleph}(\wp) = \overset{-}{\aleph}(\wp)$$

$$\ell * \overset{-}{\aleph}(\ell) * \overset{+}{\aleph}(\ell) = \ell * \overset{-}{\aleph}(\ell) \text{ and } \wp * \overset{-}{\aleph}(\wp) * \overset{+}{\aleph}(\wp) = \wp * \overset{-}{\aleph}(\wp)$$

or

$$[\ell * \overset{-}{\aleph}(\ell)] * \overset{+}{\aleph}(\ell) = \overset{+}{\aleph}(\ell) \text{ and } [\wp * \overset{-}{\aleph}(\wp)] * \overset{+}{\aleph}(\wp) = \overset{+}{\aleph}(\wp)$$

or

$$\overset{+}{\aleph}(\ell) * \overset{+}{\aleph}(\ell) = \overset{+}{\aleph}(\ell) \text{ and } \overset{+}{\aleph}(\wp) * \overset{+}{\aleph}(\wp) = \overset{+}{\aleph}(\wp)$$

Again multiply by ℓ and $\wp$ to the L.H.S, we have:

$$\ell * \overset{+}{\aleph}(\ell) * \overset{+}{\aleph}(\ell) = \ell * \overset{+}{\aleph}(\ell) \text{ and } \wp * \overset{+}{\aleph}(\wp) * \overset{+}{\aleph}(\wp) = \wp * \overset{+}{\aleph}(\wp)$$

or

$$[\ell * \overset{+}{\aleph}(\ell)] * \overset{+}{\aleph}(\ell) = \ell \text{ and } [\wp * \overset{+}{\aleph}(\wp)] * \overset{+}{\aleph}(\wp) = \wp$$

or

$$\ell * \overset{+}{\aleph}(\ell) = \ell \text{ and } \wp * \overset{+}{\aleph}(\wp) = \wp$$

Pythagorean neutrosophic triplet group (PNTG)

Definition If $(P_{\aleph},*)$ be a PNT set. Then, $P_{\aleph}$ is said be a PNTG with the properties.

(i). If $(P_{\aleph},*)$ is well-defined, i.e. for any $\ell, \wp \in P_{\aleph}$, one has $\ell * \wp \in \aleph$.

(ii). If $(P_{\aleph},*)$ is associative i.e. $(\ell_1 * \wp_1) * \hbar_1 = \ell_1 * (\wp_1 * \hbar_1)$ and $(\ell_2 * \wp_2) * \hbar_2 = \ell_2 * (\wp_2 * \hbar_2)$ for all $\ell_1, \ \ell_2, \ \wp_1, \ \wp_2, \ \hbar_1, \ \hbar_2 \in P_{\aleph}$,.

In general, the PNTG is different from the classical algebraic group.

The Pythagorean neutrosophic neutrals is considered as the classical unitary element and the Pythagorean neutrosophic opposites is considered as the classical inverse elements.

Example Consider $(Z_5,*)$, where $*$ is defined as $\ell * \wp = \ell(\text{mod } 5)$. Then, $(Z_5,*)$ is a PNTG with the given table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4

It is associative, i.e. $(\ell_1 * \wp_1) * \hbar_1 = \ell_1 * (\wp_1 * \hbar_1)$ and $(\ell_2 * \wp_2) * \hbar_2 = \ell_2 * (\wp_2 * \hbar_2)$. Now take L. H. S to prove the R. H. S, so

$$\begin{aligned}(\ell_1 * \wp_1) * \hbar_1 &= \ell_1 * \hbar_1 \\ &= \ell_1\end{aligned}$$

also,

$$\begin{aligned}\ell_1 * (\wp_1 * \hbar_1) &= \ell_1 * \wp_1 \\ &= \ell_1.\end{aligned}$$

From (1) and (2), we have

$$(\ell_1 * \wp_1) * \hbar_1 = \ell_1 * (\wp_1 * \hbar_1).$$

Similarly,

$$\begin{aligned}(\ell_2 * \wp_2) * \hbar_2 &= \ell_2 * \hbar_2 \\ &= \ell_2\end{aligned}$$

also,

$$\begin{aligned}\ell_2 * (\wp_2 * \hbar_2) &= \ell_2 * \wp_2 \\ &= \ell_2.\end{aligned}$$

From (1) and (2), we have

$$(\ell_2 * \wp_2) * \hbar_2 = \ell_2 * (\wp_2 * \hbar_2).$$

The Pythagorean triplets are:

$$\left\{\begin{array}{l}[(0,1),(0,1),(0,1)]: \text{ such that } 0*1 = \overset{+}{\aleph}(0) \\ [(1,2),(1,2),(1,2)]: \text{ such that } 1*2 = \overset{+}{\aleph}(1) \\ [(4,3),(4,3),(4,3)]: \text{ such that } 4*3 = \overset{+}{\aleph}(4)\end{array}\right\}.$$

Thus, all the elements of Z_5 give rise to a PNT.

Example Consider $(Z_3,*)$, where $*$ is defined as $\ell * \wp = \ell + \wp + 3(\text{mod}\, 5)$. Then, $(Z_3,*)$ is a PNTG under the B-Operation $*$ with the below table.

*	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

It is associative, i.e. $(\ell_1 * \wp_1) * \hbar_1 = \ell_1 * (\wp_1 * \hbar_1)$ and $(\ell_2 * \wp_2) * \hbar_2 = \ell_2 * (\wp_2 * \hbar_2)$.

Also, the Pythagorean triplets are:

$$\left\{\begin{array}{l}[(0,1),(0,0),(0,2)]: \text{ such that } 0*1 = \overset{+}{\aleph}(0) \\ [(1,2),(0,1),(2,2)]: \text{ such that } 1*2 = \overset{+}{\aleph}(1)\end{array}\right\}$$

Thus, all the elements of Z_3 give rise to a PNT.

Remark Let $(P_{\aleph},*)$ be a PNTG under $*$ and let $[(\ell,\wp),(\overset{+}{\aleph}(\ell),\overset{+}{\aleph}(\wp)),(\overset{-}{\aleph}(\ell),\overset{-}{\aleph}(\wp))]$ such that $\ell*\wp=\overset{+}{\aleph}(\ell)$, be a PNT. Then, $\overset{+}{\aleph}(\ell)$ and $\overset{+}{\aleph}(\wp)$ is not unique in $P_{\aleph}$, and also ne $\overset{+}{\aleph}(\ell)$, $\overset{+}{\aleph}(\wp)$ depends on the elements ℓ and $\wp$, and the B-Operation $*$.

We consider the below Example to prove the above remark.

In Example 2, consider the PNT, $[(1,2),(1,2),(1,2)]$: such that $1*2=\overset{+}{\aleph}(1)$, so the $\overset{+}{\aleph}(1)=0,1,2,3,4$. Similarly, $\overset{+}{\aleph}(\wp)=0,1,2,3,4$. Thus, $\overset{+}{\aleph}(\ell)$ and $\overset{+}{\aleph}(\wp)$ is not unique in $P_{\aleph}$.

Remark Let $(P_{\aleph},*)$ be a PNTG under $*$ and let $[(\ell,\wp),(\overset{+}{\aleph}(\ell),\overset{+}{\aleph}(\wp)),(\overset{-}{\aleph}(\ell),\overset{-}{\aleph}(\wp))]$ such that $\ell*\wp=\overset{+}{\aleph}(\ell)$, be a PNT. Then, $\overset{-}{\aleph}(\ell)$ and $\overset{-}{\aleph}(\wp)$ is not unique in $P_{\aleph}$, and also $\overset{-}{\aleph}(\ell)$, $\overset{-}{\aleph}(\wp)$ depends on the elements ℓ and $\wp$, and the B-Operation $*$.

We consider the below Example to prove the above remark.

In Example 2, consider the PNT, $[(2,3),(2,3),(2,3)]$: such that $2*3=\overset{-}{\aleph}(2)$, so $\overset{-}{\aleph}(2)=0,1,2,3,4$. Similarly, $\overset{+}{\aleph}(\wp)=0,1,2,3,4$. Thus, $\overset{+}{\aleph}(\ell)$ and $\overset{+}{\aleph}(\wp)$ is not unique in $P_{\aleph}$.

Theorem All PNTG is a NTG.

Proof Let $P_{\aleph}$ be a PNTG. Then it satisfies the given conditions.

(i). If $(P_{\aleph},*)$ is well-defined, i.e. for any $\ell,\wp\in P_{\aleph}$, one has $\ell*\wp\in\aleph$.

(ii). If $(P_{\aleph},*)$ is associative i.e. $(\ell_1*\wp_1)*\hbar_1=\ell_1*(\wp_1*\hbar_1)$ and $(\ell_2*\wp_2)*\hbar_2=\ell_2*(\wp_2*\hbar_2)$ for all ℓ_1, ℓ_2, $\wp_1$, $\wp_2$, $\hbar_1$, $\hbar_2\in P_{\aleph}$,.

From conditions (i) and (ii), we have that $P_{\aleph}$ is a NT.

Example In example 1, $(Z_{10},*)$ is a NT but not a PNTG because the PNT $[(2,4),(2,2),(2,6)]$ does not implies that $2*4=\overset{+}{\aleph}(2)$, as $2*4=24\equiv 4(\mathrm{mod}10)\neq\overset{+}{\aleph}(2)$. Hence, not all the elements of Z_{10} give rise to a PNT.

Definition Let $(P_{\aleph},*)$ be a PNTG. Then, $P_{\aleph}$ is called a commutative PNTG if for all $\ell,\wp\in P_{\aleph}$, we have $\ell*\wp=\wp*\ell$.

Example In example 2, $(Z_5,*)$ is a PNTG but not a commutative PNTG because $0*1\neq 1*0$. Similarly, $1*2\neq 2*1$.

Theorem The idempotent elements ℓ and $\wp$ give rise to PNT under the operation $*$ such that $\ell * \wp = \ell$.

Solution Let ℓ and $\wp$ be idempotent elements. Then, by definition, $\ell^2 = \ell$ and $\wp^2 = \wp,$ which clearly implies that $\overset{+}{\aleph}(\ell) = \ell,$ $\overset{-}{\aleph}(\ell) = \ell$ and $\overset{+}{\aleph}(\wp) = \wp,$ $\overset{-}{\aleph}(\wp) = \wp.$ Also, $\ell * \wp = \ell = \overset{+}{\aleph}(\ell)$ or $\wp * \ell = \wp = \overset{+}{\aleph}(\wp).$ Hence ℓ and $\wp$ give rise to a PNT, that is,

$[(\ell, \wp), (\ell, \wp), (\ell, \wp)] : \ell * \wp = \ell.$

Theorem Let $(P_{\aleph}, *)$ be a PNTG with respect to $*$ and let $[(\ell_1, \wp_1), (\ell_2, \wp_2), (\ell_3, \wp_3)] : \ell_1 * \wp_1 = \overset{+}{\aleph}(\ell_1),$ be a PNT then

$(i).$ $\ell_1 * \ell_2 = \ell_1 * \ell_3$ and $\wp_1 * \wp_2 = \wp_1 * \wp_3$ if and only if $\overset{+}{\aleph}(\ell_1) * \ell_2 = \overset{+}{\aleph}(\ell_1) * \ell_3$ and $\overset{+}{\aleph}(\wp_1) * \wp_2 = \overset{+}{\aleph}(\wp_1) * \wp_3.$

$(ii).$ $\ell_2 * \ell_1 = \ell_3 * \ell_1$ and $\wp_2 * \wp_1 = \wp_3 * \wp_1$ if and only if $\overset{+}{\aleph}(\ell_2) * \ell_1 = \overset{+}{\aleph}(\ell_3) * \ell_1$ and $\overset{+}{\aleph}(\wp_2) * \wp_1 = \overset{+}{\aleph}(\wp_3) * \wp_1.$

Proof $(i).$ Suppose that $\ell_1 * \ell_2 = \ell_1 * \ell_3$ and $\wp_1 * \wp_2 = \wp_1 * \wp_3.$ Since $P_{\aleph}$ is a PNTG, so $\overset{-}{\aleph}(\ell_1), \overset{-}{\aleph}(\wp_1) \in P_{\aleph}.$ Multiply $\overset{-}{\aleph}(\ell_1)$ to the L.H.S with $\ell_1 * \ell_2 = \ell_1 * \ell_3,$ we get:

$$\begin{aligned} \overset{-}{\aleph}(\ell_1) * \ell_1 * \ell_2 &= \overset{-}{\aleph}(\ell_1) * \ell_1 * \ell_3 \\ [\overset{-}{\aleph}(\ell_1) * \ell_1] * \ell_2 &= [\overset{-}{\aleph}(\ell_1) * \ell_1] * \ell_3 \\ \overset{+}{\aleph}(\ell_1) * \ell_2 &= \overset{+}{\aleph}(\ell_1) * \ell_3. \end{aligned}$$

Similarly, multiply $\overset{-}{\aleph}(\wp_1)$ to the left side with $\wp_1 * \wp_2 = \wp_1 * \wp_3,$ we get:

$$\begin{aligned} \overset{-}{\aleph}(\wp_1) * \wp_1 * \wp_2 &= \overset{-}{\aleph}(\wp_1) * \wp_1 * \wp_3 \\ [\overset{-}{\aleph}(\wp_1) * \wp_1] * \wp_2 &= [\overset{-}{\aleph}(\wp_1) * \wp_1] * \wp_3 \\ \overset{+}{\aleph}(\wp_1) * \wp_2 &= \overset{+}{\aleph}(\wp_1) * \wp_3. \end{aligned}$$

$(ii).$ The proof is similar to $(i).$

Theorem Let $(P_{\aleph}, *)$ be a PNTG with respect to $*$ and let $[(\ell_1, \wp_1), (\ell_2, \wp_2), (\ell_3, \wp_3)] : \ell_1 * \wp_1 = \overset{+}{\aleph}(\ell_1),$ be a PNT then

(i). $\overset{-}{\aleph}(\ell_1)*\ell_2=\overset{-}{\aleph}(\ell_1)*\ell_3$ and $\overset{-}{\aleph}(\wp_1)*\wp_2=\overset{-}{\aleph}(\wp_1)*\wp_3$ then $\overset{+}{\aleph}(\ell_1)*\ell_2=\overset{+}{\aleph}(\ell_1)*\ell_3$ and $\overset{+}{\aleph}(\wp_1)*\wp_2=\overset{+}{\aleph}(\wp_1)*\wp_3$.

(ii). $\ell_2*\overset{-}{\aleph}(\ell_1)=\ell_3*\overset{-}{\aleph}(\ell_1)$ and $\wp_2*\overset{-}{\aleph}(\wp_1)=\wp_3*\overset{-}{\aleph}(\wp_1)$ then $\ell_2*\overset{+}{\aleph}(\ell_1)=\ell_3*\overset{+}{\aleph}(\ell_1)$ and $\wp_2*\overset{+}{\aleph}(\wp_1)=\wp_3*\overset{+}{\aleph}(\wp_1)$.

Proof (i). Suppose that $\overset{-}{\aleph}(\ell_1)*\ell_2=\overset{-}{\aleph}(\ell_1)*\ell_3$ and $\overset{-}{\aleph}(\wp_1)*\wp_2=\overset{-}{\aleph}(\wp_1)*\wp_3$. Since $P_{\aleph}$ is a PNTG with respect to *, so ℓ_1, $\wp_1\in P_{\aleph}$. Multiply ℓ_1 to the left side with $\overset{-}{\aleph}(\ell_1)*\ell_2=\overset{-}{\aleph}(\ell_1)*\ell_3$, we get:

$$\begin{aligned}\ell_1*\overset{-}{\aleph}(\ell_1)*\ell_2 &=\ell_1*\overset{-}{\aleph}(\ell_1)*\ell_3\\ [\ell_1*\overset{-}{\aleph}(\ell_1)]*\ell_2 &=[\ell_1*\overset{-}{\aleph}(\ell_1)]*\ell_3\\ \overset{+}{\aleph}(\ell_1)*\ell_2 &=\overset{+}{\aleph}(\ell_1)*\ell_3.\end{aligned}$$

Similarly, multiply $\wp_1$ to the left side with $\overset{-}{\aleph}(\wp_1)*\wp_2=\overset{-}{\aleph}(\wp_1)*\wp_3$, we get:

$$\begin{aligned}\wp_1*\overset{-}{\aleph}(\wp_1)*\wp_2 &=\wp_1*\overset{-}{\aleph}(\wp_1)*\wp_3\\ [\wp_1*\overset{-}{\aleph}(\wp_1)]*\wp_2 &=[\wp_1*\overset{-}{\aleph}(\wp_1)]*\wp_3\\ \overset{+}{\aleph}(\wp_1)*\wp_2 &=\overset{+}{\aleph}(\wp_1)*\wp_3.\end{aligned}$$

(ii). The proof is same as (i).

Theorem Let $(P_{\aleph},*)$ be a PNTG with respect to * and let $[(\ell_1,\wp_1),(\ell_2,\wp_2),(\ell_3,\wp_3)]:\ell_1*\wp_1=\overset{+}{\aleph}(\ell_1)$, be a PNT then $\overset{+}{\aleph}(\ell_1)*\overset{+}{\aleph}(\wp_1)=\overset{+}{\aleph}(\ell_1*\wp_1)$.

Proof Consider L.H.S, $\overset{+}{\aleph}(\ell_1)*\overset{+}{\aleph}(\wp_1)$. Now, multiply to the L.H.S with ℓ_1 and to the R.H.S with $\wp_1$, we get:

$$\begin{aligned}\ell_1*\overset{+}{\aleph}(\ell_1)*\overset{+}{\aleph}(\wp_1)*\wp_1 &=[\ell_1*\overset{+}{\aleph}(\ell_1)]*[\overset{+}{\aleph}(\wp_1)*\wp_1]\\ &=\ell_1*\wp_1.\end{aligned}$$

Now consider R.H.S, we have $\overset{+}{\aleph}(\ell_1*\wp_1)$. Again multiply to the L.H.S with ℓ_1 and to the R.H.S with $\wp_1$, we get:

$$\ell_1*\overset{+}{\aleph}(\ell_1*\wp_1)*\wp_1=[\ell_1*\wp_1]*\overset{+}{\aleph}(\ell_1*\wp_1),$$

as $*$ is associative, so we have

$$\ell_1 * \overset{+}{\aleph}(\ell_1 * \wp_1) * \wp_1 = \ell_1 * \wp_1.$$

Hence proved.

Theorem Let $(P_\aleph, *)$ be a PNTG with respect to $*$ and let $[(\ell_1, \wp_1), (\ell_2, \wp_2), (\ell_3, \wp_3)] : \ell_1 * \wp_1 = \overset{+}{\aleph}(\ell_1),$ be a PNT then $\overset{+}{\aleph}(\ell_1) * \overset{+}{\aleph}(\wp_1) = \overset{+}{\aleph}(\ell_1 * \wp_1).$

Proof Consider L.H.S, $\overset{-}{\aleph}(\ell_1) * \overset{-}{\aleph}(\wp_1)$. Now multiply to the L.H.S with ℓ_1 and to the R.H.S with $\wp_1$, we get:

$$\begin{aligned} \ell_1 * \overset{-}{\aleph}(\ell_1) * \overset{-}{\aleph}(\wp_1) * \wp_1 &= [\ell_1 * \overset{-}{\aleph}(\ell_1)] * [\overset{-}{\aleph}(\wp_1) * \wp_1] \\ &= \overset{+}{\aleph}(\ell_1) * \overset{+}{\aleph}(\wp_1) \\ &= \overset{+}{\aleph}(\ell_1 * \wp_1), \text{ By above theorem.} \end{aligned}$$

Now consider R.H.S, we have $\overset{-}{\aleph}(\ell_1 * \wp_1)$. Again multiply to the L.S with ℓ_1 and to the R.S with $\wp_1$, we get:

$$\ell_1 * \overset{-}{\aleph}(\ell_1 * \wp_1) * \wp_1 = [\ell_1 * \wp_1] * \overset{-}{\aleph}(\ell_1 * \wp_1),$$

as $*$ is associative, so we have

$$\ell_1 * \overset{-}{\aleph}(\ell_1 * \wp_1) * \wp_1 = \overset{+}{\aleph}(\ell_1 * \wp_1).$$

Hence proved.

Theorem Let $(P_\aleph, *)$ be a commutative PNTG with respect to $*$ and let $[(\ell_1, \wp_1), (\ell_2, \wp_2), (\ell_3, \wp_3)] : \ell_1 * \wp_1 = \overset{+}{\aleph}(\ell_1),$ be a PNT then

(i). $\overset{+}{\aleph}(\ell_1) * \overset{+}{\aleph}(\wp_1) = \overset{+}{\aleph}(\wp_1) * \overset{+}{\aleph}(\ell_1).$

(ii). $\overset{-}{\aleph}(\ell_1) * \overset{-}{\aleph}(\wp_1) = \overset{-}{\aleph}(\wp_1) * \overset{-}{\aleph}(\ell_1).$

Proof Consider the L.H.S $\overset{+}{\aleph}(\ell_1) * \overset{+}{\aleph}(\wp_1).$ According to Theorem 7, we have

$$\begin{aligned} \overset{+}{\aleph}(\ell_1) * \overset{+}{\aleph}(\wp_1) &= \overset{+}{\aleph}(\ell_1 * \wp_1) \\ &= \overset{+}{\aleph}(\wp_1 * \ell_1), \end{aligned}$$

as $P_\aleph$ is commutative, so

$$\overset{+}{\aleph}(\ell_1) * \overset{+}{\aleph}(\wp_1) = \overset{+}{\aleph}(\wp_1) * \overset{+}{\aleph}(\ell_1),$$

again by Theorem 7.

Thus $\overset{+}{\aleph}(\ell_1) * \overset{+}{\aleph}(\wp_1) = \overset{+}{\aleph}(\wp_1) * \overset{+}{\aleph}(\ell_1).$

(ii). The proof is same as (i).

Definition Let $(P_{\aleph}, *)$ be a PNTG with respect to $*$, and let $P_{\aleph}' \subseteq P_{\aleph}$. Then, $P_{\aleph}'$ is said to be a PNT subgroup of $P_{\aleph}$ if $P_{\aleph}'$ itself a PNTG with respect to $*$.

Example Consider $(Z_5, *)$, where $*$ is defined as $\ell * \wp = \ell (\text{mod } 5)$. Then, $(Z_5, *)$ is a PNTG under the binary operation $*$, and $P_{\aleph}' = \{0, 1, 2, 3\}$ be a subset of Z_5. Then, clearly $P_{\aleph}'$ is a PNT subgroup of Z_5.

Proposition Let $(P_{\aleph}, *)$ be a PNTG and $P_{\aleph}' \subseteq P_{\aleph}$. Then $P_{\aleph}'$ is a PNT subgroup of N $\Leftrightarrow$ the following properties are satisfied.

(i). $\ell_1 * \wp_1 \in P_{\aleph}'$ for all ℓ_1, $\wp_1 \in P_{\aleph}'$.

(ii). $\overset{+}{\aleph}(\ell_1)$, $\overset{+}{\aleph}(\wp_1) \in P_{\aleph}'$ for all ℓ_1, $\wp_1 \in P_{\aleph}'$.

(iii). $\overset{-}{\aleph}(\ell_1)$, $\overset{-}{\aleph}(\wp_1) \in P_{\aleph}'$ for all ℓ_1, $\wp_1 \in P_{\aleph}'$.

Proof It is easy to prove.

Definition Let $P_{\aleph}$ be a PNTG and let $\ell \in P_{\aleph}$. The smallest positive integer $n \geq 1$ such that $\ell^n = \overset{+}{\aleph}(\ell)$ is called PNT order. It is denoted by $pnto(\ell)$.

Example Consider $(Z_5, *)$, where $*$ is defined as $\ell * \wp = \ell (\text{mod } 5)$ then

$$pnto(1) = 1, \; pnto(2) = 1, \; pnto(3) = 1, \; pnto(4) = 1.$$

Theorem Let $(P_{\aleph}, *)$ be a PNTG with respect to $*$ and let $\alpha \in P_{\aleph}$. Then $\overset{+}{\aleph}(\alpha) * \overset{+}{\aleph}(\alpha) = \overset{+}{\aleph}(\alpha)$. In general $(\overset{+}{\aleph}(\alpha))^n = \overset{+}{\aleph}(\alpha)$, where $n \geq 1$.

Proof Consider $\overset{+}{\aleph}(\alpha) * \overset{+}{\aleph}(\alpha) = \overset{+}{\aleph}(\alpha)$. Multiply α to the left side, we get;

$$\alpha * \overset{+}{\aleph}(\alpha) * \overset{+}{\aleph}(\alpha) = \alpha * \overset{+}{\aleph}(\alpha)$$
$$[\alpha * \overset{+}{\aleph}(\alpha)] * \overset{+}{\aleph}(\alpha) = [\alpha * \overset{+}{\aleph}(\alpha)]$$
$$\alpha * \overset{+}{\aleph}(\alpha) = \alpha$$
$$\alpha = \alpha$$

Similarly, we can easily see that $(\overset{+}{\aleph}(\alpha))^n = \overset{+}{\aleph}(\alpha)$ for a nonzero positive integer n.

Definition Let $P_{\aleph}$ be a PNTG and $\alpha \in P_{\aleph}$. Then, $P_{\aleph}$ is called Pythagorean neutro-cyclic triplet group if $P_{\aleph} = \langle \alpha \rangle$. We say that α is a generator part of the PNT.

5. Pythagorean neutro-homomorphism (PN-h)

In this section, we introduce PN-h and Pythagorean neutro-isomorphisms (PN-i) for the PNTGs.

Definition Let $(P_{\aleph_1}, *)$ and $(P_{\aleph_2}, \#)$ be two PNTGs. Let $F : P_{\aleph_1} \to P_{\aleph_2}$ be mapping. Then, F is called PN-h if for all ℓ, $\wp \in P_{\aleph_1}$, we have

(i). $F(\ell * \wp) = F(\ell) \ \# \ F(\wp).$

(ii). $F(\overset{+}{\aleph}(\ell)) = \overset{+}{\aleph}(F(\ell)).$

(iii). $F(\overset{-}{\aleph}(\ell)) = \overset{-}{\aleph}(F(\ell)).$

Example Let $P_{\aleph_1}$ be a PNTGs with respect to $*$ modulo 10 in $(Z_{10}, *)$, where $*$ is defined as: $\ell * \wp = \ell(\text{mod}\,10)$, and $Z_{10} = \{0, 1, 2, 3, ..., 9\}$. Let $F : P_{\aleph_1} \to P_{\aleph_1}$ be a mapping defined as:

$$F(0) = 0,\ F(1) = 1,\ F(2) = 2,\ F(3) = 3,\ F(4) = 4,\ F(5) = 5,\ F(6) = 6,$$
$$F(7) = 7,\ F(8) = 8,\ F(9) = 9.$$

Then, clearly F is a PN-h because conditions (i), (ii) and (iii) are satisfied easily.

Definition A PN-h is called PN-i if it is one--one and onto.

Example Let $P_{\aleph_1}$ be a PNTGs with respect to $*$ modulo 5 in $(Z_5, *)$, where $*$ is defined as: $\ell * \wp = \ell(\text{mod}\,5)$, and $Z_5 = \{0, 1, 2, 3, 4\}$. Let $F : P_{\aleph_1} \to P_{\aleph_1}$ be a mapping defined as:

$$F(0) = 0,\ F(1) = 1,\ F(2) = 2,\ F(3) = 3,\ F(4) = 4.$$

Then, clearly F is a PN-i.

Note that F further defines that,

(i). F is called Pythagorean neutro-endomorphism of $P_{\aleph_1}$, if $P_{\aleph_1} = P_{\aleph_2}$.

(ii). F is called Pythagorean neutro-epimorphism if F is onto.

(iii). F is called Pythagorean neutro-monomorphism if F is $(1-1)$.

(iv). A Pythagorean neutro-endomorphism F of a Pythagorean triplet group $P_{\aleph}$ is called a Pythagorean neutro-automorphism of G if F is $(1-1)$ and onto.

Definition Two given PNTGs $(P_{\aleph_1}, *)$ and $(P_{\aleph_2}, \#)$ are said to be Pythagorean neutro-isomorphic to each other if there exists a PN-i between $P_{\boxplus_1}$ and $P_{\boxplus_2}$.

They are written as $P_{\aleph_1} \cong P_{\aleph_2}$, and read as "the PNTG $P_{\aleph_1}$ is Pythagorean neutro-isomorphic to the PNTG $P_{\aleph_2}$ ".

Theorem The relation "$\cong$" of Pythagorean neutro-isomorphism over the set of all the Pythagorean neutrosophic triplet group is an equivalence relation.

Proof Let $P_{\aleph_1}$, $P_{\aleph_2}$ and $P_{\aleph_3}$ be three Pythagorean neutrosophic triplet groups. Then $P_{\aleph_1} \cong P_{\aleph_1}$ by the identity Pythagorean neutro-isomorphism.

For symmetry, suppose $P_{\aleph_1} \cong P_{\aleph_2}$. Then there exists a Pythagorean neutro-isomorphism $F : P_{\aleph_1} \to P_{\aleph_2}$. Since F is $(1-1)$ and onto Pythagorean neutro-homomorphism from $P_{\aleph_1}$ onto $P_{\aleph_2}$, therefore $F^{-1} : P_{\aleph_2} \to P_{\aleph_1}$ is a Pythagorean neutro-isomorphism. Thus $P_{\aleph_2} \cong P_{\aleph_1}$.

For transitivity, suppose, $P_{\aleph_1} \cong P_{\aleph_2}$ and $P_{\aleph_2} \cong P_{\aleph_3}$. Since $P_{\aleph_1}$ is Pythagorean neutro-isomorphism to $P_{\aleph_2}$ and $P_{\aleph_2}$ is Pythagorean neutro-isomorphism to $P_{\aleph_3}$, therefore, there exists, Pythagorean neutro-isomorphism F and g such that $F : P_{\aleph_1} \to P_{\aleph_2}$ and $g : P_{\aleph_2} \to P_{\aleph_3}$.

Since $g \circ F : P_{\aleph_1} \to P_{\aleph_3}$ is a Pythagorean neutro-isomorphism from $P_{\aleph_1}$ into $P_{\aleph_3}$. It proves that $P_{\aleph_1} \cong P_{\aleph_3}$.

Hence the relation "$\cong$" is an equivalence relation over the set of all Pythagorean neutrosophic triplet groups.

6. Distinctions and comparison

The distinctions between NT [20] and PNTG are:

(i). In the NT, any single element $\ell \in \aleph$, gives rise to a neutrosophic triplet, while in the PNTG, two elements ℓ_1, $\wp_1 \in P_{\aleph}$ give rise to a PNT with the extra condition, that is, $\ell_1 * \wp_1 = \overset{+}{\aleph}(\ell_1)$.

(ii). The structure of neutrosophic triplet is $(\ell, \wp, \hbar)$, where $\wp = \overset{+}{\aleph}(\ell)$, and $\hbar = \overset{-}{\aleph}(\ell)$, while the structure of PNT is

$$[(\ell_1, \wp_1), (\ell_2, \wp_2), (\hbar_1, \hbar_2)]: \text{ such that } \ell_1 * \wp_1 = \overset{+}{\aleph}(\ell_1),$$

where $\ell_2 = \overset{+}{\aleph}(\ell_1)$, $\ell_3 = \overset{-}{\aleph}(\ell_1)$, $\wp_2 = \overset{+}{\aleph}(\wp_1)$, and $\wp_3 = \overset{-}{\aleph}(\wp_1)$.

Clearly, the NTG has a weaker structure than the PNTG.

7. Conclusions

Inspiring from the neutrosophic triplet, we developed the notion of PNT which is a group of three ordered pairs that satisfy certain properties with some binary operation. The purpose of this paper is first to introduce the PNT and then used these PNTs to introduce the novel concept of a PNTG, which differs from a classical group in terms of structural features. The PNTG is completely different from the neutrosophic group. We discussed some properties, basic results, and particular examples of these novel concepts. We further studied PN-h, PN-i, etc., for NTs. Moreover, we discussed the main distinctions between the NTG and the PNTG.

Conflict of Interest

The authors declare that they have no conflict of interests.

Acknowledgments: This work is financially supported by the Higher Education Commission of Pakistan (Grant No: 7750/Federal/NRPU/R&D/HEC/2017).

Authorship Contributions: All authors contributed equally.

References

[1]. E.O. Adeleke, A.A.A. Agboola, F. Smarandache. Refined Neutrosophic Rings I. International Journal of Neutrosophic Science, 2(2) (2020), 77-81.

[2]. E.O. Adeleke, A.A.A. Agboola, F. Smarandache, Refined Neutrosophic Rings II. International Journal of Neutrosophic Science, 2(2) (2020), pp. 89-94.

[3]. K.D. Ahmad. On Imperfect Duplets In Some refined Neutrosophic Rings. *journal of neutrosophic and fuzzy systems, 2022*.

[4]. M. Ali, F. Smarandache, M. Khan. Study on the development of neutrosophic triplet ring and neutrosophic triplet field. *Mathematics*, *6*(4) (2018), 46.

[5]. A.M.D.J.S. Alkouri, A.R. Salleh. Complex intuitionistic fuzzy sets. In AIP conference proceedings, 1482(1) (2012), 464-470.

[6]. T.K. Atanassov. Intuitionistic fuzzy sets. Fuzzy Sets Syst 20 (1986), 87-96.

[7]. M. Bal, M.M. Shalla, N. Olgun. Neutrosophic triplet cosets and quotient groups. *Symmetry*, *10*(4) (2018), 126.

[8]. D.S. Dummit, R.M. Foote. Abstract algebra. 3rd edn. John Viley & Sons Inc, New Jersey, 2004.

[9]. I.N. Herstein. Topics in algebra. Xerox college publishing, Lexington, 1975.

[10]. M. Hu, X. Zhang. On cyclic associative semihypergroups and neutrosophic extended triplet cyclic associative semihypergroups. *Mathematics*, *10*(4) (2022), 535.

[11] W.V. Kandasamy, F. Smarandache. Some neutrosophic algebraic structures and neutrosophic n-algebraic structures. Hexis, Frontigan, (2006), 219.

[12]. W.V. Kandasamy F. Smarandache. N-algebraic structures and s-n-algebraic structures. Hexis, Phoenix, (2006), 209.

[13]. W.V. Kandasamy F. Smarandache. Basic neutrosophic algebraic structures and their applications to fuzzy and neutrosophic models. Hexis, Frontigan, (2004), 149.

[14]. Q. Li Y. Ma, X. Zhang, J. Zhang. Neutrosophic extended triplet group based on neutrosophic quadruple numbers. *Symmetry*, *11*(5) (2019), 696.

[15]. M.R. Molaei. Generalized groups. Bul Inst Politehn Ia, si Sect I 45(49) (1999), 21-24.

[16]. X. Peng Y. Yang. Some results for Pythagorean fuzzy sets. International Journal of Intelligent Systems, 30(11) (2015), 1133-1160.

[17]. M. Sahin, K. Abdullah. Neutrosophic triplet normed space. Open Phys. 2017, 15, 697–704.

[18]. M. Şahin, A. Kargın. *Neutrosophic Triplet Inner Product*. Infinite Study, 2017.

[19]. M. Şahin, A. Kargın. Neutrosophic triplet v-generalized metric space. *Axioms*, *7*(3) (2018), 67.

[20]. F. Smarandache, M. Ali. Neutrosophic triplet group. Neural Computing and Applications, 29(7) (2018), 595-601.

[21]. F. Smarandache. A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic. Rehoboth: American Research Press, 1999.

[22]. F. Smarandache. Neutrosophic set, a generalization of the intuitionistic fuzzy set, In: 2006 IEEE international conference on granular computing, 10-12 May 2006, 38--42. doi:10.1109/GRC.2006.1635754.

[23]. F. Samarandache. Neutrosophic Extenede Triplets. Special collection,Arizona State University: Tempe, AZ, USA, 2016.

[24]. D.B. Surowski. The uniqueness aspect of the fundamental theorem of finite Abelian groups. Amer Math Monthly 102 (1995), 162-163.

[25]. L.A. Zadeh. Fuzzy sets. Inform Control 8 (1965), 338--353.

[26]. X. Zhang, F. Smarandache, M. Ali, X. Liang. Commutative neutrosophic triplet group and neutro-homomorphism basic theorem. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, 2022, 215.

[27]. X. Zhang, Q. Hu, F. Smarandache, X. An. On neutrosophic triplet groups: Basic properties, NT-subgroups, and some notes. *Symmetry*, *10*(7) (2018), 289.

[28]. X. Zhang, X. Wang, F. Smarandache, T.G. Jaíyéolá, T. Lian. Singular neutrosophic extended triplet groups and generalized groups. *Cognitive Systems Research*, *57 (2019)*, 32-40.

[29]. X. Zhou, X.L. Xin. Ideals on neutrosophic extended triplet groups. *AIMS Mathematics*, *7*(3) (2022), 4767-4777.

Received: Feb 9, 2024. Accepted: Apr 29, 2024

Neutrosophic Sets and Systems, Vol. 67, 2024
University of New Mexico

Utility of neurodynamic techniques in Diabetes Mellitus from the plitogenic statistical approach

Lisbeth Josefina Reales Chacón[1], Josselyn G. Bonilla Ayala [2], Andrea Carolina Peñafiel Luna[3], Paul Adrian Arias cordova[4] Angela Campos Moposita[5]

1 Universidad Nacional del Chimborazo; Riobamba, 170515.Ecuador. E-mail lisbeth.reales@unach.edu.ec
2 Universidad Técnica de Ambato, Ambato,180103, Ecuador. E-mail jbonilla3898@uta.edu.ec
3 Universidad Técnica de Ambato, Ambato, 180103.,Ecuador. E-mail ac.penafiel@uta.edu.ec
4 Universidad Técnica de Ambato, Ambato, 180103.Ecuador. E-mail pa.arias@uta.edu.ec
5 Universidad Técnica de Ambato, Ambato, 180103.Ecuador. E-mail ap.campos@uta.edu.ec

Abstract: Diabetes mellitus type 2 is an illness condition in which the glucose level is abnormal and causes harm to the peripheral nervous fibers originating the diabetic neuropathy. The major frequent damage occurs in the sensory-motor system, whose symptoms include alteration in tactile perception, temperature, and loss of sensitivity in hands and feet. The evidence for treatments that include neural mobilization is limited, so this study aims to determine if intervention strategies with neural mobilization improve both sensory and functional responses in the neuropathy diabetic peripheral. We selected plithogenic statistics to carry out the statistical study because this disease is dynamically influenced by multiple factors of different natures. Plithogenic statistics studies the random events that occur for the multiple random variables or parameters that determine them. Due to it being based on plithogeny, it is a generalization of dialectics where the neutral is incorporated. The advantage of using this tool is that we yield a more complete result of the study if we compare it to classical statistics.

Keywords: Plithogenic Statistics, T-test, Diabetic Peripheral Neuropathy, Neural Mobilizations, Sensory Responses, Functional Responses.

1 Introduction

Diabetes mellitus is considered a public health problem. International Diabetes Federation estimates that there are 450 million people aged from 18 to 99 with a diagnosis of diabetes worldwide. From 85% to 95% of people have diabetes mellitus type 2, which frequently occurs in adults older than 40 years old, where women are more prone to this condition since in both, developed and underdeveloped countries this index is higher in this sex. According to the World Health Organization (WHO), in Latin America and the Caribbean, there are approximately 62 million people with type 2 diabetes. The prevalence is increasing rapidly in countries of scarce and medium resources in relationship to the developed countries.

According to the National Health and Nutrition Survey of Ecuador (ENSANUT in Spanish), the diabetes percentage in the country corresponds between 7.1% and 7.8% of its population. A very common consequence of diabetes is neuropathy peripheral diabetes, which has a prevalence corresponding between 40% and 50% of this population worldwide. This aggravation affects sensory, motor, and autonomic systems. Symptoms include functional deficits, paresthesia, hyperesthesia, and dysesthesia. The factors of risk are age, smoking, obesity, arterial hypertension, and deficient glycemic control; so it is essential to consider these factors for checking diabetes, and then to prevent the serious consequences of neuropathy peripheral diabetes like ulcerations, amputation, social burden, expensive treatments, state of depression, sleeping disorders, and anxiety. Several authors point out that the treatments advised by the physicians include manual therapy, neurodynamics, electrotherapy, strength, and cardiovascular exercises.

Neurodynamic techniques must be considered for treating neuropathies, and peripheral compression is utilized to improve the symptoms and to enhance the speed of driving of nervous, and then to decrease the rate of appearance of ulcers and amputations, as well as to improve sensitivity and functionality.

The present research is about the neuropathy peripheral diabetic, a condition that is important to treat for the improvement of the symptomatology of the patient who suffers from it. To this end, we apply the instrument called Michigan Neuropathy Screening Instrument (MNSI) consisting of a questionnaire and a physical examination. The sensitivity evaluation corresponds to the tactile and thermal sensitivity in the dermatomes of the superior member (C5, C6, C7, C8, T1) and lower limb (L2, L3, L4, L 5, S1, S2), sensitivity to vibration (paresthesia) in the olecranon, radial styloid, external malleolus and hallux. The maximum isometric strength is assessed with emphasis on finger flexors and muscles of bilateral plantar flexion.

These instruments help us to analyze the damage at a sensitivity level and the muscular strength of the upper and lower limbs of the patients. After the intervention with neural mobilization, a new assessment helps us to determine if there are changes in the affected areas. Hypothetically, the intervention with neural mobilizations in patients with diabetic peripheral neuropathy is effective and improves the symptoms of this condition.

One characteristic of this disease is its relationship with multiple factors that dynamically interact with each other and have origins of different natures. We can find educational, nutritional, biological, and genetic factors, among others. A quite recent theory that generalizes dynamical system theory is Plithogeny. While dialectics only takes into account the interaction of contrary concepts, plithogeny also incorporates the neutral aspect [1]. Therefore, the dynamics of the phenomena are studied more completely. Some applications of plithogenic theory are read in [2-7]

On the other hand, the Plithogenic Statistic studies the data according to the Plithogenic Probability [8-11]. The Plithogenic Probability is a multidimensional probability that studies random events that occur for all the random variables (parameters) that determine it. Application of plithogenic statistics can be found in [12-14]

In this paper, we study the effectiveness of intervention strategies with neural mobilizations to improve sensory and functional responses in Diabetic Peripheral Neuropathy. For the study, we based on Neutrosophic Plithogenic Statistic tools.

The paper is made up of the following structure: a Materials and Methods section where the main physiological methods used to study sensory and functional responses are explained. Furthermore, this section presents the basic notions of plithogenic statistics. The results section presents what was obtained from this study. The last section is dedicated to giving the conclusions.

2 Materials and Methods

2.1 Physiological methods for studying sensory and functional responses

Michigan Neuropathy Screening Instrument (MNSI) was created by Feldmann and his colleagues in 1994. It is a questionnaire, practical to apply, whose purpose is the detection of peripheral neuropathy diabetic. It consists of two phases. The first one is a questionnaire of 15 items related to the symptoms presented in the previous week, with responses of YES and NO where we assign the values of 0 and 1, respectively. If they sum up a maximum of 13, it is positive for peripheral neuropathy**.**

The second part consists of a physical exam to observe the appearance of the feet of both members with specific features such as hammer toes, fingers overlapping, hallux valgus, subluxation of the joint, prominent head goal tarsal, and medial convexity, dry skin, infections, cracks or presence of ulcerations. This responds to two variables, “Present” with a value of 0 and “Absent” with a value of 1. The Achilles reflex and perception of vibration in the big toe reply to the “Present” parameter with a value of 0, “Present with reinforcement” equal to 0.5, and “Absent” equal to 1. The final score is obtained by adding the bilateral data and we consider there is peripheral neuropathy when the score is greater than 2/8.

According to medical reports, MNSI has a sensitivity of 79% and a specificity of 94% for the detection of Peripheral Neuropathy. The evaluation of the superficial and deep sensitivity is considered as a neurological clinical method evaluation which consists of two parts. The first one is the assessment of the superficial tactile and thermal sensitivity of the dermatomes (C5, C6, C7, C8, T1, L2, L3, L4, L5, S1, S2) with direction distal to proximal and the second one is the sensitivity to the vibration (paresthesia) in the olecranon, styloid of the radius, external malleolus and hallux. So, the following variables correspond to the numerical values of Absent = 0, Altered = 1, Normal = 2, and NE= Not Evaluated.

The evaluation of the maximum isometric force with the dynamometer yields a value with emphasis on the finger flexors and the plantiflexion muscles. Each muscular proof is run 3 times. Between each repetition, we wait for twenty seconds, and finally, we choose the higher and less variational result.

For the evaluation of the muscle flexors of the finger, the patient recumbent supine, the elbow is flexing 90° and the patient has to hold and pressure with maximum force. For the evaluation of the plantiflexion muscles, the patient will be in a recumbent position, supine and the dynamometer will be placed at the height of the metatarsal region and he/she should do the flexion plant with a maximum force.

2.2 Basic Notions of Neutrosophic Statistics

Plithogenic Statistics (PS) comprises the analysis and observations of the events studied by the Plithogenic Probability [9].

Plithogenic Statistics generalizes classical MultiVariate Statistics, and in turn, allows an analysis of many output variables that are neutrosophic or indeterminate. It is also a multi-indeterminate statistic.

Various Subclasses of Plithogenic Statistics are as follows :

- Multivariate Statistics,
- Plithogenic Neutrosophic Statistics,
- Plithogenic Indeterminate Statistics,
- Plithogenic Intuitionistic Fuzzy Statistics,
- Plithogenic Picture Fuzzy Statistics,
- Plithogenic Spherical Fuzzy Statistics,
- and in general: Plithogenic (fuzzy-extension) Statistics,
- and Plithogenic Hybrid Statistics.

On the other hand, Plithogenic Refined Statistics are the most general form of statistics that studies the analysis and observations of events described by Plithogenic Refined Probability.

In classic inference statistics, the population's average of the variable is estimated from the sample's average.

When we have a classic random variable, the exact size of the sample is known and all the elements in the sample belong 100% to the population. However, this does not reflect the dynamics of a population such as the students in a college, where there is the fluctuation of them within the courses; in addition to the fact that the membership of each student varies depending on whether they are studying in a full-time, part-time or over-time course.

In a Neutrosophic Population, each element has a triple probability of membership such that $0 \leq T_j + I_j + F_j \leq 3$.

If we assume that $n \geq 2$ where n is the sample size, then the average probability for all elements in the sample is calculated by Equation 1.

$$\frac{1}{n}\sum_{j=1}^{n}(T_j, I_j, F_j) = \left(\frac{\sum_{j=1}^{n} T_j}{n}, \frac{\sum_{j=1}^{n} I_j}{n}, \frac{\sum_{j=1}^{n} F_j}{n}\right) \quad (1)$$

3 Results

The present investigation was carried out in the Atahualpa parish of the Ambato Canton and the “Aire Libre” neighborhood of the Cevallos Canton. It began with the socialization of the project, where the topic, objectives, and methodology that will be used in this study were made known. Therefore, the informed consent was presented and delivered, which is an important document to carry out the research. Immediately the study was carried out at strategic points in both Cantons. It was divided into 2 phases: the first phase of collecting information to fill out the clinical history and the second phase consisting of executing the Michigan test, evaluation of sensitivity, and the assessment of the muscular force with a duration of 25 minutes by patient.

The population of this study was made up of 34 individuals with diabetic peripheral neuropathy, of which 17 participants completed the 8-week intervention by applying an evaluation before and after it. The remaining participants were excluded from the research work due to their non-attendance. The data were calculated with a 95% confidence interval and 5% margin of error for patients with diabetic peripheral neuropathy.

The inclusion criteria were:

- Signing of the informed consent.
- People with type ll diabetes mellitus.
- Indistinct sex.
- Autonomous and independent patients.

The exclusion criteria were:

- Patients with severe cardiac pathologies.
- Patients with cognitive impairment of any level.
- Patients undergoing major surgery in the last 3 months,
- Patients with recent fractures in both the upper and lower limbs.

- Patients with varicose ulcers.

The following procedure was carried out:

Phase I (Survey)

Personal data sheet

Data collection was carried out for the individual medical history prepared to collect the sociodemographic data considered most important for this study.

Phase II (Evaluation)

The Michigan Test (MNSI)

Before the execution, the patients were explained what this assessment consists of. They were asked to lie down on the stretcher for more comfort and the questions from the questionnaire were asked about the symptoms of the previous week. After this, a physical examination of both lower limbs was carried out to see the appearance of the foot, the presence of ulcerations, Achilles reflex assessed with the hammer for reflexes in both extremities where there were observed the absence, presence or presence with reinforcement. Perception of vibration on the big toe was also observed and valued with the 128 Hz tuning fork.

Sensitivity assessment

To do this, the evaluated patient was asked to keep his eyes closed and completely relaxed. Where tactile sensitivity was assessed with a cotton pad through the dermatomes C5, C6, C7, C8, T1, L2, L3, L4, L5, S1, and S2 in a proximal to distal direction. Thermal sensitivity was evaluated with ice, taking care of the integrity and physical health of the patients from proximal to distal through the bilateral dermatomes already mentioned. Sensitivity to vibration (paresthesia) was evaluated with the 128 Hz tuning fork in the olecranon, radial styloid, external malleolus, and hallux with caution, asking what sensation they perceived.

Evaluation of maximum isometric strength with dynamometer during hand flexion and plantar flexion.

Phase III (treatment)

Neural mobilization exercises for the upper limb (Radial Neurodynamics, Ulnar Neurodynamics, Median Neurodynamics).

Neural mobilization exercises for the lower limb (Sciatic Neurodynamics, Peroneus Neurodynamics).

To gain greater accuracy in the results of the study, the different classic tests were adapted. Instead of adding points for each response, the results of each individual are grouped into triads of the type (P, I, N), which means "Positive", "Indeterminate", and "Negative". The value "Indeterminate" covers cases of indeterminacy due to some reason the patient's status could not be identified as positive or negative.

Below there are the tables that summarize the personal data of the patients. Table 1 corresponds to the sociodemographic data and Table 2 to the personal pathological data:

	Frequency	**Percentage**
SEX		
Male	4	23.5
Female	13	76.5
AGE		
Older adults	13	76.5
Adults	4	23.5
ICM		
Low weight	4	23.5

Overweight	5	29.4
Obesity	8	47.1

Table 1. Sociodemographic data

Background Pathological Personal		
	Frequency	Percentage
None	6	35.3
HTA	2	11.8
Hypothyroidism	6	35.3
Respiratory	3	17.6

Table 2. Personal pathological history

Table 3 shows the results of the original Michigan Test.

Michigan Questionnaire Initial Category		
Results	**Frequency**	**Percentage**
Normal	13	76.5
Altered	4	23.5
Michigan Questionnaire Final Category		
Results	**Frequency**	**Percentage**
Normal	17	100

Table 3. Results of the classical Michigan Test

The results of applying the method proposed in this article to the Michigan Test adapted to (P, I, N), using the average of the sum of points of the respondents before and after the treatment are $(0.35294, 0.29412, 11.58824)$ and $(0.00, 0.00, 13)$.

Table 4 shows the results of the sensitivity test in its classic variant:

Variable	**State**	**Frequency**	**%**
Sensitivity Tactile Member Superior Initial	Normal	17	100
Sensitivity Tactile Member Superior Final	Normal	17	100
Sensitivity Thermal Member Superior Initial	Altered Normal	1 16	5.9 94.1
Sensitivity Thermal Member Superior Final	Normal	17	100
Sensitivity Tactile Member Lower Initial	Altered Normal	2 15	11.8 88.2
Sensitivity Tactile Member Lower Final	Normal	17	100
Sensitivity Thermal Member Lower Initial	Altered	4	23.5

	Normal	13	76.5
Thermal Sensitivity Member Lower Final	Normal	17	100

Table 4. Results of the classic sensitivity test

In the proposed method in this article, the results were those shown in Table 5:

Variable	**State**
Tactile Sensitivity Superior Member Initial	(2,0,0)
Tactile Sensitivity Superior Member Final	(2,0,0)
Thermal Sensitivity Superior Member Initial	(1.88235,0.0,0.11765)
Thermal Sensitivity Superior Member Final	(2,0,0)
Tactile Sensitivity Lower Member Initial	(1.76471, 0.00,0.23529)
Tactile Sensitivity Lower Member Final	(2,0,0)
Thermal Sensitivity Lower Member Initial	(1.52941, 0.00, 0.47059)
Sensitivity Thermal Lower Member Final	(2,0,0)

Table 5. Results of the adapted sensitivity test

We must note that the results in Table 5 are given based on the triple (P, I, N), but in this case, the desirable result is given in reverse order to the previous one. That is, while for the MNSI the desired result is that all of them are negative, in the sensitivity test a desired result is that all of them are positive.

Below, Table 6 shows the results of vibration according to the classical method.

Vibration Superior Member Initial		
	Frequency	Percentage
Absent	2	11.8
Altered	5	29.4
Normal	10	58.8
Vibration Superior Member Final		
	Frequency	Percentage
Normal	17	100.0
Vibration Lower Member Initial		
	Frequency	Percentage
Absent	2	11.8
Altered	3	17.6
Normal	12	70.6

Vibration Lower Member Final		
	Frequency	Percentage
Altered	2	11.8
Normal	15	88.2

Table 6. Vibration evaluation according to the classical method

Finally, Table 7 contains the results applied to the method proposed in this article, based on the percentages in Table 6.

Variable	**State**
Vibration Superior Member Initial	(0.588,0.294,0.118)
Vibration Superior Member Final	(1,0,0)
Vibration Lower Member Initial	(0.706,0.176,0.118)
Vibration Lower Member Final	(0.882,0.118,0)

Table 7. Vibration evaluation according to the proposed method

Although we could apply statistics according to traditional methods, we prefer to use them based on the data summarized in Tables 5 and 7 in addition to what was obtained for the MNSI before and after. It is evident that each applied physiological test showed an improvement in patients. To prove that this improvement was statistically significant, the t-test was used for each triple before and after [15, 16].

The results are shown below:

Physiological test	**Triple of p-values in t-test**
MNSI	$(4.6610 \times 10^{-29}, 4.4833 \times 10^{-25}, 1.8120 \times 10^{-45})$
Tactile Sensitivity Superior Member	(NN, NN, NN)
Thermal Sensitivity Superior Member	$(5.8069 \times 10^{-12}, NN, 1.5987 \times 10^{-14})$
Tactile Sensitivity Lower Member	$(2.6017 \times 10^{-22}, NN, 0.00)$
Thermal Sensitivity Lower Member	$(7.3437 \times 10^{-31}, NN, 0.00)$
Vibration Superior Member	$(1.9620 \times 10^{-30}, 0.00, 6.7526 \times 10^{-12})$
Vibration Lower Member	$(2.2287 \times 10^{-30}, 2.5083 \times 10^{-7}, 2.1578 \times 10^{-12})$

Table 8. Results of the tests with the Plithogenic Statistic. NN means this is not a number.

From Table 7 it is evident that all the numerical values of the triads of p values are less than 0.05 and therefore the null hypothesis is rejected, which means that the improvement is statistically significant. NN values are obtained in cases where all patients had the desired outcome at the maximum level before the treatments and remained the same afterward.

Conclusion

Any progress that is achieved in terms of knowledge about Diabetic Peripheral Neuropathy is important since this is a scourge that affects an increasingly growing number of people in the world, including Ecuador. In this study, we worked with a group of 17 patients who suffer from this disease and to whom a treatment was applied to improve the effects caused by the disease. The effectiveness of the treatment was evaluated with the help of the t-test in the field of neutrosophic plithogenic statistics. The advantage of using this tool is that we obtain greater accuracy in the results because we are working with triples of values instead of only one. The medical values obtained from this work were the following:

- This research presented several results with a pre and post-evaluation after the intervention, where the sliding neural mobilization strategies for the upper limb (median nerve, ulnar radial) and lower limb (sciatic and peroneal nerve) are optimal to reduce symptoms, such as cramps, numbness, loss of sensitivity to touch, temperature, and vibration. Especially in the hands and feet of the studied population with diabetic peripheral neuropathy.

- The tactile and thermal perception and vibratory sensitivity of an altered and absent state reached a state of normality in those who complied with the 8-week treatment plan, thus obtaining encouraging sensory responses that are directly related to neurodynamics.
- These strategies do not infer on foot morphology, or skin condition such as dryness and cracks, they also involve personal care and the progression of the disease.

References

[1] Smarandache, F. (2021). Introducción a la lógica plitogénica. Neutrosophic Computing and Machine Learning, 18, 1-6.
[2] Smarandache, F. (2023). An overview of Plithogenic set and symbolic Plithogenic algebraic structures. Journal of fuzzy extension and applications, 4, 48-55.
[3] Singh, P. K. (2021). Dark data analysis using Intuitionistic Plithogenic graphs. International Journal of Neutrosophic Sciences, 16(2), 80-100.
[4] Singh, P. K. (2022). Complex plithogenic set. International Journal of Neutrosophic Sciences, 18, 57-72.
[5] Singh, P. K. (2021). Single-valued Plithogenic graph for handling multi-valued attribute data and its context. International Journal of Neutrosophic Science, 15, 98-112.
[6] A New Method to Assess Entrepreneurship Competence in University Students Using Based on Plithogenic Numbers and SWOT Analysis.
[7] Martin, N., Smarandache, F., and Sudha, S. (2023). A novel method of decision making based on plithogenic contradictions. Neutrosophic Systems with Applications, 10, 12-24.
[8] Smarandache, F. (2022). Neutrosophic Statistics is an extension of Interval Statistics, while Plithogenic Statistics is the most general form of statistics (second version) (Vol. 2). Infinite Study.
[9] Smarandache, F. (2021). Plithogenic probability & statistics are generalizations of multivariate probability & statistics. Infinite Study.
[10] Smarandache, F. (2017). Plithogeny, plithogenic set, logic, probability, and statistics. Infinite Study.
[11] Singh, P. K. (2020). Plithogenic set for multi-variable data analysis. Infinite Study.
[12] Zeina, M. B., Altounji, N., Abobala, M., and Karmouta, Y. (2023). Introduction to Symbolic 2-Plithogenic Probability Theory. Infinite Study.
[13] Smarandache, F. (2023). Introduction and advances to neutrosophic probability and statistics and plithogenic probability and statistics and their applications in bioinformatics and other fields (review chapter). In Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics (pp. 1-23). Academic Press.
[14] Castro Sánchez, F., Almeida Blacio, J. H., Flores Bracho, M. G., Andrade Santamaria, D. R., and Sánchez Casanova, R. (2021). Neutrosophic and Plithogenic Statistical Analysis in Educational Development. Neutrosophic Sets and Systems, 44, 223-234.
[15] Manoukian, E. B. (2022). Mathematical nonparametric statistics. Taylor & francis.
[16] Alita, D., Putra, A. D., and Darwis, D. (2021). Analysis of classic assumption test and multiple linear regression coefficient test for employee structural office recommendation. IJCCS (Indonesian Journal of Computing and Cybernetics Systems), 15, 295-306.

Received: Feb 9, 2024. Accepted: Apr 29, 2024

Neutrosophic Sets and Systems, Vol. 67, 2024
University of New Mexico

Analysis of Scientific Production on Neutrosophy: A Latin American Perspective

Oscar E. Velázquez-Soto [1], Eduardo Enrique Chibas Muñoz [2]*, Maykel Yelandi Leyva Vazquez [3], Lee Yang Díaz Chieng [2] and Jesús Estupiñan Ricardo [4]

[1] Editorial Ciencias Médicas, Centro Nacional de Información de Ciencias Médicas, Habana, Cuba; oevs04@gmail.com
[2] Universidad de Ciencias Médicas de Guantánamo, Guantánamo, Cuba; eechibas@gmail.com, leeyangdiazchieng1973@gmail.com
[3] Universidad de Guayaquil, Guayas, Ecuador , mleyvaz@gmail.com
[4] Instituto Superior Tecnológico de Investigación Científica e Innovación (ISTICI), Quito Ecuador , jestupinan2728@gmail.com

* Correspondence: eechibas@gmail.com

Abstract: Neutrosophy is a branch of philosophy proposed in 1998 by Romanian philosopher and mathematician F. Smarandache, which studies the origin, nature, and scope of neutrality and its interaction with different spectra of ideas. Because it is a relatively new field, specific scientific production on this subject may be limited, but as it develops, new research and contributions are likely to emerge. This is why it is becoming more necessary to undertake studies to understand the evolution, impact, and scope of research, especially in a context where it had never been examined in depth before: in the Latin American region. This research aims to describe the scientific production of Latin American authors in the Scopus database referring to Neutrosophic Science in the period 2019-2023 by carrying out a bibliometric study of a descriptive nature. The scientific production in Neutrosophy assumes patterns of behavior that vary from international patterns with a marked social focus. Although productivity levels have not increased in recent years, the impact and recognition from the international community have grown significantly, meaning the evolution of a process of transformation.

Keywords: Neutrosophy; bibliometric analysis; Latin America; scientific production

1. Introduction

Neutrosophy is a branch of philosophy proposed in 1998 by Romanian philosopher and mathematician F. Smarandache, which studies the origin, nature, and scope of neutrality and its interaction with different spectra of ideas. Neutrosophy examines the relationship between a proposition, theory, event, concept, or entity with its opposite and its neutrality. Since then, this logic has been applied in several fields of science. [1]

The dynamics of the opposites and their neutrals is an extension of dialectics that is the dynamic of opposites only. Neutrosophy is the basis of neutrosophic logic, neutrosophic probability, the neutrosophic set, and neutrosophic statistics. It is important to emphasize that neutrosophy allows the representation of information in a more complete and real way, allowing to embrace not only veracity or falsehood but also ambiguity, ignorance, contradiction, neutrality and saturation. [1, 2]

Although it is a relatively new concept, Neutrosophy has a background in other philosophical and epistemological currents, such as paradox, trivalent and multivalent logic, and Eastern philosophy. These backgrounds have contributed to shaping the fundamental principles and concepts of Neutrosophy as an innovative and complex philosophical theory. [3] His studies have given way to a unique research method by constituting a unified field of logic for a transdisciplinary study that transcends the boundaries between the natural and social sciences. This science allows multidisciplinary work with other related, which allows an impact on the results of research. [4]

The scientific evolution of this knowledge enables a broader and more flexible view of the world, fosters tolerance, and interdisciplinary dialogue, facilitates conflict resolution, stimulates creativity and innovation, and enriches philosophical reflection on fundamental aspects of human existence. [5]

Because it is a relatively new field in philosophy and academic research, specific scientific production on this subject may be limited compared to other more established fields. However, there are some authors and academics who have contributed to the literature on Neutrosophy and have explored its philosophical and epistemological implications and its relationship with other disciplines [6-9]. Similarly, several studies have analyzed the evolution of scientific activity on the subject through bibliometry, where a growth in recent years of the global positioning of this discipline is reflected. [10 - 12]

While statistical analyses have been carried out that enable the obtaining of reliable indicators, associated with the quality, visibility and performance of the scientific production in Neutrosophy, no in-depth research of this phenomenon has been developed in Latin America. It is precisely in this region that from the year 2018 an initiative to promote the research, dissemination and development of Neutrosophy in the territory, the Latin American Association for Neutrosophic Sciences is emerging. (ALCN). This academic and scientific organization aims to promote academic exchange, collaboration among researchers and dissemination of knowledge, as well as support in the spread of neutrosophic thinking, but from an approach oriented to the solution of social problems. [13] The creation of the ALCN opened a new door for the development of research in the region and the growth of scientific publications on this subject.

As Neutrosophic Sciences continue to develop, new research and contributions are likely to emerge that will enrich our understanding of these and their implications for contemporary philosophy. This is why it is increasingly necessary to undertake studies to understand the evolution, impact and scope of research, as well as to foster collaboration among researchers and promote their recognition in the academic community. These types of studies can evaluate the relevance and visibility of scientific publications, especially in a context where they have never been examined in depth before: in the Latin American region.

The objective of this research is to describe the scientific production of Latin American authors in the Scopus database concerning Nuetrosophical Sciences in the period 2019-2023

2. Materials and Methods

This article presents a bibliometric analysis of the scientific output of Latin American scholars in the field of Neutrosophical Sciences. The study focuses on the articles indexed in the Scopus database from 2019 to 2023.

2.1 Characterization of information sources:

Scopus® (Elsevier, Netherlands): Multidisciplinary database, considered one of the most extensive in terms of coverage of bibliographic references and summaries of peer-reviewed scientific articles. More than 45,000 arbitrated journals are indexed and more than 29,000 are currently active. This database covers topics from various disciplines such as science, technology, medicine, social sciences, arts, and humanities, among others.

2.2 Dataset extraction:

The search was performed using the terms neutrosophy, neutrosophic, and their respective variants in the Spanish language in the fields Title, Abstract and Keyword to get as many results as possible associated with this discipline. Following the search, a primary filtering process was applied to delimit the sample into original and review articles, published only in scientific journals within the period 2019-2023 and whose authorship was declared by researchers affiliated to Latin American institutions. This process resulted in the following formula:

TITLE-ABS-KEY (neutrosophy) OR TITLE-ABS-KEY (neutrosophic) OR TITLE-ABS-KEY (neutrosofía) OR TITLE-ABS-KEY (neutrosófic*) AND (LIMIT-TO (PUBYEAR , 2019) OR LIMIT-TO (PUBYEAR , 2020) OR LIMIT-TO (PUBYEAR , 2021) OR LIMIT-TO (PUBYEAR , 2022) OR LIMIT-TO (PUBYEAR , 2023)) AND (LIMIT-TO (DOCTYPE , "ar") OR LIMIT-TO (DOCTYPE , "re")) AND (LIMIT-TO (AFFILCOUNTRY , "Ecuador") OR LIMIT-TO (AFFILCOUNTRY , "Peru") OR LIMIT-TO (AFFILCOUNTRY , "Colombia") OR LIMIT-TO (AFFILCOUNTRY , "Cuba") OR LIMIT-TO (AFFILCOUNTRY , "Mexico") OR LIMIT-TO (AFFILCOUNTRY , "Chile") OR LIMIT-TO (AFFILCOUNTRY , "Brazil") OR LIMIT-TO (AFFILCOUNTRY , "Dominican Republic") OR LIMIT-TO (AFFILCOUNTRY , "Argentina") OR LIMIT-TO (AFFILCOUNTRY , "Paraguay") OR LIMIT-TO (AFFILCOUNTRY , "Uruguay") OR LIMIT-TO (AFFILCOUNTRY , "Dominica") OR LIMIT-TO (AFFILCOUNTRY , "Panama"))*

As a result of the execution of the final formula, a total of 361 items were obtained. The search and retrieval of the data was carried out in May 2024.

2.2 Dataset processing:

The results obtained were exported in Comma-Separated Values (CSV) format, prioritizing the compilation of metadata with bibliographic information, information about quotes, and keywords from the article set. These elements were normalized and processed using OpenRefine, a tool that allows you to manage large volumes of data and perform analysis, cleaning, converting and reusing them.

To perform the visual analysis of information, generating graphs, tables, structural networks and co-occurrence maps were used the software Microsoft Excel, SCImago Graphica and VOSViewer, tools that allow to analyze, visualize and share large volumes of information through the elaboration of graphs and tables. In turn, VOSViewer allows you to build and visualize bibliometric networks.

2.2 Dataset analysis:

2.2.1 Bibliometric indicators

	Indicator	Name	Definition
Production	Ndoc	Number of documents	Number of documents published in journals indexed in Scopus
	Tdoc	Document type	Document typology according to Scopus classification for publication sections (Article, Review, Conference paper, Book, Editorial, etc.)
	%Orig	Originality percentage	Percentage that original research represents with respect to the total number of documents in a given set.

	Cit	Number of citations	Number of citations received by documents published in journals indexed in Scopus. Only citations obtained from Scopus are recorded.
	CpD	Cites per Document	Average number of citations received in 1 year for the total number of documents published in journals indexed in Scopus. Only citations obtained from Scopus are recorded.
	H-Index	Hirsch Index	The h index expresses the journal's number of articles (h) that have received at least h citations. It quantifies both journal scientific productivity and scientific impact and it is also applicable to scientists, countries, etc.
Impact	SJR	SCImago Journal Rank	It expresses the average number of weighted citations received in the selected year by the documents published in the selected journal in the three previous years, --i.e. weighted citations received in year X to documents published in the journal in years X-1, X-2 and X-3
	Q	Quartiles	The quartile distribution (Q1, Q2, Q3 and Q4) refers to citable papers indexed in journals occupying these quartile positions. Based on the SJR value, journals occupy a position from the division of their categories. This distribution determines the degree of visibility of the journals belonging to each quartile, the highest visibility is found in the first quartile, and it will decrease as it moves away in position.
	SNIP	Source Normalized Impact per paper	Is the ratio of a source's average citation count per paper and the citation potential of its subject field.
	CiteScore	Cite Score Metric	Calculating the CiteScore is based on the number of citations to documents (articles,

			reviews, conference papers, book chapters, and data papers) by a journal over four years, divided by the number of the same document types indexed in Scopus and published in those same four years.
	FWCI	Field-Weighted Citation Impact	Is the ratio of citations received relative to the expected world average for the subject field, publication type, and publication year.
Collaboration	%Icol	International Collaboration percentege	Document ratio whose affiliation includes more than one country address

2.2.2 Main elements discussed in each type of analysis

Analysis	Elements Included
Dataset overview	Scientific production evolution Scientific production impact Documents type Language of documents Average years from publication Average citations per document Average citations per year per document Keywords Analysis Thematic and subject areas
Source analysis	Document per source Journal Quartile Most relevant journals Bradford's law on source clustering Core Journals' growth (cumulative) based on the number of papers
Author analysis	Top authors based on number of documents Top authors based on number of cites Authors per document Documents per author Most relevant affiliations Collaboration index Most relevant corresponding author's Scientific production based on country Top countries with the most author Country collaboration map Authors' collaboration network

Analysis	Elements Included

3. Results

3.1 Dataset Overview

The behavioral patterns of the Latin American scientific production on Neutrosophical Sciences at Scopus have remained stable over the last 5 years, taking into account the annual publication volumes as shown in Figure 1. With a total of 361 articles, the average annual publication reaches 72.2 and highlights the year 2020 as the period that most articles were published.

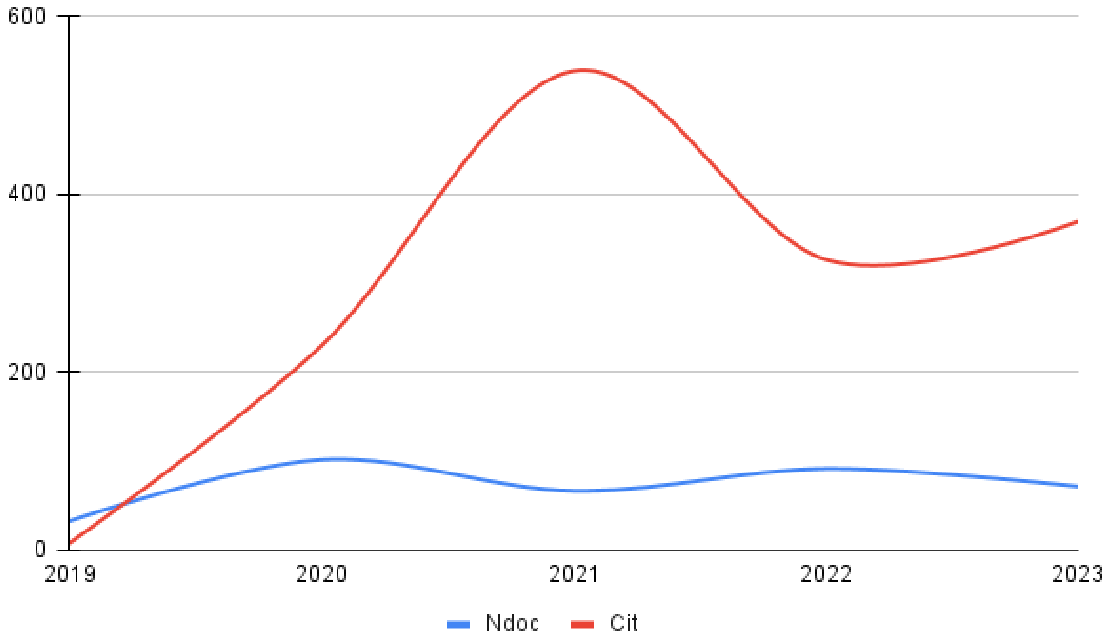


Figure 1: Evolution and impact of annual scientific production (2019-2023).

These results differ from general trends and analyses at the international level. Other studies a steady growth in the publication on Neutrosophy while forecasting an exponential increase in production, these forecasts are not fulfilled for the Latin American region. [4, 10 – 12] However, Delcea et al. (2023) describe in a more recent study that for the period 2021-2023 these values were stable and that the increase in previous years was also conditioned by the pandemic of COVID-19 which was boosted scientific production in all fields of research, including Neutrosophy. The analysis of Delcea et al. (2023) identifies similarities in behavior between patterns of scientific productivity in the Latin American region and the world.

Regardless of growth patterns, it is noteworthy that the volume of production in a relatively new discipline reflects the interest of the scientific community in its contributions to the region. Of course, nothing compared to other sciences with topics of study more historically positioned in the area, such as Biomedical and Biological Sciences. [14]

Nevertheless, the level of strength that this discipline has acquired in recent years is accompanied by an exponential increase in the impact of its publications. The evolution of quotation levels shows an increase for the year 2021, where 36.6% of the total of 1469 quotations achieved were received. This impact-associated pattern had an average of 293.8 appointments per year. In this case, a notable coincidence was identified between the behavior regarding the growth of citation in the region and the global in the same period, as well as in the fulfillment of a trend towards the increase of this phenomenon. We emphasize that the similarity lies in growth variables and not in pure values, since levels of international impacts differ greatly from those of the region. [10-12]

The progress of the quotations received by the works associated with the examined discipline is a demonstration of the positive increase in quality in the evaluation from external and internal factors of the scientific community itself about the research and social activity of this branch. And it is that, although currently measuring the impact of a discipline is much more complex than carrying out a quotation analysis [15], no doubt the growth of quotations received denotes a development and solidification of Neutrosophy in Latin America.

The analysis of the distribution by publication quarters also allows approximations of the impact and visibility indices associated with the performance of the scientific publication. For this reason, Figure 2 shows this behavior in the case under study, taking into account the expected years.

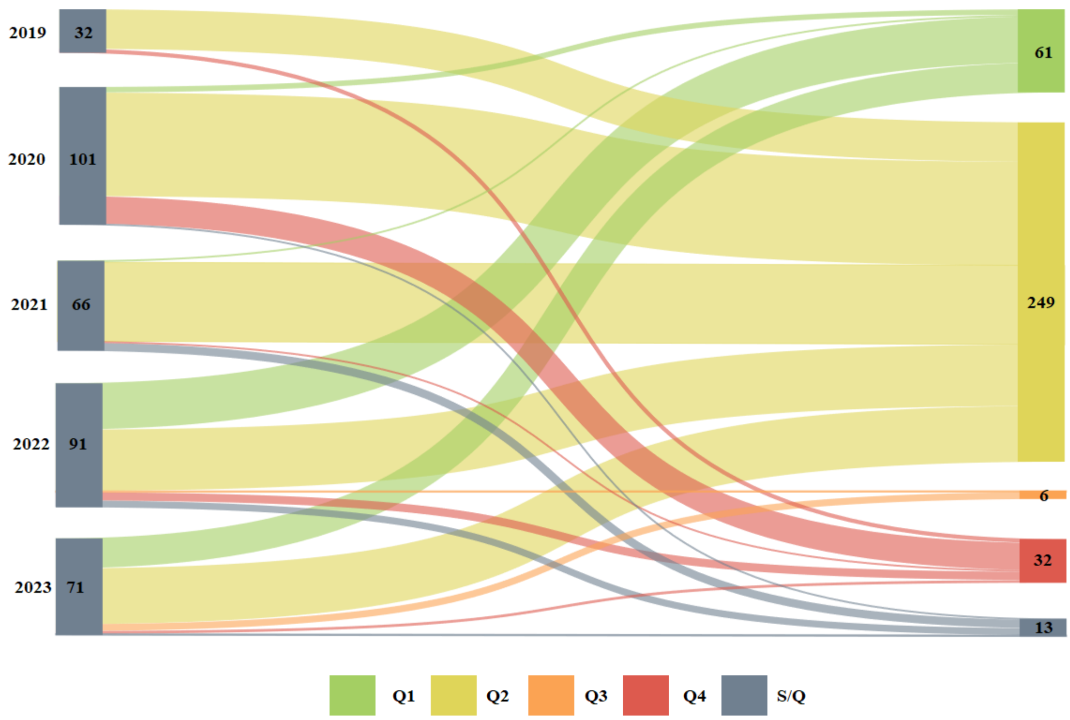


Figure 2**:** Distribution of Ndoc by SJR quartiles (2019-2023).

The highest percentage of papers 69% was published in Q2 journals, followed by 16.7% in Q1 which demonstrates that most of Latin American production on Neutrosophia was carried out in high impact sources. Similarly, it was possible to identify that this pattern was visualized to a greater extent during the last two years analyzed. Oppositely was the behavior of the items included in Q3, the year 2020 where the most work was concentrated which was decreasing over the years.

According to Zacca-González et al. (2015) publication in Q1 journals is a common pattern associated with quality of production and leadership that translates into high capacities of the scientific potential to conduct research. In other words, the content is of greater international interest and enables the dissemination and transfer of new knowledge. [16]

In this regard, this study demonstrates the positive progress in terms of the level of visibility and quality of the publications on Neutrosophia in Latin America, reflected in the gradual and sustained positioning of more than 70% of the articles in journals of greater importance concerning their areas of knowledge.

Figure 3 provides another perspective on the performance of the publishing volume, this time analyzed from the average of Cites per Document received which generally averages about 4.33 quotes received for each published document. Again the year 2021 stands out with the highest recorded average (1,49) followed by the years 2023, 2022, and 2020 respectively. These indicators do not reach high values when compared with the international context, which shows a moderate impact at the article level. [11]

According to data consulted on the SJR portal (2024), the median of CpD in Latin America is 19.3. These values vary depending on the field of study, for example for the most published discipline in the region (Medicine) [14] there is 20.79 while for Mathematics this value drops to 7.9 quotes per document. These values are reduced even further if a sub-item such as Logic is analyzed specifically, where documents are given a median of 5.57. As we can see, these values are more similar to those identified for Neutrosophy, demonstrating that although they are not comparable to the same phenomenon internationally it is an expected value for the region.

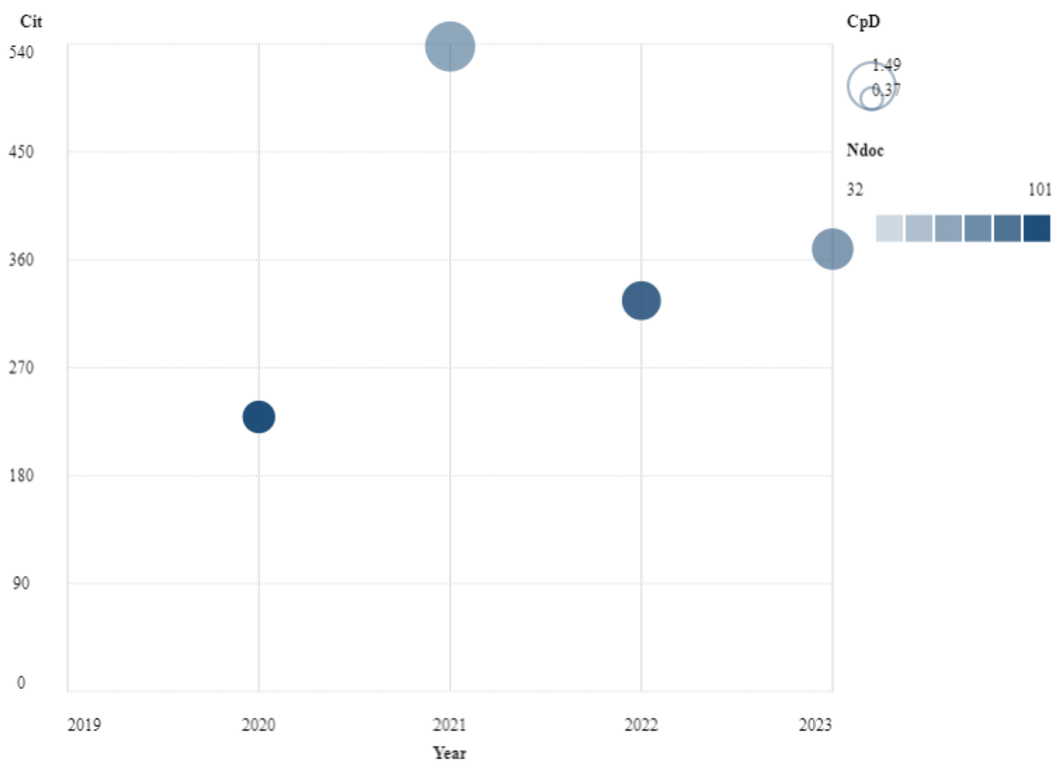


Figure 3: Distribution of CpD according to number of Cit and Ndoc published per year (2019-2023).

The largest number of documents were published in English 90.8%. This behavior shown in Table 1 is relevant, as it differs from other scientific disciplines where the publication trends in Latin America are more inclined towards the Spanish language. [17]

Table 1: Distribution and impact of scientific production according to publication language (2019-2023).

	Ndoc		**Cit**		**CpD**		**%Orig**	
Year	English	Spanish	English	Spanish	English	Spanish	English	Spanish
2023	71	0	346	23	1	0,8	98,5	100
2022	87	3	281	45	0,8	1	100	100
2021	59	19	490	48	1,5	1	100	100
2020	82	7	211	18	0,6	0,4	100	100
2019	29	4	7	0	0,09	0,09	100	0
Total	**328**	**33**	**1335**	**134**	**4,38**	**4,57**	**99,6**	**100**

As expected, the highest percentages and averages of quotes and quotes for documents received were associated with English-language production. This is the language with the greatest weight in the processes of globalization and dissemination of scientific knowledge and its use implies advantages that allow for achieving a greater likelihood of generating visibility and impact. [18]

Another important indicator is the percentage of originality in research as the original articles constitute the fundamental material of scientific development and, at the same time, the main vehicle of scientific communication. [19] After its application, it resulted that the Latin American scientific production on Neutrosophy was composed of 99.7% of research that made new contributions to this scientific discipline.

Due to the recent creation of the scientific discipline, Neutrosophical Sciences are not included in any of the Scopus Subject Areas classifications. However, it was found as a result of the analysis of subjects that Mathematics was the one that concentrated the most work. It is for this reason that in the representation of the topics addressed in the documents Figure 4 you can see how 83.3% respond to this area. Nevertheless, after the analysis of the titles, summaries, and keywords it was confirmed that are Logic (81.1%), Applied Mathematics (76.3%), Statistic and Probability (63%) and Computational mathematics (56.8%) the sub-themes most dealt with within the Mathematics. In that sense, the rest of the themes analyzed that are addressed in the works studied are related to Social Science (3.4%) and Business, Management, and Accounting (3.2%).

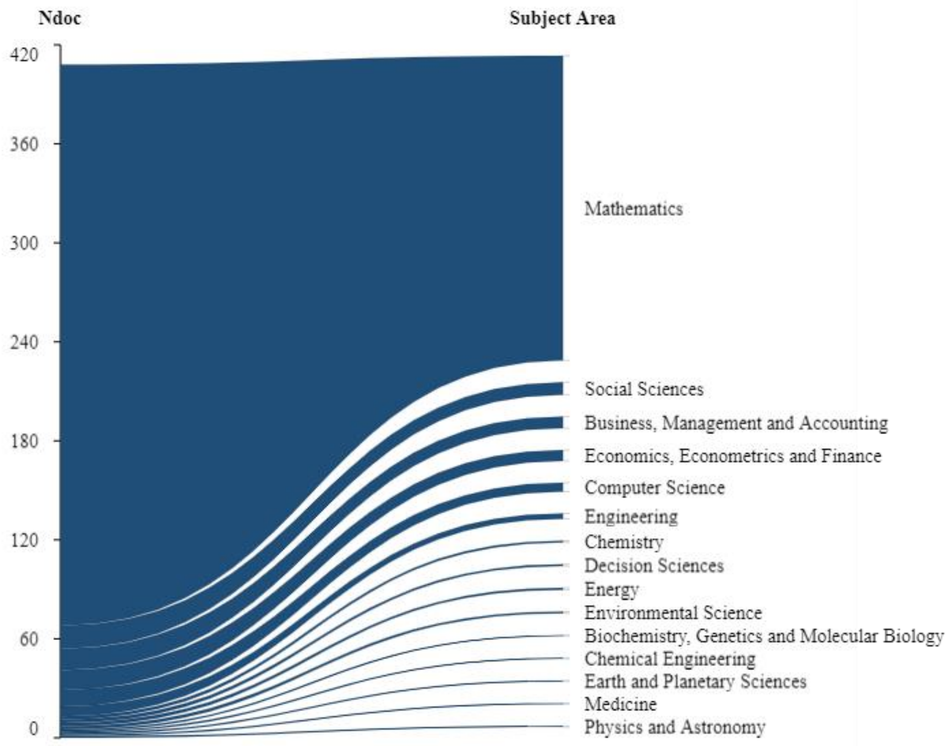


Figure 4: Distribution of Ndoc by Subject area.

Furthermore, a keyword analysis was carried out based on the co-occurrence and relationship of the terms used by the authors which can be seen in Figure 5. Pérez et al. argue that networks or graphs are the main tools for visually representing the relationships of the object studied, which are transformed into maps of knowledge that allow the measurement of the phenomenon analyzed [20].

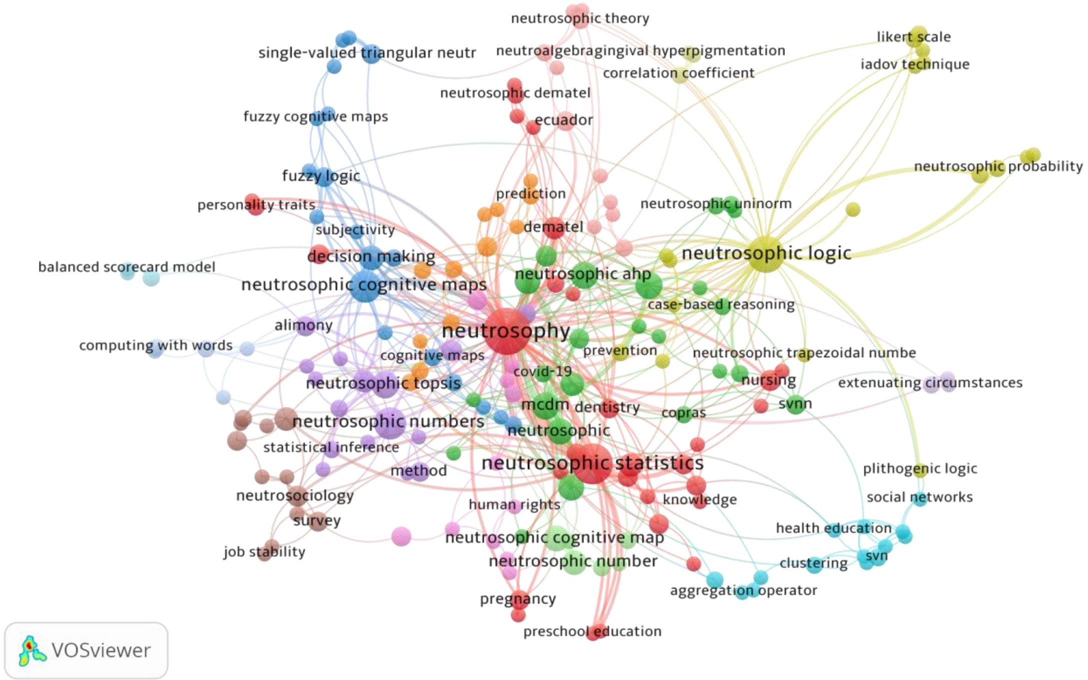


Figure 5: Co-occurrence map based on authors' keywords

Following a process of standardization and exclusion of terms based on criteria such as duplicate and unrelated words, a total of 899 items were identified. The presented co-occurrence network consists of 35 clusters, 1257 links and a total link strength of 1320. The term with the most occurrences (63) and links (101) was neutrosophy, followed by neutrosophic statistics: 31 occurrences and 58 links and of neutrosophic logic: 38 occurrences and 42 links.

This map casts results that coincide with the previous analysis because the most used terms and their relationships with higher frequencies are presented precisely in terms associated with themes of Mathematics, Logic, Statistics and Probability which represent the strongest dimensions of research within the main domain.

In this context, it is possible to delineate several zones consisting of nodes with weak links by their distance relationship with the mainstream. The words found in this case could be associated with relevant topics of Social Sciences, thus signifying a divorce between these disciplines. But this analysis is more complex and needs to be approached from another perspective.

It is that both the themes and the network of co-occurrence of keywords quantitatively show a strong tendency towards themes related to Mathematics because the very principles and statutes of Neutrosophy are based mainly on theories and concepts related to this discipline from its own conception [1, 2]. Therefore, the results obtained are inevitably observed in this way, especially if within the Scopus database itself Neutrosophic Science are not recognized as an independent discipline. That is why a rigorous analysis of the contents of the articles was carried out which produced even more conclusive results.

Within the documents classified under the domain Mathematics (340 Ndoc), let us remember that these represent 83.3% of the total, a 85.7% of articles that respond to the identification, analysis, and resolution of social problems by applying neutrosophic methods were detected. Even multidisciplinary studies were detected where several dynamic factors from other disciplines such as medicine and education are linked.

It is also identifiable a negative behavior of the authors about the use of keywords that are associated with the topics investigated, giving more protagonism to Neutrosophy, than to the very conception of the problem that they are trying to give it an answer or that are analyzing. This leads to a ghost behavior in the analysis of themes but it is important to identify them as it completely affects the analysis.

With this result, it is possible to establish a pattern that shows how the scientific production in Neutrosophy addresses social issues specific to Latin American philosophy. It is possible to state that in the Latin American regional context, the tools and knowledge neutrosophic are used for the identification, analysis and resolution of social problems which has unique approaches or distinctive contributions to the field of neutrosophy, possibly influenced by its cultural and philosophical context.

3.2 Source analysis

There were 10 scientific journals where Latin American production on Neutrosophia was distributed, as shown in Table 2. 27.27% of the journals are ranked in Q1, also with the same percentage in the case of Q2 and Q3 and to a lesser extent Q4 with 9,09%.

Table 2: Distribution and impact of scientific production according to source of publication.

Source	Ndoc	%Ndoc	Q	Cit	H-Index
Neutrosophic Sets and Systems	241	66,7	Q2	1202	33
International Journal of Neutrosophic Science	56	15,5	Q1	100	23
Investigacion Operacional	31	8,5	Q4	128	16
Universidad y Sociedad	12	3,3	S/Q	24	11
Journal of Intelligent and Fuzzy Systems	2	0,5	Q2	2	82
International Journal of Fuzzy Systems	2	0,5	Q2	1	60
Advances in the Theory of Nonlinear Analysis and its Applications	2	0,5	Q2	1	13
Management Decision	1	0,2	Q1	1	126
Sustainability (Switzerland)	1	0,2	Q1	1	169
Process Integration and Optimization for Sustainability	1	0,2	Q2	1	18
Computers and Industrial Engineering	1	0,2	Q1	1	161
Gulf Journal of Mathematics	1	0,2	Q3	1	5

Annals of the University of Craiova, Mathematics and Computer Science Series	1	0,2	Q3	0	14
Asian-European Journal of Mathematics	1	0,2	Q3	0	20
Yugoslav Journal of Operations Research	1	0,2	Q3	1	24
Revista Cubana de Obstetricia y Ginecologia	1	0,2	Q4	1	10
Axioms	1	0,2	S/Q	0	33
Symmetry	1	0,2	Q2	1	90
Journal of Cloud Computing	1	0,2	Q1	1	44
Hacettepe Journal of Mathematics and Statistics	1	0,2	Q3	1	36
Expert Systems with Applications	1	0,2	Q1	1	271
Iraqi Journal of Science	1	0,2	Q3	0	16

Concerning the impact indicators of journals, a well-known presence of sources that stand out for a higher level of quality and prestige in their areas of expertise is appreciable. This impact is seen through their H-Index where we can observe in at least 4 cases that rise to 3 digits. However, as mentioned in previous lines, sources of lower performance were also found, including two that were removed from Scopus.

The application of the Bradford Law allows for the delineation of the most quoted journals within the analyzed domain, effectively separating them from others that have had a relatively minor impact. [11,21] Its use enables a relational analysis on the dispersion between journals, articles and quotes that classifies the sources into 3 categories according to the number of published articles. Because of this, the application of this law to the set of sources obtained resulted in the core journal of the scientific production on Neutrosophy in Latin America is Neutrosophic Set and Systems(NSS). The other journals that stand out, but to a lesser extent, are the International Journal of Neutrosophic Science (IJNS) and Operational Research (IO in Spanish).

Precisely, it is remarkable that the dispersion of quotes received is concentrated mainly in NSS representing 81.8% of the total. The journal NSS from its inception emerged as a specialized journal on the subject of neutrosophy and its application in the various sciences, it focuses on the publication of original articles, has in its editorial team with the eminent pioneer professor of neutrosophy Florentin Smarandache and presents a strong revision process to its articles, all these constitute factors that favor the excellent position that the journal holds. This highlights the importance of this journal as the organ of dissemination and reference of neutrosophic thought in Latin America.

In this sense, the analysis of these sources is crucial to understanding the behavior of the publication trends and performance of the field studied. Figure 6 shows a comparison of the evolution of the annual growth of these journals and other indicators associated with impact and visibility.

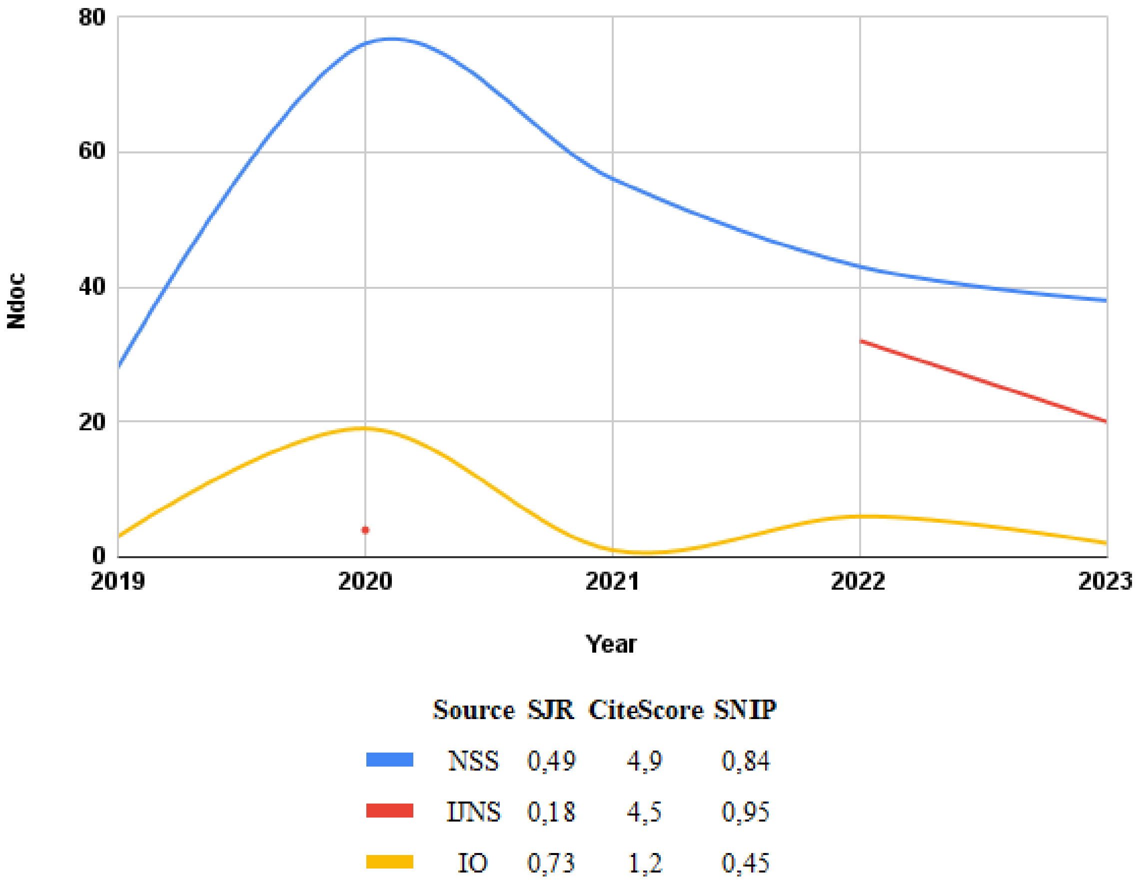


Figure 6: Annual growth of journals and other indicators associated with impact and visibility (2019–2023).

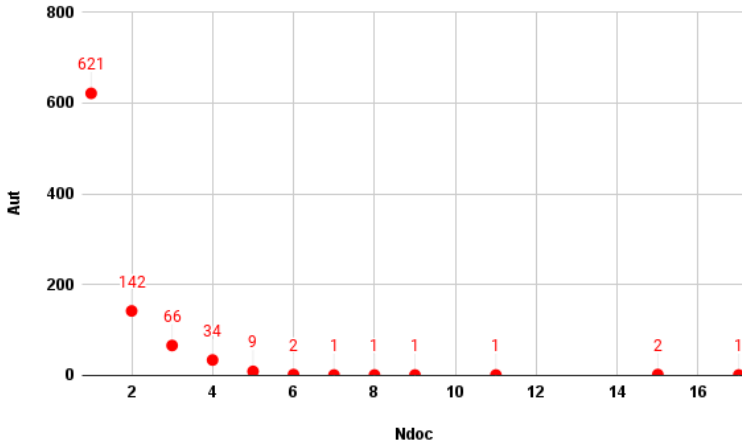
The 3 journals assume different publishing patterns in terms of volume, being notable in the image a peak of productivity in 2020 for NSS and IO however the peak in IJNS was presented in the year 2022. We can also appreciate that the IJNS starts to stand out in the same year 2022 where it assumes a greater number of articles.

As far as the impact values are concerned, we can see that there is a distribution among the journals as to the highest indicators. IO has the highest SJR, IJNS stands out as SNIP and NSS has the higher CiteScore. This not only demonstrates that the quality of the publications is internationally recognized, but can vary depending on the perspective of how it is measured.

These analyses allow us to state that the core of the production is published in sources with a medium to high impact and a high level of international prestige in their respective themes. Through this impact, the dissemination and progressive recognition of neutrosophic knowledge conceived from the Latin American region is possible.

3.3 Author analysis

A total of 882 authors were counted for the 361 articles found, distributed according to the number of published articles as shown in Figure 7. Similarly, authors were counted by paper resulting in a greater distribution of documents with multiple authorships. Figure 8.



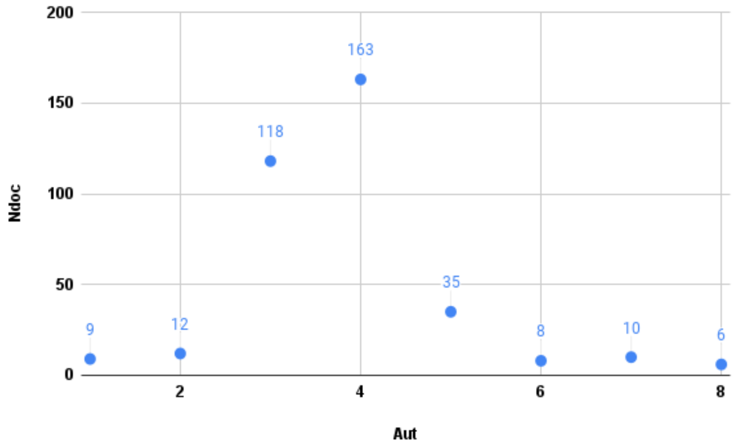


Figure 7: Document per author

Figure 8: Author per document

Chamorro (2021) highlights in his research that multiple authorship is not condemnable and maybe the future behavior in scientific research due to this high trend in recent years. Similarly, the number of authors per document may vary depending on various factors such as the scientific discipline being investigated. [22] In that sense, and given that we have identified in this study a strong tendency towards multidisciplinary publication, it is justifiable the behavior of the author in the documentary set analyzed.

To analyze the distribution of authors according to their productivity, the Lotka Law was applied. This suggests that the number of authors (An), who publish (n) works on a subject is inversely proportional to (n2).[21, 23] On the basis of the Lotka Index, it is possible to distribute authors into three levels of productivity: small producers (with a single job or productiveness index equal to 0), medium producers (2 to 9 jobs with a productive index greater than 0 and less than 1) and large producers (10 or more jobs and productive Index equal or higher than 1) [23].

The authors were divided by productivity as follows: 4 large producers, 257 medium-sized producers, and 621 small producers. As is obvious, it highlights a group of authors who have led the production of articles in the region, and not only in terms of productivity but also regarding impact. These authors can be found in Table 3 and Table 4.

Table 3: Top 5 Latin-American authors based on number of documents and citations.

Author	Country	Ndoc	Cit	H-Index	FWCI
Carlos Granados	Colombia	17	127	8	1,29
Maykel Y. Leyva Vázquez	Ecuador	15	114	18	1,47
Jesús Estupiñan Ricardo	Ecuador	15	322	14	3,19
Ariel Romero Fernández	Ecuador	11	64	5	1,54
Noel Batista Hernández	Ecuador	6	179	9	2,68

Table 4: Top 5 Latin-American leading-authors

Author	Country	Ndoc(lead)	%Ndoc(lead)
Carlos Granados	Colombia	6	35,2

Jesús Estupiñan Ricardo	Ecuador	4	26,6
Maykel Y. Leyva Vázquez	Ecuador	4	26,6
Noel Batista Hernández	Ecuador	4	66,6
Ariel Romero Fernández	Ecuador	3	27,2

The first table shows the 5 most productive regional authors and the second table the 5 authors leading research on the subject. It is no coincidence that these names coincide in the two tables, nor is it that most of these names are associated with the ALCN. Before analyzing this relationship with this organization, it is worth noting that the above-mentioned authors present mostly very favorable indicators in terms of their levels of scientific performance.

First identified Colombian mathematician Carlos Granados who has based his production on the analysis and application of neutrosophic and mathematical methods for decision-making. This scientist also stands out in terms of leadership in the research carried out with a 35.2% performance as the lead author.

Furthermore, the rest of the researchers are closely linked to the leading and representative organization of Neutrosophic Science in the region. Based on the promotion of research and technological development from Neutrosophic theories and their applications, this partnership has been building strong academic relationships with research institutions throughout the region. Its objective is the dissemination of neutrosophic thinking but also the solution of social problems that affect the region from the science itself. Because of this, it is to be expected that the main exponents of research in this branch will be part of or linked to the ALCN as is the case of the prominent Ecuadorian researcher Ariel Romero Fernández who is acting as Research Director at a prestigious Andean university.

A co-authorship network was also developed to determine the levels of relationship between authors, enabling the identification of major associations. This map can be seen in Figure 9.

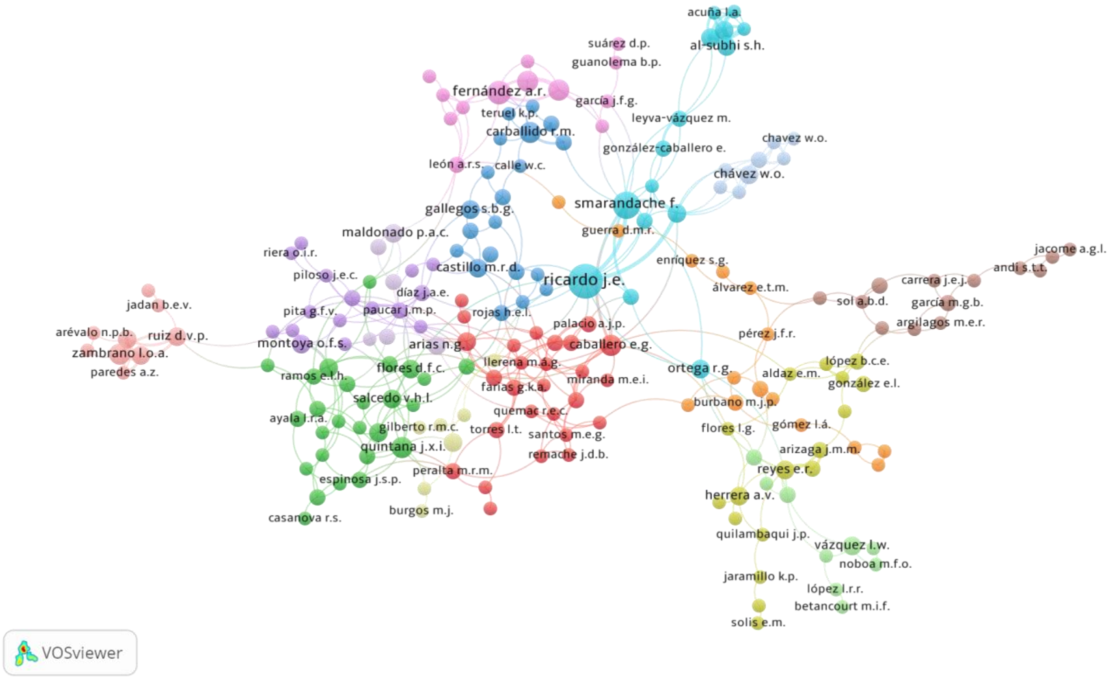


Figure 9: Co-authorship analysis network.

The group consisted of 261 authors, after which researchers with less than 2 papers and those who did not submit links with others were excluded. The final result was distributed among 14 collaborative clusters, 415 links, and 476 total link strength. The collaborative density rests with the above-mentioned researchers, reinforcing the key role played by ALCN in the region.

It is also interesting that the leading author in this branch, Smarandache F., and some other prominent exponents of Neutrosophy worldwide [11] have participated in collaborative research with Latin American authors. This shows the linkage and positioning of this discipline in the region by the international scientific community. It also shows the acceptance of the diversity of trends and fields of Neutrosophy research, and how, unlike the international context, in Latin America its study is applied primarily for social benefit. This diversity, rather than becoming a limitation to the development of science, according to the levels of relationship between authors has proven to be one of the most influential components in the substantial progress of Neutrosophic Science and their global positioning.

In this respect, Figure 9 shows the collaboration between countries according to the authors of the documents. All countries were taken into account and there were no restrictions on the presentation of these data. The network was formed as follows: 33 Items were found, grouped into 5 Clusters, and associated with 56 Link. The total link strength was 143.

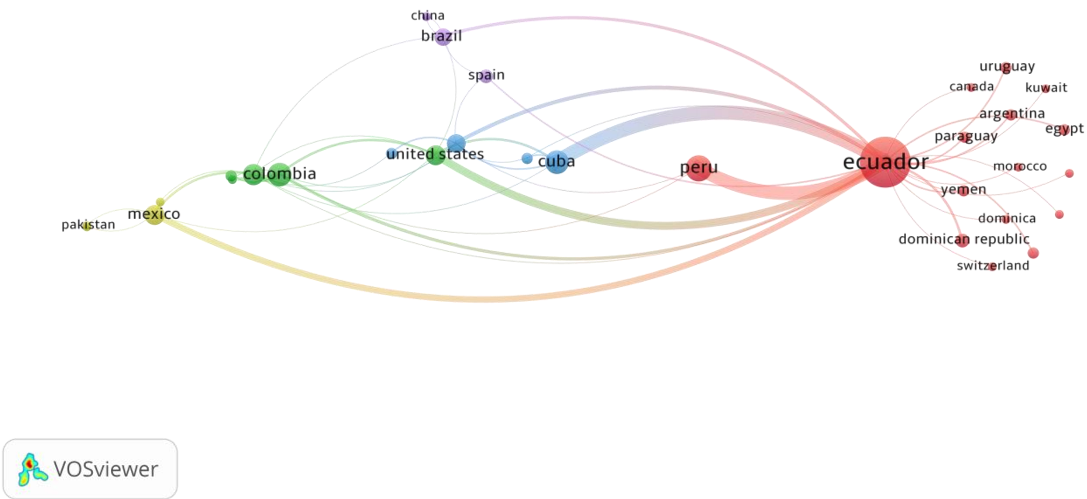


Figure 9: Country collaboration network.

The leading country in terms of documents and collaborations is Ecuador with a total of 312 articles. Ecuador acts as the "embassy" of Neutrosophy in Latin America due to the ALCN's roots in that territory. But other relevant countries could also be identified, such as Peru with 33 articles and Colombia and Cuba with 23 in each case. Collaborative links were not only detected between Latin American countries but from other latitudes, as we mentioned earlier, joint efforts were made, such as India, the United States and Spain.

These collaborations can be translated into values if %Icol measurement is applied. It is possible to calculate the annual percentage of this phenomenon which is summarized in a 31.3% overall as shown in Figure 10.

The collaboration a pattern of growth, except in 2022 when its percentage decreased considerably but recovered rapidly in the following year marking the highest value of the period analyzed. It is possible to associate this phenomenon with other results, as the levels of quotations seen earlier also declined in that same year, which may well be justified to a lesser extent by the lack of international collaboration.

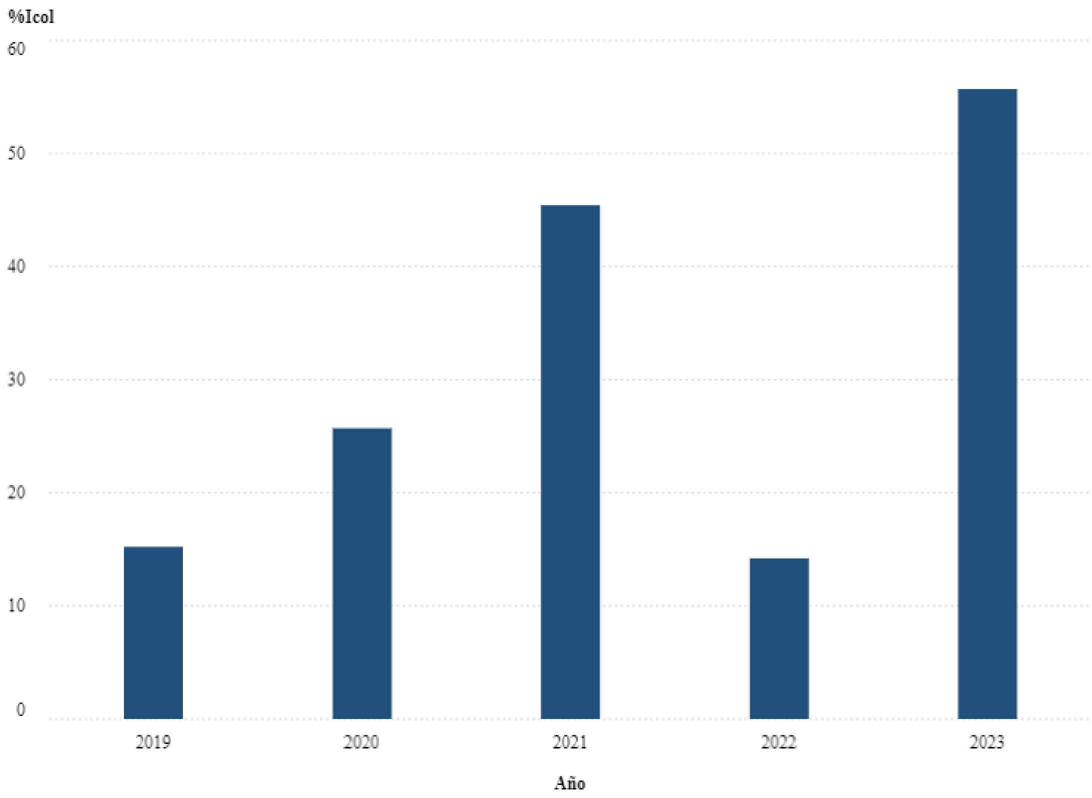


Figure 10: Evolution of annual %Icol (2019-2023).

As we explained earlier lines, Ecuador stands out for the largest share in the volume of production because it is where the ALCN officially lies. It is precisely the institutions in this country that have the highest performance in terms of research contribution, as described in Table 5.

Table 5: Top 10 Latin-American institutions based on number of documents (2019 – 2023)

Affiliation	Country	Ndoc	%Ndoc	Cit
Universidad Regional Autónoma de los Andes	Ecuador	269	74,5	1014
Universidad Nacional Mayor de San Marcos	Peru	14	3,9	28
Universidad de Guayaquil	Ecuador	13	3,6	167
Universidad de Antioquia	Colombia	10	2,77	51
Universidad Técnica de Babahoyo	Ecuador	9	2,4	111
Universidad Nacional Intercultural de la Amazonìa	Ecuador	8	2,2	4
Universidad de La Habana	Cuba	7	1,9	46
Universidad Catolica de Santiago de Guayaquil	Ecuador	7	1,9	65
Universidad Politécnica Salesiana, Cuenca	Ecuador	5	1,3	99
Universidad Nacional del Centro del Perú	Peru	5	1,3	0

The Regional Autonomous University of the Andes (UNIANDES) accounts for 74.5% of the total output. UNIANDES has emerged as one of the top 20 universities in Ecuador in terms of scientific impact.[24] Therefore, the commitment of this university as one of the main institutions sponsoring

Neutrosophy in the region has a positive impact on this outcome. It is also the leading institution in quotation values. As members of the ALCN the teachers and students of this house of higher studies have developed this theme in their research.

Other prestigious Latin American universities include the National University of San Marcos (UNMSM), the University of Guayaquil (UG), University of Antioquia (UDEA) and University of Havana (UH).

While the spread of neutrosophic thinking in the region is recognized, research in this field in institutions outside Ecuador is still limited, which poses a challenge to the scientific community.

5. Conclusions

In conclusion, we can say that the scientific production in Neutrosophy assumes patterns of behavior that vary from the international pattern associated with this topic. Mainly, it distinguishes the Latin American branch with a marked approach to the social. The application of these methods has enabled us to understand from a new perspective the individual and collective context of the region, based on neutrosophic knowledge for the identification, analysis, and resolution of social problems which is reflected in scientific research.

Although productivity levels have not increased in recent years, the impact and recognition from the international community have grown significantly, meaning the evolution of a process of transformation. The quality in the sources where these new contributions are shared is outstanding, with the publication concentrating on the main impact quarters and focusing mainly on the NSS magazine. International collaboration played a key role in the production and countries such as Ecuador, Peru, Colombia, and Cuba stand out.

ALCN is a key factor that positively influences the behavior of scientific production on Neutrsofia in the Latin American region. It has served as a means of dissemination for research in this discipline and has emerged as an agent of social change in the academic and scientific spheres. It has also contributed to the development of collaboration between the main international core of the branch and the emerging elements that are forming around the study and application of this science in the region.

. **Funding:** “This research received no external funding”

Acknowledgments: This study has been supported (not financed) by the Scientific Collaboration Agreement between the Asociación Latinoamericana de Ciencias Neutrosóficas (ALCN) and the Universidad de Ciencias Médicas de Guantánamo.

Conflicts of Interest: “The authors declare no conflict of interest.”

References

1. A. Rezaei , T. Oner, T. Katican, F. Smarandache, N. Gandotra. A short history of fuzzy, intuitionistic fuzzy, neutrosophic and plithogenic sets. Journal of [Internet]. 2022; 18 (1): 99-116. https://doi.org/10.54216/IJNS.180109
2. Smarandache Florentin. (2017). Fundamentos de la lógica y los conjuntos neutrosóficos y su aplicación a la fusión de la Información. http://dx.doi.org/10.13140/RG.2.2.11097.01124
3. Balladares-Burgos, J. (2018). Una etnofilosofía como fundamento de las etnociencias. *Nuevo Pensamiento*, *8*(11). http://p3.usal.edu.ar/index.php/nuevopensamiento/article/view/6147

4. Leyva-Vázquez M, Estupiñan J, Smarandache F. Neutrosofía en Latinoamérica, avances y perspectivas. University of New Mexico [Internet]. 2020;14:1-7. Disponible en: https://zenodo.org/record/4384978
5. Análisis neutrosófico estadístico sobre el conocimiento de la educación interprofesional como necesidad del siglo XXI. Publicación Científica del La Asociación Latinoamericana de Ciencias Neutrosóficas (ALCN) [Internet]. 2023 Abr. ;27:201-10. Disponible en: http://fs.unm.edu/NCML2/index.php/112/article/view/372
6. Ye, J. (2014). A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. Journal of Intelligent & Fuzzy Systems, 26(5), 2459-2466. https://doi.org/10.3233/IFS-130916
7. Smarandache, F. (2002). Neutrosophy, a new Branch of Philosophy. Infinite Study.
8. Vázquez, M. L., & Smarandache, F. (2018). Neutrosofía: Nuevos avances en el tratamiento de la incertidumbre. Infinite Study.
9. Estupiñán-Ricardo J, Leyva-Vázquez M, Álvarez-Gómez S, Alfonso-Manzanet J, Velázquez-Soto O, Rodríguez-Guzmán A. (2023). La aplicación de la neutrosofía en las ciencias médicas: una revisión bibliográfica narrativa. Revista Cubana de Información en Ciencias de la Salud, 34. Recuperado de https://acimed.sld.cu/index.php/acimed/article/view/2599
10. Peng, X., Dai, J. A bibliometric analysis of neutrosophic set: two decades review from 1998 to 2017. Artif Intell Rev 53, 199–255 (2020). https://doi.org/10.1007/s10462-018-9652-0
11. Delcea, C.; Domenteanu, A.; Ioanăș, C.; Vargas, V.M.; Ciucu-Durnoi, A.N. Quantifying Neutrosophic Research: A Bibliometric Study. Axioms 2023, 12, 1083. https://doi.org/10.3390/axioms12121083
12. Leyva-Vázquez MY, Estupiñán-Ricardo J, Batista-Hernández N. Investigación científica: perspectiva desde la neutrosofía y productividad. RUS [Internet]. 2022; 14(S5):640-649. Disponible en: https://rus.ucf.edu.cu/index.php/rus/article/view/3334
13. Neutrosofia [Internet]. Neutrosofia |. Neutrosofia; 2019 [citado el 11 de mayo de 2024]. Disponible en: https://neutrosofia.org/
14. Turpo-Gebera, Osbaldo; Limaymanta, César H.; Sanz-Casado, Elías (2021). "Producción científica y tecnológica de Perú en el contexto sudamericano: Un análisis cienciométrico". Profesional de la información, v. 30, n. 5, e300515. https://doi.org/10.3145/epi.2021.sep.15
15. Bornmann, L., De Moya Anegón, F., Leydesdorff, L. (2012) The new Excellence Indicator in the World Report of the SCImago Institutions Rankings 2011. Journal of Informetrics, 6 (2), pp. 333-335. http://dx.doi.org/10.1016/j.joi.2011.11.006
16. Zacca-González, G., Chinchilla-Rodríguez, Z., Vargas-Quesada, B., & de Moya-Anegón, F. (2015). Patrones de comunicación e impacto de la producción científica cubana en salud pública. Revista Cubana de Salud Pública, 41, 200-216. https://www.scielosp.org/pdf/rcsp/2015.v41n2/200-216/es
17. González Parias, C. H., Londoño Arias, J. A., & Giraldo Mejía, W. A. (2022). Evolución de la producción científica en América Latina indexada en Scopus. 2010-2021. http://revistas.bnjm.sld.cu/index.php/BAI/article/view/507
18. Beltrán-Santoyo G, Ruíz-Huerta EA, Gómez-Bernal JM. La importancia e influencia del idioma inglés dentro del campo científico . lc [Internet]. 2021; 3(5):46-1. Disponible en: https://repository.uaeh.edu.mx/revistas/index.php/lc/article/view/7499
19. Díaz-Rodríguez YL, Rodríguez-Venegas E de la C, Vitón-Castillo AA, Hernández-García OL, Díaz-Chieng LY. Análisis de artículos sobre COVID-19 publicados en revistas médicas pertenecientes a universidades médicas cubanas. Rev. inf. cient. [Internet]. 99(5): 412-424. Disponible en: http://scielo.sld.cu/scielo.php?script=sci_arttext&pid=S1028-99332020000500412&lng=es.
20. Sosa Pérez, N. de la C. ., Cuba Rodríguez, Y. ., & Bouza Betancourt, O. . (2024). Visualización del comportamiento de la innovación en el sector empresarial desde el análisis de co-ocurrencia de palabras. E-Ciencias De La Información, 14(1). https://doi.org/10.15517/eci.v14i1.56018
21. Diodato, V. P., & Gellatly, P. (2013). Dictionary of bibliometrics. Routledge.
22. Chamorro, A. . (2021). ¿Quién es autor? Las autorías múltiples, criterios y lineamientos. Revista Española De Documentación Científica, 44(2), e290. https://doi.org/10.3989/redc.2021.2.1758
23. Vitón Castillo, A., Diaz-Samada, R., & Martínez Pozo, Y. (2019). Indicadores bibliométricos aplicables a la producción científica individual. Universidad Médica Pinareña, 15(2), 279-285. Recuperado de https://revgaleno.sld.cu/index.php/ump/article/view/352

24. Estupiñán-Ricardo, J., Romero-Fernández, A., Alfonso Sánchez, I., Portelles-Cobas, D., & Velázquez-Soto, O. (2023). Producción científica y visibilidad de investigadores UNIANDES en SCOPUS: estudio bibliométrico retrospectivo en Ecuador. Revista Cubana de Información en Ciencias de la Salud, 34. Recuperado de https://acimed.sld.cu/index.php/acimed/article/view/2594

Received: Feb 1, 2024. Accepted: April 30, 2024

Neutrosophic Sets and Systems (NSS) is an academic journal, published quarterly online and on paper, that has been created for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics etc. and their applications in any field.

All submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

ISSN (print): 2331-6055, ISSN (online): 2331-608X
Impact Factor: 1.739

NSS has been accepted by SCOPUS. Starting with Vol. 19, 2018, all NSS articles are indexed in Scopus.

NSS is also indexed by Google Scholar, Google Plus, Google Books, EBSCO, Cengage Thompson Gale (USA), Cengage Learning, ProQuest, Amazon Kindle, DOAJ (Sweden), University Grants Commission (UGC) - India, International Society for Research Activity (ISRA), Scientific Index Services (SIS), Academic Research Index (ResearchBib), Index Copernicus (European Union),CNKI (Tongfang Knowledge Network Technology Co., Beijing, China), etc.

Google Dictionary has translated the neologisms "neutrosophy" (1) and "neutrosophic" (2), coined in 1995 for the first time, into about 100 languages.

FOLDOC Dictionary of Computing (1, 2), Webster Dictionary (1, 2), Wordnik (1), Dictionary.com, The Free Dictionary (1),Wiktionary (2), YourDictionary (1, 2), OneLook Dictionary (1, 2), Dictionary / Thesaurus (1), Online Medical Dictionary (1, 2), and Encyclopedia (1, 2) have included these scientific neologisms.

DOI numbers are assigned to all published articles.

Registered by the Library of Congress, Washington DC, United States, https://lccn.loc.gov/2013203857.

Recently, NSS was also approved for Emerging Sources Citation Index (ESCI) available on the Web of Science platform, starting with Vol. 15, 2017.

Editors-in-Chief:

Prof. Dr. Florentin Smarandache, Postdoc, Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA, Email: smarand@unm.edu.
Dr. Mohamed Abdel-Basset, Faculty of Computers and Informatics, Zagazig University, Egypt, Email: mohamed.abdelbasset@fci.zu.edu.eg.
Dr. Said Broumi, Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco, Email: s.broumi@flbenmsik.ma.

23316055

$39.95