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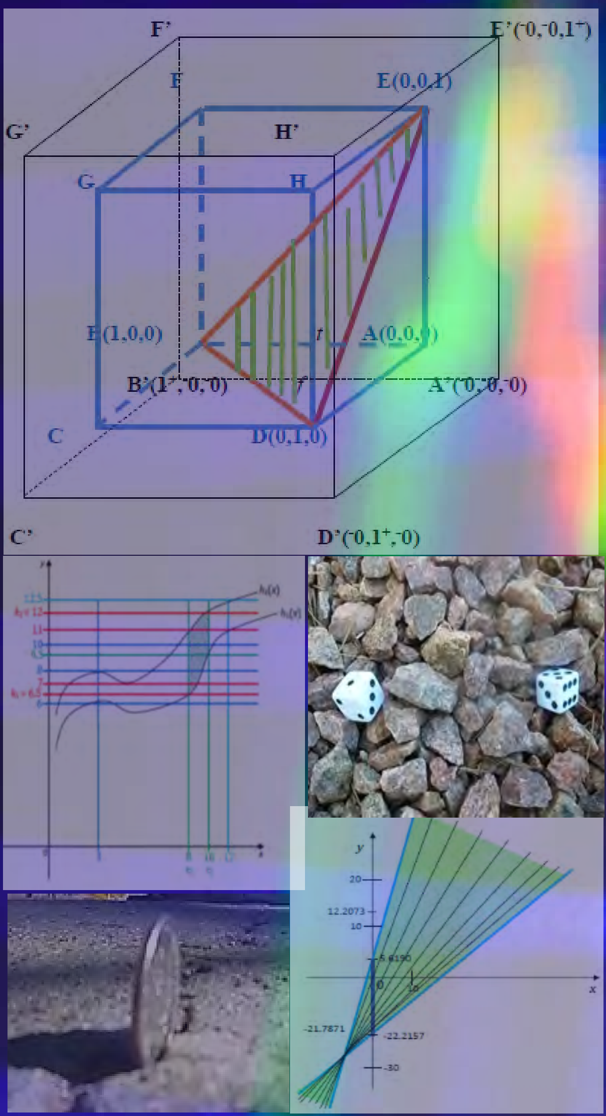

<A> <neutA> <antiA>

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Study of the relationship among economic variables in cattle production in a region of Peru, based on Plithogenic Statistics

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Abstract. This paper is an in-depth study that starts from a preliminary one, where we surveyed 141 ranchers from the town of Coto-Coto in Peru to obtain as much information as possible about the relationship between two economic variables, Activity Cost and Financial Management. Furthermore, we study the correlation of the first one of them with three other economic variables. To do this, we process the survey data using logical operators with the support of the plithogenic statistics theory. These results given in the form of plithogenic values are logically aggregated and were converted to crisp values and studied through the use of statistical tools, specifically Kendall's Tau b. The study of economic variables within the field of cattle farming is of great importance because it allows us to improve the productivity of this economic sector, which is also part of the most accepted human diet worldwide. A substantial advantage in the use of plithogeny is that linguistic values that are more natural for livestock farmers were processed, in addition, uncertainty and indeterminacy were taken into account by the use of neutrosophic numbers.

Keywords: Activity costs, financial management, profitability, resource optimization, plithogenic set, plithogenic statistics, Kendall's Tau b.

1 Introduction

Financial management, which is the process of decision-making and data analysis that seeks the optimal administration and use of the company's financial resources to achieve certain objectives, is relevant in the study that we present here because it allows us to contextualize the role of the producer, who is every cattle rancher in the planning, organization, direction, and control of economic activities that will generate effective flows as a result of the application of a cost system.

On the other hand, according to the Ministry of Agriculture and Irrigation, in 2020 he maintains that two investment programs have been started for the genetic improvement of cows. With a budget of 2.5 million soles, the National Institute of Agrarian Innovation (INIA) carries out the initiative to increase the availability of high-value bovine genetic material in seven areas of the country, benefiting 34,230 dairy farmers. This initiative will use genetic nuclei and will lead to the production of 1,800 embryos and 600,000 sperm straws.

The second project, which has a budget of 2.5 million soles and will serve 21,060 beef farmers, aims to increase the availability and access to the genetic material of cattle through the application of reproductive biotechnology. In this study, 2240 embryos and 110,000 sperm straws will be obtained. Both operations will mean a 20% increase in milk and beef production.

The livestock industry in the Junín Region has problems that could be solved. However, the lack of technical guidance in raising livestock for the production and marketing of milk, as well as the lack of knowledge of the costs incurred in each of its production processes, is a factor that limits the optimal exploitation of the product. and, therefore, prevents achieving a reasonable standard of competitiveness.

Likewise, the cost of activities is broken down into different support tools to carry out the process of improving cattle, among them, there is labor, which also requires considering feeding, since it is the fundamental basis for improving milk and meat production is mainly based on fodder production; cultivated pastures; natural pastures; preserved forages (silage and hay), balanced feed, commercial concentrates, and mineral salts. On the other hand, it is important to establish the type of farm that is going to be built and the objective of our breeding. Additionally, there are economic limitations, such as the availability of equipment, labor costs, and transportation costs, both for supplies and food.

The animals (biological activities) are raised, fed, cultivated, and ultimately sold to a livestock farm, all while receiving care to obtain by-products. To create products that satisfy the needs of a market, a profitable company, like any other, has to account for a series of production costs derived from the purchase and use of inputs. Both operating costs, which appear in the operating income statement, and process costs, which arise throughout the manufacturing process.

The purpose of this study is to delve deeper into the relationship between the variables of Activity Cost and Financial Management. In a previous initial study, we determined that there is a positive relationship between both of them. Due to the importance of the topic, we are considering delving deeper into it and conducting a more detailed survey of the ranchers at the Coto-Coto Fair in Peru. To do this, we have 141 ranchers to whom we applied a survey with a series of questions, in addition to representing the data in the form of neutrosophic numbers within the plithogeny framework. In this case, we use statistical methods to avoid bias, which is why these farmers were selected randomly.

On this occasion, we once again use the tools offered by the theory of Neutrosophy to solve problems related to uncertainty and indeterminacy. We need to keep these components within the study because it guarantees greater accuracy. Additionally, we study other economic variables.

Plithogenic sets generalize the theory of neutrosophic sets since they deal with an appurtenance function that can be either fuzzy, intuitionistic fuzzy, or neutrosophic, in addition to allowing variables of different natures to be combined [1]. The fundamental idea is to model the dynamics of systems, where in addition to the interaction of the concept <A> with its opposite <AntiA> and the neutral <NeutA> , we also model its interactions with other concepts such as <B>, <C>, etc., or their opposites < AntiB >, < AntiC >, etc., or their neutrals < NeutB >, < NeutC >, etc.

Additionally, plithogenic statistics allow the combination of statistical methods with logical elements and operations [2]. Specifically, in this work, we propose to logically model the answers given by the respondents, aggregate them with logical plithogenic aggregators, and also process them with statistical methods.

In this study, in addition to the two economic variables mentioned above, we include others such as “Optimization of financial resources”, “achievement of objectives in beef cattle producers”, and “greater profitability in beef cattle producers”.

This paper has the following structure: we continue with a Materials and Methods section where we explain the fundamental concepts of the proposed financial variables, the plithogenic sets, and the plithogenic statistics. Next, the following section is dedicated to presenting the results of this study. The last section is dedicated to Conclusions.

2 Materials and Methods

This section is an approach to the main theories used in this study. We begin with an explanation of the economic variables used in this article. This is essential to be able to understand the economic essence with which we are dealing. The following subsection is a summary of the basic concepts within plithogeny theory and the concepts and methods used concerning the plithogenic statistics.

2.1. Main economic and financial indicators used in this study

Activity cost accounting is a system that accumulates the indirect costs of each of an organization's activities before allocating those costs to the products or services and other cost objectives that result from that activity.

Similarly, Ramos et al. ([3]) define activity costing as a tool to determine the costs and distribution of actual indirect costs incurred by each activity, which is a crucial step in the decision-making process.

The Activity cost is considered a methodology that assigns costs to the inputs necessary to execute the various activities of a production process, identified as those relevant to obtaining a certain cost object, calculating the cost of these inputs through cost absorption mechanisms of the activities.

Escobar et al. ([4]) consider activity costing as a method “to determine a real cost to produce a product or the provision of a service that guarantees more reasonable and competitive market prices for the consumer.”

Likewise, financial management is defined as a process that, through the use of financial tools and indicators, seeks to make decisions to improve the financial situation of the company. This is conceived as a form of strategic planning and management that implements certain parameters to evaluate and diagnose the financial system, and to make decisions in favor of the organization.

Also, financial management represents the underlying processes that give companies their competitive advantage, allowing them to effectively manage their resources to keep capital moving and increase their profits. Also, López et al. ([5]) mention that financial management is based "on the optimization of working capital and value creation that promotes improved profitability."

The term "resource optimization" refers to the process of finding methods to improve a company's resources to achieve greater efficiency and effectiveness. Due to the high volume of customer interactions, the quality of services provided by the industry requires continuous innovation in both, resources and management.

It is a means to talk about how to improve something. From this, we can deduce that the definition of resource optimization is the search for means that allow the resources of a company to be improved for the sake of greater performance.

As a consequence of the continuous connection with customers, companies in the service sector must constantly work to improve their management and infrastructure to maintain a high level of service. Given that the scope of services includes not only restaurants but also industrial and hospital canteens, in which the quality of service must be excellent so that this does not harm other aspects of those canteens, investing in high-quality resources is essential to offer excellent customer service.

Resource optimization can be defined as the way to profitably use resources to seek the best results, greater efficiency, and continuous improvement of the organization.

The achievement of an organization's objectives is related to financial planning since it plays an essential role in growth and sustainable development, implementing mechanisms to correct possible errors in the course of achieving objectives.

Profitability is the efficiency of an activity that can be measured by comparing the surplus generated for distribution to shareholders with the total resources invested in the activity.

Greater profitability is a term that refers to any economic activity in which material, human, and/or financial resources are mobilized to achieve an objective. It is any economic benefit derived from the use of certain resources. Accounting defines two forms of profitability: economic profitability and financial profitability. This is a percentage ratio that reflects the return on investment for each unit of resource invested over time. Also, Profitability can be defined as the percentage increase in value over the original investment plus possible cash distributions. This is the relationship between income and expenses.

Profitability ratios are indices that allow us to evaluate the ability of a company to generate profits, whether through its own or external resources so that they are significant to the extent that they allow evaluating the result of the efficiency of economic and financial resource management of the company. They are indicators that help determine the company's ability to obtain profits.

It is a term that refers to any economic activity in which material, human, and/or financial resources are mobilized to achieve an objective. It is any economic benefit derived from the use of certain resources. Accounting defines two forms of profitability: economic profitability and financial profitability.

2.2. Some notions on Plithogenic theory

According to F. Smarandache, "Plithogeny is the genesis or origination, creation, formation, development, and evolution of new entities from dynamics and organic fusions of contradictory and/or neutrals and/or non-contradictory multiple old entities. Plithogeny pleads for the connections and unification of theories and ideas in any field. As "entities" in this study, we take the "knowledge" in various fields, such as soft sciences, hard sciences, arts and letters theories, etc."([1,2,6,7]).

A *Plithogenic Set* is a non-empty set P whose elements within the domain of discourse $U(P \subseteq U)$ are characterized by one or more attributes $A_1, A_2, \cdots, A_m$, $m \geq 1$, where each attribute can have a set of possible values within the spectrum S of values (states), such that S can be a finite, infinite, discrete, continuous, open, or closed set.

Each element $x \in P$ is characterized by all possible values of the attributes that are within the set $V = \{v_1, v_2, \cdots, v_n\}$. The value of an attribute has a *degree of appurtenance* $d(x, v)$ of an element x, in the set P, about a certain given criterion. The degree of appurtenance can be fuzzy, intuitionistic fuzzy, or neutrosophic, among others.

Thus,

$$\forall x \in P, d: P \times V \to \mathcal{P}([0,1]^z) \qquad (1)$$

Where $d(x, v) \subseteq [0,1]^z$ and $\mathcal{P}([0,1]^z)$ is the power set of $[0,1]^z$. $z = 1$ (for the fuzzy degree of appurtenance), $z = 2$ (for the intuitionistic fuzzy degree of appurtenance), or $z = 3$ (for the neutrosophic degree of appurtenance).

Whether the cardinality of V is greater than or equal to 1, $c: V \times V \rightarrow [0,1]$ is called *attributes value contradiction degree function* between any pair of attributes v_a, v_b, which satisfies the following axioms:

- $c(v_a, v_a) = 0$,
- $c(v_a, v_b) = c(v_b, v_a)$.

c defined as above, is denoted by c_F to indicate that this is a function called *fuzzy attributes value contradiction degree function*. It is generally defined $c_{IF}: V \times V \rightarrow [0,1]^2$ as an *intuitionistic attributes value contradiction function* and $c_N: V \times V \rightarrow [0,1]^3$ to indicate a *neutrosophic attributes value contradiction function*.

So, the Plithogenic Set is characterized by (P, a, V, d, c), consisting of the set P, the set a of attributes, the set V of values, d is the appurtenance function and c is the function attribute value contradiction degree function.

The contradiction function in practice is applied to compare the contradiction of all attributes concerning a dominant attribute if any, which is the most important compared to the others.

Definition 1. ([1, 8]) Given a plithogenic set (P, A, V, d, c), a *Plithogenic Neutrosophic Aggregation Operator* is defined as in Equation 2:

$$(a_1, a_2, a_3)AND_p(b_1, b_2, b_3) = \left((1-\bar{c})(a_1 \wedge_F b_1) + \bar{c}(a_1 \vee_F b_1), \frac{1}{2}[a_1 \wedge_F b_1 + a_1 \vee_F b_1], (1-\bar{c})(a_1 \vee_F b_1) + \bar{c}(a_1 \wedge_F b_1)\right) \quad (2)$$

Where $\bar{c} \in [0,1]$, $\wedge_F$ is a t-norm and $\vee_F$ is a t-conorm, see [9].

It is a *Plithogenic Neutrosophic Intersection* when $\bar{c} = 0$ and it is a *Plithogenic Neutrosophic Union* when $\bar{c} = 1$, [1, 2, 8]. This aggregator is more accurate than both the n-norms and n- conorms between neutrosophic sets ([1]),

A plithogenic neutrosophic set can be converted into a crisp value using the following formula, ([1]):

$$\mathcal{S}(T, I, F) = \frac{1}{3}(2 + T - I - F) \quad (3)$$

There are many applications of plithogeny in many sciences ([10-15]).

On the other hand, (U, a, V, d, c) is called *Plithogenic Probability*, where U is the event space of E. A Plithogenic Probability is the probability that an event will occur in all the random variables that determine it. Where each random variable can be classical, (T,I,F)-neutrosophic, I-neutrosophic, (T,F)-intuitionistic fuzzy, (T,N,F)-picture fuzzy, (T,N,F)-spherical fuzzy, or (another fuzzy extension) distribution function. In this way, the Plithogenic Probability generalizes the classical Multivariate Probability.

Additionally, Plithogenic Statistics comprises the analysis and observations obtained through Plithogenic methods of Probability. Plithogenic Probability generalizes the classical Multivariate Statistics.

The Refined Probabilities are decomposed into more than one element of truthfulness, more than one element of indeterminacy, or more than one element of falsity ([16]). That is, they are of the form $(T_1, T_2, \cdots, T_p, I_1, I_2, \cdots, I_q, F_1, F_2, \cdots, F_r)$, where at least one of the indices p, q, or r is strictly greater than 1.

3 Results

141 livestock farmers participating in the Coto-Coto Fair were selected at random, to apply the properties of plithogenic statistics.

We calculate the size and composition of the sample given the quantitative nature of the research population; the sample size is chosen using a statistical formula for the definition of the sample in finite populations ([17]).

$$n = \frac{N.Z^2.p.q}{e^2(N-1) + Z^2.p.q} \quad (4)$$

Where:

n= sample size

N= population size

Z: (critical coefficient depends on the confidence level) α = 95%; Z= 1.96,

P: (controlled N ratio) = 50%,

q = (1-p) = 50%,

e = (margin of error allowed) = 5%,

Applying the formula, we have:

N = 320, Z = 1.96, p = q = 0.5, e = 0.05, and n = 141.

The sample size is 141.

The selection of participants was carried out in proportion to the gender of the livestock farmers in the population, as shown in Table 1.

Gender	Amount	hi	hi*n subsamples
Men	180	0.6	84.6
Women	120	0.4	56.4
Total	300	1	141

Table 1. Livestock farmers participating in the Coto–Coto Sunday fair. Note: The table is prepared based on the information provided by the Huancayo charity.

For their convenience, they were asked to offer their opinion according to a linguistic scale, as shown in Table 2.

Linguistic Expression	Plithogenic number (T, I, F)
Strongly disagree	(0.10, 0.75, 0.85)
Disagree	(0.40, 0.70, 0.50)
Neutral	(0.50, 0.40, 0.60)
Agree	(0.65, 0.30, 0.45)
Strongly agree	(0.95, 0.05, 0.05)

Table 2: Linguistic values associated with plithogenic numbers for expert evaluation.

In Table 2 each linguistic value is associated with a plithogenic number, which is why each linguistic value is replaced by its plithogenic equivalent number when the calculations are performed. The surveys applied are detailed, as shown below.

PERUVIAN UNIVERSITY LOS ANDES
GRADUATE SCHOOL
QUESTIONNAIRE

"Cost for Activities and Financial Management in Beef Cattle Producers. Coto-Coto Livestock Fair - Chilca"

GENERAL DATA	
Survey location: …………………………… **Survey date: ……………………………** **Age: ……………………………**	**Gender:** **Male** ☐ **Female** ☐

Instructions: Below, we present several propositions. We ask you to express your personal opinion in front of them, marking with an (X) the one that best expresses your point of view according to the following rating scale:				
1. Totally Disagree	**2. Disagree**	**3. Neutral**	**4. Agree**	**5. Totally agree**

PART I: COST PER ACTIVITY

DIMENSION/ITEMS	**QUALIFICATION**				
DIMENSION: ATTRIBUTION OF DIRECT COSTS	**1**	**2**	**3**	**4**	**5**
1. You consider that the farmers at the fair produce more than one product to improve their income.					

2. You consider that the farmers at the fair could take advantage of alternatives that allow them to produce more livestock products to improve their income.					
3. You consider that the farmers at the fair know what the indirect costs are, and that allows them to obtain greater income.					
4. You consider that ranchers would be able to identify indirect costs so that they can obtain greater income.					

DIMENSION: INDIRECT COST FOR EACH ACTIVITY	1	2	3	4	5
5. You consider that ranchers should know what commercial costs are, to improve their income.					
6. You consider allowing ranchers to have talks about managing commercial costs to improve their income.					
7. You consider that livestock farmers can organize themselves to improve their relations between the producers in the stable and the intermediaries who come to the fair.					

DIMENSION: COSTS PER ACTIVITY IN GLOBAL FORM	1	2	3	4	5
8. You consider that ranchers should know what the financial costs are, that allow them to improve their income.					
9. You consider that ranchers should work with financial entities so that they have capital that allows them to improve their production.					
10. You consider that ranchers should know the administrative costs that allow them to improve their income.					
11. You consider that ranchers identify their production costs that allow them to know how much they invest in raising their cattle to have real prices at the fair.					

PART II: FINANCIAL MANAGEMENT

DIMENSION: OPTIMIZATION OF FINANCIAL RESOURCES	1	2	3	4	5
12. You consider that ranchers should know the sources of financing that allow them to have capital to improve their production in raising their livestock.					
13. You consider that ranchers are capable of managing sources of financing from private companies that allow them to improve their income.					
14. You believe that ranchers should know the investment they make in raising their livestock so that their results improve.					
15. You consider that ranchers should have technical-financial support that allows them to improve their results in the different purchase and sale negotiations.					

DIMENSION: ACHIEVEMENT OF OBJECTIVES	1	2	3	4	5
16. You consider that ranchers set objectives to improve results in livestock raising.					
17. You consider that ranchers achieve their long-term objectives so that their results improve.					
18. You consider that ranchers achieve their short-term objectives to improve their results.					

DIMENSION: GREATER PROFITABILITY	1	2	3	4	5
19. You consider that ranchers should know better how to manage the profitability they have in the production of their cattle.					
20. You consider that ranchers determine the profitability of the investment they make in raising their livestock so that their results improve.					
21. You consider that ranchers should be well aware of the profitability that their assets offer about the amount of cattle they produce.					

22. You consider that ranchers determine the profitability of their assets, which is limited to the amount of livestock they count with for-profit that allows them to improve their results.					

Before administering the survey, its degree of reliability was measured using Cronbach's Alpha coefficient ([18]). This coefficient is used to ensure that there is no inconsistency within the survey questions. Once applied, it was equal to 0.858, which means that the reliability is high and the survey can be applied.

Data processing was carried out with the help of formula (1) of the plithogenic $\mathrm{AND_p}$, where we used $\wedge_F = min$ and $\vee_F = max$; in addition to that $\bar{c} = 0.5$ because it cannot be determined which of the two operators has more importance, either the conjunction or the disjunction.

This formula was applied repeatedly to each item answered by each respondent. For example, if $X = \{x_1, x_2, \dots, x_{141}\}$ is the set of the 141 respondents, let us denote by q_j the set of 22 items for $j \in \{1,2,\dots,22\}$. If a_{ij} is the answer given by the farmer x_i to the question q_j, where a_{ij} is one of the plithogenic numbers that appear in Table 2 associated with the linguistic response answered by the ith interviewee.

To calculate the aggregate responses for each of the dimensions for the ith interviewee, the following logical identities are used:

$\mathrm{D}_{1i} = a_{i1}\mathrm{AND_p}a_{i2}\mathrm{AND_p}a_{i3}\mathrm{AND_p}a_{i4}$ (5)

It is the response of the ith interviewee on the dimension ATTRIBUTION OF DIRECT COSTS.

$\mathrm{D}_{2i} = a_{i5}\mathrm{AND_p}a_{i6}\mathrm{AND_p}a_{i7}$ (6)

It is the response of the ith interviewee on the dimension INDIRECT COST FOR EACH ACTIVITY.

$\mathrm{D}_{3i} = a_{i8}\mathrm{AND_p}a_{i9}\mathrm{AND_p}a_{i10}\mathrm{AND_p}a_{i11}$ (7)

It is the response of the ith interviewee on the dimension COSTS PER ACTIVITY IN GLOBAL FORM.

$\mathrm{D}_{4i} = a_{i12}\mathrm{AND_p}a_{i13}\mathrm{AND_p}a_{i14}\mathrm{AND_p}a_{i15}$ (8)

It is the response of the ith interviewee on the dimension OPTIMIZATION OF FINANCIAL RESOURCES.

$\mathrm{D}_{5i} = a_{i16}\mathrm{AND_p}a_{i17}\mathrm{AND_p}a_{i18}$ (9)

It is the response of the ith interviewee on the dimension ACHIEVEMENT OF OBJECTIVES.

$\mathrm{D}_{6i} = a_{i19}\mathrm{AND_p}a_{i20}\mathrm{AND_p}a_{i21}\mathrm{AND_p}a_{i22}$ (10)

It is the response of the ith interviewee on the dimension of GREATER PROFITABILITY.

$\mathrm{D}_1 = \mathrm{D}_{1,1}\mathrm{AND_p}\mathrm{D}_{1,2}\mathrm{AND_p} \dots \mathrm{AND_p}\mathrm{D}_{1,141}$ (11)

It is the total aggregated value on the dimension ATTRIBUTION OF DIRECT COSTS.

$\mathrm{D}_2 = \mathrm{D}_{2,1}\mathrm{AND_p}\mathrm{D}_{2,2}\mathrm{AND_p} \dots \mathrm{AND_p}\mathrm{D}_{2,141}$ (12)

It is the total aggregated value on the dimension INDIRECT COST FOR EACH ACTIVITY.

$\mathrm{D}_3 = \mathrm{D}_{3,1}\mathrm{AND_p}\mathrm{D}_{3,2}\mathrm{AND_p} \dots \mathrm{AND_p}\mathrm{D}_{3,141}$ (13)

It is the total aggregated value on the dimension COSTS PER ACTIVITY IN GLOBAL FORM.

$\mathrm{D}_4 = \mathrm{D}_{4,1}\mathrm{AND_p}\mathrm{D}_{4,2}\mathrm{AND_p} \dots \mathrm{AND_p}\mathrm{D}_{4,141}$ (14)

It is the total aggregated value on the dimension OPTIMIZATION OF FINANCIAL RESOURCES.

$\mathrm{D}_5 = \mathrm{D}_{5,1}\mathrm{AND_p}\mathrm{D}_{5,2}\mathrm{AND_p} \dots \mathrm{AND_p}\mathrm{D}_{5,141}$ (15)

It is the total aggregated value on the dimension ACHIEVEMENT OF OBJECTIVES.

$\mathrm{D}_6 = \mathrm{D}_{6,1}\mathrm{AND_p}\mathrm{D}_{6,2}\mathrm{AND_p} \dots \mathrm{AND_p}\mathrm{D}_{6,141}$ (16)

It is the total aggregated value on the dimension GREATEST PROFITABILITY.

$\mathrm{C} = \mathrm{D}_1\mathrm{AND_p}\mathrm{D}_2\mathrm{AND_p}\mathrm{D}_3$ (17)

It is the total aggregated value of “Activity cost”.

$\mathrm{M} = \mathrm{D}_4\mathrm{AND_p}\mathrm{D}_5\mathrm{AND_p}\mathrm{D}_6$ (18)

It is the total aggregated value of "Financial Management".

The values in Equations 11-18 are used to measure the behavior of these aspects in general. For the statistical processing, the results of Equations 5-10 are used for each of the dimensions in particular, and also the following two results for the variables “Activity cost” for each respondent (Equation 19) and “Financial Management” for each respondent (Equation 20).

$\mathrm{C}_i = \mathrm{D}_{1i}\mathrm{AND_p}\mathrm{D}_{2i}\mathrm{AND_p}\mathrm{D}_{3i}$ (19)

$\mathrm{M}_i = \mathrm{D}_{4i}\mathrm{AND_p}\mathrm{D}_{5i}\mathrm{AND_p}\mathrm{D}_{6i}$ (20)

For statistical processing, the crisp values are used to apply Equation 3 to the dimension or variable that we wish to study, which are Equations 5-10, 19, and 20.

For example, $\mathcal{S}(\mathrm{T}_{\mathrm{C}_i}, \mathrm{I}_{\mathrm{C}_i}, \mathrm{F}_{\mathrm{C}_i}) \in [0, 1]$ is the crisp value of the ith interviewee's opinion on the variable “Activity cost”, where $i = 1,2,\dots,141$. This is repeated for the other variables and dimensions.

The next step is to apply traditional statistical methods to these values converted from linguistic values for each item and each rancher, to crisp values for each dimension or variable for each interviewee.

The results are shown in the following Tables:

			Activity cost	Financial management
Kendall's tau_b	Activity cost	Correlation coefficient	1.000	.779 **
		Sig. (bilateral)	.	.000
		N	141	141
	Financial management	Correlation coefficient	.779 **	1.000
		Sig. (bilateral)	.000	.
		N	141	141

**. The correlation is significant at the 0.01 level (two-sided).

Table 3. The result of calculating Kendall's Tau_b on the correlation between "Activity cost" (C_i) and "Financial management" (M_i).

The result of the coefficient is τ = 0.779 and p = 0.000, the p-value is less than the level of significance, $p < 0.05$, therefore the coefficient is significant, and consequently there is a high correlation; demonstrating a notable relationship between the variables for activity cost and financial management.

			Activity cost	Optimization of Financial Resources
Kendall's tau_b	Activity cost	Correlation coefficient	1.000	.655 **
		Sig. (bilateral)	.	.000
		N	141	141
	Optimization of Financial Resources	Correlation coefficient	.655 **	1.000
		Sig. (bilateral)	.000	.
		N	141	141

**. The correlation is significant at the 0.01 level (two-sided).

Table 4. The result of calculating Kendall's Tau_b on the correlation between "Activity cost" (C_i) and "Optimization of Financial Resources" (D_{4i}).

The result of the coefficient is τ = 0.655 and p = 0.000, the p-value or $p < 0.05$, therefore the coefficient is significant, and thus there is a moderate correlation; demonstrating a considerable relationship between the cost variables for activities and optimization of financial resources.

			Activity cost	Goal Achievement
Kendall's Tau_b	Activity cost	Correlation coefficient	1.000	.480 **
		Sig. (bilateral)	.	.000
		N	141	141
	Goal Achievement	Correlation coefficient	.480 **	1.000
		Sig. (bilateral)	.000	.
		N	141	141

**. The correlation is significant at the 0.01 level (two-sided).

Table 5. The result of calculating Kendall's Tau_b on the correlation between "Activity cost" (C_i) and "Goal achievement" (D_{5i}).

The result of the coefficient is τ = 0.480 and p = 0.000, the p-value is less than the level of significance, $p < 0.05$, therefore the coefficient is significant, so there is a moderate correlation; demonstrating a considerable relationship between the variables activity cost and achievement of objectives.

			Activity cost	Greater Profitability
Kendall's tau_b	Activity cost	Correlation coefficient	1.000	.462 **
		Sig. (bilateral)	.	.000
		N	141	141
	Greater Profitability	Correlation coefficient	.462 **	1.000
		Sig. (bilateral)	.000	.
		N	141	141

**. The correlation is significant at the 0.01 level (two-sided).

Table 6. The result of calculating Kendall's Tau_b on the correlation between "Activity cost" (C_i) and "Greater profitability" (D_{6i}).

The result of the coefficient is τ = 0.462 and p = 0.000, the p-value is less than the level of significance, $p < 0.01$, therefore the coefficient is significant, and consequently there is a moderate correlation; demonstrating a considerable relationship between the variables activity cost and greater profitability.

Additionally, it was obtained from Equations 17 and 18 that the Activity Cost $\mathcal{S}(T_C, I_C, F_C) = 0.5063$, and $\mathcal{S}(T_M, I_M, F_M) = 0.5749$.

Conclusion

Plithogenic statistics is a new generalization of statistical theory since it generalizes both, Neutrosophic statistics and Interval statistics. In this work, we use the benefits of this new mathematical tool in the study of the statistical relationships between pairs of variables that are important to measure because they have to do with the behavior of profitability in the production and marketing of beef and milk in the town of Junín in Peru. For this purpose, a survey was applied to 141 randomly selected ranchers to study the relationship between the pairs Activity cost-Financial management, Activity cost-Optimization of financial resources, Activity cost-Goal achievement, Activity cost-Greater profitability. The advantage of using this tool is that respondents, who are not mathematics specialists, only had to respond on an easy-to-understand linguistic scale. In turn, the association of these values with neutrosophic numbers allowed us to capture the indeterminacy and uncertainty intrinsic to each opinion that is expressed. The results were that between each pair of variables studied above, there is a significant positive correlation from moderate to strong. Therefore, it is necessary to improve each of them to improve the others. Furthermore, the different values of the variables studied were shown to be in a situation of "more or less" to "slightly favorable", therefore, all of them are feasible to be improved.

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Optimization of triangular neutrosophic based economic order quantity model under preservation technology and power demand with shortages

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Abstract: The primary objective of this article is to develop a mathematical model and determine the optimal policies of an inventory system involving power demand and controlled deterioration through preservation technology. This model comes in handy in a power demand-oriented inventory system with demand high at the end of the period. The model incorporates backlogged shortages and linear holding cost. The triangular neutrosophic numbers (TNN's) are used for a nuanced representation of uncertain and imprecise inventory-related expenses. An efficient algorithm is constructed to minimize the total cost, and obtain optimal positive inventory time, optimum cycle time and minimum preservation technology investment. Few numerical examples are used to illustrate and validate the model. The comparative study conducted between models with and without preservation technology investment reveals a significant reduction in total inventory costs facilitated by the preservation facility. Also, the numerical results obtained in crisp and neutrosophic environment are compared. Specific previously obtained results are discussed to illustrate the theoretical findings. Sensitivity analysis of the model provides managerial insights replicating reality.

Keywords: economic order quantity; power demand; deteriorating items; complete backlogging; preservation technology; triangular neutrosophic number

1. Introduction

Inventory systems in the modern days face a huge challenge due to uncertain conditions and stiff competition in the product markets. This calls for new strategies or technologies to sustain in the global scenario. Deterioration of items is a common phenomenon in any inventory system. Many consequences arise due to unforeseen deterioration or breakage of items in stock. Unplanned deterioration often leads to unexpected costs for replacements, repairs, or disposal of obsolete items. Hence, stock preserving policies could be considered to reduce these costs and also avoid upscaling of shortages due to unexpected deterioration. The impact of deterioration can be significantly reduced by implementing preservation methods. A strategy is designed to minimize the cost of deterioration while making preservation investments. Moreover, stock preserving policies play a crucial role in reducing the financial burden associated with inventory maintenance.

Demand is one of the factors that influences the working mechanism of inventory management. The demand for most products is inherently time-dependent, influenced by critical factors such as freshness, seasonality, and the introduction of new products. Power demand represents a specific form of time-dependent demand. The power pattern of demand can be visualized in situations where the demand is high either at the beginning or end of a cycle. Inventory systems with variable demand like power demand are likely to face shortages in supply. In such cases, to retain the goodwill the demand is backlogged. This approach helps businesses retain customer goodwill by acknowledging demand during shortages, preventing immediate customer dissatisfaction. It allows organizations to manage variable demand effectively while maintaining positive customer relationships.

In general, the costs associated with the inventory are uncertain. Neutrosophic numbers are utilized to account for the imprecision in systems. Due to the uncertainty of many parameters in real-life scenarios, neutrosophic numbers are invaluable for mitigating uncertainty.

This article attempts to define new strategies by considering the implementation of preservation technology in an inventory system with power demand. The proposed model is particularly suited for items that align with the power pattern of demand. Consequently, the inventory model used in this study can be beneficial for products like: (i) Fresh vegetables, Bakery items, Milk-based products, or Seafoods, which experience higher demand at the beginning of the inventory period compared to the end, since fresh or new food products are preferred by the customers. (ii) Marginally discounted products including frozen meat and Ice creams which experience higher demand towards the end of the expiry period and also, newly introduced products which are assessed for their performance at the beginning and purchased towards the end of the period when the performance is promising. (iii) Clothing and apparel, Consumer electronics, Exercise equipment, Stationery and office supplies, which maintain a nearly constant demand throughout the inventory cycle. With the implementation of a well-suited preservation strategy, these items experience a significantly prolonged shelf life.

The remaining part of the article is structured in the following manner: Section 2 addresses the literature relevant to the current investigation and the study's contributions. As for Section 3, it presents the concept of neutrosophic numbers and their de-neutrosophication technique. Section 4 contains nomenclature and assumptions. The model is formulated and developed for more discussion under crisp and neutrosophic environment in Section 5. Section 6 presents iterative algorithms for determining the optimal solution. In Section 7, some particular inventory models derived from proposed model. In Section 8, a sensitivity analysis of a few system parameters is offered along with a numerical demonstration for testing the model. In Section 9, conclusions are provided.

2. Literature Review

In literature, researchers have examined various inventory models focusing on the deterioration of items. Whitin [1] was the first to introduce the idea of deterioration while modelling the inventory system. Ghare & Schrader [2] developed a mathematical model considering constant deterioration of items in stock. Sachan [3] revised these models to include shortages in a deteriorating inventory environment. Shah et al. [4] later incorporated object degradation after a certain period. Recently, Hatibaruah and Saha [5] suggested a model for managing items that deteriorate with a two-parameter Weibull distribution.

The effect of investing in preservation technologies on an inventory system for deteriorating items has been investigated by several researchers. Selected works of preservation-related inventory models shown in table. 1. Hsu et al. [6] initially proposed the preservation technology principle. With the preservation technologies, they established an inventory model based on constant demand. Later, Dye and Hsieh [7] developed a model of economic production quantity (EPQ) based on preservation-based approach, considering time-reliant demand. He and Huang [8] suggested an inventory policy for degrading items accounting for the rate of deterioration which is negatively exponentially proportional to the amount spent on preservation technologies.

Table 1: Selected preservation-related inventory models from 2010

Authors	Demand type	Deterioration	Preservation Technology	Permitted Shortages	Backlog's nature	Cost environment	Objective function
Hsu et al. (2010) [6]	Constant	✓	✓	✓	Full	Crisp	Maximize profit
Dye & Hsieh (2012) [7]	Constant	✓	✓	✓	Partial	Crisp	Maximize profit
He & Huang (2013) [8]	Price- reliant	✓	✓	×	-	Crisp	Maximize profit
Singh & Sharma (2013) [9]	Ramp-Type	✓	✓	✓	Partial	Crisp	Minimize cost
Zhang et al. (2015) [10]	Price-reliant	✓	✓	×	-	Crisp	Maximize profit
Mishra et al. (2017) [11]	Price and stock reliant	✓	✓	✓	Partial & Full	Crisp	Maximize profit
Li et al. (2019) [12]	Price-reliant	✓	✓	✓	Partial	Crisp	Maximize profit
Das et al. (2020) [13]	Price- reliant	✓	✓	✓	Partial	Crisp	Maximize profit
Khanna et al. (2020) [14]	Stock reliant	✓	✓	×	-	Crisp	Minimize cost
Bhawaria and Rathore (2021) [15]	Price and stock reliant	✓	✓	✓	Partial & Full	Crisp	Minimize cost
Mahapatra et al. (2022) [27]	Uncertain with promotional effort	✓	✓	✓	Full	Crisp & fuzzy	Minimize cost
Mohanta et al. (2023) [35]	Selling price, promotional effort, downstream trade credit	✓	✓	×	-	Triangular neutrosophic numbers	Maximize profit
This Paper	Time-reliant power demand pattern	✓	✓	✓	Full	crisp & Triangular neutrosophic numbers	Minimize cost

Singh and Sharma [9] introduced a preservation technology based two stage trade credit-financing model. Zhang et al. [10] suggested a supply chain model incorporating preservation technologies where demand is reliant on stock. Mishra et al. [11] created a preservation inventory model under shortages, considering demand to be dependent on both cost and stock levels. Price reliant inventory models with preservation strategy have been proposed by Li et al. [12] & Das et al. [13]. A stock-dependent demand inventory model based on preservation technologies was developed by Khanna et al. [14]. Bhawaria and Rathore [15] presented a controllable deteriorating inventory model with Hybrid-Type demand.

In the literature, few inventory models have been studied by considering deteriorating items with power pattern of demand. Naddor [16] was the first to identify and formulate this pattern of demand. Datta and Pal [17] investigated the power demand inventory model with a varying deterioration. Lee and Wu [18] accounted for power demand and shortages in their model. Dye [19] expanded the model to incorporate backlogging in relation to time spent. Later, Rajeswari and Vanjikkodi [20] analyzed an inventory model with power demand when deterioration is constant. San-José et al. [21] introduced an inventory system that focuses on maximizing the return on inventory investment in the context of time-reliant power demand. San-José et al. [22] recently devised a sustainable inventory system for a product with demand exhibiting a power pattern over time, wherein shortages are entirely backlogged.

Several researchers have developed their inventory models by considering the complete backlog of shortages. Posner and Yansouni [23] drew insight from the impatience of customers and associated it with backorders. Abad [24] initiated the thought that the part of backlogged demand can be expressed as a function of waiting time. Valliathal and Uthayakumar [25] presented a deteriorating stocking model with two-warehouse and partial, fully backlog of shortages. Mashud [26] introduced an economic order quantity (EOQ) model that considers deterioration, price and stock depend demand, incorporating a complete backlog of shortages. Mahapatra et al. [27] have recently put forward an inventory model dealing with uncertain demand in the presence of complete backlog for shortages.

A few inventory models have been developed in the literature by treating the cost parameters as triangular neutrosophic numbers (TNNs). The neutrosophic theory was first developed by Smarandache [28]. It efficiently expresses uncertain, contradictory, and incomplete information. Classical inventory models rely on crisp values, which cannot accurately reflect the inherent uncertainties and inaccuracies connected to actual inventory systems. Mullai and Broumi [29] proposed an inventory model that treats demand and ordering cost as TNNs to address this. Mullai and Surya [30] proposed a price break EOQ model under a neutrosophic environment. Pal and Chakraborty [31] created a time-discounted triangular neutrosophic-based degrading inventory model. Mondal et al. [32] created Logistic-growth demand-dependent EOQ model with neutrosophic coefficients under trade credit. Sugapriya et al. [33] presented power demand dependent two-warehouse deteriorating inventory model under trapezoidal bipolar neutrosophic environment. Recently, numerous researchers (Bhavani et al. [34], and Mohanta et al. [35]) established inventory models under a triangular neutrosophic environment.

To the best of our knowledge, no researcher has explored the combined impact of preservation technology on a deteriorating inventory model where demand follows a power pattern over time, while considering linear holding costs and neutrosophic cost parameters. The present study aims to fill this research gap by investigating the application of preservation technology to EOQ models with demand following a power pattern under neutrosophic environment.

Unique Contribution of this Study:

The subsequent contributions emphasize the novelty of this study:

1. Customers' demand size during the entire cycle follows power pattern of time.

2. Preservation technology is employed to mitigate the deterioration rate to fulfil customer demand, especially when it peaks towards the end of the scheduled period.
3. Shortages are allowed and that are completely backlogged. The nature of the holding cost is linear function of time.
4. The proposed model calculates the total cost within a neutrosophic environment, considering cost parameters as triangular neutrosophic numbers.
5. We have determined the optimal preservation technology cost, cycle length, and the time of shortage that minimize the total cost per unit time in the proposed model.

3. Preliminaries

Definition 3.1 [28]

Suppose X is the universal set. A "neutrosophic set (NS)" Ũ in X is defined by a truth, hesitation, false membership functions $\psi_{\tilde{U}}, \zeta_{\tilde{U}}, \nu_{\tilde{U}}$ respectively. Here, $\psi_{\tilde{U}}, \zeta_{\tilde{U}}$ and $\nu_{\tilde{U}}$ are real-valued parameters in the interval [0,1]. The NS Ũ can be expressed as $\tilde{U} = \left\{ \left\langle x; \left[\psi_{\tilde{U}}(x), \zeta_{\tilde{U}}(x), \nu_{\tilde{U}}(x) \right] \right\rangle : x \in X \,\&\, \psi_{\tilde{U}}(x), \zeta_{\tilde{U}}(x), \nu_{\tilde{U}}(x) \in]0^{-}, 1^{+}[\right\}$. The sum of the three membership functions is not constrained, allowing for flexibility in the membership functions $0^{-} \le \psi_{\tilde{U}}(x) + \zeta_{\tilde{U}}(x) + \nu_{\tilde{U}}(x) \le 3^{+}$.

Definition 3.2 [36]

If a set Ũ in the universal discourse X satisfies the condition $\tilde{A} = \left\{ \left\langle x; \left[\psi_{\tilde{U}}(x), \zeta_{\tilde{U}}(x), \nu_{\tilde{U}}(x) \right] \right\rangle : x \in X \right\}$, then it is considered to be a "Single-Valued neutrosophic set" of a Single-Valued independent variable x. Here, truth, hesitation, false membership functions are $\psi_{\tilde{U}}(x) : X \to [0,1]$, $\zeta_{\tilde{U}}(x) : X \to [0,1]$, $\nu_{\tilde{U}}(x) : X \to [0,1]$ respectively. These functions are used by the decision maker to represent their degree of belief in the variable. Additionally, $\psi_{\tilde{U}}(x) + \zeta_{\tilde{U}}(x) + \nu_{\tilde{U}}(x)$ is constrained to lie within the interval [0,3].

Definition 3.3:

Suppose we have three variables, ʊ, ʋ, and ʌ, such that $\psi_{\tilde{U}}(\square) = 1, \zeta_{\tilde{U}}(\upsilon) = 1, \nu_{\tilde{U}}(\Lambda) = 1$. In this case, the set Ũ is classified as "neutro-normal".

Definition 3.4:

The set Ũ is considered "neutro-convex" if the following criteria are met

i. $\psi_{\tilde{U}}(\omega p + (1-\omega) q) \ge \min(\psi_{\tilde{U}}(p), \psi_{\tilde{U}}(q))$

ii. $\varsigma_{\tilde{U}}(\omega p + (1-\omega) q) \ge \min(\varsigma_{\tilde{U}}(p), \varsigma_{\tilde{U}}(q))$

iii. $\nu_{\tilde{U}}(\omega p + (1-\omega) q) \ge \min(\nu_{\tilde{U}}(p), \nu_{\tilde{U}}(q))$

Here, $\omega \in [0,1]$ while the variables p and q are real numbers.

Definition 3.5 [37]

A "triangular single valued neutrosophic number" (Ũ) can be expressed as Ũ=<(q_1,q_2,q_3:ψ),(r_1,r_2,r_3:ς),(s_1,s_2,s_3:ν)>. Here, $\psi_{\tilde{U}} : R \to [0,1]$, $\zeta_{\tilde{U}} : R \to [0,1]$, $\nu_{\tilde{U}} : R \to [0,1]$ are truth, hesitation, and false membership function, respectively, that are defined as follows:

$$\psi_{\tilde{U}}(x) = \begin{cases} \dfrac{x - q_1}{q_2 - q_1}, & \text{for } q_1 \le x < q_2 \\ 1, & \text{for } x = q \\ \dfrac{q_3 - x}{q_3 - q_2}, & \text{for } q_2 < x \le q_2 \\ 0, & \text{otherwise} \end{cases}$$

$$\varsigma_{\tilde{U}}(x)=\begin{cases}\dfrac{x-q_1}{q_2-q_1}, & \text{for } q_1\le x<q_2\\ 0, & \text{for } x=q\\ \dfrac{q_3-x}{q_3-q_2}, & \text{for } q_2<x\le q_2\\ 1, & \text{otherwise}\end{cases}$$

$$\nu_{\tilde{U}}(x)=\begin{cases}\dfrac{x-q_1}{q_2-q_1}, & \text{for } q_1\le x<q_2\\ 0, & \text{for } x=q\\ \dfrac{q_3-x}{q_3-q_2}, & \text{for } q_2<x\le q_2\\ 1, & \text{otherwise}\end{cases}$$

Definition 3.6: De-neutrosophic Technique

The removal of area approach [37] has been utilized in this model to compute the de-neutrosophic value for the triangular single-valued neutrosophic number $\tilde{U}$=<$(q_1,q_2,q_3:\psi)$, $(r_1,r_2,r_3:\varsigma)$, $(s_1,s_2,s_3:\nu)$>. The resulting de-neutrosophic value of $\tilde{U}$ is expressed as,

$$D(\tilde{U})=\frac{1}{12}(q_1+2q_2+q_3+r_1+2r_2+r_3+s_1+2s_2+s_3) \quad (1)$$

4. Nomenclature and assumptions

4.1 Nomenclature

Parameters

- π_o – The cost of each order placed.
- π_p – cost per unit of purchase.
- π_b – Backordered cost of each unit per short-time unit.
- π_d – Deteriorating cost of each unit.
- y_0 – The rate of deterioration.

Decision Variable

- t_1 – The instant that the level of inventory is zero, $t_1 \ge 0$.
- T –The duration of the cycle, $(T=t_1+t_2)$.
- ξ – Investing in preservation technologies per unit of time.

Other variables

- Q – Order volume throughout a cycle of length T, (Q=MI+MB).

Functions

- t_2 – a period in the cycle time where shortages are permitted, $t_2 \ge 0$.
- h(t) – (h+bt), The cost per unit and per time unit for keeping inventory.
- MI – The highest amount of inventory during [0, T].
- MB – The highest number of units backordered during a stockout period.
- $I_1(t)$ – At time t, the amount of positive inventory, $0 \le t \le t_1$.
- $I_2(t)$ – At time t, the amount of negative inventory, $t_1 \le t \le T$.
- TC – The overall cost per unit of time.

4.2 Assumptions

- The demand rate is formulated as $D(t)=\dfrac{dt^{\frac{1}{\eta}-1}}{\eta T^{\frac{1}{\eta}-1}}$ at any time t, where T is the planning horizon, η can be any positive number, and d is a positive constant. In this expression, when $\eta > 1$, most of the demand occurs at the beginning; when $\eta=1$, demand is constant; when $\eta < 1$, most of the demand occurs at end shown in figure 1.

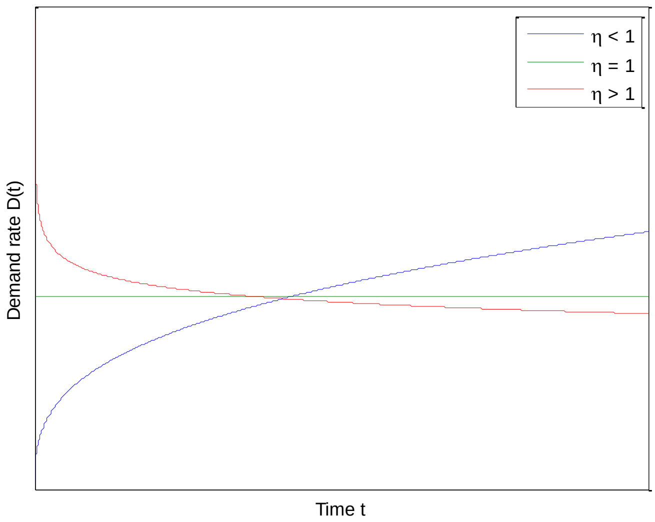

Figure 1. Demand depiction

- The deterioration rate y_0 is constant, $0 < y_0 << 1$, this can be controlled by investing in preservation strategy $y(\xi) = y_0 e^{-m\xi}$ which satisfies the condition $\frac{dy(\xi)}{d\xi} < 0$, $\frac{d^2 y(\xi)}{d\xi^2} > 0$ and $y(0)=y_0$, where m is the investment's sensitivity parameter. $0 < m < 1$.
- It is considered that the time-dependent holding cost, $h(t)=h+bt$.
- A single kind of item makes up the inventory.
- The replenishment rate is considered to be infinite.
- There is no limit to the planning horizon.
- Lead time for delivery is zero.
- The shortage is permitted, and it is fully backordered.

5. Mathematical Model

Figure 2 shows the level of available inventory at any given time.

Inventory level prior to the shortage

Inventory level between $[0, t_1]$ is depends on demand and deterioration. The differential equation can be used to depict the inventory amount during $[0, t_1]$ is

$$\frac{dI_1(t)}{dt} + y(\xi) I_1(t) = -\frac{d t^{\frac{1}{\eta}-1}}{\eta T^{\frac{1}{\eta}-1}}, \qquad 0 \le t \le t_1 \tag{2}$$

with $I_1(t_1) = 0$ as the boundary condition.

Equation (1)'s solution is provided by

$$I_1(t) = \frac{d}{T^{\frac{1}{\eta}-1}}\left[(1-y(\xi)t)\left(t_1^{\frac{1}{\eta}} - t^{\frac{1}{\eta}}\right) + \frac{y(\xi)}{1+\eta}\left(t_1^{\frac{1+\eta}{\eta}} - t^{\frac{1+\eta}{\eta}}\right)\right], \quad 0 \le t \le t_1 \tag{3}$$

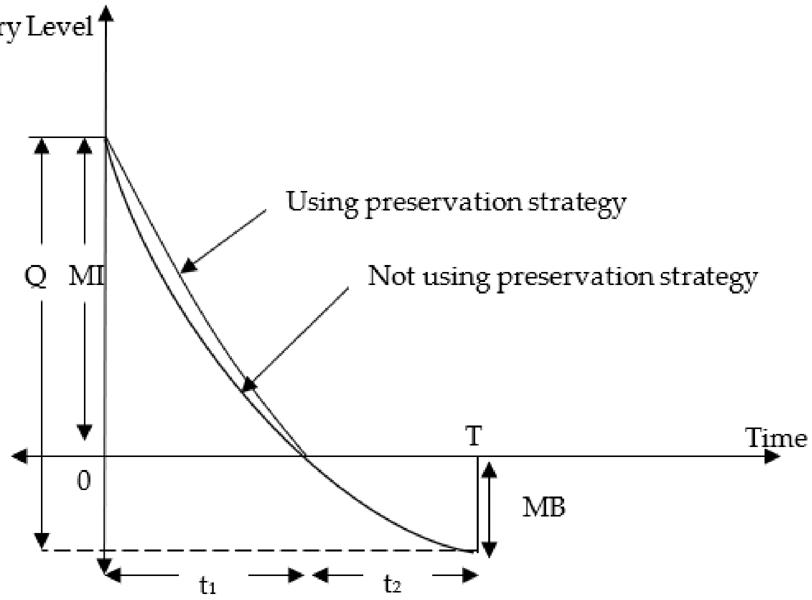

Figure 2. Inventory system depiction

Inventory level throughout stockout period

Inventory level between [t_1, T] is depends on demand. The differential equation can be used to depict the inventory amount during [t_1, T] is

$$\frac{dI_2(t)}{dt} = -\left(\frac{dt^{\frac{1}{\eta}-1}}{\eta T^{\frac{1}{\eta}-1}}\right), \qquad t_1 \le t \le T \tag{4}$$

with $I_2(t_1) = 0$ as a boundary condition.

$$I_2(t) = -\frac{d}{T^{\frac{1}{\eta}-1}}\left(t^{\frac{1}{\eta}} - t_1^{\frac{1}{\eta}}\right), \qquad t_1 \le t \le T \tag{5}$$

The highest amount of inventory during [0, T] is

$$MI = I_1(0) = \frac{d}{T^{\frac{1}{\eta}-1}}\left[t_1^{\frac{1}{\eta}} + \frac{y(\xi)t_1^{\frac{1+\eta}{\eta}}}{1+\eta}\right] \tag{6}$$

The number of backordered units are

$$MB = -I_2(T) = \frac{d}{T^{\frac{1}{\eta}-1}}\left(T^{\frac{1}{\eta}} - t_1^{\frac{1}{\eta}}\right) \tag{7}$$

Hence, the purchase volume during the time span [0, T] is Q = MI + MB.

$$Q = \frac{d}{T^{\frac{1}{\eta}-1}}\left[T^{\frac{1}{\eta}} + \frac{y(\xi)t_1^{\frac{1+\eta}{\eta}}}{1+\eta}\right] \tag{8}$$

Cost components:

The following cost elements make up each replenishment cycle's overall cost.

Ordering cost

$$OC = \pi_o \tag{9}$$

Holding cost

$$HC=\int_0^{t_1} h(t)I_1(t)dt$$

$$HC=\frac{hd}{T^{\frac{1}{\eta}-1}}\left[\frac{t_1^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{y(\xi)t_1^{\frac{1+2\eta}{\eta}}}{2(1+2\eta)}\right]+\frac{bd}{2T^{\frac{1}{\eta}-1}}\left[\frac{t_1^{\frac{1+2\eta}{\eta}}}{1+2\eta}+\frac{y(\xi)t_1^{\frac{1+3\eta}{\eta}}}{3(1+3\eta)}\right] \tag{10}$$

Backordered cost

$$BC=\pi_b\int_{t_1}^{T}\left(-I_2(t)\right)dt$$

$$BC=\frac{\pi_b d}{T^{\frac{1}{\eta}-1}}\left[-Tt_1^{\frac{1}{\eta}}+\frac{\eta T^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{t_1^{\frac{1+\eta}{\eta}}}{1+\eta}\right] \tag{11}$$

Deterioration Cost

$$DC=\pi_d\left\{Q-\int_0^{t_1}\left(\frac{dt^{\frac{(1-\eta)}{\eta}}}{\eta T^{\frac{1}{\eta}-1}}\right)dt-\int_{t_1}^{T}\left(\frac{dt^{\frac{1-\eta}{\eta}}}{\eta T^{\frac{1}{\eta}-1}}\right)dt\right\}$$

$$DC=\pi_d\frac{dy(\xi)t_1^{\frac{1+\eta}{\eta}}}{(1+\eta)T^{\frac{1}{\eta}-1}} \tag{12}$$

Purchase cost

$$PC=\pi_p\times Q$$

$$PC=\frac{\pi_p d}{T^{\frac{1}{\eta}-1}}\left[T^{\frac{1}{\eta}}+\frac{y(\xi)t_1^{\frac{1+\eta}{\eta}}}{1+\eta}\right] \tag{13}$$

Preservation technology cost

$$PTC=\xi T$$

Therefore, the overall cost per unit of time is

$$TC\left(t_1,T,\xi\right)=\frac{1}{T}\left[OC+HC+BC+DC+PC+PTC\right]$$

$$TC\left(t_1,T,\xi\right)=\frac{1}{T}\left\{\pi_o+\frac{hd}{T^{\frac{1}{\eta}-1}}\left[\frac{t_1^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{y(\xi)t_1^{\frac{1+2\eta}{\eta}}}{2(1+2\eta)}\right]+\frac{bd}{2T^{\frac{1}{\eta}-1}}\left[\frac{t_1^{\frac{1+2\eta}{\eta}}}{1+2\eta}+\frac{y(\xi)t_1^{\frac{1+3\eta}{\eta}}}{3(1+3\eta)}\right]+\frac{\pi_d dy(\xi)t_1^{\frac{1+\eta}{\eta}}}{(1+\eta)T^{\frac{1}{\eta}-1}}\right.$$
$$\left.+\frac{\pi_b d}{T^{\frac{1}{\eta}-1}}\left[-Tt_1^{\frac{1}{\eta}}+\frac{\eta T^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{t_1^{\frac{1+\eta}{\eta}}}{1+\eta}\right]+\frac{\pi_p d}{T^{\frac{1}{\eta}-1}}\left[T^{\frac{1}{\eta}}+\frac{y(\xi)t_1^{\frac{1+\eta}{\eta}}}{1+\eta}\right]+\xi T\right\} \tag{14}$$

5.1 Inventory model under triangular neutrosophic domain

In real markets, cost parameters are often uncertain, to overcome this neutrosophic numbers are used to represent various costs because they have truth, hesitation, and falsity membership functions that can address all types of parameter uncertainties. Specifically, this proposed inventory model utilizes TNNs to represent holding cost (h), purchase cost (π_p), ordering cost (π_o), deterioration cost (π_d), backordered cost (π_b). The format of the TNNs $\tilde{\pi}_{oN},\tilde{\pi}_{pN},\tilde{h}_N,\tilde{\pi}_{dN},\tilde{\pi}_{bN}$ is as follows:

$$\tilde{\pi}_{oN}=\langle(o_{11},o_{12},o_{13}),(o_{21},o_{22},o_{23}),(o_{31},o_{32},o_{33})\rangle$$
$$\tilde{\pi}_{pN}=\langle(p_{11},p_{12},p_{13}),(p_{21},p_{22},p_{23}),(p_{31},p_{32},p_{33})\rangle$$
$$\tilde{h}_N=\langle(h_{11},h_{12},h_{13}),(h_{21},h_{22},h_{21}),(h_{31},h_{32},h_{33})\rangle$$
$$\tilde{\pi}_{dN}=\langle(d_{11},d_{12},d_{13}),(d_{21},d_{22},d_{23}),(d_{31},d_{32},d_{33})\rangle$$
$$\tilde{\pi}_{bN}=\langle(b_{11},b_{12},b_{13}),(b_{21},b_{22},b_{23}),(b_{31},b_{32},b_{33})\rangle$$

By applying the removal area technique (1) to above neutrosophic costs, the resulting de-neutrosophize costs are $D(\tilde{\pi}_{oN}),D(\tilde{\pi}_{pN}),D(\tilde{h}_N),D(\tilde{\pi}_{dN}),D(\tilde{\pi}_{bN})$.

To calculate the total cost within the neutrosophic domain $\tilde{T}C_N$, we can substitute the de-neutrosophic values into equation (14), resulting in:

$$\tilde{T}C_N=\frac{1}{T}\left\{D(\tilde{\pi}_{oN})+\frac{D(\tilde{h}_N)d}{T^{\frac{1}{\eta}-1}}\left[\frac{t_1^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{y(\xi)t_1^{\frac{1+2\eta}{\eta}}}{2(1+2\eta)}\right]+\frac{bd}{2T^{\frac{1}{n}-1}}\left[\frac{t_1^{\frac{1+2\eta}{\eta}}}{1+2\eta}+\frac{y(\xi)t_1^{\frac{1+3\eta}{\eta}}}{3(1+3\eta)}\right]+\frac{D(\tilde{\pi}_{dN})dy(\xi)t_1^{\frac{1+\eta}{\eta}}}{(1+\eta)T^{\frac{1}{\eta}-1}}\right.$$
$$\left.+\frac{D(\tilde{\pi}_{bN})d}{T^{\frac{1}{\eta}-1}}\left[-Tt_1^{\frac{1}{\eta}}+\frac{\eta T^{\frac{1+\eta}{\eta}}}{1+n}+\frac{t_1^{\frac{1+\eta}{\eta}}}{1+\eta}\right]+\frac{D(\tilde{\pi}_{pN})d}{T^{\frac{1}{\eta}-1}}\left[T^{\frac{1}{\eta}}+\frac{y(\xi)t_1^{\frac{1+\eta}{\eta}}}{1+\eta}\right]+\xi T\right\} \quad (15)$$

6. Optimal Solution

In this section the necessary and sufficiency conditions for total cost's optimality are derived.

Necessary conditions:

By solving the equations $\frac{\partial TC}{\partial t_1}=0$, $\frac{\partial TC}{\partial T}=0$ and $\frac{\partial TC}{\partial \xi}=0$, the optimum value of t_1^*, T^*, ξ^*, and thereby the minimum average total cost per unit of time (TC^*), Q^* can be determined.

where,

$$\frac{\partial TC}{\partial t_1}=\frac{1}{\eta T^{\frac{1}{\eta}}}\left\{hd\left[t_1^{\frac{1}{\eta}}+\frac{1}{2}y_0e^{-m\xi}t_1^{\frac{1+\eta}{\eta}}\right]+\frac{bd}{2}\left[t_1^{\frac{1+\eta}{\eta}}+\frac{1}{3}y_0e^{-m\xi}t_1^{\frac{1+2\eta}{\eta}}\right]+\pi_d dy_0e^{-m\xi}t_1^{\frac{1}{\eta}}\right.$$
$$\left.+\pi_b d\left[-Tt_1^{\frac{1-\eta}{\eta}}+t_1^{\frac{1}{\eta}}\right]+\pi_p d\left[y_0e^{-m\xi}t_1^{\frac{1}{\eta}}\right]\right\}=0 \quad (16)$$

$$\frac{\partial TC}{\partial T}=-\frac{1}{\eta T^{\frac{1}{\eta}+1}}\left\{\pi_o T^{\frac{1}{\eta}-1}+hd\left[\frac{t_1^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{y(\xi)t_1^{\frac{1+2\eta}{\eta}}}{2(1+2\eta)}\right]+\frac{bd}{2}\left[\frac{t_1^{\frac{1+2\eta}{\eta}}}{1+2\eta}+\frac{y(\xi)t_1^{\frac{1+3\eta}{\eta}}}{3(1+3\eta)}\right]+\frac{\pi_d dy(\xi)t_1^{\frac{1+\eta}{\eta}}}{(1+\eta)}\right.$$
$$\left.+\pi_b d\left[-Tt_1^{\frac{1}{\eta}}+\frac{\eta T^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{t_1^{\frac{1+\eta}{\eta}}}{1+\eta}\right]+\pi_p d\left[T^{\frac{1}{\eta}}+\frac{1}{1+\eta}\left(y(\xi)t_1^{\frac{1+\eta}{\eta}}\right)\right]+\xi T^{\frac{1}{\eta}}\right\}$$

$$+\frac{1}{T^{\frac{1}{\eta}}}\left\{\frac{(1-\eta)\pi_o T^{\frac{1}{\eta}-2}}{\eta}+\pi_b d\left[T^{\frac{1}{\eta}}-t_1^{\frac{1}{\eta}}\right]+\frac{\pi_p d T^{\frac{1}{\eta}-1}}{\eta}+\frac{\xi T^{\frac{1}{\eta}-1}}{\eta}\right\}=0 \quad (17)$$

$$\frac{\partial TC}{\partial \xi}=1-\frac{m y_0 e^{-m\xi}}{T^{\frac{1}{\eta}}}\left\{\frac{hdt_1^{\frac{1+2\eta}{\eta}}}{2(1+2\eta)}+\frac{bdt_1^{\frac{1+3\eta}{\eta}}}{6(1+3\eta)}+\frac{\pi_d dt_1^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{\pi_p d}{1+\eta}t_1^{\frac{1+\eta}{\eta}}\right\}=0 \quad (18)$$

Sufficiency conditions:

A sufficient condition for t_1^*, T^*, ξ^* to be minimum point of TC is that the Hessian matrix $H(t_1,T,\xi)=\begin{pmatrix}\frac{\partial^2 TC}{\partial t_1^2} & \frac{\partial^2 TC}{\partial t_1\partial T} & \frac{\partial^2 TC}{\partial t_1\partial \xi}\\ \frac{\partial^2 TC}{\partial T\partial t_1} & \frac{\partial^2 TC}{\partial T^2} & \frac{\partial^2 TC}{\partial T\partial \xi}\\ \frac{\partial^2 TC}{\partial \xi\partial t_1} & \frac{\partial^2 TC}{\partial \xi\partial T} & \frac{\partial^2 TC}{\partial \xi^2}\end{pmatrix}$ evaluated at t_1^*, T^* and ξ^*, is positive definite.

The Hessian matrix H (t_1, T, ξ) is said to be positive definite if the signs of the principal minor determinants of H (t_1, T, ξ) are positive. (i.e., D_1, D_2, D_3 >0).
where

$$D_1=\frac{\partial^2 TC}{\partial t_1^2},\quad D_2=\begin{vmatrix}\frac{\partial^2 TC}{\partial t_1^2} & \frac{\partial^2 TC}{\partial t_1\partial T}\\ \frac{\partial^2 TC}{\partial T\partial t_1} & \frac{\partial^2 TC}{\partial T^2}\end{vmatrix}\text{ and }D_3=\begin{vmatrix}\frac{\partial^2 TC}{\partial t_1^2} & \frac{\partial^2 TC}{\partial t_1\partial T} & \frac{\partial^2 TC}{\partial t_1\partial \xi}\\ \frac{\partial^2 TC}{\partial T\partial t_1} & \frac{\partial^2 TC}{\partial T^2} & \frac{\partial^2 TC}{\partial T\partial \xi}\\ \frac{\partial^2 TC}{\partial \xi\partial t_1} & \frac{\partial^2 TC}{\partial \xi\partial T} & \frac{\partial^2 TC}{\partial \xi^2}\end{vmatrix}$$

$$\frac{\partial^2 TC}{\partial t_1^2}=\frac{1}{\eta^2 T^{\frac{1}{\eta}}}\left\{hd\left[t_1^{\frac{1-\eta}{\eta}}+(1+\eta)\frac{y_0}{2}e^{-m\xi}t_1^{\frac{1}{\eta}}\right]+\frac{bd}{2}\left[(1+\eta)t_1^{\frac{1}{\eta}}+\frac{(1+2\eta)}{3}y_0e^{-m\xi}t_1^{\frac{1+\eta}{\eta}}\right]\right.$$

$$\left.+\pi_b d\left[-(1-\eta)Tt_1^{\frac{1-2\eta}{\eta}}+t_1^{\frac{1-\eta}{\eta}}\right]+\pi_d d y_0 e^{-m\xi}t_1^{\frac{1-\eta}{\eta}}+\pi_p d\left[y_0e^{-m\xi}t_1^{\frac{1-\eta}{\eta}}\right]\right\},$$

$$\frac{\partial^2 TC}{\partial T^2}=\frac{(1+\eta)}{\eta^2 T^{\frac{1}{\eta}+2}}\left\{\pi_o T^{\frac{1}{\eta}}+hd\left[\frac{t_1^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{y(\xi)t_1^{\frac{1+2\eta}{\eta}}}{2(1+2\eta)}\right]+\frac{bd}{2}\left[\frac{t_1^{\frac{1+2\eta}{\eta}}}{1+2\eta}+\frac{y(\xi)t_1^{\frac{1+3\eta}{\eta}}}{3(1+3\eta)}\right]+\frac{\pi_d dy(\xi)t_1^{\frac{1+\eta}{\eta}}}{(1+\eta)}\right.$$

$$\left.+\pi_b d\left[-Tt_1^{\frac{1}{\eta}}+\frac{\eta T^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{t_1^{\frac{1+\eta}{\eta}}}{1+\eta}\right]+\pi_p d\left[T^{\frac{1}{\eta}}+\frac{1}{1+\eta}\left(y(\xi)t_1^{\frac{1+\eta}{\eta}}\right)\right]+\xi T^{\frac{1}{\eta}}\right\}$$

$$-\frac{2}{\eta T^{\frac{1}{\eta}+1}}\left\{\frac{(1-\eta)\pi_o T^{\frac{1}{\eta}-2}}{\eta}+\pi_b d\left[T^{\frac{1}{\eta}}-t_1^{\frac{1}{\eta}}\right]+\frac{\pi_p d T^{\frac{1}{\eta}-1}}{\eta}+\frac{\xi T^{\frac{1}{\eta}-1}}{\eta}\right\}$$

$$+\frac{1}{\eta^2 T^{\frac{1}{\eta}}}\left\{(1-\eta)(1-2\eta)\pi_o T^{\frac{1}{\eta}-3}+\pi_b d\eta T^{\frac{1}{\eta}-1}+(1-\eta)\pi_p d T^{\frac{1}{\eta}-2}+(1-\eta)\xi T^{\frac{1}{\eta}-2}\right\}$$

$$\frac{\partial^2 TC}{\partial \xi^2}=\frac{m^2 y_0 e^{-m\xi}}{T^{\frac{1}{\eta}}}\left\{\frac{hdt_1^{\frac{1+2\eta}{\eta}}}{2(1+2\eta)}+\frac{bdt_1^{\frac{1+3\eta}{\eta}}}{6(1+3\eta)}+\frac{\pi_d dt_1^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{\pi_p d}{1+\eta}t_1^{\frac{1+\eta}{\eta}}\right\}$$

$$\frac{\partial TC}{\partial t_1 \partial T}=\frac{\partial TC}{\partial T \partial t_1}=-\frac{1}{\eta^2 T^{\frac{1}{\eta}+1}}\left\{hd\left[t_1^{\frac{1}{\eta}}+\frac{1}{2}y_0 e^{-m\xi}t_1^{\frac{1+\eta}{\eta}}\right]+\frac{bd}{2}\left[t_1^{\frac{1+\eta}{\eta}}+\frac{1}{3}y_0 e^{-m\xi}t_1^{\frac{1+2\eta}{\eta}}\right]+\pi_d d y_0 e^{-m\xi}t_1^{\frac{1}{\eta}}\right.$$

$$\left.+\pi_b d\left[-Tt_1^{\frac{1-\eta}{\eta}}+t_1^{\frac{1}{\eta}}\right]+\pi_p d\left[y_0 e^{-m\xi}t_1^{\frac{1}{\eta}}\right]\right\}-\frac{\pi_b dt_1^{\frac{1-\eta}{\eta}}}{\eta T^{\frac{1}{\eta}}}$$

$$\frac{\partial TC}{\partial T \partial \xi}=\frac{\partial TC}{\partial \xi \partial T}=\frac{my_0 e^{-m\xi}}{\eta T^{\frac{1}{\eta}+1}}\left\{\frac{hdt_1^{\frac{1+2\eta}{\eta}}}{2(1+2\eta)}+\frac{bdt_1^{\frac{1+3\eta}{\eta}}}{6(1+3\eta)}+\frac{\pi_d dt_1^{\frac{1+\eta}{\eta}}}{1+\eta}+\frac{\pi_p d}{1+\eta}t_1^{\frac{1+\eta}{\eta}}\right\}$$

$$\frac{\partial TC}{\partial t_1 \partial \xi}=\frac{\partial TC}{\partial \xi \partial t_1}=-\frac{my_0 e^{-m\xi}}{\eta T^{\frac{1}{\eta}}}\left\{\frac{hdt_1^{\frac{1+\eta}{\eta}}}{2}+\frac{bdt_1^{\frac{1+2\eta}{\eta}}}{6}+\pi_d dt_1^{\frac{1}{\eta}}+\pi_p dt_1^{\frac{1}{\eta}}\right\}$$

Algorithm:

Step 1: Initialize the values for d, η, π_o, h, b, π_p, π_d, π_b, δ, y_0 and m.

Step 2: Evaluate TC (t_1, T, ξ)

Step 3: Evaluate $\frac{\partial TC}{\partial t_1}$, $\frac{\partial TC}{\partial T}$ and $\frac{\partial TC}{\partial \xi}$.

Step 4: Solve simultaneous equations $\frac{\partial TC}{\partial t_1}=0$, $\frac{\partial TC}{\partial T}=0$ and $\frac{\partial TC}{\partial \xi}=0$.

Step 5: Using the results from step 4, check the sufficiency conditions.

Step 6: If the computed value in step 5 is greater than zero, then move on to step 7 else, move on to step 4.

Step 7: Evaluate TC^* and Q^* using equations (14) and (8) respectively.

Step 8: Stop.

7. Particular cases

Next, we demonstrate how the suggested model may be used to get specific situations for several inventory models developed by other authors.

1. When t_1 = T, $\xi \to 0$, η=1, and $b \to 0$, then the model is reduced to an EOQ model with constant demand, constant holding cost and no shortages.
2. When η = 1, indicating a constant demand function, the proposed model simplifies to the one presented by Dye and Hsieh [9], particularly when their model incorporates complete backlogging.
3. If $\xi \to 0$, $b \to 0$, $\pi_d \to 0$, then the reduced system coincides with the model analyzed by Rajeswari and Vanjikkodi [23] when, in their model, complete backlogging is considered.

8. Numerical illustration and sensitivity analysis

8.1 Crisp Environment

A numerical example is presented here to validate the aforementioned theoretical model. Analyzing the outcomes can offer essential information to a decision-maker. Consider the following inventory system parameters for a certain type of cake, which may deteriorate over time.

$d = 100$ kilograms (kg), $\eta = 2$, $\pi_o = \$ 250$/order, $h = \$ 1$ /kg/week, $b = \$ 6$ /kg/week, $\pi_p = \$12$/kg, $\pi_d = \$15$/kg/week, $\pi_b = \$18$/kg/week, $y_0 = 0.1$, $m=0.05$.

The derivatives (16), (17) and (18) are computed. Solving the resulting highly nonlinear equation in MATLAB, yields $t_1^* = 1.0292$ weeks, $T^* = 1.3015$ weeks and investment in preservation technology $\xi^* = \$ 28.867$. Then, the order quantity $Q^* = 131.09$ kg and the overall expense per unit of time is obtained as $TC^* = \$ 1555.6473$ by using (8) and (14) respectively. Total cost function's convexity shown in figure 3 – 5.

Sufficiency condition:

$$D_1 = 1119.44 > 0, \quad D_2 = \begin{vmatrix} 1119.44 & -777.62 \\ -777.62 & 824.79 \end{vmatrix} = 318597.68 > 0,$$

$$D_3 = \begin{vmatrix} 1119.44 & -777.63 & -1.50 \\ -777.63 & 824.79 & 0.3842 \\ -1.50 & 0.3842 & 0.050 \end{vmatrix} = 14805.06 > 0$$

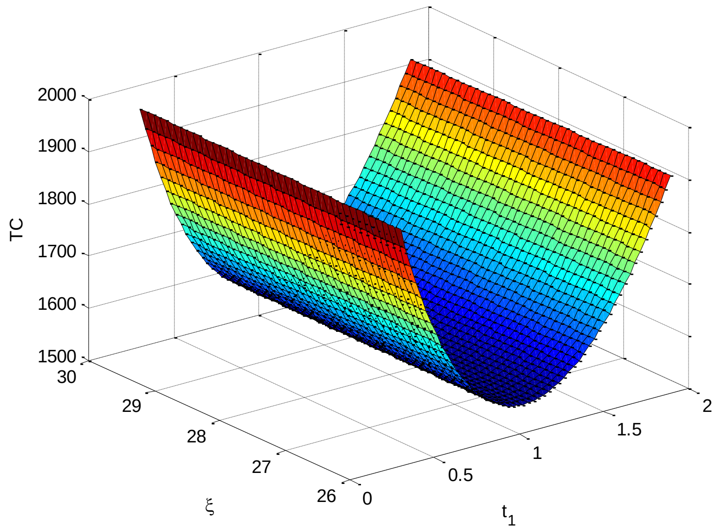

Figure 3. Total cost Vs. ξ and t_1 for fixed T

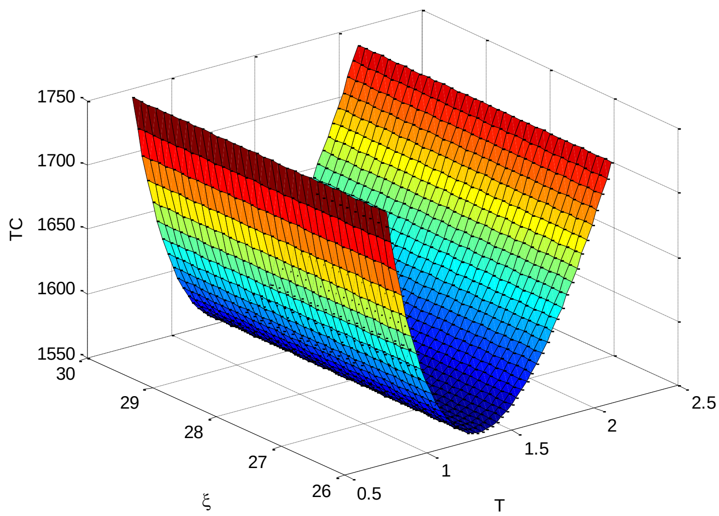

Figure 4. Total cost Vs. ξ and T for fixed t_1

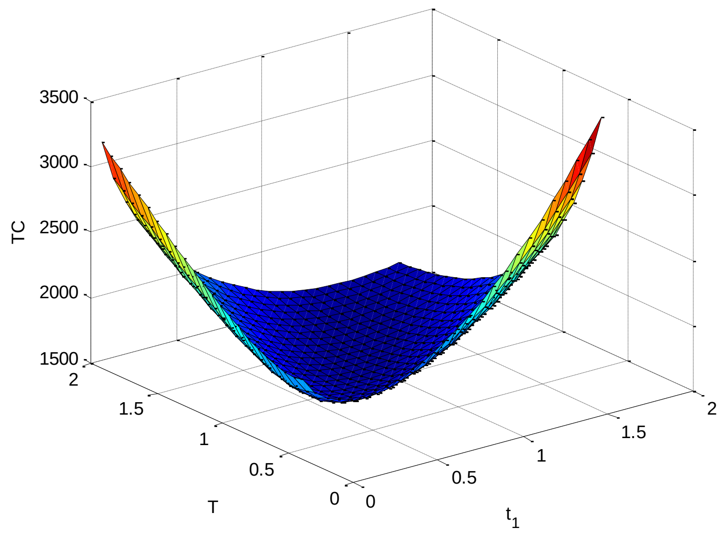

Figure 5. Total cost Vs. t_1 and T for fixed ξ

8.2 Comparitive Study

Cases	With using preservation technology					Without using preservation technology			
	t_1^*(weeks)	**T^*(weeks)**	**ξ ($)**	**Q^* (kg)**	**TC ($)**	**t_1^*(weeks)**	**T^*(weeks)**	**Q^* (kg)**	**TC ($)**
(i) Present model	1.0292	1.3015	28.867	131.09	1555.6	0.86583	1.1744	120.35	1584.9
(ii) Constant demand (η=1)	0.8996	1.1125	32.342	112.05	1615.5	0.75084	1.0031	103.13	1653.9

By comparing the total cost of the preservation technology model with the non-preservation technology model, it becomes evident that preservation technology lowers costs while extending positive inventory time.

8.3 Triangular Neutrosophic Environment

A numerical illustration has been given utilising TNNs to manifest the impact of imprecise cost parameters on the presented inventory system.

$$\tilde{\pi}_{dN} = \langle (180,250,310),(200,260,320),(150,220,290) \rangle$$

$$\tilde{\pi}_{pN} = \langle (8,12,15),(10,13,16),(6,9,11) \rangle$$

$$\tilde{h}_{N} = \langle (0.8,1,1.5),(0.9,1.3,2),(0.6,0.8,1.4) \rangle$$

$$\tilde{\pi}_{dN} = \langle (10,15,20),(12,17,21),(8,13,16) \rangle$$

$$\tilde{\pi}_{bN} = \langle (15,18,21),(16,20,23),(12,16,18) \rangle$$

Using (1) obtain the de-neutrosophic costs and substitute the obtained values in (16), (17) and (18). And solving the resulting simultaneous equations we get, t_1^* = 1.0166 weeks, T^*= 1.2947 weeks and ξ^* = $ 28.084.

From (8) and (15) the order quantity Q^* = **130.42** kg and the overall expense per unit of time is $\tilde{TC}_N^*$ = $ **1525.43**.

The neutrosophic environment has lower optimal inventory costs than a crisp environment.

8.4 Sensitivity Analysis

Sensitivity analysis of the formulated model is done out for various input parameters in Table 2. Sensitivity analysis is performed to measure the impact of each model limiting factors on the model's outcome. It is done by adjusting the parameters' value from -30% to +30%.

Table 2. Variations in parameter 'y_0', 'm', 'h', 'b', 'π_o', 'π_p', 'π_b', 'π_d'.

	% change	t_1^*(weeks)	T^*(weeks)	ξ^* ($)	Q^* (kg)	TC^* ($)	% change in TC^*
y_0	-30	1.0292	1.3015	21.733	131.09	1548.5138	-0.4586
	-20	1.0292	1.3015	24.404	131.09	1551.1844	-0.2869
	-10	1.0292	1.3015	26.76	131.09	1553.5400	-0.1355
	0	1.0292	1.3015	28.867	131.09	1555.6473	0
	+10	1.0292	1.3015	30.773	131.09	1557.5535	+0.1225
	+20	1.0292	1.3015	32.513	131.09	1559.2937	+0.2344
	+30	1.0292	1.3015	34.114	131.09	1560.8945	+0.3373
m	-30	1.0018	1.28	30.095	129.32	1566.2577	+0.6821
	-20	1.0133	1.289	30.024	130.09	1562.2375	+0.4236
	-10	1.0221	1.296	29.544	130.64	1558.7540	+0.1997
	0	1.0292	1.3015	28.867	131.09	1555.6473	0
	+10	1.0349	1.306	28.101	131.45	1552.9498	-0.1734
	+20	1.0397	1.3098	27.304	131.75	1550.5823	-0.3256
	+30	1.0438	1.3129	26.508	132.00	1548.4834	-0.4605
h	-30	1.0579	1.3239	29.477	133.30	1546.7272	-0.5734
	-20	1.0482	1.3163	29.273	132.55	1549.6328	-0.3866
	-10	1.0386	1.3089	29.07	131.82	1552.6073	-0.1954
	0	1.0292	1.3015	28.867	131.09	1555.6473	0
	+10	1.0199	1.2942	28.665	130.36	1558.7528	+0.1996
	+20	1.0107	1.2871	28.463	129.66	1561.9210	+0.4033
	+30	1.0017	1.280	28.261	128.96	1565.1528	+0.6110
b	-30	1.1486	1.4042	31.385	141.28	1543.4880	-0.7816
	-20	1.1033	1.365	30.465	137.39	1546.2535	-0.6039
	-10	1.0639	1.3311	29.631	134.02	1550.4057	-0.3369
	0	1.0292	1.3015	28.867	131.09	1555.6473	0
	+10	0.9983	1.2753	28.161	128.49	1561.6983	+0.3890
	+20	0.9705	1.2518	27.504	126.16	1568.3684	+0.8177
	+30	0.9453	1.2306	26.89	124.07	1575.5177	+1.2773
π_o	-30	0.8996	1.1226	26.203	113.25	1503.9951	-3.3203
	-20	0.9466	1.1868	27.211	119.65	1519.4851	-2.3246
	-10	0.9896	1.2462	28.09	125.57	1536.9220	-1.2037
	0	1.0292	1.3015	28.867	131.09	1555.6473	0
	+10	1.0661	1.3535	29.564	136.27	1575.2259	+1.2586

	+20	1.1006	1.4025	30.195	141.16	1595.3640	+2.5531
	+30	1.1332	1.4491	30.771	145.81	1615.8696	+3.8712
π_p	-30	1.0288	1.3012	26.081	131.20	1192.8696	-23.3200
	-20	1.029	1.3013	27.054	131.16	1313.8390	-15.5439
	-10	1.0291	1.3014	27.981	131.12	1434.7635	-7.7706
	0	1.0292	1.3015	28.867	131.09	1555.6473	0
	+10	1.0293	1.3016	29.715	131.06	1676.4936	+7.7682
	+20	1.0294	1.3017	30.529	131.03	1797.3055	+15.5343
	+30	1.0295	1.3017	31.312	130.99	1918.0858	+23.2982
π_b	-30	0.9892	1.3579	27.218	136.83	1545.2022	-0.6714
	-20	1.0052	1.3349	27.886	134.49	1549.2555	-0.4109
	-10	1.0183	1.3165	28.424	132.61	1552.6855	-0.1904
	0	1.0292	1.3015	28.867	131.09	1555.6473	0
	+10	1.0384	1.2889	29.238	129.81	1558.2484	+0.1672
	+20	1.0462	1.2783	29.553	128.73	1560.5661	+0.3162
	+30	1.053	1.2692	29.825	127.80	1562.6564	+0.4506
π_d	-30	1.0287	1.3011	25.32	131.23	1552.1105	-0.2274
	-20	1.0289	1.3012	26.573	131.17	1553.3602	-0.1470
	-10	1.0291	1.3014	27.753	131.13	1554.5364	-0.0714
	0	1.0292	1.3015	28.867	131.09	1555.6473	0
	+10	1.0293	1.3016	29.922	131.05	1556.6997	+0.0677
	+20	1.0294	1.3017	30.924	131.02	1557.6995	+0.1319
	+30	1.0295	1.3018	31.879	130.99	1558.6517	+0.1931

Graphical representations of the sensitivity of positive inventory time, total cycle time, order quantity, preservation technology investment and total cost to various factors are provided in Figures 6 – 13.

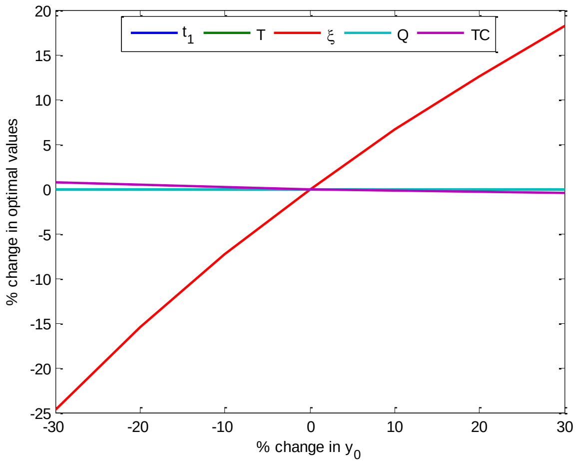

Figure 6. Effect of ‘y_0’ on optimal values

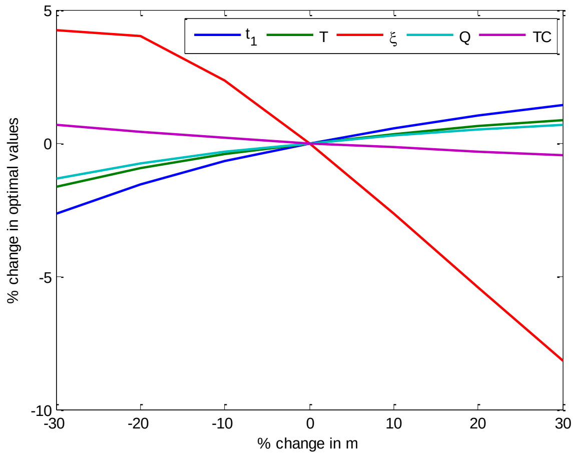

Figure 7. Effect of ‘m’ on optimal values

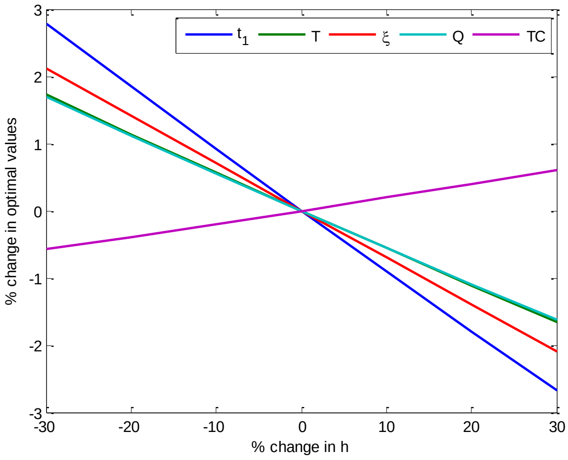

Figure 8. Effect of 'h' on optimal values

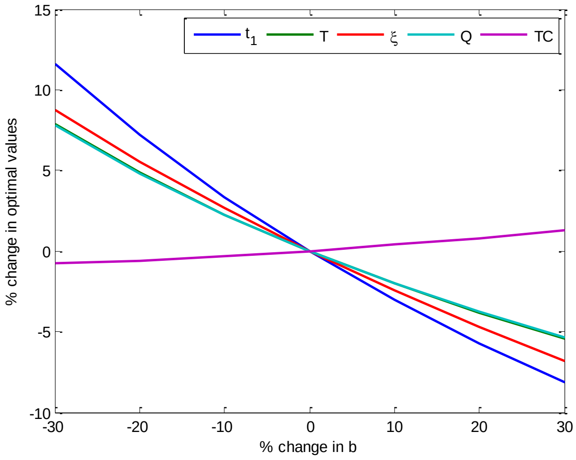

Figure 9. Effect of 'b' on optimal values

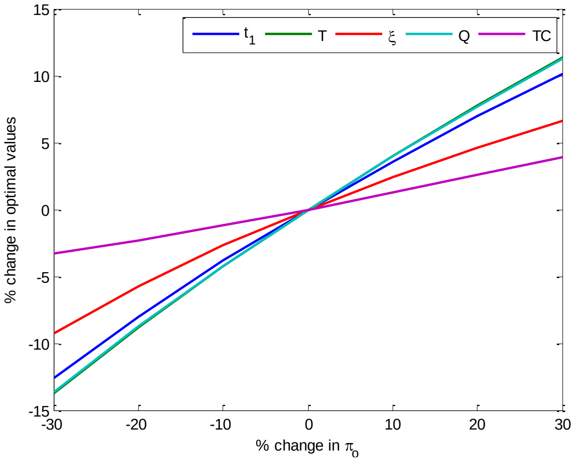

Figure 10. Effect of 'π_o' on optimal values

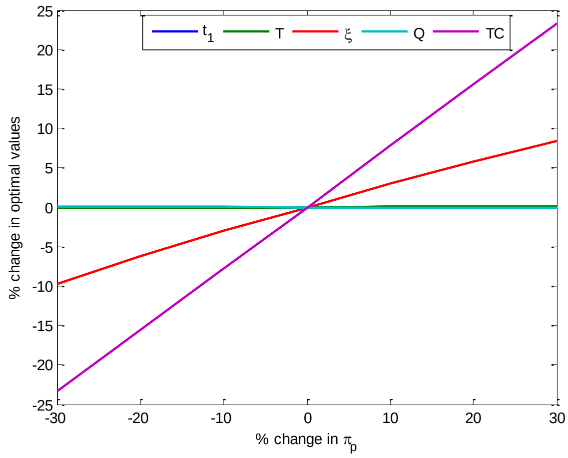

Figure 11. Effect of 'π_p' on optimal values

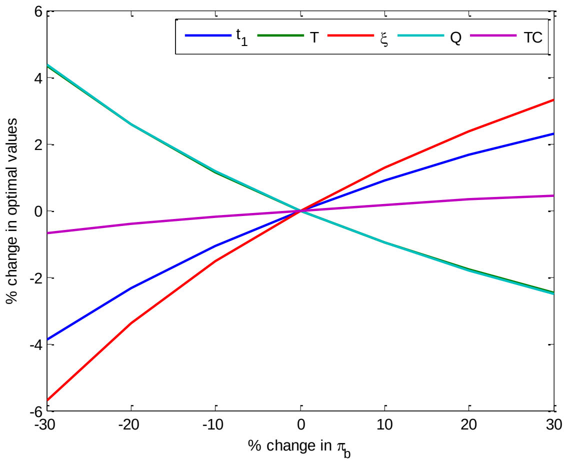

Figure 12. Effect of 'π_b' on optimal values

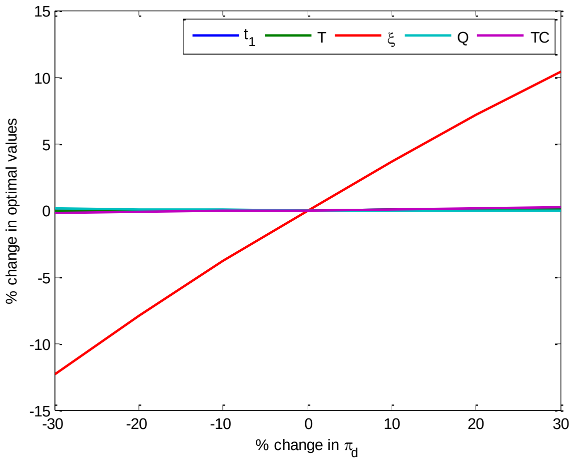

Figure 13. Effect of 'π_d' on optimal values

8.5 Results and discussion:

The following can be seen by carefully examining the table above:

1. As the parameters' values y_0, h, b, π_o, π_p, π_b, π_d increase, the optimal $TC^*(t_1, T, \xi)$ also increases. Conversely, when the corresponding parameter values y_0, h, b, π_o, π_p, π_b, π_d decrease, $TC^*(t_1, T, \xi)$ also decreases. A reduction in the parameter m leads to an increase in $TC^*(t_1, T, \xi)$, whereas the optimal $TC^*(t_1, T, \xi)$ decreases when the parameter m increases.

2. The parameters π_o and π_p have a high degree of sensitivity with regard to changes in $TC^*(t_1, T, \xi)$. While the parameters y_0, m, h, b, π_b and π_d are not as sensitive to changes in $TC^*(t_1, T, \xi)$.
3. As the parameters' values y_0, π_o, π_p, π_b, and π_d increase, the optimal ξ^* also increases. Conversely, when the corresponding parameter values y_0, π_o, π_p, π_b, and π_d decreases, ξ^* also decreases. As the parameters' values m, h, and b increase, the optimal ξ^* decreases, whereas the optimal ξ^* increases when the parameters m, h, and b increase.
4. As the parameters' values m, π_o, π_p, π_d increase, positive inventory time interval (t_1^*) and cycle time (T^*) also increase. Conversely, when the corresponding parameter values m, π_o, π_p, π_d decrease, t_1^*, T^* also decreases. As the parameters h and b increase, the optimal t_1^*, T^* decreases. Conversely, the optimal t_1^*, T^* increases when the parameters h and b decrease. As the parameter π_b increase, the optimal t_1^* increases and T^* decreases. t_1^*, T^* remain constant regardless of any variation in the parameter y_0.
5. As the parameters' values m and π_o increase, optimal Q^* also increases. Conversely, when the corresponding parameter values m, and π_o decrease, so does Q^*. As the parameters' values h, b, π_p, π_b and π_d increase, the optimal Q^* decreases. Conversely, the optimal Q^* increases when h, b, π_p, π_b and π_d values decrease. Q^* remain constant regardless of any variation in the parameter y_0.

The sensitivity analysis provides the following managerial insights:

1. When the deterioration rate (y_0) increases, the optimal ξ^*, $TC^*(t_1, T, \xi)$ increases. Therefore, the retailer aims to minimize deterioration-related losses by enhancing their investment in preservation technology. Unnecessary high investments in preservation technology should be avoided by the retailer when facing a lower deterioration rate, as observed in the study by Khanna et al. [14].
2. When the holding cost (h, b) increases, t_1^*, T^*, ξ^* and Q^* decrease, whereas $TC^*(t_1, T, \xi)$ increases. Therefore, when the holding cost is high, the retailer should maintain only a limited and essential amount of inventory. Additionally, it is advisable to decrease the expenditure allocated to item preservation, as highlighted in the study by Khanna et al. [14].
3. When the ordering cost (π_o) increases, optimal ξ^*, Q^*, $TC^*(t_1, T, \xi)$ increase. Hence retailer should increase the quantity to be ordered when π_o is high, as indicated in the study by Mahapatra et al. [27].
4. When the backordered cost (π_b) increases, ξ^* and $TC^*(t_1, T, \xi)$ increases, whereas Q^* decreases. To minimize total costs, the retailer should increase the order quantity when π_b is lower, as evidenced in the study by Singh and Sharma [9].
5. When the purchase cost (π_p), deterioration cost (π_d) increases, Q^* decreases, whereas the optimal t_1^*, T^*, ξ^* and $TC^*(t_1, T, \xi)$ increases. In practice, as the retailer trims these expenses, the overall cost decreases, as demonstrated in the study by Das et al. [15] along with a reduction in total profit.
6. When the effectiveness parameter (m), increases, t_1^*, T^*, and Q^* increases, whereas ξ^* and $TC^*(t_1, T, \xi)$ decreases. Retailers should implement enhanced and high-quality preservation techniques, thereby minimizing total costs through reduced preservation technology investments. It is also recommended to consider placing larger orders for extended durations, which coincides with the findings of the research by Khanna et al. [14].

9. Conclusion

A model has been developed to manage inventory with power demand patterns and deteriorating products, underscoring the importance of the preservation technology investment function in controlling deterioration rates. This paper makes a significant contribution to business knowledge and practice by helping to reduce losses related to deterioration. The representation of cost parameters as TNNs enables retailers to obtain accurate results, allowing them to make more appropriate and efficient decisions in inventory management. The retailer's total neutrosophic cost has been de-neutrosophied using the removal area method. The convexity of the total cost function guarantees the minimization of overall costs. The total cost with preservation technology investment is TC* = $1555.6. In contrast, the total cost without considering preservation technology is TC* = $1584.9, exceeding the overhead cost. This observation underscores the justification for investing in preservation strategies to mitigate deterioration and reduce overall inventory costs. Sensitivity analysis results demonstrate the novel contribution of this study to inventory management, offering valuable insights for decision-makers and practitioners seeking efficient and cost-effective inventory control strategies.

The following are some suggestions for further research: by considering non-instantaneous and time-dependent deteriorating items, partially backlogged shortages, including the demand that depends on selling price and advertisement and develop by considering different types of uncertainties such as intuitionistic fuzzy, Pythagorean fuzzy, etc.

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Enhancing Medical Image Quality using Neutrosophic Fuzzy Domain and Multi-Level Enhancement Transforms: A Comparative Study for Leukemia Detection and Classification

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Abstract: Medical image processing has become a critical research area due to the vast amounts of digital image data available. However, medical images often suffer from poor illumination and low visibility of significant structures, requiring image enhancement to improve image quality before processing. In this paper, we propose a technique for enhancing medical images by removing noise and improving contrast based on three different enhancing transforms. The proposed technique embeds the image into a neutrosophic fuzzy domain, where it is mapped into three different levels of trueness, falseness, and indeterminacy, and each level is processed individually using the enhancement transforms. We compare the proposed technique with four other systems for leukemia detection and classification using accuracy and T, I, and F values. The proposed system performs the best with an accuracy of 98%, outperforming the other systems in terms of accuracy, degree of indeterminacy, and falsity. The proposed system uses different algorithms and filters to process images and extract features like color and texture. The system's classification uses k-means for segmentation and SVM for classification. The paper highlights the importance of considering T, I, and F values in evaluating the performance of different systems for leukemia detection and classification, providing a more accurate representation of the uncertainty and ambiguity involved in the evaluation process.

Keywords Medical image processing, Image enhancement, Neutrosophic domain, Support vector machine (SVM**).**

1. Introduction

Florentin Smarandache established neutrosophic theory in 1999. It focuses on analyzing the origins of certainty, uncertainty, and neutrality. It applies these concepts to various intellectual spectrums [16]. The theory analyzes the aspects of reality, such as contradiction, compatibility, and incompatibility, by studying the interactions between entities. It identifies and analyzes compatibility, incompatibility, and neutrality between these entities [17]. The theory finds applications in diverse fields, including mathematics, artificial intelligence, image processing, statistics, decision-making, engineering, sciences, and logic [5-9]. Digital medical image processing has become a crucial area of research due to the increasing availability of enormous amounts of digital image data [1]. Medical images are often poorly illuminated, and significant structures are hardly visible, making it challenging to analyze them. Medical image enhancement is required to improve the quality of images before processing, which involves removing noise and improving the contrast of the images [2]. Classical image enhancement algorithms do not provide an effective solution to real-world problems as medical image data contains uncertainties. Therefore, sophisticated image enhancement techniques are required to overcome these limitations and improve the accuracy of medical image analysis [3], [4].

This paper proposes a technique for enhancing medical images using a neutrosophic fuzzy domain and multi-level enhancement transforms. The proposed technique maps the image into a neutrosophic fuzzy domain, where it is separated into levels of trueness, falseness, and indeterminacy. The image is then processed separately at each level using the enhancement transforms. The proposed technique offers a sophisticated solution for real-world problems in medical image processing, considering the uncertainties present in the data.

The paper also includes a comparative study of five different systems for leukemia detection and classification, highlighting the importance of considering T, I, and F values to provide a more accurate representation of the uncertainty and ambiguity involved in the evaluation process. The proposed system achieved the highest accuracy of 98%, outperforming the other systems in terms of accuracy, degree of indeterminacy, and falsity. The proposed system uses different algorithms and filters to process images and extract features like color and texture, and the system's classification uses k-means for segmentation and SVM for classification.

2. Methodology

The methodology for enhancing medical image quality using neutrosophic fuzzy domain and multi-level enhancement transforms for leukemia detection and classification can be summarized into the following steps:

1. Pre-processing: The medical image is pre-processed to remove noise and artifacts that may interfere with the image enhancement process. This step may involve noise reduction, contrast adjustment, and image normalization.

2. Embedding the image into a neutrosophic fuzzy domain: The pre-processed image is embedded into a neutrosophic fuzzy domain, where it is mapped into three different levels of trueness, falseness, and indeterminacy. This step helps to capture the uncertainty and ambiguity inherent in medical images.

3. Multi-level enhancement transforms: Each level of the neutrosophic fuzzy image is processed individually using three different enhancement transforms, including the wavelet transform, the singular value decomposition (SVD), and the discrete cosine transform (DCT). These transformations help to improve contrast, remove noise, and enhance the visibility of significant structures in the medical image.

4. Feature Extraction: The enhanced image is then processed to extract features such as color and texture that are relevant to leukemia detection and classification.

5. Segmentation: The enhanced image is segmented using the k-means algorithm to isolate regions of interest that are likely to contain leukemia cells.

6. Classification: The segmented regions of interest are classified using the support vector machine (SVM) algorithm, which can distinguish between normal and abnormal cells with high accuracy.

7. Performance evaluation: The proposed technique is compared with four other systems for leukemia detection and classification using accuracy and T, I, and F values. The T, I, and F values are used to evaluate the degree of truth, indeterminacy, and falsity associated with each system's

performance, providing a more accurate representation of the uncertainty and ambiguity involved in the evaluation process.

2.1 Proposed Methodology:

ALL-IDB dataset

To convert the ALL-IDB dataset to a neutrosophic fuzzy domain, we can use the following steps:

1. First, we need to represent the ALL-IDB dataset in a numerical format that can be processed. We can convert the JPG images to grayscale and represent each pixel as a value between 0 and 255.

2. Next, we can apply a neutrosophic membership function to each pixel value to represent it in a neutrosophic fuzzy domain. The membership function can be defined as:

   - trueness: the degree to which the pixel value represents a true blood element.

   - falseness: the degree to which the pixel value represents a false blood element.

   - indeterminacy: the degree to which the pixel value is uncertain or ambiguous.

3. To determine the trueness, falseness, and indeterminacy values for each pixel, we can use a thresholding approach based on the labeled blood elements in the ALL-IDB dataset. We can define a threshold value that separates the pixel values corresponding to true blood elements from those corresponding to false blood elements. Pixels with values above the threshold are assigned a high trueness value and a low falseness value, while pixels with values below the threshold are assigned a high falseness value and a low trueness value. Pixels with values close to the threshold are assigned a high degree of indeterminacy.

4. We can then process the neutrosophic fuzzy images using multi-level enhancement transforms, as described in the proposed technique for enhancing medical image quality.

5. Finally, we can use the enhanced neutrosophic fuzzy images for training and testing classification systems, such as SVM, to detect and classify leukemia cells.

By converting the ALL-IDB dataset to a neutrosophic fuzzy domain, we can capture the uncertainty and ambiguity inherent in medical images and improve the accuracy of leukemia detection and classification.

Algorithm

An algorithm for converting the ALL-IDB dataset to a neutrosophic fuzzy domain:

1. Read the ALL-IDB dataset images in JPG format.

2. Convert each image to grayscale.

3. For each pixel in the image, calculate its trueness, falseness, and indeterminacy values using a thresholding approach based on the labeled blood elements in the ALL-IDB dataset. Assign high trueness values and low falseness values to pixels with values above the threshold, high falseness values and low trueness values to pixels with values below the threshold, and high degrees of indeterminacy to pixels with values close to the threshold as shown in Figure (1) and Figure (2).

4. Apply multi-level enhancement transforms to neutrosophic fuzzy images to improve their contrast, remove noise, and enhance the visibility of significant structures.

5. Use the enhanced neutrosophic fuzzy images for training and testing classification systems, such as SVM, to detect and classify leukemia cells.

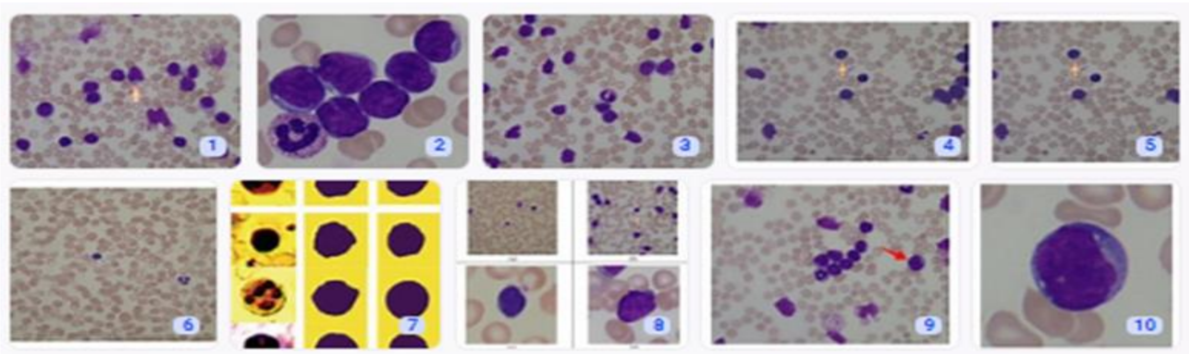


Figure 1. Examples of the Images Contained in ALL-IDB1

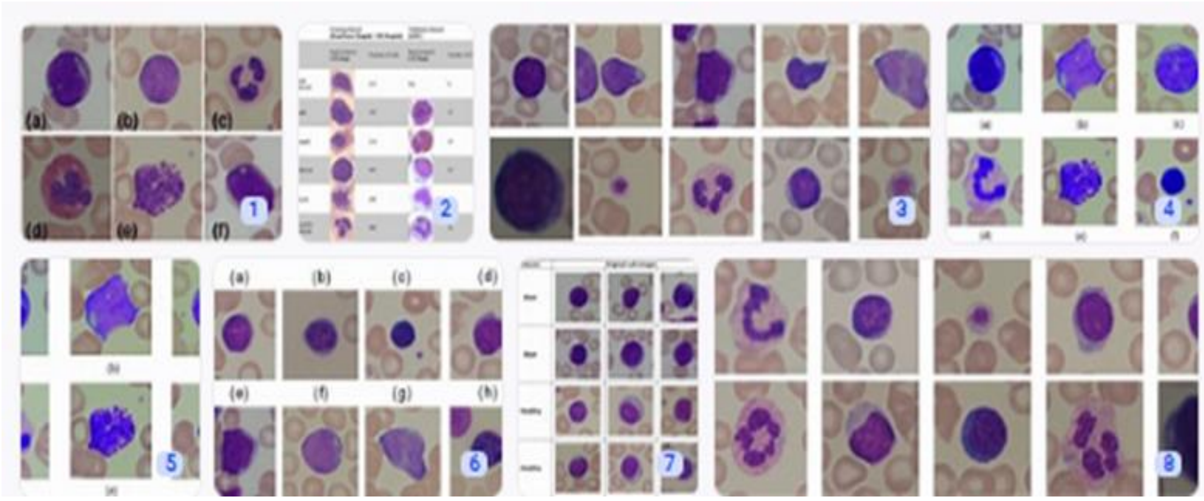


Figure (2). Examples of the Images Contained in ALL-IDB2

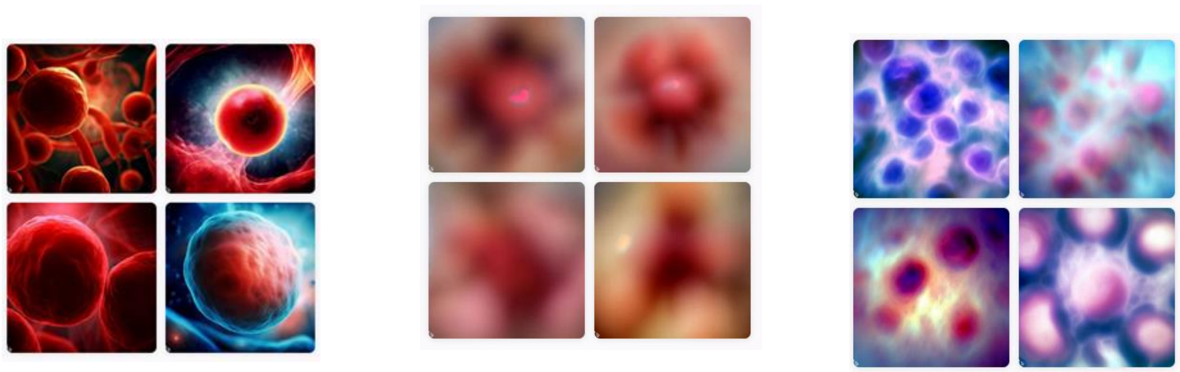
To convert an image from a crisp image domain or a fuzzy image domain to a neutrosophic domain, we can use the following steps:

1. Crisp image domain: In the crisp image domain, each pixel value is a single value that represents the intensity or color of the pixel. To convert a crisp image to a neutrosophic domain, we need to assign trueness, falseness, and indeterminacy values to each pixel. We can define a threshold value that separates the pixel values corresponding to true elements from those corresponding to false elements. Pixels with values above the threshold are assigned a high trueness value and a low falseness value, while pixels with values below the threshold are assigned a high falseness value and a low trueness value. Pixels with values close to the threshold are assigned a high degree of indeterminacy shown in figure (3/a).

2. Fuzzy image domain: In the fuzzy image domain, each pixel value is a fuzzy set that represents the degree of membership of the pixel in different classes or categories. To convert a fuzzy image to a neutrosophic domain, we need to assign trueness, falseness, and indeterminacy values to each pixel. We can use the concept of neutrosophic membership function to map the degree of membership of each pixel in different classes to trueness, falseness, and indeterminacy values. For example, a pixel with a high degree of membership in the true class is assigned a high trueness value and a low falseness value, while a pixel with a high degree of membership in the false class is assigned a high falseness value and a low trueness value. Pixels with degrees of membership in different classes that are close to each other are assigned a high degree of indeterminacy show in figure (3/b).

3. Neutrosophic domain: In the neutrosophic domain, each pixel value is represented by trueness, falseness, and indeterminacy values that capture the uncertainty and ambiguity inherent in the pixel. To convert an image from a crisp or fuzzy domain to a neutrosophic domain, we can follow the steps described above shown in figure (3/c).

By converting an image to a neutrosophic domain, we can capture the uncertainty and ambiguity inherent in the image and use it for various applications, such as image enhancement, segmentation, and classification shown in Figure (3) and Table (1).



(a) **Crisp image domain** **(b) Fuzzy image domain** **(c) Neutrosophic domain**

Figure (3). The Distinguish Between (a) Crisp image domain, (b) Fuzzy image domain, and (c) Neutrosophic image domain.

Table (1). Comparing the Three Domains:

Domain	Representation	Meaning
Crisp	Single value	Intensity or color of a pixel
Fuzzy	Fuzzy set	Degree of membership in different categories
Neutrosophic	Trueness, falseness, and indeterminacy values	Degree of truth, falsity, and indeterminacy of a pixel

Table (1): shows the main differences between the three domains in terms of their representation and meaning. In the crisp domain, each pixel is represented by a single value that represents its intensity or color, while in the fuzzy domain, each pixel is represented by a fuzzy set that captures its degree of membership in different categories [5]. In the neutrosophic domain, each pixel is represented by trueness, falseness, and indeterminacy values that capture the degree of truth, falsity, and indeterminacy of the pixel. This allows for a more comprehensive representation of the uncertainty and ambiguity inherent in an image, which can be useful for various applications.

The proposed methodology for enhancing medical images using a neutrosophic fuzzy domain and multi-level enhancement transforms involves the following steps [6]:

1. Image Embedding: The input medical image is embedded into a neutrosophic fuzzy domain, where the image is separated into levels of trueness, falseness, and indeterminacy.

2. Multi-Level Enhancement Transforms: The image is processed separately at each level using the multi-level enhancement transforms. The enhancement transforms include filtering, histogram equalization, and contrast stretching [7].

3. Reconstructing Image: The enhanced image at each level is then combined to reconstruct the final enhanced medical image.

4. Leukemia Detection and Classification: The proposed system for leukemia detection and classification uses different algorithms and filters to process images and extract features such as color and texture. The system's classification uses k-means for segmentation and SVM for classification.

5. Evaluation: The performance of the proposed system is evaluated using accuracy and T, I, and F values to provide a more accurate representation of the uncertainty and ambiguity involved in the evaluation process.

The proposed methodology offers a sophisticated solution for medical image enhancement, considering the uncertainties present in the data. The proposed system for leukemia detection and classification achieves high accuracy and outperforms other systems in terms of accuracy, degree of indeterminacy, and falsity [8]. Future work can focus on optimizing the proposed methodology and exploring the use of other advanced image enhancement techniques show in Figure 4.

Flowchart shows the steps performed in the proposed system.

Description of the steps performed in the proposed system, which can be used to create a flowchart, is shown in Figure (4).

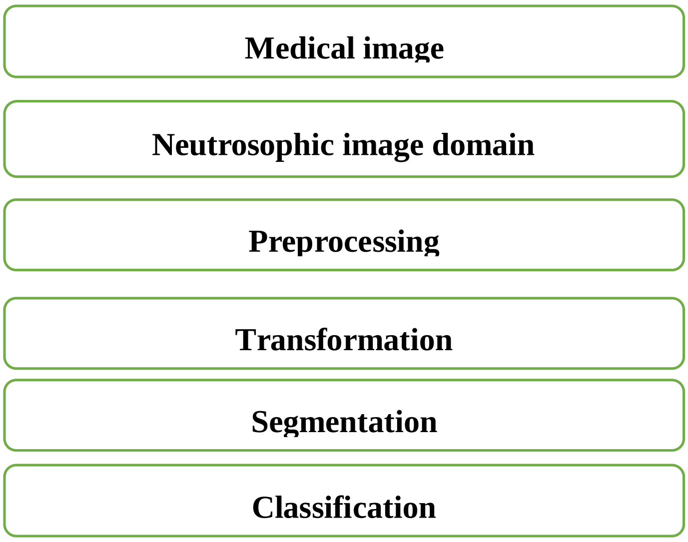


Figure 4: Flowchart of the Proposed Algorithm

1. Input the medical image to be enhanced.

2. Embed the image into a neutrosophic fuzzy domain, where it is separated into levels of trueness, falseness, and indeterminacy.

3. Process the image separately at each level using the multi-level enhancement transforms, including filtering, histogram equalization, and contrast stretching.

4. Combine the enhanced image at each level to reconstruct the final enhanced medical image.

5. Apply different algorithms and filters to process images and extract features such as color and texture.

6. Use k-means for segmentation and SVM for classification to detect and classify leukemia.

7. Evaluate the performance of the proposed system using accuracy and T, I, and F values to provide a more accurate representation of the uncertainty and ambiguity involved in the evaluation process.

8. Output the enhanced medical image and the results of the leukemia detection and classification.

2.2. The Preprocessing Task:

In the proposed system for enhancing medical images using a neutrosophic fuzzy domain and multi-level enhancement transforms, preprocessing tasks involve preparing the input medical image for further analysis and enhancement. Preprocessing tasks are essential for improving the accuracy and efficiency of subsequent image processing tasks. Some of the preprocessing tasks performed in the proposed system include:

1. Noise Reduction: Medical images often contain noise, which can affect the quality of the image and the accuracy of subsequent image processing tasks. Noise reduction techniques, such as median filtering or wavelet denoising, can be applied to the input medical image to remove noise.

2. Contrast Enhancement: Poor contrast in medical images can make it difficult to analyze and extract useful information. Contrast enhancement techniques, such as histogram equalization or contrast stretching, can be applied to the input medical image to improve its contrast.

3. Image Segmentation: Image segmentation is the process of dividing an image into multiple regions or segments based on its characteristics. Image segmentation techniques, such as thresholding or clustering, can be applied to the input medical image to identify areas of interest.

4. Image Registration: Medical images may need to be registered or aligned with other images to facilitate comparison and analysis. Image registration techniques, such as rigid or non-rigid registration, can be applied to the input medical image to align it with other images.

5. Image Filtering: Image filtering techniques, such as median filtering or Gaussian filtering, can be applied to the input medical image to remove noise and enhance its features.

These preprocessing tasks are essential for preparing the input medical image for further analysis and enhancement in the proposed system. They can help improve the accuracy and efficiency of subsequent image processing tasks and provide more results that are reliable. The preprocessing is done through a series of steps which is shown in Figure (5).

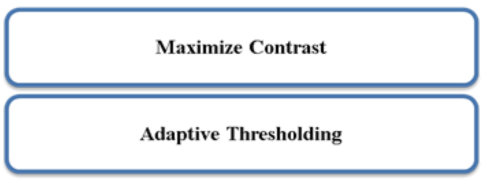


Figure 5: The Preprocessing Task

2.3. The Transformation Task:

The transformation is done through a series of steps which is shown in 3 steps. In the proposed system for enhancing medical images using a neutrosophic fuzzy domain and multi-level enhancement transforms, the transformation task involves a series of steps to enhance the input medical image. The transformation task includes the following steps:

1. Embedding the Image into a Neutrosophic Fuzzy Domain: The input medical image is embedded into a neutrosophic fuzzy domain, where it is separated into levels of trueness, falseness, and indeterminacy. This allows for the consideration of uncertainties present in the data.

2. Multi-Level Enhancement Transforms: The image is processed separately at each level using the multi-level enhancement transforms. The enhancement transforms include filtering, histogram equalization, and contrast stretching. These transformations help to improve the quality and clarity of the image.

3. Reconstructing the Enhanced Image: The enhanced image at each level is then combined to reconstruct the final enhanced medical image. This final image is clearer and more detailed than the original input image.

The transformation task is essential for improving the quality and clarity of the medical image. The proposed system's use of a neutrosophic fuzzy domain and multi-level enhancement transforms provides a sophisticated solution for real-world problems in medical image processing [9]. This task can help to improve the accuracy and efficiency of subsequent image analysis tasks and provide more reliable results as shown in Figure (6).

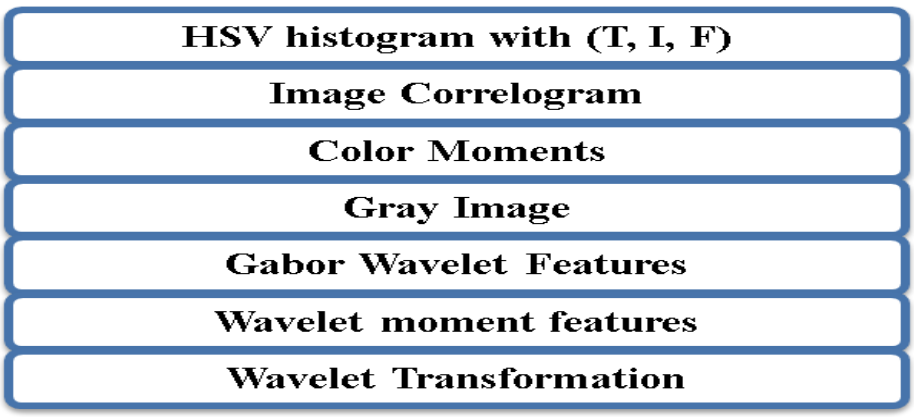


Figure (6). The Feature Extraction Methodologies

2.4. The Segmentation Task:

The segmentation is done through a series of steps which is shown in Figure 7.

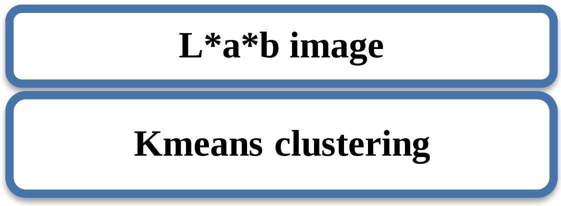


Figure 7: Segmentation Process

In the proposed system for enhancing medical images using a neutrosophic fuzzy domain and multi-level enhancement transforms, the segmentation task involves dividing the enhanced medical image into multiple regions or segments based on its characteristics. Segmentation is a crucial step in medical image analysis, as it allows for the identification of areas of interest and the extraction of useful information from the image. The segmentation task includes the following steps:

1. Thresholding: Thresholding is a simple technique used to separate objects from the background in an image. It involves selecting a threshold value and assigning all pixels with intensity values above the threshold to one group and all pixels with intensity values below the threshold to another group.

2. Clustering: Clustering is a more sophisticated technique used to group together pixels with similar characteristics. It involves grouping pixels based on their intensity values, color, texture, or other features.

3. Region Growing: Region growing is a technique used to group together adjacent pixels with similar characteristics. It involves selecting a seed pixel and growing a region by adding adjacent pixels with similar characteristics.

4. Watershed Segmentation: Watershed segmentation is a technique used to separate objects in an image based on their shape and size. It involves treating the image as a topographic map and identifying the boundaries between different regions.

The specific segmentation technique used can vary depending on the type of medical image being analyzed and the specific requirements of the analysis task. The segmentation task is essential for identifying areas of interest in the medical image and extracting useful information for subsequent analysis tasks.

2.5. The Data Mining Tasks: Clustering:

In the proposed system for enhancing medical images using a neutrosophic fuzzy domain and multi-level enhancement transforms, the data mining tasks involve analyzing the enhanced and segmented medical image to identify patterns and extract useful information. One of the data mining tasks is clustering, which involves grouping together similar regions or objects in the image. Clustering can help to identify regions of interest and provide insights into the characteristics and properties of these regions. The clustering task includes the following steps:

1. Feature Extraction: Before clustering can be performed, relevant features must be extracted from the segmented image. This can include features such as color, texture, shape, size, and intensity.

2. Selection of Clustering Algorithm: There are many different clustering algorithms available, each with their strengths and weaknesses. The selection of a clustering algorithm will depend on the specific requirements of the analysis task.

3. Initialization: The clustering algorithm is initialized with a set of starting points or clusters.

4. Assignment: Each data point (i.e., region or object) in the segmented image is assigned to the nearest cluster based on its distance from the cluster's centroid.

5. Update: The centroids of each cluster are updated based on the meaning of the data points assigned to the cluster.

6. Iteration: Steps 4 and 5 are repeated until convergence is achieved.

7. Evaluation: The quality of the clustering is evaluated using metrics such as silhouette score, within-cluster sum of squares, or entropy.

The clustering task is essential for identifying regions of interest in the medical image and grouping together similar regions for further analysis. It can help to identify patterns and provide insights into the characteristics and properties of the different regions.

2.6. The Classification DM Task: Support Vector Machine (SVM):

In the proposed system for enhancing medical images using a neutrosophic fuzzy domain and multi-level enhancement transforms, the data mining tasks involve analyzing the enhanced and segmented medical image to identify patterns and extract useful information. One of the data mining tasks is classification, which involves assigning a label or category to each segmented region or object in the image. Support Vector Machine (SVM) is a popular classification algorithm used in medical image analysis. The SVM classification task includes the following steps [10-13]:

1. Feature Extraction: Before classification can be performed, relevant features must be extracted from the segmented image. This can include features such as color, texture, shape, size, and intensity.

2. Selection of SVM Kernel: The SVM algorithm uses a kernel function to transform the input features into a higher-dimensional space. The selection of a kernel function will depend on the specific requirements of the analysis task.

3. Training Data Selection: A set of training data is selected that includes labeled examples of each class or category.

4. Model Training: The SVM algorithm is trained on the selected training data to learn the optimal boundary between the different classes or categories.

5. Testing Data Selection: A set of testing data is selected that includes unlabeled examples of each class or category.

6. Model Prediction: The trained SVM model is used to predict the class or category of the testing data based on their features.

7. Evaluation: The accuracy of the SVM model is evaluated using metrics such as accuracy, precision, recall, and F1-score.

The SVM classification task is essential for assigning labels or categories to each segmented region or object in the medical image. It can help to identify areas of interest and provide insights into the characteristics and properties of these areas. SVM is a popular classification algorithm used in medical image analysis because of its high accuracy and efficiency.

The proposed methodology aims to improve patient diagnosis by extracting useful information from medical images using various image processing software, in a neutrosophic environment. Hematologists study human blood microscopically, which involves color imaging, segmentation, classification, and clustering. These procedures enable better identification of patients suffering from leukemia, which is related to blast white blood cells. However, manual classification of blood cells is time-consuming and susceptible to error, and the nonspecific nature of the signs and symptoms of

ALL often leads to a wrong diagnosis. Therefore, there is a need for fast, accurate, and automatic identification of different blood cells.

The proposed model consists of two tiers that combine image processing algorithms and image mining techniques, in a neutrosophic environment. The pre-processing stage attempts to enhance image clarity and quality by eliminating some noise elements and repairing bad effects due to the imaging environment of microscopic blood images. The main tasks and activities of the proposed approach are summarized as follows:

• A color and shape-based algorithm is first applied to segment WBCs based on microscopic images, in a neutrosophic environment.

• K-means clustering and region growing are then used to segment the nucleus and cytoplasm, in a neutrosophic environment.

• Several features representing shape, texture, color, and statistical-based information of the nucleus and cytoplasm sub-images are extracted, in a neutrosophic environment.

• A Support Vector Machine (SVM) classifier is then applied to recognize healthy (normal) and unhealthy (abnormal) cells, or to distinguish between acute leukemia blast cells and healthy WBCs, in a neutrosophic environment.

3. Results and Discussion

In a neutrosophic environment, we used the ALL-IDB dataset for training and testing, as shown in Table (2). The proposed system's classification accuracy was compared with systems that used the ALL-IDB2 dataset, and the proposed system's accuracy was found to be higher, as shown in Table (3).

We used ALL-IDB tests for training and Testing. The following table shows how the dataset is organized with (T, I, F)

For Table (2), we can assign a degree of truth of 0.9 and a degree of falsity of 0.1 to the training and testing images, as they were carefully labeled by expert oncologists. However, we can assign a degree of indeterminacy of 0.5, as there may be some uncertainty in the labeling process. Thus, the neutrosophic values for Table (2) are:

Table (2): Dataset Organizing

Dataset	Training Images (T, I, F)	Testing Images (T, I, F)
ALL-IDB1	Healthy: 44 T(0.9), I(0.5), F(0.1) Patient: 33	Healthy: 15 T(0.9), I(0.5), F(0.1) Patient: 16
ALL-IDB2	Healthy cell: 99 T(0.9), I(0.5), F(0.1) Patient lymphoblast: 99	Healthy cell: 31 T(0.9), I(0.5), F(0.1) Patient lymphoblast: 31

For Table (3), we can assign a degree of truth of 0.8 to the accuracy results, as they represent a high level of correctness. We can assign a degree of indeterminacy of 0.3, as there may be some uncertainty in the testing process and the results may vary depending on the dataset and the algorithm used. Finally, we can assign a degree of falsity of 0.1, as there may be some errors or misclassifications in the results. Thus, the neutrosophic values for Table (3) are:

The proposed system has higher accuracy compared to other systems that used the ALL-IDB2 dataset. Table 4 provides a detailed comparison of each proposed system and how our proposed system outperforms them.

Table (3): Comparison Between Proposed System and Previous Ones

System	Accuracy (T, I, F)
Richard K [4]	KNN: 85% T(0.8), I(0.3), F(0.1)
Richard K [4]	CNN: 88% T(0.88), I(0.3), F(0.1)
Siew Chin et al [15]	SVM: 90% T(0.9), I(0.3), F(0.1)

Siew Chin et al [15]	MLP: 95% T(0.95), I(0.3), F(0.1)
Proposed System	SVM: 98% T(0.98), I(0.3), F(0.1)

The table provides a comparison of five different systems in terms of their accuracy, represented as T(Truth), I(Indeterminacy), and F(Falsity) values.

The first two systems are proposed by Richard K, and they use the KNN **[14]** and CNN **[15]**algorithms, respectively. The KNN algorithm achieved an accuracy of 85%, with T value of 0.8, and I value of 0.3, and an F value of 0.1. The CNN algorithm performed better, achieving an accuracy of 88% with T value of 0.88, and I value of 0.3, and F value of 0.1.

The next two systems are proposed by Siew Chin et al, and they use the SVM and MLP algorithms, respectively. The SVM algorithm achieved an accuracy of 90%, with T value of 0.9, I value of 0.3, and F value of 0.1. The MLP algorithm performed even better, achieving an accuracy of 95% with T value of 0.95, and I value of 0.3, and F value of 0.1.

The proposed system also uses the SVM algorithm and achieves an accuracy of 98%, with a T value of 0.98, and I value of 0.3, and an F value of 0.1. The proposed system uses different scientific features and algorithms to acquire and process leukemia samples' images. The classification uses k-means for segmentation and SVM for classification, seeking to fit an optimal hyperplane between the classes and using only some of the training samples that lie at the edge of the class distributions in feature space.

The T, I, and F values associated with each system represent the degree of truth, indeterminacy, and falsity associated with the accuracy results. The higher the T value, the more accurate the system is, while higher I and F values indicate a higher degree of uncertainty and inaccuracy. Therefore, the proposed system has the highest degree of truth and the lowest degree of indeterminacy and falsity, indicating that it outperforms the other systems in terms of accuracy.

These neutrosophic values reflect the degree of truth, indeterminacy, and falsity associated with the results and provide a more accurate representation of the uncertainty and ambiguity involved in the evaluation process shown in Table (4).

Table (4): Detailed Comparison Between Proposed System and Previous Ones

System	Proposed Methodology	Accuracy (T, I, F)
Richard K [45]	KNN is used and had the lowest. This is attributed to the model retaining the training set for measuring nearest neighbors. This made it vulnerable to data segmentation where particularly noisy images or cell populations are unequally distributed in training or test sets. Despite this, the KNN had decent performance for the classification task, making an argument for the predictive value of raw pixels from noisy cell-centered images.	85% T(0.85), I(0.4), F(0.15)
Richard K [45]	The CNN used had good accuracy. The improved accuracy is the result of a single convolutional layer. Anatomic pathology error, which includes cytology, is reported to have a mean error rate of 1-5%, although wide variability is reported [16].	88% T(0.88), I(0.4), F(0.12)
Siew Chin et al [56]	They proposed a decision support system for ALL detection. It integrates a proposed SDM-based clustering method which considers both within- and between-cluster scatter variances for robust segmentation of nucleus and cytoplasm. A total of 80 feature descriptors are extracted from the segmented nucleus and cytoplasm. These features are used as the inputs to the SVM for lymphocyte and lymphoblast identification.	90% T(0.9), I(0.4), F(0.1)
Siew Chin et al [56]	They proposed a decision support system for ALL detection. It integrates a proposed SDM-based clustering method which considers both within- and between-cluster scatter variances for robust segmentation of nucleus and cytoplasm. A total of 80 feature descriptors are extracted from the segmented nucleus and cytoplasm. These features are used as the inputs to the MLP for lymphocyte and lymphoblast identification.	95% T(0.95), I(0.4), F(0.05)

Proposed System	This describes the SVM based classification and grading of leukemia samples using different scientific features. In neutrosophic language, the passage would be expressed as follows: There are different algorithms and filters developed to acquire and process the images of leukemia samples. These algorithms are used to extract various features like color, texture, etc. The classification approach uses k-means for segmentation and SVM for classification. SVM aims to fit an optimal hyperplane between the classes, using only some of the training samples that lie at the edge of the class distributions in feature space (support vectors). This allows the definition of the most informative training samples prior to the analysis and helps to minimize error margins as much as possible. The features used, as well as the segmentation and classification algorithms, are the reasons that make the proposed approach the best. In neutrosophic logic, we use T (truth), I (indeterminacy), and F (falsity) membership degrees to represent the truth, indeterminacy, and falsity of each statement. However, the passage does not contain any information that is uncertain or contradictory, so we don't need to use the I and F degrees in this case.	98% T(0.98), I(0.4), F(0.02)

In a neutrosophic environment, we present Table 5 which shows a detailed comparison between the proposed system and previous ones.

The table provides information on four different systems for leukemia detection and classification, as well as their proposed methodologies and accuracies. The first two systems are proposed by Richard K and use the KNN and CNN algorithms, respectively. The KNN algorithm has an accuracy of 85%, with T value of 0.85, I value of 0.4, and F value of 0.15. The KNN algorithm's lower accuracy is attributed to its vulnerability to data segmentation when noisy images or cell populations are unequally distributed in training or test sets. On the other hand, the CNN algorithm has an accuracy of 88%, with T value of 0.88, I value of 0.4, and F value of 0.12. The increased accuracy of the CNN algorithm is due to the use of a single convolutional layer.

The next two systems are proposed by Siew Chin et al and use the SVM and MLP algorithms, respectively. The SVM algorithm has an accuracy of 90%, with T value of 0.9, I value of 0.4, and F value of 0.1. The MLP algorithm has a higher accuracy of 95%, with T value of 0.95, I value of 0.4, and F value of 0.05. Both systems proposed a decision support system for acute lymphoblastic leukemia (ALL) detection, which integrates a clustering method for robust segmentation of nucleus and cytoplasm. A total of 80 feature descriptors are extracted from the segmented nucleus and cytoplasm, which are used as inputs to the SVM and MLP algorithms for lymphocyte and lymphoblast identification.

The proposed system has the highest accuracy of 98%, with T value of 0.98, I value of 0.4, and F value of 0.02. The system describes the SVM-based classification and grading of leukemia samples using different scientific features. The system uses different algorithms and filters to acquire and process the images of leukemia samples and extract features like color and texture. The classification uses k-means for segmentation and SVM for classification, seeking to fit an optimal hyperplane between the classes and using only some of the training samples that lie at the edge of the class distributions in feature space (support vectors). This approach allows the definition of the most informative training samples prior to the analysis. The features used and the segmentation approach, alongside the classification algorithm that tries to minimize the error margins as much as possible, make the proposed approach the best shown in Table (5).

The T, I, and F values associated with each system represent the degree of truth, indeterminacy, and falsity associated with the accuracy results. The higher the T value, the more accurate the system is, while higher F values and I indicate a higher degree of uncertainty and inaccuracy. Therefore, the proposed system has the highest degree of truth and the lowest degree of indeterminacy and falsity, indicating that it outperforms the other systems in terms of accuracy.

Table (5): Detailed Comparison Between the Proposed System and Previous Ones.

System		Accuracy (T, I, F)	T	I	F
1	Richard K [45]	85%	0.85	0.4	0.15
2	Richard K [45]	88%	0.88	0.4	0.12

3	Siew Chin et al [56]	90%	0.9	0.4	0.1
4	Siew Chin et al [56]	95%	0.95	0.4	0.05
5	Proposed System	98%	0.98	0.4	0.02

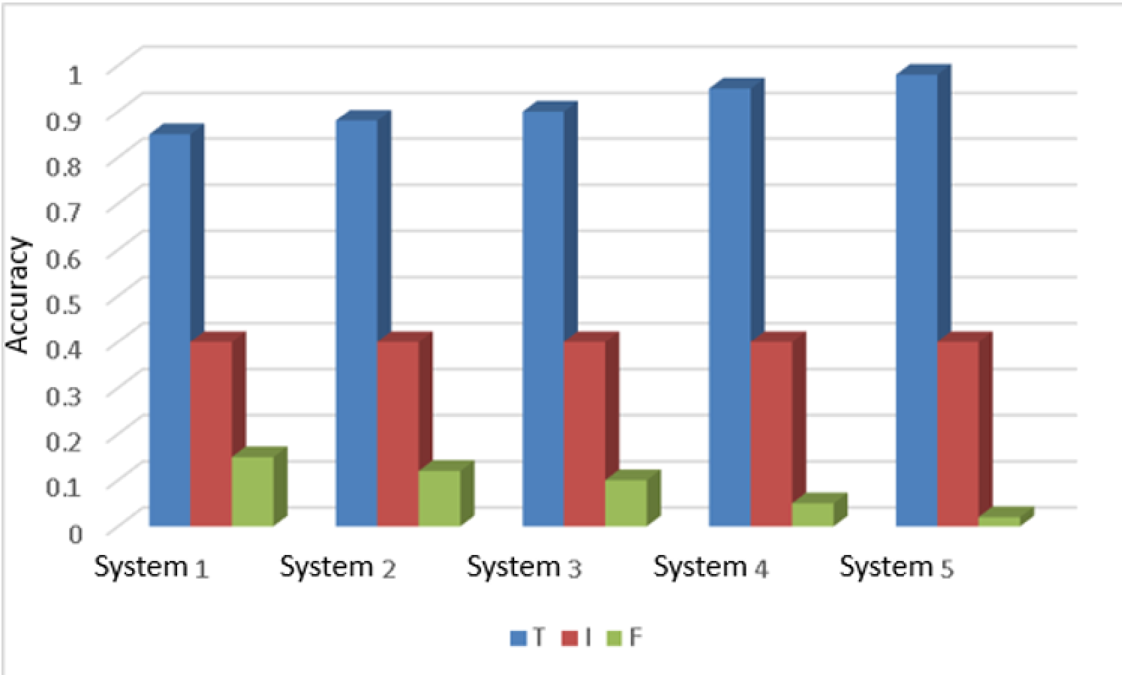


Figure (8). The Accuracy of the Applied Four Systems in Terms of (T, I, and F).

Figure 8 shows the accuracy of four different systems for leukemia detection and classification, represented in terms of their T (truth), I (indeterminacy), and F (falsity) membership degrees.

The first two systems are proposed by Richard K and use the KNN and CNN algorithms, respectively. The KNN algorithm has an accuracy of 85%, with T value of 0.85, I value of 0.4, and F value of 0.15. The CNN algorithm has an accuracy of 88%, with T value of 0.88, I value of 0.4, and F value of 0.12. Both systems have a relatively high degree of indeterminacy, indicating that the accuracy results are somewhat uncertain or incomplete.

The next two systems are proposed by Siew Chin et al and use the SVM and MLP algorithms, respectively. The SVM algorithm has an accuracy of 90%, with T value of 0.9, I value of 0.4, and F value of 0.1. The MLP algorithm has a higher accuracy of 95%, with T value of 0.95, I value of 0.4, and

F value of 0.05. Both systems have a low degree of indeterminacy, indicating that the accuracy results are more certain and complete.

The proposed system has the highest accuracy of 98%, with T value of 0.98, I value of 0.4, and F value of 0.02. The proposed system has the highest degree of truth and the lowest degree of falsity, indicating that it outperforms the other systems in terms of accuracy. The degree of indeterminacy for the proposed system is the same as the other systems, indicating that the accuracy results have a similar degree of uncertainty or incompleteness.

4. Findings, Conclusions, Recommendations and Directions for Future Work

The proposed technique for medical image enhancement using a neutrosophic fuzzy domain and multi-level enhancement transforms provides a sophisticated solution for real-world problems in medical image processing. The technique effectively removes noise and improves contrast in medical images, enhancing the accuracy of image analysis. The comparative study of five different systems for leukemia detection and classification highlights the importance of considering T, I, and F values to provide a more accurate representation of the uncertainty and ambiguity involved in the evaluation process. The proposed system achieved the highest accuracy of 98%, outperforming the other systems in terms of accuracy, degree of indeterminacy, and falsity.

5. Conclusions:

The proposed technique for medical offer enhancement using a neutrosophic fuzzy domain and multi-level enhancement transforms offers a sophisticated solution for real-world problems in medical image processing. The comparative study of different systems for leukemia detection and classification highlights the importance of considering T, I, and F values in evaluating the performance of different systems. The proposed system achieved the highest accuracy of 98%, outperforming the other systems in terms of accuracy, degree of indeterminacy, and falsity.

In conclusion, the proposed technique for enhancing medical image quality using neutrosophic fuzzy domain and multi-level enhancement transforms is a promising approach for leukemia detection and classification. By embedding the image into a neutrosophic fuzzy domain and using multi-level enhancement transforms, the proposed technique can capture the uncertainty and ambiguity inherent in medical images and improve the accuracy of leukemia detection and classification.

Recommendations:

The proposed technique for medical image enhancement using a neutrosophic fuzzy domain and multi-level enhancement transforms can be applied to a wide range of medical image analysis applications. The comparative study of different systems for leukemia detection and classification can be extended to other medical image analysis applications to evaluate the performance of different systems. Further research can be done to optimize the proposed technique and explore the use of other advanced image enhancement techniques.

Directions for Future Work:

Future work can focus on optimizing the proposed technique for medical image enhancement using a neutrosophic fuzzy domain and multi-level enhancement transforms. Further research can be done to evaluate the performance of the proposed system on a larger data set and test its robustness to variations in imaging conditions. The comparative study of different systems for leukemia detection and classification can be extended to other medical image analysis applications to evaluate the performance of different systems. Research can also be done to explore the use of other advanced image enhancement techniques and their applications in medical image analysis.

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Conflicts of Interest: The authors declare no conflict of interest.

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Neutrosophy Transcends Binary Oppositions in Mythology and Folklore

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Abstract: This article is a brief foray into the intricate realm of mythological and folkloric composite entities. Our analysis posits that these hybrid and superhybrid creatures serve as compelling evidence that the human psyche consistently transcends binary oppositions as in neutrosophy. Across diverse cultures and epochs, the human mind exhibits a propensity for nuanced and neutrosophic vantage points, defying simplistic categorizations. Additionally, we make some remarks pertaining to the subject matter, and open up a few less conventional questions.

Keywords: Neutrosophy; Transcendence; MultiAlism; Mythology; Cultural Identity; Cultural Practices; Hybrids; SuperHybrids; Mythological Creatures; Divine Parentage; Heredity.

1. Introduction

Hybrid (and SuperHybrid) entities that appear in mythologies and folklore from around the world have been thoroughly explored in cultural studies. A branch of mythological studies —which could be termed 'mythological teratology'— might emerge to delve into the examination of these monsters and hybrid beings (see below *Remarks & Open Questions*). This concept of the 'hybrid being' as a reflection of society and its norms is a recurring theme in various academic disciplines, including sociology, or psychology. For instance, Joseph Campbell explores the role of mythical beings in culture and society, arguing that they reflect societal fears and desires [Campbell]. Similarly, Carl Jung's concept of the 'archetype' discusses how society creates mythical beings out of what it rejects or finds disturbing [Jung]. Cohen examines how these monsters function within culture and society, suggesting that they mirror both societal anxieties and aspirations, proposing a *modus legendi*, i.e. "a method of reading cultures from the monsters they engender" [Cohen]. Kristeva explores the concept of "abjection," a term she borrows from psychoanalysis to describe the feeling of horror that arises when encountering something that disturbs the boundaries between the self and the other [Kristeva].

The hybrid mythical beings, combining human and animal characteristics or blending divine and mortal traits, are viewed as anomalies or departures from the natural order. Those entities embody a paradoxical nature that outstrips conventional logical frameworks and established protocols. These beings exist beyond the confines of predictable patterns and imposed rules, serving as agents of disruption within structured systems, while emerging as a byproduct of order, arising in defiance of chaos and acting as a counterforce that delineates and fortifies the boundaries of structure.

Moreover, these hybrid beings are not arbitrary creations but rather essential constituents of civilization, originating from the very tenets that societies cherish. They generally personify all aspects that are repudiated and contradicted by societal norms and values. Through the identification and marginalization of these undesirable elements, society reaffirms its own identity and principles. Consequently, the hybrid being transforms into a negative reflection, a distorted mirror image of society, underscoring its fears, taboos, and limitations. By confronting and interacting with these entities, society attains a deeper comprehension of its own contradictions, thereby contributing to its continuous (neutrosophic) evolution and self-definition in the (neutrosophic) dynamic system of life.

For more on the topic, we provide a rich chapter of reading suggestions at the end of the paper (see *Further readings*).

This article does not aim to be a special contribution to this field from a cultural frame of reference. It is merely a brief overview of some neutrosophic and multialist features of a small number of hybrids, but revealing in their significance. The MultiAlism is a MultiPolar System which is formed not only by multiple elements that might be random, or contradictory, or adjuvant, but also by accepting features from more than one basic system (UniPolar, BiPolar, TriPolar, or PluriPolar systems). This article is also an encouragement to specialized researchers to interrogate the 'mythological teratology' using neutrosophic tools and perspectives, and it is —why not?— an attempt to transcend some common questions about the emergence and perpetuation of these quasi-general creatures in worldwide mythologies.

2. Neutrosophic Identities: Mythical Hybrids and Mythical SuperHybrids

Let us select a few examples from the vast array of imaginative fields, then briefly, but systematically analyze how these hybrids are composed. As a general structure of the hybrids, one might categorize as 'Mythical Hybrids' the creatures that might be formed by the union of two entities (be it part animal + part human, or part animal + part deity, or part human + part deity, or part demon + part human, or part demon + part animal, or part demon + part deity) and as 'Mythical SuperHybrids', or 'MultiAlist Hybrids' the creatures that might formed by union of at least three entities (e.g. part deity + part human + part animal), or by combinations within different types of the same entity (e.g. $Animal_1 + Animal_2 + \ldots + Animal_n$).

- In Mesopotamian mythology, *Gilgamesh* stands as a demigod, embodying a fusion of divine and mortal heritage. Described as two-thirds divine and one-third mortal, Gilgamesh's case presents a complex blend of maternal and paternal divinity. The duality of his nature, being both divine and mortal, becomes a central theme in the narrative, influencing his actions, struggles, and ultimate quest for meaning and immortality. The ambiguity surrounding his eventual fate in later traditions introduces an element of incompleteness, allowing for diverse interpretations [Gilgamesh].
- *Lamassu* is a protective deity in the Mesopotamian mythology with the body of a bull or lion, wings of an eagle, and a human head. These colossal beings often served as guardians at the entrances of palaces and temples [Lamassu].
- *Bhima* is one of the central characters in the ancient Indian epic, the *Mahabharata*. He is the second of the five Pandava brothers, born to Kunti, the queen of Hastinapura, and the wind god Vayu. Bhima is known for his exceptional physical strength, courage, and prowess in battle. Bhima plays a crucial role in various events, including the Pandavas' exile to the forest, the gambling match at the court of Hastinapura, and the Kurukshetra War, the epic battle between the Pandavas and the Kauravas. He is often portrayed as a larger-than-life figure, capable of extraordinary acts of valor and heroism. Despite his formidable strength, Bhima is also depicted as a compassionate and kind-hearted individual, especially towards those who are oppressed or marginalized. His journey is marked by moments of triumph and adversity, ultimately culminating in his role as a key figure in the establishment of dharma on the throne of Hastinapura [Bhima].
- *Achilles*, son of a sea nymph and a mortal king, encapsulates the dichotomy of the demigod in Greek mind. Immersed in the River Styx for invincibility, his vulnerable heel becomes a symbol of incompleteness. The neutrosophic nature unfolds in his death, orchestrated by a mortal's arrow guided by a god. This intersection of mortal susceptibility and divine vengeance underscores the intricacies of Achilles' fate [Achilles].
- *Freyr* is a prominent figure in Norse mythology, often depicted as a handsome and benevolent deity, associated with bountiful harvests, love, and abundance. Despite being considered a god in Norse mythology, Freyr's status as a demigod is underscored by his dual parentage, with one parent being a god (Njord) and the other a giantess named Skadi. Freyr's most famous possession

is his magical sword, known as "Freyr's Sword" or "Sumarbrandr." According to legend, Freyr traded away his sword to win the hand of the giantess Gerd, with whom he fell deeply in love. This act of sacrifice ultimately leads to Freyr's downfall, as he is left defenseless during the final battle of Ragnarok, the apocalyptic event in Norse mythology. [Freyr].

- In Islamic tradition, *Burāq* is a mythical creature described as a steed with the head of a woman, the wings of an eagle, and the tail of a peacock. It is said to have transported the Prophet Muhammad during the Night Journey. [Burāq]
- *Banshee*, known as the "woman of the fairies," is a supernatural being deeply ingrained in Irish and other Celtic folklore. Described as a female spirit or fairy, the Banshee is often associated with specific families or clans, serving as a harbinger of death or an omen of impending misfortune. One of the most distinctive features of the Banshee is her mournful wail, a chilling cry that is said to be heard when someone within the family she watches over is about to die. The Banshee is typically depicted as a solitary figure, often appearing as an old woman with long, flowing hair dressed in a grey or white gown, having features reminiscent of otherworldly or supernatural beings, such as elves, witches, or spectral apparitions. [Banshee]
- *Huli Jing*, or fox spirits, are shape-shifting beings in Chinese folklore. They can transform into beautiful women, but their true form is that of a fox. [Huli]
- The hybrid form of *Abraxas*, with the body of a human, the head of a rooster or lion, and serpentine legs, is laden with symbolic significance.[1] The human body represents consciousness and intellect, while the animal features symbolize primal instincts and cosmic forces. The rooster, associated with the dawn and awakening, signifies spiritual enlightenment, while the serpent is a potent symbol of wisdom, renewal, and transformation. Abraxas embodies the concept of unity within duality, symbolizing the reconciliation of opposites such as good and evil, creation and destruction, light and darkness. This neutrosophic fusion of contradictory elements reflects the Gnostic worldview, which conceived the material world as a realm of duality and illusion, with the ultimate goal of spiritual liberation through gnosis, or divine knowledge. [Abraxas]
- The concept of the *Homunculus* has historical roots in alchemical and philosophical traditions. In alchemy, the homunculus was believed to be created through various arcane processes, symbolizing the desire to artificially create life.[2] The most prevalent ingredient associated with the creation of the homunculus was seminal fluid, believed to contain the essence of life. In addition to semen, other bodily fluids, such as menstrual blood, were sometimes implied as ingredients. Alchemists often implied the use of various alchemical substances, such as salts, minerals, and herbal extracts, believed to possess transformative properties. Using animal seed or other animal-derived substances was suggested. The instructions for creating a homunculus would be as follows: 'Mix the semen and sun stone and inseminate the cow or ewe. Carefully plug the animal's vagina with the sun stone. Smear the animal's genitals with the blood of another animal. Place the artificially inseminated animal inside a dark house where the sun never shines.' And so on. [Lugt] From a neutrosophic perspective, the homunculus can be explored as a metaphor for the inherent uncertainties, contradictions, and complexities within biological, philosophical, and symbolic dimensions. The homunculus, as a symbol for the artificial creation of life, embodies ethical and existential implications. Though not entirely human, this entity represents a rational animal, adding another fictional chapter to humanity's aspiration to control the boundaries of life and death. [Homunculus].

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[1] References to Abraxas can be found in various Gnostic texts, including the Nag Hammadi Library, a collection of ancient Gnostic scriptures discovered in Egypt in 1945. These texts often depict Abraxas as a divine being or archon, sometimes associated with the highest God or as an emanation from the divine realm.

[2] One of the most famous accounts of creating a homunculus comes from the writings of the Swiss alchemist Paracelsus, who claimed that a homunculus could be created through the manipulation of seminal fluid. His supposed recipe involved placing semen in a sealed glass vessel along with specific alchemical substances and then incubating the mixture in a warm, dark environment for several weeks or months. Allegedly, after the incubation period, a miniature humanoid creature would form within the vessel.

- The *Balaur* occupies a prominent role within Romanian folklore, emerging as a multi-headed dragon or serpent,[3] often intertwined with turbulent weather phenomena and disruptive chaos. Intriguingly, Romanian legends infuse the Balaur's saliva with a peculiar trait, believed to possess the transformative ability to crystallize into 'diamonds'. Symbolically, the Balaur embodies primal energies, encapsulating the essence of chaos and the untamed wilderness, evoking a complex interplay of reverence and trepidation within cultural narratives.
- The *Zmeu*, another enthralling figure in Romanian folklore, assumes the guise of a shapeshifting monstrosity, blending traits of both ogre and dragon. Its versatility in assuming diverse forms epitomizes unpredictability, perpetuating an aura of enigmatic elusiveness. Frequently depicted as a malevolent force, the Zmeu embodies themes of fear, guile, and imminent danger, serving as a poignant cautionary motif within cultural tales. Furthermore, its portrayal reflects societal apprehensions, symbolizing latent threats, inner conflicts, and the enduring struggle between opposing moral forces.
- Among the spirits of Romanian folklore, the *Zburător* (The One That Flies) emerges as a seductive entity, akin to the incubus archetype.[4] Appearing in the form of a charismatic man, the Zburător seduces unsuspecting maidens, symbolizing forbidden desires and the allure of the unknown. Symbolically traversing the delicate threshold between dreams and reality, the Zburător evokes a blend of fear, fascination, and profound introspection within the human psyche. [Chelariu]

Let us try now to integrate these entities into a coherent system.

3. Offspring of Gods and Mortals: Part Deity + Part Human Hybrids

In the rich tapestry of world mythology, the entities formed of both divine and mortal lineage, often referred to as demigods or demigoddesses, embody a complex blend of traits inherited from their divine parentage and their mortal heritage. The mythologists generally agree that the hybrid beings of such type serve to bridge the gap between the celestial and terrestrial realms, reflecting humanity's innate desire to understand the nature of existence and our place within the cosmos.

Demigods possess a diverse array of characteristics and abilities: they may inherit superhuman strength, agility, or intellect from their divine lineage, while also facing the trials and tribulations of mortality. Despite their extraordinary abilities, demigods are often depicted as flawed and vulnerable beings, grappling with the complexities of their dual nature and the expectations placed upon them by both gods and mortals.

To explore several mythologies, let's illustrate the characters of three representative figures of such beings:

- *Krishna*. In Hindu mythology, Krishna is revered as an avatar of the god Vishnu, born as the son of Devaki, the sister of king Kamsa, and Vasudeva.[5] [Vishnu] Krishna declined to take up arms in the monumental conflict between the Kauravas and the Pandavas. Instead, he presented an option: to provide personal guidance to one side and lend his army to the other. A tragic altercation erupted among the Yadava chiefs, resulting in the loss of Krishna's brother and son. Deeply saddened, Krishna retreated to the forest. There, he met his demise when a huntsman, mistaking him for prey, fatally wounded him in his only vulnerable spot—the heel. [Krishna].
- *Hercules*, the son of Zeus and Alcmene, a mortal princess, is a cornerstone figure in Greek mythology, epitomizing the complexities of demigod existence. His divine parentage, coupled with mortal struggles, forms the essence of his narrative. His legendary Twelve Labors, undertaken as penance for killing his wife and children in a fit of madness induced by the goddess Hera, showcase his extraordinary strength, courage, and determination. Zeus, recognizing Hercules' deeds, grants

[3] Legends describe the Balaur as a monstrous being with sharp claws, scales covering its body, and fiery breath.

[4] Embedded within the broader folklore of incubi and succubi, the Zburător embodies the enigmatic male spirit seeking nocturnal liaisons with slumbering women.

[5] Vasudeva is the patronymic of the deity Krishna, a son of Vasudeva. The worshippers of Vasudeva-Krishna formed one of the earliest theistic devotional movements within Hinduism.

him immortality, emphasizing the dynamic interplay between mortality and the divine in the demigod's journey. [Hercules]

- *Dagda*, "The Good God," is prominent figure in Irish mythology, revered as a powerful and benevolent (semi)deity. He is often depicted as a father figure, a wise leader, and a skilled warrior, embodying the ideals of strength, wisdom, and generosity. Dagda is considered the father or chief of the Tuatha Dé Danann, a mythical race of supernatural beings. He possesses powerful magical artifacts, including a magical club known as the "lorg mór" or "the great staff," which can both kill and resurrect with one end and control the weather with the other. He also possesses a magical cauldron called the "coire ansic" or "the cauldron of plenty," which provides an endless supply of food and drink. He is known for his fondness for indulgence and pleasure, often portrayed as a lover of food, drink, and music. [Dagda]

3.1. Hindu Devas: Celestial Intermediaries of Cosmic Balance

In the intricate cosmology of Hindu mythology, demigods, or 'Devas,' occupy a pivotal role as celestial beings who straddle the realms of divinity and humanity. These demi-deities embody a complex interplay of attributes within the framework of neutrosophy, where the concepts of certainty, uncertainty, and indeterminacy converge in a dynamic equilibrium. [Deva]

The origins of demigods in Hindu mythology are shrouded in ambiguity, mirroring the inherent indeterminacy of the cosmic order. Born from the intermingling of divine and mortal lineage, demigods embody the neutrosophic principle of uncertainty, where their existence defies conventional categorization. They inhabit the liminal space between the certainty of divine essence and the uncertainty of mortal flesh.

Demigods in Hindu mythology hold profound cultural and symbolic significance, serving as allegorical representations of the cosmic balance between order and chaos. Through their stories and legends, demigods inspire introspection, reflection, and a deeper understanding of the neutrosophic principles that govern the universe. They offer timeless insights into the complexities of existence and the eternal quest for equilibrium and harmony.

Several demigods in Hindu mythology exemplify the principles of neutrosophy through their complex and multifaceted nature.

- *Indra*, the king of the Devas, embodies the neutrosophic principle of opposition as he wages war against the forces of darkness while facing internal conflicts and moral dilemmas.
- *Agni*, the god of fire, symbolizes the neutrosophic concept of indeterminacy as he serves as both a purifier and a destroyer, embodying the dual nature of fire as both creator and destroyer.
- *Varuna*, the god of cosmic waters, represents the neutrosophic notion of partial truth as he upholds the cosmic order while grappling with his own limitations and imperfections.
- *Surya*, the god of the sun, embodies the neutrosophic principle of ambiguity as he illuminates the universe with his radiant light while casting shadows of doubt and uncertainty.
- *Vayu*, the god of the wind, symbolizes the neutrosophic concept of complementarity as he breathes life into all living beings while also carrying the seeds of destruction and change.

3.2. Greek Demigods: Exploring Heroic Archetypes

In the vast and intricate tapestry of Greek mythology, demigods stand as remarkable figures, occupying a unique space between gods and mortals. Born of unions between divine beings and humans, these hybrid heroes possess extraordinary abilities, courageous hearts, and complex destinies that shape the course of myth and legend. This inherent duality reflects the neutrosophic principle of indeterminacy, where demigods exist in a state of perpetual flux, neither fully divine nor entirely mortal.

Among the pantheon of Greek demigods, several figures stand out as exemplars of heroic archetypes, each embodying the principles of neutrosophy in their own unique way.

- Heracles (Hercules), the son of Zeus and Alcmene, epitomizes the struggle against adversity and the quest for redemption. His Twelve Labors symbolize the neutrosophic concept of opposition, where seemingly contradictory forces coexist and interact in a dynamic equilibrium.
- Jason, the son of two mortals though conceived by Zeus, demonstrates leadership, charisma, and diplomacy as the leader of the Argonauts on their quest for the Golden Fleece. He navigates political intrigue and personal challenges with tact and resilience, embodying the principle of balance and harmony in relationships and endeavors.
- Another prominent demigod, Perseus, born of Zeus and a mortal princess, represents the neutrosophic principle of indeterminacy through his quest to slay the Gorgon Medusa and rescue Princess Andromeda. His journey is fraught with uncertainty and ambiguity, yet he perseveres through cunning, resourcefulness, and sheer determination. Perseus embodies the neutrosophic notion of partial truth, where reality is inherently subjective and open to interpretation. Perseus, embarks on legendary feats, including the slaying of the Gorgon Medusa. His death lacks a singular narrative, illustrating the incompleteness inherent in the portrayal of demigod destinies. Whether Perseus meets his end in battle or through a discus throw remains a subject of interpretation, adding an indeterminate layer to his demigod status. [Kerenyi]

3.3. Tuatha Dé Danann and Fomoire: Divine-Human Lineage in Celtic Mythology

Within the fabric of Celtic mythology, the Tuatha Dé Danann[6] emerge as enigmatic figures, embodying a complex fusion of divine and mortal lineage. The earliest accounts depict their banishment from heaven due to their profound knowledge, after which they descended upon Ireland enveloped in a shroud of mist. [Tuatha].

The ancient enemies of the Tuatha Dé Danann were the Fomoire, another group of supernatural beings in Irish mythology. The Fomoire are a race of monstrous, semi-divine beings, often described as a chaotic and malevolent force, associated with darkness, chaos, and the destructive forces of nature. They are portrayed as monstrous sea creatures or giants [Fomoire].

According to Irish mythology, the Fomoire were among the earliest inhabitants of Ireland, predating the arrival of the Tuatha Dé Danann. They were said to have arrived in Ireland from distant lands and waged war against the Tuatha Dé Danann for control of the island. The battles between the Fomoire and the Tuatha Dé Danann are depicted as cosmic struggles between the forces of chaos and order, with the Fomoire representing chaos and darkness,[7] and the Tuatha Dé Danann representing light and civilization.

The Tuatha Dé Danann are skilled in magic, shape-shifting, and other forms of arcane knowledge, yet they also experience human emotions, desires, and vulnerabilities. Several key figures among the Tuatha Dé Danann exemplify the neutrosophic themes of certainty, uncertainty, and indeterminacy through their complex parentage and lineage:

- The *Dagda*, for example, is sometimes portrayed as the son of the goddess Danu and the mortal prince Elatha, reflecting the intertwining of divine and human elements within his lineage.
- Similarly, *Lugh* is depicted as the son of a mortal man, Cian, and a supernatural being, Ethniu, highlighting his hybrid nature and the ambiguity of his identity.

4. Hybridization of Humanity and Demonology: Part Demon + Part Human Hybrids

Mythology often features beings that are hybrids part demon and part human. While the specific traits and appearances of these beings can vary widely across different cultures and mythologies, here are a few examples:

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[6] Translating to "the people of the goddess Danu," who represents the primordial waters of creation and fertility, these semi-divine beings are revered for their wisdom, magic, and profound connection to the land before the arrival of the Milesians, who are considered the ancestors of the modern Irish

[7] Despite their malevolent reputation, some stories depict individual Fomoire in a more sympathetic light, portraying them as complex characters with their own desires and motivations. In some accounts, Fomoire are depicted as skilled craftsmen and warriors, capable of great feats of strength and magic.

- *Cambions*: In European folklore, cambions are offspring of a demon and a human. They are often depicted as possessing some of the supernatural powers of their demonic parent, such as shapeshifting, telepathy, or dark magic, along with the physical appearance of humans. Cambions are sometimes portrayed as seductive and manipulative figures, using their powers to influence or deceive mortals.
- *Nephilim*: In Judeo-Christian mythology, nephilim are the offspring of angels[8] and humans. They are described in ancient texts like the Book of Genesis as giants or mighty warriors. Nephilim are often associated with themes of divine punishment and corruption, as their existence is seen as a perversion of the natural order.
- *Oni-Human Hybrids*: In Japanese folklore, oni are malevolent spirits or demons often depicted as large, ogre-like creatures with horns and wild hair. Sometimes, stories feature oni-human hybrids, typically resulting from unions between an oni and a human. These hybrids may inherit some of the oni's physical traits, such as horns or strength, as well as their mischievous or malicious nature.

These are just a few examples, and there are many other variations of partially demon and partially human beings found in mythologies around the world. They often serve as compelling figures in storytelling, embodying themes of the struggle between good and evil, the supernatural and the mundane, and the complexities of identity and heritage.

4.1. Cambions: A Hybrid of Demon and Human

Human imagination has conjured a plethora of mythical beings, among which *cambions* hold a particularly intriguing place. The concept of cambions finds its roots in Western European folklore, particularly within the framework of Christian demonology.[9] Cambions are creatures born of the union between a demon and a human, embodying a unique blend of the supernatural and the mortal. These unions are typically portrayed as acts of seduction, coercion, or temptation, reflecting broader themes of moral ambiguity and the struggle between good and evil. [Cambion]

The characteristics attributed to cambions vary across different mythological traditions, but they commonly possess a blend of supernatural powers and human vulnerabilities, with physical traits reminiscent of their demonic parentage, such as horns, fangs, or unnaturally colored eyes, but mostly possessing supernatural abilities, including shapeshifting, telepathy, or control over dark magic. In addition to their supernatural powers, cambions are often portrayed having charismatic and seductive demeanors, using their allure to manipulate and deceive mortals. This aspect of their character reflects themes of temptation and moral corruption, as cambions —neither fully human nor purely demonic— navigate the complexities of their dual heritage.

- *Merlin*, the legendary wizard from Arthurian mythology, is sometimes depicted as a cambion.[10] According to some versions of the tale, Merlin's father was an incubus who seduced his mother, a mortal woman. This union resulted in Merlin's birth, granting him his magical abilities.
- In William Shakespeare's play "The Tempest," *Caliban* is a half-human, half-demon creature who serves as one of the primary antagonists. While the exact nature of Caliban's parentage is not explicitly stated in the play, his origins are described as being monstrous and unnatural. Caliban's character embodies themes of colonization, power dynamics, and the clash between civilization and the wild.

As hybrid beings, cambions occupy a liminal space between the supernatural and the mortal. In Christian demonology, cambions are often viewed as embodiments of sin and moral decay, reflecting

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[8] Sometimes interpreted as fallen angels, i.e., demons.

[9] The word "cambion" is believed to derive from the Late Latin term "cambuca," which referred to a vessel used by Roman soldiers. Over time, it evolved to denote a vessel or receptacle for spirits, eventually coming to signify the offspring of demons.

[10] The story of Merlin as a cambion is explored in various medieval texts, including Geoffrey of Monmouth's *Historia Regum Britanniae*.

the consequences of succumbing to worldly desires. They serve as cautionary figures, warning against the dangers of indulgence and spiritual corruption.

4.2. Hybrid Giants and Divine Judgment: The Tale of the Nephilim

The *nephilim*[11] are mentioned in ancient texts such as the Bible and various apocryphal works [Barker], are shrouded in mystery and controversy, occupying a unique place in the tapestry of mythological lore, embodying themes of hybridity, rebellion, and divine judgment.

The nephilim are mentioned specifically in Genesis,[12] where they are described as the offspring of unions between "the sons of God" and "the daughters of men."[13]

In some interpretations, they are depicted as literal giants, possessing immense size and strength. Other sources describe them as beings of great wickedness or spiritual corruption, whose presence on Earth threatened the order established by God.

According to the biblical narrative, the presence of the nephilim on Earth prompted divine intervention, leading to the Great Flood as a means of cleansing the world of their corruption. Only Noah and his family were spared, as they were deemed righteous in the eyes of God. The story of the Nephilim serves as a cautionary tale about the consequences of moral corruption and divine judgment.

Two representative characters are:

- *Goliath*, the legendary giant from the biblical story of David and Goliath,[14] is often interpreted as a Nephilim or descendant of the Nephilim. Goliath is described in the biblical text as a giant, towering over his opponents with his imposing stature. His armor alone is said to have weighed hundreds of shekels of bronze, underscoring his formidable presence on the battlefield.
- *Og*, the king of Bashan, is a figure mentioned in the Hebrew Bible, specifically in the Old Testament. According to the biblical narrative, Og was one of the last remaining Rephaites, a group of giants who were known for their great stature and strength. The Rephaites were believed to be an ancient race of people who inhabited the land of Canaan before the Israelites arrived. Og's defeat is recounted as one of the victories achieved by the Israelites under the leadership of Moses.[15] [Nephilim]

The story of the nephilim has left a lasting impact on religious and cultural narratives throughout history. In Jewish and Christian traditions, they are often interpreted as symbols of rebellion, sin, and divine judgment, serving as cautionary figures, warning against the dangers of pride, corruption, and moral decay.

4.3. Oni-Human Hybrids: Intersections of Humanity and Demonology in Japanese Folklore

In Japanese folklore, the *oni* are formidable and malevolent spirits or demons known for their monstrous appearance and malicious behavior. Often depicted as horned, ogre-like creatures with wild hair and fearsome expressions, oni embody the darker aspects of the supernatural realm.

Oni-human hybrids represent a multifaceted archetype within Japanese folklore. Born from the union of humanity and demonology, these hybrids embody themes of power, temptation, and the moral transgressions. Their role as malevolent spirits capable of wreaking havoc upon humans reflects broader cultural anxieties surrounding the forces of darkness and chaos.

Oni-human hybrids, sometimes referred to as "half-oni" or "oni-kijo," inherit traits from both their demonic and human heritage. They may possess the physical characteristics of oni, such as horns, fangs, and exaggerated features, while also retaining elements of their human ancestry. These

[11] The term "Nephilim" is derived from the Hebrew word "nephiyl," which translates to "giants" or "fallen ones."
[12] The Holy Bible: Genesis 6:1-4.
[13] Their precise identity is a subject of debate among scholars, with interpretations ranging from fallen angels to divine beings or rulers.
[14] The Holy Bible: Samuel 1:17.
[15] The book of Deuteronomy provides further details about Og and his kingdom. In Deuteronomy 3:11, it is mentioned that Og's bed was made of iron and was more than thirteen feet long and six feet wide, indicating his enormous size.

hybrids are often depicted as powerful and fearsome beings, capable of both great strength and cunning intelligence. Despite their monstrous appearance, oni-human hybrids may exhibit complex emotions and motivations.

Two representative characters are:

- *Shuten-doji* is a legendary oni king who terrorized the ancient capital of Kyoto. According to folklore, Shuten-doji was believed to be a half-oni, born from the union between a human woman and the king of the oni. [Shuten]
- *Ibaraki-doji* is a female oni who is often depicted as a vengeful spirit seeking retribution for past injustices. [Ibaraki]

These and other oni-human hybrids populate the Japanese mythology, embodying themes of power, vengeance, and the struggle between humanity and the supernatural. From traditional folk tales and kabuki theater to modern manga and anime, oni-human hybrids remain enduring symbols of the supernatural and the fantastic. Their stories serve as cautionary tales, reminding audiences of the dangers of succumbing to temptation and the importance of moral integrity in the face of adversity.

5. Transcending Species: Part Human + Part Animal Hybrids

Part-human, part-animal mythological hybrids have fascinated cultures throughout history, appearing in myths, legends, and folklore around the world. These hybrids embody a fusion of human and animal traits, blurring the boundaries between the human and non-human realms. Here are some common examples of such hybrids:

- The Egyptian god *Anubis*, depicted with the body of a man and the head of a jackal, serves as a guide and protector of the dead, symbolizing the transition between life and death.
- In Greek mythology, the Centaur is a creature with the upper body of a human and the lower body of a horse. Centaurs are often depicted as possessing superhuman strength and agility, as well as a wild and untamed nature. They are associated with Dionysus, the god of wine and revelry, and are often depicted as participants in his ecstatic rites and celebrations. [Centaur]
- The *Minotaur*, another creature from Greek mythology, possesses the body of a human and the head of a bull. Confined within the labyrinth of Crete, the Minotaur symbolizes brute strength, primal aggression, and the darker aspects of human nature. [Minotaur]
- In Greek and Roman mythology, the *Harpy* is a creature with the body of a bird and the head of a woman. Often depicted as fierce and predatory, harpies symbolize chaos, violence, and the destructive forces of nature. [Harpy]
- Found in various cultures worldwide, the *werewolf* is a creature that can transform from human to wolf form, often associated with themes of lycanthropy and shapeshifting. [Werewolf]
- *Mermaids* and *mermen*, creatures with the upper body of a human and the lower body of a fish, appear in folklore and mythology from cultures around the world. They are often associated with the sea, symbolizing mystery, allure, and the unknown depths of the ocean. [Mermaid]

These part-human, part-animal hybrids serve as powerful symbols in mythology, representing a wide range of themes including the relationship between humanity and nature, the complexity of human identity, and the struggle between civilization and the primal instincts.

5.1. The Winged Man: Part Human + Part Bird Hybrids

The motif of the winged man, a figure with both human and avian characteristics, has appeared in various forms throughout mythology, folklore, and art across different cultures. This hybrid creature often symbolizes a fusion of earthly and celestial elements, embodying themes of freedom, transcendence, and the duality of human nature.

Let us explore the motif of the winged man in different cultural contexts, noting that some of them can be classified as SuperHybrids as well due to their associated divine nature:

- *Garuda* is a divine being in Hindu mythology, often depicted with a human upper body and wings, while the lower body resembles an eagle or bird. As the mount of the god Vishnu, Garuda symbolizes power, strength, and the ability to soar to great heights.
- *Horus* (Egyptian Mythology), the god of the sky and kingship, is sometimes depicted with the head of a falcon and the body of a man.
- *Icarus* (Greek Mythology) is perhaps the most iconic representation of the winged man. Alongside his father Daedalus, Icarus escapes imprisonment using wings crafted from feathers and wax. However, his disobedience leads to his tragic downfall as he flies too close to the sun, melting the wax and causing him to fall.
- *Eros/Cupid* (Greco-Roman Mythology): the Greek god of love, and his Roman counterpart Cupid, are occasionally portrayed with wings. This representation aligns with their association with the flighty and unpredictable nature of love.
- *Phoenix* (Various Cultures), while not a traditional winged man, the mythical bird that cyclically regenerates or reborn is sometimes depicted with human-like characteristics, especially in art and literature.
- *Fenghuang* (Chinese Mythology), also known as the Chinese phoenix, is a mythical bird with a mix of avian and human features. Often considered a symbol of harmony and balance, the Fenghuang embodies the union of opposites.
- *Angels* (Various Cultures), are often depicted as winged beings with a human-like appearance. In Christianity, angels are messengers of God, and artistic representations frequently portray them with wings, symbolizing their celestial nature.
- *Shangó* (Yoruba Mythology), the god of thunder and lightning, is sometimes depicted with wings. The wings emphasize his connection to the sky and his ability to move swiftly across the heavens.

The motif of the winged man resonates across cultures, illustrating humanity's fascination with the idea of transcending earthly limitations and reaching for higher realms. Whether representing divine messengers, mythical heroes, or symbolic creatures, the winged man motif captivated the human imagination and conveyed universal themes of aspiration and transcendence.

6. Gods and Beasts: Part God + Part Animal Hybrids

The concept of beings that are partially god and partially animal, with no human attributes or representations, is less common in mythology and folklore compared to those with human-like characteristics. However, there are still some examples from various cultural traditions around the world where such beings are found. These creatures often embody a unique blend of divine and animalistic qualities, serving as symbols of power, transformation. Here are a few examples:

- *Azure Dragon*: In Chinese mythology, the Azure Dragon is one of the four celestial guardians, representing the east and the spring season. The Azure Dragon is sometimes depicted as a SuperHybrid, a dragon with the body of a snake and the claws of a tiger, symbolizing power, vitality, and the cosmic forces of nature. He is associated with the element of wood and serves as a protector of the heavens.
- *Pegasus*: In Greek mythology, Pegasus is a divine winged horse,[16] born from the blood of the Gorgon Medusa after she was slain by the hero Perseus. He is associated with the god Poseidon and serves as a mount for heroes such as Bellerophon.
- In Norse mythology, *Fenrir* is a monstrous wolf, the offspring of the god Loki and the giantess Angrboða. Fenrir is depicted as a fearsome and powerful creature, destined to bring about the end of the world during Ragnarok.
- *Thunderbird*: In Native American mythology, the Thunderbird is a powerful and mythical bird, often depicted as a large bird of prey with the wingspan of an eagle and the feathers of a hawk

[16] Pegasus is often depicted as a majestic white horse with wings, symbolizing swiftness, freedom, and the divine realm. While Pegasus is not a deity himself, he is closely associated with the gods, particularly Zeus, the king of the gods, and Athena, the goddess of wisdom and war.

or owl. The Thunderbird is associated with thunderstorms, lightning, and the forces of the sky, serving as a symbol of power, transformation, and the spiritual connection between humans and nature.

These examples illustrate the diverse range of beings that embody the concept of entities partially god and partially animal, with no human attributes or representations, in mythology and folklore, serving as symbols of divine power, guardianship, or natural forces.

6.1. The Mystical Azure Dragon: A Chinese Celestial Guardian

The Azure Dragon, known as *Qinglong* in Chinese, is one of the four celestial guardians in Chinese mythology, along with the Vermilion Bird, the White Tiger, and the Black Tortoise. It is often depicted as a dragon with the body of a snake and the claws of a tiger, symbolizing the convergence of different animal attributes, in such case categorized as a SuperHybrid entity. The Azure Dragon is associated with the element of wood, the direction of east, and the season of spring, representing vitality, growth, and renewal. [Azure]

The Azure Dragon holds profound cultural and symbolic significance in Chinese mythology and society, serving as a protector of the heavens and a symbol of imperial power and authority.[17] It is closely associated with the Emperor of China and the concept of the Mandate of Heaven, representing the divine sanction of rulership and the cosmic order of the universe.

Through its hybrid form and multifaceted attributes, it invites contemplation of the neutrosophic principles of ambiguity, uncertainty, and indeterminacy. As a symbol of neutrosophic balance, the Azure Dragon navigates the complexities of existence, embodying the cyclical rhythms of nature and the interconnectedness of all living beings.

6.2. Thunderbird: A Native American Mythical Entity

In the Native American mythology, including the Ojibwe, Lakota, and Haida peoples, the *Thunderbird* emerges as a powerful and enigmatic symbol of the natural world and spiritual realms. Representing thunderstorms, lightning, and the forces of the sky, this mythical creature embodies the dynamic interplay between earthly and celestial forces.

It is often depicted as a large bird of prey, resembling an eagle or hawk, with wings spanning the heavens and feathers crackling with lightning. The Thunderbird is associated with thunderstorms, lightning, and the life-giving rains that nourish the earth. Its powerful presence symbolizes the awesome and unpredictable forces of nature. [Thunderbird]

It is a benevolent and awe-inspiring creature, bringing blessings of rain and prosperity to the land. In the traditions of the Ojibwe and other tribes, the Thunderbird is believed to inhabit the highest mountains and cliffs, from which it watches over the earth and sends forth lightning and thunder to cleanse and purify the land. Ceremonial dances and songs are performed to honor the Thunderbird and invoke its protection and guidance.[18]

Therefore, the Thunderbird embodies the mysteries of the natural world and spiritual realms in Native American mythology. As a symbol of divine power and natural forces, the Thunderbird constitues a timeless reminder of the interconnectedness of worlds and the cyclical rhythms of nature.

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[17] In Chinese mythology and folklore, the Azure Dragon is celebrated in various legends, rituals, and festivals. It is often depicted as a guardian deity, protecting sacred sites such as temples, palaces, and ancestral tombs. During the Qingming Festival (Tomb-Sweeping Day), offerings are made to the Azure Dragon to honor ancestors and seek blessings for the coming year. In art and literature, the Azure Dragon is depicted as a symbol of strength, resilience, and celestial beauty, inspiring awe and reverence among the Chinese people for centuries.

[18] The Thunderbird holds profound cultural and symbolic significance in Native American mythology and spirituality, serving as a guardian of the natural world and a messenger of the gods. It is closely associated with rituals, ceremonies, and traditions related to rainmaking, agriculture, and hunting. The Thunderbird is revered as a protector of the tribe and a symbol of strength, resilience, and spiritual renewal. Its presence in Native American art, dance, and oral traditions reflects the enduring reverence and awe inspired by this mythical creature.

6.3. Wings of Ambiguity: Pegasus

In the vast expanse of Greek mythology, *Pegasus* emerges as a symbol of boundless freedom, transcending the earthly realm with his majestic wings and divine grace. According to legend, Pegasus emerged from the blood of the slain Gorgon Medusa, born of the union between the earth and the sea. With his pristine white coat and wings of pure light, Pegasus embodies the ideal of divine beauty and grace. He is often depicted as a symbol of inspiration, carrying the thunderbolts of Zeus or the muses of Mount Helicon on his celestial journeys.[19] [Pegasus]

Pegasus, typically portrayed as a magnificent winged horse, embodies the mysteries of divine beauty and transcendence. Through his hybrid form and multifaceted attributes, he invites contemplation of the neutrosophic principles.

7. Mythical MultiAlist Entities: Part God + Part Human + Part Animal SuperHybrids

These hybrid creatures embody a complex blend of divine, mortal, and animalistic attributes, serving ─in mythologists' opinion─ as symbols of transformation, power, and the interconnection between different realms of existence. Here are some examples from different mythologies:

- In Egyptian mythology, *Thoth* is often depicted as a deity with the body of a human and the head of an ibis or a baboon. As the god of wisdom, writing, and magic, Thoth embodies the divine intellect and creative power of the gods, while also possessing human-like qualities such as intelligence and compassion. His hybrid form symbolizes the synthesis of divine knowledge and mortal understanding, serving as a guide and mediator between gods and humans. [Thoth]
- In Hindu mythology, *Hanuman* is a deity with the body of a human, but a monkey face, and the intelligence and powers of a god. He is revered as the devoted companion of Lord Rama and a symbol of strength, courage, and devotion. Hanuman's hybrid nature reflects his divine lineage as the son of the wind god Vayu and a celestial nymph, as well as his close association with the natural world and the animal kingdom.[20] [Hanuman]
- In Japanese folklore, *Tengu* are kite-like beings taking a human-like form, but retaining avian wings, heads, or beaks, and emdowed with the intelligence and powers of a god. Tengu are associated with mountains and forests, where they serve as guardians and tricksters, testing the virtues of travelers and monks. [Tengu]

These cases highlight the diverse range of beings that embody the concept of being partially god, partially human, and partially animal in mythology and folklore, thus being associated with multialistic features [MultiAlist]. In a neutrosophic context, such beings represent the inherent ambiguity and paradoxical nature of existence, existing in a state of *both-and*, rather than *either-or*.

7.1. The Sphinx: The Egyptian Guardian

In the timeless sands of Egyptian mythology and history, the Sphinx stands as a testament to the enigmatic blending of human and animal attributes, endowed with divine powers. Carved from the living rock, this iconic creature embodies a profound symbolism, serving as a guardian of knowledge, mystery, and cosmic balance.

The Sphinx finds its origins in the ancient Egyptian concept of the "shesep ankh," or "living image." It is typically depicted as a recumbent lion with a human head, often bearing the likeness of a pharaoh. This hybrid form symbolizes the union of divine kingship (represented by the lion) with

[19] Pegasus holds profound cultural and symbolic significance in Greek mythology and society, serving as a symbol of divine inspiration, creativity, and transcendence. He is closely associated with the muses of Mount Helicon, who were said to have nurtured him with the waters of the Pierian Spring. Pegasus is also linked to the hero Bellerophon, whom he aided in his quest to slay the monstrous Chimera. Through his mythic adventures and legendary feats, Pegasus continues to inspire artists, poets, and dreamers to reach for the stars and pursue their loftiest aspirations.

[20] Hanuman is depicted with five faces, symbolizing his divine power and illustrating a narrative from one of his tales. In an episode where he aids Rama (specifically, rescuing Rama from the demon Ahiravana, Ravana's brother), Hanuman needed to extinguish five lamps simultaneously to defeat Ahiravana. To accomplish this task, he manifested five heads, each facing a different direction where the lamps were located.

human intelligence and wisdom (embodied by the human head). The Sphinx serves as a guardian of sacred spaces, such as the entrance to temples or the avenues leading to royal tombs, as well as a protector of cosmic order and the cycle of life and death.[21] [Sphinx]

7.2. Mythic Meld: The Intersection of Gods, Humans, and Animals in Hindu mythology

The concept of multialist hybrids —beings that are partially god, partially human, and partially animal—, is richly depicted in various tales and legends of Hindu mythology. These hybrid entities embody the intricate interplay between the divine, human, and animal realms, thus offering another neutrosophic context.

One prominent example of such hybrid in Hindu mythology is *Hanuman*, the monkey-faced deity known for his unwavering devotion to Lord Rama. Hanuman is revered as the epitome of loyalty, strength, and courage, possessing divine attributes as well as animalistic traits. His physical appearance, with a human body adorned with a monkey's face and tail, reflects his multialistic nature as both a divine being and a mixed creature of the natural world.

Another multialist hybrid is *Narasimha*, the half-man, half-lion incarnation of Lord Vishnu. According to Hindu mythology, Narasimha emerged to protect his devotee Prahlada from his tyrannical father, the demon king Hiranyakashipu. With the body of a man and the head and claws of a lion, Narasimha embodies the ferocity and power of the animal kingdom, combined with the intellect and compassion of humanity. [Narasimha]

In a neutrosophic context, these multi-alist hybrids challenge conventional notions of identity and categorization, existing at the intersection of multiple domains of existence. They embody the paradoxical nature of reality, simultaneously embodying divine, human, and animal attributes, transcending binary distinctions and embracing the multialist possibilities of the universe.

7.3. Centzon Totochtin: Aztec Rabbit Deities

In the vibrant tapestry of Aztec mythology, the *Centzon Totochtin* stand as enigmatic figures, embodying the complex interplay between divine, human and animal realms. Translating to "Four Hundred Rabbits"[22] in Nahuatl, the language of the Aztecs, these divine rabbit represents fertility, abundance, and the celebration of life. The Centzon Totochtin trace their origins to the union of the goddess Mayahuel, the deity of maguey plants, and the god Patecatl, the deity of pulque.

The 400 gods are often depicted as humanoid figures with rabbit-like features, such as long ears, whiskers, and sometimes a fluffy tail. [Centzon]

One notable example is the rabbit deity *Ometotchtli*, who presides over drunkenness and revelry, embodying the festive spirit of Aztec culture.[23] Another example is *Tepoztecatl*, the rabbit god of pulque, who oversees the fermentation and consumption of the sacred beverage.

8. Remarks & Open Questions

[21] The Sphinx is closely associated with the god Atum-Ra, the sun god and creator deity. One famous example is the Great Sphinx of Giza, which stands in front of the Pyramid of Khafre and is believed to embody the pharaoh himself, serving as his eternal protector and guide in the afterlife. Another example is the Sphinx of Amenemhat II, which guards the entrance to the temple of the god Amun-Ra at Tanis.

22 In Aztec culture, the number 400 held significant symbolism, particularly in relation to time and calendrical systems. The Aztecs used a complex calendar system composed of several interlocking cycles, one of which was the "xiuhpohualli," or the agricultural calendar, which consisted of 18 months of 20 days each, plus an additional 5 "unlucky days" at the end. The number 400 is relevant because it corresponds to the length of one "xiuhpohualli" cycle, which is comprised of 20 "veintenas" (cycles of 20 days), each lasting 20 days. When multiplied together, 20 veintenas x 20 days = 400 days. After completing one cycle of 400 days, the calendar would restart, beginning a new cycle. This cyclical nature of time represented by the number 400 was significant in Aztec cosmology and rituals, as it reflected the continuous cycle of life, death, and rebirth observed in the natural world. Additionally, the number 400 was associated with concepts of completion, renewal, and the cyclical nature of existence in Aztec belief systems.

[23] They participate in festive celebrations, such as the Huey Tozoztli festival, where offerings of food, drink, and flowers are made in their honor. The legacy of the Centzon Totochtin continues to resonate in Mexican culture today, where rabbits are revered as symbols of fertility, abundance, and the renewal of life.

8.1. Hybrid Beasts and SuperHybrid Beasts: Part $Animal_1$ + Part $Animal_2$ (+ … + Part $Animal_n$) Hybrids

Hybrid Beasts and SuperHybrid Beasts are fantastical creatures with a combination of features from different animals, abounding in folklore and mythology across cultures. Here are some examples of such beasts:

- *Anzû* (Sumerian Mythology) is a divine storm bird, often depicted as an eagle with a lion's head. It is associated with the heavens and sometimes considered a symbol of chaos.
- *Ammit* (Egyptian Mythology), also known as the "Devourer of the Dead," is a creature with the head of a crocodile, the forelimbs of a lion, and the hind limbs of a hippopotamus. It is said to devour the hearts of the unworthy during the judgment of the dead.
- *Chimera* (Greek Mythology) is a fire-breathing SuperHybrid monster with the body of a lion, the head of a goat, and a serpent's tail.
- *Griffin* (Various Cultures) is a legendary creature with the body of a lion and the head of an eagle, often associated with guarding treasures.
- *Hippogriff* (European Mythology) is a legendary creature with the front half of an eagle and the hind half of a horse.
- *Qilin* (Chinese Mythology) is a mythical creature with the body of a deer, tail of an ox, hooves of a horse, and sometimes features like a dragon or lion.
- *Baku* (Japanese Mythology) is a supernatural creature that is part elephant, part lion, and part tiger. It is believed to devour nightmares.
- *Nue* (Japanese Folklore) is a chimera-like creature, featuring the head of a monkey, the body of a tanuki (raccoon dog), the limbs of a tiger, and a snake for a tail. It is associated with ill omens.
- *Camahueto* (Mapuche Mythology, South America) is a creature with the body of a calf and a spiral-shaped horn. It is considered a powerful and sacred being.
- *Jackalope* (North American Modern Folklore) is a whimsical creature with the body of a jackrabbit and antlers like those of an antelope or deer. It is a product of American tall tales.

These creatures often embody the blending of different species and frequently serve as symbols, metaphors, or explanations for natural phenomena, embodying cultural beliefs and values.

8.2. Hybrids and SuperHybrids in Art & Fiction

The fascination with hybrids and superhybrids permeates various forms of art and fiction, captivating audiences across cultures and genres. From classical paintings to contemporary literature, these fantastical beings have seized the imagination of creators and audiences alike, transcending boundaries of time and medium.

In the realm of visual arts, depictions of hybrids and superhybrids have adorned canvases for centuries, often serving as symbols of the extraordinary and the otherworldly. Artists throughout history have been drawn to the concept of merging disparate elements from the natural world to create creatures that defy conventional classification. Whether it's the centaurs of Greek mythology, the sphinxes of ancient Egypt, or the futuristic cyborgs of science fiction, artists have explored the boundaries of imagination through their portrayals of these hybrid beings.

Similarly, in the world of literature and fiction, hybrids and superhybrids have emerged as popular subjects, enriching narratives with their complex characters and fantastical worlds. From ancient myths and legends to modern-day novels and comics, these beings inhabit stories that explore themes of identity, transformation, and the interplay between humanity and the unknown. Whether they're portrayed as heroes, villains, or something in between, hybrids and superhybrids challenge readers to question the nature of existence and the limits of imagination.

For example, *Pegasus*, the majestic winged horse of Greek mythology, occupies a prominent place in the collective imagination, celebrated across numerous legends, tales, and artistic renderings throughout history, from ancient vase paintings to modern-day literature. These artistic depictions often portrayed him in full flight, his powerful wings outstretched as he soared through the heavens, e.g. the mosaic of Pegasus found at the House of Dionysus in Paphos, Cyprus. In this mosaic, Pegasus is depicted with stunning detail and elegance, carrying the muses on his back as he ascends into the sky. In addition to visual art, Pegasus has left an indelible mark on literature, inspiring some of the greatest poets and writers of antiquity. In the epic works of Homer, Hesiod, and Pindar, Pegasus is celebrated for his mythic beauty, grace, and divine lineage.

In literature, the concept of the *homunculus* has been reimagined and adapted in various ways. It has appeared in works of fiction, such as Mary Shelley's “Frankenstein” where the scientist Victor Frankenstein creates a humanoid creature through scientific experimentation. Similarly, in Johann Wolfgang von Goethe's “Faust,” the character of Faust conjures a homunculus through magical means.

In modern literature, film, and popular culture, *cambions* continue to captivate audiences with their enigmatic allure and complex motivations. From Anne Rice's “The Witching Hour” to the television series “Supernatural,” created by Eric Kripke, cambions have been reimagined and reinvented, each iteration offering new insights into their mythological origins and cultural significance.

The *Tuatha Dé Danann* and *Fomoire* have left a lasting impact on Irish culture and folklore. Their stories and legends continue to be celebrated in literature, art, and popular culture. Many landmarks and geographical features in Ireland are associated with these mythical beings, contributing to the cultural landscape of the country. The novel “A Portrait of the Artist as a Young Man” by James Joyce includes references to these entitities. Also, the novel “An Only Child” by Frank O'Connor features such elements of Celtic folklore and mythology, and the same in “American Gods” by Neil Gaiman. The animated film “The Secret of Kells”, directed by Tomm Moore and Nora Twomey, draws inspiration from the same mythological aspects. The character Hellboy in “Hellboy” comic series by Mike Mignola encounters creatures inspired by Celtic mythology. And so forth.

In today's digital age, the popularity of hybrids and superhybrids shows no signs of waning. They continue to inspire artists, writers, filmmakers, and creators across various mediums, fueling a creative renaissance that pushes the boundaries of storytelling and artistic expression.

Certainly, contemporary writers and artists possess the creative potential to craft a diverse array of Hybrids and SuperHybrids, leveraging modern tools such as generative artificial intelligence to push the boundaries of imagination.

One intriguing possibility lies in the creation of novel beings that blend elements from disparate realms, such as the envisioned God-Human-Demon SuperHybrid. This entity embodies a fusion of divine, human, and demonic attributes, offering a complex and multifaceted character ripe for exploration in both fictional narratives and visual art.

Moreover, artists and writers can employ varying degrees of composition to construct these imaginative entities, allowing for a nuanced approach to their creation. For instance, one could specify the proportions of each constituent component, delineating the precise makeup of the hybrid being. This approach introduces a level of granularity and specificity, enabling creators to tailor the characteristics of their creations according to their artistic vision.

For example, a hypothetical hybrid might be described as 30% Demigod, 47% Demon, 3% Cambion, 15% Animal, and 5% Human. Each percentage represents a distinct aspect of the hybrid's nature, contributing to its overall identity and narrative significance. Through this detailed approach, creators can imbue their creations with depth and complexity, inviting audiences to contemplate the interplay of different forces and identities within these fantastical beings.

In essence, *the creative possibilities afforded by contemporary tools and techniques enable artists and writers to explore new frontiers in the realm of mythology and fantasy*. By harnessing the power of imagination and innovation, they can breathe life into a diverse cast of characters. *Through experimentation with the new AI tools, Hybrids and SuperHybrids will continue to evolve*.

8.3. ‘Mythological Teratology’ and Open Questions

Teratology[24] has traversed a fascinating journey through history, evolving from a discourse on prodigies and marvels to a scientific field that explores congenital malformations and their causes. This interdisciplinary realm intersects with developmental biology, embryology, and genetics, delving into the study of abnormalities in physiological development. In the modern context, teratology encompasses the medical examination of teratogenesis, congenital malformations, and individuals with significant malformations. The principles of teratogenesis provide a foundational framework for understanding the effects of environmental agents on developing organisms. These principles consider factors such as genotype, exposure timing, and environmental interactions, guiding research in teratogenic agents.

However, the roots of teratology extend deep into antiquity, where figures like Phlegon of Tralles, a prominent paradoxographer from the first and second centuries CE, meticulously chronicled extraordinary narratives in his magnum opus, "Peri thaumasion" ("Book of Wonders") [Hatzopoulos]. Phlegon's accounts, along with those of Pliny the Elder and other ancient scholars, offer glimpses into a world where anomalies were observed with a blend of astonishment and intellectual curiosity. Immersing oneself in Phlegon's narratives offers a journey into a realm where anomalies are not merely observed but chronicled, encompassing accounts of hermaphrodites, individuals undergoing sex transformations, and instances of unusual births. [Nutton] Accounts from travelers as documented in the *Natural History* of Pliny the Elder further elaborate on the existence of fantastical beings in distant lands, such as individuals with a dog's head resembling baboons, those with a single tall foot (sciapodes), or beings with faces embedded in their chests (referred to as acephala).

Ancient narratives often described individuals with anatomical anomalies, such as hermaphrodites or individuals lacking mouths or noses. These accounts, along with the tapestry of mythical monsters found in global folklore, including giants, cyclops, centaurs, and so forth, raise intriguing questions about the relationship between medical abnormalities and mythical creatures.

Could ancient accounts of marvels and hybrids have emerged as reflections of medical abnormalities? Could modern teratology provide insights into the emergence and perpetuation of mythical monsters? The concept of "Mythological Teratology" arises as a prospective pathway for exploring this intersection between myth and reality, bridging the gap between ancient lore and modern science.

Utilizing generative artificial intelligence to compare current knowledge on malformations with mythical Hybrids and SuperHybrids opens new avenues for research and exploration. By delving into this mixed study, researchers may uncover insights into the origins of mythical creatures and gain a deeper understanding of the intricate relationship between myth and reality.

8.4. Exploring the Intricacies of Heredity: Attributing Divine Paternity and Other Open Questions

Investigating the hereditary lineage of hybrid beings throughout diverse historical periods and cultural contexts unveils a fluid and intriguing terrain, prompting some bold inquiries that may challenge conventional notions.

As previously emphasized, in Greek mythology, divine paternity is a recurrent theme, exemplified by the numerous instances of gods fathering demigods with mortal women. Zeus, the king of the gods, is particularly renowned for his amorous escapades with mortals, leading to the birth of heroes like Hercules, Perseus, and Helen of Troy. These demigods inherit traits and abilities from their divine fathers, creating a neutrosophic blend of mortal and divine characteristics.

Similarly, in Roman mythology, the god Mars is considered the divine father of Romulus and Remus, the legendary founders of Rome.[25] This divine paternity adds a sacred dimension to the

[24] Originating from the Greek word "τέρας" meaning "sign sent by the gods, portent, marvel, monster".

[25] As mentioned in Livy's *History*, Rhea Silvia claimed that Mars was the father of her twins. Attributing divine paternity was not new. Alcmene, the mother of Heracles (Hercules), attributed the paternity of her son to Zeus. According to the myth, Zeus disguised himself as Alcmene's husband, Amphitryon, leading to the birth of the heroic demigod Heracles. Danaë, the mother of Perseus, claimed that Zeus impregnated her in the form of a shower of gold. Semele, mother of

origin of the Roman civilization, emphasizing the divine guidance and protection bestowed upon the city through its founding figures.

In Hinduism, the concept of divine paternity is embodied in stories from the ancient scriptures. For instance, Lord Rama, a revered deity, is believed to be the son of King Dasharatha and the result of divine intervention. Lord Krishna, another significant deity in Hinduism, is said to be born to mortal parents but with a divine purpose, emphasizing the divine's direct involvement in human affairs.

Ancient Egyptian mythology also features the concept of divine paternity, with stories of pharaohs being considered divine descendants of gods. The pharaohs were believed to be the offspring of deities like Ra or Osiris, highlighting their divine right to rule and connecting the earthly and divine realms.

The concept of divine paternity is not confined to classical mythologies; it also finds expression in indigenous beliefs and modern religions. In Native American cultures, for instance, there are stories of gods or spirits fathering heroes or important figures.

Christianity, with its foundational story of the Virgin Mary conceiving Jesus through the Holy Spirit, embodies a unique form of divine paternity. Jesus is considered the Son of God, and this divine parentage holds profound theological significance within Christian doctrine.

In each cultural context, the concept of divine paternity serves various purposes. It can explain the extraordinary qualities or destinies of certain individuals, reinforce the divine connection between gods and humans, or legitimize the rule of certain lineages. While the specifics vary, the overarching theme remains a fascinating exploration of the intersection between 'non-natural' and 'natural' beings, shaping cultural narratives, religious beliefs, and societal structures across diverse civilizations.

On the other hand, in the play "Eumenides" by Aeschylus, the god Apollo claims the father's share in heredity is 100%, reflecting an ancient belief in paternal dominance. Aeschines tells of the Amphictyons who cursed perpetrators of sacrilege by wishing upon them the birth of children that do not resemble their parents, but monsters.

To look at other aspects as well, in the southwestern state of Kerala in India, matrilineal communities coexist until nowadays with the prevalent patrilineal system. Lineage and inheritance are traced through the female line, challenging the notion of a standardized understanding of heredity within the same cultural and national context. The Mosuo people, a small ethnic group in China, practice as well a form of matrilineal society where lineage and family property are passed down through the female line. The absence of formal marriages adds another layer to their cultural variation, challenging the notion of a standardized understanding of heredity prevalent in ancient times.

In Laurence Sterne's "Tristram Shandy," there is a mockery of the homunculus theory, emphasizing the lack of empirical knowledge and the speculative nature of theories regarding the transmission of traits from parent to offspring. The evolution of scientific knowledge, including the discovery of chromosomes, meiosis, and fertilization, has significantly transformed our understanding of heredity.

Advancements in genomics have revealed that modern humans carry a (neutrosophic) percentage of their DNA inherited from Neanderthals, suggesting interbreeding between the two species. This genetic legacy is embedded in the DNA of contemporary humans, offering a tangible link to a shared ancestry that extends beyond the Homo sapiens lineage.

We wonder if this interbreeding has left traces in the collective memory. A navigation to the intricate interplay of neutrosophic elements within the realms of mythology —e.g. the hybridization between 'non-natural' beings and 'natural' beings, such as 'demigods' and 'cambions'—might share thematic elements reflective of the genetic interbreeding narrative. These hybrid beings often grapple with

Dionysus, insisted that Zeus was the father of her child. The god revealed his true form to her, but the divine radiance proved too much for Semele, leading to her demise. And so on.

complex identities, straddling different worlds, and possess extraordinary abilities that set them apart from ordinary humans.

What if the parallel narratives of hybrid beings, with obvious neutrosophic traits, born of 'non-natural'-'natural' unions, might have served as cultural echoes of our complex ancestry, as a cultural metaphor for the intricate mingling of distinct human species? What if the Neanderthal-Sapiens interbreeding, as well as other humanoid types' interbreeding, substantiated by genetic evidence, is prolonged in folklore, possibly finding a captivating reflection in the mythical realms of hybrid beings and identities? Is it possible for some parallels between scientific discoveries and mythological narratives to underline the enduring human fascination with the mysteries of our origins, and the imaginative ways in which we weave tales to make sense of our genetic heritage?

9. Conclusions

Different cultures perceive Hybrid and SuperHybrid mythological forms in varied ways. From Greek centaurs to Hindu Gandharvas, each mythological tradition contributes unique perspectives on the blending of human and non-human attributes. The symbolism associated with hybrid humans gains depth when analyzed through a neutrosophic lens. Whether viewed as symbols of chaos and monstrosity or as representations of harmony between different realms, the contradictory elements within these beings offer rich material for neutrosophic interpretation. The quest for identity is complex, as these beings navigate their existence on the blurred edges of humanity. These Hybrid and SuperHybrid entitities are powerful vehicles for philosophical contemplation and cultural exploration within the framework of neutrosophy.

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A Novel Method for Solving the Time-Dependent Shortest Path Problem under Bipolar Neutrosophic Fuzzy Arc Values

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Abstract. The Shortest path problem is highly relevant in our daily lives, addressing uncertainties like traffic conditions and weather variations. To handle such uncertainties, we utilize Fuzzy Numbers. This paper focuses on Bipolar Neutrosophic Fuzzy Numbers, which have dual positive and negative aspects. They provide a robust framework for representing arc (node/edge) weights, signifying uncertain travel times between nodes. Importantly, these weights can change over time in bipolar neutrosophic fuzzy graphs. Our study introduces an extended Bellman-Ford Algorithm for identifying optimal paths and minimum times with time-dependent Bipolar Neutrosophic Fuzzy arc weights. We demonstrate its effectiveness through a step-by-step numerical example and conduct a comparative analysis to evaluate its efficiency.

Keywords: Shortest Path Problem; Bipolar Neutrosophic Fuzzy Arc Weights; Bellman-ford Algorithm; Time-dependent Shortest path problem.

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1. **Introduction**

The Shortest Path Problem (SPP) is a fundamental concept in graph theory and optimization, focusing on finding the most efficient route between two points in a network. It has broad applications in various fields, including transportation, logistics, telecommunications, and computer science. At its core, the Shortest Path Problem aims to identify the path with the minimum total cost or distance among all possible routes connecting two nodes in a graph. The "cost" could represent various factors, such as time, distance, financial expenses, or any other relevant metric depending on the specific context.

The need for evaluation methods in solving the Shortest Path Problem arises from its pervasive applicability and the desire to optimize resource utilization, minimize travel time, and enhance overall efficiency. Evaluation methods assess the effectiveness and performance of algorithms designed to solve the Shortest Path Problem in different scenarios. Evaluation methods for the Shortest Path Problem involve assessing the accuracy, efficiency, and scalability of algorithms developed to solve it. These methods contribute not only to practical applications but also to the broader field of algorithmic research, promoting advancements in optimization techniques and algorithm design. Real-world scenarios often introduce complexities like traffic congestion and adverse weather conditions. To address these challenges, Lotfi Zadeh (1965) pioneered the concept of fuzzy set theory [1]. Zadeh's groundbreaking work extended traditional set theory to fuzzy sets (FS), which encompass membership functions ranging from 0 to 1. He further introduced the notion of linguistic variables, representing values in the form of natural languages. When these linguistic terms are expressed using fuzzy sets defined over a universal set, they form what is known as a fuzzy linguistic variable [2]. Fuzzy sets, which allow elements to have degrees of belongingness to a set, were generalized by Atanassov [4] who introduced intuitionistic fuzzy sets (IFS). IFSs have membership and non-membership functions that add up to at most one for each element. Atanassov [4] also proposed interval-valued intuitionistic fuzzy sets (IVIFS), which are a further extension of IFSs with intervals as membership and non-membership values. IVIFS have a geometric representation and various operations defined on them. IFSs and IVIFS are widely used in many practical problems. However it does not provide the neutral or indeterminacy, when there is need in neutrality or lack of knowledge in expressing. Neutrosophic sets are a concept developed by Florentin Smarandache [8] to deal with problems that involve neutrality or indeterminacy as a key factor. Neutrosophic sets have three components: membership, non-membership, and indeterminacy. Single valued neutrosophic set (SVNS) which is a subclass of neutrosophic set, where the truth, indeterminacy, and falsity membership functions take values in the standard unit interval $[0, 1]$ and its applications is proposed by Sujit Das et al. [9]. Then, the interval-valued neutrosophic fuzzy sets (IVNFS) was implemented by Broumi et al. [10] and its relational operators were discussed. Bipolar fuzzy sets by Lee [34] - m-polarFS, built upon established fuzzy theorems, serve as a valuable framework for addressing the dual aspects of positive and negative behavior within the human mind. Later, there arise a drawback for positive and negative membership, Ali [11] proposed Bipolar Neutrosophic fuzzy sets (BNFS) and its operations in decsion-making. The application of fuzzy set theory has proven highly efficient in handling data characterized by imprecision, inaccuracy, and vagueness. One significant problem it addresses is the Fuzzy Shortest Path Problem (FSPP), which involves finding optimal paths among nodes in a graph while optimizing an objective function in a fuzzy environment. In pioneering work, Dubois [13] proposed an algorithm to solve FSPP and determine optimal weights. Subsequently, Klein [15] analyzed FSPP in terms of fuzzy mathematical programming, paving the way for further research and extensions of the concept. Expanding on these foundations, Okada and Soper [19] introduced the Multiple Label Method for large random networks, offering a solution for FSPP. To address the limitations of conventional non-interactive approaches, the concept of the degree of possibility was proposed by Okada [20] , representing arc lengths using fuzzy numbers. Nayeem et al. [18] considered networks with interval-number and triangular fuzzy numbers, providing an algorithm that accommodates both types of uncertain numbers.

Recognizing the computational complexity of FSPP, Hernandes et al. [14] presented an repetitive method utilizing a generic index ranking function to compare fuzzy numbers. This approach also accounted for graphs with negative parameters. Building on these methods, Kumar [17] tackled interval-valued fuzzy numbers within FSPP and introduced an algorithm capable of solving both fuzzy shortest path length and crisp shortest path length problems. A compartive study between the floyd warshall and rectangular algorithm under fuzzy environment is implemetnted by Vidhya et al. [24].

In a different direction, Baba [16] illustrated a technique for the Intuitionistic Fuzzy SPP. Mukherjee [21] implemented Dijkstra's algorithm for solving the shortest path with intuitionistic fuzzy arc weights in a graph. Subsequently, Broumi et al. [10] conducted a comprehensive comparative study of all existing FSPP approaches, ultimately identifying the most suitable methods for uncertain environments. Dijkstra's algorithm was expanded to address the Neutrosophic Fuzzy Shortest Path Problem (NFSPP) by Broumi et al., as documented in [22]. The arc weights are expressed as neutrosophic numbers in this extension. An interval-valued neutrosophic set (IVNFS) was introduced to expand the representation beyond single-valued neutrosophic sets (SVNS). An algorithm was devised to handle arc values of this type, as detailed in the work by Dey et al. [23] Numerous studies have been conducted on the Neutrosophic Fuzzy Shortest Path Problem (NFSPP), including works such as [26] to [31]. Janani et al. introduced the concept of bipolar neutrosophic refined sets in their work [29]. Additionally, Broumi et al. addressed the shortest path problem within the framework of interval-valued Fermatean fuzzy numbers in their study [32].

Cakir et al. proposed widening the Dijkstra algorithm in the context of the Bipolar Neutrosophic Fuzzy Shortest Path Problem (BNFSPP), as presented in [33]. This extension was exemplified with a practical example and pseudocode.

This study aims to extend the concept of the bellman-ford algorithm under bipolar neutrosophic numbers for the time-dependent SPP. The Bellman-Ford can detect the presence of negative cycles in a graph, which is a critical feature in applications where identifying and addressing negative processes, such as in-network routing, is essential to prevent instability. The

bipolar neutrosophic numbers (BNN) are used in optimization problems where the objective has both positive and negative aspects. It is applicable in various engineering and operational optimization contexts. The development of these methods has motivated us to explore the application of time-dependent SPP with bipolar neutrosophic arc values using the Bellman-Ford algorithm, which, interestingly, has not been previously employed in the context of BNFS.
This work offers several key contributions:

(i) In this paper, we incorporate Bipolar Neutrosophic values as arc values, enriching the problem domain.

(ii)Furthermore, we extend the application of time-dependent Bipolar Neutrosophic numbers (TD-BNN) to the Bellman-Ford algorithm, enhancing its versatility.

(iii)To illustrate the practicality of our approach, we present a numerical example where we successfully identify optimal results.

(iv) Additionally, we conduct a comparative analysis to demonstrate the superior efficiency of our proposed method when compared to existing approaches.

The remaining sections of the paper are structured as follows: Section 2 introduces the fundamental definitions and concepts of BNFS. In Section 3, we delve into the mathematical formulation of the problem of the SPP in a bipolar neutrosophic fuzzy context. Our preferred technique is illustrated in Section 4. A numerical example in Section 5 demonstrates how our approach can applied in practice. Section 6 compares our method with other existing methods and identifies its drawbacks. The paper concludes with Section 7.

2. **Literature Review**

This section introduces the fundamental concepts of the time-dependent shortest path problem and provides an overview of the relevant literature associated with this study.

Time-Dependent Shortest Path Problem

The time-dependent shortest path problem (TD-SPP) involves finding the most efficient route between two points in a network, considering variable travel times at different points in time. TD-SPP algorithms need to be adaptable to different contexts and modes of transportation. Researchers work on models that can effectively represent and predict time-dependent changes in travel times. Researchers work on models that can effectively describe and forecast time-dependent changes in travel times. The study on time-dependent shortest path problem, its theoretical aspects, and an algorithm is proposed by Dean and Brain [36]. Androutsopoulos offers a method for solving k-SPP with time-dependent cost attributes [40]. A time-dependent algorithm for vehicle routing problems is introduced by Rabie et al. [44]. Then Huang et al. [45] proposed a method for the time-dealy neural network scenario. After several studies, Hansknecht [47] proposed a dynamic approach for TD-SPP and executed it computationally to handle significant circumstances. The constrained reliable shortest path problem in stochastic time-dependent networks (CRSP-STD) extends the reliable, the time-dependent, and the constrained shortest path problem is implemented by Matthias [48].

Fuzzy Time-dependent Shortest Path Problem.

Fuzzy time-dependent models provide a means to optimize routes under uncertain conditions. Decision-makers can consider the variability and fuzziness in travel times to make more robust and reliable decisions.

Huang et al. [42] proposed the fuzzy programming model for the travel time represented by fuzzy sets and time-dependent. Then, Huang [43] prolonged it for mixed fuzzy numbers for time-dependent networks. The label correcting method for the TD-SPP is extended by Kolovsky et al. [39]. Then, Liao et al. [41] devised a genetic algorithm to solve fuzzy constrained shortest path problems. In many real-world situations, opinions, attitudes, or sentiments are positive and negative. Bipolar neutrosophic (BN) sets provide a framework to model and represent such bipolarity, capturing both positive and negative aspects of information. Real-world networks are dynamic, and the conditions of edges may change over time. The bipolar neutrosophic approach can be applied to model the uncertainty associated with dynamic environments where edge weights are subject to change. The concept of bipolar neutrosophic graphs(BNG), their properties, and their classes of single-valued BNG is introduced by Hassan et al. [38]. Broumi et al. [46] suggested an approach for Bipolar Neutrosophic SPP. The minimal spanning tree for Bipolar neutrosophic graphs is proposed by Reddy et al. [37]. Additionally, a time-dependent Dijkstra algorithm was introduced for BNFSPP by Cakir et al. in [7]. While there is a significant body of literature dedicated to the FSPP and TD-SPP, there has been comparatively limited research focused on the intersection of these domains, known as Time-Dependent Fuzzy Shortest Path Problem(TD-FSPP). The literature in this specific area is relatively sparse, indicating a gap in the exploration of solutions that simultaneously consider both FSPP constraints and time-dependent SPP. This points to a potential avenue for further investigation and development in the field of bipolar neutrosophic network optimization.

3. Preliminaries

This section introduces the basic definitions of Intuitionistic fuzzy sets, Neutrosophic fuzzy sets, and Bipolar Neutrosophic fuzzy sets. It also examines their main properties and discusses arithmetic operations related to these sets.

Definition 3.1. An IFS $\mathfrak{I}$ on the universe $\mathfrak{Q}$ is defined by: $\mathfrak{I} = \{\mathfrak{x}, \mathfrak{m}(\mathfrak{x}), \mathfrak{n}(\mathfrak{x}) \mid \mathfrak{x} \in \mathfrak{Q}\}$ where $\mathfrak{m} : \mathfrak{Q} \to [0,1]$ and $\mathfrak{n} : \mathfrak{Q} \to [0,1]$ represents the membership and non-membership of each

$\mathfrak{x} \in \mathfrak{Q}$, respectively. Such that $0 \leq \mathfrak{m}(\mathfrak{x}) + \mathfrak{n}(\mathfrak{x}) \leq 1$ for all $\mathfrak{x} \in \mathfrak{Q}$. Hesitation or indeterminacy part can be calculated as $\pi(\mathfrak{x}) = 1 - (\sigma(\mathfrak{x}) + \rho(\mathfrak{x})$. In some of the real-life scenarios IFS can't work when $\sigma(\mathfrak{x}) + \rho(\mathfrak{x}) > 1$ [?] named as PFS, which is also known as IFS type 2 by [3].

Definition 3.2. [6] Let $\mathfrak{X}$ be the universe of discourse. Then $\mathfrak{N} = \{\langle \mathfrak{x}, \mathfrak{T}_N(\mathfrak{x}), \mathfrak{I}_N(\mathfrak{x}), \mathfrak{F}_N(\mathfrak{x})\rangle : \mathfrak{x} \in \mathfrak{X}\}$ is defined as Neutrosophic Fuzzy Set (NFS), where the truth-membership function is represented as "$\mathfrak{T}_N : \mathfrak{X} \to]^-0, 1^+[$", an interdeterminacy-membership function "$\mathfrak{I}_N : \mathfrak{X} \to]^-0, 1^+[$" and the falsity-membership function "$\mathfrak{F}_N : \mathfrak{X} \to]^-0, 1^+[$". The sum of $\mathfrak{T}(\mathfrak{X})$, $\mathfrak{I}(\mathfrak{X})$, $\mathfrak{F}(\mathfrak{X})$ has no restrictions, So

$$0 \leq \sup\, \mathfrak{T}_N(\mathfrak{x}) + \sup\, \mathfrak{I}_N(\mathfrak{X}) + \sup\, \mathfrak{F}_N(\mathfrak{X}) \leq 3 \tag{1}$$

Definition 3.3. [6]

Let $\mathfrak{X}$ be a universe of discourse and $\mathfrak{N}$ be a Bipolar Neutrosophic Fuzzy Set (BNFS) in $\mathfrak{X}$. Then $\mathfrak{N}$ can be expressed as $\mathfrak{N} = \{\langle \mathfrak{x}, \mathfrak{T}^+_N(\mathfrak{x}), \mathfrak{I}^+_N(\mathfrak{x}), \mathfrak{F}^+_N(\mathfrak{x}), \mathfrak{T}^-_N(\mathfrak{x}), \mathfrak{I}^-_N(\mathfrak{x}), \mathfrak{F}^-_N)\rangle : \mathfrak{x} \in \mathfrak{X}\}$ Where $\mathfrak{T}^+, \mathfrak{I}^+, \mathfrak{F}^+ : \mathfrak{X} \to [0,1]$ and $\mathfrak{T}^-, \mathfrak{I}^-, \mathfrak{F}^- : \mathfrak{X} \to [-1,0]$. The positive-membership degrees $\mathfrak{T}^+(\mathfrak{x})$", "$\mathfrak{I}^+(\mathfrak{x})$, $\mathfrak{F}^+(\mathfrak{x})$ represent the degree of truth, indeterminacy, and falsity of the element $\mathfrak{x}$ in the BNFS $\mathfrak{N}$. The negative-membership degrees $\mathfrak{T}^-(\mathfrak{x})$, "$\mathfrak{I}^-(\mathfrak{x})$, $\mathfrak{F}^-(\mathfrak{x})$ measure the degree of truth, indeterminacy, and falsity of the element $\mathfrak{x}$ in the opposite characteristic sets related to the BNFS $\mathfrak{N}$.

Definition 3.4. [6] Let $\tilde{\mathfrak{N}}_1 = (\mathfrak{T}^+_1, \mathfrak{I}^+_1, \mathfrak{F}^+_1, \mathfrak{T}^-_1, \mathfrak{I}^-_1, \mathfrak{F}^-_1)$ and $\tilde{\mathfrak{N}}_2 = (\mathfrak{T}^+_2, \mathfrak{I}^+_2, \mathfrak{F}^+_2, \mathfrak{T}^-_2, \mathfrak{I}^-_2, \mathfrak{F}^-_2)$ be two BNFS. The operations of BNFS are :

$$\tilde{\mathfrak{N}}_1 \oplus \tilde{\mathfrak{N}}_2 = \langle \mathfrak{T}^+_1 + \mathfrak{T}^+_2 - \mathfrak{T}^+_1\mathfrak{T}^+_2, \mathfrak{I}^+_1\mathfrak{I}^+_2, \mathfrak{F}^+_1\mathfrak{F}^+_2, -\mathfrak{T}^-_1\mathfrak{T}^-_2, -(\mathfrak{I}^-_1 - \mathfrak{I}^-_2 - \mathfrak{I}^-_1\mathfrak{I}^-_2), -(\mathfrak{F}^-_1 - \mathfrak{F}^-_2 - \mathfrak{F}^-_1\mathfrak{F}^-_2)\rangle \tag{2}$$

$$\tilde{\mathfrak{N}}_1 \otimes \tilde{\mathfrak{N}}_2 = \langle \mathfrak{T}^+_1\mathfrak{T}^+_2, \mathfrak{I}^+_1 + \mathfrak{I}^+_2 - \mathfrak{I}^+_1\mathfrak{I}^+_2, -\mathfrak{F}^+_1 + \mathfrak{F}^+_2 - \mathfrak{F}^+_1\mathfrak{F}^+_2, -(\mathfrak{T}^-_1 - \mathfrak{T}^-_2 - \mathfrak{T}^-_1\mathfrak{T}^-_2) - \mathfrak{I}^-_1\mathfrak{I}^-_2, -\mathfrak{F}^-_1\mathfrak{F}^-_2\rangle \tag{3}$$

where $\lambda \geq 0$

Definition 3.5. [6] Let $\tilde{\mathfrak{N}}_1 = (\mathfrak{T}^+_1, \mathfrak{I}^+_1, \mathfrak{F}^+_1, \mathfrak{T}^-_1, \mathfrak{I}^-_1, \mathfrak{F}^-_1)$ be the BNFS. The score function "$S(\tilde{\mathfrak{N}}_1)$" and the accuracy function "$\S(\tilde{\mathfrak{N}}_1)$ of a BNFS are defined as follows:

$$S(\tilde{\mathfrak{N}}_1) = \frac{\mathfrak{T}^+_1 + 1 - \mathfrak{I}^+_1 + 1 - \mathfrak{F}^+_1 + 1 - \mathfrak{T}^-_1 - \mathfrak{I}^-_1 - \mathfrak{F}^-_1}{6} \tag{4}$$

$$A(\tilde{\mathfrak{N}}_1) = \mathfrak{T}^+_1 - \mathfrak{F}^+_1 + \mathfrak{T}^-_1 - \mathfrak{F}^+_1 \tag{5}$$

Definition 3.6. [6] Let $\tilde{\mathfrak{N}}_1 = (\mathfrak{T}_1^+, \mathfrak{I}_1^+, \mathfrak{F}_1^+, \mathfrak{T}_1^-, \mathfrak{I}_1^-, \mathfrak{F}_1^-)$ and $\tilde{\mathfrak{N}}_2 = (\mathfrak{T}_2^+, \mathfrak{I}_2^+, \mathfrak{F}_2^+, \mathfrak{T}_2^-, \mathfrak{I}_2^-, \mathfrak{F}_2^-)$ be two BNFS. The comparison of two BNFS is defined as follows:

- If $S(\tilde{\mathfrak{N}}_1) > S(\tilde{\mathfrak{N}}_2)$ then $\tilde{\mathfrak{N}}_1 \succ \tilde{\mathfrak{N}}_2$
- If $S(\tilde{\mathfrak{N}}_1) < S(\tilde{\mathfrak{N}}_2)$ then $\tilde{\mathfrak{N}}_1 \prec \tilde{\mathfrak{N}}_2$
- If $S(\tilde{\mathfrak{N}}_1) = S(\tilde{\mathfrak{N}}_2)$ then $\tilde{\mathfrak{N}}_1 = \tilde{\mathfrak{N}}_2$

(1) f $A(\tilde{\mathfrak{N}}_1) > A(\tilde{\mathfrak{N}}_2)$ then $\tilde{\mathfrak{N}}_1 \succ \tilde{\mathfrak{N}}_2$
(2) If $A(\tilde{\mathfrak{N}}_1) < A(\tilde{\mathfrak{N}}_2)$ then $\tilde{\mathfrak{N}}_1 \prec \tilde{\mathfrak{N}}_2$
(3) If $A(\tilde{\mathfrak{N}}_1) = A(\tilde{\mathfrak{N}}_2)$ then $\tilde{\mathfrak{N}}_1 = \tilde{\mathfrak{N}}_2$

Remark 3.7. [6] A bipolar fuzzy set is a special case of a BNFS, which is a more general concept.

4. Bipolar Neutrosophic Fuzzy Shortest Path Problem

In this section, we outline the mathematical formulation of the Bipolar Neutrosophic Fuzzy Shortest Path Problem (BNFSPP).

Consider a directed graph, denoted as $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$, where $\mathfrak{V} = \mathfrak{s} = 1, 2, ..., \mathfrak{e} = \mathfrak{m}$ represents the set of vertices, and $\mathfrak{E} = (i, j) : i, j \in \mathfrak{V}, i \neq j$ represents the set of edges. In this representation, the ordered pairs (i, j) signify connections between distinct vertices within the graph, with both i and j belonging to the set of vertices $\mathfrak{V}$. It's important to note that in a connected network, there exists only one path from node i to node j, denoted as $\mathfrak{p}_{ij}$. This path comprises a sequence of arcs: $\mathfrak{p}_{ij} = (i, i_1), (i_1, i_2), ..., (i_k, j)$. Crucially, each arc starts at its source node and ends at its terminal/destination node.

The main objective is to find the best route from node $\mathfrak{S}$ (the origin/source) to node$\mathfrak{D}$ (the target/destination), considering various factors related to travel times that vary depending on the time. In the context of BNN, this parameter is represented as $\mathfrak{c}_{ij} = \langle \mathfrak{T}^+, \mathfrak{I}^+, \mathfrak{F}^+, \mathfrak{T}^-, \mathfrak{I}^-, \mathfrak{F}^- \rangle$. Here, the positive membership degrees $\mathfrak{T}^+$, $\mathfrak{I}^+$, and $\mathfrak{F}^+$ correspond to truth membership, indeterminate membership, and false membership, respectively. Similarly, the negative membership degrees $\mathfrak{T}^-$, $\mathfrak{I}^-$, and $\mathfrak{F}^-$ indicate truth membership, indeterminate membership, and false membership concerning the arc i-j. These memberships are associated with the shortest path in terms of travel time along the arc i-j. The values assigned to the arc under consideration represent the parameters associated with each edge from i to j. In this study, BNFN represent the imprecise parameters of the SPP. Consequently, the resulting problem is the BNFSPP. The mathematical model of the problem is can be expressed as:

$$\min \tilde{Z} = \sum_{i=1}^{m}\sum_{j=1}^{m} \tilde{\mathfrak{C}}^{P} x_{ij}$$
$$\text{s.t.} \sum_{j=1}^{m} x_{ij} - \sum_{k=1}^{m} x_{ki} = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1, m \\ -1 & i = m \end{cases} \tag{6}$$
$$x_{ij} \geq 0, i, j = 1, 2....m$$

The arc (i, j) is in the path if and only if $x_{ij} = 1$, otherwise $x_{ij} = 0$. The set of all paths from node $\mathfrak{s}$ to node $\mathfrak{t}$ is denoted by$T_{\mathfrak{st}}$. The Bipolar Neutrosophic fuzzy travel time of the path from the node $\mathfrak{u}$ to node $\mathfrak{v}$ is $\tilde{\mathfrak{C}}_{ij}^{P}$.

5. Proposed Algorithm

The Bellman dynamic programming is used to determine the shortest path by the forward pass calculation. The Extended Bellman-Ford with time-dependent dynamic programming under Bipolar Neutrosophic fuzzy numbers is formulated as:

Step 1: Set the distance from the source vertex as (Departure time).

$$\text{Initialization Step: Set the Source node as } \mathfrak{t}(1) = \tilde{t}_s(\text{Departure time})$$
$$\text{Main Step: } \mathfrak{t}(\alpha) = \min_{\alpha<\beta}[\mathfrak{t}(\alpha) + \mathfrak{w}_{\alpha\beta}] \tag{7}$$

Here $\mathfrak{w}_{\alpha\beta}$ is the directed Bipolar neutrosophic fuzzy time with nodes, $\mathfrak{t}(\alpha)$ is the BNF time of the SP from $\mathfrak{S}$ to $\mathfrak{D}$. Figure 1 illustrates the flowchart of the proposed method.

5.1. *Step-by-Step Procedure*

Here is a step by step procedure for the proposed algorithm:

(1) Set the distance of the source vertex to its departure time $\tilde{t}_s$ and set the distance of all other vertices to infinity.

(2) Repeat the following steps for the number of vertices minus one times in the Bipolar Neutrosophic Graph (BNG).

(3) For each edge (α, β) in the BNG, calculate the minimum value using the score function: $\mathfrak{t}(\alpha) = \min_{\alpha<\beta}[\mathfrak{t}(\alpha) + \mathfrak{w}_{\alpha\beta}]$

(4) Relaxation:
for i from 1 to V-1: for each edge$(\alpha, \beta) in the graph : if distance[\alpha] + weight(\alpha, \beta) < distance[\beta] : distance[\beta] = distance[\alpha] + weight(\alpha, \beta)$

(5) Check for Negative Cycles:

After N-1 iterations, check for any negative cycles in the BNG. For each edge (α, β) in the BNG:

If the distance of α plus the weight of the edge is less than that of β, report an error indicating the presence of a negative cycle.

(6) Output:

If no negative cycles are found, return the distance and previous arrays as the output. The distance array contains the Bipolar Neutrosophic shortest path distance from the source to each vertex, and the previous array contains the predecessor of each vertex in the Bipolar Neutrosophic shortest path.

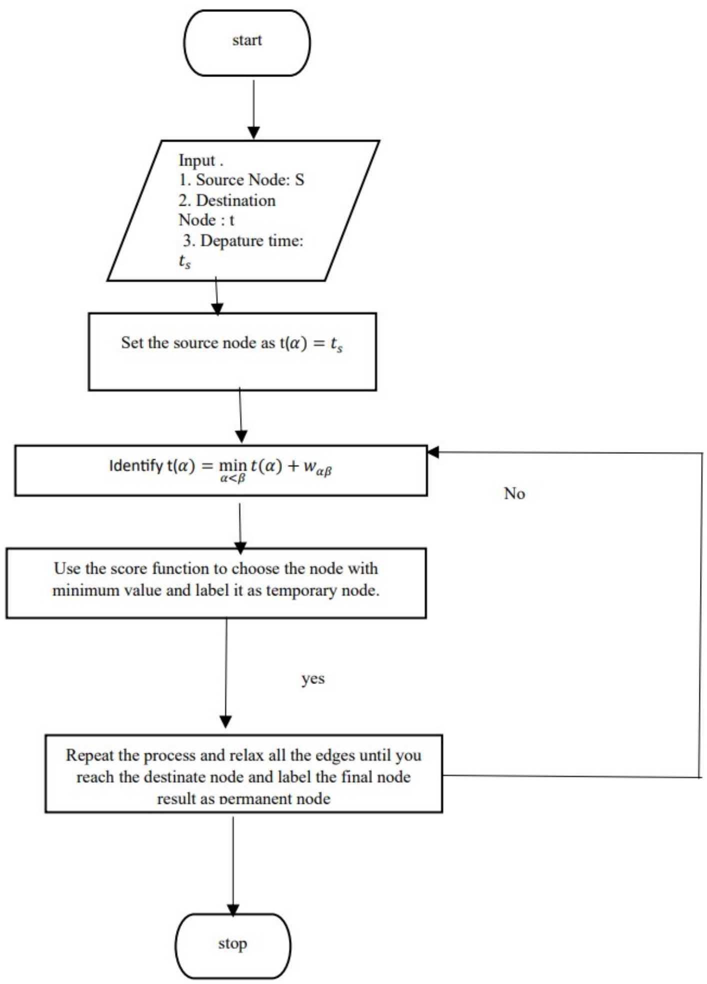

Figure 1. Flowchart for the proposed algorithm

5.2. *Improved Bellman-Ford Algorithm Pseudeocode*

Assume a graph G: with collections of vertices and edges. Let S be the source/origin vertex, and $\mathfrak{t}$ denote the distance array, which stores the shortest distances from S to all other vertices. Consider an array P to keep track of the predecessors of each vertex in the shortest path from S to that vertex. Fianlly , $\mathfrak{w}(U, V)$ denotes the weight of the edge connecting vertices U and V.

The pseudeocode for the proposed algorithm is given below in table 1:

Pesudeocode
function BellmanFord(G, S)
1. Initialize $\mathfrak{t}[S]$ *to* $\mathfrak{t}_s$ and $\mathfrak{t}[V]$ to infinity for all other vertices V in G
2. for each vertex V in G
3. if $V == S$ then
4. $\mathfrak{t}[V] = 0$
5. else
6.$\mathfrak{t}[V]$ = infinity
// Initialize P[V] to null for all vertices V in G for each vertex V in G
7. $P[V] = null$
8. Set a counter C to 0
9. $C = 0$
10. Repeat $\|V\| - 1$ times, where $\|V\|$ is the number of vertices in G while $C < \|V\| - 1$
11. For each edge (U, V) in G, check if $\mathfrak{t}[V]$ can be improved by using (U, V)
12. for each edge (U, V) in G
13. If $\mathfrak{t}[V] > \mathfrak{t}[U] + \mathfrak{w}(U, V)$ then update $\mathfrak{w}[V]$ and P[V]
14. if$\mathfrak{t}[V] > \mathfrak{t}[U] + \mathfrak{w}(U, V)$ then
15. $\mathfrak{t}[V] > \mathfrak{t}[U] + \mathfrak{w}(U, V)$
16. $P[V] = U$
17. Increment C by 1
18. $C = C + 1$
19. Check for negative cycles by relaxing the edges one more time for each edge (U, V) in G
20. If $\mathfrak{t}[V] > \mathfrak{t}[U] + \mathfrak{w}(U, V)$ then there is a negative weight cycle and shortest path does not exists
21. if $\mathfrak{t}[V] > \mathfrak{t}[U] + \mathfrak{w}(U, V)$ then
22. return "Negative cycle found"
23. Return the distance and parent arrays
24. return D, P

TABLE 1. Pesudecode for the Proposed Algorithm

6. Numerical Example

A numerical example from [7] involves a time-dependent network graph where BNFN denote the weights and the departure time $\mathfrak{t}_s = (0.2, 0.4, 0.5, -0.5, -0.7, -0.3)$

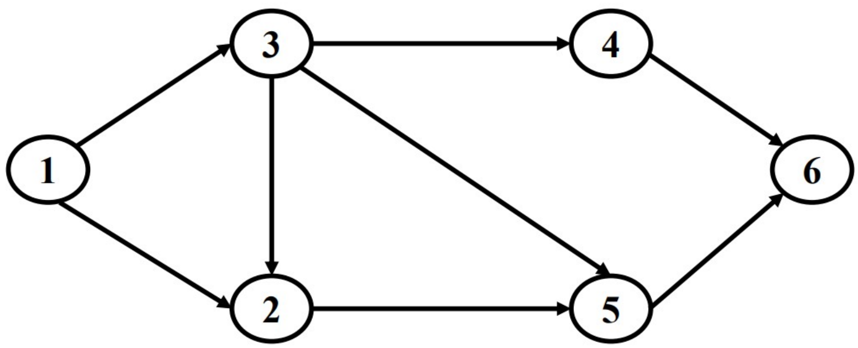

FIGURE 2. A Bipolar Neutrosophic Fuzzy Network Graph

TABLE 2. Arc values for Figure 2

Edges	**Time-dependent Bipolar Neutrosophic Fuzzy Arc Values**
$1 \to 2$	$(0.4, 0.6, 0.3, -0.5, -0.4, -0.3)$
$1 \to 3$	$(0.3, 0.8, 0.6, -0.7, -0.4, -0.2)$
$3 \to 2$	$(0.5, 0.3, 0.7, -0.4, -0.5, -0.4) - t$
$2 \to 5$	$(0.6, 0.8, 0.4, -0.7, -0.3, -0.2) * t$
$3 \to 4$	$(0.5, 0.3, 0.7, -0.4, -0.5, -0.1)$
$3 \to 5$	$(0.85, 0.3, 0.1, -0.2, -0.7, -0.8) + t$
$4 \to 6$	t
$5 \to 6$	$(0.7, 0.6, 0.2, -0.4, -0.4, -0.8)$

The best/optimal route from the $\mathfrak{S}$ to $\mathfrak{D}$ is described as follows using the suggested Algorithm 7.

Iteration 1: Begin with the source node. Assign the source node be t_s

$\mathfrak{t}(1) = (0.2, 0.4, 0.5, -0.5, -0.7, -0.3)$ and label the node as $\mathfrak{t}(1) = [(0.2, 0.4, 0.5, -0.5, -0.7, -0.3), 1]$

Iteration 2: Designate node 1 as α and node 2 as β. Proceed to ease the edges leading to node 2 employing the formula [7]. Utilize the scoring function [4] to select the minimum score and assign it as the temporary node.

$$\mathfrak{t}(2) = \min_{\alpha<2}\{\mathfrak{t}(\alpha) + \mathfrak{w}_{\alpha 2}\} = \mathfrak{t}(1) + \mathfrak{w}_{12} = (0.2, 0.4, 0.5, -0.5, -0.7, -0.3) + (0.4, 0.6, 0.3, -0.5, -0.4, -0.3)$$
$$= (0.52, 0.24, 0.15, -0.25, -0.82, -0.51)$$

$S(0.52, 0.24, 0.15, -0.25, -0.82, -0.51) = 0.615.$

Therefore, Label $\mathfrak{t}(2) = [(0.52, 0.24, 0.15, -0.25, -0.82, -0.51), 1 \to 2]$

Iteration 3: Iterate through the aforementioned procedure for node 3, relaxing all edges by applying the equation 7.

$$\mathfrak{t}(3) = \min_{\alpha<3}\{\mathfrak{t}(\alpha) + \mathfrak{w}_{\alpha 3}\} = \mathfrak{t}(1) + \mathfrak{w}_{13} = (0.2, 0.4, 0.5, -0.5, -0.7, -0.3) + (0.3, 0.8, 0.6, -0.7, -0.4, -0.2)$$
$$= (0.44, 0.8, 0.6, -0.7, -0.4, -0.2)$$

$S(0.44, 0.8, 0.6, -0.7, -0.4, -0.2) = 0.728.$

Therefore, Label $\mathfrak{t}(3) = [(0.44, 0.8, 0.6, -0.7, -0.4, -0.2), 1 \to 3]$

Iteration 4: Execute a similar sequence for node 4, wherein the edges reaching node 4 are relaxed using the equation denoted as [7]. Apply the score function identified as [4] to determine the minimum score. Subsequently, designate the node corresponding to this minimum score as the 'temporary node' for further analysis.

$$\mathfrak{t}(4) = \min_{\alpha<4}\{\mathfrak{t}(\alpha) + \mathfrak{w}_{\alpha 4}\} = \mathfrak{t}(3) + \mathfrak{w}_{34} = (0.44, 0.8, 0.6, -0.7, -0.4, -0.2) + (0.5, 0.3, 0.7, -0.4, -0.5, -0.1)$$
$$= (0.72, 0.096, 0.21, -0.14, -0.91, -0.49)$$

$S(0.72, 0.096, 0.21, -0.14, -0.91, -0.49) = 0.82.$

Therefore, Label $\mathfrak{t}(3) = [(0.72, 0.096, 0.21, -0.14, -0.91, -0.49), 1 \to 3 \to 4]$

Iteration 5: Apply the identical procedure to node 5, relaxing all edges towards node 4 with the equation [7]. Utilize the score function [4] to identify the minimum score, designating the node linked to this minimal score as the 'temporary node' for further analysis.

$$\mathfrak{t}(5) = \min_{\alpha<5}\{\mathfrak{t}(\alpha) + \mathfrak{w}_{\alpha 5}\}$$
$$= min\{\mathfrak{t}(3) + \mathfrak{w}_{35}, \mathfrak{t}(2) + \mathfrak{w}_{25}\}$$
$$= min\{(0.44, 0.8, 0.6, -0.7, -0.4, -0.2) + ((0.8, 0.3, 0.1, -0.2, -0.7, -0.8) + t),$$
$$(0.52, 0.24, 0.15, -0.25, -0.82, -0.51) + ((0.6, 0.8, 0.4, -0.7, -0.3, -0.2) * t)\}$$
$$= min\{(0.91, 0.038, 0.02, -0.04, -0.91, -0.92), (0.57, 0.21, 0.11, -0.21, -0.86, -0.54)\}$$

$S(0.91, 0.038, 0.02, -0.04, -0.91, -0.92) = 0.95.$

$S(0.57, 0.21, 0.11, -0.21, -0.86, -0.54) = 0.81.$

Therefore, the minimum values is choosen and Labeled $\mathfrak{t}(5) = [(0.57, 0.21, 0.11, -0.21, -0.86, -0.54), 1 \rightarrow 2 \rightarrow 5]$

Iteration 6: Ease all edges accessible from vertex 6 by employing the equation labeled as [7]. To determine the temporary node, apply the score function denoted as [4].

$$
\begin{aligned}
\mathfrak{t}(6) &= \min_{\alpha<6}\{\mathfrak{t}(\alpha) + \mathfrak{w}_{\alpha 6}\} \\
&= min\{\mathfrak{t}(4) + \mathfrak{w}_{46}, \mathfrak{t}(5) + \mathfrak{w}_{56}\} \\
&= min\{(0.72, 0.096, 0.21, -0.14, -0.91, -0.49) + (0.2, 0.4, 0.5, -0.5, -0.7, -0.3), \\
&(0.57, 0.21, 0.11, -0.21, -0.86, -0.54 + (0.7, 0.6, 0.2, -0.4, -0.4, -0.8)\} \\
&= min\{(0.78, 0.039, 0.11, -0.07, -0.97, -0.64), (0.87, 0.13, 0.02, -0.08, -0.92, -0.91)\}
\end{aligned}
$$

$S(0.78, 0.039, 0.11, -0.07, -0.97, -0.64) = 0.089.$

$S(0.57, 0.21, 0.11, -0.21, -0.86, -0.54) = 0.93.$

Therefore, the minimum values is choosen and Labeled $\mathfrak{t}(6) = [(0.78, 0.039, 0.11, -0.07, -0.97, -0.64), 1 \rightarrow 3 \rightarrow 4 \rightarrow 6]$

Hence the Optimal Path (node 1 to node 6) is $1 \rightarrow 3 \rightarrow 4 \rightarrow 6$) (Figure 3)along with travel time is (0.78,0.039,0.11,-0.07,-0.97,-0.64).

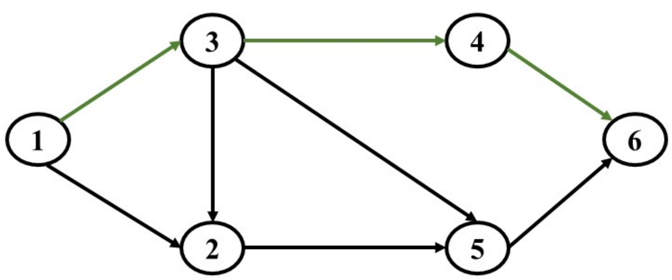

FIGURE 3. The shortest Path With Travel time using the proposed method

TABLE 3. Optimal Results of Shortest Travel Time for 2

Node	SP	BNFSP Travel Time	Score Value
2	$1 \rightarrow 2$	(0.52,0.24,0.15,-0.25,-0.82,-0.51)	0.615
3	$1 \rightarrow 3$	(0.44,0.8,0.6,-0.7,-0.4,-0.2)	0.738
4	$\rightarrow 3 \rightarrow 4$	(0.72,0.096,0.21,-0.14,-0.91,-0.49)	0.82
5	$1 \rightarrow 2 \rightarrow 5$	(0.57,0.21,0.11,-0.21,-0.86,-0.54)	0.81
6	$1 \rightarrow 3 \rightarrow 4 \rightarrow 6$	(0.78,0.039,0.11,-0.07,-0.97,-0.64)	0.089

7. Sensitivity Analysis

A sensitivity analysis was conducted to assess the performance of the proposed algorithm under various scenarios involving a time-dependent source node and variations in the arc values. The following cases were examined:

Case 1: Time-Dependent Source Node.
The algorithm was applied to a graph with a time-dependent source node.

Case 2: Zero Time-Dependent Value and Equal Arc Values
The time-dependent value was set to zero, and the arc values were kept constant as per the provided example.

Case 3: Zero Time-Dependent Value, Interchanged Arc Values.
The time-dependent value was set to zero, and the arc values (3,5) and (3,4) were interchanged.

Case 4: Time-Dependent Source Node with Interchanged Arc Values.
The source node had a time-dependent value, and the arc values (3,5) and (3,4) were interchanged.

Case 5: Time-Dependent Source Node with Interchanged Arc Value (1,2)
The source node had a time-dependent value, and the arc value (1,2) was considered in place of (2,5) and vice versa.

Case 6: Zero Time-Dependent Value with Interchanged Arc Value (1,2)
The time-dependent value was set to zero, and the arc value (1,2) was considered in place of

(2,5) and vice versa.

Table 4. Senstivity Analysis based on changing the values

Cases	SP	Bipolar Neutrosophic Number	Score Value
Case 1	$1 \to 3 \to 4 \to 6$	(0.78,0.039,0.11,-0.07,-0.97,-0.64)	0.089
Case 2	$1 \to 3 \to 4 \to 6$	(0.84,0.32,0.04,-0.17,-0.72,-0.87)	0.83
Case 3	$1 \to 3 \to 4 \to 6$	(0.91,0.04,0.02,-0.04,-0.91,-0.92)	0.79
Case 4	$1 \to 2 \to 5 \to 6$	(0.93,0.015,0.01,-0.02,-0.97,-0.94)	0.97
Case 5	$1 \to 3 \to 4 \to 6$	(0.78,0.039,0.11,-0.07,-0.97,-0.64)	0.79
Case 6	$1 \to 3 \to 4 \to 6$	(0.84,0.32,0.04,-0.17,-0.72,-0.87)	0.83

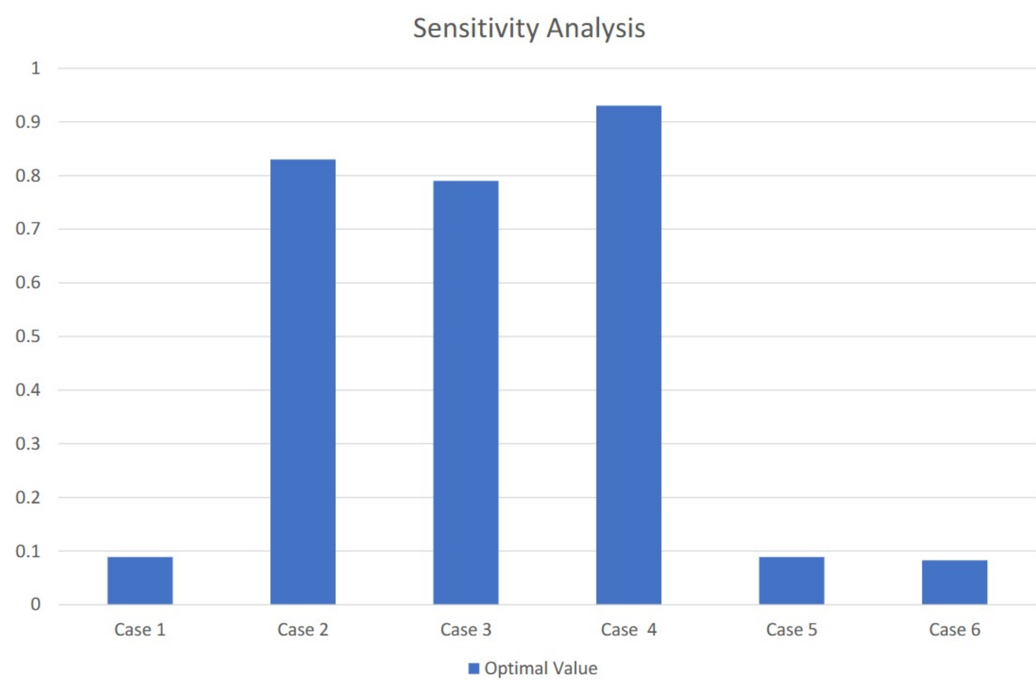

Figure 4. Senstivity Analysis for the example problem

Figure 4 illustrates the diagrammatic representation corresponding to various scenarios outlined in Table 4. The purpose of this sensitivity analysis is to assess the performance of the proposed algorithm under different conditions, encompassing variations in the time-dependent source node and alterations in arc values. Each case serves as a means to gain insights into the algorithm's robustness and adaptability across diverse input scenarios.

The outcomes derived from these individual cases play a pivotal role in comprehending the algorithm's behavior. They aid in identifying both the strengths and limitations of the algorithm when confronted with a spectrum of input configurations. This sensitivity analysis contributes valuable information for assessing the algorithm's efficacy and refining its application to different and challenging scenarios.

8. **Comparison Analysis**

The proposed algorithm consistently demonstrates superior performance compared to the method outlined in [7] and outperforms the existing approach presented by Broumi et al. in [46]. Notably, our method not only surpasses the performance of the cited methods but also provides accurate travel time scores, a feature absent in the existing methodology. Below table 5 shows the comparison result.

Table 5. Comparison for the proposed method

Methods	SP	Shortest Travel Time	Score of travel time
Time-Dependent Dijkstra Algorithm [7]	$1 \to 2 \to 5 \to 6$	(0.901,0.122,0.15,-0.078,-0.919,-0.912)	0.92
Exisitng Technique [46]	$1 \to 2 \to 5 \to 6$	(0.60,0.06,0.04,-0.024,-0.76,-0.85)	-
Proposed Method	$1 \to 3 \to 4 \to 6$	(0.78,0.039,0.11,-0.07,-0.97,-0.64)	0.089

The incorporation of an innovative approach for addressing uncertainty and variability in edge weights over time, utilizing time-dependent bipolar neutrosophic weights, presents a novel level of adaptability to dynamic environments. Utilizing bipolar neutrosophic numbers allows for the effective representation of imprecision and uncertainty in weight values, a crucial feature in dynamic environments where obtaining precise values may be challenging. A algorithm's capacity to manage uncertain weights in a time-dependent manner contributes to a more realistic modeling of real-world systems, accurately reflecting fluctuations in weights due to changing conditions. When the extended Bellman-Ford algorithm is modified to accommodate the time-dependent bipolar neutrosophic weights, it gains the capability to dynamically adapt to evolving network conditions. This adaptability proves advantageous in scenarios where edges undergo variations in travel times or costs owing to factors such as traffic conditions, weather, or other temporal influences. The proposed algorithm, featuring time-dependent bipolar neutrosophic weights, introduces a novel level of adaptability to dynamic environments. It exhibits the ability to dynamically respond to fluctuations in edge weights, making it well-suited for applications in which network conditions undergo frequent changes. The key advantage lies in its adaptability to changing environments, , particularly valuable in domains like transportation networks or communication networks where temporal variations can occur, and precise information may be elusive.

9. Applications of Bipolar Neutrosophic Time-Dependent Shortest Path Problem

Some applications of solving the Bipolar Neutrosophic Time-dependent Shortest Path Problem (BNTDSPP) using the Bellman-Ford algorithm:

(1) Transportation and Traffic Management:
By employing BNTDSP with the Bellman-Ford algorithm, transportation systems gain the ability to factor in unpredictable elements such as traffic uncertainty, accidents, and road closures. Additionally, the incorporation of time-dependent attributes allows for the consideration of varying traffic congestion levels at different times of the day.

(2) Network Routing in Telecommunications:
Telecommunication networks can optimize data transmission pathways by employing the Bellman-Ford algorithm adapted for BNTDSP. This optimization ensures the selection of the most dependable and time-efficient routes, considering variables like network congestion and the variable quality of network links.

(3) Autonomous Vehicles and Robotics:
Autonomous vehicles and robotic systems can enhance their navigational capabilities by incorporating BNTDSP within the Bellman-Ford framework. This adaptation enables these entities to navigate dynamic environments while accounting for obstacles that may arise unexpectedly, thus enabling real-time path adjustments for safe and efficient operations.

(4) Public Transportation Optimization:
Public transportation systems can streamline their operations by optimizing routes for buses, trams, or subways. This optimization encompasses the dynamic nature of passenger demand and varying traffic conditions, ultimately leading to more efficient and responsive public transportation networks.

The application of the Bellman-Ford algorithm adapted for BNTDSP in these scenarios empowers decision-makers to make well-informed choices by accommodating both bipolar neutrosophic elements and time-dependent attributes. This approach provides practical and adaptable solutions to real-world challenges across various domains.

9.1. *Benefits and Limitations of the proposed method*

Benefits

The proposed algorithm can handle graphs with both positive and negative weights, making it a versatile choice when the decision-maker uncertain about the weights in your graph.The optimal solution is guaranteed by the algorithm when it stops, regardless of the presence of negative weights, if there are no cycles with negative weight that can be reached from the source node. It can be elongated to

- Interval-Bipolar Neutrosophic numbers
- Fermatean Neutrosophic Fuzzy numbers
- interval-valued fermatean neutrosophic fuzzy number and so on.

Limitations

- The algorithm may require significant memory usage, especially for large graphs, which can be a constraint in memory-constrained environments.
- In dense graphs (graphs with a large number of edges), the prposed algorithm can be particularly slow due to its time complexity.
- BNN are capable of handling uncertainty, they may face challenges in effectively representing and manipulating temporal aspects or time-dependent variations in data, which are crucial in many real-world scenarios.

10. **Conclusion**

This paper introduces a novel methodology for determining optimal routes in a Bipolar Neutrosophic Graph, where travel times depend on the current time. In this unique scenario, vertex weights are expressed as fuzzy numbers to capture the inherent uncertainty and variability associated with travel times. To address this innovative problem, we extend the Bellman-Ford algorithm, originally designed for traditional graphs, to accommodate BNN. These specialized fuzzy numbers allow the representation of both positive and negative degrees of membership, thereby enhancing the algorithm's versatility. Our proposed method is applicable in transportation systems, where travel times are subject to uncertainty due to factors like traffic conditions, weather, or accidents. The BNSPP can also find applications in communication networks, supply chain optimization, critical infrastructure networks, financial networks, and more. The efficacy of our approach is evaluated through its application to a real-world case, with comparisons made against existing methods.

Notably, our method demonstrates computational efficiency, even when dealing with large graphs. However, its essential to acknowledge the limitations of our method, particularly in terms of the complexity in the representation of BNN. The three components—positive, neutral, and negative values—along with their respective membership functions, introduce intricacies in calculations and decision-making processes compared to simpler number systems. Additionally, while BNN adeptly handle uncertainty, challenges may arise in effectively representing and manipulating temporal aspects or time-dependent variations in data, crucial in many real-world scenarios. Looking ahead, we identify potential avenues for future research and applications in this promising direction, recognizing the need for further exploration and refinement.

Conflicts of Interest: The authors assert that they have no financial or affiliative interests that could present a conflict of interest regarding the submitted work.

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Neutrosophic Doubt Fuzzy Bi-ideal of BS-Algebras

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Abstract. In this research paper, our aim is to introduce the new concept of neutrosophic doubt fuzzy bi-ideal of BS-algebras as an extension of doubt fuzzy bi-ideal of BS-algebras and investigated its algebraic nature. Neutrosophic doubt fuzzy bi-ideal of BS-algebras is also applied in Cartesian product. Finally, we also provide the homomorphic behaviour of Neutrosophic doubt fuzzy bi-ideal of BS-algebras.

Keywords: BS-algebras, Neutrosophic doubt fuzzy bi-ideal, Homomorphism.

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1. Introduction

The fuzzy subsets was first introduced by L.A.Zadeh[8]. In 1966, Imai and Iseki gave the idea of BCK-algebras and BCI-algebras[3]. J.Neggers and H.S. Kim initiated the notion of B-algebras[4] which is a generalisation of BCK-algebras. We launched the notion of BS-algebras which is a generalisation of B-algebras and established the notion of Doubt fuzzy bi-ideal of BS-algebras[1]. We also innovated the notion of Neutrosophic fuzzy bi-ideal of BS-algebras[2]. F. Smarandache[5] extented the concept of fuzzy logic to neutrosophic logic which includes indeterminancy. Neutrosophic set theory played a major role in decision making problem, medical diagnosis, robotics, image processing, etc.

The main objective of this paper is to putforth the notion of Neutrosophic Doubt Fuzzy Bi-ideal(NDFB) of BS-algebras and studied their algebraic properties. We obtained the product of neutrosophic doubt fuzzy bi-ideal for BS-algebras. Finally, we studied how to deal with homomorphism of neutrosophic doubt fuzzy bi-ideal for BS-algebras.

2. Preliminaries:

In this Section, some basic definitions are given that are necessary for this paper. Throughout this paper, let $\mathfrak{B}$ denotes BS-algebra.

Definition 2.1 [1] A set BS-algebra $\mathfrak{B}\neq\phi$ with 1 as constant and * as binary operation satisfying the following axioms

(i) $\alpha*\alpha=1$

(ii) $\alpha*1=\alpha$

(iii) $(\alpha*\beta)*\gamma = \alpha*(\gamma*(1*\beta))\ \forall\ \alpha,\beta,\gamma \in \mathfrak{B}$

Definition 2.2 A fuzzy subset Ƒ of $\mathfrak{B}$ is called the fuzzy ideal of $\mathfrak{B}$ if it satisfies

(i) $Ƒ(1) \geq Ƒ(\alpha)$

(ii) $Ƒ(\beta) \geq \{Ƒ(\alpha) \wedge Ƒ(\beta*\alpha)\}\ \forall\ \alpha,\beta \in \mathfrak{B}$

Definition 2.3 [1] A fuzzy subset Ƒ of $\mathfrak{B}$ is called the fuzzy bi-ideal of $\mathfrak{B}$ if it satisfies

(i) $Ƒ(1) \geq Ƒ(\alpha)$

(ii) $Ƒ(\beta*\gamma) \geq \{Ƒ(\alpha) \wedge Ƒ(\alpha*(\beta*\gamma))\}\ \forall\ \alpha,\beta,\gamma \in \mathfrak{B}$

Definition 2.4 A fuzzy set Ƒ of $\mathfrak{B}$ is called the doubt fuzzy ideal of $\mathfrak{B}$ if it satisfies

(i) $Ƒ(1) \leq Ƒ(\alpha)$

(ii) $Ƒ(\beta) \leq \{Ƒ(\alpha) \vee Ƒ(\beta*\alpha))\}\ \forall\ \alpha,\beta \in \mathfrak{B}$

Definition 2.5 [1] A fuzzy set Ƒ of $\mathfrak{B}$ is called the Doubt Fuzzy Bi- ideal(DF) of $\mathfrak{B}$ if it satisfies

(i) $Ƒ(1) \leq Ƒ(\alpha)$

(ii) $Ƒ(\beta*\gamma) \leq \{Ƒ(\alpha) \vee Ƒ(\alpha*(\beta*\gamma))\}\ \forall\ \alpha,\beta,\gamma \in \mathfrak{B}$

Example 2.6 [1] Let $\mathfrak{B}$ = {1, u, v, w} be the set with the following Cayley table

*	1	u	v	w
1	1	u	v	w
u	u	1	w	v
v	v	w	1	u
w	w	v	u	1

Then ($\mathfrak{B}$, *, 1) is a BS-algebra. Then the fuzzy set $Ƒ:\mathfrak{B}\rightarrow[0,1]$ is defined by $Ƒ(1)= Ƒ(\alpha)=0.7$ and $Ƒ(\beta)= Ƒ(\gamma)=0.9$, which is a Doubt Fuzzy(DF) bi-ideal of $\mathfrak{B}$.

Definition 2.7 [6] A Neutrosophic fuzzy set $\mathcal{N}$ on the Universe of discourse X characterised by a truth membership function $Ṯ_{\mathcal{N}}(\alpha)$, an indeterminacy function $\mathcal{J}_{\mathcal{N}}(\alpha)$ and a falsity membership function $Ƒ_{\mathcal{N}}(\alpha)$ is defined as $\mathcal{N}=\{<\alpha, Ṯ_{\mathcal{N}}(\alpha), \mathcal{J}_{\mathcal{N}}(\alpha), Ƒ_{\mathcal{N}}(\alpha)>: \alpha \in X\}$ where $Ṯ_{\mathcal{N}}, \mathcal{J}_{\mathcal{N}}, Ƒ_{\mathcal{N}} : X\longrightarrow[0,1]$ and $0 \leq Ṯ_{\mathcal{N}}+ \mathcal{J}_{\mathcal{N}}+ Ƒ_{\mathcal{N}} \leq 3$

Definition 2.8 [6] Let $\mathcal{M}$ and $\mathcal{N}$ be the two neutrosophic fuzzy set of X. Then $\alpha \in X$

i) $\mathcal{M}\cup\mathcal{N}$={<$\alpha$, $\mathrm{T}_{\mathcal{M}\cup\mathcal{N}}(\alpha)$, $\mathcal{J}_{\mathcal{M}\cup\mathcal{N}}(\alpha)$, $\mathrm{F}_{\mathcal{M}\cup\mathcal{N}}(\alpha)$>}, where

$\mathrm{T}_{\mathcal{M}\cup\mathcal{N}}(\alpha) = (\mathrm{T}_{\mathcal{M}}(\alpha) \vee \mathrm{T}_{\mathcal{N}}(\alpha))$; $\mathcal{J}_{\mathcal{M}\cup\mathcal{N}}(\alpha) = (\mathcal{J}_{\mathcal{M}}(\alpha) \wedge \mathcal{J}_{\mathcal{N}}(\alpha))$; $\mathrm{F}_{\mathcal{M}\cup\mathcal{N}}(\alpha) = (\mathrm{F}_{\mathcal{M}}(\alpha) \wedge \mathrm{F}_{\mathcal{N}}(\alpha))$

ii) $\mathcal{M}\cap\mathcal{N}$={<$\alpha$, $\mathrm{T}_{\mathcal{M}\cap\mathcal{N}}(\alpha)$, $\mathcal{J}_{\mathcal{M}\cap\mathcal{N}}(\alpha)$, $\mathrm{F}_{\mathcal{M}\cap\mathcal{N}}(\alpha)$>}, where

$\mathrm{T}_{\mathcal{M}\cap\mathcal{N}}(\alpha) = (\mathrm{T}_{\mathcal{M}}(\alpha) \wedge \mathrm{T}_{\mathcal{N}}(\alpha))$; $\mathcal{J}_{\mathcal{M}\cap\mathcal{N}}(\alpha) = (\mathcal{J}_{\mathcal{M}}(\alpha) \vee \mathcal{J}_{\mathcal{N}}(\alpha))$; $\mathrm{F}_{\mathcal{M}\cap\mathcal{N}}(\alpha) = (\mathrm{F}_{\mathcal{M}}(\alpha) \vee \mathrm{F}_{\mathcal{N}}(\alpha))$

Definition 2.9 A Neutrosophic Fuzzy Set $\mathcal{N}$ of BS-algebra $\mathfrak{B}$ is called the Neutrosophic Fuzzy Ideal of $\mathfrak{B}$ if $\forall\, a, \beta, \gamma \in \mathfrak{B}$

(i) $\mathrm{T}_{\mathcal{N}}(1) \geq \mathrm{T}_{\mathcal{N}}(\alpha)$; $\mathcal{J}_{\mathcal{N}}(1) \leq \mathcal{J}_{\mathcal{N}}(\alpha)$; $\mathrm{F}_{\mathcal{N}}(1) \leq \mathrm{F}_{\mathcal{N}}(\alpha)$;

(ii) $\mathrm{T}_{\mathcal{N}}(\beta) \geq \{\mathrm{T}_{\mathcal{N}}(\alpha) \wedge \mathrm{T}_{\mathcal{N}}(\beta*\alpha)\}$;

$\mathcal{J}_{\mathcal{N}}(\beta) \leq \{\mathcal{J}_{\mathcal{N}}(\alpha) \vee \mathcal{J}_{\mathcal{N}}(\beta*\alpha)\}$;

$\mathrm{F}_{\mathcal{N}}(\beta) \leq \{\mathrm{F}_{\mathcal{N}}(\alpha) \vee \mathrm{F}_{\mathcal{N}}(\beta*\alpha)\}$

Definition 2.10 [2] A Neutrosophic fuzzy set $\mathcal{N}$ of BS-algebra $\mathfrak{B}$ is called the Neutrosophic Fuzzy Bi-ideal of $\mathfrak{B}$ if $\forall\, a, \beta, \gamma \in \mathfrak{B}$

(i) $\mathrm{T}_{\mathcal{N}}(1) \geq \mathrm{T}_{\mathcal{N}}(\alpha)$; $\mathcal{J}_{\mathcal{N}}(1) \leq \mathcal{J}_{\mathcal{N}}(\alpha)$; $\mathrm{F}_{\mathcal{N}}(1) \leq \mathrm{F}_{\mathcal{N}}(\alpha)$;

(ii) $\mathrm{T}_{\mathcal{N}}(\beta*\gamma) \geq \{\mathrm{T}_{\mathcal{N}}(\alpha) \wedge \mathrm{T}_{\mathcal{N}}(\alpha*(\beta*\gamma))\}$;

$\mathcal{J}_{\mathcal{N}}(\beta*\gamma) \leq \{\mathcal{J}_{\mathcal{N}}(\alpha) \vee \mathcal{J}_{\mathcal{N}}(\alpha*(\beta*\gamma))\}$;

$\mathrm{F}_{\mathcal{N}}(\beta*\gamma) \leq \{\mathrm{F}_{\mathcal{N}}(\alpha) \vee \mathrm{F}_{\mathcal{N}}(\alpha*(\beta*\gamma))\}$

Definition 2.11 A Neutrosophic fuzzy set $\mathcal{D}$ of BS-algebra $\mathfrak{B}$ is called the Neutrosophic Doubt Fuzzy Ideal of $\mathfrak{B}$ if $\forall\, a, \beta, \gamma \in \mathfrak{B}$

(i) $\mathrm{T}_{\mathcal{D}}(1) \leq \mathrm{T}_{\mathcal{D}}(\alpha)$; $\mathcal{J}_{\mathcal{D}}(1) \geq \mathcal{J}_{\mathcal{D}}(\alpha)$; $\mathrm{F}_{\mathcal{D}}(1) \geq \mathrm{F}_{\mathcal{D}}(\alpha)$;

(ii) $\mathrm{T}_{\mathcal{D}}(\beta) \leq \{\mathrm{T}_{\mathcal{D}}(\alpha) \vee \mathrm{T}_{\mathcal{D}}(\beta*\alpha)\}$;

$\mathcal{J}_{\mathcal{D}}(\beta) \geq \{\mathcal{J}_{\mathcal{D}}(\alpha) \wedge \mathcal{J}_{\mathcal{D}}(\beta*\alpha)\}$;

$\mathrm{F}_{\mathcal{D}}(\beta) \geq \{\mathrm{F}_{\mathcal{D}}(\alpha) \wedge \mathrm{F}_{\mathcal{D}}(\beta*\alpha)\}$

3. NEUTROSOPHIC DOUBT FUZZY BI-IDEAL (NDFB) OF BS-ALGEBRAS

In this Section, the concept of doubt fuzzy bi-ideal of $\mathfrak{B}$ can be extented to Neutrosophic doubt fuzzy bi-ideal of $\mathfrak{B}$. We proved that the union of two NDFB of $\mathfrak{B}$ is again a NDFB of $\mathfrak{B}$. We also proved that the intersection of two NDFB of $\mathfrak{B}$ is again a NDFB of $\mathfrak{B}$.

Definition 3.1 A Neutrosophic fuzzy set $\mathcal{D}$ of BS-algebra $\mathfrak{B}$ is called the Neutrosophic Doubt Fuzzy Bi-ideal (NDFB) of $\mathfrak{B}$ if $\forall\, a, \beta, \gamma \in \mathfrak{B}$

($\mathcal{D}_1$) $\mathrm{T}_{\mathcal{D}}(1) \leq \mathrm{T}_{\mathcal{D}}(\alpha)$; $\mathcal{J}_{\mathcal{D}}(1) \geq \mathcal{J}_{\mathcal{D}}(\alpha)$; $\mathrm{F}_{\mathcal{D}}(1) \geq \mathrm{F}_{\mathcal{D}}(\alpha)$;

($\mathcal{D}_2$) $\mathrm{T}_{\mathcal{D}}(\beta*\gamma) \leq \{\mathrm{T}_{\mathcal{D}}(\alpha) \vee \mathrm{T}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}$;

$\mathcal{J}_{\mathcal{D}}(\beta*\gamma) \geq \{\mathcal{J}_{\mathcal{D}}(\alpha) \wedge \mathcal{J}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}$;

$\mathrm{F}_{\mathcal{D}}(\beta*\gamma) \geq \{\mathrm{F}_{\mathcal{D}}(\alpha) \wedge \mathrm{F}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}$

Theorem 3.2 Let $\mathcal{C}$ and $\mathcal{D}$ be two NDFB of $\mathfrak{B}$. Then $\mathcal{C}\cup\mathcal{D}$ is a NDFB of $\mathfrak{B}$.

Proof

Let $\mathcal{C}$ and $\mathcal{D}$ be two NDFB of $\mathfrak{B}$. For any $a, \beta, \gamma \in \mathfrak{B}$

i) $\mathrm{T}_{\mathcal{C}\cup\mathcal{D}}(1) = \{\mathrm{T}_{\mathcal{C}}(1) \vee \mathrm{T}_{\mathcal{D}}(1)\}$

$\leq\{\mathrm{T}_{\mathcal{C}}(\alpha) \vee \mathrm{T}_{\mathcal{D}}(\alpha)\}$
$= \mathrm{T}_{\mathcal{C}\cup\mathcal{D}}(\alpha)$

Therefore, $\mathrm{T}_{\mathcal{C}\cup\mathcal{D}}(1) \leq \mathrm{T}_{\mathcal{C}\cup\mathcal{D}}(\alpha)$

and $\mathcal{J}_{\mathcal{C}\cup\mathcal{D}}(1) = \{\mathcal{J}_{\mathcal{C}}(1) \wedge \mathcal{J}_{\mathcal{D}}(1)\}$
$\geq \{\mathcal{J}_{\mathcal{C}}(\alpha) \wedge \mathcal{J}_{\mathcal{D}}(\alpha)\}$
$= \mathcal{J}_{\mathcal{C}\cup\mathcal{D}}(\alpha)$

Therefore, $\mathcal{J}_{\mathcal{C}\cup\mathcal{D}}(1) \geq \mathcal{J}_{\mathcal{C}\cup\mathcal{D}}(\alpha)$

and $\mathrm{F}_{\mathcal{C}\cup\mathcal{D}}(1) = \{\mathrm{F}_{\mathcal{C}}(1) \wedge \mathrm{F}_{\mathcal{D}}(1)\}$
$\geq \{\mathrm{F}_{\mathcal{C}}(\alpha) \wedge \mathrm{F}_{\mathcal{D}}(\alpha)\}$
$= \mathrm{F}_{\mathcal{C}\cup\mathcal{D}}(\alpha)$

Therefore, $\mathrm{F}_{\mathcal{C}\cup\mathcal{D}}(1) \geq \mathrm{F}_{\mathcal{C}\cup\mathcal{D}}(\alpha)$

ii) $\mathrm{T}_{\mathcal{C}\cup\mathcal{D}}(\beta*\gamma) = \{\mathrm{T}_{\mathcal{C}}(\beta*\gamma) \vee \mathrm{T}_{\mathcal{D}}(\beta*\gamma)\}$
$\leq \{\{\mathrm{T}_{\mathcal{C}}(\alpha) \vee \mathrm{T}_{\mathcal{C}}(\alpha*(\beta*\gamma))\} \vee \{\mathrm{T}_{\mathcal{D}}(\alpha) \vee \mathrm{T}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}\}$
$= \{\{\mathrm{T}_{\mathcal{C}}(\alpha) \vee \mathrm{T}_{\mathcal{D}}(\alpha)\} \vee \{\mathrm{T}_{\mathcal{C}}(\alpha*(\beta*\gamma)) \vee \mathrm{T}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}\}$
$= \{\mathrm{T}_{\mathcal{C}\cup\mathcal{D}}(\alpha) \vee \mathrm{T}_{\mathcal{C}\cup\mathcal{D}}(\alpha*(\beta*\gamma))\}$

Therefore, $\mathrm{T}_{\mathcal{C}\cup\mathcal{D}}(\beta*\gamma) \leq \{\mathrm{T}_{\mathcal{C}\cup\mathcal{D}}(\alpha) \vee \mathrm{T}_{\mathcal{C}\cup\mathcal{D}}(\alpha*(\beta*\gamma))\}$

and $\mathcal{J}_{\mathcal{C}\cup\mathcal{D}}(\beta*\gamma) = \{\mathcal{J}_{\mathcal{C}}(\beta*\gamma) \wedge \mathcal{J}_{\mathcal{D}}(\beta*\gamma)\}$
$\geq \{\{\mathcal{J}_{\mathcal{C}}(\alpha) \wedge \mathcal{J}_{\mathcal{C}}(\alpha*(\beta*\gamma))\} \wedge \{\mathcal{J}_{\mathcal{D}}(\alpha) \wedge \mathcal{J}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}\}$
$= \{\{\mathcal{J}_{\mathcal{C}}(\alpha) \wedge \mathcal{J}_{\mathcal{D}}(\alpha)\} \wedge \{\mathcal{J}_{\mathcal{C}}(\alpha*(\beta*\gamma)) \wedge \mathcal{J}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}\}$
$= \{\mathcal{J}_{\mathcal{C}\cup\mathcal{D}}(\alpha) \wedge \mathcal{J}_{\mathcal{C}\cup\mathcal{D}}(\alpha*(\beta*\gamma))\}$

Therefore, $\mathcal{J}_{\mathcal{C}\cup\mathcal{D}}(\beta*\gamma) \geq \{\mathcal{J}_{\mathcal{C}\cup\mathcal{D}}(\alpha) \wedge \mathcal{J}_{\mathcal{C}\cup\mathcal{D}}(\alpha*(\beta*\gamma))\}$

and $\mathrm{F}_{\mathcal{C}\cup\mathcal{D}}(\beta*\gamma) = \{\mathrm{F}_{\mathcal{C}}(\beta*\gamma) \wedge \mathrm{F}_{\mathcal{D}}(\beta*\gamma)\}$
$\geq \{\{\mathrm{F}_{\mathcal{C}}(\alpha) \wedge \mathrm{F}_{\mathcal{C}}(\alpha*(\beta*\gamma))\} \wedge \{\mathrm{F}_{\mathcal{D}}(\alpha) \wedge \mathrm{F}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}\}$
$= \{\{\mathrm{F}_{\mathcal{C}}(\alpha) \wedge \mathrm{F}_{\mathcal{D}}(\alpha)\} \wedge \{\mathrm{F}_{\mathcal{C}}(\alpha*(\beta*\gamma)) \wedge \mathrm{F}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}\}$
$= \{\mathrm{F}_{\mathcal{C}\cup\mathcal{D}}(\alpha) \wedge \mathrm{F}_{\mathcal{C}\cup\mathcal{D}}(\alpha*(\beta*\gamma))\}$

Therefore, $\mathrm{F}_{\mathcal{C}\cup\mathcal{D}}(\beta*\gamma) \geq \{\mathrm{F}_{\mathcal{C}\cup\mathcal{D}}(\alpha) \wedge \mathrm{F}_{\mathcal{C}\cup\mathcal{D}}(\alpha*(\beta*\gamma))\}$

Hence, $\boldsymbol{\mathcal{C}}\cup\boldsymbol{\mathcal{D}}$ is a NDFB of $\mathfrak{B}$

Theorem 3.3 Let $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$ be two NDFB of $\mathfrak{B}$. Then $\boldsymbol{\mathcal{C}}\cap\boldsymbol{\mathcal{D}}$ is a NDFB of $\mathfrak{B}$.

Proof

Let $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$ be two NDFB of $\mathfrak{B}$. For any $\alpha, \beta, \gamma \in \mathfrak{B}$

(i) $\mathrm{T}_{\mathcal{C}\cap\mathcal{D}}(1) = \{\mathrm{T}_{\mathcal{C}}(1) \wedge \mathrm{T}_{\mathcal{D}}(1)\}$
$\leq \{\mathrm{T}_{\mathcal{C}}(\alpha) \wedge \mathrm{T}_{\mathcal{D}}(\alpha)\}$
$= \mathrm{T}_{\mathcal{C}\cap\mathcal{D}}(\alpha)$

Therefore, $\mathrm{T}_{\mathcal{C}\cap\mathcal{D}}(1) \leq \mathrm{T}_{\mathcal{C}\cap\mathcal{D}}(\alpha)$

and $\mathcal{J}_{\mathcal{C}\cap\mathcal{D}}(1) = \{\mathcal{J}_{\mathcal{C}}(1) \vee \mathcal{J}_{\mathcal{D}}(1)\}$
$\geq \{\mathcal{J}_{\mathcal{C}}(\alpha) \vee \mathcal{J}_{\mathcal{D}}(\alpha)\}$
$= \mathcal{J}_{\mathcal{C}\cap\mathcal{D}}(\alpha)$

Therefore, $\mathcal{J}_{\mathcal{C}\cap\mathcal{D}}(1) \geq \mathcal{J}_{\mathcal{C}\cap\mathcal{D}}(\alpha)$

and $\mathrm{F}_{\mathcal{C}\cap\mathcal{D}}(1) = \{\mathrm{F}_{\mathcal{C}}(1) \vee \mathrm{F}_{\mathcal{D}}(1)\}$
$\geq \{\mathrm{F}_{\mathcal{C}}(\alpha) \vee \mathrm{F}_{\mathcal{D}}(\alpha)\}$
$= \mathrm{F}_{\mathcal{C}\cap\mathcal{D}}(\alpha)$

Therefore, $\mathrm{F}_{\mathcal{C}\cap\mathcal{D}}(1) \geq \mathrm{F}_{\mathcal{C}\cap\mathcal{D}}(\alpha)$

(ii) $\mathfrak{T}_{\mathcal{C}\cap\mathcal{D}}(\beta*\gamma) = \{\mathfrak{T}_{\mathcal{C}}(\beta*\gamma) \wedge \mathfrak{T}_{\mathcal{D}}(\beta*\gamma)\}$

$\leq \{\{\mathfrak{T}_{\mathcal{C}}(\alpha) \vee \mathfrak{T}_{\mathcal{C}}(\alpha*(\beta*\gamma))\} \wedge \{\mathfrak{T}_{\mathcal{D}}(\alpha) \vee \mathfrak{T}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}\}$

$= \{\{\mathfrak{T}_{\mathcal{C}}(\alpha) \wedge \mathfrak{T}_{\mathcal{D}}(\alpha)\} \vee \{\mathfrak{T}_{\mathcal{C}}(\alpha*(\beta*\gamma)) \wedge \mathfrak{T}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}\}$

$= \{\mathfrak{T}_{\mathcal{C}\cap\mathcal{D}}(\alpha) \vee \mathfrak{T}_{\mathcal{C}\cap\mathcal{D}}(\alpha\leq*(\beta*\gamma))\}$

Therefore, $\mathfrak{T}_{\mathcal{C}\cap\mathcal{D}}(\beta*\gamma) \leq \{\mathfrak{T}_{\mathcal{C}\cap\mathcal{D}}(\alpha) \vee \mathfrak{T}_{\mathcal{C}\cap\mathcal{D}}(\alpha*(\beta*\gamma))\}$

and $\mathcal{J}_{\mathcal{C}\cap\mathcal{D}}(\beta*\gamma) = \{\mathcal{J}_{\mathcal{C}}(\beta*\gamma) \vee \mathcal{J}_{\mathcal{D}}(\beta*\gamma)\}$

$\geq \{\{\mathcal{J}_{\mathcal{C}}(\alpha) \wedge \mathcal{J}_{\mathcal{C}}(\alpha*(\beta*\gamma))\} \vee \{\mathcal{J}_{\mathcal{D}}(\alpha) \wedge \mathcal{J}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}\}$

$= \{\{\mathcal{J}_{\mathcal{C}}(\alpha) \vee \mathcal{J}_{\mathcal{D}}(\alpha)\} \wedge \{\mathcal{J}_{\mathcal{C}}(\alpha*(\beta*\gamma)) \vee \mathcal{J}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}\}$

$= \{\mathcal{J}_{\mathcal{C}\cap\mathcal{D}}(\alpha) \wedge \mathcal{J}_{\mathcal{C}\cap\mathcal{D}}(\alpha*(\beta*\gamma))\}$

Therefore, $\mathcal{J}_{\mathcal{C}\cap\mathcal{D}}(\beta*\gamma) \geq \{\mathcal{J}_{\mathcal{C}\cap\mathcal{D}}(\alpha) \wedge \mathcal{J}_{\mathcal{C}\cap\mathcal{D}}(\alpha*(\beta*\gamma))\}$

Similarly, $\mathfrak{F}_{\mathcal{C}\cap\mathcal{D}}(\beta*\gamma) \geq \{\mathfrak{F}_{\mathcal{C}\cap\mathcal{D}}(\alpha) \wedge \mathfrak{F}_{\mathcal{C}\cap\mathcal{D}}(\alpha*(\beta*\gamma))\}$

Hence, $\mathcal{C}\cap\mathcal{D}$ is a NDFB of $\mathfrak{B}$

Corollary 3.4 Let $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_n$ are NDFB of $\mathfrak{B}$, then $\mathcal{D} = \bigcap_{i=1}^{n} \mathcal{D}_i$ is also a NDFB of $\mathfrak{B}$

Proof

Straight forward using theorem 3.3

Lemma 3.5 For all s, t ∈ I and i be any positive integer, if s = t, then

i) $s^i \leq t^i$

ii) $[(s \wedge t)]^i = (s^i \wedge t^i)$

iii) $[(s \vee t)]^i = (s^i \vee t^i)$

Theorem 3.6 Let $\mathcal{D}$ be a NDFB of $\mathfrak{B}$, then $\mathcal{D}^i = \{<\alpha, \mathfrak{T}_{\mathcal{D}^i}(\alpha), \mathcal{J}_{\mathcal{D}^i}(\alpha), \mathfrak{F}_{\mathcal{D}^i}(\alpha)>: \alpha \in \mathfrak{B}\}$ is a NDFB of $\mathfrak{B}^i$, where i is any positive integer and $\mathfrak{T}_{\mathcal{D}^i}(\alpha) = (\mathfrak{T}_{\mathcal{D}}(\alpha))^i$, $\mathcal{J}_{\mathcal{D}^i}(\alpha) = (\mathcal{J}_{\mathcal{D}}(\alpha))^i$, $\mathfrak{F}_{\mathcal{D}^i}(\alpha) = (\mathfrak{F}_{\mathcal{D}}(\alpha))^i$

Proof

Let $\mathcal{D}$ be a NDFB of $\mathfrak{B}$. For any $\alpha, \beta, \gamma \in \mathfrak{B}$

i) $\mathfrak{T}_{\mathcal{D}^i}(1) = (\mathfrak{T}_{\mathcal{D}}(1))^i$

$\leq (\mathfrak{T}_{\mathcal{D}}(\alpha))^i$

$= \mathfrak{T}_{\mathcal{D}^i}(\alpha)$

Therefore, $\mathfrak{T}_{\mathcal{D}^i}(1) \leq \mathfrak{T}_{\mathcal{D}^i}(\alpha)$

and $\mathcal{J}_{\mathcal{D}^i}(1) = (\mathcal{J}_{\mathcal{D}}(1))^i$

$\geq (\mathcal{J}_{\mathcal{D}}(\alpha))^i$

$= \mathcal{J}_{\mathcal{D}^i}(\alpha)$

Therefore, $\mathcal{J}_{\mathcal{D}^i}(1) \geq \mathcal{J}_{\mathcal{D}^i}(\alpha)$

and $\mathfrak{F}_{\mathcal{D}^i}(1) = (\mathfrak{F}_{\mathcal{D}}(1))^i$

$\geq (\mathfrak{F}_{\mathcal{D}}(\alpha))^i$

$= \mathfrak{F}_{\mathcal{D}^i}(\alpha)$

Therefore, $\mathfrak{F}_{\mathcal{D}^i}(1) \geq \mathfrak{F}_{\mathcal{D}^i}(\alpha)$

ii) $\mathfrak{T}_{\mathcal{D}^i}(\beta*\gamma) = (\mathfrak{T}_{\mathcal{D}}(\beta*\gamma))^i$

$\leq [\{\mathfrak{T}_{\mathcal{D}}(\alpha) \vee \mathfrak{T}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}]^i$

$= \{[\mathfrak{T}_{\mathcal{D}}(\alpha)]^i \vee [\mathfrak{T}_{\mathcal{D}}(\alpha*(\beta*\gamma))]^i\}$

$= \{\mathfrak{T}_{\mathcal{D}^i}(\alpha) \vee \mathfrak{T}_{\mathcal{D}^i}(\alpha*(\beta*\gamma))\}$

Therefore, $\mathfrak{T}_{\mathcal{D}^i}(\beta*\gamma) \leq \{\mathfrak{T}_{\mathcal{D}^i}(\alpha) \vee \mathfrak{T}_{\mathcal{D}^i}(\alpha*(\beta*\gamma))\}$

and $\mathcal{J}_{\mathcal{D}^i}(\beta*\gamma) = (\mathcal{J}_{\mathcal{D}}(\beta*\gamma))^i$
$\geq [\{\mathcal{J}_{\mathcal{D}}(\alpha) \wedge \mathcal{J}_{\mathcal{D}}(\alpha*(\beta*\gamma))\}]^i$
$= \{[\mathcal{J}_{\mathcal{D}}(\alpha)]^i \wedge [\mathcal{J}_{\mathcal{D}}(\alpha*(\beta*\gamma))]^i\}$
$= \{\mathcal{J}_{\mathcal{D}^i}(\alpha) \wedge \mathcal{J}_{\mathcal{D}^i}(\alpha*(\beta*\gamma))\}$

Therefore, $\mathcal{J}_{\mathcal{D}^i}(\beta*\gamma) \geq \{\mathcal{J}_{\mathcal{D}^i}(\alpha) \wedge \mathcal{J}_{\mathcal{D}^i}(\alpha*(\beta*\gamma))\}$

Similarly, we can prove that $\digamma_{\mathcal{D}^i}(\beta*\gamma) \geq \{\digamma_{\mathcal{D}^i}(\alpha) \wedge \digamma_{\mathcal{D}^i}(\alpha*(\beta*\gamma))\}$

Hence $\mathcal{D}^i$ is a NDFB of $\mathfrak{B}^i$

4. PRODUCT OF NEUTROSOPHIC DOUBT FUZZY BI IDEAL OF BS-ALGEBRAS

In this section, the product of NDFB of $\mathfrak{B}$ are defined and corresponding theorems are investigated.

Definition 4.1 Let $\mathcal{C}$ and $\mathcal{D}$ be two neutrosophic doubt fuzzy subsets of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively. Then the direct product of neutrosophic doubt fuzzy subsets of BS-algebra $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is defined by $\mathcal{C}\mathrm{x}\mathcal{D}$: $\mathfrak{B}_1 \times \mathfrak{B}_2 \longrightarrow [0,1]$ such that

$\mathcal{C}\mathrm{x}\mathcal{D} = \{<(\alpha,\beta), \Game_{\mathcal{C}x\mathcal{D}}(\alpha,\beta), \mathcal{J}_{\mathcal{C}x\mathcal{D}}(\alpha,\beta), \digamma_{\mathcal{C}x\mathcal{D}}(\alpha,\beta)>: \alpha\in\mathfrak{B}_1, \beta\in\mathfrak{B}_2\}$, where

$\Game_{\mathcal{C}x\mathcal{D}}(\alpha,\beta) = (\Game_{\mathcal{C}}(\alpha) \vee \Game_{\mathcal{D}}(\beta))$; $\Game_{\mathcal{C}x\mathcal{D}}(\alpha,\beta) = (\mathcal{J}_{\mathcal{C}}(\alpha) \wedge \mathcal{J}_{\mathcal{D}}(\beta))$; $\digamma_{\mathcal{C}x\mathcal{D}}(\alpha,\beta) = (\digamma_{\mathcal{C}}(\alpha) \wedge \digamma_{\mathcal{D}}(\beta))$

Definition 4.2 Let $\mathcal{C}$ and $\mathcal{D}$ be two neutrosophic doubt fuzzy subsets of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively. Then $\mathcal{C}\mathrm{x}\mathcal{D}$ is a NDFB of $\mathfrak{B}_1 \times \mathfrak{B}_2$ if it satisfies the following conditions

i) $\Game_{\mathcal{C}x\mathcal{D}}(1,1) \leq \Game_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2)$; $\mathcal{J}_{\mathcal{C}x\mathcal{D}}(1,1) \geq \mathcal{J}_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2)$; $\digamma_{\mathcal{C}x\mathcal{D}}(1,1) \geq \digamma_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2)$;

ii) $\Game_{\mathcal{C}x\mathcal{D}}((\beta_1, \beta_2)*(\gamma_1, \gamma_2)) \leq \{\Game_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2) \vee \Game_{\mathcal{C}x\mathcal{D}}((\alpha_1, \alpha_2)*((\beta_1, \beta_2)*(\gamma_1, \gamma_2)))\}$;
$\mathcal{J}_{\mathcal{C}x\mathcal{D}}((\beta_1, \beta_2)*(\gamma_1, \gamma_2)) \geq \{\mathcal{J}_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2) \wedge \mathcal{J}_{\mathcal{C}x\mathcal{D}}((\alpha_1, \alpha_2)*((\beta_1, \beta_2)*(\gamma_1, \gamma_2)))\}$;
$\digamma_{\mathcal{C}x\mathcal{D}}((\beta_1, \beta_2)*(\gamma_1, \gamma_2)) \geq \{\digamma_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2) \wedge \digamma_{\mathcal{C}x\mathcal{D}}((\alpha_1, \alpha_2)*((\beta_1, \beta_2)*(\gamma_1, \gamma_2)))\}$.

Theorem 4.3 Let $\mathcal{C}$ and $\mathcal{D}$ be two NDFB of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively. Then $\mathcal{C}\mathrm{x}\mathcal{D}$ is a NDFB of $\mathfrak{B}_1 \times \mathfrak{B}_2$

Proof

Let $\mathcal{C}$ and $\mathcal{D}$ be two NDFB of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively.

Let $(\alpha_1, \alpha_2), (\beta_1, \beta_2), (\gamma_1, \gamma_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2$

i) We have $\Game_{\mathcal{C}x\mathcal{D}}(1,1) = \{\Game_{\mathcal{C}}(1) \vee \Game_{\mathcal{D}}(1)\}$
$\leq \{\Game_{\mathcal{C}}(\alpha_1) \vee \Game_{\mathcal{D}}(\alpha_2)\}$
$= \Game_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2)$

Therefore, $\Game_{\mathcal{C}x\mathcal{D}}(1,1) \leq \Game_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2)$

and $\mathcal{J}_{\mathcal{C}x\mathcal{D}}(1,1) = \{\mathcal{J}_{\mathcal{C}}(1) \wedge \mathcal{J}_{\mathcal{D}}(1)\}$
$\geq \{\mathcal{J}_{\mathcal{C}}(\alpha_1) \wedge \mathcal{J}_{\mathcal{D}}(\alpha_2)\}$
$= \mathcal{J}_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2)$

Therefore, $\mathcal{J}_{\mathcal{C}x\mathcal{D}}(1,1) \geq \mathcal{J}_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2)$

and $\digamma_{\mathcal{C}x\mathcal{D}}(1,1) = \{\digamma_{\mathcal{C}}(1) \wedge \digamma_{\mathcal{D}}(1)\}$
$\geq \{\digamma_{\mathcal{C}}(\alpha_1) \wedge \digamma_{\mathcal{D}}(\alpha_2)\}$
$= \digamma_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2)$

Therefore, $\digamma_{\mathcal{C}x\mathcal{D}}(1,1) \geq \digamma_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2)$

ii) Then $\Game_{\mathcal{C}x\mathcal{D}}((\beta_1, \beta_2)*(\gamma_1, \gamma_2)) = \Game_{\mathcal{C}x\mathcal{D}}(\beta_1*\gamma_1, \beta_2*\gamma_2)$
$= \{\Game_{\mathcal{C}}(\beta_1*\gamma_1) \vee \Game_{\mathcal{D}}(\beta_2*\gamma_2)\}$
$\leq [\{\Game_{\mathcal{C}}(\alpha_1) \vee \Game_{\mathcal{C}}(\alpha_1*(\beta_1*\gamma_1))\} \vee \{\Game_{\mathcal{D}}(\alpha_2) \vee \Game_{\mathcal{D}}(\alpha_2*(\beta_2*\gamma_2))\}]$
$= [\{\Game_{\mathcal{C}}(\alpha_1) \vee \Game_{\mathcal{D}}(\alpha_2)\} \vee \{\Game_{\mathcal{C}}(\alpha_1*(\beta_1*\gamma_1)) \vee \Game_{\mathcal{D}}(\alpha_2*(\beta_2*\gamma_2))\}]$

$=\{\daleth_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2) \vee \daleth_{\mathcal{C}x\mathcal{D}}(\alpha_1*(\beta_1*\gamma_1)), (\alpha_2*(\beta_2*\gamma_2))\}$

$= \{\daleth_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2) \vee \daleth_{\mathcal{C}x\mathcal{D}}((\alpha_1, \alpha_2)*((\beta_1, \beta_2)* (\gamma_1, \gamma_2)))\}$

Therefore, $\daleth_{\mathcal{C}x\mathcal{D}}((\beta_1, \beta_2)* (\gamma_1, \gamma_2)) \leq \{\daleth_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2) \vee \daleth_{\mathcal{C}x\mathcal{D}}((\alpha_1, \alpha_2)*((\beta_1, \beta_2)* (\gamma_1, \gamma_2)))\}$

and $\mathcal{J}_{\mathcal{C}x\mathcal{D}}((\beta_1, \beta_2)* (\gamma_1, \gamma_2)) = \mathcal{J}_{\mathcal{C}x\mathcal{D}}(\beta_1*\gamma_1, \beta_2*\gamma_2)$

$= \{\mathcal{J}_{\mathcal{C}}(\beta_1*\gamma_1) \wedge \mathcal{J}_{\mathcal{D}}(\beta_2*\gamma_2)\}$

$\geq [\{\mathcal{J}_{\mathcal{C}}(\alpha_1) \wedge \mathcal{J}_{\mathcal{C}}(\alpha_1*(\beta_1*\gamma_1))\} \wedge \{\mathcal{J}_{\mathcal{D}}(\alpha_2) \wedge \mathcal{J}_{\mathcal{D}}(\alpha_2*(\beta_2*\gamma_2))\}]$

$= [\{\mathcal{J}_{\mathcal{C}}(\alpha_1) \wedge \mathcal{J}_{\mathcal{D}}(\alpha_2)\} \wedge \{\mathcal{J}_{\mathcal{C}}(\alpha_1*(\beta_1*\gamma_1)) \wedge \mathcal{J}_{\mathcal{D}}(\alpha_2*(\beta_2*\gamma_2))\}]$

$=\{\mathcal{J}_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2)\} \wedge \{\mathcal{J}_{\mathcal{C}x\mathcal{D}}(\alpha_1*(\beta_1*\gamma_1)) \wedge (\alpha_2*(\beta_2*\gamma_2))\}$

$= \{\mathcal{J}_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2)\} \wedge \{\mathcal{J}_{\mathcal{C}x\mathcal{D}}((\alpha_1, \alpha_2)*((\beta_1, \beta_2)* (\gamma_1, \gamma_2)))\}$

Therefore, $\mathcal{J}_{\mathcal{C}x\mathcal{D}}((\beta_1, \beta_2)* (\gamma_1, \gamma_2)) \geq \{\mathcal{J}_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2) \wedge \mathcal{J}_{\mathcal{C}x\mathcal{D}}((\alpha_1, \alpha_2)*((\beta_1, \beta_2)* (\gamma_1, \gamma_2)))\}$

Similarly we can easily prove that,

$\digamma_{\mathcal{C}x\mathcal{D}}((\beta_1, \beta_2)* (\gamma_1, \gamma_2)) \geq \{\digamma_{\mathcal{C}x\mathcal{D}}(\alpha_1, \alpha_2) \wedge \digamma_{\mathcal{C}x\mathcal{D}}((\alpha_1, \alpha_2)*((\beta_1, \beta_2)* (\gamma_1, \gamma_2)))\}$

Hence $\mathcal{C}x\mathcal{D}$ is a NDFB of $\mathfrak{B}_1 \times \mathfrak{B}_2$

5. HOMOMORPHISM OF NDFB OF $\mathfrak{B}$

In this section, the homomorphic behaviour of NDFB of $\mathfrak{B}$ are defined and related theorems are discussed.

Definition 5.1 Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two BS-algebras and $h:\mathfrak{B}_1 \longrightarrow \mathfrak{B}_2$ be a function.

i) If $\mathcal{D}$ is a NDFB in $\mathfrak{B}_2$, then the preimage of $\mathcal{D}$ under h denoted by $h^{-1}(\mathcal{D})$ is the NDFB in $\mathfrak{B}_1$ is defined by $h^{-1}(\mathcal{D})=\{<(\alpha), h^{-1}(\daleth_{\mathcal{D}}(\alpha)), h^{-1}(\mathcal{J}_{\mathcal{D}}(\alpha)), h^{-1}(\digamma_{\mathcal{D}}(\alpha))>: \alpha\in\mathfrak{B}\}$,

where $h^{-1}(\daleth_{\mathcal{D}}(\alpha)) = \daleth_{\mathcal{D}}(h(\alpha))$; $h^{-1}(\mathcal{J}_{\mathcal{D}}(\alpha)) = \mathcal{J}_{\mathcal{D}}(h(\alpha))$; $h^{-1}(\digamma_{\mathcal{D}}(\alpha)) = \digamma_{\mathcal{D}}(h(\alpha))$;

Theorem 5.2 Let h: $\mathfrak{B}_1 \rightarrow \mathfrak{B}_2$ be an epimorphism of BS-algebras if $\mathcal{D}$ is a NDFB of $\mathfrak{B}_2$, then the pre image of $\mathcal{D}$ under h is also a NDFB of $\mathfrak{B}_1$.

Proof

Let $\mathcal{D}$ is a NDFB of $\mathfrak{B}_2$. Let $\alpha,\beta,\gamma \in \mathfrak{B}_1$

Now, $h^{-1}(\daleth_{\mathcal{D}}(1)) = \daleth_{\mathcal{D}}(h(1))$

$\leq \daleth_{\mathcal{D}}(h(\alpha))$

$= h^{-1}(\daleth_{\mathcal{D}}(\alpha))$

Therefore $h^{-1}(\daleth_{\mathcal{D}}(1)) \leq h^{-1}(\daleth_{\mathcal{D}}(\alpha))$

and $h^{-1}(\mathcal{J}_{\mathcal{D}}(1)) = \mathcal{J}_{\mathcal{D}}(h(1))$

$\geq \mathcal{J}_{\mathcal{D}}(h(\alpha))$

$= h^{-1}(\mathcal{J}_{\mathcal{D}}(\alpha))$

Therefore $h^{-1}(\mathcal{J}_{\mathcal{D}}(1)) \geq h^{-1}(\mathcal{J}_{\mathcal{D}}(\alpha))$

and $h^{-1}(\digamma_{\mathcal{D}}(1)) = \digamma_{\mathcal{D}}(h(1))$

$\geq \digamma_{\mathcal{D}}(h(\alpha))$

$= h^{-1}(\digamma_{\mathcal{D}}(\alpha))$

Therefore $h^{-1}(\digamma_{\mathcal{D}}(1)) \geq h^{-1}(\digamma_{\mathcal{D}}(\alpha))$

ii) Again, $h^{-1}(\daleth_{\mathcal{D}}(\beta*\gamma)) = \daleth_{\mathcal{D}}(h(\beta*\gamma))$

$= \daleth_{\mathcal{D}}(h(\beta)*h(\gamma))$

$\leq \{\daleth_{\mathcal{D}}(h(\alpha)) \vee \daleth_{\mathcal{D}}(h(\alpha)*[h(\beta)*h(\gamma)])\}$

$= \{\daleth_{\mathcal{D}}(h(\alpha)) \vee \daleth_{\mathcal{D}}(h(\alpha*(\beta*\gamma)))\}$

Therefore, $h^{-1}(\daleth_{\mathcal{D}}(\beta*\gamma)) \leq \{h^{-1}(\daleth_{\mathcal{D}}(\alpha)) \vee h^{-1}(\daleth_{\mathcal{D}}(\alpha*(\beta*\gamma)))\}$

and $h^{-1}(\mathcal{J}_{\mathcal{D}}(\beta*\gamma)) = \mathcal{J}_{\mathcal{D}}(h(\beta*\gamma))$

$= \mathcal{J}_{\mathcal{D}}(h(\beta)*h(\gamma))$

$\geq \{\mathcal{J}_{\mathcal{D}}(h(\alpha)) \wedge \mathcal{J}_{\mathcal{D}}(h(\alpha)*[h(\beta)*h(\gamma)])\}$

$= \{\mathcal{J}_{\mathcal{D}}(h(\alpha)) \wedge \mathcal{J}_{\mathcal{D}}(h(\alpha*(\beta*\gamma)))\}$

Therefore, $h^{-1}(\mathcal{J}_{\mathcal{D}}(\beta*\gamma)) \geq \{h^{-1}(\mathcal{J}_{\mathcal{D}}(\alpha)) \wedge h^{-1}(\mathcal{J}_{\mathcal{D}}(\alpha*(\beta*\gamma)))\}$

and $h^{-1}(\digamma_{\mathcal{D}}(\beta*\gamma)) = \digamma_{\mathcal{D}}(h(\beta*\gamma))$

$= \digamma_{\mathcal{D}}(h(\beta)*h(\gamma))$

$\geq \{\digamma_{\mathcal{D}}(h(\alpha)) \wedge \digamma_{\mathcal{D}}(h(\alpha)*[h(\beta)*h(\gamma)])\}$

$= \{\digamma_{\mathcal{D}}(h(\alpha)) \wedge \digamma_{\mathcal{D}}(h(\alpha*(\beta*\gamma)))\}$

Therefore, $h^{-1}(\digamma_{\mathcal{D}}(\beta*\gamma)) \geq \{h^{-1}(\digamma_{\mathcal{D}}(\alpha)) \wedge h^{-1}(\digamma_{\mathcal{D}}(\alpha*(\beta*\gamma)))\}$

Hence $h^{-1}(\boldsymbol{\mathcal{D}})$is a NDFB of $\mathfrak{B}_1$.

Definition 5.3 [1] Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two BS-algebras h: $\mathfrak{B}_1 \rightarrow \mathfrak{B}_2$ be a homomorphism. Then h(1) =1

Theorem 5.4 Let h: $\mathfrak{B}_1 \rightarrow \mathfrak{B}_2$ be a homomorphism of BS-algebras if $\boldsymbol{\mathcal{D}}$ is a NDFB of $\mathfrak{B}_1$, then $h(\boldsymbol{\mathcal{D}})$is a NDFB of $\mathfrak{B}_2$.

Proof

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathfrak{B}_1$ and $\beta_1, \beta_2, \beta_3 \in \mathfrak{B}_2$ such that $h(\alpha_1)= \beta_1$, $h(\alpha_2)= \beta_2$, $h(\alpha_3)= \beta_3$

Now, $\Tau_{\mathcal{D}}(\beta_1)= \Tau_{\mathcal{D}}(h(\alpha_1))$

$= h^{-1}(\Tau_{\mathcal{D}}(\alpha_1))$

$\geq h^{-1}(\Tau_{\mathcal{D}}(1))$

$= \Tau_{\mathcal{D}}(h(1))$

$= \Tau_{\mathcal{D}}(1)$

Therefore, $\Tau_{\mathcal{D}}(\beta_1) \geq \Tau_{\mathcal{D}}(1)$

And $\mathcal{J}_{\mathcal{D}}(\beta_1)= \mathcal{J}_{\mathcal{D}}(h(\alpha_1))$

$= h^{-1}(\mathcal{J}_{\mathcal{D}}(\alpha_1))$

$\leq h^{-1}(\mathcal{J}_{\mathcal{D}}(1))$

$= \mathcal{J}_{\mathcal{D}}(h(1))$

$= \mathcal{J}_{\mathcal{D}}(1)$

Therefore, $\mathcal{J}_{\mathcal{D}}(\beta_1) \leq \mathcal{J}_{\mathcal{D}}(1)$

And $\digamma_{\mathcal{D}}(\beta_1)= \digamma_{\mathcal{D}}(h(\alpha_1))$

$= h^{-1}(\digamma_{\mathcal{D}}(\alpha_1))$

$\leq h^{-1}(\digamma_{\mathcal{D}}(1))$

$= \digamma_{\mathcal{D}}(h(1))$

$= \digamma_{\mathcal{D}}(1)$

Therefore, $\digamma_{\mathcal{D}}(\beta_1) \leq \digamma_{\mathcal{D}}(1)$

ii) Again, $\Tau_{\mathcal{D}}(\beta_2* \beta_3) = \Tau_{\mathcal{D}}(h(\alpha_2)* h(\alpha_3))$

$= h^{-1}(\Tau_{\mathcal{D}}(\alpha_2*\alpha_3))$

$\leq \{h^{-1}(\Tau_{\mathcal{D}}(\alpha_1)) \vee h^{-1}(\Tau_{\mathcal{D}}(\alpha_1*(\alpha_2*\alpha_3)))\}$

$=\{\Tau_{\mathcal{D}}(h(\alpha_1)) \vee \Tau_{\mathcal{D}}(h(\alpha_1*(\alpha_2*\alpha_3)))\}$

$=\{\Tau_{\mathcal{D}}(h(\alpha_1)) \vee \Tau_{\mathcal{D}}(h(\alpha_1)*(h(\alpha_2)*h(\alpha_3)))\}$

$=\{\Tau_{\mathcal{D}}(\beta_1) \vee \Tau_{\mathcal{D}}(\beta_1*(\beta_2*\beta_3))\}$

Therefore, $\Tau_{\mathcal{D}}(\beta_2* \beta_3) \leq \{\Tau_{\mathcal{D}}(\beta_1) \vee \Tau_{\mathcal{D}}(\beta_1*(\beta_2*\beta_3))\}$

And $\mathcal{J}_{\mathcal{D}}(\beta_2* \beta_3) = \mathcal{J}_{\mathcal{D}}(h(\alpha_2)* h(\alpha_3))$

$= h^{-1}(\mathcal{J}_{\mathcal{D}}(\alpha_2*\alpha_3))$
$\geq \{h^{-1}(\mathcal{J}_{\mathcal{D}}(\alpha_1)) \wedge h^{-1}(\mathcal{J}_{\mathcal{D}}(\alpha_1*(\alpha_2*\alpha_3)))\}$
$=\{\mathcal{J}_{\mathcal{D}}(h(\alpha_1)) \wedge \mathcal{J}_{\mathcal{D}}(h(\alpha_1*(\alpha_2*\alpha_3)))\}$
$=\{\mathcal{J}_{\mathcal{D}}(h(\alpha_1)) \wedge \mathcal{J}_{\mathcal{D}}(h(\alpha_1)*(h(\alpha_2)*h(\alpha_3)))\}$
$=\{\mathcal{J}_{\mathcal{D}}(\beta_1) \wedge \mathcal{J}_{\mathcal{D}}(\beta_1*(\beta_2*\beta_3))\}$

Therefore, $\mathcal{J}_{\mathcal{D}}(\beta_2* \beta_3) \geq \{\mathcal{J}_{\mathcal{D}}(\beta_1) \wedge \mathcal{J}_{\mathcal{D}}(\beta_1*(\beta_2*\beta_3))\}$

Similarly, $F_{\mathcal{D}}(\beta_2* \beta_3) \geq \{F_{\mathcal{D}}(\beta_1) \wedge F_{\mathcal{D}}(\beta_1*(\beta_2*\beta_3))\}$

Hence $h(\mathcal{D})$ is a NDFB of $\mathfrak{B}_2$.

Conclusion

In this research paper, the notion of Neutrosophic doubt fuzzy bi-ideal (NDFB) of BS-algebras $\mathfrak{B}$ are introduced and studied their algebraic properties. We obtained the Cartesian product of neutrosophic doubt fuzzy bi-ideal(NDFB) for BS-algebras $\mathfrak{B}$. Finally, we studied how to deal with homomorphism in neutrosophic doubt fuzzy bi-ideal(NDFB) for BS-algebras $\mathfrak{B}$.

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Neutrosophic Vision of the Expected Opportunity Loss Criterion (NEOL) Decision Making Under Risk

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Abstract:

One of the major challenges facing decision-makers at the present time is obtaining complete information about the issue under study, due to the unstable conditions of the work environment that are beyond the control of decision-makers, which requires them to reach an optimal decision in light of these circumstances and fluctuations and to benefit from the data that is collected. Collected by specialists to determine the appropriate probability distribution corresponding to random cases of nature, here we are faced with the issue of making a decision in the event of risk because the probability distribution is a distribution linked to the data controlled by the conditions of the work environment, which entails a great risk. Decision makers bear the responsibility of choosing the optimal decision that reduces This risk is achieved and the greatest possible profit and the least possible loss are achieved. The issue of decision-making becomes more complex as the number of events increases, and we are in dire need of an ideal study of the issue that takes into account all the circumstances of the work environment. The concept of missed opportunity is very useful in analyzing the decision making under risk, after making the decision and the occurrence of events, the decision makers may regret and wish they had chosen actions different from those they chose at the beginning. To reduce the regret of the decision makers and minimize the expected lost opportunity, researchers in the field of classical operations research presented the criterion of the expected lost opportunity through which the decision can be determined. The ideal with the least percentage of regret. In this research, we present a neutrosophic vision of the expected opportunity loss criterion by taking the data of the issue under study. neutrosophic values are ranges whose lowest limit expresses profit in the worst conditions, and only the highest represents profit in the best conditions.

key words:

Operations research; decision-making theory; decision making under risk; neutrosophic science; neutrosophic decision-making theory; neutrosophic missed opportunity criterion

Introduction:

Decision-making theory depends on the data provided by specialists in collecting data on the issue under study and the type of this data: whether it is confirmed data, uncertain data, or random data repeated according to a certain probability distribution law, through which the methods that must be followed to obtain the optimal decision are determined. The need to know the results of any decision before making it when the data is uncertain or the data is random and the decision maker does not know anything about the state that nature will take or even about the chances of any of it occurring, so the decision maker uses the primary information to obtain additional information that reduces the risk. In classical logic, a group of methods were used that helped the decision maker to make the ideal decision, and since this decision depends on specific classical values that do not take into account the changes that may occur in the work environment, and in light of the changes, fluctuations, and challenges that the decision maker faces in all fields, they increase day by day. Day after day, there is a need for a new study that relies on data that has a margin of freedom and takes into account all circumstances, from the best to the worst. Therefore, many researchers and those interested in studying operations research methods have presented many papers, including [9-1], which is considered a new vision, a neutrosophical vision for this. The methods are based on the concepts and foundations laid down by the founder of this logic, see [10]. Through the indeterminacy of the elements of the profit matrix and used in the decision-making process, an ideal neutrosophic decision is obtained. In this research, we will take the elements of the profit (or loss) matrix and the probabilities are neutrosophic values in the form of ranges, the lowest of which corresponds to the worst states of nature and the highest of which corresponds to the best states of nature, to formulate the expected opportunity loss criterion used to choose the optimal decision under risk. We will apply this criterion to the example that was presented in the research [9], where it was done. Using other criteria to make decisions under risk. By comparing the results provided by each criterion, decision makers can use the appropriate criterion for the issue under study.

Discussion:

The risk lies in the view of decision making when the movement of nature is random and subject to a probability distribution that may not be completely known, but rather a distribution assumed by experts, or by the decision maker himself, which reflects its effects on the decision itself. To reduce the risk, the probabilities are calculated or estimated. Through practical facts or a statistical study taken from previous experiments and studies, classical operations research methods presented a study of decision-making theory in its three cases: the case of confirmed data, uncertain data, and random data, where appropriate criteria were set for each case so that decision makers can make decisions that limit losses. But these decisions are appropriate for work conditions similar to the work conditions in which the data was collected, and any change may cause a large and unexpected loss. Therefore, in previous research [8] we presented a neutrosophical vision of some standards for decision-making in the case of uncertain data, and in another research [9]. We presented some criteria for decision-making under risk, as a complement to what we presented from the neutrosophical study of decision-making under risk. In this research, we present a neutrosophical vision for the expected opportunity loss criterion.

1- The classic general formulation of the decision problem:

The decision maker has alternatives $A(a_1, a_2, \dots, a_m)$where m is the number of alternatives available to the decision maker, and the states that nature can take in the future $\theta(\theta_1, \theta_2, \dots, \theta_n)$ where n is the number of states that nature can take at Its movement, and the amount of profit or loss that the decision maker will achieve is $X(a_i, \theta_j)$, or by short code X_{ij} .Then the profit matrix is given by the following table:

States of nature / Alternatives	θ_1	θ_2	...	θ_n
a_1	X_{11}	X_{12}	...	X_{1n}
a_2	X_{21}	X_{22}	...	X_{2n}
...	...	...	...	...
a_m	X_{m1}	X_{m2}	...	X_{mn}
$P(\theta_j)$	$P(\theta_1)$	$P(\theta_2)$	...	$P(\theta_n)$

Table No. (1) Classic general data for the decision-making issue (profit matrix)

2- The general neutrosophic formulation of the decision-making problem:

The decision maker has the alternatives $A(a_1, a_2, \dots, a_m)$where m is the number of alternatives available to the decision maker, and the states that nature can take in the future are $\theta(\theta_1, \theta_2, \dots, \theta_n)$where n is the number of states that nature can take when they move (they are independent of each other), We symbolize the amount of profit or loss that the decision maker will achieve $NX = X(a_i, \theta_j) \pm \varepsilon_{ij}$, or by short code NX_{ij} These are neutrosophic values, and ε_{ij} ,it is indeterminacy, it can be$\varepsilon_{ij} \in [\lambda_1, \lambda_2]$or $\varepsilon_{ij} \in \{\lambda_1, \lambda_2\}$. $^{-}0$

Also, the law of probability distribution to which the possible states of nature are subject, we take it as a neutrosophic number series or a neutrosophic mathematical function that corresponds to each state of nature with the probability of its occurrence:

$NP(\theta_j) = P(\theta_j) + \mu_j$. Where $^{-}0 \leq \sum_{j=1}^{n} NP(\theta_j) \leq 3^{+}$ and $0 \leq P(\theta_j) \leq 1$.

μ_j it is indeterminacy that can by $\mu_j \in [\delta_1, \delta_2]$ or $\mu_j \in \{\delta_1, \delta_2\}$. Based on the previous data, the goal is to choose the optimal alternative according to the available states of nature in order to obtain the greatest possible profit or the least possible loss. We organize the previous information in the following table:

States of nature / Alternatives	θ_1	θ_2	...	θ_n
a_1	$X_{11} \pm \varepsilon_{11}$	$X_{12} \pm \varepsilon_{12}$	...	$X_{1n} \pm \varepsilon_{1n}$
a_2	$X_{21} \pm \varepsilon_{21}$	$X_{22} \pm \varepsilon_{22}$	...	$X_{2n} \pm \varepsilon_{2n}$
...	...	...	...	...
a_m	$X_{m1} \pm \varepsilon_{m1}$	$X_{m2} \pm \varepsilon_{m2}$	...	$X_{mn} \pm \varepsilon_{mn}$
$P(\theta_j)$	$P(\theta_1)$	$P(\theta_2)$	...	$P(\theta_n)$

Table No. (2) Neutrosophic general data for the decision-making issue (profit matrix)

3- **In a previous study [9], we presented a neutrosophical vision of three criteria used to determine the optimal decision under risk:**

a. Neutrosophic aspiration level criterion.
b. Neutrosophic most likely criterion.
c. Neutrosophic largest expected values criterion.

We chose the appropriate alternative for the following question:

Example 1:

We have the following table of alternatives and states of nature:

States of nature / Alternatives	θ_1	θ_2	θ_3
a_1	$[300,350]$	$[100,150]$	$[400,450]$
a_2	$[-220,-170]$	$[170,220]$	$[500,550]$
a_3	$[-400,-350]$	$[200,250]$	$[300,350]$
a_4	$[160,210]$	$[300,350]$	$[200,250]$
$P(\theta_j)$	$[0.3,0.45]$	$[0.1,0.25]$	$[0.6,0.75]$

Table No. (3) Neutrosophic profit matrix table

It is required to determine the appropriate alternative using:

Neutrosophic aspiration level criterion:

According to the following data, the level of ambition of the decision maker:

The profit belongs to the range $M \in [300,350]$.

The loss belongs to the range $N \in [200,250]$.

The appropriate alternative that achieves the level of ambition of the decision maker in profit and loss is alternative a_1.

Neutrosophic most likely criterion:

From the table we notice that the most likely case is case θ_3, then the issue will lead to a decision in case of confirmation according to the following table:

Most likely case / Alternative	θ_3
a_1	$[400,450]$
a_2	$[500,550]$
a_3	$[300,350]$
a_4	$[200,250]$
$P(\theta_3)$	$[0.6,0.75]$

Table No. (4): Table of the most likely neutrosophic states

We choose the largest value in the condition column θ_3, which is $[500,550]$ corresponding to the alternative a_2, and a_2 is the appropriate alternative.

Largest expected values criterion:

States of nature / Alternatives	θ_1	θ_2	θ_3	$E(a_i)$
a_1	$[300,350]$	$[100,150]$	$[400,450]$	$[340,532.5]$
a_2	$[-220,-170]$	$[170,220]$	$[500,550]$	$[251,391]$
a_3	$[-400,-350]$	$[200,250]$	$[300,350]$	$[80,167.5]$
a_4	$[160,210]$	$[300,350]$	$[200,250]$	$[198,369.5]$
$P(\theta_j)$	$[0.3,0.45]$	$[0.1,0.25]$	$[0.6,0.75]$	

Table No. (5): Table of expected neutrosophic values

By comparing the elements of column $E(a_i)$, we notice that the largest expected values are $[340,532.5]$ corresponding to alternative a_1. Alternative a_1 is the appropriate alternative according to this criterion.

4- **In this research, we present a neutrosophical vision of another of the criteria used to choose the appropriate alternative decision making under risk:**

Expected opportunity loss criterion (EOL):

Based on the information contained in references [11-13], we find that the minimum expected lost opportunity criterion depends on choosing the decision that guarantees us the least regret, i.e., the decision with the lost opportunity, and it is calculated according to the following steps:

From the profit matrix:

we choose the largest profit value corresponding to each state of nature, θ_j, and let M_j be:

$$M_j = \underbrace{Max}_{i} X_{ij}$$

We form the Regret Matrix from the following relation:

$$X'_{ij} = M_j - X_{ij}$$

We obtain the following regret matrix:

States of nature / Alternatives	θ_1	θ_2	...	θ_n
a_1	X'_{11}	X'_{12}	...	X'_{1n}
a_2	X'_{21}	X'_{22}	...	X'_{2n}
...	...	...	...	...
a_m	X'_{m1}	X'_{m2}	...	X'_{mn}
$P(\theta_j)$	$P(\theta_1)$	$P(\theta_2)$	...	$P(\theta_n)$

Table No. (6) Classic regret matrix

c. We calculate the expected value corresponding to each alternative:

$$E(a_i) = \sum_{j=1}^{n} P(\theta_j) \,. X'_{ij} \quad ; i = 1,2, \dots , m$$

We symbolize the appropriate alternative, through which we will determine the optimal decision, with the symbol E_K, and it is calculated from the following relation:

$$E_K = \underbrace{Min}_{i} [E(a_i)] \quad ; i = 1,2, \dots , m$$

Neutrosophic Vision of the Expected Opportunity Loss Criterion (NEOL):

Using the data in the general neutrosophic formulation of the risk decision problem we find: we choose the largest profit value corresponding to each state of nature, θ_j, and let NM_j be:

$$NM_j = \underbrace{Max}_{i} NX_{ij}$$

We form the Regret Matrix from the following relationship:

$$NX'_{ij} = NM_j - NX_{ij}$$

We obtain the following neutrosophic regret matrix:

States of nature / Alternatives	θ_1	θ_2	...	θ_n
a_1	NX'_{11}	NX'_{12}	...	NX'_{1n}
a_2	NX'_{21}	NX'_{22}	...	NX'_{2n}
...	...	...	...	...
a_m	NX'_{m1}	NX'_{m2}	...	NX'_{mn}
$NP(\theta_j)$	$NP(\theta_1)$	$NP(\theta_2)$	...	$NP(\theta_n)$

Table No. (7) Neutrosophic Regret Matrix

We calculate the expected neutrosophic value corresponding to each alternative in the regret matrix:

$$NE(a_i) = \sum_{j=1}^{n} NP(\theta_j) \, . NX'_{ij} \quad ; i = 1,2, \dots, m$$

We symbolize the optimal expected neutrosophic minimum value, through which we will determine the appropriate alternative, with the symbol NE_K, and it is calculated from the following relation:

$$NE_K = \underbrace{Min}_{i} [NE(a_i)] \quad ; i = 1,2, \dots, m$$

Example 2:

We apply the neutrosophic expected value criterion to the following data in Example No. (1):

States of nature / Alternatives	θ_1	θ_2	θ_3
a_1	$[300,350]$	$[100,150]$	$[400,450]$
a_2	$[-220,-170]$	$[170,220]$	$[500,550]$
a_3	$[-400,-350]$	$[200,250]$	$[300,350]$
a_4	$[160,210]$	$[300,350]$	$[200,250]$
$P(\theta_j)$	$[0.3,0.45]$	$[0.1,0.25]$	$[0.6,0.75]$

Table No. (8) Neutrosophic profit matrix table for the problem

What is required is to determine the appropriate alternative using the expected lost opportunity criterion:

We form the neutrosophic regret matrix using the following relation:

$$NX'_{ij} = NM_j - NX_{ij}$$

We obtain the following matrix:

States of nature / Alternatives	θ_1	θ_2	θ_3	$E(a_i)$
a_1	0	200	100	$[80,125]$
a_2	520	130	0	$[169,266.5]$
a_3	700	100	200	$[340,490]$
a_4	140	0	300	$[222,288]$
$P(\theta_j)$	$[0.3,0.45]$	$[0.1,0.25]$	$[0.6,0.75]$	

Table No. (9) Regret matrix table and expected value of the neutrosophic minimum

In our example, we find that the alternative a_1 corresponding to the minimum expected value of neutrosophic $[80,125]$is the appropriate alternative that achieves the lowest value of regret and achieves a profit whose expected value is $[340,532.5]$.

Conclusion and results:

Through the previous study, we presented a neutrosophic vision of one of the important criteria used for decision-making under risk, which can be used in many life issues to reduce the regret resulting from making a decision on a specific issue by using neutrosophic value data, and to help decision makers in choosing an example decision in the case of risk, we present the following comparison is between the results of some of the criteria used to make a decision that suits all circumstances in the event of risk. We leave them to choose the criterion that can be relied upon and is appropriate for the issue under study through the following table.

Alternative and profit / criterion	Alternative	profit
Neutrosophic aspiration level criterion	a_1	[400,450]
Neutrosophic most likely criterion	a_2	[500,550]
Neutrosophic largest expected values criterion.	a_1	[340,532.5]
Neutrosophic expected opportunity loss criterion	a_1	[340,532.5]

Table No. (10): Comparison table

We note that alternative a_1 is the appropriate alternative according to three criteria. The criterion of the most likely neutrosophic state determines alternative a_2, best alternative.

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Production Planning with The Neutrosophic Fuzzy Multi-Objective Optimization Technique

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Abstract: Businesses want to realize their production at the desired time and quality. In addition, businesses aim to use their existing resources efficiently and increase their earnings. The realization of more than one purpose is achieved by multi-purpose planning of production. This study considered the multi-objective production planning of a company that manufactures spare parts for household appliances. The management of the company wants to minimize its cost, cycle time, defective rates, and material wastage while maximizing its profits in the process of producing all 121 different products. This study presents a solution for this multi-item multi-objective production problem using the intuitionistic fuzzy and neutrosophic fuzzy multi-objective optimization models. The study gives a step-by-step explanation of the methods used to achieve the solution and a comparison detailing the problems of the business and the solutions obtained. The results show that the neutrosophic fuzzy multi-objective optimization model, which is capable of independently handling uncertainty, produces better results than the intuitionistic fuzzy multi-objective optimization model.

Keywords: Multi objective programming, Neutrosophic fuzzy set, Multi-objective optimization, Neutrosophic fuzzy multi-objective optimization, Production planning.

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1. Introduction

Production refers to the entirety of the process that brings together companies, organizations, and individuals that produce, supply, and demand goods and services, the outcome of which are products that create economic benefits. E-commerce and virtual marketplaces that have emerged from the prevailing world economic order rife with increasing competition and technological advances continue to make for a difficult environment for companies. Businesses have to operate under multiple and often conflicting purposes at any given time.

To stay ahead of the competition, businesses have to satisfy the demands of their customers by making their products available in the desired features and dimensions, and at the desired time. This can be achievable through robust production planning. Planning plays a vital role in the organization's race to stay alive and competitive by showing the business its present position and outlining the best way to make use of the available resources to attain organizational goals. Production planning, in its simplest form, is the set of activities involved in determining the product to be manufactured, when and in what form. The production system faces uncertainties resulting from various reasons like the changes in the needs and preferences of people as well as deviations in the production process. In the event of such uncertainty, businesses are faced with the task of overcoming the confusion caused by the unpredictability while also striving to continue their activities under multiple objectives. The occurrence of these uncertainties in the course of operation is an inevitable part of doing business.

Uncertainty refers to a situation in which the information available about a certain situation is incomplete, inaccurate, or doubtful. The basis of uncertainty is the information that people receive from the outside world. Uncertainty causes doubt about the value of a variable, phenomenon, the decision to be made and the conclusion to be drawn. It hinders the functioning of the decision mechanism, and the success of a business is greatly proportional to how well it can handle uncertainties. Under the uncertain circumstances of doing business, it is almost impossible to meet the supply and demand expectations using only the classical approaches. This limitation of classical approaches can be complemented by fuzzy structures that can make uncertainty known in the best possible way.

Fuzzy structures provide an environment in which decision-makers are able to consider the uncertainty in all its aspects leading to more accurate and timely decisions. Fuzzy structures, that have increasingly complemented the limitations of classical approaches in the process of making and activating production plans, have also shown a developing sequence from fuzzy sets to intuitionistic fuzzy sets and neutrosophic fuzzy sets.

This study aims to show that production planning in a production company can be achieved using the neutrosophic fuzzy multi-objective optimization technique. The study thus used neutrosophic fuzzy sets with a multi-objective optimization technique, which has been touted in the literature as a generalization of fuzzy and intuitionistic fuzzy sets. The application followed three different approaches of the neutrosophic fuzzy multi-objective optimization technique; Model I, Model II and Model III. After the three models, the intuitionistic fuzzy multi-objective optimization technique was used to measure the effectiveness of the neutrosophic fuzzy multi-objective optimization technique. The results obtained from both applications were then compared.

2. Literature Review

Businesses are faced with the task of fulfilling the demands of the customers and delivering the desired product attributes in a timely manner. Those that are able to carry out the production process within a certain plan stand to attain a competitive edge in the market. Planning refers to the process of outlining how to achieve the company goals by making the best use of available resources. Production planning enables businesses to coordinate their activities in daily, weekly, monthly and annual timeframes in line with business objectives [1]. The competition that is the result of the various developments going on in the world has made planning an integral part of the business. The optimal of business resources like human, machine, material, and time is tied to the quality of the plan [2].

Businesses are often in operation for more than one purpose. They are faced with multiple objectives such as profit maximization and resource minimization instead of a single objective like cost minimization. The multi-objective nature of the goals combined with product diversity turns production into a complex multi-objective problem. Cheng and Xiao-Bing looked at the production planning problem with reference to the delivery time, production balance, stock, and overtime purposes [3]. Yazdani et al. (2021) examined the minimization of the sustainability function, which includes the total production and process costs and the amount of harmful environmental and social components [4].

Most production planning problems are multi-objective with varying degrees of uncertainty. The fuzzy logic system introduced by Zadeh in the 1960s has proved useful for decision-makers in many areas. Zadeh (1965) stated that the human thought structure, to a great extent, isn't clear, but fuzzy [5]. People express their thoughts in linguistic terms rather than numerical data. This leads to different ways of interpreting different thoughts leading to different decisions and hence uncertainty. Fuzzy systems offer the opportunity to easily model uncertainty. Various researchers have used fuzzy sets to solve multi-objective optimization problems. Wang and Liang developed a fuzzy multi-objective linear programming model to solve multi-item multi-objective aggregate production planning decision problems. In the proposed model, a solution is presented to minimize the total production cost, labor turnover rate, transportation, and ordering costs [6]. Kumawat et al. (2021) came up with a fuzzy multi-objective optimization model that is intended to minimize carbon emissions and energy consumption in sustainable production planning [7]. Komsiyah et al. (2018) employed a fuzzy goal programming technique in furniture production to maximize profit and minimize production and raw material cost [8]. In 1986, Atanassov presented intuitionistic fuzzy sets as an alternative to fuzzy sets. Intuitionistic fuzzy sets, unlike fuzzy sets, define uncertainty using the degree of membership and non-membership, as well as the degree of hesitation. Neutrosophic fuzzy sets by Smarandache in 1995 brought a new approach to solving problems involving uncertainty [9]. Many researchers have contributed to the development of neutrosophic fuzzy sets and the solution to different problems. Some of the studies are given in Table 1.

Table 1 Summary of the Literature

Author	**Technique**	**Goal**	**Explanation**
Bharati and Singh [10]	Intuitionistic Fuzzy Optimization	Multi-objective, production and profit maximization	Agricultural production problem
Ali et al. [11]	Intuitionistic Fuzzy Optimization, Fuzzy Goal Programming	Multi-objective, Profit maximization, minimization carrying cost	Inventory modeling
Hussian et al. [12]	Neutrosophic Linear Programming	Single objective - profit maximization	production enterprise
Abdel-Baset et al. [13]	Neutrosophic Integer Programming Problems	Single objective	Neutrosophic integer programming with numerical examples
Ahmad and Adhami [14]	Fuzzy optimization, Intuitionistic optimization, Neutrosophic optimization	Multi-objective, transportation cost, labor cost, safety cost	Transportation problems
Hu et al. [15]	Intuitionistic Fuzzy programming, Neutrosophic Programming	Multi-objective, normal time production cost, and overtime production cost, inventory cost, order cost, labor cost.	Production Planning Problem
Khan et al. [16]	Intuitionistic Fuzzy Optimization, Neutrosophic Fuzzy Optimization	Multi-profit maximization, production, and carrying cost minimization	Production planning problem in a hardware company
Mondal et al. [17]	Neutrosophic Geometric Programming, Neutrosophic Non-Linear Programming	Single objective - cost minimization	Economic order quantity problem
Roy and Das [18]	Neutrosophic multi-objective linear programming,	Multi-profit, quality, employee satisfaction	Production Planning

	Intuitionistic fuzzy optimization,		

3. Basic Concepts

In this section, are mentioned basic concepts of fuzzy sets, intuitionistic fuzzy sets, and neutrosophic fuzzy.

Definition 1 (Fuzzy Set): The fuzzy set $\tilde{A}$, consisting of x elements defined in the universal set X, is presented as, $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$. The degree of membership function $\mu_{\tilde{A}}(x)$ is also called the degree of accuracy. The degree of membership function takes values between 0 and 1 and is defined as $\mu_{\tilde{A}}$: X→[0,1] [19].

Definition 2 (Intuitionistic Fuzzy Set): Intuitionistic fuzzy sets, which are the generalization of fuzzy sets, are defined based on two subsets; the membership function $\mu_{A^{\tilde{I}}}(x)$ and the non-membership function $\nu_{A^{\tilde{I}}}(x)$. The intuitionistic fuzzy defined in the universal set X is in the form of $A^{\tilde{I}} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) \mid x \in X\}$. $\mu_{A^{\tilde{I}}}$: X →[0,1] and $\nu_{A^{\tilde{I}}}$: X →[0,1]

Intuitionistic fuzzy sets are defined by the degree of hesitation as well as the degree of membership and non-membership. The hesitation index or degree of hesitation is denoted

$\pi_{A^{\tilde{I}}}$. The degree of hesitation is calculated by subtracting the sum of the degrees of membership and non-membership from one. $\pi_{A^{\tilde{I}}} = 1 - \mu_{A^{\tilde{I}}}(x) - \nu_{A^{\tilde{I}}}(x)$ [19].

Definition 3. (Neutrosophic Fuzzy Set): Let X be the universal set and x a general element in this set. The neutrosophic fuzzy set A defined in X is characterized by the membership functions of truth, uncertainty, and inaccuracy.

$$\mu_{A^{\tilde{N}}}(x);\ x \to]0^-, 1^+[$$

$$\sigma_{A^{\tilde{N}}}(x) :\ x \to]0^-, 1^+[$$

$$\nu_{A^{\tilde{N}}}(x) :\ x \to]0^-, 1^+[$$

The Neutrosophic fuzzy sets, whether real, standard, or non-standard subsets in the $\mu_{A^{\tilde{N}}}(x)$, $\sigma_{A^{\tilde{N}}}(x)$, $\nu_{A^{\tilde{N}}}(x)$ $]0^-, 1^+[$ range are shown as follows.

$$A^{\tilde{N}} = \{< x, \mu_{\tilde{A}}(x), \sigma_{A^{\tilde{N}}}(x), \nu_{\tilde{A}}(x) > \mid x \in X\}$$

There is no limit to $\mu_{A^{\tilde{N}}}(x)$, $\sigma_{A^{\tilde{N}}}(x)$, $\nu_{A^{\tilde{N}}}(x)$ totals and

$$0^- \leq \sup \mu_{A^{\tilde{N}}}(x) + \sup \sigma_{A^{\tilde{N}}}(x) + \sup \nu_{A^{\tilde{N}}}(x) \leq 3^+$$

Definition 4. (Single Valued Neutrosophic sets): Let X be the universal set and x a general element in this set. The single-valued neutrosophic set is defined by the truth membership function $\mu_{A^{\tilde{N}}}(X)$, indeterminacy membership function $\sigma_{A^{\tilde{N}}}(X)$ and the falsity membership function $\nu_{A^{\tilde{N}}}(X)$ [20].

$$A^{\tilde{N}} = \{< x, \mu_{\tilde{A}}(x), \sigma_{A^{\tilde{N}}}(x), \nu_{\tilde{A}}(x) > \mid x \in X\}$$

$$0 \leq \mu_{A^{\tilde{N}}}(x) + \sigma_{A^{\tilde{N}}}(x) + \nu_{A^{\tilde{N}}}(x) \leq 3,\ \mu_{A^{\tilde{N}}}(x), \sigma_{A^{\tilde{N}}}(x) \text{ and } \nu_{A^{\tilde{N}}}(x) \in [0,1]$$

4. Multi-Objective Optimization Solution Methods

Multi-objective optimization problems arise when there is a need for simultaneous optimization of more than one objective. A general multi-objective optimization model is mathematically represented as [21]:

$Max(Min)f(x) = \{Z_1(x), Z_2(x), \ldots. Z_n(x)\}$

Constraints

$g(x) \leq 0, x \in X$

$g(x) \geq 0, x \in X$

$g(x) = 0, x \in X$

$x \geq 0, x \in X$

There are multiple solution techniques for multi-objective optimization problems including neutrosophic fuzzy and intuitionistic fuzzy multi-objective optimization techniques which will be outlined in the next section.

A. Intuitionistic Fuzzy Multi-Objective Optimization Technique

The intuitionistic fuzzy multi-objective optimization model is denoted as follows [22].

$max = [f_1{}^I(x), f_2{}^I(x), \ldots f_{k1}{}^I(x)]$

$min = [f_{k1+1,}{}^I(x), f_{k1+2}{}^I(x), \ldots f_k{}^I(x)]$

Constraints

$g_i(x) \leq c_i, \quad$ i= $1,2,\ldots, m_1$

$g_i(x) \geq c_i, \quad$ i= $m_1 + 1, m_1 + 2, \ldots, m_2$

$g_i(x) = c_i, \quad$ i= $m_2 + 1, m_2 + 2, \ldots, m,$

$x \geq 0, \quad$ **(1)**

Where;

x: decision variable,

$f_j{}^I(x)$: Objective function,

$g_l(x)$: Constraint function

The following steps are followed when solving a multi-objective intuitionistic fuzzy optimization problem [10]:

Step 1: The intuitionistic fuzzy multi-objective optimization problem is set up as given in equation (1).

Step 2: One of the objective functions of the problem is chosen randomly and solved using a single-objective classical linear programming technique respecting all the constraints of the problem, and ignoring the other objectives.

Step 3: Step 2 is repeated for all other objective functions. The values of the objective functions and decision variables are obtained. The optimal solutions obtained for each objective function are then used to determine the solution vector $x_1, x_2\ x_3, \ldots, x_k$.

Step 4: The solution vectors obtained are then used to get the values of each of the objective functions as in the pay-off matrix shown in Table 2 below [23].

Table 2 *Pay-off Matrix*

Step 5: The lower and upper limits of the membership functions are determined using the values obtained from the pay-off matrix. The lower and upper limits of membership and non-membership functions are calculated differently for different objective functions. Table 3 shows how the lower and upper limits of the membership functions are calculated when the objective functions have a maximum structure [10].

Table 3 *Determination of the Upper and Lower Limits of Intuitionistic Fuzzy Membership Functions with a Maximization Structure*

Objective Function Type	**Objective Function with Maximization Type**	
Membership Function Type	**Lower Limit**	**Upper Limit**
Membership Function (μ)	$L_k^{\mu} = min\{f_k(x_k)\}$	$U_k^{\mu} = max\{f_k(x_k)\}$
Non-Membership Function (ν)	$L_k^{\nu} = L_k^{\mu}$	$U_k^{\nu}=U_k^{\mu} - \lambda(U_k^{\mu} - L_k^{\mu})$

Table 4 shows how to calculate the lower and upper limits of the membership functions where the objective functions have a minimization type [10].

Table 4 *Determination of the Upper and Lower Limits of Intuitionistic Fuzzy Membership Functions with a Minimization Type*

Objective Function Type	Objective Function with Minimization Type	
Membership Function Type	**Lower Limit**	**Upper Limit**
Membership Function (μ)	$L_k^{\mu} = min\{f_k(x_k)\}$	$U_k^{\mu} = max\{f_k(x_k)\}$
Non-Membership Function (v)	$L_k^{v}=L_k^{\mu}+\lambda(U_k^{\mu} - L_k^{\mu})$	$U_k^{v} = U_k^{\mu}$

The U_k shown in the equations indicates the upper limit of the relevant objective, in other words, the highest value it can take. L_k indicates the lower limit of the relevant objective, in other words, the lowest value it can take. U_k indicates the best value that the objective function can take in problems with a maximization type as well as the worst value that the objective function will take in problems with a minimization type. The λ value given in the table is determined by the decision maker, provided that it is in the range of $0 < \lambda < 1$ [24]. In our study, the value λ=0,3 was used.

Step 6: Membership and non-membership functions are created with the lower and upper limits obtained in step 5. The determination of membership and non-membership functions differs according to the type of objective functions.

If the objective function has a maximization type, the membership function is defined as shown in equation (2).

$$\mu_{k\tilde{\imath}}(x)= \begin{cases} 0, & if \quad f_k(x) \leq L_k^{\mu} \\ \frac{f_k(x)-L_k^{\mu}}{U_k^{\mu}-L_k^{\mu}}, & if \quad L_k^{\mu} \leq f_k(x) \leq U_k^{\mu} \\ 1, & if \quad f_k(x) \geq U_k^{\mu} \end{cases} \tag{2}$$

If the objective function has a maximization type, the non-membership function is defined as shown in equation (3).

$$v_{k\tilde{\imath}}(x)= \begin{cases} 0, & if \quad f_k(x) \geq U_k^{v} \\ \frac{U_k^{v}(x)-f_k(x)}{U_k^{v}-L_k^{v}}, & if \quad L_k^{v} \leq f_k(x) \leq U_k^{v} \\ 1, & if \quad f_k(x) \leq L_k^{v} \end{cases} \tag{3}$$

Where;

L_k^{μ}: Lower limit of membership function

U_k^{μ}: Upper limit of membership function

L_k^{ν}: Lower limit of a non-membership function

U_k^{ν}: Upper limit of a non-membership function

$f_k(x)$: k. Function.

Figure 1 shows the membership and non-membership functions for intuitionistic fuzzy sets with a maximization type [10].

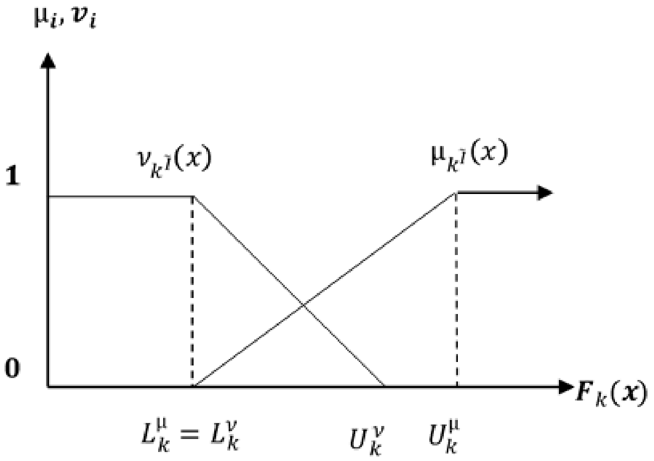

Figure 1. Membership Functions in Intuitionistic Fuzzy Sets with a Maximization Type

If the objective function has a minimization type, the membership function is defined as shown in equation (4).

$$\mu_{k^I}(x) = \begin{cases} 1, & if \quad f_k(x) \le L_k^{\mu} \\ \frac{U_k^{\mu} - f_k(x)}{U_k^{\mu} - L_k^{\mu}}, & if \quad L_k^{\mu} \le f_k(x) \le U_k^{\mu} \\ 0, & if \quad f_k(x) \ge U_k^{\mu} \end{cases} \tag{4}$$

If the objective function has a minimization type, the non-membership function is defined as shown in equation (5).

$$\nu_{k^I}(x) = \begin{cases} 1, & if \quad f_k(x) \ge U_k^{\nu} \\ \frac{f_k(x) - L_k^{\nu}(x)}{U_k^{\nu} - L_k^{\nu}} & if \quad L_k^{\nu} \le f_k(x) \le U_k^{\nu} \\ 0, & if \quad f_k(x) \le L_k^{\mu} \end{cases} \tag{5}$$

Figure 2 shows the membership and non-membership functions for intuitionistic fuzzy sets with a minimization type [25].

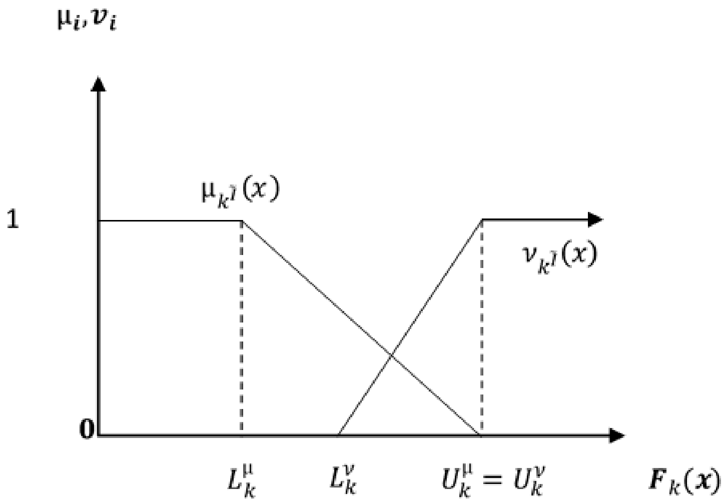

Figure 2. Membership Functions in Intuitionistic Fuzzy Sets with a Minimization Type

In an optimization problem, where there are objectives with a maximization type, the acceptance level, in other words, the degree of membership increases as each function approaches its highest value (U_k^μ). Therefore, the decision maker is completely satisfied when the relevant objective reaches the highest value. However, in cases where the objectives have a minimizing type, the satisfaction level increases as each function approaches its lowest value (L_k^μ). The decision maker is completely satisfied if all objectives reach their lowest values.

Step 7: After the membership and non-membership functions have been determined, the intuitionistic fuzzy optimization model is shown as in equation (6). The membership function is maximized and the non-membership function is minimized.

$$\begin{aligned}
&Maximum\ \ \mu_{k^{\tilde{I}}}(f_k(x)) \\
&Minimum\ \nu_{k^{\tilde{I}}}(f_k(x)) \\
&Constraints \\
&\mu_k^I\big(f_k(x)\big) + \nu_k^I\big(f_k(x)\big) \le 1 \\
&\mu_k^I\big(f_k(x)\big) \ge \nu_k^I\big(f_k(x)\big) \\
&\nu_{k^{\tilde{I}}}(f_k(x)) \ge 0 \\
&g_j\ \le b_j \\
&x \ge 0 \\
&k\ = 1,2, \dots, k. \\
&j = 1,2, \dots, m.
\end{aligned} \tag{6}$$

This model is then transformed into the classical linear programming model as shown in equation (7). This transformation is accomplished by pairing the membership function with an additional variable, like α, and the non-membership function with an additional variable, like β.

$$Maximum\,(\alpha-\beta)$$

$$Constraints$$

$$\mu_k^I\big(f_k(x)\big)\geq\alpha$$

$$v_k^I\big(f_k(x)\big)\leq\beta$$

$$\alpha+\beta\leq 1$$

$$\alpha\geq\beta$$

$$\beta\geq 0$$

$$g_j\ \leq b_j$$

$$x\geq 0$$

$$k\,=1,2,\dots,k.$$

$$j=1,2,\dots,m. \quad \textbf{(7)}$$

Variables α and β are defined in the range [0,1]. α represents the smallest membership degree, while β represents the largest non-membership degree. α maximizes the smallest membership degree and β minimizes the largest non-membership degree. Thus, thanks to the linear model obtained, the problems can be easily solved.

B. **Neutrosophic Fuzzy Multi-Objective Optimization Technique**

The Neutrosophic fuzzy multi-objective linear programming problems are shown in equation (8) [26].

$$max=[f_1^N(x),f_2^N(x),\dots,f_{k1}^N(x)\,]$$

$$min=[f_{k_1+1}^N(x),f_{k_1+2}^N(x),\dots,f_k^N(x)\,]$$

$$Constraints$$

$$g_i(x)\leq c_i,\quad i=1,2,\dots,m_1$$

$$g_i(x)\geq c_i,\quad i=m_1+1,m_1+2,\dots,m_2 \quad \textbf{(8)}$$

$g_i(x) = c_i, \quad i = m_2 + 1, m_2 + 2, \dots, m_2$

$x \geq 0$

The neutrosophic fuzzy multi-objective optimization model, the neutrosophic fuzzy decision set D^N, is defined as the combination of neutrosophic fuzzy objectives ($G_k{}^N$) and neutrosophic fuzzy constraints ($C_j{}^N$) as in equation (9) [27].

$$D^N = \left(\bigcap_{k=1}^{p} G_k{}^N\right) \cap \left(\bigcap_{k=1}^{q} C_j{}^N\right) = \{x, \mu_{D^N}(x), \sigma_{D^N}(x), \nu_{D^N}(x)\} \qquad \textbf{(9)}$$

For $\forall x \in X$, the membership functions of the neutrosophic fuzzy decision set are defined as shown in equation (10).

$$\mu_{D^{\tilde{N}}}(x) = min(\mu_{G1}{}^N(x), \mu_{G2}{}^N(x), \dots, \mu_{\mu_{Gp}}{}^N(x);\ \mu_{C1}{}^N(x), \mu_{C2}{}^N(x), \dots, \mu_{Cq}{}^N(x))$$

$$\sigma_{D^{\tilde{N}}}(x) = min(\sigma_{G1}{}^N(x), \sigma_{G2}{}^N(x), \dots, \sigma_{Gq}{}^N(x);\ \sigma_{C1}{}^N(x), \sigma_{C2}{}^N(x), \dots, \sigma_{Cq}{}^N(x))$$

$$\nu_{D^{\tilde{N}}}(x) = max(\nu_{G1}{}^N(x), \nu_{G2}{}^N(x), \dots, \nu_{Gp}{}^N(x);\ \nu_{C1}{}^N(x), \nu_{C2}{}^N(x), \dots, \nu_{Cq}{}^N(x)) \qquad \textbf{(10)}$$

Where;

$\mu_{D^{\tilde{N}}}(x)$: Decision set for the truth membership function,

$\sigma_{D^{\tilde{N}}}(x)$: Decision set for the indeterminacy membership function,

$\nu_{D^{\tilde{N}}}(x)$: Decision set for the falsity membership function,

$\mu_{Gp}{}^{\tilde{N}}(x)$: Goal set for the truth membership function,

$\sigma_{Gp}{}^{\tilde{N}}(x)$: Goal set for the indeterminacy membership function,

$\nu_{Gp}{}^{\tilde{N}}(x)$: Goal set for the falsity membership function,

$\mu_{Cj}{}^{\tilde{N}}(x)$: Constraint set for the truth membership function,

$\sigma_{Cj}{}^{\tilde{N}}(x)$: Constraint set for the indeterminacy membership function,

$\nu_{Cj}{}^{\tilde{N}}(x)$: Constraint set for the falsity membership function.

The neutrosophic fuzzy multi-objective optimization problems given in Equation (10) are transformed into the model specified in Equation (11) by adding variables like α,γ,β to the model after the membership functions of the objectives and constraints are determined.

$$
\begin{aligned}
&Max\ \alpha \\
&Max\ \gamma \\
&Min\ \beta \\
&Constraints \\
&\mu_{G_k^N}(x) \geq \alpha \\
&\mu_{C_k^N}(x) \geq \alpha \\
&\sigma_{G_k^N}(x) \geq \gamma \\
&\sigma_{C_k^N}(x) \geq \gamma \\
&\nu_{G_k^N}(x) \leq \beta \\
&\nu_{C_k^N}(x) \leq \beta \\
&k = 1,2,\dots,p. \\
&\alpha + \gamma + \beta \leq 3 \\
&\alpha \geq \beta \\
&\alpha \geq \gamma \\
&\alpha,\gamma,\beta \in [0,1]
\end{aligned}
\tag{11}
$$

The α,γ,β variables added to the model can be defined as the degree of satisfaction of the membership functions of the objective functions. These variables are useful in transforming the neutrosophic fuzzy multi-objective optimization model into a single-objective optimization model making it easy to solve neutrosophic fuzzy multi-objective programming problems with linear programming techniques [18].

The steps followed in solving the multi-objective neutrosophic fuzzy optimization problem are outlined hereunder [18, 16].

Step 1: The neutrosophic fuzzy multi-objective optimization problem is formulated as shown in equation (8).

Step 2: Any one of the objective functions of the problem is selected and solved using the classical linear programming technique following all the constraints of the problem until ideal solutions are obtained while ignoring the other objectives.

Step 3: Step 2 is repeated for all other objective functions, and the values of the objective functions and decision variables are determined.

Step 4: To create membership functions, goals, and constraints are first determined. For this, the pay-off matrix (pay-off table) is obtained using the ideal solutions obtained in Step 2 [18].

$$\begin{vmatrix} f_1(x^1)^* \ f_2(x^1) \dots \dots \dots f_p(x^1) \\ f_1(x^2) \ f_2(x^2)^* \dots \dots \dots f_p(x^2) \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ f_1(x^p) \ f_2(x^p) \dots \dots \dots f_p(x^p)^* \end{vmatrix}$$

The diagonal values of the pay-off matrix given above show the best value that each objective can get. In multi-objective optimization problems, it is easy to find the best value an objective can get. On the contrary, it is not easy to find the worst solution value of a goal. The pay-off matrix, which is obtained by optimizing each objective independently, allows one to easily find the worst value that an objective can get. In this way, the intervals in which the objective functions are found can be easily calculated.

Step 5: The best and worst values for each objective function are obtained from the pay-off matrix. These values are then used to determine the lower and upper limits of the membership functions.

The lower and upper limits of the truth, indeterminacy, and falsity functions differ according to the type of the objective function. Table 5 gives the formulas for calculating the lower and upper limit values of the membership functions for objective functions with a maximum type.

Table 5 *Determination of the Upper and Lower Limits of Neutrosophic Fuzzy Membership Functions with a Maximization Type*

Objective Function Type	Objective Function with a Maximization Type	
Membership Function	Lower Limit	Upper Limit
Truth Membership Function (μ)	$L_k^{\mu}= \min\{f_k(x_r^*)\}$	$U_k^{\mu}=\max\{f_k(x_r^*)\}$
Indeterminacy Membership Function (σ)	$L_k^{\sigma}=L_k^{\mu}+\lambda_1(U_k^{\mu}-L_k^{\mu})$	$U_k^{\sigma}=U_k^{\mu}$
Falsity Membership Function (ν)	$L_k^{\nu}=L_k^{\mu}$	$U_k^{\nu}=L_k^{\mu}+\lambda_2(U_k^{\mu}-L_k^{\mu})$

Table 6 gives the formulas for calculating the lower and upper limit values of the membership functions for objective functions with a minimum type.

Table 6 *Determination of the Upper and Lower Limits of Neutrosophic Fuzzy Membership Functions with a Minimization Type*

Objective Function Type	Objective Function with a Minimization Type	
Membership Function	Lower Limit	Upper Limit
Truth Membership Function (μ)	$L_k^{\mu}= \min\{f_k(x_r^*)\}$	$U_k^{\mu} = \max\{f_k(x_r^*)\}$
Indeterminacy Membership Function (σ)	$L_k^{\sigma}=L_k^{\mu}$	$U_k^{\sigma}=L_k^{\mu}+\lambda_1(U_k^{\mu}-L_k^{\mu})$"
Falsity Membership Function (ν)	$L_k^{\nu}=L_k^{\mu}+\lambda_2(U_k^{\mu}-L_k^{\mu})$	$U_k^{\nu}=U_k^{\mu}$

The L_k and U_k in the equations show the lower and upper limits of each objective. U_k (upper), shows the best value for maximization problems, and L_k (lower) shows the best value for minimization problems. Regardless of whether the objective function is in the maximization or minimization type, each objective value falls between the lower limit and the upper limit, as shown in equation (12) [28].

$$L_k \leq f_k(x) \leq U_k \tag{12}$$

The λ_1 and λ_2 values in the equations given in Tables 5 and 6 are the tolerance variables chosen by the decision maker to determine the indeterminacy and falsity membership functions, respectively [16]. These values are in the range [0,1] and are taken to be λ_1= 0,5 and λ_2=0,3 for this study.

Step 6: The truth, indeterminacy, and falsity functions can be formed using the lower and upper limits obtained in Table 5 and Table 6. The determination of the membership functions is done according to the type of objective functions as shown in the equations below [18].

For objective functions with a maximization type, the truth membership function is defined as shown in equation (13).

$$\mu_k(f_k(x)) = \begin{cases} 0 & , \quad f_k(x) \leq L_k^{\mu} \quad ise \\ \dfrac{f_k(x) - L_k^{\mu}}{U_k^{\mu} - L_k^{\mu}} & , \quad L_k^{\mu} \leq f_k(x) \leq U_k^{\mu} \quad ise \\ 1 & , \quad f_k(x) \geq U_k^{\mu} \quad ise \end{cases} \tag{13}$$

If the objective function has a maximization type, the indeterminacy membership function is defined as shown in equation (14).

$$\sigma_k(f_k(x)) = \begin{cases} 0 & , \quad f_k(x) \leq L_k^{\sigma} \quad ise \\ \dfrac{f_k(x) - L_k^{\sigma}}{U_k^{\sigma} - L_k^{\sigma}} & , \quad L_k^{\sigma} \leq f_k(x) \leq U_k^{\sigma} \quad ise \\ 1 & , \quad f_k(x) \geq U_k^{\sigma} \quad ise \end{cases} \tag{14}$$

For objective functions with a maximization type, the falsity membership function is defined as shown in equation (15).

$$\nu_k(f_k(x)) = \begin{cases} 1 & , \quad f_k(x) \leq L_k^{\nu} \quad ise \\ \dfrac{U_k^{\nu} - f_k(x)}{U_k^{\nu} - L_k^{\nu}} & , \quad L_k^{\nu} \leq f_k(x) \leq U_k^{\nu} \quad ise \\ 0 & , \quad f_k(x) \geq U_k^{\nu} \quad ise \end{cases} \tag{15}$$

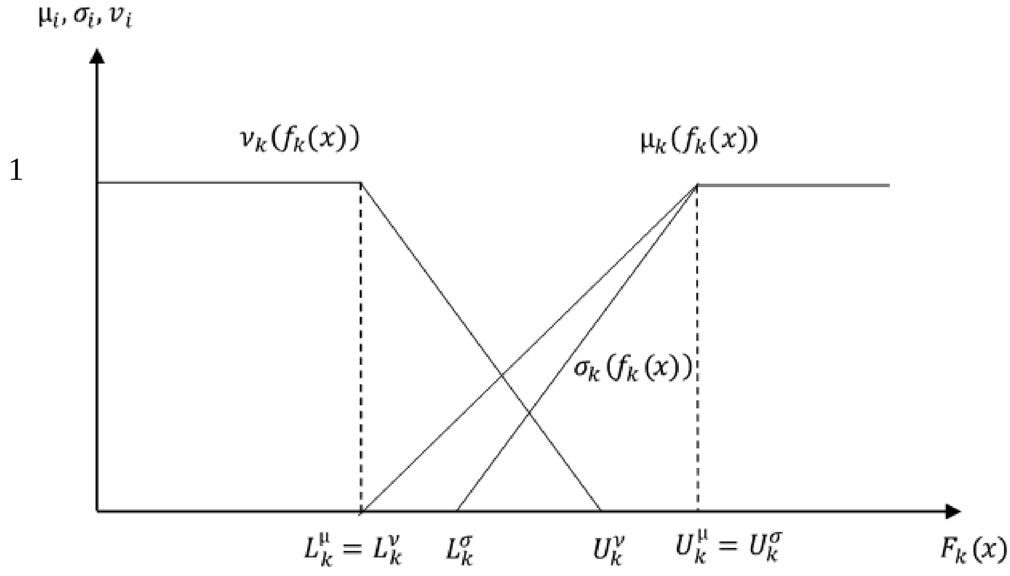

Figure 3 *Membership Functions in Neutrosophic Fuzzy Sets with Maximization Type*

Figure 3 shows the truth, 'indeterminacy, and falsity membership functions for neutrosophic fuzzy sets with a maximization type [29].

If the objective function has a minimization type, the membership functions are defined as shown in the equations below [16].

The truth membership function where the objective function has a minimization type is defined as shown in equation (16).

$$\mu_k(f_k(x)) = \begin{cases} 1 & , \quad f_k(x) \leq L_k^{\mu} \quad ise \\ \dfrac{U_k^{\mu} - f_k(x)}{U_k^{\mu} - L_k^{\mu}} & , \quad L_k^{\mu} \leq f_k(x) \leq U_k^{\mu} \quad ise \\ 0 & , \quad f_k(x) \geq U_k^{\mu} \quad ise \end{cases} \tag{16}$$

The indeterminacy membership function where the objective function has a minimization type is defined as shown in equation (17).

$$\sigma_k(f_k(x)) = \begin{cases} 1 & , \quad f_k(x) \leq L_k^{\sigma} \quad ise \\ \dfrac{U_k^{\sigma} - f_k(x)}{U_k^{\sigma} - L_k^{\sigma}} & , \quad L_k^{\sigma} \leq f_k(x) \leq U_k^{\sigma} \quad ise \\ 0 & , \quad f_k(x) \geq U_k^{\sigma} \quad ise \end{cases} \tag{17}$$

The falsity membership function where the objective function has a minimization type is defined as shown in equation (18).

$$\nu_k(f_k(x)) = \begin{cases} 0 & , \quad f_k(x) \leq L_k^{\nu} \quad ise \\ \dfrac{f_k(x) - L_k^{\nu}}{U_k^{\nu} - L_k^{\nu}} & , \quad L_k^{\nu} \leq f_k(x) \leq U_k^{\nu} \quad ise \\ 1 & , \quad f_k(x) \geq U_k^{\nu} \quad ise \end{cases} \tag{18}$$

Where;

L_k^{μ}: The lower limit for the truth membership function

U_k^{μ}: The upper limit for the truth membership function

L_k^{σ}: The lower limit for the indeterminacy membership function

U_k^{σ} : The upper limit for the indeterminacy membership function

L_k^{ν}: The lower limit for the falsity membership function

U_k^{ν}: The upper limit for the falsity membership function

$f_k(x): k.$ Goal function

Figure 4 shows the truth, indeterminacy, and falsity membership functions for neutrosophic fuzzy sets with a minimization type [16].

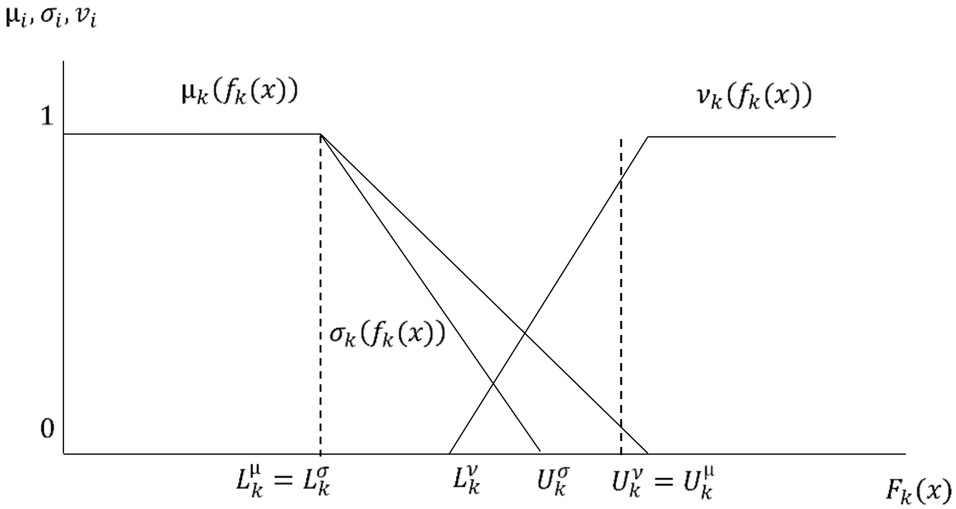

Figure 4 *Membership Functions for Neutrosophic Fuzzy Sets with a Minimization Type*

Step 7: After determining the neutrosophic fuzzy membership functions, the neutrosophic fuzzy multi-objective optimization model is transformed into a single-objective linear optimization model. This transformation is achieved by adding α, γ, β variables to the model. There are different ways of optimizing the membership functions in the neutrosophic fuzzy multi-objective optimization technique

In the neutrophic fuzzy multi-objective optimization technique, membership functions can be optimized in different ways. In this study, we used three models, Model I, Model II, and Model III which are described below.

Model I

According to the Model I approach in Equation (19), the truth membership and indeterminacy membership functions are maximized and the falsity membership function is minimized when solving neutrosophic fuzzy multi-objective optimization problems [30].

$$Max\ (\alpha + \gamma - \beta)$$
$$Constraints$$
$$\mu_k(f_k(x)) \geq \alpha$$
$$\sigma_k(f_k(x)) \geq \gamma$$
$$\nu_k(f_k(x)) \leq \beta$$
$$0 \leq \sigma + \gamma + \beta \leq 3$$
$$\alpha \geq \gamma$$
$$\alpha \geq \beta$$
$$\alpha, \beta, \gamma \in [0,1]$$
$$x \geq 0 \quad \textbf{(19)}$$

Model II

According to the Model II approach in Equation (20), the truth membership function is maximized while indeterminacy membership and the falsity membership functions are minimized when solving neutrosophic fuzzy multi-objective optimization problems [30].

$$Max\ (\alpha - \gamma - \beta)$$
$$Constraints$$
$$\mu_k(f_k(x)) \geq \alpha$$
$$\sigma_k(f_k(x)) \leq \gamma$$
$$\nu_k(f_k(x)) \leq \beta$$
$$0 \leq \alpha + \gamma + \beta \leq 3$$
$$\alpha \geq \gamma$$
$$\alpha \geq \beta$$
$$\alpha, \gamma, \beta \in [0,1]$$
$$Ax \leq b$$
$$x \geq 0 \quad \textbf{(20)}$$

Model III

According to the Model III approach in Equation (21), the truth, the indeterminacy, and the falsity membership functions are maximized when solving neutrosophic fuzzy multi-objective optimization problems [18].

$$Max\ (\alpha + \gamma + \beta)$$

$$Constraints$$

$$\mu_k(f_k(x)) \geq \alpha$$

$$\sigma_k(f_k(x)) \geq \gamma$$

$$\nu_k(f_k(x)) \leq \beta$$

$$0 \leq \alpha + \gamma + \beta \leq 3$$

$$\alpha \geq \gamma$$

$$\alpha \geq \beta$$

$$\alpha, \gamma, \beta \ \in [0,1]$$

$$Ax \leq b$$

$$x \geq 0 \tag{21}$$

5. Multi-Objective Production Planning Problem in a Manufacturing Enterprise

Businesses can stay ahead of the competition by consistently fulfilling the demands of the customers promptly and with the desired features. Production planning plays an integral part in the success of this goal. Planning helps a business take stock of its current situation and decide on the best way to achieve its goals using the available resources. Production planning refers to the set of activities entailed in the determination of the product, the quantity, and the time of production.

This study sought to draw up a production schedule for a company located in the Turkish province of Eskişehir, and manufacturing parts for household appliances. The company was founded in 1994 and uses plastic as its main raw material. The production process requires the use of 62 different pieces of plastic materials which the company secures from different suppliers. The plastic pieces are loaded into different injection machines depending on the product to be made. The company has 53 injection machines to turn plastic particles into the parts that are needed. The flow chart in Figure 5 below shows the production flow for the company.

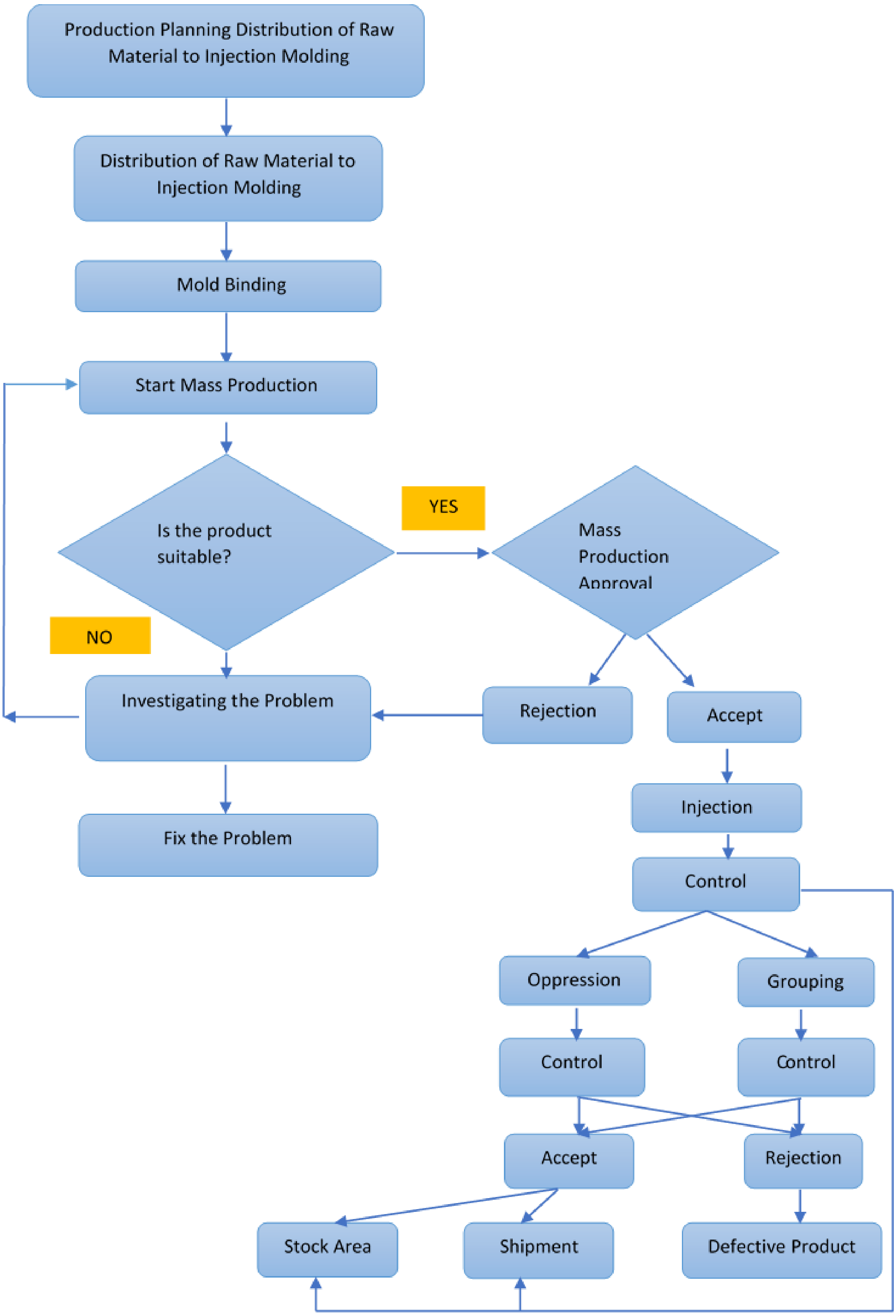
Production Planning Distribution of Raw Material to Injection Molding
Distribution of Raw Material to Injection Molding
Mold Binding
Start Mass Production
Is the product suitable?
YES
Mass Production Approval
NO
Investigating the Problem
Rejection
Accept
Fix the Problem
Injection
Control
Oppression
Grouping
Control
Control
Accept
Rejection
Stock Area
Shipment
Defective Product

Figure 5 *Work Flow Diagram of a Household Appliances Manufacturing Plant*

Terminology

Indices

j : Product type ($j = 1,2, \ldots ,121$)

i: Raw material index ($i = 1,2, \ldots ,62$)

r: Machine index ($r = 1,2, \ldots ,53$)

Decision Variables

x_j :Manufactured items

Parameters

k_j = The profit per unit for one unit x_j (currency)

c_j = production cost of one unit x_j of the product (currency)

s_j = necessary time for the production of one unit x_j of the product (second)

d_j = Defective product rate

f_j = amount of material waste in the production one unit x_j (gram)

T_j = j. The demand for the product

N_i =i. The amount of raw material (gram)

h_{ij} = One unit j. necessary for the product N amount of raw materials (gram)

g_j = To group one unit x_j product necessary time (second)

b_j = To print one unit x_j product necessary time (second)

M_r= r. Machine production capacity (unit)

G = Grouping workshop total labor time

B= Print workshop total labor time

Problem Objectives

The production planning for the production facility takes multiple objectives as its basis. The objectives are outlined below.

1. Production Cost (Z_1): The management of the company where this study was conducted wants the production process to run in a way that minimizes the total cost of production. The general representation of the objective function Z_1 is as shown in equation (22).

$$Min Z_1 = \sum_{j=1}^{n} c_j x_j \tag{22}$$

2. Profit (Z_2): The management wants to achieve a production process that maximizes the profit it gets from the sale of its products. The general representation of the objective function Z_2 is as shown in equation (23).

$$Max Z_2 = \sum_{j=1}^{n} k_j x_j \tag{23}$$

3. Cycle Time (Z_3): Cycle time, in its simplest form, is the time for a product to emerge from the production line. In this study, cycle time refers to the time a product takes to go through the production process, i.e. raw materials go through the injection machine and come out as a product.

The management wants to minimize the total cycle time of the products. The objective function Z_3 is generally represented as shown in equation (24).

$$MinZ_3 = \sum_{j=1}^{n} s_j x_j \tag{24}$$

4. Defective Rates. (Z_4): Breakdown of machinery and equipment, changes in the amount and quality of raw materials, and shortcomings in the competence and skills of the personnel among other reasons may hinder the production process from achieving the desired quality of products, leading to defective products. It is the objective of management to minimize the rate of defective products leaving the process. The general representation of the objective function Z_4 is as shown in equation (25).

$$MinZ_4 = \sum_{j=1}^{n} d_j x_j \tag{25}$$

5. Material Waste Amount (Z_5): The management wants the material loss in the production process to be at the lowest possible level. The general representation of the objective function Z_5 is as shown in equation (26).

$$MinZ_5 = \sum_{j=1}^{n} f_j x_j \tag{26}$$

Constraints:

The production model of the company has 237 constraints under 5 groups. The constraints are outlined below.

1. Demand Constraints

$$x_j \geq T_j$$

2. Raw Material Constraints

$$h_{ij} x_j \leq N_i$$

3. Machine Constraints

$$x_j \leq M_r$$

4. Labor Time Constraint at the Grouping Workshop

$$g_j x_j \leq G$$

5. Print Workshop Total Labor Time Constraint

$$b_j x_j \leq B$$

After these explanations, the multi-objective linear production model for the company is theoretically established as shown in equation (27).

$$\begin{aligned} MinZ_1 &= \textstyle\sum_{j=1}^{n} c_j x_j \\ MinZ_2 &= \textstyle\sum_{j=1}^{n} k_j x_j \\ MinZ_3 &= \textstyle\sum_{j=1}^{n} s_j x_j \\ MinZ_4 &= \textstyle\sum_{j=1}^{n} d_j x_j \end{aligned} \tag{27}$$

$MinZ_5 = \sum_{j=1}^{n} f_j x_j$

$Constraints$

$x_j \geq T_j$

$h_{ij} x_j \leq N_i$

$m_j x_j \leq M_r$

$g_j x_j \leq G$

$b_j x_j \leq B$

and $x_j \geq 0$ and x_j integer j=1,2,…,121.

To solve the multi-objective optimization model, where indeterminacy is in question, using the neutrosophic fuzzy multi-objective optimization technique, the model needs to be converted to a neutrosophic fuzzy type. Each of the objectives in the production model was independently solved, without including the others, using the LINGO 19 package program, and concerning all the constraints of the production model. The constraints of the objective functions were found as follows.

$2.329.133 \leq MinZ_1 \leq 2.742.388$

$1.599.267 \leq MaxZ_2 \leq 1.760.236$

$25.501.660 \leq MinZ_3 \leq 27.526.357$

$99.861{,}89 \leq MinZ_4 \leq 104.081{,}90$

$5.574{,}678 \leq MinZ_5 \leq 5.790{,}102$

The pay-off matrix obtained from solving each objective in the production model independently of the other objectives is given in Table 7.

Table 7 *The Pay-off Matrix for the Production Plan*

	Z_1	Z_2	Z_3	Z_4	Z_5
Min Z_1	2.329.133	1.599.267	25.501.660	99.861,89	5.574,678
Max Z_2	2.742.388	1.760.236	27.526.357	104.081,90	5.790,102
Min Z_3	2.329.133	1.599.267	25.501.660	99.861,89	5.574,678
Min Z_4	2.329.133	1.599.267	25.501.660	99.861,89	5.574,678
Min Z_5	2.329.133	1.599.267	25.501.660	99.861,89	5.574,678

To use the neutrosophic programming with the obtained constraints, the truth, indeterminacy, and falsity membership functions of each objective were constructed as follows.

$$\mu_1(Z_1(x)) = \begin{cases} 1 & if \quad Z_1(x) \leq 2.329.133 \\ \dfrac{2.742.388 - Z_1(x)}{413.255} & if \quad 2.329.133 \leq Z_1(x) \leq 2.742.388 \\ 0 & if \quad Z_1(x) \geq 2.742.388 \end{cases}$$

$$\sigma_1(Z_1(x)) = \begin{cases} 1 & if \quad Z_1(x) \leq 2.329.133 \\ \dfrac{2.535.760{,}5 - Z_1(x)}{206.627.5} & if \quad 2.329.133 \leq Z_1(x) \leq 2.535.760{,}5 \\ 0 & if \quad Z_1(x) \geq 2.535.760{,}5 \end{cases}$$

$$\nu_1(Z_1(x)) = \begin{cases} 0 & if \quad Z_1(x) \leq 2.453.109.5 \\ \dfrac{Z_1(x) - 2.453109,5}{289.278,5} & if \quad 2.453.109,5 \leq Z_1(x) \leq 2.742.388 \\ 1 & if \quad Z_1(x) \geq 2.742.388 \end{cases}$$

$$\mu_2(Z_2(x)) = \begin{cases} 0 & , \quad if \quad Z_2(x) \leq 1.599.267 \\ \dfrac{Z_2(x) - 1.599.267}{160.969} & , \quad if \quad 1.599.267 \leq Z_2(x) \leq 1.760.236 \\ 1 & , \quad if \quad Z_2(x) \geq 1.760.236 \end{cases}$$

$$\sigma_2(Z_2(x)) = \begin{cases} 0, & if \quad Z_2(x) \leq 1,679,751,5 \\ \dfrac{Z_2(x) - 1.679.751,5}{80.484,5} , & if \quad 1.679.751,5 \leq Z_2(x) \leq 1.760.236 \\ 1, & if \quad Z_2(x) \geq 1.760.236 \end{cases}$$

$$\nu_2(Z_2(x)) = \begin{cases} 1 & , \quad if \quad Z_2(x) \leq 1.599.267 \\ \dfrac{1.647.557,7 - Z_2(x)}{48.290,7} & , \quad if \quad 1.599.267 \leq Z_2(x) \leq 1.647.557,7 \\ 0 & , \quad if \quad Z_2(x) \geq 1.647.557,7 \end{cases}$$

$$\mu_3(Z_3(x)) = \begin{cases} 1 & , \quad if \quad Z_3(x) \leq 25.501.660 \\ \dfrac{27.526.357 - Z_3(x)}{2.024.697} & , \quad if \quad 25.501.660 \leq Z_3(x) \leq 27.526.357 \\ 0 & , \quad if \quad Z_3(x) \geq 27.526.357 \end{cases}$$

$$\sigma_3(Z_3(x)) = \begin{cases} 1, & if \quad Z_3(x) \leq 25.501.660 \\ \dfrac{26.514.008,5 - Z_3(x)}{1.012.348,5} , & if \quad 25.501.660 \leq Z_3(x) \leq 26.514.008,5 \\ 0, & if \quad Z_3(x) \geq 25.501.660 \end{cases}$$

$$\nu_3(Z_3(x)) = \begin{cases} 0 & , \quad if \quad Z_3(x) \leq 26.109.0691 \\ \dfrac{Z_3(x) - 26.109.0691}{1.417.287,9} & , \quad if \quad 26.109.0691 \leq Z_3(x) \leq 27.526.357 \\ 1 & , \quad if \quad Z_3(x) \geq 27.526.357 \end{cases}$$

$$\mu_4(Z_4(x)) = \begin{cases} 1 & , \ if \quad Z_4(x) \leq 99.861,89 \\ \dfrac{104.081,90 - Z_4(x)}{4.220,01} & , \ if \quad 99.861,89 \leq Z_4(x) \leq 104.081,90 \\ 0 & , \ if \quad Z_4(x) \geq 104.081,90 \end{cases}$$

$$\sigma_4(Z_4(x)) = \begin{cases} 1, & if \quad Z_4(x) \leq 99.861,89 \\ \dfrac{101.971,895 - Z_4(x)}{2.110,005}, & if \quad 99.861,89 \leq Z_4(x) \leq 101.127,895 \\ 0, & if \quad Z_4(x) \geq 101.127,895 \end{cases}$$

$$\nu_4(Z_4(x)) = \begin{cases} 0 & , \ if \quad Z_4(x) \leq 101.127,893 \\ \dfrac{Z_4(x) - 101.127,893}{2.954,007} & , \ if \quad 101.127,893 \leq Z_4(x) \leq 104.081,90 \\ 1 & , \ if \quad Z_4(x) \geq 104.081,90 \end{cases}$$

$$\mu_5(Z_5(x)) = \begin{cases} 1 & , \ if \quad Z_5(x) \leq 5.574,678 \\ \dfrac{5.790,102 - Z_5(x)}{215,424} & , \ if \quad 5574,678 \leq Z_5(x) \leq 5.790,102 \\ 0 & , \ if \quad Z_5(x) \geq 5.790,102 \end{cases}$$

$$\sigma_5(Z_5(x)) = \begin{cases} 1, & if \quad Z_5(x) \leq 5.574,678 \\ \dfrac{5.682,39 - Z_5(x)}{107,712}, & if \quad 5.574,678 \leq Z_5(x) \leq 5.682,39 \\ 0, & if \quad Z_5(x) \geq 5.682,39 \end{cases}$$

$$\nu_5(f_5(x)) = \begin{cases} 0 & , \ if \quad Z_5(x) \leq 5.639,3052 \\ \dfrac{Z_5(x) - 5.639,3052}{150,7968} & , \ if \quad 5.639,3052 \leq Z_5(x) \leq 5.790,102 \\ 1 & , \ if \quad Z_5(x) \geq 5.790,102 \end{cases}$$

The neutrosophic fuzzy multi-objective optimization model was transformed into the classical linear programming model using the created membership functions. After determining the neutrosophic fuzzy membership functions, the neutrosophic fuzzy multi-objective optimization problems were solved independently using the Model I, Model II, and Model III approaches outlined in the previous section.

6. Problem Solution and Findings

In this section, the solution of the production problem with different approaches will be discussed.

a- Neutrosophic Fuzzy Multi-Objective Optimization Model I Approach

According to the Neutrophic fuzzy multi-objective optimization technique Model I approach, the truth membership and indeterminacy membership functions are maximized and the falsity membership function is minimized as follows.

$\boldsymbol{Max\ (\alpha + \gamma - \beta)}$

$Constraints$

$\mu_1(Z_1(x)) \geq \alpha$

$\sigma_1(Z_1(x)) \geq \gamma$

$\nu_1(Z_1(x)) \leq \beta$

$\mu_2(Z_2(x)) \geq \alpha$

$\sigma_2(Z_2(x)) \geq \gamma$

$\nu_2(Z_2(x)) \leq \beta$

$\mu_3(Z_3(x)) \geq \alpha$

$\sigma_3(Z_3(x)) \geq \gamma$

$\nu_3(Z_3(x)) \leq \beta$

$\mu_4(Z_4(x)) \geq \alpha$

$\sigma_4(Z_4(x)) \geq \gamma$

$\nu_4(Z_4(x)) \leq \beta$

$\mu_5(Z_5(x)) \geq \alpha$

$\sigma_5(Z_5(x)) \geq \gamma$

$\nu_5(Z_5(x)) \leq \beta$

$x_j \geq T_j$

$h_{ij}x_j \leq N_i$

$m_j x_j \leq M_r$

$g_j x_j \leq G$

$b_j x_j \leq B$

$0 \leq \sigma + \gamma + \beta \leq 3$

$\alpha \geq \gamma$

$\alpha \geq \beta$

$\alpha, \beta, \gamma \in [0,1]$

and $x_j \geq 0$ and x_j integer j=1,2,…,121

b- Neutrosophic Fuzzy Multi-Objective Optimization Model II Approach

According to the Model II approach, which is a neutrosophic fuzzy multi-objective optimization technique, the constraints of the membership functions of the production model do not change, rather it is the degree of optimization of the membership functions that changes. Accordingly, the maximization of the truth membership function and the minimization of the indeterminacy and the falsity functions are shown below.

$\boldsymbol{Max\ (\alpha - \gamma - \beta)}$

$Constraints$

$\mu_1(Z_1(x)) \geq \alpha$

$\sigma_1(Z_1(x)) \leq \gamma$

$\nu_1(Z_1(x)) \leq \beta$

$\mu_2(Z_2(x)) \geq \alpha$

$\sigma_2(Z_2(x)) \leq \gamma$
$\nu_2(Z_2(x)) \leq \beta$
$\mu_3(Z_3(x)) \geq \alpha$
$\sigma_3(Z_3(x)) \leq \gamma$
$\nu_3(Z_3(x)) \leq \beta$
$\mu_4(Z_4(x)) \geq \alpha$
$\sigma_4(Z_4(x)) \leq \gamma$
$\nu_4(Z_4(x)) \leq \beta$
$\mu_5(Z_5(x)) \geq \alpha$
$\sigma_5(Z_5(x)) \leq \gamma$
$\nu_5(Z_5(x)) \leq \beta$
$x_j \geq T_j$
$h_{ij} x_j \leq N_i$
$m_j x_j \leq M_r$
$g_j x_j \leq G$
$b_j x_j \leq B$
$0 \leq \sigma + \gamma + \beta \leq 3$
$\alpha \geq \gamma$
$\alpha \geq \beta$
$\alpha, \beta, \gamma \in [0,1]$
and $x_j \geq 0$ and x_j integer j=1,2,…,121

c- Neutrosophic Fuzzy Multi-Objective Optimization Model III Approach

According to the Neutrosophic fuzzy multi-objective optimization technique Model III approach, the maximization of the truth, indeterminacy, and falsity membership functions are obtained as follows.

$\boldsymbol{Max\ (\alpha + \gamma + \beta)}$
$Constraints$
$\mu_1(Z_1(x)) \geq \alpha$
$\sigma_1(Z_1(x)) \geq \gamma$
$\nu_1(Z_1(x)) \leq \beta$
$\mu_2(Z_2(x)) \geq \alpha$
$\sigma_2(Z_2(x)) \geq \gamma$
$\nu_2(Z_2(x)) \leq \beta$
$\mu_3(Z_3(x)) \geq \alpha$
$\sigma_3(Z_3(x)) \geq \gamma$
$\nu_3(Z_3(x)) \leq \beta$
$\mu_4(Z_4(x)) \geq \alpha$
$\sigma_4(Z_4(x)) \geq \gamma$
$\nu_4(Z_4(x)) \leq \beta$
$\mu_5(Z_5(x)) \geq \alpha$
$\sigma_5(Z_5(x)) \geq \gamma$
$\nu_5(Z_5(x)) \leq \beta$

$x_j \geq T_j$
$h_{ij} x_j \leq N_i$
$m_j x_j \leq M_r$
$g_j x_j \leq G$
$b_j x_j \leq B$
$0 \leq \sigma + \gamma + \beta \leq 3$
$\alpha \geq \gamma$
$\alpha \geq \beta$
$\alpha, \beta, \gamma \in [0,1]$
and $x_j \geq 0$ and x_j integer j=1,2,…,121

d- Intuitionistic Fuzzy Multi-Objective Optimization

The intuitionistic fuzzy membership and non-membership functions of the objectives of the production problem obtained from the pay-off matrix in Table 7 are given below.

$$\mu_1(Z_1(x)) = \begin{cases} 1 & if \quad Z_1(x) \leq 2.329.133 \\ \dfrac{2.742.388 - Z_1(x)}{413.255} & if \quad 2.329.133 \leq Z_1(x) \leq 2.742.388 \\ 0 & if \quad Z_1(x) \geq 2.742.388 \end{cases}$$

$$\nu_1(Z_1(x)) = \begin{cases} 0 & if \quad Z_1(x) \leq 2.453.109.5 \\ \dfrac{Z_1(x) - 2.453109,5}{289.278,5} & if \quad 2.453.109,5 \leq Z_1(x) \leq 2.742.388 \\ 1 & if \quad Z_1(x) \geq 2.742.388 \end{cases}$$

$$\mu_2(Z_2(x)) = \begin{cases} 0 & , \ if \quad Z_2(x) \leq 1.599.267 \\ \dfrac{Z_2(x) - 1.599.267}{160.969} & , \ if \quad 1.599.267 \leq Z_2(x) \leq 1.760.236 \\ 1 & , \ if \quad Z_2(x) \geq 1.760.236 \end{cases}$$

$$\nu_2(Z_2(x)) = \begin{cases} 1 & , \ if \quad Z_2(x) \leq 1.599.267 \\ \dfrac{1.711.945,3 - Z_2(x)}{48.290,7} & , \ if \quad 1.599.267 \leq Z_2(x) \leq 1.711.945,3 \\ 0 & , \ if \quad Z_2(x) \geq 1.711.945,3 \end{cases}$$

$$\mu_3(Z_3(x)) = \begin{cases} 1 & , \ if \quad Z_3(x) \leq 25.501.660 \\ \dfrac{27.526.357 - Z_3(x)}{2.024.697} & , \ if \quad 25.501.660 \leq Z_3(x) \leq 27.526.357 \\ 0 & , \ if \quad Z_3(x) \geq 27.526.357 \end{cases}$$

$$\nu_3(Z_3(x)) = \begin{cases} 0 & , \ if \quad Z_3(x) \leq 26.109.0691 \\ \dfrac{Z_3(x) - 26.109.069,1}{1.417.287,9} & , \ if \quad 26.109.0691 \leq Z_3(x) \leq 27.526.357 \\ 1 & , \ if \quad Z_3(x) \geq 27.526.357 \end{cases}$$

$$\mu_4(Z_4(x)) = \begin{cases} 1 & , \ if \quad Z_4(x) \le 99.861,89 \\ \dfrac{104.081,90 - Z_4(x)}{4.220,01} & , \ if \quad 99.861,89 \le Z_4(x) \le 104.081,90 \\ 0 & , \ if \quad Z_4(x) \ge 104.081,90 \end{cases}$$

$$v_4(f_4(x)) = \begin{cases} 0 & , \ if \quad Z_4(x) \le 101.127,893 \\ \dfrac{Z_4(x) - 101.127,893}{2.954,007} & , \ if \quad 101.127,893 \le Z_4(x) \le 104.081,90 \\ 1 & , \ if \quad Z_4(x) \ge 104.081,90 \end{cases}$$

$$\mu_5(Z_5(x)) = \begin{cases} 1 & , \ if \quad Z_5(x) \le 5.574,678 \\ \dfrac{5.790,102 - Z_5(x)}{215,424} & , \ if \quad 5574,678 \le Z_5(x) \le 5.790,102 \\ 0 & , \ if \quad Z_5(x) \ge 5.790,102 \end{cases}$$

$$v_5(f_5(x)) = \begin{cases} 0 & , \ if \quad Z_5(x) \le 5.639,3052 \\ \dfrac{Z_5(x) - 5.639,3052}{150,7968} & , \ if \quad 5.639,3052 \le Z_5(x) \le 5.790,102 \\ 1 & , \ if \quad Z_5(x) \ge 5.790,102 \end{cases}$$

$\boldsymbol{Max\ (\alpha - \beta)}$

$Constraints$

$\mu_1(Z_1(x)) \ge \alpha$

$v_1(Z_1(x)) \le \beta$

$\mu_2(Z_2(x)) \ge \alpha$

$v_2(Z_2(x)) \le \beta$

$\mu_3(Z_3(x)) \ge \alpha$

$v_3(Z_3(x)) \le \beta$

$\mu_4(Z_4(x)) \ge \alpha$

$v_4(Z_4(x)) \le \beta$

$\mu_5(Z_5(x)) \ge \alpha$

$v_5(Z_5(x)) \le \beta$

$x_j \ge T_j$

$h_{ij}x_j \le N_i$

$m_j x_j \le M_r$

$g_j x_j \le G$

$b_j x_j \le B$

$0 \le \sigma + \beta \le 1$

$\alpha \ge \beta$

and $x_j \ge 0$ and x_j integer j=1,2,…,121

The production problem is created and solved with intuitionistic fuzzy sets as given in equation 7. Comparison results are given in Table 8.

Table 8 *Comparison of the Results of the Objective Functions from Different Solution Techniques*

Solution Techniques / Objective Functions	Neutrosophic Fuzzy Multi-Objective Optimization Technique Model I Approach	Neutrosophic Fuzzy Multi-Objective Optimization Technique Model II Approach	Neutrosophic Fuzzy Multi-Objective Optimization Technique Model III Approach	Intuitionistic Fuzzy Multi-Objective Optimization Technique
Z_1	2.517.602	2.517.905	2.517.602	2.546.347
Z_2	1.686.825	1.686.707	1.686.825	1.696.381
Z_3	26.425.010	26.426.520	26.425.010	26.565.750
Z_4	101.786,4	101.789,6	101.786,4	102.079,4
Z_5	5.658,624	5.673,082	5.658.620	5.680,644

According to the solution results in Table 8, the neutrosophic fuzzy multi-objective optimization technique under Models I-II-III gave more optimal results for the objective functions than the intuitionistic fuzzy multi-objective optimization technique. However, the objective function Z_2, which seeks profit maximization, was found to be more optimal under the intuitionistic fuzzy multi-objective optimization technique. The analysis further shows that Model I and Model III in which the indeterminacy membership function is maximized produced more effective results than Model II in which the indeterminacy membership function is minimized.

7. CONCLUSION

This study examined the multi-objective planning problem and sought a solution for a manufacturing company that produces parts for household appliances. The study drew up a one-month multi-objective, multi-item production planning problem and formulated a solution using intuitionistic fuzzy and neutrosophic multi-objective programming. The results of the study show that the neutrosophic fuzzy programming approach provides a better solution than the intuitionistic fuzzy approach.

The study demonstrates the applicability of intuitionistic fuzzy and neutrosophic fuzzy set approaches in multi-objective, multi-item complex production planning problems. Future studies could try different types of fuzzy sets to solve similar and complex multi-objective problems.

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Evaluation of Sustainable Waste Valorization using TreeSoft Set with Neutrosophic Sets

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Abstract
This study proposed a neutrosophic set framework with TreeSoft Set for sustainable waste valorization selection. The neutrosophic set is used to overcome uncertainty and vague information in the evaluation process. The neutrosophic set has three membership degrees: truth, indeterminacy, and falsity. The multi-criteria decision-making (MCDM) methodology deals with various criteria to evaluate waste valorization. The VIKOR method is an MCDM method used to rank the alternatives. The numerical example was created with 12 criteria and 10 alternatives. Three decision-makers and experts are invited to evaluate the requirements and options. We used the bipolar neutrosophic numbers to replace the opinions of experts.

Keywords: TreeSoft Set, Neutrosophic Set; Waste Valorization, Sustainability.

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1. Introduction

The expanding number of people and the handling of waste both show positive correlations, with the former rising due to the increased use of goods to meet demands. The increasing prevalence of trash and substantial waste, which includes both organic and inorganic materials, poses significant environmental risks, including releasing greenhouse gases into the atmosphere, contaminating land, and contaminating underground water supplies. The World Bank predicts that 2.01 billion tons of municipal solid waste (MSW) were generated globally in 2018, with 33% of that garbage requiring disposal that is ecologically friendly[1].

According to projections, by 2050, there will be 3.40 billion tonnes of MSW worldwide. Conventional waste disposal techniques, including landfills and incineration, are used worldwide but have an unsustainable quality regarding the financial, ecological, and social aspects because of the significant emissions generated throughout the decomposition process[2]–[4]. Moreover, problems emerging from the waste treatment life cycle, which includes gathering, transferring, treatment, disposal, and the creation of byproducts, exacerbate sustainability concerns. It is crucial to choose a suitable and sustainable waste breakdown method to mitigate the aforementioned difficulties and guarantee the long-term sustainability of the procedure for handling waste[5]–[7].

Multicriteria decision-making (MCDM) approaches are well suited for comparing the outcomes of the options as the prioritization process encompasses many alternatives and several criteria in the evaluation stages. Numerous physical and intangible criteria may be included in MCDM situations. Due to the ambiguity of the language evaluations, using crisp data for intangible criteria in traditional MCDM approaches may result in an adequate evaluation[8]–[11]. As a result, fuzzy sets have been added to these techniques to improve their suitability in unpredictable environments. Zadeh invented fuzzy sets to indicate an element's partial membership in a set[12], [13].

In 1995, Florentin Smarandache extended intuitionistic fuzzy sets to create neurosophic sets to express uncertainty in the data and the decision maker's indecision. Truth, Indeterminacy, and Falsity, which represent the corresponding degrees of truthiness, indeterminacy, and falsity, make up neutrosophic sets. The preciseness of the information is expressed in the neutrosophic sets by truth (degrees of belongingness), falsehood (non-belongingness) values, and indeterminacy (degree of hesitation) values, which indicate the decision maker's hesitancy. Neutronosophic sets enable insufficient data to be characterized in subsets that may be used to discriminate between relativity and completeness. These characteristics indicate uncertainty and indeterminacy[14]–[16].

2. Methodology

This section introduces some definitions of IVNSs.

Definition 1

Let Y be a universe of discourse with a generic element in Y denoted by y. We can define the neutrosophic variable y as $y = (T, I, F)$ where T,I, and F refer to the degrees of truth, indeterminacy, and falsity membership.

$$0 \leq \sup\big(T(y)\big) + \sup\big(I(y)\big) + \sup\big(F(y)\big) \leq 3 \tag{1}$$

We can define the IVNSs as:

$$y = ([T^L, T^U], [I^L, I^U], [F^L, F^U]) \tag{2}$$

Definition 2

We can deneutrosophication of IVNNs as:

$$D(y) = \left(\frac{T_y^L + T_y^U}{2} + \left(1 - \frac{\left(I_y^L + I_y^U\right)}{2}\right) * \ I_y^U - \left(\frac{F_y^L + F_y^U}{2}\right) * \left(1 - F_y^U\right) \right) \tag{3}$$

Definition 3

Let $y_1 = \left(\left[T_{y_1}^L, T_{y_1}^U\right], \left[I_{y_1}^L, I_{y_1}^U\right], \left[F_{y_1}^L, F_{y_1}^U\right]\right) \, and \, y_2 = \left(\left[T_{y_2}^L, T_{y_2}^U\right], \left[I_{y_2}^L, I_{y_2}^U\right], \left[F_{y_2}^L, F_{y_2}^U\right]\right)$ two interval-valued neutrosophic numbers, then some mathematical equations can be defined as:

$$y_1^c = \left(\left[F_{y_1}^L, F_{y_1}^U\right], \left[1 - I_{y_1}^U, 1 - I_{y_1}^L\right], \left[T_{y_1}^L, T_{y_1}^U\right]\right) \tag{4}$$

$$y_1 \oplus y_2 = \begin{pmatrix} \left[T_{y_1}^L + T_{y_2}^L - T_{y_1}^L T_{y_2}^L, T_{y_1}^U + T_{y_2}^U - T_{y_1}^U T_{y_2}^U\right], \\ \left[I_{y_1}^L I_{y_2}^L, I_{y_1}^U I_{y_2}^U\right], \\ \left[F_{y_1}^L F_{y_2}^L, F_{y_1}^U F_{y_2}^U\right] \end{pmatrix} \tag{5}$$

$$y_1 \otimes y_2 = \begin{pmatrix} [T^L_{y_1} T^L_{y_2}, T^U_{y_1} T^U_{y_2}], \\ [I^L_{y_1} + I^L_{y_2} - I^L_{y_1} I^L_{y_2}, I^U_{y_1} + I^U_{y_2} - I^U_{y_1} I^U_{y_2}], \\ [F^L_{y_1} + F^L_{y_2} - F^L_{y_1} F^L_{y_2}, F^U_{y_1} + F^U_{y_2} - F^U_{y_1} F^U_{y_2}] \end{pmatrix} \quad (6)$$

Definition 4

We can define the bipolar neutrosophic sets (BNSs) [17]–[19]as:

$$A = \{< x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x) >\} \quad (7)$$

Let $y_1 = (T_1^+(x), I_1^+(x), F_1^+(x), T_1^-(x), I_1^-(x), F_1^-(x))$ and $y_2 = (T_2^+(x), I_2^+(x), F_2^+(x), T_2^-(x), I_2^-(x), F_2^-(x))$

$$y_1 + y_2 = \begin{pmatrix} T_1^+ + T_2^+ - T_1^+ T_2^+, I_1^+ I_2^+, F_1^+ F_2^+, \\ -T_1^- T_2^-, -(-I_1^- - I_2^- - I_1^- I_2^-), -(-F_1^- - F_2^- - F_1^- F_2^-) \end{pmatrix} \quad (8)$$

$$y_1 \cdot y_2 = \begin{pmatrix} T_1^+ T_2^+, I_1^+ + I_2^+ - I_1^+ I_2^+ + F_1^+ + F_2^+ - F_1^+ F_2^+, \\ -(-T_1^- - T_2^- - T_1^- T_2^-), -I_1^- I_2^-, -F_1^- F_2^- \end{pmatrix} \quad (9)$$

Let U be a universe disclosure and H a non-empty subset of U, with $P(H)$ $be\ a\ powerset\ of\ H$.

Let TSR be a set of attributes of the problem (criteria),

$$TSR = \{TSR_1, TSR_2, \dots, TSR_{\mathrm{n}}\}, n \geq 1 \quad (10)$$

Where $TSR_1, TSR_2, \dots, TSR_{\mathrm{n}}$ are criteria of the first level of the tree.

Each attribute $TSR_1, 1 \leq i \leq n, is\ formed\ by\ sub-attributes$:

$TSR_1 = \{TSR_{1,1}, TSR_{1,2}. \dots, \}$

$TSR_2 = \{TSR_{2,1}, TSR_{2,2}. \dots, \}$

.

.

$TSR_{\mathrm{n}} = \{TSR_{\mathrm{n},1}, TSR_{\mathrm{n},2}. \dots, \}$

Where $TSR_{\mathrm{i,j}}$ are sub-attributes.

The TreeSoft set can be formed by:

$$F: P(Tree(TSR)) \to P(H) \quad (11)$$

$Tree(TSR)$ is the set of all nodes and leaves from level 1 to level m and $P(Tree(TSR))$ is the power set of the $Tree(TSR)$.

$$Tree(TSR) = \{TSR_i |\ i_1 = 1,2,3, \dots\} \cup \{TSR_i |\ i_1, i_2 = 1,2,3, \dots\} \cup \{TSR_i |\ i_1, i_2, i_3 = 1,2,3, \dots\} \cup \dots \cup \{TSR_i | i_1, i_2, \dots\ i_m = 1,2,3, \dots\} \quad (12)$$

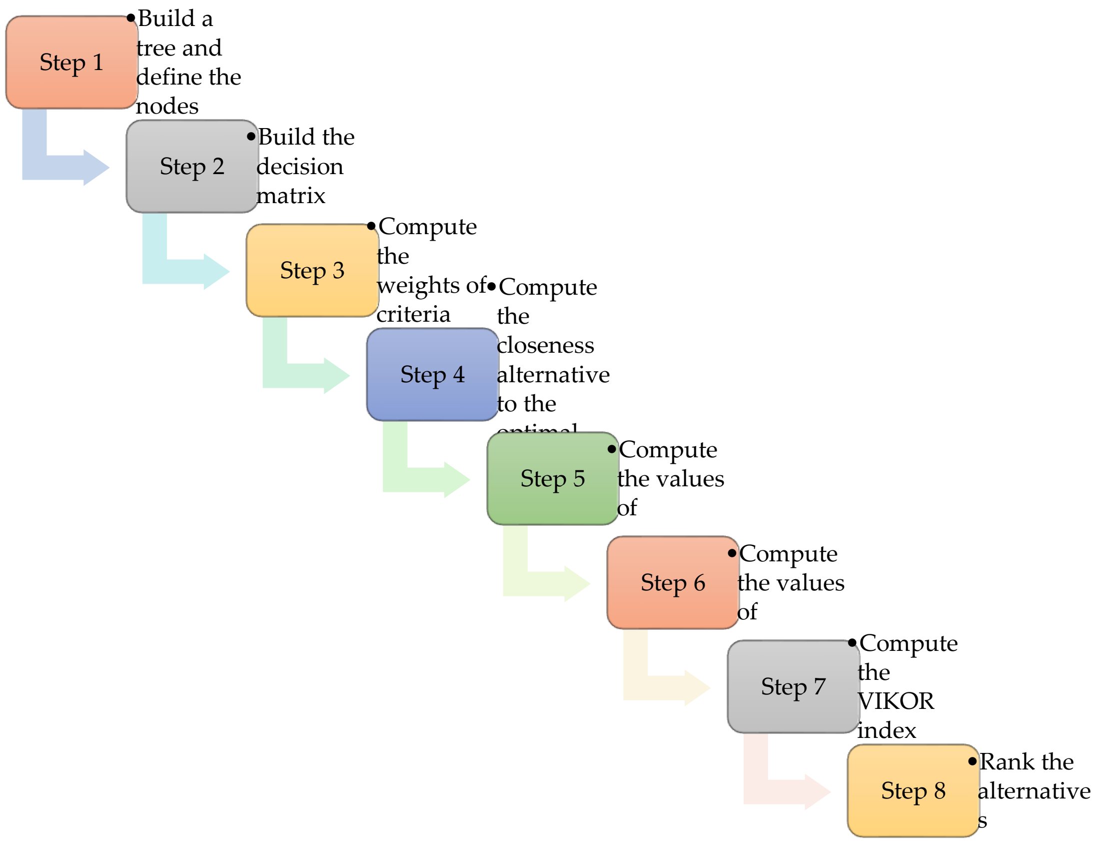

Figure 1. The steps of the proposed methodology.

The next steps of the neutrosophic TreeSoft Set with the VIKOR method as shown in Figure 1.

Step 1. Build a tree and define the nodes [20]–[23].

The tree has more than one level, in the first level, the main criteria and introduced as $SWM_1, SWM_2 \dots, SWM_n$

In the second level, the sub-criteria are introduced as $SWM_{1.1}, SWM_{1.2}, \dots$. And $SWM_{2.1}, SWM_{2.2}, \dots$.

Step 2. Build the decision matrix

The decision matrix is built by using the information of decision makers and experts between criteria and alternatives.

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}; i = 1,2, \dots, m; j = 1,2, \dots, n \quad (13)$$

Step 3. Compute the weights of the criteria.

The weights of the criteria are computed by using the average method.

$$\sum_{j=1}^{n} w_j = 1 \quad (14)$$

Step 4. Compute the closeness alternative to the optimal solution.

$$U_i = \left\{ \sum_{j=1}^{n} \left[\frac{w_j (r_j^* - r_{ij})}{(r_j^* - r_j^-)} \right]^P \right\}^{\frac{1}{p}} i = 1,2, \dots, m; j = 1,2, \dots, n; 1 \leq P \leq \infty \quad (15)$$

Where r_j^* is the best and r_j^- is the worst

$$\begin{cases} r_j^* = \max_i x_{ij} \\ r_j^- = \min_i x_{ij} \end{cases} (positive\ criteria) i = 1,2, \dots, m; j = 1,2, \dots, n \quad (16)$$

$$\begin{cases} r_j^* = \min_i x_{ij} \\ r_j^- = \max_i x_{ij} \end{cases} (negative\ criteria) i = 1,2, \dots, m; j = 1,2, \dots, n \quad (17)$$

Step 5. Compute the values of S_i

$$S_i = \sum_{j=1}^{n} w_j \frac{(r_j^* - r_{ij}^-)}{(r_j^* - r_j^-)} \; i = 1,2, \dots, m; j = 1,2, \dots, n \quad (18)$$

Step 6. Compute the values of R_i

$$R_i = \max_j \left[w_j \frac{(r_j^* - r_{ij}^-)}{(r_j^* - r_j^-)} \right] i = 1,2, \dots, m; j = 1,2, \dots, n \quad (19)$$

Step 7. Compute the VIKOR index.

$$Q_i = t * \left[\frac{(S_i - S^*)}{S^- - S^*} \right] + (1 - t) * \left[\frac{(R_i - R^*)}{R^- - R^*} \right] i = 1,2, \dots, m; j = 1,2, \dots, n \quad (20)$$

$$S^* = \min_i S_i, S^- = \max_i S_i, R^* = \min_i R_i, R^- = \max_i R_i \quad (21)$$

Where $t = 0.5$

Step 8. Rank the alternatives.

The alternatives are ranked as descending values of Q_i.

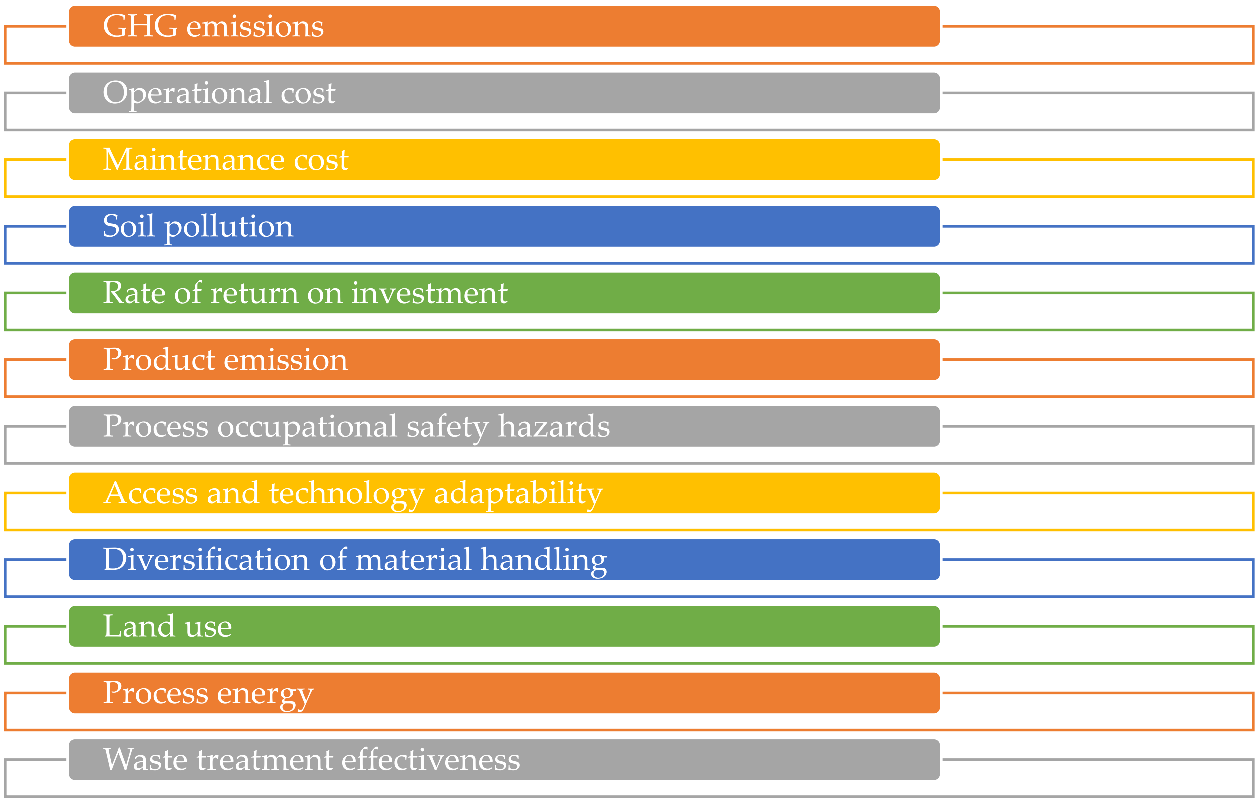

Figure 2. The list of used criteria.

3. Numerical Example

This section shows the results of TreeSoft with the BNS and VIKOR methods. This study used 12 criteria as shown in Figure 2 and ten alternatives. Three experts used the bipolar neutrosophic numbers (BNNs) to evaluate the criteria and alternatives.

Step 1. Build a tree and define the nodes.

The tree has more than one level, in the first level, the main criteria and introduced as $SWM_1, SWM_2 \dots, SWM_n$

In the second level, the sub-criteria are introduced as $SWM_{1.1}, SWM_{1.2}$, ... And $SWM_{2.1}, SWM_{2.2}$, ...

Step 2. Build the decision matrix using Eq. (13). Table 1 shows the decision matrix.

Table 1. The decision matrix.

	SWM_1	SWM_2	SWM_3	SWM_4	SWM_5	SWM_6	SWM_7	SWM_8	SWM_9	SWM_{10}	SWM_{11}	SWM_{12}
SWA_1	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]
SWA_2	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]
SWA_3	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]
SWA_4	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]
SWA_5	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]
SWA_6	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]
SWA_7	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]
SWA_8	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]
SWA_9	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]
SWA_{10}	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]

	SWM_1	SWM_2	SWM_3	SWM_4	SWM_5	SWM_6	SWM_7	SWM_8	SWM_9	SWM_{10}	SWM_{11}	SWM_{12}
SWA_1	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]
SWA_2	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]
SWA_3	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]
SWA_4	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]
SWA_5	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]
SWA_6	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]
SWA_7	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]
SWA_8	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]
SWA_9	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]
SWA_{10}	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]

	SWM_1	SWM_2	SWM_3	SWM_4	SWM_5	SWM_6	SWM_7	SWM_8	SWM_9	SWM_{10}	SWM_{11}	SWM_{12}
SWA_1	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]
SWA_2	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]
SWA_3	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]
SWA_4	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]
SWA_5	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]
SWA_6	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]
SWA_7	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]
SWA_8	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]
SWA_9	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]
SWA_{10}	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.90, 0.10, 0.10, −0.40, −0.80, −0.90]	[0.80, 0.50, 0.50, −0.30, −0.80, −0.80]	[0.50, 0.50, 0.50, −0.50, −0.50, −0.50]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]	[0.10, 0.80, 0.70, −0.90, −0.20, −0.10]	[0.40, 0.20, 0.70, −0.50, −0.20, −0.10]

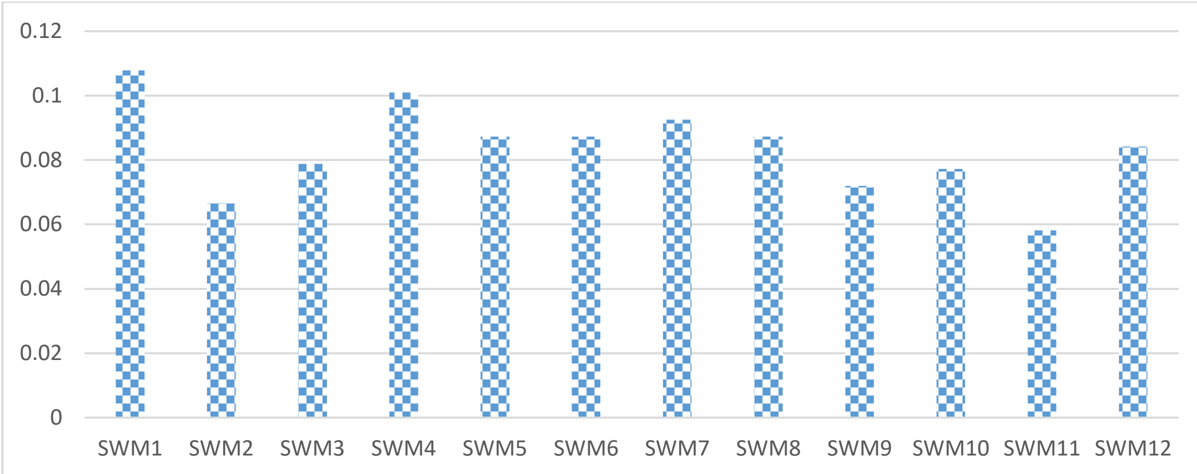

Figure 3. The weights of criteria.

Step 3. Compute the weights of criteria as shown in Figure 3.

Step 4. Compute the closeness alternative to the optimal solution using Eq. (15).

Step 5. Compute the values of S_i using Eq. (18).

Step 6. Compute the values of R_i using Eq. (19).

Step 7. Compute the VIKOR index using Eqs. (20 and 21)

Step 8. Rank the alternatives as shown in Figure 4. Alternative 6 is the best and alternative 8 is the worst.

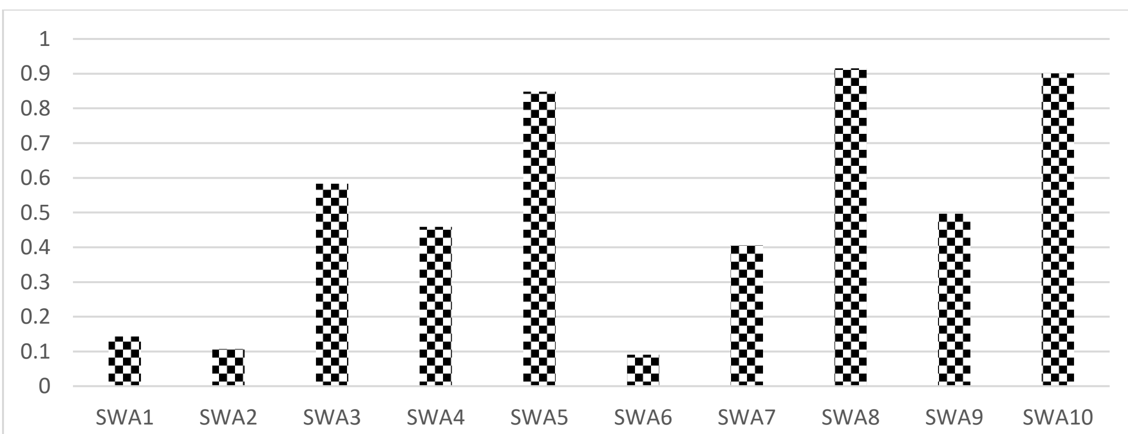

Figure 4. The values of the VIKOR index.

4. Conclusions

This study used the MCDM methodology to evaluate waste valorization. The MCDM methodology is used to deal with various criteria. The VIKOR method is used to rank the alternatives. The MCDM methodology is integrated with a neutrosophic set to deal with uncertainty in the evaluation process. The neutrosophic set and MCDM methodology integrated with TreeSoft Set in the evaluation process. Three decision-makers and experts are invited to rank the criteria and alternatives. We used the BNNs to replace the opinions of experts. Three decision matrices are created using the VIKOR method. We obtain the crisp value in each decision matrix and combine it to get one matrix. We used 12 criteria and 10 alternatives in this study.

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Applications of sets and functions by using an open sets in Fuzzy neutrosophic topological spaces

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Abstract: The definitions provided by the authors of the current study are offered together with a discussion of the recent advances that they have contributed. We begin with an introduction to fn-$\mathrm{Fr}_{\#b_{\varrho_N}}$, which includes the concepts of closed and open sets. We explore characteristics in fn-$\beta d^{\#b_{\varrho_N}}$ and fn-$\mathrm{e}_{b_{\varrho_N}}(\varrho_N)$, and provide an idea of obtained results by adding the notion of $\mathrm{FNb}_{\varrho_N}\mathrm{OS}$ and analyzing a few of their properties in $fnts$. We've researched the contrasts between the derived, exterior, and frontier notions that are provided. We also looked at the ideas of $\langle \mathcal{T}^{b_{\varrho_N}} \rangle C$-functions and Γ^{S}-segregated functions and examined and determined the traits.

Keywords: $\mathrm{Fr}_{\#b_{\varrho_N}}$, $\beta d^{\#b_{\varrho_N}}$, $\mathrm{e}_{b_{\varrho_N}}(\varrho_N)$, $fnb_{\varrho_N}\mathrm{OS}$, derived, exterior, and frontier

1. Introduction

Uncertainties are a major source of real-world difficulties in the fields of business, finance, medicine, engineering, and the behavioural sciences. Using conventional mathematical methods to solve the uncertainties for these data presents challenges. To avoid problems while working with ambiguous data, there exist methods like fuzzy sets, rough sets, fuzzy sets with intuitionistic properties, and vague sets that may be used as mathematical tools. Due to the inadequate parametrization tools, all of these techniques implicitly face difficulties when attempting to solve problems involving inconsistent and indeterminate data. The characteristics of n-closed sets, interior operators, closure operators, and open sets determine how neutrosophic is used in topology. Topologists explored sets next to neutrosophic closed and open sets.

L. A. Zadeh [29] proposed fuzzy sets in 1965 sand investigated various aspects of their features, A fuzzy set is a class of elements with an assortment of membership grades. Such a collection is characterised by a membership (or feature) function that assigns a membership grade, ranging from zero to one, to each item. He extended the notions of inclusion, union, intersection, complement, connection, convexity, etc. to these sets and demonstrated various aspects of these notions in relation to fuzzy sets. In particular, a separation theorem for convex fuzzy sets is proved that does not need the fuzzy sets to be disjoint.

Atanassov[14, 15, 16] have created intuitionistic fuzzy sets and looked through numerous outcomes, he presented the concept of the "Generalised Net" and examined its fundamental characteristics along with a few of its uses in the fields of artificial intelligence, systems theory, health, economics, transportation, and the chemical industry.

He spearheaded most of the applied research in the field of generalised nets and was the driving force behind its theoretical investigation. Many of the operations and interactions he has established over generalised nets have parallels in the theory of regular Petri nets. Nevertheless, there is no counterpart in Petri net theory for the topological and logical operators he has presented. Atanassov's other primary area of study is fuzzy sets, originally established by Zadeh, which he developed further by presenting the concept of "Intuitionistic Fuzzy Sets" and investigating the elements that make up its foundation. He is also recognised as a pioneer in the use of intuitionistic fuzzy sets to expert systems, systems theory, decision-making, and other domains.

F.Smarandache [9, 10, 24] examined the idea of using a neutrosophic set as a technique for resolving problems involving persistent, unpredictable, and unreliable data. He also noted the features of the generalisation of intuitionistic fuzzy logic. The study of the nature, origin, and scope of neutralities as well as their interactions with other ideational spectra is done within a branch of neutrosophy called the neutrosophic set. The neutrosophic set is a robust universal formal framework that was introduced lately. However, from a technical point of view, the neutrosophic set has to be specified.

P. Basker and Broumi Said [5, 6, 7] Investigators investigated the idea of $N\psi^{\#}_{\alpha}\ ^{0}$ and $N\psi^{\#}_{\alpha}\ ^{1}$-spaces and neutrosophic functions in neutrosophic topological spaces, and neutrosophic homeomorphisms from which the notion of $(\beta_{\rho n})$-OS in pythagorean neutrosophic topological spaces

Neurosophic topological spaces and the resulting neutrosophic set were studied in 2012 by A. A. Salama and S. A. Alblowi [23]. The concepts of fuzzy neutrosophic topological spaces and fuzzy neutrosophic sets were examined in 2014 by I. Arockiarani and J. Martina Jency [4]. In 2018, Fatimah M. Mohammed, Anas A. Hijab, and Shaymaa F. Matar [8] implemented fuzzy neutrosophic weakly-generalized closed sets in fuzzy neutrosophic topological spaces.

The concept of sharp, weakly neutrosophic closed functions was introduced by Ali Hussein Mahmood Al Obaidi, Qays Hatem Imran, and Murtadha Mohammed Abdulkadhim [1]. Hypersoft topological spaces were employed by Sagvan Y. Musa and Baravan A. Asaad [22] to connect the concepts.

In 2023, the neutrosophic soft generalised b-closed sets in neutrosophic soft topological spaces were created by Alkan Özkan, \eyda Yazgan, and Sandeep Kaur [2], Muthumari G et al. [20] the neutrosophic over topologized graphs' homomorphism and isomorphism were derived, Tomasz Witczak [27], Interior and closure of anti-minimal and anti-biminimal areas in the framework of anti-topology. The authors developed and examined a novel class of neutrosophic open and closed maps in neutrosophic topological spaces. P.Anbarasi Rodrigo et al. [3] and P. Thangaraja et al. [28]. Separation Axioms, Neighbourhood and Continuity were discussed in [21, 25, 26]. A few descriptions of both new and Neutrosophic objects were covered in [11, 12, 13]. An application of neutrosophic theory and computation of neutrosophic were generalized in [17, 18, 19].

This paper's Section 1 lists the definitions cited by the authors as well as recent advances that they have provided. We introduce the concept of FNb_{ϱ_N} OS in Section 2 using $fnts$. FNb_{ϱ_N} have determined $Fr_{\#b_{\varrho_N}}, \beta d^{\#b_{\varrho_N}}, e_{b_{\varrho_N}}(\varrho_N)$ and $E^{FNb_{\varrho_N}}(\varrho_N)$ studied some of their properties by using the above concepts we have derived the applications of fn-open and closed sets. In this study, FNS, FNTS, MN and MX stand for fn-set, fn-Topological Spaces, Minimum and Maximum respectively.

The following are the main novelties of this paper.

- fn-open and closed sets
- FNb_{ϱ_N}- point of interior
- fn-b_{ϱ_N}-border
- fn-b_{ϱ_N}-frontier
- fn-b_{ϱ_N}-exterior
- fn-b_{ϱ_N}-derived
- fn-Γ^S-segregated
- fn-b_{ϱ_N}-Totally-Continuous

The essential definitions listed below will aid in understanding this research work.

Definition 1.1.[4] A $\boldsymbol{fn}$-set $\boldsymbol{A}$ on X is defined as A is equal to $\langle \boldsymbol{\varpi}, \boldsymbol{I_A(\varpi)}, \boldsymbol{J_A(\varpi)}, \boldsymbol{K_A(\varpi)}\rangle$, $\boldsymbol{\varpi}$ belongs to $\boldsymbol{X}$ where $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$ from $\boldsymbol{X}$ to$[\mathbf{0}, \mathbf{1}]$ and $\mathbf{0} \leq \boldsymbol{sum\ \ of}\ \ \{\boldsymbol{I_A(\varpi)},\ \boldsymbol{J_A(\varpi)},\ \boldsymbol{K_A(\varpi)}\} \leq \mathbf{3}$.

Definition 1.2. [4] A $\boldsymbol{fn}$-set, A belongs to the subset of a $\boldsymbol{fn}$-set B (i.e.,) $\boldsymbol{A} \subseteq \boldsymbol{B}$ $\forall \boldsymbol{\varpi}$ if $\boldsymbol{I_A(\varpi)} \leq \boldsymbol{I_B(\varpi)}$ $\boldsymbol{J_A(\varpi)} \leq \boldsymbol{J_B(\varpi)}$ $\boldsymbol{K_A(\varpi)} \geq \boldsymbol{L_B(\varpi)}$

Definition 1.3. [4] Let $\boldsymbol{X}$ must represent a non-empty set., and $\boldsymbol{A} = \langle \boldsymbol{\varpi}, \boldsymbol{I_A(\varpi)}, \boldsymbol{J_A(\varpi)}, \boldsymbol{K_A(\varpi)}\rangle$, $\boldsymbol{B} = \langle \boldsymbol{\varpi}, \boldsymbol{I_B(\varpi)}, \boldsymbol{J_B(\varpi)}, \boldsymbol{K_B(\varpi)}\rangle$ be two $\boldsymbol{fn}$-set. Then
Union of $\boldsymbol{A}$ and $\boldsymbol{B}$ is $\langle \boldsymbol{\varpi}, \mathbf{MX\ \ of}\{\boldsymbol{I_A(\varpi)}, \boldsymbol{I_B(\varpi)}\}, \mathbf{MN\ \ of}\{\boldsymbol{J_A(\varpi)}, \boldsymbol{J_B(\varpi)}\}, \mathbf{MN\ \ of}\{\boldsymbol{K_A(\varpi)}, \boldsymbol{K_B(\varpi)}\}\rangle$ and Intersection of $\boldsymbol{A}$ and $\boldsymbol{B}$ is $\langle \boldsymbol{\varpi}, \mathbf{MN\ \ of}\{\boldsymbol{I_A(\varpi)}, \boldsymbol{I_B(\varpi)}\}, \mathbf{MN\ \ of}\{\boldsymbol{J_A(\varpi)}, \boldsymbol{J_B(\varpi)}\}, \mathbf{MX\ \ of}\{\boldsymbol{K_A(\varpi)}, \boldsymbol{K_B(\varpi)}\}\rangle$.

Definition 1.4. [4] The difference between two $\boldsymbol{fn}$-set A and B is defined as
Differ from $\boldsymbol{A}$ to $\boldsymbol{B}$ is $\langle \boldsymbol{\varpi}, \mathbf{MN\ \ of}\{\boldsymbol{I_A(\varpi)}, \boldsymbol{K_B(\varpi)}\}, \mathbf{MN\ \ of}\{\boldsymbol{J_A(\varpi)}, \mathbf{1} - \boldsymbol{J_B(\varpi)}\}, \mathbf{MN\ \ of}\{\boldsymbol{K_A(\varpi)}, \boldsymbol{I_B(\varpi)}\}\rangle$.

Definition 1.5. [4] A $\boldsymbol{fn}$-set it is said that $\boldsymbol{A}$ over the universe $\boldsymbol{X}$ equals
- Null or empty $\boldsymbol{fn}$-set if $\mathbf{0}_N = \langle \boldsymbol{\varpi}, \mathbf{0}, \mathbf{0}, \mathbf{1}\rangle$ $\forall \boldsymbol{\varpi} \in \boldsymbol{X}$.
- Absolute (universe) $\boldsymbol{fn}$-set if $\mathbf{1}_N = \langle \boldsymbol{\varpi}, \mathbf{1}, \mathbf{1}, \mathbf{0}\rangle$ $\forall \boldsymbol{\varpi} \in \boldsymbol{X}$.

Definition 1.6. [4] $\boldsymbol{A^c}$ represents the complement of a $\boldsymbol{fn}$-set $\boldsymbol{A}$, which is defined as $\boldsymbol{A^c} = \langle \boldsymbol{\varpi}, \boldsymbol{I_{(A^c)}(\varpi)}, \boldsymbol{J_{(A^c)}(\varpi)}, \boldsymbol{K_{(A^c)}(\varpi)}\rangle$, Where $\boldsymbol{I_{(A^c)}(\varpi)} = \boldsymbol{K_A(\varpi)}, \boldsymbol{J_{(A^c)}(\varpi)} = \mathbf{1} - \boldsymbol{J_A(\varpi)}, \boldsymbol{K_{(A^c)}(\varpi)} = \boldsymbol{I_A(\varpi)}$.
Another way to define the complement of a $\boldsymbol{fn}$-set A is as $\boldsymbol{A^c} = \mathbf{1}_N - \boldsymbol{A}$.

2. Applications of Fuzzy Neutrosophic open and closed sets

Definition 2.1. A fns, $\varrho_N = \langle \mathrm{H}, \zeta_{\varrho_N}, \eta_{\varrho_N}, \theta_{\lambda_N}\rangle$ in a $fnts$ Γ is to be
(i) fn-b_{ϱ_N}-OS (FNbOS), $\mathrm{FNi}(\mathrm{FNc}(\varrho_N)) \cup \mathrm{FNc}(\mathrm{FNi}(\varrho_N)) \supseteq \varrho_N$
(ii) fn-b_{ϱ_N}-CS (FNbCS), $\mathrm{FNi}(\mathrm{FNc}(\varrho_N)) \cap \mathrm{FNc}(\mathrm{FNi}(\varrho_N)) \subseteq \varrho_N$
We'll utilize shortened versions of FNb_{ϱ_N}-$\mathbb{N}$bhd, for the word FNb_{ϱ_N}-neighbourhood

Definition 2.2. Let Γ be an $fnts$ and let $\varrho_n^1 \in \Gamma$. A part of $\mathbb{N}$ of Γ is FNb_{ϱ_N}-$\mathbb{N}$bhd of ϱ_n^1, if $\exists$ a FNb_{ϱ_N}-OS, E such that $\varrho_n^1 \in \mathrm{E} \subset \mathbb{N}$.

Definition 2.3. Let $\boldsymbol{\varrho_N}$ be a subset of $\boldsymbol{\Gamma}$. Then, if $\boldsymbol{\varrho_N}$ is a $\mathbf{FNb}_{\varrho_N}$-Nbhd of $\boldsymbol{\varrho_n^1}$, then $\boldsymbol{\varrho_n^1} \in \boldsymbol{\varrho_N}$ is to be $\mathbf{FNb}_{\varrho_N}$-point of interior $\boldsymbol{\varrho_N}$. $\mathbf{FNb}_{\varrho_N}$-interior $\boldsymbol{\varrho_N}$ is the whole set $\mathbf{FNb}_{\varrho_N}$-point of interior $\boldsymbol{\varrho_N}$, and it is $\mathbf{b}_{\varrho_N}$-int$(\boldsymbol{\varrho_N})$, $\mathbf{IN}_{\mathbf{b}_{\varrho_N}}(\boldsymbol{\varrho_N}) = \cup\{\mathbf{E}: \mathbf{E\ \ is\ \ FNb}_{\varrho_N}\mathbf{OS},\ \mathbf{E} \subset \boldsymbol{\varrho_N}\}$

Let be the part of a space $\boldsymbol{\varrho_N}$ Γ. The meeting point for all $\mathbf{FNb}_{\varrho_N}$-closed sets containing $\boldsymbol{\varrho_N}$ is defined as the $\mathbf{FNb}_{\varrho_N}$-closure of $\boldsymbol{\varrho_N}$, $\mathbf{CL}_{\mathbf{b}_{\varrho_N}}(\boldsymbol{\varrho_N}) = \cap\{\mathbf{E}: \boldsymbol{\varrho_N} \subset \mathbf{E} \in \mathbf{FNb}_{\varrho_N}(\boldsymbol{\Gamma})\}$

Definition 2.4. An ϱ_N be a space that has a group of individuals. Γ, an element $\varrho_n^1 \in \Gamma$ is to be b_{ϱ_N}-point of ϱ_N if for all b_{ϱ_N}-OS, Γ_1 containing ϱ_n^1, $\Gamma_1 \cap (\varrho_N - \{\varrho_N^1\}) \neq \phi$. The whole set b_{ϱ_N}-point of ϱ_N is b_{ϱ_N}-derived (briefly. $\mathrm{e}_{b_{\varrho_N}}$) of ϱ_N as indicated by $\mathrm{e}_{b_{\varrho_N}}(\varrho_N)$.

Example 2.5. Let $\Gamma = \{\alpha, \beta, \gamma\}$ and $\Upsilon = \{0_N, 1_N, \varrho_{N_1}, \varrho_{N_2}, \varrho_{N_3}, \varrho_{N_4}\}$ where
$\varrho_{N_1} = \{\langle\Gamma(\alpha)0.82, \Gamma(\alpha)0.79, \Gamma(\alpha)0.59\rangle, \langle\Gamma(\beta)0.4, \Gamma(\beta)0.61, \Gamma(\beta)0.4\rangle, \langle\Gamma(\gamma)0.39, \Gamma(\gamma)0.4, \Gamma(\gamma)0.5\rangle\}$,
$\varrho_{N_2} = \{\langle\Gamma(\alpha)0.69, \Gamma(\alpha)0.59, \Gamma(\alpha)0.39\rangle, \langle\Gamma(\beta)0.78, \Gamma(\beta)0.2, \Gamma(\beta)0.3\rangle, \langle\Gamma(\gamma)0.99, \Gamma(\gamma)0.39, \Gamma(\gamma)0.19\rangle\}$,
$\varrho_{N_3} = \{\langle\Gamma(\alpha)0.82, \Gamma(\alpha)0.78, \Gamma(\alpha)0.49\rangle, \langle\Gamma(\beta)0.8, \Gamma(\beta)0.51, \Gamma(\beta)0.4\rangle, \langle\Gamma(\gamma)0.9, \Gamma(\gamma)0.7, \Gamma(\gamma)0.2\rangle\}$,
$\varrho_{N_4} = \{\langle\Gamma(\alpha)0.69, \Gamma(\alpha)0.59, \Gamma(\alpha)0.59\rangle, \langle\Gamma(\beta)0.59, \Gamma(\beta)0.21, \Gamma(\beta)0.4\rangle, \langle\Gamma(\gamma)0.39, \Gamma(\gamma)0.3, \Gamma(\gamma)0.4\rangle\}$.
Here ϱ_{N_3} be a subset of a space Γ and a point $\alpha \in \Gamma$ and Γ_1 a b_{ϱ_N}-OS, then it is a b_{ϱ_N}-point of ϱ_N is $\mathrm{e}_{b_{\varrho_N}}(\{\langle\Gamma(\alpha)0.82, \Gamma(\alpha)0.79, \Gamma(\alpha)0.49\rangle, \langle\Gamma(\beta)0.8, \Gamma(\beta)0.51, \Gamma(\beta)0.4\rangle, \langle\Gamma(\gamma)0.9, \Gamma(\gamma)0.7, \Gamma(\gamma)0.2\rangle\})$.

Theorem 2.6. As for segments ϱ_{N_1}, ϱ_{N_2} of a space Γ, all of the following claims are true::
If $\varrho_{N_2} \supset \varrho_{N_1}$, then

a) $\mathrm{e}_{b_{\varrho_N}}(\varrho_{N_2}) \supset \mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1})$
b) $\mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1} \cup \varrho_{N_2}) \supset \mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1}) \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_{N_2})$
c) $\mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1}) \supset \mathrm{e}_{b_{\varrho_N}}\left(\mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1})\right) - \varrho_{N_1}$
d) $\varrho_{N_1} \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1}) \supset \mathrm{e}_{b_{\varrho_N}}\left(\varrho_{N_1} \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1})\right)$.

Proof. (a) It is obvious. (b) It is an immediate consequence of (c).

(c) If $\varrho_n^1 \in \mathrm{e}_{b_{\varrho_N}}\left(\mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1})\right) - \varrho_{N_1}$ and Γ_1 is a b_{ϱ_N}-OS, offering ϱ_n^1, $\Gamma_1 \cap \left(\mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1}) - \{\varrho_n^1\}\right) \neq \phi$. Permit $\varrho_n^2 \in \Gamma_1 \cap \left(\mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1}) - \{\varrho_n^1\}\right)$. Then due to the fact $\varrho_n^2 \in \mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1})$, $\varrho_n^2 \in \Gamma_1$, $\Gamma_1 \cap (\varrho_{N_1} - \{\varrho_n^2\}) \neq \phi$. Permit $\Gamma^\# \in \Gamma_1 \cap (\varrho_{N_1} - \{\varrho_n^2\})$, $\Gamma^\# \neq \varrho_n^1$ to be for $\Gamma^\# \in \varrho_{N_1}$, $\varrho_n^1 \notin \varrho_{N_1}$. Accordingly $\Gamma_1 \cap (\varrho_{N_1} - \{\varrho_n^1\}) \neq \phi$. Consequently $\varrho_n^1 \in \mathrm{e}_{b_{\varrho_N}}(\varrho_{N_1})$.

(d) Let's Take $\varrho_n^1 \in \mathrm{e}_{b_{\varrho_N}}\left(\varrho_N \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_N)\right)$. If $\varrho_n^1 \in \varrho_N$, The ultimate result is clear. Let $\varrho_n^1 \in \mathrm{e}_{b_{\varrho_N}}\left(\varrho_N \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_N)\right) - \varrho_N$, for b_{ϱ_N}-OS, $\Gamma_1 \subset \varrho_n^1$, $\Gamma_1 \cap \left(\varrho_N \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_N) - \{\varrho_n^1\}\right) \neq \phi$. Consequently $\Gamma_1 \cap (\varrho_N - \{\varrho_n^1\}) \neq \phi$ or $\Gamma_1 \cap \left(\mathrm{e}_{b_{\varrho_N}}(\varrho_N) - \{\varrho_n^1\}\right) \neq \phi$. It eventually follows (c) that $\Gamma_1 \cap (\varrho_N - \{\varrho_n^1\}) \neq \phi$. So $\varrho_n^1 \in \mathrm{e}_{b_{\varrho_N}}(\varrho_N)$. So, whatever the circumstance, $\varrho_N \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_N) \supset \mathrm{e}_{b_{\varrho_N}}\left(\varrho_N \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_N)\right)$.

Theorem 2.7. In any subset that exists ϱ_N of a Γ, $b_{\varrho_N}\mathrm{CLof}(\varrho_N) = \varrho_N \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_N)$.

Proof. Since $\mathrm{e}_{b_{\varrho_N}}(\varrho_N) \subset b_{\varrho_N}\mathrm{CLof}(\varrho_N)$, $\varrho_N \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_N) \subset b_{\varrho_N}\mathrm{CLof}(\varrho_N)$. As opposed to that, let $\varrho_n^1 \in b_{\varrho_N}\mathrm{CLof}(\varrho_N)$. If $\varrho_n^1 \in \varrho_N$, then the evidence is conclusive. If $\varrho_n^1 \notin \varrho_N$, then every single b_{ϱ_N}-OS $\Gamma_1 \subset \varrho_n^1 \cap \varrho_N$ at something different from ϱ_n^1 .Consequently $\varrho_n^1 \in \mathrm{e}_{b_{\varrho_N}}(\varrho_N)$. Thus $\varrho_N \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_N) \supset b_{\varrho_N}\mathrm{CLof}(\varrho_N) \Longrightarrow b_{\varrho_N}\mathrm{CLof}(\varrho_N) = \varrho_N \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_N)$. This concludes the evidence to be presented.

Observation 2.8. In any subset that exists $\varrho_{N_1}, \varrho_{N_2}$ of Γ, These statements are all accurate:

a) $\mathrm{IN}_{b_{\varrho_N}}(\varrho_{N_1})$ being the biggest b_{ϱ_N}-OS $\subset \varrho_{N_1}$.
b) ϱ_{N_1} is b_{ϱ_N}-OS $\Leftrightarrow \varrho_{N_1} = \mathrm{IN}_{b_{\varrho_N}}(\varrho_{N_1})$.
c) $\mathrm{IN}_{b_{\varrho_N}}\left(\mathrm{IN}_{b_{\varrho_N}}(\varrho_{N_1})\right) = \mathrm{IN}_{b_{\varrho_N}}(\varrho_{N_1})$.
d) $\Gamma - \mathrm{CL}_{b_{\varrho_N}}(\varrho_{N_1}) = \mathrm{IN}_{b_{\varrho_N}}(\Gamma - \varrho_{N_1})$.

e) $\varrho_{N_1} \subset \varrho_{N_2}$, then $IN_{b_{\varrho N}}(\varrho_{N_2}) \supset IN_{b_{\varrho N}}(\varrho_{N_1})$.
f) $IN_{b_{\varrho N}}(\varrho_{N_1} \cup \varrho_{N_2}) \supset IN_{b_{\varrho N}}(\varrho_{N_1}) \cup IN_{b_{\varrho N}}(\varrho_{N_2})$.

Theorem 2.9. In any of the subsets $\varrho_{N_1}, \varrho_{N_2}$ of Γ, All of these claims are true:
a) $IN_{b_{\varrho N}}(\varrho_{N_1}) = \varrho_{N_1} - \mathrm{e}_{b_{\varrho N}}(\Gamma - \varrho_{N_1})$.
b) $\Gamma - IN_{b_{\varrho N}}(\varrho_{N_1}) = CL_{b_{\varrho N}}(\Gamma - \varrho_{N_1})$.

Proof.

(a) Let $\varrho_n^1 \in \varrho_{N_1} - \mathrm{e}_{b_{\varrho N}}(\Gamma - \varrho_{N_1}) \Longrightarrow \varrho_n^1 \notin \mathrm{e}_{b_{\varrho N}}(\Gamma - \varrho_{N_1})$ and so $\exists$ a $b_{\varrho N}$-OS, Γ_1 containing ϱ_n^1 such that $\Gamma_1 \cap (\Gamma - \varrho_{N_1}) = \phi$. Then $\varrho_n^1 \in \Gamma_1 \subset \varrho_{N_1}$ and hence $\varrho_n^1 \in b_{\varrho N}INTof(\varrho_{N_1})$, i.e., $\varrho_{N_1} - \mathrm{e}_{b_{\varrho N}}(\Gamma - \varrho_{N_1}) \subset b_{\varrho N}INTof(\varrho_{N_1})$. As opposed to that, if $\varrho_n^1 \in b_{\varrho N}INTof(\varrho_{N_1}) \Longrightarrow \varrho_n^1 \notin \mathrm{e}_{b_{\varrho N}}(\Gamma - \varrho_{N_1})$. Since $b_{\varrho N}INTof(\varrho_{N_1})$ is $b_{\varrho N}$-open and $b_{\varrho N}INTof(\varrho_{N_1}) \cap (\Gamma - \varrho_{N_1}) = \phi$. Hence $b_{\varrho N}INTof(\varrho_{N_1}) = \varrho_{N_1} - \mathrm{e}_{b_{\varrho N}}(\Gamma - \varrho_{N_1})$.

(b) $\Gamma - b_{\varrho N}INTof(\varrho_{N_1}) = \Gamma - \left(\varrho_{N_1} - \mathrm{e}_{b_{\varrho N}}(\Gamma - \varrho_{N_1})\right) = (\Gamma - \varrho_{N_1}) \cup \mathrm{e}_{b_{\varrho N}}(\Gamma - \varrho_{N_1}) = b_{\varrho N}CLof(\Gamma - \varrho_{N_1})$.

Definition 2.10. In any of the subsets ϱ_N of Γ, $\beta d^{\#b_{\varrho N}}(\varrho_N) = \varrho_N - b_{\varrho N}INTof(\varrho_N)$ It has been stated to have $b_{\varrho N}$-border about ϱ_N.

Example 2.11. Let $\Gamma = \{\alpha, \beta, \gamma\}$ and $\Upsilon = \{0_N, 1_N, \varrho_{N_1}, \varrho_{N_2}, \varrho_{N_3}, \varrho_{N_4}\}$ where
$\varrho_{N_1} = \{\langle\Gamma(\alpha)0.71, \Gamma(\alpha)0.69, \Gamma(\alpha)0.5\rangle, \langle\Gamma(\beta)0.3, \Gamma(\beta)0.52, \Gamma(\beta)0.43\rangle, \langle\Gamma(\gamma)0.29, \Gamma(\gamma)0.29, \Gamma(\gamma)0.29\rangle\}$,
$\varrho_{N_2} = \{\langle\Gamma(\alpha)0.59, \Gamma(\alpha)0.61, \Gamma(\alpha)0.36\rangle, \langle\Gamma(\beta)0.76, \Gamma(\beta)0.23, \Gamma(\beta)0.33\rangle, \langle\Gamma(\gamma)0.89, \Gamma(\gamma)0.29, \Gamma(\gamma)0.29\rangle\}$,
$\varrho_{N_3} = \{\langle\Gamma(\alpha)0.62, \Gamma(\alpha)0.68, \Gamma(\alpha)0.39\rangle, \langle\Gamma(\beta)0.18, \Gamma(\beta)0.61, \Gamma(\beta)0.74\rangle, \langle\Gamma(\gamma)0.19, \Gamma(\gamma)0.23, \Gamma(\gamma)0.43\rangle\}$,
$\varrho_{N_4} = \{\langle\Gamma(\alpha)0.39, \Gamma(\alpha)0.49, \Gamma(\alpha)0.39\rangle, \langle\Gamma(\beta)0.62, \Gamma(\beta)0.24, \Gamma(\beta)0.14\rangle, \langle\Gamma(\gamma)0.23, \Gamma(\gamma)0.31, \Gamma(\gamma)0.32\rangle\}$
Here ϱ_{N_2} be a subset of a space Γ and
$\beta d^{\#b_{\varrho N}}(\{\langle\Gamma(\alpha)0.59, \Gamma(\alpha)0.61, \Gamma(\alpha)0.36\rangle, \langle\Gamma(\beta)0.76, \Gamma(\beta)0.23, \Gamma(\beta)0.33\rangle, \langle\Gamma(\gamma)0.89, \Gamma(\gamma)0.29, \Gamma(\gamma)0.29\rangle\}) =$
$\{\langle\Gamma(\alpha)0.59, \Gamma(\alpha)0.61, \Gamma(\alpha)0.36\rangle, \langle\Gamma(\beta)0.76, \Gamma(\beta)0.23, \Gamma(\beta)0.33\rangle, \langle\Gamma(\gamma)0.89, \Gamma(\gamma)0.29, \Gamma(\gamma)0.29\rangle\} -$
$IN_{b_{\varrho N}}(\{\langle\Gamma(\alpha)0.59, \Gamma(\alpha)0.61, \Gamma(\alpha)0.36\rangle, \langle\Gamma(\beta)0.76, \Gamma(\beta)0.23, \Gamma(\beta)0.33\rangle, \langle\Gamma(\gamma)0.89, \Gamma(\gamma)0.29, \Gamma(\gamma)0.29\rangle\})$

Observation 2.12. In any of the subsets ϱ_N of Γ, All of these claims are true:
a) $\varrho_N = IN_{b_{\varrho N}}(\varrho_N) \cup \beta d^{\#b_{\varrho N}}(\varrho_N)$.
b) $IN_{b_{\varrho N}}(\varrho_N) \cap \beta d^{\#b_{\varrho N}}(\varrho_N) = \phi$.
c) ϱ_N a $b_{\varrho N}$-OS $\Leftrightarrow \beta d^{\#b_{\varrho N}}(\varrho_N) = \phi$.
d) $\beta d^{\#b_{\varrho N}}\left(IN_{b_{\varrho N}}(\varrho_N)\right) = \phi$.
e) $IN_{b_{\varrho N}}\left(\beta d^{\#b_{\varrho N}}(\varrho_N)\right) = \phi$.

Theorem 2.13. In any of the subsets ϱ_N of Γ, All of these claims are correct:
a) $\beta d^{\#b_{\varrho N}}\left(\beta d^{\#b_{\varrho N}}(\varrho_N)\right) = \beta d^{\#b_{\varrho N}}(\varrho_N)$.
b) $\beta d^{\#b_{\varrho N}}(\varrho_N) = \varrho_N \cap CL_{b_{\varrho N}}(\Gamma - \varrho_N)$.
c) $\beta d^{\#b_{\varrho N}}(\varrho_N) = \mathrm{e}_{b_{\varrho N}}(\Gamma - \varrho_N)$.

Proof.

(a) If $\varrho_n^1 \in \mathrm{IN}_{b_{\varrho_N}}\left(\beta d^{\#b_{\varrho_N}}(\varrho_N)\right)$, then $\varrho_n^1 \in \beta d^{\#b_{\varrho_N}}(\varrho_N)$. As opposed to that, $\beta d^{\#b_{\varrho_N}}(\varrho_N) \subset \varrho_N$, $\varrho_n^1 \in \mathrm{IN}_{b_{\varrho_N}}\left(\beta d^{\#b_{\varrho_N}}(\varrho_N)\right) \subset \mathrm{IN}_{b_{\varrho_N}}(\varrho_N)$. Hence $\varrho_n^1 \in \mathrm{IN}_{b_{\varrho_N}}(\varrho_N) \cap \beta d^{\#b_{\varrho_N}}(\varrho_N)$ which contradicts (c). Thus $\cap\, of\ \mathrm{IN}_{b_{\varrho_N}}(\varrho_N)$ & $\beta d^{\#b_{\varrho_N}}(\varrho_N)$ is ϕ.

(b) $\beta d^{\#b_{\varrho_N}}(\varrho_N) =$ difference of $\mathrm{IN}_{b_{\varrho_N}}(\varrho_N)$ from $\varrho_N = \varrho_N - \left(\Gamma - \mathrm{CL}_{b_{\varrho_N}}(\Gamma - \varrho_N)\right) = \varrho_N \cap \mathrm{CL}_{b_{\varrho_N}}(\Gamma - \varrho_N)$.

(c) $\beta d^{\#b_{\varrho_N}}(\varrho_N) =$ difference of $\mathrm{IN}_{b_{\varrho_N}}(\varrho_N)$ from $\varrho_N = \varrho_N - \left(\varrho_N - \mathfrak{D}\varepsilon''_{\alpha\delta}(\Gamma - \varrho_N)\right) = \mathrm{e}_{b_{\varrho_N}}(\Gamma - \varrho_N)$.

Definition 2.14. A b_{ϱ_N}-frontier of any of the subsets ϱ_N of Γ is $\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N) = \cap\, of\ \mathrm{CL}_{b_{\varrho_N}}(\varrho_N) \& \mathrm{CL}_{b_{\varrho_N}}(\Gamma \backslash \varrho_N)$.

Example 2.15. Let $\Gamma = \{\alpha, \beta, \gamma\}$ and consider the family $\Upsilon = \{0_N, 1_N, \varrho_{N_1}, \varrho_{N_2}\}$ where
$\varrho_{N_1} = \{\langle\Gamma(\alpha)0.6, \Gamma(\alpha)0.5, \Gamma(\alpha)0.3\rangle, \langle\Gamma(\beta)0.3, \Gamma(\beta)0.7, \Gamma(\beta)0.3\rangle, \langle\Gamma(\gamma)0.1, \Gamma(\gamma)0.2, \Gamma(\gamma)0.6\rangle\}$,
$\varrho_{N_2} = \{\langle\Gamma(\alpha)0.9, \Gamma(\alpha)0.1, \Gamma(\alpha)0.3\rangle, \langle\Gamma(\beta)0.6, \Gamma(\beta)0.2, \Gamma(\beta)0.3\rangle, \langle\Gamma(\gamma)0.9, \Gamma(\gamma)0.9, \Gamma(\gamma)0.2\rangle\}$,
Here ϱ_{N_1} be a subset of a space Γ and
$\mathrm{Fr}_{\#b_{\varrho_N}}(\{\langle\Gamma(\alpha)0.6, \Gamma(\alpha)0.5, \Gamma(\alpha)0.3\rangle, \langle\Gamma(\beta)0.3, \Gamma(\beta)0.7, \Gamma(\beta)0.3\rangle, \langle\Gamma(\gamma)0.1, \Gamma(\gamma)0.2, \Gamma(\gamma)0.6\rangle\})$ is equal to
$\mathrm{CL}_{b_{\varrho_N}}(\{\langle\Gamma(\alpha)0.6, \Gamma(\alpha)0.5, \Gamma(\alpha)0.3\rangle, \langle\Gamma(\beta)0.3, \Gamma(\beta)0.7, \Gamma(\beta)0.3\rangle, \langle\Gamma(\gamma)0.1, \Gamma(\gamma)0.2, \Gamma(\gamma)0.6\rangle\})$
$\cap \mathrm{CL}_{b_{\varrho_N}}(\{\langle\Gamma(\alpha)0.6, \Gamma(\alpha)0.5, \Gamma(\alpha)0.3\rangle, \langle\Gamma(\beta)0.3, \Gamma(\beta)0.7, \Gamma(\beta)0.3\rangle, \langle\Gamma(\gamma)0.1, \Gamma(\gamma)0.2, \Gamma(\gamma)0.6\rangle\})$.

Theorem 2.16. In any of the subsets ϱ_N of Γ, All of these claims are true:

a) $\mathrm{CL}_{b_{\varrho_N}}(\varrho_N) = \mathrm{IN}_{b_{\varrho_N}}(\varrho_N) \cup \mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N)$

b) $\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N) = \beta d^{\#b_{\varrho_N}}(\varrho_N) \cup \mathrm{e}_{b_{\varrho_N}}(\varrho_N)$.

c) $\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N) = \mathrm{CL}_{b_{\varrho_N}}(\varrho_N) \cap \mathrm{CL}_{b_{\varrho_N}}(\Gamma \backslash \varrho_N)$.

d) $\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N)$ is b_{ϱ_N}-closed

Theorem 2.17. In any of the subsets ϱ_N of Γ, All of these claims are correct:

a) $\mathrm{IN}_{b_{\varrho_N}}(\varrho_N) \cap \mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N) = \phi$.

b) $\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N) \supset \beta d^{\#b_{\varrho_N}}(\varrho_N)$.

c) ϱ_N is b_{ϱ_N}-open set iff $\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N) = \mathrm{e}_{b_{\varrho_N}}(\varrho_N)$

d) $\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N) = \mathrm{Fr}_{\#b_{\varrho_N}}(\Gamma \backslash \varrho_N)$

e) $\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N) \supset \mathrm{CL}_{b_{\varrho_N}}\left(\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N)\right)$.

f) $\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N) \supset \mathrm{Fr}_{\#b_{\varrho_N}}\left(\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N)\right)$.

g) $\mathrm{Fr}_{\#b_{\varrho_N}}(\varrho_N) \supset \mathrm{Fr}_{\#b_{\varrho_N}}\left(\mathrm{CL}_{b_{\varrho_N}}(\varrho_N)\right)$

h) $IN_{b_{\varrho_N}}(\varrho_N) = \varrho_N - Fr_{\#b_{\varrho_N}}(\varrho_N)$.

Proof.

(a) $IN_{b_{\varrho_N}}(\varrho_N) \cup Fr_{\#b_{\varrho_N}}(\varrho_N) = IN_{b_{\varrho_N}}(\varrho_N) \cup \left(CL_{b_{\varrho_N}}(\varrho_N) - IN_{b_{\varrho_N}}(\varrho_N)\right) = CL_{b_{\varrho_N}}(\varrho_N)$.

(b) $IN_{b_{\varrho_N}}(\varrho_N) \cap Fr_{\#b_{\varrho_N}}(\varrho_N) = IN_{b_{\varrho_N}}(\varrho_N) \cap \left(CL_{b_{\varrho_N}}(\varrho_N) - IN_{b_{\varrho_N}}(\varrho_N)\right) = \phi$.

(c) Since $IN_{b_{\varrho_N}}(\varrho_N) \cup Fr_{\#b_{\varrho_N}}(\varrho_N) = IN_{b_{\varrho_N}}(\varrho_N) \cup \beta d^{\#b_{\varrho_N}}(\varrho_N) \cup e_{b_{\varrho_N}}(\varrho_N)$,
$Fr_{\#b_{\varrho_N}}(\varrho_N) = \beta d^{\#b_{\varrho_N}}(\varrho_N) \cup e_{b_{\varrho_N}}(\varrho_N)$

(d) $Fr_{\#b_{\varrho_N}}(\varrho_N) = CL_{b_{\varrho_N}}(\varrho_N) - IN_{b_{\varrho_N}}(\varrho_N) = CL_{b_{\varrho_N}}(\varrho_N) \cap CL_{b_{\varrho_N}}(\Gamma \backslash \varrho_N)$.

(e) $CL_{b_{\varrho_N}}\left(Fr_{\#b_{\varrho_N}}(\varrho_N)\right) = CL_{b_{\varrho_N}}\left(CL_{b_{\varrho_N}}(\varrho_N) \cap CL_{b_{\varrho_N}}(X \backslash \varrho_N)\right)$
$\subset CL_{b_{\varrho_N}}\left(CL_{b_{\varrho_N}}(\varrho_N)\right) \cap CL_{b_{\varrho_N}}\left(CL_{b_{\varrho_N}}(\Gamma \backslash \varrho_N)\right) = Fr_{\#b_{\varrho_N}}(\varrho_N)$.
Hence $Fr_{\#b_{\varrho_N}}(\varrho_N)$ is b_{ϱ_N}-closed.

(f) $Fr_{\#b_{\varrho_N}}\left(Fr_{\#b_{\varrho_N}}(\varrho_N)\right) = CL_{b_{\varrho_N}}$ of $Fr_{\#b_{\varrho_N}}(\varrho_N) \cap CL_{b_{\varrho_N}}\left(\Gamma - Fr_{\#b_{\varrho_N}}(\varrho_N)\right)$
$\subset CL_{b_{\varrho_N}}$ of $Fr_{\#b_{\varrho_N}}(\varrho_N) = Fr_{\#b_{\varrho_N}}(\varrho_N)$

(g) $Fr_{\#b_{\varrho_N}}\left(CL_{b_{\varrho_N}}(\varrho_N)\right) = CL_{b_{\varrho_N}}\left(CL_{b_{\varrho_N}}(\varrho_N)\right) - IN_{b_{\varrho_N}}\left(CL_{b_{\varrho_N}}(\varrho_N)\right) = CL_{b_{\varrho_N}}(\varrho_N) - IN_{b_{\varrho_N}}\left(CL_{b_{\varrho_N}}(\varrho_N)\right) =$
$CL_{b_{\varrho_N}}(\varrho_N) - IN_{b_{\varrho_N}}(\varrho_N) = Fr_{\#b_{\varrho_N}}(\varrho_N)$.

(h) $\varrho_N - Fr_{\#b_{\varrho_N}}(\varrho_N) = \varrho_N - \left(CL_{b_{\varrho_N}}(\varrho_N) - IN_{b_{\varrho_N}}(\varrho_N)\right) = IN_{b_{\varrho_N}}(\varrho_N)$.

Within the ensuing theorem $FNb_{\varrho_N}^{(C)}$ indicate the group of points ϱ_n^1 of Γ which a function is used $q: (\Gamma_1, \xi_1) \to (\Gamma_2, \xi_2)$ is not FNb_{ϱ_N}-C.

Theorem 2.18. The $\cup(FNb_{\varrho_N})$-frontiers of the mirror reflections of FNb_{ϱ_N}-OS that includes $q(\varrho_n^1)$ is $\Leftrightarrow$ to $FNb_{\varrho_N}^{(C)}$.

Proof. Proceed to consider q is not FNb_{ϱ_N}-at a point, continuous ϱ_n^1 of $\Gamma_1 \Longrightarrow \exists$ an OS, $J \subset \Gamma_2$ containing $q(\varrho_n^1) \mid q(I)$ is not a portion of J $\forall I \in FNb_{\varrho_N}O(\Gamma_1)$ containing ϱ_n^1. Hence we've $I \cap (\Gamma_1 - q^{-1}(J)) \neq \phi$, $\forall$ $I \in FNb_{\varrho_N}O(\Gamma_1)$ containing ϱ_n^1. It follows that $\varrho_n^1 \in CL_{b_{\varrho_N}}(\Gamma_1 - q^{-1}(\varrho_N))$. Additionally, we have $\varrho_n^1 \in q^{-1}(J) \subset CL_{b_{\varrho_N}}(q^{-1}(\varrho_N))$. Thus, it follows that $\varrho_n^1 \in Fr_{\#b_{\varrho_N}}(q^{-1}(J))$. Now, let q be FNb_{ϱ_N}-Cont. at $\varrho_n^1 \in \Gamma_1$ and $J \subset \Gamma_2$ be any OS containing $q(\varrho_n^1)$. Then $\varrho_n^1 \in q^{-1}(J)$ is a FNb_{ϱ_N}-open set of Γ_1. Thus $\varrho_n^1 \in IN_{b_{\varrho_N}}(q^{-1}(J))$ and therefore $\varrho_n^1 \notin Fr_{\#b_{\varrho_N}}(q^{-1}(J))$ for every OS, J containing $q(\varrho_n^1)$.

Definition 2.19. In any of the subsets ϱ_N of a Γ, $E^{FNb_{\varrho_N}}$ of ϱ_N is b_{ϱ_N} INT of $\Gamma - \varrho_N$ this will eventually take place. FNb_{ϱ_N}-exterior regarding ϱ_N.

Example 2.20. Let $\Gamma = \{\alpha, \beta, \gamma\}$ and consider the family $\Upsilon = \{0_N, 1_N, \varrho_{N_1}, \varrho_{N_2}\}$ where
$\varrho_{N_1} = \{\langle\Gamma(\alpha)0.4, \Gamma(\alpha)0.5, \Gamma(\alpha)0.4\rangle, \langle\Gamma(\beta)0.6, \Gamma(\beta)0.6, \Gamma(\beta)0.4\rangle, \langle\Gamma(\gamma)0.3, \Gamma(\gamma)0.4, \Gamma(\gamma)0.7\rangle\}$,
$\varrho_{N_2} = \{\langle\Gamma(\alpha)0.8, \Gamma(\alpha)0.3, \Gamma(\alpha)0.2\rangle, \langle\Gamma(\beta)0.4, \Gamma(\beta)0.3, \Gamma(\beta)0.2\rangle, \langle\Gamma(\gamma)0.3, \Gamma(\gamma)0.2, \Gamma(\gamma)0.3\rangle\}$,

Here ϱ_{N_1} be a subset of a space Γ and
$E^{FNb_{\varrho_N}}(\{\langle\Gamma(\alpha)0.8,\Gamma(\alpha)0.3,\Gamma(\alpha)0.2\rangle,\langle\Gamma(\beta)0.4,\Gamma(\beta)0.3,\Gamma(\beta)0.2\rangle,\langle\Gamma(\gamma)0.3,\Gamma(\gamma)0.2,\Gamma(\gamma)0.3\rangle\}) =$
$IN_{b_{\varrho_N}}(\Gamma-\{\langle\Gamma(\alpha)0.8,\Gamma(\alpha)0.3,\Gamma(\alpha)0.2\rangle,\langle\Gamma(\beta)0.4,\Gamma(\beta)0.3,\Gamma(\beta)0.2\rangle,\langle\Gamma(\gamma)0.3,\Gamma(\gamma)0.2,\Gamma(\gamma)0.3\rangle\})$.

Observation 2.21. In any of the subsets ϱ_N of Γ, All of these claims are true:

a) $E^{FNb_{\varrho_N}}(\varrho_N)$ is FNb_{ϱ_N}-OS.

b) $E^{FNb_{\varrho_N}}(\varrho_N) = IN_{b_{\varrho_N}}(\Gamma-\varrho_N) = \Gamma - CL_{b_{\varrho_N}}(\varrho_N)$.

c) If $\binom{\varrho_N}{1} \subset \binom{\varrho_N}{2} \Rightarrow E^{FNb_{\varrho_N}}\binom{\varrho_N}{1} \supset E^{FNb_{\varrho_N}}\binom{\varrho_N}{2}$.

d) $E^{FNb_{\varrho_N}}\left(\binom{\varrho_N}{1} \cup \binom{\varrho_N}{2}\right) \subset E^{FNb_{\varrho_N}}\binom{\varrho_N}{1} \cup E^{FNb_{\varrho_N}}\binom{\varrho_N}{2}$.

e) $E^{FNb_{\varrho_N}}(\Gamma) = \phi$.

f) $E^{FNb_{\varrho_N}}(\phi) = \Gamma$.

g) $\Gamma = IN_{b_{\varrho_N}}(\varrho_N) \cup E^{FNb_{\varrho_N}}(\varrho_N) \cup Fr_{\#b_{\varrho_N}}(\varrho_N)$.

Theorem 2.22. In any of the subsets ϱ_N of Γ, All of these claims are correct:

a) $E^{FNb_{\varrho_N}}\left(E^{FNb_{\varrho_N}}(\varrho_N)\right) = IN_{b_{\varrho_N}}\left(CL_{b_{\varrho_N}}(\varrho_N)\right)$.

b) $E^{FNb_{\varrho_N}}(\varrho_N) = E^{FNb_{\varrho_N}}\left(\Gamma - E^{FNb_{\varrho_N}}(\varrho_N)\right)$.

c) $IN_{b_{\varrho_N}}(\varrho_N) \subset E^{FNb_{\varrho_N}}\left(E^{FNb_{\varrho_N}}(\varrho_N)\right)$.

Proof.

(a) $E^{FNb_{\varrho_N}}\left(E^{FNb_{\varrho_N}}(\varrho_N)\right) = E^{FNb_{\varrho_N}}\left(\Gamma - CL_{b_{\varrho_N}}(\varrho_N)\right)$
$= IN_{b_{\varrho_N}}\left(\Gamma - \left(\Gamma - CL_{b_{\varrho_N}}(\varrho_N)\right)\right) = IN_{b_{\varrho_N}}\left(CL_{b_{\varrho_N}}(\varrho_N)\right)$.

(b) $E^{FNb_{\varrho_N}}\left(\Gamma - E^{FNb_{\varrho_N}}(\varrho_N)\right) = E^{FNb_{\varrho_N}}\left(\Gamma - IN_{b_{\varrho_N}}(\Gamma-\varrho_N)\right)$
$= IN_{b_{\varrho_N}}\left(\Gamma - \left(\Gamma - IN_{b_{\varrho_N}}(\Gamma-\varrho_N)\right)\right) = IN_{b_{\varrho_N}}\left(IN_{b_{\varrho_N}}(\Gamma-\varrho_N)\right) = IN_{b_{\varrho_N}}(\Gamma-\varrho_N) = E^{FNb_{\varrho_N}}(\varrho_N)$.

(c) $IN_{b_{\varrho_N}}(\varrho_N) \subset IN_{b_{\varrho_N}}\left(CL_{b_{\varrho_N}}(\varrho_N)\right) = IN_{b_{\varrho_N}}\left(\Gamma - IN_{b_{\varrho_N}}(\Gamma-\varrho_N)\right)$
$= IN_{b_{\varrho_N}}\left(\Gamma - E^{FNb_{\varrho_N}}(\varrho_N)\right) = E^{FNb_{\varrho_N}}\left(E^{FNb_{\varrho_N}}(\varrho_N)\right)$

Definition 2.23. Γ be an $fnts$ and let $\varrho_n^1 \in \Gamma$. A subset $\mathbb{N}$ of Γ is fn-b_{ϱ_N}-$\mathbb{N}$bhd of ϱ_n^1, if $\exists$ a fn-b_{ϱ_N}-OS, $E \mid \varrho_n^1 \in E \subset \mathbb{N}$.

Definition 2.24. An ϱ_N be a $\subset \Gamma$, $\varrho_n^1 \in \varrho_N$ meant to be fn-b_{ϱ_N}- innermost point ϱ_N if ϱ_N is a fn-b_{ϱ_N}-$\mathbb{N}$bhd of ϱ_n^1. The entire set fn-b_{ϱ_N}- point of interior ϱ_N is fn-b_{ϱ_N}-interior ϱ_N and it is $IN_{b_{\varrho_N}}(\varrho_N)$, $IN_{b_{\varrho_N}}(\varrho_N)$ is union of $\{L: L \text{ is } fnb_{\varrho_N}OS,\ L \subset \varrho_N\}$

A ϱ_N be a section of a space. Γ We define FNb_{ϱ_N}-closure of ϱ_N to serve as a junction for all FNb_{ϱ_N}-closed sets made of ϱ_N, b_{ϱ_N} CL of $\varrho_N = \cap\{\mathrm{L}: \varrho_N \subset \mathrm{L} \in \mathrm{fnb}_{\varrho_N}(\Gamma)\}$

Definition 2.25. ϱ_N an area where a number of elements are present. Γ, an element $\varrho_n^1 \in \Gamma$ is to be b_{ϱ_N}-point of ϱ_N if $\forall$ b_{ϱ_N}-OS, Γ_1 containing ϱ_n^1, $\Gamma_1 \cap (\varrho_N - \{\varrho_N^1\}) \neq \phi$. The whole set b_{ϱ_N}-point of ϱ_N is b_{ϱ_N}-derived (briefly. $\mathrm{e}_{\mathrm{b}_{\varrho_N}}$) a bunch of ϱ_N as indicated by $\mathrm{e}_{\mathrm{b}_{\varrho_N}}(\varrho_N)$.

Definition 2.26. In any subset $\exists$ ϱ_N of a Γ, $\mathrm{E}^{\mathrm{FNb}_{\varrho_N}}(\varrho_N)$ is b_{ϱ_N} Int of $\Gamma - \varrho_N$ this will occur FNb_{ϱ_N}-exterior regarding ϱ_N.

Definition 2.27. Let $(\mathrm{F}, \Gamma_\mathrm{F})$ be an FNTS. Two never empty FNS's ϱ_{N_1} and ϱ_{N_2} of Γ are regarded as Γ^S-segregated if $\varrho_{N_1} \cap \mathrm{b}_{\varrho_N}$ CL of$(\varrho_{N_2}) = \phi_N$ $\varrho_{N_1} \cap \mathrm{b}_{\varrho_N}$ CL of$(\varrho_{N_2}) = \phi_N$ and b_{ϱ_N} CL of$(\varrho_{N_1}) \cap \varrho_{N_2} = \phi_N$. Both of these circumstances are comparable to the one condition. $\left(\varrho_{N_1} \cap \mathrm{b}_{\varrho_N}\ CL\ \text{of}(\varrho_{N_2})\right) \cup \left(\mathrm{b}_{\varrho_N}\ CL\ \text{of}(\varrho_{N_1}) \cap \varrho_{N_2}\right) = \phi_N$.

Definition 2.28. Let a FNTS be (F, Γ_F). If G is a FN subset of F, then the collection Γ^S of G is $\{G \cap U: U \in \Gamma\}$ G is referred to be a FN subspace topology on F if is a FNT on G.

Observation 2.29. FN disjoint is any two FN separated sets. FN, however, does not necessarily divide two independent sets of FN.

Theorem 2.30. A $(G, \Gamma^S(G))$ be a FNTS's FN subspace. $(\mathrm{F}, \Gamma_\mathrm{F})$, $\varrho_{N_1}, \varrho_{N_2}$ be 2 NF sets of G. Then $\varrho_{N_1}, \varrho_{N_2}$ a FN Γ^S-segregated $\Leftrightarrow$ they are FN $\Gamma^S(G)$-segregated.

Proof: By concept, $\mathrm{CL}_{\mathrm{b}_{\varrho_N}} G(\varrho_{N_1}) = \mathrm{CL}_{\mathrm{b}_{\varrho_N}} F(\varrho_{N_1}) \cap G$ and $\mathrm{CL}_{\mathrm{b}_{\varrho_N}} G(\varrho_{N_2}) = \mathrm{CL}_{\mathrm{b}_{\varrho_N}} F(\varrho_{N_2}) \cap G$.

Now $\left(\mathrm{CL}_{\mathrm{b}_{\varrho_N}} G(\varrho_{N_1}) \cap \varrho_{N_2}\right) \cup \left(\varrho_{N_1} \cap \mathrm{CL}_{\mathrm{b}_{\varrho_N}} G(\varrho_{N_2})\right) = \left(\mathrm{CL}_{\mathrm{b}_{\varrho_N}} F(\varrho_{N_1}) \cap G \cap \varrho_{N_2}\right) \cup \left(\varrho_{N_1} \cap \mathrm{CL}_{\mathrm{b}_{\varrho_N}} F(\varrho_{N_2}) \cap G\right) = \left(\mathrm{CL}_{\mathrm{b}_{\varrho_N}} F(\varrho_{N_1}) \cap \varrho_{N_2}\right) \cup \left(\varrho_{N_1} \cap \mathrm{CL}_{\mathrm{b}_{\varrho_N}} F(\varrho_{N_2})\right)$.

Hence $\left(\mathrm{CL}_{\mathrm{b}_{\varrho_N}} G(\varrho_{N_1}) \cap \varrho_{N_2}\right) \cup \left(\varrho_{N_1} \cap \mathrm{CL}_{\mathrm{b}_{\varrho_N}} G(\varrho_{N_2})\right) = \phi_N$

$\Leftrightarrow \left(\mathrm{CL}_{\mathrm{b}_{\varrho_N}} F(\varrho_{N_1}) \cap \varrho_{N_2}\right) \cup \left(\varrho_{N_1} \cap \mathrm{CL}_{\mathrm{b}_{\varrho_N}} F(\varrho_{N_2})\right) = \phi_N$, because $\varrho_{N_1}, \varrho_{N_2} \subset G$.

It follows that $\varrho_{N_1}, \varrho_{N_2}$ are FN Γ^S-segregated if and only if they are FN $\Gamma^S(G)$-segregated.

Theorem 2.31. If ϱ_{N_1} and ϱ_{N_2} are Γ^S-segregated sets of an FNTS $(\mathrm{F}, \Gamma_\mathrm{F})$ and $C_1 \subset \varrho_{N_1}$ and $C_2 \subset \varrho_{N_2}$, then C_1 and C_2 are also $\Gamma^S(G)$-segregated.

Proof: Given $C_1 \subset \varrho_{N_1} \Rightarrow \mathrm{b}_{\varrho_N}$ CL of$(C_1) \subset \mathrm{b}_{\varrho_N}$ CL of(ϱ_{N_1}) and $C_2 \subset \varrho_{N_2} \Rightarrow \mathrm{b}_{\varrho_N}$ CL of$(C_2) \subset \mathrm{b}_{\varrho_N}$ CL of(ϱ_{N_2}). Since $\varrho_{N_1} \cap \mathrm{b}_{\varrho_N}$ CL of$(\varrho_{N_2}) = \phi_N$ and b_{ϱ_N} CL of$(\varrho_{N_1}) \cap \varrho_{N_2} = \phi_N$. It follows that $C_1 \cap \mathrm{b}_{\varrho_N}$ CL of$(C_2) = \phi_N$ and b_{ϱ_N} CL of$(C_1) \cap C_2 = \phi_N$. Hence C_1 and C_2 are $\Gamma^S(G)$-segregated.

Theorem 2.32. Two FNC(FNO) sets ϱ_{N_1} and ϱ_{N_2} of an FNTS are Γ^S-segregated $\Leftrightarrow$ They don't make appropriate.

Proof: Given that any 2 Γ^S-segregated sets don't match. If ϱ_{N_1} and ϱ_{N_2} are both disjoint and FN closed, then $\varrho_{N_1} \cap \varrho_{N_2} = \phi_N$, b_{ϱ_N} CL of $(\varrho_{N_1}) = \varrho_{N_1}$ and b_{ϱ_N} CL of $(\varrho_{N_2}) = \varrho_{N_2}$. So b_{ϱ_N} CL of$(\varrho_{N_1}) \cap \varrho_{N_2} = \phi_N$ and b_{ϱ_N} CL of$(\varrho_{N_2}) \cap \varrho_{N_1} = \phi_N$ implies ϱ_{N_1} and ϱ_{N_2} are Γ^S-segregated. If ϱ_{N_1} and ϱ_{N_2} are both disjoint and FN open, then $\varrho_{N_1}(c)$ and $\varrho_{N_2}(c)$ are both FN closed so that b_{ϱ_N} CL of$\left(\varrho_{N_1}(c)\right)$ and b_{ϱ_N} CL of$\left(\varrho_{N_2}(c)\right)$.

Also $\varrho_{N_1} \cap \varrho_{N_2} = \phi_N \Rightarrow \varrho_{N_1} \subset \varrho_{N_2}(c)$ and $\varrho_{N_2} \subset \varrho_{N_1}(c) \Rightarrow b_{\varrho N}$ CL of$(\varrho_{N_1}) \subset$ $b_{\varrho N}$ CL of$\left(\varrho_{N_2}(c)\right) = \varrho_{N_2}(c)$ and $b_{\varrho N}$ CL of$(\varrho_{N_2}) \subset b_{\varrho N}$ CL of$\left(\varrho_{N_1}(c)\right) = \varrho_{N_1}(c) \Rightarrow$ $b_{\varrho N}$ CL of$(\varrho_{N_1}) \cap \varrho_{N_2} = \phi_N$ and $b_{\varrho N}$ CL of$(\varrho_{N_2}) \cap \varrho_{N_1} = \phi_N \Rightarrow \varrho_{N_1}$ and ϱ_{N_2} are Γ^S-segregated.

Theorem 2.33. Two FN disjoint sets ϱ_{N_1} and ϱ_{N_2} are Γ^S-segregated in an FNTS(F, Γ_F) $\Leftrightarrow$ they are both FNO & FNC in the FN subspace $\varrho_{N_1} \cup \varrho_{N_2}$.

Proof: Let the disjoint FN sets ϱ_{N_1} and ϱ_{N_2} be Γ^S-segregated in Γ, so that $\varrho_{N_1} \cap \mathrm{CL}_{b_{\varrho N}}\Gamma(\varrho_{N_2}) = \phi_N$ and $\varrho_{N_2} \cap \mathrm{CL}_{b_{\varrho N}}\Gamma(\varrho_{N_1}) = \phi_N$. Let $L = \varrho_{N_1} \cup \varrho_{N_2}$, $\mathrm{CL}_{b_{\varrho N}}L(\varrho_{N_1}) = \mathrm{CL}_{b_{\varrho N}}\Gamma(\varrho_{N_1}) \cap L = \mathrm{CL}_{b_{\varrho N}}\Gamma(\varrho_{N_1}) \cap$ $(\varrho_{N_1} \cup \varrho_{N_2}) = \left[\mathrm{CL}_{b_{\varrho N}}\Gamma(\varrho_{N_1}) \cap \varrho_{N_1}\right] \cup \left[\mathrm{CL}_{b_{\varrho N}}\Gamma(\varrho_{N_1}) \cap \varrho_{N_2}\right] = \varrho_{N_1} \cup \phi_N = \varrho_{N_1}$ [because $\varrho_{N_1} \subset$ $\mathrm{CL}_{b_{\varrho N}}\Gamma(\varrho_{N_1})$ and $\mathrm{CL}_{b_{\varrho N}}\Gamma(\varrho_{N_1}) \cap \varrho_{N_2} = \phi_N$]. A is FNC in the FN subspace $\varrho_{N_1} \cup \varrho_{N_2}$, by the definition of FNC. Similarly ϱ_{N_2} is FNC in $\varrho_{N_1} \cup \varrho_{N_2}$. Again $\varrho_{N_1} \cap \varrho_{N_2} = \phi_N$, they are complements of each other in L and hence they are both FNO in L. Conversely, let the disjoint FN sets ϱ_{N_1} and ϱ_{N_2} be both FNO and FNC in L . So $\varrho_{N_1} = b_{\varrho N}$ CL of $L(\varrho_{N_1}) = [b_{\varrho N}$ CL of $\Gamma(\varrho_{N_1}) \cap L] =$ $b_{\varrho N}$ CL of $\Gamma(\varrho_{N_1}) \cap (\varrho_{N_1} \cup \varrho_{N_2}) = [b_{\varrho N}$ CL of $\Gamma(\varrho_{N_1}) \cap \varrho_{N_1}] \cup [b_{\varrho N}$ CL of $\Gamma(\varrho_{N_1}) \cap \varrho_{N_2}] = \varrho_{N_1} \cup$ $[b_{\varrho N}$ CL of $\Gamma(\varrho_{N_1}) \cap \varrho_{N_2}]$ because $\varrho_{N_1} \subset b_{\varrho N}$ CL of $\Gamma(\varrho_{N_1})$ $\longrightarrow$ (1). Since $\varrho_{N_1} \cap \varrho_{N_2} = \phi_N \Rightarrow$ $\varrho_{N_1} \cap \left(b_{\varrho N}\ CL \text{ of } \Gamma(\varrho_{N_1}) \cap \varrho_{N_2}\right) = \phi_N$, it follows from (1) that is $\left(\mathrm{CL}_{b_{\varrho N}}\Gamma(\varrho_{N_1}) \cap \varrho_{N_2}\right) = \phi_N$. Similarly $(\mathrm{CL}_{b_{\varrho N}}\Gamma(\varrho_{N_2}) \cap \varrho_{N_1}) = \phi_N$. Hence ϱ_{N_1} and ϱ_{N_2} are Γ^S-segregated in Γ.

Definition 2.34. Let Γ be a FNTS. A set $Y \subset \Gamma$ is said to be $b_{\varrho N}$-Sat if for every $\gamma \in Y$ it follows $b_{\varrho N}$ CL $of(\{\gamma\}) \subset Y$. The grouping of all $b_{\varrho N}$-saturated sets in Γ, we indicate by $\mathcal{S}at^{b_{\varrho N}}(\Gamma)$.

Theorem 2.35. Let Γ, a FNTS. Then $\delta^{b_{\varrho N}}(\Gamma)$ is a whole algebraic Boolean set.

Proof. We'll demonstrate that every combination and complement of each element in $\delta^{b_{\varrho N}}(\Gamma)$ are members of $\delta^{b_{\varrho N}}(\Gamma)$. Of course, the only proof that is not trivial is the one using the complements. Let $Y \in \delta^{b_{\varrho N}}(\Gamma)$ and suppose that $b_{\varrho N}$ CL $of(\{\gamma_1\})$ does not contained in $\Gamma - Y$ for some $\gamma_1 \in \Gamma -$ Y. Then there exists $\gamma_2 \in Y$ such that $\gamma_2 \in b_{\varrho N}$ CL $of(\{\gamma_1\})$. It follows that γ_1, γ_2 possess no disjoint neighbourhoods. Then $\gamma_1 \in b_{\varrho N}$ CL $of(\{\gamma_2\})$. However, this is in conflict with the notion of $\delta^{b_{\varrho N}}(\Gamma)$ we have $b_{\varrho N}$ CL $of(\{\gamma_2\}) \subset Y$. Hence, $b_{\varrho N}$ CL $of(\{\gamma_1\}) \subset \Gamma - Y$ for every $\gamma_1 \in \Gamma - Y$, which implies $\Gamma - Y \in \delta^{b_{\varrho N}}(\Gamma)$.

Corollary 2.36. $\delta^{b_{\varrho N}}(\Gamma)$ includes each intersection and union of $b_{\varrho N}$-CS and $b_{\varrho N}$-OS's in Γ.

Definition 2.37. A function $\alpha: (\Gamma_1, \varrho_1) \to (\Gamma_2, \varrho_2)$ is referred to as

a) $b_{\varrho N}(C\#)$ if $\alpha^{-1}(Q_2)$is $b_{\varrho N}$-CS in (Γ_1, ϱ_1) for every CS Q_2 of (Γ_2, ϱ_2).

b) $b_{\varrho N}$-Totally-Continuous (briefly. $\langle\mathcal{T}^{b_{\varrho N}}\rangle C$) at a point $\gamma_1 \in \Gamma_1$ if for each open subset Q_2 in Γ_2 containing $\alpha(\gamma_1)$, there exists a $b_{\varrho N}$-clopen subset Q_1 in Γ_1 containing γ_1 such that $\alpha(Q_1) \subset Q_2$

c) $\langle\mathcal{T}^{b_{\varrho N}}\rangle C$ if it has this property at each point of Γ_1.

Theorem 2.38. The following statements are equivalent for a function $\alpha: (\Gamma_1, \varrho_1) \to (\Gamma_2, \varrho_2)$:

a) α is $\langle \mathcal{T}^{b_{\varrho N}} \rangle C$;
b) $\forall$OS, Q_2 of Γ_2 , $\alpha^{-1}(Q_2)$ is $b_{\varrho_N} CLOS$ in Γ_1;

Proof. $(a) \Rightarrow (b)$ Let Q_2 be an OS of a Γ_2 and let $\gamma \in \alpha^{-1}(Q_2)$. Since $(\gamma) \in Q_2$, by (a), $\exists$ a b_{ϱ_N}-CLOS Q_{1_γ} in Γ_1 containing γ such that $Q_{1_\gamma} \subset \alpha^{-1}(Q_2)$. We obtain $\alpha^{-1}(Q_2) = \cup_{\gamma \in \alpha^{-1}(Q_2)} Q_{1_\gamma}$. Thus, $\alpha^{-1}(Q_2)$ is b_{ϱ_N}-CLOS in Γ_1.

$(b) \Rightarrow (a)$ Clear.

Remark 2.39. Every $\langle \mathcal{T}^{b_{\varrho N}} \rangle C \Longrightarrow b_{\varrho_N}(C\#)$.

Definition 2.40. A space (Γ_1, ϱ_1) is said to be $b_{\varrho_N} < \sim S >$ if every b_{ϱ_N}-OS of Q_1 is OS in Q_1.

Remark 2.41. If a function $\alpha: (\Gamma_1, \varrho_1) \to (\Gamma_2, \varrho_2)$ is totally continuous and Q_1 is a $b_{\varrho_N} < \sim S >$, then α is $\langle \mathcal{T}^{b_{\varrho N}} \rangle C$.

Definition 2.42. An FNTS (Γ_1, ϱ_1) is said to be $b_{\varrho_N} \ll \mathfrak{Con}$ if the combination of two nonempty disjoint b_{ϱ_N}-OS cannot be expressed in writing.

Theorem 2.43. If α is a $\langle \mathcal{T}^{b_{\varrho N}} \rangle C$-function from a $b_{\varrho_N} \ll \mathfrak{Con}$-space Q_1 onto any space Q_2, then Q_2 is an indiscrete space.

Proof. If possible, suppose that Q_2 is not indiscrete. Let L be a valid OS of Γ_2 that isn't empty. Then $\alpha^{-1}(L)$ is a valid non-empty b_{ϱ_N}-CLOS of (Γ_1, ϱ_1), it is a contradiction to the fact that Γ_1 is $b_{\varrho_N} \ll \mathfrak{Con}$-space.

Theorem 2.44. The set of all points $\gamma \in X$ wherein a function $\alpha: (\Gamma_1, \varrho_1) \to (\Gamma_2, \varrho_2)$ is not $\langle \mathcal{T}^{b_{\varrho N}} \rangle C$ is the $\cup$ of $Fr_{\#b_{\varrho_N}}$ of the open sets' inverted images that include $\alpha(\gamma)$.

Proof. Suppose that α is not $\langle \mathcal{T}^{b_{\varrho N}} \rangle C$ at $\gamma \in Q_1$ $\Longrightarrow \exists$ an OS Q_2 of Γ_2 containing $\alpha(\gamma)$ such that $\alpha(Q_1)$ is not contained in Q_2 for each $Q_1 \in b_{\varrho_N} O(\Gamma_1)$ containing γ and hence $\gamma \in b_{\varrho_N}\ CL\ of(\Gamma_1 \backslash \alpha^{-1}(Q_2))$. On the other hand, $\Gamma_1 \in \alpha^{-1}(Q_2) \subset b_{\varrho_N}\ CL\ of(f^{-1}(Q_2))$ and hence $\Gamma_1 \in Fr_{\#b_{\varrho_N}}(\alpha^{-1}(Q_2))$.

Conversely, suppose that α is $\langle \mathcal{T}^{b_{\varrho N}} \rangle C$ at $\gamma \in \Gamma_1$ and let Q_2 be an OS of Γ_2 containing $\alpha(\gamma) \Longrightarrow \exists\ Q_1 \in b_{\varrho_N} O(\Gamma_1)$ containing γ such that $Q_1 \subset \alpha^{-1}(Q_2)$. Hence $\gamma \in b_{\varrho_N}\ INT\ of(\alpha^{-1}(Q_2))$. Therefore, $\Gamma_1 \in Fr_{\#b_{\varrho_N}}(\alpha^{-1}(Q_2))$ for each open set Q_2 of Γ_2 containing $\alpha(\gamma)$.

Conclusion: We have given an introduction to fn-$Fr_{\#b_{\varrho_N}}$, including the ideas of closed and open sets. We examined features in fn-$\beta d^{\#b_{\varrho N}}$ and fn-$\mathrm{e}_{b_{\varrho_N}}(\varrho_N)$, and we evaluated some of their features in fn-topological spaces to provide an idea of the findings we gained by adding the concept of fn-b_{ϱ_N}OS. We have produced a comparisons between the provided concepts of border, exterior, and derived. Additionally, we studied and identified the features of $\langle \mathcal{T}^{b_{\varrho N}} \rangle C$-functions and Γ^S-

segregated functions. In the future, we want to investigate more findings derived from the aforementioned principles and endeavour to provide applications.

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Measurement of the effectiveness of an educational program inspired by indigenous knowledge for forest management, applied to Forest Engineering students at the National University of Central Peru, using the neutrosophic 2-tuple linguistic method

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Abstract. The "guide to Asháninka knowledge for forest management" is a document prepared by academics and scientists from the National University of Central Peru (NUCP) in the Faculty of Forestry Engineering, which includes the knowledge of this indigenous people about forest nature. Despite being ancestral knowledge, we believe that it can serve as a complement to the scientific study of the subject since it is based on knowledge accumulated over centuries of experience. The main purpose of this article is to determine the influence of the "Ashaninka knowledge guide for forest management" on the learning level of NUCP Forest Engineering students. To accomplish these objectives, Asháninka knowledge was compiled, and the learning guide was designed and applied to the students. 36 students from the faculty were taken as part of the study, to whom two tests were administered, one before studying the guide and another after. The tests consisted of questionnaires on knowledge about the guide and the forests. A neutrosophic linguistic scale was used for respondents to answer both tests. The advantage of this methodology is that experts could evaluate more easily with the help of natural language, in addition to the fact that the incorporation of neutrosophy helps to take into account indeterminacy and therefore there is more accuracy. Specifically, the Neutrosophic 2-tuple linguistic method was used. The results were converted to crisp numbers and the evaluations of the two tests were compared with the help of the Wilcoxon test.

Keywords: Knowledge of indigenous peoples, Asháninka people, Forest Engineering, neutrosophic 2-tuples linguistic method, Wilcoxon range test.

1 Introduction

The United Nations Conference on Environment and Development (UNCED, 1992), held in Brazil, was a defining moment for the promotion of the rights of indigenous peoples about the environment.

A series of legal standards were adopted, such as the Rio Declaration, the Convention on Biological Diversity (CBD), and Agenda 21, and international legal standards were established to protect the traditional knowledge and practices of indigenous peoples in matters of management and environmental Conservation.

Indigenous peoples are the agents of the greatest diversity in the world, they host not only biological diversity but also cultural, linguistic, and landscape diversity, etc. Their modes of existence vary from one place to another, of the 6,000 cultures that exist in the world (approximate figure), 4,000 to 5,000 cultures are indigenous. 6,000 languages in the world are spoken by indigenous peoples, most importantly, most of the Earth's biological diversity is inhabited by these peoples. The countries with the greatest indigenous presence are Australia, Brazil, China, Colombia, Ecuador, the United States of America, the Philippines, India, Indonesia, Madagascar, and Malaysia.

This research is intended to incorporate ancestral knowledge in the training and professional practice of the Forest Engineer, in a learning guide, with the research problem consisting of determining how the "Asháninkas Knowledge (AK) Guide" influences the management of forests in the learning level of Forest Engineering students at the National University of Central Peru (NUCP).

AK and scientists do not exclude each other but rather complement each other. The use of Asháninka knowledge in forest management contributes to improving the implementation of a project, learning module, workshop, or guide by providing valuable information about the local context, both at the landscape and ecosystem levels. Furthermore, it may be able to increase long-term sustainability, while strengthening the self-esteem of communities, helping them participate in local and national development.

The incorporation of ancestral knowledge in the curricular plans of professional and technical careers in our country would have positive impacts, such as changes in attitude, thoughts, and environmental awareness. At the National University of Central Peru, two faculties have professional courses in Forest Engineering, but neither of them has incorporated a module, guide, or course related to ancestral knowledge. For the reasons mentioned, the general motivation of the research is: to determine the influence of the "AK guide for forest management" on the learning level of NUCP Forest Engineering students, for which, it was taken into account the following specific objectives: Compile information from AK focused on forest management; design the "AK guide to forest management"; and apply the guide to NUCP Forest Engineering students.

To quantitatively measure the result of this learning methodology there are some challenges, first of all, the student's learning must be measured, which is why we proposed to carry out a test where the students show the learning acquired, first before studying the guide and a test for later. The second challenge is to measure results that are subjective and contain indeterminacy, for this, we use the neutrosophic 2-tuple linguistic model [1-3]. Finally, to process the data we measure the general situation with the support of the non-parametric Wilcoxon sign test [4, 5].

The neutrosophic method generalizes the well-known 2-tuple linguistic method from the fuzzy outline to the neutrosophic framework [6-8]. The advantages of this model are its simplicity, effectiveness, and veracity, since the respondent expresses his or her opinion on a linguistic measurement scale, which is understandable to everyone. This method is an example of Computing with Words (CWW) introduced by L. Zadeh, where calculation with words is preferred over calculation with numbers, which corresponds to the usual way that humans evaluate [9-11]. Additionally, the neutrosophic framework helps to incorporate indeterminacy as part of the evaluation.

The Wilcoxon signed-rank test is a nonparametric test to compare the mean rank of two related samples and determine if there are differences between them [4, 5]. It is used as an alternative to the Student t-test when the normality of these samples cannot be assumed. In the article, we convert the result of the 2-tuple linguistic methods into numerical crisp values and we apply the Wilcoxon sign test on them.

This paper is divided into a Materials and Methods section, where the basic notions of the neutrosophic 2-tuple linguistic method are explained. Section 3 consists of the presentation of the results obtained in this study. The final section contains the Conclusions. An Annex is also included with the applied tests for the students and the data collected.

2 Materials and Methods

This section contains the basic concepts of the neutrosophic 2-tuple linguistic model and notions of the Wilcoxon rank test.

Definition 1 ([6]). Let $S = \{s_0, s_1, \dots, s_g\}$ be a set of linguistic terms and $\beta \epsilon [0, g]$ is a value that represents the result of a symbolic operation, then the *linguistic 2-tuple* that expresses the information equivalent to β, is obtained using the following function:

$\Delta: [0, g] \to S \times [-0.5, 0.5)$
$\Delta(\beta) = (s_i, \alpha)$ (1)

Where s_i is such that $i = \text{round}(\beta)$ and $\alpha = \beta - i$, $\alpha \in [-0.5, 0.5)$ and “round” is the usual rounding operator, s_i is the index label closest to β and α is the value of the *symbolic translation*.

It should be noted that $\Delta^{-1}: \langle S \rangle \to [0, g]$ is defined as $\Delta^{-1}(s_i, \alpha) = i + \alpha$. Thus, a linguistic 2-tuple $\langle S \rangle$ is identified with its numerical value in $[0, g]$.

Suppose that $S = \{s_0, \dots, s_g\}$ is a *2-Tuple Linguistic Set* (2TLS) with odd cardinality g+1. It is defined for $(s_T, a), (s_I, b), (s_F, c) \in L$ and a, b, c ∈[0, g], where $(s_T, a), (s_I, b), (s_F, c) \in L$ independently express the degree of truthfulness, indeterminacy, and falsehood by 2TLS. *2-Tuple Linguistic Neutrosophic Number* (2TLNN) is defined as follows ([1-3, 12-16]):

$l_j = \{(s_{T_j}, a), (s_{I_j}, b), (s_{F_j}, c)\}$ (2)

Where $0 \le \Delta^{-1}(s_{T_j}, a) \le g$, $0 \le \Delta^{-1}(s_{I_j}, b) \le g$, $0 \le \Delta^{-1}(s_{F_j}, c) \le g$, and $0 \le \Delta^{-1}(s_{T_j}, a) + \Delta^{-1}(s_{I_j}, b) + \Delta^{-1}(s_{F_j}, c) \le 3g$.

The *scoring* and *accuracy functions* allow us to rank 2TLNN.

Let $l_1 = \{(s_{T_1}, a), (s_{I_1}, b), (s_{F_1}, c)\}$ be a 2TLNN in L, the scoring and accuracy functions in l_1 are defined as follows, respectively:

$$\mathcal{S}(l_1) = \Delta\left\{\frac{2g + \Delta^{-1}(s_{T_1}, a) - \Delta^{-1}(s_{I_1}, b) - \Delta^{-1}(s_{F_1}, c)}{3}\right\}, \ \Delta^{-1}(\mathcal{S}(l_1)) \in [0, g] \qquad (3)$$

$$H(l_1) = \Delta\left\{\frac{g + \Delta^{-1}(s_{T_1}, a) - \Delta^{-1}(s_{F_1}, c)}{2}\right\}, \ \Delta^{-1}(H(l_1)) \in [0, g] \qquad (4)$$

Formula of Wilcoxon test is Equation 5 ([4, 5]):

$$W = \sum_{i=1}^{N_r} [sgn(x_{2,i} - x_{1,i}) \cdot R_i] \qquad (5)$$

Where:
W: is the test statistic,
N_r: is the size of the sample, excluding values $x_1 = x_2$,
sgn: is the sign function,
$x_{1,i}, x_{2,i}$: are the pairs of related ranges of two different distributions,
R_i: range i.

This test is used to determine if there are differences between both populations. The objective is to know if there is a significant improvement in the student's knowledge of forest engineering science when they study the Asháninka knowledge guide. The null hypothesis of the test is that the mean difference between both populations is 0, while the alternative hypothesis is that there is a significant difference between both and therefore both populations are different.

3 Results of the study

Next, we describe the method followed to carry out the present study. Firstly, we define the linguistic scale on which we will base the tests. This is the one shown below:

1. An initial linguistic scale is defined on which the evaluations will be based, this is: $S = \{"Deficient", "Regular", "Good", "Very\ Good", "Excellent"\}$ or equivalently $S = \{s_0, s_1, s_2, s_3, s_4\}$, this grade is the one that will be given for each of the aspects to be evaluated in the test to each of the evaluated students. That is, each of the 36 students e_i $i \in \{1, 2, \dots, 36\}$ answers a questionnaire of 13 questions. The evaluator gives a grade for each student according to a triple of linguistic values on the scale S, as shown below:

$\rho_{ij} = (s_{\rho ijT}, s_{\rho ijI}, s_{\rho ijF})$ for the pre-test and $\theta_{ij} = (s_{\theta ijT}, s_{\theta ijI}, s_{\theta ijF})$ for the post-test, where the elements of the triples belong to the scale S such that $s_{\rho ijT}$ and $s_{\theta ijT}$ are the linguistic values of truthfulness, $s_{\rho ijI}$ and $s_{\theta ijI}$ are the linguistic values of indeterminacy, and $s_{\rho ijF}$ and $s_{\theta ijF}$ are the linguistic values of falsity.

2. Each student e_i has a final grade $\rho_i = (s_{\rho iT}, s_{\rho iI}, s_{\rho iF})$ and $\theta_i = (s_{\theta iT}, s_{\theta iI}, s_{\theta iF})$, obtained as the arithmetic mean of their results according to the assessment given by the experts regarding certain aspects of the answers. Details of the surveys appear in the Annex.

   That is, these triples are obtained from values $\beta_i^{\rho} = (\beta_{iT}^{\rho}, \beta_{iI}^{\rho}, \beta_{iF}^{\rho})$, $\beta_i^{\theta} = (\beta_{iT}^{\theta}, \beta_{iI}^{\theta}, \beta_{iF}^{\theta})$ where:

   $\beta_{iT}^{\rho} = \frac{\sum_{j=1}^{9} k_{ijT}^{\rho}}{9}$, $\beta_{iT}^{b} = \frac{\sum_{j=1}^{9} k_{ijT}^{\theta}}{9}$;

   $\beta_{iI}^{\rho} = \frac{\sum_{j=1}^{9} k_{ijI}^{\rho}}{9}$, $\beta_{iI}^{b} = \frac{\sum_{j=1}^{9} k_{ijI}^{\theta}}{9}$;

   $\beta_{iF}^{\rho} = \frac{\sum_{j=1}^{9} k_{ijF}^{\rho}}{9}$, $\beta_{iF}^{b} = \frac{\sum_{j=1}^{9} k_{ijF}^{\theta}}{9}$;

   Where:

   $k_{ijT}^{\rho}, k_{ijT}^{\theta} \in \{0, 1, 2, 3, 4\}$ are the indices of truthfulness evaluations for each answer given in the survey by the student e_i, in the pre-test and post-test, respectively.

   $k_{ijI}^{\rho}, k_{ijI}^{\theta} \in \{0, 1, 2, 3, 4\}$ are the indices of the indeterminacy evaluations for each answer given in the survey by the student e_i, in the pre-test and post-test, respectively.

   $k_{ijF}^{\rho}, k_{ijF}^{\theta} \in \{0, 1, 2, 3, 4\}$ are the indices of falsehood evaluations for each answer given in the survey by the student e_i, in the pre-test and post-test, respectively.

3. For each $\beta_i^{\rho} = (\beta_{iT}^{\rho}, \beta_{iI}^{\rho}, \beta_{iF}^{\rho})$y $\beta_i^{\theta} = (\beta_{iT}^{\theta}, \beta_{iI}^{\theta}, \beta_{iF}^{\theta})$ we get a triple symbolic translations denoted by $\alpha_i^{\rho} = (\alpha_{iT}^{\rho}, \alpha_{iI}^{\rho}, \alpha_{iF}^{\rho})$ and $\alpha_i^{\theta} = (\alpha_{iT}^{\theta}, \alpha_{iI}^{\theta}, \alpha_{iF}^{\theta})$. This is how we obtain the pairs $l_i^{\rho} = ((s_{\rho iT}, \alpha_{iT}^{\rho}), (s_{\rho iI}, \alpha_{iI}^{\rho}), (s_{\rho iF}, \alpha_{iF}^{\rho}))$ and $l_i^{\theta} = ((s_{\theta iT}, \alpha_{iT}^{\theta}), (s_{\theta iI}, \alpha_{iI}^{\theta}), (s_{\theta iF}, \alpha_{iF}^{\theta}))$.

4. $x_i = \Delta^{-1}(\mathcal{S}(l_i^{\rho}))$ and $y_i = \Delta^{-1}(\mathcal{S}(l_i^{\theta}))$ are obtained.

5. The Wilcoxon test is applied to determine if both populations are equal. If the p-value satisfies $p \leq 0.05$ then the null hypothesis is rejected, which means that there is a significant improvement in the students' mastery of the knowledge that appears in the guide.

   Otherwise, it is interpreted that there is no significant improvement after studying the guide.

Next, we present the results obtained from the calculations.

Once the survey questions were answered by each of the students, an expert on the subject who is a professor at the Faculty of Forestry Engineering in charge of these topics was asked to evaluate the students' results. The nine criteria to measure the results of the survey are shown in Table 1.

Criterion to evaluate	Explanation	Deficient	Regular	Good	Very good	Excellent
1. CLARITY	Respond in an appropriate language		To be filled out by the interviewer			
2. OBJECTIVITY	Responses are expressed in observable behaviors		To be filled out by the interviewer			
3. CURRENT NEWS	The answers are appropriate to the advancement of science and technology		To be filled out by the interviewer			
4. ORGANIZATION	There is a logical organization in the answers		To be filled out by the interviewer			
5. SUFFICIENCY	Understand aspects of quantity and quality		To be filled out by the interviewer			
6. INTENTIONALITY	Suitable for im-		To be filled out by the interviewer			

	provement and attitudes towards environmental conservation.	
7. CONSISTENCY	Based on theoretical–scientific aspects of Forest Sciences	To be filled out by the interviewer
8. COHERENCE	There is consistency between the answers to each of the questions asked	To be filled out by the interviewer
9. METHODOLOGY	The strategy responds to the purpose of the diagnosis	To be filled out by the interviewer

Table 1: Criteria to measure in student's evaluation.

For greater ease for the evaluator regarding the use of the neutrosophic 2-tuples, a triple was suggested for each type of evaluation, although he had the possibility of changing each of the values if he considered it necessary.

Evaluation/Triple	(s_T, a)	(s_I, b)	(s_F, c)
Deficient	$(s_0, 0)$	$(s_0, 0)$	$(s_4, 0)$
Regular	$(s_2, 0)$	$(s_2, 0)$	$(s_2, 0)$
Good	$(s_3, 0)$	$(s_1, 0)$	$(s_1, 0)$
Very good	$(s_3, 0)$	$(s_0, 0)$	$(s_0, 0)$
Excellent	$(s_4, 0)$	$(s_0, 0)$	$(s_0, 0)$

Table 2: Values suggested to the evaluator as a linguistic triple.

Figures 1 and 2 show the absolute frequency of the responses, before and after the students' training, respectively.

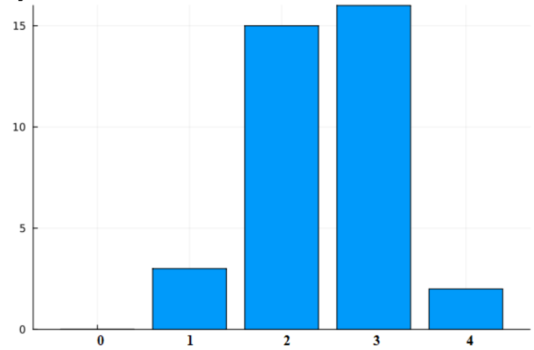

Figure 1: Bar graph with the results of the final evaluative indices in the pre-test. The abscissa shows the indices and the ordinate shows the absolute frequency.

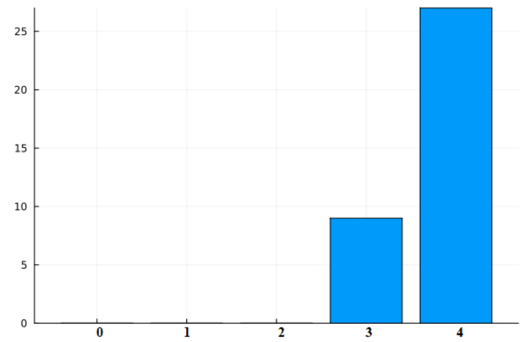

Figure 2: Bar graph with the results of the final evaluative indices in the post-test. The abscissa shows the indices and the ordinate shows the absolute frequency.

According to the results of Figures 1 and 2 the shift in the distribution of frequencies from the pre-test to the post-test, with a higher concentration at the best performance level after the training, indicates that the training program successfully enhanced the students' abilities. This can be seen as a positive outcome of the training intervention.

. To determine if this improvement is significant, we applied the Wilcoxon test, the results of which were the following, according to Table 3:

Z	-5,169[b]
Asymptotic sig. (bilateral)	0.000

b. Wilcoxon signed rank test

c. It is based on negative ranges.

Table 3: Wilcoxon non-parametric test, for related samples, of the pre-test and post-test scores.

In Table 3, from the non-parametric Wilcoxon statistical test, it is observed that the p-value is $0.00 < 0.05$, allowing us to reject H_0 of the equality of grades means, the obtained grades by the students in the pre- and post-tests (H_0: $\mu_1 = \mu_2$), consequently, there is a highly significant difference in the means of the grades or scores obtained in the pre- and post-test. The post-test group scores are higher than the pre-test scores. Based on the obtained results, consequently, there is statistical evidence to approve Ha (Ha: $\mu_2 > \mu_1$), so, there is a highly significant influence of the "AK guide for forest management" on the learning level of the NUCP Forest Engineering students, after their application.

Conclusion

The "AK guide to forest management" is a pedagogical instrument that collects the knowledge of the Peruvian indigenous people about the vegetation and life in these ecosystems. With this paper we wanted to demonstrate the validity of this knowledge, even today, and that it can complement current scientific knowledge. In the article, a test was carried out on the guide to a group of 36 Forest Engineering students, who studied it. The results of the pre-test were compared with the post-test using the Wilcoxon method of paired samples and the conclusion was that there is a significant improvement in the scientific knowledge of the students who studied the guide. That is why it is considered a valuable teaching instrument for the Forest Engineering career at the National University of Central Peru.

To ensure greater reliability in the results, we used a linguistic scale for the expert to carry out the evaluation, since human beings find it easier to evaluate in linguistic terms than in numbers. This principle was established by Zadeh who called this Computing with words rather than numbers, and an example of a technique is the 2-tuples linguistic method. In our case, we decided to gain greater accuracy with the use of the neutrosophic 2-tuple that allows the evaluator to give linguistic values not only for truthfulness, but also for indeterminacy and falsity, which allows the expert to express his opinion more accurately, and also incorporate inconsistency, contradictions, and ignorance.

Further research is encouraged to refine the neutrosophic 2-tuple linguistic method, potentially incorporating advanced computational techniques such as machine learning to analyze linguistic data more effectively. This could lead to the development of more sophisticated evaluation tools that can capture the complexity and nuance of human judgments in educational assessments.

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Annex

The Annex contains the results of the data collection and the surveys applied.

Student	Pre-test (x_i)	Post-test (y_i)
e_1	3.4	4
e_2	2	2.8
e_3	3.4	4
e_4	3.2	3.8
e_5	1.4	3.6
e_6	2.4	4
e_7	3	4
e_8	3.2	3.8
e_9	3.2	3.8
e_{10}	1.8	2.8
e_{11}	2.6	3.8
e_{12}	2.2	3.8
e_{13}	0.8	3
e_{14}	3	3.4
e_{15}	2.4	4
e_{16}	2.2	3.4

e_{17}	1.8	3.4
e_{18}	1.4	2.6
e_{19}	3.8	4
e_{20}	4	4
e_{21}	2.8	3.8
e_{22}	1.8	3.6
e_{23}	2.2	2.6
e_{24}	3	4
e_{25}	2.4	3.8
e_{26}	2.4	3.8
e_{27}	3	3.8
e_{28}	3	3.8
e_{29}	2.2	3.4
e_{30}	3.2	4
e_{31}	3.4	4
e_{32}	3	3.6
e_{33}	2.2	4
e_{34}	2.2	4
e_{35}	2.2	4
e_{36}	3	4

Table 4: Values of x_i and y_i obtained by each student.

Table 5 shows the questionnaire applied for the pre-test and post-test.

Format 01.
DIAGNOSTIC EVALUATION QUESTIONNAIRE
ASHÁNINKAS KNOWLEDGE GUIDE FOR FOREST MANAGEMENT APPLIED AT NUCP FOREST ENGINEERING STUDENTS"

Fisrt and Second names: Cycle:
Career:

1. Have you heard about ancestral knowledge?
   Yes /No
   If your answer is Yes. Mention the ancestral knowledge that you know
   …………………………………………………………………………………………………… …………
   ……………………………………………… ……. ……………………………………………………………
2. Ashaninka knowledge? Yes /No
   If your answer is Yes. Mention the Asháninka knowledge that you know.
   ………
   ……………………… …………………………………………………………………………………
3. Is Asháninka knowledge important for Sustainable Forest Management? Yes /No
   If your answer is Yes. Comment:
   ………
   ……………………… ……………………………………………………………………………………
4. Could Asháninka's knowledge be included in the first stage "negotiation for forest exploitation", required by the competent authority, now SERFOR? Yes /No
5. Forestry exploitation is contemplated in the Forestry and Wildlife Law No. 29763, taking into account the sustainable forest management of forests and wildlife. Should Asháninka knowledge be incorporated into activities such as planning, camp construction, laying down, sectioning, coding, and transportation? Yes /No
6. If your answer is Yes, in what activities should AKs be incorporated? Correct/incorrect
   a) Planning,
   b) Construction of camps,
   c) Lying down,
   d) Sectioned,
   e) Coding,
   f) Transportation,
   g) All the mentioned,
   h) None
7. Is there a connection or link between biotic and abiotic factors and humans? Yes /No.

8. If your answer is Yes, do you affirm that AKs are born and preserved from this interconnection? Yes /No If your answer is Yes, please comment on it …… ……
9. Why do the Asháninkas value their knowledge and apply it in their daily lives? Mark the correct answer. a) Because they know that they coexist with nature and through knowledge they could carry out activities with a sustainable approach that allows them to conserve and preserve their resources. b) They are learned from generation to generation c) For them, nature and all living beings have an owner and they must ask permission to use their resources, this limits their consumption. d) All are valid
10. The productive activities (hunting, gathering, fishing) of the Asháninkas are carried out for subsistence. With forestry extraction their poverty problems should end, however, the opposite is true. The causes are: Write F if false and T if true. a) Deterioration of their ecosystems due to forestry extraction activities b) Loss of the AK, their worldview being forgotten c) Malpractice of the Forest Engineer d) Inefficient legislation e) AKs are not included in the study curricula of Forest Engineering
11. Is the Assembly Minutes a determining document for the commercialization of forestry products? Yes /No If your answer is Yes. Comment on the importance of the Communal Assembly Act for the Asháninkas ………………………………………………………………………………… ……………………… ……
12. Is the role that community leaders play in contracting with logging companies decisive? Yes /No If your answer is Yes, what is your opinion about it? …… ……………………… ………………………………………………………………………………
13. Is it necessary to revalue AK, taking into account that communities sell their forest resources at negligible prices? Yes /No If your answer is Yes, say Why? …… ……………………… ………………………………………………………………………………

Table 5: Questionnaire applied for the pre-test and post-test.

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Modeling Influenced Criteria in Classifiers' Imbalanced Challenges Based on TrSS Bolstered by The Vague Nature of Neutrosophic Theory

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Abstract: Because of the advancements in technology, classification learning has become an essential activity in today's environment. Unfortunately, through the classification process, we noticed that the classifiers are unable to deal with the imbalanced data, which indicates there are many more instances (majority instances) in one class than in another. Identifying an appropriate classifier among the various candidates is a time-consuming and complex effort. Improper selection can hinder the classification model's ability to provide the right outcomes. Also, this operation requires preference among a set of alternatives by a set of criteria. Hence, multi-criteria decision-making (MCDM) methodology is the appropriate methodology can deploy in this problem. Accordingly, we applied MCDM and supported it through harnessing neurotrophic theory as motivators in uncertainty circumstances. Single value Neutrosophic sets (SVNSs) are applied as branch of Neutrosophic theory for evaluating and ranks classifiers and allows experts to select the best classifier So, to select the best classifier (alternative), we use MCDM method called Multi-Attributive Ideal-Real Comparative Analysis (MAIRAC) and the criteria weight calculation method called Stepwise Weight Assessment Ratio Analysis (SWARA) where these methods consider single-value neutrosophic sets (SVNSs) to improve and boost these techniques in uncertain scenarios. All these methods are applied after modeling criteria and its sub-criteria through a novel technique is Tree Soft Sets (TrSS). Ultimately, the findings of leveraging these techniques indicated that the hybrid multi-criteria meta-learner (HML)-based classifier is the best classifier compared to the other compared models.

Keywords: Neutrosophic theory; Multi-Criteria Decision Making; Class Imbalance; Meta-Learner; Ranking Classifiers; Single Values Neutrosophic Sets, Tree Soft Set (Trss).

1. Introduction

Currently, Artificial Intelligence (AI) techniques have been applied in several spheres. As in [1] where AI techniques are leveraged in healthcare, Fiscal fraud [2], and agriculture[3]. As well, machine learning (ML) techniques subset of AI are gleaning valuable knowledge from massive, complicated, diverse, and hierarchical data[4]. Also, ML techniques can be used as a classifier. Just like [5] described classification as ML techniques wherein a computer program learns from historical data and then applies that knowledge to forecast the class label for data that hasn't yet been observed. These techniques are represented in k Nearest Neighbors (KNN), Support Vector Machines (SVM), and random forest (RF) [5]. Moreover, Johnson et al.[6] affirmed that the majority or minority class is forecasted using binary classification. Besides that [7] categorized data in binary classification into balanced or imbalanced. And [8] demonstrated that the majority of classifiers travail optimally when the response variable's distribution in the dataset is balanced. In spite of that, the techniques mentioned in [5] encounter significant difficulties due to imbalanced data. Due to [9] one class has more instances than the other classes in imbalanced data , and the distribution of classes is skewed toward that class. From perspective of [10] positive instances are typically referred to as the minority class, whereas negative instances are typically called the majority class. Also, [11] indicated that in handling an imbalanced hurdle, the conventional algorithms exhibit a bias in favor of the dominant class.

Hence, [12] exhibited that class imbalance is still a perplexing problem that needs further study to be properly understood and skillfully managed. All that motivated[13] for developing algorithms or techniques that are very dependable and efficient is essential to properly addressing the problems brought on by the imbalanced datasets.

As per the prior literature [13, 14], several techniques have been suggested to tackle the problem of class disparity, and these may be roughly categorized into (1) algorithm-level methods [15]included cost-sensitive learning which employs the expenses of incorrect classifying samples and make an effort to improve the classifiers' favorability for the minority class by adding various cost variables into the algorithms. (2) sampling methods [15] which encompasses over sampling, random under sampling, synthetic minority oversampling technique (SMOTE), and edited nearest neighbour.(3) ensemble learning, this technique applied boosted based methods, pre-processing ensemble, and boosted imbalanced data solving toward endeavor to improve the unbalanced data classification's accuracy by fusing many classifiers to produce a novel, more potent classifier. Additionally, Chamlal et al. [13] proposed Hybrid Multi-criteria Meta-learner (HML) which includes an ensemble-based meta-learner component and a multi-objective optimization component as its two primary parts.

General speaking, selection of optimal and suitable classifier for treating with imbalanced data amongst these techniques is crucial process.

1.1 Motivation of Study

According to the surveys conducted for prior studies. From perspective of [13] it’s difficult to foresee the unpredictable, notably when tackling the problem of class imbalance, which occurs when the training data's class distribution is biased in favor of one particular class. In the same vein [16] demonstrated that the majority and minority samples are included in the imbalance datasets. Comparatively speaking, there are much less sample instances in the minority class than in the majority class. Accordingly, severe skews in the distribution of classes and inadequate rendition of specific data are persistent challenges in many domains as medicine [17], predicting defects for software[18],and in financial services [19].Hence, [20] stated that the performance of conventional classifiers may suffer when there is an imbalanced distribution of classes in a dataset.

Another aspect discussed by [21] in classification issues, depending on only a single criteria is widely used for evaluation. However, the evaluation of only one aspect may mis select the best performance classifier. To select the best performance classifier, several evaluation criteria, including proficiency, time consuming, uniformity, and others, need to be utilized. The multicriteria evaluation aims to achieve a balance between these criteria instead of depending on only one criterion [22] . So, we need an efficient multi-criteria decision-making method that assesses and ranks classifiers and allows experts to select the best classifier for their applications by using the previously mentioned criteria.

1.2 Contribution

Herein, we are evaluating the optimal classifier based on a set of criteria. For conducting this process, we are leveraging MCDM techniques which have ability to treat with such problems. Especially, Stepwise Weight Assessment Ratio Analysis (SWARA) for obtaining criteria's weights. Also, Multi-Attributive Ideal-Real Comparative Analysis (MAIRAC) is applied for ranking the alternatives of classifiers and select optimal classifier.

Neutrosophic theory is deploying in this study and contribute to MCDM techniques for bolstering and supporting expert in ambiguity situations as incomplete data and uncertainty [23]. Due to, ability of this theory to measure membership function as truth (T), also non-membership function false (F) whilst take into consideration indeterminacy (I). Thereby, single value neutrosophic sets (SVNSs) as type of Neutrosophic is implementing in evaluation process.

Also, Tree soft set (TrSS) is leveraging in this problem to model the identified criteria and its sub-criteria in set of nodes which resident into set of levels. TrSS is introduced Smarandache [24] who is founder of this approach as well as introduced Neutrosophic theory.

1.3 Study Outline

This study is organized into a set of sections; each section exhibits the benefits of our study and the followed steps toward achieving study's objectives.

Section one: illustrated the main idea of our study, motivations and the main contributions which are provided through our study. For completing our objectives' study, we conducted survey for prior techniques and studies in section 2. Through the conducted surveys, we determined the effective techniques to treat our problem through conducting SDMM. To validate the accuracy of this model, it forced us to apply the constructed model on real case study in section four. Finally, we recorded the results and conclusions which we reached in this study research through section five.

2. Literature review

In this section we exhibited the earlier studies which related to our study's objectives. Therefore, this section divides into set of sub-sections. Each sub-section introduces previous studies and techniques have been harnessed.

2.1 Around classification of imbalanced data

Several strategies [22, 25] have been mentioned by researchers to deal with the imbalanced data problem. These strategies can be categorized into data-level approaches, algorithm-level approaches, cost-sensitive strategies, and boosting strategies. The first strategy rebalances the data, utilizing the resampling technique to improve accuracy. In the second strategy, the standard classifiers are biased towards the minor class by adjusting their methodology. The third strategy gathers data-level and algorithm-level strategies by giving higher costs to positive samples and decreasing these costs. The fourth strategy combines multiple learners and then aggregates their predictions.

The Random Over-Sampling (ROS)[26] technique is the simplest over-sampling approach that randomly generates minor instances from the imbalanced data set until the class distribution more balanced. The Random Under-Sampling Strategy (RUS) is the simplest under-sampling approach that picks negative examples at random and discards them from the dataset until the class distribution is balanced. Wang et al. [27] used the SMOTE method to synthesize data with the Tomek Links technique to eliminate some of the majority of cases. Wang et al. [28] introduced Focal-XGBoost and Weighted-XGBoost, which blend the XGBoost algorithm with focal and weighted strategies to cope with imbalanced classification issues by minimizing the significance of well-classified cases. Ref [29] applied integrated optimization and sampling presumptions to address class imbalances. The researchers employed simulated annealing to choose the optimal subset of negative class records based on F-score. The under-sampled training set was then trained using several core classifiers, including SVM, KNN, DA, and DT. Boosted Random Forest [30] is constructed from two components: the boosting technique and the random forest classifier, in which each decision tree in the forest is created based on misclassification penalties. HICD [31] depends on data density; it is a hybrid, unbalanced classification model. It creates subsets for various instance classes, builds ensemble models, splits the data space using a density-based resampling technique, and chooses suitable models according to the instance distribution. Liu et al. [32] developed the fuzzy SVM algorithm and began dealing with borderline noise by employing a new strategy of measuring distance and gaussian fuzzy to decrease the influence of this noise. Zhang et al. [33]trained several classifiers on balanced subsets obtained by POS (perturbation-based oversampling) and used majority voting for ensemble learning. Barua et al. [34]presented the MWMOTE technique, which addresses imbalanced learning through determining key minority class instances, providing penalties based on proximity to the majority class, and creating synthetic instances from the minority class. Choudhary et al. [35] provided a method that employs a fuzzy clustering technique to segment the complex imbalance challenge into smaller issues before allocating ratings to each sub-classifier for a majority vote.

2.2 Influenced Neutrosophic Theory in Evaluation Process

In the recent studies, Neutrosophic with its various types are emerged in various vital domains toward supporting the stakeholders with valuable decisions in anxiety ambience. As Elhenawy et al. [36] employed neutrosophic especially, Triangular Neutrosophic Sets (TriNSs) for weighting criteria which contributed to evaluate alternatives of Metaverse in healthcare domain.Also, SVNSs are merged with TrSS in [37] for modeling criteria and its sub-criteria based blockchain technology (BCT).Portfolio selection model is established in [38] through adopting neutrosophic theory and entropy objective function and this model is applied on real time case study to validate the model accuracy. The optimal warehouse management software is selected through utilizing various MCDM with SVNSs for evaluating warehouse management software programs and select optimal toward achieving sustainable logistics systems [39]

As mentioned previously, we need efficient MCDM techniques to be implemented under authority of SVNSs and modeling the identified criteria and sub-criteria using TrSS which applied to model the criteria and sub-criteria in various applications as indicate the best location for solar hydrogen production [40]. Also, TrSS is leveraged in [41] for recommending the secure enterprise based on modeling blockchain criteria and its sub-criteria using TrSS.

Overall, These techniques are leveraged for constructing a robust decision-making model for assessing and ranking classifiers and allow experts to select the best classifier. So, we will use HML and MESA (boost ensemble imbalanced learning using a meta-sampler) with other familiar solutions based on sampling and cost-sensitive techniques for selecting the best classifier.

3. Soft Decision-Making Model (SDMM)

The objective of this section is to cover the following points:

- o How is the evaluation process conducting?
- o What techniques are used for serving the objective of model and study? And what is the role of each technique?
- o What are the influenced criteria which impact on the quality and performance of classifiers?

The previous questions will be answered through the following sub-sections.

3.1 Preliminaries

The utilized techniques and its basic concept are exhibited in this sub-section.

3.1.1 Tree Soft Sets [37]

A novel technique of TrSS is proposed by Smarandache who is founder for Neutrosophic theory Smarandache [24]. This technique has several concepts which described as:

- Assuming that $\aleph$ be a universe of discourse which includes ϱ a non-empty as subset of $\aleph$, thus the powerset of ϱ expressed as p (ϱ).
- Let TrSS encompasses set of levels, each one has a multitude of nodes as:
  - ➢ Level 1: consists of a multitude of nodes where each node represents main criteria, then expressed as: C={C1, C2,..Cn} for integer n ≥ 1.
  - ➢ Level 2: includes several sub-nodes of {C1, C2,..Cn} and stated as { C1-1, …C1-n } branched of C1 , and { C2-1, …C2-n } branched of C2 , finally { Cn-m, …Cn-m } branched of Cn .
  - We call the leaves of the graph-tree, all terminal nodes (nodes that have no descendants). Then, Tree Soft Set: F: P (Tree (C)) → p (ϱ).
  - Tree (C) is the set of all nodes and leaves (from level 1 to level n) of the graph-tree, and P (Tree(δ)) is the powerset of the Tree (Ind). All node sets of TrSS of level n as: Tree (C) = {C nm | nm= 1, 2, ...}.

3.1.2 Single-Valued Neutrosophic Sets (SVNSs)[42]

SVNSs are a branch of Neutrosophic theory that originated from Smarandache's work. Whilst SVNSs consider three measurement and probabilities as Truth (ϑ), Falsity (ν), and Indeterminacy (δ). Hence, three measurements are deployed and represented as:

- Assume that χ is universal set and κ is element in χ and this element is formed as: $\vartheta_\kappa(\omega)$, $\nu_\kappa(\omega)$, $\delta_\kappa(\omega)$.

- $0 \leq sup\ \vartheta_{\kappa}(\omega)$+sup $\nu_{\kappa}(\omega)$+ sup$\delta_{\kappa}(\omega) \leq 3$.
- The operations in SVNSs formed as:
  - Addition of two sets : $\widetilde{\mathrm{Ne}_1}+\widetilde{\mathrm{Ne}_2}=$
  $\langle(\tau_1 + \tau_2 - \tau_1 \tau_2, \gamma_1, \gamma_2, \varphi_1\varphi_2)\rangle$
  - Multiplication of two sets : $\widetilde{\mathrm{Ne}_1}+\widetilde{\mathrm{Ne}_2}=$
  $\langle(\tau_1 \tau_2, \gamma_1 + \gamma_2 - \gamma_1 \gamma_2, \varphi_1 + \varphi_2 - \varphi_1 \varphi_2)\rangle$

3.2 Development of SDMM: Evaluation classifiers and selecting the best classifier in the imbalanced data problem

Developing SDMM for evaluating determined alternatives of classifiers required follow set of steps to implement the mentioned techniques.

Step 1: Structuring criteria and sub-criteria into set of levels.

1.1 Determining set of alternatives of classifiers which involve into evaluation process.

1.2 Determining the influenced criteria and sub-criteria which contribute to evaluating process.

1.3 Modeling and structuring these criteria and its sub-criteria as nodes into several levels.

1.4 DMs panel is formed for rating enterprises based on modelled criteria and sub-criteria.

Step 2: SVNSs based SWARA for generating weighting [43].

2.1 SWARA is deployed for obtaining criteria weights as:

- Expert panel is rating criteria through using SVN scale. Each decision maker (DM) rates the criteria in his/her decision matrix.
- Deneutrosophic rating of each DM according to Eq.(1).

$$\mathrm{De_j} = \frac{2+\vartheta-\nu-\delta}{3} \quad (1)$$

- Where: ϑ, ν , δ indicated to truth, false, and indeterminacy respectively.
- The constructed deneutrosophic matrices are aggregated into an aggregated matrix through employing Eq.(2).

$$\wp_{\mathbf{j}} = \frac{\sum_{\mathbf{j}}^{\mathrm{n}} \mathrm{De}_{\mathbf{j}}}{\mathbf{U}} \quad (2)$$

- Where: **U** refers to number of DMs
- Comparative importance of Average value (Sj) is obtaining according to Eq.(3) [44]. The values of Sj facilitate obtaining the values of coefficient (Kj) according to Eq.(4).

$$\mathbf{S_j} = \begin{cases} \mathbf{0} & \mathbf{j = 1} \\ \wp_{\mathbf{j-1}} - \wp_{\mathbf{j}} & \mathbf{j > 1} \end{cases} \quad (3)$$

$$\mathbf{K_j} = \begin{cases} \mathbf{1} & \mathbf{j = 1} \\ \mathbf{s_j} & \mathbf{j > 1} \end{cases} \quad (4)$$

- Generating recalculated weights (qj) through implementing Eq.(5).

$$\mathbf{q_j} = \begin{cases} \mathbf{1} & \mathbf{j = 1} \\ \dfrac{\mathbf{q_j - 1}}{\mathbf{k_j}} & \mathbf{j > 1} \end{cases} \quad (5)$$

- Accordingly, qj contributed to obtain final weights ($\boldsymbol{w_j}$)based on Eq.(6).

$$\boldsymbol{w_j} = \frac{\boldsymbol{q_j}}{\sum_{k=1}^{n} \boldsymbol{q_k}} \quad (6)$$

2.2 **SWARA is deployed for obtaining sub- criteria weights.** We follow the steps of 2.1 to generate sub- criteria weights.

Step 3: Recommending Optimal classifier based on MAIRAC and SVNSs.

3.1 Constructing Neutrosophic decision matrix for DM based on SVN scale.

3.2 Denutrosophic decision matrix for each DM according to Eq.(2). Also, Eq.(3) has vital role for aggregating these matrices into an aggregated matrix.

3.3 Eq.(6) responsible for calculating theoretical evaluation matrix(TP) toward estimating preferences of alternatives.

$$P_{A_j} = \frac{1}{m} \quad (6)$$

Where: m indicates the number of alternatives

3.4 Calculating real evaluation matrix (TR) according to Eq.s(7),(8).

$$\mathbf{tr_{ij}} = \mathbf{tp_{ij}} \left(\frac{\mathbf{x_{ij}} - \bar{\mathbf{x}}_{\mathbf{i}}}{\mathbf{x_i^+} - \mathbf{x_i^-}}\right), \text{ for maximum} \quad (7)$$

$$\mathbf{tr_{ij}} = \mathbf{tp_{ij}} \left(\frac{\mathbf{x_{ij}} - \mathbf{x_i^+}}{\mathbf{x_i^-} - \mathbf{x_i^+}}\right), \text{ for minimum} \quad (8)$$

3.5 Calculating criteria function (Q) based on Eq.(9).

$$Q_i = \sum_{i=1}^{m} g_{ij} \quad (9)$$

Where,

$$g_{ij} = t_{p_{ij}} - t_{r_{ij}} \quad (10)$$

4 Empirical Case Study.

We are implementing our SDMM for evaluating and ranking classifiers. Herein,we are leveraging five classifiers (alternatives) are Smote-Tomek Link (STL), Focal-XGBoost (FGB), Boosted Random Forest (BRF), MESA, and HML. The evalaution for five alternatives are conducting based on four criteria used for evaluation are described in Table 1.

The evalaution is conducting through implementing the steps are aforementioned.

4.1 Assigning each criterion to certain node and also, sub-criteria through leveraging TrSS technique as in Figure 1.

4.2 Three DMs are rating modeled criteria and sub-criteria through utilizing the SVN scale in Table 1.

4.2.1 For main criteria, DMs are rating the four criteria through SVN scale in Table 1.

- DMs'preferences and transform these preferences into deneutrosophic values according to Eq.(1). Eq.(2) used to aggregate these preferences as in Table 2.

- Sorting the criteria in descending order according toaggreget values of criteria.According to aggregated values in Table 2 , C4 is the more important than C2.Also, C2 is more important than C3.Accordingly, C3 is more important than C1.

- Employing Eq.(3) for genertaing Sj values .also, q_j and criteria weights are obtaining through Eq.s(5),(6).Table 3 involves the findings of applied Eq.s.Figure 2 showcases final weights for criteria.

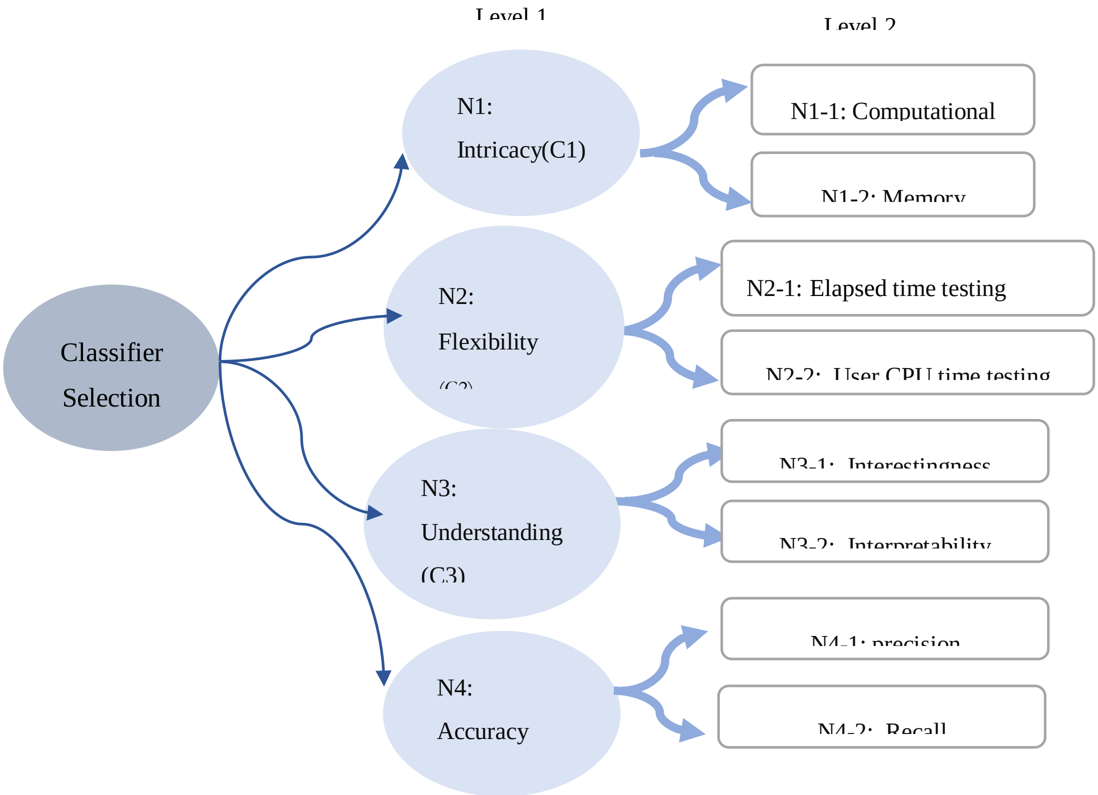

Figure 1.Tree soft for determined criteria and sub-criteria

4.2.2 For Sub- criteria, DMs are rating for each criterion is branched from main criterion according to structure of TrSS in Figure 1. SVN scale in Table 1 is utilized to rate the sub-criteria.

- The steps in 4.2.1 are followed for obtaining weights for these sub-criteria. Final findings formed in Figure 3,4,5,6 for sub-criteria of main criteria.

4.3 Three DMs are utilized SVN scale in Table 1 for second time for rating five alternatives of classifiers .

4.4 Aggregated decision matrix is generated through employing Eq.(2) after convreting three matrices form neutrosophic to deneutrosophic. Table 4 aggregets the three matrices into single matrix.

4.5 Calculating calculating theoretical evaluation matrix(TP) through Eq.(6) and represented in Table 5.

4.6 Calculating real evaluation matrix (TR)through employing Eq.(7) for maximum whlist Eq.(8) for minimum. The findings recorded in Table 6.

4.7 Calculating criteria function (Q) based on Eq.s (9),(10) and final ranking for alternatives is shown in Figure 7. This figure indicated that alternitive 5 (A5) HML is the optimal classifier. In contrast, alternative 2(A2) is the worst.

Table 1. Scale of SVN

Synonmy	Acronym	Scale		
		T	I	F
Extremly Weak	EW	0.00	1.00	1.00
Absolutely Weak	AW	0.10	0.90	0.90
Very Weak	VW	0.20	0.85	0.80
Weak	W	0.30	0.75	0.70
Fairly Weak	FW	0.40	0.65	0.60
Fairly	F	0.50	0.50	0.50
Fairly Well	FW	0.60	0.35	0.40
Well	W	0.70	0.25	0.30
Very Well	VW	0.80	0.15	0.20
Absolutely Well	AW	0.90	0.10	0.10
Extremly Well	EW	1.00	0.00	0.00

Table 2. Denutrosophic Matrix

Criteria	Expert Panel			Aggregeted values
	DM1	**DM2**	**DM3**	
Intricacy(C1)	0.5	0.62	0.9	0.672222222
Flexibility (C2)	0.62	0.82	1	0.811111111

Understanding (C3)	0.82	0.9	0.5	0.738888889
Accuracy (C4)	0.9	1	0.82	0.905555556

Table 3. Final criteria weights

Criteria	**Sj**	**Kj**	**qj**	**wj**
Accuracy (C4)	0	1	1	0.280523
Flexibility (C2)	0.094444	1.094444	0.913706	0.256316
Understanding (C3)	0.072222	1.072222	0.852161	0.239051
Intricacy(C1)	0.066667	1.066667	0.798901	0.22411
		Sum	3.564767	1

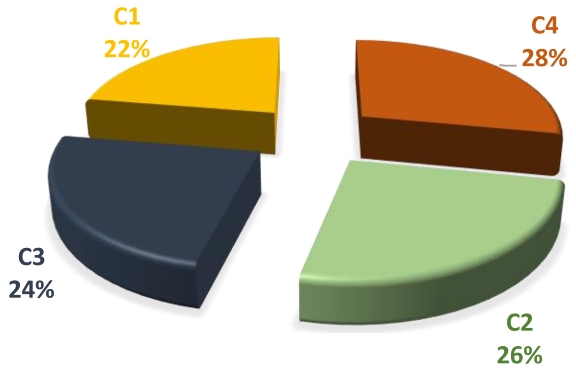

Figure 2. Final criteria weights

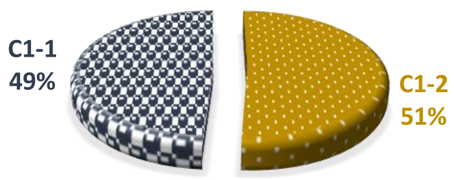

Figure 3.Sub-ceiteria of main criteria 1 weights

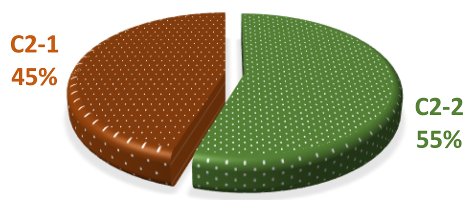

Figure 4. Sub-ceiteria of main criteria 2 weights

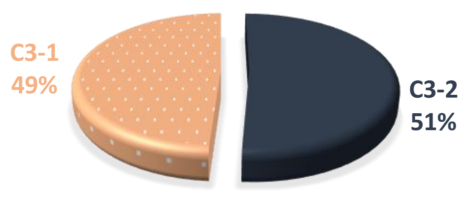

Figure 5.Sub-ceiteria of main criteria 3 weights

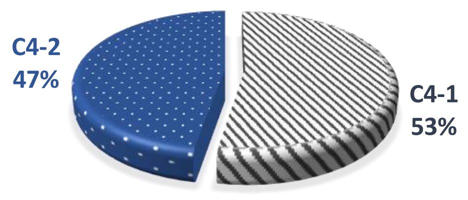

Figure 6.Sub-ceiteria of main criteria 4 weights

Table 4. Aggregeted matrix

Alternatives / criteria	Intricacy(C1)	Flexibility (C2)	Understanding (C3)	Accuracy (C4)
SMOTE Tomek Link	0.5	0.572222222	0.5	0.461111111
FGB	0.65	0.283333333	0.538888889	0.5
BRF	0.816666667	0.5	0.75	0.5
MESA	0.816666667	0.716666667	0.75	0.75
HML	0.9	0.75	0.844444444	0.844444444

Table 5. Theoretical Evaluation Matrix(TP)

Alternatives / criteria	Intricacy(C1)	Flexibility (C2)	Understanding (C3)	Accuracy (C4)
SMOTE Tomek Link	0.04482204	0.051263133	0.047810176	0.056104651
FGB	0.04482204	0.051263133	0.047810176	0.056104651
BRF	0.04482204	0.051263133	0.047810176	0.056104651
MESA	0.04482204	0.051263133	0.047810176	0.056104651
HML	0.04482204	0.051263133	0.047810176	0.056104651

Table 6. Real Evaluation Matrix (TR)

Alternatives / criteria	Intricacy(C1) MIN	Flexibility (C2) MAX	Understanding (C3) MAX	Accuracy (C4) MAX
SMOTE Tomek Link	0.04482204	0.03173432	0	0
FGB	0.028013775	0	0.005397923	0.005691776
BRF	0.009337925	0.02380074	0.034700934	0.005691776
MESA	0.009337925	0.047601481	0.034700934	0.005691776
HML	0	0.051263133	0.047810176	0.056104651

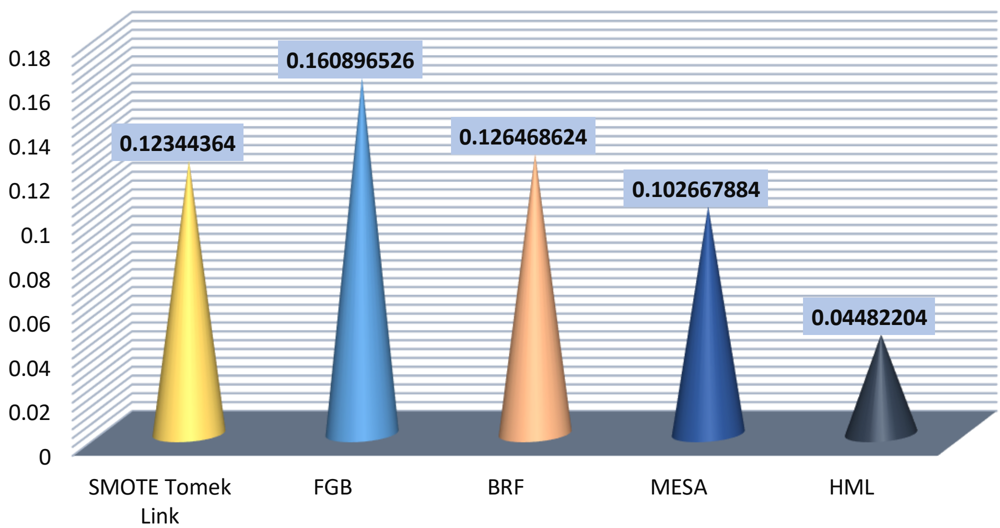

Figure 7. Final Ranking for Alternatives

5. Conclusions

Classification learning is a vital process. However, most classifiers can't deal with the imbalanced data problem, which indicates there are many more instances (majority instances) in one class than in another. Unfortunately, algorithms usually concentrate on only one criterion when facing this problem. Moreover, this became the catalyst for conducting this study and constructing SDMM based on MCDM techniques to compare different classifiers according to multiple dimensions in the case of an imbalanced data problem based on set of influenced criteria and sub-criteria. Hence, that motivates us for modeling and structuring these criteria and its sub-criteria to illustrate the relation between each other. Accordingly, SWARA as technique of MCDM, is applied to obtain criteria and sub-criteria weights, with the assistance of SVNSs. The findings of SVNSs based SWARA showcased in Figures 2 for main criteria whereas sub-criteria's weights illustrated in Figures 3,4,5,6. After that role of MAIRAC based on SVNSs initializes for rating five alternatives of classifiers through leveraging generated criteria's weights from SWARA -SVNSs. The findings recommended that A5 HML is the optimal classifier otherwise, A2 is the worst as in Figure 7.

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A Python Framework for Neutrosophic Sets and Mappings

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Abstract. In this paper we present an open source framework developed in Python and consisting of three distinct classes designed to manipulate in a simple and intuitive way both symbolic representations of neutrosophic sets over universes of various types as well as mappings between them. The capabilities offered by this framework extend and generalize previous attempts to provide software solutions to the manipulation of neutrosophic sets such as those proposed by Salama et al. [21], Saranya et al. [23], El-Ghareeb [7], Topal et al. [29] and Sleem [26]. The code is described in detail and many examples and use cases are also provided.

Keywords: neutrosophic set; neutrosophic mapping; Python; class; framework.

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1. **Introduction**

Since the notion of neutrosophic set was introduced in 1999 by Smarandache [27] as a generalization of both the notions of fuzzy set introduced by Zadeh [31] in 1965 and intuitionistic fuzzy set proposed by Atanassov [4] in 1983, neutrosophic set theory had a rapid development and has been profitably used in many fields of pure Mathematics [1, 12, 13, 15, 16] as well as in several areas of applied sciences such as Graph Theory [5], Decision Making [14], Medicine [6], Statistics [10, 24, 28], Image Analysis [9, 32] Machine Learning [8, 25], etc.

In numerous instances, especially when dealing with applications stemming from real-world issues, manually manipulating neutrosophic sets that possess a finite yet consistent number of elements, along with their associated mappings, can be quite laborious and challenging. Consequently, there exists a significant demand for a system that can streamline the automation of key neutrosophic operations, including union, intersection, neutrosophic difference, and the calculation of neutrosophic images or counterimages by mappings. Previous attempts to address this need were undertaken by Salama et al., who initially employed tools like Microsoft Excel [20] and later transitioned to using C# [21]. Another software application for processing neutrosophic sets developed in C# was described by Saranya et al. [23]. More recently, El-Ghareeb introduced a Python package designed to handle both single and interval-valued neutrosophic numbers and sets [7]. Unfortunately, however, in the latter paper the two classes concerning neutrosophic sets are described incompletely and summarily than those concerning neutrosophic numbers. Version 0.0.5 of this software that we consulted does not appear to provide adequate functionality even for the main neutrosophic operations and in any case the related repository on GitHub of the source code mentioned in the article does not appear to be available. Furthermore, other Python-based software solutions for handling neutrosophic numbers and matrices have been proposed by Topal et al. [29] and Sleem [26]. However, the authors are not aware of any other Python software specifically designed for the manipulation of neutrosophic sets is currently known.

This underscores the ongoing requirement for a set of well-structured Python classes, ideally available under an Open Source license, that enable automated and interactive manipulation of symbolic representations of neutrosophic sets, along with their associated mappings. Additionally, there is a need for comprehensive documentation and user-friendly design to facilitate straightforward integration for future implementations.

For this reason, we intended to design and develop a modern framework that extends and generalizes the above software solutions overcoming some of their limitations and offering greater flexibility in their use, including interactive, aimed at the manipulation of neutrosophic sets and functions. The structure of the entire framework has been carefully described by means of of Unified Modeling Language (UML for short), a modeling and specification description language very popular in Software Engineering. The underlying given structures as well as the most significant methods of each of the classes of which the the framework have been explained in detail in order to allow for further future refinements of both a theoretical and applicative nature.

We are confident that the necessity mentioned has been effectively tackled through the Python framework outlined in this paper. The complete source code for this framework has been released under the Open Source GNU General Public License version 3.0 (or GPL-3.0) and is freely accessible at the url github.com/giorgionordo/pythonNeutrosophicSets.

In particular, Section 2 describes the general structure of the framework, the dependency relationships among the various classes that comprise it as well as the reasons that suggested the use of the Puython language and the decision to release the entire code produced under an Open Source license.

Section 3 introduces some useful notations for extending some data structures typical of Python in order to create a flexible substandard for describing neutrosophic sets and functions between them. In addition, some utility functions are briefly called that later will be invoked by some classes in the framework.

Section 4 contains a description of the properties and methods of the class `NSuniverse` used to represent the universe sets on which the neutrosophic sets will be defined.

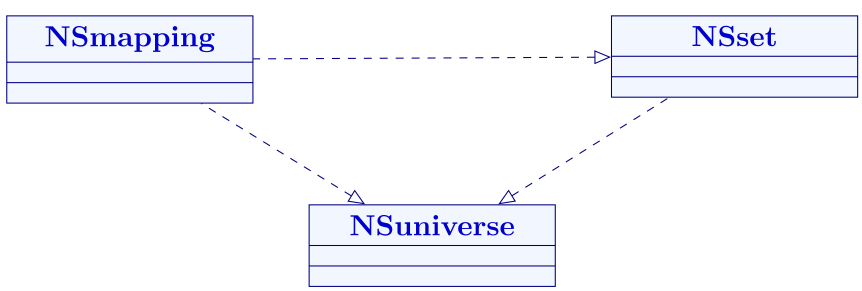
Section 5 describes in great detail the properties and methods that make up the `NSset` class, the main class of the framework, used for the representation of neutrosophic sets and, in addition, numerous examples of practical use are also provided both in both traditional and interactive environments.

Section 6 is devoted to the description of the class `NSmapping` by which functions between two neutrosophic sets are represented. The properties and methods of this class are described in detail and illustrated with several practical examples.

Finally, some final remarks are made in Section 7, highlighting the strengths of the framework presented in this paper and inviting other researchers to continue, extend and improve the development of the code described here.

2. The framework PYNS

The PYthon Neutrosophic Sets framework (`PYNS` for short) described in the present paper consists of three classes designed to manage respectively universe sets (the `NSuniverse` class), neutrosophic sets (the `NSset` class) and functions between them (the `NSmapping` class. As is natural, the class `NSset` depends on (i.e. uses) the class `NSuniverse` class) while the `NSmapping` class uses the other two as described in the following UML diagram of classes.



where the dashed arrow means "uses".

These three classes are respectively contained in the Python files `ns_universe.py`, `ns_set.py` and `ns_mapping.py` which are located in the package directory `pyns`. The same directory contains also the file `ns_util.py` where are defined some utility functions external to the classes but employed by them. The structure of the package is described by the following diagram.

```
/pyns
 ├── ns_universe.py ............... (contains the NSuniverse class)
 ├── ns_set.py ............................ (contains the NSset class)
 ├── ns_mapping.py.................. (contains the NSmapping class)
 └── ns_util.py .................. (contains common utility functions)
```

The choice of programming language is a crucial aspect in the development of a scientific framework, since it determines the performance, flexibility and ease of use of the entire system. In the specific case of our framework, the choice to implement it using the Python language is based on several factors:

- Clear and expressive syntax: Python is known for its simple and readable syntax, which makes the code more intuitive to write and understand. This feature is especially relevant for a scientific framework, as it facilitates the creation and manipulation of complex data structures such as neutrosophic sets and mappings between them.
- Extensive standard library: Python offers an extensive standard library covering multiple scientific and mathematical domains. This allows developers to easily use existing functions and tools to implement complex algorithms and optimize the performance of the framework.
- Easy integration with other technologies: Python is known for its ability to integrate with other programming languages and external libraries. This is particularly useful in a scientific context, where it may be necessary to use specialized libraries or existing computational tools.
- Quick learning: Python is often considered one of the most accessible languages even for programming novices. Its relatively smooth learning curve allows students, researchers and less experienced developers to tackle the framework with greater ease, thus encouraging its dissemination and adoption in the scientific domain.

In summary, the choice of Python as the main language for the `PYNS` framework was crucial in making the entire system more accessible, flexible, and powerful. It allowed developers to focus more on scientific challenges and mathematics specifics, rather than the complexities of the programming language, thus accelerating the development and adoption of this important research tool.

Just as the Python language is released under an open source license approved by the Open Source Initiative (OSI), the framework described in this paper is also made available as open code. This offers numerous benefits and incentives for both developers and the scientific community as a whole, including:

- Knowledge sharing: The release of the framework under an open source license promotes the sharing of knowledge and scientific discoveries. By allowing anyone to access the source code, developers and researchers can learn from others, build on others' work, and contribute improvements and new ideas.
- Collaboration: The open source license encourages collaboration among experts and researchers from from different academic institutions, organizations and countries. This synergy can lead to faster developments, new discoveries, and innovative solutions to complex problems.
- Transparency and verifiability: The availability of source code allows for greater transparency in the implementation of the framework. The scientific community can verify and validate the results obtained, increasing confidence towards the framework and the results obtained through it.
- Adaptability and customization: Users can adapt the framework to their own specifics and customize it to address unique problems. This flexibility results in a greater number of possible applications and uses of the framework in various scientific contexts.
- Cost reduction: Releasing the framework as open source eliminates the costs associated of purchasing licenses or copyrights. This allows academic institutions and organizations with limited resources to free access to advanced scientific analysis tools and uncertain data.
- Community growth: The adoption of the open source license attracts a community of developers, researchers and enthusiasts interested in the field of neutrosophic sets that can contribute to the evolution of the framework by providing feedback, reporting bugs, and participating in the development of new features.
- Continuity and longevity: The open source model can ensure greater longevity of the framework, as it is not dependent on a single developer or institution. The community can take care of the project over time, ensuring that it is always updated and supported, even if there are changes in the original organization.

In particular, our framework `PYNS` is available under GNU General Public License version 3.0 (or GPL-3.0), a generic software license developed by the Free Software Foundation (FSF) that provides users with a set of rights and freedoms to use, modify, and distribute the software covered by the license. More specifically, GPL-3.0 allows:

- Freedom of Use: the software may be used for any purpose, whether personal or commercial.

- Freedom to Study and Modify: you can analyze and study the source code of the software to understand how it works and make changes to it according to your own needs.
- Freedom of Distribution: you can distribute copies of modified or unmodified software to anyone, while complying with the requirements of GPL-3.0.
- Sharing of Changes: if you distribute modified software, you have an obligation to make available the source code of your changes as well.
- Compatibility with Derivative Works (copyleft): any derivative work based on software covered by GPL-3.0 must also be released under GPL-3.0 or a compatible license.

In conclusion, from the evaluation of all these aspects, it follows that releasing the `PYNS` framework under GPL-3.0 license represents a strategic choice that promotes innovation, collaboration and the dissemination of knowledge in the field of neutrosophic theory, thus promoting the advancement of scientific research in this area.

3. Conventions and utility functions

In the following we will make extensive use of Python's `dict` (dictionary) data structure both for the internal representation of neutrosophic sets and for the definition of functions between universe sets. To make it even easier and more streamlined to use such structures both in interactive mode as well as in writing client code based on such classes, it was chosen to also allow their representation as a string and in free format, i.e., leaving the user free to:

- indifferently use not only the usual symbol `:` (colon) but also alternatively the strings `->` (arrow) and `|->` (maps-to) as separators between keys and values
- indifferently use not only the usual symbol `,` (comma) but also `;` (semicolon) as separators of the value-key pairs

in any combination thereof, and we will refer to this type of representation by the name *extended dictionary*. In other words, while a classical Python dictionary has a form like:

$$\{key_1 : value_1, key_2 : value_2, \ldots key_n : value_n\} \ ,$$

an extended dictionary can be expressed as strings of the type:

$$\texttt{"}key_1\texttt{->}value_1\texttt{,}\,key_2\texttt{|->}value_2\texttt{;}\ \ldots key_n\texttt{->}value_n\texttt{"}\ .$$

The already mentioned file `NS_util.py` contains some general utility functions that will be used repeatedly in the classes we will describe later. More specifically, these functions are:

- `NSreplace(`*text*`,`*sostituz*`)` which performs a series of substitutions on the string *text* by replacing each key in the *sostituz* dictionary with its corresponding value; in particular, if that value is the null string `""` the effect will be to remove all occurrences of the key,
- `NSstringToTriplesList(`*text*`)` which converts the string *text* containing a list of triples into the corresponding data structure by using the function `findall` contained in the module `re` (regular expression) and the function `literal_eval` contained in the module `ast` (Abstract Syntax Trees) which allows interpreting data expressions contained in a string,
- `NSisExtDict(`*obj*`)` that checks whether the object *obj* passed as parameter is a string representing an extended dictionary and returns the Boolean value `True` if it is,
- `NSstringToDict(`*text*`)` which converts the string *text* containing an extended dictionary in a real Python dictionary,
- `NSsplitText(`*text*`,`*max_length*`)` which returns the string *text* splitted into multiple lines of length not exceeding the value *max_length*.

The complete code for these functions is given in the following listing.

```python
from re import findall
from ast import literal_eval

def NSreplace(text, sostituz):
    for k in sostituz:
        text = text.replace(k, sostituz[k])
    return text

def NSstringToTriplesList(text):
    pattern = r'\[.*?\]|\(.*?\)'
    str_list = findall(pattern, text)
    tpl_list = [tuple(literal_eval(s)) for s in str_list]
    return tpl_list

def NSisExtDict(obj):
    result = False
    if type(obj) == str:
        result = (":" in obj) or ("->" in obj)

def NSstringToDict(text):
    sostituz = {"'": "", '"': "", "(": "", ")": "", "[": "", "]": "",
                "{": "", "}": "", " ": ",", ";": ",", ",,": ",",
                "|->": ":", "->": ':'}
    text = NSreplace(text, sostituz)
    listcouples = text.split(',')
    diz = dict()
    for couple in listcouples:
        key, value = couple.split(':')
        diz[key] = value
    return diz
```

```python
def NSsplitText(text, max_length):
    words = text.split()
    lines = []
    current_line = ""
    for word in words:
        if len(current_line) + len(word) <= max_length:
            current_line += word + " "
        else:
            lines.append(current_line.strip())
            current_line = word + " "
    lines.append(current_line.strip())
    result = "\n".join(lines)
    return result
```

4. The **NSuniverse** class

The universe set is the fundamental notion on which the definition of a neutrosophic set is founded on. We have chosen to represent it by means of a list of strings.
The corresponding class which implements such a notion is shortly described in the following UML class diagram.

NSuniverse
__universe : list of strings
__init__(*args) : constructor with generic argument get() : returns the list of elements of the universe set cardinality() : returns the number of elements of the current universe set isSubset(unv) : checks if the current universe set is contained in another one __eq__() : checks if two universe sets are equal overloading the == operator __ne__() : checks if two universe sets are different overloading the ! = operator __iter__() : initializes iterator on elements of the current universe set __next__() : returns the iterated element of the current universe set __str__() : returns the current universe set in string format __format__(spec) : returns the formatted string of the universe respect to a specifer __repr__() : returns a detailed representation of the universe set

In order to ensure maximum usability and versatility in the use of this class, the constructor method accepts string, lists, tuples, lists of elements of any length or another object `NSuniverse` and proceeds to transform them into strings and store them in a list. The basic steps of this method, expressed in pseudo-code, are described in the following algorithm.

Constructor method of the class `NSuniverse`

Function `__init__`(*args*):
- Get the *length* of *args*
- **if** *length* = 0 **then**
  - Raise an Exception
- **else if** *length* = 1 **then**
  - **if** *args* **is** *a list, a tuple or an object of the class* **then**
    - Converts *args* appropriately and stores it in *universe*
  - **else if** *args* **is** *a string* **then**
    - Removes parentheses, commas and semicolons from *args*, splits and gets a list of strings to store in *universe*
  - **else if** *args* **is** *a set* **then**
    - Raise an Exception
  - **else**
    - Converts *args* to string and creates a list with only this element to be stored in *universe*
- **else**
  - Converts *args* to a list of strings to be stored in *universe*
- **if** *universe has repeated elements* **then**
  - Raise an Exception
- Stores *universe* in the property *__universe*

The corresponding Python code of the constructor method of the `NSuniverse` class is given below.

```
from .ns_util import NSreplace

class NSuniverse:

    def __init__(self, *args):
        universe = list()
        length = len(args)
        if length == 0:
            raise IndexError("the universe set must contain at least an element")
        elif length == 1:
            elem = args[0]
            if type(elem) in [list, tuple]:
                universe = [str(e) for e in elem]
            elif type(elem) == NSuniverse:
                universe = elem.get()
            elif type(elem) == str:
                sostituz = { "{":"", "}":"", "[":"", "]":"", "(":"", ")":"",
                             ",":" ", ";":" " }
                universe = NSreplace(elem, sostituz).split()
            elif type(elem) == set:
                raise ValueError("type set is not suitable because the elements of the universe set must be assigned in a specific order")
            else:
                universe = [str(elem)]
        else:
```

```
            for i in range(length):
                universe.append(str(args[i]))
        univset = set(universe)
        if len(universe) != len(univset):
            raise ValueError("the universe set cannot contain repeated elements")
        self.__universe = universe
```

The constructor method also intercepts potential error situations in the definition of universe sets such as attempting to define an empty set by calling it without any parameters or that of inserting repeated elements (which conflicts with the usual set definition) and in each of these cases raises an appropriate exception.

Let us observe that the flexibliity of the constructor method allows us to define a universe set using various formats such as lists, tuples, strings, or simple enumerations of elements without worrying about maintaining a rigid or uniform notation, which is particularly useful to facilitate usability in interactive use. For example, the universe set $\mathbb{U} = \{1, 2, 3, 4, 5\}$ can be defined as an object of the class `NSuniverse` in any of the following ways mutually equivalent:

- `U=NSuniverse([1,2,3,4,5])` as a list,
- `U=NSuniverse((1,2,3,4,5))` as a tuple,
- `U=NSuniverse("1,2,3,4,5")` as a string of elements comma separated,
- `U=NSuniverse("1;2;3;4;5")` as a string of elements separated by semicolon,
- `U=NSuniverse("1 2 3 4 5")` as a string of elements separated by spaces,
- `U=NSuniverse("1,2 3 4;5")` as a string of elements separated in various ways,
- `U=NSuniverse("{1,2,3,4,5}")` as a string representing a set,
- `U=NSuniverse("[1,2,3,4,5]")` as a string representing a list,
- `U=NSuniverse("(1,2,3,4,5)")` as a string representing a tuple,
- `U=NSuniverse("(1;2;3;4;5)")` using semicolon as separator,
- `U=NSuniverse(1,2,3,4,5)` as a listing of numerical values only,
- `U=NSuniverse("1",2,"3",4,"5")` in a mixed form,

as well as in different combinations of them.

However, it is not allowed to define a universe set by means of the `set` type of the Python language (i.e., expressions such as `NSuniverse({1,2,3,4,5})` are not accepted) since it is an unordered data collection and for a precise design choice the elements must be listed in a specific order, feature this will prove valuable in simplifying and making consistent definitions of both neutrosophic sets and mappings between them.

The class `NSuniverse` is equipped with very few basic methods, that is `get()` which returns the list of strings corresponding to the instance of the universe set exactly as it is stored internally in the class and `cardinality()` which returns the number of elements present in the object instantiated by the class.

```python
    def get(self):
        return self.__universe

    def cardinality(self):
        return len(self.__universe)
```

The method `isSubset()` checks whether the current universe set is contained in a second universe set passed as parameter and returns the Boolean value `True` in the positive case.

```python
    def isSubset(self, unv):
        setself = set(self.get())
        setunv = set(unv.get())
        result = setself.issubset(setunv)
        return result
```

To facilitate the comparison of two objects of type `NSuniverse`, the equality == and diversity != operators have been overloaded using their corresponding special methods.

```python
    def __eq__(self, unv):
        equal = (self.get() == unv.get())
        return equal

    def __ne__(self, unv):
        different = not (self == unv)
        return different
```

In order to be able to easily print on the screen objects of type `NSuniverse` in text format and to provide a complete representation of them, the special methods `__str__()` and `__repr__()` were defined as follows by using the overloading.

```python
    def __str__(self):
        list_string_elements = [str(e) for e in self.__universe]
        s = "{ " + ", ".join(list_string_elements) + " }"
        return s

    def __repr__(self):
        return f"Universe set: {str(self)}"
```

As an example, we show how the methods and operators described above can be used not only in a client code, but also in the interactive mode by means of the Python console:

```python
>>> from pyns.ns_universe import NSuniverse
>>> U = NSuniverse("1", 2, 3, "4")
>>> print(U)
{ 1, 2, 3, 4 }
>>> print( U != NSuniverse([1,3,5]))
True
>>> V = NSuniverse(" ( a b c , d ; e )")
>>> print(V.cardinality())
5
>>> print(V.get())
['a', 'b', 'c', 'd', 'e']
>>> print(V)
{ a, b, c, d, e }
```

Since in the following we will also need to make formatted prints of objects of the universe set type according to a certain format specifier, it is also necessary to redefine by overloading the special method `__format__()`.

```
    def __format__(self, spec):
        unvstr = str(self)
        result = f"{unvstr:{spec}}"
        return result
```

Finally, to simplify the code of other classes devoted to neutrosophic sets and mappings, it is useful to establish an iterator over the objects of the `NSuniverse` class. This iterator should sequentially provide all and only the elements within a given universe set. This is achieved by introducing a new property, *self.__i*, to serve as the internal index for the current element and redefining by overloading the special methods `__iter__()` and `__next__()` which are respectively intended to initialize the index of the iterator and yield the element associated with the current index.

```
    def __iter__(self):
        self.__i = 0
        return self

    def __next__(self):
        if self.__i < len(self.__universe):
            elem = self.__universe[self.__i]
            self.__i +=1
            return elem
        raise StopIteration
```

Thanks to the introduction of the iterator on the class `NSuniverse` it will be, for example, possible to handle loops directly on objects of type universe set, exactly as happens with other standard Python types such as lists and tuples. This approach will contribute to make the syntax of our code leaner and more understandable, as highlighted in the following example.

```
from pyns.ns_universe import NSuniverse

U = NSuniverse(" ( a b c , d ; e )")
for i, u in enumerate(U):
    print(f"- the {i}-th element is {u}")
```

which produces output of the type:

```
- the 0-th element is a
- the 1-th element is b
- the 2-th element is c
- the 3-th element is d
- the 4-th element is e
```

5. The **NSset** class

The representation of neutrosophic sets on a given universe set is done by means of the `NSset` class which, obviously, uses the `NSuniverse` class.

The original definition of neutrosophic set, given in 1999 by Smarandache [27], refers to the interval $]0^-, 1^+[$ of the nonstandard real numbers and although it is consistent from a philosophical point of view, unfortunately, it is not suitable to be used for approaching real-world problems. For such a reason, in 2010, the same author, jointly with Wang, Zhang and Sunderraman [30], also introduced the notion of single valued neutrosophic set which, referring instead to the unit interval $[0, 1]$ of the usual set of real numbers $\mathbb{R}$, can be usefully used in scientific and engineering applications. In the following we will refer exclusively to single valued neutrosophic sets.

Definition 5.1. [30] Let $\mathbb{U}$ be an universe set and $A \subseteq \mathbb{U}$, a *single valued neutrosophic set* over $\mathbb{U}$ (*SVN-set* for short), denoted by $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$, is a set of the form:

$$\widetilde{A} = \{(u, \mu_A(u), \sigma_A(u), \omega_A(u)) : u \in \mathbb{U}\}$$

where $\mu_A : \mathbb{U} \to I$, $\sigma_A : \mathbb{U} \to I$ and $\omega_A : \mathbb{U} \to I$ are the *membership function*, the *indeterminacy function* and the *non-membership function* of A, respectively and $I = [0, 1]$ be the unit interval of the real numbers. For every $u \in \mathbb{U}$, $\mu_A(u)$, $\sigma_A(u)$ and $\omega_A(u)$ are said the *degree of membership*, the *degree of indeterminacy* and the *degree of non-membership* of u, respectively.

Definition 5.2. [27, 30] Let $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ and $\widetilde{B} = \langle \mathbb{U}, \mu_B, \sigma_B, \omega_B \rangle$ be two SVN-sets over the universe set $\mathbb{U}$, we say that $\widetilde{A}$ is a *neutrosophic subset* (or simply a subset) of $\widetilde{B}$ and we write $\widetilde{A} \Subset \widetilde{B}$ if, for every $u \in \mathbb{U}$, it results $\mu_A(u) \le \mu_B(u)$, $\sigma_A(u) \le \sigma_B(u)$ and $\omega_A(u) \ge \omega_B(u)$. We also say that $\widetilde{A}$ is contained in $\widetilde{B}$ or that $\widetilde{B}$ contains $\widetilde{A}$ and we write $\widetilde{B} \Supset \widetilde{A}$ to denote that $\widetilde{B}$ is a *neutrosophic superset* of $\widetilde{A}$.

Definition 5.3. [27, 30] Let $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ and $\widetilde{B} = \langle \mathbb{U}, \mu_B, \sigma_B, \omega_B \rangle$ be two SVN-sets over the universe set $\mathbb{U}$. We say that $\widetilde{A}$ is a *neutrosophically equal* (or simply equal) to $\widetilde{B}$ and we write $\widetilde{A} \equiv \widetilde{B}$ if $\widetilde{A} \Subset \widetilde{B}$ and $\widetilde{B} \Subset \widetilde{A}$.

Notation 1. *Let $\mathbb{U}$ be a set, $I = [0, 1]$ the unit interval of the real numbers, for every $r \in I$, with $\underline{r}$ we denote the constant mapping $\underline{r} : \mathbb{U} \to I$ defined by $\underline{r}(u) = r$, for every $u \in \mathbb{U}$.*

Definition 5.4. [30] The SVN-set $\langle \mathbb{U}, \underline{0}, \underline{0}, \underline{1} \rangle$ is said to be the *neutrosophic empty set* over $\mathbb{U}$ and it is denoted by $\widetilde{\emptyset}$, or more precisely by $\widetilde{\emptyset}_{\mathbb{U}}$ in case it is necessary to specify the corresponding universe set.

Definition 5.5. [30] The SVN-set $\langle \mathbb{U}, \underline{1}, \underline{1}, \underline{0} \rangle$ is said to be the *neutrosophic absolute set* over $\mathbb{U}$ and it is denoted by $\widetilde{\mathbb{U}}$.

In our class, the data structure used to represent a SVN-set is a dictionary (also called associative array) which uses the elements of the universe set as keys and associates them with a list of three floating-point numbers corresponding to the degrees of membership, indeterminacy and non-membership respectively. This dictionary, referred as the `__neutrosophicset` property, is stored in conjunction with the universe set, referred as the `__universe` property, to which it is inseparably linked. Indeed, it is no coincidence that the class `NSuniverse` does not provide any method for allowing modifications (such as insertions or removals) of the elements of an object of type universe set since such operations could disrupt the consistency of the SVN-sets defined on it. The `NSset` class is described by the following UML diagram.

NSset
__universe : object of the class NSuniverse __neutrosophicset : dictionary of lists
__init__(*args) : constructor with generic arguments __setDegree(u,i,r) : assigns r to the i-th degree of element u setMembership(u, mu) : assigns the membership degree to u setIndeterminacy(u, sigma) : assigns the indeterminacy degree to u setNonMembership(u, omega) : assigns the non-membership degree to u setElement(u, triple) : assigns simultaneously the three degrees to u getUniverse() : returns the universe of the SVN-set get() : returns the dictionary containing the degrees of each element getElement(u) : returns the triple the degrees of the element u __getDegree(u,i) : returns the i-th degree of element u getMembership(u) : returns the membership degree of u getIndeterminacy(u) : returns the indeterminacy degree of u getNonMembership(u) : returns the non-membership degree of u setEmpty() : makes equal to the neutrosophic empty set setAbsolute() : makes equal to the absolute SVN-set cardinality() : returns the number of elements of the SVN-set isNSsubset(nset) : checks if is a neutrosophic subset of another one isNSsuperset(nset) : checks if is a neutrosophic superset of another one NSunion(nset) : returns the neutrosophic union with another one NSintersection(nset) : returns the neutrosophic intersection with another one isNSdisjoint(nset) : checks if is neutrosophically disjoint with another one NScomplement() : returns the neutrosophic complement NSdifference(nset) : returns the neutrosophic difference with another one __eq__() : checks if two SVN-sets are equal overloading the == operator __ne__() : checks if two SVN-sets are different overloading the != operator __add__(nset) : neutrosophic union overloading the + operator __and__(nset) : neutrosophic intersection overloading the & operator __invert__() : neutrosophic complement overloading the ∼ operator __sub__(nset) : neutrosophic difference overloading the − operator __le__(nset) : neutrosophic subset overloading the <= operator __ge__(nset) : neutrosophic superset overloading the >= operator __str__() : returns the SVN-set in string format __format__(spec) : returns the formatted string of the SVN-set respect to a specifer __repr__() : returns a detailed representation of the SVN-set

The constructor method of this class accepts one or two parameters and allows us to define a SVN-set in several different ways:

- in the form `NSset(`*universe set*`)` assigning to it as its only parameter the universe set (expressed as a list, tuple, or string) over which it is defined and thus creating an empty SVN-set $\widetilde{\emptyset}$,
- in the form `NSset(`*universe set*, *values*`)` passing two parameters, the first of which is a universe set and the second an enumeration (expressed as a list, tuple, or string) of triples of real values representing the degree of membership, indeterminacy and non-membership of all the elements of the universe set, or
- in the form `NSset(`*neutrosophic set*`)` by copying another object of the type `NSset`.

The basic steps of this method, expressed in pseudo-code, are described in the following algorithm.

Constructor method of the class `NSset`

```
Function __init__(args):
    Create a dictionary neutrosophicset
    Get the length of args
    if length = 1 then
        if args is a list, a tuple, a string or an object of the class NSuniverse then
            Create universe from args and set neutrosophicset empty
        else if args is an object of the class NSset then
            Get universe and neutrosophicset from args
        else
            Raise an Exception
    else if length = 2 then
        Use the same constructor with the first parameter of args to obtain an object of
          type NSset from which to derive universe and set the list values equal to the
          second parameter of args
        if values is a list or a tuple then
            if length of values is different from the length of universe then
                Raise an Exception
            Assigns to each element of universe the values of the corresponding triple of
              values
        else if args is a string then
            Converts args to a list of triples and uses the same constructor with universe
              and such a list to obtain an object of type NSset from which to take
              neutrosophicset
        else
            Raise an Exception
    else
        Raise an Exception
    Stores universe and neutrosophicset in the properties __universe and
      __neutrosophicset respectively
```

Note how every possible error condition – such as, for example, attempting to pass a variable of type other than `NSuniverse` or a list of values not consisting of triples of real numbers included in the $[0,1]$ interval or, again, using it with fewer than one or more than two parameters – is intercepted in the code and reported to the client by raising an appropriate exception.

The Python code corresponding to the constructor method of the `NS_set` class is given below.

```python
from .ns_universe import NSuniverse
from .ns_util import NSreplace, NSstringtoTriplesList, NSsplitText

class NSset:

    degreename = ["membership", "indeterminacy", "non-membership"]
    reprmaxlength = 64

    def __init__(self, *args):
        neutrosophicset = dict()
        length = len(args)
        if length == 1:
            element = args[0]
            if type(element) in [list, tuple, str, NSuniverse]:
                universe = NSuniverse(element)
                for e in universe.get():
                    neutrosophicset[e] = [0,0,1]
            elif type(element) == NSset:
                universe = element.getUniverse()
                for e in universe:
                    neutrosophicset[e] = element.getElement(e)
            else:
                raise ValueError("value not compatible with the type universe set")
        elif length == 2:
            nset = NSset(args[0])
            universe = nset.getUniverse()
            values = args[1]
            if type(values) in [list ,tuple]:
                if len(values) != len(universe):
                    raise IndexError("the number of value triples does not correspond with the number of elements")
                for i in range(len(universe)):
                    elem = universe[i]
                    t = values[i]
                    if type(t) not in [tuple,list] or len(t) !=3:
                        raise IndexError("the second parameter of the constructor method must contain only triple")
                    t = [float(t[j]) for j in range(3)]
                    for j in range(3):
                        if not 0 <= t[j] <= 1:
                            raise ValueError(f"incompatible {self.degreename[j]} degree value")
                    neutrosophicset[elem] = t
            elif type(values) == str:
                tpl_list = NSstringtoTriplesList(values)
                nset = NSset(universe, tpl_list)
```

```
                neutrosophicset = nset.get()
            else:
                raise ValueError("the second parameter of the constructor method must contain a list of triples of real numbers")
        else:
            raise IndexError("the number of parameters do not match those of the constructor method")
        self.__universe = NSuniverse(universe)
        self.__neutrosophicset = neutrosophicset
```

Let us observe that to enable the parsing of strings containing triples with the values of membership degrees, indeterminacy and non-membership, variously expressed, were used the function `NSstringToTriplesList` contained in the utility file `NS_util.py`.

Similar to what we have seen previously for objects of type universe set, the constructor method of the class `NSset` also allows for a multiplicity of expressions thanks to which we can define neutrosophic sets with a direct and informal notation by means of lists, tuples or strings of elements separated indifferently by commas or semicolons.
For example, the SVN-set $\langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ over the universe set $\mathbb{U} = \{a, b, c\}$ defined by:

$$\left\langle \frac{a}{(0.5, 0.3, 0.2)}, \frac{b}{(0.6, 0.2, 0.3)}, \frac{c}{(0.4, 0.2, 0.7)} \right\rangle$$

can be defined as an object of the class `NSset` in any of the following ways mutually equivalent:

- `A=NSset("a,b,c",[(0.5,0.3,0.2),(0.6,0.2,0.3),(0.4,0.2,0.7)])` as a list of tuples,
- `A=NSset("a,b,c",[[0.5,0.3,0.2],[0.6,0.2,0.3],[0.4,0.2,0.7]])` as a list of lists,
- `A=NSset("a,b,c",[[0.5,0.3,0.2],(0.6,0.2,0.3),[0.4,0.2,0.7]])` as a mixed list of lists and tuples,
- `A=NSset("a,b,c",([0.5,0.3,0.2],[0.6,0.2,0.3],[0.4,0.2,0.7]))` as a tuple of lists,
- `A=NSset("a,b,c",((0.5,0.3,0.2),(0.6,0.2,0.3),(0.4,0.2,0.7)))` as a tuple of tuples,
- `A=NSset("a,b,c",((0.5,0.3,0.2),[0.6,0.2,0.3],(0.4,0.2,0.7)))` as a mixed tuple of tuples and lists,
- `A=NSset("a,b,c","[0.5,0.3,0.2],(0.6,0.2,0.3);[0.4,0.2,0.7]")` as a string containing lists and tuples

where the universe set can also be expressed in any equivalent form as `U=NSuniverse("{a,b,c}")`, `U=NSuniverse("[a,b;c]")` or `U=NSuniverse("(a;b,c)")` so that it can be used later in the definition of the SVN-set in the form:

- `A=NSset(U,"[0.5,0.3,0.2],(0.6,0.2,0.3);[0.4,0.2,0.7]")`

A SVN-set $\langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ already defined over a universe set $\mathbb{U}$ can be subsequently modified by assigning to each of its generic elements $u \in \mathbb{U}$ its degrees of membership $mu = \mu_A(u)$, indeterminacy $sigma = \sigma_A(u)$ and non-membership $omega = \omega_A(u)$) separately through the methods:

- `setMembership(self,` u`,` mu`)`
- `setIndeterminacy(self,` u`,` $sigma$`)`, and
- `setNonMembership(self,` u`,` $omega$`)`

or by assigning in one shot the entire $triple = (\mu_A(u), \sigma_A(u), \omega_A(u))$ of values using the method:

- `setElement(self,` u`,` $triple$`)`

all of which are based on the private method `__setDegree(self,` u`,` i`,` v`)` that assigns the value v to the i-th degree (for $i = 0, 1, 2$, which correspond in the order to membership, indeterminacy and non-membership degree) of a given element u of the current SVN-set. Obviously, in the latter method we take into account the fact that the element u must belong to the corresponding universe set and that the membership degrees must be real values included in the unit interval I and if not, appropriate exceptions will be raised.

```
    def __setDegree(self, u, i, r):
        u = str(u)
        if u not in self.getUniverse():
            raise IndexError('non-existent element')
        r = float(r)
        if not (0 <= r <= 1):
            raise ValueError(f"incompatible {self.degreename[i]} degree value")
        self.__neutrosophicset[u][i] = r

    def setMembership(self, u, mu):
        self.__setDegree(u, 0, mu)

    def setIndeterminacy(self, u, sigma):
        self.__setDegree(u, 1, sigma)

    def setNonMembership(self, u, omega):
        self.__setDegree(u, 2, omega)

    def setElement(self, u, triple):
        if type(triple) == str:
            sostituz = { "(":"", ")":"", ",":" ", ";":" " }
            triple = NSreplace(triple, sostituz).split()
        else:
            triple = list(triple)
        if len(triple) != 3:
            raise ValueError('error in the number of parameters passed')
        triple = [float(e) for e in triple]
        for i in range(3):
            self.__setDegree(u, i, triple[i])
```

Three basic methods called `getUniverse()`, `get()` and `getElement(`u`)` respectively return to us the universe set of a given SVN-set as a string list of its elements, the neutrosophic set itself as a dictionary having for keys the elements of the universe and for values the triples of the degrees of membership, indeterminacy and non-membership as well as the triple of the degrees of a given element $u \in \mathbb{U}$.

```
    def getUniverse(self):
        return self.__universe.get()

    def get(self):
        return self.__neutrosophicset

    def getElement(self, u):
        u = str(u)
        if u not in self.getUniverse():
            raise IndexError('non-existent element')
        return self.__neutrosophicset[u]
```

Given a SVN-set $\langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ over a universe set $\mathbb{U}$, The values of the degrees of membership $\mu_A(u)$, indeterminacy $\sigma_A(u)$ and non-membership $\omega_A(u)$ of a generic element $u \in \mathbb{U}$ can be obtained by the methods:

- `getMembership(`u`)`,
- `getIndeterminacy(`u`)`, and
- `getNonMembership(`u`)`

which are all based on the private method `__getDegree(self,`u`,`i`)` that returns the value of the i-th degree (for $i = 0, 1, 2$, which correspond in the order to membership, indeterminacy and non-membership degree) of a given element u of the current SVN-set. Let us note that in the latter method, we consider the prerequisite that the element u must belong to the corresponding universe set and if this condition is not satisfied a suitable exception will be raised.

```
    def __getDegree(self, u, i):
        u = str(u)
        if u not in self.getUniverse():
            raise IndexError('non-existent element')
        return self.__neutrosophicset[u][i]

    def getMembership(self, u):
        return self.__getDegree(u, 0)

    def getIndeterminacy(self, u):
        return self.__getDegree(u, 1)

    def getNonMembership(self, u):
        return self.__getDegree(u, 2)
```

In case we need the empty neutrosophic set $\widetilde{\emptyset}$ or the neutrosophic absolute set $\widetilde{\mathbb{U}}$ over an universe set $\mathbb{U}$ we can refer to the methods `setEmpty()` and `setAbsolute()`, respectively.

```
    def setEmpty(self):
        for e in self.__universe.get():
            self.__neutrosophicset[e] = [0, 0, 1]

    def setAbsolute(self):
        for e in self.__universe.get():
            self.__neutrosophicset[e] = [1, 1, 0]
```

The on-screen printing in text format of objects of type `NSset` as well as their complete representation is achieved by overloading the special methods `__str__` and `__repr__` as follows.

```
    def __str__(self, tabularFormat=False):
        if tabularFormat == True:
            (dashes, elemwidth, valwidth) = ("-"*64, 10, 14)
            s = "\n            |   membership    |  indeterminacy | non-membership |\n" + dashes + "\n"
            for e in self.getUniverse():
                (mu, sigma, omega) = self.getElement(e)
                s += f" {str(e):{elemwidth}} | {mu:{valwidth}} | {sigma:{valwidth}} | {omega:{valwidth}} |\n"
            s += dashes + "\n"
        else:
            elems = []
            for e in self.getUniverse():
                (mu, sigma, omega) = self.getElement(e)
                elems.append(f"{e}/({mu},{sigma},{omega})")
            s = "< " + ", ".join(elems) + " >"
            s = NSsplitText(s, self.reprmaxlength)
        return s

    def __repr__(self):
        return f"Neutrosophic set: {str(self)}"
```

In particular, the special method `__str__` allow us to print on the screen a SVN-set in both the simplified representation (which is the default option) and in the clearer and more extensive tabular representation.

Furthermore, in order to be able to choose to print a SVN-set in the simplified representation or in the tabular one even in interactive use or writing client code, it was chosen to redefine the special method `__format__` so that it recognizes the new custom format specifier `t` that corresponds to printing in tabular format objects of type `NSset`.

```
    def __format__(self, spec):
        if spec == "t":
            result = self.__str__(tabularFormat=True)
        else:
            result = self.__str__(tabularFormat=False)
        return result
```

Thanks to the redefinition by overloading of the special methods `__eq__` and `__ne__` we can use the operators of equality == and diversity != directly to objects of type `NSset`.

```
    def __eq__(self, nset):
        if self.getUniverse() != nset.getUniverse():
            raise ValueError("the two neutrosophic sets cannot be defined on different universe sets")
        equal = self.isNSsubset(nset) and nset.isNSsubset(self)
        return equal

    def __ne__(self, nset):
        if self.getUniverse() != nset.getUniverse():
            raise ValueError("the two neutrosophic sets cannot be defined on different universe sets")
        different = not (self == nset)
        return different
```

In order to better illustrate how the above methods are used, let us consider the following example of code executed interactively in the Python console.

```
>>> from pyns.ns_universe import NSuniverse
>>> from pyns.ns_set import NSset
>>> U = NSuniverse("a,b,c")
>>> A = NSset(U)
>>> A.setElement('a', (0.8,0.2,0.1))
>>> A.setElement('c', (0.3,0.2,0.4))
>>> A.getMembership('a')
0.8
>>> A.getNonMembership('c')
0.4
>>> print(A.getElement(c))
[0.3, 0.2, 0.4]
>>> A.setIndeterminacy('b',0.9)
>>> print(A)
< a/(0.8,0.2,0.1), b/(0.0,0.9,1.0), c/(0.3,0.2,0.4) >
>>> print(f"{A:t}")
            |   membership    |  indeterminacy | non-membership |
----------------------------------------------------------------
 a          |             0.8 |            0.2 |            0.1 |
 b          |               0 |            0.9 |              1 |
 c          |             0.3 |            0.2 |            0.4 |
----------------------------------------------------------------
>>> A.setAbsolute()
>>> print(A)
< a/(1,1,0), b/(1,1,0), c/(1,1,0) >
```

From the example above, one might assume that creating universe sets and SVN-sets requires manual definition. However, the open structure of our framework actually enables us to define objects of type `NSuniverse` and `NSset` dynamically within the code, commonly referred to as defining them 'on the fly'. This dynamic approach is especially advantageous when dealing with SVN-sets of considerable cardinality, as illustrated in the following Python code.

```
from pyns.ns_universe import NSuniverse
from pyns.ns_set import NSset
from random import random

```

```python
lst = [(i,j) for i in range(1,6) for j in range(1,4)]

U = NSuniverse(lst)
A = NSset(U)

for u in U.get():
    triple = [round(random(), 2) for k in range(3)]
    A.setElement(u, triple)

print(f"The following SVNS-set has cardinality {A.cardinality()}: {A:t}")
```

which produces output of the type:

```
The following SVNS-set has cardinality 15:
             |   membership    |  indeterminacy | non-membership |
-----------------------------------------------------------------
 (1, 1)      |            0.55 |            0.1 |            0.5 |
 (1, 2)      |            0.39 |           0.92 |           0.09 |
 (1, 3)      |            0.33 |           0.29 |           0.25 |
 (2, 1)      |            0.16 |            0.9 |           0.43 |
 (2, 2)      |            0.71 |           0.52 |           0.33 |
 (2, 3)      |            0.65 |           0.38 |           0.04 |
 (3, 1)      |            0.95 |           0.14 |           0.94 |
 (3, 2)      |            0.74 |           0.02 |           0.01 |
 (3, 3)      |            0.77 |           0.63 |           0.19 |
 (4, 1)      |            0.18 |           0.75 |           0.15 |
 (4, 2)      |            0.49 |           0.92 |           0.75 |
 (4, 3)      |            0.34 |           0.17 |           0.88 |
 (5, 1)      |            0.88 |            0.6 |           0.83 |
 (5, 2)      |             0.5 |           0.56 |            0.8 |
 (5, 3)      |            0.38 |           0.47 |            0.2 |
-----------------------------------------------------------------
```

To verify that a SVN-set expressed as an object of type `NSset` is neutrosophically contained in another SVN-set, we may resort to the method `isNSsubset(`*nset*`)` which, similarly to the built-in `issubset()` method available for objects of type `set`, returns the Boolean value `True` if the current SVN-set is neutrosophically contained in the second SVN-set *nset* passed as a parameter or the value `False` otherwise.

```python
    def isNSsubset(self, nset):
        if self.getUniverse() != nset.getUniverse():
            raise ValueError("the two neutrosophic sets cannot be defined on different universe sets")
        if self.getUniverse() != nset.getUniverse():
            return False
        else:
            result = True
            for e in self.getUniverse():
                (muA, sigmaA, omegaA) = self.getElement(e)
                (muB, sigmaB, omegaB) = nset.getElement(e)
                if (muA > muB) or (sigmaA > sigmaB) or (omegaA < omegaB):
                    result = False
                    break
            return result
```

Based on `isNSsubset`, it is then immediate to define the method `isNSsuperset(`*nset*`)` (analogous to the built-in `issuperset` method) which returns `True` if the current SVN-set neutrosophically contains the second SVN-set *nset* passed as a parameter or `False` otherwise.

```
def isNSsuperset(self, nset):
    if self.getUniverse() != nset.getUniverse():
        raise ValueError("the two neutrosophic sets cannot be defined on different universe sets")
    return nset.isNSsubset(self)
```

In both cases, it is preliminarily verified that the two SVN-sets are defined on the same universe set and if not an appropriate exception is raised.

The following code executed in interactive mode in the Python console illustrates the use of the methods just described.

```
>>> from pyns.ns_universe import NSuniverse
>>> from pyns.ns_set import NSset
>>> A = NSset("a, b, c", "(0.3,0,0.5), (0.7,0.2,0.2), (0.1,0.5,0.4)")
>>> print(A)
< a/(0.3,0.0,0.5), b/(0.7,0.2,0.2), c/(0.1,0.5,0.4) >
>>> B = NSset("a, b, c", "(0.4,0.2,0.3), (0.8,0.3,0.1), (0.2,0.5,0.2)")
>>> print(B)
< a/(0.4,0.2,0.3), b/(0.8,0.3,0.1), c/(0.2,0.5,0.2) >
>>> print(A.isNSsubset(B))
True
>>> print(A.isNSsuperset(B))
False
```

Definition 5.6. [19] The *neutrosophic union* of two SVN-sets $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ and $\widetilde{B} = \langle \mathbb{U}, \mu_B, \sigma_B, \omega_B \rangle$, denoted by $\widetilde{A} \Cup \widetilde{B}$, is the neutrosophic set defined by $\langle \mathbb{U}, \mu_A \vee \mu_B, \sigma_A \vee \sigma_B, \omega_A \wedge \omega_B \rangle$.

Definition 5.7. [19] The *neutrosophic intersection* of two SVN-sets $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ and $\widetilde{B} = \langle \mathbb{U}, \mu_B, \sigma_B, \omega_B \rangle$, denoted by $\widetilde{A} \Cap \widetilde{B}$, is the neutrosophic set defined by $\langle \mathbb{U}, \mu_A \wedge \mu_B, \sigma_A \wedge \sigma_B, \omega_A \vee \omega_B \rangle$.

Definition 5.8. [30] Let $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ and $\widetilde{B} = \langle \mathbb{U}, \mu_B, \sigma_B, \omega_B \rangle$ be two SVN-sets over $\mathbb{U}$. We say that $\widetilde{A}$ and $\widetilde{B}$ are *neutrosophically disjoint* if $\widetilde{A} \Cap \widetilde{B} \equiv \widetilde{\emptyset}$. On the contrary, if $\widetilde{A} \Cap \widetilde{B} \not\equiv \widetilde{\emptyset}$ we say that $\widetilde{A}$ *neutrosophically meets* $\widetilde{B}$ (or that $\widetilde{A}$ and $\widetilde{B}$ neutrosophically meet each other).

Within our `NSset` class, the neutrosophic union and neutrosophic intersection were implemented through the methods `NSunion()` and `NSintersection()`, respectively. These methods mirror the built-in Python methods `union` e `intersection` for objects of type `set` and they are both based on the private method `__NSoperation(self,` *nset*, *fm*, *fs*, *fo*`)` corresponding to a generic operation and that returns the neutrosophic set obtained from the current SVN-set and the second SVN-set *nset* by applying the three functions *fm*, *fs*, *fo* passed as parameters to their membership, indeterminacy and non-membership degrees respectively.

```
    def __NSoperation(self, nset, fm, fs, fo):
        if self.getUniverse() != nset.getUniverse():
            raise ValueError("the two neutrosophic sets cannot be defined on different universe sets")
        if callable(fm) == False or callable(fs) == False or callable(fo) == False:
            raise  ValueError("the last three parameters must be functions")
        C = NSset(self.__universe)
        for e in self.getUniverse():
            (muA, sigmaA, omegaA) = self.getElement(e)
            (muB, sigmaB, omegaB) = nset.getElement(e)
            triple = [fm(muA, muB), fs(sigmaA, sigmaB), fo(omegaA, omegaB)]
            C.setElement(e, triple)
        return C

    def NSunion(self, nset):
        C = self.__NSoperation(nset, max, max, min)
        return C

    def NSintersection(self, nset):
        C = self.__NSoperation(nset, min, min, max)
        return C
```

We illustrate the above methods with an example of code executed interactively in the Python console.

```
>>> from pyns.ns_universe import NSuniverse
>>> from pyns.ns_set import NSset
>>> U = NSuniverse("a, b, c")
>>> A = NSset(U,"(0.3,0,0.5), (0.7,0.2,0.2), (0.1,0.5,0.4)")
>>> print(A)
< a/(0.3,0.0,0.5), b/(0.7,0.2,0.2), c/(0.1,0.5,0.4) >
>>> B = NSset(U,"(0.4,0.2,0.3), (0.8,0.3,0.1), (0.2,0.5,0.2)")
>>> print(B)
a/(0.4,0.2,0.3), b/(0.8,0.3,0.1), c/(0.2,0.5,0.2) >
>>> C = A.NSunion(B)
>>> print(C)
< a/(0.4,0.2,0.3), b/(0.8,0.3,0.1), c/(0.2,0.5,0.2) >
>>> D = A.NSintersection(B)
>>> print(D)
< a/(0.3,0.0,0.5), b/(0.7,0.2,0.2), c/(0.1,0.5,0.4) >
```

The method `isNSdisjoint(nset)` returns the Boolean value `True` if the current SVN-set is neutrosophically disjoint from the second SVN-set passed as parameter or the value `False` otherwise.

```
    def isNSdisjoint(self, nset):
        nsempty = NSset(self.__universe)
        disjoint = self.NSintersection(nset) == nsempty
        return disjoint
```

An example of the use of this method is provided in the following code executed in the console Python interactive.

```
>>> from pyns.ns_universe import NSuniverse
>>> from pyns.ns_set import NSset
>>> A = NSset("a, b, c", "(0.3,0,0.5), (0.7,0,1), (0,0.5,1)")
>>> B = NSset("a, b, c", "(0,0.8,1), (0,0.1,0.2), (0.1,0.0,0.4)")
>>> print(A.isNSdisjoint(B))
True
```

Definition 5.9. [27,30] Let $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ be a SVN-set over the universe set $\mathbb{U}$, the *neutrosophic complement* (or, simply, the complement) of $\widetilde{A}$, denoted by $\widetilde{A}^{\complement}$, is the SVN-set $\widetilde{A}^{\complement} = \langle \mathbb{U}, \omega_A, \underline{1} - \sigma_A, \mu_A \rangle$ that is $\widetilde{A}^{\complement} = \{(u, \omega_A(u), 1 - \sigma_A(u), \mu_A(u)) : u \in \mathbb{U}\}$.

Although the neutrosophic difference of two SVN-sets $\widetilde{A}$ and $\widetilde{B}$ can be defined (in analogy with ordinary sets) as the neutrosophic intersection of the first set with the neutrosophic complement of the second set, that is, as $\widetilde{A} \Cap \widetilde{B}^{\complement}$, for our purposes it is preferable to provide an explicit and operational definition.

Definition 5.10. The *neutrosophic difference* of two SVN-sets $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ and $\widetilde{B} = \langle \mathbb{U}, \mu_B, \sigma_B, \omega_B \rangle$, denoted by $\widetilde{A} \setminus\!\setminus \widetilde{B}$, is the neutrosophic set defined by $\langle \mathbb{U}, \mu_A \wedge \omega_B, \sigma_A \wedge (\underline{1} - \sigma_B), \omega_A \vee \mu_B \rangle$.

The latter operations have also been implemented in the class `NSset` through the methods `NScomplement()` and `NSdifference()`.

```
    def NScomplement(self):
        C = NSset(self.__universe)
        for e in self.getUniverse():
            (muA, sigmaA, omegaA) = self.getElement(e)
            triple = [omegaA, 1 - sigmaA, muA]
            C.setElement(e, triple)
        return C

    def NSdifference(self, nset):
        if self.getUniverse() != nset.getUniverse():
            raise ValueError("the two neutrosophic sets cannot be defined on
    different universe sets")
        C = NSset(self.__universe)
        for e in self.getUniverse():
            (muA, sigmaA, omegaA) = self.getElement(e)
            (muB, sigmaB, omegaB) = nset.getElement(e)
            triple = [min(muA,omegaB), min(sigmaA,1-sigmaB), max(omegaA,muB)]
            C.setElement(e, triple)
        return C
```

The following code example executed interactively in the Python console illustrates the use of the two methods that have been just described.

```
>>> from pyns.ns_universe import NSuniverse
>>> from pyns.ns_set import NSset
>>> U = NSuniverse('a','b','c')
```

```
>>> A = NSset(U, "(0.5,0.3,0.2), (0.6,0.2,0.3), (0.4,0.2,0.7)")
>>> print(A)
< a/(0.5,0.3,0.2), b/(0.6,0.2,0.3), c/(0.4,0.2,0.7) >
>>> B = NSset(U, "(0.2,0.2,0.2), (0.4,0.1,0.6), (0.8,0.3,0.1)")
>>> print(B)
< a/(0.2,0.2,0.2), b/(0.4,0.1,0.6), c/(0.8,0.3,0.1) >
>>> C = A.NScomplement()
>>> print(C)
< a/(0.2,0.7,0.5), b/(0.3,0.8,0.6), c/(0.7,0.8,0.4) >
>>> D = A.NSdifference(B)
>>> print(D)
< a/(0.2,0.3,0.2), b/(0.6,0.2,0.4), c/(0.1,0.2,0.8) >
```

To facilitate even more streamlined and intuitive use of neutrosophic set operations, especially in the interactive use of the framework, overloading was then used to redefine the special methods `__add__`, `__and__`, `__invert__`, `__sub__`, `__le__`, and `__ge__`, respectively referred to the operators $+$, &, $\sim$, $-$, $<=$, and $>=$ for use with objects of type `NSset` by making them coincide with the methods `NSunion()`, `NSintersection()`, `NScomplement()`, `NSdifference()`, `isNSsubset()` and `isNSsuperset()`, thus obtaining the correspondence summarized in the following table.

	class method	**symbol**	**operator**
neutrosophic union	`NSunion()`	$\Cup$	$+$
neutrosophic intersection	`NSintersection()`	$\Cap$	&
neutrosophic complement	`NScomplement()`	$\complement$	$\sim$
neutrosophic difference	`NSdifference()`	$\setminus\!\!\setminus$	$-$
neutrosophic subset	`isNSsubset()`	$\Subset$	$<=$
neutrosophic superset	`isNSsuperset()`	$\Supset$	$>=$

The following example illustrates how the methods and operators defined above can be easily and profitably used in the interactive mode by means of the Python console.

```
>>> from pyns.ns_universe import NSuniverse
>>> from pyns.ns_set import NSset
>>> U = NSuniverse('a','b','c')
>>> A = NSset(U, "(0.5,0.3,0.2), (0.6,0.2,0.3), (0.4,0.2,0.7)")
>>> print(A)
< a/(0.5,0.3,0.2), b/(0.6,0.2,0.3), c/(0.4,0.2,0.7) >
>>> B = NSset(U, "(0.2,0.2,0.2), (0.4,0.1,0.6), (0.8,0.3,0.1)")
>>> print(B)
< a/(0.2,0.2,0.2), b/(0.4,0.1,0.6), c/(0.8,0.3,0.1) >
>>> print(A + B)
< a/(0.5,0.3,0.2), b/(0.6,0.2,0.3), c/(0.8,0.3,0.1) >
>>> print(A & B)
< a/(0.2,0.2,0.2), b/(0.4,0.1,0.6), c/(0.4,0.2,0.7) >
>>> print(~A)
< a/(0.2,0.7,0.5), b/(0.3,0.8,0.6), c/(0.7,0.8,0.4) >
>>> F = A - B
>>> print(F)
< a/(0.2,0.3,0.2), b/(0.6,0.2,0.4), c/(0.1,0.2,0.8) >
>>> print(F <= A)
True
>>> print(F == A & ~B)
True
```

6. The `NSmapping` class

The mappings between two universe sets and the main operations involving them are represented and handled through the `NSmapping` class that uses both the `NSuniverse` class and the `NSset` class.

For every mapping $f : \mathbb{U} \to \mathbb{V}$, the class stores the domain $\mathbb{U}$ and the codomain $\mathbb{V}$ as objects of type `NSuniverse` in the properties `__domain` and `__codomain` respectively, as well as the correspondence between each generic element $u \in \mathbb{U}$ and its value $f(u) \in \mathbb{V}$ by a dictionary corresponding to the property `__map = {u: f(u) for u in 𝕌}`.
The class is briefly described in the following UML diagram.

NSmapping
__domain : object of the class NSuniverse __codomain : object of the class NSuniverse __map : dictionary with keys in domain and values in codomain
__init__(*args) : constructor with generic arguments getDomain() : returns the universe set corresponding to the domain getCodomain() : returns the universe set corresponding to the codomain getMap() : returns the dictionary containing the element-value pairs setValue(u,v) : assigns the value v to the element u getValue(u) : returns the value of the element u by the mapping getFibre(v) : returns the fibre of v as a list of elements of the domain NSimage(nset) : returns the neutrosophic image of a SVN-set by the mapping NScounterimage(nset) : returns the neutrosophic inverse image of a SVN-set __eq__() : checks if two mappings are equal overloading the == operator __ne__() : checks if two mappings are different overloading the ! = operator __str__() : returns the mapping in string format __repr__() : returns a detailed representation of the mapping

The constructor method accepts one or three arguments and allows us to define a mapping in several different ways:

- in the form `NSmapping(`*domain*`,` *codomain*`,` *values*`)` where *domain* and *codomain* are both universe sets expressed in any of the ways already seen above, namely as tuples, lists, strings, or instances of the class `NSuniverse`, while *values* is an enumeration of codomain values neatly corresponding to domain values which can be expressed indifferently as a tuple, list, string, dictionary or extended dictionary,
- in the form `NSmapping(`*values*`)` where *values* is either a regular Python dictionary or an extended dictionary; in this case the universe sets related to the domain and codomain will be created automatically by collecting respectively the keys and values of the dictionary passed as parameter, without repeating their values and checking that no error condition occurs,
- in the form `NSmapping(`*mapping*`)` by copying another object of the type `NSmapping`.

The basic steps of this method are described in the following algorithm.

Constructor method of the class `NSmapping`

Function `__init__`(*args*)**:**
- Create a dictionary *map*
- Get the *length* of *args*
- **if** *length* $= 0$ **then**
  - Raise an Exception
- **else if** *length* $= 1$ **then**
  - **if** *args* ***is*** *an object of type* `NSmapping` **then**
    - Copies the properties in the current object
  - **else if** *args* ***is*** *a dictionary* **then**
    - Copy *args* to *map* and gets *domain* and *codomain* as keys and values of *args*, respectively
  - **else if** *args* ***is*** *an extended dictionary* **then**
    - Gets the dictionary from the string and passes it to the same constructor to obtain an object of type `NSmapping` from which to derive *domain*, *codomain* and the dictionary of correspondence *map*
- **else if** *length* $= 3$ **then**
  - Attempts to take the first three parameters of *args* to assign them respectively to the universe sets *domain*, *codomain* and the object *values*
  - **if** *values* ***is*** *a dictionary or an extended dictionary* **then**
    - Passes *values* to the same constructor to obtain an object of type `NSmapping` and, if its domain and codomain are compatible with those passed as parameters, derive the dictionary of correspondences *map*
  - **else if** *values* ***is*** *a list, a tuple or a string* **then**
    - **if** *values* ***is*** *a list or a tuple* **then**
      - Converts *values* to a list of strings;
    - **else**
      - Split *values* and turns it into a string list;
    - **if** *the length of values* $\neq$ *cardinality of the domain* **then**
      - Raise an Exception
    - **if** *the set of values* ***is not*** *contained in the codomain* **then**
      - Raise an Exception
    - Neatly stores the elements of *values* as values of the *map* dictionary whose keys are the elements of the *domain*
  - **else**
    - Raise an Exception
- **else**
  - Raise an Exception
- Stores *domain*, *domain* and *map* in the properties *__domain*, *__codomain* and *__map* respectively

Let us note how every possible error condition – such as recalling it with only one parameter that is not an object `NSmapping`, with a number of parameters other than one and three or, again, passing parameters that are not the two universe sets and a list of values however expressed – is intercepted in the code and reported to the client by raising an appropriate exception.

The Python code corresponding to the constructor method of this class is given below.

```
from .ns_universe import NSuniverse
from .ns_set import NSset
from .ns_util import NSreplace, NSstringToDict, NSisExtDict

class NSmapping:

    def __init__(self, *args):
        map = dict()
        length = len(args)
        if length == 0:
            raise ValueError("constructor method must have at least one parameter")
        elif length == 1:
            if type(args[0]) == NSmapping:
                domain = args[0].getDomain()
                codomain = args[0].getDomain()
                map = args[0].getMap()
            elif type(args[0]) == dict:
                map = args[0]
                domain = NSuniverse(list(map.keys()))
                codomain = NSuniverse(list(set(map.values())))
            elif type(args[0]) == str:
                try:
                    map_dict = NSstringToDict(args[0])
                except:
                    raise ValueError("invalid parameter")
                nsmap = NSmapping(map_dict)
                domain = nsmap.getDomain()
                codomain = nsmap.getCodomain()
                map = nsmap.getMap()
            else:
                raise ValueError("the type of the parameter do not match those of the constructor method")
        elif length == 3:
            try:
                domain = NSuniverse(args[0])
            except:
                raise ValueError("the first parameter of the constructor method must be a universe set")
            try:
                codomain = NSuniverse(args[1])
            except:
                raise ValueError("the second parameter of the constructor method must be a universe set")
            values = args[2]
            card_domain = domain.cardinality()
            if type(values)==dict or NSisExtDict(values)==True:
                nsmap = NSmapping(values)
                if set(nsmap.getDomain()) != set(domain):
                    raise ValueError("the indicated domain is incompatible with the definition of the mapping")
                if nsmap.getCodomain().isSubset(codomain) == False:
                    raise ValueError("the indicated codomain is incompatible with the definition of the mapping")
                map = nsmap.getMap()
```

```
            elif type(values) in [list, tuple, str]:
                if type(values) in [list, tuple]:
                    values = [str(e) for e in values]
                else:
                    sostituz = {"[": "", "]": "", "(": "", ")": "",
                                ",": " ", ";": " "}
                    values = NSreplace(values, sostituz).split()
                if len(values) != card_domain:
                    raise IndexError("the number of values passed does not coincide with the cardinality of the declared domain")
                values_set = set(values)
                codomain_set = set(codomain.get())
                if not values_set.issubset(codomain_set):
                    raise ValueError("one or more values do not belong to the declared codomain")
                for i in range(card_domain):
                    map[domain.get()[i]] = values[i]
            else:
                raise ValueError("the third parameter of the constructor method must express a obj match")
        else:
            raise IndexError("the number of parameters do not match those of the constructor method")
        self.__domain = domain
        self.__codomain = codomain
        self.__map = map
```

Note that, as in the case of the definition of the universe set constructor method, it is excluded that the third parameter, corresponding to the enumeration of values can be an object of type `set` since by its nature as an unordered collection of data would provide an ambiguous formulation of the mapping.

As in the case of the objects `NSuniverse` and `NSset`, for the mappings represented by objects of the type `NSmapping` much attention was paid to the usability and flexibility of the syntax which allows us to define mappings between universe sets in a variety of possible forms. For example, the mappings $f : \mathbb{U} \to \mathbb{V}$ between the universe sets $\mathbb{U} = \{a, b, c\}$ and $\mathbb{V} = \{1, 2\}$ and such that $f(a) = f(c) = 2$ and $f(b) = 1$ can be defined as an object of the class `NSmapping` in any of the following ways mutually equivalent:

- `NSmapping(['a','b','c'], [1,2], [2,1,2])` by using lists,
- `NSmapping(('a','b','c'), (1,2), (2,1,2))` by using tuples,
- `NSmapping("a,b,c", "1;2", "2,1,2")` by using strings,
- `NSmapping("(a,b,c)", "{1,2}", "(2;1;2)")` by using strings containing lists or tuples,
- `NSmapping(['a','b','c'], (1,2), "2,1,2"))` in a mixed form of lists, tuples and strings,
- `NSmapping({'a':2, 'b':1, 'c':2})` by using dictionaries,

- `NSmapping("'a':2, 'b':1, 'c':2")` by using a string that contains a matching of values in a dictionary format,
- `NSmapping("a->2, b->1, c->2")` by using an extended dictionary with the arrow notation,
- `NSmapping("a|->2, b|->1, c|->2")` by using an extended dictionary with the the "maps to" notation,
- `NSmapping("'a'->2, b|->1; (c->2)")` by using an extended dictionary in a mixed form,
- `NSmapping("a,b,c", "1,2", "a->2, b->1, c->2")` by declaring domain and codomain and using an extended dictionary,
- `NSmapping("a,b,c", "1,2", "c->2, a|->2; b|->1")` by declaring domain and codomain and using an extended dictionary in a mixed form and without a precise order,

as well as in different combinations of them or, again, by preliminarily defining one or both of the universe sets in any of the forms already seen above, by setting, for example, `U=NSuniverse("{a,b,c}")` and `V=NSuniverse((1,2)")`, so that they can be used later in the definition of the mapping in the form like:

- `NSmapping(U, V, [2,1,2])`, or
- `NSmapping(U, V, "a->2, b->1, c->2")`.

Three basic methods called `getDomain()`, `getCodomain()` and `getMap()` respectively return us the domain and codomain of the mapping as objects of type `NSuniverse` as well as the dictionary containing all the element-value pairs that define the mapping.

```
    def getDomain(self):
        return self.__domain.get()

    def getCodomain(self):
        return self.__codomain.get()

    def getMap(self):
        return self.__map
```

The method `setValue(`u, v`)` assigns a single value v by the current mapping to a specific element u of the domain.

```
    def setValue(self, u, v):
        u = str(u)
        v = str(v)
        if u not in self.__domain.get():
            raise IndexError('non-existent element in the domain of the
mapping')
        if v not in self.__codomain.get():
```

```
            raise IndexError('non-existent element in the codomain of the mapping')
        self.__map[u] = v
```

Instead, the method `getValue(u)` returns the value corresponding to an element u of the domain by the current mapping.

```
    def getValue(self, u):
        u = str(u)
        if u not in self.__domain.get():
            raise IndexError('non-existent element in the domain of the mapping')
        return self.__map[u]
```

In order to be able to easily print on the screen objects of type `NSmapping` in text format and to provide a complete representation of them, the special methods `__str__` and `__repr__` were overloaded as follows.

```
    def __str__(self):
        unvwidth = 28
        totwidth = unvwidth*2 + 8
        s = f"\n {str(self.__domain):>{unvwidth}}   ->   {str(self.__codomain):<{unvwidth}}\n"+"-"*totwidth+"\n"
        for e in self.__domain:
            s += f" {e:>{unvwidth}}  |->  {self.__map[e]:<{unvwidth}}\n"
        return s

    def __repr__(self):
        return f"Neutrosophic mapping: {str(self)}"
```

Thanks to the redefinition by overloading of the special methods `__eq__` and `__ne__` we can apply the operators of equality == and diversity != directly to objects of type `NSmapping`.

```
    def __eq__(self, g):
        if self.getDomain() != g.getDomain() or self.getCodomain() != g.getCodomain():
            return False
        else:
            equal = True
            for e in self.getDomain():
                if self.getValue(e) != g.getValue(e):
                    equal = False
                    break
            return equal

    def __ne__(self, g):
        different = not (self == g)
        return different
```

The following code executed in interactive mode in the Python console illustrates the use of the methods just described.

```
>>> from pyns.ns_universe import NSuniverse
>>> from pyns.ns_set import NSset
>>> from pyns.ns_mapping import NSmapping
>>> U = NSuniverse("a,b,c")
>>> V = NSuniverse(1,2)
>>> f = NSmapping(U,V, (2,1,2))
>>> print(f)
                  { a, b, c }   ->   { 1, 2 }
----------------------------------------------------------------
                            a  |->  2
                            b  |->  1
                            c  |->  2
>>> print(f.getValue('a'))
2
>>> g = NSmapping("a->2  b->1  c->2")
>>> print( f==g )
True
>>> print(g)
                  { a, b, c }   ->   { 1, 2 }
----------------------------------------------------------------
                            a  |->  2
                            b  |->  1
                            c  |->  2
>>> print(h.getDomain())
{ a, b, c }
>>> print(h.getCodomain())
{ 1, 2 }
>>> print(f.getMap())
{'a': '2', 'b': '1', 'c': '2'}
>>> h = NSmapping("a,b,c", "1,2", "a->2  b->1  c->2")
>>> print( f==h )
True
```

The method `getFibre(v)` returns the fibre of an element v of the codomain by the current mapping $f : \mathbb{U} \to V$, that is, the set of all elements of the domain whose value is v, i.e. $f^{-1}(\{v\}) = \{u \in \mathbb{U} : f(u) = v\}$. The corresponding code is given below.

```
def getFibre(v):
    v = str(v)
    if v not in self.__codomain.get():
        raise IndexError('non-existent element in the codomain of the mapping')
    fibre = list()
    for e in self.__map:
        if self.__map[e] == v:
            fibre.append(e)
    return fibre
```

Definition 6.1. [11, 22] Let $f : \mathbb{U} \to \mathbb{V}$ be a mapping between two universe sets $\mathbb{U}$ and $\mathbb{V}$, and $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$ be a SVN-set over $\mathbb{U}$. The *neutrosophic image* of $\widetilde{A}$ by f, denoted by

$\widetilde{f}\left(\widetilde{A}\right)$, is the SVN-set over $\mathbb{V}$ defined by:

$$\widetilde{f}\left(\widetilde{A}\right) = \left\langle \mathbb{V}, \mu_{f(A)}, \sigma_{f(A)}, \omega_{f(A)} \right\rangle$$

where the mappings $\mu_{f(A)} : \mathbb{V} \to I$, $\sigma_{f(A)} : \mathbb{V} \to I$ and $\omega_{f(A)} : \mathbb{V} \to I$ are defined respectively by:

$$\mu_{f(A)}(v) = \begin{cases} \sup\limits_{u \in f^{-1}(\{v\})} \mu_A(u) & \text{if } f^{-1}(\{v\}) \neq \emptyset \\ 1 & \text{otherwise} \end{cases},$$

$$\sigma_{f(A)}(v) = \begin{cases} \sup\limits_{u \in f^{-1}(\{v\})} \sigma_A(u) & \text{if } f^{-1}(\{v\}) \neq \emptyset \\ 1 & \text{otherwise} \end{cases},$$

$$\omega_{f(A)}(v) = \begin{cases} \inf\limits_{u \in f^{-1}(\{v\})} \omega_A(u) & \text{if } f^{-1}(\{v\}) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for every $v \in \mathbb{V}$.

The method `NSimage(self, nset)` returns the image of a SVN-set *nset* over the domain by the current mapping. The corresponding code is given below.

```
def NSimage(self, nset):
    result = NSset(self.__codomain)
    for v in self.getCodomain():
        fibre = self.getFibre(v)
        if fibre == []:
            triple = [1,1,0]
        else:
            mu_values = list()
            sigma_values = list()
            omega_values = list()
            for u in fibre:
                mu_values.append(nset.getMembership(u))
                sigma_values.append(nset.getIndeterminacy(u))
                omega_values.append(nset.getNonMembership(u))
            triple = [max(mu_values),max(sigma_values),min(omega_values)]
        result.setElement(v, triple)
    return result
```

Definition 6.2. [11, 22] Let $f : \mathbb{U} \to \mathbb{V}$ be a mapping between two universe sets $\mathbb{U}$ and $\mathbb{V}$, and $\widetilde{B} = \langle \mathbb{V}, \mu_B, \sigma_B, \omega_B \rangle$ be a SVN-set over $\mathbb{V}$. The *neutrosophic inverse image* of $\widetilde{B}$ by f, denoted by $\widetilde{f}^{-1}\left(\widetilde{B}\right)$, is the SVN-set over $\mathbb{U}$ defined by:

$$\widetilde{f}^{-1}\left(\widetilde{B}\right) = \left\langle \mathbb{U}, \mu_{f^{-1}(B)}, \sigma_{f^{-1}(B)}, \omega_{f^{-1}(B)} \right\rangle$$

where the mappings $\mu_{f^{-1}(B)} : \mathbb{U} \to I$, $\sigma_{f^{-1}(B)} : \mathbb{U} \to I$ and $\omega_{f^{-1}(B)} : \mathbb{U} \to I$ are defined respectively by:

$$\mu_{f^{-1}(B)}) = \mu_B \circ f, \quad \sigma_{f^{-1}(B)} = \sigma_B \circ f, \quad \text{and} \quad \omega_{f^{-1}(B)} = \omega_B \circ f\,.$$

The method `NScounterimage(self, nset)` returns the counter image of a SVN-set *nset* over the codomain by the current mapping. The corresponding code is provided below.

```
    def NScounterimage(self, nset):
        result = NSset(self.__domain)
        for u in self.getDomain():
            value = self.getValue(u)
            triple = nset.getElement(value)
            result.setElement(u, triple)
        return result
```

The following code executed in interactive mode in the Python console summarizes and explicates the use of the methods just described along with those already seen in the other two classes.

```
>>> U = NSuniverse("a,b,c,d,e")
>>> V = NSuniverse(1,2,3,4)
>>> f = NSmapping(U, V, (1,3,1,2,1))
>>> print(f)
            { a, b, c, d, e }   ->   { 1, 2, 3, 4 }
----------------------------------------------------------------
                            a   |->  1
                            b   |->  3
                            c   |->  1
                            d   |->  2
                            e   |->  1
>>> A = NSset(U, "(0.7,0.3,0.1), (0.4,0.6,0.9), (0,0,1), (0.1,0.4,0.5), (0.2,0.2
    ,0.3)")
>>> print(f"{A:t}")
            |   membership   |  indeterminacy | non-membership |
----------------------------------------------------------------
 a          |            0.7 |            0.3 |            0.1 |
 b          |            0.4 |            0.6 |            0.9 |
 c          |            0.0 |            0.0 |            1.0 |
 d          |            0.1 |            0.4 |            0.5 |
 e          |            0.2 |            0.2 |            0.3 |
----------------------------------------------------------------
>>> B = f.NSimage(A)
>>> print(B)
< 1/(0.7,0.3,0.1), 2/(0.1,0.4,0.5), 3/(0.4,0.6,0.9),
4/(1.0,1.0,0.0) >
>>> C = f.NScounterimage(B)
< a/(0.7,0.3,0.1), b/(0.4,0.6,0.9), c/(0.7,0.3,0.1),
d/(0.1,0.4,0.5), e/(0.7,0.3,0.1) >
>>> print(A.isNSsubset(C))
True
```

7. Conclusions

In this paper we have presented `PYNS`, an open source framework developed in Python and consisting of three distinct classes designed to manipulate in a simple and intuitive way symbolic representations of neutrosophic sets over universes of various types as well as mappings between them.

The codebase of this framework, currently comprising approximately 1200 lines of code, empowers us with the capability to seamlessly define, represent, and manipulate universe sets, neutrosophic sets, and functions operating between neutrosophic sets. This is facilitated through a comprehensive set of operations, including neutrosophic union, neutrosophic intersection, neutrosophic difference, as well as the computation of image and back-image of a neutrosophic set by means of a function, among others. These operations operate at various levels, impacting the values of the membership, indeterminacy and non-membership degree of each individual element.

The capabilities offered by this framework extend and generalize previous attempts to provide software solutions for the manipulation of neutrosophic sets already undertaken in recent years by several authors such as. Salama et al. [21], Saranya et al. [23], El-Ghareeb [7], Topal et al. [29] and Sleem [26].

Furthermore, the modular structure of `PYNS` not only facilitates interactive usage for experimentation and counterexample searches within the neutrosophic domain, making efficient use of a simple and intuitive notation, but also enables easy integration into more complex Python projects that can take advantage of robust and extensively tested methods for operations on neutrosophic sets that this framework provides.

Both the code and the underlying data structures of the three classes `NSuniverse`, `NSset` and `NSmapping` with particular regard to their properties and methods have been explained in detail in the previous sections and also concrete examples of using the introduced objects and methods have been given.

The attention given to the usability of these classes and the extensive documentation provided with a rich assortment of examples and use cases, gives us confidence that, in addition to being used for the exploration of uncertain data and practical applications, it can be the subject of further study and expansion opening up new research perspectives in various scientific and applied disciplines that use the tools of neutrosophic set theory. In particular, the authors believe that interesting developments in the medical field may come from the application and extension of this framework to neutrosophic hypersoft mappings that have proven to be useful in the diagnosis of hepatitis [17] or its eventual adaptation to fuzzy hypersoft mappings [2] which have proven to be useful in the diagnosis of HIV and tuberculosis [3, 18].

The complete Python framework PYNS including the source code of all the classes described in this paper as well as a selection of example programs that use them are available at the url github.com/giorgionordo/pythonNeutrosophicSets.

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New Fixed Point Results in Neutrosophic Metric Spaces

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Abstract: In this manuscript, we give the generalization of banach's, Kannan's and Chatterjee's fixed Point theorems in neutrosophic metric spaces by using new (TS-IFα) contractive mappings. Also, we establish common fixed point results in neutrosophic metric space by using Occasionally weakly compatible maps for integral type inequalities.

Keywords: Fuzzy metric space; Neutrosophic metric space; Banach; Kannan; Chatterjee; Fixed point theorems.

1. Introduction

Fixed point theory is an important tool to find the existence and uniqueness of solution of integral and differential equations. Researchers in [1-9] worked on several generalized fixed point results. After being given the notion of fuzzy sets by L. A. Zadeh [10], a large number of researchers provide many generalizations. In this continuation, Kramosil and Michalek [11] originated the approach of fuzzy metric spaces, George and Veeramani [12] diversify the approach of fuzzy metric spaces. Garbiec [13] tossed the fuzzy interpretation of Banach contraction principle in fuzzy metric spaces.
The approach of intuitionistic fuzzy metric spaces (IFMS) was tossed by Park in [14]. Kirişci and Simsek [5] tossed the approach of neutrosophic metric space (NMS). Simsek and Kirişci [6] and Sowndrarajan et al. [1] proved some fixed point (FP) results in the setting of NMS. Kiran and Khatoon [15] proved Kannan's and Chatterjee's FP results in the sense of IFMS by using (TS-IFα) mappings for $\alpha: \Omega \rightarrow [0,1)$. Patel et al. [9] prove common fixed point (FP) results in the sense of

Occasionally weakly compatible (OWC) maps on IFMS for integral type inequality. Authors in [16-20] worked on different generalizations of NMSs and proved several fixed point results.
In this manuscript, we give the generalization of banach's, Kannan's and Chatterjee's FP theorems in NMS by using new (TS-IFα) mappings and for $\alpha: \Omega \times \Omega \to [0,1)$. Also, we establish common fixed point results in NMS by using Occasionally weakly compatible maps for integral type inequality. Our results are more generalized in the existing literature.

Theorem 1.1 [8] Let $(\Omega, \psi, \phi, *, \circ)$ be complete IFMS and a mapping $h: \Omega \to \Omega$ is named an intuitionistic fuzzy contraction depends on α (IFC_α) if there exists a mapping $\alpha: \Omega \to [0,1)$ where $\alpha(h\varkappa) \le \alpha(\varkappa)$ such that

$$\frac{1}{\psi(h\varkappa, h\mathrm{y}, t)} - 1 \le \alpha(\varkappa)\left(\frac{1}{\psi(\varkappa, \mathrm{y}, t)}\right)$$

and

$$\phi(h\varkappa, h\mathrm{y}, t) \le \alpha(\varkappa)\phi(\varkappa, \mathrm{y}, t)$$

$\forall\, \varkappa, \mathrm{y} \in \Omega$ and $t > 0$. Then h has a unique FP.

Theorem 1.2 [7] Let $(\Omega, \psi, \phi, *, \circ)$ be IFMS and a mapping $h: \Omega \to \Omega$ is named to be $(DC - IF)$ contraction mapping if $\exists\, \delta \in (0,1)$ such that

$$\delta\psi(h\varkappa, h\mathrm{y}, t) \ge \psi(\varkappa, \mathrm{y}, t), \qquad \frac{1}{\delta}\phi(h\varkappa, h\mathrm{y}, t) \le \phi(\varkappa, \mathrm{y}, t)$$

$\forall\, \varkappa, \mathrm{y} \in \Omega$ and $t > 0$. Then h has a unique FP.

Definition 1.1 [5] Suppose $\Omega \ne \emptyset$, assume a six tuple $(\Omega, \psi, \phi, \theta, *, \circ)$ where $*$ is a continuous t-norm (CTN), $\circ$ is a continuous t-conorm (CTCN), ψ, ϕ and θ NS on $\Omega \times \Omega \times (0, \infty)$. If $(\Omega, \psi, \phi, \theta, *, \circ)$ meet the below circumstances for all $\varkappa, \mathrm{y}, v \in \Omega$ and $t, s > 0$:

(NS1) $\psi(\varkappa, \mathrm{y}, t) + \phi(\varkappa, \mathrm{y}, t) + \theta(\varkappa, \mathrm{y}, t) \le 3$,

(NS2) $0 \le \psi(\varkappa, \mathrm{y}, t) \le 1$,

(NS3) $\psi(\varkappa, \mathrm{y}, t) = 1 \iff \varkappa = \mathrm{y}$,

(NS4) $\psi(\varkappa, \mathrm{y}, t) = \psi(\mathrm{y}, \varkappa, t)$, (NS5) $\psi(\varkappa, v(t+s)) \geq \psi(\varkappa, \mathrm{y}, t)^{*}\psi(\mathrm{y}, v, s)$,

(NS6) $\psi(\varkappa, \mathrm{y}, \cdot): [0, \infty) \rightarrow [0,1]$ is a continuous,

(NS7) $\lim_{t\rightarrow\infty} \psi(\varkappa, \mathrm{y}, t) = 1$,

(NS8) $0 \leq \phi(\varkappa, \mathrm{y}, t) \leq 1$,

(NS9) $\phi(\varkappa, \mathrm{y}, t) = 0 \Leftrightarrow \varkappa = \mathrm{y}$,

(NS10) $\phi(\varkappa, \mathrm{y}, t) = \phi(\mathrm{y}, \varkappa, t)$,

(NS11) $\phi(\varkappa, v, (t+s)) \leq \phi(\varkappa, \mathrm{y}, t) \circ \phi(\mathrm{y}, v, s)$,

(NS12) $\phi(\varkappa, \mathrm{y}, \cdot): [0, \infty) \rightarrow [0,1]$ is a continuous,

(NS13) $\lim_{t\rightarrow\infty} \phi(\varkappa, \mathrm{y}, t) = 0$,

(NS14) $0 \leq \theta(\varkappa, \mathrm{y}, t) \leq 1$,

(NS15) $\theta(\varkappa, \mathrm{y}, t) = 0 \Leftrightarrow \varkappa = \mathrm{y}$,

(NS16) $\theta(\varkappa, \mathrm{y}, t) = \theta(\mathrm{y}, \varkappa, t)$,

(NS17) $\theta(\varkappa, v, (t+s)) \leq \theta(\varkappa, \mathrm{y}, t) \circ \theta(\mathrm{y}, v, s)$,

(NS18) $\theta(\varkappa, \mathrm{y}, \cdot): [0, \infty) \rightarrow [0,1]$ is a continuous,

(NS19) $\lim_{t\rightarrow\infty} \theta(\varkappa, \mathrm{y}, t) = 0$,

(NS20) If $t \leq 0$ then $\psi(\varkappa, \mathrm{y}, t) = 0, \phi(\varkappa, \mathrm{y}, t) = 1, \theta(\varkappa, \mathrm{y}, t) = 1$.

Then $(\Omega, \psi, \phi, \theta)$ Neutrosophic metric on Ω and $(\Omega, \psi, \phi, \theta, *, \circ)$ is an NMS.

Definition 1.2 [6] Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be an NMS. Then

(i) a sequence $\{\varkappa_n\}$ in Ω is named to be G-Cauchy sequence (GCS) if and only if for all

$t > 0$ and $\mathrm{m} > 0$,

$\lim_{n\to\infty}\psi(\varkappa_n,\varkappa_{n+m},t)=1$, $\lim_{n\to\infty}\phi(\varkappa_n,\varkappa_{n+m},t)=0$ and $\lim_{n\to\infty}\theta(\varkappa_n,\varkappa_{n+m},t)=0$

.

(ii) a sequence $\{\varkappa_n\}$ in Ω is named to be G-convergent (GC) to $\varkappa$ in Ω, if and only if for all $t>0$,

$$\lim_{n\to\infty}\psi(\varkappa_n,\varkappa,t)=1,\ \lim_{n\to\infty}\phi(\varkappa_n,\varkappa,t)=0 \text{ and } \lim_{n\to\infty}\theta(\varkappa_n,\varkappa,t)=0.$$

(iii) an ENMS is named to be complete iff each GCS is convergent.

Definition 1.3 [2] A self mappings pair (h,g) of IFMS is named to be weakly compatible if they commute at the coincident points i.e. $hu=gu$ for some $u\in\Omega$, then $hgu=ghu$.

Definition 1.4 [2] Let $(\Omega,\psi,\phi,*,\circ)$ be an IFMS. $h,g:\Omega\to\Omega$. A point $\varkappa\in\Omega$ is called coincident point of h and g if and only if $h\varkappa=g\varkappa$.

Definition 1.5 [3] A self mappings pair (h,g) of IFMS is named to be OWC iff there is a point $\varkappa\in\Omega$ is coincident point if it commutes at the coincident points h and g at which h and g commute.

Lemma 1.1 [3] Let $(\Omega,\psi,\phi,*,\circ)$ be an IFMS. $h,g:\Omega\to\Omega$ and h,g have unique coincident point, $w=h\varkappa=g\varkappa$, then h and g has a unique common FP w.

Lemma 1.2 [4] Let $(\Omega,\psi,\phi,*,\circ)$ be an IFMS and $\forall\ \varkappa,y\in\Omega,t>0$ and $k\in(0,1)$ such that $\psi(\varkappa,y,kt)\geq\psi(\varkappa,y,t)$ and $\phi(\varkappa,y,kt)\leq\phi(\varkappa,y,t)$ then $\varkappa=y$.

2. Main Section-I

In this section, we give some theorems by using new (TS-IFα) mapping, also examine a result with example.

Theorem 2.1 Let $(\Omega,\psi,\phi,\theta,*,\circ)$ be a G-complete NMS and $h:\Omega\to\Omega$ be (TS-IFα) mapping, i.e., if $\exists\ \alpha:\Omega\times\Omega\to[0,1)$ where

$\alpha(h\varkappa, h\mathrm{y}) \leq \alpha(\varkappa, \mathrm{y})$

such that

$$\alpha(\varkappa,\mathrm{y})\psi(h(\varkappa),h(\mathrm{y}),t) \geq \psi(\varkappa,\mathrm{y},t),$$

$$\frac{1}{\alpha(\varkappa,\mathrm{y})}\phi(h(\varkappa),h(\mathrm{y}),t) \leq \phi(\varkappa,\mathrm{y},t)$$

and

$$\frac{1}{\alpha(\varkappa,\mathrm{y})}\theta(h(\varkappa),h(\mathrm{y}),t) \leq \theta(\varkappa,\mathrm{y},t)$$

$\forall\, t > 0$. Then h has a unique fixed point (FP).

Proof: Let $\varkappa_0 \in \Omega$ be a random point. We build a sequence $\varkappa_m \in \Omega$ by

$$\varkappa_m = h^m\varkappa_0 = h\varkappa_{m-1}$$

For $m \in \mathbb{N}$. Now $\forall\, t > 0$ we obtain

$$\psi(\varkappa_{m+1},\varkappa_m,t) \geq \frac{1}{\left(\alpha(\varkappa_1,\varkappa_0)\right)^m}\psi(\varkappa_1,\varkappa_0,t)$$

Now for $m,n \in \mathbb{N}$ such that $n \geq m,$ we deduce

$$\psi\left(\varkappa_m,\varkappa_{n+p},t\right) \geq \frac{1}{\left(\alpha(\varkappa_1,\varkappa_0)\right)^m}\psi\left(\varkappa_1,\varkappa_0,\frac{t}{p}\right)$$

Hence, we get

$$1 \leq \lim_{m\to\infty}\left(\frac{1}{\left(\alpha(\varkappa_1,\varkappa_0)\right)^m}\psi\left(\varkappa_1,\varkappa_0,\frac{t}{p}\right)\right) \leq \lim_{m\to\infty}\psi\left(\varkappa_m,\varkappa_{n+p},t\right) \leq 1$$

That is

$$\lim_{m\to\infty}\psi\left(\varkappa_m,\varkappa_{n+p},t\right) = 1.$$

Again for $m \in \mathbb{N}$. Now $\forall\, t > 0$ we obtain

$$\phi(\varkappa_{m+1},\varkappa_m,t) \leq \left(\alpha(\varkappa_1,\varkappa_0)\right)^m\phi(\varkappa_1,\varkappa_0,t)$$

Now for $m,n \in \mathbb{N}$ such that $n \geq m,$ we deduce

$$\phi(\varkappa_m, \varkappa_{n+p}, t) \leq \frac{1}{(\alpha(\varkappa_1, \varkappa_0))^m} \phi\left(\varkappa_1, \varkappa_0, \frac{t}{p}\right)$$

Hence, we get

$$0 \geq \lim_{m \to \infty} \left(\frac{1}{(\alpha(\varkappa_1, \varkappa_0))^m} \phi\left(\varkappa_1, \varkappa_0, \frac{t}{p}\right) \right) \geq \lim_{m \to \infty} \phi(\varkappa_m, \varkappa_{n+p}, t) \geq 0$$

This implies

$$\lim_{m \to \infty} \phi(\varkappa_m, \varkappa_{n+p}, t) = 0$$

Similarly, for $m \in \mathbb{N}$. Now $\forall\, t > 0$ we obtain

$$\theta(\varkappa_{m+1}, \varkappa_m, t) \leq (\alpha(\varkappa_1, \varkappa_0))^m \theta(\varkappa_1, \varkappa_0, t)$$

Now for $m, n \in \mathbb{N}$ such that $n \geq m$, we deduce

$$\theta(\varkappa_m, \varkappa_{n+p}, t) \leq \frac{1}{(\alpha(\varkappa_1, \varkappa_0))^m} \theta\left(\varkappa_1, \varkappa_0, \frac{t}{p}\right)$$

Hence, we get

$$0 \geq \lim_{m \to \infty} \left(\frac{1}{(\alpha(\varkappa_1, \varkappa_0))^m} \theta\left(\varkappa_1, \varkappa_0, \frac{t}{p}\right) \right) \geq \lim_{m \to \infty} \theta(\varkappa_m, \varkappa_{n+p}, t) \geq 0$$

This implies

$$\lim_{m \to \infty} \theta(\varkappa_m, \varkappa_{n+p}, t) = 0.$$

Hence, $\{\varkappa_m\} \in \Omega$ GSC sequence. Due to the completeness of Ω there exists $v \in \Omega$ such that $\varkappa_m \to v$ as $m \to \infty$. We have

$$\lim_{m \to \infty} \psi(\varkappa_m, v, t) = 1,$$

$$\lim_{m \to \infty} \phi(\varkappa_m, v, t) = 0$$

And

$$\lim_{m \to \infty} \theta(\varkappa_m, v, t) = 0.$$

Now we examine that v is a FP of h. Therefore, h is (TS-IFα) $\forall\, m \in \mathbb{N}$ we obtain

$$\psi(hv, h\varkappa_m, t) \geq \frac{1}{\alpha(v, \varkappa_m)} \psi(v, \varkappa_m, t)$$

$$\lim_{m\to\infty}\psi(hv,h\varkappa_m,t)\geq\lim_{m\to\infty}\frac{1}{\alpha(v,\varkappa_m)}\psi(v,\varkappa_m,t)=\frac{1}{\alpha(v,v)}>1$$

This implies

$$1<\lim_{m\to\infty}\psi(hv,h\varkappa_m,t)\leq 1$$

Hence

$$\lim_{m\to\infty}\psi(hv,h\varkappa_m,t)=1$$

Again, we have

$$\phi(hv,h\varkappa_m,t)\leq\alpha(v,\varkappa_m)\phi(v,\varkappa_m,t)$$

Now for $t>0,\forall\, m\in\mathbb{N}$, we get

$$\lim_{m\to\infty}\phi(hv,h\varkappa_m,t)=0$$

Similarly, we have

$$\theta(hv,h\varkappa_m,t)\leq\alpha(v,\varkappa_m)\theta(v,\varkappa_m,t)$$

Now for $t>0,\forall\, m\in\mathbb{N}$, we get

$$\lim_{m\to\infty}\theta(hv,h\varkappa_m,t)=0.$$

Hence, $h\varkappa_m\to hv$, this implies $v=hv$. Now we examine the uniqueness of FP. Let v_1 be another FP. We have

$$\psi(v,v_1,t)=\psi(hv,hv_1,t)\geq\frac{1}{\left(\alpha(v,v_1)\right)^m}\psi(v,v_1,t)\to 1\text{ as }m\to\infty$$

This implies

$$1<\lim_{m\to\infty}\frac{1}{\left(\alpha(v,v_1)\right)^m}\psi(v,v_1,t)\leq\psi(v,v_1,t)\leq 1$$

Hence,

$$\psi(v,v_1,t)=1$$

Again, we have

$$\phi(v,v_1,t)=\phi(hv,hv_1,t)\leq\left(\alpha(v,v_1)\right)^m\phi(v,v_1,t)\to 0\text{ as }m\to\infty$$

This implies

$$0\leq\phi(v,v_1,t)\leq\lim_{m\to\infty}\left(\alpha(v,v_1)\right)^m\phi(v,v_1,t)<0$$

Hence

$\phi(v, v_1, t) = 0$

Similarly, we have

$$\theta(v, v_1, t) = \theta(hv, hv_1, t) \leq \left(\alpha(v, v_1)\right)^m \theta(v, v_1, t) \to 0 \text{ as } m \to \infty$$

This implies

$$0 \leq \theta(v, v_1, t) \leq \lim_{m \to \infty} \left(\alpha(v, v_1)\right)^m \theta(v, v_1, t) < 0$$

Hence

$\theta(v, v_1, t) = 0.$

This examine that $v = v_1$.

Example 2.1 Let $\Omega = [0,1]$. Define $\alpha: \Omega \times \Omega \to [0,1)$ by

$$\alpha(\varkappa) = \begin{cases} 0 & \text{if } \varkappa = 0 \text{ or } y = 0, \\ \dfrac{1}{(\max\{\varkappa, y\})^2} & \text{if otherwise} \end{cases} \quad \forall \varkappa, y \in \Omega$$

and define a G-complete NMS in [1] by

$$\psi(\varkappa, y, t) = \frac{t}{t + |\varkappa - y|}, \phi(\varkappa, y, t) = \frac{|\varkappa - y|}{t + |\varkappa - y|} \text{ and } \theta(\varkappa, y, t) = \frac{|\varkappa - y|}{t}$$

Also define $h: \Omega \to \Omega$ by

$$h(\varkappa) = \begin{cases} 0 & \text{if } \quad \varkappa = 0, \\ \dfrac{1}{\varkappa} & \text{if } \varkappa \in (0,1] \end{cases}$$

Then the mapping is a new (TS-IFα) contractive mapping.

We have for cases:

(i) If $\varkappa = y = 0$, then $h\varkappa = hy = 0$;

(ii) If $\varkappa = 0$ and $y \in (0,1]$, then $h\varkappa = 0$ and $hy = \frac{1}{y}$;

(iii) If $y = 0$ and $\varkappa \in (0,1]$, then $hy = 0$ and $h\varkappa = \frac{1}{\varkappa}$;

(iv) If $\varkappa, y \in (0,1]$, then $h\varkappa = \frac{1}{\varkappa}$ and $hy = \frac{1}{y}$;

Then all the circumstances of theorem 2.1 are fulfilled and 0 is a unique FP of h.

Theorem 2.2 Let $(\Omega,\psi,\phi,\theta,*,\circ)$ be a G-complete NMS and $h:\Omega\to\Omega$ be a contractive mapping such that ∃ $\alpha:\Omega\times\Omega\to\left(0,\frac{1}{2}\right)$, where

$$\alpha(h\varkappa,h\mathrm{y})\leq\alpha(\varkappa,\mathrm{y})$$

Such that

$$\alpha(\varkappa,\mathrm{y})\psi(h(\varkappa),h(\mathrm{y}),t)\geq[\psi(\varkappa,h\varkappa,t)+\psi(\mathrm{y},h\mathrm{y},t)],$$

$$\frac{1}{\alpha(\varkappa,\mathrm{y})}\phi(h(\varkappa),h(\mathrm{y}),t)\leq[\phi(\varkappa,h\varkappa,t)+\phi(\mathrm{y},h\mathrm{y},t)]$$

And

$$\frac{1}{\alpha(\varkappa,\mathrm{y})}\theta(h(\varkappa),h(\mathrm{y}),t)\leq[\theta(\varkappa,h\varkappa,t)+\theta(\mathrm{y},h\mathrm{y},t)]$$

$\forall\, t>0.$ Then h has a unique FP.

Theorem 2.3 Let $(\Omega,\psi,\phi,\theta,*,\circ)$ be a G-complete NMS and $h:\Omega\to\Omega$ be a contractive mapping such that ∃ $\alpha:\Omega\times\Omega\to\left(0,\frac{1}{2}\right)$, where

$$\alpha(h\varkappa,h\mathrm{y})\leq\alpha(\varkappa,\mathrm{y})$$

Such that

$$\alpha(\varkappa,\mathrm{y})\psi(h(\varkappa),h(\mathrm{y}),t)\geq[\psi(\varkappa,h\mathrm{y},t)+\psi(\mathrm{y},h\varkappa,t)],$$

$$\frac{1}{\alpha(\varkappa,\mathrm{y})}\phi(h(\varkappa),h(\mathrm{y}),t)\leq[\phi(\varkappa,h\mathrm{y},t)+\phi(\mathrm{y},h\varkappa,t)]$$

And

$$\frac{1}{\alpha(\varkappa,\mathrm{y})}\theta(h(\varkappa),h(\mathrm{y}),t)\leq[\theta(\varkappa,h\mathrm{y},t)+\theta(\mathrm{y},h\varkappa,t)]$$

$\forall\, t>0.$ Then h has a unique FP.

Theorem 2.4 Let $(\Omega,\psi,\phi,\theta,*,\circ)$ be a G-complete NMS and $h:\Omega\to\Omega$ be a contractive mapping such that ∃ $\alpha:\Omega\times\Omega\to[0,1)$, where

$$\alpha(h\varkappa,h\mathrm{y})\leq\alpha(\varkappa,\mathrm{y})$$

Such that

$$\frac{1}{\psi(h\varkappa,h\mathrm{y},t)}-1\leq\alpha(\varkappa,\mathrm{y})\left(\frac{1}{\psi(\varkappa,\mathrm{y},t)}\right),$$

$\phi(h\varkappa, hy, t) \leq \alpha(\varkappa, y)\phi(\varkappa, y, t)$

and

$\theta(h\varkappa, hy, t) \leq \alpha(\varkappa, y)\theta(\varkappa, y, t)$

$\forall\ \varkappa, y \in \Omega$ and $t > 0$. Then h has a unique FP.

Theorem 2.2 and 2.3 are the generalizations of Kannan's and Chatterjee's FP theorems in NMS. Theorem 2.4 is neutrosophic contraction mapping theorem. We can prove easily by using theorem 2.1.

3. Main Section-II

Definition 3.1 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be a NMS. $h, g: \Omega \to \Omega$. A point $\varkappa \in \Omega$ is called coincident point of h and g if and only if $h\varkappa = g\varkappa$.

Definition 3.2 A self mappings pair (h, g) of a NMS is named to be weakly compatible if they commute at the coincident points i.e. $hu = gu$ for some $u \in \Omega$, then $hgu = ghu$.

Definition 3.3 A self mappings pair (h, g) of NMS is named to be OWC iff there is a point $\varkappa \in \Omega$ is coincident point if it commutes at the coincident points h and g at which h and g commute.

Lemma 3.1 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be a NMS. $h, g: \Omega \to \Omega$ and h, g have unique coincident point, $w = h\varkappa = g\varkappa$, then h and g has a unique common FP w.

Proof easily follows from [9] and [3].

Lemma 3.2 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be an NMS and $\forall\ \varkappa, y \in \Omega, t > 0$ and $k \in (0,1)$ such that $\psi(\varkappa, y, kt) \geq \psi(\varkappa, y, t), \phi(\varkappa, y, kt) \leq \phi(\varkappa, y, t)$ and $\theta(\varkappa, y, kt) \leq \theta(\varkappa, y, t)$ then $\varkappa = y$.

Proof easily follows from [4] and [9].

Theorem 3.1 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be an NMS with CTM $*$ and CTCN $\circ$. Let $A, B, C, D: \Omega \to \Omega$ and the pairs (A, C) and (B, D) are OWC. If $\exists\ k \in (0,1)$ such that

$$\int_0^{\psi(A\varkappa, By, kt)} f(t)dt \geq \int_0^{\min\left\{\begin{array}{c}\psi(C\varkappa, Dy, t), \psi(By, C\varkappa, t), \psi(C\varkappa, A\varkappa, t), \psi(By, Dy, t), \\ \psi(A\varkappa, Dy, t). \left(\frac{\psi(C\varkappa, A\varkappa, t)}{\psi(By, Dy, t)}\right)\end{array}\right\}} f(t)dt,$$

$$\int_0^{\phi(A\varkappa,By,kt)} f(t)dt \le \int_0^{max\left\{\begin{matrix}\phi(C\varkappa,Dy,t),\phi(By,C\varkappa,t),\phi(C\varkappa,A\varkappa,t),\phi(By,Dy,t),\\ \phi(A\varkappa,Dy,t).\left(\frac{\phi(C\varkappa,A\varkappa,t)}{\phi(By,Dy,t)}\right)\end{matrix}\right\}} f(t)dt$$

And

$$\int_0^{\theta(A\varkappa,By,kt)} f(t)dt \le \int_0^{max\left\{\begin{matrix}\theta(C\varkappa,Dy,t),\theta(By,C\varkappa,t),\theta(C\varkappa,A\varkappa,t),\theta(By,Dy,t),\\ \theta(A\varkappa,Dy,t).\left(\frac{\theta(C\varkappa,A\varkappa,t)}{\theta(By,Dy,t)}\right)\end{matrix}\right\}} f(t)dt$$

$\forall\ \varkappa, y \in \Omega$ and $t > 0$. *Then, there exists a unique FP of* A, B, C and D.

Proof: *Because pairs* (A, C) and (B, D) are OWC, so there exist $\varkappa, y \in \Omega$ such that $A\varkappa = C\varkappa$ and $By = Dy$. We claim that $A\varkappa = By$, we have

$$\int_0^{\psi(A\varkappa,By,kt)} f(t)dt \ge \int_0^{min\left\{\begin{matrix}\psi(A\varkappa,By,t),\psi(By,A\varkappa,t),\psi(A\varkappa,A\varkappa,t),\psi(By,By,t),\\ \psi(A\varkappa,By,t).\left(\frac{\psi(A\varkappa,A\varkappa,t)}{\psi(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\int_0^{\psi(By,By,kt)} f(t)dt \ge \int_0^{min\left\{\begin{matrix}\psi(By,By,t),\psi(By,By,t),\psi(By,By,t),\psi(By,By,t),\\ \psi(By,By,t).\left(\frac{\psi(By,By,t)}{\psi(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\ge \int_0^{\psi(By,By,t)} f(t)dt,$$

$$\int_0^{\phi(A\varkappa,By,kt)} f(t)dt \le \int_0^{max\left\{\begin{matrix}\phi(A\varkappa,By,t),\phi(By,A\varkappa,t),\phi(A\varkappa,A\varkappa,t),\phi(By,By,t),\\ \phi(A\varkappa,By,t).\left(\frac{\phi(A\varkappa,A\varkappa,t)}{\phi(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\int_0^{\phi(By,By,kt)} f(t)dt \le \int_0^{max\left\{\begin{matrix}\phi(By,By,t),\phi(By,By,t),\phi(By,By,t),\phi(By,By,t),\\ \phi(By,By,t).\left(\frac{\phi(By,By,t)}{\phi(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\le \int_0^{\phi(By,By,t)} f(t)dt$$

And

$$\int_0^{\theta(A\varkappa,By,kt)} f(t)dt \le \int_0^{max\left\{\begin{matrix}\theta(A\varkappa,By,t),\theta(By,A\varkappa,t),\theta(A\varkappa,A\varkappa,t),\theta(By,By,t),\\ \theta(A\varkappa,By,t).\left(\frac{\theta(A\varkappa,A\varkappa,t)}{\theta(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\int_0^{\theta(By,By,kt)} f(t)dt \le \int_0^{max\left\{\begin{matrix}\theta(By,By,t),\theta(By,By,t),\theta(By,By,t),\theta(By,By,t),\\ \theta(By,By,t).\left(\frac{\theta(By,By,t)}{\theta(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\le \int_0^{\theta(By,By,t)} f(t)dt$$

Hence, from lemma 3.2, $A\varkappa = By$ *i.e* $A\varkappa = C\varkappa = By = Dy$. *Assume* v *be another point such that* $Av = Cv$ *then, we obtain* $Av = Cv = By = Dy$ *so* $A\varkappa = Av$ *and* $w = A\varkappa = C\varkappa$ *is the unique point of* A *and* C. *From lemma 3.1, only* w *is common FP of* A *and* C. *Likewise, there is* $v \in \Omega$ *a unique point such that* $v = Bv = Dv$. *Now we examine that* $v = w$.

$$\int_0^{\psi(w,v,kt)} f(t)dt = \int_0^{\psi(Aw,Bv,kt)} f(t)dt$$

$$\geq \int_0^{\min\left\{\begin{matrix}\psi(Cw,Dv,t),\psi(Bv,Cw,t),\psi(Cw,Aw,t),\psi(Bv,Dv,t),\\ \psi(Aw,Dv,t).\left(\frac{\psi(Cw,Aw,t)}{\psi(Bv,Dv,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$= \int_0^{\min\left\{\begin{matrix}\psi(w,v,t),\psi(v,w,t),\psi(w,w,t),\psi(v,v,t),\\ \psi(w,v,t).\left(\frac{\psi(w,w,t)}{\psi(v,v,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$= \int_0^{\min\{\psi(w,v,t),\psi(v,w,t),1,1,\psi(w,v,t)\}} f(t)dt = \int_0^{\psi(w,v,t)} f(t)dt,$$

$$\int_0^{\phi(w,v,kt)} f(t)dt = \int_0^{\phi(Aw,Bv,kt)} f(t)dt$$

$$\leq \int_0^{\max\left\{\begin{matrix}\phi(Cw,Dv,t),\phi(Bv,Cw,t),\phi(Cw,Aw,t),\phi(Bv,Dv,t),\\ \phi(Aw,Dv,t).\left(\frac{\phi(Cw,Aw,t)}{\phi(Bv,Dv,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$= \int_0^{\max\left\{\begin{matrix}\phi(w,v,t),\phi(v,w,t),\phi(w,w,t),\phi(v,v,t),\\ \phi(w,v,t).\left(\frac{\phi(w,w,t)}{\phi(v,v,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$= \int_0^{\max\{\phi(w,v,t),\phi(v,w,t),1,1,\phi(w,v,t)\}} f(t)dt = \int_0^{\phi(w,v,t)} f(t)dt$$

And

$$\int_0^{\theta(w,v,kt)} f(t)dt = \int_0^{\theta(Aw,Bv,kt)} f(t)dt$$

$$\leq \int_0^{\max\left\{\begin{matrix}\theta(Cw,Dv,t),\theta(Bv,Cw,t),\theta(Cw,Aw,t),\theta(Bv,Dv,t),\\ \theta(Aw,Dv,t).\left(\frac{\theta(Cw,Aw,t)}{\theta(Bv,Dv,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$= \int_0^{\max\left\{\begin{array}{c}\theta(w,v,t),\theta(v,w,t),\theta(w,w,t),\theta(v,v,t),\\ \theta(w,v,t).\left(\frac{\theta(w,w,t)}{\theta(v,v,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\max\{\theta(w,v,t),\theta(v,w,t),1,1,\theta(w,v,t)\}} f(t)dt = \int_0^{\theta(w,v,t)} f(t)dt$$

Hence, from lemma 3.2, $w = v$. *That is,* v *is a common FP of* A, B, C *and* D. *Now assume another common FP* u *of* A, B, C *and* D *for examining the uniqueness. Then*

$$\int_0^{\psi(u,v,kt)} f(t)dt = \int_0^{\psi(Au,Bv,kt)} f(t)dt$$

$$\geq \int_0^{\min\left\{\begin{array}{c}\psi(Cu,Dv,t),\psi(Bv,Cu,t),\psi(Cu,Au,t),\psi(Bv,Dv,t),\\ \psi(Au,Dv,t).\left(\frac{\psi(Cu,Au,t)}{\psi(Bv,Dv,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\min\left\{\begin{array}{c}\psi(u,v,t),\psi(v,u,t),\psi(u,u,t),\psi(v,v,t),\\ \psi(u,v,t).\left(\frac{\psi(u,u,t)}{\psi(v,v,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\min\{\psi(u,v,t),\psi(v,u,t),1,1,\psi(u,v,t)\}} f(t)dt = \int_0^{\psi(u,v,t)} f(t)dt,$$

$$\int_0^{\phi(u,v,kt)} f(t)dt = \int_0^{\phi(Au,Bv,kt)} f(t)dt$$

$$\leq \int_0^{\max\left\{\begin{array}{c}\phi(Cu,Dv,t),\phi(Bv,Cu,t),\phi(Cu,Au,t),\phi(Bv,Dv,t),\\ \phi(Au,Dv,t).\left(\frac{\phi(Cu,Au,t)}{\phi(Bv,Dv,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\max\left\{\begin{array}{c}\phi(u,v,t),\phi(v,u,t),\phi(u,u,t),\phi(v,v,t),\\ \phi(u,v,t).\left(\frac{\phi(u,u,t)}{\phi(v,v,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\max\{\phi(u,v,t),\phi(v,u,t),1,1,\phi(u,v,t)\}} f(t)dt = \int_0^{\phi(u,v,t)} f(t)dt$$

And

$$\int_0^{\theta(u,v,kt)} f(t)dt = \int_0^{\theta(Au,Bv,kt)} f(t)dt$$

$$\leq \int_0^{\max\left\{\begin{array}{c}\theta(Cu,Dv,t),\theta(Bv,Cu,t),\theta(Cu,Au,t),\theta(Bv,Dv,t),\\ \theta(Au,Dv,t).\left(\frac{\theta(Cu,Au,t)}{\theta(Bv,Dv,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{max\left\{\begin{array}{c}\theta(u,v,t),\theta(v,u,t),\theta(u,u,t),\theta(v,v,t),\\ \theta(u,v,t).\left(\frac{\theta(u,u,t)}{\theta(v,v,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{max\{\theta(u,v,t),\theta(v,u,t),1,1,\theta(u,v,t)\}} f(t)dt = \int_0^{\theta(u,v,t)} f(t)dt$$

Hence, from lemma 3.2, $v = w$. Hence v is a unique common FP.

Corollary 3.1 Let $(\Omega,\psi,\phi,\theta,*,\circ)$ be a NMS with CTM $a * b = \min\{a,b\}$ and CTCN $a \circ b = \max\{a,b\}$. Let $A,B,C,D:\Omega \to \Omega$ and the pairs (A,C) and (B,D) are OWC. If $\exists\ k \in (0,1)$ such that

$$\int_0^{\psi(Ax,By,kt)} f(t)dt \geq \int_0^{min\left\{\begin{array}{c}\psi(Cx,Dy,t)*\psi(By,Cx,t)*\psi(Cx,Ax,t)*\psi(By,Dy,t)*\\ \psi(Ax,Dy,t).\left(\frac{1+\psi(Cx,Ax,t)}{1+\psi(By,Dy,t)}\right)\end{array}\right\}} f(t)dt,$$

$$\int_0^{\phi(Ax,By,kt)} f(t)dt \leq \int_0^{max\left\{\begin{array}{c}\phi(Cx,Dy,t)\circ\phi(By,Cx,t)\circ\phi(Cx,Ax,t)\circ\phi(By,Dy,t)\circ\\ \phi(Ax,Dy,t).\left(\frac{1+\phi(Cx,Ax,t)}{1+\phi(By,Dy,t)}\right)\end{array}\right\}} f(t)dt$$

And

$$\int_0^{\theta(Ax,By,kt)} f(t)dt \leq \int_0^{max\left\{\begin{array}{c}\theta(Cx,Dy,t)\circ\theta(By,Cx,t)\circ\theta(Cx,Ax,t)\circ\theta(By,Dy,t)\circ\\ \theta(Ax,Dy,t).\left(\frac{1+\theta(Cx,Ax,t)}{1+\theta(By,Dy,t)}\right)\end{array}\right\}} f(t)dt$$

$\forall\ x,y \in \Omega$ *and* $t > 0$. *Then, there exists a unique FP of A, B, C and D.*

Proof: *Easily can prove on the lines of theorem 3.1.*

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Exploring Neutrosophic Numeral System Algorithms for Handling Uncertainty and Ambiguity in Numerical Data: An Overview and Future Directions

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Abstract: The Neutrosophic Numeral System Algorithms are a set of techniques designed to handle uncertainty and ambiguity in numerical data. These algorithms use Neutrosophic Set Theory, a mathematical framework that deals with incomplete, indeterminate, and inconsistent information. In this paper, we provide an overview of different approaches used in Neutrosophic Numeral System Algorithms, including Neutrosophic Binary System, Neutrosophic Decimal System, and Neutrosophic Octal System. These systems use different bases and representations to account for degrees of truth, indeterminacy, and falsity in numerical data. We also explore the relationship between Neutrosophic Numeral System Algorithms and Number Neutrosophic Systems, which are another type of Neutrosophic System used for representing numerical data. Number Neutrosophic Systems use Neutrosophic Numbers to represent degrees of truth, indeterminacy, and falsity in numerical data, and they can be used in conjunction with Neutrosophic Numeral System Algorithms to handle uncertainty and ambiguity in decision-making and artificial intelligence applications. Moreover. We discuss the advantages and disadvantages of each algorithm and their potential applications in various fields. Finally, we highlight the importance of Neutrosophic cryptography in addressing uncertainty and ambiguity in decision making and artificial intelligence and discuss future research directions. Understanding Neutrosophic Numeral System Algorithms and their relationship with Number Neutrosophic Systems is crucial for developing effective techniques for handling uncertainty and ambiguity in numerical data in decision-making, pattern recognition, and artificial intelligence applications.

Keywords: Numerical Systems; Neutrosophic Mathematics; Neutrosophic Numerical Systems

1. Introduction

Neutrosophic systems are a powerful tool for representing and analyzing numerical data that are uncertain or indeterminate in various domains and applications. They are a generalization of classical fuzzy logic, based on the philosophical concept of Neutrosophy, which was introduced by Florentin Smarandache in 1995. Neutrosophic number systems allow for the representation of truth, falsity, and indeterminacy as (T, I, F), where the sum of T, F, and I is always equal to 1.0. They have practical applications in decision-making, pattern recognition, image processing, and artificial intelligence, where they can represent uncertain or incomplete data and model human reasoning and decision-making processes. Salama et al.'s work [1-3] likely includes their contributions to the development and application of neutrosophic systems.

This paper provides an overview of Neutrosophic Numeral System Algorithms and their relationship with Number Neutrosophic Systems. We begin by introducing the concept of Neutrosophic Set Theory and Neutrosophic Numbers, and then discuss the different approaches used in Neutrosophic Numeral System Algorithms, including Neutrosophic Binary System, Neutrosophic Decimal System, and Neutrosophic Octal System. These systems use different bases and representations to account for degrees of truth, indeterminacy, and falsity in numerical data. We also explore the relationship between Neutrosophic Numeral System Algorithms and Number Neutrosophic Systems, which are another type of Neutrosophic System used for representing numerical data. Number Neutrosophic Systems use Neutrosophic Numbers to represent degrees of truth, indeterminacy, and falsity in numerical data, and they can be used in conjunction with Neutrosophic Numeral System Algorithms to handle uncertainty and ambiguity in decision-making and artificial intelligence applications [5-8].

This paper presents a new approach to numerical systems and cryptography based on neutrosophic mathematics. It examines various types of neutrosophic number systems, such as decimal, binary, octal, and hexadecimal systems, and explores a chart of neutrosophic number systems for better understanding and presents examples of neutrosophic number systems. It also discusses the conversion of numbers between different systems.

1.1 Terminologies

This paper reviews some related work on neutrosophy and neutrosophic number systems from [9-15]. Neutrosophy and neutrosophic set theory are concerned with the study of neutralities and their mathematical representation. Neutrosophic logic and neutrosophic number systems extend classical logic and number systems by allowing for a third value of indeterminacy or neutrality. Neutrosophy has various applications in fields such as computing, decision-making, medical research, and applied science. Neutrosophy also influences algebra, where many neutrosophic algebraic structures have been defined and studied extensively.

1.2 Methodology

The search terms used included "Neutrosophic Numeral System Algorithms", "Neutrosophic Binary System", "Neutrosophic Decimal System", "Neutrosophic Octal System", "Number Neutrosophic Systems", "Neutrosophic Numbers", "Decision-Making", "Pattern Recognition", and "Artificial Intelligence". Finally, the analysis involved providing examples of applications of Neutrosophic Numeral System Algorithms and discussing future directions for research in this area. The methodology used in this paper is intended to provide a comprehensive overview of Neutrosophic

Numeral System Algorithms and their relationship with Number Neutrosophic Systems and to identify areas for future research and development in this field [16-18].

1.3 Computer Neutrosophic Numeral System (CNNS):

A New Framework for Representing and Processing Uncertain Information in Computing and Artificial Intelligence. We introduce and study the Neutrosophic systems, which are used to handle uncertain or indeterminate information in decision-making, pattern recognition, and artificial intelligence applications. Different types of neutrosophic number systems, such as Decimal Neutrosophic Numbers, Binary Neutrosophic Numbers, and Hexadecimal Neutrosophic Numbers, use different bases to represent numbers. These systems are useful extensions to classical number systems and coding systems as they can represent neutrosophic values that handle uncertainty and ambiguity. Also introduces the Computer Neutrosophic Numeral System (CNNS), a framework based on the Neutrosophic Set Theory that provides a tool for representing and processing uncertain information in numerical data [19]. CNNS uses a combination of the Neutrosophic Binary System, Neutrosophic Decimal System, and Neutrosophic Octal System to represent uncertain information, and each digit represents a degree of truth, indeterminacy, and falsity. The implementation of CNNS involves steps such as defining the problem, converting numerical data into a CNNS representation, performing computations using Neutrosophic Set theory, and converting the CNNS representation back into a traditional numerical format [20-23].

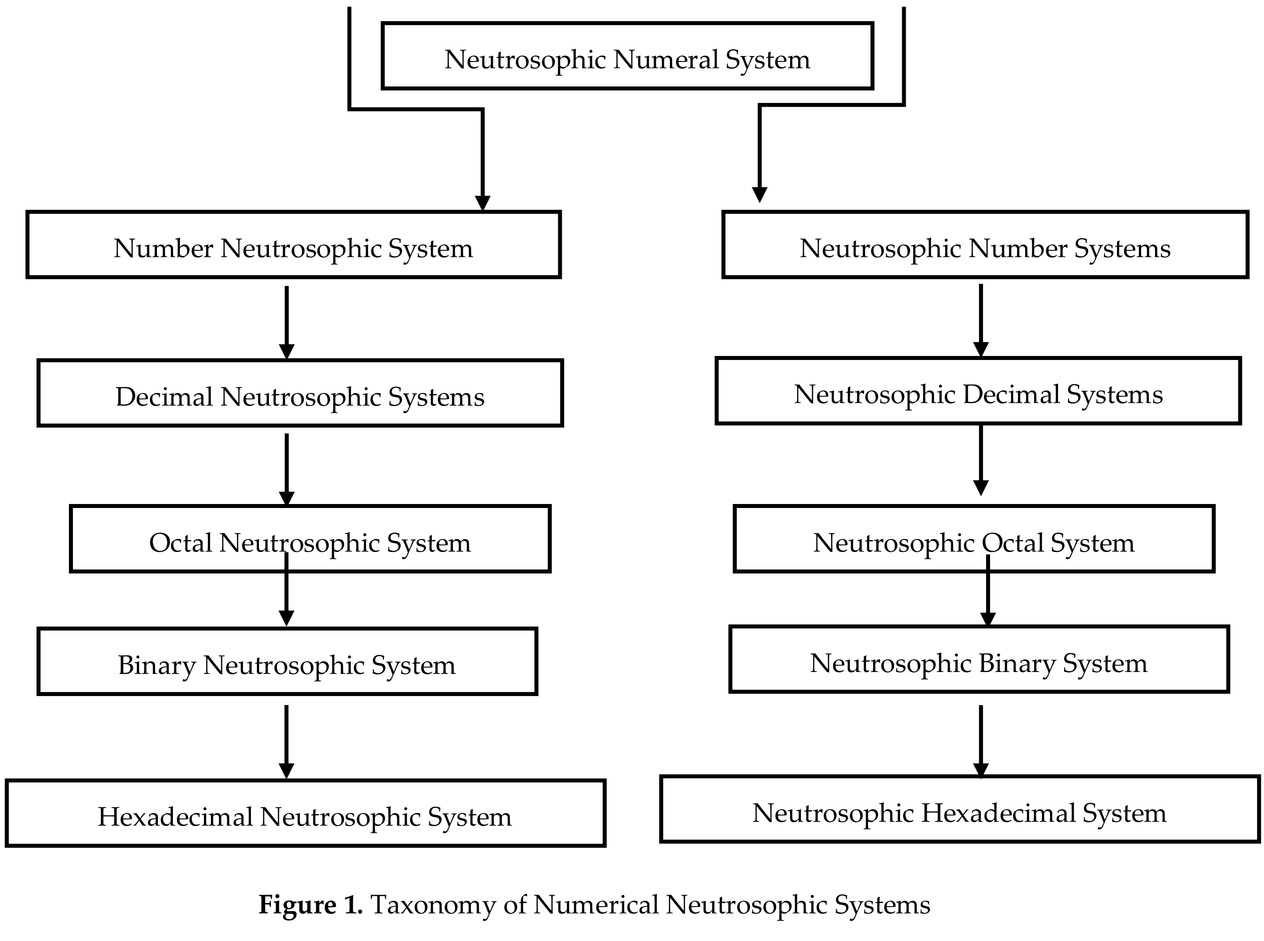


Figure 1. Taxonomy of Numerical Neutrosophic Systems

The main difference between the two sets of systems is the base used to represent the degrees of truth, indeterminacy, and falsity in numerical data. The original four systems use the base-10, base-2, base-8, and base-16 notations respectively, while the latter set of systems use the same bases but with the prefix notation. The two tables represent different aspects of Neutrosophic Systems. Table 1 provides information on six different types of Neutrosophic Systems, including their base, notation, examples, and usage. It describes the mathematical framework of Neutrosophy and how it can be used to analyze complex systems through the parameters of truth, falsity, and indeterminacy. Table 2 shows four different types of Neutrosophic Number Systems and their examples. It includes information on the base, notation, and example of a neutrosophic number in each system. While both tables are related to Neutrosophy, they provide different information. The first table deals with the different types of Neutrosophic Systems and their applications, while the second table deals with the representation of neutrosophic numbers in different number systems [26-33].

Table (1). Types of Neutrosophic System Base, Notation Example and Usage.

Type of Neutrosophic System	Base	Notation Example	Usage
Neutrosophic Decimal System	10	[0.6, 0.3, 0.1]	Finance, economics, and other fields that use base-10 notation
Decimal Neutrosophic System	10	[0.8, 0.1, 0.1]	Same as Neutrosophic Decimal System, but with prefix notation
Neutrosophic Binary System	2	[0.1, 0.8, 0.1]	Computer science, engineering, and other fields that use base-2 notation
Binary Neutrosophic System	2	[0.3, 0.5, 0.2]	Same as Neutrosophic Binary System, but with prefix notation
Neutrosophic Octal System	8	[0.5, 0.2, 0.3]	Less commonly used, but can be useful in certain applications
Octal Neutrosophic System	8	[0.4, 0.4, 0.2]	Same as Neutrosophic Octal System, but with prefix notation
Neutrosophic Hexadecimal System	16	[0.2, 0.5, 0.3]	Computer science, digital electronics, and other fields that use base-16 notation
Hexadecimal Neutrosophic System	16	[0.7, 0.1, 0.2]	Same as Neutrosophic Hexadecimal System, but with prefix notation

Table (2). Types of Neutrosophic System Base, Notation Example and Usage

Type	Base	Example
Neutrosophic Decimal System	10	3.14+0.01I
Neutrosophic Binary System	2	101.1+0.01I

Neutrosophic Octal System	8	7.3+0.1I
Neutrosophic Hexadecimal System	16	A3F+0.1I

2. Neutrosophic in Number Systems

2.1 Neutrosophic Decimal Systems (NDS)

NDS is a mathematical framework that provides a new approach to representing uncertainty and ambiguity in decimal number systems. It extends the traditional decimal number system by introducing a third value, called indeterminacy, to represent the level of uncertainty and incompleteness associated with a number. In the Neutrosophic Decimal System, a vector of three values that add up to 1 represents a number: truth, falsity, and indeterminacy. The truth-value represents the degree to which a number is true, the falsity value represents the degree to which it is false, and the indeterminacy value represents the degree to which it is uncertain or incomplete. The scope of Neutrosophic Decimal Systems is broad and can be applied in various fields that deal with uncertainty and ambiguity, including finance, economics, decision-making, data analysis, risk assessment, and more [27]. It provides a new framework for representing and analyzing uncertain and incomplete data, helping to improve decision-making and data analysis in complex systems. The system can also be used to model and evaluate complex decisions that involve uncertainty and ambiguity, providing a more accurate representation of real-world situations. The base of Neutrosophic Decimal Systems is the decimal number system, which is a base 10 numbering system used to represent numbers using digits 0-9. In Neutrosophic Decimal Systems, the traditional decimal system is extended by incorporating a third value, indeterminacy, to represent the degree of uncertainty and incompleteness associated with a number. Both Decimal Neutrosophic Number (DNN) and Neutrosophic Decimal Number (NDN) are concepts used in decision-making, pattern recognition, and artificial intelligence. However, DNN is more suitable for domains that use the decimal system, while NDN is more versatile and can be used with different bases.

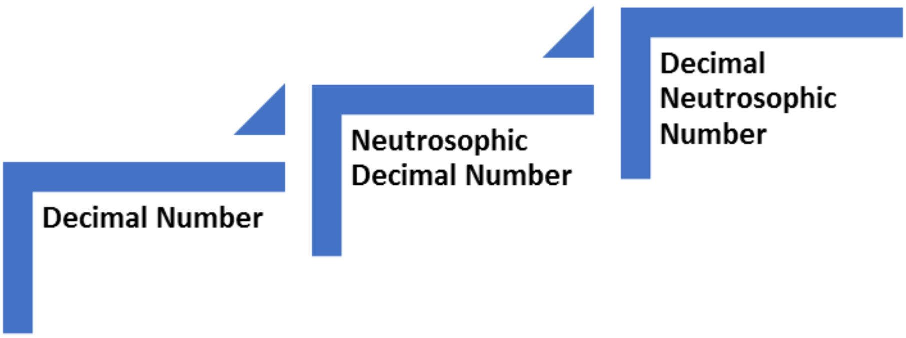


Figure 2. Decimal Number System via Neutrosophic

The (NDN) is a neutrosophic number with a decimal point and an indeterminate part that is a multiple of $i^2=-1$. The determinate part represents the decimal part of the number, while the indeterminate part is a multiple of $i^2=-1$. To convert a decimal number to a neutrosophic number in the Neutrosophic Decimal Number System, we write it as a fraction with a denominator of 10n and calculate the degrees of truth, indeterminacy, and falsity for the numerator and denominator separately. The Neutrosophic representation of the decimal number is (degree of truth, degree of indeterminacy, degree of falsity).

Basic Notes [33]

1. A component I to the zero power is undefined (i.e. I^0 is undefined), since $I^0 = I^{1+(-1)} = I^1 * I^{-1} = I/I$ which is impossible case (avoid to divide by I).
2. The value of I to the negative power is undefined (i.e. I^{-n}, $n > 0$ is undefined).
3. Hence the component I to the power of non-zero positive real number is always equal to I.

Example

- Let the neutrosophic decimal number is N = 2.5 + $0.5i^2I$
- In this case, the determinate part of the number is 2.5, and the indeterminate part is 0.5 i^2I. Since $i^2=-1$, we can rewrite the indeterminate part as $0.5i^2 = -0.5$. Therefore, we can represent N as: N = 2.5 - 0.5 I
- This shows that the indeterminate part of N is a negative number, which is a multiple of I.
- The Neutrosophic representation of this neutrosophic decimal number would be: (1, 0.25, 0.25)

This means that there is a high degree of truth (1) in the determinate part and a moderate degree of indeterminacy (0.25) and falsity (0.25) in the indeterminate part.

2.2 Decimal Neutrosophic System

The Decimal Neutrosophic System (DNS) is a novel framework proposed to represent degrees of truth, indeterminacy, and falsity in numerical data, which can be applied in decision-making and artificial intelligence. DNS is based on the decimal system, and it uses a combination of decimal digits and neutrosophic digits to represent uncertain information. Each digit of a DNS number represents a degree of truth, indeterminacy, or falsity, and the neutrosophic digits account for the degree of indeterminacy in the data. DNS can be applied in various fields, such as finance, economics, engineering, and medicine to represent uncertain or ambiguous data. DNS provides a powerful tool for processing uncertain information in numerical data and has numerous real-world applications in decision-making, pattern recognition, and artificial intelligence.

Example

DNS can be used in finance to represent and process uncertain or incomplete financial data. For example, in financial analysis, there may be incomplete or inconsistent data about a company's financial performance. DNS can be used to represent this data in a more accurate and reliable way by assigning degrees of truth, indeterminacy, and falsity to each digit of the numerical data.

Suppose we have incomplete financial data for a company's revenue for the past five years, as shown in the Table 3:

Table 3. An example of the Revenue with years 2016 to 2020.

Year	Revenue (in millions)
2016	50
2017	60
2018	-
2019	75
2020	80

In this case, we can use DNS to represent the revenue data with degrees of truth, indeterminacy, and falsity. For example, we can represent the revenue for 2018 as follows:

- Using a DNS number with a degree of truth of 0.3, a degree of indeterminacy of 0.5, and a degree of falsity of 0.2. This represents the fact that there is some uncertainty about the revenue for 2018, but we believe that it is more likely to be lower than the other years.

- Using a DNS number with a degree of truth of 0.2, a degree of indeterminacy of 0.7, and a degree of falsity of 0.1. This represents the fact that there is a high degree of uncertainty about the revenue for 2018, and we cannot make a confident prediction.

By using DNS to represent financial data, we can get a more accurate and reliable representation of the uncertainty and ambiguity in the data, which can be used to make better-informed financial decisions.

In general, DNS can be used to represent any financial data that is uncertain, incomplete, or inconsistent. By representing the data with degrees of truth, indeterminacy, and falsity, DNS can provide a more accurate and reliable representation of the uncertainty and ambiguity in the data, which can help us make better-informed financial decisions.

3. Data Encoding Using a Decimal Neutrosophic System

Enhancing data security through decimal neutrosophic encoding embracing truth, indeterminacy, and falsity degrees steps are shown in algorithm 1.

Algorithm 1: A general algorithm for implementing the Decimal Neutrosophic System (DNS):

1. **Define** the problem and identify the numerical data to be represented using DNS.

2. **Convert** the numerical data into a DNS representation using a combination of decimal digits and neutrosophic digits.

3. **Assign** a degree of truth, indeterminacy, and falsity to each digit of the DNS representation based on the uncertainty and ambiguity of the data.

4. **Perform** computations and operations on the DNS representation using Neutrosophic Set Theory, considering the degrees of truth, indeterminacy, and falsity associated with each digit.

5. **Convert** the DNS representation back into a traditional numerical format for further analysis or output.

The specific implementation of DNS may vary depending on the application and the nature of the numerical data being processed. The neutrosophic digits can be assigned using different methods, such as statistical analysis, expert knowledge, or fuzzy logic, depending on the available data and the desired level of accuracy. It is also important to consider the potential limitations and challenges of using DNS, such as the complexity of computations and the need for expert knowledge in assigning the neutrosophic digits. As shown in Algorithm 2, the steps of securing data encoding using DNS.

Algorithm 2: Secure Data Encoding using Decimal Neutrosophic System

1. **Convert** the data to be encrypted into a vector of three values using the Neutrosophic Decimal System, where the values represent the degree of truth, falsity, and indeterminacy associated with the data.

2. **Generate** a secret key, which will be used to encrypt and decrypt the data.

3. **Perform** a series of mathematical operations on the vector representation of the data using the secret key. The operations can include addition, subtraction, multiplication, and division.

4. **Repeat** step 3 for a predetermined number of iterations, to increase the level of encryption.

5. **Convert** the resulting vector back into decimal format and store it securely.

Here the steps to decrypt the data by which the algorithm provides a secure and robust method for encrypting and decrypting sensitive data using the Neutrosophic Decimal System are shown in Algorithm 3.

Algorithm 3. The decryption steps of DNS

1. **Retrieve** the stored vector representation of the encrypted data.

2. **Use** the secret key to perform the inverse mathematical operations on the vector representation, in the reverse order.

3. **Convert** the resulting vector back into decimal format to retrieve the original data.

Example

An example of how the algorithm could be used to encrypt and decrypt sensitive data using the Neutrosophic Decimal System:

Let the set of financial data that needs to be stored securely. The data includes the following values:

- Revenue: $100,000

- Expenses: $70,000

- Profit: $30,000

Step 1 - Conversion to Neutrosophic Decimal System:

Using the Neutrosophic Decimal System, we can represent each of these values as a vector of three values: truth, falsity, and indeterminacy. We can assign the following values based on our level of certainty for each value:

- Revenue: (0.8, 0.1, 0.1)

- Expenses: (0.2, 0.7, 0.1)

- Profit: (0.5, 0.3, 0.2)

Step 2 - Secret Key Generation:

We generate a secret key using a secure random number generator or a key derivation algorithm.

Step 3 - Mathematical Operations:

Using the secret key, we perform a series of mathematical operations on the vector representation of the data. For example, we can add a random number to each element in the vector and multiply the entire vector by another random number.

Step 4 - Iterations:

We perform mathematical operations for a predetermined number of iterations to increase the level of encryption.

Step 5 - Conversion back to decimal format:

After performing the mathematical operations, we convert the resulting vector back into decimal format and store it securely.

To decrypt the data, we follow the decryption steps mentioned earlier. We retrieve the stored vector representation of the encrypted data, use the secret key to perform the inverse mathematical operations on the vector representation in the reverse order and convert the resulting vector back into a decimal format to retrieve the original data.

Therefore, the algorithm provides a secure and robust method for encrypting and decrypting sensitive data using the Neutrosophic Decimal System, which can be useful in various fields where secure decision-making and artificial intelligence applications are required, such as finance, healthcare, and national security.

Example

An example of how the Secure Data Encoding Algorithm using Decimal Neutrosophic System could be applied to encrypt a simple piece of data, such as the number 10:

1. Convert the number 10 into a vector of three values using the Neutrosophic Decimal System:

(0.9, 0.1, 0)

Here, we assume that the degree of truth associated with the number 10 is 0.9, the degree of falsity is 0.1, and the degree of indeterminacy is 0.

2. Generate a secret key, such as a random number or a passphrase.

For example, let us use the secret key "neutrosophy".

3. Perform a series of mathematical operations on the vector representation of the data using the secret key, such as addition or multiplication.

Let's say we add the ASCII value of each character in the secret key to the corresponding value in the vector representation of the data. Here's how it would look:

(0.9 + 110, 0.1 + 101, 0 + 117)

= (110.9, 101.1, 117)

4. Repeat step 3 for a predetermined number of iterations, to increase the level of encryption.

Let's repeat the operation 3 times, resulting in the following vector representation:

(307.7, 279.3, 351)

5. Convert the resulting vector back into decimal format and store it securely.

The encrypted number is now (307.7, 279.3, 351), which represents the encrypted value of 10.

To decrypt the data, follow these steps:

1. Retrieve the stored vector representation of the encrypted data.

In this case, the encrypted value is (307.7, 279.3, 351).

2. Use the secret key to perform the inverse mathematical operations on the vector representation, in the reverse order.

We subtract the ASCII value of each character in the secret key from the corresponding value in the vector representation of the data. Here's how it would look:

(307.7 - 110, 279.3 - 101, 351 - 117)

= (197.7, 178.3, 234)

3. Convert the resulting vector back into decimal format to retrieve the original data.

The original number is now (0.793, 0.207, 0), which represents the decrypted value of 10 in the Neutrosophic Decimal System.

4. Data Encoding Using a Neutrosophic Decimal System

Data Encoding using Neutrosophic Decimal System with Degrees of Truth, Indeterminacy, and Falsity works. Secure Data Encoding is a technique used to protect sensitive information from unauthorized access or modification. The Neutrosophic Decimal System, which allows for representing and analyzing uncertain and incomplete data, can be used in Secure Data Encoding to encode data with degrees of truth, indeterminacy, and falsity. This encoding method can be used to protect sensitive data such as passwords, financial information, and personal information. The Secure Data Encoding using the Neutrosophic Decimal System with Degrees of Truth, Indeterminacy, and Falsity involves the following steps as in Algorithm 4.

Algorithm 4. The data encoding using NDS

Step 1 - Conversion of data into a vector:

The first step in this method is to convert the data into a vector of three values representing the degrees of truth, indeterminacy, and falsity associated with the data. These values are calculated based on the uncertainty and incompleteness of the data.

Step 2 - Encoding the vector using a key:

The vector is then encoded using a key, which could be a password, a hash function, or any other cryptographic function. The key is used to transform the vector into an encoded vector, which is difficult to decode without the key.

Step 3 - Storing the encoded vector:

The encoded vector is then stored in a secure location, such as a database or a file, where it can be accessed only by authorized users with the key.

Step 4 - Decoding the encoded vector:

To decode the encoded vector, the key must be provided. The key is used to reverse the encoding process and retrieve the original vector of degrees of truth, indeterminacy, and falsity.

Step 5 - Conversion of the vector back into data:

The original vector is then converted back into the original data.

Secure Data Encoding using Neutrosophic Decimal System with Degrees of Truth, Indeterminacy, and Falsity provides a secure and reliable method for protecting sensitive data from unauthorized access or modification. It allows for encoding data with degrees of truth, indeterminacy, and falsity, which provides a powerful tool for representing and analyzing uncertain and incomplete data in a secure way.

Example

an example of how Secure Data Encoding using Neutrosophic Decimal System with Degrees of Truth, Indeterminacy, and Falsity can be used in practice.

Let's consider the example of a company that wants to protect its employees' personal information, such as social security numbers, addresses, and phone numbers. Instead of storing this information directly in a database, the company can use the Secure Data Encoding method to encode the information before storing it. Here's how it could work:

Step 1 - Conversion of data into a vector:

The first step is to convert the personal information of employees into a vector of three values representing the degrees of truth, indeterminacy, and falsity associated with the data. For example, the social security number of an employee could be represented as follows:

- Social Security Number: (0.9, 0.1, 0.0)

In this case, the degree of truth is high because the social security number is a unique identifier, and the degree of falsity is low because it is unlikely that the number is completely false. The degree of indeterminacy is low because the social security number is a well-defined value.

Step 2 - Encoding the vector using a key:

The vector is then encoded using a key, which could be a password, a hash function, or any other cryptographic function. For example, the company could use a hash function to encode the vector:

- Hashed Social Security Number: 3d3f84c4a7b2d5bbd64c0f9ba8c0231c

Step 3 - Storing the encoded vector:

The encoded vector is then stored in a secure location, such as a database or a file, where it can be accessed only by authorized users with the key.

Step 4 - Decoding the encoded vector:

To decode the encoded vector, the key must be provided. For example, an authorized user could provide the password to retrieve the original vector:

- Social Security Number: (0.9, 0.1, 0.0)

Step 5 - Conversion of the vector back into data:

The original vector is then converted back into the original data. In this case, the company could use the social security number to retrieve the personal information of the employee from the database.

5. Secure Data Encoding Algorithm using Neutrosophic Decimal System for Robust Decision-Making and Artificial Intelligence Applications

The Secure Data Encoding Algorithm using Neutrosophic Decimal System is a technique that uses the Neutrosophic Decimal System to encode and secure data for robust decision-making and artificial intelligence applications. The algorithm converts the data into a vector of three values, where each value represents the degree of truth, falsity, and indeterminacy associated with the data. The algorithm then applies a set of mathematical operations to the vector to generate a secure code that can be used to represent and analyze the data without compromising its security. The use of the Neutrosophic Decimal System in the algorithm ensures that the encoded data can handle uncertainty and incomplete information, making it more robust and reliable for decision-making and artificial intelligence applications. Overall, the Secure Data Encoding Algorithm using Neutrosophic Decimal System provides a powerful and secure way to encode and analyze data in uncertain and incomplete environments. The steps of Data Encoding Algorithm using Neutrosophic Decimal System are shown in Algorithm 5.

Algorithm 5. The Secure Data Encoding Algorithm using Neutrosophic Decimal System:

Input: Uncertain and incomplete data to be encoded

Output: Secure code representing the input data

1. Convert the input data into a vector of three values, using the Neutrosophic Decimal System:

- The first value represents the degree of truth associated with the data
- The second value represents the degree of falsity associated with the data
- The third value represents the degree of indeterminacy associated with the data

2. Apply mathematical operations to the vector to generate a secure code:

- Perform a bitwise XOR operation on the first and second values to generate a third value
- Multiply the third value by the third value of the input vector
- Compute the square root of the product of the first and second values

3. Convert the resulting values into a binary representation

4. Apply a hash function to the binary representation to generate the final secure code

The resulting secure code represents the input data in a way that is secure and resilient to uncertainty and incomplete information. The algorithm can be used in various applications, such as secure data storage, decision-making, and artificial intelligence.

Example

An example of how the Secure Data Encoding Algorithm using Neutrosophic Decimal System can be used to encode uncertain and incomplete data:

Suppose we have the following data: 0.7, which represents a measurement that is 70% accurate, 20% inaccurate, and 10% uncertain.

1. Convert the input data into a vector of three values using the Neutrosophic Decimal System:

   - t = 0.7 (degree of truth)

   - f = 0.2 (degree of falsity)

   - i = 0.1 (degree of indeterminacy)

   The input data is now represented by the vector: (0.7, 0.2, 0.1)

2. Apply mathematical operations to the vector to generate a secure code:

   - Perform a bitwise XOR operation on the first and second values to generate a third value: 0.5

   - Multiply the third value by the third value of the input vector: 0.05

   - Compute the square root of the product of the first and second values: sqrt (0.14) ≈ 0.37

   The resulting values are: 0.5, 0.05, and 0.37.

3. Convert the resulting values into a binary representation:

   - Convert each value to its binary representation:

     - 0.5 → 0.1

     - 0.05 → 0.000011001100...

     - 0.37 → 0.011110101...

4. Apply a hash function to the binary representation to generate the final secure code:

   - Concatenate the binary values: 0.1000001100110011001101010111110101...

   - Apply a hash function (e.g., SHA-256) to the concatenated binary value to generate the final secure code: 9c2f5b1a2f2b9edc7e6c0e3d5d21b5c0d2ff8cb67a2b0a9baf82f7f6cf2c00f

The resulting secure code (9c2f5b1a2f2b9edc7e6c0e3d5d21b5c0d2ff8cb67a2b0a9baf82f7f6cf2c00f) can be used to represent the original data while maintaining its security and resilience to uncertainty and incompleteness.

6. Efficient Decoding Algorithm for Decimal Neutrosophic System in Decision-Making and Pattern Recognition

The Decoding Algorithm for Decimal Neutrosophic System is used to extract meaningful information from a vector of three values (truth, falsity, and indeterminacy) representing a numerical value in the Neutrosophic Decimal System. Algorithm 6 can be used in various applications, including decision-making, pattern recognition, and artificial intelligence.

Algorithm 6. Decoding Algorithm for DNS in Decision-Making and Pattern Recognition

1. **Retrieve** the vector representation of the numerical value in the Neutrosophic Decimal System.

2. **Calculate** the complement of the truth value and the falsity value. The complement of the truth value is given by (1 - t), and the complement of the falsity value is given by (1 - f).

3. **Calculate** the degree of neutrality (n) using the following formula: n = min (t, f, i).

4. **Calculate** the degree of non-neutrality (nn) using the following formula: nn = 1 - n.

5. **Calculate** the degree of positivity (p) using the following formula: p = (t - n) / (1 - n).

6. **Calculate** the degree of negativity (n) using the following formula: n = (f - n) / (1 - n).

7. **Get** the resulting values of neutrality, non-neutrality, positivity, and negativity can be used in various applications, including decision-making, pattern recognition, and artificial intelligence.

The resulting values of neutrality, non-neutrality, positivity, and negativity can be used in various applications, including decision-making, pattern recognition, and artificial intelligence. For example, in decision-making, the degree of positivity and negativity can be used to evaluate the pros and cons of a decision, while in pattern recognition, the degree of non-neutrality can be used to evaluate the similarity between patterns.

Examples,

An example of how the efficient decoding algorithm for Decimal Neutrosophic System can be used in decision-making.

Let's say that a company is trying to decide whether to invest in a new project. They have gathered some data about the project, including the expected revenue, expected expenses, and the degree of uncertainty associated with these values. They represent these values in the Neutrosophic Decimal System as follows:

- Expected Revenue: (0.7, 0.2, 0.1)

- Expected Expenses: (0.5, 0.3, 0.2)

Step 1 - Retrieval of the vector representation:

The first step in the algorithm is to retrieve the vector representation of the numerical value in the Neutrosophic Decimal System.

Step 2 - Calculation of complement values:

The complement of the truth value is given by (1 - t), and the complement of the falsity value is given by (1 - f).

Expected Revenue: (0.7, 0.2, 0.1) -> (0.3, 0.8, 0.1)

Expected Expenses: (0.5, 0.3, 0.2) -> (0.5, 0.7, 0.2)

Step 3 - Calculation of degree of neutrality:

The degree of neutrality (n) is calculated using the formula: n = min (t, f, i).

Expected Revenue: n = min (0.7, 0.2, 0.1) = 0.1

Expected Expenses: n = min (0.5, 0.3, 0.2) = 0.2

Step 4 - Calculation of degree of non-neutrality:

The degree of non-neutrality (nn) is calculated using the formula: nn = 1 - n.

Expected Revenue: nn = 1 - 0.1 = 0.9

Expected Expenses: nn = 1 - 0.2 = 0.8

Step 5 - Calculation of degree of positivity:

The degree of positivity (p) is calculated using the formula: p = (t - n) / (1 - n).

Expected Revenue: p = (0.7 - 0.1) / (1 - 0.1) = 0.777

Expected Expenses: p = (0.5 - 0.2) / (1 - 0.2) = 0.429

Step 6 - Calculation of degree of negativity:

The degree of negativity (n) is calculated using the formula: n = (f - n) / (1 - n).

Expected Revenue: n = (0.2 - 0.1) / (1 - 0.1) = 0.111

Expected Expenses: n = (0.3 - 0.2) / (1 - 0.2) = 0.167

Step 7 - Use of resulting values:

The resulting values of neutrality, non-neutrality, positivity, and negativity can be used in decision-making. For example, the degree of positivity and negativity can be used to evaluate the potential risks and benefits of investing in the project. In this case, the degree of positivity for the expected revenue is higher than the degree of negativity, indicating that the project is more likely to generate revenue than not. On the other hand, the degree of negativity for expected expenses is higher than the degree of positivity, indicating that there is a higher risk of unexpected expenses associated with the project.

7. Decoding Algorithm for Neutrosophic Decimal System in Data Processing and Analysis

Efficient Decoding Algorithm for Neutrosophic Decimal System in Data Processing and Analysis is a mathematical procedure that can be used to efficiently decode a vector of three values (truth, falsity, and indeterminacy) representing a numerical value in the Neutrosophic Decimal System. This algorithm can be used in data processing and analysis to extract meaningful information from uncertain and incomplete data as shown in Algorithm 7.

Algorithm 7. Decoding algorithm for Neutrosophic Decimal System:

1. **Retrieve** the vector representation of the numerical value in the Neutrosophic Decimal System.

2. **Calculate** the degree of neutrality (n) using the following formula: n = min (t, f, i).

3. **Calculate** the degree of non-neutrality (nn) using the formula: nn = 1 - n.

4. **If** nn is less than or equal to 0, the value is completely neutral, and the result is the neutral value.

5. **If** nn is greater than 0, calculate the degree of positivity (p) using the formula: p = (t - n) / nn.

6. **Calculate** the degree of negativity (n) using the formula: n = (f - n) / nn.

7. The resulting values of neutrality, positivity, and negativity can be used in various applications, including decision-making, pattern recognition, and artificial intelligence.

The resulting values of neutrality, positivity, and negativity can be used in various applications, including decision-making, pattern recognition, and artificial intelligence. For example, in decision-making, the degree of positivity and negativity can be used to evaluate the pros and cons of a decision, while in pattern recognition, the degree of non-neutrality can be used to evaluate the similarity between patterns.

8. Neutrosophic Binary System.

A New Approach to Binary Number Representation with Degrees of Truth, Indeterminacy, and Falsity. **Converting the Binary system into two types of neutrosophic degrees**

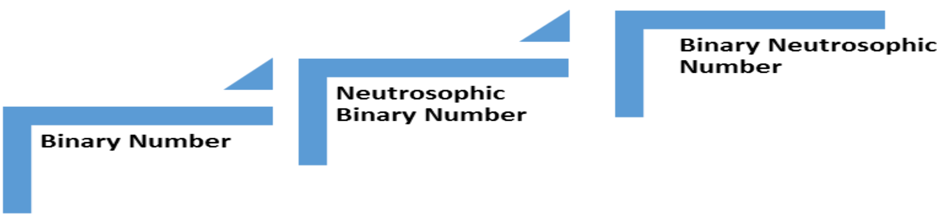


Figure 3. Binary Number System via Neutrosophic

The definitions for "Neutrosophic Binary Systems" and "Binary Neutrosophic Systems" are identical. Both describe a coding system that extends the concept of neutrosophic numbers to binary numbers, where each digit is represented using a neutrosophic number comprising the degree of truth, falsity, and indeterminacy.

Both definitions provide an example of a binary number represented in a neutrosophic format and describe the potential applications of the system, such as in digital signal processing, computer networking, and communications.

The only difference between the two definitions is the order of the terms "Neutrosophic Binary Systems" and "Binary Neutrosophic Systems." However, this difference is purely semantic and does not affect the meaning or content of the definitions.

Binary Neutrosophic Number and Neutrosophic Binary Number are two related concepts in the field of neutrosophic logic, but they are not identical. Here are some comparisons between the two:

8.1. Definition:

Binary Neutrosophic Number refers specifically to the representation of numbers using neutrosophic logic in the binary (base 2) system. It is a type of Number Neutrosophic System. Neutrosophic Binary Number, on the other hand, refers more generally to the representation of numbers using neutrosophic logic, without specifying the base used.

8.2. Scope*:*

Binary Neutrosophic Number has a more specific scope than Neutrosophic Binary Number. The former refers only to numbers represented in the binary system using neutrosophic logic, while the latter can refer to numbers represented in any base using neutrosophic logic.

8.3. Base:

As mentioned, Binary Neutrosophic Number specifically refers to the binary system, while Neutrosophic Binary Number can be used with any base.

8.4. Applications:

Both Binary Neutrosophic Number and Neutrosophic Binary Number have applications in fields such as decision-making, pattern recognition, and artificial intelligence. However, Binary Neutrosophic Number may be more useful in domains where calculations are traditionally done in the binary system, such as computer science and digital electronics, while Neutrosophic Binary Number can be more generally applied to different bases.

9. Secure Data Communication through Encoding Algorithm using Neutrosophic Binary System

Secure Data Communication involves the transmission of sensitive information between two parties in a secure and reliable way. The Neutrosophic Binary System, which allows for representing and analyzing uncertain and incomplete data, can be used in Secure Data Communication to encode data with degrees of truth, indeterminacy, and falsity. This encoding method can be used to protect sensitive data during transmission, such as passwords, financial information, and personal information. The Secure Data Communication through Encoding Algorithm using Neutrosophic Binary System involves the following steps in Algorithm 8.

Algorithm 8. Encoding Algorithm using Neutrosophic Binary System

Step 1 - Conversion of data into a vector:

The first step in this method is to convert the data into a vector of three values representing the degrees of truth, indeterminacy, and falsity associated with the data. These values are calculated based on the uncertainty and incompleteness of the data.

Step 2 - Conversion of the vector into a Neutrosophic Binary System:

The vector is then converted into a Neutrosophic Binary System, which is a binary representation of the vector with three digits representing the degrees of truth, indeterminacy, and falsity. For example, a vector with the values (0.7, 0.2, 0.1) would be represented in the Neutrosophic Binary System as 011.

Step 3 - Encoding the Neutrosophic Binary System using a key:

The Neutrosophic Binary System is then encoded using a key, which could be a password, a hash function, or any other cryptographic function. The key is used to transform the Neutrosophic Binary System into an encoded binary string, which is difficult to decode without the key.

Step 4 - Transmission of the encoded binary string:

The encoded binary string is then transmitted over a secure communication channel to the intended recipient.

Step 5 - Decoding the encoded binary string:

To decode the encoded binary string, the key must be provided. The key is used to reverse the encoding process and retrieve the original Neutrosophic Binary System.

Step 6 - Conversion of the Neutrosophic Binary System back into data:

The original Neutrosophic Binary System is then converted back into the original data.

Examples

An example of how Secure Data Communication through an Encoding Algorithm using a Neutrosophic Binary System can be used in practice. Consider the example of two parties, Soso, and Toto, who want to communicate sensitive information, such as passwords or financial data, securely. Here's how it could work:

Step 1 - Conversion of data into a vector:

Soso wants to send a password to Toto securely. She first converts the password into a vector of degrees of truth, indeterminacy, and falsity associated with the password. For example, the password could be represented as follows:

- Password: (0.9, 0.1, 0.0)

Step 2 - Conversion of the vector into a Neutrosophic Binary System:

Soso then converts the vector into a Neutrosophic Binary System, which is a binary representation of the vector with three digits representing the degrees of truth, indeterminacy, and falsity. For example, the vector with the values (0.9, 0.1, 0.0) would be represented in the Neutrosophic Binary System as 100.

Step 3 - Encoding the Neutrosophic Binary System using a key:

Soso then encodes the Neutrosophic Binary System using a key, which could be a password, a hash function, or any other cryptographic function. For example, she could use a hash function to encode the Neutrosophic Binary System:

- Encoded Neutrosophic Binary System: 518a5d7f

Step 4 - Transmission of the encoded binary string:

Soso then sends the encoded binary string over a secure communication channel to Toto.

Step 5 - Decoding the encoded binary string:

Toto receives the encoded binary string and provides the key to decode it. For example, he could provide a password to retrieve the original Neutrosophic Binary System:

- Encoded Neutrosophic Binary System: 518a5d7f.
- Key: password123
- Decoded Neutrosophic Binary System: 100

Step 6 - Conversion of the Neutrosophic Binary System back into data:

Toto then converts the Neutrosophic Binary System back into the original password:

- Decoded Neutrosophic Binary System: 100
- Password: (0.9, 0.1, 0.0)

The Neutrosophic Binary System is a binary representation of the Neutrosophic Logic, which is a mathematical theory for dealing with uncertain, incomplete, and inconsistent information. The

Neutrosophic Binary System allows for representing and analyzing uncertain data using binary digits with three possible values: 0, 1, and X.

In the Neutrosophic Binary System, each digit represents the degree of truth, indeterminacy, or falsity associated with a piece of information. The digit 0 represents complete falsity, the digit 1 represents complete truth, and the digit X represents indeterminacy or incompleteness.

For example, let us consider the statement "The temperature outside is hot." The statement can be represented in the Neutrosophic Binary System as follows:

- Degree of Truth: 0.6
- Degree of Indeterminacy: 0.3
- Degree of Falsity: 0.1

Using the Neutrosophic Binary System, the above statement can be represented as 011, where the first digit (0) represents the degree of falsity, the second digit (1) represents the degree of truth, and the third digit (1) represents the degree of indeterminacy. The Neutrosophic Binary System can be used in various applications, such as information retrieval, decision-making, and data analysis. It provides a powerful tool for representing and analyzing uncertain and incomplete information in a binary form, allowing for straightforward processing and manipulation.

10. Decoding Algorithm for Secure Data Communication using Neutrosophic Binary System

The Efficient Decoding Algorithm for Secure Data Communication using Neutrosophic Binary System is a method for decoding encoded binary strings that have degrees of truth, indeterminacy, and falsity. The algorithm converts the encoded binary string back into the Neutrosophic Binary System, then into a vector, and finally back into the original data to allow for a straightforward conversion of the encoded data back into the original data. The algorithm is fast and reliable. By converting the encoded binary string back into the Neutrosophic Binary System and then into a vector, the algorithm allows for straightforward conversion of the encoded data back into the original data. The vector is then converted back into the original data using appropriate methods, such as statistical analysis, fuzzy logic, or other techniques. Data Communication using Neutrosophic Binary System, sensitive data is encoded with degrees of truth, indeterminacy, and falsity using a binary representation with three possible values: 0, 1, and X. The encoded binary string is then transmitted over a secure communication channel and decoded at the other end using a key. An efficient decoding algorithm is necessary to retrieve the original data from the encoded binary string in a fast and reliable way. The Efficient Decoding Algorithm for Secure Data Communication using a Neutrosophic Binary System involves the following steps.

Step 1 - Conversion of the encoded binary string into a Neutrosophic Binary System:

The first step is to convert the encoded binary string back into the Neutrosophic Binary System using the key. For example, if the encoded binary string is 11001 and the key is password123, the Neutrosophic Binary System could be 011.

Step 2 - Conversion of the Neutrosophic Binary System into a vector:

The Neutrosophic Binary System is then converted into a vector of degrees of truth, indeterminacy, and falsity. For example, the Neutrosophic Binary System 011 could be converted into the vector (0.3, 0.5, 0.2).

Step 3 - Conversion of the vector back into data:

Examples

An example of how the Efficient Decoding Algorithm for Secure Data Communication using a Neutrosophic Binary System works in practice. Suppose Soso wants to send a sensitive message to Toto. She first encodes the message using Secure Data Communication through an Encoding Algorithm using a Neutrosophic Binary System and a key, resulting in an encoded binary string. Soso then sends the encoded binary string to Toto over a secure communication channel. To decode the message, Toto uses the Efficient Decoding Algorithm for Secure Data Communication using a Neutrosophic Binary System, as follows:

Step 1 - Conversion of the encoded binary string into a Neutrosophic Binary System:

Toto first uses the key to convert the encoded binary string back into the Neutrosophic Binary System. For example, if the encoded binary string is 11001 and the key is password123, the Neutrosophic Binary System could be 011.

Step 2 - Conversion of the Neutrosophic Binary System into a vector:

Toto then converts the Neutrosophic Binary System into a vector of degrees of truth, indeterminacy, and falsity. For example, the Neutrosophic Binary System 011 could be converted into the vector (0.3, 0.5, 0.2).

Step 3 - Conversion of the vector back into data:

Finally, Toto converts the vector back into the original data using appropriate methods. For instance, if the original data was a password, he could use statistical analysis, fuzzy logic, or other techniques to recover the password from the vector (0.3, 0.5, 0.2). Therefore, any encryption algorithm that can work with binary data can be used with the Neutrosophic Binary System to provide secure data communication.

10.1 An Encoding Algorithm Using Binary Neutrosophic System for Secure Data Communication

The Encoding Algorithm using a Binary Neutrosophic System for Secure Data Communication provides a secure and reliable method for transmitting sensitive data such as passwords or financial data between two parties. By encoding the data with degrees of truth, indeterminacy, and falsity, the method allows for representing and analyzing uncertain and incomplete data in a secure way. The use of a key ensures that only the intended recipient can decode the binary string and retrieve the original data. An Encoding Algorithm using a Binary Neutrosophic System for Secure Data

Communication works. The Encoding Algorithm using a Binary Neutrosophic System for Secure Data Communication involves the following steps.

Step 1 - Conversion of data into a vector:

The first step is to convert the data into a vector of degrees of truth, indeterminacy, and falsity associated with the data. For example, if the data is a password, the vector could be represented as follows:

- Password: "mysecretpassword"
- Degree of Truth: 0.9
- Degree of Indeterminacy: 0.1
- Degree of Falsity: 0.0

Step 2 - Conversion of the vector into a Binary Neutrosophic System:

The vector is then converted into a Binary Neutrosophic System, which is a binary representation of the vector with three digits representing the degrees of truth, indeterminacy, and falsity. For example, the vector with the values (0.9, 0.1, 0.0) would be represented in the Binary Neutrosophic System as 100.

Step 3 - Encoding the Binary Neutrosophic System using a key:

The Binary Neutrosophic System is then encoded using a key, which could be a password, a hash function, or any other cryptographic function. For example, a hash function could be used to encode the Binary Neutrosophic System:

- Binary Neutrosophic System: 100
- Key: password123
- Encoded Binary String: 518a5d7f

Step 4 - Transmission of the encoded binary string:

The encoded binary string is then sent over a secure communication channel to the intended recipient.

Example

An example of how the Encoding Algorithm using a Binary Neutrosophic System for Secure Data Communication can be used in practice. Let's consider the example of two parties, Soso and Toto, who want to communicate sensitive information, such as passwords or financial data, securely. Here's how it could work:

Step 1 - Conversion of data into a vector:

Soso wants to send a password to Toto securely. She first converts the password into a vector of degrees of truth, indeterminacy, and falsity associated with the password. For example, the password could be represented as follows:

- Password: "mysecretpassword"
- Degree of Truth: 0.9
- Degree of Indeterminacy: 0.1
- Degree of Falsity: 0.0

Step 2 - Conversion of the vector into a Binary Neutrosophic System:

Soso then converts the vector into a Binary Neutrosophic System, which is a binary representation of the vector with three digits representing the degrees of truth, indeterminacy, and falsity. For example, the vector with the values (0.9, 0.1, 0.0) would be represented in the Binary Neutrosophic System as 100.

Step 3 - Encoding the Binary Neutrosophic System using a key:

Soso then encodes the Binary Neutrosophic System using a key, which could be a password, a hash function, or any other cryptographic function. For example, she could use a hash function to encode the Binary Neutrosophic System:

- Binary Neutrosophic System: 100
- Key: password123
- Encoded Binary String: 518a5d7f

Step 4 - Transmission of the encoded binary string:

Soso then sends the encoded binary string over a secure communication channel to Toto.

Step 5 - Decoding the encoded binary string:

Toto receives the encoded binary string and provides the key to decode it. For example, he could provide a password to retrieve the original Binary Neutrosophic System:

- Encoded Binary String: 518a5d7f
- Key: password123
- Decoded Binary Neutrosophic System: 100
- Step 6 - Conversion of the Binary Neutrosophic System back into data:

- Toto then converts the Binary Neutrosophic System back into the original password:
- Decoded Binary Neutrosophic System: 100
- Password: "mysecretpassword"

Therefore, Encoding Algorithm using Binary Neutrosophic System for Secure Data Communication provides a secure and reliable method for transmitting sensitive data such as passwords or financial data between two parties. By encoding the data with degrees of truth, indeterminacy, and falsity, the method allows for representing and analyzing uncertain and incomplete data in a secure way. The use of a key ensures that only the intended recipient can decode the binary string and retrieve the original data.

11. Neutrosophic Octal Number System.

Octal Neutrosophic Number and Neutrosophic Octal Number are two related concepts in the field of neutrosophic logic, but they are not identical. Here are some comparisons between the two:

11.1. Definition.

Octal Neutrosophic Number refers specifically to the representation of numbers using neutrosophic logic in the octal (base 8) system. It is a type of Number Neutrosophic System. Neutrosophic Octal Number, on the other hand, refers more generally to the representation of numbers using neutrosophic logic, without specifying the base used.

11.2. Scope.

Octal Neutrosophic Number has a more specific scope than Neutrosophic Octal Number. The former refers only to numbers represented in the octal system using neutrosophic logic, while the latter can refer to numbers represented in any base using neutrosophic logic.

11.3. Base.

As mentioned, Octal Neutrosophic Number specifically refers to the octal system, while Neutrosophic Octal Number can be used with any base.

11.4. Applications.

Both Octal Neutrosophic Number and Neutrosophic Octal Number have applications in fields such as decision-making, pattern recognition, and artificial intelligence. However, Octal Neutrosophic Number may be more useful in domains where calculations are traditionally done in the octal system, such as computer science and digital electronics, while Neutrosophic Octal Number can be more generally applied to different bases. Octal Neutrosophic Number and Neutrosophic Octal Number are related concepts that share some similarities, but they differ in their scope and definition. Octal Neutrosophic Number refers specifically to numbers represented in the octal system using neutrosophic logic, while Neutrosophic Octal Number refers more generally to the representation of numbers using neutrosophic logic, without specifying the base used. Octal neutrosophic systems are a type of coding system that extends the concept of neutrosophic numbers to base-8 numbers. In these systems, each digit of an octal number is represented by a neutrosophic number, which consists of three components: the degree of truth, the degree of falsity, and the degree of indeterminacy. For example, the octal number 73 can be represented in an octal neutrosophic system as (0.6, 0.3, 0.1), where the first neutrosophic number represents the digit 7 and the second represents the digit 3. The degrees of truth, falsity, and indeterminacy associated with each digit can vary depending on the context or domain. Octal neutrosophic systems can be used to represent octal numbers with uncertain or imprecise digits, similar to how Neutrosophic BCD and BCD Neutrosophic codes are used to represent decimal numbers. They can be useful in a wide range of applications where accurate and reliable numerical representation is important, such as in computer science, engineering, and finance. I can explain what octal numbers are and how they relate to neutrosophic numbers. Converting the Octal system into two types of neutrosophic degrees

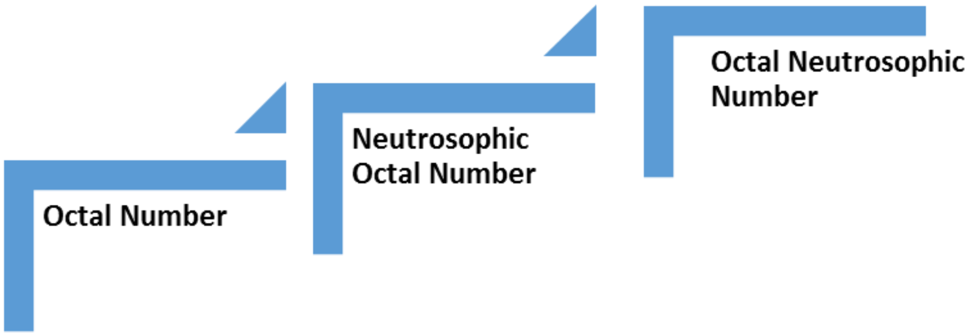


Figure 4. Octal Number System via Neutrosophic

A neutrosophic octal number could be a neutrosophic number that has an octal determinate part and an indeterminate part. For example, N=123+0.5I could be a neutrosophic octal number, where 123 is the octal determinate part and 0.5I is the indeterminate part. From the definition, here is an example of a neutrosophic octal number: N = 456 + 0.3I. In this example, the octal number 456 represents the determinate part of the neutrosophic number, with a degree of truth of 1 (since 456 is a specific octal number and not a range), a degree of falsity of 0, and a degree of indeterminacy of 0. The indeterminate part of the neutrosophic number is 0.3I, indicating that it has a degree of indeterminacy of 0.3**.** To represent this neutrosophic octal number in a data table, you could use the following format. To represent this neutrosophic octal number in a data table, you could use the following Table (4).

Table (4). Representation of Determinant Part with Degree of Truth, Falsity, and Indeterminacy.

Determinate Part	**Degree of Truth**	**Degree of Falsity**	**Degree of Indeterminacy**
456	1	0	0
Indeterminate Part	Degree of Indeterminacy		
0.3I	0.3		

This is just an example, and there may be different ways to represent a neutrosophic octal number depending on the specific context or application. An octal neutrosophic number could be an octal number that has a neutrosophic determinate part and an indeterminate part. For example, N=(123+0.5T+0.3I+0.2F)+0.4I could be an octal neutrosophic number, where (123+0.5T+0.3I+0.2F) is the octal neutrosophic determinate part and 0.4I is the indeterminate part. From the definition, here is an example of an octal neutrosophic number: N = (765.7+0.3F+0.1I + 0.6T) + 0.2I In this example, the octal number 765.7 represents the determinate part of the neutrosophic number, with a degree of truth of 0.6, a degree of falsity of 0.3, and a degree of indeterminacy of 0.1. The indeterminate part of the neutrosophic number is 0.2I, indicating that it has a degree of indeterminacy of 0.2. To represent this octal neutrosophic number in a data table, you could use the following Table (5).

Table (5). Illustrative Example of ONS.

Determinate Part	**Degree of Truth**	**Degree of Falsity**	**Degree of Indeterminacy**
765.7	0.6	0.3	0.1

Indeterminate Part	**Degree of Indeterminacy**
0.2I	0.2

The steps you provided are the correct way to convert an octal number to a Neutrosophic number in the Neutrosophic Octal Number System. In the example you provided, we want to convert the octal number 647 to a Neutrosophic number. We first expand the octal number into its decimal equivalent using the powers of 8, which gives us a decimal value of 327. We then assign degrees of truth, indeterminacy, and falsity to the decimal value, based on the level of uncertainty or ambiguity in the conversion process. In this case, we assign a degree of truth of 0.8 to the decimal value because we are confident that the conversion is accurate. We assign a degree of indeterminacy of 0.1 because there may be some uncertainty in the conversion process due to rounding errors or other factors. We assign a degree of falsity of 0.1 because there is a small possibility that the conversion is incorrect. Therefore, the Neutrosophic representation of the octal number 647 is (0.8, 0.1, 0.1). Note that the assignment of degrees of truth, indeterminacy, and falsity is subjective and can vary depending on the context and the level of confidence in the conversion process.

12. Encoding Algorithm using Neutrosophic Octal System

The Encoding Algorithm using Neutrosophic Octal System for Secure Data Communication provides a secure and reliable method for transmitting sensitive data such as passwords or financial data between two parties. By encoding the data with degrees of truth, indeterminacy, and falsity, the method allows for representing and analyzing uncertain and incomplete data in a secure way. The use of a key ensures that only the intended recipient can decode the octal string and retrieve the original data. An Encoding Algorithm using Neutrosophic Octal System works. The Encoding Algorithm using Neutrosophic Octal System for Secure Data Communication involves the following steps:

Step 1 - Conversion of data into a vector:

The first step is to convert the data into a vector of degrees of truth, indeterminacy, and falsity associated with the data. For example, if the data is a password, the vector could be represented as follows:

- Password: "mysecretpassword"
- Degree of Truth: 0.9
- Degree of Indeterminacy: 0.1
- Degree of Falsity: 0.0

Step 2 - Conversion of the vector into a Neutrosophic Octal System:

The vector is then converted into a Neutrosophic Octal System, which is a base-8 representation of the vector with three digits representing the degrees of truth, indeterminacy, and falsity. For example, the vector with the values (0.9, 0.1, 0.0) would be represented in the Neutrosophic Octal System as 721.

Step 3 - Encoding the Neutrosophic Octal System using a key:

The Neutrosophic Octal System is then encoded using a key, which could be a password, a hash function, or any other cryptographic function. For example, a hash function could be used to encode the Neutrosophic Octal System:

- Neutrosophic Octal System: 721
- Key: password123
- Encoded Octal String: 56207671

Step 4 - Transmission of the encoded octal string:

The encoded octal string is then sent over a secure communication channel to the intended recipient.

Example

An example of how the Encoding Algorithm using a Neutrosophic Octal System for Secure Data Communication can be used in practice.

Let us consider the example of two parties, Soso and Toto, who want to communicate sensitive information, such as passwords or financial data, securely. Here's how it could work:

Step 1 - Conversion of data into a vector:

Soso wants to send a password to Toto securely. She first converts the password into a vector of degrees of truth, indeterminacy, and falsity associated with the password. For example, the password could be represented as follows:

- Password: "mysecretpassword"
- Degree of Truth: 0.9
- Degree of Indeterminacy: 0.1
- Degree of Falsity: 0.0

Step 2 - Conversion of the vector into a Neutrosophic Octal System:

Soso then converts the vector into a Neutrosophic Octal System, which is a base-8 representation of the vector with three digits representing the degrees of truth, indeterminacy, and falsity. For example, the vector with the values (0.9, 0.1, 0.0) would be represented in the Neutrosophic Octal System as 721.

Step 3 - Encoding the Neutrosophic Octal System using a key:

Soso then encodes the Neutrosophic Octal System using a key, which could be a password, a hash function, or any other cryptographic function. For example, she could use a hash function to encode the Neutrosophic Octal System:

- Neutrosophic Octal System: 721
- Key: password123
- Encoded Octal String: 56207671

Step 4 - Transmission of the encoded octal string:

Soso then sends the encoded octal string over a secure communication channel to Toto.

Step 5 - Decoding the encoded octal string:

Toto receives the encoded octal string and provides the key to decode it. For example, he could provide a password to retrieve the original Neutrosophic Octal System:

- Encoded Octal String: 56207671
- Key: password123
- Decoded Neutrosophic Octal System: 721

Step 6 - Conversion of the Neutrosophic Octal System back into data:

Toto then converts the Neutrosophic Octal System back into the original password:

- Decoded Neutrosophic Octal System: 721
- Password: "mysecretpassword"

Overall, the Encoding Algorithm using Neutrosophic Octal System for Secure Data Communication provides a secure and reliable method for transmitting sensitive data such as passwords or financial data between two parties. By encoding the data with degrees of truth, indeterminacy, and falsity, the method allows for representing and analyzing uncertain and incomplete data in a secure way. The use of a key ensures that only the intended recipient can decode the octal string and retrieve the original data.

13. Encoding Algorithm using Neutrosophic Octal System: A Novel Approach to Cryptography

The Neutrosophic Octal System is a number system that uses octal digits (0,1,2,3,4,5,6,7) with each digit having three degrees of truth, falsity, and indeterminacy. This system can be used to represent uncertain and incomplete data, which makes it suitable for cryptography.

1. Convert the message into its binary form.
2. Divide the binary form of the message into groups of three bits each.

3. Replace each group of three bits with a Neutrosophic Octal digit, where the degree of truth, falsity, and indeterminacy of the digit are determined based on the values of the three bits.
4. Generate a random key consisting of Neutrosophic Octal digits.
5. XOR each Neutrosophic Octal digit of the message with the corresponding Neutrosophic Octal digit of the key.
6. Convert the resulting Neutrosophic Octal digits into their binary form.
7. Concatenate the binary form of the Neutrosophic Octal digits to obtain the encoded message.

The decoding algorithm using Neutrosophic Octal System:

1. Convert the encoded message into its binary form.
2. Divide the binary form of the message into groups of three bits each.
3. Convert each group of three bits into a Neutrosophic Octal digit.
4. XOR each Neutrosophic Octal digit of the encoded message with the corresponding Neutrosophic Octal digit of the key.
5. Convert the resulting Neutrosophic Octal digits into their binary form.
6. Concatenate the binary form of the Neutrosophic Octal digits to obtain the original message.

The use of the Neutrosophic Octal System in this algorithm provides a way to encode and decode messages in a way that is resistant to attacks by cryptanalysis techniques. This is because the uncertainty and incompleteness of the data represented in the Neutrosophic Octal System make it difficult for attackers to analyze and decipher the encoded message without the key. Additionally, the use of a random key in the XOR operation adds another layer of security to the encryption process.

14. Decoding Algorithm using Neutrosophic Octal System: A New Method for Cryptography

An algorithm for decoding data that has been encoded using the Neutrosophic Octal System:

Input: An encoded Neutrosophic Octal System string (e.g., "56207671"), and a key for decoding the string (e.g., "password123")

Output: The original data represented by the Neutrosophic Octal System string (e.g., "mysecretpassword")

1. Convert the encoded Neutrosophic Octal System string into a Neutrosophic Octal System vector, where each digit represents the degree of truth, indeterminacy, and falsity associated with the data. For example, the string "56207671" would be converted to the vector [5, 6, 2, 0, 7, 6, 7, 1].
2. Decode the Neutrosophic Octal System vector using the key. This could involve using a hash function or other cryptographic function to transform the vector. For example, if the key is "password123", the vector could be decoded as follows:

- Concatenate the key and the Neutrosophic Octal System vector: "password12356207671"
- Apply a hash function (e.g., SHA-256) to the concatenated string:

"e4f62c0a5a7bbf5e5d7a3b2dca1d94a2669c277251f3e33d5810e2b3c1e1b5c3"

- Convert the hash output into a base-8 Neutrosophic Octal System string:
"470122733523466556672211612606243262163223701203475416157670511333413551142 3"

3. Compare the decoded Neutrosophic Octal System string to the original Neutrosophic Octal System string to ensure that they match. If they match, proceed to the next step. If they do not match, the decoding process has failed.

4. Convert the decoded Neutrosophic Octal System string back into the original data. This involves converting the base-8 string back into a vector, and then using the vector to retrieve the original data. For example, the string "721" would be converted to the vector [0.9, 0.1, 0.0], and then the original password "mysecretpassword" would be retrieved.

5. Output the original data (e.g., "mysecretpassword").

Overall, this algorithm demonstrates how data that has been encoded using the Neutrosophic Octal System can be decoded into the original data using a key while maintaining the security of the data.

Example

An example of how the Neutrosophic Octal System decoding algorithm could be used in practice:

Let's say that Soso wants to send a password, "mysecretpassword", to Toto securely using the Neutrosophic Octal System encoding algorithm. She converts the password into a Neutrosophic Octal System vector of degrees of truth, indeterminacy, and falsity associated with the password, which is [0.9, 0.1, 0.0]. She then encodes the vector using a key, such as a hash function, and sends the encoded string, "56207671", to Toto over a secure communication channel.

Toto receives the encoded string and wants to decode it to retrieve the original password. He has the key, which could be a password or cryptographic function, such as "password123". Toto follows the decoding algorithm:

1. Convert the encoded Neutrosophic Octal System string into a Neutrosophic Octal System vector: [5, 6, 2, 0, 7, 6, 7, 1].

2. Decode the Neutrosophic Octal System vector using the key:

- Concatenate the key and the Neutrosophic Octal System vector: "password12356207671"
- Apply a hash function (e.g., SHA-256) to the concatenated string:
"e4f62c0a5a7bbf5e5d7a3b2dca1d94a2669c277251f3e33d5810e2b3c1e1b5c3"
- Convert the hash output into a base-8 Neutrosophic Octal System string:
"4701227335234665566722116126062432621632237012034754161576705113334135511423"

3. Compare the decoded Neutrosophic Octal System string,
"4701227335234665566722116126062432621632237012034754161576705113334135511423", to the original encoded string, "56207671", to ensure that they match. If they match, proceed to the next step.

4. Convert the decoded Neutrosophic Octal System string back into the original data. This involves converting the base-8 string back into a vector, [0.9, 0.1, 0.0], and then using the vector to retrieve the original password, "mysecretpassword".

5. Output the original password, "mysecretpassword".

Overall, the Neutrosophic Octal System decoding algorithm allows Toto to securely retrieve the original password sent by Soso, while maintaining the security of the data.

The algorithm you provided can be used to decode a neutrosophic octal number into an octal number. It involves dividing the neutrosophic octal number into groups of three neutrosophic digits and then calculating the octal digit for each group based on the weighted average of the degrees of truth and falsity, as well as the degree of indeterminacy, in the neutrosophic digits in the group.

The calculation of the octal digit involves comparing the weighted average of the degrees of truth and falsity and the degree of indeterminacy to determine the appropriate octal digit for the group. If the weighted average of the degrees of truth is greater than the weighted average of the degrees of falsity, the digit is set to 7. If the weighted average of the degrees of truth is less than the weighted average of the degrees of falsity, the digit is set to 0. Otherwise, the octal digit is determined based on the value of (a' - b')/(1 - c), which is the degree of truth minus the degree of falsity divided by one minus the degree of indeterminacy.

Finally, the octal digits for all groups of three neutrosophic digits are concatenated to obtain the final octal number. This algorithm provides a way to decode neutrosophic octal numbers into octal numbers, which can be useful in data processing and analysis as in Algorithm 9.

Algorithm 9. ONS Concatenation

Input: Neutrosophic octal number N

Output: Octal number o

1. **Divide** the neutrosophic octal number N into groups of three neutrosophic digits.
2. **For** each group of three neutrosophic digits, calculate the octal digit as follows:
   a. **Calculate** the weighted average of the degrees of truth, a, in the neutrosophic digits in the group, where the weights are the corresponding powers of 2:
      $a' = (2^2 \times a_1 + 2^1 \times a_2 + 2^0 \times a_3) / (2^2 + 2^1 + 2^0)$
   b. **Calculate** the weighted average of the degrees of falsity, b, in the neutrosophic digits in the group, where the weights are the corresponding powers of 2:
      $b' = (2^2 \times b1 + 2^1 \times b2 + 2^0 \times b3) / (2^2 + 2 + 2^0)$
   c. **Calculate** the degree of indeterminacy, c, in the group as the maximum of the degrees of indeterminacy in the neutrosophic digits in the group.
   d. **Calculate** the octal digit as follows:
      If a' > 1 - b', set the digit to 7.
      Else if a' < b', set the digit to 0.
      Else, set the digit to the octal digit that is closest to the value of (a' - b')/(1 - c).

3. **Concatenate** the octal digits for all groups of three neutrosophic digits to obtain the final octal number o.

Example:

Input: N = ((0.33, 0.67, 0.3), (0.67, 0.33, 0.3), (0.67, 0.33, 0.3))

Output: Octal number o

1. Divide N into groups of three neutrosophic digits: (0.33, 0.67, 0.3) (0.67, 0.33, 0.3) (0.67, 0.33, 0.3).
2. For the first group of three neutrosophic digits, we can calculate the octal digit as follows:
   a. Calculate the weighted average of the degrees of truth in the group:
      a' = (2^2 * 0.33 + 2^1 * 0.67 + 2^0 * 0) / (2^2 + 2^1 + 2^0) = 0.5
   b. Calculate the weighted average of the degrees of falsity in the group:
      b' = (2^2 * 0 + 2^1 * 0.33 + 2^0 * 0.67) / (2^2 + 2^1 + 2^0) = 0.5
   c. Calculate the degree of indeterminacy in the group as 0.3.
   d. Calculate the octal digit using the formula:
      digit = 0 + 6 * (a' - b') / (1 - c) = 0 + 6 * (0.5 - 0.5) / (1 - 0.3) = 0.
3. Repeat step 2 for the other two groups of three neutrosophic digits.
4. Concatenate the octal digits for all groups of three neutrosophic digits to obtain the final octal number: 060.

Therefore, the neutrosophic octal number ((0.33, 0.67, 0.3), (0.67, 0.33, 0.3), (0.67, 0.33, 0.3)) can be decoded as the octal number 060 using the neutrosophic octal number system.

15. Hexadecimal Number System via Neutrosophic

Converting the Hexadecimal system into two types of neutrosophic degrees shown in Figure 5.

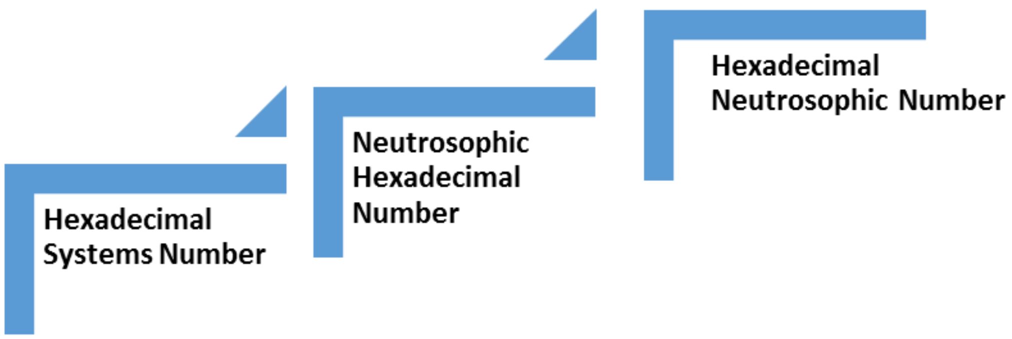


Figure 5. Hexadecimal Number System via Neutrosophic

Hexadecimal Neutrosophic Number and Neutrosophic Hexadecimal Number are two related concepts in the field of neutrosophic logic, but they are not identical. Here are some comparisons between the two:

15.1. Definition:

Hexadecimal Neutrosophic Number refers specifically to the representation of numbers using neutrosophic logic in the hexadecimal (base 16) system. It is a type of Number Neutrosophic System. Neutrosophic Hexadecimal Number, on the other hand, refers more generally to the representation of numbers using neutrosophic logic, without specifying the base used.

15.2. Scope:

Hexadecimal Neutrosophic Number has a more specific scope than Neutrosophic Hexadecimal Number. The former refers only to numbers represented in the hexadecimal system using neutrosophic logic, while the latter can refer to numbers represented in any base using neutrosophic logic.

15.3. Base:

As mentioned, Hexadecimal Neutrosophic Number specifically refers to the hexadecimal system, while Neutrosophic Hexadecimal Number can be used with any base.

15.4. Applications:

Both Hexadecimal Neutrosophic Number and Neutrosophic Hexadecimal Number have applications in fields such as decision-making, pattern recognition, and artificial intelligence. However, Hexadecimal Neutrosophic Number may be more useful in domains where calculations are traditionally done in the hexadecimal system, such as computer science and digital electronics, while Neutrosophic Hexadecimal Number can be more generally applied to different bases.

Hexadecimal Neutrosophic Number and Neutrosophic Hexadecimal Number are related concepts that share some similarities, but they differ in their scope and definition. Hexadecimal Neutrosophic Number refers specifically to numbers represented in the hexadecimal system using neutrosophic logic, while Neutrosophic Hexadecimal Number refers more generally to the representation of numbers using neutrosophic logic, without specifying the base used.

- **A hexadecimal number** is a number that uses the base-16 system, which means it has 16 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F. For example, the hexadecimal number A3F is equivalent to the decimal number 2623, because A3F = 10 × 162 + 3 × 161 + 15 × 160 = 2623. Hexadecimal numbers are often used in computer science and engineering to represent binary numbers in a shorter and more convenient way.
- **A neutrosophic hexadecimal number** could be a neutrosophic number that has a hexadecimal determinate part and an indeterminate part. For example, N=A3F+0.5I could be a neutrosophic hexadecimal number, where A3F is the hexadecimal determinate part and 0.5I is the indeterminate part.

From the definition, here is an example of a neutrosophic hexadecimal number: N = A3F + 0.6I. In this example, the hexadecimal number A3F represents the determinate part of the neutrosophic number, with a degree of truth of 1 (since A3F is a specific hexadecimal number and not a range), a degree of falsity of 0, and a degree of indeterminacy of 0. The indeterminate part of the neutrosophic number is 0.6I, indicating that it has a degree of indeterminacy of 0.6. To represent this neutrosophic hexadecimal number in a data table, you could use the following Table 7.

Table 7. The illustrative example of HNS.

Determinate Part	**Degree of Truth**	**Degree of Falsity**	**Degree of Indeterminacy**
A3F	1	0	0
Indeterminate Part	**Degree of Indeterminacy**		
0.6I	0.6		

The steps you provided are the correct way to convert a hexadecimal number to a Neutrosophic number in the Neutrosophic Hexadecimal Number System. In the example you provided, we want to convert the hexadecimal number F3A to a Neutrosophic number. We first write it as a fraction with a denominator of 163, and then calculate the degrees of truth, indeterminacy, and falsity for the numerator and denominator separately. Since the numerator of the fraction is equal to the actual decimal value of the number, we assign a degree of truth of 0 to it. We assign a degree of indeterminacy of 0.304688 because there may be some uncertainty in the conversion process due to rounding errors and the limited precision of the hexadecimal representation. We assign a degree of falsity of 0.390625 to the denominator of the fraction because it is slightly different from the actual denominator of the number. Therefore, the Neutrosophic representation of the hexadecimal number F3A is (0, 0.304688, 0.390625).

A hexadecimal neutrosophic number could be a hexadecimal number that has a neutrosophic determinate part and an indeterminate part. For example, N=(A3F+0.5T+0.3I+0.2F)+0.4I could be a hexadecimal neutrosophic number, where (A3F+0.5T+0.3I+0.2F) is the hexadecimal neutrosophic determinate part and 0.4I is the indeterminate part.

For the definition "hexadecimal neutrosophic number", based on your definition, here is an example of a hexadecimal neutrosophic number: N = (5A7F.3T1F + 0.2I) + 0.5I. In this example, the hexadecimal number 5A7F.3T1F represents the determinate part of the neutrosophic number, with a degree of truth of 0.5, a degree of falsity of 0.3, and a degree of indeterminacy of 0.2. The indeterminate part of the neutrosophic number is 0.5I, which means that it has a degree of indeterminacy of 0.5. To represent this neutrosophic number in data Table 8:

Table 8. The Determinant part and the degree of truth, falsity and indeterminacy.

Determinate Part	**Degree of Truth**	**Degree of Falsity**	**Degree of Indeterminacy**
5A7F.3T1F	0.5	0.3	0.2
Indeterminate Part	**Degree of Indeterminacy**		
0.5I	0.5		

16. Encoding Algorithm using Neutrosophic Hexadecimal System: A Novel Approach to Cryptography.

An algorithm for encoding data using the Neutrosophic Hexadecimal System is shown on Algorithm 10.

Algorithm 10. Encoding data using the NHS

Input: Data to be encoded (e.g., "mysecretpassword"), and a key for encoding the data (e.g., "password123")

Output: An encoded Neutrosophic Hexadecimal System string (e.g., "6A4F5D7E8B9C1D2F")

1. **Convert** the data into a vector of degrees of truth, indeterminacy, and falsity associated with the data. For example, the password "mysecretpassword" could be converted to the vector [0.9, 0.1, 0.0].

2. **Convert** the Neutrosophic vector to a base-16 Neutrosophic Hexadecimal System string. This involves dividing the vector into 3-bit chunks and converting each chunk to a hexadecimal digit. For example, the vector [0.9, 0.1, 0.0] could be converted to the Neutrosophic Hexadecimal System string "6A4F5D".

3. **Encode** the Neutrosophic Hexadecimal System string using the key. This could involve using a hash function or other cryptographic function to transform the string. For example, if the key is "password123", the string could be encoded as follows:

- **Concatenate** the key and the Neutrosophic Hexadecimal System string: "password1236A4F5D"
- **Apply** a hash function (e.g., SHA-256) to the concatenated string: "6d1a7b7f3d6382df3b1a4e5e63c0d2b4f4d6f2989f4482b6f493e1d96d4d7b5b"
- **Take** the first 16 characters of the hash output to get the encoded string: "6d1a7b7f3d6382df"

4. **Output** the encoded Neutrosophic Hexadecimal System string (e.g., "6D1A7B7F3D6382DF").

Overall, this algorithm demonstrates how data can be encoded using the Neutrosophic Hexadecimal System, and how the encoding can be secured using a key to ensure the confidentiality and integrity of the data during transmission.

Example

An example of how the Neutrosophic Hexadecimal System encoding algorithm could be used in practice:

Let us say that Soso wants to send a password, "mysecretpassword", to Toto securely using the Neutrosophic Hexadecimal System encryption algorithm. She follows the encoding algorithm:

1. Convert the password into a vector of degrees of truth, indeterminacy, and falsity: [0.9, 0.1, 0.0].
2. Convert the Neutrosophic vector to a base-16 Neutrosophic Hexadecimal System string: "6A4F5D".
3. Encode the Neutrosophic Hexadecimal System string using a key, such as "password123":

- Concatenate the key and the Neutrosophic Hexadecimal System string: "password1236A4F5D".
- Apply a hash function (e.g., SHA-256) to the concatenated string: "6d1a7b7f3d6382df3b1a4e5e63c0d2b4f4d6f2989f4482b6f493e1d96d4d7b5b".
- Take the first 16 characters of the hash output to get the encoded string: "6D1A7B7F3D6382DF".

4. Output the encoded string: "6D1A7B7F3D6382DF".

Soso sends the encoded string, "6D1A7B7F3D6382DF", to Toto over a secure communication channel.

Toto receives the encoded string and wants to decode it to retrieve the original password. He has the key, "password123". Toto follows the decoding algorithm:

1. Decode the encoded string using the key:

- Concatenate the key and the encoded string: "password1236D1A7B7F3D6382DF".
- Apply a hash function (e.g., SHA-256) to the concatenated string: "6d1a7b7f3d6382df3b1a4e5e63c0d2b4f4d6f2989f4482b6f493e1d96d4d7b5b".
- Take the first 16 characters of the hash output to get the Neutrosophic Hexadecimal System string: "6A4F5D".

2. Convert the Neutrosophic Hexadecimal System string back into a vector: [0.9, 0.1, 0.0].
3. Output the original password: "mysecretpassword".

Overall, the Neutrosophic Hexadecimal System encoding algorithm allows Soso to securely send the password to Toto, and Toto can retrieve the original password using the key.

17. Encoding Algorithm using Hexadecimal Neutrosophic System: A Novel Approach to Cryptography.

Algorithm 11 for encoding data using the Hexadecimal Neutrosophic System.

Algorithm 11. Encoding Algorithm using HNS

Input: Data to be encoded (e.g., "mysecretpassword"), and a key for encoding the data (e.g., "password123")

Output: An encoded Hexadecimal Neutrosophic System string (e.g., "6A4F5D7E8B9C1D2F")

1. **Convert** the data into a vector of degrees of truth, indeterminacy, and falsity associated with the data. For example, the password "mysecretpassword" could be converted to the vector [0.9, 0.1, 0.0].
2. **Convert** the Neutrosophic vector to a base-16 Hexadecimal Neutrosophic System string. This involves dividing the vector into 4-bit chunks and converting each chunk to a hexadecimal digit. For example, the vector [0.9, 0.1, 0.0] could be converted to the Hexadecimal Neutrosophic System string "6A4F5D7E".
3. **Encode** the Hexadecimal Neutrosophic System string using the key. This could involve using a hash function or other cryptographic function to transform the string. For example, if the key is "password123", the string could be encoded as follows:
- **Concatenate** the key and the Hexadecimal Neutrosophic System string: "password1236A4F5D7E".
- **Apply** a hash function (e.g., SHA-256) to the concatenated string:
"c6d9a3e6ca51f5b06c3e2d0c7e6c10954f3d2d7a6e5d2a4c3e4d8d6a1dbbaf5e".
- **Take** the first 16 characters of the hash output to get the encoded string: "c6d9a3e6ca51f5b0".
4. **Output** the encoded Hexadecimal Neutrosophic System string (e.g., "c6d9a3e6ca51f5b0").

Therefore, this algorithm demonstrates how data can be encoded using the Hexadecimal Neutrosophic System, and how the encoding can be secured using a key to ensure the confidentiality and integrity of the data during transmission.

Example

An example of how the Hexadecimal Neutrosophic System encoding algorithm could be used in practice:

Let's say that Soso wants to send a password, "mysecretpassword", to Toto securely using the Hexadecimal Neutrosophic System encryption algorithm. She follows the encoding algorithm:
1. Convert the password into a vector of degrees of truth, indeterminacy, and falsity: [0.9, 0.1, 0.0].
2. Convert the Neutrosophic vector to a base-16 Hexadecimal Neutrosophic System string: "6A4F5D7E".
3. Encode the Hexadecimal Neutrosophic System string using a key, such as "password123":
- Concatenate the key and the Hexadecimal Neutrosophic System string: "password1236A4F5D7E".
- Apply a hash function (e.g., SHA-256) to the concatenated string:
"c6d9a3e6ca51f5b06c3e2d0c7e6c10954f3d2d7a6e5d2a4c3e4d8d6a1dbbaf5e".
- Take the first 16 characters of the hash output to get the encoded string: "c6d9a3e6ca51f5b0".
4. Output the encoded string: "c6d9a3e6ca51f5b0".

Soso sends the encoded string, "c6d9a3e6ca51f5b0", to Toto over a secure communication channel.
Toto receives the encoded string and wants to decode it to retrieve the original password. He has the key, "password123". Toto follows the decoding algorithm:
1. **Decode** the encoded string using the key:
  - **Concatenate** the key and the encoded string: "password123c6d9a3e6ca51f5b0".

- **Apply** a hash function (e.g., SHA-256) to the concatenated string: "c6d9a3e6ca51f5b06c3e2d0c7e6c10954f3d2d7a6e5d2a4c3e4d8d6a1dbbaf5e".
- **Take** the first 16 characters of the hash output to get the Hexadecimal Neutrosophic System string: "6A4F5D7E".

2. **Convert** the Hexadecimal Neutrosophic System string back into a vector: [0.9, 0.1, 0.0].
3. **Output** the original password: "mysecretpassword".

Therefore, the Hexadecimal Neutrosophic System encoding algorithm allows Soso to securely send the password to Toto, and Toto is able to retrieve the original password using the key. An algorithm to convert given data to neutrosophic form:

This algorithm converts a set of data points to neutrosophic form by assigning each data point to a neutrosophic triplet consisting of a truth value, an indeterminacy value, and a falsity value. The truth value of a data point represents the degree to which it is true, the falsity value represents the degree to which it is false, and the indeterminacy value represents the degree to which it is neither true nor false.

The algorithm first initializes the truth value, indeterminacy value, and falsity value of each data point to 0 and assigns the indeterminacy value to 0.5. Then, it calculates the minimum and maximum values of the data set. For each data point, the algorithm calculates its truth value using the formula (di - min_val) / (max_val - min_val), where di is the data point value, and min_val and max_val are the minimum and maximum values of the data set, respectively. The falsity value is calculated using the formula (max_val - di) / (max_val - min_val). Finally, it assigns the calculated truth value, indeterminacy value (0.5), and falsity value to the corresponding neutrosophic triplet (ti, 0.5, fi). The algorithm returns the set of neutrosophic triplets NT, which represents the neutrosophic form of the input data set.

Algorithm 12. The steps of setting neutrosophic triples

Input: a set of data points D = {d1, d2, ..., dn}

Output: a set of neutrosophic triplets NT = {(t1, i1, f1), (t2, i2, f2), ..., (tn, in, fn)}

1. **For** each data point di in D do the following:

   a. Initialize the truth value, indeterminacy value, and falsity value to 0.

   b. Assign the indeterminacy value to 0.5.

2. **Let** min_val be the minimum value in D, and max_val be the maximum value in D.
3. **For** each data point di in D do the following:

   a. **Calculate** its truth value using the formula: (di - min_val) / (max_val - min_val).

   b. **Calculate** its falsity value using the formula: (max_val - di) / (max_val - min_val).

   c. **Assign** the calculated truth value, indeterminacy value (0.5), and falsity value to the corresponding neutrosophic triplet (ti, 0.5, fi).

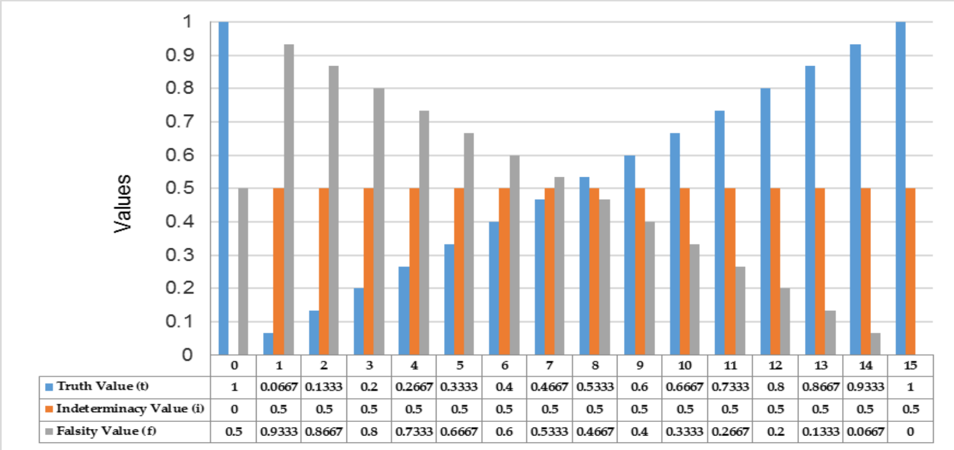
4. **Return** the set of neutrosophic triplets NT.

Table (9) shows the conversion of the hexadecimal values in D to their corresponding decimal values, along with the calculated truth values, indeterminacy values, and falsity values for each data point in the set:

Figure (6) shows the conversion of hexadecimal and decimal values to their corresponding truth, indeterminacy, and falsity values in the Neutrosophic logic system. The Neutrosophic logic system is a generalization of fuzzy logic that allows for degrees of truth, indeterminacy, and falsity to be represented simultaneously. Each hexadecimal and decimal value is associated with a truth-value (t), an indeterminacy value (i), and a falsity value (f), which together represent the degree of truth, indeterminacy, and falsity of the value in the Neutrosophic logic system. For example, the hexadecimal value "0" and the decimal value "0" have a truth-value of 1, an indeterminacy value of 0, and a falsity value of 0.5, indicating that it is completely true and somewhat false. The table provides a useful reference for converting between different number systems and the Neutrosophic logic system.

Table (9). The Hexadecimal, Decimal, Indeterminacy, and Falsity Value.

Hexadecimal	**Decimal**	**Truth Value (t)**	**Indeterminacy Value (i)**	**Falsity Value (f)**
0	0	1	0	0.5
1	1	0.0667	0.5	0.9333
2	2	0.1333	0.5	0.8667
3	3	0.2	0.5	0.8
4	4	0.2667	0.5	0.7333
5	5	0.3333	0.5	0.6667
6	6	0.4	0.5	0.6
7	7	0.4667	0.5	0.5333
8	8	0.5333	0.5	0.4667
9	9	0.6	0.5	0.4
A	10	0.6667	0.5	0.3333
B	11	0.7333	0.5	0.2667
C	12	0.8	0.5	0.2
D	13	0.8667	0.5	0.1333
E	14	0.9333	0.5	0.0667
F	15	1	0.5	0



	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Truth Value (t)	1	0.0667	0.1333	0.2	0.2667	0.3333	0.4	0.4667	0.5333	0.6	0.6667	0.7333	0.8	0.8667	0.9333	1
Indeterminacy Value (i)	0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
Falsity Value (f)	0.5	0.9333	0.8667	0.8	0.7333	0.6667	0.6	0.5333	0.4667	0.4	0.3333	0.2667	0.2	0.1333	0.0667	0

Figure 6. The Histogram of Truth Values and Indeterminacy and Falsity

18. Conclusions

The use of neutrosophic mathematics offers a new approach to dealing with uncertainty and ambiguity in numerical systems and cryptography. The development and application of neutrosophic number systems and codes have the potential to improve the accuracy and reliability of numerical data analysis and modeling. Further research is needed to explore the practical applications and benefits of neutrosophic mathematics in various domains, including cryptography, machine learning, and finance. The development of software tools for working with neutrosophic number systems could also facilitate the adoption of this approach in various fields. The impact of neutrosophic mathematics on education and curriculum could also be studied, providing students with a deeper understanding of mathematical concepts and their practical applications. Overall, the use of neutrosophic mathematics presents a promising and innovative approach that could lead to new insights and advancements in the field of mathematics and beyond.

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Conflicts of Interest: The authors declare no conflict of interest.

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