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Exploring Negative-Valued $\mathcal{N}$eutrosophic Structures in the Context of Subalgebras and Ideals in BF-algebras

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Abstract. This scholarly inquiry comprehensively examines Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebras and Negative-Valued $\mathcal{N}$eutrosophic BF-ideals in the context of BF-algebras, aiming to scrutinize their intrinsic characteristics and reveal intricate interrelationships. Employing a systematic and rigorous approach, this study significantly enhances our understanding of these elements within the broader context of algebraic structures, serving as a cornerstone for the advancement of mathematical knowledge in this area and providing a robust framework for future investigations. The findings offer valuable insights, laying the groundwork for further research in this specialized domain and contributing significantly to ongoing academic discourse. By conducting a thorough examination of Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebras and Negative-Valued $\mathcal{N}$eutrosophic BF-ideals, this study facilitates a deeper understanding within the broader landscape of algebraic structures and plays a pivotal role in advancing mathematical knowledge in this specialized field, fostering continued exploration and innovation.

Keywords: BF-algebra; Negative-Valued $\mathcal{N}$eutrosophic Structure; Negative-Valued $\mathcal{N}$eutrosophic BF-Subalgebra; Negative-Valued $\mathcal{N}$eutrosophic BF-ideal.

1. Introduction

A groundbreaking shift in set theory, known as the introduction of fuzzy sets by Zadeh[16] in 1995, marked a significant turning point. In 2002, Neggers and Kim[12] introduced the innovative concept of B-algebra, leading to a multitude of consequential outcomes. Walendziak[15] further extended this framework to formulate BF-algebra, a more general version of B-algebra, and conducted an

extensive investigation into the properties of ideals and normal ideals within BF-algebra.

Atanassov[4] made a significant contribution by introducing the notion of the measure of non-inclusion or falsity (f) and providing an interpretation of intuitionistic fuzzy sets. The term "$\mathcal{N}$eutrosophic", signifying neutrality in thought, was coined by Smarandache, where the primary differentiation is fuzzy/intuitionistic fuzzy logic/sets and $\mathcal{N}$eutrosophic logic/sets lies in the introduction of a third/neutral component. He pioneered the introduction of an autonomous element, representing the level of ambiguity or neutrality, established the $\mathcal{N}$eutrosophic set relies on a triad of constituents, namely (t, i, f), which correspond to authenticity, ambiguity, and falsification. This demonstrates its practical applicability in diverse sectors [1, 2, 3, 8, 14]. Jun et al.[9] introduced a novel mapping characterized by negative-values and developed N-structures. Khan et al.[10] introduced the concept of $\mathcal{N}$eutrosophic N-Structure and employed it within the context of a semi-group. Additionally, Muralikrishna et al. [11] first introduced the concept of Structuere N-ideal within the context of BF-algebra.

Seok-Zun Song et al.[13] Pioneered the idea of $\mathcal{N}$eutrosophic N-ideal in BCK-algebras and conducted an extensive exploration of its various attributes, culminating in the establishment of characterizations for $\mathcal{N}$eutrosophic N-ideal. To set the stage for our discussion, we first provide definitions from [5,6,15] that are essential for the context of this paper.

2. Main contributions to this work

Introducing and extensively examining the concept of Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebras and Negative-Valued $\mathcal{N}$eutrosophic BF-ideals in the context of BF-algebras.

Providing a thorough analysis of the inherent characteristics of Negative-Valued $\mathcal{N}$eutrosophic BF-Subalgebras and Negative-Valued $\mathcal{N}$eutrosophic BF-ideals.

Elucidating the intricate relationships that exist among Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebras and Negative-Valued $\mathcal{N}$eutrosophic BF-ideals.

Conducting a meticulous exploration of the unique properties associated with Negative-Valued $\mathcal{N}$eutrosophic BF-ideals.

Advancing the understanding of BF-algebras and broadening the utility of Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebras and Negative-Valued

$\mathcal{N}$eutrosophic BF-ideals for managing uncertainty in Negative - valued $\mathcal{N}$eutrosophic soft sets.

3. Prerequisites

Notations: Throughout this article, we use the following notations.

Table 1

BF-algebra	$\mathcal{BFA}$
Negative-Valued $\mathcal{N}$eutrosophic Structure	$\mathcal{NNS}$
Negative-Valued $\mathcal{N}$eutrosophic BF-ideal	$\mathcal{NNI}$
Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebra	$\mathcal{NNSA}$

Definition 3.1 (15)**.** A $\mathcal{BFA}$ is a structure $\mathrm{S} := (\mathrm{S} \neq \phi, \otimes, 0) \in \mathrm{K}(\tau)$

(I)$\mathrm{t}_1 \otimes \mathrm{t}_1 = 0, - - - - - - - - -(1)$

(II)$\mathrm{t}_1 \otimes 0 = \mathrm{t}_1, - - - - - - -(2)$

(III)$0 \otimes (\mathrm{t}_1 \otimes \mathrm{t}_2) = \mathrm{t}_2 \otimes \mathrm{t}_1, \forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S} - - - - - - -(3)$

Example 3.2 (15)**.** The set $(S = \{0, 1, 2, 3\}, \otimes, 0)$ having the composition table

Table 2

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	2
3	3	0	2	0

is a $\mathcal{BFA}$.

Example 3.3 (15)**.** Let $S = (R, \otimes, 0)$where $\otimes$ is given by $t_1 \otimes t_2 = \begin{cases} t_1, \mathrm{if} t_2 = 0 \\ t_2, \mathrm{if} t_1 = 0 \\ 0, otherwise \end{cases}$

and set of real numbers(R) is a $\mathcal{BFA}$.

Example 3.4 (6)**.** The set $(S = \{0, 1, 2, 3\}, \otimes, 0)$ having the composition table

Table 3

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

is a $\mathcal{BFA}$.

Example 3.5 (15)**.** Let $S = [0, \infty), \otimes$ is defined on S as $t_1 \otimes t_2 = |t_1 - t_2|, \forall t_1, t_2 \in S$ is a $\mathcal{BFA}$.

Note 3.6 (7)**.** *Let* $S = (R, \otimes, 0)$ *where* $\otimes$ *is defined as* $t_1 \otimes t_2 = \begin{cases} t_1, \text{if} t_2 = 0 \\ 0, \text{if} t_1 = 0, t_1 = t_2 \\ t_2 \otimes t_1, otherwise \end{cases}$

is not a $\mathcal{BFA}$.

Definition 3.7 (7, 11)**.** A relation '$\leq$ ' on S is a partial ordering satisfying

$(\forall t_1, t_2 \in S)$, $t_1 \leq t_2 \Leftrightarrow t_1 \otimes t_2 = 0$ − − − − − − − − − (4)

Note 3.8 (15)**.** *In any* $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$, *the following holds:*

$(\forall \mathrm{t}_1 \in \mathrm{S})(0 \otimes (0 \otimes \mathrm{t}_1)) = \mathrm{t}_1$ − − − − − − −(5)

$(\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S})(0 \otimes \mathrm{t}_1) = (0 \otimes t_2)$ *iff* $t_1 = t_2$ − − − − − − −(6)

$(\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S})(\mathrm{t}_2 \otimes t_1 = 0)$, *if* $t_1 \otimes t_2 = 0$− − − − − − − (7)

Definition 3.9 (15)**.** Consider a $\mathcal{BFA}$, $S := (\mathrm{S} \neq \phi, \otimes, 0)$. $M(\neq \phi) \subseteq \mathrm{S}$ is said to be a subalgebra if $\mathrm{t}_1 \otimes \mathrm{t}_2 \in \mathrm{M}, \forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{M}$. − − − − − − − −(8)

Note 3.10 (15)**.** *It is clear that if M is a subalgebra of S then* $0 \in \mathrm{M}$.

Example 3.11 (15)**.** Consider a $\mathcal{BFA}$, $(\mathrm{S} = \{0, 1, 2, 3\}, \otimes, 0)$ having the composition table

Table 4

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

The set $\mathrm{M} = \{0, 1\}$ is a subalgebra of S.

Definition 3.12 (15)**.** Consider a $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$. $M(\neq \phi) \subseteq S$ is said to be ideal of S if $0 \in M$ $- - - -$ (9)

$(\forall t_1, t_2 \in S)(t_1 \otimes t_2 \in M, t_2 \in M \Rightarrow t_1 \in M)$ - - - (10)

Example 3.13 (15)**.** Consider a $\mathcal{BFA}$, (S = $\{0, 1, 2, 3\}$, $\otimes$, 0) having the composition table 2

Clearly, $\{0\}$ and S are ideals of S and M = $\{0, 3\} \subseteq$ S is not an ideal of S. ($1 \otimes 3 = 0 \in$ M and $3 \in$ M $\Rightarrow 1 \notin$ M)

4. **Negative-Valued $\mathcal{N}$eutrosophic concept on BF-algebra**

Represent by γ(S, $[-1, 0]$) be the family of mappings from a set S to $[-1, 0]$ (called, A Negative-Valued mapping on S). A $\mathcal{NNS}$ is denoted by (S, g) of S and g is a Negative-Valued mapping on S. A $\mathcal{NNS}$ over a universe $S \neq \phi$ (see [9]) is

$S_{\mathcal{N}} = \frac{S}{(\aleph_{\mathcal{N}}, I_{\mathcal{N}}, \Psi_{\mathcal{N}})} = \left\{ \frac{t_1}{\aleph_{\mathcal{N}}(t_1), I_{\mathcal{N}}(t_1), \Psi_{\mathcal{N}}(t_1)} / t_1 \in S \right\}$

where $\aleph_{\mathcal{N}}$,$I_{\mathcal{N}}$ and $\Psi_{\mathcal{N}}$ are Negative-Valued mappings on S termed as the "Non-positive truth membership" mapping, the "non-positive indeterminacy membership" mapping and the "non-positive falsity membership" mapping, resp., on S.

A $\mathcal{NNS}$, $S_{\mathcal{N}}$ over S holds:

$(\forall \mathrm{t}_1 \in \mathrm{S})(-3 \leq \aleph_N(\mathrm{t}_1) + \mathrm{I}_N(\mathrm{t}_1) + \Psi_N(\mathrm{t}_1) \leq 0)$

Let us represent $\forall t_1, t_2 \in S$ **,** $\mathrm{t}_1 \mathrm{v} \mathrm{t}_2$ **denotes** $\max\{\mathrm{t}_1, \mathrm{t}_2\}$ **and** $t_1 \wedge t_2$ **denotes** $\min\{t_1, t_2\}$

Definition 4.1. A $\mathcal{NNS}$, $\mathrm{S}_{\mathcal{N}}$ over a $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$, is a $\mathcal{NNSA}$ if

$i)\aleph_{\mathcal{N}}(t_1 \otimes t_2) \leq \vee \{\aleph_{\mathcal{N}}(t_1), \aleph_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - -(11)$

$ii)I_{\mathcal{N}}(t_1 \otimes t_2) \geq \wedge \{I_{\mathcal{N}}(t_1), I_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - -(12)$

$\mathrm{iii})\Psi_{\mathcal{N}}(t_1 \otimes t_2) \leq \vee \{\Psi_{\mathcal{N}}(t_1), \Psi_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - (13)$

Example 4.2. Consider a $\mathcal{BFA}$, (S = $\{0, 1, 2, 3\}$,$\otimes$, 0) having the table 3.

The $\mathcal{NNSA}$ of S is

$S_N = \left\{ \frac{0}{-0.8, -0.1, -0.8}, \frac{1}{-0.8, -0.8, -0.4}, \frac{2}{-0.8, -0.9, -0.4}, \frac{3}{-0.8, -0.9, -0.6} \right\}$

Definition 4.3. A $\mathcal{NNS}$, $S_{\mathcal{N}}$ over a $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$ is a $\mathcal{NNI}$ of S if

$(i)\aleph_{\mathcal{N}}(0) \leq \aleph_{\mathcal{N}}(t_1) \leq \vee \{\aleph_{\mathcal{N}}(t_1 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - -(14)$

$(ii)I_{\mathcal{N}}(0) \geq I_{\mathcal{N}}(t_1) \geq \wedge \{I_{\mathcal{N}}(t_1 \otimes t_2), I_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - - - (15)$

$(iii)\Psi_{\mathcal{N}}(0) \leq \Psi_{\mathcal{N}}(t_1) \leq \vee \{\Psi_{\mathcal{N}}(t_1 \otimes t_2), \Psi_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - (16)$

Example 4.4. Consider a $\mathcal{BFA}$, (S = $\{0, 1, 2, 3\}$,$\otimes$, 0) having the table 3.

The $\mathcal{NNI}$ of S is

$S_{\mathcal{N}} = \left\{ \frac{0}{-0.7, -0.1, -0.8}, \frac{1}{-0.2, -0.8, -0.4}, \frac{2}{-0.6, -0.9, -0.4}, \frac{3}{-0.2, -0.9, -0.6} \right\}$

Proposition 4.5. *If $S_{\mathcal{N}}$ is a $\mathcal{NNI}$ over a $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$ with $\mathrm{t}_1 \leq \mathrm{t}_2$, $\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S}$ then*

(i)$\aleph_{\mathcal{N}}(t_1) \leq \aleph_{\mathcal{N}}(t_2)(\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S})$, *i.e* $\aleph_{\mathcal{N}}$ *is order preserving.- - - -(17)*

(ii)$\mathrm{I}_{\mathcal{N}}(t_1) \geq I_{\mathcal{N}}(t_2)(\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S})$, *i.e* $I_{\mathcal{N}}$ *is order reversing.- - - - - (18)*

(iii)$\Psi_{\mathcal{N}}(t_1) \leq \Psi_{\mathcal{N}}(t_2)(\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S})$, *i.e* $\Psi_{\mathcal{N}}$ *is order preserving.- - - -(19)*

Proof.

Given $S_{\mathcal{N}}$ *is a* $\mathcal{NNI}$ *over a* $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$ *with* $\mathrm{t}_1 \leq \mathrm{t}_2$, $\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S}$

$\Rightarrow \mathrm{Since} \mathrm{t}_1 \leq \mathrm{t}_2 \Rightarrow \mathrm{t}_1 \otimes \mathrm{t}_2 = 0(\mathrm{by}(4))$

To prove i) : $S_{\mathcal{N}}$ *is a* $\mathcal{NNI}$

$\Rightarrow \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_1 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\}$ *(by(14))*

$\Rightarrow \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(0), \aleph_{\mathcal{N}}(t_2)\}$

$\Rightarrow \aleph_{\mathcal{N}}(t_1) \leq \aleph_{\mathcal{N}}(t_2)$ *(by(14))*

$\Rightarrow \aleph_{\mathcal{N}}$ *is order preserving.*

To prove ii) : $S_{\mathcal{N}}$ *is a* $\mathcal{NNI}$

$\Rightarrow \mathrm{I}_{\mathcal{N}}(t_1) \geq \wedge\{\mathrm{I}_{\mathcal{N}}(t_1 \otimes t_2), \mathrm{I}_{\mathcal{N}}(t_2)\}$ *(by(15))*

$\Rightarrow \mathrm{I}_{\mathcal{N}}(t_1) \geq \wedge\{\mathrm{I}_{\mathcal{N}}(0), \mathrm{I}_{\mathcal{N}}(t_2)\}$

$\Rightarrow \mathrm{I}_{\mathcal{N}}(t_1) \geq \mathrm{I}_{\mathcal{N}}(t_2)$ *(by(15))*

$\Rightarrow \mathrm{I}_{\mathcal{N}}$ *is order reversing.*

To prove iii) : $S_{\mathcal{N}}$ *is a* $\mathcal{NNI}$

$\Rightarrow \Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_1 \otimes t_2), \Psi_{\mathcal{N}}(t_2)\}$ *(by(16))*

$\Rightarrow \Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(0), \Psi_{\mathcal{N}}(t_2)\}$

$\Rightarrow \Psi_{\mathcal{N}}(t_1) \leq \Psi_{\mathcal{N}}(t_2)$ *(by(16))*

$\Rightarrow \Psi_{\mathcal{N}}$ *is order preserving.*

Theorem 4.6. *If* $S_{\mathcal{N}}$ *is a* $\mathcal{NNI}$ *over a* $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$ *then* $S_{\mathcal{N}}$ *is a* $\mathcal{NNSA}$ *of S.*

Proof.

Let $S_{\mathcal{N}}$ **be a** $\mathcal{NNI}$ **of S,** $\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S}$

$\Rightarrow \aleph_{\mathcal{N}}(0) \leq \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_1 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\}$ **(by (14))**

$\Rightarrow \mathrm{I}_{\mathcal{N}}(0) \geq \mathrm{I}_{\mathcal{N}}(t_1) \geq \wedge\{\mathrm{I}_{\mathcal{N}}(t_1 \otimes t_2), \mathrm{I}_{\mathcal{N}}(t_2)\}$**(by (15))**

$\Rightarrow \Psi_{\mathcal{N}}(0) \leq \Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_1 \otimes t_2), \Psi_{\mathcal{N}}(t_2)\}$**(by (16))**

Put $t_1 = t_1 \otimes t_2$ **in (14)**

$\Rightarrow \aleph_{\mathcal{N}}(t_1 \otimes t_2) \leq \vee\{\aleph_{\mathcal{N}}(t_1 \otimes t_2 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\}$

$\Rightarrow \aleph_{\mathcal{N}}(t_1 \otimes t_2) \leq \vee\{\aleph_{\mathcal{N}}(t_1), \aleph_{\mathcal{N}}(t_2)\}(by(1)\&(2))$

Similarly we can prove for $I_{\mathcal{N}}$ **and** $\Psi_{\mathcal{N}}$ **also Hence,** $S_{\mathcal{N}}$ **is a** $\mathcal{NNSA}$ **of S**

Note 4.7. *The Converse of the above theorem need not be true.*

Example 4.8. Suppose we have a $\mathcal{BFA}$[5], $(\mathrm{S} = \{0, 1, 2\}, \otimes, 0)$ having the Composition table

TABLE 5

$\otimes$	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

The $\mathcal{NNS}$ of S is,
$S_{\mathcal{N}} = \left\{ \frac{0}{-0.5, 0, -0.9}, \frac{1}{-0.5, 0, 0}, \frac{2}{0, 0, -0.5} \right\}$
is not a $\mathcal{NNI}$ but $\mathcal{NNSA}$.

Since $\aleph_{\mathcal{N}}(t_1) = \aleph_{\mathcal{N}}(2) = 0 \nleq \vee\{\aleph_{\mathcal{N}}(2 \otimes 1) = -0.5, \aleph_{\mathcal{N}}(1) = -0.5\}$

The following theorem is an adequate condition for $\mathcal{NNSA}$ to be $\mathcal{NNI}$.

Theorem 4.9. *If* S_N *be a* $\mathcal{NNSA}$ *over a* $\mathcal{BFA}$ $S := (S \neq \phi, \otimes, 0)$ *with* $t_1 \otimes t_2 \leq t_3, \forall \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in \mathrm{S}$ *and*

$\aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_2), \aleph_{\mathcal{N}}(t_3)\}$ $(\forall \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in \mathrm{S})$

$I_{\mathcal{N}}(t_1) \geq \wedge\{I_{\mathcal{N}}(t_2), I_{\mathcal{N}}(t_3)\}$ $(\forall \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in \mathrm{S})$

$\Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_2), \Psi_{\mathcal{N}}(t_3)\}$ $(\forall \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in \mathrm{S})$

then $S_{\mathcal{N}}$ *is a* $\mathcal{NNI}$ *of S*

Proof. *Let* $S_{\mathcal{N}}$ *be a* $\mathcal{NNSA}$ *of S with* $t_1 \otimes t_2 \leq t_3, \forall t_1, t_2, t_3 \in S$

$\Rightarrow \aleph_{\mathcal{N}}(t_1 \otimes t_2) \leq \vee\{\aleph_{\mathcal{N}}(t_1), \aleph_{\mathcal{N}}(t_2)\}$ *(by (11))*

Put $\mathrm{t}_1 = \mathrm{t}_2$

$\Rightarrow \aleph_{\mathcal{N}}(t_1 \otimes t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_1), \aleph_{\mathcal{N}}(t_1)\}$

$\Rightarrow \aleph_{\mathcal{N}}(0) \leq \aleph_{\mathcal{N}}(t_1)$ *(by(1))*

and $\aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_1 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\} \Leftrightarrow \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_3), \aleph_{\mathcal{N}}(t_2)\}$ *(by(17))*

and $\mathrm{I}_{\mathcal{N}}(t_1 \otimes t_2) \geq \wedge\{\mathrm{I}_{\mathcal{N}}(t_1), I_{\mathcal{N}}(t_2)\}$ *(by(12))*

Put $\mathrm{t}_1 = \mathrm{t}_2$

$\Rightarrow \mathrm{I}_{\mathcal{N}}(t_1 \otimes t_1) \geq \wedge\{\mathrm{I}_{\mathcal{N}}(t_1), I_{\mathcal{N}}(t_1)\}$

$\Rightarrow \mathrm{I}_{\mathcal{N}}(0) \geq I_{\mathcal{N}}(t_1)$ *(by(1))*

and $\mathrm{I}_{\mathcal{N}}(t_1) \geq \wedge\{\mathrm{I}_{\mathcal{N}}(t_1 \otimes t_2), I_{\mathcal{N}}(t_2)\} \Leftrightarrow I_{\mathcal{N}}(t_1) \geq \wedge\{I_{\mathcal{N}}(t_3), I_{\mathcal{N}}(t_2)\}$ *(by(18))*

Similarly, we can prove for $\Psi_{\mathcal{N}}$ *also.*

Hence $S_{\mathcal{N}}$*is a* $\mathcal{NNI}$ *of S.*

Theorem 4.10. *If* $S_{\mathcal{N}}$ *is a* $\mathcal{NNI}$ *over a* $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$ *with* $t_1 \otimes t_2 \leq t_3, \forall t_1, t_2, t_3 \in S$ *then i)*$\aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_2), \aleph_{\mathcal{N}}(t_3)\}$

ii)$I_{\mathcal{N}}(t_1) \geq \wedge\{I_{\mathcal{N}}(t_2), I_{\mathcal{N}}(t_3)\}$

$iii)\Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_2), \Psi_{\mathcal{N}}(t_3)\}$

Proof. *Given* $\mathrm{t}_1 \otimes \mathrm{t}_2 \leq \mathrm{t}_3, \forall \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in S$

To prove (i)*:*$S_{\mathcal{N}}$ *is* $\mathcal{NNI}$

$\Rightarrow \aleph_{\mathcal{N}}(0) \leq \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_1 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\}$ *(by(14))*

$\Rightarrow \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_3), \aleph_{\mathcal{N}}(t_2)\}$*(by Proposition 4.5)*

To prove (ii)*:*$S_{\mathcal{N}}$ *is* $\mathcal{NNI}$

$\Rightarrow I_{\mathcal{N}}(0) \geq I_{\mathcal{N}}(t_1) \geq \wedge\{I_{\mathcal{N}}(t_1 \otimes t_2), I_{\mathcal{N}}(t_2)\}$ *(by(15))*

$\Rightarrow I_{\mathcal{N}}(t_1) \geq \wedge\{I_{\mathcal{N}}(t_3), I_{\mathcal{N}}(t_2)\}$ *(by Proposition 4.5)*

To prove (iii)*:*$S_{\mathcal{N}}$ *is* $\mathcal{NNI}$

$\Rightarrow \Psi_{\mathcal{N}}(0) \leq \Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_1 \otimes t_2), \Psi_{\mathcal{N}}(t_2)\}$ *(by(16))*

$\Rightarrow \Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_3), \Psi_{\mathcal{N}}(t_2)\}$ *(by Proposition 4.5)*

Note 4.11. *Applying induction on n and from the Theorem 4.10, we have*

Theorem 4.12. *If* $S_{\mathcal{N}}$ *is a* $\mathcal{NNI}$ *over a* $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$ *then for any* $\mathrm{p}, \mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, ..., \mathrm{a}_\mathrm{n} \in \mathrm{S}$ *and*

$(....((\mathrm{p} \otimes \mathrm{a}_1) \otimes \mathrm{a}_2) \otimes ....) \otimes \mathrm{a}_\mathrm{n} = 0$ *implies*

$i)\aleph_{\mathcal{N}}(p) \leq \vee\{\aleph_{\mathcal{N}}(a_1), \aleph_{\mathcal{N}}(a_2), ..., \aleph_{\mathcal{N}}(a_n)\}$

$ii)I_{\mathcal{N}}(p) \geq \wedge\{I_{\mathcal{N}}(a_1), I_{\mathcal{N}}(a_2), ..., I_{\mathcal{N}}(a_n)\}$

$iii)\Psi_{\mathcal{N}}(p) \leq \vee\{\Psi_{\mathcal{N}}(a_1), \Psi_{\mathcal{N}}(a_2), ..., \Psi_{\mathcal{N}}(a_n)\}$

5. **Conclusions:**

The investigation of $\mathcal{NNSA}$ and $\mathcal{NNI}$ within the context of $\mathcal{BFA}$ has led to several conclusions.

Firstly, the study has provided a thorough analysis of the inherent characteristics of $\mathcal{NNSA}$ and $\mathcal{NNI}$. This analysis has helped in understanding the properties and behaviors of these structures within BF-algebras.

Secondly, the investigation has revealed the intricate relationships that exist between $\mathcal{NNSA}$ and $\mathcal{NNI}$. By exploring these relationships, researchers have gained insights into how these structures interact and influence each other within the broader context of algebraic structures.

Furthermore, the study has delved into the unique properties associated with $\mathcal{NNI}$. By examining these properties, researchers have enhanced their understanding of $\mathcal{NNI}$ and its potential applications in managing uncertainty in Negative-Valued $\mathcal{N}$eutrosophic soft sets.

Overall, the investigation of $\mathcal{NNSA}$ and $\mathcal{NNI}$ within the context of $\mathcal{BFA}$ has contributed significantly to the field. It has expanded our comprehension of these

structures and their relationships, paving the way for further research and advancements in this specialized domain of mathematics

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