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Solution of First Order Initial Value Problem using Analytical and Numerical Method in Neutrosophic Environment

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Abstract: In this paper, the solution of a first-order linear non-homogeneous fuzzy differential equation with an initial condition is described in a neutrosophic environment. For this purpose, using triangular neutrosophic numbers, the neutrosophic analytical method, and the fourth-order Runge-Kutta numerical method have been introduced for solving fuzzified first order differential equation. We also observed solutions at the (α,β,γ)-cut with varied time scales. In addition, the error between the analytical and numerical solution obtained on the (α,β,γ)-cut is evaluated and illustrated using tables with varying time. A good amount of agreement is seen using closed form and numerical solutions.

Keywords: Differential equation, Fuzzy set, Triangular Fuzzy numbers, Neutrosophic, Runge-Kutta 4th order.

1. Introduction

We are often faced with many ambiguous situations because of the limited, vague and uncertain knowledge available in our daily lives. It becomes impossible to depict and characterize any phenomenon in precise manner. In order to deal with these circumstances, Zadeh proposed fuzzy set theory in 1965 [1]. In numerous situations, we all employ intellectual terms such as "easy," "hard," "extremely easy," "very hard," and so on. These are ambiguous words, and the information derived from them differs in many ways. In fuzzy set theory, each intellectual word is assigned a membership grade, and all of these intellectual forms may then be simply fitted into the fuzzy environment. Basically, fuzzy set theory allows each element of a set "A" to have a specific degree of membership, represented by $\mu_A(x)$, which denotes that each element x of set A has a membership value lying in the closed interval [0, 1]. When we want to fit distinct intellectual words into a fuzzy set then we assign a numerical value to them between 0 and 1, and call them fuzzy numbers. Chang and Zadeh developed fuzzy numbers in 1972 [2], while Dubois and Prade studied generalization of fuzzy numbers in 1978 [3].

In practice, we normally consider the membership value, although this is insufficient. In such cases, the non-membership value must also be taken into account. But fuzzy sets are established solely for membership values, they do not take non-membership values into account. Atanassov presented the intuitionistic fuzzy set (IFS) in 1986 [4-5], which is an extension of fuzzy sets that encompassed both situations. Because it contains information that belongs to the set as well as information that does not belong to the set, intuitionistic fuzzy sets are regarded as an extension of fuzzy sets.

In the real-life uncertainty, there is also the possibility of a different situation, known as indeterminacy. When the knowledge on which items belong to the set and do not belong to the set is insufficient, a neutral state condition known as indeterminacy arises. In order to comprehend this scenario in real life, Florentin Smarandache was the first to establish neutrosophic set theory which consider truth value, indeterminate value, and false value in 2006 [6]. In Neutrosophic set, grade of membership of Truth values (T), Indeterminate values (I) and False values (F) has been defined within the non-standard interval -]0,1[+. Non-standard intervals of the neutrosophic set theory works good in the concept of philosophy. In reality if we deal with engineering and science problems it is impossible to fit data in the non-standard interval. To solve such problems, Wang et al. created single-valued Neutrosophic sets by considering the unit interval [0,1] in its standard form in 2010 [7]. Furthermore, many researchers, including Aal SIA et al., Deli and Subas , and Chakraborty et al. have defined single-valued neutrosophic number [8-10]. Similarly, Ye defined Trapezoidal Neutrosophic numbers and its application in the field of decision-making [11]. Using this approach, lots of work is going on by considering several real-life issues (see for instance [12-20]). For example, Abdel-Basset et al. presented type-2 neutrosophic numbers for decision making problems, results on recent pandemic COVID-19, supply chain model, industrial and management problems. Similar applications and other generalization of the theory are discussed in the research articles, viz; [16-21]. Researchers must use certain methodologies, particularly differential equations, in order to initiate a discussion about modelling any phenomena and study its behaviour. In the modelling of any phenomenon, the data we receive is incomplete, imprecise, and uncertain., Kaleva proposed fuzzy differential equations to better grasp such ambiguity in real life in 1986 [22]. To develop the area of fuzzy differential equation some researchers extended the concept of calculus in fuzzy environment. Dubois & Prade, Goetschel & Voxman, Puri & Ralescu and others pioneered the fuzzy derivative and its extended theory [23-26]. They solved an initial value problem for a first order differential equation by employing the notion of fuzzy derivatives. Similarly, Buckley et al. proposed the solution of an nth order ordinary differential equation using fuzzy initial conditions [27-28]. The generalized Hukuhara differentiability for fuzzy-valued functions plays the most important role in the development of the fuzzy differential equation, which was presented by Bede and Seikkala [29-31]. There are many methods available in the literature to solve fuzzy differential equations, such as analytical, semi-analytical, and numerical methods, which have been used by various researchers, for example, Nieto et al. and Ghazanfari et al. who used Numerical method, namely Euler approximation and Runge-Kutta method of order 4 for solving first order linear fuzzy differential equations [32-33]. Lots of works have been done for the development of FDE (see for instance [34-39]). Because intuitionistic fuzzy set is a generalization of fuzzy set, fuzzy differential equation is likewise generalized to an intuitionistic environment. Researchers Ben et al. and many others have discussed analytical and numerical techniques for the solution of Intuitionistic fuzzy differential equation [40-41].

In this paper, we describe how to solve differential equations in a neutrosophic setting using calculus features of the neutrosophic set, which was discussed by Smarandache in 2015 [42]. He was first introduced neutrosophic derivative which is an extension of fuzzy derivative. Neutrosophic derivative has new type of the granular derivative (gr-derivative) which was introduced by Son et al. [43]. Also, he gave the gr-partial derivative of neutrosophic-valued several variable functions and investigated the *if and only if condition* for the existence of gr-derivative of neutrosophic-valued function. In the recent time, a lot of effort is done in the neutrosophic environment to describe many real-life occurrences using differential equations. For example, Sumanthi et al. has discussed the solution of neutrosophic differential equation using trapezoidal neutrosophic numbers, Parikh and Sahni discussed about the second order differential using Sumudu transform in neutrosophic environment, Moi discussed boundary value problem for second order differential equation in neutrosophic environment, and many other researchers discussed similar problems [44-47]. In this study, we addressed theory for the solution of first order differential equations using numerical approach, namely, Runge Kutta of 4th order in neutrosophic environment, which was inspired by these researches.

1.1 Motivation:

Our review of the literature revealed that there has been little research on Neutrosophic differential equations. Thus, there is a lot of scope for progress in this area. So, in order to proceed in this direction, we must first define the basic theory of first-order differential equations in a neutrosophic environment. As a result, the development of a technique for finding a solution to a differential equation, which has previously been done in a classical and fuzzy environment, has prompted us to consider similar forms of expansion in a neutrosophic environment.

1.2 Uniqueness of paper

This research article presents the theory of first order neutrosophic initial value problems in order to find a solution to a first order differential equation in a neutrosophic environment. The aims of this paper are as follows:

- To define fundamental preliminary concept in the neutrosophic environment.
- To offer an analytical approach for solving first-order differential equations using triangular neutrosophic numbers.
- To propose a numerical approach for solving first-order differential equations using triangular neutrosophic numbers.
- To solve the problem using both analytical and numerical methods, and to obtain the solution using neutrosophic triangular number.
- Interpret the solution obtained using analytical and numerical method and calculate the difference between them.

1.3 Structure of the paper

The structure of the paper is as follow: In Section 2, certain mathematical preliminaries are provided, which is relevant to our study. The development of a first order differential equation employing neutrosophic triangular numbers, lemma, and theorems is covered in Section 3, which includes both analytical and numerical theory. In Section 4, the neutrosophic initial-value problem is established and validated using a classical solution. Section 5 contains the results and discussions which is depicted graphically. Finally, a brief conclusion about this article has been given in Section 6.

2. Mathematical Preliminaries

Definition 2.1 Fuzzy set [3]: A membership function $\mu_A(x)$ is defined on an element x in the universal set X, for every $x \in X$, which can also be expressed as $\mu_A(x) \in [0,1]$. The fuzzy set A is defined as, $A = \{(x, \mu_A(x)) | \forall x \in X\}$.

Definition 2.2 α-level of Fuzzy set [3]: The α-level of set A is defined as $A_\alpha = \{\mu_A(x) \geq \alpha, \alpha \in [0,1]\}$, where x ϵ X. This set includes all the elements of X with membership values in A that are greater than or equal to α.

Definition 2.3 Intuitionistic fuzzy set (IFS) [2]: An Intuitionistic fuzzy set B over universal set of X is represented by $B = \{(x, \mu_B(x), v_B(x)) | \forall x \in X\}$, where value $\mu_B(x)$ represent membership value of x in B, and value $v_B(x)$ represent non-membership value of x in B.

Definition 2.4 α, β - level of Intuitionistic Fuzzy set [2]: For any Intuitionistic fuzzy set B with the α, β-level set which is defined as i.e., $B_{\alpha,\beta} = \{x: \mu_B(x) \geq \alpha, v_B(x) \leq \beta, \forall x \in X, \alpha, \beta \in [0,1]\}$ with $\alpha + \beta \leq 1$, where X is universal set.

Definition 2.5 Neutrosophic set (NS) [6]: A neutrosophic set defined as N = {$T_N(x)$, $I_N(x)$, $F_N(x)$: $|\forall x \in X$}, where $T_N(x)$, $I_N(x)$, $F_N(x)$ are from universal set X →] $^-0$, 1^+ [, which represents the truth membership grade ($T_N(x)$), indeterminacy membership grade ($I_N(x)$), and false membership grade ($F_N(x)$) of the element x ∈ X, with the condition $^-0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3^+$.

Definition 2.6 Single-Valued Neutrosophic Set (SVNS) [6]: Let N be any single-valued Neutrosophic Set which is defined as N= {$T_N(x)$, $I_N(x)$, $F_N(x)$: $\forall x \in X$}, where $T_N(x)$, $I_N(x)$, $F_N(x)$ are from universal set X → [0,1] represents the truth membership ($T_N(x)$), indeterminacy membership ($I_N(x)$), and false membership ($F_N(x)$) of the element x ∈ X, with the condition $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3$.

Definition 2.7 Neutrosophic Number [6]: A neutrosophic set N defined over the universal single valued set of real numbers R is said to be neutrosophic number if it has the following properties:

1) N is normal: if $\exists\, x_0 \in R$, such that $T_N(x_0) = 1$ ($I_N(x_0) = F_N(x_0) = 0$).
2) N is convex set for the truth function $T_N(x)$, i.e., $T_N(\mu x_1 + (1-\mu)x_2) \geq \min(T_N(x_1), T_N(x_2))$, $\forall x_1, x_2 \in R$, $\mu \in [0,1]$.
3) N is concave set for the indeterminacy function ($I_N(x)$) and false function ($F_N(x)$), i.e., $I_N(\mu x_1 + (1-\mu)x_2) \geq \max(I_N(x_1), I_N(x_2))$, $\forall x_1, x_2 \in R$, $\mu \in [0,1]$, $F_N(\mu x_1 + (1-\mu)x_2) \geq \max(F_N(x_1), F_N(x_2))$, $\forall x_1, x_2 \in R$, $\mu \in [0,1]$.

Definition 2.8 (α, β, γ)-level of Neutrosophic set [46]: A neutrosophic set with (α, β, γ)-level of X is denoted by G (α,β,γ), where $\alpha,\beta,\gamma \in [0,1]$, and is defined as G($\alpha$,β,γ) ={$T_N(x)$, $I_N(x)$, $F_N(x)$: $\forall x \in X$, $T_N(x) \geq \alpha$, $I_N(x) \leq \beta$, $F_N(x) \leq \gamma$}, where $0 \leq \alpha + \beta + \gamma \leq 3$.

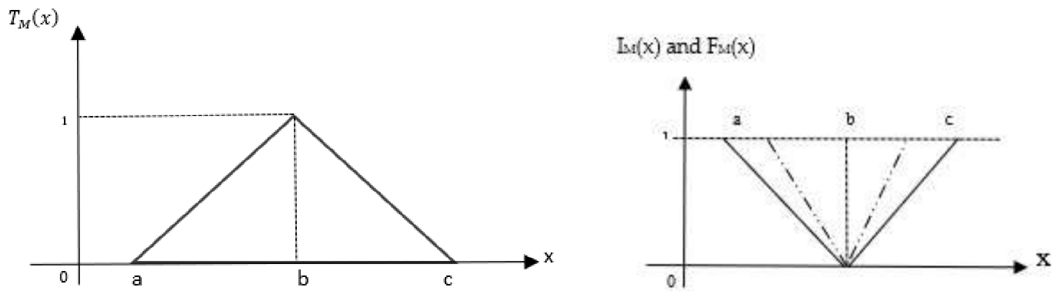
Definition 2.9 Triangular Neutrosophic Number [46]: Let N be a single valued neutrosophic set (SVNS) having Truth $T_N(x)$, Indeterminacy $I_N(x)$ and False $F_N(x)$ membership function over universal set X, then the triangular neutrosophic number is defined as

$$T_N(x) = \begin{cases} \left(\frac{x-a}{b-a}\right) & for\ a \le x < b \\ 1 & for\ x = b \\ \left(\frac{c-x}{c-b}\right) & for\ b < x \le c \\ 0 & otherwise \end{cases}$$

$$I_N(x) = \begin{cases} \left(\frac{b-x}{b-a}\right) & for\ a \le x < b \\ 0 & for\ x = b \\ \left(\frac{x-b}{c-b}\right) & for\ b < x \le c \\ 1 & otherwise \end{cases}$$

$$F_M(x) = \begin{cases} \left(\frac{b-x}{b-a}\right) & for\ a \le x < b \\ 0, & for\ x = b \\ \left(\frac{x-c}{c-b}\right) & for\ b < x \le c \\ 1 & otherwise \end{cases}$$

where $a \le b \le c$ and $a, b, c \in R$. Triangular neutrosophic number are denoted as $N_T\langle(a, b, c)\rangle$ where the truth membership function (T_N (x)) increases in a linear way for x ∈ $[a, b]$ and decrease in a linear form for x ∈ $[b, c]$ for I_N (x) and F_N (x) inverse behavior is seen from the truth membership for x ∈ $[a, b]$ and for x ∈ $[b, c]$ which is depicted in figure 1.



(a) T_M(x) as triangular form **(b)** I_M(x) and F_M(x) as triangular form

Figure 1: Graph of Triangular Neutrosophic Number.

Definition 2.10 (α, β, γ)-cut of a Triangular Neutrosophic Number [46]: A Triangular neutrosophic set with (α, β, γ)-cut is denoted by $A_{TN(\alpha,\beta,\gamma)}$, where α, β, γ ∈ [0,1], and is defined as $A_{TN(\alpha,\beta,\gamma)}$ = {T_N(x), I_N(x), F_N(x): $T_N(x) \ge \alpha$, $I_N(x) \le \beta$, $F_N(x) \le \gamma$, x ∈ X }. Here $0 \le \alpha + \beta + \gamma \le 3$ and

$$A_{TN(\alpha,\beta,\gamma)} = \left[\left(a + \alpha(b-a)\right), \left(c - \alpha(c-b)\right)\right],$$
$$\left[\left(b - \beta(b-a)\right), \left(b + \beta(c-b)\right)\right],$$
$$\left[\left(b - \gamma(b-a)\right), \left(b + \gamma(c-b)\right)\right].$$

3. Ordinary Differential Equation of First Order with initial value in form of Triangular Neutrosophic numbers

Let us consider a linear non-homogeneous ordinary differential equation of first order,

$$\frac{dy(t)}{dt} = py(t) + \varepsilon\ , \quad y(t_0) = y_0 \tag{1}$$

where y is a dependent variable, t is a independent variable, p and ε are constant and t_0 is the initial value of the parameter t.

Here, we consider initial value in the form of neutrosophic environment, so we have

$$y_T(t_0) = [a + \alpha(b - a), c - \alpha(c - b)]$$
$$y_I(t_0) = [b - \beta(b - a), b + \beta(c - b)]$$
$$y_F(t_0) = [b - \gamma(b - a), b + \gamma(c - b)]$$

where $y_T(t_0), y_I(t_0)$, and $y_F(t_0)$ represents truth, indeterminacy and false membership respectively.

Case 1: When the coefficient p in differential equation (1) is positive ($p > 0$), and taking (α, β, γ)-cut, the modified differential equation (1) in neutrosophic environment can be written as

$$\frac{d\left(\left[\underline{y_T}(t,\alpha), \overline{y_T}(t,\alpha)\right];\ \left[\underline{y_I}(t,\beta), \overline{y_I}(t,\beta)\right];\left[\underline{y_F}(t,\gamma), \overline{y_F}(t,\gamma)\right]\right)}{dt}$$
$$= p\left(\left[\underline{y_T}(t,\alpha), \overline{y_T}(t,\alpha)\right];\ \left[\underline{y_I}(t,\beta), \overline{y_I}(t,\beta)\right];\left[\underline{y_F}(t,\gamma), \overline{y_F}(t,\gamma)\right]\right) + [\varepsilon,\varepsilon];[\varepsilon,\varepsilon];[\varepsilon,\varepsilon]$$

with the initial condition

$$y(t_0,\ \alpha, \beta, \gamma) = \left(\left[\underline{y_T}(t_0,\alpha), \overline{y_T}(t_0,\alpha)\right];\ \left[\underline{y_I}(t_0,\beta), \overline{y_I}(t_0,\beta)\right];\left[\underline{y_F}(t_0,\gamma), \overline{y_F}(t_0,\gamma)\right]\right).$$

Solving equation (1) analytically in neutrosophic environment and using initial condition, we obtained the solution for T, I and F as

$$\underline{y_T}(t_0,\alpha) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + a + \alpha(b - a)\right)e^{p(t-t_0)} \tag{2}$$

$$\overline{y_T}(t_0,\alpha) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + c - \alpha(c - b)\right)e^{p(t-t_0)} \tag{3}$$

$$\underline{y_I}(t_0,\beta) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + b - \beta(b - a)\right)e^{p(t-t_0)} \tag{4}$$

$$\overline{y_I}(t_0,\beta) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + b + \beta(c - b)\right)e^{p(t-t_0)} \tag{5}$$

$$\underline{y_F}(t_0,\gamma) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + b - \gamma(b - a)\right)e^{p(t-t_0)} \tag{6}$$

$$\overline{y_F}(t_0,\gamma) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + b + \gamma(c - b)\right)e^{p(t-t_0)} \tag{7}$$

where $\underline{y_T}(t_0,\alpha)$ and $\overline{y_T}(t_0,\alpha)$ represents solution in the form of lower and upper bound of truth value respectively. Similarly, $\underline{y_I}(t_0,\beta)$, $\overline{y_I}(t_0,\beta)$, $\underline{y_F}(t_0,\gamma)$ and $\overline{y_F}(t_0,\gamma)$ represents solution in the form of lower and upper bound of indeterminacy and false value respectively.

If $\varepsilon = 0$, then the solution of the differential equation reduced to,

$$\underline{y_T}(t_0,\alpha) = \big(a + \alpha(b - a)\big)e^{p(t-t_0)}$$
$$\overline{y_T}(t_0,\alpha) = \big(c - \alpha(c - b)\big)e^{p(t-t_0)}$$
$$\underline{y_I}(t_0,\beta) = \big(b - \beta(b - a)\big)e^{p(t-t_0)}$$
$$\overline{y_I}(t_0,\beta) = \big(b + \beta(c - b)\big)e^{p(t-t_0)}$$
$$\underline{y_F}(t_0,\gamma) = \big(b - \gamma(b - a)\big)e^{p(t-t_0)}$$
$$\overline{y_F}(t_0,\gamma) = \big(b + \gamma(c - b)\big)e^{p(t-t_0)}$$

Case: 2 When the coefficient p in differential equation (1) is negative (i.e $p = -m$ and $m > 0$), and taking (α, β, γ)-cut, the modified differential equation (1) in neutrosophic environment can be written as

$$\frac{d\left(\left[\underline{y_T}(t,\alpha),\overline{y_T}(t,\alpha)\right];\ \left[\underline{y_I}(t,\beta),\overline{y_I}(t,\beta)\right];\left[\underline{y_F}(t,\gamma),\overline{y_F}(t,\gamma)\right]\right)}{dt}$$

$$= p\left(\left[\underline{y_T}(t,\alpha),\overline{y_T}(t,\alpha)\right];\ \left[\underline{y_I}(t,\beta),\overline{y_I}(t,\beta)\right];\left[\underline{y_F}(t,\gamma),\overline{y_F}(t,\gamma)\right]\right) + [\varepsilon,\varepsilon];[\varepsilon,\varepsilon];[\varepsilon,\varepsilon]$$

with initial condition

$$y(t_0,\ \alpha,\beta,\gamma) = \left(\left[\underline{y_T}(t_0,\alpha),\overline{y_T}(t_0,\alpha)\right];\ \left[\underline{y_I}(t_0,\beta),\overline{y_I}(t_0,\beta)\right];\left[\underline{y_F}(t_0,\gamma),\overline{y_F}(t_0,\gamma)\right]\right)$$

where t_0 is the initial value of the parameter t.

Solving equation (1) analytically in neutrosophic environment and using initial condition, we obtained the solution for T, I and F as

$$\left[\underline{y_T}(t_0,\alpha),\overline{y}_T(t_0,\alpha)\right]$$

$$= \frac{1}{2}((a+\alpha(b-a)) - (c-\alpha(b-a))e^{m(t-t_0)} + \frac{1}{2}(((a+\alpha(b-a)) - (c-\alpha(b-a))$$

$$-\frac{2\varepsilon}{m})e^{-m(t-t_0)} + \frac{\varepsilon}{m} \tag{8}$$

$$\left[\underline{y_I}(t_0,\beta),\overline{y_I}(t_0,\beta)\right]$$

$$= \frac{1}{2}\Big((b-\beta(b-a)) - (b+\beta(c-b))\Big)e^{m(t-t_0)}$$

$$+\frac{1}{2}\left((b-\beta(b-a)) + (b+\beta(c-b)) - \frac{2\varepsilon}{m}\right)e^{-m(t-t_0)} + \frac{\varepsilon}{m} \tag{9}$$

$$\left[\underline{y_F}(t_0,\gamma),\overline{y_F}(t_0,\gamma)\right] =$$

$$= \frac{1}{2}\Big((b-\gamma(b-a)) - (b+\gamma(c-b))\Big)e^{m(t-t_0)}$$

$$+\frac{1}{2}\left((b-\gamma(b-a)) + (b+\gamma(c-b)) - \frac{2\varepsilon}{m}\right)e^{-m(t-t_0)} + \frac{\varepsilon}{m} \tag{10}$$

If $\varepsilon = 0$ in equation (1) then the solution of the differential equation from the above equation is given as,

$$\left[\underline{y_T}(t_0,\alpha),\overline{y_T}(t_0,\alpha)\right]$$

$$= \frac{1}{2}\Big((a+\alpha(b-a)) - (c-\alpha(b-a))\Big)e^{m(t-t_0)}$$

$$+\frac{1}{2}\Big((a+\alpha(b-a)) + (c-\alpha(b-a))\Big)e^{-m(t-t_0)}$$

$$\left[\underline{y_I}(t_0, \beta), \overline{y_I}(t_0, \beta)\right]$$

$$= \frac{1}{2}\Big(\big(b - \beta(b-a)\big) - \big(b + \beta(c-b)\big)\Big)\, e^{m(t-t_0)}$$

$$+ \frac{1}{2}\Big(\big(b - \beta(b-a)\big) + \big(b + \beta(c-b)\big)\Big)\, e^{-m(t-t_0)}$$

$$\left[\underline{y_F}(t_0, \gamma), \overline{y_F}(t_0, \gamma)\right]$$

$$= \frac{1}{2}\Big(\big(b - \gamma(b-a)\big) - \big(b + \gamma(c-b)\big)\Big)\, e^{m(t-t_0)}$$

$$+ \frac{1}{2}\Big(\big(b - \gamma(b-a)\big) + \big(b + \gamma(c-b)\big)\Big)\, e^{-m(t-t_0)}$$

3.1 Solution of First Order Differential Equation with initial value in the form of Triangular Neutrosophic numbers using Runge-Kutta method of 4th order

Let us consider a linear non-homogeneous ordinary differential equation of first order

$$\frac{dy(t)}{dt} = py(t) + \varepsilon \ , \quad y(t_0) = y_0 \tag{11}$$

where y is dependent variable, t is independent variable, p and ε are constant and t_0 is the initial value of the parameter t.

Here, we consider initial value in form of neutrosophic environment and we have

$$y_T(t_0) = [a + \alpha(b-a), c - \alpha(c-b)]$$
$$y_I(t_0) = [b - \beta(b-a), b + \beta(c-b)]$$
$$y_F(t_0) = [b - \gamma(b-a), b + \gamma(c-b)]$$

where $y_T(t_0), y_I(t_0)$, and $y_F(t_0)$ represents truth, indeterminacy and false membership respectively. Fuzzy numerical solution of the given differential equation denoted as

$$y(t_n)_{T,I,F} = \left[\underline{y}(t_n)_T \,, \overline{y}\,(t_n)_T, \underline{y}(t_n)_I \,, \overline{y}\,(t_n)_{I,}\, \underline{y}(t_n)_F \,, \overline{y}\,(t_n)_F\right]$$

where $y(t_n)_T = \left[\underline{y}(t_n)_T \,, \overline{y}\,(t_n)_T\right]$, $y(t_n)_I = \left[\underline{y}(t_n)_I \,, \overline{y}\,(t_n)_I\right]$, $y(t_n)_F = \left[\underline{y}(t_n)_F \,, \overline{y}\,(t_n)_F\right]$

represents function of truth , indeterminacy and false membership respectively.

Solving equation (11) numerically in neutrosophic environment and using initial condition, we obtained the solution for T, I and F as

$$\underline{y}(t_{n+1})_T = \underline{y}(t_n)_T + \textstyle\sum_{j=1}^{4} P_j\, k_{j,1}(t_n, y(t_n)_T) \tag{12}$$
$$\overline{y}(t_{n+1})_T = \overline{y}(t_n)_T + \textstyle\sum_{j=1}^{4} P_j\, k_{j,2}(t_n, y(t_n)_T) \tag{13}$$
$$\underline{y}(t_{n+1})_I = \underline{y}(t_n)_I + \textstyle\sum_{j=1}^{4} P_j\, k_{j,1}(t_n, y(t_n)_I) \tag{14}$$
$$\overline{y}(t_{n+1})_I = \overline{y}(t_n)_I + \textstyle\sum_{j=1}^{4} P_j\, k_{j,2}(t_n, y(t_n)_I) \tag{15}$$
$$\underline{y}(t_{n+1})_F = \underline{y}(t_n)_F + \textstyle\sum_{j=1}^{4} P_j\, k_{j,1}(t_n, y(t_n)_F) \tag{16}$$
$$\overline{y}(t_{n+1})_F = \overline{y}(t_n)_F + \textstyle\sum_{j=1}^{4} P_j\, k_{j,2}(t_n, y(t_n)_F) \tag{17}$$

where the P_j's are constants. Then $k_{j,1}, k_{j,2}$ for j =1, 2, 3, 4 are defined as follow for truth, indeterminacy and false membership respectively:

First we obtained equation for the truth membership, which are denoted as $k_{j,1}(t_n, y(t_n)_T)$, $k_{j,2}(t_n, y(t_n)_T)$, where j =1, 2, 3, 4, and u is function which is defined as $\{u \epsilon \left[\underline{y}'(t_n)_T, \overline{y}'(t_n)_T\right]\}$.

The coefficients $k_{1,1}(t_n, y(t_n)_T)$, $k_{4,2}(t_n, y(t_n)_T)$are defined as

$$k_{1,1}(t_n, y(t_n)_T) = \min h\{ y(t_n, u)/u\epsilon \,(p\underline{y}(t_n)_T + \varepsilon\,, p\overline{y}\,(t_n)_T + \varepsilon)\} \quad (18\text{-a})$$

$$k_{1,2}(t_n, y(t_n)_T) = \max h\{ y(t_n, u)/u\epsilon \,(p\underline{y}(t_n)_T + \varepsilon\,, p\overline{y}\,(t_n)_T + \varepsilon)\} \quad (18\text{-b})$$

$$k_{2,1}(t_n, y(t_n)_T) = \min h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \,((q_{1,1}(t_n, y(t_n))), (q_{1,2}(t_n, y(t_n)))) \} \quad (18\text{-c})$$

$$k_{2,2}(t_n, y(t_n)_T) = \max h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \,((q_{1,1}(t_n, y(t_n))), (q_{1,2}(t_n, y(t_n))))\} \quad (18\text{-d})$$

$$k_{3,1}(t_n, y(t_n)_T) = \min h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \,((q_{2,1}(t_n, y(t_n))), (q_{2,2}(t_n, y(t_n)))\} \quad (18\text{-e})$$

$$k_{3,2}(t_n, y(t_n)_T) = \max h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \,((q_{2,1}(t_n, y(t_n))), (q_{2,2}(t_n, y(t_n)))\} \quad (18\text{-f})$$

$$k_{4,1}(t_n, y(t_n)_T) = \min h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \,((q_{3,1}(t_n, y(t_n))), (q_{3,2}(t_n, y(t_n)))\} \quad (18\text{-g})$$

$$k_{4,2}(t_n, y(t_n)_T) = \max h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \,(((q_{3,1}(t_n, y(t_n))), (q_{3,2}(t_n, y(t_n)))\} \quad (18\text{-h})$$

In above equations, we define $q_{j,1}(t_n, y(t_n)))$, $q_{j,2}(t_n, y(t_n)))$ for j =1, 2, 3 as follows,

$$q_{1,1}(t_n, y(t_n)) = \underline{y}(t_n)_T + \frac{h}{2} k_{1,1}(t_n, y(t_n)_T)$$

$$q_{1,2}(t_n, y(t_n)) = \overline{y}(t_n)_T + \frac{h}{2} k_{1,2}(t_n, y(t_n)_T)$$

$$q_{2,1}(t_n, y(t_n)) = \underline{y}(t_n)_T + \frac{h}{2} k_{2,1}(t_n, y(t_n)_T)$$

$$q_{2,2}(t_n, y(t_n)) = \overline{y}(t_n)_T + \frac{h}{2} k_{2,2}\,(t_n, y(t_n)_T)$$

$$q_{3,1}(t_n, y(t_n)) = \underline{y}(t_n)_T + \frac{h}{2} k_{3,1}(t_n, y(t_n)_T)$$

$$q_{3,2}(t_n, y(t_n)) = \overline{y}(t_n)_T + \frac{h}{2} k_{3,2}(t_n, y(t_n)_T)$$

Secondly, we obtained equation for the intederminancy membership which are denoted as $k_{j,1}(t_n, y(t_n)_I)$, $k_{j,2}(t_n, y(t_n)_I)$, where j = 1, 2, 3, 4, and u is function which is defined as $\{u \epsilon \left[\underline{y}'(t_n)_I, \overline{y}'(t_n)_I\right]\}$. Thus,

$$k_{1,1}(t_n, y(t_n)_I) = \min h\{ y(t_n, u)/u\epsilon \,(p\underline{y}(t_n)_I + \varepsilon\,, p\overline{y}\,(t_n)_I + \varepsilon)\} \quad (19\text{-a})$$

$$k_{1,2}(t_n, y(t_n)_I) = \max h\{ y(t_n, u)/u\epsilon \,(p\underline{y}(t_n)_I + \varepsilon\,, p\overline{y}\,(t_n)_I + \varepsilon)\} \quad (19\text{-b})$$

$$k_{2,1}(t_n, y(t_n)_I) = \min h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \,((r_{1,1}(t_n, y(t_n))), (r_{1,2}(t_n, y(t_n)))\} \quad (19\text{-c})$$

$$k_{2,2}(t_n, y(t_n)_I) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\; ((r_{1,1}(t_n, y(t_n))), (r_{1,2}(t_n, y(t_n)))\} \quad (19\text{-d})$$

$$k_{3,1}(t_n, y(t_n)_I) = \min h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\; ((r_{2,1}(t_n, y(t_n))), (r_{2,2}(t_n, y(t_n)))\} \quad (19\text{-e})$$

$$k_{3,2}(t_n, y(t_n)_I) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\; ((r_{2,1}(t_n, y(t_n))), (r_{2,2}(t_n, y(t_n)))\} \quad (19\text{-f})$$

$$k_{4,1}(t_n, y(t_n)_I) = \min h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\; ((r_{3,1}(t_n, y(t_n))), (r_{3,2}(t_n, y(t_n)))\} \quad (19\text{-g})$$

$$k_{4,2}(t_n, y(t_n)_I) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\; ((r_{3,1}(t_n, y(t_n))), (r_{3,2}(t_n, y(t_n))) \quad (19\text{-h})$$

In above equations, we define $r_{j,1}(t_n, y(t_n))), r_{j,2}(t_n, y(t_n)))$ for j =1, 2, 3 as follows,

$$r_{1,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_I + \frac{h}{2}k_{1,1}(t_n, y(t_n)_I)$$

$$r_{1,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_I + \frac{h}{2}k_{1,1}(t_n, y(t_n)_I)$$

$$r_{1,2}\big(t_n, y(t_n)\big) = \overline{y}(t_n)_I + \frac{h}{2}k_{1,2}(t_n, y(t_n)_I)$$

$$r_{2,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_I + \frac{h}{2}k_{2,1}(t_n, y(t_n)_I)$$

$$r_{2,2}\big(t_n, y(t_n)\big) = \overline{y}(t_n)_I + \frac{h}{2}k_{2,2}\,(t_n, y(t_n)_I)$$

$$r_{3,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_I + \frac{h}{2}k_{3,1}(t_n, y(t_n)_I)$$

$$r_{3,2}\big(t_n, y(t_n)\big) = \overline{y}(t_n)_I + \frac{h}{2}k_{3,2}(t_n, y(t_n)_I)$$

Lastly, we obtained equation for the false membership which are denoted as $k_{j,1}(t_n, y(t_n)_F)$, $k_{j,2}(t_n, y(t_n)_F)$, where j = 1, 2, 3, 4, and u is function which is defined as $\{u\,\epsilon\, \left[\underline{y}'(t_n)_F\,, \overline{y}\,'\,(t_n)_F\,\right]\}$. Thus,

$$k_{1,1}(t_n, y(t_n)_F) = \min h\{\, y(t_n, u)/u\epsilon\; (\,p\underline{y}(t_n)_F + \varepsilon\,, p\overline{y}\,(t_n)_F + \varepsilon)\,\} \quad (20\text{-a})$$

$$k_{1,2}(t_n, y(t_n)_F) = \max h\{\, y(t_n, u)/u\epsilon\;\; (\,p\underline{y}(t_n)_F + \varepsilon\,, p\overline{y}\,(t_n)_F + \varepsilon)\,\} \quad (20\text{-b})$$

$$k_{2,1}(t_n, y(t_n)_F) = \min h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\; ((s_{1,1}(t_n, y(t_n))), (s_{1,2}(t_n, y(t_n)))\} \quad (20\text{-c})$$

$$k_{2,2}(t_n, y(t_n)_F) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\; ((s_{1,1}(t_n, y(t_n))), (s_{1,2}(t_n, y(t_n)))\} \quad (20\text{-d})$$

$$k_{3,1}(t_n, y(t_n)_F) = \min h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\; ((s_{2,1}(t_n, y(t_n))), (s_{2,2}(t_n, y(t_n)))\} \quad (20\text{-e})$$

$$k_{3,2}(t_n, y(t_n)_F) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\; ((s_{2,1}(t_n, y(t_n))), (s_{2,2}(t_n, y(t_n)))\} \quad (20\text{-f})$$

$$k_{4,1}(t_n, y(t_n)_F) = \min h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\; ((s_{3,1}(t_n, y(t_n))), (s_{3,2}(t_n, y(t_n)))\,\} \quad (20\text{-g})$$

$$k_{4,2}(t_n, y(t_n)_F) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((s_{3,1}(t_n, y(t_n))), (s_{3,2}(t_n, y(t_n)))\} \qquad (20\text{-h})$$

In the above equations, we define $s_{j,1}(t_n, y(t_n))), s_{j,2}(t_n, y(t_n)))$ for j =1, 2, 3 as follows,

$$s_{1,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_F + \frac{h}{2} k_{1,1}(t_n, y(t_n)_F)$$

$$s_{1,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_F + \frac{h}{2} k_{1,1}(t_n, y(t_n)_F)$$

$$s_{1,2}\big(t_n, y(t_n)\big) = \overline{y}(t_n)_F + \frac{h}{2} k_{1,2}(t_n, y(t_n)_F)$$

$$s_{2,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_F + \frac{h}{2} k_{2,1}(t_n, y(t_n)_F)$$

$$s_{2,2}\big(t_n, y(t_n)\big) = \overline{y}(t_n)_F + \frac{h}{2} k_{2,2}\,(t_n, y(t_n)_F)$$

$$s_{3,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_F + \frac{h}{2} k_{3,1}(t_n, y(t_n)_F)$$

$$s_{3,2}\big(t_n, y(t_n)\big) = \overline{y}(t_n)_F + \frac{h}{2} k_{3,2}(t_n, y(t_n)_F)$$

From the equations 18-(a to h), 19-(a to h) and 20-(a to h) we obtained the solution as follows,

$$\underline{y}(t_{n+1})_T = \underline{y}(t_n)_T \ + \tfrac{1}{6}\big[\, k_{1,2}(t_n, y(t_n)_T) + 2k_{2,2}(t_n, y(t_n)_T) + 2k_{3,2}(t_n, y(t_n)_T) + k_{4,2}(t_n, y(t_n)_T)\big] \quad (21)$$

$$\overline{y}(t_{n+1})_T = \overline{y}(t_n)_T \ + \tfrac{1}{6}\big[\, k_{1,2}(t_n, y(t_n)_T) + 2k_{2,2}(t_n, y(t_n)_T) + 2k_{3,2}(t_n, y(t_n)_T) + k_{4,2}(t_n, y(t_n)_T)\big] \quad (22)$$

$$\underline{y}(t_{n+1})_I = \underline{y}(t_n)_I \ + \tfrac{1}{6}\big[\, k_{1,1}(t_n, y(t_n)_I) + 2k_{2,1}(t_n, y(t_n)_I) + 2k_{3,1}(t_n, y(t_n)_I) + k_{4,1}(t_n, y(t_n)_I)\big] \quad (23)$$

$$\overline{y}(t_{n+1})_I = \overline{y}(t_n)_I \ + \tfrac{1}{6}\big[\, k_{1,2}(t_n, y(t_n)_I) + 2k_{2,2}(t_n, y(t_n)_I) + 2k_{3,2}(t_n, y(t_n)_I) + k_{4,2}(t_n, y(t_n)_I)\big] \quad (24)$$

$$\underline{y}(t_{n+1})_F = \underline{y}(t_n)_F \ + \tfrac{1}{6}\big[\, k_{1,1}(t_n, y(t_n)_F) + 2k_{2,1}(t_n, y(t_n)_F) + 2k_{3,1}(t_n, y(t_n)_F) + k_{4,1}(t_n, y(t_n)_F)\big] \quad (25)$$

$$\overline{y}(t_{n+1})_F = \overline{y}(t_n)_F \ + \tfrac{1}{6}\big[\, k_{1,2}(t_n, y(t_n)_F) + 2k_{2,2}(t_n, y(t_n)_F) + 2k_{3,2}(t_n, y(t_n)_F) + k_{4,2}(t_n, y(t_n)_F)\big] \quad (26)$$

where $y(t_{n+1})_T = [\underline{y}(t_{n+1})_T\,,\ \overline{y}(t_{n+1})_T]$ represent solution in the form of truth membership. Similarly, $y(t_{n+1})_I = [\underline{y}(t_{n+1})_I\,,\ \overline{y}(t_{n+1})_I]$, $y(t_{n+1})_F = [\underline{y}(t_{n+1})_F, \ \overline{y}(t_{n+1})_F$ represents solution for indeterminacy and false membership respectively.

The approximate solutions for $t_n, 0 \le t \le N$ are denoted by

$$y(t_n)_{T,I,F} = \Big[\underline{y}(t_n)_T\,, \overline{y}\,(t_n)_T, \underline{y}(t_n)_I\,, \overline{y}\,(t_n)_I, \underline{y}(t_n)_F\,, \overline{y}\,(t_n)_F\Big].$$

The solution is calculated using grid points $a = t_0 \le t_1 \le t_2 \ldots \le t_n = b$ and $h = \frac{b-a}{N} = t_{n+1} - t_n$

$$\underline{y}(t_{n+1})_T = \underline{y}(t_n)_T \ + \tfrac{1}{6} y[(t_n, y(t_n)_T)] \qquad (27)$$

$$\overline{y}(t_{n+1})_T = \overline{y}(t_n)_T + \frac{1}{6}y[(t_n, y(t_n)_T)] \tag{28}$$

$$\underline{y}(t_{n+1})_I = \underline{y}(t_n)_I + \frac{1}{6}y[(t_n, y(t_n)_I)] \tag{29}$$

$$\overline{y}(t_{n+1})_I = \overline{y}(t_n)_I + \frac{1}{6}y[(t_n, y(t_n)_I)] \tag{30}$$

$$\underline{y}(t_{n+1})_F = \underline{y}(t_n)_F + \frac{1}{6}y[(t_n, y(t_n)_F)] \tag{31}$$

$$\overline{y}(t_{n+1})_F = \overline{y}(t_n)_F + \frac{1}{6}y[(t_n, y(t_n)_F)] \tag{32}$$

where $y(t_n)_T = \left[\underline{y}(t_n)_T, \overline{y}(t_n)_T\right]$, $y(t_n)_I = \left[\underline{y}(t_n)_I, \overline{y}(t_n)_I\right]$, and $y(t_n)_F = \left[\underline{y}(t_n)_F, \overline{y}(t_n)_F\right]$ represents function of truth, indeterminacy, and false membership respectively.

4. Numerical Example:

In order to validate our development of theoretical approach we have performed numerical studies. In validation section, we summarize the results of these tests and compare the results with classical solution as well as fuzzy analytical solution and also discuss the error between them. So, for that we consider generalized Fuzzy initial value problem, which is $y'(t) = y(t)$, $y(0) = 1$ and we find the solution for y at t=1.

Solution: We apply classical method, analytical method and numerical method in an neutrosophic environment and then compare the solution as well as error between different method.

Method-1 Classical method

Given equation is $y'(t) = y(t), y(0) = 1$

Solving first order linear differntial eqution with initial condition we get following equation,

$$y(t) = e^t$$

For t=1, the solution of y(t) is 2.7183 upto four decimal places.

Method-2 Fuzzified Analytical method

Let us consider differential equation $y'(t) = y(t)$ with initial values for truth, indeterminancy,and false membership given in the form of tringular neutrosophic numbers,

$$y_T(0) = [\alpha, 2-\alpha], \quad y_I(0) = [1-0.5\beta, 1+0.5\beta], \quad y_F(0) = [1-0.25\gamma, 1+0.25\gamma]$$

Solving differential equation $y'(t) = y(t)$ with proposed fuzzified analytical theory (section 3 case 1 equations (2) to (7)), we get the following solution,

$$\underline{y}(t)_{T\alpha} = \alpha e^t \qquad \overline{y}(t)_{T\alpha} = (2-\alpha)e^t$$

$$\underline{y}(t)_{I\beta} = (1-0.5\beta)e^t \qquad \overline{y}(t)_{I\beta} = (1+0.5\beta)e^t$$

$$\underline{y}(t)_{F\gamma} = (1-0.25\gamma)e^t \qquad \overline{y}(t)_{F\gamma} = (1+0.25\gamma)e^t$$

Method-3 Fuzzy Numerical method

Let us consider differential equation $y'(t) = y(t)$ with initial values for truth, indeterminancy,and false membership which is in the form of tringular neutrosophic numbers as,

$$y(0)_T = [\alpha, 2-\alpha], y(0)_I = [1-0.5\beta, 1+0.5\beta], y(0)_F = [1-0.25\gamma, 1+0.25\gamma]$$

Solving differential equation $y'(t) = y(t)$ by proposed Runge kutta method of 4 th order (section 3.1 case -1 equations (27) to (32)), we get following solutions,

$$\underline{y}(t_1)_{T\alpha} = \alpha + \frac{1}{6}\left(\frac{41\alpha}{4}\right) \quad (33)$$

$$\overline{y}(t_1)_{T\alpha} = (2-\alpha) + \frac{1}{6}\left(\frac{82-41\alpha}{4}\right) \quad (34)$$

$$\underline{y}(t_1)_{I\beta} = (1-0.5\beta) + \frac{1}{6}(10.25 - 0.8541\beta) \quad (35)$$

$$\overline{y}(t_1)_{I\beta} = (1+0.5\beta) + \frac{1}{6}(10.25 + 0.8541\beta) \quad (36)$$

$$\underline{y}(t_1)_{F\gamma} = (1-0.25\gamma) + \frac{1}{6}(10.25 - 2.5625\gamma) \quad (37)$$

$$\overline{y}(t_1)_{F\gamma} = (1+0.25\gamma) + \frac{1}{6}(10.25 + 2.5625\gamma) \quad (38)$$

5. **Numerical observation**

Table :1 Solution of y(t) using RK 4th order at t=0.1 and h=0.1

$(\alpha,\beta,\gamma)-cut$	Lower bound of Truth value at t=0.1 $\underline{y}(t_0)_{T\alpha}$	Upper bound of Truth value at t=0.1 $\overline{y}(t_0)_{T\alpha}$	Lower bound of Indeterminacy value at t=0.1 $\underline{y}(t_0)_{I\beta}$	Upper bound of Indeterminacy value at t=0.1 $\overline{y}(t_0)_{I\beta}$	Lower bound of Falsity value at t=0.1 $\underline{y}(t_0)_{F\gamma}$	Upper bound of Falsity value at t=0.1 $\overline{y}(t_0)_{F\gamma}$
0	0.0000000000	2.2103416667	1.1051708333	1.1051708333	1.1051708333	1.1051708333
0.2	0.2210341667	1.9893075000	0.9946537500	1.2156879167	1.0499122917	1.1604293750
0.4	0.4420683333	1.7682733333	0.8841366667	1.3262050000	0.9946537500	1.2156879167
0.6	0.6631025000	1.5472391667	0.7736195833	1.4367220833	0.9393952083	1.2709464583
0.8	0.8841366667	1.3262050000	0.6631025000	1.5472391667	0.8841366667	1.3262050000
1	1.1051708333	1.1051708333	0.5525854167	1.6577562500	0.8288781250	1.3814635417

Table :2 Solution of y(t) using RK 4th order at t=0.5 and h=0.1

$(\alpha,\beta,\gamma)-cut$	Lower bound of Truth value at t=0.5 $\underline{y}(t_{0.5})_{T\alpha}$	Upper bound of Truth value at t=0.5 $\overline{y}(t_{0.5})_{T\alpha}$	Lower bound of Indeterminacy value at t=0.5 $\underline{y}(t_{0.5})_{I\beta}$	Upper bound of Indeterminacy value at t=0.5 $\overline{y}(t_{0.5})_{I\beta}$	Lower bound of Falsity value at t=0.5 $\underline{y}(t_{0.5})_{F\gamma}$	Upper bound of Falsity value at t=0.5 $\overline{y}(t_{0.5})_{F\gamma}$
0	0.0000000000	3.6442359242	1.8221179621	1.8221179621	1.8221179621	1.8221179621
0.2	0.3644235924	3.2798123318	1.6399061659	2.0043297583	1.7310120640	1.9132238602
0.4	0.5967296960	2.9153887393	1.4576943697	2.1865415545	1.6399061659	2.0043297583
0.6	1.0932707773	2.5509651469	1.2754825735	2.3687533507	1.5488002678	2.0954356564
0.8	1.1934593921	2.1865415545	1.0932707773	2.5509651469	1.4576943697	2.1865415545

1	1.8221179621	1.8221179621	0.9110589810	2.7331769431	1.3665884716	2.2776474526

Table :3 Solution of y(t) using RK 4th order at t=1 and h=0.1

$(\alpha,\beta,\gamma)-$ *cut*	Lower bound of Truth value at t=1 $\underline{y}(t_1)_{T\alpha}$	Upper bound of Truth value at t=1 $\overline{y}(t_1)_{T\alpha}$	Lower bound of Indeterminacy value at t=1 $\underline{y}(t_1)_{I\beta}$	Upper bound of Indeterminacy value at t=1 $\overline{y}(t_1)_{I\beta}$	Lower bound of Falsity value at t=1 $\underline{y}(t_1)_{F\gamma}$	Upper bound of Falsity value at t=1 $\overline{y}(t_1)_{F\gamma}$
0	0.0000000000	5.4365594883	2.7182797441	2.7182797441	2.7182797441	2.7182797441
0.2	0.4919202828	4.8929035394	2.4464517697	2.9901077185	2.5823657569	2.8541937313
0.4	0.8902158253	4.3492475906	2.1746237953	3.2619356930	2.4464517697	2.9901077185
0.6	1.6309678465	3.8055916418	1.9027958209	3.5337636674	2.3105377825	3.1260217058
0.8	1.7804316506	3.2619356930	1.6309678465	3.8055916418	2.1746237953	3.2619356930
1	2.7182797441	2.7182797441	1.3591398721	4.0774196162	2.0387098081	3.3978496802

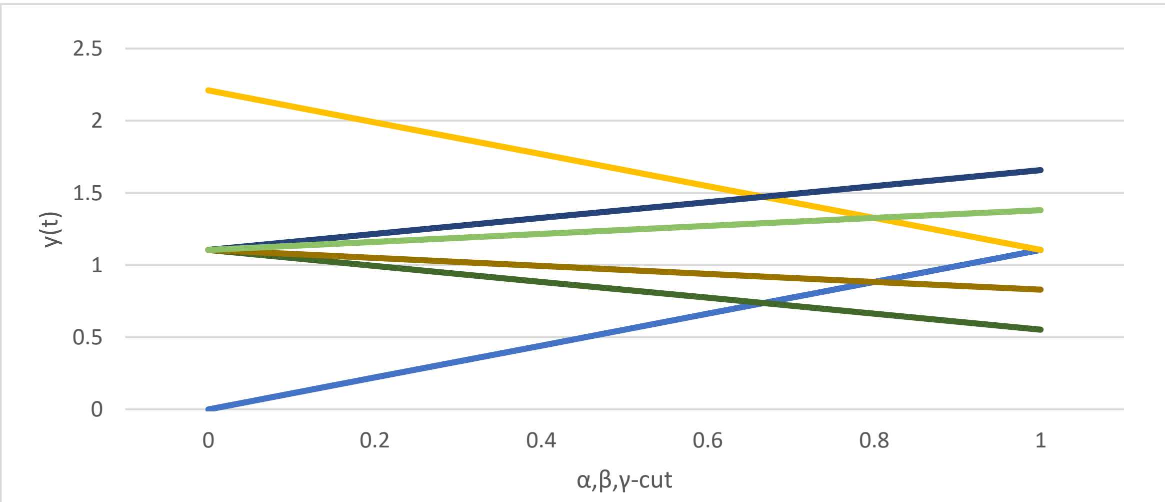


Figure 2: Solution of y(t) at t=0.1.

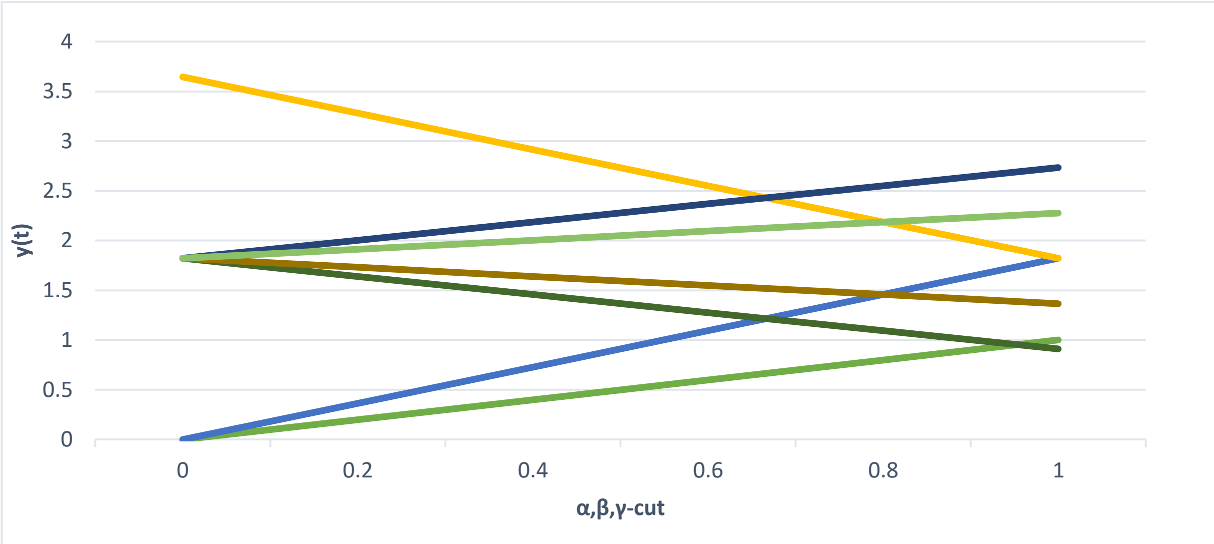


Figure 3: Solution of y(t) at t=0.5.

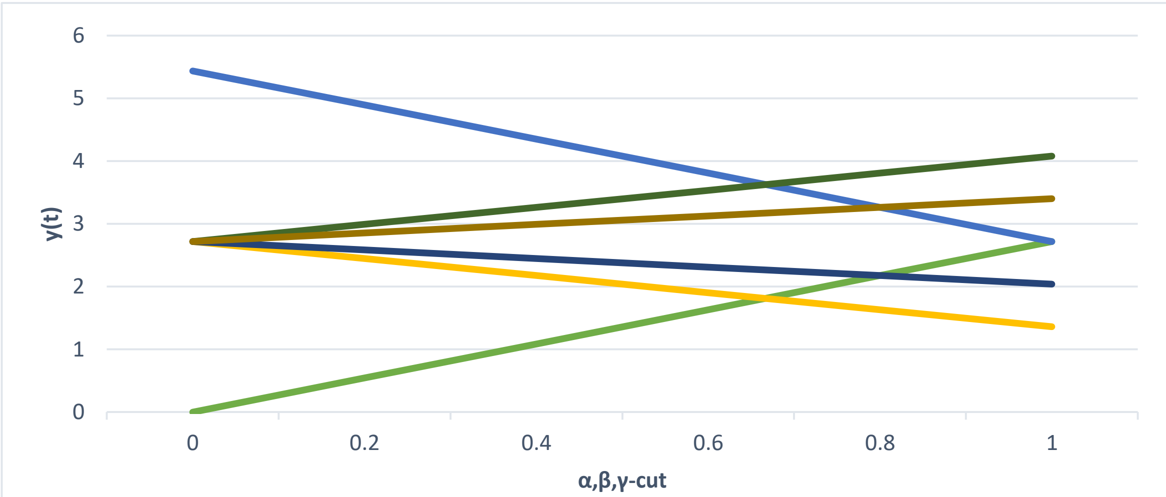


Figure 4: Solution of y(t) at t=1.

The results obtained from the calculation of equations (33) to (38) are shown in tables 1 to 3 respectively, for different (α, β,γ)-cut values with respect to the step size h= 0.2. It is clearly seen from the table 1 that the value of truth membership $y(t)_T = [\underline{y}(t_0)_{T\alpha}, \overline{y}(t_0)_{T\alpha}]$ for lower bound increases and the upper bound decreases. Similarly for indeterminacy, given by $y(t)_I = [\underline{y}(t_0)_{I\beta}, \overline{y}(t_0)_{I\beta}]$) and false membership ($y(t)_F = \left[\underline{y}(t_0)_{F\gamma}, \overline{y}(t_0)_{F\gamma}\right]$) for lower bound decreases and for the upper bound increases (depicted in the tables 2 and 3 respectively). In addition, from table 3, we observed that value of lower and upper bound of truth membership for (α, β,γ)-cut, when equal to 1 is 2.7182797441 and value of lower and upper bound for indeterminacy ($y(t)_I = [\underline{y}(t_0)_{I\beta}, \overline{y}(t_0)_{I\beta}]$) and false membership ($y(t)_F = \left[\underline{y}(t_0)_{F\gamma}, \overline{y}(t_0)_{F\gamma}\right]$) at (α, β,γ)-cut when equal to 0 are 2.7182797441, which match with exact solution. The graphs for various values for truth, indeterminacy and falsity with (α,β,γ)-cut are shown in figures 2, 3 and 4 respectively for different values of t (time). As the α-cut value increases and β,γ-cut values decrease solution approaches to the exact solution.

Table :4 Error Between RK 4th order and exact solution.

t(time)	Exact solution	Approximate solution by RK 4th order method where step size h=0.1	Error Between exact solution and solution find by RK 4th order
0.1	1.105170918	1.105170833	0.00000008467
0.2	1.221402758	1.221402571	0.00000018731
0.3	1.349858808	1.349858497	0.00000031052
0.4	1.491824698	1.491824240	0.00000045756
0.5	1.648721271	1.648720639	0.00000063210
0.6	1.822118800	1.822117962	0.00000083830
0.7	2.013752707	2.013751627	0.00000108087
0.8	2.225540928	2.225539563	0.00000136520
0.9	2.459603111	2.459601414	0.00000169738
1	2.718281828	2.718279744	0.00000208432

Furthermore, table 4 represents error between exact solution and solution obtained from Runge-Kutta 4th order. From the table 4, it is clearly seen that the exact solution at t=1 is 2.718281828 and on the other hand solution at t=1 is 2.718279744 using Runge-Kutta 4th order in neutrosophic environment for truth membership at (α, β,γ)-cut equal to 1 and the error between them is 0.00000208432.

6. Conclusion

In this paper, the first order ordinary differential equation using neutrosophic numbers with initial conditions have been solved. We have developed theory in a neutrosophic environment supplemented with an example showing the solution for first-order linear homogeneous differential equation both using analytical and numerical approach. For generalization, the (α,β,γ) - cut values are used for the neutrosophic numbers. Thus, to show the effectiveness of proposed method it has been applied to general example where the solution is given in terms of the truth, indeterminacy and falsity membership grade. We have shown the results in the form of tables for different (α,β,γ) - cut values and the graphs are also drawn. The results obtained are also discussed in details. Also, we have shown the growth of error between exact solution and approximate solution which are represented by tabulated values. This will promote the future study on higher order differential equations with neutrosophic numbers using numerical method which will help to decrease the error.

Conflict of interest

The Authors have no conflict of interest.

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