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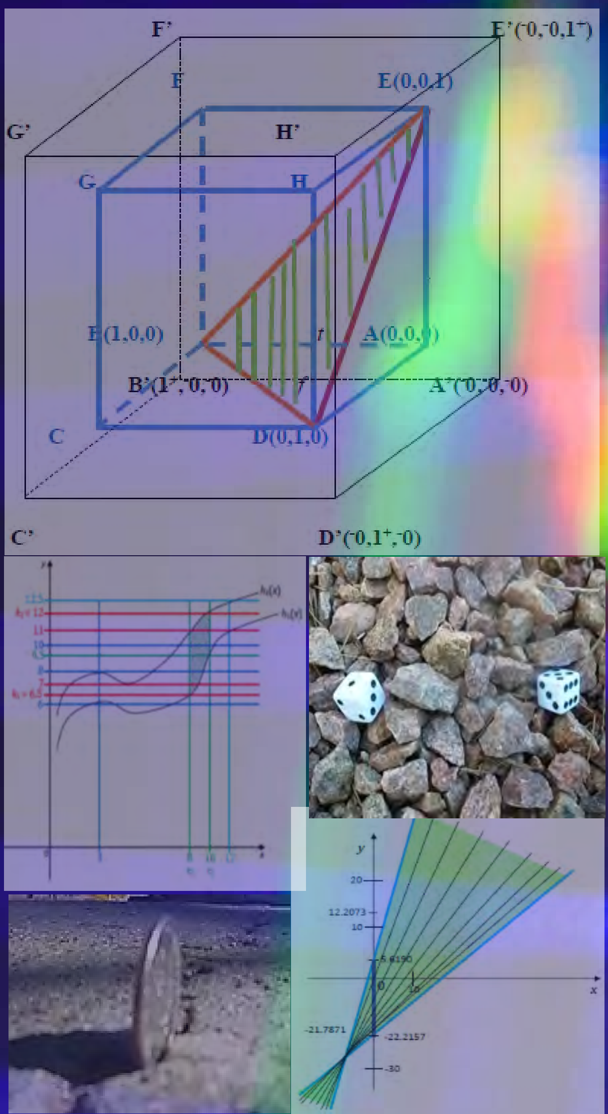

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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On Neutrosophic Crisp Sets and Neutrosophic Crisp Mathematical Morphology

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Abstract: In this paper, we analyze the basic algebraic operations of neutrosophic crisp sets and their properties, show that some operations of neutrosophic crisp sets with type 2 and type 3 are not closed by counter examples, and give some new operations. Then, discuss some morphological operators in neutrosophic crisp mathematical morphology, and study the application to edge extraction of color images, and give some Python programs and related experimental results.

Keywords: Fuzzy Set; Neutrosophic Set; Neutrosophic Crisp Set; Mathematical Morphology; Neutrosophic Crisp Mathematical Morphology

1. Introduction and Preliminaries

The theory of neutrosophic set is established by F. Smarandache, and it is applied to many areas such as non-classical logics, decision science, image processing, algebraic systems and so on [1–8]. In 2014, A.A.Salama and F.Smarandache introduced the new concept of neutrosophic crisp set, and studied the basic operations of neutrosophic crisp sets [9, 10]. On the other hand, mathematical morphology is a branch of image processing, which arose in 1964 [11, 12], and it is generalized to fuzzy mathematical morphology [13]. In 2017, the two research areas are associated, and the new notion of neutrosophic crisp mathematical morphology is firstly proposed [14]. Then, some new articles on this research direction are published [15, 16, 17]. This paper will carry out exploratory research along this direction, mainly discussing the operation and properties of neutrosophic crisp sets, and studying the application of neutrosophic crisp mathematical morphology in color image processing (previously only applied research on binary images and gray images).

At first, let's review some basic concepts in neutrosophic crisp set and neutrosophic crisp mathematical morphology.

Definition 1.1 ([9, 10]) Let X be a non-empty fixed sample space. A neutrosophic crisp set (NCS for short) A is an object having the form $\langle A_1, A_2, A_3\rangle$ where A_1, A_2 and A_3 are subsets of X.

Remark 1.1. In this paper, the set of all neutrosophic crisp sets of X will be denoted NCS(X).

Definition 1.2 ([9, 10]) The object having the form $A=\langle A_1, A_2, A_3\rangle$ is called:

(a) A neutrosophic crisp set of Type 1 (NCS-Type 1) if satisfying

$$A_1\cap A_2=\varnothing,\ A_1\cap A_3=\varnothing \text{ and } A_2\cap A_3=\varnothing.$$

(b) A neutrosophic crisp set of Type 2 (NCS-Type 2) if satisfying

◎ ◎◎ $A_1 \cap A_2 = \varnothing$, $A_1 \cap A_3 = \varnothing$, $A_2 \cap A_3 = \varnothing$, and $A_1 \cup A_2 \cup A_3 = X$.

(c) A neutrosophic crisp set of Type 3 (NCS-Type 3) if satisfying

$$A_1 \cap A_2 \cap A_3 = \varnothing \text{ and } A_1 \cup A_2 \cup A_3 = X.$$

Remark 1.2. In this paper, the set of all neutrosophic crisp sets of Type 1, Type 2 and Type 3 of X will be denoted NCS1(X), NCS2(X) and NCS3(X), respectively.

Definition 1.3 ([9, 10]) Let $A=\langle A_1, A_2, A_3\rangle$ be a NCS in X, then the complement of the set A may be defined as three kinds of complements:

(C1) $A^{C1}=\langle A_1{}^C, A_2{}^C, A_3{}^C\rangle$, where $A_1{}^C$, $A_2{}^C$ and $A_3{}^C$ are the complement of the set $A_1{}^C$, $A_2{}^C$ and A_3;

(C2) $A^{C2}=\langle A_3, A_2, A_1\rangle$;

(C3) $A^{C3}=\langle A_3, A_2{}^C, A_1\rangle$, where $A_2{}^C$ is the complement of the set A_2.

Definition 1.4 ([9, 10]) Let X be a non-empty set, and the NCSs A and B be in the form $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle$. The inclusion relation may be defined as two types:

Type 1. $A \subseteq_1 B$ if and only if $A_1 \subseteq B_1$, $A_2 \subseteq B_2$ and $A_3 \supseteq B_3$;

Type 2. $A \subseteq_2 B$ if and only if $A_1 \subseteq B_1$, $A_2 \supseteq B_2$ and $A_3 \supseteq B_3$.

Definition 1.5 ([9, 10, 14]) Let X be a non-empty set, and the NCSs A and B be of the form $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle$. Then

(1) the intersection of A and B may be defined as two types:

Type 1. $A \cap_1 B = \langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3\rangle$;

Type 2. $A \cap_2 B = \langle A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3\rangle$.

(2) the union of A and B may be defined as two types:

Type 1. $A \cup_1 B = \langle A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3\rangle$;

Type 2. $A \cup_2 B = \langle A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3\rangle$.

Remark 1.3. In the papers [9, 10], the union with type 1 of A and B is written by $A \cup_1 B = \langle A_1 \cup B_1, A_2 \cap B_2, A_3 \cup B_3\rangle$. We think that this is a typographical error. In [14], the operation has been changed to the above definition.

Definition 1.6 ([9, 10]) Let X be a non-empty set. Then

(1) φ_N may be defined as the following four types:

(a) Type 1: $\varphi_{N1}=\langle \varnothing, \varnothing, X\rangle$,

(b) Type 2: $\varphi_{N2}=\langle \varnothing, X, X\rangle$,

(c) Type 3: $\varphi_{N3}=\langle \varnothing, X, \varnothing\rangle$,

(d) Type 4: $\varphi_{N4}=\langle \varnothing, \varnothing, \varnothing\rangle$.

(2) X_N may be defined as the following four types:

(a) Type 1: $X_{N1}=\langle X, \varnothing, \varnothing\rangle$,

(b) Type 2: $X_{N2}=\langle X, X, \varnothing\rangle$,

(c) Type 3: $X_{N3}=\langle X, \varnothing, X\rangle$,

(d) Type 4: $X_{N4}=\langle X, X, X\rangle$.

Now, we introduce some basic concepts in mathematical morphology. Consider the space $E = \mathbf{R}^n$ or $\boldsymbol{Z}^n$, with origin $\boldsymbol{O}$ = (0, ..., 0). Given $A \subseteq E$, the complement of $A \subseteq E$ is $A^C = E \backslash A$, and the transpose or symmetrical of A is $A^\vee = \{-x \mid x \in A\}$. For every $p \in E$, the translation by p is the map $E \to E$: $x \to x+p$; it transforms any subset A of E into its translate by p, $A_p = \{x+p \mid x \in A\}$. Most morphological operations on sets can be obtained by combining set-theoretical operations with two basic operators, dilation and erosion. The latter arise from two set-theoretical operations, the Minkowski addition $\oplus$ (Minkowski, 1903) and subtraction Θ (Hadwiger,1950), defined as follows for any A, $B \in P(E)$:

$$A \oplus B = \bigcup_{b \in B} A_b = \bigcup_{a \in A} B_a = \{a+b \mid a \in A, b \in B\}.$$

$$A \Theta B = \bigcap_{b \in B} A_{-b} = \{p \in E \mid B_p \subseteq A\}.$$

Formally speaking, A and B play similar roles as binary operands. However, in real situations, A will stand for the image (which is big, and given by the problem), and B for the structuring element (a small shape chosen by the user), so that $A \oplus B$ and $A \Theta B$ will be transformed images.

Definition 1.7 ([11]) We define the dilation by B, δ_B: $P(E) \to P(E)$; $A \to A \oplus B$, and the erosion by B, ε_B: $P(E) \to P(E)$; $A \to A \Theta B$.

It should be noted that dilation and erosion are dual by complementation, in other words dilating a set is equivalent to eroding its complement with the symmetrical structuring element:

$$(A \oplus B)^C = A^C \Theta \check{B}; \ (A \Theta B)^C = A^C \oplus \check{B}.$$

Definition 1.8 ([14]) Let $X = \mathbf{R}^n$ or $\boldsymbol{Z}^n$, A, $B \in \mathrm{NCS}(X)$. Then we define two types of the neutrosophic crisp dilation as follows:

Type 1: $A \oplus_1 B = \langle A_1 \oplus B_1, A_2 \oplus B_2, A_3 \Theta B_3 \rangle$;

Type 2: $A \oplus_2 B = \langle A_1 \oplus B_1, A_2 \Theta B_2, A_3 \Theta B_3 \rangle$.

Definition 1.9 ([14]) Let $X = \mathbf{R}^n$ or $\boldsymbol{Z}^n$, A, $B \in \mathrm{NCS}(X)$. Then we define two types of the neutrosophic crisp erosion as follows:

Type 1: $A \Theta_1 B = \langle A_1 \Theta B_1, A_2 \Theta B_2, A_3 \oplus B_3 \rangle$;

Type 2: $A \Theta_2 B = \langle A_1 \Theta B_1, A_2 \oplus B_2, A_3 \oplus B_3 \rangle$.

2. On Some Operations of Neutrosophic Crisp Sets

For NCS-Type 3, the complement of $A = \langle A_1, A_2, A_3 \rangle \in \mathrm{NCS3}(X)$ of type 3 in [9] (Definition 3.5) or [10] (Definition 1.1.10) is defined as following:

$$A^{C3} = \langle A_3, A_2^{\,C}, A_1 \rangle, \text{ where } A_2^{\,C} \text{ is the complement of the set } A_2.$$

The following example shows that the definition above is not well, since A^{C3} may be not an NCS-Type 3.

Example 2.1 Let $X = \{a, b, c, d, e, f, g\}$, $A = \langle \{a, b, c\}, \{b, c, d, e\}, \{a, e, f, g\} \rangle$. Then A is an NCS- Type 3 in X, that is, $A \in \mathrm{NCS3}(X)$. But

$$A^{C3} = \langle \{a, e, f, g\}, \{a, f, g\}, \{a, b, c\} \rangle \notin \mathrm{NCS3}(X), \text{ since } \{a, e, f, g\} \cap \{a, f, g\} \cap \{a, b, c\} = \{a\} \neq \varnothing.$$

Moreover, the intersection and union operations of neutrosophic crisp sets are not applied to NCS-Type 2 and NCS-Type 3, since they are not closed, and some counterexamples are shown as follows.

Example 2.2 Let $X = \{a, b, c, d, e\}$, $A = \langle \{a, b\}, \{c, d\}, \{e\} \rangle$, $B = \langle \{a\}, \{b, c\}, \{d, e\} \rangle$. Then A, $B \in \mathrm{NCS2}(X)$. But

Type 1. $A \cap_1 B = \langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3 \rangle = \langle \{a\}, \{c\}, \{d, e\} \rangle \notin \mathrm{NCS2}(X)$;

Type 2. $A \cap_2 B = \langle A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3 \rangle = \langle \{a\}, \{b, c, d\}, \{d, e\} \rangle \notin \mathrm{NCS2}(X)$.

Example 2.3 Let $X = \{a, b, c, d, e, f\}$, $A = \langle \{a, b, f\}, \{c, d\}, \{d, e, f\} \rangle$, $B = \langle \{a, f\}, \{b, f\}, \{c, d, e\} \rangle$. Then A, $B \in \mathrm{NCS3}(X)$. But

Type 1. $A \cap_1 B = \langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3 \rangle = \langle \{a, f\}, \varnothing, \{c, d, e, f\} \rangle \notin \mathrm{NCS3}(X)$;

Type 2. $A \cap_2 B = \langle A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3 \rangle = \langle \{a, f\}, \{b, c, d, f\}, \{c, d, e, f\} \rangle \notin \mathrm{NCS3}(X)$.

Example 2.4 Let $X = \{a, b, c, d, e\}$, $A = \langle \{a, b\}, \{c, d\}, \{e\} \rangle$, $B = \langle \{a\}, \{b, c\}, \{d, e\} \rangle$. Then A, $B \in \mathrm{NCS2}(X)$. But

Type 1. $A \cup_1 B = \langle A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3 \rangle = \langle \{a, b\}, \{b, c, d\}, \{e\} \rangle \notin \mathrm{NCS2}(X)$;

Type 2. $A \cup_2 B = \langle A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3 \rangle = \langle \{a, b\}, \{c\}, \{e\} \rangle \notin \mathrm{NCS2}(X)$.

Example 2.5 Let $X = \{a, b, c, d, e, f\}$, $A = \langle \{a, b, f\}, \{c, d\}, \{b, e, f\} \rangle$, $B = \langle \{a, f\}, \{b, f\}, \{b, c, d, e\} \rangle$. Then A, $B \in \mathrm{NCS3}(X)$. But

Type 1. $A \cup_1 B = \langle A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3 \rangle = \langle \{a, b, f\}, \{b, c, d, f\}, \{b, e\} \rangle \notin \mathrm{NCS3}(X)$;

Type 2. $A \cup_2 B = \langle A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3 \rangle = \langle \{a, b, f\}, \varnothing, \{b, e\} \rangle \notin \mathrm{NCS3}(X)$.

In [10] (Proposition 1.1.1), the authors show that $\varphi_N \subseteq A$ and $A \subseteq X_N$ where $A \in \mathrm{NCS}(X)$. But the following examples show that this result is not true.

Example 2.6 Let $X = \{1, 2, 3, 4, 5\}$, $A=\langle\{1, 2, 3\}, \{4\}, \{5\}\rangle$. Then
$\varphi_{N1}=\langle\varnothing, \varnothing, X\rangle \not\subset_2 A$, $A \not\subset_1 X_{N1}=\langle X, \varnothing, \varnothing\rangle$;
$\varphi_{N2}=\langle\varnothing, X, X\rangle \not\subset_1 A$, $A \not\subset_1 X_{N2}=\langle X, X, \varnothing\rangle$;
$\varphi_{N3}=\langle\varnothing, X, \varnothing\rangle \not\subset_1 A$, $\varphi_{N3}=\langle\varnothing, X, \varnothing\rangle \not\subset_2 A$, $A \not\subset_1 X_{N3}=\langle X, \varnothing, X\rangle$, $A \not\subset_2 X_{N3}=\langle X, \varnothing, X\rangle$;
$\varphi_{N4}=\langle\varnothing, \varnothing, \varnothing\rangle \not\subset_1 A$, $\varphi_{N4}=\langle\varnothing, \varnothing, \varnothing\rangle \not\subset_2 A$, $A \not\subset_1 X_{N4}=\langle X, X, X\rangle$, $A \not\subset_2 X_{N4}=\langle X, X, X\rangle$.

Proposition 1.1.1 in [10] should be revised to the following result (the proof is omitted).

Proposition 2.1 Let $A=\langle A_1, A_2, A_3\rangle$ be an NCS in X, then
(1) $\varphi_{N1}=\langle\varnothing, \varnothing, X\rangle \subseteq_1 A$, and $A \subseteq_1 X_{N2}=\langle X, X, \varnothing\rangle$;
(2) $\varphi_{N2}=\langle\varnothing, X, X\rangle \subseteq_2 A$, and $A \subseteq_2 X_{N1}=\langle X, \varnothing, \varnothing\rangle$.

In [10] (Proposition 1.1.2), the authors show that De Morgan law hold for neutrosophic crisp sets. But the following examples show that this result is not true.

Example 2.7 Let $X = \{1, 2, 3, 4, 5\}$, $A=\langle\{1, 2, 3\}, \{4\}, \{5\}\rangle$, $B=\langle\{1, 2\}, \{3, 4\}, \{5\}\rangle$. Then
$(A\cap_1 B)^{C2} = \langle\{1, 2\}, \{4\}, \{5\}\rangle^{C2} = \langle\{5\}, \{4\}, \{1, 2\}\rangle$,
$A^{C2} \cup_1 B^{C2} = \langle\{5\}, \{4\}, \{1, 2, 3\}\rangle \cup_1 \langle\{5\}, \{3, 4\}, \{1, 2\}\rangle = \langle\{5\}, \{3, 4\}, \{1, 2\}\rangle \neq (A\cap_1 B)^{C2}$.

Example 2.8 Let $X = \{1, 2, 3, 4, 5\}$, $A=\langle\{1, 2, 3\}, \{4, 5\}, \{5\}\rangle$, $B=\langle\{1\}, \varnothing, \{2\}\rangle$. Then
$(A\cap_2 B)^{C1}= \langle\{1\}, \{4, 5\}, \{2, 5\}\rangle^{C1} = \langle\{2, 3, 4, 5\}, \{1, 2, 3\}, \{1, 3, 4\}\rangle$,
$A^{C1} \cup_2 B^{C1} = \langle\{4, 5\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\rangle \cup_1 \langle\{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}, \{1, 3, 4, 5\}\rangle = \langle\{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}, \{1, 3, 4\}\rangle \neq (A\cap_2 B)^{C1}$.

Proposition 1.1.2 in [10] should be revised to the following assertion.

Proposition 2.2 Let X be a non-empty set, and the NCSs A and B be of the form $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle$. Then
(1) $(A\cap_1 B)^{C1} = A^{C1} \cup_1 B^{C1}$, and $(A\cup_1 B)^{C1} = A^{C1}\cap_1 B^{C1}$;
(2) $(A\cap_1 B)^{C3} = A^{C3} \cup_1 B^{C3}$, and $(A\cup_1 B)^{C3} = A^{C3}\cap_1 B^{C3}$;
(3) $(A\cap_2 B)^{C2} = A^{C2} \cup_1 B^{C2}$, and $(A\cup_2 B)^{C2} = A^{C2}\cap_1 B^{C2}$;
(4) $(A\cap_2 B)^{C3} = A^{C3} \cup_2 B^{C3}$, and $(A\cup_2 B)^{C3} = A^{C3}\cap_2 B^{C3}$.

Proof. (1) By the definitions of the complement ^{C1}, intersection and union (see Definition 1.3 and Definition 1.5) we have
$(A\cap_1 B)^{C1}$
$= \langle A_1\cap B_1, A_2\cap B_2, A_3\cup B_3\rangle^{C1}$
$= \langle (A_1\cap B_1)^C, (A_2\cap B_2)^C, (A_3\cup B_3)^C\rangle$
$= \langle A_1^C\cup B_1^C, A_2^C\cup B_2^C, A_3^C \cap B_3^C\rangle$
$= \langle A_1^C, A_2^C, A_3^C\rangle \cup_1 \langle B_1^C, B_2^C, B_3^C\rangle$
$= \langle A_1, A_2, A_3\rangle^{C1} \cup_1 \langle B_1, B_2, B_3\rangle^{C1}$
$= A^{C1} \cup_1 B^{C1}$.
Similarly, we can get that $(A\cup_1 B)^{C1} = A^{C1}\cap_1 B^{C1}$.
(2) By the definitions of the complement ^{C3}, intersection and union (see Definition 1.3 and Definition 1.5) we have
$(A\cap_1 B)^{C3}$
$= \langle A_1\cap B_1, A_2\cap B_2, A_3\cup B_3\rangle^{C3}$
$= \langle A_3\cup B_3, (A_2\cap B_2)^C, A_1\cap B_1\rangle$
$= \langle A_3\cup B_3, A_2^C\cup B_2^C, A_1\cap B_1\rangle$
$= \langle A_3, A_2^C, A_1\rangle \cup_1 \langle B_3, B_2^C, B_1\rangle$
$= \langle A_1, A_2, A_3\rangle^{C3} \cup_1 \langle B_1, B_2, B_3\rangle^{C3}$
$= A^{C3} \cup_1 B^{C3}$.

Similarly, we can get that $(A\cup_1 B)^{C3} = A^{C3}\cap_1 B^{C3}$.

(3) By the definitions of the complement ^{C2}, intersection and union (see Definition 1.3 and Definition 1.5) we have

$(A\cap_2 B)^{C2}$
$= \langle A_1\cap B_1, A_2\cup B_2, A_3\cup B_3\rangle^{C2}$
$= \langle A_3\cup B_3, A_2\cup B_2, A_1\cap B_1\rangle$
$= \langle A_3, A_2, A_1\rangle \cup_1 \langle B_3, B_2, B_1\rangle$
$= \langle A_1, A_2, A_3\rangle^{C2} \cup_1 \langle B_1, B_2, B_3\rangle^{C2}$
$= A^{C2} \cup_1 B^{C2}$;
$(A\cup_2 B)^{C2}$
$= \langle A_1\cup B_1, A_2\cap B_2, A_3\cap B_3\rangle^{C2}$
$= \langle A_3\cap B_3, A_2\cap B_2, A_1\cup B_1\rangle$
$= \langle A_3, A_2, A_1\rangle \cap_1 \langle B_3, B_2, B_1\rangle$
$= \langle A_1, A_2, A_3\rangle^{C2} \cap_1 \langle B_1, B_2, B_3\rangle^{C2}$
$= A^{C2} \cap_1 B^{C2}$.

(4) By the definitions of the complement ^{C3}, intersection and union (see Definition 1.3 and Definition 1.5) we have

$(A\cap_2 B)^{C3}$
$= \langle A_1\cap B_1, A_2\cup B_2, A_3\cup B_3\rangle^{C3}$
$= \langle A_3\cup B_3, (A_2\cup B_2)^C, A_1\cap B_1\rangle$
$= \langle A_3\cup B_3, A_2^C\cap B_2^C, A_1\cap B_1\rangle$
$= \langle A_3, A_2^C, A_1\rangle \cup_2 \langle B_3, B_2^C, B_1\rangle$
$= \langle A_1, A_2, A_3\rangle^{C3} \cup_2 \langle B_1, B_2, B_3\rangle^{C3}$
$= A^{C3} \cup_2 B^{C3}$;
$(A\cup_2 B)^{C3}$
$= \langle A_1\cup B_1, A_2\cap B_2, A_3\cap B_3\rangle^{C3}$
$= \langle A_3\cap B_3, (A_2\cap B_2)^C, A_1\cup B_1\rangle$
$= \langle A_3\cap B_3, A_2^C\cup B_2^C, A_1\cup B_1\rangle$
$= \langle A_3, A_2^C, A_1\rangle \cap_2 \langle B_3, B_2^C, B_1\rangle$
$= \langle A_1, A_2, A_3\rangle^{C3} \cap_2 \langle B_1, B_2, B_3\rangle^{C3}$
$= A^{C3} \cap_2 B^{C3}$.

Next, we give some new operations on neutrosophic crisp sets.

Definition 2.1 [18] Let X be a non-empty set, and the NCSs A and B be of the form $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle$. Then we can define the intersection and union with type 3 as follows:

Type 3. $A\cap_3 B = \langle A_1\cap B_1, A_2\cap B_2, A_3\cap B_3\rangle$;
Type 3. $A\cup_3 B = \langle A_1\cup B_1, A_2\cup B_2, A_3\cup B_3\rangle$.

Definition 2.2 Let X be a non-empty set, and $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle\in$NCS2($X$) or NCS3($X$). Then we can define the intersection and union with star * as follows:

$A\cap^* B = \langle A_1\cap B_1, X-(A_1\cap B_1)\cup(A_3\cup B_3), A_3\cup B_3\rangle$;
$A\cup^* B = \langle A_1\cup B_1, X-(A_1\cup B_1)\cup(A_3\cap B_3), A_3\cap B_3\rangle$.

We can easily verify that the following asserts are true (the proofs are omitted).

Proposition 2.3 Let $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle$ be two NCSs in X, then

(1) $(A\cap_3 B)^{C1} = A^{C1} \cup_3 B^{C1}$, and $(A\cup_3 B)^{C1} = A^{C1} \cap_1 B^{C1}$;
(2) $(A\cap_3 B)^{C2} = A^{C2} \cap_3 B^{C2}$, and $(A\cup_3 B)^{C2} = A^{C2} \cup_3 B^{C2}$.

Proposition 2.4 Let X be a non-empty set, and $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle\in$NCS2($X$) or NCS3($X$). Then $A\cap_* B$, $A\cup_* B\in$NCS2(X) or NCS3(X), and

(1) $(A\cap^* B)^{C1} = A^{C1} \cup^* B^{C1}$, and $(A\cup^* B)^{C1} = A^{C1} \cap^* B^{C1}$;
(2) $(A\cap^* B)^{C2} = A^{C2} \cup^* B^{C2}$, and $(A\cup^* B)^{C2} = A^{C2} \cap^* B^{C2}$.

3. On Neutrosophic Crisp Mathematical Morphology and Applications

In this section, we firstly give the new definitions of neutrosophic crisp dilation and erosion, and then applies them to the edge segmentation of color images. It should be noted that the neutrosophic crisp morphology operations can only be applied to binary image processing before, and our innovative method is as follows: (1) the color image is divided into three grayscale images according to three color channels (R, G, B); (2) three grayscale images are converted to binary value images, respectively; (3) the neutrosophic crisp dilation and erosion operations are applied to them respectively (we use three kinds of operations for comparison); combine the results of the three color channels to obtain the binary value edges of the original color image.

Definition 3.1 Let $X = \mathbf{R}^n$ or $\boldsymbol{Z}^n$, A, $B\in$NCS(X). Then we define new neutrosophic crisp dilation and erosion as follows:

$A\oplus_3 B = \langle A_1\oplus B_1, A_2\oplus B_2, A_3\oplus B_3\rangle$;

$A\Theta_3 B = \langle A_1\Theta B_1, A_2\Theta B_2, A_3\Theta B_3\rangle$.

Remark 3.1. In this paper, for binary value image (as a multidimensional vector), the operations "x+y" and "x−y" will be replaced by "max{x, y}" and "min{x, 1−y}", respectively.

Now, we apply three different neutrosophic crisp morphological operators (see Definition 1.8, Definition 1.9 and Definition 3.1) to extract the edges of the color image (as shown in the figure 1).

Figure 1. The original color image.

First, the RGB three channels of the original image are separated into three grayscale images. We use Python program as follows, and the separation results are shown in Figure 2 (a), (b) and (c).

Python Program 3.1

```
from PIL import Image                     # import Image class , from PIL package
from matplotlib import pyplot as plt      # import pyplot class , rename it ply
import cv2                                # import the cv2 package
import numpy as np                        # import the numpy package and rename it np
img1 = plt.imread('yellow_duck.jpg')      # read the picture to be used
red = img1[:,:,0]                         # get the red channel of the picture
green = img1[:,:,1]                       # get the green channel of the picture
blue = img1[:,:,2]                        # get the blue channel of the picture
```

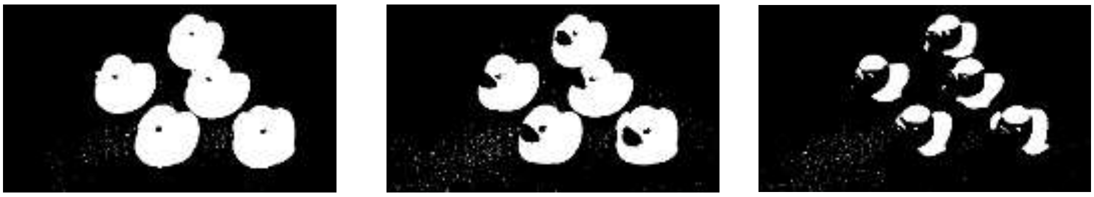
(**a**) (**b**) (**c**)

Figure 2. Separated three grayscale images: (**a**) R-channel; (**b**) G-channel; (**b**) B-channel.

Second, binarize the above three gray images to obtain three black and white images, see following Python program and Figure 3 (a), (b) and (c).

Python Program 3.2

```
# Define a function to convert grayscale image to binary image
def threshold(img, Maxvalue,choice):
    array=(cv2.THRESH_BINARY,cv2.THRESH_BINARY_INV,cv2.THRESH_TRUNC,cv2.THRESH_TOZERO,cv2.THRESH_TOZERO_INV,cv2.THRESH_BINARY)
    ret, binary = cv2.threshold(img, Maxvalue, 255, array[choice])
    return binary
# call threshold function to convert the grayscale images of the three channel into binary images respectively.
A1 = threshold(red,90,0)
A2 = threshold(green,130,0)
A3 = threshold(blue,90,0)
```



(**a**) A_1 (**b**) A_2 (**c**) A_3

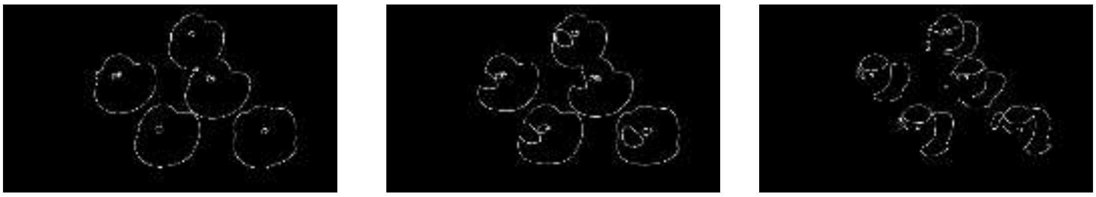
Figure 3. The binarization results of the three gray images: (**a**) A_1; (**b**) A_2; (**b**) A_3.

Third, by Definition 3.1, we obtain the binary value edges of the three channels, that is, putting $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle$, where $B_1=B_2=B_3=$ (1, 1, 1, 1, 1; 1, 1, 1, 1, 1; 1, 1, 1, 1, 1; 1, 1, 1, 1, 1; 1, 1, 1, 1, 1), $A\Theta_3 B=\langle A_1\Theta B_1, A_2\Theta B_2, A_3\Theta B_3\rangle$. $C_1= A_1-A_1\Theta B_1$, $C_2= A_2-A_2\Theta B_2$, $C_3= A_3-A_3\Theta B_3$, see following Figure 4 (a), (b) and (c). And, putting $D= C_1 +C_2 +C_3$, obtain the binary value edges of the original color image, see following Python program and Figure 5.,

Python Program 3.3

```
# Define a function to achieve the operation "x+y" replaced by "max{x, y}"
def Plus(C1,C2,C3):
    edge= C1+C2+C3
    for i in range(edge.shape[0]):
        for j in range(edge.shape[1]):
            if (edge[i, j]) >= 255:
                edge[i, j]=255
    return edge
# Define a function to achieve the dilation operation.
def dilate(binary, kernel):
    Kernel_Dilate = np.ones((kernel.shape[0], kernel.shape[1]), np.uint8)
    for i in range(kernel.shape[0]):
        for j in range(kernel.shape[1]):
            Kernel_Dilate[i,j] = kernel[i,j]
    return cv2.dilate(binary, Kernel_Dilate)
# Define a function to achieve the erosion operation
def erode(binary, kernel):
    Kernel_Erode = np.ones((kernel.shape[0], kernel.shape[1]), np.uint8)
    for i in range(kernel.shape[0]):
        for j in range(kernel.shape[1]):
```

```
            Kernel_Erode[i, j] = kernel[i, j]
    return cv2.erode(binary, Kernel_Erode)
B1 = np.ones((5,5), np.uint8)          # define an one matrix with five rows and five columns
C1 = erode(A1,np.array(B1))            # C1=A1ΘB1
C1 = A1- C1                            # C1= A1–A1ΘB1
C2 = erode(A2,np.array(B1))            # C2=A2ΘB2
C2 = A2- C2                            # C2= A2–A2ΘB2
C3 = erode(A3,np.array(B1))            # C3=A3ΘB3
C3 = A3- C3                            # C3= A3–A3ΘB3
D=Plus(C1,C2,C3)                       # D1= C1 +C2 +C3
```



(**a**) C_1 (**b**) C_2 (**c**) C_3

Figure 4. (**a**) the edge of image A_1; (**b**) the edge of image A_2; (**c**) the edge of image A_3.

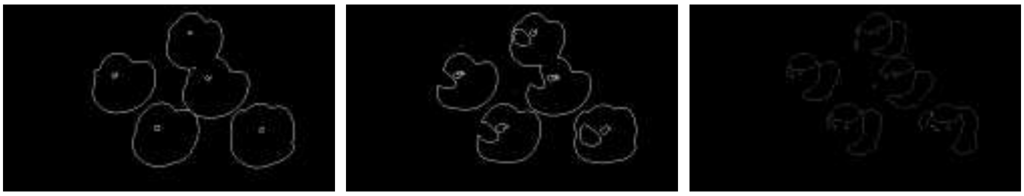
Figure 5. Merged edges of the original image (D).

Next, we can apply Definition 1.8 (Type 2) to give another edge extraction method similar to the above method. Putting $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle$, where B_1 = (0, 0, 0, 1, 0, 0, 0; 0, 0, 1, 1, 1, 0, 0; 0, 1, 1, 1, 1, 1, 0; 1, 1, 1, 1, 1, 1, 1; 0, 1, 1, 1, 1, 1, 0; 0, 0, 1, 1, 1, 0, 0; 0, 0, 0, 1, 0, 0, 0), B_2 = (0, 0, 0, 1, 0, 0, 0; 0, 0, 1, 1, 1, 0, 0; 0, 1, 1, 0, 1, 1, 0; 1, 1, 0, 0, 0, 1, 1; 0, 1, 1, 0, 1, 1, 0; 0, 0, 1, 1, 1, 0, 0; 0, 0, 0, 1, 0, 0, 0),B_3 =(0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 1, 0, 0, 0; 0, 0, 1, 1, 1, 0, 0; 0, 0, 0, 1, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0), $A\oplus_2 B=\langle A_1\oplus B_1, A_2\Theta B_2, A_3\Theta B_3\rangle$. $C_1=(A_1\oplus B_1)-A_1$, $C_2= A_2-(A_2\Theta B_2)$, $C_3=A_3-(A_3\Theta B_3)$, see following Figure 6 (a), (b) and (c). And, putting $D_1= C_1+C_2+C_3$, obtain the binary value edges of the original color image, see following Python program and Figure 7.

Python Program 3.4

```
B1 = [[0,0,0,1,0,0,0],                          # define the matrix named B1
      [0,0,1,1,1,0,0],
      [0,1,1,1,1,1,0],
      [1,1,1,1,1,1,1],
      [0,1,1,1,1,1,0],
      [0,0,1,1,1,0,0],
      [0,0,0,1,0,0,0]]
B2 = [[0,0,0,1,0,0,0],                          # define the matrix named B2
      [0,0,1,1,1,0,0],
      [0,1,1,0,1,1,0],
      [1,1,0,0,0,1,1],
      [0,1,1,0,1,1,0],
      [0,0,1,1,1,0,0],
      [0,0,0,1,0,0,0]]
```

```
B3 = [[0,0,0,0,0,0,0],                         # define the matrix named B3
      [0,0,0,0,0,0,0],
      [0,0,0,1,0,0,0],
      [0,0,1,1,1,0,0],
      [0,0,0,1,0,0,0],
      [0,0,0,0,0,0,0],
      [0,0,0,0,0,0,0]]
C1 = dilate(A1, np.array(B1))           # C1=A1⊕B1
C1 = C1- A1                             # C1=(A1⊕B1)–A1
C2 = erode(A2, np.array(B2))            # C2= A2ΘB2
C2 = A2- C2                             # C2= A2–(A2ΘB2)
C3 = erode(A3, np.array(B3))            # C3= A3ΘB3
C3 = A3- C3                             # C3= A3–(A3ΘB3)
D=Plus(C1, C2, C3)                      # D1= C1+ C2+ C3
```



(**a**) C_1 (**b**) C_2 (**c**) C_3

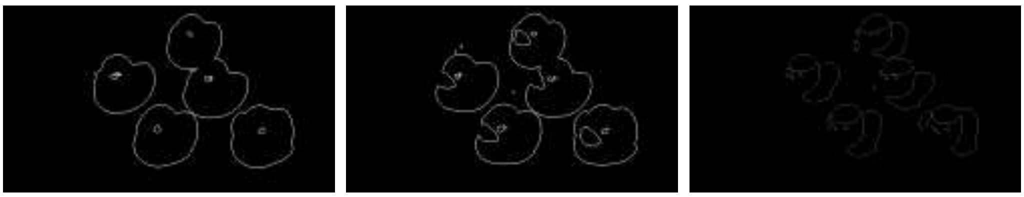
Figure 6. (**a**) the edge of image A_1; (**b**) the edge of image A_2; (**c**) the edge of image A_3.

Figure 7. Merged edges of the original image (D_1).

Finally, we can apply Definition 1.9 (Type 2) to give another edge extraction method similar to the above method. Putting $A=\langle A_1, A_2, A_3\rangle$, $B=\langle B_1, B_2, B_3\rangle$, where B_1, B_2, B_3 not change (see above), $A\Theta_2 B=\langle A_1\Theta B_1, A_2\oplus B_2, A_3\oplus B_3\rangle$. $C_1= A_1-(A_1\Theta B_1)$, $C_2= (A_2\oplus B_2)-A_2$, $C_3= (A_3\oplus B_3)-A_3$, see following Figure 8 (a), (b) and (c). And, putting $D_2= C_1+ C_2+ C_3$, see following Python program and Figure 9.

Python Program 3.5

```
C1 = erode (A1,np.array(B1))            # C1= A1ΘB1
C1 = A1- C1                             # C1= A1–(A1ΘB1)
C2 = dilate (A2,np.array(B2))           # C2= A2⊕B2
C2 = C2- A2                             # C2= (A2⊕B2)–A2
C3 = dilate (A3,np.array(B3))           # C3= A3⊕B3
C3 = C3- A3                             # C3= (A3⊕B3)–A3
D=Plus(C1, C2, C3)                      # D2= C1+ C2+ C3
```



(**a**) C_1 (**b**) C_2 (**c**) C_3

Figure 8. (**a**) the edge of image A_1; (**b**) the edge of image A_2; (**c**) the edge of image A_3.

Figure 9. Merged edges of the original image (D_2).

4. Conclusions

In this paper, some properties of the existing algebraic operations of neutrosophic crisp sets are discussed, and some new operations are given. The results shown that many different algebraic operation systems can be set up for neutrosophic crisp sets, they can be selected according to different applications. Meanwhile, this paper studied the application of neutrosophic crisp mathematical morphology in color image edge extraction, and the experimental results by Python shown that different morphological operators can be selected in this kind of application.

Because the color image binarization processing first in this paper, and then extract the edge by using morphological operator. So, the theory of neutrosophic crisp mathematical morphology need to do further research, so that we can deal directly with gray image or color image by using new morphological operators.

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New Results On Pythagorean Neutrosophic Open Sets in Pythagorean Neutrosophic Topological Spaces

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Abstract. In this paper, we introduce and study the notion of Pythagorean neutrosophic $^{\star}b$-open set on Pythagorean neutrosophic topology. Besides, we define the concepts of Pythagorean neutosophic $^{\star}b$-open function, Pythagorean neutrosophic $^{\star}b$-continuous function and Pythagorean neutrosophic $^{\star}b$-homeomorphism. Moreover, we establish some of their properties and characterizations.

Keywords: Pythagorean neutrosophic $^{\star}b$-open sets, Pythagorean neutosophic $^{\star}b$-open function, Pythagorean neutrosophic $^{\star}b$-continuous function, Pythagorean neutrosophic $^{\star}b$-homeomorphism

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1. **Introduction**

L.A. Zadeh [18] introduced the idea of fuzzy set theory in 1965. Then, Chang [4] defined the concept of fuzzy topological space and generalized some basic notions of topology. Besides, Atanassov [2,3] in 1983 introduced the concept of intuitionistic fuzzy set. Furthermore, the notion of neutrosophic set was introduced by Smarandache [14] and Wang et.al. studied te notion of interval neutrosophic set. Moreover, A.A. Salama and S.A. Albowi [13] defined the concept of crisp set and neutrosophic crisp set topological spaces. In 2013, Yager [17] introduced the concept of Pythagorean membership grades in multicriteria decision making. Later, Yager, Zahand and Xu [16] gave some basic operations for Pythagorean fuzzy number. Taking into account mentioned previously, Iswarya et.al. [11] studied the concept of neutrosophic semi-open sets and neutrosophic semi-closed sets. Further, In 2017, Imran et.al. [10]

introduced neutrosophic semi-α-open sets and studied their fundamental properties. Additionally, Arockiarani et.al. [1] defined the notion of neutrosophic semi-open (resp. pre-open and α-open) functions and investigated their relations. Later, Rao et.al. [15] introduced neutrosophic pre-open sets. Then, P.Evanzalin Ebenanjar et al. [6] defined neutrosophic b-open sets in neutrosophic topological space and investigated their properties. Recently Bromi and Smarandache defined the Haudroff distance between neutrosophic sets and

On the other hand, mathematicians have extended the notion of neutrosophic sets. In 2020, Sneha and Nirmala [12] defined the concept of pythagorean neutrosophic b-open sets and pythagorean enutrosophice semi-open sets and established some properties and notions associated to these sets, additionally they defined some variants of continuity. Simultaneously, Granados [7, 8] defined the concept of pythagorean neutrosophice pre-open sets and showed other related notions about pythagorean neutrosophic sets, furthermore he studied and established new variants of continuity on these sets. In this paper, we use these notions to extend the concept of pythagorean neutrosophic open sets and define a new notion of sets which are called Pythagorean neutrosophic $^\star b$-open set. Besides, we show some of their properties. We also define the concept of Pythagorean neutosophic $^\star b$-open function, Pythagorean neutrosophic $^\star b$-continuous function and Pythagorean neutrosophic $^\star b$-homeomorphism. Moreover, we establish some some of their properties and characterizations. Now, we procure some well-known notions which are useful for the developing of this paper. Let A be a subset of X. Then, we will denote the Pythagorean neutrosophic interior and Pythagorean neutrosophic closure of A as follows: $PNInt(A)$ and $PNCl(A)$, respectively. Additionally, if A is a Pythagorean neutrosophic open set in X, then $PNInt(A) = A$. On the other hand, the complement of a Pythagorean neutrosophic open set is called Pythagorean neutrosophic closed set, moreover if B is a Pythagorean neutrosophic closed set in X, then $PNCl(A) = A$. [12] defined the following concepts: Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic if $f^{-1}(V)$ is a Pythagorean neutrosophic in X for every Pythagorean neutrosophic open set V in Y. Besides, let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic pre-continuous if $f^{-1}(V)$ is a Pythagorean neutrosophic pre-open in X for every Pythagorean neutrosophic open set V in Y.

Definition 1.1. For any Pythagorean neutrosophic set A in a Pythagorean neutrosophic topological space (X, τ), A is said to be Pythagorean neutrosophic pre-open set [7] if $A \subseteq PNInt((PNCl(A))$.

Theorem 1.2. *[7] Every Pythagorean neutrosophic open set is a Pythagorean neutrosophic pre-open set.*

Definition 1.3. [7] Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic pre-open if $f(A)$ is Pythagorean neutrosophic pre-open set in Y for every Pythagorean neutrosophic open set A in X.

2. Pythagorean neutrosophic $^\star b$-open sets

In this section we introduce and study the notion of Pythagorean neutrosophic $^\star b$-open sets and we establish some notions associated to them.

Definition 2.1. Let X be a non-empty set. If a, b, c are real standard or non standard subsets of $]0^-, 1^+[$, then the Pythagorean neutrosophic set $x_{a,b,c}$ is said to be Pythagorean neutrosophic point (or simply, PNP) in X and it is given by:

$$x_{a,b,c}(x_p) = \begin{cases} (a, b, c) & \text{if } x = x_p \\ (0, 0, 1) & \text{if } x \neq x_p \end{cases}$$

For each $x_p \in X$ is said to be the support of $x_{a,b,c}$, where a denotes the degree of membership value, b denotes the degree of indeterminacy and c is the degree of non-membership value of $x_{a,b,c}$.

Definition 2.2. For any Pythagorean neutrosophic set A in a Pythagorean neutrosophic topological space (X, τ), A is said to be Pythagorean neutrosophic $^\star b$-open set (or simply, $PN^\star bOS$) if $A \subseteq PNInt((PNCl(A)) \cap PNCl(PNInt(A))$. The complement of a Pythagorean neutrosophic $^\star b$-open set is called Pythagorean neutrosophic $^\star b$-closed set.

Remark 2.3. The collection of all Pythagorean neutrosophic $^\star b$-open sets and Pythagorean neutrosophic $^\star b$-closed sets are denoted by $PN^\star bOS(X, \tau)$ and $PN^\star bCS(X, \tau)$, respectively.

Proposition 2.4. *Let (X, τ) be a Pythagorean neutrosophic topological space and $A \subseteq X$. Then, If A is a Pythagorean neutrosophic $^\star b$-open set, then A is Pythagorean neutrosophic pre-open set.*

The converse of the above proposition need not be true as can be seen in the following example:

Example 2.5. Let $X = \{q, w, e\}$ and $\tau = \{0_N, A, B, 1_N\}$. Then, $A = \langle(0.4, 0.5, 0.2), (0.3, 0.2, 0.1), (0.9, 0.6, 0.8)\rangle$, $B = \langle(0.2, 0.4, 0.5), (0.1, 0.1, 0.2), (0.6, 0.5, 0.8)\rangle$ and $C = \{(0.5, 0.6, 0.1), (0.4, 0.3, 0.1), (0.9, 0.8, 0.5)\rangle$. Then, we can see that C is a Pythagorean neutrosophic pre-open set, but it is not a Pythagorean neutrosophic $^\star b$-open set.

Definition 2.6. A Pythagorean neutrosophic set V in a Pythagorean neutrosophic topological space (X, τ) is said to be Pythagorean neutrosophic $^\star b$-closed (or simply, $PN^\star bCS$)) if $V \supseteq PNInt(PNCl(V)) \cap PNCl(PNInt(V))$.

Definition 2.7. Let (X, τ) be a Pythagorean neutrosophic topological space and V be a Pythagorean neutrosophic set on X. Then we define the Pythagorean neutrosophic $^\star b$-interior and Pythagorean neutrosophic $^\star b$-closure of V as:

(1) Pythagorean neutrosophic $^\star b$-interior of V (or simply, $PN^\star BINT(V)$) as the union of all Pythagorean neutrosophic $^\star b$-open sets of X contained in V. It means that $PN^\star BINT(V) = \bigcup\{A : A \text{ is a } PN^\star bOS \text{ in } X \text{ and } A \subseteq V\}$.

(2) Pythagorean neutrosophic $^\star b$-closure of V (or simply, $PN^\star BCL(V)$) as the intersection of all Pythagorean neutrosophic $^\star b$-closed set of X containing V. It means that $PN^\star BCL(V) = \bigcap\{B : B \text{ is a } PN^\star bCS \text{ in } X \text{ and } V \subseteq B\}$.

Remark 2.8. By the Definition 2.7, we can see that $PN^\star BCL(V)$ is the smallest Pythagorean neutrosophic $^\star b$-closed set of X which contains V. Besides, $PN^\star BINT(V)$ is the largest Pythagorean neutrosophic $^\star b$-open set of X which is contained in V.

Proposition 2.9. *Let V be a Pythagorean neutrosophic set in a Pythagorean neutrosophic topological space (X, τ). Then, the following statements hold:*

(1) *If V is Pythagorean neutrosophic $^\star b$-open set, then $Cl(V)$ is is a Pythagorean neutrosophic $^\star b$-closed set.*

(2) *If V is Pythagorean neutrosophic $^\star b$-closed set, then $Cl(V)$ is is a Pythagorean neutrosophic $^\star b$-open set.*

Proof: The proof is followed by the Definitions 2.2, 2.6 and 2.7.

Theorem 2.10. *Let V be a Pythagorean neutrosophic set in a Pythagorean neutrosophic topological space (X, τ). Then, the following statements hold:*

(1) $Cl(PN^\star BINT(V)) = PN^\star BCL(Cl(V))$.

(2) $Cl(PN^\star BCL(V)) = PN^\star BINT(Cl(V))$.

Proof: We begin proving (1): Let V be a Pythagorean neutrosophic set. Now, by the Definition 2.7 part (1), $PN^\star BINT(V) = \bigcup\{A : A \text{ is a } PN^\star bOS \text{ in } X \text{ and } A \subseteq V\}$, this implies that $Cl(PN^\star BINT(V)) = Cl(\bigcup\{A : A \text{ is a } PN^\star bOS \text{ in } X \text{ and } A \subseteq V\})=$ $\bigcap\{Cl(A) : Cl(A) \text{ is a } PN^\star bCS \text{ in } X \text{ and } Cl(V) \subseteq Cl(A)\}$. Now, we will replace $Cl(A)$ by B, then we have that $Cl(PN^\star BINT(V) = \bigcap\{B : B \text{ is a } PN^\star bCS \text{ in } X \text{ and } Cl(V) \subseteq B\}$, and so $Cl(PN^\star BINT(V)) = PN^\star BCL(Cl(V))$.

The proof of (2) is made similarly to (1).

Theorem 2.11. *For a Pythagorean neutrosophic topological space (X, τ) and $A, B \subseteq X$. The following statements hold:*

(1) *Every Pythagorean neutrosophic set is Pythagorean neutrosophic $\star$b-open set.*

(2) $PN^\star BINT(PN^\star BINT(A)) = PN^\star BINT(A)$.

(3) $PN^\star BCL(PN^\star BCL((A)) = PN^\star BCL(A)$.

(4) *Let* A, B *be two Pythagorean neutrosophic $\star$b-open sets, then* $PN^\star bOS(A) \cup PN^\star bOS(B) = PN^\star bOS(A \cup B)$.

(5) *Let* A, B *be two Pythagorean neutrosophic $\star$b-closed sets, then* $PN^\star bCS(A) \cap PN^\star bCS(B) = PN^\star bCS(A \cap B)$.

(6) *For any two sets* A, B, $PN^\star BINT(A) \cap PN^\star BINT(B) = PN^\star BInt(A \cap B)$.

(7) *For any two sets* A, B, $PN^\star BCL(A) \cup PN^\star BCL(B) = PN^\star BCL(A \cup B)$.

(8) *If* A *is* $PN^\star bOS(X, \tau)$, *then* $A = PN^\star BINT(A)$.

(9) *If* $A \subseteq B$, *then* $PN^\star BINT(A) \subseteq PN^\star BINT(B)$.

(10) *For any two sets* A, B, $PN^\star BINT(A) \cup PN^\star BINT(B) \subseteq PN^\star BINT(A \cup B)$.

(11) *If* A *is* $PN^\star bCS(X, \tau)$, *then* $A = PN^\star BCL(A)$.

(12) *If* $A \subseteq B$, *then* $PN^\star BCL(A) \subseteq PN^\star BCL(B)$.

(13) *For any two sets* A, B, $PN^\star BCL(A \cap B) \subseteq PN^\star BCL(A) \cap PN^\star BCL(B)$.

Proof: The proofs of (1), (2), (3), (4), (5), (9), (11) and (12) are followed by the Definitions 2.2 and 2.6. The proofs of (6), (7) and (8) are followed by the Definition 2.7 and the proofs of (10) and (13) are followed by the Definition 2.7 and parts (9) and (12) of this Theorem.

The following example shows that the equality (10) need not be hold in Theorem 2.11 .

Example 2.12. Let $X = \{q, w, e\{$ and $\tau = \{0_N$, A, B, C, D, $1_N\}$ where $A = \langle($ 0.4, 0.7, 0.1), (0.5, 0.6, 0.2), (0.9, 0.7, 0.3 $)\rangle$, $B = \langle($ 0.4, 0.6, 0.1), (0.7, 0.7, 0.2), (0.9, 0.5, 0.1) $\rangle$, $C = \langle($ 0.4, 0.7, 0.1), (0.7, 0.7, 0.2), (0.9, 0.7, 0.1 $)\rangle$, $D = \langle($ 0.4, 0.6, 0.1), (0.5, 0.6, 0.2), (0.9, 0.5, 0.3) $\rangle$. Then, τ is Pythagorean neutrosophic topological space. Now, Consider $E = \langle($ 0.7, 0.6, 0.1), (0.7, 0.6, 0.1), (0.9, 0.5, 0 $)\rangle$ and $F = \langle($ 0.4, 0.6, 0.1), (0.5, 0.7, 0.2), (1, 0.7, 0.1)$\rangle$. Then, $PN^\star BINT(E) = D$ and $PN^\star BINT(F) = D$. This implies that $PN^\star BINT(E) \cup PN^\star BINT(F) = D$. Now, $E \cup F = \langle($ 0.7, 0.6, 0.1), (0.7, 0.7, 0.1), (1, 0.7, 0) $\rangle$, it follows that $PN^\star BINT(E \cup F) = B$. Then, $PN^\star BINT(E \cup F) \nsubseteq PN^\star BINT(E) \cup PN^\star BINT(F)$.

The following example shows that the equality (13) need not be hold in Theorem 2.11 .

Example 2.13. Let $X = \{q, w, e\}$, $\tau = \{0_N$, A, B, C, D, $1_N\}$ and $C_\tau = \{1_N$, E, F, G, H, $0_N\}$ where $A = \langle($ 0.5, 0.6, 0.1), (0.6, 0.7, 0.1), (0.9, 0.5, 0.2 $)\rangle$, $B = \langle($ 0.4, 0.5, 0.2), (0.8, 0.6, 0.3), (0.9, 0.7, 0.3 $)\rangle$, $C = \langle($ 0.4, 0.5, 0.2), (0.6, 0.6, 0.3), (0.9, 0.5, 0.3 $)\rangle$, $D = \langle($ 0.5, 0.6, 0.1), (0.8, 0.7, 0.1), (0.9, 0.7, 0.2 $)\rangle$, $E = \langle($ 0.1, 0.4, 0.5), (0.1, 0.3, 0.6), (0.2, 0.5, 0.9 $)\rangle$, $F = \langle($ 0.2, 0.5, 0.4), (0.3, 0.4, 0.8), (0.3, 0.3, 0.9 $)\rangle$, $G = \langle($ 0.2, 0.5, 0.4), (0.3, 0.4, 0.6), (

0.3, 0.5, 0.9)⟩, $H = \langle(0.1, 0.4, 0.5), (0.1, 0.3, 0.8), (0.2, 0.3, 0.9)\rangle$. Then τ is Pythagorean neutrosophic topological space. Now, Consider $I = \langle(0.1, 0.2, 0.5), (0.2, 0.3, 0.7), (0.3, 0.3, 1)\rangle$ and $J = \langle(0.2, 0.4, 0.8), (0.1, 0.2, 0.8), (0.2, 0.5, 0.9)\rangle$. Then $PN^{\star}BCL(I) = G$ and $PN^{\star}BCL(J) = G$. This implies that $PN^{\star}BCL(I) \cap PN^{\star}BCL(J) = G$. Now, $I \cap J = \langle(0.1, 0.2, 0.8), (0.1, 0.2, 0.8), (0.2, 0.3, 1)\rangle$, it follows that $PN^{\star}BCL(I \cap G) = H$. Then $PN^{\star}BCL(I) \cap PN^{\star}BCL(J) \nsubseteq PN^{\star}BCL(I \cap G)$.

The following example shows that the intersection of two Pythagorean neutrosophic $^{\star}b$-open sets need not be a Pythagorean neutrosophic $^{\star}b$-open set.

Example
2.14. Let $X = \{q, w\}$, $A = \langle(0.1, 0.3, 0.5), (0.3, 0.5, 0.7)\rangle$, $B = \langle(0.1, 0.1, 0.4), (0.7, 0.5, 0.3)\rangle$, $C = \langle(0.4, 0.6, 0.9), (0.6, 03, 0.3)\rangle$ and $D = \langle(0.3, 0.5, 0.3), (0.9, 0.5, 0.9)\rangle$. Then, τ is a Pythagorean neutrosophic topological space. Now, choose $A_1 = \langle(0.3, 0.5, 0.3), (1.0, 0.1, 0.1)\rangle$ and $A_2 = \langle(1.0, 1.0, 0.4), (0.9, 0.4, 0.6)\rangle$. We can see that $A_1 \cap A_2$ is not a Pythagorean neutrosophic $^{\star}b$-open set of (X, τ).

The following examples show that the union of two Pythagorean neutrosophic $^{\star}b$-closed sets need not be a Pythagorean neutrosophic $^{\star}b$-closed set.

Example 2.15. By the example 2.14, we can imply that $A_1^c \cup A_2^c$ is not a Pythagorean neutrosophic $^{\star}b$-closed set of (X, τ).

Example 2.16. Let $X = \{q\}$ and $A = \langle(1, 0.5, 0.7)\rangle$, $B = \langle(0, 0.9, 0.2)\rangle$, $C = \langle(1, 0.9, 0.2)\rangle$ and $D = \langle(0, 0.5, 0.7)\rangle$. Then, τ is a Pythagorean neutrosophic topological space. Now, choose $A_1^c = \langle(0.4, 0.5, 1)\rangle$ and $A_2^c = \langle(0.2, 0, 0.8)\rangle$. We can see that $A_1 \cup A_2$ is not a Pythagorean neutrosophic $^{\star}b$-closed set of (X, τ).

Proposition 2.17. *Let A be a Pythagorean neutrosophic set in Pythagorean neutrosophic topological space (X, τ). If B is a Pythagorean neutrosophic $^{\star}b$-open set and $B \subseteq A \subseteq PNInt(PNCl(A)) \cap PNCl(PNInt(A))$, then A is a Pythagorean neutrosophic $^{\star}b$-open set.*

Theorem 2.18. *Arbitrary union of Pythagorean neutrosophic $^{\star}b$-open sets is a Pythagorean neutrosophic $^{\star}b$-open set.*

Proof: Let $A_1, A_2, ...A_n$ be a collection of Pythagorean neutrosophic $^{\star}b$-open sets, then by the Definition 2.2, $A_1 \subseteq PNInt(PNCl(A_1)) \cap PNCl(PNInt(A_1))$, $A_2 \subseteq PNInt(PNCl(A_2)) \cap PNCl(PNInt(A_2))$, ..., $A_n \subseteq PNInt(PNCl(A_n)) \cap PNCl(PNInt(A_n))$. Now, $A_1 \cup A_2 \cup ... \cup A_n \subseteq (PNInt(PNCl(A_1)) \cap PNCl(PNInt(A_1))) \cup (PNInt(PNCl(A_2)) \cap PNCl(PNInt(A_2))) \cup ... \cup (PNInt(PNCl(A_n)) \cap PNCl(PNInt(A_N)))$, by the Theorem 2.11 parts (7) and (10), $A_1 \cup A_2 \cup ... \cup A_n \subseteq PNInt(PNCl(A_1 \cup A_2 \cup ... \cup$

$A_n)) \cap PNCl(PNInt(A_1 \cup A_2 \cup ... \cup A_n))$. This proofs that $A_1 \cup A_2 \cup ... \cup A_n$ is a Pythagorean neutrosophic $^\star b$-open set.

Th following example shows that the intersection of two Pythagorean neutrosophic $^\star b$-open sets need not be a Pythagorean neutrosophic $^\star b$-open set.

Example 2.19.
Let $X = \{z, x\}$ and $A = \langle(0.3, 0.5, 0.4), (0.6, 0.2, 0.5)\rangle$, $B = \langle(0.2, 0.6, 0.7), (0.5, 0.3, 0.1)\rangle$, $C = \langle(0.3, 0.6, 0.4), (0.6, 0.3, 0.1)\rangle$ and $D = \langle(0.2, 0.5, 0.7), (0.5, 0.2, 0.5)\rangle$. Then, τ is a Pythagorean neutrosophic topological space. Now, take $A_1 = \langle(0.4, 0.6, 0.4), (0.8, 0.3, 0.4)\rangle$ and $A_2 = \langle(1.0, 0.9, 0.2), (0.5, 0.7, 0)\rangle$. We can see that $A_1 \cup A_2$ is not a Pythagorean neutrosophic $^\star b$-open set of (X, τ).

Remark 2.20. By the Example 2.14, we support that the arbitrary intersection of Pythagorean neotrosophic $^\star b$-open sets need not be a Pythagorean neotrosophic $^\star b$-open set.

Proposition 2.21. *Arbitrary intersection of Pythagorean neutrosophic $^\star b$-closed sets is a Pythagorean neutrosophic $^\star b$-closed set.*

Remark 2.22. By the Example 2.15, the arbitrary union of Pythagorean neotrosophic $^\star b$-closed sets need not be a Pythagorean neotrosophic $^\star b$-closed set.

Theorem 2.23. *A Pythagorean neutrosophic set A in a Pythagorean neutrosophic topological space (X, τ) is Pythagorean neutrosophic b-open if and only for every Pythagorean neutrosophic point $x_{a,b,c} \in A$ there exits a Pythagorean neutrosophic $^\star b$-open $B_{x_{a,b,c}}$ such that $x_{a,b,c} \in B_{x_{a,b,c}} \subseteq A$.*

Proof: **Necessary**: Let A be a Pythagorean neutrosophic b-open set. Then, we have that $B_{x_{a,b,c}} = A$ for each $x_{a,b,c}$.
Sufficiency: Suppose that for every Pythagorean neutrosophic point $x_{a,b,c} \in A$, there exits a neutrosophic $^\star b$-open set $B_{x_{a,b,c}}$ such that $x_{a,b,c} \in B_{x_{a,b,c}} \subseteq A$. Thus, $A = \bigcup\{x_{a,b,c} : x_{a,b,c} \in A\} \subseteq \{B_{x_{a,b,c}} : x_{a,b,c} \in A\} \subseteq A$ and then, $A = \bigcup\{B_{x_{a,b,c}} : x_{a,b,c} \in A\}$. Therefore, by the Theorem 2.18, it is a Pythagorean neutrosophic $^\star b$-open set

Definition 2.24. Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic $^\star b$-open if $f(A)$ is Pythagorean neutrosophic $^\star b$-open set in Y for every Pythagorean neutrosophic open set A in X.

Proposition 2.25. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. If f is Pythagorean neutrosophic $^\star b$-open, then, f is Pythagorean neutrosophic pre-open*

Proof: Let f be a Pythagorean neutrosophic $^\star b$-open and A be a Pythagorean neutrosophic open set in X. Then, by hypothesis $f(A)$ is a Pythagorean neutrosophic $^\star b$-open set in Y, by the Proposition 2.4, $f(A)$ is a Pythagorean neutrosophic pre-open set in X. Therefore, f is a Pythagorean neutrosophic pre-open function.

3. Pythagorean neutrosophic $^\star b$-continuous functions

In this section, we use the notion of Pythagorean neutrosophic $^\star b$-open sets to introduce and study the concepts of Pythagorean neutrosophic $^\star b$-continuous function and Pythagorean neutrosophic $^\star b$-homeomorphism. Moreover, we establish some of their properties.

Definition 3.1. Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic $^\star b$-continuous if $f^{-1}(V)$ is a Pythagorean neutrosophic $^\star b$-open set in X for every Pythagorean neutrosophic open set V in Y.

Proposition 3.2. *Every Pythagorean neutrosophic continuous function is Pythagorean neutrosophic $^\star b$-continuous function.*

Definition 3.3. Let $x_{a,b,c}$ be a Pythagorean neutrosophic point of a Pythagorean neutrosophic topological space (X, τ). A Pythagorean neutrosophic set D of X is said to be Pythagorean neutrosophic neighbourhood of $x_{a,b,c}$ if there exits a Pythagorean neutrosophic open set V in X such that $x_{a,b,c} \in V \subseteq D$

Proposition 3.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, the following statements are equivalent:*

(1) *f is a Pythagorean neutrosophic $^\star b$-continuous function.*
(2) *For each Pythagorean neutrosophic point $x_{a,b,c}$ and every Pythagorean neutrosophic A of $f(x_{a,b,c})$, there exits a Pythagorean neutrosophic $^\star b$-open set B of X such that $x_{a,b,c} \in B \subseteq f^{-1}(A)$.*
(3) *For each Pythagorean neutrosophic point $x_{a,b,c} \in X$ and every Pythagorean neutrosophic neighbourhood A of $f(x_{a,b,c})$, there exits a Pythagorean neutrosophic $^\star b$-open set B of X such that $x_{a,b,c} \in B$ and $f(B) \subseteq A$.*

Proof: (1) $\Rightarrow$ (2): Let $x_{a,b,c}$ be a Pythagorean neutrosophic point of X and let A be a Pythagorean neutrosophic neighbourhood of $f(x_{a,b,c})$. Then, there exits a Pythagorean neutrosophic open set B of Y such that $f(x_{a,b,c}) \in B \subseteq A$. Now, since f is a Pythagorean neutrosophic $^\star b$-continuous function, we have that $f^{-1}(B)$ is a Pythagorean neutrosophic $^\star b$-open set of X and $x_{a,b,c} \in f^{-1}(f(x_{a,b,c})) \subseteq f^{-1}(B) \subseteq f^{-1}(A)$ and this ends the proof.

(2) $\Rightarrow$ (3): Let $x_{a,b,c}$ be a Pythagorean neutrosophic point of X and let A be a Pythagorean neutrosophic neighbourhood of $f(x_{a,b,c})$. By hypothesis, there exits a Pythagorean neutrosophic $^\star b$-open set B of X such that $x_{a,b,c} \in B \subseteq f^{-1}(A)$ and then $x_{a,b,c} \in B$ of X such that $f(B) \subseteq f(f^{-1}(A)) \subseteq A$ and this ends the proof.

(3) $\Rightarrow$ (1): Let B be a Pythagorean neutrosophic open set of Y and let $x_{a,b,c} \in f^{-1}(B)$ and so $f(x_{a,b,c}) \in B$ and then B is a Pythagorean neutrosophic neighbourhood of $f(x_{a,b,c})$. Now, since B is a Pythagorean neutrosophic open set and by hypothesis, there exits a Pythagorean neutrosophic $^\star b$-open set A of X such that $x_{a,b,c} \in A$ and $f(A) \subseteq B$. Indeed, $x_{a,b,c} \in A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(B)$ and this implies that $f^{-1}(B)$ is a Pythagorean neutrosophic b-open set of X. Therefore, f is a Pythagorean neutrosophic $^\star b$-open continuous function.

Proposition 3.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. If f is a Pythagorean neutrosophic $^\star b$-open function, then f is a Pythagorean neutrosophic pre-continuous function.*

Definition 3.6. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean nuetrosophic $^\star b$-homeomorphism if f and f^{-1} are Pythagorean neutrosophic $^\star b$-continuous functions.

Example 3.7. Let $X = \{q, w\}$ and $Y = \{e, r\}$. Then, $\tau = \{0_N, U_1, U_2, 1_N\}$ and $\sigma = \{0_N, V, 1_N\}$ are Pythagorean neutrosophic topological spaces on X and Y respectively, where $U_1 = \langle x, (0.2, 0.4, 0.7), (0.4, 0.4, 0.4)\rangle$, $U_2 = \langle x, (0.3, 0.5, 0.6), (0.5, 0.4, 0.6)\rangle$ and $V = \langle y, (0.3, 0.5, 0.6), (0.5, 0.2, 0.7)\rangle$. Then, we define the function $f : (X, \tau) \to (Y, \sigma)$ as $f(q) = e$ and $f(w) = w$. We can see that f and f^{-1} are Pythagorean neutrosophic $^\star b$-continuous and then f is Pythagorean neutrosophic $^\star b$-homeomorphism.

Definition 3.8. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean nuetrosophic homeomorphism if f and f^{-1} are Pythagorean neutrosophic continuous functions.

Theorem 3.9. *Each Pythagorean neutrosophic homeomorphism is Pythagorean neustrosophic $^\star b$-homeomorphism.*

Proof: Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection and Pythagorean neutrosophic homeomorphism function in which f and f^{-1} are Pythagorean neutrosophic continuous functions. Since that every Pythagorean neutrosophic continuous function is Pythagorean neutrosophic $^\star b$-continuous, this implies that f and f^{-1} are Pythagorean neutrosophic $^\star b$-continuous functions. Therefore, f is a Pythagorean neutrosophic $^\star b$-homeomorphism. **Proof**: The following example shows that the converse of the above Theorem need not be true.

Example 3.10. Let $X = \{q, w\}$ and $Y = \{e, r\}$. Then, $\tau = \{0_N, U_1, U_2, 1_N\}$ and $\sigma = \{0_N, V, 1_N\}$ are Pythagorean neutrosophic topological spaces on X and Y respectively, where $U_1 = \langle x, (0.3, 0.5, 0.8), (0.4, 0.4, 0.4)\rangle$, $U_2 = \langle x, (0.1, 0.3, 0.8), (0.1, 0.5, 0.8)\rangle$ and $V = \langle y, (0.4, 0.5, 0.6), (0.1, 0.3, 0.6)\rangle$. Then, we define the function $f : (X, \tau) \to (Y, \sigma)$ as $f(q) = e$ and $f(w) = w$. We can see that f is a Pythagorean neutrosophic b-homeomorphism, but it is not a Pythagorean neutrosophic homeomorphism.

Theorem 3.11. *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, the following statements hold:*

(1) *f is Pythagorean neutrosophic $^{\star}b$-closed.*
(2) *f is Pythagorean neutrosophic $^{\star}b$-open.*
(3) *f is Pythagorean neutrosophic $^{\star}b$-homeomorphism.*

Proof: (1) $\Rightarrow$ (2) : Let f be a bijection Pythagorean neutrosophic $^{\star}b$-closed function. Then, f^{-1} is Pythagorean neutrosophic $^{\star}b$-continuous function. Now, since every Pythagorean neutrosophic open set of (X, τ) is a Pythagorean neutrosophic b-open set of (X, τ), this implies that f is a Pythagorean neutrosophic $^{\star}b$-open function.

(2) $\Rightarrow$ (3) : Let f be a bijective Pythagorean neutrosophic $^{\star}b$-open function. Then, f^{-1} is a Pythagorean neutrosophic $^{\star}b$-continuous function. Indeed, f and f^{-1} are Pythagorean neutrosophic $^{\star}b$-continuous functions. Therefore, f is a Pythagorean neutrosophic $^{\star}b$-homeomorphism.

(3) $\Rightarrow$ (1) : Let f be a Pythagorean neutrosophic $^{\star}b$-homeomorphism. Then, f and f^{-1} are Pythagorean neutrosophic $^{\star}b$-continuous functions. Since every Pythagorean neutrosophic closed set of (X, τ) is a Pythagorean neutrosophic $^{\star}b$-closed set of (X, τ), this implies that f is a Pythagorean neutrosophic $^{\star}b$-closed function.

The following example shows that the composition of two Pythagorean neutrosophic $^{\star}b$-homeomorphisms need not be a Pythagorean neutrosophic $^{\star}b$-homeomorphism.

Example 3.12. Let $X = \{q, w\}$, $Y = \{e, r\}$ and $Z = \{t, y\}$. Then, $\tau = \{0_N, U, 1_N\}$, $\sigma = \{0_N, V, 1_N\}$ and $\omega = \{0_N, W, 1_N\}$ are Pythagorean neutrosophic topological spaces on X, Y and Z respectively, where $U = \langle x, (0.1, 0.3, 0.5), (0.3, 0.5, 0.7)\rangle$, $V = \langle y, (0.2, 0.7, 0.9), (0.3, 0.6, 0.7)\rangle$ and $W = \langle z, (0.7, 0.5, 0.2), (0.7, 0.7, 0.2)\rangle$. We define the function $f : (X, \tau) \to (Y, \sigma)$ as $f(q) = e$ and $f(w) = r$. Besides, we define the function $g : (Y, \sigma) \to (Z, \omega)$ as $g(e) = t$ and $g(r) = y$. We can see that f and g are Pythagorean neutrosophic $^{\star}b$-homeomorphism, but $g \circ f$ is not a Pythagorean neutrosophic $^{\star}b$-homeomorphism.

Definition 3.13. Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic $^\star b$-irresolute if $f^{-1}(V)$ is a Pythagorean neutrosophic $^\star b$-open set in X for every Pythagorean neutrosophic $^\star b$-open set V in Y.

Definition 3.14. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic $^\star b$i-homeomorphism if f and f^{-1} are Pythagorean neutrosophic $^\star b$-irresolute functions.

Theorem 3.15. *Every Pythagorean neutrosophic $^\star bi$-homeomorphism is a Pythagorean neutrosophic $^\star b$-homeomorphism.*

Proof: Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection and Pythagorean neutrosophic $^\star b$i-homeomorphism function. Suppose that B is a Pythagorean neutrosophic closed set of (Y, σ), this implies that B is a Pythagorean neutrosophic $^\star b$-closed set of (Y, σ). Now, since f is Pythagorean neutrosophic irresolute, $f^{-1}(B)$ is a Pythagorean neutrosophic $^\star b$-closed set of (X, τ). Indeed, f is a Pythagorean neutrosophic $^\star b$-continuous function. therefore, f and f^{-1} are Pythagorean neutrosophi $^\star b$-continuous functions and then f is Pythagorean neutrosophic $^\star b$-homeomorphism.

The following example shows that the converse of the above Theorem need not be true.

Example 3.16. Let $X = \{q, w\}$ and $Y = \{e, r\}$. Then, $\tau = \{0_N, U_1, U_2, 1_N\}$ and $\sigma = \{0_N, V, 1_N\}$ are Pythagorean neutrosophic topological spaces on X and Y respectively, where $U_1 = \langle x, (0.2, 0.4, 0.6), (0.3, 0.3, 0.3)\rangle$, $U_2 = \langle x, (0.4, 0.7, 0.9), (0.1, 0.1, 0.3)\rangle$ and $V = \langle y, (0.4, 0.7, 0.9), (0.1, 0.2, 0.3)\rangle$. Then, we define the function $f : (X, \tau) \to (Y, \sigma)$ as $f(q) = e$ and $f(w) = w$. We can see that f is a Pythagorean neutrosophic $^\star b$-homeomorphism, but it is not a Pythagorean neutrosophic $^\star b$i-homeomorphism.

Theorem 3.17. *If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \omega$ are Pythagorean neutrosophic $^\star bi$-homeomorphisms, then $g \circ f : (X, \tau) \to (Z, \omega)$ is a Pythagorean neutrosophic $^\star bi$-homeomorphism.*

Proof: Let f and g be two Pythagorean neutrosophic b-homeomorphisms. Now, suppose that B is a Pythagorean neutrosophic $^\star b$-closed set of (Z, ω), then $g^{-1}(B)$ is a Pythagorean neutrosophic $^\star b$-closed set of (Y, σ). Then by hypothesis, $f^{-1}(g^{-1}(B))$ is a Pythagorean neutrosophic $^\star b$-closed set of (X, τ). Therefore, $g \circ f$ is a Pythagorean neutrosophic $^\star b$-irresolute function Now, let β be a Pythagorean neutrosophic $^\star b$-closed set of (X, τ). By assumption, $f(\beta)$ is a Pythagorean neutrosophic $^\star b$-closed set of (Y, σ). Then, by hypothesis, $g(f(\beta))$ is a Pythagorean neutrosophic $^\star b$-closed set of (Z, ω). This implies that $g \circ f$ is a

Pythagorean neutrosophic $^\star b$-irresolute function and then $g \circ f$ is a Pythagorean neutrosophic $^\star bi$-homeomorphism.

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Comparative Mathematical Model for Predicting of Financial Loans Default using Altman Z-Score and Neutrosophic AHP Methods

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Abstract: The current system in the bank depends only on the client's failure to pay monthly installments for three consecutive months to start moving and take the necessary actions towards the client. This routine system is the basic reason of happening the problem of loans default. In this paper the researcher presents a comparative mathematical model to predict the default of clients, as well as to devise a modern parallel model to measure the degree of credit risk criteria that guides the bank in the following-up of the client. Altman model is one of the famous methods for default prediction, formula is used to predict the probability of loan default by using Z-scores. The Z-score is a linear combination of five coefficient-weighted common financial criteria. The researcher applies the Neutrosophic Analytical Hierarchy Process (NAHP) model on the same five common financial criteria which the bank can using them to provide constant following-up of the uses of the granted loan to guarantee that all terms set by the bank are met. The information was gathered in the form of neutrosophic data sets and evaluated using a novel Neutrosophic Analytic Hierarchy Process (NAHP) model. The researcher applies the proposed model in the credit department of one of the private Egyptian Banks (QNB) choosing random samples of real clients.

Keywords: Credit Risk, Loans Default, Altman Model, AHP Model, Neutrosophic AHP, Decision Making.

1. Introduction

Credit managers in Egyptian banks are grappling with the issue of bad loans. Banks' exposure to real credit problems would erode trust in the banking sector, because the consequences of such problems do not affect only the distressed banks, but also the rest of the banking sector in the country and the economy as a whole.

The subject of forecasting loan defaults is one of the issues that researchers and bank executives are most interested in since it is critical in minimizing default and its negative consequences for banks, borrowers, and the overall economy. According to credit rating agencies such as Moods Institution and others, the issue of defaulting loans has a major impact on the state's credit indicators globally.**[1]**

The issue varies from one country to another, and even within a single country, from one bank to another. The issue changes from time to time, both within the bank and through the banking sector as a whole. Bank credit is typically governed by policies and guidelines aimed at minimizing potential credit risk, but no bank can achieve zero credit risk in practice because bank credit is often followed by loans risks. The first of these risks comes from the fact that the credit is based on the borrower's or project's financial statements, which are not completely covered since they will be collected in the future.

Even though banks perform studies before issuing loans in compliance with the correct rules and basics, the risk of the borrower defaulting and his failure to pay remains uncertain, even because of the probability of incidents or consequences that prevent the borrower from committing to his obligations to the bank; if this possibility is met, the bank's financial rights become in a dangerous situation. The model compares the results of the two approaches (Altman Model and NAHP Model) and calculates the weight of each sub-criteria.

The Altman Model and the Neutrosophic Analytical Hierarchy Process Model are compared in this study, these models are compatible with bank's system because they are working on the factors which the bank used to evaluate the clients. This research contributes to highlighting the Neutrosophic set's accuracy in decision-making. It also underlines the need of using multi-criteria (criteria and factors) in decision-making models, particularly in information systems with numerous factors for a single aspect.[2]. The details of Altman Model is introduced in section 2, section 3 and 4 is explaining all rules of NAHP model, the result of the model case study is discussed in section 5 and 6.

Literature Review

Kulalı applied the Altman Z-Score model on financial data to 19 companies which suffered from bankrupt when trading in the BIST in the years between 2000-2013. When applying the Altman Z-Score model to predict the financial failure of these companies, the result of financial failure was estimated by 95% one year 90% two years earlier. This presented the success of the Altman Z-Score model in predicting the financial failure. [3]

Bağcı presented a study of Altman Model to measure the financial situation of the firms in textile industry to understand the situation of these firms to can face a possible economic crisis. He used the financial data of 24 companies in the textile industry area traded on BIST between 2008-2013, the financial situation of firms was examined by employing Altman Z-Score model. Z-scores were calculated by using the financial ratios of the textile industry. According to the observed results, suggestive Z-Scores between 2008-2013 were 0.63, 0.57, 0.60, 0.62, 0.63, 0.67 respectively which showed that the industry was exposed to high risk in terms of financial failure. [4]

Mişu and et al measured the integration of Analytic Hierarchy Process (AHP) into Delphi framework in neutrosophic environment. They presented a new technique of NAHP for checking consistency and calculating consensus degree of expert's opinions. They used neutrosophic technique to overcome the confusion of experts in evaluating the available alternatives due to the multiplicity of criteria associated with those alternatives. they found that the effectiveness of the AHP can be increased by adding Delphi technology with neutrosophic theory to reduce noise resulting from individual concerns instead of focusing on solving the problem, and increasing the degree of agreements around the standards presented.[5]

Fernando and et al proposed a methodological framework design to modify trade-offs between evaluation criteria to provide decision makers with more clear mortgage risk evaluation system**.** The result of this study showed that the AHP approach has the potential to increase the existing credit scoring systems of Portuguese banking firms. Also AHP can be used to assist banking institutions in managing new evaluation criteria feature and holding type.[6]

Kaygisiz Ertuğ and Girginer presented a research to develop an evaluation integrated model to consider the quantitative and qualitative criteria for the selected firms that demanded commercial loans for both public and private banks**.** The researchers combined the AHP model and Grey Relational Analysis (GRA) into a one evaluation model**.** The results appeared that, whereas firm honesty and reports criteria are the main criteria with the highest priority, sale and marketing constructions are the main criteria with the lowest priorities for both public and private banks.[7]

After reviewing a number of previous researches in the same field that were chosen in the research, the researcher deducts that the NAHP model have been used in a specific problems of credit risks introduced by the banks. This paper provides all types of loans which the bank is offering to the clients especially medium and long-term loans, which are always the cause of a client's financial failure due to the length of the period of repayment for the loan by following up the client using the weight of credit financial indicators which are presented in the client's financial statements in the beginning of applying the loan.

2. Altman Model

Altman was one of the first researches where developed financial forecasting models. He used 33 financial ratios and examined each ratio separately. He then used the method of statistical analysis and limited his model to the five most important financial ratios: [8]

X_1 = Working Capital / Total Assets

X_2 = Retained Earnings / Total Assets

X_3 = Profit before interest and tax / Total Assets

X_4 = Market value of Equity / Total Liabilities

X_5 = Total Sales / Total Assets

He then assigned a relative weight to each element of the model, different from each other, and each ratio has its own value according to its relative importance in the model. [8]

He Used (1.2) Factor For the ratio of X_1, (1.4) Factor For the ratio of X_2, (3.3) Factor For the ratio of X_3, (0.6) Factor For the ratio of X_4 and (1.0) Factor For the ratio of X_5.

The final form of the model equation became as follows:

$$Z = 1.2 * X_1 + 1.4 * X_2 + 3.3 * X_3 + 0.6 * X_4 + 1.0 * X_5$$

Altman classified customers according to Z score as follows:

1- Green zone if $Z \leq 1.8$, which means the client is excellent and pays all his installments in their due dates.

2- Yellow Zone $2.9 > Z > 1.8$, which means the client is good although he can't pay few installments in some months but do his best to do that.

3- Red Zone $Z > 2.9$, which means the client is in a danger because he stopped to pay the installments and the bank must take an action with him. [8]

In this paper the researcher develops a new Neutrosophic AHP model to discover the client fraud by using credit risk criteria, and derive a new sub-criteria in studying the cases of clients to facilitate the function of the credit officer in detecting the manipulation of the client in the financial statements before starting to take the scheduling procedures. This procedures are vary from bank to bank and from one client to another according to credit officer evaluation.

This model aims to study and follow the position of the client from the day he got the loan till the final installment is paid. The researcher applies the Neutrosophic AHP model on the clients to can predict if they will complete the all installments to pay off the entire loans in there due dates or not. The result will compare with Altman classifying model to can judge if the model is working well or not.

3. Basic definitions of Single Value Neutrosophic Number

Neutrosophic theory is a better choice to emulate the human thinking which has the capability to handle the indeterminacy. The decision-making process still keeps to rely not only on true values, but also on false ones as well as on indeterminacy membership. Thus neutrosophic logic makes the chance to emulate the human thinking and deal with the problems which have the probability of true, false and indeterminacy at the same time, to can be applied in the real world problems. [9]

A neutrosophic set <T, I, F> is composed of three parameters which are a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, and F ∈ [0,1].

Assume that X be the space of the objects, and x ∈ X. A neutrosophic set A in X is defined by three functions: truthfulness-membership function TA(x), an indeterminacy-membership function IA(x) and falsehood-membership function FA(x).

Definition 1: Assume that $N_1 = (T_1, I_1, F_1)$ AND $N_2 = (T_2, I_2, F_2)$ are two single value neutrosophic numbers, Then, their operations are defined as follows [10]

$$N_1 + N_2 = (T_1 + T_2 - T_1T_2, I_1I_2, F_1F_2) \tag{1}$$

$$N_1 \times N_2 = (T_1T_2, I_1 + I_2 - I_1I_2, F_1 + F_2 - F_1F_2) \tag{2}$$

$$N_2 \,/\, N_1 = (T_2/T_1, I_2 - I_1/1 - I_1, F_2 - F_1/1 - F_1) \tag{3}$$

Definition 2: Assume that $N_1 = (T_1, I_1, F_1)$ is a single value neutrosophic number and A is an arbitrary positive real number, Then, their operations are defined as follows [10]

$$A \times N_1 = \left(1 - (1 - T_1)^A, {I_1}^A, {F_1}^A\right), A > 0 \tag{4}$$

$$N_1/A = \left(1 - (1 - T_1)^{\frac{1}{A}}, {I_1}^{\frac{1}{A}}, {F_1}^{\frac{1}{A}}\right), A > 0 \tag{5}$$

Definition 3: Assume that $N_1 = (T_1, I_1, F_1)$is a single value neutrosophic number, then its score function is defined as $S(N_1)$ as follows: [10]

$$S(N_1) = (3 + T_1 - 2I_1 - F_1)/4 \quad (6)$$

4. Neutrosophic Analytical Hierarchy Process

AHP which developed in the 1970s by Thomas Saaty is a decision-making method which has been designed in a structured form to analyze complex decisions. It works by dividing a problem into a hierarchy of criteria and sub-criteria which can be analyzed independently. This hierarchy chart is containing the decision goal, the alternatives for reaching it, and the criteria for evaluating the alternatives. [12]

AHP is a mathematical tool of problem solving that has been created after understanding the structure of a problem and the real limitation that managers face while solving it. .

The following phases are the procedure of the neutrosophic analytic hierarchy process:

1- The proposed NAHP method begins by defining the neutrosophic values, which correspond to the 1–9 Saaty scale and are used to compare various criteria.
2- The decision-making problem's criteria, sub-criteria, and alternatives are identified in the second phase, then starts the process of building the problem's hierarchy.
3- The neutrosophic preference is determined in the third phase by comparing each criterion and sub criterion pair-wise. Following that, the alternatives are compared under each criterion or sub-criterion.
4- The fourth phase tests the accuracy of each pair-wise comparison then the neutrosophic preference relation is constructed.
5- The neutrosophic relative weight of each preference relation is calculated, the relative weight is measured by adding each column in the matrix, then dividing each number in the matrix by the sum of its columns, and finally averaging across the rows.
6- The overall weights are evaluated in the final phase, and the best alternative is chosen by multiplying the structure of the number of alternatives by the number of criteria. [12]

Step 1: Determine the objective of your study; decompose problem hierarchy to represent the goal, criteria, and the possibility of alternatives.

Step 2: A set of linguistic variables used by decision makers and importance weight based on neutrosophic values are as shown in Table 1..

Table 1. The neutrosophic scale for comparison matrix **[12]**

Linguistic term	Neutrosophic set	Linguistic term	Reciprocal neutrosophic set
Extremely Highly Preferred	(0.90, 0.10, 0.10)	Mildly Lowly Preferred	(0.10, 0.90, 0.90)
Extremely Preferred	(0.85,0.20, 0.15)	Mildly Preferred	(0.15,0.80, 0.85)
Very Strongly to Extremely Preferred	(0.80, 0.25, 0.20)	Mildly preferred to Very Lowly Preferred	(0.20, 0.75, 0.80)
Very Strongly Preferred	(0.75,0.25, 0.25)	Very Lowly Preferred	(0.25,0.75, 0.75)
Strongly Preferred	(0.70, 0.30, 0.30)	Lowly Preferred	(0.30, 0.70, 0.70)
Moderately Highly to Strongly Preferred	(0.65, 0.30, 0.35)	Moderately Lowly Preferred to Lowly Preferred	(0.35, 0.70, 0.65)
Moderately Highly Preferred	(0.60, 0.35, 0.40)	Moderately Lowly Preferred	(0.40, 0.65, 0.60)
Equally to Moderately Preferred	(0.55, 0.40, 0.45)	Moderately to Equally Preferred	(0.45, 0.60, 0.55)
Equally Preferred	(0.50, 0.50, 0.50)	Equally Preferred	(0.50, 0.50, 0.50)

At a given level of the hierarchy, these pair-wise comparisons are stored into the following matrix.

Step3: De-neutrosophication of the neutrosophic numbers to crisp values using the score function as in Eq. (6).

Matrix M for (n=5) criteria :

$$\mathbf{M} = \begin{bmatrix} 0.5 & a_{12} & a_{13} & a_{14} & a_{15} \\ a^{-1}_{(21)} & 0.5 & a_{23} & a_{24} & a_{25} \\ a^{-1}_{(31)} & a^{-1}_{(32)} & 0.5 & a_{34} & a_{35} \\ a^{-1}_{(41)} & a^{-1}_{(42)} & a^{-1}_{(43)} & 0.5 & a_{45} \\ a^{-1}_{(51)} & a^{-1}_{(52)} & a^{-1}_{(53)} & a^{-1}_{(54)} & 0.5 \end{bmatrix} \quad (7)$$

$$\textbf{Sum(column)} \quad S_{c1} \quad S_{c2} \quad S_{c3} \quad S_{c4} \quad S_{c5}$$

Step4: Matrix M is then normalized according to:

$$a_{ji} = \frac{a_{ji}}{\sum_{i=1}^{n} a_{ji}} \quad (8)$$

For all i and j. Weights which identifying the priorities of compared elements for the specific level of the hierarchy are then calculated as:

$$W_i = \frac{\sum_{i=1}^{n} a_{ji}}{n} \quad i = 1, 2, \dots . n \qquad (9)$$

Step5: The weights are related to the pair-wise comparisons matrix M according to:

$$A * W = \lambda_{Max} * W \qquad (10)$$

Where **λmax** is a standard used as a reference index that helps indirectly to assess consistency of the values. So, a consistency index **CI** is defined as:

$$CI = \frac{\lambda_{Max} - n}{n-1} \qquad (11)$$

Step6: The consistency ratio **CR** is calculated as:

$$CR = CI / RI \qquad (12)$$

Where RI is the random index, which is a function of the number of compared elements n, as shown in Table 2. The consistency ratio is an important measure of the values' consistency. Usually, a CR is a range of less than 0.1 is showing the values of consistent **.** [11]

Table 2. Average of random inconsistency indices (RI) for n

n	1	2	3	4	5	6	7	8	9	10
RI	0.00	0.00	0.58	0.90	1.12	1.24	1.32	1.41	1.46	1.49

Once the weights value of w is calculated for each level, the values are calculated to produce a set of overall priorities for the hierarchy. This is done by multiplying the elements' weights of the given level by the weight corresponding to the parent element in the upper or main level. Then, worthiness of the potential alternatives is accepted based on the produced weights corresponding to the considered criteria. Finally, a decision is made to achieve the goal set by selecting the alternative that gets the highest weight**.**

5. Result

5.1- The Implementation of NAHP Model

Multiple and conflicting criteria of decision-making are assessed in MCDM, a sub-field of process science. MCDM is a constant technique that can be used to choose the best choice from a set of choices in order to solve any problem that a decision-maker can face involving multiple criteria.[11]

The NAHP is a selection process that consists of following steps as shown in Figure 1:

1. Define the problem and objectives.

2. Structure the factors in criteria, sub-criteria and alternatives.

3. Construct a set of all problems in a square comparison matrix in which the set of elements is compared with itself by using the fundamental scale of pair-wise comparison shown in table.

4. Calculate weighting and consistency ratio.

5. Evaluate alternatives according weighting and get ranking.

Decision making operation is a procedure of choosing the most suitable alternative between the all-suitable alternatives, the alternatives should be studying in depth for the final implementation. In such cases decision maker should answer multi criteria decision making problem.[14]

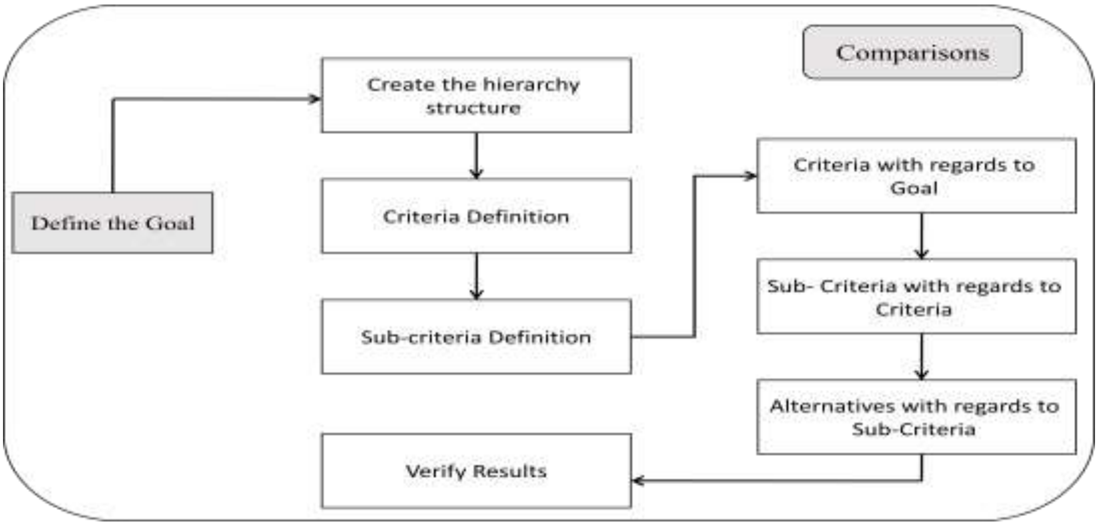

Figure 1: Steps for building an NAHP model

In this paper the researcher wants to present Neutrosophic Analytic Hierarchy Process (NAHP) as a support methodology for improving decision making processes. Also the researcher will focus on making strategy decisions in a bank with applying both basic and adjusted NAHP application models.

The researcher presents five major groups of banking rules criteria which are using to judge on clients. The NAHP provides an objective way for reaching to an optimal decision for both individual and group decision makers with a limited level of inconsistency.

It makes it possible to select the best alternative (under several criteria) shown in Figure 2 from a number of alternatives through carrying out pair-wise comparison values.[13]

Overall priorities for ranking the alternatives are being calculated on the basis of pair-wise comparisons.

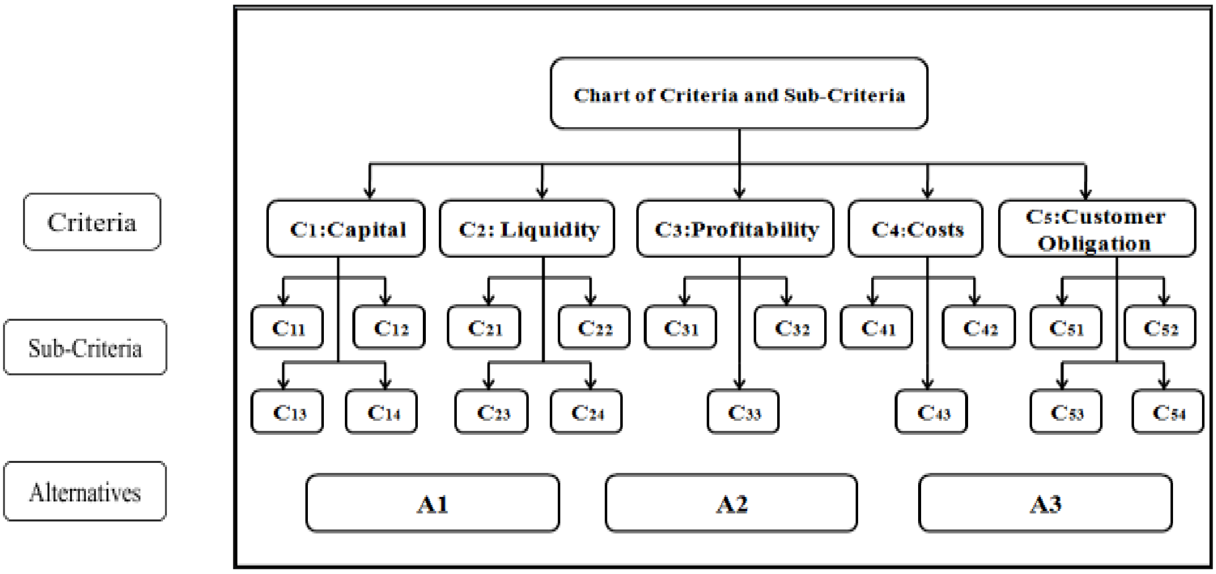

Figure 2 : The Hierarchy Chart of NAHP Model

5.2 - The Alternatives for NAHP Bank Decision Model

In the process of following up the client who obtained the loan, the decision maker in the credit sector has some alternatives which he should taking if any shortage happened from the client During the loan repayment period. They can be as following:

A1- Avoiding bad loans by following effective procedures and reliance on adequate guarantee and cash flow to repay the loan. (Green Area)

A2- Providing suggestions and alternatives to help the client in the project operations, and reducing payment terms, delay interest and scheduling loans. (Yellow Area)

A3- The bank declares the client bankruptcy immediately, and selling the pledged assets to the bank to liquidate the client's property. (Red Area)

5.3- Criteria and Sub-criteria of NAHP Model

The division of each criteria into sub-criteria are showed in the following Table 3:

Table 3. Criteria and Sub-criteria of NAHP Model.

Criteria	Sub Criteria
Working Capital(C_1)	C_{11} : Decreasing business amount and selling part of current assets. C_{12}: Using borrowing for covering the financial deficit. C_{13}: Decreasing profits annually. C_{14}: Stabilizing growth rates.
Liquidity(C_2)	C_{21} : Appearance of unplanned payment obligations in the project. C_{22} : Slow rate of assets turnover for the project. C_{23} : Constant increasing in the cost with lower sales. C_{24} : Inefficient using of production methods.
Profitability(C_3)	C_{31} : Sales decline. C_{32} : Increasing sales with lower profits. C_{33} : A gap between total profits and income net.
Costs(C_4)	C_{41} : Continues operating losses. C_{42} : High percentage of expenses to sales. C_{43} : Increasing the percentage of damaged production.
Customer Obligation(C_5)	C_{51} : Issuing checks that exceed the loan account. C_{52} : Failure to pay the due payments more than once. C_{53} : Decreasing the borrowing client accounts in the bank. C_{54} : Sudden changes to the timing of withdrawals and deposits.

To examine the related criteria of nonperforming loans problem, the researcher uses MCDM in AHP to evaluate the controlling factors of NPL in Egyptian banks and then make a comprehensive evaluation of them.

An aggregated pair-wise comparison matrix represents the average preferences and judgments of decision makers and, modeled in the form of neutrosophic scales as mentioned in Table 4.For simplicity, the aggregated pair-wise comparison matrix has been converted into crisp values using Eq. (6) and results represented in Table 5.

Table 4. Neutrosophic pair-wise comparison matrix of criteria.

Criteria	Working Capital(C_1)	Liquidity (C_2)	Profitability (C_3)	Costs (C_4)	Customer Obligation(C_5)
Working Capital(C_1)	**(0.50,0.50,0.50)**	**(0.55,0.40,0.45)**	**(0.45,0.60,0.55)**	**(0.80,0.25,0.20)**	**(0.70,0.30,0.30)**
Liquidity(C_2)	**(0.45,0.60,0.55)**	**(0.50,0.50,0.50)**	**(0.45,0.60,0.55)**	**(0.90,0.90,0.90)**	**(0.70,0.30,0.30)**
Profitability(C_3)	**(0.55,0.40,0.45)**	**(0.55,0.40,0.45)**	**(0.50,0.50,0.50)**	**(0.75,0.25,0.25)**	**(0.60,0.35,0.40)**
Costs(C_4)	**(0.50,0.50,0.50)**	**(0.50,0.50,0.50)**	**(0.50,0.50,0.50)**	**(0.50,0.50,0.50)**	**(0.30,0.70,0.70)**
Customer Obligation(C_5)	**(0.30,0.70,0.70)**	**(0.30,0.70,0.70)**	**(0.50,0.50,0.50)**	**(0.70,0.30,0.30)**	**(0.50,0.50,0.50)**

Table 5. Crisp values of judgments of neutrosophic pair-wise matrix.

Criteria	Working Capital(C_1)	Liquidity (C_2)	Profitability (C_3)	Costs (C_4)	Customer Obligation(C_5)
Working Capital(C_1)	**0.5**	**0.757**	**0.425**	**0.775**	**0.7**
Liquidity(C_2)	**0.425**	**0.5**	**0.425**	**0.9**	**0.7**
Profitability(C_3)	**0.757**	**0.757**	**0.5**	**0.75**	**0.625**
Costs(C_4)	**0.225**	**0.1**	**0.25**	**0.5**	**0.3**
Customer Obligation(C_5)	**0.3**	**0.3**	**0.375**	**0.7**	**0.5**

After that, the normalization illustrated to normalize the crisp value, the criteria's corresponding normalized weights mentioned using Eq. (9):$W_1 = 0.243, W_2 = 0.222, W_3 = 0.268, W_4 = 0.103, W_5 = 0.164$. According to the previous step, the total of criteria weights will be as the following: $\sum W_i = 1$. and the arrangement of criteria with respect to priorities is C_3, C_1, C_2, C_5 and C_4 respectively.

After calculating the weight of each sub-criteria for each main criteria, the researcher concluded that the most important criteria for the bank and which reflected the situation of the client in paying the monthly installments of the loan in their due time is profitability of the project, then working capital and the liquidity as shown in Table 6. So, the decision maker will depend on these criteria to predict the clients' condition through the following up of loan repayment.

Table 6. Rank of Main Criteria.

Criteria	**Sum of Weight of Sub-criteria**	**Rank**
C_1	0.243	**2**
C_2	0.222	**3**
C_3	0.268	**1**
C_4	0.103	**5**
C_5	0.164	**4**

5.4- Sub – Criteria of each Criteria

By applying the same steps on all sub criteria of main criteria, we concluded the following results as shown in Tables (7 – 11).

Table 7. Sub-Criteria of C_1

Capital	C_{11}	C_{12}	C_{13}	C_{14}	W
C_{11}	0.50	0.70	0.68	0.78	0.31
C_{12}	0.30	0.50	0.43	0.63	0.21
C_{13}	0.33	0.76	0.50	0.63	0.25
C_{14}	0.50	0.50	0.50	0.50	0.24

Table 8. Sub-Criteria of C_2

Liquidity	C_{21}	C_{22}	C_{23}	C_{24}	W
C_{21}	0.50	0.70	0.63	0.68	0.31
C_{22}	0.30	0.50	0.30	0.76	0.22
C_{23}	0.50	0.70	0.50	0.70	0.29
C_{24}	0.33	0.43	0.30	0.5	0.19

Table 9. Sub-Criteria of C_3

Profit	C_{31}	C_{32}	C_{33}	W
C_{31}	0.50	0.70	0.68	0.41
C_{32}	0.30	0.50	0.76	0.32
C_{33}	0.33	0.43	0.50	0.27

Table 10. Sub-Criteria of C_4

Cost	C_{41}	C_{42}	C_{43}	W
C_{41}	0.50	0.63	0.70	0.38
C_{42}	0.50	0.50	0.63	0.34
C_{43}	0.30	0.50	0.50	0.27

Table 6. Sub-Criteria of C_5

Customer	C_{51}	C_{52}	C_{53}	C_{51}	W
C_{51}	0.50	0.30	0.63	0.76	0.25
C_{52}	0.70	0.50	0.70	0.68	0.31
C_{53}	0.50	0.30	0.50	0.76	0.24
C_{54}	0.43	0.33	0.43	0.50	0.20

After calculating all equations of all NAHP process, the final weights of alternatives will be as shown in the Table 12:

Table 7. Alternatives of Bank Solutions

Alternatives	A1	A2	A3	Weight(X)
A1	0.50	0.63	0.70	0.410
A2	0.38	0.50	0.68	0.341
A3	0.30	0.33	0.50	0.249

When the researcher applies the same method which using by Altman Model, the weight of alternatives can be compared as the following :

IF X > = 0.410 Then the alternative will be the first one A = A1 (Green Area).

IF 0.410 > X > = 0.341 Then the alternative will be the second one A = A2 (Yellow Area).

IF 0.341 > X > = 0.249 Then the alternative will be the third one A = A3 (Red Area).

5.5- Applying Altman Model and NAHP Model

5.5.1- User Interface

The researcher uses the GUI tools to create, edit, and monitor the model. In the proposed model, the interface consists of a set of forms built in Visual Studio.NET 2016 because it is considered a flexible and a common software. The user can input the raw of data needed for a consultation. Figure 3 and Figure 4. Show samples of the used criteria model in application. The user may have information regarding a specific result and the interface can provide additional explanations about how the model reached to the conclusion.

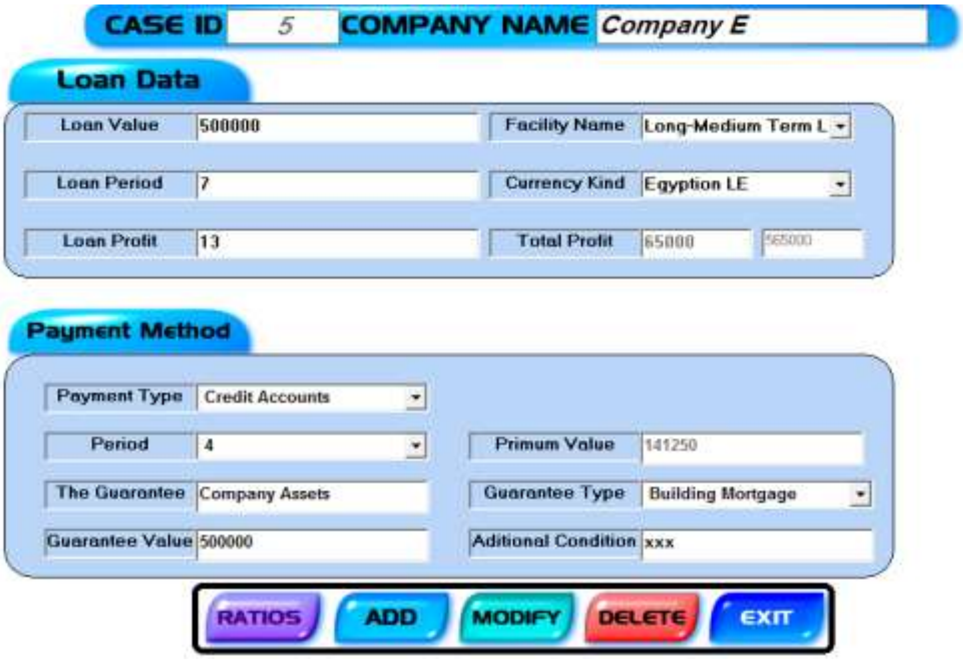

Figure 3: Snapshot of Client's Data

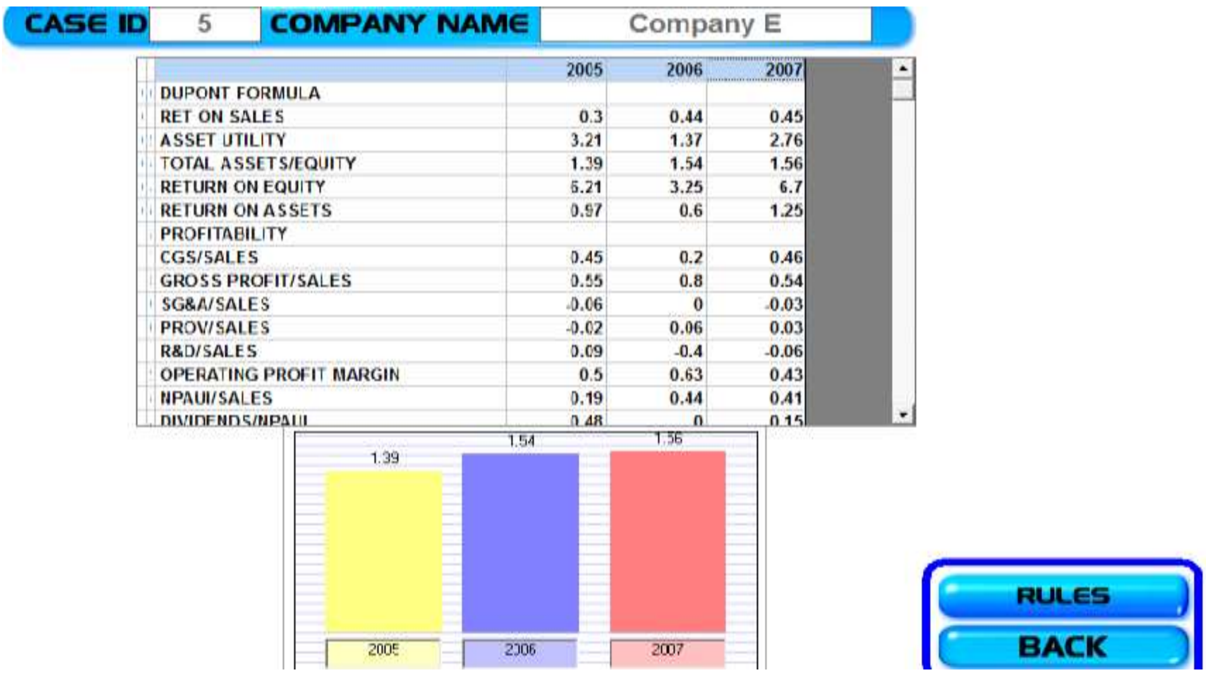

Figure 4 : Snapshot of Model criteria

5.5.2- Result of Applying Model

It was difficult to deal with any of public banks to use the proposed model practically in the credit loan department due to many considerations like laws forbidden, security issues … etc. The bank's administration allows the researcher to obtain and study the historical data of previous year, and only offers a set of available historical clients' cases (200 bank's clients). By applying the proposed model on these clients for testing the model, the researcher deducts the following results. Table 13 shows the numbers of classified clients sample and Figure 5 shows the difference between bank clients, Altman model clients and NAHP model clients.

Table 8. Numbers of Clients Samples

	Previous Actual Clients	Altman Model	NAHP Model
Green Area	100	120	140
Yellow Area	50	45	40
Red Area	50	35	20
Total of Success Payment	135	165	180
Total of Defaulted Payment	65	35	20

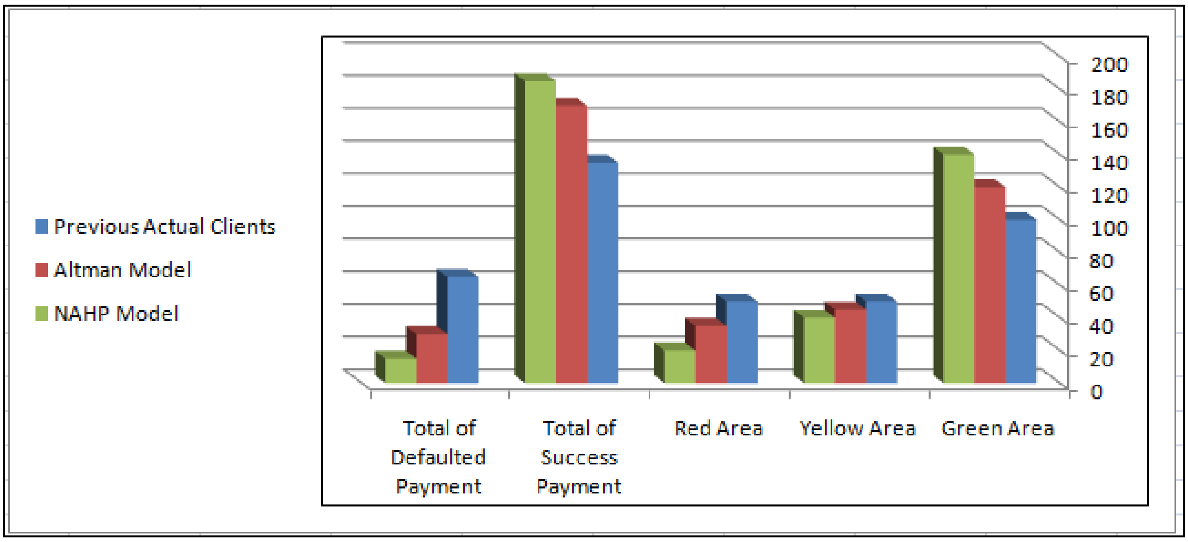

Figure 5 : Difference between bank clients, Altman model clients and NAHP model clients.

6. Discussion

Based on the analysis of the previous results that have been reached, the researcher concludes that:

The actual number of the clients which the bank approved are 200 clients, divided as follows :

1- 100 clients who reached to full success payment (Green Area).

2- 60 clients who showed payment fluctuation between the payment of monthly installments and the delay in paying some installments. (Yellow Area)

3- 40 clients who stopped to pay the monthly installments for 3 months or more (Red Area).

The total number of success payment clients are 135 clients (who repaid the total loan to the bank) ,and the total number of the clients who were unable to pay the fixed installments on their due dates are 65 clients.

When applying Altman model to the same number of actual clients specified by the bank, the previous numbers change to the following results and are divided as follows:

1- The number of clients in the (Green Area) increased to 120 clients after 20 clients increased from the (Yellow Area) as a result of close and accurate examination of the client's commitment to pay on due dates without any delay.

2- The number of clients in the (Red Area) decreased to 35 clients who moved to the yellow area, as a result of being controlled and helped to overcome the emergency crises to ensure that they repay the loan installments.

The total number of success payment clients increase to 165 clients (who repaid the total loan to the bank) ,and the total number of the clients who were unable to pay the fixed installments on their due dates decrease to 35 clients as shown in Table 13.

After applying the NAHP model to the same criteria used before, the number of clients who repaid the entire loan increased to 180 clients, being divided as shown in Table 13, and the number of defaulting clients decreased to 20 clients only, which is the highest percentage reached by the model compared to the existing system in the bank.

7.Conclusion

The study shows that all criteria which the bank is using to judge on the clients through the process of following up their obligation in paying the installments, are not used in such an effective way that can be a helpful factor to the credit officer to make the right decision at the right time.

In this paper, the researcher applies two models on these criteria, Altman Model and Neutrosophic-AHP Model. The paper provides a comparative analysis for them to show that we can use the same criteria used by the bank in very clear calculations to handle the criteria of evaluating the clients.

The paper proposes criteria for judging the clients and studies consistency of these criteria. This study also analyzes criteria and factors by calculating their weights based on the properties of the alternatives. The paper also measures the accuracy of decision by comparing the consistency of using multi-criteria and criteria for decision model.

The paper proves that Neutrosophic-AHP is more accurate rather than Altman Model and bank traditional Model. It also shows the effect of using criteria and its factors on the accuracy of the decision made.

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Analysis of Neutrosophic Multiple Regression

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Abstract: The idea of Neutrosophic statistics is utilized for the analysis of the uncertainty observation data. Neutrosophic multiple regression is one of a vital roles in the analysis of the impact between the dependent and independent variables. The Neutrosophic regression equation is useful to predict the future value of the dependent variable. This paper to predict the students' performance in campus interviews is based on aptitude and personality tests, which measures conscientiousness, and predict the future trend. Neutrosophic multiple regression is to authenticate the claim and examine the null hypothesis using the F-test. This study exhibits that Neutrosophic multiple regression is the most efficient model for uncertainty rather than the classical regression models
Keywords: Neutrosophic multiple regression; Neutrosophic regression; Neutrosophic correlation

1. Introduction

The concept of fuzzy logic was introduced by Zadeh [1], the elements in the collections are represented by the membership value in the closed interval [0,1]. Atanassov [2,3,4] introduce the intuitionistic fuzzy set that is an extension of the fuzzy set. It is useful to examine the real-life circumstances by considering membership and non-membership grades but without indeterminate membership grades. Smarandache [5, 6] extend the idea of intuitionistic fuzzy sets with the account of indeterminate membership grades, which we called Neutrosophic sets. Aftermath, Salama et al., [7] introduced the operations on Neutrosophic sets and progressed Neutrosophic sets theory in [8, 9, 10, 11, 12].

The important role of analyzing the correlation of dependent and independent variables is to estimate the strength and relation between two variables. Hanafy et al., [13] introduced the concepts of Neutrosophic correlation and its coefficients for the case of finite spaces. The Neutrosophic regression analysis is a powerful method to identify the relationships between the dependent and independent variables and also forecasting the uncertainty observation data. Some of the applications of Neutrosophic regression can be seen in literature such as Karacoska [14], Cervigon, et al., [15], Kumar & Chong [16], and Abdul et al., [17]. Smarandach [18] introduced the theory of Neutrosophic statistics that is the extension of classical statistics and also investigated Neutrosophic regression analysis. The real-time applications of Neutrosophic regression can be seen in Aslam [20], Salama et al., [21]. Prabhu et al., [22] analyzed the real-time multiple analysis. Some other contributions are in this domain have already been done by various researchers such as Tanaka & Ishibuchi [23] and Aslam [24].

Broumi & Smarandache [25] studied the weighted correlation and correlation coefficient between two interval Neutrosophic sets that were defined by Wang et al., [26]. Zhang et al., [27] explained the correlation coefficient measures and their entropy for interval Neutrosophic sets. Ye [28] proposed the two correlation coefficients between normal Neutrosophic numbers (NNSs) based on the score functions of normal Neutrosophic numbers (NNNs) and investigated their properties. He also developed a MADM method with NNSs under normal Neutrosophic numbers. Ye [29] presented a new correlation coefficient measure between dynamic single-valued Neutrosophic multisets. Karaaslan [30] studied the measures between two Neutrosophic sets; two interval-Neutrosophic sets; two Neutrosophic-refined sets and their applications of these methods are utilized in multi-criteria decision-making problems. Broumi and Smarandache [31] also proposed the correlation coefficient between interval Neutrosophic sets. Rajarajeswari and Uma [32] put forward the correlation measure for IFMS. Recently, Broumi and Smarandache [reference] defined the Hausdorff distance between Neutrosophic sets and some similarity measures based on the distance such as the set-theoretic approach and matching function to calculate the similarity degree between Neutrosophic sets. Broumi [32] explained the concept of correlation measure of Neutrosophic-refined sets that is the extension of the correlation measure of Neutrosophic sets and intuitionistic fuzzy multi-sets. Le [33] established the fuzzy decision-making method based on the weighted correlation coefficient under the intuitionistic fuzzy environment. Le [34] explained the cosine similarity measures for intuitionistic fuzzy sets and their applications. Gerstenkorn [35] studied the concept of correlation under the environment of intuitionistic fuzzy sets. Further, Hung [36] defined the correlation for intuitionistic fuzzy sets based on the centroid method. Ye [37] introduced the multicriteria decision-making method by the use of the correlation coefficient under a single-valued Neutrosophic environment. Deli [38] studied the concept of Neutrosophic-refined sets and their applications in medical diagnosis. Sahin [39] explained the correlation coefficient of single-valued Neutrosophic hesitant fuzzy sets and applied them in decision-making problems. Pramanik et al., [40] studied the multicriteria decision-making problems by applying a rough Neutrosophic correlation coefficient. Nagarajan et al., [41] explained Neutrosophic interval valued graphs. Lathamaheswari et al., [42] explained type 2 fuzzy in bio medicine. Ye [43] explained the improved correlation coefficients of single-valued Neutrosophic sets and interval Neutrosophic sets for multiple attribute decision-making problems. Liu et al., [44] established a correlation coefficient for the interval-valued Neutrosophic hesitant fuzzy sets and applied them in multiple attribute decision-making. Ye [45] studied the multi-criteria decision-making method using the correlation coefficient under a single-valued Neutrosophic environment. González-Rodríguez et al.,[46] explained ANOVA test for Fuzzy data. Jiryaei A et al.,[47] studied fuzzy random variables.

2.Preliminaries:

Regression line with dependent and one independent equation is

$$Y = a + bX + e. \qquad (1)$$

When Y is the output value on dependent, variable X is the input value of the independent variable, b is the slope, a is the intercept and e is the residual.
More than one independent variable equation as:

$$Y = a + b_1X_1 + b_2X_2 + \ldots + b_nX_n + e \qquad (2)$$

Here n number of independent variables and $b_1, b_2 \ldots b_n$ are number of slopes for each.e is the standard error . The estimation of a and b for to minimize the error of prediction equation

$$Y^{'} = a + b_1X_1 + b_2X_2 + \ldots + b_nX_n \qquad (3)$$

The equation for a with two independent variables is:

$$a = Y - b_1X_1 - b_2X_2 \qquad (4)$$

For the two-variable case:

$$b_1 = \frac{\sum x_2^{\ 2} \sum x_1 y - \sum x_1 x_2 \sum x_2 y}{\sum x_1^{\ 2} \sum x_2^{\ 2} - (\sum x_1 x_2)^2} \qquad (5)$$

$$b_2 = \frac{\sum x_1^{\ 2} \sum x_2 y - \sum x_1 x_2 \sum x_1 y}{\sum x_1^{\ 2} \sum x_2^{\ 2} - (\sum x_1 x_2)^2} \qquad (6)$$

From the above equations 5 & 6 only for two variables x_1 and x_2.

Smarandache [18] is given the Neutrosophic extended for classical statistics operation. The operations are as follows.

Let's S_1 and S_2 be two sets of numbers.

$S_1+S_2 = \{x_1+x_2 \mid x_1 \in S_1$ and $x_2 \in S_2\}$

$S_1-S_2 = \{x_1-x_2 \mid x_1 \in S_1$ and $x_2 \in S_2\}$

$S_1 \cdot S_2 = \{x_1 \cdot x_2 \mid x_1 \in S_1$ and $x_2 \in S_2\}$

$a \cdot S_1 = S_1 \cdot a = \{a \cdot x_1 \mid x_1 \in S_1\}$

$a + S_1 = S_1 + a = \{a+x_1 \mid x_1 \in S_1\}$

$a - S_1 = \{a-x_1 \mid x_1 \in S_1\}$

$S_1 - a = \{x_1 - a \mid x_1 \in S_1\}$

$S_1S_2 = \{x_1x_2 \mid x_1 \in S_1, x_2 \in S_2, x_2 \neq 0\}$

$S_1n = \{x_1n \mid x_1 \in S_1\}$

$S_1a = \{x_1a \mid x_1 \in S_1, a \neq 0\}$

$aS_1 = \{ax_1 \mid x_1 \in S_1, \quad x_1 \neq 0\}$

$\sqrt{S_1n} = \{\sqrt{x_1n} \mid x_1 \in S_1\}$

3.Numerical example

In table 1 shows the student performance campus interview based on aptitude and personality test, that measure the conscientiousness

Y is the dependent variable conscientiousness x_1 is the aptitude test and personality test as shown in the following table 1.

table:1 Database

Y	X_1	x_2
[1,3]	3	2
2	2	[2,1]
[2,4]	[1,2]	[3,2]
4	[2,3]	4
[1,4]	[2,1]	[4,4]
6	[2,3]	[4,5]
[2,4]	2	1
[10,13]	[5,6]	[6,7]
[14,15]	7	8
5	[7,1]	3

$$\sum x_1 y = \sum X_1 Y - \frac{\sum X_1 \sum Y}{N} \qquad (7)$$

$$\sum x_2 y = \sum X_2 Y - \frac{\sum X_2 \sum Y}{N} \qquad (8)$$

$$\sum x_1 x_2 = \sum X_1 X_2 - \frac{\sum X_1 \sum X_2}{N} \qquad (9)$$

Using the equation 7, 8, and 9

$$\sum x_1 y = [38,91.9], \sum x_2 y = [23,136.1] \sum x_1 x_2 = [35,19.9]$$

Matrix form of the values is corresponding to the correlation, sum of square, and cross product of the variables as shown in the following table 2.

table 2: Matrix form of the values

	Y	X_1	X_2
Y	$\sum y^2$ =[387,532]	$\sum x_1 y$ =[218,247]	$\sum x_2 y$ =[245,310]
X_1	r_{yx1} =[0.89954, 1.31925]	$\sum x_1^2$ =[156,1236]	$\sum x_1 x_2$ =[146,142]
X_2	r_{yx2} =[1.56309, 1.17855]	r_{x1x2} =[1.172603, 0.406182]	$\sum x_2^2$ =[175,189]

Using equation 5 and 6 the value of the regression coefficient

b_1= [-2.34988, 1.650734], b_2= [0.965172, 1.093934]

from equation 4 the value of the intercept is

a= [-4.29976, 10.18347]

Therefore the Neutrosophic regression equation is

Y=[-4.29976, 10.18347] + [-2.34988, 1.650734] x_1 +[0.965172, 1.093934] x_2

The proportion of variance is in the set of independent variables is R square value. The Neutrosophic R square value is

A Neutrosophic residual sum of squares is $NRSS = \sum(y - \hat{y})^2$ (9)

$$NRSS = \sum (y - \hat{y})^2 = [183,267.7]$$

A Neutrosophic total sum of squares $NTSS = \sum(y - \bar{y})^2$ (10)

$$NTSS = \sum (y - \bar{y})^2 = [2268.2,1875]$$

A Neutrosophic coefficient of determination is $NCD = 1 - \frac{NRSS}{NTSS}$ (11)

$$NCD = 1 - \frac{NRSS}{NTSS} = [0.097,0.129]$$

The Neutrosophic mean of Y is [46,50]. The Neutrosophic r square is [0.09,0.12] from the above results shows that the variation between independent and dependent variables is 9 % and 12 %. That means the student performance campus interview variation based on aptitude and personality test is between 9 % and 12 %. Hence, it is revealed that these variables are also affected by the student performance on-campus interview.

4.Significance test of R square

Using the F test for significance of R square is

$$F = \frac{R^2 / K}{(1 - R^2)(N - K - 1)} \quad (12)$$

Which is distributed as F with K and N-K-1 degrees of freedom when the null hypothesis is true. Now R^2 represents the multiple correlations rather than the single correlation.

The null hypothesis: R square value is not zero population with degrees of freedom is N-K-1

Using (12), the Neutrosophic F value is [0.007904,0093]

Comparing the tabulated value using degrees of freedom and the calculated value. It shows that the null hypothesis is accepted.

5.Regression with beta weights

Comparison of correlation and regression equation is

$$Z_Y{}' = r_{xy} Z_x \quad (13)$$

But β means a b weight when X and Y are in standard scores, so for the simple regression case, r = β , and we have:

$$Z_Y{}' = \beta Z_x \quad (14)$$

The bottom line on this is we can estimate β weights using a correlation matrix.

$$\beta_1 = \frac{r_{yx_1} - r_{yx_2} r_{x_1 x_2}}{1 - r_{x_1 x_2}{}^2} \quad (15)$$

$$\beta_2 = \frac{r_{yx_2} - r_{yx_1} r_{x_1 x_2}}{1 - r_{x_1 x_2}{}^2} \quad (16)$$

where r_{yx1} is the correlation of y with X_1, r_{yx2} is the correlation of y with X_2, and r_{12} is the correlation of x_1 with x_2. Note that the two formulas are nearly identical and the correlation matrix shows in table:3

table :3 Correlation matrix

	Y	X_1	X_2
Y	1		
X_1	[-0.89954, -1.31925	1	
X_2	[-1.56309, -1.17855]	[1.172603, 0.406182]	1

Using the equation 15 and 16 calculate the Neutrosophic beta coefficients. That is

$$\beta_1 = [-0.50399, -1.3679], \beta_2 = [-1.2303, 0.329977]$$

Note that there is a surprisingly large difference in beta weights given the magnitude of correlations.

6.The limitations on statistics

In table 4 shows that limitation on different category statistics

table:4 Limitation on Statistics

Statistics		Limitations
Classical statistics	It is applied for the analysis to determining the sample and the parameter in the population or sample space is determined.	The analysis only for the determined parameter. Testing the analysis of variance and significance under classical statistics only for determined observation.
Fuzzy statistics	The analysis using fuzzy statistics applies to the data having uncertainty. The statistics depend on Fuzzy statistics and do not consider indeterminacy.	It will be applied for observations in Fuzzy. Under fuzzy statistics testing the analysis of variance and significance only for the observations are fuzzy and uncertain.
Intuitionistic fuzzy statistics	It is the extension of fuzzy statistics and considering membership and non-membership grades.	It will apply only intervals belongs to membership and non-membership. Under Intuitionistic statistics testing the analysis of variance and significance only for the observation are membership and non-membership that belongs to the real unit interval.
Neutrosophic statistics	It is based on Neutrosophic logic and is considered the measure of indeterminacy. It is the	It is applied to an uncertain environment. Under Neutrosophic statistics testing the analysis of

	extension of intuitionistic fuzzy sets.	variance and significance when the observations are not fuzzy in the interval and it is an extension of classical and fuzzy statistics.

7. Conclusion

In this paper, we introduce the multiple regression method under the environment of Neutrosophic sets. Moreover, we proposed a method to compute the correlation coefficient of Neutrosophic sets which is given us information about the degree of the relationships between the variables based on Neutrosophic sets. Further, the method is applied to predict the students' performance in campus interviews based on aptitude and personality tests. Based on the above method the result shows that the variation between independent and dependent variables is 9% and 12%, which means that the students' performance variations based on aptitude and personality tests are between 9% and 12%. Thus, it is revealed that aptitude and personality tests are affected students' performance in campus interviews. Future work will be focused on the concept of interval Neutrosophic multiple regression analysis.

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Estimating Re-Evaluation of the Risk Report Obtained Using the Altman Z-Score Model in Mergers with Neutrosophic Numbers

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Abstract: The Altman Z-Score model, introduced by Edward Altman in 1968, is one of the most common models used in financial risk analysis. However, although it is a widely used model, many theories against the growing uncertainty of our daily lives have been replaced by theories modeled by more complex sets of numbers that are nonlinear. Therefore, the necessity of evaluating the financial uncertainty situation with different and innovative methods has emerged for the models obtained. When looking at the studies in the literature with the use of the Altman Z-score model, it shows that the predictive power of the Z-score model is quite high. This model is a model that can be applied by using the data in the balance sheet and income statement, which are among the basic financial statements in accounting. In this study, the Altman Z-score model was arranged according to the neutrosophic numbers of some data from the balance sheet and income statement of the two companies, and a nonlinear study was compared with the classical Altman Z-score model. As a result of the study, it was predicted that the results obtained with the Altman Z-score and neutrosophic numbers have more positive results in company mergers than the formula found by the discriminant method, and the instability situations will decrease as a result of the merger of the company.

Keywords: Altman Z-score; Risk Analysis; Neutrosophic Numbers

1. Introduction

Neutrosophy is a branch of philosopy, introduced by Smarandache in 1980, which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set and neutrosophic statistics in [1]. Neutrosophic logic is a general framework for unification of many existing logics such as fuzzy logic which is introduced by Zadeh in [2] and intuitionistic fuzzy logic which is introduced by Atanassov in [3]. Fuzzy set has only degree of membership, intuitionistic fuzzy set has only degree of membership and degree of nonmembership. Thus; they do not explain the indeterminancy states. But neutrosophic set has degree of membership (t), degree of indeterminacy (i) and degree of nonmembership (f) and define the neutrosophic set on three components (t, i, f). A lot of researchers have been dealing with neutrosophic set theory in [4].

The main purpose of the establishment of businesses is to make a profit. However, over time, businesses may face the reality of failure and bankruptcy, which they consider as the last point, along with deterioration in their financial structures. Today, although there are companies that have existed for years, there are companies that have to terminate their activities in a short time and go bankrupt. Edward I. Altman (1968) conducted a study on bankruptcy firms and created a model using financial ratios and discriminant analyzes to predict the insolvency of firms. In the literature, this model has been named as Altman Z-score model. Altman (1968) analyzed the financial ratios of 66 manufacturing enterprises with the method of multiple discriminant analysis and developed the Z-Score model consisting of five ratios that can be used in predicting financial failure. Altman classified the enterprises with a Z Score greater than 2,99 in the "safe zone" (Non-Bankupt). There is no failure for companies located in the safe zone. Businesses with a Z Score between 1,81 and 2,99 are classified in the gray area. Caution should be exercised when investing in businesses located in the gray zone. Businesses with a Z Score below 1,81 are determined as businesses with a high risk of financial failure. This study of Altman showed a successful classification performance of 95% for one year ago and 72% for two years ago in predicting financial failure. The Z-score has been developed over time for industry and service sectors and new models have been adapted to these sectors. Altman's Z-score model is one of the best known statistically derived models for predicting the upcoming bankruptcies of companies [5].

It is known that reasonable assurance mathematical and statistical methods are used in the establishment of audit systems and in auditing. Analysis with neutrosophic numbers can also be considered in this context. Especially when the continuity of businesses and bankruptcy risk are evaluated together, the auditor prepares the financial statements with the assumption that the enterprises have an indefinite life and can continue their activities in the foreseeable future. Financial statements are prepared to cover a period of at least 12 months. The concept of business continuity also refers to this. Within the scope of the auditor's responsibilities, it is necessary to obtain sufficient and appropriate audit evidence about the appropriateness of management's use of the going concern principle in preparing the financial statements. It is thought that whether there is a bankruptcy risk in business combinations can be measured with the concept of continuity [6].

In this study, the results obtained by adapting Altman's Z-score model, which is one of the accounting-based bankruptcy prediction models, to neutrosophic clusters in company mergers were compared with the classical Altman's Z-score model and a new perspective was tried to be gained to the literature.

2. Altman Z-skor

In this model, Altman dividing 22 different ratios into 5 groups has created a discriminant model. This model;

$$Z = 0{,}012.X_1 + 0{,}014.X_2 + 0{,}033.X_3 + 0{,}006.X_4 + 0{,}999.X_5$$

- X_1 = Working Capital / Total Assets
- X_2 = Non-Distributed Profits / Total Assets
- X_3 = Profit Before Interest and Tax / Total Assets
- X_4 = Equity / Total Debts
- X_5 = Sales / Total Assets

According to some of the model included: Firms with a Z-score below 1,81 are classified as uncertain and companies within 2,99 are classified as safe in a distressed situation between 1,81 and 2,99. This model has been criticized in terms of predicting financial failure of private firms and non-production operating service sector for its construction for public companies. Altman (2000), with a number of regulations, introduced 2 new models for the private sector manufacturing and service sector.

Altman Z model for private sector manufacturing companies;

- Z = 0,717. X_1 + 0,847. X_2 + 3,107. X_3 + 0,42. X_4 + 0,998. X_5

Altman Z model for private sector service companies;

- Z = 6,56. X_1 + 3,26. X_2 + 6,72. X_3 + 1,05. X_4

The X_5 variable was removed from the new model, and the impact of the manufacturing sector was reduced. As a result of this situation, the coefficients of the new models have changed and as a result, the Z-score intervals have changed. The ranges of all created Z-score models are given in the table below. [7,8].

Z Score Ranges for Public Manufacturing Sector Firms (Altman, 1968)	**Altman Z Score Ranges for Private Sector Manufacturing Companies (Altman, 2000)**	**Altman Z Score Ranges for Private Sector Service Companies (Altman, 2000)**
Safe if Z score > 2,99	Safe if Z score > 2,90	Safe if Z score > 2,60
Uncertain if 1,81 ≤ Z score ≤ 2,99	Uncertain if 1,23 ≤ Z score ≤ 2,90	Uncertain if 1,1 ≤ Z score ≤ 2,60
Troubled if the Z score < 1,8	Troubled if the Z score < 1,23	Troubled if Z score < 1,1

3. Method and Modeling

Data on working capital, total assets, undistributed profits, profit before interest and tax, equity and debt belonging to companies A and B, whose names are kept confidential for modeling purposes, are given below.

For Company A;

- Working Capital = 21.552.520,00
- Total Assets = 29.147.026,00
- Non-Distributed Profits = 84.157,00
- Profit Before Interest and Tax = 11.517.421,00
- Equity = 24.581.236,00
- Debts = 4.565.790,00
- Sales = 18.413.971,00

In accordance with the information provided, for company A;

- X_1 = Working Capital / Total Assets = 0,7394414785
- X_2 = Non-Distributed Profits / Total Assets = 0,0028873272
- X_3 = Profit Before Interest and Tax / Total Assets = 0,3951490969
- X_4 = Equity / Total Debt = 5,3837859385
- X_5 = Net Sales / Total Assets = 0,6317615732

As a result, the Altman-Z Score obtained for company A;

$$Z = 0{,}717.X_1 + 0{,}847.X_2 + 3{,}107.X_3 + 0{,}42.X_4 + 0{,}998.X_5 = 4{,}652041494$$

For Company B;

- Working Capital = 23.519.164,00
- Total Assets = 47.405.811,00
- Non-Distributed Profits = 954.663,00
- Profit Before Interest and Tax = 12.244.001,00
- Equity = 36.727.311,00
- Debts = 10.678.500,00
- Sales = 41.524.596,00

In accordance with the information provided, for company B;

- X_1 = Working Capital / Total Assets = 0,35572096
- X_2 = Non-Distributed Profits / Total Assets = 0,017056971
- X_3 = Profit Before Interest and Tax / Total Assets = 0,802477804
- X_4 = Equity / Total Debt = 1,444535339
- X_5 = Net Sales / Total Assets = 0,874187066

As a result, the Altman-Z Score obtained for company B;

$$Z = 0{,}717.X_1 + 0{,}847.X_2 + 3{,}107.X_3 + 0{,}42.X_4 + 0{,}998.X_5 = 3{,}49397814$$

For company C obtained as a result of the merger of A and B companies;

- Working Capital = 45.071.684,00
- Total Assets = 76.552.837,00
- Non-Distributed Profits = 1.038.820,00
- Profit Before Interest and Tax = 23.761.422,00
- Equity = 61.308.547,00
- Debts = 15.244.290,00
- Sales = 59.938.567,00

In accordance with the information provided, for company C;

- X_1 = Working Capital / Total Assets = 0,588765691
- X_2 = Non-Distributed Profits / Total Assets = 0,013569974
- X_3 = Profit Before Interest and Tax / Total Assets = 0,310392442
- X_4 = Equity / Total Debt = 4,021738435
- X_5 = Net Sales / Total Assets = 0,782969898

Altman-Z Score obtained as a result of the merger of A and B companies;

$$Z = 0{,}717.X_1 + 0{,}847.X_2 + 3{,}107.X_3 + 0{,}42.X_4 + 0{,}998.X_5 = 3{,}868562186$$

Normalization and adaptation to neutrosophic numbers for company A [9,10,11];

$$T_A: E \rightarrow]^-0, 1^+[$$

$$I_A: E \rightarrow]^-0, 1^+[$$

$$F_A: E \rightarrow]^-0, 1^+[$$

and

$T_A(x)$ $x \in E$, Accuracy Degree = 0,3766177702

$I_A(x)$ $x \in E$, Instability Degree = 0,3589821806

$F_A(x)$ $x \in E$, Inaccuracy Degree = 0,2644000492

- *T (x)= Safe Zone Z-Skore / Z-Skore, I(x)=Uncertain Zone Z-Skore / Z-Skore, F(x)= Troubled Zone Z-Skore / Z-Skore*

Normalization and adaptation to neutrosophic numbers for company B [9,10,11];

$$T_B: E \rightarrow]^-0, 1^+[$$

$$I_B: E \rightarrow]^-0, 1^+[$$

$$F_B: E \rightarrow]^-0, 1^+[$$

for;

$T_B(x)$ $x \in E$, Accuracy Degree = 0,170000531

$I_B(x)$ $x \in E$, Instability Degree = 0,477965211

$F_B(x)$ $x \in E$, Inaccuracy Degree = 0,352034257

Altman-Z Score for company C obtained as a result of the merger of A and B companies; results obtained as a result of normalization [4,5,6]

$$T_C: E \rightarrow]^-0, 1^+[$$

$$I_C: E \rightarrow]^-0, 1^+[$$

$$F_C: E \rightarrow]^-0, 1^+[$$

and

$T_C(x)$ $x \in E$, Accuracy Degree = 0,2503674853

$I_C(x)$ $x \in E$, Instability Degree = 0,4316849309

$F_C(x)$ $x \in E$, Inaccuracy Degree = 0,3179475838

- *T (x)= Safe Zone Z-Skore / Z-Skore, I(x)=Uncertain Zone Z-Skore / Z-Skore, F(x)= Troubled Zone Z-Skore / Z-Skore*

Adapting the results obtained as a result of normalizing the Altman-Z Score without merging of A and B companies to neutrosophic numbers and combining them with neutrosophic numbers. Let's use the AB index to distinguish this combination from the classic Altman-Z Score [9,10,11];

$$T_{AB}: E \rightarrow]^{-}0, 1^{+}[$$

$$I_{AB}: E \rightarrow]^{-}0, 1^{+}[$$

$$F_{AB}: E \rightarrow]^{-}0, 1^{+}[$$

and

$T_{AB}(x)$ $x \in E$, Accuracy Degree = 0,6458237826

$I_{AB}(x)$ $x \in E$, Instability Degree = 0,2296159870

$F_{AB}(x)$ $x \in E$, Inaccuracy Degree = 0,1245602304

4. Conclusion

When the data obtained for A and B companies considered in our study were compared with the results obtained with the classical Altman-Z score and neutrosophic numbers;

- While the accuracy level is lower in the classical method, it is predicted that the company will become more secure as a result of the merger as the accuracy level increases in the data obtained by using non-linear neutrosophic numbers.
- Compared to the classical method, it is seen that the degree of instability decreases by almost half as the accuracy level increases.
- It has been observed that the same situation for the degree of indecision is also valid for the degree of inaccuracy, and the degree of inaccuracy is significantly reduced.

Along with all these results, the formula obtained from the discriminant method in the classical Altman-Z model is insufficient to calculate the positive benefit of company mergers, and when we look at the results obtained with neutrosophic numbers, it is predicted that mergers will make a more positive contribution in favor of the company and the situations of negativity and indecision within the company will decrease. Therefore, linear methods increase the accuracy rate of company financial data in company mergers, resulting in more positive results for the company. When the financial data of the company are evaluated, the possibility of making correct analysis according to the linear method also increases. It was concluded that this situation increases the possibility of making a more accurate decision in terms of decision data. In addition, it is considered that these studies can contribute to the field of audit, such as the continuity of the enterprises to be carried out in the field of audit, by evaluating the analytical procedures. Finally, it was predicted that in the future, more advanced number systems such as neutrosophic numbers will be used in risk analysis instead of classical number systems.

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Novel Concept of Interval-Valued Neutrosophic Incidence Graphs with Application

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Abstract: Neutrosophic set (NS) is a framework used when the imprecision and uncertainty of an event are described based on three possible aspects, i.e., the membership degree, neutral membership degree and non-membership degree. On the other hand, neutrosophic graphs (NG) are applicable to deal with bulk information events. Furthermore, the incidence graph concept in neutrosophic sets contains a handful of problems like decision-making as well as, social and communication networks. This paper aims to propose the interval-valued neutrosophic sets to incidence graph and represent a new concept, namely interval-valued neutrosophic incidence graph (IVNIG). An IVNIG is a generalization of the concept of single-valued neutrosophic incidence graph (SVNIG). Moreover, some properties related to IVNIG, such as strong edge, strong pair, strong cut pair and neutrosophic incidence cut pair, are also discussed using suitable examples. The defined new concept of IVNIG is applied and investigated on a practical problem of safe root travelling.

Keywords: bridge; cut pair; interval-valued neutrosophic incidence graphs; strong edge; strong pair

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1. Introduction

The concept of fuzzy sets was pioneered by Zadeh [1]. Later, Zadeh introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [2], where the membership degrees' values are intervals of numbers instead of numbers itself. Interval-valued fuzzy sets provide a more comprehensive overview of uncertainty than traditional fuzzy sets. However, these may not be enough in modelling indeterminate and inconsistent information to deal with in the real world. Therefore, to encounter this problem, Smarandache [3] proposed the notion of neutrosophic sets by combining non-standard analysis. In a neutrosophic set, the membership value is associated with three components which are truth membership (t), indeterminacy membership (i) and falsity membership (f), where each membership value is a real standard or non-standard subset of the non-standard unit interval $]0^-,1^+[$ with no restriction on their sum.

Later, Smarandache and Wang et al. [4] introduced the single-valued neutrosophic sets (SVNS) to apply neutrosophic sets in real-life problems. Also, Wang et al. [5] presented the concept of an interval-valued neutrosophic set (IVNS). Representation of graphs using IVNS is more precise and more flexible than the SVNS. An IVNS is a generalization of the concept of SVNS. Three membership functions (t, i, f) are independent and their values belong to the unit interval [0, 1].

Graph theory has become a significant area of applied mathematics, commonly considered a combinatorics area. The graph is a widely used tool for solving combinatorial problems in different areas such as computer science, optimization, topology, number theory, algebra and geometry. It should be pointed out that, when there is uncertainty regarding either the set of vertices or edges, or both, the model becomes a fuzzy graph [6]. In recent years, there has been an increasing amount of literature performed on the fuzzy graph [7–11], intuitionistic fuzzy graphs [12–18] and interval-valued intuitionistic fuzzy graphs [19–22]. All of them have considered the vertex and edge sets as fuzzy and/or intuitionistic fuzzy sets. However, when the relations between nodes (or vertices) are inconsistent, the fuzzy graphs and intuitionistic fuzzy graphs fail to work.

Therefore, Smarandache [23–25] defined four main categories of neutrosophic graphs. These are based on literal indeterminacy (I), which are I-edge neutrosophic graph and I-vertex neutrosophic graph. These concepts were studied extensively and gained much attention among the researchers due to their applications in real-world problems [26–27]. The two other graph categories are based on (t, i, f) components, called (t, i, f)-edge neutrosophic graph and (t, i, f)-vertex neutrosophic graph. However, these two categories were not developed at all. Later on, Broumi et al. [28] introduced and investigated a new neutrosophic graph model called single-valued neutrosophic graph (SVNG). This model allows attaching the membership (t), indeterminacy (i) and non-membership (f) degrees to both vertices and edges. The SVNG is a generalization of fuzzy graph (FG) and intuitionistic fuzzy graph (IFG).

The same authors, Broumi et al. [29–30] introduced the concept of an interval-valued neutrosophic graph (IVNG) as a generalization of the SVNG. The properties were discussed using proof and examples. Later on, Akram and Nasir [31] showed some flaws in Broumi’s definition, which cannot be applied in network models. The authors then modified the definition of an IVNG, discussing some operations involved. Using this approach, Akram and Sitara [32] introduced IVNG structure and several concepts on interval-valued neutrosophic competition graphs were presented in [33].

Dinesh [34] first introduced the concept of unordered pairs of vertices, which are not incident with end vertices. The fuzzy incidence shows the relations between vertices and provides information about the influence of a vertex on the edge. Later, the idea of the fuzzy incidence graph was extended by Dinesh [35], and the author introduces new concepts in this regard. Moreover, Mathew and Mordeson [36] discussed the connectivity concepts in fuzzy incidence graphs. These are important in interconnection networks with influenced flows. Therefore, it is crucial to analyze their connectivity properties. Next, the fuzzy incidence graph was studied by Malik et al. [37]. The authors applied the notion of the fuzzy incidence graph in problems involving human trafficking. They discussed the role played by the vulnerability of countries and their government’s response to human trafficking. Other than that, Mathew et al. [38] studied some properties of incidence cuts and connectivity in fuzzy incidence graphs. The incidence is used to model flows in human trafficking networks. Fuzzy incidence block was defined by Mathew and Moderson [39], discussing their applications in illegal migration problems. They used fuzzy incidence graphs as a non-deterministic network model with supporting links by applying fuzzy incidence blocks to avoid the network’s vulnerable links.

In view of all that has been mentioned so far, Akram et al. [40] investigated the extension of the fuzzy incidence graph in the form of the neutrosophic environment. The authors introduced the notion of a single-valued neutrosophic incidence graph (SVNIG) and discussed the connectivity in this regard. Later, Akram et al. [41] studied the idea of bipolar neutrosophic sets to incidence graphs, and some related properties were defined. Recently, Hussain et al. [42] have presented the neutrosophic vague incidence graph and defined the edge connectivity, the vertex connectivity, and pair connectivity in neutrosophic vague incidence graph. A summary of the author's contribution toward the incidence graph is presented in Table 1.

Based on the idea of SNVIG, in this paper we propose the interval-valued neutrosophic sets with incidence graph, representing a new concept, namely interval-valued neutrosophic incidence graphs (IVNIG). The properties related to IVNIG, such as strong edge, strong pair, strong cut pair and neutrosophic incidence cut pair are also discussed with suitable examples. The rest of this paper is instructed as follows: Part 2 contains a brief background about graphs and neutrosophic set applied later. We then introduce the concept of IVNIG graph and investigate its properties in Part 3. In Part 4, we apply the proposed method in application of finding the best route. In Part 5, illustrate the comparative study and advantages of the proposed method. Finally, Part 6 outlines the conclusion together with limitations of the study and suggest an open problem for future research.

Table 1. Contribution of authors to incidence graphs

Authors	Year	Contributions
Dinesh [34]	2012	fuzzy incidence graph
Dinesh [35]	2016	extended of fuzzy incidence graph
Mathew and Mordeson [36]	2017	connectivity concepts in fuzzy incidence graph
Mathew and Mordeson [39]	2017	fuzzy incidence block
Malik et al. [37]	2018	fuzzy incidence graph in human trafficking
Akram et al. [40]	2018	single-valued neutrosophic incidence graph
Mathew et al. [38]	2019	incidence cuts and connectivity in fuzzy incidence graph
Akram et al. [41]	2019	bipolar neutrosophic incidence graph
Hussain et al. [42]	2020	neutrosophic vague incidence graph

2. Preliminaries

In this part, some basic concepts related to neutrosophic sets, single-valued neutrosophic sets, interval-valued neutrosophic sets, fuzzy graph, single-valued neutrosophic graphs and interval-valued neutrosophic graphs are presented and used in the next parts.

Definition 2.1 [43]

A **fuzzy graph** is a pair of functions $G=(\sigma,\mu)$, where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ, .i.e., $\sigma: V \to [0,1]$ and $\mu: V\times V \to [0,1]$ such that $\mu(uv) \le \sigma(u)\wedge\sigma(v)$ for all $u,v\in V$. Here, uv denotes the edge between u and v while $\sigma(u)\wedge\sigma(v)$ denotes the minimum of $\sigma(u)$ and $\sigma(v)$. σ is called the fuzzy vertex set of V, while μ is called the fuzzy edge set of E.

Definition 2.2 [36]

Let $G=(V,E)$ be a graph, σ be a fuzzy subset of V, μ be a fuzzy subset of E and ψ be a fuzzy subset of $V\times E$. If $\psi=(v,e)\le \min(\sigma(v),\mu(e))$ for all $v\in V$ and $e\in E$, then ψ is called a **fuzzy incidence** of G.

Definition 2.3 [36]

Let $G=(V,E)$ be a graph and (σ,μ) be a fuzzy subgraph of G. If ψ is a fuzzy incidence of G, then $\tilde{G}=(\sigma,\mu,\psi)$ is called a **fuzzy incidence graph** of G. Any $x\in V$ is said to be in support of σ if $\sigma(x)>0, xy\in V\times V$ is said to be in support of μ if $\mu(xy)>0$, and $(x,yz)\in V\times E$ is said to be in support of ψ if $\psi(x,yz)>0$. The supports of $\sigma,\mu,$ and ψ are denoted as

σ^*, μ^*, and ψ^*, respectively. Let $xy \in \mathrm{Supp}(\mu)$. Then, xy is an edge of the fuzzy incidence graph $\tilde{G}=(\sigma,\mu,\psi)$ and if $(x,xy),(y,xy) \in \mathrm{Supp}(\psi)$, then (x,xy) and (y,xy) are called **pairs**. Two vertices v_i and v_j joined by a path in a fuzzy incidence graph are said to be **connected**. The **incidence strength** of a fuzzy incidence graph $\tilde{G}=(\sigma,\mu,\psi)$ is defined to be $\min\{\psi(v,e) \mid (v,e) \in \mathrm{Supp}(\psi)\}$.

Definition 2.4 [44]

A **neutrosophic graph** is defined as a pair $G(V,E)$ where

i. $V=\{v_1,v_2,\ldots,v_n\}$ such that $T_1:V\to[0,1], I_1:V\to[0,1]$ and $F_1:V\to[0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function respectively, where $0 \le T_1(v)+I_1(v)+F_1(v) \le 3$

ii. $E \subseteq V\times V \to [0,1]$ where E is relation on V such that

$$T_2(uv) \le \min\{T_1(u),T_1(v)\},$$
$$I_2(uv) \le \min\{I_1(u),I_1(v)\},$$
$$F_2(uv) \le \max\{F_1(u),F_1(v)\},$$
$$\text{and } 0 \le T_2(uv)+I_2(uv)+F_2(uv) \le 3, \forall uv \in E$$

Definition 2.5 [40]

Let $G'=(V,E,I)$ be an incidence graph, where V is a vertex set of G, E is edge set of G and I is incidence of G, then a single-valued neutrosophic incidence graph is an ordered-triplet, $\tilde{G}=(A,B,C)$ such that

1. A is a single-valued neutrosophic set on V
2. B is a single-valued neutrosophic relation on V
3. C is a single-valued neutrosophic subset of $V\times E$ such that

$$T_C(x,xy) \le \min\{T_A(x),T_B(xy)\},$$
$$I_C(x,xy) \le \min\{I_A(x),I_B(xy)\},$$
$$F_C(x,xy) \le \max\{F_A(x),F_B(xy)\}, \forall x \in V, xy \in E$$

Definition 2.6 [45][46]

The interval-valued neutrosophic set A in X is defined by

$$A=\left\{\left(x,\left[t_A^l(x),t_A^u(x)\right],\left[i_A^l(x),i_A^u(x)\right],\left[f_A^l(x),f_A^u(x)\right]\right):x\in X\right\},$$

where $t_A^l(x), t_A^u(x), i_A^l(x), i_A^u(x), f_A^l(x)$ and $f_A^u(x)$ are neutrosophic subsets of X such that

$$t_A^l(x) \le t_A^u(x),$$
$$i_A^l(x) \le i_A^u(x), \text{ and}$$
$$f_A^l(x) \le f_A^u(x), \forall x \in X$$

For any two interval-valued neutrosophic sets:

$$A=\left\{\left(x,\left[t_A^l(x),t_A^u(x)\right],\left[i_A^l(x),i_A^u(x)\right],\left[f_A^l(x),f_A^u(x)\right]\right):x\in X\right\}, \text{ and}$$
$$B=\left\{\left(x,\left[t_B^l(x),t_B^u(x)\right],\left[i_B^l(x),i_B^u(x)\right],\left[f_B^l(x),f_B^u(x)\right]\right):x\in X\right\},$$

define that,

$$A\cup B=\left\{\begin{pmatrix} x,\max\left(t_A^l(x),t_B^l(x)\right),\max\left(t_A^u(x),t_B^u(x)\right),\max\left(i_A^l(x),i_B^l(x)\right), \\ \max\left(i_A^u(x),i_B^u(x)\right),\min\left(f_A^l(x),f_B^l(x)\right),\min\left(f_A^u(x),f_B^u(x)\right) \end{pmatrix}:x\in X\right\}$$

$$A\cap B=\left\{\begin{pmatrix} x,\min\left(t_A^l(x),t_B^l(x)\right),\min\left(t_A^u(x),t_B^u(x)\right),\min\left(i_A^l(x),i_B^l(x)\right), \\ \min\left(i_A^u(x),i_B^u(x)\right),\max\left(f_A^l(x),f_B^l(x)\right),\max\left(f_A^u(x),f_B^u(x)\right) \end{pmatrix}:x\in X\right\}$$

Definition 2.7 [31]

An interval-valued neutrosophic graph on a nonempty set X is a pair $G=(A,B)$, where A is an interval-valued neutrosophic set on X and B is an interval-valued neutrosophic relation on X such that

$$t_B^l(xy)\le\min\left(t_A^l(x),t_A^l(y)\right),\ t_B^u(xy)\le\min\left(t_A^u(x),t_A^u(y)\right),$$
$$i_B^l(xy)\le\min\left(i_A^l(x),i_A^l(y)\right),\ i_B^u(xy)\le\min\left(i_A^u(x),i_A^u(y)\right),$$
$$f_B^l(xy)\le\min\left(f_A^l(x),f_A^l(y)\right),\ f_B^u(xy)\le\min\left(f_A^u(x),f_A^u(y)\right),\forall x,y\in X.$$

Note that B is called symmetric relation on A.

3. Interval-Valued Neutrosophic Incidence Graphs

Definition 1

An interval-valued neutrosophic incidence graph (IVNIG) of an incidence graph $G=(V,E,I)$ is an ordered-triplet, $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$, such that

1. $\hat{X}$ is an interval-valued neutrosophic set on V
2. $\hat{Y}$ is an interval-valued neutrosophic relation on V
3. $\hat{Z}$ is an interval-valued neutrosophic subset of $V\times E$ such that

$$T_{\hat{Z}}^L(v,vw)\le\min\left\{T_{\hat{X}}^L(v),T_{\hat{Y}}^L(vw)\right\},$$
$$T_{\hat{Z}}^U(v,vw)\le\min\left\{T_{\hat{X}}^U(v),T_{\hat{Y}}^U(vw)\right\},$$
$$I_{\hat{Z}}^L(v,vw)\le\min\left\{I_{\hat{X}}^L(v),I_{\hat{Y}}^L(vw)\right\},$$
$$I_{\hat{Z}}^U(v,vw)\le\min\left\{I_{\hat{X}}^U(v),I_{\hat{Y}}^U(vw)\right\},$$
$$F_{\hat{Z}}^L(v,vw)\le\max\left\{F_{\hat{X}}^L(v),F_{\hat{Y}}^L(vw)\right\},$$
$$F_{\hat{Z}}^U(v,vw)\le\max\left\{F_{\hat{X}}^U(v),F_{\hat{Y}}^U(vw)\right\},\forall v\in V,vw\in E.$$

We now discuss an example of an IVNIG.

Example 1

Consider an incidence graph, $G=(V,E,I)$ such that $V=\{a,b,c,d\},E=\{ab,ac,bc,cd,ad\}$ and $I=\{(a,ab),(b,ab),(a,ac),(c,ac),(b,bc),(c,bc),(c,cd),(d,cd),(a,ad),(d,ad)\}$, as shown in Figure 1.

Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG associated with G, as shown in Tables 2 – 4 and Figure 2. Also, let $\hat{X}$ be an interval-valued neutrosophic incidence set on V given as:

$$\hat{X}=\left\{\begin{matrix}(a,[0.1,0.4],[0.2,0.5],[0.3,0.7]),(b,[0.3,0.5],[0.2,0.6],[0.1,0.7]), \\ (c,[0.3,0.8],[0.4,0.9],[0.5,0.9]),(d,[0.4,0.7],[0.3,0.8],[0.4,0.9])\end{matrix}\right\},$$

$\hat{Y}$ be an interval-valued neutrosophic incidence relation on V given as:

$$\hat{Y}=\left\{\begin{array}{l}(ab,[0.1,0.4],[0.2,0.5],[0.3,0.7]),(ac,[0.1,0.4],[0.2,0.5],[0.4,0.8]),(bc,[0.3,0.5],[0.2,0.6],[0.5,0.8]),\\(cd,[0.3,0.7],[0.3,0.8],[0.5,0.8]),(ad,[0.1,0.4],[0.2,0.5],[0.4,0.8])\end{array}\right\},$$

$\hat{Z}$ be an interval-valued neutrosophic incidence set on $V\times E$ given as:

$$\hat{Z}=\left\{\begin{array}{l}((a,ab),[0.1,0.3],[0.1,0.4],[0.3,0.7]),((b,ab),[0.1,0.3],[0.1,0.5],[0.2,0.6]),\\((a,ac),[0.1,0.3],[0.1,0.5],[0.3,0.8]),((c,ac),[0.1,0.4],[0.1,0.4],[0.4,0.8]),\\((b,bc),[0.2,0.5],[0.1,0.5],[0.4,0.7]),((c,bc),[0.3,0.5],[0.2,0.6],[0.4,0.8]),\\((c,cd),[0.3,0.7],[0.3,0.8],[0.4,0.9]),((d,cd),[0.3,0.6],[0.2,0.7],[0.3,0.8]),\\((a,ad),[0.1,0.4],[0.2,0.4],[0.3,0.7]),((d,ad),[0.1,0.4],[0.2,0.5],[0.4,0.8])\end{array}\right\}.$$

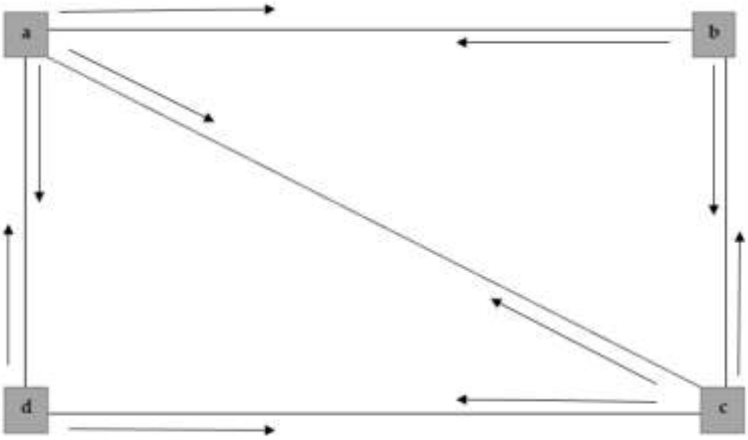

Figure 1. Incidence graph

Table 2. IVNIG set on V

	a	b	c	d
$t_{\hat{X}}$	[0.1,0.4]	[0.3,0.5]	[0.3,0.8]	[0.4,0.7]
$i_{\hat{X}}$	[0.2,0.5]	[0.2,0.6]	[0.4,0.9]	[0.3,0.8]
$f_{\hat{X}}$	[0.3,0.7]	[0.1,0.7]	[0.5,0.9]	[0.4,0.9]

Table 3. IVNIG relation on V

	ab	ac	bc	cd	ad
$t_{\hat{Y}}$	[0.1,0.4]	[0.1,0.4]	[0.3,0.5]	[0.3,0.7]	[0.1,0.4]
$i_{\hat{Y}}$	[0.2,0.5]	[0.2,0.5]	[0.2,0.6]	[0.3,0.8]	[0.2,0.5]
$f_{\hat{Y}}$	[0.3,0.7]	[0.4,0.8]	[0.5,0.8]	[0.5,0.8]	[0.4,0.8]

Table 4. IVNIG set on $V\times E$

	(a,ab)	(b,ab)	(a,ac)	(c,ac)	(b,bc)	(c,bc)	(c,cd)	(d,cd)	(a,ad)	(d,ad)
$t_{\hat{Z}}$	[0.1,0.3]	[0.1,0.3]	[0.1,0.3]	[0.1,0.4]	[0.2,0.5]	[0.3,0.5]	[0.3,0.7]	[0.3,0.6]	[0.1,0.4]	[0.1,0.4]
$i_{\hat{Z}}$	[0.1,0.4]	[0.1,0.5]	[0.1,0.5]	[0.1,0.4]	[0.1,0.5]	[0.2,0.6]	[0.3,0.8]	[0.2,0.7]	[0.2,0.4]	[0.2,0.5]
$f_{\hat{Z}}$	[0.3,0.7]	[0.2,0.6]	[0.3,0.8]	[0.4,0.8]	[0.4,0.7]	[0.4,0.8]	[0.4,0.9]	[0.3,0.8]	[0.3,0.7]	[0.4,0.8]

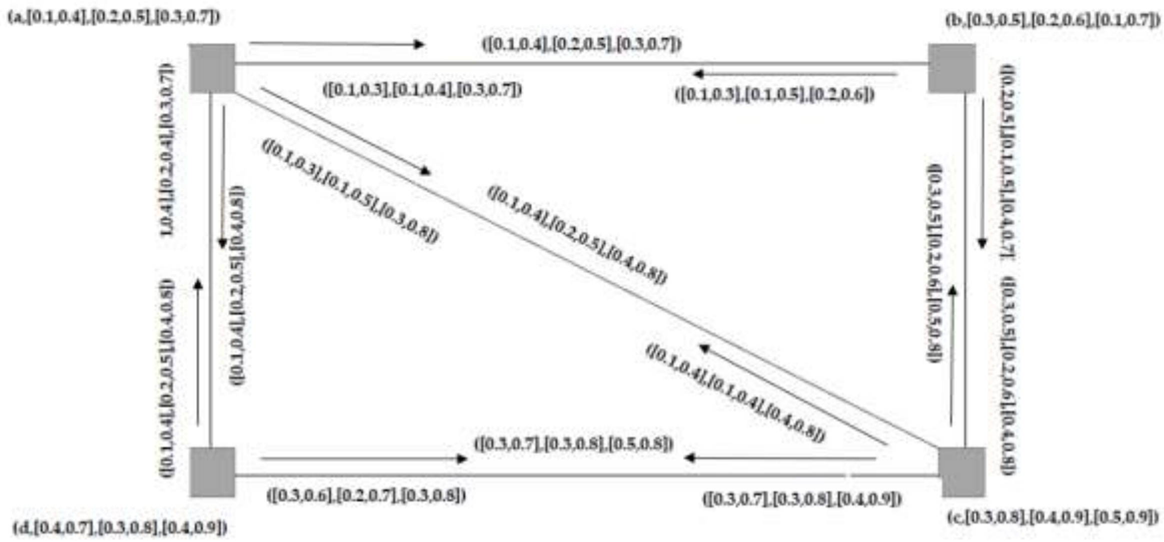

Figure 2 Interval-valued neutrosophic incidence graph

Definition 2

The support of an IVNIG $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ is denoted by $G^*=(X^*,Y^*,Z^*)$ where

$$X^*=\operatorname{supp}(\hat{X})=\{v\in V:[T_A^L(v),T_A^U(v)]>0,[I_A^L(v),I_A^U(v)]>0,[F_A^L(v),F_A^U(v)]>0\}$$

$$Y^*=\operatorname{supp}(\hat{Y})=\{vw\in E:[T_B^L(vw),T_B^U(vw)]>0,[I_B^L(vw),I_B^U(vw)]>0,[F_B^L(vw),F_B^U(vw)]>0\}$$

$$Z^*=\operatorname{supp}(\hat{Z})=\left\{\begin{matrix}(v,vw)\in I:[T_C^L(v,vw),T_C^U(v,vw)]>0,[I_C^L(v,vw),I_C^U(v,vw)]>0,\\ [F_C^L(v,vw),F_C^U(v,vw)]>0\end{matrix}\right\}$$

Definition 3

If $vw\in Y^*$, then vw is an edge of the IVNIG $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$, while if $(v,vw),(w,vw)\in Z^*$, then (v,vw) and (w,vw) are called pairs of $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$.

Definition 4

A sequence

$$Q: z_0,(z_0,z_0z_1),z_0z_1,(z_1,z_0z_1),z_1,(z_1,z_1z_2),z_1z_2,(z_2,z_1z_2),z_2,\ldots,$$
$$z_{n-1},(z_{n-1},z_{n-1}z_n),z_{n-1}z_n,(z_n,z_{n-1}z_n),z_n$$

of vertices, edges and pairs in $\hat{G}$ is known as a walk. It is a closed walk if $z_0=z_n$. In contrast, if all edges are distinct, it is a trail, while if the pairs are distinct, then it is an incidence trail. Q is called a path if the vertices are distinct. A path is called a cycle if the initial and end vertices of the path are the same. Any two vertices of $\hat{G}$ are said to be connected if a path joins them.

Example 2

In Example 1 presented earlier

$$Q_1: a,(a,ac),ac,(c,ac),c,(c,cd),cd,(d,cd),d,(d,da),da,(a,da),a$$

is known as a walk. In fact, it is a closed walk since the initial and final vertices are the same. It is not a path, but it is a trail and an incidence trail.

$$Q_2: a,(a,ac),ac,(c,ac),c,(c,cd),cd,(d,cd),d$$

On the other hand, Q_2 -is a walk, path, trail and an incidence trail.

Definition 5

Let $\hat{G}=\left(\hat{X},\hat{Y},\hat{Z}\right)$ be an IVNIG. Then, $\hat{H}=\left(\hat{L},\hat{M},\hat{N}\right)$ is an interval-valued neutrosophic incidence subgraph of $\hat{G}$ if $\hat{L}\subseteq\hat{X},\hat{M}\subseteq\hat{Y}$ and $\hat{N}\subseteq\hat{Z}$. $\hat{H}$ is an interval-valued neutrosophic incidence spanning subgraph of $\hat{G}$ if $L^{*}=X^{*}$.

Definition 6

In an IVNIG, the strength of a path, $\bar{P}$ is an ordered triplet denoted by $\bar{S}\left(\bar{P}\right)=\left(\bar{s}_1,\bar{s}_2,\bar{s}_3\right)$, where

$$\bar{s}_1=\min\left\{\left[T_Y^L\left(xy\right),T_Y^U\left(xy\right)\right]:xy\in\bar{P}\right\}$$
$$\bar{s}_2=\min\left\{\left[I_Y^L\left(xy\right),I_Y^U\left(xy\right)\right]:xy\in\bar{P}\right\}$$
$$\bar{s}_3=\max\left\{\left[F_Y^L\left(xy\right),F_Y^U\left(xy\right)\right]:xy\in\bar{P}\right\}$$

Similarly, the incidence strength of a path, $\bar{P}$ in an IVNIG is denoted by $I\bar{S}\left(\bar{P}\right)=\left(i\bar{s}_1,i\bar{s}_2,i\bar{s}_3\right)$, where

$$i\bar{s}_1=\min\left\{\left[T_Z^L\left(x,xy\right),T_Z^U\left(x,xy\right)\right]:\left(x,xy\right)\in\bar{P}\right\}$$
$$i\bar{s}_2=\min\left\{\left[I_Z^L\left(x,xy\right),I_Z^U\left(x,xy\right)\right]:\left(x,xy\right)\in\bar{P}\right\}$$
$$i\bar{s}_3=\max\left\{\left[F_Z^L\left(x,xy\right),F_Z^U\left(x,xy\right)\right]:\left(x,xy\right)\in\bar{P}\right\}$$

Example 3

Let $G=\left(V,E,I\right)$ be an incidence graph and $\hat{G}=\left(\hat{X},\hat{Y},\hat{Z}\right)$ is an IVNIG associated with G, which is shown in Tables 2 – 4. Clearly, $\bar{P}_1:a,\left(a,ac\right),ac,\left(c,ac\right),c,\left(c,cd\right),cd,\left(d,cd\right),d$ is a path in $\hat{G}$.

The strength of the path $\bar{P}_1$ is $S\left(\bar{P}_1\right)=\left(\left[0.1,0.4\right],\left[0.2,0.5\right],\left[0.5,0.8\right]\right)$ while the incidence strength of $\bar{P}_1$ is $IS\left(\bar{P}_1\right)=\left(\left[0.1,0.3\right],\left[0.1,0.4\right],\left[0.4,0.9\right]\right)$.

Definition 7

In an IVNIG, the greatest strength of the path from m to n, where $m,n\in A^{*}\cup B^{*}$ is the maximum strength of all paths from m to n. Moreover, $\bar{S}^{\infty}\left(m,n\right)$ is sometimes called the connectedness between m to n.

$$\begin{aligned}\bar{S}^{\infty}\left(m,n\right)&=\max\left\{\bar{S}\left(P_1\right),\bar{S}\left(P_2\right),\bar{S}\left(P_3\right),\ldots\right\}\\&=\left(\bar{s}_1^{\infty},\bar{s}_2^{\infty},\bar{s}_3^{\infty}\right)\\&=\left(\max\left(\bar{s}_{11},\bar{s}_{12},\bar{s}_{13},\ldots\right),\max\left(\bar{s}_{21},\bar{s}_{22},\bar{s}_{23},\ldots\right),\min\left(\bar{s}_{31},\bar{s}_{32},\bar{s}_{33},\ldots\right)\right)\end{aligned}$$

Similarly, the greatest incidence strength of the path from m to n, where $m,n\in A^{*}\cup B^{*}$ is the maximum incidence strength of all paths from m to n, given by

$$\begin{aligned}I\bar{S}^{\infty}\left(m,n\right)&=\max\left\{I\bar{S}\left(P_1\right),I\bar{S}\left(P_2\right),I\bar{S}\left(P_3\right),\ldots\right\}\\&=\left(i\bar{s}_1^{\infty},i\,\bar{s}_2^{\infty},i\,\bar{s}_3^{\infty}\right)\\&=\left(\max\left(i\bar{s}_{11},i\,\bar{s}_{12},i\,\bar{s}_{13},\ldots\right),\max\left(i\bar{s}_{21},i\,\bar{s}_{22},i\,\bar{s}_{23},\ldots\right),\min\left(i\bar{s}_{31},i\,\bar{s}_{32},i\,\bar{s}_{33},\ldots\right)\right)\end{aligned}$$

where $P_i,i=1,2,3,\ldots$ are different paths from m to n. $I\bar{S}^{\infty}\left(m,n\right)$ is sometimes referred to as the incidence connectedness between m to n.

Example 4

In the IVNIG given in Tables 2 - 4, the total paths from vertex *b* to *d* are given as follows:

$$\bar{P}_1 : b,(b,bc),bc,(c,bc),c,(c,cd),cd,(d,cd),d$$
$$\bar{P}_2 : b,(b,ab),ab,(a,ab),a,(a,ad),ad,(d,ad),d$$
$$\bar{P}_3 : b,(b,bc),bc,(c,bc),c,(c,ac),ac,(a,ac),a,(a,ad),ad,(d,ad),d$$
$$\bar{P}_4 : b,(b,ab),ab,(a,ab),a,(a,ac),ac,(c,ac),c,(c,cd),cd,(d,cd),d$$

The corresponding incidence strengths of each path are

$$IS(\bar{P}_1) = (\bar{s}_{11}, \bar{s}_{21}, \bar{s}_{31}) = ([0.3,0.5],[0.2,0.6],[0.5,0.8])$$
$$IS(\bar{P}_2) = (\bar{s}_{12}, \bar{s}_{22}, \bar{s}_{32}) = ([0.1,0.4],[0.2,0.5],[0.4,0.8])$$
$$IS(\bar{P}_3) = (\bar{s}_{13}, \bar{s}_{23}, \bar{s}_{33}) = ([0.1,0.4],[0.2,0.5],[0.5,0.8])$$
$$IS(\bar{P}_4) = (\bar{s}_{14}, \bar{s}_{24}, \bar{s}_{34}) = ([0.1,0.4],[0.2,0.5],[0.5,0.8])$$

Hence, the greatest incidence strength of the path form is calculated as follows:

$$\begin{aligned} I\bar{S}^{\infty}(b,d) &= \max\{IS(\bar{P}_1), IS(\bar{P}_2), IS(\bar{P}_3), IS(\bar{P}_4)\} \\ &= \left(\max\{i\bar{s}_{11}, i\bar{s}_{12}, i\bar{s}_{13}, i\bar{s}_{14}\}, \max\{i\bar{s}_{21}, i\bar{s}_{22}, i\bar{s}_{23}, i\bar{s}_{24}\}, \min\{i\bar{s}_{31}, i\bar{s}_{32}, i\bar{s}_{33}, i\bar{s}_{34}\}\right) \\ &= \begin{pmatrix} \max\{[0.3,0.5],[0.1,0.4],[0.1,0.4],[0.1,0.4]\}, \max\{[0.2,0.6],[0.2,0.5],[0.2,0.5],[0.2,0.5]\}, \\ \min\{[0.5,0.8],[0.4,0.8],[0.5,0.8],[0.5,0.8]\} \end{pmatrix} \\ &= ([0.3,0.5],[0.2,0.6],[0.4,0.8]). \end{aligned}$$

Definition 8

An IVNIG $\hat{G} = (\hat{X}, \hat{Y}, \hat{Z})$ is a cycle if and only if the underlying graph $G^* = (X^*, Y^*, Z^*)$ is a cycle.

Definition 9

The IVNIG $\hat{G} = (\hat{X}, \hat{Y}, \hat{Z})$ is a neutrosophic cycle if and only if, $G^* = (X^*, Y^*, Z^*)$ is a cycle, and there exists no unique edge $vw \in Y^*$, such that

$$T_Y^L(vw) = \min\{T_Y^L(xy) : xy \in Y^*\}, T_Y^U(vw) = \min\{T_Y^U(xy) : xy \in Y^*\},$$
$$I_Y^L(vw) = \min\{I_Y^L(xy) : xy \in Y^*\}, \ I_Y^U(vw) = \min\{I_Y^U(xy) : xy \in Y^*\},$$
$$F_Y^L(vw) = \max\{F_Y^L(xy) : xy \in Y^*\}, F_Y^U(vw) = \max\{F_Y^U(xy) : xy \in Y^*\}.$$

Definition 10

The IVNIG $\hat{G} = (\hat{X}, \hat{Y}, \hat{Z})$ is a neutrosophic incidence cycle if, and only if it is a neutrosophic cycle and there exists no unique pair $(v, vw) \in Z^*$, such that

$$T_Z^L(v,vw) = \min\{T_Z^L(x,xy) : (x,xy) \in Z^*\},$$
$$T_Z^U(v,vw) = \min\{T_Z^U(x,xy) : (x,xy) \in Z^*\},$$
$$I_Z^L(v,vw) = \min\{I_Z^L(x,xy) : (x,xy) \in Z^*\},$$
$$I_Z^U(v,vw) = \min\{I_Z^U(x,xy) : (x,xy) \in Z^*\},$$
$$F_Z^L(v,vw) = \max\{F_Z^L(x,xy) : (x,xy) \in Z^*\},$$
$$F_Z^U(v,vw) = \max\{F_Z^U(x,xy) : (x,xy) \in Z^*\}.$$

Example 5

Let $\hat{G}=\left(\hat{X},\hat{Y},\hat{Z}\right)$ be an IVNIG. G is a cycle since $G^*=\left(X^*,Y^*,Z^*\right)$ (support of $\hat{G}$) is a cycle.

$$\hat{X}=\left\{\begin{array}{l}\left(a,[0.1,0.4],[0.2,0.5],[0.3,0.7]\right),\left(b,[0.3,0.5],[0.2,0.6],[0.1,0.7]\right),\\ \left(c,[0.3,0.8],[0.4,0.9],[0.5,0.8]\right),\left(d,[0.4,0.7],[0.3,0.8],[0.4,0.8]\right),\\ \left(e,[0.7,0.9],[0.5,0.8],[0.3,0.5]\right)\end{array}\right\}$$

$$\hat{Y}=\left\{\begin{array}{l}\left(ab,[0.2,0.8],[0.2,0.7],[0.3,0.8]\right),\left(bc,[0.3,0.8],[0.4,0.7],[0.5,0.8]\right),\\ \left(cd,[0.4,0.8],[0.4,0.9],[0.3,0.8]\right),\left(de,[0.1,0.9],[0.5,0.8],[0.2,0.6]\right),\\ \left(ea,[0.3,0.6],[0.2,0.5],[0.2,0.5]\right)\end{array}\right\}$$

$$\hat{Z}=\left\{\begin{array}{l}\left((a,ab),[0.1,0.4],[0.2,0.5],[0.4,0.8]\right),\left((b,ab),[0.2,0.5],[0.2,0.6],[0.2,0.8]\right),\\ \left((b,bc),[0.3,0.4],[0.1,0.5],[0.5,0.7]\right),\left((c,bc),[0.2,0.5],[0.2,0.6],[0.5,0.7]\right),\\ \left((c,cd),[0.3,0.7],[0.3,0.8],[0.5,0.7]\right),\left((d,cd),[0.4,0.6],[0.2,0.7],[0.4,0.8]\right),\\ \left((d,de),[0.1,0.6],[0.3,0.8],[0.4,0.7]\right),\left((e,de),[0.1,0.8],[0.4,0.7],[0.3,0.6]\right),\\ \left((a,ea),[0.1,0.3],[0.1,0.4],[0.3,0.7]\right),\left((e,ea),[0.3,0.5],[0.2,0.4],[0.3,0.5]\right)\end{array}\right\},$$

$$\begin{array}{l}T_Y^L(ab)=0.1=\min\{T_Y^L(ab),T_Y^L(bc),T_Y^L(cd),T_Y^L(de),T_Y^L(ea)\},\\ T_Y^U(ab)=0.6=\min\{T_Y^U(ab),T_Y^U(bc),T_Y^U(cd),T_Y^U(de),T_Y^U(ea)\},\\ I_Y^L(ab)=0.2=\min\{I_Y^L(ab),I_Y^L(bc),I_Y^L(cd),I_Y^L(de),I_Y^L(ea)\},\\ I_Y^U(ab)=0.5=\min\{I_Y^U(ab),I_Y^U(bc),I_Y^U(cd),I_Y^U(de),I_Y^U(ea)\},\\ F_Y^L(ab)=0.5=\max\{F_Y^L(ab),F_Y^L(bc),F_Y^L(cd),F_Y^L(de),F_Y^L(ea)\},\\ F_Y^U(ab)=0.8=\max\{F_Y^U(ab),F_Y^U(bc),F_Y^U(cd),F_Y^U(de),F_Y^U(ea)\}.\end{array}$$

and

$$\begin{array}{l}T_Y^L(bc)=0.1=\min\{T_Y^L(ab),T_Y^L(bc),T_Y^L(cd),T_Y^L(de),T_Y^L(ea)\},\\ T_Y^U(bc)=0.6=\min\{T_Y^U(ab),T_Y^U(bc),T_Y^U(cd),T_Y^U(de),T_Y^U(ea)\},\\ I_Y^L(bc)=0.2=\min\{I_Y^L(ab),I_Y^L(bc),I_Y^L(cd),I_Y^L(de),I_Y^L(ea)\},\\ I_Y^U(bc)=0.5=\min\{I_Y^U(ab),I_Y^U(bc),I_Y^U(cd),I_Y^U(de),I_Y^U(ea)\},\\ F_Y^L(bc)=0.5=\max\{F_Y^L(ab),F_Y^L(bc),F_Y^L(cd),F_Y^L(de),F_Y^L(ea)\},\\ F_Y^U(bc)=0.8=\max\{F_Y^U(ab),F_Y^U(bc),F_Y^U(cd),F_Y^U(de),F_Y^U(ea)\}.\end{array}$$

Thus, $\hat{G}$ is an interval-valued neutrosophic cycle.

Furthermore, $\hat{G}$ is a neutrosophic incidence cycle since there is more than one pair, namely (b,ab) and (d,de) such that

$$\begin{array}{l}T_Z^L(b,ab)=0.1=\min\{T_Z^L(v,vw):(v,vw)\in Z^*\},\\ T_Z^U(b,ab)=0.3=\min\{T_Z^U(v,vw):(v,vw)\in Z^*\},\\ I_Z^L(b,ab)=0.1=\min\{I_Z^L(v,vw):(v,vw)\in Z^*\},\\ I_Z^U(b,ab)=0.4=\min\{I_Z^U(v,vw):(v,vw)\in Z^*\},\\ F_Z^L(b,ab)=0.5=\max\{F_Z^L(v,vw):(v,vw)\in Z^*\},\\ F_Z^U(b,ab)=0.8=\max\{F_Z^U(v,vw):(v,vw)\in Z^*\}\end{array}$$

and

$$T_Z^L(d,de)=0.1=\min\{T_Z^L(v,vw):(v,vw)\in Z^*\},$$
$$T_Z^U(d,de)=0.3=\min\{T_Z^U(v,vw):(v,vw)\in Z^*\},$$
$$I_Z^L(d,de)=0.1=\min\{I_Z^L(v,vw):(v,vw)\in Z^*\},$$
$$I_Z^U(d,de)=0.4=\min\{I_Z^U(v,vw):(v,vw)\in Z^*\},$$
$$F_Z^L(d,de)=0.5=\max\{F_Z^L(v,vw):(v,vw)\in Z^*\},$$
$$F_Z^U(d,de)=0.8=\max\{F_Z^U(v,vw):(v,vw)\in Z^*\}$$

The concepts of bridges, cut vertices and cut pairs in IVNIG are defined as follows:

Definition 11

Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG. An edge, vw in $\hat{G}$ is called a bridge if and only if vw is a bridge in $G^*=(X^*,Y^*,Z^*)$ which is the removal of vw disconnects G^*.

An edge vw is called a neutrosophic bridge if

$$\bar{S}'^{\infty}(a,b)<\bar{S}^{\infty}(a,b)\text{ for some } a,b\in X^*$$
$$(\bar{s}_1'^{\infty},\bar{s}_2'^{\infty},\bar{s}_3'^{\infty})<(\bar{s}_1^{\infty},\bar{s}_2^{\infty},\bar{s}_3^{\infty})$$
$$\bar{s}_1'^{\infty}<\bar{s}_1^{\infty},\bar{s}_2'^{\infty}<\bar{s}_2^{\infty},\bar{s}_3'^{\infty}>\bar{s}_3^{\infty}$$

where $\bar{S}'^{\infty}(a,b)$ and $\bar{S}^{\infty}(a,b)$ denote the connectedness between a and b in $\bar{G}=\hat{G}-\{vw\}$ and $\hat{G}$, respectively.

An edge vw is called a neutrosophic incidence bridge if

$$I\bar{S}'^{\infty}(a,b)<I\bar{S}^{\infty}(a,b)\text{ for some } a,b\in X^*$$
$$(i\bar{s}_1'^{\infty},i\bar{s}_2'^{\infty},i\bar{s}_3'^{\infty})<(i\bar{s}_1^{\infty},i\bar{s}_2^{\infty},i\bar{s}_3^{\infty})$$
$$i\bar{s}_1'^{\infty}<i\bar{s}_1^{\infty},i\bar{s}_2'^{\infty}<i\bar{s}_2^{\infty},i\bar{s}_3'^{\infty}>\bar{s}_3^{\infty}$$

where $I\bar{S}'^{\infty}(a,b)$ and $I\bar{S}^{\infty}(a,b)$ denote the incidence connectedness between a and b in $\bar{G}=\hat{G}-\{vw\}$ and $\hat{G}$, respectively.

Definition 12

Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG. A vertex, v, in $\hat{G}$ is called a cut vertex if and only if it is a cut vertex in $G^*=(X^*,Y^*,Z^*)$ where $G^*-\{v\}$ is the disconnect of G^*.

A vertex, v in an IVNIG is called a neutrosophic cut vertex if the connectedness between any two vertices in $\bar{G}=\hat{G}-\{v\}$ is less than the connectedness between the same vertices in $\hat{G}-$ that is,

$$\bar{S}'^{\infty}(a,b)<\bar{S}^{\infty}(a,b)\text{ for some } a,b\in X^*.$$

A vertex, v in an IVNIG is a neutrosophic incidence cut vertex if for any pair of vertices a and b other than v, the following condition holds:

$$\bar{S}'^{\infty}(a,b)<\bar{S}^{\infty}(a,b),$$

where $I\bar{S}'^{\infty}(a,b)$ and $I\bar{S}^{\infty}(a,b)$ denote the incidence connectedness between a and b in $\bar{G}=\hat{G}-\{vw\}$ and $\hat{G}$, respectively.

Definition 13

Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG. A pair, (v,vw) in $\hat{G}$ is called a cut pair if and only if (v,vw) is a cut pair in $G^*=(X^*,Y^*,Z^*)$ that is, after removing the pair (v,vw), there is no path between v and vw.

Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG. A pair, (v,vw) is called a neutrosophic cut pair if deleting the pair (v,vw) reduces the connectedness between $v,vw\in X^*\cup Y^*$, that is,

$$\bar{S}'^{\infty}(v,vw)<\bar{S}^{\infty}(v,vw),$$

where $\bar{S}'^{\infty}(v,vw)$ and $\bar{S}^{\infty}(v,vw)$ denote the connectedness between v and w in $\bar{G}=\hat{G}-\{v,vw\}$ and $\hat{G}$, respectively.

A pair (v,vw) is called neutrosophic incidence cut pair if

$$I\bar{S}'^{\infty}(v,vw)<I\bar{S}^{\infty}(v,vw) \text{ for } v,vw\in X^*\cup Y^*,$$

where $I\bar{S}'^{\infty}(v,vw)$ and $I\bar{S}^{\infty}(v,vw)$ denote the incidence connectedness between v and vw in $\bar{G}=\hat{G}-\{vw\}$ and $\hat{G}$, respectively.

Definition 14

Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG. An edge, vw of $\hat{G}$ is called a strong edge if

$$\bar{S}'^{\infty}(v,w)\le\left(\left[T_Y^L(vw),T_Y^U(vw)\right],\left[I_Y^L(vw),I_Y^U(vw)\right],\left[F_Y^L(vw),F_Y^U(vw)\right]\right),$$

where $\bar{S}'^{\infty}(v,w)$ represents the connectedness between v and w in $\bar{G}=\hat{G}-\{vw\}$.

In particular, an edge vw is said to be an $\alpha-$strong edge if

$$\bar{S}'^{\infty}(v,w)<\left(\left[T_Y^L(vw),T_Y^U(vw)\right],\left[I_Y^L(vw),I_Y^U(vw)\right],\left[F_Y^L(vw),F_Y^U(vw)\right]\right),$$

and it is called $\beta-$strong edge if

$$\bar{S}'^{\infty}(v,w)=\left(\left[T_Y^L(vw),T_Y^U(vw)\right],\left[I_Y^L(vw),I_Y^U(vw)\right],\left[F_Y^L(vw),F_Y^U(vw)\right]\right).$$

Definition 15

A pair (v,vw) in an IVNIG, $\hat{G}$ is called a strong pair if

$$I\bar{S}'^{\infty}(v,vw)\le\left(\left[T_Z^L(v,vw),T_Z^U(v,vw)\right],\left[I_Z^L(v,vw),I_Z^U(v,vw)\right],\left[F_Z^L(v,vw),F_Z^U(v,vw)\right]\right),$$

where $I\bar{S}'^{\infty}(v,vw)$ represents the incidence connectedness between v and vw in $\bar{G}=\hat{G}-\{(v,vw)\}$.

In particular, an edge (v,vw) is called $\alpha-$strong pair if

$$I\bar{S}'^{\infty}(v,vw)<\left(\left[T_Z^L(v,vw),T_Z^U(v,vw)\right],\left[I_Z^L(v,vw),I_Z^U(v,vw)\right],\left[F_Z^L(v,vw),F_Z^U(v,vw)\right]\right),$$

and it is called $\beta-$strong pair if

$$I\bar{S}'^{\infty}(v,vw)=\left(\left[T_Z^L(v,vw),T_Z^U(v,vw)\right],\left[I_Z^L(v,vw),I_Z^U(v,vw)\right],\left[F_Z^L(v,vw),F_Z^U(v,vw)\right]\right).$$

All edges and pairs do not need to be strong. There exist edges and pairs that are not strong in an IVNIG. Such edges and pairs are given in the following definition.

Definition 16

Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG. An edge, vw is said to be $\delta-$edge if

$$\bar{S}'^{\infty}(v,w)>\left(\left[T_Y^L(vw),T_Y^U(vw)\right],\left[I_Y^L(vw),I_Y^U(vw)\right],\left[F_Y^L(vw),F_Y^U(vw)\right]\right).$$

Similarly, a pair (v,vw) in $\hat{G}$ is called $\delta-$pair if

$$I\bar{S}^{'\infty}(v,vw) > \left(\left[T_Z^L(v,vw),T_Z^U(v,vw)\right],\left[I_Z^L(v,vw),I_Z^U(v,vw)\right],\left[F_Z^L(v,vw),F_Z^U(v,vw)\right]\right).$$

Theorem 1. Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG. If vw is a neutrosophic bridge, then vw is a strong edge in any cycle.

Proof. Let vw be a neutrosophic bridge. By contradiction, suppose that vw is not a strong edge of a cycle. Then, in this cycle, we can find an alternative path, P_1 from v to w that contains the edge vw and $S(P_1)$ is less than or equal to $S(P_2)$, where P_2 is the path that does not contain the edge vw. Thus, removing the edge of vw from $\hat{G}$ does not affect the connectedness between v and w, which is a contradiction to our assumption. Hence, vw is a strong edge in any cycle. □

Theorem 2. If (v,vw) is a neutrosophic incidence cut pair, then (v,vw) is a strong pair in any cycle.

Proof. Let (v,vw) be a neutrosophic incidence cut pair in $\hat{G}$ and by contradiction, suppose that (v,vw) is not a strong pair of a cycle. Then, we can find an alternative path from v to vw having incidence strength greater than or equal to that of the path involving the pair (v,vw). Thus, removal of the pair (v,vw) does not affect the incidence connectedness between v and vw. This is a contradiction to our assumption that (v,vw) is a neutrosophic incidence cut pair. Hence, (v,vw) is a strong pair in any cycle. □

Theorem 3. Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG. If vw is a neutrosophic bridge in $\hat{G}$, then

$$\bar{S}^{\infty}(v,w)=(\bar{s}_1^{\infty},\bar{s}_2^{\infty},\bar{s}_3^{\infty})=\left(\left[T_Y^L(vw),T_Y^U(vw)\right],\left[I_Y^L(vw),I_Y^U(vw)\right],\left[F_Y^L(vw),F_Y^U(vw)\right]\right).$$

Proof. Let $\hat{G}$ be an IVNIG and vw is a neutrosophic bridge in $\hat{G}$. By contradiction, suppose that

$$\bar{S}^{\infty}(v,w)>\left(\left[T_Y^L(vw),T_Y^U(vw)\right],\left[I_Y^L(vw),I_Y^U(vw)\right],\left[F_Y^L(vw),F_Y^U(vw)\right]\right).$$

Then, there exists a $v-w$ path, P, with

$$\bar{S}(P)>\left(\left[T_Y^L(vw),T_Y^U(vw)\right],\left[I_Y^L(vw),I_Y^U(vw)\right],\left[F_Y^L(vw),F_Y^U(vw)\right]\right)$$

and

$$\left(\left[T_Y^L(xy),T_Y^U(xy)\right],\left[I_Y^L(xy),I_Y^U(xy)\right],\left[F_Y^L(xy),F_Y^U(xy)\right]\right)>$$
$$\left(\left[T_Y^L(vw),T_Y^U(vw)\right],\left[I_Y^L(vw),I_Y^U(vw)\right],\left[F_Y^L(vw),F_Y^U(vw)\right]\right),$$

for all edges on path P. Now, P together with the edge vw forms a cycle in which vw is the weakest edge, but it is a contradiction to the fact that vw is a neutrosophic bridge. Hence,

$$\bar{S}^{\infty}(v,w)=(\bar{s}_1^{\infty},\bar{s}_2^{\infty},\bar{s}_3^{\infty})=\left(\left[T_Y^L(vw),T_Y^U(vw)\right],\left[I_Y^L(vw),I_Y^U(vw)\right],\left[F_Y^L(vw),F_Y^U(vw)\right]\right).$$

□

Theorem 4. Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG. If (v,vw) is a neutrosophic incidence cut pair in $\hat{G}$, then

$$I\bar{S}^{\infty}(v,vw)=(i\bar{s}_1^{\infty},i\bar{s}_2^{\infty},i\bar{s}_3^{\infty})$$
$$=([T_Z^L(v,vw),T_Z^U(v,vw)],[I_Z^L(v,vw),I_Z^U(v,vw)],[F_Z^L(v,vw),F_Z^U(v,vw)]).$$

Proof. Let $\hat{G}$ be an IVNIG and (v,vw) is a neutrosophic incidence cut pair in $\hat{G}$. By contradiction, suppose that

$$I\bar{S}^{\infty}(v,vw)>([T_Z^L(v,vw),T_Z^U(v,vw)],[I_Z^L(v,vw),I_Z^U(v,vw)],[F_Z^L(v,vw),F_Z^U(v,vw)]).$$

Then, there exists a $v-w$ path, P, with

$$I\bar{S}(P)>([T_Z^L(v,vw),T_Z^U(v,vw)],[I_Z^L(v,vw),I_Z^U(v,vw)],[F_Z^L(v,vw),F_Z^U(v,vw)])$$

and

$$([T_Z^L(x,xy),T_Z^U(x,xy)],[I_Z^L(x,xy),I_Z^U(x,xy)],[F_Z^L(x,xy),F_Z^U(x,xy)])>$$
$$([T_Z^L(v,vw),T_Z^U(v,vw)],[I_Z^L(v,vw),I_Z^U(v,vw)],[F_Z^L(v,vw),F_Z^U(v,vw)]),$$

for all pairs on path P. Now, P together with the pair (v,vw) forms a cycle in which (v,vw) is the weakest pair. However, it is a contradiction to the fact that (v,vw) is a neutrosophic incidence cut pair. Hence,

$$I\bar{S}^{\infty}(v,vw)=(i\bar{s}_1^{\infty},i\bar{s}_2^{\infty},i\bar{s}_3^{\infty})$$
$$=([T_Z^L(v,vw),T_Z^U(v,vw)],[I_Z^L(v,vw),I_Z^U(v,vw)],[F_Z^L(v,vw),F_Z^U(v,vw)]).$$

□

Theorem 5. Every neutrosophic incidence cut pair in IVNIG is a strong cut pair.

Proof. Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG. Let $(v,vw)\in Z^*$ be a neutrosophic incidence cut pair. Then, by Definition 12, we have

$$I\bar{S}'^{\infty}(v,vw)<I\bar{S}^{\infty}(v,vw).$$

By contradiction, suppose that (v,vw) is not a strong incidence pair. Then it follows that

$$I\bar{S}'^{\infty}(v,vw)>([T_Z^L(v,vw),T_Z^U(v,vw)],[I_Z^L(v,vw),I_Z^U(v,vw)],[F_Z^L(v,vw),F_Z^U(v,vw)]).$$

Let P be the path from v to vw in $\bar{G}=\hat{G}-\{(v,vw)\}$ with the greatest incidence strength. Then P together with (v,vw) forms a cycle in $\hat{G}$. Now, in this cycle, (v,vw) is the weakest pair. However, based on Theorem 2, this is not possible since (v,vw) is a neutrosophic incidence cut pair. This is a contradiction to our assumption, hence (v,vw) is a strong incidence pair. □

Theorem 6. Let $\hat{G}=(\hat{X},\hat{Y},\hat{Z})$ be an IVNIG. The pair (v,vw) is a neutrosophic incidence cut pair if and only if it is $\alpha-$strong.

Proof. Let (v,vw) be a neutrosophic incidence cut pair in $\hat{G}$. Based on Definition 12,

$$I\bar{S}^{\infty}(v,vw)>I\bar{S}'^{\infty}(v,vw)$$

Then, based on Theorem 4, it follows that

$$\left(\left[T_Z^L(v,vw),T_Z^U(v,vw)\right],\left[I_Z^L(v,vw),I_Z^U(v,vw)\right],\left[F_Z^L(v,vw),F_Z^U(v,vw)\right]\right)>I\bar{S}'^{\infty}(v,v\,w),$$

which is the definition of $\alpha-$strong. Hence, (v,vw) is an $\alpha-$strong pair in $\hat{G}$.

Conversely, suppose that (v,vw) is an $\alpha-$strong pair in $\hat{G}$. Then, by definition

$$\left(\left[T_Z^L(v,vw),T_Z^U(v,vw)\right],\left[I_Z^L(v,vw),I_Z^U(v,vw)\right],\left[F_Z^L(v,vw),F_Z^U(v,vw)\right]\right)>I\bar{S}'^{\infty}(v,v\,w).$$

It follows that $P:v,(v,vw),vw$ is the unique strongest incidence path from v to vw. The removal of (v,vw) reduces the incidence strength between v and vw, giving

$$I\bar{S}^{\infty}(v,vw)>I\bar{S}'^{\infty}(v,vw).$$

Hence, (v,vw) is a neutrosophic incidence cut pair. □

4. Application in Finding the Best Route

In this section, the developed approach of IVNIGs is utilized in the safe root problem dealing with the selection of the best route among some routes. Suppose Mr Manapat wants to travel from Thailand to Indonesia following all border lines between Thailand and Indonesia. There are basically three ways of doing so. The first one is a direct way, i.e., Thailand to Indonesia, the second one is Thailand to Malaysia and Malaysia to Indonesia and the last one is Thailand to Singapura, Singapura to Malaysia and Malaysia to Indonesia, as shown in Figure 3.

Let V = {Thailand (THAI), Singapura (SGPR), Malaysia (MAL), Indonesia (IDN)} be the set of countries.

Let E = {(THAI,SGPR),(SGPR,MAL),(THAI,MAL),(MAL,IDN),(THAI,IDN)} a subset of $V\times V$.

Let X be the interval-valued neutrosophic set on V, given as:

$$X=\begin{Bmatrix}(\text{THAI},[0.1,0.4],[0.2,0.5],[0.3,0.7]),(\text{SGPR},[0.3,0.5],[0.2,0.6],[0.1,0.7]),\\(\text{MAL},[0.3,0.8],[0.4,0.9],[0.5,0.9]),(\text{IDN},[0.4,0.7],[0.3,0.8],[0.4,0.9])\end{Bmatrix}.$$

Let Y be the interval-valued neutrosophic relation on V, given as:

$$Y=\begin{Bmatrix}((\text{THAI},\text{SGPR}),[0.1,0.4],[0.2,0.5],[0.3,0.7]),((\text{THAI},\text{MAL}),[0.1,0.4],[0.2,0.5],[0.4,0.8]),\\((\text{SGPR},\text{MAL}),[0.3,0.5],[0.2,0.6],[0.5,0.8]),((\text{MAL},\text{IDN}),[0.3,0.7],[0.3,0.8],[0.5,0.8]),\\((\text{THAI},\text{IDN}),[0.1,0.4],[0.2,0.5],[0.4,0.8])\end{Bmatrix}.$$

Let Z be the interval-valued neutrosophic set on $V \times E$, given as:

$$Z = \left\{ \begin{array}{l} \left(\left(\text{THAI},\left(\text{THAI},\text{SGPR}\right)\right),\left[0.1,0.3\right],\left[0.1,0.4\right],\left[0.3,0.7\right]\right), \\ \left(\left(\text{SGPR},\left(\text{THAI},\text{SGPR}\right)\right),\left[0.1,0.3\right],\left[0.1,0.5\right],\left[0.2,0.6\right]\right), \\ \left(\left(\text{THAI},\left(\text{THAI},\text{MAL}\right)\right),\left[0.1,0.3\right],\left[0.1,0.5\right],\left[0.3,0.8\right]\right), \\ \left(\left(\text{MAL},\left(\text{THAI},\text{MAL}\right)\right),\left[0.1,0.4\right],\left[0.1,0.4\right],\left[0.4,0.8\right]\right), \\ \left(\left(\text{SGPR},\left(\text{SGPR},\text{MAL}\right)\right),\left[0.2,0.5\right],\left[0.1,0.5\right],\left[0.4,0.7\right]\right), \\ \left(\left(\text{MAL},\left(\text{SGPR},\text{MAL}\right)\right),\left[0.3,0.5\right],\left[0.2,0.6\right],\left[0.4,0.8\right]\right), \\ \left(\left(\text{MAL},\left(\text{MAL},\text{IDN}\right)\right),\left[0.3,0.7\right],\left[0.3,0.8\right],\left[0.4,0.9\right]\right), \\ \left(\left(\text{IDN},\left(\text{MAL},\text{IDN}\right)\right),\left[0.3,0.6\right],\left[0.2,0.7\right],\left[0.3,0.8\right]\right), \\ \left(\left(\text{THAI},\left(\text{THAI},\text{IDN}\right)\right),\left[0.1,0.4\right],\left[0.2,0.4\right],\left[0.3,0.7\right]\right), \\ \left(\left(\text{IDN},\left(\text{THAI},\text{IDN}\right)\right),\left[0.1,0.4\right],\left[0.2,0.5\right],\left[0.4,0.8\right]\right) \end{array} \right\}.$$

Let $\left(T_{\bar{P}}^{L}\left(uv\right),T_{\bar{P}}^{U}\left(uv\right)\right)$ represent the degree of protection for travelling from country u to the country v. There are three paths from THAI to IDN, such that

$\bar{P}_1:\text{THAI},\left(\text{THAI},\left(\text{THAI},\text{IDN}\right)\right)\left(\text{THAI},\text{IDN}\right),\left(\text{IDN},\left(\text{THAI},\text{IDN}\right)\right),\text{IDN}$

$\bar{P}_2:\text{THAI},\left(\text{THAI},\left(\text{THAI},\text{MAL}\right)\right)\left(\text{THAI},\text{MAL}\right),\left(\text{MAL},\left(\text{THAI},\text{MAL}\right)\right),\text{MAL},\left(\text{MAL},\left(\text{MAL},\text{IDN}\right)\right),$
$\left(\text{MAL},\text{IDN}\right),\left(\text{IDN},\left(\text{MAL},\text{IDN}\right)\right),\text{IDN}$

$\bar{P}_3:\text{THAI},\left(\text{THAI},\left(\text{THAI},\text{SGPR}\right)\right)\left(\text{THAI},\text{SGPR}\right),\left(\text{SGPR},\left(\text{THAI},\text{SGPR}\right)\right),\text{SGPR},\left(\text{SGPR},\left(\text{SGPR},\text{MAL}\right)\right),$
$\left(\text{SGPR},\text{MAL}\right),\left(\text{MAL},\left(\text{SGPR},\text{MAL}\right)\right),\text{MAL},\left(\text{MAL},\left(\text{MAL},\text{IDN}\right)\right),\left(\text{MAL},\text{IDN}\right),\left(\text{IDN},\left(\text{MAL},\text{IDN}\right)\right),\text{IDN}$

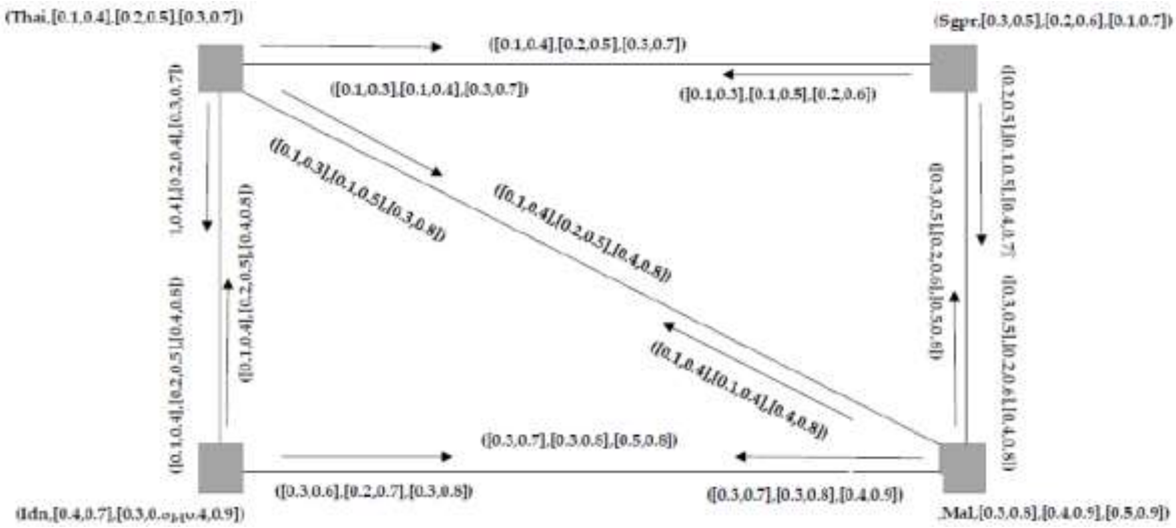

Figure 3 Model of travelling paths from Thailand to Indonesia

$\bar{IS}^{\infty}(\text{THAI}, \text{IDN})$ is the greatest incidence strength of the path between THAI and IDN. In other words, this is the safest path between THAI and IDN. To calculate the value of $\bar{IS}^{\infty}(\text{THAI}, \text{IDN})$, we need to first calculate the incidence strength of paths $\bar{P}_1, \bar{P}_2$ and $\bar{P}_3$ denoted by $\bar{IS}_{\bar{P}_1}(\text{THAI}, \text{IDN})$, $\bar{IS}_{\bar{P}_2}(\text{THAI}, \text{IDN})$ and $\bar{IS}_{\bar{P}_3}(\text{THAI}, \text{IDN})$, respectively. By calculation, we obtain

$$\bar{IS}_{\bar{P}_1}(\text{THAI}, \text{IDN}) = ([0.1, 0.4], [0.2, 0.5], [0.4, 0.8])$$
$$\bar{IS}_{\bar{P}_2}(\text{THAI}, \text{IDN}) = ([0.3, 0.7], [0.3, 0.8], [0.5, 0.9])$$
$$\bar{IS}_{\bar{P}_3}(\text{THAI}, \text{IDN}) = ([0.2, 0.4], [0.2, 0.5], [0.5, 0.9]).$$

Hence,

$$\bar{IS}^{\infty}(\text{THAI}, \text{IDN}) = ([0.3, 0.7], [0.2, 0.6], [0.4, 0.8]).$$

We see that $\left(T^{L}_{\bar{IS}^{\infty}}(\text{THAI}, \text{IDN}), T^{U}_{\bar{IS}^{\infty}}(\text{THAI}, \text{IDN})\right) = \left(T^{L}_{\bar{IS}_{\bar{P}2}}(\text{THAI}, \text{IDN}), T^{U}_{\bar{IS}_{\bar{P}2}}(\text{THAI}, \text{IDN})\right)$.

Therefore, $\bar{P}_2$ is the safest path for travelling. We present the proposed method in the following algorithm.

Algorithm:

1. Input the vertex set $\hat{V}$.
2. Input the edge set $\hat{E} \subseteq \hat{V} \times \hat{V}$.
3. Set up the interval-valued neutrosophic set X on $\hat{V}$.
4. Set up the interval-valued neutrosophic relation Y on $\hat{V}$.
5. Set up the interval-valued neutrosophic set Z on $\hat{V} \times \hat{E}$.
6. Calculate the incidence strength $\bar{IS}(x_i, y_j)$ of all possible paths from x to y such that

$$i\bar{s}_1 = \min\left\{\left[T^L_Z(x_i, x_i x_{i+1}), T^U_Z(x_i, x_i x_{i+1})\right] : (x_i, x_i x_{i+1}) \in I\right\}$$
$$i\bar{s}_2 = \min\left\{\left[I^L_Z(x_i, x_i x_{i+1}), I^U_Z(x_i, x_i x_{i+1})\right] : (x_i, x_i x_{i+1}) \in I\right\}$$
$$i\bar{s}_3 = \min\left\{\left[F^L_Z(x_i, x_i x_{i+1}), F^U_Z(x_i, x_i x_{i+1})\right] : (x_i, x_i x_{i+1}) \in I\right\}$$

7. Calculate the greatest incidence strength $\bar{IS}^{\infty}$ of the path from x to y.
8. The safest path is $S(v_k) = \min\left(T^L_{\bar{P}_i}(xy), T^U_{\bar{P}_i}(xy)\right)$ where $i = 1...k$
9. If v_k has more than one value then any path can be chosen.

5. Comparative Study and Advantages of the proposed algorithm

In this section, a comparative study based on the results of numerical computation is performed to validate the proposed method. For this purpose, we present a comparative analysis between fuzzy incidence graphs (FIG), single-valued neutrosophic incidence graph (SVNIG) and the proposed method IVNIG as presented in Table 5.

From the safest path column in Table 5, it can be seen that the safest path of our proposed method IVNIG is consistent with the FIG and SVNIG which is P_2. However, as we may notice, FIG just takes into consideration crisp membership values to represent the uncertain data. In this case, the non-membership values are directly complementing to their respective membership values. We can observe that these two elements of membership and non-membership are said to be dependent here. This approach, even though effective in dealing with uncertainty, but still cannot capture some types of uncertainties such as indeterminate and inconsistent information. Therefore, some new theories are required to overcome this problem.

Should be noted that, the proposal of IVNIG is to provide a generalization of the notion of SVNIG. The justification of this generalization lies in the following observation: sometimes it is not appropriate to assume that the degrees of (t, i, f) are exactly defined, therefore we can admit a kind of further uncertainty where the values of these components are not numbers, but interval of numbers. Clearly, SVNIG may be viewed as special cases of IVNIG here if the degrees of (t, i, f) are the only numbers. Furthermore, an IVNIG also can avoid the loss of information. Sometimes, the degree of memberships is not certainly known. Then interval-valued may better represent this kind of information. Similarly, if the three components are dependent, then IVNIG can be reduced to the FIG.

For SVNIG, the sum of the components is; $0 \le t+i+f \le 3$ when all three components are independent; $0 \le t+i+f \le 2$ when two components are dependent, while the third one is independent; $0 \le t+i+f \le 1$ when all three components are dependent. When three or two of the components T, I, F are independent, one leaves room for incomplete information (sum < 1), paraconsistent and contradictory information (sum > 1), or complete information (sum = 1). If all three components T, I, F are dependent, then similarly one leaves room for incomplete information (sum < 1), or complete information (sum = 1).

Table 5. A comparative study between FIG, SVNIG and IVNIG

	Incidence strength of path	Greatest incidence strength	Safest path
FIG	$I\bar{S}_{\bar{P}_1} = (0.1)$ $I\bar{S}_{\bar{P}_2} = (0.3)$ $I\bar{S}_{\bar{P}_3} = (0.2)$	$I\bar{S}^{\infty} = (0.3)$	$\bar{P}_2$
SVNIG	$I\bar{S}_{\bar{P}_1} = (0.1, 0.3, 0.4)$ $I\bar{S}_{\bar{P}_2} = (0.3, 0.4, 0.5)$ $I\bar{S}_{\bar{P}_3} = (0.2, 0.3, 0.5)$	$I\bar{S}^{\infty} = (0.3, 0.2, 0.5)$	$\bar{P}_2$
IVNIG (proposed method)	$I\bar{S}_{\bar{P}_1} = ([0.1, 0.4],[0.2, 0.5],[0.4, 0.8])$ $I\bar{S}_{\bar{P}_2} = ([0.3, 0.7],[0.3, 0.8],[0.5, 0.9])$ $I\bar{S}_{\bar{P}_3} = ([0.2, 0.4],[0.2, 0.5],[0.5, 0.9])$	$I\bar{S}^{\infty} = ([0.3, 0.7],[0.2, 0.6],[0.4, 0.8])$	$\bar{P}_2$

6. Conclusions

A new IVNIG has been successfully proposed. We constructed a new set for neutrosophic incidence graphs based on the definition from the previous study. An interval-valued neutrosophic set is an extension of an interval-valued fuzzy set combined with a single-valued neutrosophic set, a more powerful model to solve real-life problems. This paper has presented certain properties related to IVNIG such as strong edge, strong pair, strong cut pair and neutrosophic incidence cut pair. Also, in this work, we just limit our attention to the class of standard unit interval [0,1]. The assumption is that this unit interval may be sufficient to be applied in the real-life problems. However, further analysis can be potentially conducted on the non-standard unit interval to generalize the developed concept. Moreover, for future research, another higher order of uncertainty can be proposed to the neutrosophic set or neutrosophic incidence graph, i.e., incorporate the membership function to each element of (t, i, f) instead of interval-valued.

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Pairwise Neutrosophic *b*-Continuous Function in Neutrosophic Bitopological Spaces

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Abstract: The main focus of this article is to procurethe notions of pairwise neutrosophic continuous and pairwise neutrosophic *b*-continuous mappings in neutrosophic bitopological spaces. Then, we formulate some results on them via neutrosophic bitopological spaces.

Keywords: Neutrosophic Topology; Neutrosophic Bitopology; Pairwise Neutrosophic*b*-Interior; Pairwise Neutrosophic *b*-Closure; Pairwise Neutrosophic Continuous.

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1. Introduction

Zadeh [31] presented the notions of fuzzy set (in short FS) in the year 1965. Afterwards, Chang [4] applied the idea of topology on fuzzy sets and introduced the fuzzy topological space. In the year 2017, Dutta and Tripathy [15] studied on fuzzy *b*-θ open sets via fuzzy topological space. Later on, Smarandache [23] grounded the idea of neutrosophic set (in short N-set) in the year 1998, as anextension of the concept of intuitionistic fuzzy set (in short IF-set) [3], where every element has threeindependent memberships values namely truth, indeterminacy, and false membership values respectively. Afterwards, Salama and Alblowi [21] applied the notions of topology on N-sets and introduced neutrosophic topological space (in short NT-space) by extending the notions of fuzzy topological spaces. Salama and Alblowi [22] also defined generalized N-set and introduced the concept of generalized NT-space. Later on, Arokiarani et al. [2] introduced the ideas of neutrosophic point and studied some functions in neutrosophic topological spaces. The notions of neutrosophic pre-open (in short NP-O) and neutrosophic pre-closed (in short NP-C) sets via NT-spaces are studied by Rao and Srinivasa [20]. The idea of *b*-open sets via topological spaces was established by Andrijevic [1]. Afterwards, Ebenanjar et al. [16] presents the concept of neutrosophic *b*-open set (in short N-*b*-O-set) via NT-spaces. In the year 2020, Das and Pramanik [8] presents the generalized neutrosophic *b*-open sets in NT-spaces. The notions of neutrosophic Φ-open set and neutrosophic Φ-continuous functions via NT-spaces was also presented by Das and Pramanik [9]. The concept of neutrosophic simply soft open set in neutrosophic soft topological space was studied by Das and Pramanik [10]. In the year 2021, Das and Tripathy [14] presented the notions of neutrosophic simply *b*-open set via NT-spaces. In the year 2020, Das and Tripathy [12] grounded the notions of neutrosophic multiset and applied topology on it. In the year 2021, Das et al. [5] studied the concept of quadripartitioned neutrosophic topological spaces. The notion of bitopological space was introduced by Kelly [17] in the year 1963. In the year 2011, Tripathy and Sarma [26] studied on *b*-locally open sets via bitopological spaces. The idea of pairwise *b*-locally open and b-locally closed functions in bitopological spaces was studied by Tripathy and Sarma [27]. Tripathy and Sarma [28] also studied on weakly b-continuous mapping via bitopological spaces in the year 2013. Later on, the concept of generalized b-closed sets in ideal bitopological spaces was studied by Tripathy and Sarma [29]. Afterwards, Tripathy and Debnath [25] presented the notions of fuzzy b-locally open sets in fuzzy bitopological space. Thereafter, Ozturk and Ozkan [19] introduced the idea of neutrosophic bitopological space (in short NBi-T-space) in the year 2019. Recently, Das and Tripathy [13] presented the idea of pairwise N-b-O-sets and studied their different properties.

The main focus of this article is to procure the notions of pairwise τ_{ij}-neutrosophic-b-interior (in short P-τ_{ij}-$N_{b\text{-}int}$), pairwise τ_{ij}-neutrosophic-b-closure (in short P-τ_{ij}-$N_{b\text{-}cl}$), pairwise neutrosophic continuous mapping (in short P-N-C-mapping), pairwise neutrosophic b-continuous mapping (in short pairwise N-bC-mapping) via NBi-T-spaces.

2. Preliminaries and Definitions:

The notion of N-set is defined as follows:

Let X be a fixed set. Then, an N-set [23] L over X is denoted as follows:
$L=\{(t,T_L(t),I_L(t),F_L(t)):t\in X\}$, where T_L, I_L, F_L $:X\rightarrow[0,1]$ are called the truth-membership, indeterminacy-membership and false-membership functions and $0 \leq T_L(t) + I_L(t) + F_L(t)\leq 3$, for all $t\in X$.

The neutrosophic null set (0_N) and neutrosophic whole set (1_N) over a fixed set X are definedas follows:

(i) $0_N=\{(t,0,0,1): t\in X\}$;

(ii) $1_N=\{(t,1,0,0): t\in X\}$.

The N-sets 0_N and 1_N also has three other representations. They are given below:

$$0_N=\{(t,0,0,0): t\in X\} \;\&\; 1_N=\{(t,1,1,1): t\in X\};$$
$$0_N =\{(t,0,1,0): t\in X\} \;\&\; 1_N=\{(t,1,0,1): t\in X\};$$
$$0_N =\{(t,0,1,1): t\in X\} \;\&\; 1_N=\{(t,1,1,0): t\in X\}.$$

Let p, q, $r\in[0,1]$. An neutrosophic point (in short N-point) [2] $x_{p.q.r}$ is an N-set over X given by

$$x_{p.q.r}(y)=\begin{cases}(p,q,r), if\ x=y,\\ (0,0,1), if\ x\neq y,\end{cases}$$

where p, q, r denotes the truth, indeterminacy and false membership value of $x_{p.q.r}$.

The notion of NT-space is defined as follows:

A family τ of N-sets over X is called an [21] neutrosophic topology (in short N-topology) on X if the following axioms hold:

(i) 0_N, $1_N\in\tau$;

(ii) L_1, $L_2\in\tau\Rightarrow L_1\cap L_2\in\tau$;

(iii) $\cup L_i\in\tau$, for every $\{L_i: i\in\Delta\} \subseteq\tau$, where Δ is the support set.

Then, (X,τ) is called an NT-space. Each element of τ is an neutrosophic open set (in short NO-set). If L is an NO-set in (X, τ), then L^c is called an neutrosophic closed set (in short NC-set).

The notion of NBi-T-space is defined as follows:

Let τ_1 and τ_2 be two different N-topologies on X. Then, (X,τ_1,τ_2) is [19] called an NBi-T-space. An N-set L is called a pairwise NO-set in (X,τ_1,τ_2), if there exist an NO-set L_1 in τ_1 and an NO-set L_2 in τ_2 such that $L=L_1\cup L_2$. The complement of L i.e., L^c is called a pairwise neutrosophic closed set (in short pairwise NC-set) in (X,τ_1,τ_2).

Remark 2.1.[13] In an NBi-T-space(X,τ_1,τ_2), every τ_i-NO-set is a pairwise τ_{ij}-NO-set.

Remark 2.2. Let G be an N-set over X and (X,τ_1,τ_2) be an NBi-T-space. Then, we shall use the following notations throughout the article:

(i) $N_{cl}^i(G)$= Neutrosophic closure of G in (X,τ_i) (i=1, 2);

(ii) $N_{int}^i(G)$= Neutrosophic interior of G in (X,τ_i)(i=1, 2).

Definition 2.1.[13] Let (X,τ_1,τ_2) be an NBi-T-space. Then, P is called a

(i) τ_i-neutrosophic semi-open set (in short τ_i-NSO-set) if and only if $P\subseteq N_{cl}^i N_{int}^i(P)$;

(ii) τ_i-neutrosophic pre-open set (in short τ_i-NPO-set) if and only if $P\subseteq N_{int}^i N_{cl}^i(P)$;

(iii) τ_i-neutrosophic b-open set (in short τ_i-N-bO-set) if and only if $P\subseteq N_{cl}^i N_{int}^i(P)\cup N_{int}^i N_{cl}^i(P)$.

Remark 2.3.[13] Let (X,τ_1,τ_2) be an NBi-T-space. Then, an N-set P over X is called a τ_i-neutrosophic b-closed set (in short τ_i-N-bC-set) if and only if P^c is a τ_i-N-bO-set.

Proposition 2.1.[13] In an NBi-T-space (X,τ_1,τ_2), if P is τ_i-NSO-set (τ_i-NPO-set), then P is a τ_i-N-bO-set.

Proposition 2.2.[13] Let (X,τ_1,τ_2) be an NBi-T-space. Then, the union of any two τ_i-N-bO-sets is a τ_i-N-bO-set.

Definition 2.2.[13] Let (X,τ_1,τ_2) be an NBi-T-space. Then, P is called a

(i) τ_{ij}-neutrosophic semi-open set (in short τ_{ij}-NSO-set) if and only if $P\subseteq N_{cl}^i N_{int}^j(P)$;

(ii) τ_{ij}-neutrosophic pre-open set (in short τ_{ij}-NPO-set) if and only if $P\subseteq N_{int}^j N_{cl}^i(P)$;

(iii) τ_{ij}-neutrosophic b-open set (in short τ_{ij}-N-b-O-set) if and only if $P\subseteq N_{cl}^i N_{int}^j(P)\cup N_{int}^j N_{cl}^i(P)$.

Remark 2.4.[13] An N-set L over X is called a τ_{ij}-neutrosophic b-closed set (in short τ_{ij}-N-bC-set) if and only if L^c is a τ_{ij}-N-bO-set in (X,τ_1,τ_2).

Theorem 2.1.[13] Let (X,τ_1,τ_2) be an NBi-T-space. Then, every τ_{ij}-NSO-set (τ_{ij}-NPO-set) is a τ_{ij}-N-bO-set.

Definition 2.3.[13] An N-set L is called a pairwise τ_{ij}-NPO-set (pairwise τ_{ij}-NSO-set) in an NBi-T-space(X,τ_1,τ_2) if $L=K\cup M$, where K is a τ_{ij}-NPO-set (τ_{ij}-NSO-set) and M is a τ_{ji}-NPO-set (τ_{ji}-NSO-set) in (X,τ_1,τ_2).

Definition 2.4.[13] An N-set L is called a pairwise τ_{ij}-N-bO-set in a NBi-T-space(X,τ_1,τ_2) if $L=K\cup M$, where K is a τ_{ij}-N-bO-set and M is a τ_{ji}-N-bO-set in (X,τ_1,τ_2). If L is a pairwise τ_{ij}-N-bO-set in (X,τ_1,τ_2), then L^c is called a pairwise τ_{ij}-neutrosophic-b-closed set (in short pairwise τ_{ij}-N-bC-set) in (X,τ_1,τ_2).

Lemma 2.1.[13] In an NBi-T-space(X,τ_1,τ_2), every pairwise τ_{ij}-NPO-set (pairwise τ_{ij}-NSO-set) is a pairwise τ_{ij}-N-bO-set.

Proposition 2.3.[13] Let(X,τ_1,τ_2) be an NBi-T-space. Then, the union of two pairwise τ_{ij}-N-bO-set in (X,τ_1,τ_2) is also a pairwise τ_{ij}-N-bO-set.

Theorem 2.2. Let (X,τ_1,τ_2) be an NBi-T-space. Then, the union of two pairwise τ_{ij}-NSO-set in (X,τ_1,τ_2) is also a pairwise τ_{ij}-NSO-set.

Proof. Let L and M be two pairwise τ_{ij}-NSO-sets in an NBi-T-space(X,τ_1,τ_2). So, one can write $L=L_1\cup L_2$ and $M=M_1\cup M_2$, where L_1, M_1 are τ_{ij}-NSO-sets and L_2, M_2 are τ_{ji}-NSO-sets in (X,τ_1,τ_2). Since, L_1 and M_1 are τ_{ij}-NSO-sets, so $L_1\subseteq N_{cl}^{i}N_{int}^{j}(L_1)$ and $M_1\subseteq N_{cl}^{i}N_{int}^{j}(M_1)$. Further, Since L_2 and M_2 are τ_{ji}-NSO-sets, so $L_2\subseteq N_{cl}^{j}N_{int}^{i}(L_2)$, $M_2\subseteq N_{cl}^{j}N_{int}^{i}(M_2)$.

Now, $L\cup M=(L_1\cup L_2)\cup(M_1\cup M_2)=(L_1\cup M_1)\cup(L_2\cup M_2)$.

Therefore, $L_1\cup M_1\subseteq N_{cl}^{i}N_{int}^{j}(L_1)\cup N_{cl}^{i}N_{int}^{j}(M_1)$

$=N_{cl}^{i}(N_{int}^{j}(L_1)\cup N_{int}^{j}(M_1))$

$\subseteq N_{cl}^{i}N_{int}^{j}(L_1\cup M_1)$.

This implies, $L_1\cup M_1$ is a τ_{ij}-NSO-set in (X,τ_1,τ_2).

Similarly, it can be established that $L_2\cup M_2$ is a τ_{ji}-NSO-set in (X,τ_1,τ_2). Therefore, $L\cup M$ is a pairwise τ_{ij}-NSO-set in (X,τ_1,τ_2). Hence, the union of two pairwise τ_{ij}-NSO-set in (X,τ_1,τ_2) is again a pairwise τ_{ij}-NSO-set in (X,τ_1,τ_2).

Theorem 2.4. Let (X,τ_1,τ_2) be an NBi-T-space. Then, the union of two pairwise τ_{ij}-NPO-set in (X,τ_1,τ_2) is a pairwise τ_{ij}-NPO-set.

Proof. Let L and M be two pairwise τ_{ij}-NPO-sets in an NBi-T-space(X,τ_1,τ_2). So, one can write $L=L_1\cup L_2$ and $M=M_1\cup M_2$, where L_1, M_1 are τ_{ij}-NPO-sets and L_2, M_2 are τ_{ji}-NPO-sets in (X,τ_1,τ_2). Since, L_1 and M_1 are τ_{ij}-NPO-sets, so $L_1\subseteq N_{int}^{j}N_{cl}^{i}(L_1)$ and $M_1\subseteq N_{int}^{j}N_{cl}^{i}(M_1)$. Further, since L_2 and M_2 are τ_{ji}-NPO-sets, so $L_2\subseteq N_{int}^{i}N_{cl}^{j}(L_2)$ and $M_2\subseteq N_{int}^{i}N_{cl}^{j}(M_2)$.

Now, $L\cup M=(L_1\cup L_2)\cup(M_1\cup M_2)=(L_1\cup M_1)\cup(L_2\cup M_2)$.

Therefore, $L_1\cup M_1\subseteq N_{int}^{j}N_{cl}^{i}(L_1)\cup N_{int}^{j}N_{cl}^{i}(M_1)$

$=N_{int}^{j}(N_{cl}^{i}(L_1)\cup N_{cl}^{i}(M_1))$

$\subseteq N_{int}^{j}N_{cl}^{i}(L_1\cup M_1)$.

This implies, $L_1\cup M_1$ is a τ_{ij}-NPO-set in (X,τ_1,τ_2). Similarly, it can be established that $L_2\cup M_2$ is a τ_{ji}-NPO-set in (X,τ_1,τ_2). Therefore, $L\cup M$ is a pairwise τ_{ij}-NPO-set in (X,τ_1,τ_2). Hence, the union of two pairwise τ_{ij}-NPO-sets in (X,τ_1,τ_2) is again a pairwise τ_{ij}-NPO-set.

3. Pairwise *b*-Continuous Function:

In this section, we procure the notions of pairwise *b*-continuous functions via neutrosophic bitopological space and formulate some results on it.

Definition 3.1. Let (X,τ_1,τ_2) be an NBi-T-space. Then, the pairwise τ_{ij}-neutrosophic-*b*-interior (in short P-τ_{ij}-$N_{b\text{-}int}$) of an N-set L is the union of all pairwise τ_{ij}-N-*b*O-sets contained in L, i.e. P-τ_{ij}-$N_{b\text{-}int}(L)=\cup\{K{:}K$ is a pairwise τ_{ij}-N-*b*O-set in X and $K\subseteq L\}$.

Clearly, P-τ_{ij}-$N_{b\text{-}int}(L)$ is the largest pairwise τ_{ij}-N-*b*O-set which contained in L.

Definition 3.2. Let (X,τ_1,τ_2) be an NBi-T-space. Then, the pairwise τ_{ij}-neutrosophic-*b*-closure (in short P-τ_{ij}-$N_{b\text{-}cl}$) of an N-set L is the intersection of all pairwise τ_{ij}-N-*b*C-sets containing L, i.e. P-τ_{ij}-$N_{b\text{-}cl}(L)=\cap\{K{:}K$ is a pairwise τ_{ij}-N-*b*C-set in X and $L\subseteq K\}$.

Clearly, P-τ_{ij}-$N_{b\text{-}cl}(L)$ is the smallest pairwise τ_{ij}-N-*b*C-set which containing L.

Theorem 3.1. Let L and K be two neutrosophic subsets of an NBi-T-space (X,τ_1,τ_2). Then,

(*i*) P-τ_{ij}-$N_{b\text{-}int}(0_N)=0_N$, P-$\tau_{ij}$-$N_{b\text{-}int}(1_N)=1_N$;

(*ii*) P-τ_{ij}-$N_{b\text{-}int}(L)\subseteq L$;

(*iii*) $L\subseteq M\Rightarrow$ P-τ_{ij}-$N_{b\text{-}int}(L)\subseteq$P-$\tau_{ij}$-$N_{b\text{-}int}(M)$;

(iv) P-τ_{ij}-$N_{b\text{-}int}(L)=L$ if L is a pairwise τ_{ij}-N-bO-set.

Proof. (i) Straight forward.

(ii) By Definition 3.1, we have P-τ_{ij}-$N_{b\text{-}int}(L)=\cup\{K:K$ is a pairwise τ_{ij}-N-bO-set in X and $K\subseteq L\}$. Since, each $K\subseteq L$, so $\cup\{K:K$ is a pairwise τ_{ij}-N-bO-set in X and $K\subseteq L\}\subseteq L$, i.e. P-$\tau_{ij}$-$N_{b\text{-}int}(L)\subseteq L$. Therefore, P-$\tau_{ij}$-$N_{b\text{-}int}(L)\subseteq L$.

(iii) Let L and M be two neutrosophic subset of an NBi-T-space (X,τ_1,τ_2) such that $L\subseteq M$.

Now, P-τ_{ij}-$N_{b\text{-}int}(L)=\cup\{K:K$ is a pairwise τ_{ij}-N-bO-set in X and $K\subseteq L\}$

$\subseteq\cup\{K:K$ is a pairwise τ_{ij}-N-bO-set in X and $K\subseteq M\}$ [since $L\subseteq M$]

=P-τ_{ij}-$N_{b\text{-}int}(M)$

$\Rightarrow$ P-τ_{ij}-$N_{b\text{-}int}(L)\subseteq$P-$\tau_{ij}$-$N_{b\text{-}int}(M)$.

Therefore, $L\subseteq M\Rightarrow$P-τ_{ij}-$N_{b\text{-}int}(L)\subseteq$P-$\tau_{ij}$-$N_{b\text{-}int}(M)$.

(iv) Let L be a pairwise τ_{ij}-N-bO-set in an NBi-T-space (X,τ_1,τ_2).

Now, P-τ_{ij}-$N_{b\text{-}int}(L)=\cup\{K$: K is a pairwise τ_{ij}-N-bO-set in X and $K\subseteq L\}$. Since, L is a pairwise τ_{ij}-N-bO-set in (X,τ_1,τ_2), so L is the largest pairwise τ_{ij}-N-bO-set in (X,τ_1,τ_2), which is contained in L. Therefore, $\cup\{K:K$ is a pairwise τ_{ij}-N-bO-set in X and $K\subseteq L\}=L$. This implies, P-τ_{ij}-$N_{b\text{-}int}(L)=L$.

Theorem 3.2. Let L and K be two neutrosophic subsets of an NBi-T-space (X,τ_1,τ_2). Then,

(i) P-τ_{ij}-$N_{b\text{-}cl}(0_N)=0_N$ & P-τ_{ij}-$N_{b\text{-}cl}(1_N)=1_N$;

(ii) $L\subseteq$P-τ_{ij}-$N_{b\text{-}cl}(L)$;

(iii) $L\subseteq M\Rightarrow$ P-τ_{ij}-$N_{b\text{-}cl}(L)\subseteq$P-$\tau_{ij}$-$N_{b\text{-}cl}(M)$;

(iv) P-τ_{ij}-$N_{b\text{-}cl}(L)=L$ if L is a pairwise τ_{ij}-N-bC-set.

Proof. (i) Straightforward.

(ii) It is clear that P-τ_{ij}-$N_{b\text{-}cl}(L)=\cap\{K:K$ is a pairwise τ_{ij}-N-bC-set in X and $L\subseteq K\}$.

Since, each $L\subseteq K$, so $L\subseteq\cap\{K:K$ is a pairwise τ_{ij}-N-bC-set in X and $L\subseteq K\}$, i.e. $L\subseteq$P-τ_{ij}-$N_{b\text{-}cl}(L)$.

(iii) Let L and M be two neutrosophic subset of an NBi-T-space (X,τ_1,τ_2) such that $L\subseteq M$.

Now, P-τ_{ij}-$N_{b\text{-}cl}(L)=\cap\{K:K$ is a pairwise τ_{ij}-N-bC-set in X and $L\subseteq K\}$.

$\subseteq\cap\{K:K$ is a pairwise τ_{ij}-N-bC-set in X and $M\subseteq K\}$ [since $L\subseteq M$]

=P-τ_{ij}-$N_{b\text{-}cl}(M)$

$\Rightarrow$ P-τ_{ij}-$N_{b\text{-}cl}(L)\subseteq$P-$\tau_{ij}$-$N_{b\text{-}cl}(M)$.

Therefore, $L\subseteq M\Rightarrow$P-τ_{ij}-$N_{b\text{-}cl}(L)\subseteq$P-$\tau_{ij}$-$N_{b\text{-}cl}(M)$.

(iv) Let L be a pairwise τ_{ij}-N-bC-set in an NBi-T-space (X,τ_1,τ_2). Now, P-τ_{ij}-$N_{b\text{-}cl}(L)=\cap\{K:K$ is a pairwise τ_{ij}-N-bC-set in X and $L\subseteq K\}$. Since, L is a pairwise τ_{ij}-N-bC-set in a (X,τ_1,τ_2), so L is the smallest pairwise τ_{ij}-N-bC-set, which contains L. This implies, $\cap\{K:K$ is a pairwise τ_{ij}-N-bC-set in X and $L\subseteq K\}=L$. Therefore, P-τ_{ij}-$N_{b\text{-}cl}(L)=L$.

Proposition 3.3. Let L be a neutrosophic subset of an NBi-T-space (X,τ_1,τ_2). Then,

(i) $[$P-τ_{ij}-$N_{b\text{-}int}(L)]^c$ =P-τ_{ij}-$N_{b\text{-}cl}(L^c)$;

(ii) $[$P-τ_{ij}-$N_{b\text{-}cl}(L)]^c$ =P-τ_{ij}-$N_{b\text{-}int}(L^c)$.

Proof. (i) Let (X,τ_1,τ_2) be an NBi-T-space. Let $L=\{(w, T_L(w), I_L(w), F_L(w)): w\in X\}$ be an neutrosophic subset of (X,τ_1,τ_2).

Now, P-τ_{ij}-$N_{b\text{-}int}(L)$ $=\cup\{K$: K is a pairwise τ_{ij}-N-bO-set in X and $K\subseteq L\}$

$=\{(w,\vee T_{L_p}(w),\wedge I_{L_p}(w),\wedge F_{L_p}(w)):w\in X\}$,

where L_p is a pairwise τ_{ij}-N-bO-set in X such that $L_p\subseteq L$, for each $p\in\Delta$.

This implies, [P-τ_{ij}-$N_{b\text{-}int}$(L)]^c={(w,$\wedge T_{L_p}(w)$,$\vee I_{L_p}(w)$,$\vee F_{L_p}(w)$):$w\in X$}.

Here $\wedge T_{L_p}(w)\leq T_L(w)$,$I_{L_p}(w)\geq I_L(w)$, $F_{L_p}(w)\geq F_L(w)$, for each $w\in X$.

Therefore, P-τ_{ij}-$N_{b\text{-}int}$(L^c)={(w, $\wedge T_{L_p}(w)$, $\vee I_{L_p}(w)$,$\vee F_{L_p}(w)$): $w\in X$}

= $\cap${L_p: $p\in\Delta$ and L_p is a pairwise τ_{ij}-N-bC-set in X such that $L^c\subseteq L_p$}

Hence, [P-τ_{ij}-$N_{b\text{-}int}$(L)]^c = P-τ_{ij}-$N_{b\text{-}cl}$(L^c).

(*ii*) Let (X,τ_1,τ_2) be an NBi-T-space and L={(w, $T_L(w)$, $I_L(w)$, $F_L(w)$): $w\in X$} be a N-set over X. Then,

P-τ_{ij}-$N_{b\text{-}cl}$(L) = $\cap${K:K is a pairwise τ_{ij}-N-bC-set in X and $L\subseteq K$}

={(w,$\wedge T_{L_p}(w)$,$\vee I_{L_p}(w)$,$\vee F_{L_p}(w)$):$w\in X$},

where L_p is a pairwise τ_{ij}-N-bC-set in X such that $L\subseteq L_p$, for each $p\in\Delta$.

This implies, [P-τ_{ij}-$N_{b\text{-}cl}$(L)]^c = {(w,$\vee T_{L_p}(w)$,$\wedge I_{L_p}(w)$,$\wedge F_{L_p}(w)$):$w\in X$}.

Here, $\vee T_{L_p}(w)\geq T_L(w)$,$\wedge I_{L_p}(w)\leq I_L(w)$,$\wedge F_{L_p}(w)\leq F_L(w)$, for each $w\in X$.

Therefore, P-τ_{ij}-$N_{b\text{-}int}$(L^c)={(w,$\vee T_{L_p}(w)$,$\wedge I_{L_p}(w)$,$\wedge F_{L_p}(w)$):$w\in X$}

=$\cup${L_p: $p\in\Delta$ and L_p is a pairwise τ_{ij}-N-bO-set in X such that $L_p\subseteq L^c$}.

Hence, [P-τ_{ij}-$N_{b\text{-}cl}$(L)]^c= P-τ_{ij}-$N_{b\text{-}int}$(L^c).

Theorem 3.1. Let (X,τ_1,τ_2) be an NBi-T-space. Then, the neutrosophic null set (0_N) and the neutrosophic whole set (1_N) are both τ_{ij}-N-bO-set and τ_{ji}-N-bO-set.

Proof. Let (X,τ_1,τ_2) be an NBi-T-space. Now, $N_{cl}^iN_{int}^j(0_N)\cup N_{int}^jN_{cl}^i(0_N)=N_{cl}^i(0_N)\cup N_{int}^j(0_N)= 0_N\cup 0_N=0_N$. Therefore, $0_N\subseteq 0_N=N_{cl}^iN_{int}^j(0_N)\cup N_{int}^jN_{cl}^i(0_N)$. Hence, the neutrosophic null set (0_N) is a τ_{ij}-N-bO-set.

Similarly, it can be established that the neutrosophic null set (0_N) is a τ_{ji}-N-bO-set.

Further, one can show that the neutrosophic whole set (1_N) are both τ_{ij}-N-bO-set and τ_{ji}-N-bO-set.

Theorem 3.2. In an NBi-T-space(X,τ_1,τ_2), every τ_i-NO-set is a τ_{ji}-N-bO-set.

Proof. Let L be a τ_i-NO-set in an NBi-T-space(X,τ_1,τ_2). Therefore, $N_{int}^i(L)=L$. Now,$L\subseteq N_{cl}^j(L)= N_{cl}^jN_{int}^i(L)$. This implies, $L\subseteq N_{cl}^jN_{int}^i(L)\cup N_{int}^iN_{cl}^j(L)$. Hence, L is a τ_{ji}-N-bO-set in (X,τ_1,τ_2).

Theorem 3.3. In an NBi-T-space (X,τ_1,τ_2),

(*i*) every τ_{ij}-N-bO-set is a pairwise τ_{ij}-N-bO-set;

(*ii*) every τ_{ji}-N-bO-set is a pairwise τ_{ji}-N-bO-set;

(*iii*) every τ_{ij}-N-bC-set is a pairwise τ_{ij}-N-bC-set;

(*iv*) every τ_{ji}-N-bC-set is a pairwise τ_{ji}-N-bC-set.

Proof. (*i*) Let L be a τ_{ij}-N-bO-set in an NBi-T-space (X,τ_1,τ_2). Then, L can be expressed as L=$L\cup 0_N$, where L is a τ_{ij}-N-bO-set and 0_N is a τ_{ji}-N-bO-set in (X,τ_1,τ_2). This implies, L is a pairwise τ_{ij}-N-bO-set in (X,τ_1,τ_2).

(*ii*) Straightforward.

(*iii*) Let L be a τ_{ij}-NC-set in an NBi-T-space (X,τ_1,τ_2). Then, L can be expressed as L=$L\cap 1_N$, where L is a τ_{ij}-NC-set and 1_N is a τ_{ji}-NC-set in (X,τ_1,τ_2). This implies, L is a pairwise τ_{ij}-N-bC-set in (X,τ_1,τ_2).

(*iv*) Straightforward.

Theorem 3.4. In an NBi-T-Space (X,τ_1,τ_2), every τ_i-NO-set is a pairwise τ_{ij}-N-bO-set.

Proof. Let L be a τ_i-NO-set in an NBi-T-space(X,τ_1,τ_2). By Theorem 3.2., it is clear that L is a τ_{ji}-N-bO-set. Further, by Theorem 3.3., it is clear that L is a pairwise τ_{ij}-N-bO-set.

Theorem 3.5. Let (X,τ_1,τ_2) be an NBi-T-space. Then, 0_N and 1_N are both pairwise τ_{ij}-N-bO-set and pairwise τ_{ji}-N-bO-set.

Proof. Let (X,τ_1,τ_2) be an NBi-T-space. One can write $0_N=A\cup B$, where $A=0_N$ is a τ_{ij}-N-*b*O-set and $B=0_N$ is a τ_{ji}-N-*b*O-set in (X,τ_1,τ_2). This implies, 0_N is a pairwise τ_{ij}-N-*b*O-set in (X,τ_1,τ_2).

Similarly, it can be established that 0_N is a pairwise τ_{ji}-N-*b*O-set in (X,τ_1,τ_2).

Again, one can write $1_N=L\cup M$, where $L=1_N$ is a τ_{ij}-N-*b*O-set and $M=1_N$ is a τ_{ji}-N-*b*O-set in (X,τ_1,τ_2). This implies, 1_N is a pairwise τ_{ij}-N-*b*O-set in (X,τ_1,τ_2).

Similarly, it can be also established that 1_N is a pairwise τ_{ji}-N-*b*O-set in (X,τ_1,τ_2).

Theorem 3.6. Let (X,τ_1,τ_2) be an NBi-T-space. Then, both 0_N and 1_N are pairwise τ_{ij}-N-*b*C-set and pairwise τ_{ji}-N-*b*C-set.

Proof. By Theorem 3.5, it is clear that 0_N is both pairwise τ_{ij}-N-*b*O-set and pairwise τ_{ji}-N-*b*O-set. Hence, its complement 1_N is both pairwise τ_{ij}-N-*b*C-set and pairwise τ_{ji}-N-*b*C-set.

Similarly, from Theorem 3.5, it is clear that 1_N is both pairwise τ_{ij}-N-*b*O-set and pairwise τ_{ji}-N-*b*O-set. Hence, its complement 0_N is both pairwise τ_{ij}-N-*b*C-set and pairwise τ_{ji}-N-*b*C-set.

Remark 3.1. Throughout the article, we denote τ_{ij}^{b} as a collection of all pairwise τ_{ij}-N-*b*O-sets and τ_{ij}^{C} as a collection of all pairwise τ_{ij}-N-*b*C-sets in (X,τ_1,τ_2). The collection τ_{ij}^{b} forms an neutrosophic supra topology on X.

Definition 3.3. Let (X,τ_1,τ_2) and (Y,δ_1,δ_2) be two NBi-T-spaces. Then, an one to one and onto mapping $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is called a

(*i*) pairwise neutrosophic semi continuous mapping (in short P-NS-C-mapping) if and only if $\xi^{-1}(L)$ is a τ_i-NSO-set in X, whenever L is a pairwise δ_{ij}-NO-set in Y.

(*ii*) pairwise neutrosophic pre continuous mapping (in short P-NP-C-mapping) if and only if $\xi^{-1}(L)$ is a τ_i-NPO-set in X, whenever L is a pairwise δ_{ij}-NO-set in Y.

(*iii*) pairwise neutrosophic continuous mapping (in short P-N-C-mapping) if and only if $\xi^{-1}(L)$ is a τ_i-NO-set in X, whenever L is a pairwise δ_{ij}-NO-set in Y.

(*iv*) pairwise neutrosophic *b*-continuous mapping (in short P-N-*b*-C-mapping) if and only if $\xi^{-1}(L)$ is a τ_i-N-*b*O-set in X, whenever L is a pairwise δ_{ij}-NO-set in Y.

Theorem 3.7. Let (X,τ_1,τ_2) and (Y,δ_1,δ_2) be two NBi-T-spaces. Then, every P-N-C-mapping from (X,τ_1,τ_2) to (Y,δ_1,δ_2) is a P-NP-C-mapping (P-NS-C-mapping).

Proof. Let L be a pairwise δ_{ij}-NO-set in (Y,δ_1,δ_2). Since, $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-N-C-mapping from (X,τ_1,τ_2) to (Y,δ_1,δ_2), so $\xi^{-1}(L)$ is a τ_i-NO-set in (X,τ_1,τ_2). It is known that every τ_i-NO-set is a τ_i-NPO-set (τ_i-NSO-set). Therefore, $\xi^{-1}(L)$ is a τ_i-NPO-set (τ_i-NSO-set) in (X,τ_1,τ_2). Hence, $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-NP-C-mapping (P-NS-C-mapping).

Theorem 3.8. Let (X,τ_1,τ_2) and (Y,δ_1,δ_2) be two NBi-T-spaces. Then, every P-NS-C-mapping (P-NP-C-mapping) from (X,τ_1,τ_2) to (Y,δ_1,δ_2) is a P-N-*b*-C-mapping.

Proof. Let L be a pairwise δ_{ij}-NO-set in (Y,δ_1,δ_2). Since, $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-NS-C-mapping (P-NP-C-mapping) from (X,τ_1,τ_2) to (Y,δ_1,δ_2), so $\xi^{-1}(L)$ is a τ_i-NSO-set (τ_i-NPO-set) in (X,τ_1,τ_2). It is known that, every τ_i-NSO-set (τ_i-NPO-set) is a τ_i-N-*b*O-set. Therefore, $\xi^{-1}(L)$ is a τ_i-N-*b*O-set in (X,τ_1,τ_2). Hence, $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-N-*b*-C-mapping.

Theorem 3.9. Let (X,τ_1,τ_2) and (Y,δ_1,δ_2) be two NBi-T-spaces. Then, every P-N-C-mapping from (X,τ_1,τ_2) to (Y,δ_1,δ_2) is a P-N-*b*-C-mapping.

Proof. Let L be a pairwise δ_{ij}-NO-set in (Y,δ_1,δ_2). Since, $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-N-C-mapping from (X,τ_1,τ_2) to (Y,δ_1,δ_2), so $\xi^{-1}(L)$ is a τ_i-NO-set in (X,τ_1,τ_2). It is known that, every τ_i-NO-set is a

τ_i-N-b-O-set. Therefore, $\xi^{-1}(L)$ is a τ_i-N-b-O-set in (X,τ_1,τ_2). Hence, $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a p-N-b-C-mapping.

Theorem 3.10. If $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ and $\chi:(Y,\delta_1,\delta_2)\rightarrow(Z,\theta_1,\theta_2)$ be two P-N-C-mapping, then the composition mapping $\chi\circ\xi:(X,\tau_1,\tau_2)\rightarrow(Z,\theta_1,\theta_2)$ is also a P-N-C-mapping.

Proof. Let $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ and $\chi:(Y,\delta_1,\delta_2)\rightarrow(Z,\theta_1,\theta_2)$ be two P-N-C-mappings. Let L be a pairwise θ_{ij}-NO-set in (Z,θ_1,θ_2). Since, $\chi:(Y,\delta_1,\delta_2)\rightarrow(Z,\theta_1,\theta_2)$ is a P-N-C-mapping, so $\chi^{-1}(L)$ is a δ_i-NO-set in Y. Since, $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-N-C-mapping, so $\xi^{-1}(\chi^{-1}(L))= (\chi\circ\xi)^{-1}(L)$ is a τ_i-NO-set in X.

Theorem 3.11. If $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ be an one to one and onto mapping between two NBi-T-spaces, then the following two are equivalent:

(*i*) ξ is a P-N-b-C-mapping.

(*ii*) $\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(A)) \subseteq \tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(A))$, for every neutrosophic subset A of Y.

Proof. (*i*)⇒(*ii*)

Let $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ be a P-N-b-C-mapping. Let A be an neutrosophic subset of Y. Here, $\text{P-}\delta_{ij}\text{-}N_{int}(A)$ is a pairwise δ_{ij}-NO-set in Y and $\text{P-}\delta_{ij}\text{-}N_{int}(A)\subseteq A$. This implies, $\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(A))\subseteq\xi^{-1}(A)$. By the hypothesis, $\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(A))$ is a τ_i-N-b-O-set in X. Therefore, $\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(A))$ is a τ_i-N-b-O-set in X such that $\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(A))\subseteq\xi^{-1}(A)$. It is known that $\tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(A))$ is the largest τ_i-N-b-O-set in X, which is contained in $\xi^{-1}(A)$. Hence, $\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(A))\subseteq\tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(A))$.

(*ii*)⇒(*i*)

Let A be a pairwise δ_{ij}-NO-set in (Y,δ_1,δ_2). Therefore, $\text{P-}\delta_{ij}\text{-}N_{int}(A)=A$. By hypothesis, $\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(A))\subseteq\tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(A))$. This implies, $\xi^{-1}(A)\subseteq\tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(A))$. It is known that $\tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(A)) \subseteq\xi^{-1}(A)$. Therefore, $\tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(A))=\xi^{-1}(A)$. Hence,$\xi^{-1}(A)$ is a τ_i-N-b-O-set in (X,τ_1,τ_2). Therefore, ξ is a P-N-b-C-mapping from an NBi-T-space (X,τ_1,τ_2) to another NBi-T-space (Y,δ_1,δ_2).

Theorem 3.12. An one to one and onto mapping $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-N-b-C-mapping if and only if $\text{P-}\delta_{ij}\text{-}N_{int}(\xi(A))\subseteq\xi(\tau_i\text{-}N_{b\text{-}int}(A))$, for every N-set A over X and i, j= 1,2, and $i\neq j$.

Proof. Let $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ be a P-N-b-C-mapping. Let A be an N-set over X. Then, $\xi(A)$ is also an N-set over Y. By Theorem 3.11, we have $\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(\xi(A)))\subseteq\tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(\xi(A)))$. This implies, $\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(\xi(A)))\subseteq\tau_i\text{-}N_{b\text{-}int}(A)$. Hence, $\text{P-}\delta_{ij}\text{-}N_{int}(\xi(A))\subseteq\xi(\tau_i\text{-}N_{b\text{-}int}(A))$. Therefore, $\text{P-}\delta_{ij}\text{-}N_{int}(\xi(A))\subseteq \xi(\tau_i\text{-}N_{b\text{-}int}(A))$, for every N-set A over X and i, j= 1,2; and $i\neq j$.

Conversely, let $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ be a mapping between two NBi-T-spaces such that

$$\text{P-}\delta_{ij}\text{-}N_{int}(\xi(A))\subseteq\xi(\tau_i\text{-}N_{b\text{-}int}(A)) \qquad (1)$$

for every N-set A over$X$ and i, j = 1,2; and $i\neq j$.

Let A be an N-set over Y. Then, $\xi^{-1}(A)$ is an N-set over X. By putting $A=\xi^{-1}(A)$ in eq. (1), we have,

$\text{P-}\delta_{ij}\text{-}N_{int}(\xi(\xi^{-1}(A))) \subseteq\xi(\tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(A)))$

$\Rightarrow\text{P-}\delta_{ij}\text{-}N_{int}(A)\subseteq\xi(\tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(A)))$

$\Rightarrow\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(A)\subseteq\tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(A))$.

Therefore, $\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(A)\subseteq\tau_i\text{-}N_{b\text{-}int}(\xi^{-1}(A))$, for every N-set A of Y. Hence, by Theorem 3.11., the mapping $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-N-b-C-mapping.

Corollary 3.1. If $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is an one to one and onto mapping from an NBi-T-space (X,τ_1,τ_2) to another NBi-T-space (Y,δ_1,δ_2), then the following two are equivalent:

(*i*) ξ is a P-N-C-mapping.

(*ii*) $\xi^{-1}(\text{P-}\delta_{ij}\text{-}N_{int}(Q)) \subseteq\tau_i\text{-}N_{int}(\xi^{-1}(Q))$, for every N-set Q over$Y$.

Definition 3.4. Let (X,τ_1,τ_2) be an NBi-T-space. Let $x_{a,b,c}$ be an N-point in X. Then, an N-set Q over X is called a pairwise τ_{ij}-neutrosophic b-neighbourhood (in short P-τ_{ij}-N-b-nbd) of $x_{a,b,c}$, if there exist a pairwise τ_{ij}-N-bO-set U such that $x_{a,b,c}\in U\subseteq Q$.

Theorem 3.13. Let (X,τ_1,τ_2) be an NBi-T-space. An N-set Q over X is a pairwise τ_{ij}-N-bO-set if and only if Q is a P-τ_{ij}-N-b-nbd of all of its N-points.

Proof. Let Q be a pairwise τ_{ij}-N-bO-set in an NBi-T-space (X,τ_1,τ_2). Let $x_{a,b,c}$ be an N-point in X such that $x_{a,b,c}\in Q$. Therefore, $x_{a,b,c}\in Q\subseteq Q$. This implies, Q is a P-τ_{ij}-N-b-nbd of $x_{a,b,c}$. Hence, Q is the P-τ_{ij}-N-b-nbd of all of its N-points.

Conversely, let Q be a P-τ_{ij}-N-b-nbd of all of its N-points. Assume that $x_{a,b,c}$ be an N-point in X, such that $x_{a,b,c}\in Q$. Therefore, there exist a pairwise τ_{ij}-N-bO-set G such that $x_{a,b,c}\in G\subseteq Q$.
Now, $Q=\cup_{x_{a,b,c}\in Q} x_{a,b,c}\subseteq\cup_{x_{a,b,c}\in Q} G\subseteq\cup_{x_{a,b,c}\in Q} Q=Q$. This implies, $Q=\cup_{x_{a,b,c}\in Q} G$, which is a pairwise τ_{ij}-N-bO-set. Therefore, Q is a pairwise τ_{ij}-N-bO-set in (X,τ_1,τ_2).

Theorem 3.14. An one to one and onto mapping $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-N-b-C-mapping if and only if for every N-point $x_{a,b,c}\in Y$ and for any P-δ_{ij}-N-b-nbd V of $x_{a,b,c}$ in Y, there exist a τ_i-neutrosophic-b-neighbourhood (in short τ_i-N-b-nbd) U of $\xi^{-1}(x_{a,b,c})$ in X such that $U\subseteq\xi^{-1}(V)$.

Proof. Let $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ be a P-N-b-C-mapping. Let $x_{a,b,c}$ be an N-point in Y and V be a P-δ_{ij}-N-b-nbd of $x_{a,b,c}$. Then, there exist a pairwise δ_{ij}-NO-set G in Y such that $x_{a,b,c}\in G\subseteq V$. This implies, $\xi^{-1}(x_{a,b,c})\in\xi^{-1}(G)\subseteq\xi^{-1}(V)$. Since, $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-N-b-C-mapping, so $\xi^{-1}(G)$ is a τ_i-N-bO-set in X. By taking $U=\xi^{-1}(G)$, we see that U is a τ_i-N-bO-set in X such that $\xi^{-1}(x_{a,b,c})\in U\subseteq\xi^{-1}(V)$. Hence, $U=\xi^{-1}(G)$ is a τ_i-N-b-nbd of $\xi^{-1}(x_{a,b,c})$ and $U\subseteq\xi^{-1}(V)$.

Conversely, let for every N-point $x_{a,b,c}\in Y$ and for any P-δ_{ij}-N-nbd V of $x_{a,b,c}$ in Y, there exist a τ_i-N-b-nbd U of $\xi^{-1}(x_{a,b,c})$ in X such that $U\subseteq\xi^{-1}(V)$. Let G be a pairwise δ_{ij}-NO-set in Y and $x_{a,b,c}\in G$. By Theorem 3.13., G is a P-δ_{ij}-N-nbd of $x_{a,b,c}$. By hypothesis, there exists a τ_i-N-b-nbd H of $\xi^{-1}(x_{a,b,c})\in X$ such that $\xi^{-1}(x_{a,b,c})\in H\subseteq\xi^{-1}(G)$. This implies, $\xi^{-1}(G)$ is the τ_i-N-b-nbd of each of its N-points. Therefore, $\xi^{-1}(G)$ is a τ_i-N-bO-set in X. Hence, $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-N-b-C-mapping.

Theorem 3.15. If $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ be a P-N-b-C-mapping and $\chi:(Y,\delta_1,\delta_2)\rightarrow(Z,\theta_1,\theta_2)$ be a P-N-C-mapping, then the composition mapping $\chi\circ\xi:(X,\tau_1,\tau_2)\rightarrow(Z,\theta_1,\theta_2)$ is a P-N-b-C-mapping.

Proof. Let $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ be a P-N-b-C-mapping and $\chi:(Y,\delta_1,\delta_2)\rightarrow(Z,\theta_1,\theta_2)$ be a P-N-C-mapping. Let L be a pairwise θ_{ij}-NO-set in (Z,θ_1,θ_2). Since, $\chi:(Y,\delta_1,\delta_2)\rightarrow(Z,\theta_1,\theta_2)$ is a P-N-C-mapping, so $\chi^{-1}(L)$ is a δ_i-NO-set in Y. Now, by Lemma 2.1., it is clear that $\chi^{-1}(L)$ is a pairwise δ_{ij}-NO-set in (Y,δ_1,δ_2). Since, $\xi:(X,\tau_1,\tau_2)\rightarrow(Y,\delta_1,\delta_2)$ is a P-N-b-C-mapping, so $\xi^{-1}(\chi^{-1}(L))=(\chi\circ\xi)^{-1}(L)$ is a τ_i-NO-set in X. Since, every τ_i-NO-set is a τ_i-N-bO-set, so $(\chi\circ\xi)^{-1}(L)$ is a τ_i-N-bO-set in X. Hence, $\chi\circ\xi:(X,\tau_1,\tau_2)\rightarrow(Z,\theta_1,\theta_2)$ is a P-N-b-C-mapping.

4. Conclusion

In this article, we introduce the notion of pairwise neutrosophic-b-interior, pairwise neutrosophic-b-closure, pairw neutrosophic b-continuous mapping, we prove some propositions and theorems on NBi-T-spaces. In the future, we hope that based on these notions in NBi-T-spaces, many new investigations can be carried out.

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Tangent Similarity Measure Based MADM-Strategy under SVPNS-Environment

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Abstract: The main focus of this article is to procure a new similarity measure namely tangent similarity measure for single valued pentapartitioned neutrosophic sets (SVPNS). We formulate some results on tangent similarity measure of similarities between two SVPNSs. Then, we develop a SVPNS-MADM (SVPNS-Multi-Attribute-Decision-Making) model under the SVPNS environment based on the tangent similarity measure. Further, we validate our proposed SVPNS-MADM model by giving a numerical example.

Keywords: MADM; Pentapartitioned Neutrosophic Set; Tangent Similarity.

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1. Introduction:

In the year 1965, Zadeh [37] introduced the concept of fuzzy set (FS) theory to deal with the uncertainty events. Afterwards, Atanassov [1] exended the concept of FS by introducing the notions of intuitionistic fuzzy set (IFS). In the year 2011, Pramanik and Mukhopadhyaya [28] proposed a MADM approach based on grey relational analysis under intuitionistic fuzzy set-environment. In the year 2014, Mondal et al. [20] developed a MADM-strategy to select the quality brick under intuitionistic fuzzy environment. In the year 1998, Smarandache [30] grounded the idea of neutrosophic set (NS) by extending the notion of fuzzy set (FS) and intuitionistic fuzzy set (IFS) to deal with the uncertainty events having indeterminacy. In an NS, every element has three independent components namely truth, indeterminacy, and false membership values. Thereafter, Salama and Alblowi [29] applied the notions of topology on NSs and introduced the concept of neutrosophic topological space (NTS). Later on, many researchers around the globe gives their contributions ([6], [7], [12], etc.] in the area of NTS. Indeterminacy membership plays an important role in multi-attribute-decision-making problems of real world. In the year 2010, Wang et al. [31]

introduced the idea of SVNS (single valued neutrosophic set), which is a subclass of NS. One can represent indeterminate and incomplete information which makes trouble to take decision/selection in the real world by using a SVNS. The SVNS is more capable to deal with this situation. Later on, Mondal et al. [21] studied the role of neutrosophic logic in data mining process in the year 2016. Afterwards, many researchers of different countries studied SVNS for the formation of MADM model/algorithm in different branches of real-world such as medical diagnosis, educational problem, social problems, decision-making problems, conflict resolution, image processing, etc. Thereafter, many researchers (Biswas et al. [2], Das et al. [5], Mondal & Pramanik [16], Pramanik et al. [23], Pramanik et al. [24], etc.) used SVNS in their MCDM (multi-criteria-decision-making) models. Later on, Ye [32-35], Ye & zhang [36], Mondal & Pramanik [17], Mondal et al. [18], Mondal et al. [19] etc. established several MADM models based on similarity measures under the SVNS-environment / interval valued neutrosophic set-environment / rough neutrosophic set-environment. Pramanik et al. [25] proposed a MADM-approach under the single valued neutrosophic soft expert set environment in the year 2015. In the year 2020, Mukherjee and Das [22] presented the notions of neutrosophic bipolar vague soft set and proposed a MADM-strategy.

In the year 2020, Mallick and Pramanik [15] grounded the notions of single valued pentapartitioned neutrosophic set (SVPNS) by splitting indeterminacy into three independent components namely contradiction, ignorance, and unknown-membership. Later on, Das et al. [4] introduced the notions of pentapartitioned neutrosophic *Q*-ideals of *Q*-algebra in the year 2021. Recently, Das and Tripathy [13] applied the idea of topology on SVPNSs and defined pentapartitioned neutrosophic topological space.

In this article, we proposed a SVPNS-MADM model based on tangent similarity measure under the SVPNS environment. Also, we validate our model by a numerical example.

The rest of the paper has been split into following sections:

Section 2 recalls some relevant definitions, properties, and operations on SVPNSs. Section 3 presents the tangent similarity measure of similarities between two SVPNSs. We formulate some results on tangent similarity measure under SVPNS environment. In section 4, we present a SVPNS-MADM strategy based on tangent similarity measure under the SVPNS environment. In section 5, we have validated our proposed MADM model by a real world numerical example. Section 6 represents the concluding remarks of our work done in this study.

2. Some Relevant Definitions:

In this section, we give some basic definitions and results those are relevant to the main results of this article.

Definition 2.1. [15] Let L be a fixed set. Then P, a SVPNS over L is denoted as follows: $P=\{(\kappa,\Delta_P(\kappa),\Gamma_P(\kappa),\Pi_P(\kappa),\Omega_P(\kappa),\Phi_P(\kappa)):\ u\in L\}$, where Δ_P, Γ_P, Π_P, Ω_P, $\Phi_P : L\rightarrow]0,1[$ denotes the truth, contradiction, ignorance, unknown and falsity membership functions respectively. So

$$0 \leq \Delta_P(\kappa)+\Gamma_P(\kappa)+\Pi_P(\kappa)+\Omega_P(\kappa)+\Phi_P(\kappa) \leq 5.$$

Definition 2.2. [15] The absolute SVPNS (1_{PN}) and the null PNS (0_{PN}) over L are defined by

(*i*) $1_{PN} = \{(\kappa,1,1,0,0,0): \kappa\in L\}$;

(*ii*) 0_{PN} = {(κ,0,0,1,1,1): κ∈*L*}.

Definition 2.3.[15] Let *X*={(κ,Δ_X(κ),Γ_X(κ),Π_X(κ),Ω_X(κ),Φ_X(κ)): *q*∈*L*} and *Y* = {(κ,Δ_Y(κ),Γ_Y(κ),Π_Y(κ), Ω_Y(κ),Φ_Y(κ)): κ∈*L*} be two SVPNSs over *L*. Then, *X*⊆*Y* ⇔ Δ_X(κ)≤Δ_Y(κ), Γ_X(κ)≤Γ_Y(κ), Π_X(κ)≥Π_Y(κ), Ω_X(κ)≥Ω_Y(κ), Φ_X(κ)≥Φ_Y(κ), for all κ∈*L*.

Definition 2.4.[15] Let *X*={(κ,Δ_X(κ),Γ_X(κ),Π_X(κ),Ω_X(κ),Φ_X(κ)): κ∈*L*} and *Y*={(κ,Δ_Y(κ),Γ_Y(κ),Π_Y(κ),Ω_Y(κ), Ω_Y(κ)): κ∈*L*} be two SVPNSs over *L*. Then, *X*∪*Y*={(κ, max{Δ_X(κ),Δ_Y(κ)}, max{Γ_X(κ),Γ_Y(κ)}, min{Π_X(κ), Π_X(κ)}, min{Ω_X(κ),Ω_X(κ)}, min{Φ_X(κ),Φ_X(κ)}): κ∈*L*}.

Definition 2.5.[15] Let *X*={(κ,Δ_X(κ),Γ_X(κ),Π_X(κ),Ω_X(κ),Φ_X(κ)): κ∈*L*} and *Y*={(κ,Δ_Y(κ),Γ_Y(κ),Π_Y(κ),Ω_Y(κ), Φ_Y(κ)): κ∈*L*} be two SVPNSs over *L*. Then, *X*∩*Y*={(κ, min{Δ_X(κ),Δ_Y(κ)}, min{Γ_X(κ),Γ_Y(κ)}, max{Π_X(κ), Π_X(κ)}, max{Ω_X(κ),Ω_X(κ)}, max{Φ_X(κ),Φ_X(κ)}): κ∈*L*}.

Definition 2.6.[15] Let *X*={(κ,Δ_X(κ),Γ_X(κ),Π_X(κ),Ω_X(κ),Φ_X(κ)): κ∈*W*} be a SVPNS over *L*. Then, the complement of *X* is defined by X^c={(κ,Φ_X(κ),Ω_X(κ),1-Π_X(κ),Γ_X(κ),Δ_X(κ)): κ∈*L*}.

3. Tangent Similarity Measure under SVPNS Environment:

Definition 3.1. Suppose that *Y*={(κ,Δ_Y(κ),Γ_Y(κ),Π_Y(κ),Ω_Y(κ),Φ_Y(κ)): κ∈*L*} and *R*={(κ,Δ_R(κ),Γ_R(κ),Π_R(κ), Ω_R(κ),Φ_R(κ)): κ∈*L*} be two SVPNSs over a fixed set *L*. Then, the tangent similarity measure of similarities between *Y* and *R* is defined by:

$$T_{SVPNSM}(Y, R) = 1-\frac{1}{n}\sum_{x\in L} tan[\frac{\pi}{12}[|\Delta_Y(κ)-\Delta_R(κ)|+|\Gamma_Y(κ)-\Gamma_R(κ)|+|\Pi_Y(κ)-\Pi_R(κ)|+|\Omega_Y(κ)-\Omega_R(κ)| +|\Phi_Y(κ)-\Phi_R(κ)|]]. \quad (1)$$

Theorem 3.1. Suppose that T_{SVPNSM}(*Y*, *R*) be the tangent similarity measure of similarities between the SVPNSs *Y* and *R*. Then, the following properties hold:

(*i*) 0 ≤ T_{SVPNSM}(*Y*, *R*) ≤ 1;

(*ii*) T_{SVPNSM}(*Y*, *R*) = T_{SVPNSM}(*R*, *Y*);

(*iii*) T_{SVPNSM}(*Y*, *R*)=1 if and only if *Y*=*R*.

Proof. (*i*) It is known that, the tangent function is monotonic increasing in the interval [0, π/4]. It is also lies in the interval [0, 1]. Therefore, 0 ≤ T_{SVPNSM}(*Y*, *R*) ≤ 1.

(*ii*) From Definition 3.1., we have,

T_{SVPNSM}(*Y*, *R*)

$$=1-\frac{1}{n}\sum_{κ\in L} tan[\frac{\pi}{12}[|\Delta_Y(κ)-\Delta_R(κ)|+|\Gamma_Y(κ)-\Gamma_R(κ)|+|\Pi_Y(κ)-\Pi_R(κ)|+|\Omega_Y(κ)-\Omega_R(κ)|+|\Phi_Y(κ)-\Phi_R(κ)|]].$$

$$=1-\frac{1}{n}\sum_{κ\in L} tan[\frac{\pi}{12}[|\Delta_R(κ)-\Delta_Y(κ)|+|\Gamma_R(κ)-\Gamma_Y(κ)|+|\Pi_R(κ)-\Pi_Y(κ)|+|\Omega_R(κ)-\Omega_Y(κ)|+|\Phi_R(κ)-\Phi_Y(κ)|]]$$

= T_{SVPNSM}(*R*, *Y*).

Therefore, T_{SVPNSM}(*Y*, *R*) = T_{SVPNSM}(*R*, *Y*).

(*iii*) Let *Y* and *R* be two SVPNSs over *L* such that *Y*=*R*. Therefore, Δ_Y(κ)=Δ_R(κ), Γ_Y(κ)=Γ_R(κ), Π_Y(κ)= Π_R(κ), Ω_Y(κ)=Ω_R(κ), and Φ_Y(κ)=Φ_R(κ), for all κ∈*L*. This implies, |Δ_Y(κ)-Δ_R(κ)|=0, |Γ_Y(κ)-Γ_R(κ)|=0, |Π_Y(κ)-Π_R(κ)|=0, |Ω_Y(κ)-Ω_R(κ)|=0 and |Φ_Y(κ)-Φ_R(κ)|=0, for all κ∈*L*. Hence, T_{SVPNSM}(*Y*, *R*)=1- $\frac{1}{n}\sum_{x\in L} tan(0)$=1.

Conversely, let $T_{SVPNSM}(Y, R)$=1. Therefore, $|\Delta_Y(\kappa)-\Delta_R(\kappa)|=0$, $|\Gamma_Y(\kappa)-\Gamma_R(\kappa)|=0$, $|\Pi_Y(\kappa)-\Pi_R(\kappa)|=0$, $|\Omega_Y(\kappa)-\Omega_R(\kappa)|=0$, and $|\Phi_Y(\kappa)-\Phi_R(\kappa)|=0$, for all $\kappa\in L$. This implies, $\Delta_Y(\kappa)=\Delta_R(\kappa)$, $\Gamma_Y(\kappa)=\Gamma_R(\kappa)$, $\Pi_Y(\kappa)=\Pi_R(\kappa)$, $\Omega_Y(\kappa)=\Omega_R(\kappa)$, and $\Phi_Y(\kappa)=\Phi_R(\kappa)$, for all $\kappa\in L$. Hence, $Y=R$.

Theorem 3.2. Let Y, R and C be three SVPNSs over L. If $Y \subseteq R \subseteq C$, then $T_{SVPNSM}(Y, R) \geq T_{SVPNSM}(Y, C)$ and $T_{SVPNSM}(R, C) \geq T_{SVPNSM}(Y, C)$.

Proof. Suppose that Y, R and C be three SVPNSs over L such that $Y\subseteq R\subseteq C$. Therefore, $\Delta_Y(\kappa)\leq\Delta_R(\kappa)$, $\Gamma_Y(\kappa)\leq\Gamma_R(\kappa)$, $\Pi_Y(\kappa)\geq\Pi_R(\kappa)$, $\Omega_Y(\kappa)\geq\Omega_R(\kappa)$, $\Phi_Y(\kappa)\geq\Phi_R(\kappa)$, $\Delta_R(\kappa)\leq\Delta_C(\kappa)$, $\Gamma_R(\kappa)\leq\Gamma_C(\kappa)$, $\Pi_R(\kappa)\geq\Pi_C(\kappa)$, $\Omega_R(\kappa)\geq\Omega_C(\kappa)$, $\Phi_R(\kappa)\geq\Phi_C(\kappa)$, $\Delta_Y(\kappa)\leq\Delta_C(\kappa)$, $\Gamma_Y(\kappa)\leq\Gamma_C(\kappa)$, $\Pi_Y(\kappa)\geq\Pi_C(\kappa)$, $\Omega_Y(\kappa)\geq\Omega_C(\kappa)$, $\Phi_Y(\kappa)\geq\Phi_C(\kappa)$, for all $\kappa\in L$.

We have,

$|\Delta_Y(\kappa)-\Delta_R(\kappa)|\leq|\Delta_Y(\kappa)-\Delta_C(\kappa)|$, $|\Gamma_Y(\kappa)-\Gamma_R(\kappa)|\leq|\Gamma_Y(\kappa)-\Gamma_C(\kappa)|$, $|\Pi_Y(\kappa)-\Pi_R(\kappa)|\leq|\Pi_Y(\kappa)-\Pi_C(\kappa)|$, $|\Omega_Y(\kappa)-\Omega_R(\kappa)|\leq|\Omega_Y(\kappa)-\Omega_C(\kappa)|$, $|\Phi_Y(\kappa)-\Phi_R(\kappa)|\leq|\Phi_Y(\kappa)-\Phi_C(\kappa)|$, for all $\kappa\in L$.

Therefore,

$T_{SVPNSM}(Y, R)$

$$=1-\frac{1}{n}\sum_{\kappa\in L} tan[\frac{\pi}{12}[|\Delta_Y(\kappa)-\Delta_R(\kappa)|+|\Gamma_Y(\kappa)-\Gamma_R(\kappa)|+|\Pi_Y(\kappa)-\Pi_R(\kappa)|+|\Omega_Y(\kappa)-\Omega_R(\kappa)|+|\Phi_Y(\kappa)-\Phi_R(\kappa)|]]$$

$$\geq 1-\frac{1}{n}\sum_{\kappa\in L} tan[\frac{\pi}{12}[|\Delta_Y(\kappa)-\Delta_C(\kappa)|+|\Gamma_Y(\kappa)-\Gamma_C(\kappa)|+|\Pi_Y(\kappa)-\Pi_C(\kappa)|+|\Omega_Y(\kappa)-\Omega_C(\kappa)|+|\Phi_Y(\kappa)-\Phi_Y(\kappa)|]]$$

$= T_{SVPNSM}(Y, C)$

This implies, $T_{SVPNSM}(Y, R) \geq T_{SVPNSM}(Y, C)$.

Further, we have,

$|\Delta_R(\kappa)-\Delta_C(\kappa)|\leq|\Delta_Y(\kappa)-\Delta_C(\kappa)|$, $|\Gamma_R(\kappa)-\Gamma_C(\kappa)|\leq|\Gamma_Y(\kappa)-\Gamma_C(\kappa)|$, $|\Pi_R(\kappa)-\Pi_C(\kappa)|\leq|\Pi_Y(\kappa)-\Pi_C(\kappa)|$, $|\Omega_R(\kappa)-\Omega_C(\kappa)|\leq|\Omega_Y(\kappa)-\Omega_C(\kappa)|$, $|\Phi_R(\kappa)-\Phi_C(\kappa)|\leq|\Phi_Y(\kappa)-\Phi_C(\kappa)|$, for all $\kappa\in L$.

Therefore,

$T_{SVPNSM}(R, C)$

$$=1-\frac{1}{n}\sum_{\kappa\in L} tan[\frac{\pi}{12}[|\Delta_R(\kappa)-\Delta_C(\kappa)|+|\Gamma_R(\kappa)-\Gamma_C(\kappa)|+|\Pi_R(\kappa)-\Pi_C(\kappa)|+|\Omega_R(\kappa)-\Omega_C(\kappa)|+|\Phi_R(\kappa)-\Phi_C(\kappa)|]]$$

$$\geq 1-\frac{1}{n}\sum_{\kappa\in L} tan[\frac{\pi}{12}[|\Delta_Y(\kappa)-\Delta_C(\kappa)|+|\Gamma_Y(\kappa)-\Gamma_C(\kappa)|+|\Pi_Y(\kappa)-\Pi_C(\kappa)|+|\Omega_Y(\kappa)-\Omega_C(\kappa)|+|\Phi_Y(\kappa)-\Phi_C(\kappa)|]]$$

$= T_{SVPNSM}(Y, C)$

Hence, $T_{SVPNSM}(R, C) \geq T_{SVPNSM}(Y, C)$.

Definition 3.2. Suppose that, $Y=\{(\kappa,\Delta_Y(\kappa),\Gamma_Y(\kappa),\Pi_Y(\kappa),\Omega_Y(\kappa),\Phi_Y(\kappa)): \kappa\in L\}$ and $R=\{(\kappa,\Delta_R(\kappa),\Gamma_R(\kappa),\Pi_R(\kappa),\Omega_R(\kappa),\Phi_R(\kappa)): \kappa\in L\}$ be two SVPNSs over L. Then, the weighted tangent similarity measure of the similarities between two SVPNSs Y and R is defined by

$$T_{WSVPNSM}(Y, R) = 1-\frac{1}{n}\sum_{\kappa\in L} w_\kappa\, tan[\frac{\pi}{12}[|\Delta_Y(\kappa)-\Delta_R(\kappa)|+|\Gamma_Y(\kappa)-\Gamma_R(\kappa)|+|\Pi_Y(\kappa)-\Pi_R(\kappa)|+|\Omega_Y(\kappa)-\Omega_R(\kappa)|+|\Phi_Y(\kappa)-\Phi_R(\kappa)|]], \quad (2)$$

where, $\sum_{\kappa\in L} w_\kappa = 1$.

In view of Theorem 3.1. and Theorem 3.2., we formulate the following two Propositions.

Proposition 3.1. Suppose that $T_{WSVPNSM}(Y, R)$ be the weighted tangent similarity measure of similarities between the SVPNSs Y and R. Then,

(i) $0 \le T_{WSVPNSM}(Y, R) \le 1$;

(ii) $T_{WSVPNSM}(Y, R) = T_{WSVPNSM}(R, Y)$;

(iii) $T_{WSVPNSM}(Y, R) = 1$ if and only if $Y = R$.

Proposition 3.2. Assume that Y, R and C be three SVPNSs over L. If $Y \subseteq R \subseteq C$, then $T_{WSVPNSM}(Y, R) \ge T_{WSVPNSM}(Y, C)$ and $T_{WSVPNSM}(R, C) \ge T_{WSVPNSM}(Y, C)$.

4. SVPNS-MADM Strategy Based on Tangent Similarity Measure:

In this section, we develop a SVPNS-MADM model / algorithm under the SVPNS environment using the tangent similarity measure of similarities between two SVPNSs.

In our day to day life we face difficulty when we need to choose a suitable alternative from a set of possible alternatives. For that we should have to plan a strategy to take the appropriate decision.

Let $L = \{L_1, L_2,..., L_p\}$ be the family of possible alternatives. Let $A = \{A_1, A_2, ..., A_q\}$ be the family of attributes. Then, a group of DM (decision maker) together can give their evaluation information for each alternative L_i (i = 1, 2,..., p) against the attribute A_j (j = 1, 2, ..., q) by a SVPNS. Therefore, by using the whole evaluation information of all alternatives given by the decision makers, we can form a decision matrix.

The following are the steps of the proposed SVPNS-MADM:

Step-1: Decision Matrix Formation using SVPNS.

According to the decision makers evaluation information $E_{L_i}= \{(A_j,\Delta_{ij}(L_i, A_j),\Gamma_{ij}(L_i, A_j),\Pi_{ij}(L_i, A_j), \Omega_{ij}(L_i, A_j),\Phi_{ij}(L_i, A_j)): A_j \in A\}$ for each alternatives L_i against the attributes A_j (j = 1, 2, ..., q), we can build a decision matrix, where $(\Delta_{ij}(L_i, A_j),\Gamma_{ij}(L_i, A_j),\Pi_{ij}(L_i, A_j),\Omega_{ij}(L_i, A_j),\Phi_{ij}(L_i, A_j)) = (L_i, A_j)$ indicates the evaluation information of the alternative L_i (i = 1, 2,..., p) against the attribute A_j (j = 1, 2, ..., q).

The decision matrix (D^M) can be expressed as follows:

D^M	A_1	A_2	....	....	A_q
L_1	$(\Delta_{11}(L_1, A_1), \Gamma_{11}(L_1, A_1), \Pi_{11}(L_1, A_1), \Omega_{11}(L_1, A_1), \Phi(L_1, A_1))$	$(\Delta_{12}(L_1, A_2), \Gamma_{12}(L_1, A_2), \Pi_{12}(L_1, A_2), \Omega_{12}(L_1, A_2), \Phi_{12}(L_1, A_2))$	....	....	$(\Delta_{1q}(L_1, A_q), \Gamma_{1q}(L_1, A_q), \Pi_{1q}(L_1, A_q), \Omega_{1q}(L_1, A_q), \Phi_{1q}(L_1, A_q))$
L_2	$(\Delta_{21}(L_2, A_1), \Gamma_{21}(L_2, A_1), \Pi(L_2, A_1), \Omega_{21}(L_2, A_1), \Phi_{21}(L_2, A_1))$	$(\Delta_{22}(L_2, A_2), \Gamma_{22}(L_2, A_2), \Pi_{22}(L_2, A_2), \Omega_{22}(L_2, A_2), \Phi_{22}(L_2, A_2))$	......	....	$(\Delta_{2q}(L_2, A_q), \Gamma_{2q}(L_2, A_q), \Pi_{2q}(L_2, A_q), \Omega_{2q}(L_2, A_q), \Phi_{2q}(L_2, A_q))$
. . . .	. . . .	. . . .	. . . .	. . . .	. . . .

L_p	(Δ_{p1}(L_p, A_1), Γ_{p1}(L_p, A_1), Π_{p1}(L_p, A_1), Ω_{p1}(L_p, A_1), Φ_{p1}(L_p, A_1))	(Δ_{p2}(L_p, A_2), Γ_{p2}(L_p, A_2), Π_{p2}(L_p, A_2), Ω_{p2}(L_p, A_2), Φ_{p2}(L_p, A_2))	…… …..	…… ….	(Δ_{pq}(L_p, A_q), Γ_{pq}(L_p, A_q), Π_{pq}(L_p, A_q), Ω_{pq}(L_p, A_q), Φ_{pq}(L_p, A_q))

Step-2. Determination of the Weights for each Attribute.

Determination of the value of weights for each attributes is an important task for any multi attribute decision making model. If the weights of the attributes are completely unknown in a MADM problem, then the decision makers can use the compromise function.

The compromise function of L is defined as follows:

$\xi_j = \sum_{i=1}^{p}$ (3 + $\Delta_{ij}(L_i, A_j)$ +$\Gamma_{ij}(L_i, A_j)$ -$\Pi_{ij}(L_i, A_j)$ -$\Omega_{ij}(L_i, A_j)$ -$\Phi_{ij}(L_i, A_j)$)/5. (3)

Then, the weights of the j-th attribute is defined by $w_j = \frac{\xi_j}{\sum_{j=1}^{q}\xi_j}$ (4)

Here, $\sum_{j=1}^{q} w_j$=1.

Step-3. Selection of the Benefit-type Attributes and Cost-type Attributes.

In any MADM problems, the attributes can be divided into two types namely benefit-type attribute and cost-type attribute. In our proposed SVPNS-MADM strategy, an ideal alternative can be defined by using a minimum operator for the cost-type attributes and maximum operator for the benefit-type attributes to determine the best value of each attribute among all alternatives.

The ideal alternative is defined as follows:

$I = (C_1^+, C_2^+, C_3^+, \dots\dots, C_q^+)$ (5)

When C_j (j=1, 2, …., q) is a benefit type of attribute, then

C_j^+= (max {$\Delta_{ij}(L_i, A_j)$: i=1, 2, 3, ….., p}, max {$\Gamma_{ij}(L_i, A_j)$: i=1, 2, 3, ….., p}, min {$\Pi_{ij}(L_i, A_j)$: i=1, 2, 3, ….., p}, min {$\Omega_{ij}(L_i, A_j)$: i=1, 2, 3, ….., p}, min {$\Phi_{ij}(L_i, A_j)$: i=1, 2, 3, ….., p}). (6)

When C_j (j=1, 2, …., q) is a cost type of attribute, then

C_j^+= (min {$\Delta_{ij}(L_i, A_j)$: i=1, 2, 3, ….., p}, min {$\Gamma_{ij}(L_i, A_j)$: i=1, 2, 3, ….., p}, max {$\Pi_{ij}(L_i, A_j)$: i=1, 2, 3, ….., p}, max {$\Omega_{ij}(L_i, A_j)$: i=1, 2, 3, ….., p}, max {$\Phi_{ij}(L_i, A_j)$: i=1, 2, 3, ….., p}). (7)

Step-4. Determination of the Tangent Similarity Measure between the Ideal Alternative and Other Alternatives.

In this step, we calculate the tangent similarity measure of similarities between the ideal alternatives and the decision elements from the decision matrix by using eq. (1).

Step-5: Determination of the accumulated measure values.

To aggregate the similarity measures corresponding to each alternative we use the following accumulated measure function (AMF):

$D^{i}_{AMF} = \sum_{j=1}^{q} w_j T_{SVPNSM}((L_i, A_j), C_j^+)$ (8)

where, $(L_i, A_j) = (\Delta_{ij}(L_i, A_j), \Gamma_{ij}(L_i, A_j), \Pi_{ij}(L_i, A_j), \Omega_{ij}(L_i, A_j), \Phi_{ij}(L_i, A_j))$.

Step-6: Ranking of the alternatives.

Ranking of alternatives is prepared based on the ascending order of accumulated measure values. The alternative associated with the highest accumulated measure value is the best suitable alternatives.

Step-7: End.

The flow chart of the proposed SVPNS-MADM strategy is given below:

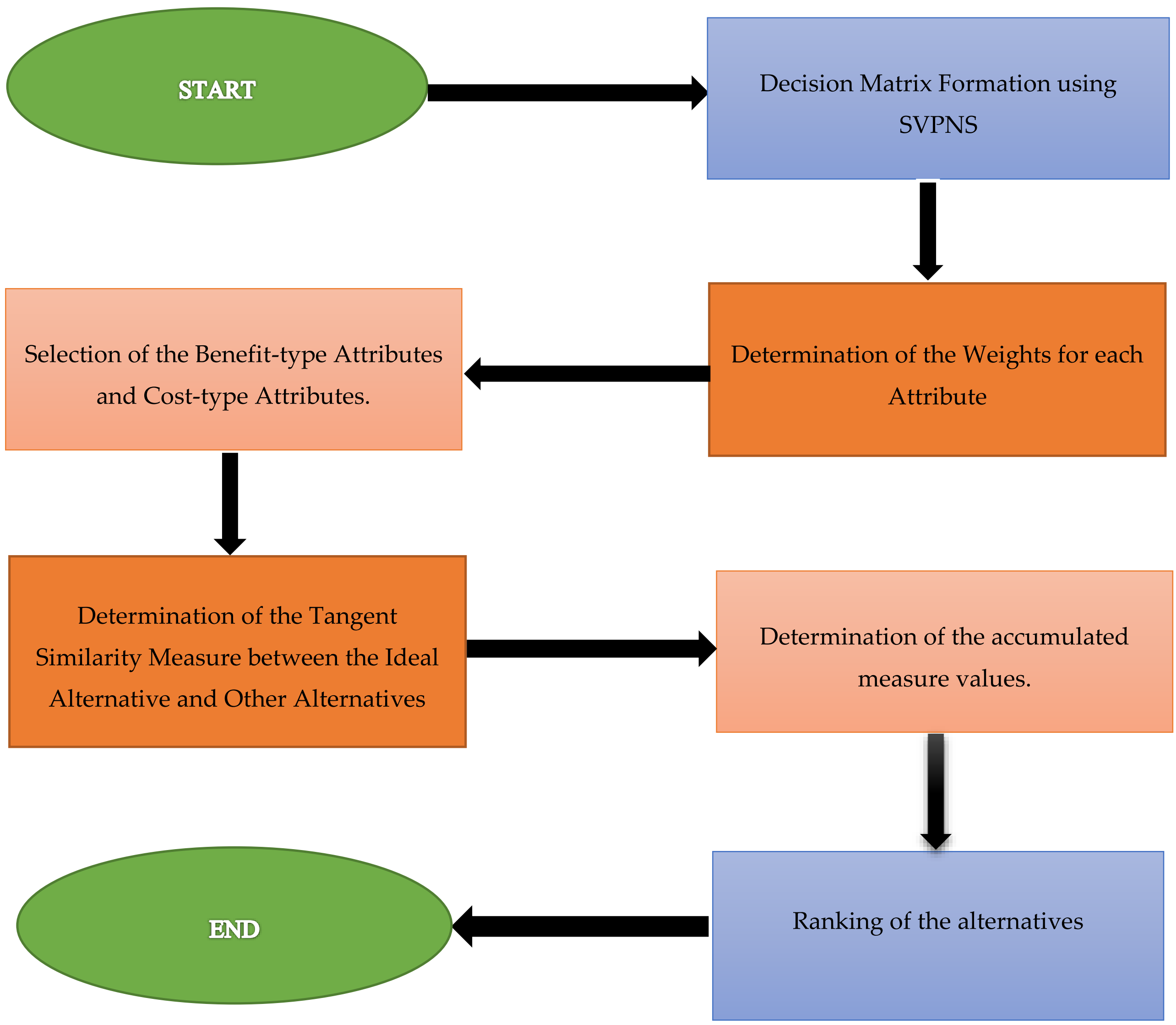

Figure- 1

5. Validation of the Proposed SVPNS-MADM Strategy.

In this section, we validate our proposed SVPNS-MADM strategy by a numerical example.

Example 5.1. Selection of Plot to Build a New House in Urban Area.

There are few other things come into account, when we are searching for a good plot to build our house. To choose a plot, lots of things we have to took in our consideration. Initially it is obscure, whether the place had good communication with the city, availability of well roads, gas-pipeline, water facility, electricity etc.

Most of the common problems to select a plot are:

(*i*) Price of the plot.
(*ii*) Well connectivity with necessary facilities.
(*iii*) Does the plot have significant slope or have to fill or cut the slope.
(*iv*) The buildings in neighbor plot, size, shape which also affect your disclosure to sunlight in your living area.
(*v*) Types of soil is also another important factor to build a house, to keep the house stable types of soil composition and reactive nature of soil are relevant. Normally, worth will high to build in more reactive area.

So, the selection of plot by a person can be considered as a MAMD problem. After initializing, the decision maker selects three major alternatives namely L_1, L_2 and L_3. For the selection of suitable place, the decision maker select four attributes such as A_1: Price of the place, A_2: Well connectivity with the other part of the city, A_3: Shape of the plot, A_4: Type of the soil of plot.

Then, the SVPNS-MADM strategy is presented as follows:

By using the evaluation information for all alternatives given by the decision makers, we prepare the decision matrix as follows:

Table-1:

	A_1	A_2	A_3	A_4
L_1	(0.8,0.3,0.2,0.4,0.3)	(0.7,0.3,0.5,0.2,0.4)	(0.8,0.2,0.3,0.4,0.2)	(0.9,0.2,0.3,0.4,0.3)
L_2	(0.9,0.1,0.4,0.2,0.3)	(0.8,0.3,0.5,0.4,0.3)	(0.7,0.2,0.2,0.2,0.3)	(0.8,0.1,0.2,0.4,0.3)
L_3	(0.8,0.4,0.3,0.2,0.2)	(0.9,0.1,0.5,0.2,0.2)	(0.6,0.1,0.2,0.3,0.1)	(0.8,0.2,0.2,0.4,0.5)

Now, we determine the weight of each attribute by using the eq. (3) and eq. (4). The weight vector for all attributes is given below.

(w_1, w_2, w_3, w_4) = (0.268097, 0.238606, 0.252011, 0.241287).

According to the expert opinion, we choose the attribute A_2, A_3, A_4 as benefit-types of attribute and the attributes A_1 as cost-type of attributes. Now, we choose the ideal alternative solution by using eq. (5), eq. (6), and eq. (7). The ideal solution I is given in the following table.

Table-2:

	A_1	A_2	A_3	A_4
L_1	(0.9,0.3,0.1,0.5,0.2)	(0.8,0.2,0.2,0.1,0.4)	(0.9,0.1,0.3,0.1,0.3)	(0.9,0.4,0.2,0.3,0.4)
L_2	(0.8,0.1,0.3,0.3,0.2)	(0.9,0.2,0.3,0.4,0.2)	(0.6,0.1,0.2,0.3,0.3)	(0.9,0.2,0.1,0.2,0.2)
L_3	(0.9,0.4,0.2,0.3,0.1)	(0.7,0.3,0.4,0.1,0.2)	(0.8,0.2,0.1,0.2,0.3)	(0.8,0.3,0.1,0.3,0.1)
I	(0.8,0.1,0.3,0.5,0.2)	(0.9,0.3,0.2,0.1,0.2)	(0.9,0.2,0.1,0.1,0.3)	(0.9,0.4,0.1,0.2,0.1)

After the formation of ideal alternative solution in Table-2, we determine the tangent similarity measure of similarities between the ideal alternative solution and the decision elements from table -1 by using eq. (1).

The aggregate tangent similarity measures corresponding to each alternative are given below:
D^1_{AMF} = 0.70862, D^2_{AMF} =0.889084, D^3_{AMF} = 0.885916.

Therefore, $D^1_{AMF} < D^3_{AMF} < D^2_{AMF}$. This implies, the alternative L_2 is the most suitable alternative (plot) for choosing to build a house.

6. Conclusions:

In the article, we have established a SVPNS-MADM strategy based on tangent similarity measure of similarities between two SVPNSs. We have also validated our proposed SVPNS-MADM strategy by solving an illustrative numerical example to demonstrate the effectiveness of the proposed SVPNS-MADM strategy.

The proposed SVPNS-MADM strategy can also be used to deal with the other decision-making problems such as tender selection [5], teacher selection [28], medical diagnosis [26, 27], weaver selection [14], brick selection [16, 20], logistic center location selection [23, 24], etc.

Conflict of Interest: The authors declare that they have no conflict of interest.

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Neutrosophic Supra Simply Open Set and Neutrosophic Supra Simply Compact Space

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Abstract: The aim of this article is to procure the notions of neutrosophic supra simply open set, neutrosophic supra simply open cover, neutrosophic supra simply compactness via neutrosophic supra topological spaces. Further, we formulate some results in the form of remarks, theorems, propositions etc.

Keywords: *Neutrosophic Supra Simply Open*; *Neutrosophic Supra Simply b-Open*; *Neutrosophic Supra Compact*; *Neutrosophic Supra Simply compact*.

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1. Introduction

Smarandache [30] grounded the notion of neutrosophic set (in short NS) in the year 1998 by extending the concept of intuitionistic fuzzy set [1]. In the year 2010, Wang et al. presented the idea of single valued neutrosophic set. Till now, many researchers around the globe applied the notions of NS and its extensions in the formation of MADM-algorithms [3, 6, 10, 18, 21, 22, 24, 25, etc.]. In the year 2012, Salama and Alblowi [28] established the idea of neutrosophic topological space (in short NTS). Further, Salama and Alblowi [29] presented generalized NS and generalized NTS. The concept of neutrosophic semi-closed set and neutrosophic semi-open set in NTSs was introduced by Iswaraya and Bageerathi [16]. Afterwards, Arokiarani et al. [2] grounded the idea of neutrosophic semi-open functions and established relation between them. The concept of generalized neutrosophic closed sets in NTSs was studied by Dhavaseelan, and Jafari [14]. The neutrosophic generalized closed sets in NTSs was established by Pushpalatha and Nandhini [26]. Thereafter, Ebenanjar et al. [15] presented the neutrosophic *b*-open sets in NTSs. In the year 2018, Maheswari et al. [19] grounded the idea of neutrosophic generalized *b*-closed sets in NTSs. In the year 2020, Das and Pramanik [7] introduced the notion of generalized neutrosophic *b*-open sets in NTSs. Das and Pramanik [8] also established the concept of neutrosophic Φ-open sets and neutrosophic Φ-continuous functions. Mallick and Pramanik [20] introduced the notions of pentapartitioned neutrosophic set and studied their several properties. Later on, Das et al. [5] presented the idea of pentapartitioned neutrosophic *Q*-ideals of *Q*-algebra. In the year 2021, Das et al. [4] grounded the notions of quadripartitioned neutrosophic topological space. Noori and Yousif [23] introduced the idea of soft simply compact space via soft topological spaces in the year 2020. Afterwards, Das and Pramanik [9] presented the concept of neutrosophic simply soft open set in neutrosophic soft topological spaces, and studied the neutrosophic simply soft compactness. Das and Tripathy [12] introduced the neutrosophic simply-b-open set in NTSs. Dhavaseelan et al. [13] grounded the notion of neutrosophic supra topology and studied the neutrosophic α-supra open sets in neutrosophic supra topological spaces (in short NSTS). Later on, Jayaparthasarathy et al. [17] presented an application of neutrosophic supra topology in data mining process.

The main focus of this article is to procure the concept of neutrosophic supra simply open set, neutrosophic supra simply compactness via NSTSs. We formulate some results on neutrosophic supra simply open set, neutrosophic supra simply compactness.

2. Preliminaries and Definitions:

Definition 2.1. [17] A family τ of NSs over a fixed set W is called an neutrosophic supra topology (in short NST) on W if the following holds:

(i) $0_N, 1_N \in \tau$;

(ii) $\cup J_i \in \tau$, for every $\{J_i : I \in \Delta\} \subseteq \tau$.

The pair (W,τ) is called an neutrosophic supra topological space (in short NSTS). If $Y \in \tau$, then Y is called an neutrosophic supra open set (in short NSO-set) and Y^c is called an neutrosophic supra closed set (in short NSC-set).

Remark 2.1. [17] The family of all NSO-sets and NSC-sets in an NSTS (W,τ) may be denoted by NSO(W) and NSC(W) respectively.

Definition 2.2. [17] Suppose that (W,τ) be an NSTS. Then, the neutrosophic supra interior (in short N_{int}^s) and neutrosophic supra closure (in short N_{cl}^s) of an NS Y over W are defined by

$N_{int}^s(Y) = \cup\{R : R$ is an NSO-set in W and $R \subseteq Y\}$;

$N_{cl}^s(Y) = \cap\{P : P$ is an NSC-set in W and $Y \subseteq P\}$.

Definition 2.3. [13] Assume that (W,τ) be an NSTS. Then Y, an NS over W is called an neutrosophic supra α-open set (in short NS-α-O-set) iff $Y \subseteq N_{int}^s(N_{cl}^s(N_{int}^s(Y)))$.

Definition 2.4. [13] Suppose that (W,τ) be an NSTS. Then Y, an NS over W is called an neutrosophic supra semi-open set if and only if $Y \subseteq N_{cl}^s(N_{int}^s(Y))$.

Definition 2.5. [13] Let (W,τ) be an NSTS. Let Y be an NS over W. Then, Y is called an neutrosophic supra pre-open set if and only if $Y \subseteq N_{int}^s(N_{cl}^s(Y))$.

Remark 2.2. Throughout the paper, NS-α-O(W), NSSO(W), NSPO(W) denotes the family of all NS-α-O-sets, NSS-O-sets, NSP-O-sets in NSTS (W,τ).

Definition 2.6. [13] Suppose that ξ be a one to one and onto mapping from an NSTS (W,τ_1) to another NSTS (M,τ_2). Then, ξ is called an

(i) neutrosophic supra continuous (in short NS-Continuous) function if $\xi^{-1}(K)$ is a NSO-set in W, whenever K is a NSO-set in M.

(ii) neutrosophic supra-α-continuous (in short NS-α-Continuous) function if $\xi^{-1}(K)$ is a NS-α-O-set in W, whenever K is a NSO-set in M.

3. Neutrosophic Supra Simply Open Set:

In this section, we procure the notions of neutrosophic supra b-open set, neutrosophic supra b-continuous mapping, neutrosophic supra simply open set, neutrosophic supra simply continuous mapping, neutrosophic supra simply b-continuous mapping, neutrosophic supra simply compactness, and neutrosophic supra simply b-compactness in NSTSs.

Definition 3.1. Suppose that (W,τ) be an NSTS. Then Y, an NS over W is called an neutrosophic supra b-open set (NS-b-O-set) iff $Y \subseteq N^s_{int}(N^s_{cl}(Y)) \cup N^s_{cl}(N^s_{int}(Y))$.

Remark 3.1. Throughout the paper, NS-b-O(W) and NS-b-C(W) denotes the family of all NS-b-O-sets and NS-b-C-sets in NSTS (W,τ). Clearly, NSOS(W) ⊆ NS-bO(W) and NSCS(W) ⊆ NS-bC(W).

Definition 3.2. Suppose that ξ be an one to one and onto mapping from an NSTS (W,τ_1) to another NSTS (M,τ_2). Then, ξ is called an neutrosophic supra-b-continuous (in short NS-b-Continuous) function if $\xi^{-1}(K)$ is an NS-b-O-set in W, whenever K is an NSO-set in M.

Definition 3.3. A collection $\{S_\alpha: \alpha\in\Delta\}$ of NSO-sets in (W,τ), where Δ is an index set, is called an neutrosophic supra open cover (in short NSO-cover) of an neutrosophic set S if $S \subseteq \cup_{\alpha\in\Delta} S_\alpha$.

Definition 3.4. A family $\{S_\alpha: \alpha = 1, 2, 3, \dots, n\}$ of NSO-sets in (W,τ) is called an neutrosophic supra open finite sub cover (in short NSO-finite sub cover) of an neutrosophic set S if $S \subseteq \cup_{\alpha=1}^{n} S_\alpha$.

Definition 3.5. An NSTS (W,τ) is called an neutrosophic supra compact space (in short NS-compact-space) if every NSO-cover of W has an NSO-finite sub cover.

Definition 3.6. An neutrosophic subset B of an NSTS (W,τ) is said to be an neutrosophic supra compact set relative to W if every NSO-cover of B has a finite sub-cover.

Definition 3.7. A family $\{S_\alpha: \alpha\in\Delta\}$ of NS-b-O-sets in (W,τ), where Δ is an index set, is called an neutrosophic supra b-open cover (in short NS-b-O-cover) of an neutrosophic set S if $S \subseteq \cup_{\alpha\in\Delta} S_\alpha$.

Definition 3.8. Suppose that (W,τ) be an NSTS. Then, (W,τ) is called an neutrosophic supra b-compact space (in short NS-b-compact-space) if every NS-b-O-cover of W has a finite sub-cover.

Definition 3.9. An neutrosophic subset B of an NSTS (W,τ) is said to be an neutrosophic supra b-compact relative to W if every NS-b-O-cover of B has a finite sub-cover.

Theorem 3.1. Every NS-b-compact-space is an NS-compact-space.

Proof. Suppose that (W,τ) be an NS-b-compact-space. Therefore, every NS-b-O-cover of (W,τ) has a finite sub-cover. Let (W,τ) may not be an NS-compact-space. Then, there exists an NSO-cover $\mathcal{H}$ (say) of W, which has no finite sub-cover. Since, every NSO-set is an NS-b-O-set, so there exists an NS-b-O-cover $\mathcal{H}$ of W, which has no finite sub-cover. This contradicts the fact that (W,τ) is an NS-b-compact-space. Therefore, (W,τ) must be an NS-compact-space.

Definition 3.10. Let (W,τ) be an neutrosophic supra topological space. Then, an NS Z over W is called an neutrosophic supra simply open set (in short NSSO-set) in (W,τ) if and only if it is an NSO-set in (W,τ) with the condition $N_{int}N_{cl}(Z) \subseteq N_{cl}N_{int}(Z)$.

Clearly, every NSSO-set is an NSO-set in (W,τ).

Remark 3.2. In an NSTS (W,τ), both 0_N and 1_N are NSSO-set.

Definition 3.11. Suppose that Z be an neutrosophic set over a fixed set W. Then, Z is called an neutrosophic supra simply b-open set (in short NSS-b-O-set) in the NSTS (W,τ) if and only if it is an NS-b-O-set in (W,τ) with the condition $N_{int}N_{cl}(Z) \subseteq N_{cl}N_{int}(Z)$.

If Y is an NSS-b-O-set, then Y^c is called an neutrosophic supra simply b-closed set (in short NSS-b-C-set). The family of all NSS-b-O-sets and NSS-b-C-sets may be denoted as NSS-b-O(W) and NSS-b-C(W) respectively.

Clearly, every NSS-b-O-set in an NSTS (W,τ), is also an NS-b-O-set.

Remark 3.3. In an NSTS (W,τ), both 0_N and 1_N are NSSO-set.

Theorem 3.2. In an NSTS (W,τ), every NSO-set is an NSS-O-set in an NSTS (W,τ).

Proof: Suppose that J be an NSO-set in an NSTS (W,τ). Therefore, $N_{int}(J) = J$. It is known that, every NSO-set is an NS-b-O-set. Therefore, J is an NS-b-O-set in (W,τ). Further, it is known that $J \subseteq N_{cl}(J)$.

Now, $J \subseteq N_{cl}(J)$

$\Rightarrow J \subseteq N_{cl}N_{int}(J)$

$\Rightarrow N_{cl}(J) \subseteq N_{cl}N_{cl}N_{int}(J)$

$=N_{cl}N_{int}(J)$ [Since $N_{cl}N_{int}(J)$ is a NSC-set in (W,τ)] (1)

Further, we have, $N_{int}N_{cl}(J) \subseteq N_{cl}(J)$ (2)

From eq. (1) and eq. (2), we have, $N_{int}N_{cl}(J) \subseteq N_{cl}N_{int}(J)$.

Therefore, J is an NS-b-O set in (W,τ) and $N_{int}N_{cl}(J) \subseteq N_{cl}N_{int}(J)$. Hence, J is an NS-b-O set in (W,τ).

Proposition 3.1. Every NSO-set is an NSS-b-O-set in an NSTS (W,τ).

Theorem 3.3. Every neutrosophic supra semi-open set is an NSS-b-O-set in an NSTS (W,τ).

Proof. Suppose that Q be an neutrosophic supra semi-open set in an NTS (W,τ). Therefore, $Q \subseteq N_{cl}N_{int}(Q)$. It is known that, every neutrosophic supra semi-open set is an NS-b-O set. This implies, Q is an NS-b-O set in (W,τ).

Now, $Q \subseteq N_{cl}N_{int}(Q)$

$\Rightarrow N_{cl}(Q) \subseteq N_{cl}N_{cl}N_{int}(Q)$

$=N_{cl}N_{int}(Q)$ [Since $N_{cl}N_{int}(Q)$ is an NSC-set in (W,τ)]

$\Rightarrow N_{cl}(Q) \subseteq N_{cl}N_{int}(Q)$ (3)

It is known that, $N_{int}N_{cl}(Q) \subseteq N_{cl}(Q)$ (4)

From eq. (3) and eq. (4), we have, $N_{int}N_{cl}(Q) \subseteq N_{cl}N_{int}(Q)$. Therefore, Q is an NS-b-O set in (W,τ) and $N_{int}N_{cl}(Q) \subseteq N_{cl}N_{int}(Q)$. Hence Q is an NSS-b-O set in (W,τ).

Theorem 3.4. If an neutrosophic set A is both neutrosophic supra pre open set and NSS-b-O-set in an NSTS (W,τ), then it is also an neutrosophic supra semi open set in (W,τ).

Proof. Let Q_1 be both neutrosophic supra pre-open set and NSS-b-O-set in an NTS (W,τ). Since, Q_1 is an neutrosophic supra pre open set, so $Q_1 \subseteq N_{int}N_{cl}(Q_1)$. Further, since Q_1 is an NSS-b-O-set, so Q_1 is an NS-b-O-set and $N_{int}N_{cl}(Q_1) \subseteq N_{cl}N_{int}(Q_1)$. This implies, $Q_1 \subseteq N_{cl}N_{int}(Q_1)$. Therefore, Q_1 is an neutrosophic supra semi open set.

Remark 3.4. Suppose that Z_1 and Z_2 be two NSS-b-O-sets. Then, $Z_1 \cap Z_2$ may not be an NSS-b-O-set.

Proof. Let Z_1 and Z_2 be two NSS-b-O-sets. Therefore, Z_1, Z_2 are NS-b-O-sets in (W,τ) such that $N_{int}N_{cl}(Z_1) \subseteq N_{cl}N_{int}(Z_1)$, $N_{int}N_{cl}(Z_2) \subseteq N_{cl}N_{int}(Z_2)$. But it is known that the intersection of two NS-b-O-sets may not be an NS-b-O-set in a NSTS (W,τ). Hence, $Z_1 \cap Z_2$ may not be an NS-b-O-set in (W,τ). Therefore, $Z_1 \cap Z_2$ may not be an NS-b-O-set in (W,τ).

Definition 3.12. An one to one and onto mapping ξ: $(W,\tau_1) \rightarrow (M,\tau_2)$ is said to be an neutrosophic supra simply continuous (in short NSS-Continuous) mapping if $\xi^{-1}(Z)$ is an NSSO-set in W whenever Z is an NSO-set in M.

Remark 3.5. Every NSS-Continuous mapping is an NS-Continuous mapping.

Definition 3.13. An one to one and onto mapping ξ: $(W,\tau_1) \rightarrow (M,\tau_2)$ is said to be an neutrosophic supra simply b-continuous (in short NSS-b-Continuous) mapping if $\xi^{-1}(Z)$ is an NSS-b-O-set in W whenever Z is an NSO-set in M.

Remark 3.6. Every NSS-b-Continuous mapping is an NS-b-Continuous mapping.

Definition 3.14. An one to one and onto mapping ξ: $(W,\tau_1) \rightarrow (M,\tau_2)$ is called an neutrosophic supra simply open mapping (in short NSS-Open-mapping) if $\xi(K)$ is an NSSO-set in M, whenever K is an NSO-set in W.

Definition 3.15. An one to one and onto mapping ξ: $(W,\tau_1)\to(M,\tau_2)$ is called an neutrosophic supra simply *b*-open mapping (in short NSS-*b*-Open-mapping) if $\xi(K)$ is an NSS-*b*-O-set in *M*, whenever *K* is an NSO-set in *W*.

Definition 3.16. An one to one and onto family $\{Z_\alpha: \alpha\in\Delta\}$, where Δ is an index set and for each $\alpha\in\Delta$, Z_α is an NS-*b*-O-set in a NTS (W,τ) is said to be an neutrosophic supra *b*-open cover of an neutrosophic set *Z* if $Z\subseteq\cup_{\alpha\in\Delta} Z_\alpha$.

Definition 3.17. An NSTS (W,τ) is called an neutrosophic supra simply compact space if every neutrosophic simply open cover of *W* has a finite sub-cover.

Definition 3.18. An neutrosophic subset *K* of an NSTS (W,τ) is called an neutrosophic supra simply compact set relative to *W* if every neutrosophic supra simply open cover of *K* has a finite sub-cover.

Theorem 3.5. Every neutrosophic supra simply closed subset of an neutrosophic supra simply compact space (W,τ) is an neutrosophic supra simply compact set relative to *W*.

Proof: Suppose that (W,τ) be an neutrosophic supra simply compact space and *K* be an neutrosophic supra simply closed set in (W,τ). Therefore, K^c is an NSS-O-set in (W,τ). Suppose that $U = \{U_i: i\in\Delta$ and $U_i\in$NSS-O$(W)\}$ be an neutrosophic supra simply open cover of *K*. Therefore, $\mathcal{H}=\{K^c\}\cup U$ is an neutrosophic supra simply open cover of *X*. Since, *X* is an neutrosophic supra simply compact space, so it has a finite sub-cover say $\{H_1, H_2, H_3,\ldots,H_n, K^c\}$. This implies, $\{H_1, H_2, H_3,\ldots, H_n\}$ is a finite neutrosophic supra simply open cover of *K*. Hence, *K* is an neutrosophic supra simply compact set relative to *W*.

Definition 3.19. An NSTS (W,τ) is called an neutrosophic supra simply *b*-compact space if each neutrosophic simply *b*-open cover of *W* has a finite sub-cover.

Definition 3.20. An neutrosophic subset *K* of (W,τ) is called an neutrosophic supra simply *b*-compact set relative to *W* if every neutrosophic supra simply *b*-open cover of *K* has a finite sub-cover.

Theorem 3.6. Every neutrosophic supra simply *b*-closed subset of an neutrosophic supra simply *b*-compact space (W,τ) is an neutrosophic supra simply *b*-compact set relative to *W*.

Proof: Suppose that (W,τ) be an neutrosophic supra simply *b*-compact space and *K* be an neutrosophic supra simply *b*-closed set in (W,τ). Therefore, K^c is an NSS-*b*-O-set in (W,τ). Suppose that $U = \{U_i: i\in\Delta$ and $U_i\in$NSS-*b*-O$(W)\}$ be an neutrosophic supra simply *b*-open cover of *K*. Therefore, $\mathcal{H}=\{K^c\}\cup U$ is an neutrosophic supra simply *b*-open cover of *X*. Since, *X* is an neutrosophic supra simply *b*-compact space, so it has a finite sub-cover say $\{H_1, H_2, H_3,\ldots,H_n, K^c\}$. This implies, $\{H_1, H_2, H_3,\ldots, H_n\}$ is a finite neutrosophic supra simply *b*-open cover of *K*. Hence, *K* is an neutrosophic supra simply *b*-compact set relative to *W*.

Theorem 3.7.

(i) Every neutrosophic supra *b*-compact space is an neutrosophic supra simply *b*-compact space.

(ii) Every neutrosophic supra simply *b*-compact space is an neutrosophic supra compact space.

Proof. (i) Let (W,τ) be an neutrosophic supra *b*-compact space. Suppose that (W,τ) is not an neutrosophic supra simply *b*-compact space. Then there exists an neutrosophic supra simply *b*-open cover $\mathcal{H}$ (say) of *W*, which has no finite sub-cover. Since, every neutrosophic supra simply *b*-open set is an neutrosophic supra *b*-open set, so we have an neutrosophic supra *b*-open cover $\mathcal{H}$ of *W*, which has no finite sub-cover. This contradicts our assumption. Hence, (W,τ) is an neutrosophic supra simply *b*-compact space.

(ii) Let (W,τ) be an neutrosophic supra simply b-compact space. Suppose that (W,τ) is not an neutrosophic supra compact space. Therefore, there exists an neutrosophic supra open cover $\mathfrak{R}$(say) of W, which has no finite sub-cover. Since, every neutrosophic supra open set is an neutrosophic supra simply b-open set, so we have an neutrosophic supra simply b-open cover $\mathfrak{R}$ of W, which has no finite sub-cover. This contradicts our assumption. Hence, (W,τ) is an neutrosophic supra compact space.

Theorem 3.8. If $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ is an neutrosophic supra open function and (M,τ_2) is an neutrosophic supra compact space, then (W,τ_1) is an neutrosophic supra compact space.

Proof. Assume that $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ be an neutrosophic supra open function and (M,τ_2) be an neutrosophic supra compact space. Suppose $\mathcal{H}$ = $\{H_i$: $i\in\Delta$ and $H_i\in NSO(W)\}$ be an neutrosophic supra open cover of W. Therefore, $\xi(\mathcal{H})=\{\xi(H_i)$: $i\in\Delta$ and $\xi(H_i)\in NSO(M)\}$ is an neutrosophic supra open cover of M. Since, (M,τ_2) is an neutrosophic supra compact space, so there exists a finite sub-cover say $\{\xi(H_1), \xi(H_2), \ldots\ldots., \xi(H_n)\}$ such that $M \subseteq\cup\{\xi(H_i)$: i=1, 2, …, $n\}$. This implies, $\{H_1, H_2, \ldots, H_n\}$ is a finite sub-cover for W. Hence, (W,τ_1) is an neutrosophic supra compact space.

Theorem 3.9. Suppose that (W,τ_1) and (M,τ_2) be two NSTSs.

(i) If $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ is an neutrosophic supra b-open function and (M,τ_2) is an neutrosophic supra b-compact space, then (W,τ_1) is an neutrosophic supra compact space.

(ii) If $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ is an neutrosophic supra simply b-open function and (M,τ_2) is an neutrosophic supra simply b-compact space, then (W,τ_1) is also an neutrosophic supra b-compact space.

Proof. (i) Let $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ be an neutrosophic supra b-open function and (M,τ_2) be an neutrosophic supra b-compact space. Let $\mathcal{H}=\{H_i$: $i\in\Delta$ and $H_i\in$N-bO$(W)\}$ be an neutrosophic supra open cover of W. Therefore, $\xi(\mathcal{H})=\{\xi(H_i)$: $i\in\Delta$ and $\xi(H_i)\in$N-bO$(M)\}$ is an neutrosophic supra b-open cover of M. Since, (M,τ_2) is an neutrosophic supra b-compact space, so there exists a finite sub-cover say $\{\xi(H_1), \xi(H_2), \ldots\ldots., \xi(H_n)\}$ such that $M \subseteq\cup\{\xi(H_i)$: i=1, 2, …, $n\}$. This implies, $\{H_1, H_2, \ldots, H_n\}$ is a finite sub-cover for W. Hence, (W,τ_1) is an neutrosophic supra compact space.

(ii) Suppose that $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ be an neutrosophic supra simply b-open function and (M,τ_2) be an neutrosophic supra simply b-compact space. Let $\mathcal{H}=\{K_i$: $i\in\Delta$ and $K_i\in$N-bO$(W)\}$ be an neutrosophic supra b-open cover of W. Therefore, $\xi(\mathcal{H})=\{\xi(K_i)$: $i\in\Delta$ and $\xi(K_i)\in$NSS-b-O$(M)\}$ is an neutrosophic supra simply b-open cover of M. Since, (M,τ_2) is an neutrosophic supra simply b-compact space, so there exists a finite sub-cover say $\{\xi(K_1), \xi(K_2), \ldots\ldots., \xi(K_n)\}$ such that $M\subseteq\cup\{\xi(K_i)$: i=1, 2, …, $n\}$. Therefore, $\{K_1, K_2, \ldots, K_n\}$ is a finite sub-cover for W. Hence, $(W,\tau_1)$is an neutrosophic supra b-compact space.

Theorem 3.10. Let (W,τ_1) and (M,τ_2) be two NSTSs.

(i) If $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ is an neutrosophic supra b-continuous function, then $\xi(Q)$ is an neutrosophic supra simply b-compact set in M whenever Q is an neutrosophic supra b-compact set relative to W.

(ii) If $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ is an neutrosophic supra b-continuous function, then $\xi(Z)$ is an neutrosophic supra compact set in M whenever Z is an neutrosophic supra b-compact set relative to W.

Proof. (i) Suppose that $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ be an neutrosophic supra b-continuous function and Q be an neutrosophic supra b-compact set relative to W. Let $\mathcal{H}=\{H_i$: $i\in\Delta$ and $H_i\in N^s$-bO$(M)\}$ be an neutrosophic supra simply b-open cover of $\xi(Q)$. Since, every NSS-b-O-set is an NS-b-O-set, so $\mathcal{H}=\{H_i$: $i\in\Delta$ and $H_i\in$NSS-b-O$(M)\}$ is an neutrosophic supra b-open cover of $\xi(Q)$. By hypothesis $\xi^{-1}(\mathcal{H})$= $\{\xi^{-1}(H_i)$: $i\in\Delta$ and $\xi^{-1}(H_i)\in$NSS-b-O$(M)\}$ is an neutrosophic supra b-open cover of $\xi^{-1}(\xi(Q))=Q$.

Since, Q is an neutrosophic supra b-compact set relative to W, so there exists a finite sub-cover of Q say $\{H_1, H_2, H_3, \ldots, H_n\}$ such that $Q\subseteq\cup_i\{H_i:i=1,2,\ldots,n\}$.

Now, $Q\subseteq\cup_i\{H_i:i=1,2,\ldots,n\}$

$\Rightarrow\xi(Q)\subseteq\cup_i\{\xi(H_i):i=1,2,\ldots,n\}$

Therefore, there exist a finite sub-cover $\{\xi(H_1), \xi(H_2), \xi(H_3),\ldots,\xi(H_n)\}$ of $\xi(Q)$ such that $\xi(Q)\subseteq\cup_i\{\xi(H_i):i=1,2,\ldots,n\}$. Hence, $\xi(Q)$ is an neutrosophic supra simply b-compact set relative to M.

(ii) Let $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ be an neutrosophic supra b-continuous function and Z be an neutrosophic supra b-compact set relative to W. Let $\mathcal{H}=\{H_i: i\in\Delta$ and $H_i\in N\text{-}bO(M)\}$ be an neutrosophic supra open cover of $\xi(Z)$. By hypothesis $\xi^{-1}(\mathcal{H})=\{\xi^{-1}(H_i): i\in\Delta$ and $\xi^{-1}(H_i)\in N\text{-}bO(M)\}$ is an neutrosophic supra b-open cover of $\xi^{-1}(\xi(Z))=Z$. Since, Z is an neutrosophic supra b-compact set relative to W, so there exists a finite sub-cover of Z say $\{H_1, H_2, H_3, \ldots, H_n\}$ such that $Z\subseteq\cup_i\{H_i:i=1,2,\ldots,n\}$.

Now, $Z\subseteq\cup_i\{H_i:i=1,2,\ldots,n\}$

$\Rightarrow\xi(Z)\subseteq\cup_i\{\xi(H_i):i=1,2,\ldots,n\}$

Therefore, there exist a finite sub-cover $\{\xi(H_1), \xi(H_2), \xi(H_3),\ldots,\xi(H_n)\}$ of $\xi(Z)$ such that $\xi(Z)\subseteq\cup_i\{\xi(H_i):i=1,2,\ldots,n\}$. Hence, $\xi(Z)$ is an neutrosophic supra compact set relative to M.

Theorem 3.11. Every neutrosophic supra simply continuous function from an NSTS (W,τ_1) to another NSTS (M,τ_2) is an neutrosophic supra continuous function.

Proof. Suppose that $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ be an neutrosophic supra simply continuous function. Let Q be an NSO-set in (M,τ_2). By hypothesis $\xi^{-1}(Q)$ is an NSSO-set in (W,τ_1). It is known that, every NSSO-set is an NSO-set, so $\xi^{-1}(Q)$ is an NSSO-set in (W,τ_2). Therefore, $\xi^{-1}(Q)$ is an NSSO-set in (W,τ_2), whenever Q is an NSO-set in (M,τ_2). Hence, $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ is an neutrosophic supra simply continuous function.

Theorem 3.12. Every neutrosophic supra continuous function from an NSTS (W,τ_1) to another NSTS (M,τ_2) is an neutrosophic supra simply b-continuous function.

Proof. Let $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ be an neutrosophic supra continuous function. Let Q be an NSO-set in (M,τ_2). By hypothesis $\xi^{-1}(Q)$ is an NSO-set in (W,τ_1). Since, every NSO-set is an NSS-b-O set, so $\xi^{-1}(Q)$ is an NSS-b-O set in (W,τ_2). Therefore, $\xi^{-1}(Q)$ is an NSS-b-O set in (W,τ_2), whenever Q is an NSO-set in (M,τ_2). Hence, $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ is an neutrosophic supra simply b-continuous function.

Theorem 3.13. Every neutrosophic supra simply b-continuous function from an NSTS (W,τ_1) to another NSTS (M,τ_2) is an neutrosophic supra b-continuous function.

Proof. Suppose that $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ be an neutrosophic supra simply b-continuous function. Let Q be an NSO-set in (M,τ_2). By hypothesis $\xi^{-1}(Q)$ is an NSS-b-O-set in (W,τ_1). Since, every NSS-b-O-set is an NS-b-O-set, so $\xi^{-1}(Q)$ is an NS-b-O-set in (W,τ_2). Therefore, $\xi^{-1}(Q)$ is an NS-b-O-set in (W,τ_2) whenever Q is an NSO-set in (M,τ_2). Hence, $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ is an neutrosophic supra b-continuous function.

Theorem 3.14. If $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ be an NSS-b-Continuous mapping and $\gamma:(M,\tau_2)\rightarrow(L,\tau_3)$ be an NS-Continuous mapping, then the composition mapping $\gamma\circ\xi: (W,\tau_1)\rightarrow(L,\tau_3)$ is an NSS-b-Continuous mapping.

Proof. Suppose that Q be an NSO-set in (L,τ_3). Since, $\gamma:(M,\tau_2)\rightarrow(L,\tau_3)$ is an NS-Continuous mapping, so $\gamma^{-1}(Q)$ is an NSO-set in (M,τ_2). Further, since $\xi:(W,\tau_1)\rightarrow(M,\tau_2)$ is an NSS-b-Continuous mapping, so $\xi^{-1}(\gamma^{-1}(Q))= (\gamma\circ\xi)^{-1}(Q)$ is an NSS-b-O-set in (W,τ_1). Hence, $(\gamma\circ\xi)^{-1}(Q)$ is an NSS-b-O-set in (W,τ_1), whenever Q is an NSO-set in (L,τ_3). Therefore, $\gamma\circ\xi: (W,\tau_1)\rightarrow(L,\tau_3)$ is an NSS-b-Continuous mapping.

4. Conclusion: In this article, we have established the notions of neutrosophic supra compactness, neutrosophic supra simply compactness via neutrosophic supra topological spaces. Further, we have proved some theorems on neutrosophic supra compactness, neutrosophic supra simply compactness. We hope that, in future based on these notions of neutrosophic supra simply open set and neutrosophic supra simply compactness many new investigations can be done.

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An Algebraic Approach to Neutrosophic Euclidean Geometry

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Abstract: The objective of this paper is to present a general definition of Neutrosophic Euclidean geometry. The mechanism of comparison between neutrosophic numbers is introduced, as well as the absolute value is extended to the neutrosophical case. In addition, the concept of neutrosophic plane with n neutrosophic dimensions is obtained. Also, Euclidean geometric concepts are extended neutrosophically such as neutrosophic distance, neutrosophic midpoint, neutrosophic vectors, neutrosophic circles, and lines. A connection between neutrosophic geometrical concepts and classical Euclidean geometry is described and established.

Keywords: Neutrosophic Plane, Neutrosophic Absolute Value, Neutrosophic Distance, Neutrosophic Midpoint, Neutrosophic Vectors, Neutrosophic Circles, Neutrosophic Lines.

1. Introduction

Neutrosophic logic is a generalization of intuitionistic fuzzy logic by adding an indeterminacy I with property $I=I^2$. Neutrosophic set concept has wide applications in different areas of science, such as decision making [7,20], health care [8,21], machine learning [9], artificial intelligence [10], soft computing [22], industry [23], and statistics [11].

On the other hand, neutrosophic sets played an interesting role in pure mathematics such as topology and analysis [12,13], spaces [1,2], and algebraic structures [3,4,5,6].

Neutrosophic spaces theory began with Agboola et.al [14], where they studied neutrosophic vector spaces and their properties. Recently, many studies have been carried out on these spaces, where AH-subspaces and homomorphisms were presented [15]. In [16,17,18,19], Hatip et. al studied neutrosophic modules (a generalized form of neutrosophic spaces) with its substructures such as homomorphisms and AH-submodules.

Euclidean geometry with two dimensions was built over points (a, b) taken from the space $R \times R$, Where constants are taken from the field of real numbers R. To extend the classical geometric system neutrosophicaly, we shall build an algebraic map which we call (AH-isometry) between the Cartesian product $R(I) \times R(I)$, and the classical space $R^2 \times R^2$.

In the classical geometry, multiplying by scalars is depending on taking these scalars from the field R , in neutrosophic geometry, multiplying is depending on neutrosophic real numbers from the neutrosophic field $R(I)$.

We show that the AH-isometry map will preserve addition and distances, in other words, it allows us to study neutrosophic vectors, points, shapes by going back to its corresponding classical concepts. This work is considered as a first step in the study of neutrosophic geometry, it has many benefits in the progression of neutrosophic studies and establishing neutrosophic functional analysis in the future, since we can define neutrosophic geometrical shapes and study their relation with classical geometrical shapes.

To reach our goals, we shall define an order relation between neutrosophic numbers as well as neutrosophic absolute values, thus we can build the concept of neutrosophic distances and norms in easy way.

This work clarifies a charming connection between neutrosophic algebra and geometry.

Motivation

We regard that there is not a geometrical system based on neutrosophic spaces, thus our motivation is to close this important research gap by defining the basic theoretical concepts of a new geometrical system based on neutrosophic numbers and spaces.

Preliminaries

Definition 2.1:[32] Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $\{x,(\mu A(x),\delta A(x),\gamma A(x)):x\in X\}$, where $\mu A(x)$, $\delta A(x)\,and\,\gamma A(x)$ represent the degree of membership, the degree of indeterminacy, and the degree of non-membership respectively of each element $x\in X$ to the set A .

Definition 2.1: [31] Classical neutrosophic number has the form $a+bI$ where a,b are real or complex numbers and I is the indeterminacy such that $0\cdot I=0$ and $I^2=I$ which results that $I^n=I$ for all positive integers n.

Definition 2.2 : [30] Let$(G,*)$be any group, the neutrosophic group is generated by I and G under * denoted by$N(G)=\{\langle G\cup I\rangle,*\}$.

Definition 2.3: [30] Let R be any ring. The neutrosophic ring $\langle R\cup I\rangle$ is also a ring generated by R and I under the operations of R.

Definition 2.4: [33] Let K be a field, the neutrosophic file generated by $\langle K\cup I\rangle$ which is denoted by $K(I)=\langle K\cup I\rangle$.

Definition 2.4: [29] Let $(M,+,.)$ be any R-module over a neutrosophic ring $R(I)$ The triple $(M(I),+,.)$ is called a strong neutrosophic R-module over a neutrosophic ring $R(I)$ generated by M and I.

3. Main concepts and discussion

In the beginning, we will define the basic concepts in neutrosophic Euclidean geometry and then we will study their relations with classical geometry.

Definition 3.1

Let $R(I)=\{a+bI;\ a,b\in R\}$ be the real neutrosophic field, we say that $a+bI\leq c+dI$ if and only if $a\leq c\ and\ a+b\leq c+d$.

Theorem 3.2

The relation defined in Definition 3.1 is a partial order relation.

Proof:

Let $x = a + bI, y = c + dI, z = m + nI \in R(I)$, we have

$x \leq x$ that is because $a \leq a$ and $a + b \leq a + b$.

Now, suppose that $x \leq y$ and $y \leq x$, then $a \leq c, a + b \leq c + d, c \leq a, c + d \leq a + b$, hence $a = c, a + b = c + d$, which means that $d = b$ and $x = y$.

Assume that $x \leq y$ and $y \leq z$, hence $a \leq c, a + b \leq c + d, c \leq m, c + d \leq m + n$, this implies that $a \leq m, a + b \leq m + n$, hence $x \leq z$. Thus $\leq$ is a partial order relation on R(I).

Remark 3.3

According to Theorem 3.2, we are able to define positive neutrosophic real numbers as follows:

$a + bI \geq 0 = 0 + 0.I$ implies that $a \geq 0, a + b \geq 0$.

Absolute value on R(I) can be defined as follows:

$|a + bI| = |a| + I[|a + b| - |a|]$, we can see that $|a + bI| \geq 0$.

We can compute the square root of a neutrosophic positive real number as follows:

$\sqrt{a + bI} = \sqrt{a} + I[\sqrt{a + b} - \sqrt{a}]$, *it is clear that* $(\sqrt{a} + I[\sqrt{a + b} - \sqrt{a}])^2 = a + bI$ *and* $\sqrt{a + bI} \geq 0$.

Example 3.4

$x = 2 - I$ is a neutrosophic positive real number, since $2 \geq 0$ *and* $(2 - 1) = 1 \geq 0$.

$2 + I \geq 2$, that is because $2 \geq 2$ *and* $(2 + 1) = 3 \geq (2 + 0) = 2$.

$|1 + 3I| = |1| + I[|1 + 3| - |1|] = 1 + 3I$.

$\sqrt{9 + 4I} = \sqrt{9} + I[\sqrt{13} - \sqrt{9}] = 3 + (\sqrt{13} - 3)I$.

Definition 3.5

We define the neutrosophic plane with n neutrosophic dimensions (N-dimensions) as follows:

$R(I) \times R(I) \times R(I) \times \ldots \times R(I)(n - times)$.

Example 3.6

$R(I) = \{a + bI; a, b \in R\}$ is a neutrosophic plane with one N-dimension.

$R(I)^2 = \{(a + bI, c + dI); a, b, c, d \in R\}$ is a neutrosophic plane with two S-dimensions.

- In the following, we will concentrate on the two N-dimensional neutrosophic plane.

Definition 3.7:

Let $A(a + bI, c + dI), B(x + yI, z + tI)$ be two neutrosophic points from $R(I)^2$, we define:

$\overrightarrow{AB} = ([x + yI] - [a + bI], [z + tI] - [c + dI])$, is called a neutrosophic vector with two N-dimensions.

Definition 3.8

Let $\vec{u} = (a + bI, c + dI)$

Be a neutrosophic vector, we define its norm as follows:

$\|\vec{u}\| = \sqrt{(a + bI)^2 + (c + dI)^2} = \sqrt{a^2 + c^2 + I[(a + b)^2 + (c + d)^2 - a^2 - c^2]}$.

It is easy to see that $\|\vec{u}\| \geq 0$, *according to Remark* 3.3.

Definition 3.9

Let $A(a + bI, c + dI), B(x + yI, z + tI)$ be two neutrosophic points from $R(I)^2$, we define:

(a) The midpoint of [AB] is $C\left(\frac{a+bI+x+yI}{2}, \frac{c+dI+z+tI}{2}\right)$.

(b) The neutrosophic distance between A and B is equal to $\|\overrightarrow{AB}\|$.

Example 3.10

Consider the following neutrosophic points $A(1+I, 2-3I), B(-I, -1+2I)$,

The neutrosophic vector $\overrightarrow{AB} = (-1-2I, -3+5I)$, the square of neutrosophic distance between A and B is $\|\overrightarrow{AB}\|^2 = 1+9+I[9+4-1-9] = 10+3I$.

Hence the neutrosophic distance is equal to $\sqrt{10+3I} = \sqrt{10} + I[\sqrt{13} - \sqrt{10}]$. We can find easily that $(\sqrt{10+3I})^2 = (\sqrt{10} + I[\sqrt{13} - \sqrt{10}])^2 = 10+3I$.

Let C be the neutrosophic midpoint of [AB], then $C\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{2}I\right)$.

Now, we list some geometrical and algebraic properties of the classical space $R^2 \times R^2$. We will need them in forthcoming sections.

Remark 3.11

Let $V = R^2 \times R^2$ be the Cartesian product of the classical Euclidean plane with itself, we have

(a) V has a module structure over the ring $R \times R$, with respect to the following operations:

Addition: $\big((a,b),(c,d)\big) + \big((x,y),(z,t)\big) = \big((a+x, b+y),(c+z, d+t)\big)$,

Multiplication by a duplet scalar from $R \times R$: $(m,n).\big((a,b),(c,d)\big) = \big((m.a, n.b),(m.c, n.d)\big)$.

(b) The norm of any vector in V can be defined as a duplet number from $R \times R$, as follows:

$$\|((a,b),(c,d))\| = \left(\sqrt{a^2+c^2}, \sqrt{b^2+d^2}\right).$$

Example 3.12

Consider the following two points from the space V , $A((1,2),(2,5)), B((-1,4),(3,-2))$, we have:(a) $\overrightarrow{AB} = \big((-2,2),(1,-7)\big)$.

(b) $\|\overrightarrow{AB}\| = \left(\sqrt{(-2)^2+(1)^2}, \sqrt{(2)^2+(-7)^2}\right) = (\sqrt{5}, \sqrt{53})$..

(c) Let $r = (5,8) \in R \times R$ be a duplet scalar, we have:

$r.\overrightarrow{AB} = \big((-10,16),(5,-56)\big)$,it is clear that $\|r.\overrightarrow{AB}\| = r.\|\overrightarrow{AB}\|$.

4. The connection between neutrosophic and classical geometry

This section is devoted to clarify the relationships between neutrosophic coordinates defined above, and between classical geometrical coordinates.

Many important questions arise according to section 3. The first one is about famous relations in classical geometry for example is the midpoint of $[AB]$ has the same neutrosophic distance from A and B?. If the answer is no, then our geometrical system is weak and has no importance because it contradicts with logical statements.

The second, do the neutrosophic points have relationships with classical points? This question is the most important one, that is because if it has a positive answer, then we are able to study geometrical shapes in neutrosophic plane.

The third is about how can we define neutrosophic lines, circles, elliptic curves,… etc.

We try to answer these important questions by using algebra, since the neutrosophic plane with two N-dimensions is a module over the ring $R(I)$.

Definition 4.1: (a) **(Two-dimensional AH-isometry)** Let $M = R(I)^2 = R(I) \times R(I), V = R^2 \times R^2$ Be the neutrosophic plane with two N-dimensions and the Cartesian product of the classical Euclidean space R^2 with itself, we define the AH-isometry map as follows:

$f: M \to V; f(a + bI, c + dI) = ((a, a + b), (c, c + d))$.

(b) (**One dimensional AH-isometry**) We can define the one-dimensional isometry between R(I) and the space R×R as follows:

$g: R(I) \to R \times R; g(a + bI) = (a, a + b)$.

Remark: The one-dimensional isometry is an isometry, i.e., an algebraic isomorphism between R(I) and $R \times R$. Also, it preserves distances on $R(I)$.

Proof: Let $a + bI, c + dI$ be two neutrosophic real numbers, then

$g(a + bI + c + dI) = g([a + c] + [b + d]I) = (a + c, a + c + b + d) = (a, a + b) + (c, c + d) = g(a + bI) + g(c + dI)$.

$g([a + bI].[c + dI]) = g(ac + I[ad + bc + bd]) = (ac, ac + ad + bc + bd) = (a, a + b).(c, c + d) = g(a + bI).g(c + dI)$.

g is a correspondence one-to-one, that is because $Ker(g) = \{0\}$, and for every pair $(a, b) \in R \times R$, there exists $a + (b - a)I \in R(I) such\ that\ g(a + [b - a]I) = (a, b)$. Thus g is an isomorphism.

The distance on $R(I)$ can be defined as follows:

Let $A = a + bI, B = c + dI$ be two neutrosophic real numbers, then $L = \|\overrightarrow{AB}\| = d[(a + bI, c + dI)] = |a + bI - (c + dI)| = |(a - c) + I(b - d)| = |a - c| + I[|a + b - c - d| - |a - c|]$.

(According to the definition of the absolute value in Remark 3.3).

On the other hand, we have:

$g(\|\overrightarrow{AB}\|) = (|a - c|, |(a + b) - (c + d)|) = (d(a, c), d(a + b, c + d)) = d[(a, a + b), (c, c + d)]$=$d(T(a + bI), T(c + dI))$

=$\|g(\overrightarrow{AB})\|$.

This implies that the distance is preserved up to isometry. i.e. $\|g(AB)\| = g(\|AB\|)$

Example 4.2

Consider the following neutrosophic point$A(1 + I, 3 - 6I)$, its isometric image is $((1,2), (3, -3))$.

Consider the following neutrosophic vector $\vec{u} = (2 - I, 4 + I), its\ isometric\ vector\ is\ \vec{v} = ((2,1), (4,5))$.

The idea behind the AH-isometry is to deal with neutrosophic points as classical points, and to explore their properties using classical Euclidean geometry.

The following theorem is considered as the fundamental theorem in neutrosophic Euclidean geometry, since it describes the relation between neutrosophic space with two N-dimensions and the classical module generated by the Cartesian product of the classical Euclidean space by itself.

Theorem 4.3: (Fundamental Theorem In neutrosophic Euclidean Geometry)

Let $f: M \to V; f(a + bI, c + dI) = ((a, a + b), (c, c + d))$ be the AH-isometry defined above, we have:

(a) f preserves addition operation between vectors.

(b) f preserves distances between points.

(c) f is a bijection one-to-one between M and V.

(d) Multiplying a neutrosophic vector by a neutrosophic real number is preserved up to isometry, i.e. The direct image of a neutrosophic vector multiplied by a neutrosophic real number is exactly equal to its AH-isometric image multiplied by the one-dimensional isometric image of the corresponding neutrosophic real number.

Proof:

(a) Let $\vec{u}=(a+bI,c+dI),\vec{v}=(x+yI,z+tI)$ be two neutrosophic vectors, we have

$f(\vec{u}+\vec{v})=f(a+x+I[b+y],c+z+I[d+t])=((a+x,a+x+b+y),(c+z,c+z+d+t))$
$=\big((a,a+b),(c,c+d)\big)+((x,x+y),(z,z+t))$=$f(\vec{u})+f(\vec{v})$

(b) We must prove that the norm of the classical vector $\overrightarrow{f(u)}$,
is exactly equal to the one-dimensional isometric image of the norm of neutrosophic vector $\vec{u}$.

$\|f(\vec{u})\|^2=(a^2+c^2,(a+b)^2+(c+d)^2)$, on the other hand, we have

$g(\|\vec{u}\|^2)=g(a^2+c^2+I[(a+b)^2+(c+d)^2-a^2-c^2]=(a^2+c^2,(a+b)^2+(c+d)^2)=\|f(\vec{u})\|^2$.

(c) Suppose that $f(a+bI,c+dI)=f(x+yI,z+tI)$, hence$\big((a,a+b),(c,c+d)\big)=\big((x,x+y),(z,z+t)\big)$, thus $x=a,b=y,z=c,d=t$, so that f is injective.

It is clear that f is surjective, thus it is a bijection.

(d) Consider the following neutrosophic vector $\vec{u}=(a+bI,c+dI)$

With the following neutrosophic real number $m+nI, we\ have$

$(m+nI).\vec{u}=((m+nI)(a+bI),(m+nI)(c+dI)=((ma+I[mb+na+nb]),(mc+I[md+nc+nd])$, on the other hand, we have

$$f\big((m+nI).\vec{u}\big)=\Big(\big(ma,(ma+mb+na+nb)\big),(mc,mc+md+nc+nd)\Big)$$

$$=(m,m+n).\big((a,a+b),(c,c+d)\big)=g(m+nI).f(a+bI,c+dI).$$

Example 4.4

Consider the following two neutrosophic points $A(1+2I,I),B(3I,-2+I)$, we have:

(a) The isometric points of A, B are $A'=\big((1,3),(0,1)\big),B'=\big((0,3),(-2,-1)\big)$.

(b) $\overrightarrow{AB}=(-1+I,-2), the\ corresponding\ isometric\ vector\ is\ \overrightarrow{A'B'}=\big((-1,0),(-2,-2)\big)=f(\overrightarrow{AB})$.

(c) The neutrosophic distance $[AB]=\sqrt{1+4+I[0+4-1-4]}$
$=\sqrt{5-I}=\sqrt{5}+I[4-\sqrt{5}]$. The classical distance between isometric images is

$$[A'B']=\Big(\sqrt{(-1)^2+(-2)^2},\sqrt{(0)^2+(-2)^2}\Big)=\big(\sqrt{5},4\big)=g([AB]).$$

Theorem 4.3 introduces an algorithm to transform any neutrosophic point to a classical Cartesian product of two classical points. The following theorem describes the inverse relation between classical coordinates and neutrosophic coordinates, i.e. It clarifies how to go back from classical coordinates to neutrosophic coordinates.

Theorem 4.5

Let $A((a,b),(c,d))$ be a Cartesian product of two classical points, then the inverse isometric image (the corresponding neutrosophic point) is

$B(a+(b-a)I,c+(d-c)I)$.

Proof:

It holds directly by taking the image of B with respect to AH-isometry, the point A is obtained.

Example 4.6:

Consider the following classical point $A((1,2),(-1,4))$, its corresponding neutrosophic point is $B(1+I,-1+5I)$.

As a result of Section 4, we can find that all geometrical famous properties is still true in neutrosophic Euclidean geometry, that is because we can transform any neutrosophic point to a corresponding classical point with preserving addition, distances, and multiplication by scalars.

5. Some neutrosophic geometrical shapes with two N-dimensions

Definition 5.1: (Neutrosophic circle)

Let $M(a+bI,c+dI)$ be a fixed neutrosophic point, we define the neutrosophic circle with centre M and radius $R=r_1+r_2I\geq 0$ to be the set of all two N-dimensional points $N(X,Y)=N(x_0+x_1I,y_0+y_1I); dist(M,N)=R=const.$

Theorem 5.2:

Let $M(a+bI,c+dI)$ be a fixed neutrosophic point, $R=r_1+r_2I$ be a neutrosophic real positive number, we have:

(a) The equation of the circle with center M and radius R is $([(x_0+x_1I]-[a+bI])^2+([y_0+Iy_1]-[c+dI])^2=R^2$.

(b) The previous neutrosophic circle is equivalent to the following direct product of two classical circles

$C_1:(x_0-a)^2+(y_o-c)^2={r_1}^2, C_2:([x_0+x_1]-[a+b])^2+([y_0+y_1]-[c+d])^2=(r_1+r_2)^2.$

Proof:

(a) By using the neutrosophic distance form defined in Definition 3.8 and Definition 3.9, we get is $([(x_0+x_1I]-[a+bI])^2+([y_0+Iy_1]-[c+dI])^2=R^2$.

(b) To obtain the classical equivalent geometrical system of the neutrosophic circle, it is sufficient to take its isometric image as follows:

$f(([(x_0+x_1I]-[a+bI])^2)+f(([y_0+Iy_1]-[c+dI])^2)=f(R^2)$,hence

$((x_0-a)^2,(x_0+x_1-[a+b])^2)+((y_0-c)^2,(y_0+y_1-[c+d])^2)=({r_1}^2,(r_1+r_2)^2)$,thus

$$\left(((x_0-a)^2+(y_0-c)^2),((x_0+x_1-[a+b])^2+(y_0+y_1-[c+d])^2)\right)=({r_1}^2,(r_1+r_2)^2),$$

Thus, we get $(x_0-a)^2+(y_o-c)^2={r_1}^2$ and $([x_0+x_1]-[a+b])^2+([y_0+y_1]-[c+d])^2=(r_1+r_2)^2$.

Example 5.3:

Consider the following neutrosophic circle: $C:(X-I)^2+(Y-(2-3I))^2=(2+I)^2$

It is equivalent to the direct product of the following two classical circles:

$C_1:(x_0-0)^2+(y_0-2)^2=2^2, C_2:([x_0+x_1]-[-1])^2+([y_0+y_1]-[-1])^2=(2+1)^2.$

Definition 5.4: (Neutrosophic line)

We define the neutrosophic line by the set of all two N-dimensional points (X,Y) with the property

$$AX+BY+C=0; X=x_0+x_1I, Y=y_0+y_1I, A=a_0+a_1I, B=b_0+b_1I, C=c_0+c_1I.$$

Theorem 5.5:

Let $AX+BY+C=0$ be an equation of a neutrosophic line d, this line is equivalent to the direct product of the following two classical lines:

$$d_1: a_0x_0+b_0y_0+c_0=0, d_2:(a_0+a_1)(x_0+x_1)+(b_0+b_1)(y_0+y_1)+c_0+c_1=0.$$

Proof:

By taking the isometric image to the equation $AX + BY + C = 0$, we get the proof.

Example 5.6:

Consider the following neutrosophic line $(1 + I)X + (2 - 4I)Y + 1 - 3I = 0$, it is equivalent to the following two classical lines

$$d_1: x_0 + 2y_0 + 1 = 0, d_2: 2\,(x_0 + x_1) - 2(y_0 + y_1) - 2 = 0.$$

Remark 5.7:

(a) If we have two classical circles $C_1: (x_0 - a)^2 + (y_0 - c)^2 = (r_1)^2, C_2: (x_1 - b)^2 + (y_1 - d)^2 = (r_2)^2$, then we can transform the set of their direct product $C_1 \times C_2$, into one neutrosophic circle by using the inverse image of the AH-isometry as follows:

$C: (X - M)^2 + (Y - N)^2 = r^2; X = x_0 + (x_1 - x_0)I, Y = y_0 + (y_1 - y_0)I, M = a + (b - a)I, N = c + (d - c)I, r = r_1 + (r_2 - r_1)I.$

The proof holds easily by taking the inverse image with respect to AH-isometry.

(b) By the same argument, if we have two classical lines:

$a_0x_0 + b_0y_0 + c_0 = 0, a_1x_1 + b_1y_1 + c_1 = 0$.We can transform the set of their direct product into one neutrosophic line as follows:

$$AX + BY + C = 0; A = a_0 + (a_1 - a_0)I, B = b_0 + (b_1 - b_0)I, X = x_0 + (x_1 - x_0)I, Y = y_0 + (y_1 - y_0)I, C = c_0 + (c_1 - c_0)I.$$

4. Conclusions

In this article, we presented a general definition of neutrosophic Euclidean geometry. We studied the mechanism of comparison between neutrosophic numbers. In the luminosity of our findings, we make a connection between neutrosophic geometrical concepts and classical Euclidean geometry. Further the concept of neutrosophic plane with n neutrosophic dimensions is obtained. Also, Euclidean geometric concepts are extended neutrosophically such as neutrosophic distance, neutrosophic midpoint, neutrosophic vectors, neutrosophic circles, and lines.

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Induced Plithogenic Cognitive Maps with Combined Connection Matrix to investigate the glitches of online learning system

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Abstract: Plithogenic Cognitive Maps (PCM) introduced by Nivetha and Smarandache are extensively applied in decision making. This research work extends PCM to Induced PCM by introducing the concept of combined connection matrix (CCM). The proposed induced PCM decision making model with CCM is applied to examine the glitches of online learning system. It was observed that expert's opinion on the associational impact between the factors considered for study in the form combined connection matrix is more advantageous on comparison with conventional connection matrix representation, as CCM is a mixture of crisp/fuzzy/intuitionistic/neutrosophic representations. The proposed model will certainly facilitate the decision makers in designing optimal solutions to the real time problems and it shall be extended based on the needs of the decision makers and employed in various other decision making environment. The shortcomings of the model are also discussed in brief.

Keywords: Plithogenic Cognitive Maps, Induced Plithogenic Cognitive maps, combined connection matrix, glitches, online learning system.

1. Introduction

The prime objective of the decision making scenario is to devise optimal solution to the problem by determining the contributing factors and its associational impacts. Cognitive maps introduced by Robert Axelrod [1] are the excellent key graphical structures that comprises of vertices representing the factors of the decision making problem and edges representing the associations. The edge weights belongs to the set {-1,0,1}. If the edge that connects two vertices assume the weight 1 then the factors have positive influence over one another, if the edge weight is -1 then the factors have negative influence and if the edge weight is 0 then the factors have no influence over one another. The association between the factors are represented in the form of a connection matrix with crisp values. Cognitive maps are applied by researchers extensively. Kwon et al [2] has described the role of Cognitive maps in influencing the decision maker's semantic and syntactic comprehension in problem solving. The approach of cognitive maps lacks the ability in handling incomplete and imprecise information. To handle uncertainty scenario in decision making, Kosko [3] extended cognitive maps to fuzzy cognitive maps (FCM). In FCM models the edge weight assume values in the range [-1,1] and the connection matrix consists of fuzzy values. FCM models are applied in pattern recognition. Gonzalo Pajares [4] applied FCM in stereovision matching . Papakostas et al [5] made the first attempt in developing the FCM model for pattern recognition. Papakostas and Koulouriotis [6] presented the classifying of patterns using FCM. Claudio Lucchiari [7] used FCM in diagnostic decisions. FCM models are extensively applied by the researchers to make optimal decisions.

Fuzzy Cognitive models are extended to intuitionistic FCM models in which the connection matrix has intuitionistic values that comprises of membership and non-membership values. Papageorgiou and Dimitris [8] extended CM models to Intuitionistic Fuzzy Cognitive Maps (IFCM) and applied in process control and decision support applications. Luo et al [9] discussed time series predictions based on IFCM and this model is also applied in making decisions on target business strategies. Intuitionistic Fuzzy Cognitive models are extended to neutrosophic cognitive map (NCM) models in which the connection matrix is neutrosophic in nature. The neutrosophic sets consists of truth, indeterminacy and falsity membership values which is comprehensive than intuitionistic sets. Neutrosophic sets are used in resolve the complications in IOT entrprises [10], appraise green SCM [11], choosing optimal the supplier [12], smart medical devise [13] , project [14], designing feasible solutions to the problem of resource levelling [15]. The extended NCM's are first introduced by Vasantha Kandasamy and Smarandache [16]. Aasim Zafar and Mohd Anas [17] applied NCM in situation analysis. Al-Subhi et al [18] extended NCM models to new Neutrosophic Cognitive Maps model. Ferreira [19] has applied NCM to make decision on supply chain management. The researchers of FCM have extended these decision making models based on the circumstances of decision making and also made need based customization. The another comprehensive extension of FCM, IFCM and NCM models is Plithogenic Cognitive Maps (PCM) and it was developed by Nivetha Martin and Smarandache [20]. The concept of plithogeny introduced by Smarandace [21-22] was discussed in various fields and different concepts such as concentric plithogenic hypergraph, plithogenic hyper soft sets, combined plithogenic hypersoft sets, plithogenic fuzzy whole hypersoft set have been evolved. The efficiency of the proposed concepts Plithogenic sets are widely applied in several decision making scenarios of supply chain sustainability [23], multi attribute decision making, medical diagnosis [24]. In the developed PCM model the connection matrix is plithogenic in nature and the contradiction degree of the factors was considered. Sujatha et al [25] applied PCM models in making feasible decisions in analysis on Novel Corona virus by considering contradiction degree of the experts. Nivetha et al [26] developed a COVID-19 diagnostic model to investigate the mediating effects of the factors with new plithogenic sub cognitive maps approach. The PCM models are gaining momentum amidst the researchers at recent times.

Another comprehensive extension of FCM model is Induced Fuzzy Cognitive Maps model which has the similar approach of FCM. The induced FCM models differ from FCM models in determining the fixed point of the dynamical system. Induced FCM models are applied by different researchers as follows: Ritha et al [27] to determine the predictor's interest in cosmetic surgery, Narayanamoorthy et al [28] to make decisions on the problems faced by hand loom workers, Devadoss et al [29-31] to study the miracles in Holy Bible, work-life imbalance and Periyar philosophy of self-respect. Pathinathan et al [32-33] to explore the hazards of plastic pollution, road accidents by adolescences and problems faced by the farmers. Charles et al [34] to investigate the health of women in Chennai slums. Thirusangu et al [35] developed new induced bidirectional associative FCM model. Saraswathi et al [36] used the approach of fuzzy matrix analysis to study induced FCM models. Dhrubajyoti Ghosh et al [37-39] proposed induced Fuzzy Bi-Model to analyse the industrial relationship between employee and employer, real world problems and the impact of social networking in students. Sujatha et al [40] to model the traffic flow, Lily et al [41] to examine the symptoms of migraine. Induced FCM models with intuitionistic and neutrosophic representations are also developed by researchers. To mention a few, Induced FCM is applied in the field of agriculture to identify the effects of Endosulfan [42], to examine the causes of road accidents [43], to investigate the disappearance of house sparrow [44], to study the symptoms of tuberculosis, cancer [45-46], to explore the concepts of semantics extraction [47].

The feasibility of Plithogenic Cognitive Maps has motivated the authors to develop Induced Plithogenic Cognitive maps model. The developed model is the first attempt of formulating induced PCM and it is applied to examine the glitches in online learning system. In general the connection matrix representing the relation between the factors contains only same kind of values which may be crisp/fuzzy/intuitionistic/ neutrosophic in nature and on other hand it many contain linguistic values and it may be quantified using various kind of fuzzy numbers. But in this paper the concept of combined connection matrix is introduced in which the connection matrix comprises of a combination of crisp, fuzzy, intuitionistic and neutrosophic values. The combined connection matrix is more comprehensive in nature and it is highly reflective in sense, the association between the factors say F1 and F2 may be crisp in nature, F3 and F4 may be fuzzy in nature but in conventional connection matrix the association between all the factors are of same kind. The opinion of the experts are unconstrained in combined connection matrix and it is restricted to one kind of value in conventional matrix. In this proposed model four expert's opinion are considered and the aggregate combined connection matrix is obtained using plithogenic aggregate operators by taking contradiction degree of the experts into account.

The paper is structured as follows: section 2 presents the basic preliminaries; section 3 consists of the methodology; section 4 comprises of the application of the developed model; section 4 discusses the results and the last section concludes the work.

2. Preliminaries

This section comprises of the basic definitions related to the research work.

2.1. Definition [20]

Plithogenic Cognitive Maps (PCM) is a directed graph with nodes, edges and contradiction degree. The nodes are represented as $D_1,D_2,D_3,..D_n$ and the edge weights as e_{ij} respectively. The connection matrix or the adjacency matrix comprises of the plithogenic edge weight between the directed edge D_iD_j.

2.2. Definition [20]

The instantaneous vector V = $(a_1,a_2,\ldots a_n)$, $a_i \in \{0,1\}$. If a_i = 1 or 0 then it indicates the ON/OFF position of the node at a particular instant of time respectively.

2.3. Definition [20]

PCM with directed cycles is called as cyclic and it is also called as dynamical system if the causal relations flow through the cycle in revolutionary manner and the attainment of equilibrium state is called as the fixed point.

2.4. Definition [20]

The settling of the vector in a PCM of the form $V_1 \to V_2 \to V_3 \to \cdots . V_i \to .. V_1$

indicates the dynamical system has limit cycle.

2.5. Definition [20]

The plithogenic aggregate operators are defined as a $\wedge_P$ b = (1-c) [a $\wedge_F$ b] + c [a $\vee_F$ b], where c represents the contradiction degree, a $\wedge_F$ b is the t_{norm} defined as ab and a $\vee_F$ b is the t_{conorm} defined as a+b-ab.

2.6. Definition [22]

The neutrosophic set N_A of the form (T, I, F) is transformed to intuitionistic fuzzy set (T, f) by the method of impression membership method, where f is computed as follows

$$f_A = \begin{cases} F_A + \frac{[1-F_A-I_A][1-F_A]}{[F_A+I_A]} & if\ F_A = 0 \\ F_A + \frac{[1-F_A-I_A][F_A]}{[F_A+I_A]} & if\ 0 < F_A \le 0.5 \\ F_A + [1-F_A-I_A]\left[0.5 + \frac{F_A-0.5}{F_A+I_A}\right] & if\ 0.5 < F_A \le 1 \end{cases}$$

By using median membership fuzzy values are computed as $<\Delta(A)> = <\frac{T_A}{[T_A+f_A]}>$.

3. Methodology of Plithogenic Induced Cognitive Maps with Combined Connection matrix

Plithogenic Induced Cognitive Maps are similar to the approach of Plithogenic Cognitive Maps and this section presents the steps involved in the proposed method.

Step 1: The factors of the problem and its causal relationship are determined with the help of the expert's opinion.

Step 2: The combined connection matrix M represents the associations and inter impact between the factors. The values in the matrix may be crisp, fuzzy, intuitionistic and neutrosophic. The combined connection matrix reflects the expert's perception on the association between the factors. The different combined connection matrices of the experts considering contradiction degree is aggregated using plithogenic aggregate operators as defined in and defuzzified using

Step 3: Let C be the instantaneous state vector and it is passed on to the combined connection matrix M, the new resultant vector C1 is obtained as discussed briefly in [20]. The same procedure is applied to the resultant vector until the fixed point is determined in Plithogenic Cognitive Maps, but in induced Plithogenic Cognitive Maps, the newly obtained resultant vector C1 is subjected to component wise computations. The vector C1 is threshold by assigning 1 to the values greater than or equal to 0.5 and 0 to the values lesser than 0.5. If the vector C1 is of the form (1 00 1001001) then the components of vector C1 are taken as (1000000000), (0001000000), (0000001000), (0000000001) and each of the component is passed on the M with the same PCM approach and the resultant threshold vector with maximum 1's is taken as the next new vector and the procedure is repeated to find the fixed point.

Thus the procedure of plithogenic induced cognitive maps differ from plithogenic cognitive maps in determining the fixed point.

4. Factors contributing to the glitches of online learning system

This section presents the factors that hinders the online learning system based on four expert's opinion and determines its associational impacts. The combined connection matrix represent the association and the interrelational impacts between the factors. It is combined in nature, the elements in the matrix are of varied kinds such as crisp, fuzzy, intuitionistic and neutrosophic and it reflects the association existing between factors in the perception of the experts. The factors are as follows

O1 High rate of difficulties in adapting to the new learning system

O2 Largely Confined to Elite Class of society

O3 Lack of proper network channels

O4 Deficit of computer literacy

O5 Monotonous content delivery

O6 No space for enhancing social skills

O7 The affective domain of the students is kept refrained

O9 Unable to cater the diverse needs of the learners

O10 Longer exposure to digital gadgets detains health

O11 Weaker interaction with faculty and peer

O12 Holistic development of learners has less scope

The combined connection matrix of the first expert

	O 1	**O 2**	**O 3**	**O 4**	**O 5**	**O 6**	**O 7**	**O 8**	**O 9**	**O 10**	**O 11**	**O 12**
O1	0	0.8	0	1	1	0	0	1	0	1	1	0
O2	1	0	(0.2,0.7)	1	0	0	1	1	1	0	0	1
O3	0	0	0	0	0	(0.8,0.1)	0	0	0	0	0	0.7

O4	(0.7,0.2)	1	0	0	0	0	1	0	0	0	1	0.4
O5	1	0	0	0	0	1	0	0	0	0	0.6	1
O6	0	0	1	0	1	0	0	0	0	0	0	0
O7	0	1	0	0.8	0	0	0	1	0	0	0.8	1
O 8	0.9	1	0	0	0	0	(0.5,0.3)	0	0	1	1	1
O 9	0	0.7	0	0	0	0	0	0	0	0	1	1
O 10	1	0	0	0	0	0	0	1	0	0	1	(0.8,0.1,0.1)
O 11	0.9	0	0	(0.6,0.1,0.3)	1	0	1	1	1	(0.7,0.1,0.2)	0	1
O 12	0.8	0.8	1	0.9	1	0	1	0.6	1	1	0.7	0

Fig.4.1 represents the graphical representation of first expert in various forms

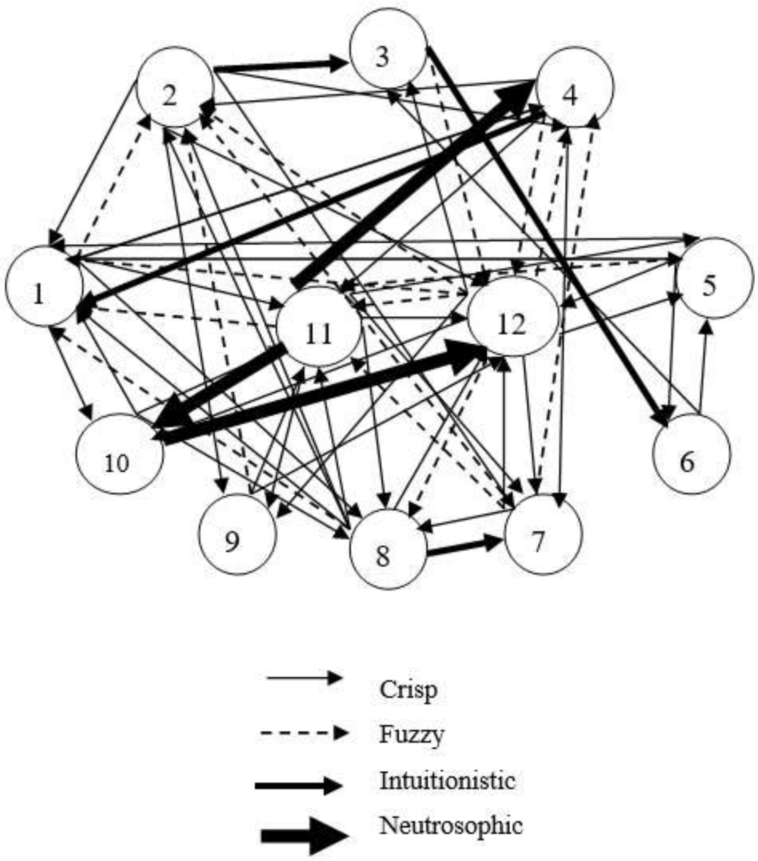

Fig.4.1 Graphical representation of first expert in various forms

The combined connection matrix of the second expert

	O 1	O 2	O 3	O 4	O 5	O 6	O 7	O 8	O 9	O 10	O 11	O 12
O1	0	(0.8,.2)	0	1	.5	0	0	1	0	1	1	0
O2	1	0	(0.2,0.7, .3)	1	0	0	1	.4	1	0	0	1
O3	0	0	0	0	0	(0.5,0.1)	0	0	0	0	0	0.7
O4	(.5,.2)	1	0	0	0	0	1	0	0	0	1	0.5
O5	1	0	0	0	0	1	0	0	0	0	(.6,.2, .3)	1
O6	0	0	1	0	1	0	0	0	0	0	0	0
O7	0	1	0	1	0	0	0	1	0	0	0.8	1
O 8	0.9	1	0	0	0	0	0.3	0	0	1	1	1
O 9	0	0.7	0	0	0	0	0	0	0	0	.8	1
O 10	1	0	0	0	0	0	0	.5	0	0	1	(.8,.1,.3)
O 11	0.5	0	0	(.3,.4)	1	0	.4	1	1	(.7,.2)	0	.3
O 12	0	(.8,.3)	1	(.5,.3)	1	0	.3	1	1	.2	0.4	0

Fig.4.2 represents the graphical representation of second expert in various forms

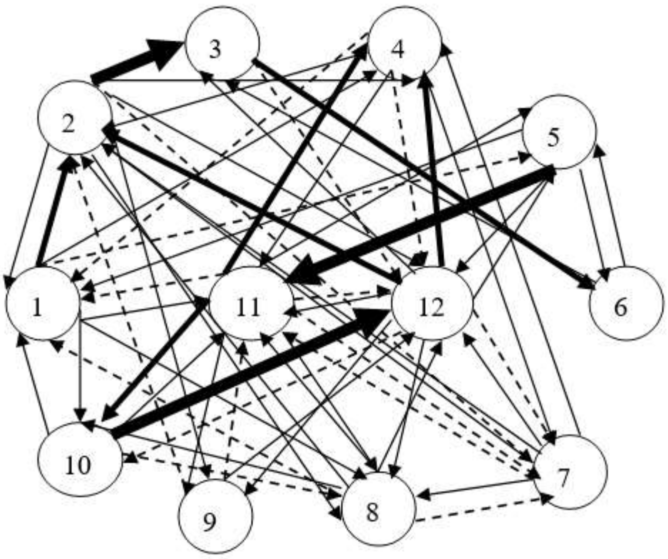

Fig.4.2 Graphical representation of second expert

The combined connection matrix of the third expert

	O 1	O 2	O 3	O 4	O 5	O 6	O 7	O 8	O 9	O 10	O 11	O 12
O1	0	(0.8,.2,.5)	0	1	.3	0	0	1	0	1	1	0
O2	1	0	(0.2,0.1)	1	0	0	.3	.4	1	0	0	.4
O3	0	0	0	0	0	1	0	0	0	0	0	(0.3,.4)
O4	(.5,.2,.1)	1	0	0	0	0	1	0	0	0	.5	(0.3,.2)
O5	.6	0	0	0	0	1	0	0	0	0	(.4,.3)	.4
O6	0	0	.8	0	1	0	0	0	0	0	0	0
O7	0	1	0	1	0	0	0	1	0	0	0.8	1
O 8	(0.5,.3)	1	0	0	0	0	(0.3,.2)	0	0	1	.7	1
O 9	0	(0.5,.3)	0	0	0	0	0	0	0	0	(.8..2)	1
O 10	1	0	0	0	0	0	0	.5	0	0	1	(.8,.1,.3)
O 11	0.5	0	0	.3	1	0	.6	1	.6	.2	0	.3
O 12	(0.7,0.2, 0.5)	(.4,.3)	1	.5	1	0	(.3,.2)	1	.5	(.3,.2)	0.5	0

Fig.4.3 represents the graphical representation of third expert in various form

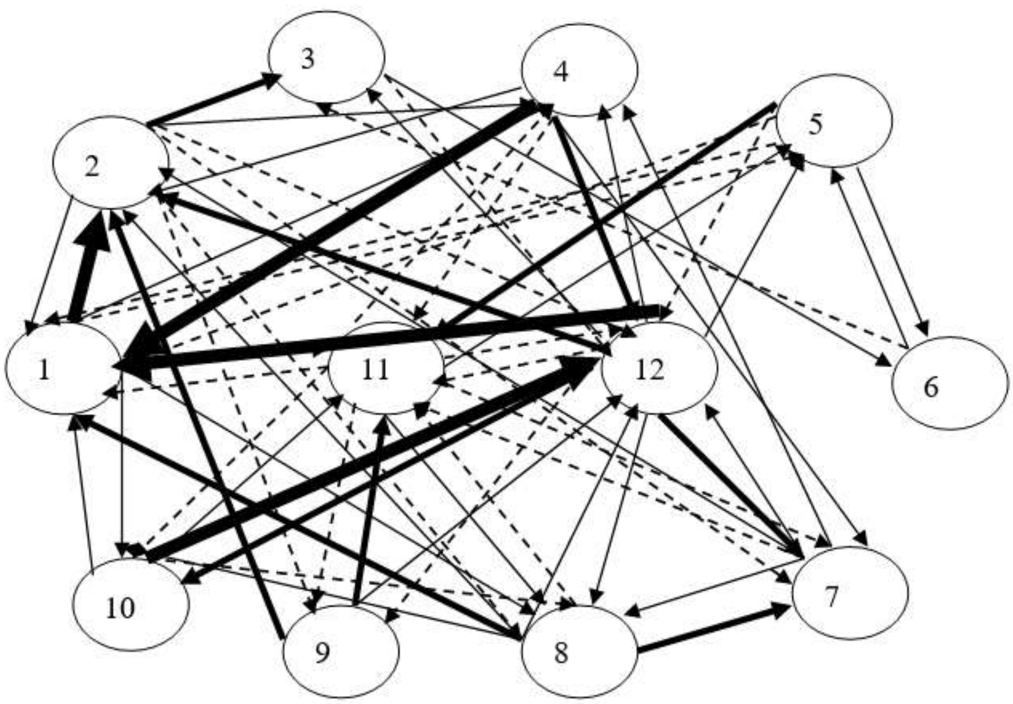

Fig.4.3 Graphical Representation of Third Expert

The combined connection matrix of the fourth expert

	O 1	**O 2**	**O 3**	**O 4**	**O 5**	**O 6**	**O 7**	**O 8**	**O 9**	**O 10**	**O 11**	**O 12**
O1	0	.3	0	1	.5	0	0	.8	0	1	.2	0
O2	1	0	(0.7,.4)	1	0	0	.3	.3	1	0	0	.5
O3	0	0	0	0	0	.6	0	0	0	0	0	(0.3,.4)
O4	(.3,.2,.5)	1	0	0	0	0	.6	0	0	0	1	1
O5	1	0	0	0	0	1	0	0	0	0	(.5,.3)	1
O6	0	0	(.5,.6)	0	1	0	0	0	0	0	0	0
O7	0	1	0	1	0	0	0	(.3,.2)	0	0	1	1
O 8	1	1	0	0	0	0	(0.2,.3)	0	0	1	1	1
O 9	0	(0.5,.6)	0	0	0	0	0	0	0	0	1	1
O 10	.6	0	0	0	0	0	0	.5	0	0	1	(.3,.7)
O 11	0.5	0	0	(.5,.4)	1	0	.4	.5	1	(.3,.2)	0	.3
O 12	1	(.4,.3,.1)	1	(.2,.3)	1	0	.7	1	1	.3	1	0

Fig.4.4 represents the graphical representation of fourth expert in various forms

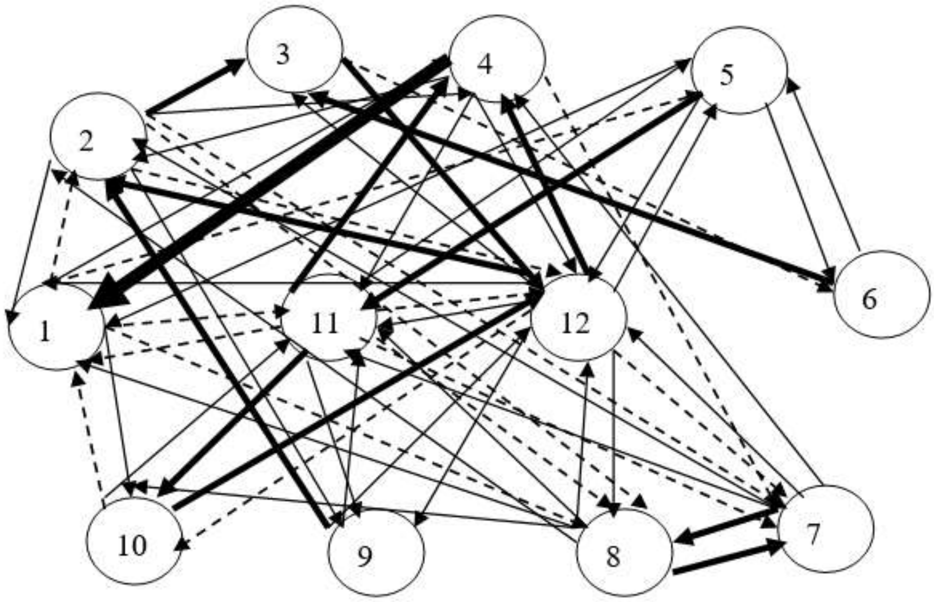

Fig.4.4 Graphical Representation of fourth Expert

The contradiction degree of the experts is presented as follows

E1	E 2	E 3	E 4
0	¼	2/4	¾

The aggregate defuzzified connection matrix M_G obtained from four combined connection matrix is

	O 1	O 2	O 3	O 4	O 5	O 6	O 7	O 8	O 9	O 10	O 11	O 12
O1	0	.994	0	1	1	0	0	1	0	1	1	0
O2	1	0	.899	1	0	0	1	1	1	0	0	1
O3	0	0	0	0	0	1	0	0	0	0	0	.968
O4	.983	1	0	0	0	0	1	0	0	0	1	1
O5	1	0	0	0	0	1	0	0	0	0	.968	1
O6	0	0	.4	0	1	0	0	0	0	0	0	0
O7	0	1	0	1	0	0	0	1	0	0	1	1
O 8	1	1	0	0	0	0	.922	0	0	1	1	1
O 9	0	.976	0	0	0	0	0	0	0	0	1	1
O 10	1	0	0	0	0	0	0	1	0	0	1	.983
O 11	0.987	0	0	.899	1	0	1	1	1	.968	0	1
O 12	1	.986	1	.988	1	0	1	1	1	1	1	0

Let us keep the first factor in ON position

C_1=(1 0 0 0 0 0 0 0 0 0 0 0)

$C_1 M_G$ =(0 .994 0 1 1 0 0 1 0 1 1 0)

→(1 1 0 1 1 0 0 1 0 1 1 0)= C_1^1

$C_1^1 M_G$ =(1 0 0 0 0 0 0 0 0 0 0 0)

=(0 .994 0 1 1 0 0 1 0 1 1 0)

→(0 1 0 1 1 0 0 1 0 1 1 0)

$C_1^1 M_G$ =(0 1 0 0 0 0 0 0 0 0 0 0)

=(1 0 .899 1 0 0 1 1 1 0 0 1)

→(1 0 1 1 0 0 1 1 1 0 0 1)

$C_1^1 M_G$ =(0 0 0 1 0 0 0 0 0 0 0 0)

=(0.983 1 0 0 0 0 1 0 0 0 1 1)

→(1 1 0 0 0 0 1 0 0 0 1 1)

$C_1^1 M_G$ =(0 0 0 0 1 0 0 0 0 0 0 0)

=(1 0 0 0 0 1 0 0 0 0 .968 1)

→(1 0 0 0 0 1 0 0 0 0 1 1)

$C_1^1 M_G$ =(0 0 0 0 0 0 0 1 0 0 0 0)

=(1 1 0 0 0 0 .922 0 0 1 1 1)

→(1 1 0 0 0 0 1 0 0 1 1 1)

$C_1^1 M_G$ =(0 0 0 0 0 0 0 0 0 0 1 0 0)

=(1 0 0 0 0 0 0 1 0 0 1 .983)

→(1 0 0 0 0 0 0 1 0 0 1 1)

$C_1^1 M_G$ =(0 0 0 0 0 0 0 0 0 0 0 1 0)

=(.987 0 0 .899 1 0 1 1 1 .968 0 1)

→(1 0 0 1 1 0 1 1 1 1 0 1)

Therefore C_2= (1 0 0 1 1 0 1 1 1 1 0 1)

$C_2 M_G$ =(4.98 5.96 1 2.988 2 0 2.922 4 1 3 7.968 5.983)

→(1 1 1 1 1 0 1 1 1 1 1 1)= C_2^1

$C_2^1 M_G$ =(1 0 0 0 0 0 0 0 0 0 0 0)

=(0 .994 0 1 1 0 0 1 0 1 1 0)

→(0 1 0 1 1 0 0 1 0 1 1 0)

$C_2^1 M_G$ =(0 1 0 0 0 0 0 0 0 0 0 0)

=(1 0 .899 1 0 0 1 1 1 0 0 1)

→(1 0 1 1 0 0 1 1 1 0 0 1)

$C_2^1 M_G$ =(0 0 1 0 0 0 0 0 0 0 0 0)

=(0 0 0 0 0 1 0 0 0 0 0 .968)

→(0 0 0 0 0 1 0 0 0 0 0 1)

$C_2^1 M_G$ =(0 0 0 1 0 0 0 0 0 0 0 0)

(.983 1 0 0 0 0 1 0 0 0 1 1)

→(1 1 0 0 0 0 1 0 0 0 1 1)

$C_2^1 M_G$ =(0 0 0 0 1 0 0 0 0 0 0 0)

=(1 0 0 0 0 1 0 0 0 0 .968 1)

→(1 0 0 0 0 1 0 0 0 0 1 1)

$C_2^1 M_G$ =(0 0 0 0 0 0 1 0 0 0 0 0)

=(0 1 0 1 0 0 0 1 0 0 1 1)

→(0 1 0 1 0 0 0 1 0 0 1 1)

$C_2^1 M_G$ =(0 0 0 0 0 0 0 1 0 0 0 0)

=(1 1 0 0 0 0 .922 0 0 1 1 1)

→(1 1 0 0 0 0 1 0 0 1 1 1)

$C_2^1 M_G$ =(0 0 0 0 0 0 0 0 1 0 0 0)

=(0 .976 0 0 0 0 0 0 0 0 1 1)

→(0 1 0 0 0 0 0 0 0 0 1 1)

$C_2^1 M_G$ =(0 0 0 0 0 0 0 0 0 1 0 0)

= (1 0 0 0 0 0 0 1 0 0 1 .983)

→(1 0 0 0 0 0 0 1 0 0 1 1)

$C_2^1 M_G$ =(0 0 0 0 0 0 0 0 0 0 1 0)

=(.987 0 0 .899 1 0 1 1 1 .968 0 1)

→(1 0 0 1 1 0 1 1 1 1 0 1)

$C_2^1 M_G$ =(0 0 0 0 0 0 0 0 0 0 0 1)

=(1 .986 1 .988 1 0 1 1 1 1 1 0)

→(1 1 1 1 1 0 1 1 1 1 1 0)

Therefore C_3=(1 1 1 1 1 0 1 1 1 1 1 0)

C_3x M_G =(5.97 4.97 .899 3.89 2 2 3.922 5 2 2.968 6.968 8.95)

$C_3^1 M_G$ =(1 0 0 0 0 0 0 0 0 0 0 0)

(1 1 1 1 1 1 1 1 1 1 1 1)= C_3^1

$C_3^1 M_G$ =(1 0 0 0 0 0 0 0 0 0 0 0)

=(0 .994 0 1 1 0 0 1 0 1 1 0)

→(0 1 0 1 1 0 0 1 0 1 1 0)

$C_3^1 M_G$ =(0 1 0 0 0 0 0 0 0 0 0 0)

=(1 0 .899 1 0 0 1 1 1 0 0 1)

→(1 0 1 1 0 0 1 1 1 0 0 1)

$C_3^1 M_G$ =(0 0 1 0 0 0 0 0 0 0 0 0)

=(0 0 0 0 0 1 0 0 0 0 0 0 .968)

→(0 0 0 0 0 1 0 0 0 0 0 1)

$C_3^1 M_G$ =(0 0 0 1 0 0 0 0 0 0 0 0)

=(.983 1 0 0 0 0 1 0 0 0 1 1)

→(1 1 0 0 0 0 1 0 0 0 1 1)

$C_3^1 M_G$ =(0 0 0 0 1 0 0 0 0 0 0 0)

=(1 0 0 0 0 1 0 0 0 0 .968 1)

→(1 0 0 0 0 1 0 0 0 0 1 1)

$C_3^1 M_G$ =(0 0 0 0 0 1 0 0 0 0 0 0)

=(0 0 .4 0 1 0 0 0 0 0 0 0)

→(0 0 0 0 1 0 0 0 0 0 0 0)

$C_3^1 M_G$ =(0 0 0 0 0 0 1 0 0 0 0 0)

=(0 1 0 1 0 0 0 1 0 0 1 1)

→(0 1 0 1 0 0 0 1 0 0 1 1)

$C_3^1 M_G$ =(0 0 0 0 0 0 0 1 0 0 0 0)

=(1 1 0 0 0 0 .922 0 0 1 1 1)

→(1 1 0 0 0 0 1 0 0 1 1 1)

$C_3^1 M_G$ =(1 0 0 0 0 0 0 0 1 0 0 0)

=(0 .976 0 0 0 0 0 0 0 0 1 1)

→(0 1 0 0 0 0 0 0 0 0 1 1)

$C_3^1 M_G$ =(1 0 0 0 0 0 0 0 0 1 0 0)

=(1 0 0 0 0 0 0 1 0 0 1 .983)

→(1 0 0 0 0 0 0 1 0 0 1 1)

$C_3^1 M_G$ =(1 0 0 0 0 0 0 0 0 0 1 0)

=(.987 0 0 .899 1 0 1 1 1 .968 0 1)

→(1 0 0 1 1 0 1 1 1 1 0 1)

C_3^1 M_G =(0 0 0 0 0 0 0 0 0 0 0 1)

=(1 .986 1 .988 1 0 1 1 1 1 1 0)

→(1 1 1 1 1 0 1 1 1 1 1 0) = C_4

Hence C_3=C_4

By repeating in the same fashion the limit points shall be obtained for other ON position of the factors.

The same procedure of induced PCM shall be applied to the conventional aggregate fuzzy connection matrix. The aggregate expert's matrix is conventional in nature as it comprises of only fuzzy values. The same decision making problem can be dealt with PCM procedure as discussed by Nivetha and Florentin. The limit points are determined in both the cases of aggregate combined connection matrix and conventional fuzzy connection matrix. By considering the below aggregate conventional fuzzy connection matrix, the limit points obtained for various cases are presented in Table 4.1

	O 1	**O 2**	**O 3**	**O 4**	**O 5**	**O 6**	**O 7**	**O 8**	**O 9**	**O 10**	**O 11**	**O 12**
O1	0	1	0	1	1	0	0	1	0	1	1	0
O2	1	0	1	1	0	0	1	1	1	0	0	1
O3	0	0	0	0	0	1	0	0	0	0	0	1
O4	1	1	0	0	0	0	1	0	0	0	1	1
O5	1	0	0	0	0	1	0	0	0	0	1	1
O6	0	0	.0	0	1	0	0	0	0	0	0	0
O7	0	1	0	1	0	0	0	1	0	0	1	1
O 8	1	1	0	0	0	0	1	0	0	1	1	1
O 9	0	1	0	0	0	0	0	0	0	0	1	1
O 10	1	0	0	0	0	0	0	1	0	0	1	1
O 11	1	0	0	1	1	0	1	1	1	1	0	1
O 12	1	1	1	1	1	0	1	1	1	1	1	0

Table 4.1 Limit points of Induced PCM & PCM

	Case (a)	Case (b)	Case (c)	Case (d)
On position of the state vector	**Limit point By Induced PCM with aggregate combined connection matrix**	**Limit point By Induced PCM with aggregate Conventional Fuzzy connection matrix**	**Limit point By PCM with aggregate combined connection matrix**	**Limit point By PCM with aggregate conventional Fuzzy connection matrix**
(100000000000)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)
(010000000000)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)
(001000000000)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)
(000100000000)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)
(000010000000)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)
(000001000000)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)
(000000100000)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)
(000000010000)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)
(000000001000)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)
(000000000100)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)
(000000000010)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)
(000000000001)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 0 1 1 1 1 1 0)	(1 1 1 1 1 1 1 1 1 1 1 1)	(1 1 1 1 1 1 1 1 1 1 1 1)

5. Sensitivity Analysis

In induced PCM ,by positioning the first factor O1 in ON state, the resultant limit point obtained is (1 1 1 1 1 0 1 1 1 1 1 0). The limit point states that the factors influenced by the O1 High rate of difficulties in adapting to the new learning system. Similarly the influence of all other factors can be determined. On comparing with Plithogenic Cognitive Maps model, the limit point obtained by keeping the first factor in ON position is (1 1 1 1 1 1 1 1 1 1 1 1).The limit point obtained indicates that the factor O1 has influence on all the factors but the factor that plays a key role is not represented in it. But in induced PCM the flow of limit points is found and the pattern is determined and it is represented in Table 4.2 and Fig.4.5.

Table 4.2. Induced Triggering Pattern

Factors in ON state	**Triggering pattern**
(1 0 0 0 0 0 0 0 0 0 0 0)	$C_1 \rightarrow C_{11} \rightarrow C_{12} \rightarrow C_{12}$
(0 1 0 0 0 0 0 0 0 0 0 0)	$C_2 \rightarrow C_{12} \rightarrow C_{12}$
(0 0 1 0 0 0 0 0 0 0 0 0)	$C_3 \rightarrow C_{12} \rightarrow C_{12}$
(0 0 0 1 0 0 0 0 0 0 0 0)	$C_4 \rightarrow C_{12} \rightarrow C_{12}$
(0 0 0 0 1 0 0 0 0 0 0 0)	$C_5 \rightarrow C_{12} \rightarrow C_{12}$
(0 0 0 0 0 1 0 0 0 0 0 0)	$C_6 \rightarrow C_5 \rightarrow C_{12} \rightarrow C_{12}$
(0 0 0 0 0 0 1 0 0 0 0 0)	$C_7 \rightarrow C_{12} \rightarrow C_{12}$
(0 0 0 0 0 0 0 1 0 0 0 0)	$C_8 \rightarrow C_{12} \rightarrow C_{12}$
(0 0 0 0 0 0 0 0 1 0 0 0)	$C_9 \rightarrow C_{12} \rightarrow C_{12}$
(0 0 0 0 0 0 0 0 0 1 0 0)	$C_{10} \rightarrow C_{12} \rightarrow C_{12}$
(0 0 0 0 0 0 0 0 0 0 1 0)	$C_{11} \rightarrow C_{12} \rightarrow C_{12}$
(0 0 0 0 0 0 0 0 0 0 0 1)	$C_{12} \rightarrow C_{11} \rightarrow C_{12} \rightarrow C_{12}$

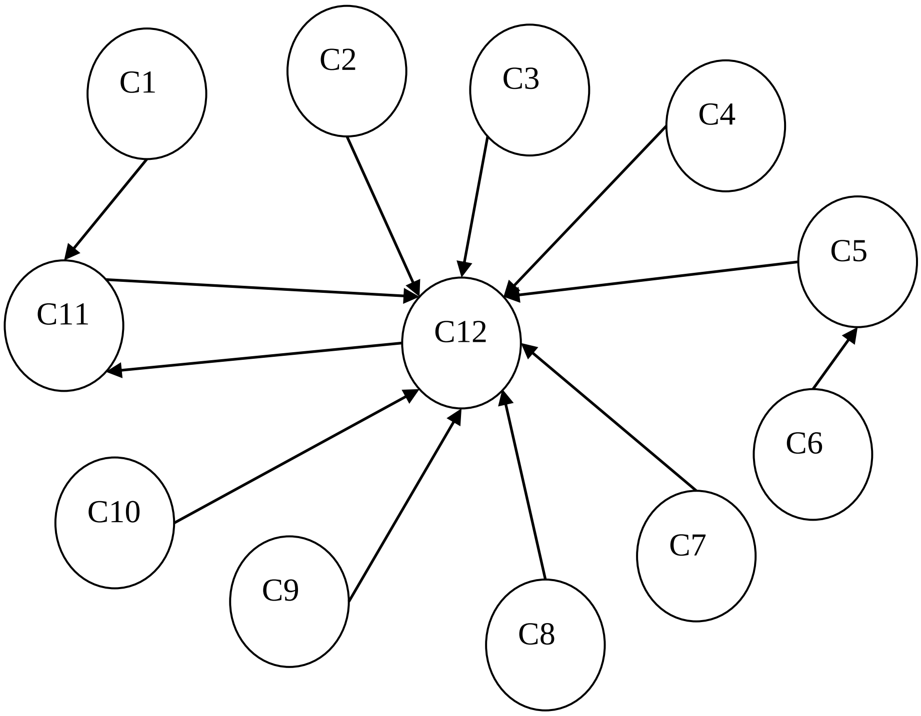

Fig.4.5 Graphical Representation of Induced PCM

Fig.4.5 clearly envisages the factor C12 is the core and intermediate factor that hurdles the online learning system. The limit point in the former case (a) is more reflective than the limit point obtained in case (c). The discussion is also made under PCM with aggregate combined and conventional matrices and the limit points obtained are same and the Table.4.1 is self-explanatory.

5. Conclusion

This paper introduces a new decision making model based on Plithogenic Induced Cognitive Maps. The concept of combined connection matrix and its significance in representing the associational impacts between the factors from expert's outlook is discussed. The validity of the proposed model is discussed in analysing the glitches in the online learning system. The sensitivity analysis vividly explicates the efficiency of the proposed model over the earlier developed models. Induced PCM models will certainly benefit the decision makers to arrive at optimal decisions. The developed model can be discussed by considering the contradiction degree of the factors and linguistic combined connection matrix. This approach can be applied in different decision making scenarios and in various contexts. The proposed method has certain shortcomings, one such is the construction of the combined connection matrix as sometimes it is not always certain to have mixture of values as at many instances the nature of the values decides the nature of the connection matrix; another limitation of the proposed model is the defuzzification methods used to determine the aggregate connection matrix of a single kind. The first limitation can be handled by the right choice of the experts and the second by the appropriate method of defuzzification.

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Neutrosophic Fuzzy Threshold Graph

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Abstract: The aim of this paper is to introduce the extension of intuitionistic fuzzy threshold graph to Neutrosophic field of approach. A new structure called Neutrosophic fuzzy threshold graph (NFTG) have been portrayed and the Neutrosophic fuzzy alternating 4-cycle have been explained using suitable examples. Two parameters namely, Neutrosophic fuzzy threshold dimension and Neutrosophic fuzzy threshold partition number have been illustrated with examples. Theorems based on proposed concepts were proved and one application has been discussed to insist the utility of Neutrosophic fuzzy threshold graph in resource management technique.

Keywords: threshold graph; Neutrosophic graph; threshold dimension; partition number; fuzzy alternating 4- cycle; fuzzy threshold graph.

1. Introduction

A graphical structure using fuzzy concept was first introduced in 1975[8]. Professor A Rosenfeld proposed the concept when he studied about uncertainty of systems. This new approach attracts several researchers as it is applicable to almost all real world problems where uncertainty plays a major role. The term Intuitionistic fuzzy set came into existence in 1986 and it was proposed by professor K. Atanassov [2,3]. The graphical representation of Intuitionistic fuzzy set was framed by Akram M., Akmal R [1] and they elaborate the work with basic operations of set theory as well as using matrix representations. Each vertex and edges of such graphs were labeled with two values that define the membership and non-membership degrees.

Adding one more parameter called indeterminacy value along with the existing two parameters of Intuitionistic fuzzy , a new concept called Neutrosophic fuzzy set was introduced by professor Florentin Smarandache [9] and he developed the graphical structures[10,11]. The term Threshold graph was coined by V. Chvatal and P. L. Hammer in 1973[4] while they do research on packaging problems and later in 1985 E. T. Ordman utilized the concept in resource allocation techniques[7]. In 2015, Sovan Samanta and Madhumangal Pal, professors from Vidyasagar University framed fuzzy threshold graphs[12] and it was generalized to intuitionistic threshold graphs by professors Lanzhen Yang and Hua Mao in 2019[6].

Contribution

Our country is facing challenges regarding Covid-19 vaccine, since we have limited number of doses. Optimum allocation should be needed to meet the needs. The challenges of supply, storage, and delivery of vaccines must take place under strict sanitary conditions is unavoidable too. It is also difficult to reach some remote areas or minorities due to the unavailability of storage requirements and safe delivery. To identify the effective allocation of the COVID-19 vaccine for priority groups, decision-makers must involve experts from multiple fields to get benefit from their experiences in setting priorities and principle guidelines. The neutrosophic, like other fields, contributed to the understanding and analyzing COVID-19 pandemic too. In this paper, we extend the concept of threshold graph by embedding the neutrosophic set properties and we will prove some basic theorems related to the concept. We also define two parameters called threshold dimension and threshold partition number of Neutrosophic fuzzy threshold graphs.

Motivation

A neutrosophic set plays an important role in uncertainty modeling. The development of uncertainty theory plays a fundamental role in formulation of real-life scientific mathematical model, structural modeling in engineering field, medical diagnoses problem etc. In this current decade, researchers have exposed their considerations to make progress with the theories related to neutrosophic area and constantly try to endorse its sufficient scope applications in dissimilar branches of neutrosophic domain. However, our main objective is to support the theory efficiently with these following points.

1. Introduction of Neutrosophic fuzzy threshold graph.
2. Extension of fuzzy threshold graphs and its concepts using neutrosophic fuzzy graph.
3. Application of neutrosophic fuzzy threshold graph in optimum resource allocation.

2. Preliminaries

This section gives a brief preview about the existing concepts which will be utilized in section 3.

Definition 2.1: A graph $G = (P, Q)$ is called a fuzzy graph if there exist function $\mu_P : V^* \to [0, 1]$ and $\mu_Q : V^* \times V^* \to [0, 1]$ called membership function such that for all (v_i, v_j) in $V^* \times V^*$, $\mu_Q(v_i, v_j) \le \min\{\mu_Q(v_i), \mu_Q(v_j)\}$, where $V^* = \{v_1, v_2, v_3, \cdots, v_r\}$ is the vertex set of G.

Definition 2.2: A graph $G = (P, Q)$ is called a neutrosophic fuzzy graph (NFG) with Vertex set $V^* = \{v_1, v_2, v_3, \cdots, v_r\}$, whose membership, non-member ship and Indeterminacy functions satisfy the following conditions:

(i) $\mu_P : V^* \to [0, 1], \upsilon_P : V^* \to [0, 1]$ and $\sigma_P : V^* \to [0, 1]$ denote the degree of truth-membership function, falsity-membership function and indeterminacy-membership function of the vertex $v_i \in V^*$ respectively, and $0 \le \mu_P(v) + \upsilon_P(v) + \sigma_P(v) \le 3$, $\forall\, v \in V^*$ $(i = 1, 2, 3 \ldots r)$.

(ii) $\mu_Q : V^* \times V^* \to [0, 1], \upsilon_Q : V^* \times V^* \to [0, 1]$ and $\sigma_Q : V^* \times V^* \to [0, 1]$ denote the degree of truth-membership function, falsity-membership function and indeterminacy-membership function of the edge (v_i, v_j) respectively such that $\mu_Q(v_i, v_j) \le \min\{\mu_Q(v_i), \mu_Q(v_j)\}$,

$\sigma_Q(v_i, v_j) \leq \min\{\sigma_Q(v_i), \sigma_Q(v_j)\}$,

$\upsilon_Q(v_i, v_j) \leq \max\{\upsilon_Q(v_i), \upsilon_Q(v_j)\}$

and $0 \leq \mu_Q(v_i, v_j) + \upsilon_Q(v_i, v_j) + \sigma_Q(v_i, v_j) \leq 3$ for every (v_i, v_j),

where the sets P and Q be the Neutrosophic fuzzy subsets defined on V^* and E^* respectively[5].

Definition 2.3: The vertex cardinality of a Neutrosophic fuzzy graph, $G = (P, Q)$ denoted by $|V|_N$ and it is defined as $|V|_N = \sum_{v \in V} \frac{1 + \mu_P(v) + \sigma_P(v) - \upsilon_P(v)}{3}$.

Definition 2.4: The stability number of a Neutrosophic fuzzy graph $G = (P, Q)$ is defined as the order of largest stable set of G and it is denoted by $\zeta(G)$.

3. Neutrosophic fuzzy threshold graphs

In this section we introduce the definition of Neutrosophic fuzzy threshold graph (NFTG), Neutrosophic fuzzy threshold dimension $\eta(G)$ and Neutrosophic fuzzy threshold partition number $\eta_p(G)$ and we prove some theorem based on the concepts stated.

Definition 3.1: A graph $G = (P, Q)$ is called a Neutrosophic fuzzy threshold graph (NFTG) if there exist $\tau_1 > 0, \tau_2 > 0$ and $\tau_3 > 0$ such that

$$\sum_{u \in U} \mu_P(u) \leq \tau_1 \text{ , } \sum_{u \in U} (1 - \upsilon_P(u)) \leq \tau_2 \text{ and } \sum_{u \in U} \sigma_P(u) \leq \tau_3 \qquad (1)$$

Provided that $U \subseteq V^*$ is an independent set in G. NFTG is generally denoted as $G = (P, Q; \tau_1, \tau_2, \tau_3)$.

Remark 3.1: The notion $U \subseteq V^*$ is an independent set in G is same as that the notion $U \subseteq V^*$ is an independent set in G^*. If $G = (P, Q; \tau_1, \tau_2, \tau_3)$ and $U \subseteq V^*$ is a dependent set in G, then we have at least one of the condition (3.1) does not hold. For this case,

$$\sum_{u \in U} \mu_P(u) > \tau_1 \text{ or } \sum_{u \in U} (1 - \upsilon_P(u)) > \tau_2 \text{ or } \sum_{u \in U} \sigma_P(u) > \tau_3.$$

Example 3.1: Let $G^* = (V^*, E^*)$ be a graph whose vertex and edge set is $V^* = \{m, n, o, p, q\}$ and $E^* = \{(m, n), (n, o), (o, p), (p, n), (n, q)\}$ respectively and the sets P and Q be the Neutrosophic fuzzy subsets defined on V^* and E^* respectively (see Table). Based on the data, the NFTG for this graph is given as $G = (P, Q; 0.6, 0.8, 0.5)$.

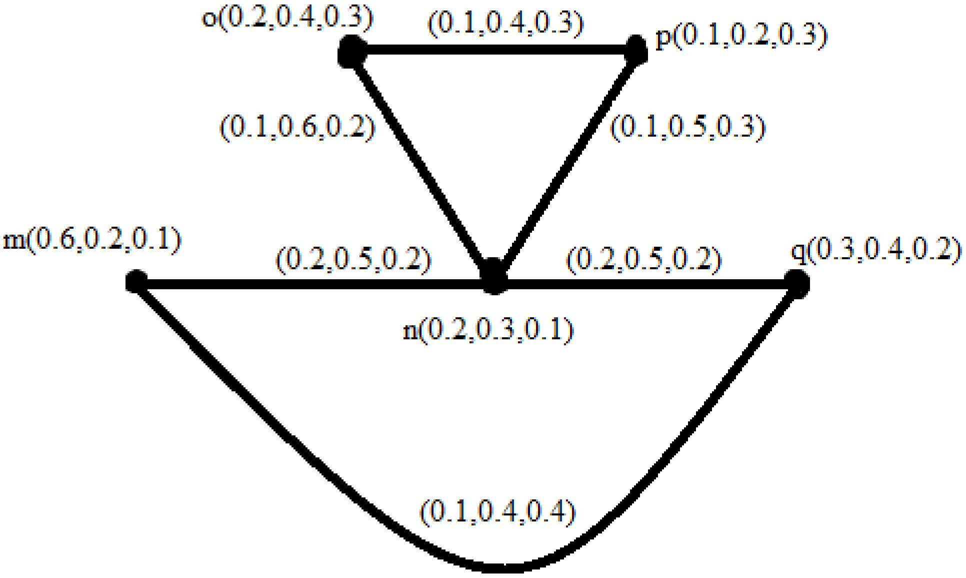


Figure 1. Neutrosophic fuzzy threshold graph

Table 1: Membership, Non-membership and Indeterminacy degrees for Vertices and Edges

Vertices	**m**	**n**	**o**	**p**	**q**	
μ_P	0.6	0.2	0.2	0.1	0.3	
υ_P	0.2	0.3	0.4	0.2	0.4	
σ_P	0.1	0.1	0.2	0.3	0.2	
Edges	**(m,n)**	**(n,o)**	**(o,p)**	**(p,n)**	**(m,q)**	**(n,q)**
μ_Q	0.2	0.1	0.1	0.1	0.1	0.2
υ_Q	0.5	0.6	0.4	0.5	0.4	0.5
σ_Q	0.2	0.2	0.3	0.3	0.4	0.2

Proposition 3.1: If $G^*=(V^*, E^*)$ is an underlying graph for a Neutrosophic fuzzy graph $G=(P,Q)$ and if $W \subseteq V^*$ is an independent set in NFTG, $G=(P,Q;\tau_1,\tau_2,\tau_3)$ then, the cardinality of W satisfies the following relation:

$$|W|_N \le \sum_{w\in W} \frac{\tau_1+\tau_2+\tau_3}{3}$$

Proof:

Given that $G=(P,Q;\tau_1,\tau_2,\tau_3)$ is and NFTG. Then by definition (3.1), we have

$$\sum_{w\in W} \mu_P(w) \le \tau_1 \text{ , } \sum_{w\in W} \left(1-\upsilon_P(w)\right) \le \tau_2 \text{ and } \sum_{w\in W} \sigma_P(w) \le \tau_3 \tag{2}$$

If r denotes the number of vertices in W, then

$$\sum_{w\in W} \left(1-\upsilon_P(w)\right) \le \tau_2 \quad \Rightarrow r - \sum_{w\in W} \upsilon_P(w) \le \tau_2 \tag{3}$$

Thus,

$$-\sum_{w\in W}\upsilon_P(w)\le\tau_2-r \quad \Rightarrow \sum_{w\in W}\upsilon_P(w)\ge r-\tau_2.$$

Now from definition (2.3), we have

$$|W|_N=\sum_{w\in W}\frac{1+\mu_P(w)+\sigma_P(w)-\upsilon_P(w)}{3} \tag{4}$$

Substituting (2), (3) in (4) we get

$$|W|_N=\sum_{w\in W}\frac{\mu_P(w)}{3}+\sum_{w\in W}\frac{1-\upsilon_P(w)}{3}+\sum_{w\in W}\frac{\sigma_P(w)}{3}$$
$$\le\frac{1}{3}(\tau_1+\tau_2+\tau_3)$$

Therefore, $|W|_N\le\sum_{w\in W}\frac{\tau_1+\tau_2+\tau_3}{3}$.

Proposition 3.2: A fuzzy threshold graph is a special case of Neutrosophic fuzzy threshold graph.
Proof:
Let $G=(P,Q)$ be a fuzzy threshold graph, then there exist $\tau_1>0$, such that $\sum_{u\in U}\mu_P(u)\le\tau_1$, where U is an independent set contained in vertex set V of G. Since the non-membership and indeterminacy degree values for a fuzzy threshold graph is zero, we can choose $\tau_2=r$ and $\tau_3=1$. Thus there exist $\tau_1>0,\tau_2>0$ and $\tau_3>0$ such that
$\sum_{u\in U}\mu_P(u)\le\tau_1$, $\sum_{u\in U}(1-\upsilon_P(u))\le\tau_2$ and $\sum_{u\in U}\sigma_P(u)\le\tau_3$,
Where r denotes the number of vertices and hence $G=(P,Q;\tau_1,\tau_2,\tau_3)$ forms a Neutrosophic fuzzy threshold graph.

Definition 3.2: Let $G=(P,Q)$ be a Neutrosophic fuzzy graph with $V^*=\{m,n,o,p\}$, we say that the four vertices constitute a Neutrosophic fuzzy alternating 4-cycle if it satisfies the following four conditions:

(i) $(\mu_Q(m,n),\upsilon_Q(m,n),\sigma_Q(m,n))\ne(0,0,0)$

(ii) $(\mu_Q(o,p),\upsilon_Q(o,p),\sigma_Q(o,p))\ne(0,0,0)$

(iii) $(\mu_Q(m,o),\upsilon_Q(m,o),\sigma_Q(m,o))=(0,0,0)$ and

(iv) $(\mu_Q(n,p),\upsilon_Q(n,p),\sigma_Q(n,p))=(0,0,0)$.

Remark 3.2: The following three graphs may form a sub graph for a Neutrosophic fuzzy alternating 4-cycle:

(i) A Neutrosophic fuzzy path P_4:
$(\mu_Q(m,p),\upsilon_Q(m,p),\sigma_Q(m,p))=(0,0,0)$ and
$(\mu_Q(n,o),\upsilon_Q(n,o),\sigma_Q(n,o))\ne(0,0,0)$.
Or

$\left(\mu_Q(n,o), \upsilon_Q(n,o), \sigma_Q(n,o)\right) = (0,0,0)$ and

$\left(\mu_Q(m,p), \upsilon_Q(m,p), \sigma_Q(m,p)\right) \neq (0,0,0).$

(ii) A Neutrosophic fuzzy square C_4:

$\left(\mu_Q(m,p), \upsilon_Q(m,p), \sigma_Q(m,p)\right) \neq (0,0,0)$ and

$\left(\mu_Q(n,o), \upsilon_Q(n,o), \sigma_Q(n,o)\right) \neq (0,0,0).$

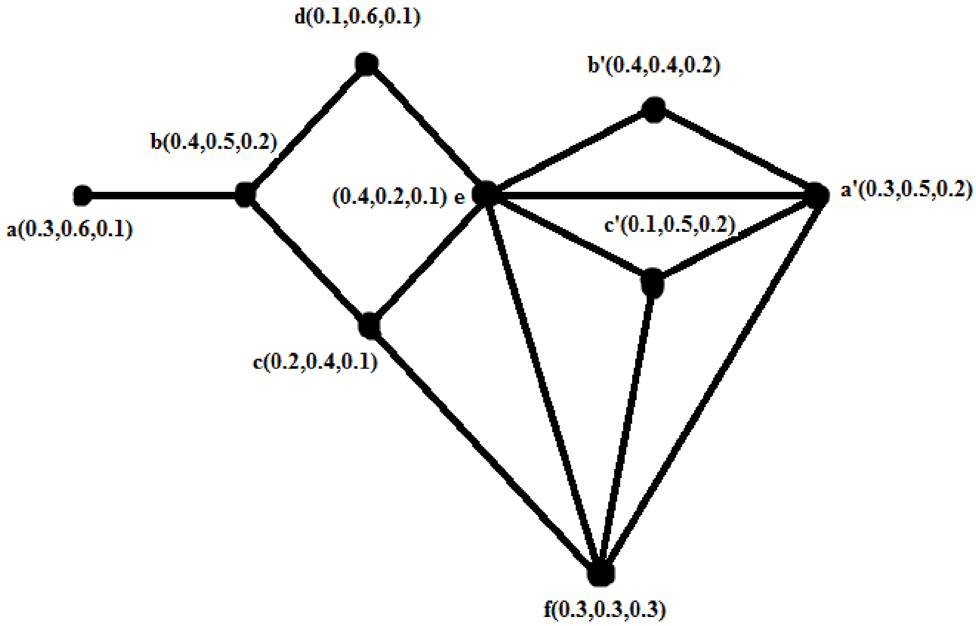
(iii) A Neutrosophic fuzzy matching $2K_2$:

$\left(\mu_Q(m,p), \upsilon_Q(m,p), \sigma_Q(m,p)\right) = (0,0,0)$ and

$\left(\mu_Q(n,o), \upsilon_Q(n,o), \sigma_Q(n,o)\right) = (0,0,0).$

Definition 3.3: A Neutrosophic fuzzy threshold graph $G = (P, Q; \tau_1, \tau_2, \tau_3)$ is said to have a threshold dimension $\eta(G)$ if there exists η(G) number of Neutrosophic fuzzy threshold sub graphs whose union covers the edge set E* of $G = (P, Q; \tau_1, \tau_2, \tau_3)$, provided that such partition is minimal.

Definition 3.4: If $G_1, G_2, \ldots. G_p$ is p Neutrosophic fuzzy threshold sub graphs whose union covers the edge set E* of a Neutrosophic fuzzy threshold graph $G = (P, Q; \tau_1, \tau_2, \tau_3)$ and does not have common arcs , then p is said to be the Neutrosophic fuzzy threshold partition number of G and it is denoted by $\eta_p(G)$.



For this graph, $\eta(G) = 2, \eta_p(G) = 3$

Figure 2. Threshold dimension and partition number of NFTG

Proposition 3.3: Let $G = (P, Q; \tau_1, \tau_2, \tau_3)$ be a Neutrosophic fuzzy threshold graph. Then its threshold dimension $\eta(G)$ satisfies the relation, $\eta(G) \le r - \zeta(G)$, where r denotes the number of vertices in V* of G. In particular if $G = (P, Q; \tau_1, \tau_2, \tau_3)$ is a triangular free graph, then $\eta(G) = \eta_p(G) = r - \zeta(G)$.

Proof:

Let U* be the stable set with maximum number of vertices of G. Then each star having center at $u \in V^* - U^*$ forms a Neutrosophic fuzzy threshold graph of G. The union of all such stars along with weak arcs of stable set U* forms a covering for the edge set E* of G. Therefore, $\eta(G) \leq |V^* - U^*|$. Given that r denotes the number of vertices in V* of G and by definition (2.4), we have $\eta(G) \leq r - \zeta(G)$. In particular if $G = (P, Q; \tau_1, \tau_2, \tau_3)$ is a triangular free graph, then each Neutrosophic fuzzy threshold graph of G is a star or a star with weak edge and therefore $\eta(G) = r - \zeta(G)$. Also such partition does not have common arcs among them. Hence, if G is triangle free then $\eta(G) = r - \zeta(G)$ $= \eta_p(G)$.

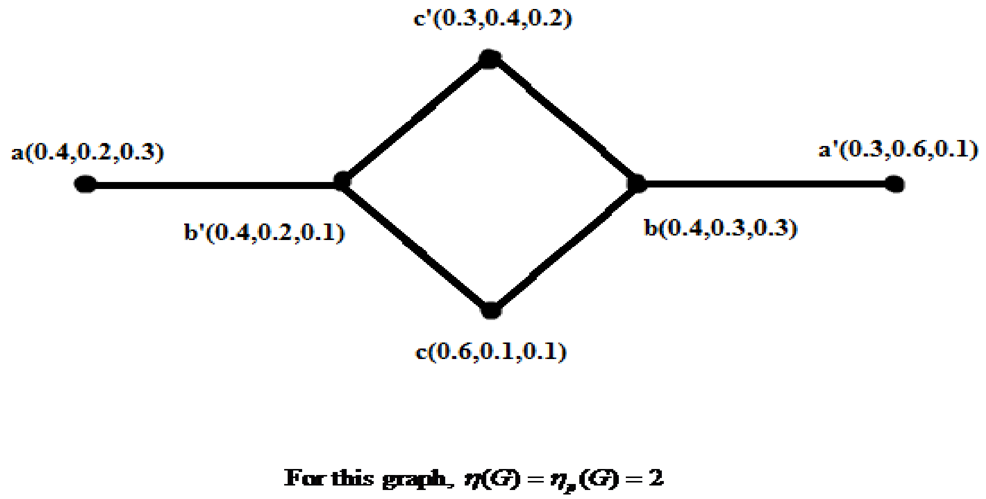


Figure 3. Threshold dimension and partition number of triangle free NFTG

Table 2. Comparison table on properties of NFTG and FTG

S.No.	Neutrosophic fuzzy threshold graph	Intuitionistic Fuzzy threshold graph
1	G*=(V*, E*) is an underlying graph for a Neutrosophic fuzzy graph $G = (P, Q)$ and if $W \subseteq V^*$ is an independent set in NFTG, $G = (P, Q; \tau_1, \tau_2, \tau_3)$ then, the cardinality of W satisfies the following relation: $\lvert W \rvert_N \leq \sum_{w \in W} \frac{\tau_1 + \tau_2 + \tau_3}{3}$	*Let* $G = (P, Q; \tau_1, \tau_2)$ and $U \subseteq V$ be an independent set in Intuitionistic Fuzzy threshold graph G, then $\lvert U \rvert_{IF} \leq \frac{\tau_1 + \tau_2}{3}$
2	A fuzzy threshold graph is a special case of Neutrosophic fuzzy threshold graph.	A fuzzy threshold graph is a special case of Intuitionistic Fuzzy threshold graph.
3	Let $G = (P, Q; \tau_1, \tau_2, \tau_3)$ be a Neutrosophic fuzzy threshold graph. Then its threshold dimension $\eta(G)$ satisfies the relation, $\eta(G) \leq r - \zeta(G)$, where r denotes the number of vertices in V* of G. In particular if $G = (P, Q; \tau_1, \tau_2, \tau_3)$ is a triangular free graph, then $\eta(G) = \eta_p(G) = r - \zeta(G)$.	If $G = (P, Q; \tau_1, \tau_2)$ is a triangle free IFG, then $t(G)=t_p(G) = n - \alpha(G)$, where $\alpha(G)$ is the number of vertices of the maximum independent set of G, and n is the number of vertices of G.

4. Resource Management Technique using Neutrosophic fuzzy threshold graph:

This section discusses the application of Neutrosophic Fuzzy Threshold graph in resource management technique.

Resource management plays a vital role in production field and it one of the emerging topics of optimization techniques. Neutrosophic Fuzzy Threshold graph find its unique way of solving the problems faced during best resource allocation. Let us elaborate one such application here.

Invention of Covid-19 vaccine is the big challenge faced by almost all countries of the world now. Several researches were effectively undertaken to reach the goal. India, which is in top second position of affected people count in the world, is at the last stage of testing and will soon release the vaccine for Covid-19. At the same time, being a developing country it does not have enough resource to supply the medicine to all people in the country immediately. Definitely, the resource controlling becomes necessary, so that the vaccine must reach the needed ones at proper time. Clinics must be situated at optimized places which ensure enough supply of medical resources to the cities. On the other hand, the medical resource couldn't get wasted by supplying it to the peoples with good immune and were in safe zone, who really doesn't need that. Let us analyze such situation using a Neutrosophic Fuzzy Threshold graph G= (P, Q), in which the vertices denote the cities and the clinic providing medical aids as given below:

Suppose that 3 clinics C1,C2 and C2 were supplying medical aids to peoples of six cities a, b, c, e, f and g. and The labeling values of Graph G=(P,Q) represents the requirement and supply of medical resource . For example,

- For the city g, $\mu_P(g)$ denotes the Covid-19 positive people who required immediate medicine, $\upsilon_P(g)$ denotes the people who were under safe zone and required medicine only for precaution and $\sigma_P(g)$ denoted the people who were asymptotic and those were the ones who need medical attention so that they couldn't spread disease further.
- In case of clinics, $(\mu_P(C1), \upsilon_P(C1), \sigma_P(C1))$ denoted the supply, storage and the sudden unexpected demand of medical resources respectively.
- The meaning of triplet $(\mu_Q(g, C1), \upsilon_Q(g, C1), \sigma_Q(g, C1))$ is that, it is the actual amount of medical aids provided to the three categories of people in city from the clinic C1.

Since the medical resource utilized by the people is dominated by the one in the clinics, the Neutrosophic threshold dimension can easily be determined from the number of clinics. It is clear that the Neutrosophic threshold dimension of graph in Figure (4) is 3, that is we can induce three Neutrosophic fuzzy threshold sub graphs as given in figure (5, 6 & 7), where the triplet (τ_1, τ_2, τ_3) denotes the limitation of amount of medicine provided to three categories of people corresponding to the clinic C.

- Figure (5) gives the Neutrosophic fuzzy threshold graph with $(\tau_1 = 0.6, \tau_2 = 2.92, \tau_3 = 0.34)$, where the clinic C1 supplies 0.6 amount of medicine to the affected people in three cities {a,b,c}, where only $\mu_P(a) + \mu_P(b) + \mu_P(c) = 0.37$ is required , 0.34 amount of medicine to the one who were asymptotic and 0.08(3-2.92) amount to the people in safe zone, whose actual requirement is only 0.055.
- Figure (6) gives the Neutrosophic fuzzy threshold graph with $(\tau_1 = 0.62, \tau_2 = 4.93, \tau_3 = 0.2)$, where the clinic C2 supplies 0.62 amount of medicine to the affected people in five cities {b,c,e,f,g}, where only $\mu_P(b) + \mu_P(c) + \mu_P(e) + \mu_P(g) + \mu_P(f) = 0.6$ is required ,0.2 amount of medicine to the one who were asymptotic and 0.07(5-4.93) amount to the people in safe zone, whose actual requirement is only 0.06.

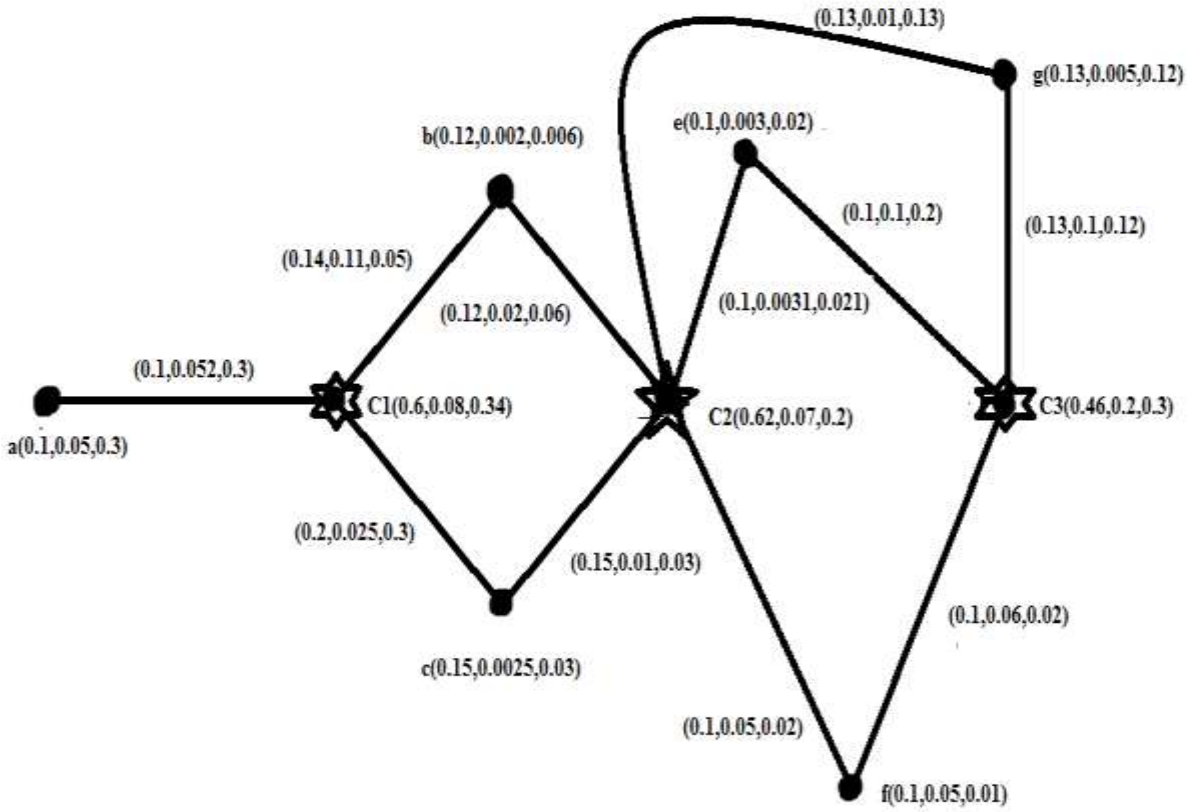


Figure 4. Neutrosophic Fuzzy threshold graph

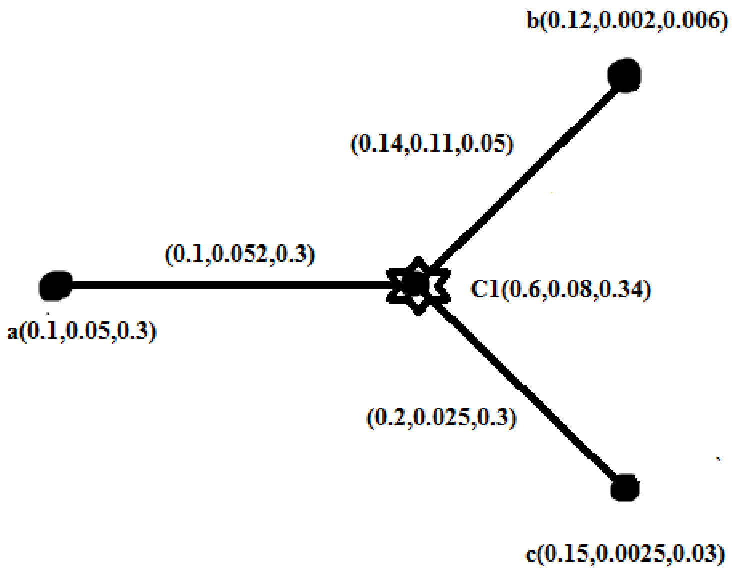


Figure 3. Clinic C1

- Figure (7) gives the Neutrosophic fuzzy threshold graph with $\left(\tau_1 = 0.46, \tau_2 = 2.8, \tau_3 = 0.3\right)$, where the clinic C3 supplies 0.46 amount of medicine to the affected people in three cities {e,f,g}, where only $\mu_P(e)+\mu_P(g)+\mu_P(f)=0.33$ is required , 0.3 amount of medicine to the one who were asymptotic and 0.2(3-2.8) amount to the people in safe zone, whose actual requirement is only 0.058.

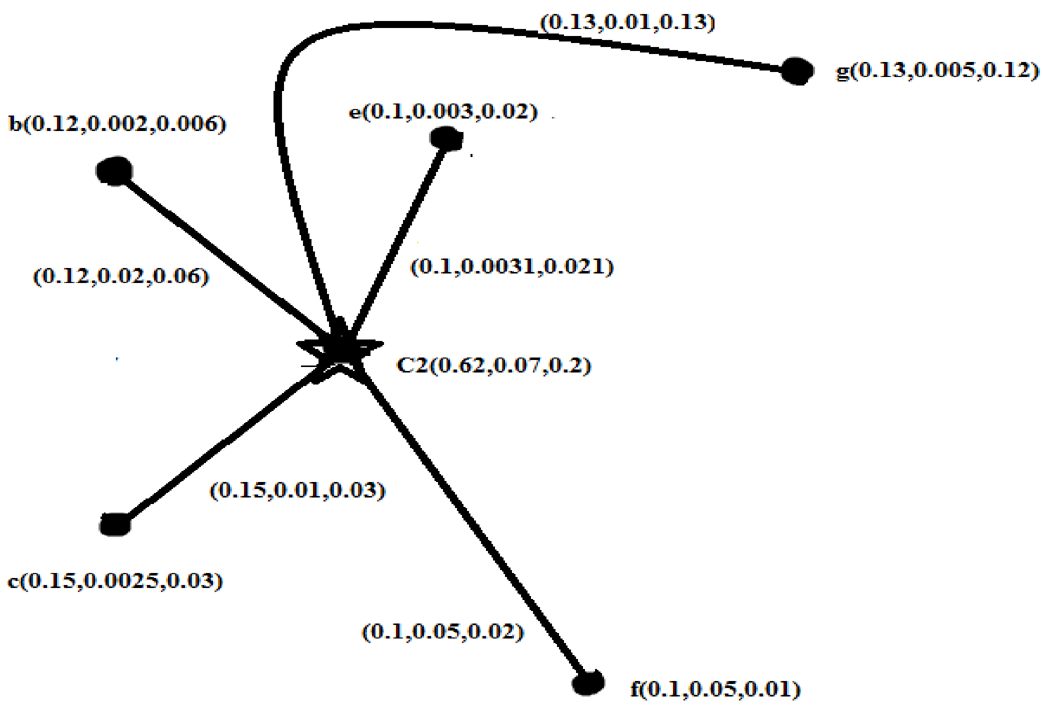


Figure 6. Clinic C2

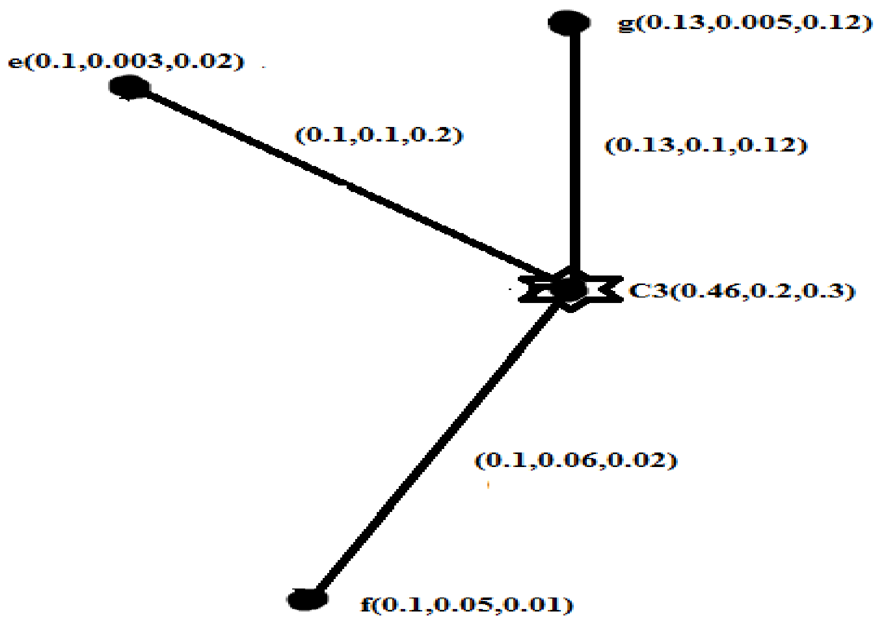


Figure 7. Clinic C3

Thus Neutrosophic fuzzy threshold graphs will give better comparison and proper details in resource analysis. Using Neutrosophic fuzzy threshold graph we can get more accurate results than intuitionistic fuzzy threshold graphs.

5. Conclusions

Neutrosophic graphs will give more accurate results in case of uncertainty. Even though, we obtain some basic information using intuitionistic fuzzy graphs, the value of indeterminacy will provide a clear cut results in resource allocation process. As an extension of intuitionistic fuzzy threshold graphs, Neutrosophic fuzzy threshold graph was introduced and Neutrosophic fuzzy alternating 4- cycle, threshold dimensions of Neutrosophic fuzzy threshold graphs were defined. We proved that Neutrosophic fuzzy threshold graph is the generalized case of Fuzzy threshold graph. Proper examples were given for each proposed concepts and theorems based on concepts were proved. We also gave one application that illustrates how Neutrosophic fuzzy threshold graph and threshold dimension were utilized in allocation of medical resource from clinics to people in cities.

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Conflicts of Interest

Authors declare that there is no conflict of interest.

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Neutrosophic point and its neighbourhood structure

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Abstract. In this article we define neutrosophic point, neutrosophic crisp point, neighbourhood of a neutrosophic point and investigate some properties of neutrosophic point as well as neighbourhood of a neutrosophic point. We also study the characterization of neutrosophic topological space in terms of neighbourhoods.

Keywords: Neutrosophic set ; Neutrosophic point; Neutrosophic crisp point ; Neighbourhood of a neutrosophic point.

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1. Introduction

In the year 1965, L.A.Zadeh [18] introduced a revolutionary concept, Fuzzy set theory. But after some decades a new branch of philosophy, known as Neutrosophic set theory, was developed and studied by Florentin Smarandache [10–12]. Smarandache [12] proved that neutrosophic set was a generalisation of intuitionistic fuzzy set which was developed by K.Atanassov [1] in 1986 as a generalisation of fuzzy set. In an intuitionistic fuzzy set an element belonging to the universe of discourse has the degree of membership and the degree of non-membership. But in case of neutrosophic set an element has another grade of membership known as degree of indeterminacy besides the degree of membership and the degree of non-membership. After Smarandache had introduced the concept of neutrosophy, it was studied by many researchers [13, 14, 17]. In the year 2002, Smarandache [11] introduced the notion of neutrosophic topology on the non-standard interval. F.G.Lupiáñez [6, 8, 9] studied and investigated many properties of neutrosophic topological space. In [6] F.G.Lupiáñez showed that an intuitionistic fuzzy topology may not be a neutrosophic topology. The author [7] also developed the concept of interval neutrosophic sets and topology. A.A.Salma and S.Alblowi [13, 14] studied neutrosophic topological space and generalised neutrosophic topological space. A.A.Salma et.al. [15, 16] also investigated neutrosophic filters and neutrosophic continuous functions. Later, neutrosophic topology was studied by many mathematicians [2–4]. In the year 2016, Serkan Karatas and Cemil Kuru [5] redefined the set operations and introduced a new neutrosophic topology and then investigated some important properties of general topology on the redefined neutrosophic topological space. Since in neutrosphic set theory, the indeterminacy-membership is given the same importance as the truth-membership and falsehood-membership and since all the three neutrosophic components are independent of one another, so this theory is more flexible and effective than all the previous set theories. For this reason this theory is attracting the researchers throughout the world and is very useful not only in the developement of science and technology but also in various other fields. For instance, Abdel-Basset et.al. [19–23] studied the applications of neutrosophic theory in various scientific fields. Pramanik and Roy [25] in 2014 studied on the conflict between India and Pakistan over Jammu-Kashmir through neutrosophic game Theory. Mondal and Pramanik [26] studied the problems of eunuchs in West Bengal(India) based on neutrosophic cognitive maps. Very recently some studies on COVID-19 [21, 24] had been done under neutrosophic environment. But the theory still has many concepts to be developed.

In this article we try to introduce neutrosophic point and its neighbourhood structure on the neutrosophic topological space defined by Serkan Karatas and Cemil Kuru [5]. We investigate some results on neutrosophic points, neighbourhood of a neutrosophic point and study the characterization of neutrosophic topological space in terms of the neighbourhoods of neutrosophic points.

2. **Preliminaries**

In this section we discuss some concepts related with neutrosophic sets.

2.1. **Definition: [5]**

Let X be the universe of discourse. A neutrosophic set (NS for short) A over X is defined as $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x)\rangle : x \in X\}$, where $\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A$ are functions from X to $[0, 1]$ and $0 \leq \mathcal{T}_A(x) + \mathcal{I}_A(x) + \mathcal{F}_A(x) \leq 3$.

The set of all neutrosophic sets over X is denoted by $\mathcal{N}(X)$.

2.2. **Definition: [5]**

Let $A, B \in \mathcal{N}(X)$. Then

(i) (Inclusion): If $\mathcal{T}_A(x) \leq \mathcal{T}_B(x), \mathcal{I}_A(x) \geq \mathcal{I}_B(x), \mathcal{F}_A(x) \geq \mathcal{F}_B(x)$ for all $x \in X$ then A is said to be a neutrosophic subset of B and which is denoted by $A \subseteq B$.

(ii) (Equality): If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(iii) (Intersection): The intersection of A and B, denoted by $A \cap B$, is defined as $A \cap B = \{\langle x, \mathcal{T}_A(x) \wedge \mathcal{T}_B(x), \mathcal{I}_A(x) \vee \mathcal{I}_B(x), \mathcal{F}_A(x) \vee \mathcal{F}_B(x)\rangle : x \in X\}$.

(iv) (Union): The union of A and B, denoted by $A \cup B$, is defined as $A \cup B = \{\langle x, \mathcal{T}_A(x) \vee \mathcal{T}_B(x), \mathcal{I}_A(x) \wedge \mathcal{I}_B(x), \mathcal{F}_A(x) \wedge \mathcal{F}_B(x)\rangle : x \in X\}$.

(v) (Complement): The complement of the neutrosophic set A, denoted by A^c, is defined as $A^c = \{\langle x, \mathcal{F}_A(x), 1 - \mathcal{I}_A(x), \mathcal{T}_A(x)\rangle : x \in X\}$

(vi) (Universal Set): If $\mathcal{T}_A(x) = 1, \mathcal{I}_A(x) = 0, \mathcal{F}_A(x) = 0$ for all $x \in X$ then A is said to be neutrosophic universal set and which is denoted by $\tilde{X}$.

(vii) (Empty Set): If $\mathcal{T}_A(x) = 0, \mathcal{I}_A(x) = 1, \mathcal{F}_A(x) = 1$ for all $x \in X$ then A is said to be neutrosophic empty set and which is denoted by $\tilde{\emptyset}$.

2.3. **Definition: [14]**

Let $\{A_i : i \in \triangle\} \subseteq \mathcal{N}(X)$, where $\triangle$ is an index set. Then

(i) $\cup_{i\in\triangle} A_i = \{\langle x, \vee_{i\in\triangle}\mathcal{T}_{A_i}(x), \wedge_{i\in\triangle}\mathcal{I}_{A_i}(x), \wedge_{i\in\triangle}\mathcal{F}_{A_i}(x)\rangle : x \in X\}$.
i.e., $\cup_{i\in\triangle} A_i = \{\langle x, \sup_{i\in\triangle} \mathcal{T}_{A_i}(x), \inf_{i\in\triangle} \mathcal{I}_{A_i}(x), \inf_{i\in\triangle} \mathcal{F}_{A_i}(x)\rangle : x \in X\}$.

(ii) $\cap_{i\in\triangle} A_i = \{\langle x, \wedge_{i\in\triangle}\mathcal{T}_{A_i}(x), \vee_{i\in\triangle}\mathcal{I}_{A_i}(x), \vee_{i\in\triangle}\mathcal{F}_{A_i}(x)\rangle : x \in X\}$.
i.e., $\cap_{i\in\triangle} A_i = \{\langle x, \inf_{i\in\triangle} \mathcal{T}_{A_i}(x), \sup_{i\in\triangle} \mathcal{I}_{A_i}(x), \sup_{i\in\triangle} \mathcal{F}_{A_i}(x)\rangle : x \in X\}$.

2.4. **Neutrosophic topological space : [5]**

2.4.1. **Definition: [5]**

Let $\tau \subseteq \mathcal{N}(X)$. Then τ is called a neutrosophic topology on X if

(i) $\tilde{\emptyset}$ and $\tilde{X}$ belong to τ.

(ii) The union of any number of neutrosophic sets in τ belongs to τ.

(iii) The intersection of any two neutrosophic sets in τ belongs to τ.

If τ is a neutrosophic topology on X then the pair (X, τ) is called a neutrosophic topological space (NTS for short) over X. The members of τ are called neutrosophic open sets in X. If for a neutrosophic set A, $A^c \in \tau$ then A is said to be a neutrosophic closed set in X.

2.4.2. **Theorem: [5]**

Let (X, τ) be a neutrosophic topological space over X. Then

(i) $\tilde{\emptyset}$ and $\tilde{X}$ are neutrosophic closed sets over X.

(ii) The intersection of any number of neutrosophic closed sets is a neutrosophic closed set over X.

(iii) The union of any two neutrosophic closed sets is a neutrosophic closed set over X.

3. Main Results

In this section we introduce and study the following concepts. Throughout this discussion we have considered the neutrosophic topological space defined by Serkan Karatas and Cemil Kuru [5] in the year 2016.

3.1. Definition:

Let $\mathcal{N}(X)$ be the set of all neutrosophic sets over X. A NS $P = \{\langle x, \mathcal{T}_P(x), \mathcal{I}_P(x), \mathcal{F}_P(x)\rangle : x \in X\}$ is called a neutrosophic point (NP for short) iff for any element $y \in X$, $\mathcal{T}_P(y) = \alpha, \mathcal{I}_P(y) = \beta, \mathcal{F}_P(y) = \gamma$ for $y = x$ and $\mathcal{T}_P(y) = 0, \mathcal{I}_P(y) = 1, \mathcal{F}_P(y) = 1$ for $y \neq x$, where $0 < \alpha \leq 1, 0 \leq \beta < 1, 0 \leq \gamma < 1$.

A neutrosophic point $P = \{\langle x, \mathcal{T}_P(x), \mathcal{I}_P(x), \mathcal{F}_P(x)\rangle : x \in X\}$ will be denoted by $P^x_{\alpha,\beta,\gamma}$ or $P < x, \alpha, \beta, \gamma >$ or simply by $x_{\alpha,\beta,\gamma}$. For the NP $x_{\alpha,\beta,\gamma}$, x will be called its support.

The complement of the NP $P^x_{\alpha,\beta,\gamma}$ will be denoted by $(P^x_{\alpha,\beta,\gamma})^c$ or by $x^c_{\alpha,\beta,\gamma}$.

A NS $P = \{\langle x, \mathcal{T}_P(x), \mathcal{I}_P(x), \mathcal{F}_P(x)\rangle : x \in X\}$ is called a neutrosophic crisp point (NCP for short) iff for any element $y \in X$, $\mathcal{T}_P(y) = 1, \mathcal{I}_P(y) = 0, \mathcal{F}_P(y) = 0$ for $y = x$ and $\mathcal{T}_P(y) = 0, \mathcal{I}_P(y) = 1, \mathcal{F}_P(y) = 1$ for $y \neq x$.

3.2. Definition:

Let A be a neutrosophic set over X. Also let $x_{\alpha,\beta,\gamma}$ and $y_{\alpha^/,\beta^/,\gamma^/}$ be two neutrosophic points in X. Then

(i) $x_{\alpha,\beta,\gamma}$ is said to be contained in A, denoted by $x_{\alpha,\beta,\gamma} \subseteq A$, iff $\alpha \leq \mathcal{T}_A(x), \beta \geq \mathcal{I}_A(x), \gamma \geq \mathcal{F}_A(x)$.

(ii) $x_{\alpha,\beta,\gamma}$ is said to belong to A, denoted by $x_{\alpha,\beta,\gamma} \in A$, iff $\alpha \leq \mathcal{T}_A(x), \beta \geq \mathcal{I}_A(x), \gamma \geq \mathcal{F}_A(x)$.

(iii) $x_{\alpha,\beta,\gamma}$ is said to be contained in $y_{\alpha^/,\beta^/,\gamma^/}$, denoted by $x_{\alpha,\beta,\gamma} \subseteq y_{\alpha^/,\beta^/,\gamma^/}$, iff $x = y$ and $\alpha \leq \alpha^/, \beta \geq \beta^/, \gamma \geq \gamma^/$.

(iv) $x_{\alpha,\beta,\gamma}$ is said to belong to $y_{\alpha^/,\beta^/,\gamma^/}$, denoted by $x_{\alpha,\beta,\gamma} \in y_{\alpha^/,\beta^/,\gamma^/}$, iff $x = y$ and $\alpha \leq \alpha^/, \beta \geq \beta^/, \gamma \geq \gamma^/$.

(v) A NCP $x_{1,0,0} \subseteq A$ iff $\mathcal{T}_A(x) = 1, \mathcal{I}_A(x) = 0, \mathcal{F}_A(x) = 0$.

(vi) A NCP $x_{1,0,0} \in A$ iff $\mathcal{T}_A(x) = 1, \mathcal{I}_A(x) = 0, \mathcal{F}_A(x) = 0$.

3.3. Remark:

In 3.2, the definitions of inclusion and belongingness are being the same between a NP and a NS as well as between two neutrosophic points.The clarification behind that is given below :

Suppose the inclusion and belongingness between a NP and a NS are defined as follows :

$$x_{\alpha,\beta,\gamma} \subseteq A \text{ iff } \alpha \leq \mathcal{T}_A(x), \beta \geq \mathcal{I}_A(x), \gamma \geq \mathcal{F}_A(x).$$
$$x_{\alpha,\beta,\gamma} \in A \text{ iff } \alpha < \mathcal{T}_A(x), \beta > \mathcal{I}_A(x), \gamma > \mathcal{F}_A(x).$$

Let $X = \{x, y\}$ and $A = \{\langle x, 0.5, 0.4, 0.3\rangle, \langle y, 0.4, 0.5, 0.6\rangle\}$. Also we consider the neutrosophic points $P\langle x, 0.5, 0.4, 0.3\rangle$ and $Q\langle x, 0.3, 0.6, 0.7\rangle$. According to the above definitions, $P \subseteq A$, $Q \subseteq A$, $Q \in A$, but $P \notin A$.

Having observed the neutrosophic points P and Q and the neutrosophic set A, it is really difficult to accept that $P \notin A$ whereas $Q \in A$. Similar is the case between two neutrosophic points.

3.4. **Proposition:**

Every NS $A \in \mathcal{N}(X)$ can be expressed as the union of all neutrosophic points contained in A.

Proof: Let $B = \bigcup\{x_{\alpha,\beta,\gamma} : x_{\alpha,\beta,\gamma} \in A\}$, where $x \in X$. If $\mathcal{T}_A(x) \neq 0, \mathcal{I}_A(x) \neq 1, \mathcal{F}_A(x) \neq 1$ then

$$\begin{aligned}\mathcal{T}_A(x) &= \sup\{\alpha : x_{\alpha,\beta,\gamma} \text{ is a NP and } \alpha \leq \mathcal{T}_A(x), \beta \geq \mathcal{I}_A(x), \gamma \geq \mathcal{F}_A(x)\}\\ &= \mathcal{T}_{\cup x_{\alpha,\beta,\gamma}}(x)\\ &= \mathcal{T}_B(x).\end{aligned}$$

$$\begin{aligned}\mathcal{I}_A(x) &= \inf\{\beta : x_{\alpha,\beta,\gamma} \text{ is a NP and } \alpha \leq \mathcal{T}_A(x), \beta \geq \mathcal{I}_A(x), \gamma \geq \mathcal{F}_A(x)\}\\ &= \mathcal{I}_{\cup x_{\alpha,\beta,\gamma}}(x)\\ &= \mathcal{I}_B(x).\end{aligned}$$

$$\begin{aligned}\mathcal{F}_A(x) &= \inf\{\gamma : x_{\alpha,\beta,\gamma} \text{ is a NP and } \alpha \leq \mathcal{T}_A(x), \beta \geq \mathcal{I}_A(x), \gamma \geq \mathcal{F}_A(x)\}\\ &= \mathcal{F}_{\cup x_{\alpha,\beta,\gamma}}(x)\\ &= \mathcal{F}_B(x).\end{aligned}$$

Therefore $A = B$, i.e., $A = \bigcup\{x_{\alpha,\beta,\gamma} : x_{\alpha,\beta,\gamma} \in A\}$. Hence proved.

3.5. **Proposition:**

Let $A, B \in \mathcal{N}(X)$. Then $A = B$ iff $P \in A \Longleftrightarrow P \in B$ for every NP $P \in \mathcal{N}(X)$.

Proofs: Let $A = B$ and let $x_{\alpha,\beta,\gamma}$ be a NP. Then

$$
\begin{aligned}
& x_{\alpha,\beta,\gamma} \in A \\
\Leftrightarrow & \alpha \leq \mathcal{T}_A(x), \beta \geq \mathcal{I}_A(x), \gamma \geq \mathcal{F}_A(x) \\
\Leftrightarrow & \alpha \leq \mathcal{T}_B(x), \beta \geq \mathcal{I}_B(x), \gamma \geq \mathcal{F}_B(x) [\because \text{A = B}] \\
\Leftrightarrow & x_{\alpha,\beta,\gamma} \in B
\end{aligned}
$$

Therefore the proposition is necessary.
Converse part : From proposition 3.4 we can write $A = \bigcup\{x_{\alpha,\beta,\gamma} : x_{\alpha,\beta,\gamma} \in A\}$ and $B = \bigcup\{y_{\alpha^/,\beta^/,\gamma^/} : y_{\alpha^/,\beta^/,\gamma^/} \in B\}$. Now

$$
\begin{aligned}
& x_{\alpha,\beta,\gamma} \in A \\
\Rightarrow & x_{\alpha,\beta,\gamma} \in B \text{ [by the hypothesis]} \\
\Rightarrow & \cup\, x_{\alpha,\beta,\gamma} \subseteq B \\
\Rightarrow & A \subseteq B
\end{aligned}
$$

Exactly in the same manner we can show that $B \subseteq A$. Therefore $A = B$, i.e.,the proposition is sufficient.

Hence proved.

3.6. **Proposition:**

If $x_{\alpha,\beta,\gamma} \in A$ and $A \subseteq B$, where $A, B \in \mathcal{N}(X)$ then $x_{\alpha,\beta,\gamma} \in B$.

Proof: Since $A \subseteq B$, so $\mathcal{T}_A(x) \leq \mathcal{T}_B(x), \mathcal{I}_A(x) \geq \mathcal{I}_B(x), \mathcal{F}_A(x) \geq \mathcal{F}_B(x)$.

$$
\begin{aligned}
& x_{\alpha,\beta,\gamma} \in A \\
\Rightarrow & \alpha \leq \mathcal{T}_A(x), \beta \geq \mathcal{I}_A(x), \gamma \geq \mathcal{F}_A(x) \\
\Rightarrow & \alpha \leq \mathcal{T}_B(x), \beta \geq \mathcal{I}_B(x), \gamma \geq \mathcal{F}_B(x) \\
\Rightarrow & x_{\alpha,\beta,\gamma} \in B
\end{aligned}
$$

Hence proved.

3.7. **Proposition:**

Let $\{A_i : i \in \triangle\} \subseteq \mathcal{N}(X)$, where $\triangle$ is an index set. Let $x_{\alpha,\beta,\gamma}$ and $y_{\alpha^/,\beta^/,\gamma^/}$ be any two neutrosophic points over X. Then the following hold good.

(i) $x_{\alpha,\beta,\gamma} \in \bigcap\{A_i : i \in \triangle\} \Longleftrightarrow x_{\alpha,\beta,\gamma} \in A_i \, \forall \, i \in \triangle$.

(ii) If $x_{\alpha,\beta,\gamma} \in A_i$ for some $i \in \triangle$ then $x_{\alpha,\beta,\gamma} \in \bigcup\{A_i : i \in \triangle\}$.

(iii) If $x_{\alpha,\beta,\gamma} \in \bigcup\{A_i : i \in \triangle\}$ then there exists a NS $A(x_{\alpha,\beta,\gamma})$ such that $x_{\alpha,\beta,\gamma} \in A(x_{\alpha,\beta,\gamma}) \subseteq \bigcup\{A_i : i \in \triangle\}$.

(iv) If $x_{\alpha,\beta,\gamma} \in A$, where $A \in \mathcal{N}(X)$, then there exist $\alpha^{/}, \beta^{/}, \gamma^{/}$ such that $\alpha \leq \alpha^{/}, \beta \geq \beta^{/}, \gamma \geq \gamma^{/}$ and $x_{\alpha^{/},\beta^{/},\gamma^{/}} \in A$.

Proofs: (i)

$$\begin{aligned}
&x_{\alpha,\beta,\gamma} \in \bigcap\{A_i : i \in\triangle\}.\\
\Leftrightarrow &x_{\alpha,\beta,\gamma} \in \{\langle x, \wedge_{i\in\triangle}\mathcal{T}_{A_i}(x), \vee_{i\in\triangle}\mathcal{I}_{A_i}(x), \vee_{i\in\triangle}\mathcal{F}_{A_i}(x)\rangle : x \in X\}.\\
\Leftrightarrow &\alpha \leq \wedge_{i\in\triangle}\mathcal{T}_{A_i}(x), \beta \geq \vee_{i\in\triangle}\mathcal{I}_{A_i}(x), \gamma \geq \vee_{i\in\triangle}\mathcal{F}_{A_i}(x).\\
\Leftrightarrow &\alpha \leq \inf_{i\in\triangle}\mathcal{T}_{A_i}(x), \beta \geq \sup_{i\in\triangle}\mathcal{I}_{A_i}(x), \gamma \geq \sup_{i\in\triangle}\mathcal{F}_{A_i}(x).\\
\Leftrightarrow &\alpha \leq \mathcal{T}_{A_i}(x), \beta \geq \mathcal{I}_{A_i}(x), \gamma \geq \mathcal{F}_{A_i}(x)\, \forall\, i \in\triangle\ .\\
\Leftrightarrow &x_{\alpha,\beta,\gamma} \in A_i\, \forall\, i \in\triangle\ .
\end{aligned}$$

Hence Proved.

(ii)

$$\begin{aligned}
&x_{\alpha,\beta,\gamma} \in A_i \text{ for some } i \in\triangle\ .\\
\Rightarrow &x_{\alpha,\beta,\gamma} \in \{\langle x, \mathcal{T}_{A_i}(x), \mathcal{I}_{A_i}(x), \mathcal{F}_{A_i}(x)\rangle : x \in X\} \text{ for some } i \in\triangle\ .\\
\Rightarrow &\alpha \leq \mathcal{T}_{A_i}(x), \beta \geq \mathcal{I}_{A_i}(x), \gamma \geq \mathcal{F}_{A_i}(x) \text{ for some } i \in\triangle\ .\\
\Rightarrow &\alpha \leq \sup_{i\in\triangle}\mathcal{T}_{A_i}(x), \beta \geq \inf_{i\in\triangle}\mathcal{I}_{A_i}(x), \gamma \geq \inf_{i\in\triangle}\mathcal{F}_{A_i}(x).\\
\Rightarrow &x_{\alpha,\beta,\gamma} \in \{\langle x, \vee_{i\in\triangle}\mathcal{T}_{A_i}(x), \wedge_{i\in\triangle}\mathcal{I}_{A_i}(x), \wedge_{i\in\triangle}\mathcal{F}_{A_i}(x)\rangle : x \in X\}.\\
\Rightarrow &x_{\alpha,\beta,\gamma} \in \bigcup\{A_i : i \in\triangle\}.
\end{aligned}$$

Hence Proved.

(iii)

$$\begin{aligned}
&x_{\alpha,\beta,\gamma} \in \bigcup\{A_i : i \in\triangle\}\\
\Rightarrow &\alpha \leq \sup_{i\in\triangle}\mathcal{T}_{A_i}(x), \beta \geq \inf_{i\in\triangle}\mathcal{I}_{A_i}(x), \gamma \geq \inf_{i\in\triangle}\mathcal{F}_{A_i}(x).\\
\Rightarrow &\alpha \leq \mathcal{T}_{A_r}(x), \beta \geq \mathcal{I}_{A_s}(x), \gamma \geq \mathcal{F}_{A_t}(x) \text{ for some } r, s, t \in\triangle\ .\\
\Rightarrow &x_{\alpha,\beta,\gamma} \in A_r \cup A_s \cup A_t.\\
\Rightarrow &x_{\alpha,\beta,\gamma} \in A(x_{\alpha,\beta,\gamma}), \text{where } A(x_{\alpha,\beta,\gamma}) = A_r \cup A_s \cup A_t.
\end{aligned}$$

Obviously $A(x_{\alpha,\beta,\gamma}) = A_r \cup A_s \cup A_t \subseteq \bigcup\{A_i : i \in\triangle\}$.
Thus $x_{\alpha,\beta,\gamma} \in A(x_{\alpha,\beta,\gamma}) \subseteq \bigcup\{A_i : i \in\triangle\}$.
Hence proved.

(iv) Since $A \in \mathcal{N}(x)$, so from the proposition 3.4, $A = \bigcup\{x_{p,q,r} : x_{p,q,r} \in A\}$. Now

$$\begin{aligned} & x_{\alpha,\beta,\gamma} \in A. \\ \Rightarrow & \alpha \leq \mathcal{T}_A(x),\ \beta \geq \mathcal{I}_A(x),\ \gamma \geq \mathcal{F}_A(x) \\ \Rightarrow & \alpha \leq \sup_{x_{p,q,r} \in A} p,\ \beta \geq \inf_{x_{p,q,r} \in A} q,\ \gamma \geq \inf_{x_{p,q,r} \in A} r. \end{aligned}$$

Let $\sup\{p : x_{p,q,r} \in A\} = \alpha^{/}$, $\inf\{q : x_{p,q,r} \in A\} = \beta^{/}$, $\inf\{r : x_{p,q,r} \in A\} = \gamma^{/}$. Then $\alpha \leq \alpha^{/}, \quad \beta \geq \beta^{/}, \quad \gamma \geq \gamma^{/}$. Obviously $x_{\alpha^{/},\beta^{/},\gamma^{/}} \in A$

Thus there exist $\alpha^{/}, \beta^{/}, \gamma^{/}$ such that $\alpha \leq \alpha^{/}, \beta \geq \beta^{/}, \gamma \geq \gamma^{/}$ and $x_{\alpha^{/},\beta^{/},\gamma^{/}} \in A$. Hence proved.

3.8. **Remark:**

The converse of the proposition 3.7(ii) is not true.We shall establish it by the following counter example.

Let $X = \{x, y\}$ and $A = \{\langle x, 0.5, 0.7, 0.6\rangle, \langle y, 0.6, 0.7, 0.7\rangle\}, B = \{\langle x, 0.4, 0.6, 0.6\rangle, \langle y, 0.3, 0.8, 0.7\rangle\}$ and $C = \{\langle x, 0.3, 0.4, 0.5\rangle, \langle y, 0.6, 0.1, 0.7\rangle\}$ be three neutrosophic sets over X.Then $A \cup B \cup C = \{\langle x, 0.5, 0.4, 0.5\rangle, \langle y, 0.6, 0.1, 0.7\rangle\}$ Let us consider the neutrosophic point $P\langle x, 0.4, 0.6, 0.5\rangle$. It is clear that $P\langle x, 0.4, 0.6, 0.5\rangle \in A \cup B \cup C$ but $P\langle x, 0.4, 0.6, 0.5\rangle$ belongs to neither of the neutrosophic sets A, B and C.

3.9. **Definition:**

Let (X, τ) be a neutrosophic topological space. A NS $A \in \mathcal{N}(X)$ is called a neutrosophic neighbourhood or simply neighbourhood (nhbd for short) of a NP $x_{\alpha,\beta,\gamma}$ iff there exists a NS $B \in \tau$ such that $x_{\alpha,\beta,\gamma} \in B \subseteq A$.

A neighbourhood A of the NP $x_{\alpha,\beta,\gamma}$ is said to be a neutrosophic open neighbourhood of $x_{\alpha,\beta,\gamma}$ if A is a neutrosophic open set.

The family consisting of all the neighbourhoods of the NP $x_{\alpha,\beta,\gamma}$ is called the system of neighbourhoods(or neighbourhood system) of $x_{\alpha,\beta,\gamma}$. This family is denoted by $\mathbf{N}(x_{\alpha,\beta,\gamma})$.

3.10. **Proposition:**

A NS in a NTS is neutrosophic open iff it is a nhbd of each of its neutrosophic points.

Proof: Let (X, τ) be a NTS and let $A \in \mathcal{N}(X)$.

Suppose that A is a τ-open set. Then for every NP $x_{\alpha,\beta,\gamma} \in A$, we have $x_{\alpha,\beta,\gamma} \in A \subseteq A$ and so A is a nhbd of $x_{\alpha,\beta,\gamma}$.Thus A is nhbd of each of its neutrosophic points.

Next suppose that A is a nhbd of each of its neutrosophic points. If $A = \tilde{\emptyset}$ then A is open as $\tilde{\emptyset} \in \tau$. But if $A \neq \tilde{\emptyset}$ then for each $x_{\alpha,\beta,\gamma} \in A$ there exists a τ-open set $B(x_{\alpha,\beta,\gamma})$ such that

$x_{\alpha,\beta,\gamma} \in B(x_{\alpha,\beta,\gamma}) \subseteq A$. Obviously $A = \cup B(x_{\alpha,\beta,\gamma})$ and so A is τ-open set, being a union of τ-open sets.

Hence proved.

3.11. **Proposition:**

Two neutrosophic topologies on the same set are identical iff they admit the same neighbourhoods.

Proof: Necessary part is very obvious. Conversely suppose that τ_1 and τ_2 are two neutrosophic topologies on X having the same neighbourhood system of the neutrosophic points over X. Let $A \in \mathcal{N}(X)$. Now

$$\begin{aligned} & A \text{ is a } \tau_1 - \text{open set} \\ \Leftrightarrow & A \text{ is a } \tau_1 - \text{neighbourhood of each of its neutrosophic points} \\ \Leftrightarrow & A \text{ is a } \tau_2 - \text{neighbourhood of each of its neutrosophic points} \\ \Leftrightarrow & A \text{ is a } \tau_2 - \text{open set} \end{aligned}$$

Therefore $\tau_1 = \tau_2$.
Hence proved.

3.12. **Remark:**

It is very clear that for every NP $x_{\alpha,\beta,\gamma}$, $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X) \Longleftrightarrow x_{\alpha,\beta,\gamma} \in \tilde{X}$.

3.13. **Properties of neutrosophic neighbourhoods:**

Let (X, τ) be a neutrosophic topological space and let $x \in X$.If $\mathbf{N}(x_{\alpha,\beta,\gamma})$ be the collection of all nhbds of the neutrosophic point $x_{\alpha,\beta,\gamma}$ then

N1) $\mathbf{N}(x_{\alpha,\beta,\gamma}) \neq \emptyset$ for every NP $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$.

N2) $N \in \mathbf{N}(x_{\alpha,\beta,\gamma}) \Rightarrow x_{\alpha,\beta,\gamma} \in N$.

N3) $N \in \mathbf{N}(x_{\alpha,\beta,\gamma}), N \subseteq M \Rightarrow M \in \mathbf{N}(x_{\alpha,\beta,\gamma})$.

N4) $M, N \in \mathbf{N}(x_{\alpha,\beta,\gamma}) \Rightarrow M \cap N \in \mathbf{N}(x_{\alpha,\beta,\gamma})$.

N5) $N \in \mathbf{N}(x_{\alpha,\beta,\gamma}) \Rightarrow$ there exists a $M \in \mathbf{N}(x_{\alpha,\beta,\gamma})$ such that $M \subseteq N$ and $M \in \mathbf{N}(y_{\alpha',\beta',\gamma'})$ for all $y_{\alpha',\beta',\gamma'} \in M$.

Proofs:

N1) Since $\tilde{X}$ is an open set, so it is a nhbd of every NP $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$. Thus there exists at least one nhbd for every NP $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$. Therefore $\mathbf{N}(x_{\alpha,\beta,\gamma}) \neq \emptyset$ for every NP $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$.

N2) $N \in \mathbf{N}(x_{\alpha,\beta,\gamma}) \Rightarrow N$ is a nhbd of $x_{\alpha,\beta,\gamma} \Rightarrow x_{\alpha,\beta,\gamma} \in N$.

N3)

$$N \in \mathbf{N}(x_{\alpha,\beta,\gamma})$$
$$\Rightarrow N \text{ is a neighbourhood of } x_{\alpha,\beta,\gamma}$$
$$\Rightarrow \exists \text{ an open set } G \text{ such that } x_{\alpha,\beta,\gamma} \in G \subseteq N.$$
$$\Rightarrow \exists \text{ an open set } G \text{ such that } x_{\alpha,\beta,\gamma} \in G \subseteq M.[\because N \subseteq M]$$
$$\Rightarrow M \text{ is a neighbourhood of } x_{\alpha,\beta,\gamma}$$
$$\Rightarrow M \in \mathbf{N}(x_{\alpha,\beta,\gamma})$$

N4) $M, N \in \mathbf{N}(x_{\alpha,\beta,\gamma}) \Rightarrow M, N$ are neighbourhoods of $x_{\alpha,\beta,\gamma} \Rightarrow \exists\ G_1, G_2 \in \tau$ such that $x_{\alpha,\beta,\gamma} \in G_1 \subseteq N$ and $x_{\alpha,\beta,\gamma} \in G_2 \subseteq M$. But $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$. Therefore $x_{\alpha,\beta,\gamma} \in G_1 \cap G_2 \subseteq M \cap N$ and so $M \cap N$ is a nhbd of $x_{\alpha,\beta,\gamma}$, i.e., $M \cap N \in \mathbf{N}(x_{\alpha,\beta,\gamma})$.

N5) Since $N \in \mathbf{N}(x_{\alpha,\beta,\gamma})$, so there exists a τ-open set M such that $x_{\alpha,\beta,\gamma} \in M \subseteq N$. Since M is an open set and since $x_{\alpha,\beta,\gamma} \in M \subseteq M$, so $M \in \mathbf{N}(x_{\alpha,\beta,\gamma})$. Thus $M \in \mathbf{N}(x_{\alpha,\beta,\gamma})$ and $M \subseteq N$.

Again since M is an open set, so M is a nhbd of each of its neutrosophic points. Therefore $M \in \mathbf{N}(y_{\alpha',\beta',\gamma'})$ for all $y_{\alpha',\beta',\gamma'} \in M$.

Hence proved.

3.14. **Characterization of NTS in terms of neutrosophic neighbourhoods:**

Let X be the universe of discourse and $x \in X$. Let $\mathbf{N}(x_{\alpha,\beta,\gamma})$ be a family of neutrosophic sets over X satisfying the following five conditions :

N1) $\mathbf{N}(x_{\alpha,\beta,\gamma}) \neq \emptyset$ for every NP $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$.

N2) $N \in \mathbf{N}(x_{\alpha,\beta,\gamma}) \Rightarrow x_{\alpha,\beta,\gamma} \in N$.

N3) $N \in \mathbf{N}(x_{\alpha,\beta,\gamma}), N \subseteq M \Rightarrow M \in \mathbf{N}(x_{\alpha,\beta,\gamma})$.

N4) $M, N \in \mathbf{N}(x_{\alpha,\beta,\gamma}) \Rightarrow M \cap N \in \mathbf{N}(x_{\alpha,\beta,\gamma})$.

N5) $N \in \mathbf{N}(x_{\alpha,\beta,\gamma}) \Rightarrow$ there exists a $M \in \mathbf{N}(x_{\alpha,\beta,\gamma})$ such that $M \subseteq N$ and $M \in \mathbf{N}(y_{\alpha',\beta',\gamma'})$ for all $y_{\alpha',\beta',\gamma'} \in M$.

Then there exists a unique neutrosophic topology τ on X in such a way that if $\mathbf{N}^*(x_{\alpha,\beta,\gamma})$ is the collection of all nhbds of the NP $x_{\alpha,\beta,\gamma}$, defined by the topology τ, then $\mathbf{N}(x_{\alpha,\beta,\gamma}) = \mathbf{N}^*(x_{\alpha,\beta,\gamma})$.

Proof: We define τ as follows :

A NS $G \in \tau$ iff $G \in \mathbf{N}(x_{\alpha,\beta,\gamma})$ for every NP $x_{\alpha,\beta,\gamma} \in G$.

We claim that τ is a neutrosophic topology on X.

T1) $\tilde{\emptyset} \in \tau$ as $\tilde{\emptyset}$ contains no NP. By (N1) $\mathbf{N}(x_{\alpha,\beta,\gamma}) \neq \emptyset$ for every NP $x_{\alpha,\beta,\gamma} \in \mathcal{N}(X)$. Therefore there exists a $G(x_{\alpha,\beta,\gamma}) \in \mathbf{N}(x_{\alpha,\beta,\gamma})$ for every NP $x_{\alpha,\beta,\gamma} \in \tilde{X}$. Since $G(x_{\alpha,\beta,\gamma}) \subseteq \tilde{X}$, so by (N3), $\tilde{X} \in \mathbf{N}(x_{\alpha,\beta,\gamma})$ for every $x_{\alpha,\beta,\gamma} \in \tilde{X}$. Therefore $\tilde{X} \in \tau$. Thus $\tilde{\emptyset}, \tilde{X} \in \tau$.

T2) Suppose $G_1, G_2 \in \tau$. Then

$$G_1 \in \mathbf{N}(x_{\alpha,\beta,\gamma})\ \forall\ x_{\alpha,\beta,\gamma} \in G_1 \text{ and } G_2 \in \mathbf{N}(x_{\alpha,\beta,\gamma})\ \forall\ x_{\alpha,\beta,\gamma} \in G_2$$

$$\Rightarrow G_1 \in \mathbf{N}(x_{\alpha,\beta,\gamma}) \text{ and } G_2 \in \mathbf{N}(x_{\alpha,\beta,\gamma})\ \forall\ x_{\alpha,\beta,\gamma} \in G_1 \cap G_2$$

$$\Rightarrow G_1 \cap G_2 \in \mathbf{N}(x_{\alpha,\beta,\gamma})\ \forall\ x_{\alpha,\beta,\gamma} \in G_1 \cap G_2 \text{ [by (N4)]}$$

$$\Rightarrow G_1 \cap G_2 \in \tau \text{ [by the definition of } \tau]$$

T3) Suppose $\{G_i : i \in \Delta\} \subseteq \tau$. We show that $\cup\{G_i : i \in \Delta\} \in \tau$. Now

$$G_i \in \tau\ \forall\ i \in \Delta$$

$$\Rightarrow G_i \in \mathbf{N}(x_{\alpha,\beta,\gamma})\ \forall\ x_{\alpha,\beta,\gamma} \in G_i \text{ and } \forall\ i \in \Delta$$

$$\Rightarrow \cup \{G_i : i \in \Delta\} \in \mathbf{N}(x_{\alpha,\beta,\gamma})\ \forall\ x_{\alpha,\beta,\gamma} \in \cup\{G_i : i \in \Delta\} \text{[by (N2)and (N3)]}$$

$$\Rightarrow \cup \{G_i : i \in \Delta\} \in \tau \text{ [by the definition of } \tau]$$

Therefore τ is a neutrosophic topology on X.

We now show that $\mathbf{N}(x_{\alpha,\beta,\gamma}) = \mathbf{N}^*(x_{\alpha,\beta,\gamma})$, i.e., $N \in \mathbf{N}(x_{\alpha,\beta,\gamma}) \Leftrightarrow N$ is a nhbd of $x_{\alpha,\beta,\gamma}$.

Let $N \in \mathbf{N}(x_{\alpha,\beta,\gamma})$. Then by (N5) there exists $M \in \mathbf{N}(x_{\alpha,\beta,\gamma})$ such that $M \subseteq N$ and $M \in \mathbf{N}(y_{\alpha^/,\beta^/,\gamma^/})$ for all $y_{\alpha^/,\beta^/,\gamma^/} \in M$. Now $M \in \mathbf{N}(x_{\alpha,\beta,\gamma}) \Rightarrow x_{\alpha,\beta,\gamma} \in M$ [by (N2)]. Also $M \in \mathbf{N}(y_{\alpha^/,\beta^/,\gamma^/})$ for all $y_{\alpha^/,\beta^/,\gamma^/} \in M \Rightarrow M \in \tau$. Thus M is a τ-open set such that $x_{\alpha,\beta,\gamma} \in M \subseteq N$. Therefore N is a neighbourhood of $x_{\alpha,\beta,\gamma}$, i.e., $N \in \mathbf{N}^*(x_{\alpha,\beta,\gamma})$, i.e., $\mathbf{N}(x_{\alpha,\beta,\gamma}) \subseteq \mathbf{N}^*(x_{\alpha,\beta,\gamma})$. Conversely let $N \in \mathbf{N}^*(x_{\alpha,\beta,\gamma})$ so that N is a nhbd of $x_{\alpha,\beta,\gamma}$. Then there exists a τ-open set G such that $x_{\alpha,\beta,\gamma} \in G \subseteq N$. Now $G \in \tau \Rightarrow G \in \mathbf{N}(x_{\alpha,\beta,\gamma})$ for all $x_{\alpha,\beta,\gamma} \in G$. But $G \in \mathbf{N}(x_{\alpha,\beta,\gamma})$ and $G \subseteq N$ together imply by (N3) that $N \in \mathbf{N}(x_{\alpha,\beta,\gamma})$. Therefore $\mathbf{N}^*(x_{\alpha,\beta,\gamma}) \subseteq \mathbf{N}(x_{\alpha,\beta,\gamma})$.Therefore $\mathbf{N}(x_{\alpha,\beta,\gamma}) = \mathbf{N}^*(x_{\alpha,\beta,\gamma})$.

Next we show the uniqueness of the topology.

Le τ and $\tau^/$ be two topologies on X having the same system of neighbourhoods. Let $G \in \mathcal{N}(X)$. Then

$$G \in \tau.$$

$$\Leftrightarrow G \text{ is a } \tau - \text{open set.}$$

$$\Leftrightarrow G \text{ is a } \tau - \text{neighbourhood of } x_{\alpha,\beta,\gamma} \text{ for all NP } x_{\alpha,\beta,\gamma} \in G.$$

$$\Leftrightarrow G \text{ is a } \tau^/ - \text{neighbourhood of } x_{\alpha,\beta,\gamma} \text{ for all NP } x_{\alpha,\beta,\gamma} \in G.$$

$$\Leftrightarrow G \text{ is a } \tau^/ - \text{open set.}$$

$$\Leftrightarrow G \in \tau^/$$

Therefore $\tau = \tau^/$. Thus the topology is unique.

Hence proved.

4. **Conclusion**

Like fuzzy and intuitionistic fuzzy set theories, neutrosophic set theory also deals with imprecise situation. But neutrosophic theory also handles the situation of neutrality which keeps this theory ahead of those theories. In this article we tried to introduce the concept of neutrosophic point and neighbourhood of a neutrosophic point. We discussed some properties of neutrosophic points and their neighbourhoods. We also studied about the characterization of neutrosophic topological space in terms of the neighbourhoods of the neutrosophic points.

5. **Conflict of Interest**

We certify that there is no actual or potential conflict of interest in relation to this article.

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On Some Properties of Plithogenic Neutrosophic Hypersoft Almost Topological Group

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Abstract: The main objective of this study is to introduce the notion of plithogenic neutrosophic hypersoft almost topological group. We have defined some new concepts and investigated properties of regularly open set and regularly closed set and then we observed the definitions of plithogenic neutrosophic hypersoft closed mapping, open mapping and finally we have defined the definition of plithogenic neutrosophic hypersoft almost continuous mapping. By observing the definition of plithogenic neutrosophic hypersoft almost continuous mapping we have studied neutrosophic hypersoft topological group and plithogenic neutrosophic hypersoft almost topological group and some of their properties.

Keywords: Soft Set; Neutrosophic Hypersoft Set; Plithogenic Neutrosophic Hypersoft Set; Neutrosophic Hypersoft Topological Group; Plithogenic Neutrosophic Hypersoft Almost Topological Group.

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1. Introduction

In 1965, the fuzzy set (FS) theory concept was first defined by Zadeh [1]. With the help of FS, defined the concept of membership function and explained the idea of uncertainty. The concept of FS was generalized by Atanassov [2] and introduced the degree of non-membership as a component and proposed the intuitionistic fuzzy set (IFS). After that many researchers defined various new concepts on a generalization of FS. Smarandache [3] introduced neutrosophic set (NS) theory which are generalizations of IFS and FS and introduced the degree of indeterminacy as an independent component and discovered the neutrosophic set. Rana et. al. [16] discussed on plithogenic fuzzy whole hypersoft set of decision-making techniques.

The notion of soft set (SS) theory is one more fundamental set theory that was introduced by Molodtsov [3] in 1999. Now a day, SS theory is used in many branches of Science and Technology and SS has become one of the most popular branches in mathematics for its huge areas of applications in various research fields. Gradually, with the help of SS theory, many researchers have been introduced the notions of fuzzy SS [5], intuitionistic SS [6], neutrosophic SS [8] theory, etc. The concept of Hypersoft Set (HS) [14] theory was introduced by Smarandeche which is a generalization of SS theory. And also extended and introduced the concept of HS in the plithogenic environment and generalized it. Saqlain et. al [15] discussed the generalization of TOPSIS for Neutrosophic

Hypersoft set (NHS). Rahman et. al. [18] defined the development of Hybrids of HS with Complex FS, Complex IFS, and Complex NS, and also Rahman et. al. [19] discussed Convex and Concave HSs with their some properties. Saeed et. al [20] studied the fundamentals of HS theory and Abbas et. al. [21] discussed the basic operations on hypersoft sets and hypersoft points. Saqlain et. al. [22, 23] discussed aggregate operators of the neutrosophic hypersoft set and also single and multi-valued neutrosophic hypersoft set. Singh [24] worked on a plithogenic set (PS) for multi-variable data analysis and tried to develop new mathematical theories for precise representation through PS. Alkhazaleh [25] studied the concept of plithogenic soft set (PSS), also defined some properties of PSS. Zulqarnain et. al. [26] generalization of aggregated operators on NHS. Sankar et. al. [27] discussed Covid-19 by using PS. Khan et. al [28] studied the measures of linear and nonlinear interval-valued hexagonal fuzzy number. Haque et. al [29] discussed the multi-criteria group decision-making problems by exponential operational law in a generalised spherical fuzzy environment. Chakraborty et. al [30] studied the classification of trapezoidal bipolar neutrosophic numbers, de-bipolarization and implementation in cloud service based MCGDM problem. Zulqarnain et. al. [31] done work in solving decision-making problems using the TOPSIS method under an intuitionistic fuzzy hypersoft environment based on correlation coefficient and aggregation operators. Zulqarnain et. al. [32] discussed on operations of interval-valued NHS.

In this paper, we study the concept of the neutrosophic hypersoft topological group. Next, we introduce some definitions related to the neutrosophic hypersoft topological group and then we have introduced the definition of the Plithogenic Neutrosophic Hypersoft Almost Topological Group and discussed some related propositions.

2. Materials and Methods

2.1. Definition [3]

Let $\mathcal{U}$ be a universal set. A neutrosophic set (NS) A of $\mathcal{U}$ is denoted as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in \mathcal{U}\}$, where $T_A(x), I_A(x), F_A(x) : \mathcal{U} \longrightarrow [0,1]$ are the corresponding degree of truth, indeterminacy, and falsity of any $x \in \mathcal{U}$. Note that $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

2.2. Definition [7, 8]

The component of neutrosophic set A is denoted by A^C and is defined as

$$A^c(x) = \{\langle \mathrm{x}, T_{A^c}(x) = F_A(x), I_{A^c}(x) = 1 - I_A(x), F_{A^c}(x) = T_A(x)\rangle : x \in \mathcal{U}\}.$$

2.3. Definition [7, 8]

Let $\mathcal{U}$ be a non-empty set and $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in \mathcal{U}\}$, $B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle : x \in \mathcal{U}\}$, are neutrosophic sets. Then the neutrosophic set-theoretic operations are defined as follows:

(i) $A \cap B = \{\langle x, min(T_A(x), T_B(x)), min(I_A(x), I_B(x)), max(F_A(x), F_B(x))\rangle : x \in \mathcal{U}\}$

(ii) $A \cup B = \{\langle x, max(T_A(x), T_B(x)), max(I_A(x), I_B(x)), min(F_A(x), F_B(x))\rangle : x \in \mathcal{U}\}$

(iii) $A \le B$ if for each $x \in X$, $T_A(x) \le T_B(x)$, $I_A(x) \le I_B(x)$, $F_A(x) \ge F_B(x)$.

2.4. Definition [3]

Let $\mathcal{U}$ be a crisp group (CG) and A be an NS of $\mathcal{U}$. Then A is said to be a Neutrosophic Subgroup (NSG) of $\mathcal{U}$ if and only if the following conditions are satisfied:

(i) $A(xy) \ge \min\{A(x), A(y)\}$

$i.e., T_A(xy) \ge T_A(x) \cap T_A(y),\ I_A(xy) \ge I_A(x) \cap I_A(y),\ F_A(xy) \ge F_A(x) \cap F_A(y)$

(ii) $A(x^{-1}) \ge A(x)$

$i.e., T_A(x^{-1}) \ge T_A(x),\ I_A(x^{-1}) \ge I_A(x)$ and $F_A(x^{-1}) \le F_A(x)$.

2.5. Definition [7]

Suppose X be a non-empty set and a neutrosophic topology is a family τ_N of neutrosophic subsets of X satisfying the following axioms:

(i) $0_N, 1_N \in \tau_N$

(ii) $G_{N_1} \cap G_{N_2} \in \tau_N$ for any $G_{N_1}, G_{N_2} \in \tau_N$

(iii) $\cup G_{N_i} \in \tau_N$; $\forall \{G_{N_i}: i \in J\} \subseteq \tau_N$

In this case, the pair (X, τ_N) is said to be a neutrosophic topological space and any neutrosophic set in τ_N is called a neutrosophic open set. The element of τ_N are known as open neutrosophic sets, a neutrosophic set F is a neutrosophic closed set if and only if it F^C is a neutrosophic open set.

2.6. Definition [9]

Let X be a group and $\mathcal{G}$ be a neutrosophic group on X. Let $\tau^{\mathcal{G}}$ be a neutrosophic topology on $\mathcal{G}$ and then $(\mathcal{G}, \tau^{\mathcal{G}})$ is said to be a neutrosophic topological group if the following conditions are satisfied:

(1) The mapping $\psi: (\mathcal{G}, \tau^{\mathcal{G}}) \times (\mathcal{G}, \tau^{\mathcal{G}}) \to (\mathcal{G}, \tau^{\mathcal{G}})$ defined by $\psi(x, y) = xy$, for all $x, y \in X$, is relatively neutrosophic continuous.

(2) The mapping $\mu: (\mathcal{G}, \tau^{\mathcal{G}}) \to (\mathcal{G}, \tau^{\mathcal{G}})$ defined by $\mu(x) = x^{-1}$, for all $x \in X$, is relatively neutrosophic continuous.

2.7. Definition: [5]

Let $\mathcal{U}$ be a universal set (US), let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and E be the set of attributes values. Then the ordered pair of $(F, \mathcal{U})$ is said to be Soft Set (SS) over $\mathcal{U}$, where $F: E \to \mathcal{P}(\mathcal{U})$.

2.8. Definition: [4, 5]

Let $\mathcal{U}$ be a universal set (US) and $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$.

Let $a_1, a_2, a_3, \dots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attributes value are respectively the sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n$, with $\mathcal{A}_i \cap \mathcal{A}_j = \phi$, for $i \neq j$ and $i, j \in \{1, 2, 3, \dots, n\}$. Let $E_\alpha = \mathcal{A}_1 \times \mathcal{A}_2, \times \dots \times \mathcal{A}_n$. Then the ordered pair (F, E_α) is called a Hypersoft Set (HS) of $\mathcal{U}$, where $F: E_\alpha \to \mathcal{P}(\mathcal{U})$.

2.9. Definition [4, 6]

Let $\mathcal{U}$ be a universal set (US) and P$\subseteq \mathcal{U}$. A plithogenic set (PS) is denoted by $P_r = (P, \alpha, E_\alpha, p, q)$ where α be an attribute, E_α is the respective range of attributes values, $p: P \times E_\alpha \to [0,1]^r$ is the degree of appurtenance function (DAF) and $q: E_\alpha \times E_\alpha \to [0,1]^s$ is the corresponding degree of contradiction function (DCF), where $r, s \in \{1,2,3\}$.

2.10. Definition [3, 4]

Let $\mathcal{U}_\mathcal{N}$ be the US termed as a neutrosophic universal set if for all $x \in \mathcal{U}_\mathcal{N}$, x has truth belongingness, indeterminacy belongingness, and falsity belongingness to $\mathcal{U}_\mathcal{N}$, i.e., membership of x belonging to $[0,1] \times [0,1] \times [0,1]$.

2.11. Definition [3, 4]

Let $\mathcal{U}_P$ be plithogenic universal set over an attribute value set α is termed as the plithogenic US if for all $x \in \mathcal{U}_P$, x belongs to $\mathcal{U}_P$ with some degree on the basis of each attribute value. This degree can be crisp, fuzzy, intuitionistic fuzzy or neutrosophic.

2.12. Definition [4, 5]

Let $\mathcal{U}_\mathcal{N}$ be a neutrosophic universal set and $\alpha = \{a_1, a_2, a_3, \dots, a_n\}$ be a set of attributes with attribute value sets respectively as $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n$, with $\mathcal{A}_i \cap \mathcal{A}_j = \phi$, for $i \neq j$ and $i, j \in \{1, 2, 3, \dots, n\}$. Also, let $E_\alpha = \mathcal{A}_1 \times \mathcal{A}_2, \times \dots \times \mathcal{A}_n$. Then (F, E_α), where $F: E_\alpha \to \mathcal{P}(\mathcal{U}_\mathcal{N})$ is said to be a Neutrosophic Hypersoft Set (NHS) over $\mathcal{U}_\mathcal{N}$.

2.13. Definition [4]

Let $\mathcal{U}_P$ be a plithogenic universal set and $\alpha = \{a_1, a_2, a_3, \dots, a_n\}$ be a set of attributes with attribute value sets respectively as $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n$, with $\mathcal{A}_i \cap \mathcal{A}_j = \phi$, for $i \neq j$ and $i, j \in \{1, 2, 3, \dots, n\}$. Also, let $E_\alpha = \mathcal{A}_1 \times \mathcal{A}_2, \times \dots \times \mathcal{A}_n$. Then (F, E_α), where $F: E_\alpha \to \mathcal{P}(\mathcal{U}_P)$ is said to be a plithogenic Hypersoft Set (PHS) over $\mathcal{U}_P$.

2.14. Definition [4]

The ordered pair (F, E_α) is said to be a plithogenic neutrosophic Hypersoft Set (PNHS) if for all $B \in$ range (F) and for all $i \in \{1,2,\ldots,n\}$, there exists $f_{\mathcal{N}_i}: B \times R_i \rightarrow [0,1] \times [0,1] \times [0,1]$ such that for all $(b,r) \in B \times R_i$, $f_{\mathcal{N}_i}(b,r) \in [0,1] \times [0,1] \times [0,1]$.

A set of all the PNHSs over a set $\mathcal{U}$ is denoted by PNHS$(\mathcal{U})$.

2.15. Definition [4]

Let the ordered pair (F, E_α) be a plithogenic neutrosophic Hypersoft Set (PNHS) of a crisp group $\mathcal{U}$. where $E_\alpha = \mathcal{A}_1 \times \mathcal{A}_2, \times \ldots \times \mathcal{A}_n$ and for all $i \in \{1,2,\ldots,n\}$, R_i are crisp groups. Then (F, E_α) is said to be a plithogenic neutrosophic Hypersoft Subgroup (PNHSG) of $\mathcal{U}$ if and only if for all $B \in$ range (F); for all $(b_1,r_1),(b_2,r_2) \in B \times R_i$ and for all $f_{\mathcal{N}_i}: B \times R_i \rightarrow [0,1] \times [0,1] \times [0,1]$; with $f_{\mathcal{N}_i}(b,r) = \{< (b,R), f^T_{\mathcal{N}_i}(b,r), f^I_{\mathcal{N}_i}(b,r), f^F_{\mathcal{N}_i}(b,r) >: (b,r) \in B \times R_i\}$, the following subsequent conditions are satisfied:

(i) $f^T_{\mathcal{N}_i}\left((b_1,r_1).(b_2,r_2)^{-1}\right) \geq \min\{f^T_{\mathcal{N}_i}(b_1,r_1), f^T_{\mathcal{N}_i}(b_2,r_2)\}$

(ii) $f^T_{\mathcal{N}_i}(b_1,r_1)^{-1} \geq f^T_{\mathcal{N}_i}(b_1,r_1)$

(iii) $f^I_{\mathcal{N}_i}\left((b_1,r_1).(b_2,r_2)^{-1}\right) \geq \min\{f^I_{\mathcal{N}_i}(b_1,r_1), f^I_{\mathcal{N}_i}(b_2,r_2)\}$

(iv) $f^I_{\mathcal{N}_i}(b_1,r_1)^{-1} \geq f^I_{\mathcal{N}_i}(b_1,r_1)$

(v) $f^F_{\mathcal{N}_i}\left((b_1,r_1).(b_2,r_2)^{-1}\right) \leq \max\{f^F_{\mathcal{N}_i}(b_1,r_1), f^F_{\mathcal{N}_i}(b_2,r_2)\}$

(vi) $f^F_{\mathcal{N}_i}(b_1,r_1)^{-1} \leq f^F_{\mathcal{N}_i}(b_1,r_1)$.

A set of all the PNHSG of a crisp group $\mathcal{U}$ is denoted by PNHSG$(\mathcal{U})$.

3. Main Results

3.1. Definition

Let $NHS(\mathcal{U}_\mathcal{N}, E) = N$ be the family of all NHS over $\mathcal{U}_\mathcal{N}$ via attributes in E and $\tau_{\mathcal{U}_\mathcal{N}} \subseteq NHS(\mathcal{U}_\mathcal{N}, E)$. Then $\tau_{\mathcal{U}_\mathcal{N}}$ is said to be neutrosophic hypersoft topology (NHT) on N if the following conditions hold:

(i) $\phi_{\mathcal{U}_\mathcal{N}}, 1_{\mathcal{U}_\mathcal{N}} \in \tau_{\mathcal{U}_\mathcal{N}}$

(ii) The intersection of any finite number of members of $\tau_{\mathcal{U}_\mathcal{N}}$ also belongs to $\tau_{\mathcal{U}_\mathcal{N}}$.

(iii) The union of any collection of members of $\tau_{\mathcal{U}_\mathcal{N}}$ belongs to $\tau_{\mathcal{U}_\mathcal{N}}$.

Then $(N, \tau_{\mathcal{U}_\mathcal{N}})$ is said to be neutrosophic hypersoft topological space (NHTS). Every member of $\tau_{\mathcal{U}_\mathcal{N}}$ is called $\tau_{\mathcal{U}_\mathcal{N}}$- open neutrosophic hypersoft set. An NHS is called $\tau_{\mathcal{U}_\mathcal{N}}$- closed if and only if its complement is called $\tau_{\mathcal{U}_\mathcal{N}}$- open.

3.2. Definition

Let the pair $(F, E_\alpha) = H$ be a neutrosophic hypersoft group (NHG) of a crisp group (CG) $\mathcal{U}$. Let $\tau_{\mathcal{U}_G}$ be the neutrosophic hypersoft topology on H then $(H, \tau_{\mathcal{U}_G})$ is said to be neutrosophic hypersoft topological group (NHTG) if the following conditions are satisfied:

(1) The mapping $\psi: (H, \tau_{\mathcal{U}_G}) \times (H, \tau_{\mathcal{U}_G}) \rightarrow (H, \tau_{\mathcal{U}_G})$ such that $\psi(x,y) = xy$, for all $x, y \in H = (F, E_\alpha)$, is relatively neutrosophic hypersoft continuous.

(2) The mapping $\mu: (H, \tau_{\mathcal{U}_G}) \rightarrow (H, \tau_{\mathcal{U}_G})$ such that $\mu(x) = x^{-1}$, for all $x \in H = (F, E_\alpha)$, is relatively neutrosophic hypersoft continuous.

where $x = (b_1, r_1)$ and $y = (b_2, r_2)$. Then the pair $(H, \tau_{\mathcal{U}_G})$ is known as NHTG.

3.3. Definition

Let the pair $(F, E_\alpha) = H$ be an NHG of a crisp group (CG) $\mathcal{U}$. Let $\tau_{\mathcal{U}_G}$ be the neutrosophic hypersoft topology group on H. Then for fixed $\sigma = (a_1, a_2) \in H$, the left translation $l_\sigma: (H, \tau_{\mathcal{U}_G}) \to (H, \tau_{\mathcal{U}_G})$ is defined by $l_\sigma(x) = \sigma x,\ \forall\, x \in H$,

$$\sigma x = \{\langle \sigma, T_{\mathcal{U}_G}(\sigma x), I_{\mathcal{U}_G}(\sigma x), F_{\mathcal{U}_G}(\sigma x)\rangle : x \in H = (F, E_\alpha)\}.$$

Similarly, the right translation $r_\sigma: (H, \tau_{\mathcal{U}_G}) \to (H, \tau_{\mathcal{U}_G})$ is defined by $r_\sigma(x) = x\sigma\ \forall\, x \in H$,

$$x\sigma = \{\langle \sigma, T_{\mathcal{U}_G}(x\sigma), I_{\mathcal{U}_G}(x\sigma), F_{\mathcal{U}_G}(x\sigma)\rangle : x \in H = (F, E_\alpha)\}.$$

3.1. Lemma

Suppose $(F, E_\alpha) = H$ be an NHG of a crisp group (CG) $\mathcal{U}$. Let $\tau_{\mathcal{U}_G}$ be an NHTG in H. Then for each $\sigma = (a_1, a_2) \in \mathcal{G}_\mathrm{e}$, the translations l_σ and r_σ respectively neutrosophic hypersoft homomorphism of $(H, \tau_{\mathcal{U}_G})$ into itself.

Proof: From Proposition 3.11 [10], we have $l_\sigma[H] = \mathcal{G}$ and $r_\sigma[H] = H$, for all $\sigma \in \mathrm{H_e}$ and let $\pi: (H, \tau_{\mathcal{U}_G}) \to (H, \tau_{\mathcal{U}_G}) \times (H, \tau_{\mathcal{U}_G})$ defined by $\pi(x) = (\sigma, x)$ for each $x \in H$. Then $r_\sigma: \beta o \pi$. Since $\sigma \in H_\mathrm{e}$, $T_{\mathcal{U}_G}(\sigma) = T_{\mathcal{U}_G}(e)$, $I_{\mathcal{U}_G}(\sigma) = I_{\mathcal{U}_G}(e)$ and $F_{\mathcal{U}_G}(\sigma) = F_{\mathcal{U}_G}(e)$. Thus $T_{\mathcal{U}_G}(\sigma) \supseteq T_{\mathcal{U}_G}(x)$, $I_{\mathcal{U}_G}(\sigma) \supseteq I_{\mathcal{U}_G}(x)$ and $F_{\mathcal{U}_G}(\sigma) \subseteq F_{\mathcal{U}_G}(x)$, for each $x \in H$. It follows from Proposition 3.34 [11] that $\pi: (H, \tau_{\mathcal{U}_G}) \to (H, \tau_{\mathcal{U}_G}) \times (H, \tau_{\mathcal{U}_G})$ is relatively neutrosophic hypersoft continuous. By the hypothesis, β is relatively neutrosophic hypersoft continuous. So, r_σ is relatively neutrosophic hypersoft continuous. Moreover ${r_\sigma}^{-1} = r_{\sigma^{-1}}$. Similarly, we are shown the relatively neutrosophic hypersoft continuous of ${l_\sigma}^{-1} = l_{\sigma^{-1}}$.

3.4. Definition

Let $PNHS(\mathcal{U}_\mathcal{P}, E) = P$ be the family of all PNHS over $\mathcal{U}_\mathcal{P}$ via attributes in E and $\tau_{\mathcal{U}_\mathcal{P}} \subseteq PNHS(\mathcal{U}_\mathcal{P}, E)$. Then $\tau_{\mathcal{U}_\mathcal{P}}$ is said to be plithogenic neutrosophic hypersoft topology (PNHT) on P if the following conditions are satisfied:

(i) $\phi_{\mathcal{U}_\mathcal{P}}, 1_{\mathcal{U}_\mathcal{P}} \in \tau_{\mathcal{U}_\mathcal{P}}$

(ii) The intersection of any two neutrosophic hypersoft sets in $\tau_{\mathcal{U}_\mathcal{P}}$ belongs to $\tau_{\mathcal{U}_\mathcal{P}}$.

(iii) The union of neutrosophic hypersoft sets in $\tau_{\mathcal{U}_\mathcal{P}}$ belongs to $\tau_{\mathcal{U}_\mathcal{P}}$.

Then $(P, \tau_{\mathcal{U}_\mathcal{P}})$ is said to be plithogenic neutrosophic hypersoft topological space (PNHTS).

3.5. Definition

The complement $\mathcal{A}^c$ of a plithogenic neutrosophic hypersoft open set (PNHOS) in an NHTS $(P, \tau_{\mathcal{U}_\mathcal{P}})$ is said to be plithogenic neutrosophic hypersoft closed set (PNHCoS) in $(P, \tau_{\mathcal{U}_\mathcal{P}})$.

3.6. Definition

Let the pair $(F, E_\alpha) = M$ be a PNHS of a crisp group (CG) $\mathcal{U}$. Let $\tau_{\mathcal{U}_G}$ [from definition 2.15] be the plithogenic neutrosophic hypersoft topology on M then $(M, \tau_{\mathcal{U}_G})$ is said to be plithogenic neutrosophic hypersoft topological group (PNHTG) if the following conditions are satisfied:

(1) The mapping $\psi: (M, \tau_{\mathcal{U}_G}) \times (M, \tau_{\mathcal{U}_G}) \to (M, \tau_{\mathcal{U}_G})$ such that $\psi(x, y) = xy$, for all $x, y \in M = (F, E_\alpha)$, is relatively plithogenic neutrosophic hypersoft continuous.

(2) The mapping $\mu: (M, \tau_{\mathcal{U}_G}) \to (M, \tau_{\mathcal{U}_G})$ such that $\mu(x) = x^{-1}$, for all $x \in M = (F, E_\alpha)$, is relatively plithogenic neutrosophic hypersoft continuous.

where $x = (b_1, r_1)$ and $y = (b_2, r_2)$. Then the pair $(M, \tau_{\mathcal{U}_G})$ is called a PNHTG.

3.7. Definition

Let the pair (F, E_α) be a PNHS of a crisp group (CG) $\mathcal{U}$, where $E_\alpha = \mathcal{A}_1 \times \mathcal{A}_2, \times \ldots \times \mathcal{A}_n$ and $i = \{1, 2, \ldots, n\}$, $\mathcal{A}_i$ are crisp groups. Let U, V be two PNHS in (F, E_α). We define the product of UV PNHS U, V and V^{-1} of V as follows:

$$UV(z) = \{< z, T_{UV}(z), I_{UV}(z), F_{UV}(z) >: z = (b, r) \in (F, E_\alpha)\}$$

where

$$T_{UV}(z) = \sup\{\min\{T_U(x), T_V(y)\}\}$$
$$I_{UV}(z) = \sup\{\min\{I_U(x), I_V(y)\}\}$$

$F_{UV}(z) = \sup\{\min\{F_U(x), F_V(y)\}\}$

where $z = x.y$ and $x = (b_1, r_1)$; $y = (b_2, r_2)$ and for $V = \{< z, T_V(z), I_V(z), F_V(z) >: z = (b, r) \in (F, E_\alpha)\}$,

we have $V^{-1} = \{< z, T_V(z^{-1}), I_V(z^{-1}), F_V(z^{-1}) >: z = (b, r) \in (F, E_\alpha)\}$.

3.8. Definition

Let the ordered pair (F, E_α) be a plithogenic neutrosophic hypersoft set, where $E_\alpha = \mathcal{A}_1 \times \mathcal{A}_2, \times \ldots \times \mathcal{A}_n$. Let $(P, \tau_{\mathcal{U}_\mathcal{P}})$ be a PNHTS and $\mathcal{A} = \{\langle x, T_\mathcal{A}(x), I_\mathcal{A}(x), F_\mathcal{A}(x)\rangle: x \in (F, E_\alpha)\}$ be a PNHS in $(P, \tau_{\mathcal{U}_\mathcal{P}})$, then the plithogenic neutrosophic hypersoft interior of $\mathcal{A}$ is defined as

$$PNH - int(\mathcal{A}) = \cup\{G: G \text{ is an } PNHOS \text{ in } X \text{ and } G \subseteq \mathcal{A}\}.$$

3.9. Definition

Let the ordered pair (F, E_α) be a plithogenic neutrosophic hypersoft set, where $E_\alpha = \mathcal{A}_1 \times \mathcal{A}_2, \times \ldots \times \mathcal{A}_n$. Let $(P, \tau_{\mathcal{U}_\mathcal{P}})$ be a PNHTS and $\mathcal{A} = \{\langle x, T_\mathcal{A}(x), I_\mathcal{A}(x), F_\mathcal{A}(x)\rangle: x \in (F, E_\alpha)\}$, be a PNHS in $(P, \tau_{\mathcal{U}_\mathcal{P}})$, then the plithogenic neutrosophic hypersoft closure of $\mathcal{A}$ is defined as

$$PNH - cl(\mathcal{A}) = \cap\{K: K \text{ is an } PNHCoS \text{ in } X \text{ and } K \supseteq \mathcal{A}\}.$$

3.10. Definition

A mapping $\phi: (P, \tau_{\mathcal{U}_{\mathcal{P}1}}) \longrightarrow (K, \tau_{\mathcal{U}_{\mathcal{P}2}})$ is a plithogenic neutrosophic hypersoft continuous if the pre-image of each open plithogenic neutrosophic hypersoft set in $(K, \tau_{\mathcal{U}_{\mathcal{P}2}})$ is open plithogenic neutrosophic hypersoft set in $(P, \tau_{\mathcal{U}_{\mathcal{P}1}})$.

3.11. Definition

Let $\mathcal{A}$ be a PNHS of a PNHTS $(P, \tau_{\mathcal{U}_\mathcal{P}})$, then $\mathcal{A}$ is called a plithogenic neutrosophic hypersoft semi-open set (PNHSOS) of $(P, \tau_{\mathcal{U}_\mathcal{P}})$ if there exists a $\mathcal{B} \in \tau_{\mathcal{U}_\mathcal{P}}$ such that $\mathcal{A} \subseteq PNH - Cl(\mathcal{B})$.

3.12. Definition

Let $\mathcal{A}$ be a PNHS of a PNHTS $(P, \tau_{\mathcal{U}_\mathcal{P}})$, then $\mathcal{A}$ is called a plithogenic neutrosophic hypersoft semi-closed set (PNHSCoS) of $(P, \tau_{\mathcal{U}_\mathcal{P}})$ if there exists a $\mathcal{B}^c \in \tau_{\mathcal{U}_\mathcal{P}}$ such that $PNH - Int(\mathcal{B}) \subseteq \mathcal{A}$.

3.13. Definition

A PNHS $\mathcal{A}$ of a PNHTS $(P, \tau_{\mathcal{U}_\mathcal{P}})$ is said to be a plithogenic neutrosophic hypersoft regularly open set (PNHROS) of $(P, \tau_{\mathcal{U}_\mathcal{P}})$ if $PNH - int(PNH - cl(\mathcal{A})) = \mathcal{A}$.

3.14. Definition

A PNHS $\mathcal{A}$ of a PNHTS $(P, \tau_{\mathcal{U}_\mathcal{P}})$ is said to be a plithogenic neutrosophic hypersoft regularly closed set (PNHRCoS) of $(P, \tau_{\mathcal{U}_\mathcal{P}})$ if $PNH - cl(PNH - int(\mathcal{A})) = \mathcal{A}$.

3.1. Theorem: (i) The intersection of any two PNHROSs is a PNHROS, and
(ii) The union of any two PNHRCoSs is a PNHRCoS.

Proof:

(i) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be any two PNHROSs of a PNHTS $(P, \tau_{\mathcal{U}_\mathcal{P}})$. Since $\mathcal{A}_1 \cap \mathcal{A}_2$ is PNHOS, we have $\mathcal{A}_1 \cap \mathcal{A}_2 \subseteq PNH - int(PNH - cl(\mathcal{A}_1 \cap \mathcal{A}_2)$. Now, $PNH - int(PNH - cl(\mathcal{A}_1 \cap \mathcal{A}_2)) \subseteq PNH - int(PNH - cl(\mathcal{A}_1)) = \mathcal{A}_1$ and $PNH - int(PNH - cl(\mathcal{A}_1 \cap \mathcal{A}_2)) \subseteq PNH - int(PNH - cl(\mathcal{A}_2)) = \mathcal{A}_2$ implies that $PNH - int(PNH - cl(\mathcal{A}_1 \cap \mathcal{A}_2)) \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$. Hence the theorem.

(ii) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be any two PNHROSs of a PNHTS $(P, \tau_{\mathcal{U}_\mathcal{P}})$. Since $\mathcal{A}_1 \cup \mathcal{A}_2$ is PNHOS, we have $\mathcal{A}_1 \cup \mathcal{A}_2 \supseteq PNH - cl(PNH - int(\mathcal{A}_1 \cup \mathcal{A}_2))$. Now, $PNH - cl(PNH - int(\mathcal{A}_1 \cup \mathcal{A}_2)) \supseteq PNH - cl(PNH - int(\mathcal{A}_1)) = \mathcal{A}_1$ and $PNH - cl(PNH - int(\mathcal{A}_1 \cup \mathcal{A}_2)) \supseteq PNH - cl(PNH - int(\mathcal{A}_2)) = \mathcal{A}_2$ implies that $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq PNH - cl(PNH - int(\mathcal{A}_1 \cup \mathcal{A}_2))$. Hence the theorem.

3.15. Definition

Let $\phi: (P, \tau_{\mathcal{U}_{\mathcal{P}1}}) \longrightarrow (K, \tau_{\mathcal{U}_{\mathcal{P}2}})$ be a mapping from a PNHTS $(P, \tau_{\mathcal{U}_{\mathcal{P}1}})$ to another PNHTS $(K, \tau_{\mathcal{U}_{\mathcal{P}2}})$, then ϕ is called a Plithogenic neutrosophic hypersoft continuous mapping (PNHCM), if $\phi^{-1}(\mathcal{A}) \in \tau_{\mathcal{U}_{\mathcal{P}1}}$ for each $\mathcal{A} \in \tau_{\mathcal{U}_{\mathcal{P}2}}$; or equivalently $\phi^{-1}(\mathcal{B})$ is a PNHCoS of $(P, \tau_{\mathcal{U}_{\mathcal{P}1}})$ for each PNHCoS $\mathcal{B}$ of $(K, \tau_{\mathcal{U}_{\mathcal{P}2}})$.

3.16. Definition

Let $\phi: (P, \tau_{\mathcal{U}_{\mathcal{P}1}}) \longrightarrow (K, \tau_{\mathcal{U}_{\mathcal{P}2}})$ be a mapping from a PNHTS $(P, \tau_{\mathcal{U}_{\mathcal{P}1}})$ to another PNHTS $(K, \tau_{\mathcal{U}_{\mathcal{P}2}})$, then ϕ is called a plithogenic neutrosophic hypersoft open mapping (PNHOM), if $\phi(\mathcal{A}) \in \tau_{\mathcal{U}_{\mathcal{P}2}}$ for each $\mathcal{A} \in \tau_{\mathcal{U}_{\mathcal{P}1}}$.

3.17. Definition

Let $\phi: (P, \tau_{\mathcal{U}_{\mathcal{P}1}}) \longrightarrow (K, \tau_{\mathcal{U}_{\mathcal{P}2}})$ be a mapping from a PNHTS $(P, \tau_{\mathcal{U}_{\mathcal{P}1}})$ to another PNHTS $(K, \tau_{\mathcal{U}_{\mathcal{P}2}})$, then ϕ is called a plithogenic neutrosophic hypersoft closed mapping (PNHCoM) if $\phi(\mathcal{B})$ is a PNHCoS of $(K, \tau_{\mathcal{U}_{\mathcal{P}2}})$ for each PNHCoS $\mathcal{B}$ of $(P, \tau_{\mathcal{U}_{\mathcal{P}1}})$.

3.18. Definition

Let $\phi: (H, \tau_{\mathcal{U}_{\mathcal{P}1}}) \longrightarrow (K, \tau_{\mathcal{U}_{\mathcal{P}2}})$ be a mapping from a PNHTS $(H, \tau_{\mathcal{U}_{\mathcal{P}1}})$ to another PNHTS $(K, \tau_{\mathcal{U}_{\mathcal{P}2}})$, then ϕ is called a plithogenic neutrosophic hypersoft Semi-Continuous Mapping (PNHSCM), if $\phi^{-1}(\mathcal{A})$ is a plithogenic neutrosophic hypersoft semi-open set of $(H, \tau_{\mathcal{U}_{\mathcal{P}1}})$, for each $\mathcal{A} \in \tau_{\mathcal{U}_{\mathcal{P}2}}$.

3.19. Definition

Let $\phi: (P, \tau_{\mathcal{U}_{\mathcal{P}1}}) \longrightarrow (K, \tau_{\mathcal{U}_{\mathcal{P}2}})$ be a mapping from a PNHTS $(P, \tau_{\mathcal{U}_{\mathcal{P}1}})$ to another PNHTS $(K, \tau_{\mathcal{U}_{\mathcal{P}2}})$, then ϕ is called a Plithogenic neutrosophic hypersoft semi-open mapping (PNHSOM) if $\phi(\mathcal{A})$ is a PNHSOS for each $\mathcal{A} \in \tau_{\mathcal{U}_{\mathcal{P}1}}$.

3.20. Definition

Let $\phi: (P, \tau_{\mathcal{U}_{\mathcal{P}1}}) \longrightarrow (K, \tau_{\mathcal{U}_{\mathcal{P}2}})$ be a mapping from a PNHTS $(P, \tau_{\mathcal{U}_{\mathcal{P}1}})$ to another PNHTS $(K, \tau_{\mathcal{U}_{\mathcal{P}2}})$, then ϕ is called a Plithogenic neutrosophic hypersoft semi-closed mapping (PNHSCoM) if $\phi(\mathcal{B})$ is a PNHSCoS for each PNHCoS $\mathcal{B}$ of $(P, \tau_{\mathcal{U}_{\mathcal{P}1}})$.

3.21. Definition

A mapping $\phi: (M, \tau_{\mathcal{U}_{\mathcal{P}1}}) \longrightarrow (K, \tau_{\mathcal{U}_{\mathcal{P}2}})$ is said to be a plithogenic neutrosophic hypersoft almost continuous mapping (PNHACM), if $\phi^{-1}(\mathcal{A}) \in (M, \tau_{\mathcal{U}_{\mathcal{P}1}})$ for each plithogenic neutrosophic hypersoft regularly open set $\mathcal{A}$ of $(K, \tau_{\mathcal{U}_{\mathcal{P}2}})$.

3.22. Definition

Let the pair $(F, E_\alpha) = M$ be a PNHS of a crisp group (CG) $\mathcal{U}$. Let $\tau_{\mathcal{U}_G}$ [from definition 2.15] be the plithogenic neutrosophic hypersoft topology on M then $(M, \tau_{\mathcal{U}_G})$ is said to be plithogenic neutrosophic hypersoft almost topological group (PNHATG) if the following conditions are satisfied:

(1) The mapping $\psi: (M, \tau_{\mathcal{U}_G}) \times (M, \tau_{\mathcal{U}_G}) \to (M, \tau_{\mathcal{U}_G})$ such that $\psi(x, y) = xy$, for all $x, y \in M = (F, E_\alpha)$, is relatively plithogenic neutrosophic hypersoft almost continuous.

(2) The mapping $\mu: (M, \tau_{\mathcal{U}_G}) \to (M, \tau_{\mathcal{U}_G})$ such that $\mu(x) = x^{-1}$, for all $x \in M = (F, E_\alpha)$, is relatively plithogenic neutrosophic hypersoft almost continuous.

where $x = (b_1, r_1)$ and $y = (b_2, r_2)$. Then the pair $(M, \tau_{\mathcal{U}_G})$ is known as PNHATG.

3.2. Theorem:

Let $(M, \tau_{\mathcal{P}_G})$ be a PNHATG and let $\sigma = (a_1, a_2) \in M$ be any element. Then

(i) A mapping $g_\sigma: (M, \tau_{\mathcal{U}_G}) \longrightarrow (H, \tau_{\mathcal{U}_G})$ such that $g_\sigma(x) = \sigma x$, for all $x \in M$, is PNHACM;

(ii) A mapping $h_\sigma: (M, \tau_{\mathcal{U}_G}) \longrightarrow (M, \tau_{\mathcal{U}_G})$ such that $h_\sigma(x) = x\sigma$, for all $x \in M$, is PNHACM.

Proof:

(i) Let $\delta = (a_3, a_4) \in M$ and let W be a PNHROS containing $\sigma\delta$ in M. From Definition 3.22, ∃ plithogenic neutrosophic hypersoft open nbds $\mathcal{U}, \mathcal{V}$ of σ, δ in M so that $\mathcal{U}\mathcal{V} \subseteq W$. Especially, $\sigma\mathcal{V} \subseteq W$ that is $g_\sigma(\mathcal{V}) \subseteq W$. This shows that g_σ is PNHACM at δ and therefore g_σ is PNHACM.

(ii) Suppose $\delta = (a_3, a_4) \in M$ and $W \in PNHROS(M)$ containing $\delta\sigma$. Then ∃ PNHOSs $\delta \in \mathcal{U}$ and $\sigma \in \mathcal{V}$ in M so that $\mathcal{U}\mathcal{V} \subseteq W$. This shows $\mathcal{U}_\sigma \subseteq W$, i.e., $h_\sigma(\mathcal{U}) \subseteq W$. This implies h_σ is PNHACM at δ. As arbitrary element δ is in M, therefore h_σ is PNHACM.

3.3. Theorem:

Let U be PNHROS in a PNHATG $(M, \tau_{\mathcal{U}_G})$. Then the following conditions hold good, where $\sigma = (a_1, a_2)$

(1) $\sigma U \in PNHROS(M)$, for all $\sigma \in M$.

(2) $U\sigma \in PNHROS(M)$, for all $\sigma \in M$.

(3) $U^{-1} \in PNHROS(M)$.

Proof:

(1) First, we have to prove that $\sigma U \in \tau_{\mathcal{U}_G}$. Let $\delta = (a_3, a_4) \in \sigma U$. Then from Definition 3.22 of PNHATGs, ∃ PNHOSs $\sigma^{-1} \in W_1$ and $\delta \in W_2$in M so that $W_1 W_2 \subseteq U$. Especially, $\sigma^{-1}W_2 \subseteq U$. i.e., equivalently, $W_2 \subseteq \sigma U$. This shows that $\delta \in PNH - int(\sigma U)$ and thus, $PNH - int(\sigma U) = \sigma U$. i.e., $\sigma U \in \tau_{\mathcal{U}_G}$. Consequently, $\sigma U \subseteq PNH - int(PNH - cl(\sigma U))$.

Now, we have to prove that $PNH - int(PNH - cl(\sigma U)) \subseteq \sigma U$. Since U is PNHOS, $PNH - cl(U) \in PNHRCoS(M)$. From Theorem 3.2, $g_{\sigma^{-1}}: (M, \tau_{\mathcal{U}_G}) \longrightarrow (M, \tau_{\mathcal{U}_G})$ is PNHACM and therefore, $\sigma PNH - cl(U)$ is PNHCoS. Thus, $PNH - int\big(PNH - cl(\sigma U)\big) \subseteq PNH - cl(\sigma U) \subseteq \sigma PNH - cl(U)$. i.e., $\sigma^{-1}PNH - int(PNH - cl(\sigma U)) \subseteq PNH - cl(U)$. Since $PNH - int(PNH - cl(\sigma U))$ is PNHROS, it follows that $\sigma^{-1}PNH - int\big(PNH - cl(\sigma U)\big) \subseteq PNH - int\big(PNH - cl(U)\big) = U$, i.e., $PNH - int(PNH - cl(\sigma U)) \subseteq \sigma U$. Thus $\sigma U = PNH - int(PNH - cl(\sigma U))$. This shows that $\sigma U \in PNHROS(M)$.

(2) Following Theorem 3.3 (1), the proof is straightforward.

(3) Let $x \in U^{-1}$, then ∃ PNHOS $\delta \in W$ in H so that $W^{-1} \subseteq U \Rightarrow W \subseteq U^{-1}$. Therefore U^{-1} has interior-point δ. Thus, U^{-1} is PNHOS. i.e., $U^{-1} \subseteq PNH - int\big(PNH - cl(U^{-1})\big)$. Now we have to prove that $PNH - int\big(PNH - cl(U^{-1})\big) \subseteq U^{-1}$. Since U is PNHOS, $PNH - cl(U)$ is PNHRCoS and hence $PNH - cl(U)^{-1}$ is PNHCoS in M. Therefore, $PNH - int\big(PNH - cl(U^{-1})\big) \subseteq PNH - cl(U^{-1}) \subseteq PNH - cl(U)^{-1} \Rightarrow PNH - int\big(PNH - cl(U^{-1})\big) \subseteq (PNH - cl(U))^{-1} \subseteq U^{-1}$. Thus, $U^{-1} = PNH - int\big(PNH - cl(U^{-1})\big)$. This shows that $U^{-1} \in PNHROS(H)$.

3.1. Corollary

Let Q be any PNHRCoS in a PNHATG in M. Then

(1) $\sigma Q \in PNHRCoS(M)$, for each $\sigma \in M$.

(2) $Q^{-1} \in PNHRCoS(M)$.

3.4. Theorem:

Let U be any PNHROS in a PNHATG M. Then

(1) $PNH - cl(U\sigma) = PNH - cl(U)\sigma$, for each $\sigma \in M$, where $\sigma = (a_1, a_2)$

(2) $PNH - cl(\sigma U) = \sigma PNH - cl(U)$, for each $\sigma \in M$.

(3) $PNH - cl(U^{-1}) = PNH - cl(U)^{-1}$.

Proof:

(1) Taking $\delta = (a_3, a_4) \in PNH - cl(U\sigma)$ and consider $q = \delta\sigma^{-1}$. Let $q \in W$ be PNHOS in M. Then ∃ PNHOSs $\sigma^{-1} \in V_1$ and $\delta \in V_2$ in M, so that $V_1V_2 \subseteq PNH - int(PNH - cl(W))$. By assumption, there is $g \in U\sigma \cap V_2 \Rightarrow g\sigma^{-1} \in U \cap V_1 V_2 \subseteq U \cap PNH - int(PNH - cl(W)) \Rightarrow U \cap PNH -$

$int(PNH-cl(W)) \neq \phi_{\mathcal{U}_{\mathcal{P}}} \Rightarrow U \cap (PNH-cl(W)) \neq \phi_{\mathcal{U}_{\mathcal{P}}}$. Since U is PNHOS, $U \cap W \neq \phi_{\mathcal{U}_{\mathcal{P}}}$. i.e., $x \in PNH-cl(U)\sigma$.

Conversely, let $q \in PNH-cl(U)\sigma$. Then $q = \delta g$ for some $\delta \in PNH-cl(U)$.

To prove $PNH-cl(U)a \subseteq PNH-cl(Ua)$.

Let $\delta g \in W$ be an PNHOS in M. Then $\exists$ PNHOSs $\sigma \in V_1$ in M and $\delta \in V_2$ in M so that $V_1V_2 \subseteq PNH-int(PNH-cl(W))$. Since $\delta \in PNH-cl(U)$, $U \cap V_2 \neq \phi_{\mathcal{U}_{\mathcal{P}}}$. There is $g \in U \cap V_2$. This gives $g\sigma \in (U\sigma) \cap PNH-int(PNH-cl(W)) \Rightarrow (U\sigma) \cap (PNH-cl(W)) \neq \phi_{\mathcal{U}_{\mathcal{P}}}$. From Theorem 3.2, $U\sigma$ is PNHOS and thus $(U\sigma) \cap W \neq \phi_{\mathcal{U}_{\mathcal{P}}}$, therefore $q \in PNH-cl(U\sigma)$.

Therefore $PNH-cl(U\sigma) = PNH-cl(U)\sigma$.

(2) Following Theorem 3.4 (1), prove is straightforward.

(3) Since $PNH-cl(U)$ is PNHRCoS, $PNH-cl(U)^{-1}$ is PNHCoS in M. So, $U^{-1} \subseteq PNH-cl(U)^{-1}$ this implies $PNH-cl(U^{-1}) \subseteq PNH-cl(U)^{-1}$. Next, let $q \in PNH-cl(U)^{-1}$. Then $q = \delta^{-1}$, for some $\delta \in PNH-cl(U)$. Let $q \in V$ be any PNHOS in M. Then $\exists$ PNHOS U in M so that $\delta \in U$ with $U^{-1} \subseteq PNH-int(PNH-cl(V))$. Also, there is $\sigma \in \mathcal{A} \cap U$ which implies $\sigma^{-1} \in \mathcal{A}^{-1} \cap PNH-int(PNH-cl(V))$. That is, $\mathcal{A}^{-1} \cap PNH-int(PNH-cl(V)) \neq \phi_{\mathcal{U}_{\mathcal{P}}} \Rightarrow U^{-1} \cap PNH-cl(V) \neq \phi_{\mathcal{U}_{\mathcal{P}}} \Rightarrow \mathcal{A}^{-1} \cap V \neq \phi_{\mathcal{U}_{\mathcal{P}}}$, since U^{-1} is PNHOS. Therefore, $q \in PNH-cl(U)^{-1}$. Hence $PNH-cl(U^{-1}) \subseteq PNH-cl(U)^{-1}$.

3.5. Theorem:

Let Q be PNHRCo subset in a PNHATG M. Then the following statements are satisfied:

(1) $PNH-int(\sigma Q) = \sigma PNH-int(Q)$, for all $\sigma \in M$, where $\sigma = (a_1, a_2)$

(2) $PNH-int(Q\sigma) = PNH-int(Q)\sigma$, for all $\sigma \in M$.

(3) $PNH-int(Q^{-1}) = PNH-int(Q)^{-1}$.

Proof:

(1) Since Q is PNHRCoS, $PNH-int(Q)$ is PNHROS in M. Consequently, $\sigma PNH-int(Q) \subseteq PNH-int(\sigma Q)$. Conversely, let q be an arbitrary element of $PNH-int(\sigma Q)$. Assume that $q = \sigma\delta$, for some $\delta = (a_3, a_4) \in Q$. By assumption, this shows σQ is PNHCoS and that is $PNH-int(\sigma Q)$ is PNHROS in M. Suppose $\sigma \in U$ and $\delta \in V$ be PNHOSs in M, so that $UV \subseteq PNH-int(\sigma Q)$. Then $\sigma V \subseteq \sigma Q$, which it follows that $\sigma V \subseteq \sigma PNH-int(Q)$. Thus, $PNH-int(\sigma Q) \subseteq \sigma PNH-int(Q)$. Hence the statement follows.

(2) Following Theorem 3.5 (1), prove is straightforward.

(3) Since $PNH-int(Q)$ is PNHROS, so $PNH-int(Q)^{-1}$ is PNHOS in M. Therefore, $Q^{-1} \subseteq PNH-int(Q)^{-1}$ implies that $PNH-int(Q^{-1}) \subseteq PNH-int(Q)^{-1}$. Next, let q be an arbitrary element of $PNH-int(Q)^{-1}$. Then $q = \delta^{-1}$, for some $\delta \in PNH-int(Q)$. Let $q \in V$ be PNHOS in M. Then $\exists$ PNHOS U is in M so that $\delta \in U$ with $U^{-1} \subseteq PNH-cl(PNH-int(V))$. Also, there is $g \in Q \cap U$ which implies $g^{-1} \in Q^{-1} \cap PNH-cl(PNH-int(V))$. That is $Q^{-1} \cap PNH-cl(PNH-int(V)) \neq \phi_{\mathcal{U}_{\mathcal{P}}} \Rightarrow Q^{-1} \cap PNH-int(V) \neq \phi_{\mathcal{U}_{\mathcal{P}}} \Rightarrow Q^{-1} \cap V \neq \phi_{\mathcal{U}_{\mathcal{P}}}$, since Q^{-1} is PNHCoS. Hence $PNH-int(Q^{-1}) = PNH-int(Q)^{-1}$.

3.6. Theorem:

Let $\mathcal{A}$ be any PNHSOS in a PNHATG M. Then

(1) $PNH-cl(\sigma\mathcal{A}) \subseteq \sigma PNH-cl(\mathcal{A})$, for all $\sigma \in M$, where $\sigma = (a_1, a_2)$

(2) $PNH-cl(\mathcal{A}\sigma) \subseteq PNH-cl(\mathcal{A})\sigma$, for all $\sigma \in M$.

(3) $PNH-cl(\mathcal{A}^{-1}) \subseteq PNH-cl(\mathcal{A})^{-1}$.

Proof:

(1) As $\mathcal{A}$ is PNHSOS, $PNH - cl(\mathcal{A})$ is PNHRCoS. From Theorem 3.2, $g_{\sigma^{-1}}: (M, \tau_{\mathcal{U}_G}) \longrightarrow (M, \tau_{\mathcal{U}_G})$ is PNHACM. So, $\sigma PNH - cl(\mathcal{A})$ is PNHCoS. Hence $PNH - cl(\sigma\mathcal{A}) \subseteq \sigma PNH - cl(\mathcal{A})$.

(2) As $\mathcal{A}$ is PNHSOS, $PNH - cl(\mathcal{A})$ is PNHRCoS. From Theorem 3.2, $h_{\sigma^{-1}}: (M, \tau_{\mathcal{U}_G}) \longrightarrow (M, \tau_{\mathcal{U}_G})$ is PNHACM. So, $PNH - cl(\mathcal{A})\sigma$ is PNHCoS. Thus, $PNH - cl(\mathcal{A}\sigma) \subseteq PNH - cl(\mathcal{A})\sigma$.

(3) Since $\mathcal{A}$ is PNHSOS, so, $PNH - cl(\mathcal{A})$ is PNHRCoS and hence $PNH - cl(\mathcal{A})^{-1}$ is PNHCoS. Consequently, $PNH - cl(\mathcal{A}) \subseteq PNH - cl(\mathcal{A})^{-1}$.

4. **Limitation:** Every Plithogenic Neutrosophic Topological Group is Plithogenic Neutrosophic Hypersoft Almost Topological Group but the converse is not true.

5. Conclusion

In this paper, we have studied the concept of the Plithogenic Neutrosophic Hypersoft Almost Topological Group (PNHATG). To study PNHATG we have introduced some definitions related to PNHATG such as regularly open set and regularly closed set and then we observed the definitions of plithogenic neutrosophic hypersoft closed mapping, open mapping and finally, we have defined the definition of plithogenic neutrosophic hypersoft almost continuous mapping and then we have defined PNHATG and proved some theorems on PNHATG. We hope our work will encourage the reader for future work. In the future, we try to extend our work to study closed subgroups of Plithogenic Interval-valued Neutrosophic Hypersoft Almost Topological Group.

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n-ary Fuzzy Hypersoft Expert Sets

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Abstract. In 2018, Smarandache introduced the concept of hypersoft set by replacing the approximate function of the Molodtsov's soft sets with the multi-argument approximate function. Moreover, the fuzzy hybrid model of hypersoft set was developed and thus the theory of fuzzy hypersoft set was initiated. This chapter is devoted to introduce the concept of n-ary fuzzy hypersoft set extending the fuzzy hypersoft set with multiple set of universes (or n-dimension universal sets), the concept of fuzzy hypersoft expert set that presents the opinions of all experts in one fuzzy hypersoft set model without any operations, and the concept of n-ary fuzzy hypersoft expert set that exhibits the opinions of all experts in one n-ary fuzzy hypersoft set model without any operations. Apparently, the n-ary fuzzy hypersoft expert sets include both n-ary fuzzy hypersoft sets and fuzzy hypersoft expert set. Some basic operations of each of these extended fuzzy hypersoft sets are derived and their structural properties are investigated. Finally, an application of ternary fuzzy hypersoft expert set (i.e., n=3) in real-life problem are given.

Keywords: Hypersoft set; fuzzy hypersoft set; n-ary fuzzy hypersoft set; fuzzy hypersoft expert set; n-ary fuzzy hypersoft expert set

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1. Introduction

Many fields deal with uncertain data that cannot be successfully modeled by ordinary mathematics. Fuzzy sets [40], intuitionistic fuzzy sets [10] and neutrosophic sets [38] are well-known and often useful approaches for describing uncertainty. Many generalized types of these uncertain sets were proposed (see [2, 5, 17–19, 33]), and are currently being studied on new extended types. In 1999, Molodtsov [28] developed soft sets as a new mathematical model for dealing with uncertainty-based parametric data. Moreover, many researchers studied basic operations of the soft sets [6, 11, 16, 20, 26]. In the last decade, it was discussed the extended types of soft sets such as fuzzy soft sets [12, 25], intuitionistic fuzzy soft sets [13], neutrosophic soft sets [23, 24] and N-soft sets [15]. In [14, 21, 22, 29–32], the theoretical aspects on these hybrid models of soft sets were studied. A soft set can be considered as is a subset of parameterized family of a universal set. Açkgöz and Taş [3] initiated the theory of binary soft set based on two universal sets and a parameter set and emphasized that it can be adapted for n-dimension universal sets. Alkhazaleh and Salleh [7] proposed the idea of soft expert set, an extension of soft set, containing more than one expert opinion. A few years later, they generalized the soft expert set to fuzzy soft expert set, and argued that these sets are more effective and useful than soft expert set [8]. The approximate function in the structure of a soft set is defined from a parameter set to the power set of a universal set. In 2018, Smarandache [37] proposed defining the approximate function of a soft set from the cartesian product of n different sets of parameters to the power set of a universal set. Thus, Smarandache [37] conceptualized hypersoft set as a generalization of soft set, and then presented fuzzy hypersoft set sets as a fuzzy hybrid model of hypersoft sets. Abbas et al. [1] presented some basic operations like complement, union, intersection, difference of (fuzzy) hypersoft sets. Saeed et al. [34] studied of the fundamentals of hypersoft set theory. UrRahman et al. [39] developed a conceptual framework of convexity and concavity on the hypersoft sets. In [27, 35, 36], the authors proposed the extensions of hypersoft sets to make them more functional in various directions. In recent years, the research on the hypersoft sets and extensions have been progressing actively and rapidly.

This chapter aims to propose new extensions of fuzzy hypersoft sets called n-ary fuzzy hypersoft set, fuzzy hypersoft expert set and n-ary fuzzy hypersoft expert set. Simply, n-ary fuzzy hypersoft set is a fuzzy hypersoft set over the multiple set of universes, fuzzy hypersoft expert set is a fuzzy hypersoft set containing the opinions of experts, and n-ary fuzzy hypersoft expert set is a fuzzy hypersoft set over the multiple set of universes and contains the opinions of experts. Moreover, it intends to present the operations of complement, intersection and union on the n-ary fuzzy hypersoft sets, fuzzy hypersoft expert sets and n-ary fuzzy hypersoft expert sets. Also, the solution of a problem under the ternary fuzzy hypersoft expert set environment from the real world scene is addressed. This chapter organized as follows: Section 2 presents some fundamental concepts of fuzzy sets, soft sets, hypersoft sets, and fuzzy hypersoft sets. Sections 3, 4 and 5 are devoted to the theories of n-ary fuzzy hypersoft sets, fuzzy hypersoft expert sets and n-ary fuzzy hypersoft expert sets, respectively. Section 6 presents an real-life application of n-ary fuzzy hypersoft expert sets. The last section is the conclusions.

2. **Preliminaries**

In this section, some basic notions related to the fuzzy sets, soft sets, binary soft sets, soft expert sets, hypersoft sets, fuzzy hypersoft sets and fuzzy hypersoft set operations are recalled.

2.1. *Fuzzy Sets*

Definition 2.1. ([40]) Let A be a nonempty finite set. A fuzzy set $\mathcal{F}$ in A is defined as

$$\mathcal{F} = \{ \,^{(\mu_{\mathcal{F}}(a))}a : a \in A\} \tag{1}$$

where $\mu_{\mathcal{F}} : A \rightarrow [0,1]$ is called a membership function for $\mathcal{F}$ and $\mu_{\mathcal{F}}(a)$ represents the membership degree of a in $\mathcal{F}$. The set of all fuzzy sets in A is dented by $\mathfrak{F}(A)$.

Example 2.2. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of houses. According to the membership "cheap", one can create the fuzzy set

$$\mathcal{F} = \{\ {}^{0.2}a_1,\ {}^{0.6}a_2,\ {}^{0}a_3,\ {}^{1}a_1,\ {}^{0.3}a_1\}.$$

Definition 2.3. ([40]) Let F be a fuzzy set in A.

(a): If $\mu_{\mathcal{F}}(a) = 0$ for all $a \in A$ then it is called null (empty) fuzzy set and denoted by $\widehat{\emptyset}$.

(b): If $\mu_{\mathcal{F}}(a) = 1$ for all $a \in A$ then it is called absolute (universal) fuzzy set and denoted by $\widehat{A}$.

Definition 2.4. ([40]) Let $\mathcal{F}$ and $\mathcal{G}$ be two fuzzy sets in A. Then, we have the following operational laws.

(a): $\mathcal{F}$ is a fuzzy subset of $\mathcal{G}$ if $\mu_{\mathcal{F}}(a) \leq \mu_{\mathcal{G}}(a)$ for all $a \in A$, and denoted by $\mathcal{F} \subseteq_f \mathcal{G}$.

(b): The fuzzy sets $\mathcal{F}$ and $\mathcal{G}$ are equal if $\mu_{\mathcal{F}}(a) = \mu_{\mathcal{G}}(a)$ for all $a \in A$, and denoted by $\mathcal{F} = \mathcal{G}$.

(c): The complement of $\mathcal{F}$ is denoted and defined by $\mathcal{F}^r$, where $\mu_{\mathcal{F}^r}(a) = 1 - \mu_{\mathcal{F}}(a)$ for all $a \in A$.

(d): The intersection $\mathcal{F}$ and $\mathcal{G}$ is denoted and defined $\mathcal{F} \cap_f \mathcal{G}$, where $\mu_{(\mathcal{F} \cap_f \mathcal{G})}(a) = \min\{\mu_{\mathcal{F}}(a), \mu_{\mathcal{G}}(a)\} = \mu_{\mathcal{F}}(a) \wedge \mu_{\mathcal{G}}(a)$ for all $a \in A$.

(e): The union $\mathcal{F}$ and $\mathcal{G}$ is denoted and defined $\mathcal{F} \cup_f \mathcal{G}$, where $\mu_{(\mathcal{F} \cup_f \mathcal{G})}(a) = \max\{\mu_{\mathcal{F}}(a), \mu_{\mathcal{G}}(a)\} = \mu_{\mathcal{F}}(a) \vee \mu_{\mathcal{G}}(a)$ for all $a \in A$.

2.2. *Soft Sets*

Let A be a universal set, and the power set of A is denoted by $P(A)$.

Definition 2.5. ([28]) Let X be a set of parameters and $Y \subseteq X$. A soft set $(\mathcal{S}, Y)$ over A is defined as

$$(\mathcal{S}, Y) = \{\ (x, \mathcal{S}(x)) : x \in Y \text{ and } \mathcal{S}(x) \in P(A)\} \tag{2}$$

where $\mathcal{S} : Y \rightarrow P(A)$.

Example 2.6. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of suite rooms. Also, $X = \{x_1 = \textit{cheap},\ x_2 = \textit{modern},\ x_3 = \textit{beautiful}\}$ is the set of parameters, which describe the attractiveness of the suite rooms, and $Y = X$. Then, one can create the soft set

$$(\mathcal{S}, Y) = \{(x_1, \{a_1, a_4, a_5\}), (x_2, \{a_4, a_5\}), (x_3, \{a_1, a_2, a_3, a_4\})\}.$$

Definition 2.7. ([3]) Let A_1 and A_2 be two universal sets such that $A_1 \cap A_2 = \emptyset$, and $P(A_1)$ $P(A_2)$ are power sets of A_1 and A_2, respectively. Also, let X be a set of parameters and $Y \subseteq X$. A binary soft set $(\mathcal{S}_2, Y)$ over $\mathfrak{A} = \{A_1, A_2\}$, is defined as

$$(\mathcal{S}_2, Y) = \{\ (x, \mathcal{S}_2(x)) : x \in Y \text{ and } \mathcal{S}_2(x) \in P(A_1) \times P(A_2)\} \tag{3}$$

Example 2.8. Let $A_1 = \{a_1^1, a_2^1, a_3^1, a_4^1, a_5^1\}$ and $A_2 = \{a_1^2, a_2^2, a_3^2, a_4^2\}$ be the sets of suite rooms and king rooms. Also, $X = \{x_1 = cheap,\ x_2 = modern,\ x_3 = beautiful\}$ is the set of parameters, which describe the attractiveness of the rooms, and $Y = X$. Then, one can create the binary soft set

$$(\mathcal{S}_2, Y) = \{(x_1, (\{\{a_1^1, a_4^1, a_5^1\}, \{a_3^2, a_4^2\}\})), (x_2, (\{a_4^1, a_5^1\}, \{a_1^2, a_2^2, a_4^2\})), (x_3, (\{a_1^1, a_2^1, a_3^1, a_4^1\}, \{a_4^2, a_5^2\}))\}.$$

Definition 2.9. ([7]) Let X be a set of parameters, $\mathcal{E}$ be a set of experts and $\mathcal{O}$ be a set of opinion. Also, let $\mathcal{P} = X \times \mathcal{E} \times \mathcal{O}$ and $\mathcal{Q} \subseteq \mathcal{P}$. A soft expert set $(\mathcal{S}, \mathcal{Q})$ over A is defined as

$$(\mathcal{S}, \mathcal{Q}) = \{\ ((x, e, o), \mathcal{S}((x, e, o))) : (x, e, o) \in \mathcal{Q} \subseteq X \times \mathcal{E} \times \mathcal{O} \text{ and } \mathcal{S}(x) \in P(A)\} \tag{4}$$

where $\mathcal{S} : \mathcal{Q} \to P(A)$.

Example 2.10. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of suite rooms. Also, $X = \{x_1 = cheap,\ x_2 = modern,\ x_3 = beautiful\}$ is the set of parameters, which describe the attractiveness of the suite rooms, and $\mathcal{E} = \{e_1, e_2\}$ is the set of experts and $\mathcal{O} = \{o_1 = agree(1)\ o_2 = disagree(1)\}$ is the set of opinions. For $Q = \{(x_1, e_1, 1), (x_1, e_2, 1), (x_1, e_2, 1), (x_2, e_1, 1), (x_1, e_1, 0), (x_2, e_2, 0)\} \subseteq X \times \mathcal{E} \times \mathcal{O}$, one can create the soft expert set

$$(\mathcal{S}, \mathcal{Q}) = \left\{ \begin{array}{l} ((x_1, e_1, 1), \{a_1, a_2\}), ((x_1, e_2, 1), \{a_4, a_5\}), \\ ((x_1, e_2, 1), \emptyset), ((x_2, e_1, 1), \{a_1, a_3, a_4, a_5\}), \\ ((x_1, e_1, 0), \{a_3, a_4, a_5\}), ((x_2, e_2, 0), \{a_1\}) \end{array} \right\}.$$

2.3. *Hypersoft Sets*

Throughout this chapter, $X_1, X_2, ..., X_m$ are the pairwise disjoint sets of parameters (i.e., $X_i \cap X_{i'} = \emptyset$ for each $i, i' \in I = \{1, 2, ..., m\}$ and $i \neq i'$), and $\mathbf{X} = \prod_{i \in I} X_i = X_1 \times X_2 \times ..., X_m$. Generally, the parameters are attributes, characteristics, properties of the objects.

Definition 2.11. ([37]) Let Y_i be the nonempty subset of X_i for each $i \in I = \{1, 2, ..., m\}$ and $\mathbf{Y} = \prod_{i \in I} Y_i = Y_1 \times Y_2 \times ..., Y_m$. Then, the pair $(\mathcal{H}, \mathbf{Y})$ is called a hypersoft set over A, where $\mathcal{H}$ is mapping given by

$$\mathcal{H} : \mathbf{Y} \to P(A) \tag{5}$$

Also, x^i is an element of Y_i and $(x^i)_{i \in I} = (x^1, x^2, ..., x^m)$ is an element of $\mathbf{Y} = Y_1 \times Y_2 \times ..., Y_m$.

Note 1. In this chapter, we use the notation $\mathbf{x^I} = (x^i)_{i \in I}$.

Example 2.12. Assume that a person wants to buy a car and, for this purpose, visits to a car showroom where cars of the same segment are exhibited. Let $A = \{a_1, a_2, a_3\}$ be a universe containing cars in the same segment. The characteristics or attributes of these cars must be analyzed so that a decision can be made. The pairwise disjoint sets of attributes (parameters) are X_1, X_2 and X_3 and describe image-prestige, performance and economy, respectively. These sets are $X_1 = \{x_1^1 = safe,\ x_2^1 = comfortable,\ x_3^1 = design - aesthetic\}$ $X_2 = \{x_1^2 = engine\ power,\ x_2^2 = torque\}$, and

$X_3 = \{x_1^3 = fuel\ consumption,\ x_2^3 = tax,\ x_3^3 = sale\ price\}$. He/she determines the attributes (parameters) to be used in evaluating the cars as $Y_1 = X_1$, $Y_2 = X_2$ and $Y_3 = \{x_1^3, x_3^3\} \subseteq X_3$ (i.e., $\mathbf{Y} = Y_1 \times Y_2 \times Y_3$). As a result of the evaluation, it is created the following hypersoft set.

$$(\mathcal{H}, \mathbf{Y}) = \left\{ \begin{array}{l} ((x_1^1, x_1^2, x_1^3), \{a_1, a_2\}), \\ ((x_1^1, x_1^2, x_3^3), \{a_3\}), \\ ((x_1^1, x_2^2, x_1^3), \{a_1, a_3\}), \\ ((x_1^1, x_2^2, x_3^3), \emptyset), \\ ((x_2^1, x_1^2, x_1^3), \{a_2, a_3\}), \\ ((x_2^1, x_1^2, x_3^3), \{a_1\}), \\ ((x_2^1, x_2^2, x_1^3), \{a_2\}), \\ ((x_2^1, x_2^2, x_3^3), \{a_3\}), \\ ((x_3^1, x_1^2, x_1^3), \emptyset), \\ ((x_3^1, x_1^2, x_3^3), A), \\ ((x_3^1, x_2^2, x_1^3), \{a_2\}), \\ ((x_3^1, x_2^2, x_3^3), \{a_1\}) \end{array} \right\}.$$

2.4. *Fuzzy Hypersoft Sets*

Definition 2.13. ([37]) Let Y_i be the nonempty subset of X_i for each $i \in I = \{1, 2, ..., m\}$ and $\mathbf{Y} = \prod_{i \in I} Y_i = Y_1 \times Y_2 \times ..., Y_m$. Also, let $\mathfrak{F}(A)$ be the set of all fuzzy sets in A. Then, the pair $(\widetilde{\mathcal{H}}, \mathbf{Y})$ is called a fuzzy hypersoft set over A, where $\widetilde{\mathcal{H}}$ is mapping given by

$$\widetilde{\mathcal{H}} : \mathbf{Y} \rightarrow \mathfrak{F}(A) \tag{5}$$

Note 2. The collection of all fuzzy hypersoft set over the universal set A for $\mathbf{X}$ is denoted by $\mathfrak{C}\langle A, \mathbf{X} \rangle$.

Example 2.14. Consider the problem in Example 2.12. As a result of the evaluation under the fuzzy environment, it is created the following fuzzy hypersoft set.

$$(\mathcal{H}, \mathbf{Y}) = \left\{ \begin{array}{l} ((x_1^1, x_1^2, x_1^3), \{^{(0.4)}a_1^1, ^{(0.3)}a_2^1, ^{(0.6)}a_3^1\}), \\ ((x_1^1, x_1^2, x_3^3), \{^{(0.5)}a_1^1, ^{(0.6)}a_2^1, ^{(0.6)}a_3^1\}), \\ ((x_1^1, x_2^2, x_1^3), \{^{(0.3)}a_1^1, ^{(0.3)}a_2^1, ^{(0.2)}a_3^1\}), \\ ((x_1^1, x_2^2, x_3^3), \{^{(0.1)}a_1^1, ^{(0.4)}a_2^1, ^{(0)}a_3^1\}), \\ ((x_2^1, x_1^2, x_1^3), \{^{(0)}a_1^1, ^{(0)}a_2^1, ^{(0)}a_3^1\}), \\ ((x_2^1, x_1^2, x_3^3), \{^{(1)}a_1^1, ^{(1)}a_2^1, ^{(1)}a_3^1\}), \\ ((x_2^1, x_2^2, x_1^3), \{^{(0.5)}a_1^1, ^{(0.7)}a_2^1, ^{(0.4)}a_3^1\}), \\ ((x_2^1, x_2^2, x_3^3), \{^{(0.2)}a_1^1, ^{(0.1)}a_2^1, ^{(0.6)}a_3^1\}), \\ ((x_3^1, x_1^2, x_1^3), \{^{(0.8)}a_1^1, ^{(0.6)}a_2^1, ^{(0.4)}a_3^1\}), \\ ((x_3^1, x_1^2, x_3^3), \{^{(0.2)}a_1^1, ^{(0.4)}a_2^1, ^{(0.6)}a_3^1\}), \\ ((x_3^1, x_2^2, x_1^3), \{^{(0.9)}a_1^1, ^{(0.3)}a_2^1, ^{(1)}a_3^1\}), \\ ((x_3^1, x_2^2, x_3^3), \{^{(0.4)}a_1^1, ^{(0.7)}a_2^1, ^{(0.5)}a_3^1\}) \end{array} \right\}.$$

Definition 2.15. ([1]) Let $(\widetilde{\mathcal{H}}, \mathbf{Y}) \in \mathfrak{C}\langle A, \mathbf{X}\rangle$.

(a): If $\widetilde{\mathcal{H}}(\mathbf{x^I}) = \widehat{\emptyset}$ for each $\mathbf{x^I} \in \mathbf{Y}$ then it is said to be a relative null fuzzy hypersoft set (with respect to $\mathbf{Y}$), denoted by $\widehat{\emptyset}_{\mathbf{Y}}$. If $\mathbf{Y} = \mathbf{X}$ then it is called a null fuzzy hypersoft set and denoted by $\widehat{\emptyset}_{\mathbf{X}}$.

(b): If $\widetilde{\mathcal{H}}(\mathbf{x^I}) = \widehat{A}$ for each $\mathbf{x^I} \in \mathbf{Y}$ then it is said to be a relative whole fuzzy hypersoft set (with respect to $\mathbf{Y}$), denoted by $\widehat{A}_{\mathbf{Y}}$. If $\mathbf{Y} = \mathbf{X}$ then it is called an absolute fuzzy hypersoft set and denoted by $\widehat{A}_{\mathbf{X}}$.

Note 3. $\mathbf{Y} \subseteq \mathbf{Z}$ (i.e., $(Y_1 \times Y_2 \times ... \times Y_m) \subseteq (Z_1 \times Z_2 \times ... \times Z_m)$) iff $Y_i \subseteq Z_i$ for all $i \in I$.

Definition 2.16. ([1]) Let $(\widetilde{\mathcal{H}}, \mathbf{Y}), (\widetilde{\mathcal{K}}, \mathbf{Z}) \in \mathfrak{C}\langle A, \mathbf{X}\rangle$.

(a): $(\widetilde{\mathcal{H}}, \mathbf{Y})$ is called a fuzzy hypersoft subset of $(\widetilde{\mathcal{K}}, \mathbf{Z})$, denoted by $(\widetilde{\mathcal{H}}, \mathbf{Y}) \sqsubseteq (\widetilde{\mathcal{K}}, \mathbf{Z})$, if $\mathbf{Y} \subseteq \mathbf{Z}$ and $\widetilde{\mathcal{H}}(\mathbf{x^I}) \subseteq_f \widetilde{\mathcal{K}}(\mathbf{x^I})$ for each $\mathbf{x^I} \in \mathbf{Y}$.

(b): The fuzzy hypersoft sets $(\widetilde{\mathcal{H}}, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}, \mathbf{Z})$ are called equal, denoted by $(\widetilde{\mathcal{H}}, \mathbf{Y}) = (\widetilde{\mathcal{K}}, \mathbf{Z})$, if $(\widetilde{\mathcal{H}}, \mathbf{Y}) \sqsubseteq (\widetilde{\mathcal{K}}, \mathbf{Z})$ and $(\widetilde{\mathcal{K}}, \mathbf{Z}) \sqsubseteq (\widetilde{\mathcal{H}}, \mathbf{Y})$.

Definition 2.17. ([1]) Let $(\widetilde{\mathcal{H}}, \mathbf{Y}) \in \mathfrak{C}\langle A, \mathbf{X}\rangle$. Then, the relative complement of fuzzy hypersoft set $(\widetilde{\mathcal{H}}, \mathbf{Y})$, denoted by $(\widetilde{\mathcal{H}}, \mathbf{Y})^r$, is defined as

$$(\widetilde{\mathcal{H}}, \mathbf{Y})^r = (\widetilde{\mathcal{H}}^r, \mathbf{Y}), \tag{5}$$

where $\widetilde{\mathcal{H}}^r(\mathbf{x^I})$ is the fuzzy complement of $\widetilde{\mathcal{H}}(\mathbf{x^I})$ for each $\mathbf{x^I} \in \mathbf{Y}$.

Note 4. It is clear that $\mathbf{T} = \mathbf{Y} \cap \mathbf{Z} = (Y_1 \times Y_2 \times ... \times Y_m) \cap (Z_1 \times Z_2 \times ... \times Z_m) = (Y_1 \cap Z_1) \times (Y_2 \cap Z_2) \times ... \times (Y_m \cap Z_m)$ and $\mathbf{T} = \mathbf{Y} \cup \mathbf{Z} = (Y_1 \times Y_2 \times ... \times Y_m) \cup (Z_1 \times Z_2 \times ... \times Z_m) = (Y_1 \cup Z_1) \times (Y_2 \cup Z_2) \times ... \times (Y_m \cup Z_m)$. If $Y_i \cap Z_i = \emptyset$ for some $i \in I$ then $\mathbf{T} = \mathbf{Y} \cap \mathbf{Z} = \emptyset$. From now on, we assume that $\mathbf{T} = \mathbf{Y} \cap \mathbf{Z} \neq \emptyset$. (Similarly, $\mathfrak{S} = \mathfrak{Q} \cap \mathfrak{R} \neq \emptyset$ in Sections 4 and 5).

Definition 2.18. ([1]) Let $(\widetilde{\mathcal{H}}, \mathbf{Y}), (\widetilde{\mathcal{K}}, \mathbf{Z}) \in \mathfrak{C}\langle A, \mathbf{X}\rangle$. Then, the restricted intersection of fuzzy hypersoft sets $(\widetilde{\mathcal{H}}, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}, \mathbf{Z})$ is denoted and defined by $(\widetilde{\mathcal{L}}, \mathbf{T}) = (\widetilde{\mathcal{H}}, \mathbf{Y}) \Cap (\widetilde{\mathcal{K}}, \mathbf{Z})$ where $\mathbf{T} = \mathbf{Y} \cap \mathbf{Z}$ and

$$\widetilde{\mathcal{L}}(\mathbf{x^I}) = \widetilde{\mathcal{H}}(\mathbf{x^I}) \cap_f \widetilde{\mathcal{K}}(\mathbf{x^I}) \tag{6}$$

for each $\mathbf{x^I} \in \mathbf{T}$.

Definition 2.19. ([1]) Let $(\widetilde{\mathcal{H}}, \mathbf{Y}), (\widetilde{\mathcal{K}}, \mathbf{Z}) \in \mathfrak{C}\langle A, \mathbf{X}\rangle$. Then, the extended intersection of fuzzy hypersoft sets $(\widetilde{\mathcal{H}}, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}, \mathbf{Z})$ is denoted and defined by $(\widetilde{\mathcal{L}}, \mathbf{T}) = (\widetilde{\mathcal{H}}, \mathbf{Y}) \sqcap (\widetilde{\mathcal{K}}, \mathbf{Z})$ where $\mathbf{T} = \mathbf{Y} \cup \mathbf{Z}$ and

$$\widetilde{\mathcal{L}}(\mathbf{x^I}) = \begin{cases} \widetilde{\mathcal{H}}(\mathbf{x^I}), & \text{if } \mathbf{x^I} \in \mathbf{Y}, \\ \widetilde{\mathcal{K}}(\mathbf{x^I}), & \text{if } \mathbf{x^I} \in \mathbf{Z}, \\ \widetilde{\mathcal{H}}(\mathbf{x^I}) \cap_f \widetilde{\mathcal{K}}(\mathbf{x^I}), & \text{if } \mathbf{x^I} \in \mathbf{Y} \cap \mathbf{Z}, \end{cases} \tag{7}$$

for each $\mathbf{x^I} \in \mathbf{T}$.

Definition 2.20. ([1]) Let $(\widetilde{\mathcal{H}}, \mathbf{Y}), (\widetilde{\mathcal{K}}, \mathbf{Z}) \in \mathfrak{C}\langle A, \mathbf{X}\rangle$. Then, the restricted union of fuzzy hypersoft sets $(\widetilde{\mathcal{H}}, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}, \mathbf{Z})$ is denoted and defined by $(\widetilde{\mathcal{L}}, \mathbf{T}) = (\widetilde{\mathcal{H}}, \mathbf{Y}) \Cup (\widetilde{\mathcal{K}}, \mathbf{Z})$ where $\mathbf{T} = \mathbf{Y} \cap \mathbf{Z}$ and

$$\widetilde{\mathcal{L}}(\mathbf{x^I}) = \widetilde{\mathcal{H}}(\mathbf{x^I}) \cup_f \widetilde{\mathcal{K}}(\mathbf{x^I}) \tag{8}$$

for each $\mathbf{x^I} \in \mathbf{T}$.

Definition 2.21. ([1]) Let $(\widetilde{\mathcal{H}}, \mathbf{Y}), (\widetilde{\mathcal{K}}, \mathbf{Z}) \in \mathfrak{C}\langle A, \mathbf{X}\rangle$. Then, the extended union of fuzzy hypersoft sets $(\widetilde{\mathcal{H}}, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}, \mathbf{Z})$ is denoted and defined by $(\widetilde{\mathcal{L}}, \mathbf{T}) = (\widetilde{\mathcal{H}}, \mathbf{Y}) \sqcup (\widetilde{\mathcal{K}}, \mathbf{Z})$ where $\mathbf{T} = \mathbf{X} \cup \mathbf{Y}$ and

$$\widetilde{\mathcal{L}}(\mathbf{x^I}) = \begin{cases} \widetilde{\mathcal{H}}(\mathbf{x^I}), & \text{if} \ \ \mathbf{x^I} \in \mathbf{Y}, \\ \widetilde{\mathcal{K}}(\mathbf{x^I}), & \text{if} \ \ \mathbf{x^I} \in \mathbf{Z}, \\ \widetilde{\mathcal{H}}(\mathbf{x^I}) \cup_f \widetilde{\mathcal{K}}(\mathbf{x^I}), & \text{if} \ \ \mathbf{x^I} \in \mathbf{Y} \cap \mathbf{Z}, \end{cases} \tag{9}$$

for each $\mathbf{x^I} \in \mathbf{T}$.

3. n-ary Fuzzy Hypersoft Sets

In this section, we introduce the notion of n-ary fuzzy hypersoft set and derive its fundamental operations.

Let $\{A_j : j \in J = \{1, 2, ..., n\}\}$ be a collection of universal sets such that $A_j \cap A_{j'} = \emptyset$ for each $j, j' \in J = \{1, 2, ..., n\}$ and $j \neq j'$. Also, let $\mathfrak{F}(\mathfrak{A}) = \prod\limits_{j \in J} \mathfrak{F}(A_j) = \mathfrak{F}(A_1) \times \mathfrak{F}(A_2) \times ... \times \mathfrak{F}(A_n)$, where $\mathfrak{F}(A_j)$ denotes the set of all fuzzy sets in A_j.

Definition 3.1. A pair $(\widetilde{\mathcal{H}}_n, \mathbf{Y})$ is said to be an n-ary fuzzy hypersoft set over $\mathfrak{A} = \{A_1, A_2, ..., A_n\}$, where $\widetilde{\mathcal{H}}_n$ is mapping given by

$$\widetilde{\mathcal{H}}_n : \mathbf{Y} \to \mathfrak{F}(\mathfrak{A}). \tag{10}$$

Simply, an n-ary fuzzy hypersoft set is described as the following:

$$\begin{aligned} (\widetilde{\mathcal{H}}_n, \mathbf{Y}) &= \{(\mathbf{x^I}, \widetilde{\mathcal{H}}_n(\mathbf{x^I})) : \mathbf{x^I} \in \mathbf{Y} \text{ and } \widetilde{\mathcal{H}}_n(\mathbf{x^I}) \in \mathfrak{A}\} \\ &= \left\{ \left(\mathbf{x^I}, \begin{pmatrix} \{^{(\mu_{\widetilde{\mathcal{H}}_n(\mathbf{x^I})}(a^1))}a^1 : a^1 \in A_1\}, \\ \{^{(\mu_{\widetilde{\mathcal{H}}_n(\mathbf{x^I})}(a^2))}a^2 : a^2 \in A_2\}, \\ . \\ . \\ . \\ \{^{(\mu_{\widetilde{\mathcal{H}}_n(\mathbf{x^I})}(a^n))}a^n : a^n \in A_n\} \end{pmatrix} \right) : \mathbf{x^I} \in \mathbf{Y} \right\}, \end{aligned}$$

where $\widetilde{\widetilde{\mathcal{H}}}_{n(j)}(\mathbf{x^I}) = \{^{(\mu_{\widetilde{\mathcal{H}}_n(\mathbf{x^I})}(a^j))}a^j : a^j \in A_j\}$ for $j = 1, 2, ..., n$ and it is termed to be an A_j-part of $\widetilde{\mathcal{H}}_n(\mathbf{x^I})$.

Especially, if $n = 2, 3, 4$ and 5 then it is called a binary fuzzy hypersoft set, ternary fuzzy hypersoft set, quaternary fuzzy hypersoft set, quinary fuzzy hypersoft set, respectively.

Note 5. The set of all n-ary fuzzy hypersoft sets over $\mathfrak{A} = \{A_1, A_2, ..., A_n\}$ for $\mathbf{X}$ is denoted by $\mathfrak{C}_{N_n}\langle \mathfrak{A}, \mathbf{X}\rangle$.

Example 3.2. We consider the problem in Examples 2.12 and 2.14. However, he/she aims to determine the optimal car in each segment by evaluating cars in different segments simultaneously. Assume that $A_1 = \{a_1^1, a_2^1, a_3^1\}$, $A_2 = \{a_1^2, a_2^2\}$ and $A_3 = \{a_1^3, a_2^3, a_3^3\}$ are sets of cars in the B-segment (Supermini family), C-segment (Small family) and D-segment(Large family), respectively. Considering the parameter subsets $Y_1 = X_1$, $Y_2 = X_2$ and $Y_3 = \{x_1^3, x_3^3\} \subseteq X_3$, he/she evaluates the cars in different segments, and thus constructs the following ternary fuzzy hypersoft set.

$$(\widetilde{\mathcal{H}}_3, \mathbf{Y}) = \left\{ \begin{array}{l}
((x_1^1, x_1^2, x_1^3), (\{^{(0.4)}a_1^1, ^{(0.3)}a_2^1, ^{(0.6)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.5)}a_1^3, ^{(0.5)}a_2^3, ^{(0.4)}a_3^3\})),\\
((x_1^1, x_1^2, x_3^3), (\{^{(0.5)}a_1^1, ^{(0.6)}a_2^1, ^{(0.6)}a_3^1\}, \{^{(0.7)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.3)}a_1^3, ^{(0.4)}a_2^3, ^{(0.5)}a_3^3\})),\\
((x_1^1, x_2^2, x_1^3), (\{^{(0.3)}a_1^1, ^{(0.3)}a_2^1, ^{(0.2)}a_3^1\}, \{^{(0.8)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.6)}a_1^3, ^{(0.3)}a_2^3, ^{(0.4)}a_3^3\})),\\
((x_1^1, x_2^2, x_3^3), (\{^{(0.1)}a_1^1, ^{(0.4)}a_2^1, ^{(0)}a_3^1\}, \{^{(0.2)}a_1^2, ^{(0.4)}a_2^2\}, \{^{(0.6)}a_1^3, ^{(0.4)}a_2^3, ^{(0.4)}a_3^3\})),\\
((x_2^1, x_1^2, x_1^3), (\{^{(0)}a_1^1, ^{(0)}a_2^1, ^{(0)}a_3^1\}, \{^{(1)}a_1^2, ^{(0)}a_2^2\}, \{^{(0.2)}a_1^3, ^{(0.6)}a_2^3, ^{(0.8)}a_3^3\})),\\
((x_2^1, x_1^2, x_3^3), (\{^{(1)}a_1^1, ^{(1)}a_2^1, ^{(1)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.5)}a_1^3, ^{(0.4)}a_2^3, ^{(0.3)}a_3^3\})),\\
((x_2^1, x_2^2, x_1^3), (\{^{(0.5)}a_1^1, ^{(0.7)}a_2^1, ^{(0.4)}a_3^1\}, \{^{(0.1)}a_1^2, ^{(0.5)}a_2^2\}, \{^{(0.2)}a_1^3, ^{(0.4)}a_2^3, ^{(0.6)}a_3^3\})),\\
((x_2^1, x_2^2, x_3^3), (\{^{(0.2)}a_1^1, ^{(0.1)}a_2^1, ^{(0.6)}a_3^1\}, \{^{(0.4)}a_1^2, ^{(0.6)}a_2^2\}, \{^{(0.8)}a_1^3, ^{(0.4)}a_2^3, ^{(0.1)}a_3^3\})),\\
((x_3^1, x_1^2, x_1^3), (\{^{(0.8)}a_1^1, ^{(0.6)}a_2^1, ^{(0.4)}a_3^1\}, \{^{(0.6)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0)}a_1^3, ^{(0.2)}a_2^3, ^{(0)}a_3^3\})),\\
((x_3^1, x_1^2, x_3^3), (\{^{(0.2)}a_1^1, ^{(0.4)}a_2^1, ^{(0.6)}a_3^1\}\{^{(0.7)}a_1^2, ^{(0.4)}a_2^2\}, \{^{(0.4)}a_1^3, ^{(0.6)}a_2^3, ^{(0.3)}a_3^3\})),\\
((x_3^1, x_2^2, x_1^3), (\{^{(0.9)}a_1^1, ^{(0.3)}a_2^1, ^{(1)}a_3^1\}, \{^{(0.2)}a_1^2, ^{(0.2)}a_2^2\}, \{^{(0.4)}a_1^3, ^{(0.2)}a_2^3, ^{(0.4)}a_3^3\})),\\
((x_3^1, x_2^2, x_3^3), (\{^{(0.4)}a_1^1, ^{(0.7)}a_2^1, ^{(0.5)}a_3^1\}, \{^{(0.6)}a_1^2, ^{(0.8)}a_2^2\}, \{^{(0.8)}a_1^3, ^{(0.8)}a_2^3, ^{(0.5)}a_3^3\}))
\end{array} \right\}.$$

Definition 3.3. Let $(\widetilde{\mathcal{H}}_n, \mathbf{Y}) \in \mathfrak{C}_{N_n}\langle \mathfrak{A}, \mathbf{X}\rangle$.

- **(a):** If $\widetilde{\mathcal{H}}_n(\mathbf{x^I}) = (\widehat{\emptyset}, \widehat{\emptyset}, ..., \widehat{\emptyset})$ (i.e., $\widetilde{\widetilde{\mathcal{H}}}_{n(j)}(\mathbf{x^I}) = \widehat{\emptyset} \ \ \forall j \in J$) for each $\mathbf{x^I} \in \mathbf{Y}$ then it is said to be a relative null n-ary fuzzy hypersoft set (with respect to $\mathbf{Y}$), denoted by $\widehat{\emptyset}_{\mathbf{Y}}^{N_n}$. If $\mathbf{Y} = \mathbf{X}$ then it is called a null n-ary fuzzy hypersoft set and denoted by $\widehat{\emptyset}_{\mathbf{X}}^{N_n}$.
- **(b):** If $\widetilde{\mathcal{H}}_n(\mathbf{x^I}) = (\widehat{A}_1, \widehat{A}_2, ..., \widehat{A}_n)$ (i.e., $\widetilde{\widetilde{\mathcal{H}}}_j(\mathbf{x^I}) = \widehat{A}_j \ \ \forall j \in J$) for each $\mathbf{x^I} \in \mathbf{Y}$ then it is called a relative whole n-ary fuzzy hypersoft set (with respect to $\mathbf{Y}$), denoted by $\widehat{\mathfrak{A}}_{\mathbf{Y}}^{N_n}$. If $\mathbf{Y} = \mathbf{X}$ then it is said to be an absolute n-ary fuzzy hypersoft set and denoted by $\widehat{\mathfrak{A}}_{\mathbf{X}}^{N_n}$.

Definition 3.4. Let $(\widetilde{\mathcal{H}}_n, \mathbf{Y}), (\widetilde{\mathcal{K}}_n, \mathbf{Z}) \in \mathfrak{C}_{N_n}\langle \mathfrak{A}, \mathbf{X}\rangle$.

- **(a):** $(\widetilde{\mathcal{H}}_n, \mathbf{Y})$ is termed a fuzzy hypersoft subset of $(\widetilde{\mathcal{K}}_n, \mathbf{Z})$, denoted by $(\widetilde{\mathcal{H}}_n, \mathbf{Y}) \sqsubseteq_{N_n} (\widetilde{\mathcal{K}}_n, \mathbf{Z})$, if $\mathbf{Y} \subseteq \mathbf{Z}$ and

$$\widetilde{\mathcal{H}}_{n(j)}(\mathbf{x^I}) \subseteq_f \widetilde{\mathcal{H}}_{n(j)}(\mathbf{x^I}) \ \ \forall j \in J \tag{11}$$

for each $\mathbf{x^I} \in \mathbf{Y}$.

(b): The n-ary fuzzy hypersoft sets $(\widetilde{\mathcal{H}}_n, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}_n, \mathbf{Z})$ are called equal, denoted by $(\widetilde{\mathcal{H}}_n, \mathbf{Y}) =_{N_n} (\widetilde{\mathcal{K}}_n, \mathbf{Z})$, if $(\widetilde{\mathcal{H}}_n, \mathbf{Y}) \sqsubseteq_{N_n} (\widetilde{\mathcal{K}}_n, \mathbf{Z})$ and $(\widetilde{\mathcal{K}}_n, \mathbf{Z}) \sqsubseteq_{N_n} (\widetilde{\mathcal{H}}_n, \mathbf{Y})$.

Example 3.5. Consider the ternary fuzzy hypersoft set $(\widetilde{\mathcal{H}}_3, \mathbf{Y})$ in Example 3.2. Also, we assume that the disjoint parameter subsets $Z_1 = \{x_1^1, x_3^1\} \subseteq X_1$, $Z_2 = X_2$, $Z_3 = \{x_1^3, x_3^3\} \subseteq X_3$ (i.e., $\mathbf{Z} = Z_1 \times Z_2 \times Z_3$) and

$$(\widetilde{\mathcal{K}}_3, \mathbf{Z}) = \left\{ \begin{array}{l} ((x_1^1, x_1^2, x_1^3), (\{^{(0.4)}a_1^1, ^{(0.1)}a_2^1, ^{(0.4)}a_3^1\}, \{^{(0.1)}a_1^2, ^{(0.6)}a_2^2\}, \{^{(0.5)}a_1^3, ^{(0.3)}a_2^3, ^{(0.1)}a_3^3\})), \\ ((x_1^1, x_1^2, x_3^3), (\{^{(0.2)}a_1^1, ^{(0)}a_2^1, ^{(0.1)}a_3^1\}, \{^{(0.1)}a_1^2, ^{(0.4)}a_2^2\}, \{^{(0.1)}a_1^3, ^{(0.1)}a_2^3, ^{(0.1)}a_3^3\})), \\ ((x_1^1, x_2^2, x_1^3), (\{^{(0.2)}a_1^1, ^{(0.3)}a_2^1, ^{(0.2)}a_3^1\}, \{^{(0.8)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.2)}a_1^3, ^{(0.2)}a_2^3, ^{(0.1)}a_3^3\})), \\ ((x_1^1, x_2^2, x_3^3), (\{^{(0.1)}a_1^1, ^{(0.3)}a_2^1, ^{(0)}a_3^1\}, \{^{(0.1)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0.2)}a_1^3, ^{(0.1)}a_2^3, ^{(0.1)}a_3^3\})), \\ ((x_3^1, x_1^2, x_1^3), (\{^{(0.4)}a_1^1, ^{(0.4)}a_2^1, ^{(0.4)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0)}a_1^3, ^{(0.2)}a_2^3, ^{(0)}a_3^3\})), \\ ((x_3^1, x_1^2, x_3^3), (\{^{(0.1)}a_1^1, ^{(0.1)}a_2^1, ^{(0.1)}a_3^1\}\{^{(0.3)}a_1^2, ^{(0.4)}a_2^2\}, \{^{(0.3)}a_1^3, ^{(0.2)}a_2^3, ^{(0.2)}a_3^3\})), \\ ((x_3^1, x_2^2, x_1^3), (\{^{(0.3)}a_1^1, ^{(0.3)}a_2^1, ^{(1)}a_3^1\}, \{^{(0.1)}a_1^2, ^{(0.1)}a_2^2\}, \{^{(0.3)}a_1^3, ^{(0.1)}a_2^3, ^{(0.1)}a_3^3\})), \\ ((x_3^1, x_2^2, x_3^3), (\{^{(0.2)}a_1^1, ^{(0.2)}a_2^1, ^{(0.2)}a_3^1\}, \{^{(0.6)}a_1^2, ^{(0.8)}a_2^2\}, \{^{(1)}a_1^3, ^{(1)}a_2^3, ^{(0.5)}a_3^3\})) \end{array} \right\}.$$

Then, we have $\mathbf{Z} \subseteq \mathbf{Y}$ (by considering Note 3) but $(\widetilde{\mathcal{K}}_3, \mathbf{Z})$ is not a ternary fuzzy hypersoft subset $(\widetilde{\mathcal{H}}_3, \mathbf{Y})$ since $\widetilde{\mathcal{K}}_{3(3)}((x_3^1, x_2^2, x_3^3)) \nsubseteq_f \widetilde{\mathcal{H}}_{3(3)}((x_3^1, x_2^2, x_3^3))$. If we take $\widetilde{\mathcal{K}}_{3(3)}((x_3^1, x_2^2, x_3^3)) \subseteq_f \{^{(0.8)}a_1^3, ^{(0.8)}a_2^3, ^{(0.5)}a_3^3\}$ then we can say that $(\widetilde{\mathcal{K}}_3, \mathbf{Z})$ is an n-ary fuzzy hypersoft subset $(\widetilde{\mathcal{H}}_3, \mathbf{Y})$ (i.e., $(\widetilde{\mathcal{H}}_3, \mathbf{Y}) \sqsubseteq_{N_3} (\widetilde{\mathcal{K}}_3, \mathbf{Z})$).

Definition 3.6. Let $(\widetilde{\mathcal{H}}_n, \mathbf{Y}) \in \mathfrak{C}_{N_n}\langle \mathfrak{A}, \mathbf{X} \rangle$. Then, the relative complement of n-ary fuzzy hypersoft set $(\widetilde{\mathcal{H}}_n, \mathbf{Y})$, denoted by $(\widetilde{\mathcal{H}}_n, \mathbf{Y})^{r_{N_n}}$, is defined as

$$(\widetilde{\mathcal{H}}_n, \mathbf{Y})^{r_{N_n}} = (\widetilde{\mathcal{H}}_n^r, \mathbf{Y}), \tag{12}$$

where $\widetilde{\mathcal{H}}_{n(j)}^r(\mathbf{x^I})$ is the fuzzy complement of $\widetilde{\mathcal{H}}_{n(j)}(\mathbf{x^I})$ $(\forall j \in J)$ for each $\mathbf{x^I} \in \mathbf{Y}$.

Example 3.7. The complement of the ternary fuzzy hypersoft set $(\widetilde{\mathcal{H}}_3, \mathbf{Y})$ in Example 3.2 is

$$(\widetilde{\mathcal{H}}_3, \mathbf{Y})^{r_{N_3}} = \left\{ \begin{array}{l} ((x_1^1, x_1^2, x_1^3), (\{^{(0.6)}a_1^1, ^{(0.7)}a_2^1, ^{(0.4)}a_3^1\}, \{^{(0.7)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0.5)}a_1^3, ^{(0.5)}a_2^3, ^{(0.6)}a_3^3\})), \\ ((x_1^1, x_1^2, x_3^3), (\{^{(0.5)}a_1^1, ^{(0.4)}a_2^1, ^{(0.4)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0.7)}a_1^3, ^{(0.6)}a_2^3, ^{(0.5)}a_3^3\})), \\ ((x_1^1, x_2^2, x_1^3), (\{^{(0.7)}a_1^1, ^{(0.7)}a_2^1, ^{(0.8)}a_3^1\}, \{^{(0.2)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0.4)}a_1^3, ^{(0.7)}a_2^3, ^{(0.6)}a_3^3\})), \\ ((x_1^1, x_2^2, x_3^3), (\{^{(0.9)}a_1^1, ^{(0.6)}a_2^1, ^{(1)}a_3^1\}, \{^{(0.8)}a_1^2, ^{(0.6)}a_2^2\}, \{^{(0.4)}a_1^3, ^{(0.6)}a_2^3, ^{(0.6)}a_3^3\})), \\ ((x_2^1, x_1^2, x_1^3), (\{^{(1)}a_1^1, ^{(1)}a_2^1, ^{(1)}a_3^1\}, \{^{(0)}a_1^2, ^{(1)}a_2^2\}, \{^{(0.8)}a_1^3, ^{(0.4)}a_2^3, ^{(0.2)}a_3^3\})), \\ ((x_2^1, x_1^2, x_3^3), (\{^{(0)}a_1^1, ^{(0)}a_2^1, ^{(0)}a_3^1\}, \{^{(0.7)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0.5)}a_1^3, ^{(0.6)}a_2^3, ^{(0.7)}a_3^3\})), \\ ((x_2^1, x_2^2, x_1^3), (\{^{(0.5)}a_1^1, ^{(0.3)}a_2^1, ^{(0.6)}a_3^1\}, \{^{(0.9)}a_1^2, ^{(0.5)}a_2^2\}, \{^{(0.8)}a_1^3, ^{(0.6)}a_2^3, ^{(0.4)}a_3^3\})), \\ ((x_2^1, x_2^2, x_3^3), (\{^{(0.8)}a_1^1, ^{(0.9)}a_2^1, ^{(0.4)}a_3^1\}, \{^{(0.6)}a_1^2, ^{(0.4)}a_2^2\}, \{^{(0.2)}a_1^3, ^{(0.6)}a_2^3, ^{(0.9)}a_3^3\})), \\ ((x_3^1, x_1^2, x_1^3), (\{^{(0.2)}a_1^1, ^{(0.4)}a_2^1, ^{(0.6)}a_3^1\}, \{^{(0.4)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(1)}a_1^3, ^{(0.8)}a_2^3, ^{(1)}a_3^3\})), \\ ((x_3^1, x_1^2, x_3^3), (\{^{(0.8)}a_1^1, ^{(0.6)}a_2^1, ^{(0.4)}a_3^1\}\{^{(0.3)}a_1^2, ^{(0.6)}a_2^2\}, \{^{(0.6)}a_1^3, ^{(0.4)}a_2^3, ^{(0.7)}a_3^3\})), \\ ((x_3^1, x_2^2, x_1^3), (\{^{(0.1)}a_1^1, ^{(0.7)}a_2^1, ^{(0)}a_3^1\}, \{^{(0.8)}a_1^2, ^{(0.8)}a_2^2\}, \{^{(0.6)}a_1^3, ^{(0.8)}a_2^3, ^{(0.6)}a_3^3\})), \\ ((x_3^1, x_2^2, x_3^3), (\{^{(0.6)}a_1^1, ^{(0.3)}a_2^1, ^{(0.5)}a_3^1\}, \{^{(0.4)}a_1^2, ^{(0.2)}a_2^2\}, \{^{(0.2)}a_1^3, ^{(0.2)}a_2^3, ^{(0.5)}a_3^3\})) \end{array} \right\}.$$

Proposition 3.8. *Let $(\widetilde{\mathcal{H}}_n, \mathbf{Y}) \in \mathfrak{C}_{N_n}\langle \mathfrak{A}, \mathbf{X} \rangle$. Then, we have the following.*

(i): $((\widetilde{\mathcal{H}}_n, \mathbf{Y})^{r_{N_n}})^{r_{N_n}} =_{N_n} (\widetilde{\mathcal{H}}_n, \mathbf{Y})$.
(ii): $(\widehat{\mathfrak{A}}^{N_n}_{\mathbf{Y}})^{r_{N_n}} =_{N_n} \widehat{\emptyset}^{N_n}_{\mathbf{Y}}$.
(iii): $(\widehat{\emptyset}^{N_n}_{\mathbf{Y}})^{r_{N_n}} =_{N_n} \widehat{\mathfrak{A}}^{N_n}_{\mathbf{Y}}$.

Proof. The proofs are straightforward. □

Definition 3.9. Let $(\widetilde{\mathcal{H}}_n, \mathbf{Y}), (\widetilde{\mathcal{K}}_n, \mathbf{Z}) \in \mathfrak{C}_{N_n}\langle \mathfrak{A}, \mathbf{X}\rangle$. Then, the restricted intersection of n-ary fuzzy hypersoft sets $(\widetilde{\mathcal{H}}_n, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}_n, \mathbf{Z})$ is denoted and defined by $(\widetilde{\mathcal{L}}_n, \mathbf{T}) = (\widetilde{\mathcal{H}}_n, \mathbf{Y}) \Cap_{N_n} (\widetilde{\mathcal{K}}_n, \mathbf{Z})$ where $\mathbf{T} = \mathbf{Y} \cap \mathbf{Z}$ and

$$\widetilde{\mathcal{K}}_{n(j)}(\mathbf{x^I}) = \widetilde{\mathcal{H}}_{n(j)}(\mathbf{x^I}) \cap_f \widetilde{\mathcal{K}}_{n(j)}(\mathbf{x^I}) \quad \forall j \in J \tag{13}$$

for each $\mathbf{x^I} \in \mathbf{T}$.

Definition 3.10. Let $(\widetilde{\mathcal{H}}_n, \mathbf{Y}), (\widetilde{\mathcal{K}}_n, \mathbf{Z}) \in \mathfrak{C}_{N_n}\langle \mathfrak{A}, \mathbf{X}\rangle$. Then, the extended intersection of n-ary fuzzy hypersoft sets $(\widetilde{\mathcal{H}}_n, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}_n, \mathbf{Z})$ is denoted and defined by $(\widetilde{\mathcal{L}}_n, \mathbf{T}) = (\widetilde{\mathcal{H}}_n, \mathbf{Y}) \sqcap_{N_n} (\widetilde{\mathcal{K}}_n, \mathbf{Z})$ where $\mathbf{T} = \mathbf{Y} \cup \mathbf{Z}$ and

$$\widetilde{\mathcal{L}}_n(\mathbf{x^I}) = \begin{cases} \widetilde{\mathcal{H}}_n(\mathbf{x^I}), & \text{if } \ \mathbf{x^I} \in \mathbf{Y}, \\ \widetilde{\mathcal{K}}_n(\mathbf{x^I}), & \text{if } \ \mathbf{x^I} \in \mathbf{Z}, \\ \widetilde{\mathcal{H}}_n(\mathbf{x^I}) \cap_f \widetilde{\mathcal{K}}_n(\mathbf{x^I}), & \text{if } \ \mathbf{x^I} \in \mathbf{Y} \cap \mathbf{Z}, \end{cases} \tag{14}$$

for each $\mathbf{x^I} \in \mathbf{T}$, where Eq. (13) is applied to obtain $\widetilde{\mathcal{H}}_n(\mathbf{x^I}) \cap_f \widetilde{\mathcal{K}}_n(\mathbf{x^I})$.

Example 3.11. Consider the ternary fuzzy hypersoft set $(\widetilde{\mathcal{H}}_3, \mathbf{Y})$ in Example 3.2. Also, we suppose that the disjoint parameter subsets $Z_1 = \{x^1_2\} \subseteq X_1$, $Z_2 = X_2$, $Z_3 = \{x^3_2, x^3_3\} \subseteq X_3$ (i.e., $\mathbf{Z} = Z_1 \times Z_2 \times Z_3$) and

$$(\widetilde{\mathcal{K}}_3, \mathbf{Z}) = \left\{ \begin{array}{l} ((x^1_2, x^2_1, x^3_2), (\{^{(0.4)}a^1_1, ^{(0.6)}a^1_2, ^{(0.5)}a^1_3\}, \{^{(0.2)}a^2_1, ^{(0.6)}a^2_2\}, \{^{(0.4)}a^3_1, ^{(0.5)}a^3_2, ^{(0.4)}a^3_3\})), \\ ((x^1_2, x^2_1, x^3_3), (\{^{(0.8)}a^1_1, ^{(0.4)}a^1_2, ^{(0.6)}a^1_3\}, \{^{(0.3)}a^2_1, ^{(0.7)}a^2_2\}, \{^{(0.6)}a^3_1, ^{(0.3)}a^3_2, ^{(0.5)}a^3_3\})), \\ ((x^1_2, x^2_2, x^3_2), (\{^{(0.4)}a^1_1, ^{(0.4)}a^1_2, ^{(0.2)}a^1_3\}, \{^{(0.5)}a^2_1, ^{(0.5)}a^2_2\}, \{^{(0.2)}a^3_1, ^{(0.4)}a^3_2, ^{(0.7)}a^3_3\})), \\ ((x^1_2, x^2_2, x^3_3), (\{^{(0.5)}a^1_1, ^{(0.4)}a^1_2, ^{(0.2)}a^1_3\}, \{^{(0.7)}a^2_1, ^{(0.7)}a^2_2\}, \{^{(0.2)}a^3_1, ^{(0.2)}a^3_2, ^{(0.1)}a^3_3\})), \end{array} \right\}.$$

Then, the restricted intersection and extended intersection of the ternary fuzzy hypersoft sets $(\widetilde{\mathcal{H}}_3, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}_3, \mathbf{Z})$ are respectively

$$(\widetilde{\mathcal{H}}_3, \mathbf{Y}) \Cap_{N_3} (\widetilde{\mathcal{K}}_3, \mathbf{Z}) = \left\{ \begin{array}{l} ((x^1_2, x^2_1, x^3_3), (\{^{(0.8)}a^1_1, ^{(0.4)}a^1_2, ^{(0.6)}a^1_3\}, \{^{(0.3)}a^2_1, ^{(0.3)}a^2_2\}, \{^{(0.5)}a^3_1, ^{(0.3)}a^3_2, ^{(0.3)}a^3_3\})), \\ ((x^1_2, x^2_2, x^3_3), (\{^{(0.2)}a^1_1, ^{(0.1)}a^1_2, ^{(0.2)}a^1_3\}, \{^{(0.4)}a^2_1, ^{(0.6)}a^2_2\}, \{^{(0.2)}a^3_1, ^{(0.2)}a^3_2, ^{(0.1)}a^3_3\})) \end{array} \right\},$$

and

$$(\widetilde{\mathcal{H}}_3, \mathbf{Y}) \sqcap_{N_3} (\widetilde{\mathcal{K}}_3, \mathbf{Z}) = \left\{ \begin{array}{l} ((x_1^1, x_1^2, x_1^3), (\{^{(0.4)}a_1^1, ^{(0.3)}a_2^1, ^{(0.6)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.5)}a_1^3, ^{(0.5)}a_2^3, ^{(0.4)}a_3^3\})), \\ ((x_2^1, x_1^2, x_2^3), (\{^{(0.4)}a_1^1, ^{(0.6)}a_2^1, ^{(0.5)}a_3^1\}, \{^{(0.2)}a_1^2, ^{(0.6)}a_2^2\}, \{^{(0.4)}a_1^3, ^{(0.5)}a_2^3, ^{(0.4)}a_3^3\})), \\ ((x_2^1, x_1^2, x_3^3), (\{^{(0.8)}a_1^1, ^{(0.4)}a_2^1, ^{(0.6)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0.5)}a_1^3, ^{(0.3)}a_2^3, ^{(0.3)}a_3^3\})), \\ ((x_1^1, x_2^2, x_1^3), (\{^{(0.3)}a_1^1, ^{(0.3)}a_2^1, ^{(0.2)}a_3^1\}, \{^{(0.8)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.6)}a_1^3, ^{(0.3)}a_2^3, ^{(0.4)}a_3^3\})), \\ ((x_2^1, x_2^2, x_2^3), (\{^{(0.4)}a_1^1, ^{(0.4)}a_2^1, ^{(0.2)}a_3^1\}, \{^{(0.5)}a_1^2, ^{(0.5)}a_2^2\}, \{^{(0.2)}a_1^3, ^{(0.4)}a_2^3, ^{(0.7)}a_3^3\})), \\ ((x_2^1, x_2^2, x_3^3), (\{^{(0.2)}a_1^1, ^{(0.1)}a_2^1, ^{(0.2)}a_3^1\}, \{^{(0.4)}a_1^2, ^{(0.6)}a_2^2\}, \{^{(0.2)}a_1^3, ^{(0.2)}a_2^3, ^{(0.1)}a_3^3\})), \\ ((x_2^1, x_1^2, x_1^3), (\{^{(0)}a_1^1, ^{(0)}a_2^1, ^{(0)}a_3^1\}, \{^{(1)}a_1^2, ^{(0)}a_2^2\}, \{^{(0.2)}a_1^3, ^{(0.6)}a_2^3, ^{(0.8)}a_3^3\})), \\ ((x_2^1, x_1^2, x_3^3), (\{^{(1)}a_1^1, ^{(1)}a_2^1, ^{(1)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.5)}a_1^3, ^{(0.4)}a_2^3, ^{(0.3)}a_3^3\})), \\ ((x_2^1, x_2^2, x_1^3), (\{^{(0.5)}a_1^1, ^{(0.7)}a_2^1, ^{(0.4)}a_3^1\}, \{^{(0.1)}a_1^2, ^{(0.5)}a_2^2\}, \{^{(0.2)}a_1^3, ^{(0.4)}a_2^3, ^{(0.6)}a_3^3\})), \\ ((x_2^1, x_2^2, x_3^3), (\{^{(0.2)}a_1^1, ^{(0.1)}a_2^1, ^{(0.6)}a_3^1\}, \{^{(0.4)}a_1^2, ^{(0.6)}a_2^2\}, \{^{(0.8)}a_1^3, ^{(0.4)}a_2^3, ^{(0.1)}a_3^3\})), \\ ((x_3^1, x_1^2, x_1^3), (\{^{(0.8)}a_1^1, ^{(0.6)}a_2^1, ^{(0.4)}a_3^1\}, \{^{(0.6)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0)}a_1^3, ^{(0.2)}a_2^3, ^{(0)}a_3^3\})), \\ ((x_3^1, x_1^2, x_3^3), (\{^{(0.2)}a_1^1, ^{(0.4)}a_2^1, ^{(0.6)}a_3^1\}\{^{(0.7)}a_1^2, ^{(0.4)}a_2^2\}, \{^{(0.4)}a_1^3, ^{(0.6)}a_2^3, ^{(0.3)}a_3^3\})), \\ ((x_3^1, x_2^2, x_1^3), (\{^{(0.9)}a_1^1, ^{(0.3)}a_2^1, ^{(1)}a_3^1\}, \{^{(0.2)}a_1^2, ^{(0.2)}a_2^2\}, \{^{(0.4)}a_1^3, ^{(0.2)}a_2^3, ^{(0.4)}a_3^3\})), \\ ((x_3^1, x_2^2, x_3^3), (\{^{(0.4)}a_1^1, ^{(0.7)}a_2^1, ^{(0.5)}a_3^1\}, \{^{(0.6)}a_1^2, ^{(0.8)}a_2^2\}, \{^{(0.8)}a_1^3, ^{(0.8)}a_2^3, ^{(0.5)}a_3^3\})) \end{array} \right\}.$$

Definition 3.12. Let $(\widetilde{\mathcal{H}}_n, \mathbf{Y}), (\widetilde{\mathcal{K}}_n, \mathbf{Z}) \in \mathfrak{C}_{N_n}\langle \mathfrak{A}, \mathbf{X} \rangle$. Then, the restricted union of n-ary fuzzy hypersoft sets $(\widetilde{\mathcal{H}}_n, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}_n, \mathbf{Z})$ is denoted and defined by $(\widetilde{\mathcal{L}}_n, \mathbf{T}) = (\widetilde{\mathcal{H}}_n, \mathbf{Y}) \Cup_{N_n} (\widetilde{\mathcal{K}}_n, \mathbf{Z})$ where $\mathbf{T} = \mathbf{Y} \cap \mathbf{Z}$ and

$$\widetilde{\mathcal{K}}_{n(j)}(\mathbf{x^I}) = \widetilde{\mathcal{H}}_{n(j)}(\mathbf{x^I}) \cup_f \widetilde{\mathcal{K}}_{n(j)}(\mathbf{x^I}) \quad \forall j \in J \tag{15}$$

for each $\mathbf{x^I} \in \mathbf{T}$.

Definition 3.13. Let $(\widetilde{\mathcal{H}}_n, \mathbf{Y}), (\widetilde{\mathcal{K}}_n, \mathbf{Z}) \in \mathfrak{C}_{N_n}\langle \mathfrak{A}, \mathbf{X} \rangle$. Then, the extended union of n-ary fuzzy hypersoft sets $(\widetilde{\mathcal{H}}_n, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}_n, \mathbf{Z})$ is denoted and defined by $(\widetilde{\mathcal{L}}_n, \mathbf{T}) = (\widetilde{\mathcal{H}}_n, \mathbf{Y}) \sqcup_{N_n} (\widetilde{\mathcal{K}}_n, \mathbf{Z})$ where $\mathbf{T} = \mathbf{X} \cup \mathbf{Y}$ and

$$\widetilde{\mathcal{L}}_n(\mathbf{x^I}) = \begin{cases} \widetilde{\mathcal{H}}_n(\mathbf{x^I}), & \text{if} \quad \mathbf{x^I} \in \mathbf{Y}, \\ \widetilde{\mathcal{K}}_n(\mathbf{x^I}), & \text{if} \quad \mathbf{x^I} \in \mathbf{Z}, \\ \widetilde{\mathcal{H}}_n(\mathbf{x^I}) \cup_f \widetilde{\mathcal{K}}_n(\mathbf{x^I}), & \text{if} \quad \mathbf{x^I} \in \mathbf{Y} \cap \mathbf{Z}, \end{cases} \tag{16}$$

for each $\mathbf{x^I} \in \mathbf{T}$, where where Eq. (15) is applied to obtain $\widetilde{\mathcal{H}}_n(\mathbf{x^I}) \cup_f \widetilde{\mathcal{K}}_n(\mathbf{x^I})$.

Example 3.14. Consider the ternary fuzzy hypersoft sets $(\widetilde{\mathcal{H}}_3, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}_3, \mathbf{Z})$ in Examples 3.2 and 3.11. Then, the restricted union and extended union of the ternary fuzzy hypersoft sets $(\widetilde{\mathcal{H}}_3, \mathbf{Y})$ and $(\widetilde{\mathcal{K}}_3, \mathbf{Z})$ are respectively

$$(\widetilde{\mathcal{H}}_3, \mathbf{Y}) \Cup_{N_3} (\widetilde{\mathcal{K}}_3, \mathbf{Z}) = \left\{ \begin{array}{l} ((x_2^1, x_1^2, x_3^3), (\{^{(1)}a_1^1, ^{(1)}a_2^1, ^{(1)}a_3^1\}, \{^{(0.7)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.6)}a_1^3, ^{(0.4)}a_2^3, ^{(0.5)}a_3^3\})), \\ ((x_2^1, x_2^2, x_3^3), (\{^{(0.5)}a_1^1, ^{(0.4)}a_2^1, ^{(0.6)}a_3^1\}, \{^{(0.7)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.8)}a_1^3, ^{(0.4)}a_2^3, ^{(0.1)}a_3^3\})) \end{array} \right\},$$

and

$$(\widetilde{\mathcal{H}}_3,\mathbf{Y}) \sqcup_{N_3} (\widetilde{\mathcal{K}}_3,\mathbf{Z}) = \left\{ \begin{array}{l}
((x_1^1,x_1^2,x_1^3),(\{^{(0.4)}a_1^1,^{(0.3)}a_2^1,^{(0.6)}a_3^1\},\{^{(0.3)}a_1^2,^{(0.7)}a_2^2\},\{^{(0.5)}a_1^3,^{(0.5)}a_2^3,^{(0.4)}a_3^3\})),\\
((x_2^1,x_1^2,x_2^3),(\{^{(0.4)}a_1^1,^{(0.6)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.2)}a_1^2,^{(0.6)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.5)}a_2^3,^{(0.4)}a_3^3\})),\\
((x_2^1,x_1^2,x_3^3),(\{^{(1)}a_1^1,^{(1)}a_2^1,^{(1)}a_3^1\},\{^{(0.7)}a_1^2,^{(0.7)}a_2^2\},\{^{(0.6)}a_1^3,^{(0.4)}a_2^3,^{(0.5)}a_3^3\})),\\
((x_1^1,x_2^2,x_1^3),(\{^{(0.3)}a_1^1,^{(0.3)}a_2^1,^{(0.2)}a_3^1\},\{^{(0.8)}a_1^2,^{(0.7)}a_2^2\},\{^{(0.6)}a_1^3,^{(0.3)}a_2^3,^{(0.4)}a_3^3\})),\\
((x_2^1,x_2^2,x_2^3),(\{^{(0.4)}a_1^1,^{(0.4)}a_2^1,^{(0.2)}a_3^1\},\{^{(0.5)}a_1^2,^{(0.5)}a_2^2\},\{^{(0.2)}a_1^3,^{(0.4)}a_2^3,^{(0.7)}a_3^3\})),\\
((x_2^1,x_2^2,x_3^3),(\{^{(0.5)}a_1^1,^{(0.4)}a_2^1,^{(0.6)}a_3^1\},\{^{(0.7)}a_1^2,^{(0.7)}a_2^2\},\{^{(0.8)}a_1^3,^{(0.4)}a_2^3,^{(0.1)}a_3^3\})),\\
((x_2^1,x_1^2,x_1^3),(\{^{(0)}a_1^1,^{(0)}a_2^1,^{(0)}a_3^1\},\{^{(1)}a_1^2,^{(0)}a_2^2\},\{^{(0.2)}a_1^3,^{(0.6)}a_2^3,^{(0.8)}a_3^3\})),\\
((x_2^1,x_1^2,x_3^3),(\{^{(1)}a_1^1,^{(1)}a_2^1,^{(1)}a_3^1\},\{^{(0.3)}a_1^2,^{(0.7)}a_2^2\},\{^{(0.5)}a_1^3,^{(0.4)}a_2^3,^{(0.3)}a_3^3\})),\\
((x_2^1,x_2^2,x_1^3),(\{^{(0.5)}a_1^1,^{(0.7)}a_2^1,^{(0.4)}a_3^1\},\{^{(0.1)}a_1^2,^{(0.5)}a_2^2\},\{^{(0.2)}a_1^3,^{(0.4)}a_2^3,^{(0.6)}a_3^3\})),\\
((x_2^1,x_2^2,x_3^3),(\{^{(0.2)}a_1^1,^{(0.1)}a_2^1,^{(0.6)}a_3^1\},\{^{(0.4)}a_1^2,^{(0.6)}a_2^2\},\{^{(0.8)}a_1^3,^{(0.4)}a_2^3,^{(0.1)}a_3^3\})),\\
((x_3^1,x_1^2,x_1^3),(\{^{(0.8)}a_1^1,^{(0.6)}a_2^1,^{(0.4)}a_3^1\},\{^{(0.6)}a_1^2,^{(0.3)}a_2^2\},\{^{(0)}a_1^3,^{(0.2)}a_2^3,^{(0)}a_3^3\})),\\
((x_3^1,x_1^2,x_3^3),(\{^{(0.2)}a_1^1,^{(0.4)}a_2^1,^{(0.6)}a_3^1\}\{^{(0.7)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.6)}a_2^3,^{(0.3)}a_3^3\})),\\
((x_3^1,x_2^2,x_1^3),(\{^{(0.9)}a_1^1,^{(0.3)}a_2^1,^{(1)}a_3^1\},\{^{(0.2)}a_1^2,^{(0.2)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.2)}a_2^3,^{(0.4)}a_3^3\})),\\
((x_3^1,x_2^2,x_3^3),(\{^{(0.4)}a_1^1,^{(0.7)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.6)}a_1^2,^{(0.8)}a_2^2\},\{^{(0.8)}a_1^3,^{(0.8)}a_2^3,^{(0.5)}a_3^3\}))
\end{array} \right\}.$$

Proposition 3.15. *Let* $(\widetilde{\mathcal{H}}_n,\mathbf{Y}),(\widetilde{\mathcal{K}}_n,\mathbf{Z}),(\widetilde{\mathcal{L}}_n,\mathbf{T}) \in \mathfrak{C}_{N_n}\langle\mathfrak{A},\mathbf{X}\rangle$*. Then, we have the following equalities.*

- **(i):** $(\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond (\widetilde{\mathcal{K}}_n,\mathbf{Z}) =_{N_n} (\widetilde{\mathcal{K}}_n,\mathbf{Z}) \diamond (\widetilde{\mathcal{H}},\mathbf{Y})$ *for each* $\diamond \in \{\Cap_{N_n},\Cup_{N_n}\}$*.*
- **(ii):** $(\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond ((\widetilde{\mathcal{K}}_n,\mathbf{Z}) \diamond (\widetilde{\mathcal{L}}_n,\mathbf{T})) =_{N_n} ((\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond (\widetilde{\mathcal{K}}_n,\mathbf{Z})) \diamond (\widetilde{\mathcal{L}}_n,\mathbf{T})$ *for each* $\diamond \in \{\Cap_{N_n},\Cup_{N_n}\}$*.*
- **(iii):** $(\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond ((\widetilde{\mathcal{K}}_n,\mathbf{Z}) \circ (\widetilde{\mathcal{L}}_n,\mathbf{T})) =_{N_n} ((\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond (\widetilde{\mathcal{K}}_n,\mathbf{Z})) \circ ((\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond (\widetilde{\mathcal{L}}_n,\mathbf{T}))$ *for each* $\diamond,\circ \in \{\Cap_{N_n},\Cup_{N_n}\}$*.*
- **(iv):** $((\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond (\widetilde{\mathcal{K}}_n,\mathbf{Z}))^{r_{N_n}} =_{N_n} (\widetilde{\mathcal{H}}_n,\mathbf{Y})^{r_N} \circ (\widetilde{\mathcal{K}}_n,\mathbf{Z})^{r_{N_n}}$ *for each* $\diamond,\circ \in \{\Cap_{N_n},\Cup_{N_n}\}$ *and* $\diamond \neq \circ$*.*

Proof. The proofs are straightforward. □

Proposition 3.16. *Let* $(\widetilde{\mathcal{H}}_n,\mathbf{Y}),(\widetilde{\mathcal{K}}_n,\mathbf{Z}),(\widetilde{\mathcal{L}}_n,\mathbf{T}) \in \mathfrak{C}_{N_n}\langle\mathfrak{A},\mathbf{X}\rangle$*. Then, we have the following equalities.*

- **(i):** $(\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond (\widetilde{\mathcal{K}}_n,\mathbf{Z}) =_{N_n} (\widetilde{\mathcal{K}}_n,\mathbf{Z}) \diamond (\widetilde{\mathcal{H}},\mathbf{Y})$ *for each* $\diamond \in \{\sqcap_{N_n},\sqcup_{N_n}\}$*.*
- **(ii):** $(\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond ((\widetilde{\mathcal{K}}_n,\mathbf{Z}) \diamond (\widetilde{\mathcal{L}}_n,\mathbf{T})) =_{N_n} ((\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond (\widetilde{\mathcal{K}}_n,\mathbf{Z})) \diamond (\widetilde{\mathcal{L}}_n,\mathbf{T})$ *for each* $\diamond \in \{\sqcap_{N_n},\sqcup_{N_n}\}$*.*
- **(iii):** $(\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond ((\widetilde{\mathcal{K}}_n,\mathbf{Z}) \circ (\widetilde{\mathcal{L}}_n,\mathbf{T})) =_{N_n} ((\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond (\widetilde{\mathcal{K}}_n,\mathbf{Z})) \circ ((\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond (\widetilde{\mathcal{L}}_n,\mathbf{T}))$ *for each* $\diamond,\circ \in \{\sqcap_{N_n},\sqcup_{N_n}\}$*.*
- **(iv):** $((\widetilde{\mathcal{H}}_n,\mathbf{Y}) \diamond (\widetilde{\mathcal{K}}_n,\mathbf{Z}))^{r_{N_n}} =_{N_n} (\widetilde{\mathcal{H}}_n,\mathbf{Y})^{r_{N_n}} \circ (\widetilde{\mathcal{K}}_n,\mathbf{Z})^{r_{N_n}}$ *for each* $\diamond,\circ \in \{\sqcap_{N_n},\sqcup_{N_n}\}$ *and* $\diamond \neq \circ$*.*

Proof. The proofs are straightforward. □

4. **Fuzzy Hypersoft Expert Sets**

In this section, we define the concept of fuzzy hypersoft expert set and give its basic operations with the properties.

Throughout this section, A is a universal set, $X_1, X_2, ..., X_m$ are the pairwise disjoint sets of parameters (i.e., $X_i \cap X_{i'} = \emptyset$ for each $i, i' \in I = \{1, 2, ..., m\}$ and $i \neq i'$), and $\mathbf{X} = \prod_{i \in I} X_i = X_1 \times X_2 \times ... \times X_m$. Also, $\mathcal{E}$ is a set of experts, $\mathcal{O}$ is a set of opinions, $\mathfrak{P} = \mathbf{X} \times \mathcal{E} \times \mathcal{O}$ and $\mathfrak{Q} \subseteq \mathfrak{P}$.

Definition 4.1. A pair $(\widetilde{\mathcal{H}}, \mathfrak{Q})$ is said to be a fuzzy hypersoft expert set over A, where $\widetilde{\mathcal{H}}$ is mapping given by

$$\widetilde{\mathcal{H}} : \mathfrak{Q} \to \mathfrak{F}(A). \tag{17}$$

Note 6. In this chapter, we suppose two-valued opinions only in the set $\mathcal{O}$, i.e., $\mathcal{O} = \{o_1 = agree(1),\ o_2 = disagree(0)\}$. However, the multi-valued opinions may be supposed as well.

Note 7. The set of all fuzzy hypersoft expert set over the universal set A for $\mathfrak{P}$ is denoted by $\mathfrak{C}_E\langle A, \mathfrak{P}\rangle$.

Example 4.2. Consider the problem in Example 2.14. Assume that he/se seeks the opinions of 3 experts with the intention of determining the optimal car(s) to buy. The set of experts is $\mathcal{E} = \{e_1, e_2, e_3\}$ and the set of opinions is $\mathcal{O} = \{o_1 = agree(1),\ o_2 = disagree(0)\}$. For

$$\mathfrak{Q} = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 1), (((x_1^1, x_1^2, x_1^3), e_2, 1), (((x_1^1, x_1^2, x_1^3), e_3, 1), (((x_1^1, x_1^2, x_3^3), e_1, 1), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), (((x_1^1, x_1^2, x_3^3), e_3, 1), (((x_1^1, x_1^2, x_1^3), e_1, 0), (((x_1^1, x_1^2, x_1^3), e_2, 0), \\ (((x_1^1, x_1^2, x_1^3), e_3, 0), (((x_1^1, x_1^2, x_3^3), e_1, 0), (((x_1^1, x_1^2, x_3^3), e_2, 0), (((x_1^1, x_1^2, x_3^3), e_3, 0) \end{array} \right\} \subseteq \mathbf{X} \times \mathcal{E} \times \mathcal{O},$$

it is created the following fuzzy hypersoft expert set.

$$(\widetilde{\mathcal{H}}, \mathfrak{Q}) = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 1), \{ ^{(0.5)}a_1, ^{(0.6)} a_2, ^{(0.3)} a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_2, 1), \{ ^{(0.2)}a_1, ^{(0.4)} a_2, ^{(0.7)} a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_3, 1), \{ ^{(0.4)}a_1, ^{(0.3)} a_2, ^{(0.3)} a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 1), \{ ^{(0.7)}a_1, ^{(0.2)} a_2, ^{(0.4)} a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), \{ ^{(1)}a_1, ^{(0.2)} a_2, ^{(0.9)} a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_3, 1), \{ ^{(0.3)}a_1, ^{(0.3)} a_2, ^{(0.6)} a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_1, 0), \{ ^{(0.4)}a_1, ^{(0.2)} a_2, ^{(0.5)} a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_2, 0), \{ ^{(0.5)}a_1, ^{(0.6)} a_2, ^{(0.3)} a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_3, 0), \{ ^{(0.7)}a_1, ^{(0.7)} a_2, ^{(0.7)} a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 0), \{ ^{(0.1)}a_1, ^{(0.6)} a_2, ^{(0.5)} a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 0), \{ ^{(0.1)}a_1, ^{(0.5)} a_2, ^{(0.3)} a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_3, 0), \{ ^{(0.7)}a_1, ^{(0.6)} a_2, ^{(0.5)} a_3\}) \end{array} \right\}.$$

Definition 4.3. Let $(\widetilde{\mathcal{H}}, \mathfrak{Q}) \in \mathfrak{C}_E\langle A, \mathfrak{P}\rangle$. Then,

- **(a):** $(\widetilde{\mathcal{H}}, \mathfrak{Q})^1 = \{(q, \widetilde{\mathcal{H}}(q)) : q \in \mathbf{X} \times \mathcal{E} \times \{1\} \subseteq \mathfrak{Q}\}$ is termed to be an agree-hypersoft expert set over A.
- **(b):** $(\widetilde{\mathcal{H}}, \mathfrak{Q})^0 = \{(q, \widetilde{\mathcal{H}}(q)) : q \in \mathbf{X} \times \mathcal{E} \times \{0\} \subseteq \mathfrak{Q}\}$ is termed to be a disagree-hypersoft expert set over A.

From the definition, it is obvious that $(\widetilde{\mathcal{H}}, \mathfrak{Q})^1 \cup (\widetilde{\mathcal{H}}, \mathfrak{Q})^0 = (\widetilde{\mathcal{H}}, \mathfrak{Q})$.

Example 4.4. We consider the fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}, \mathfrak{Q})$ given in Example 4.2. Then, the agree-fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}, \mathfrak{Q})^1$ and the disagree-fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}, \mathfrak{Q})^0$ are respectively

$$(\widetilde{\mathcal{H}}, \mathfrak{Q})^1 = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 1), \{ {}^{(0.5)}a_1, {}^{(0.6)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_2, 1), \{ {}^{(0.2)}a_1, {}^{(0.4)}a_2, {}^{(0.7)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_3, 1), \{ {}^{(0.4)}a_1, {}^{(0.3)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 1), \{ {}^{(0.7)}a_1, {}^{(0.2)}a_2, {}^{(0.4)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), \{ {}^{(1)}a_1, {}^{(0.2)}a_2, {}^{(0.9)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_3, 1), \{ {}^{(0.3)}a_1, {}^{(0.3)}a_2, {}^{(0.6)}a_3\}) \end{array} \right\}.$$

and

$$(\widetilde{\mathcal{H}}, \mathfrak{Q})^0 = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 0), \{ {}^{(0.4)}a_1, {}^{(0.2)}a_2, {}^{(0.5)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_2, 0), \{ {}^{(0.5)}a_1, {}^{(0.6)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_3, 0), \{ {}^{(0.7)}a_1, {}^{(0.7)}a_2, {}^{(0.7)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 0), \{ {}^{(0.1)}a_1, {}^{(0.6)}a_2, {}^{(0.5)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 0), \{ {}^{(0.1)}a_1, {}^{(0.5)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_3, 0), \{ {}^{(0.7)}a_1, {}^{(0.6)}a_2, {}^{(0.5)}a_3\}) \end{array} \right\}.$$

Definition 4.5. Let $(\widetilde{\mathcal{H}}, \mathfrak{Q}) \in \mathfrak{C}_E\langle A, \mathfrak{P}\rangle$.

(a): If $\widetilde{\mathcal{H}}((\mathbf{x^I}, e, o)) = \widehat{\emptyset}$ for each $(\mathbf{x^I}, e, o) \in \mathfrak{Q}$ then it is called a relative null fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{\emptyset}^E_{\mathfrak{Q}}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is said to be a null fuzzy hypersoft expert set and denoted by $\widehat{\emptyset}^E_{\mathfrak{P}}$.

(b): If $\widetilde{\mathcal{H}}^1((\mathbf{x^I}, e, 1)) = \widehat{\emptyset}$ for each $(\mathbf{x^I}, e, 1) \in \mathfrak{Q}$ then it is termed to be a relative null agree-fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{\emptyset}^{E^1}_{\mathfrak{Q}}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is named a null agree-fuzzy hypersoft expert set and denoted by $\widehat{\emptyset}^{E^1}_{\mathfrak{P}}$.

(c): If $\widetilde{\mathcal{H}}^0((\mathbf{x^I}, e, 0)) = \widehat{\emptyset}$ for each $(\mathbf{x^I}, e, 0) \in \mathfrak{Q}$ then it is termed to be a relative null disagree-fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{\emptyset}^{E^0}_{\mathfrak{Q}}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is called a null disagree-fuzzy hypersoft expert set and denoted by $\widehat{\emptyset}^{E^0}_{\mathfrak{P}}$.

(d): If $\widetilde{\mathcal{H}}((\mathbf{x^I}, e, o)) = \widehat{A}$ for each $(\mathbf{x^I}, e, o) \in \mathfrak{Q}$ then it is said to be a relative whole fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{A}^E_{\mathfrak{Q}}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is named an absolute fuzzy hypersoft expert set and denoted by $\widehat{A}^E_{\mathfrak{P}}$.

(e): If $\widetilde{\mathcal{H}}^1((\mathbf{x^I}, e, 1)) = \widehat{A}$ for each $(\mathbf{x^I}, e, 1) \in \mathfrak{Q}$ then it is said to be a relative whole agree-fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{A}^{E^1}_{\mathfrak{Q}}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is named an absolute agree-fuzzy hypersoft expert set and denoted by $\widehat{A}^{E^1}_{\mathfrak{P}}$.

(f): If $\widetilde{\mathcal{H}}^0((\mathbf{x^I}, e, 0)) = \widehat{A}$ for each $(\mathbf{x^I}, e, 0) \in \mathfrak{Q}$ then it is called a relative whole disagree-fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{A}^{E^0}_{\mathfrak{Q}}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is called an absolute disagree-fuzzy hypersoft expert set and denoted by $\widehat{A}^{E^0}_{\mathfrak{P}}$.

Example 4.6. The following hypersoft expert sets are given as examples of relative null agree-fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$) and relative whole disagree-fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$) over A.

$$\widehat{\emptyset}_{\mathfrak{Q}}^{E^1} = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 1), \{ {}^{(0)}a_1, {}^{(0)}a_2, {}^{(0)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_2, 1), \{ {}^{(0)}a_1, {}^{(0)}a_2, {}^{(0)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_3, 1), \{ {}^{(0)}a_1, {}^{(0)}a_2, {}^{(0)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 1), \{ {}^{(0)}a_1, {}^{(0)}a_2, {}^{(0)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), \{ {}^{(0)}a_1, {}^{(0)}a_2, {}^{(0)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_3, 1), \{ {}^{(0)}a_1, {}^{(0)}a_2, {}^{(0)}a_3\}) \end{array} \right\},$$

and

$$\widehat{A}_{\mathfrak{Q}}^{E^0} = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 0), \{ {}^{(1)}a_1, {}^{(1)}a_2, {}^{(1)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_2, 0), \{ {}^{(1)}a_1, {}^{(1)}a_2, {}^{(1)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_3, 0), \{ {}^{(1)}a_1, {}^{(1)}a_2, {}^{(1)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 0), \{ {}^{(1)}a_1, {}^{(1)}a_2, {}^{(1)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 0), \{ {}^{(1)}a_1, {}^{(1)}a_2, {}^{(1)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_3, 0), \{ {}^{(1)}a_1, {}^{(1)}a_2, {}^{(1)}a_3\}) \end{array} \right\}.$$

Definition 4.7. Let $(\widetilde{\mathcal{H}}, \mathfrak{Q}), (\widetilde{\mathcal{K}}, \mathfrak{R}) \in \mathfrak{C}_E\langle A, \mathfrak{P}\rangle$.

- **(a):** $(\widetilde{\mathcal{H}}, \mathfrak{Q})$ is termed to be a fuzzy hypersoft expert subset of $(\widetilde{\mathcal{K}}, \mathfrak{R})$, denoted by $(\widetilde{\mathcal{H}}, \mathfrak{Q}) \sqsubseteq_E (\widetilde{\mathcal{K}}, \mathfrak{R})$, if $\mathfrak{Q} \subseteq \mathfrak{R}$ and $\widetilde{\mathcal{H}}((\mathbf{x}^{\mathbf{I}}, e, o)) \subseteq_f \widetilde{\mathcal{K}}((\mathbf{x}^{\mathbf{I}}, e, o))$ for each $(\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{Q}$.
- **(b):** The fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}, \mathfrak{Q})$ and $(\widetilde{\mathcal{K}}, \mathfrak{R})$ are named equal, denoted by $(\widetilde{\mathcal{H}}, \mathfrak{Q}) =_E (\widetilde{\mathcal{K}}, \mathfrak{R})$, if $(\widetilde{\mathcal{H}}, \mathfrak{Q}) \sqsubseteq_E (\widetilde{\mathcal{K}}, \mathfrak{R})$ and $(\widetilde{\mathcal{K}}, \mathfrak{R}) \sqsubseteq_E (\widetilde{\mathcal{H}}, \mathfrak{Q})$.

Example 4.8. Consider $(\widetilde{\mathcal{H}}, \mathfrak{Q})$ in Example 4.2 and the $(\widetilde{\mathcal{H}}, \mathfrak{Q})^1$ and $(\widetilde{\mathcal{H}}, \mathfrak{Q})^0$ Example 4.4. It is obvious that $(\widetilde{\mathcal{H}}, \mathfrak{Q})^1$ and $(\widetilde{\mathcal{H}}, \mathfrak{Q})^0$ are fuzzy hypersoft expert subsets of $(\mathcal{H}, \mathfrak{Q})$. Moreover, $\widehat{\emptyset}_{\mathfrak{Q}}^{E^1}$ is fuzzy hypersoft expert subset of $(\widetilde{\mathcal{H}}, \mathfrak{Q})$.

Definition 4.9. Let $(\widetilde{\mathcal{H}}, \mathfrak{Q}) \in \mathfrak{C}_E\langle A, \mathfrak{P}\rangle$. Then, the relative complement of fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}, \mathfrak{Q})$, denoted by $(\widetilde{\mathcal{H}}, \mathfrak{Q})^{r_E}$, is defined as

$$(\widetilde{\mathcal{H}}, \mathfrak{Q})^{r_E} = (\widetilde{\mathcal{H}}^r, \mathfrak{Q}), \tag{18}$$

where $\widetilde{\mathcal{H}}^r((\mathbf{x}^{\mathbf{I}}, e, o))$ is the fuzzy complement of $\widetilde{\mathcal{H}}((\mathbf{x}^{\mathbf{I}}, e, o))$ for each $(\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{Q}$.

Example 4.10. We consider the fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}, \mathfrak{Q})$ in Example 4.2. Then, the relative complement of fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}, \mathfrak{Q})$ is

$$(\widetilde{\mathcal{H}}, \mathfrak{Q})^{r_E} = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 1), \{ {}^{(0.5)}a_1, {}^{(0.4)}a_2, {}^{(0.7)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_2, 1), \{ {}^{(0.8)}a_1, {}^{(0.6)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_3, 1), \{ {}^{(0.6)}a_1, {}^{(0.7)}a_2, {}^{(0.7)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 1), \{ {}^{(0.3)}a_1, {}^{(0.8)}a_2, {}^{(0.6)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), \{ {}^{(0)}a_1, {}^{(0.8)}a_2, {}^{(0.1)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_3, 1), \{ {}^{(0.7)}a_1, {}^{(0.7)}a_2, {}^{(0.4)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_1, 0), \{ {}^{(0.6)}a_1, {}^{(0.8)}a_2, {}^{(0.5)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_2, 0), \{ {}^{(0.5)}a_1, {}^{(0.4)}a_2, {}^{(0.7)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_3, 0), \{ {}^{(0.3)}a_1, {}^{(0.3)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 0), \{ {}^{(0.9)}a_1, {}^{(0.4)}a_2, {}^{(0.5)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 0), \{ {}^{(0.9)}a_1, {}^{(0.5)}a_2, {}^{(0.7)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_3, 0), \{ {}^{(0.3)}a_1, {}^{(0.4)}a_2, {}^{(0.5)}a_3\}) \end{array} \right\}.$$

Proposition 4.11. *Let $(\widetilde{\mathcal{H}}, \mathfrak{Q}) \in \mathfrak{C}_E\langle A, \mathfrak{P}\rangle$. Then, we have the following.*

- **(i):** $((\widetilde{\mathcal{H}}, \mathfrak{Q})^{r_E})^{r_E} =_E (\widetilde{\mathcal{H}}, \mathfrak{Q})$.
- **(ii):** $(\widehat{A}^E_{\mathfrak{Q}})^{r_E} =_E (\widehat{\emptyset}^E_{\mathfrak{Q}})$.
- **(iii):** $(\widehat{\emptyset}^E_{\mathfrak{Q}})^{r_E} =_E (\widehat{A}^E_{\mathfrak{Q}})$.

Proof. The proofs are straightforward. □

Definition 4.12. Let $(\widetilde{\mathcal{H}}, \mathfrak{Q}), (\widetilde{\mathcal{K}}, \mathfrak{R}) \in \mathfrak{C}_E\langle A, \mathfrak{P}\rangle$. Then, the restricted intersection of fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}, \mathfrak{Q})$ and $(\widetilde{\mathcal{K}}, \mathfrak{R})$ is denoted and defined by $(\widetilde{\mathcal{L}}, \mathfrak{S}) = (\widetilde{\mathcal{H}}, \mathfrak{Q}) \Cap_E (\widetilde{\mathcal{K}}, \mathfrak{R})$ where $\mathfrak{S} = \mathfrak{Q} \cap \mathfrak{R}$ and

$$\widetilde{\mathcal{L}}((\mathbf{x}^{\mathbf{I}}, e, o)) = \widetilde{\mathcal{H}}((\mathbf{x}^{\mathbf{I}}, e, o)) \cap_f \widetilde{\mathcal{K}}((\mathbf{x}^{\mathbf{I}}, e, o)) \tag{19}$$

for each $(\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{S}$.

Definition 4.13. Let $(\widetilde{\mathcal{H}}, \mathfrak{Q}), (\widetilde{\mathcal{K}}, \mathfrak{R}) \in \mathfrak{C}_E\langle A, \mathfrak{P}\rangle$. Then, the extended intersection of fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}, \mathfrak{Q})$ and $(\widetilde{\mathcal{K}}, \mathfrak{R})$ is denoted and defined by $(\widetilde{\mathcal{L}}, \mathfrak{S}) = (\widetilde{\mathcal{H}}, \mathfrak{Q}) \sqcap_E (\widetilde{\mathcal{K}}, \mathfrak{R})$ where $\mathfrak{S} = \mathfrak{Q} \cup \mathfrak{R}$ and

$$\widetilde{\mathcal{L}}((\mathbf{x}^{\mathbf{I}}, e, o)) = \begin{cases} \widetilde{\mathcal{H}}((\mathbf{x}^{\mathbf{I}}, e, o)), & \text{if } (\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{Q}, \\ \widetilde{\mathcal{K}}((\mathbf{x}^{\mathbf{I}}, e, o)), & \text{if } (\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{R}, \\ \widetilde{\mathcal{H}}((\mathbf{x}^{\mathbf{I}}, e, o)) \cap_f \widetilde{\mathcal{K}}((\mathbf{x}^{\mathbf{I}}, e, o)), & \text{if } (\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{Q} \cap \mathfrak{R}, \end{cases} \tag{20}$$

for each $(\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{S}$.

Example 4.14. Consider the fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}, \mathfrak{Q})$ in Example 4.2. Also, we assume that

$$\mathfrak{R} = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_2^3), e_1, 1), (((x_1^1, x_1^2, x_2^3), e_3, 1), (((x_1^1, x_1^2, x_3^3), e_1, 1), (((x_1^1, x_1^2, x_3^3), e_2, 1), \\ (((x_1^1, x_1^2, x_2^3), e_1, 0), (((x_1^1, x_1^2, x_2^3), e_3, 0), (((x_1^1, x_1^2, x_3^3), e_1, 0), (((x_1^1, x_1^2, x_3^3), e_2, 0) \end{array} \right\} \subseteq \mathbf{X} \times \mathcal{E} \times \mathcal{O},$$

and

$$(\widetilde{\mathcal{K}}, \mathfrak{R}) = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_2^3), e_1, 1), \{ {}^{(0.4)}a_1, {}^{(0)}a_2, {}^{(0.7)}a_3\}), \\ (((x_1^1, x_1^2, x_2^3), e_3, 1), \{ {}^{(0.2)}a_1, {}^{(0.6)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 1), \{ {}^{(0.7)}a_1, {}^{(0.4)}a_2, {}^{(0.4)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), \{ {}^{(0.1)}a_1, {}^{(0.1)}a_2, {}^{(0.5)}a_3\}), \\ (((x_1^1, x_1^2, x_2^3), e_1, 0), \{ {}^{(0.5)}a_1, {}^{(0.6)}a_2, {}^{(0.5)}a_3\}), \\ (((x_1^1, x_1^2, x_2^3), e_3, 0), \{ {}^{(0.7)}a_1, {}^{(0.3)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 0), \{ {}^{(0.3)}a_1, {}^{(0)}a_2, {}^{(0.4)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 0), \{ {}^{(0.2)}a_1, {}^{(0.3)}a_2, {}^{(0.5)}a_3\}) \end{array} \right\}.$$

Then, the restricted intersection and extended intersection of fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}, \mathfrak{Q})$ and $(\widetilde{\mathcal{K}}, \mathfrak{R})$ are respectively

$$(\widetilde{\mathcal{H}}, \mathfrak{Q}) \Cap_E (\widetilde{\mathcal{K}}, \mathfrak{R}) = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_3^3), e_1, 1), \{ {}^{(0.7)}a_1, {}^{(0.2)}a_2, {}^{(0.4)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), \{ {}^{(0.1)}a_1, {}^{(0.1)}a_2, {}^{(0.5)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 0), \{ {}^{(0.1)}a_1, {}^{(0)}a_2, {}^{(0.4)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 0), \{ {}^{(0.1)}a_1, {}^{(0.3)}a_2, {}^{(0.3)}a_3\}) \end{array} \right\},$$

and

$$(\widetilde{\mathcal{H}}, \mathfrak{Q}) \sqcap_E (\widetilde{\mathcal{K}}, \mathfrak{R}) = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 1), \{ {}^{(0.5)}a_1, {}^{(0.6)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_2, 1), \{ {}^{(0.2)}a_1, {}^{(0.4)}a_2, {}^{(0.7)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_3, 1), \{ {}^{(0.4)}a_1, {}^{(0.3)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_2^3), e_1, 1), \{ {}^{(0.4)}a_1, {}^{(0)}a_2, {}^{(0.7)}a_3\}), \\ (((x_1^1, x_1^2, x_2^3), e_3, 1), \{ {}^{(0.2)}a_1, {}^{(0.6)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 1), \{ {}^{(0.7)}a_1, {}^{(0.2)}a_2, {}^{(0.4)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), \{ {}^{(0.1)}a_1, {}^{(0.1)}a_2, {}^{(0.5)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_3, 1), \{ {}^{(0.3)}a_1, {}^{(0.3)}a_2, {}^{(0.6)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_1, 0), \{ {}^{(0.4)}a_1, {}^{(0.2)}a_2, {}^{(0.5)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_2, 0), \{ {}^{(0.5)}a_1, {}^{(0.6)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_1^3), e_3, 0), \{ {}^{(0.7)}a_1, {}^{(0.7)}a_2, {}^{(0.7)}a_3\}), \\ (((x_1^1, x_1^2, x_2^3), e_1, 0), \{ {}^{(0.5)}a_1, {}^{(0.6)}a_2, {}^{(0.5)}a_3\}), \\ (((x_1^1, x_1^2, x_2^3), e_3, 0), \{ {}^{(0.7)}a_1, {}^{(0.3)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_1, 0), \{ {}^{(0.1)}a_1, {}^{(0)}a_2, {}^{(0.4)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_2, 0), \{ {}^{(0.1)}a_1, {}^{(0.3)}a_2, {}^{(0.3)}a_3\}), \\ (((x_1^1, x_1^2, x_3^3), e_3, 0), \{ {}^{(0.7)}a_1, {}^{(0.6)}a_2, {}^{(0.5)}a_3\}) \end{array} \right\}.$$

Definition 4.15. Let $(\widetilde{\mathcal{H}},\mathfrak{Q}),(\widetilde{\mathcal{K}},\mathfrak{R}) \in \mathfrak{C}_E\langle A,\mathfrak{P}\rangle$. Then, the restricted union of fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}},\mathfrak{Q})$ and $(\widetilde{\mathcal{K}},\mathfrak{R})$ is denoted and defined by $(\widetilde{\mathcal{L}},\mathfrak{S}) = (\widetilde{\mathcal{H}},\mathfrak{Q}) \Cup_E (\widetilde{\mathcal{K}},\mathfrak{R})$ where $\mathfrak{S} = \mathfrak{Q} \cap \mathfrak{R}$ and

$$\widetilde{\mathcal{L}}((\mathbf{x^I},e,o)) = \widetilde{\mathcal{H}}((\mathbf{x^I},e,o)) \cup_f \widetilde{\mathcal{K}}((\mathbf{x^I},e,o)) \tag{21}$$

for each $(\mathbf{x^I},e,o) \in \mathfrak{S}$.

Definition 4.16. Let $(\widetilde{\mathcal{H}},\mathfrak{Q}),(\widetilde{\mathcal{K}},\mathfrak{R}) \in \mathfrak{C}_E\langle A,\mathfrak{P}\rangle$. Then, the extended union of fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}},\mathfrak{Q})$ and $(\widetilde{\mathcal{K}},\mathfrak{R})$ is denoted and defined by $(\widetilde{\mathcal{L}},\mathfrak{S}) = (\widetilde{\mathcal{H}},\mathfrak{Q}) \sqcup_E (\widetilde{\mathcal{K}},\mathfrak{R})$ where $\mathfrak{S} = \mathfrak{Q} \cup \mathfrak{R}$ and

$$\widetilde{\mathcal{L}}((\mathbf{x^I},e,o)) = \begin{cases} \widetilde{\mathcal{H}}((\mathbf{x^I},e,o)), & \text{if } (\mathbf{x^I},e,o) \in \mathfrak{Q}, \\ \widetilde{\mathcal{K}}((\mathbf{x^I},e,o)), & \text{if } (\mathbf{x^I},e,o) \in \mathfrak{R}, \\ \widetilde{\mathcal{H}}((\mathbf{x^I},e,o)) \cup_f \widetilde{\mathcal{K}}((\mathbf{x^I},e,o)), & \text{if } (\mathbf{x^I},e,o) \in \mathfrak{Q} \cap \mathfrak{R}, \end{cases} \tag{22}$$

for each $(\mathbf{x^I},e,o) \in \mathfrak{S}$.

Example 4.17. We consider the fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}},\mathfrak{Q})$ and $(\widetilde{\mathcal{K}},\mathfrak{R})$ in Examples 4.2 and 4.14, respectively. Then, the restricted union and extended union of fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}},\mathfrak{Q})$ and $(\widetilde{\mathcal{K}},\mathfrak{R})$ are respectively

$$(\widetilde{\mathcal{H}},\mathfrak{Q}) \Cup_E (\widetilde{\mathcal{K}},\mathfrak{R}) = \left\{ \begin{array}{l} (((x_1^1,x_1^2,x_3^3),e_1,1),\{\,^{(0.7)}a_1,^{(0.4)}a_2,^{(0.4)}a_3\}), \\ (((x_1^1,x_1^2,x_3^3),e_2,1),\{\,^{(1)}a_1,^{(0.2)}a_2,^{(0.9)}a_3\}), \\ (((x_1^1,x_1^2,x_3^3),e_1,0),\{\,^{(0.3)}a_1,^{(0.6)}a_2,^{(0.5)}a_3\}), \\ (((x_1^1,x_1^2,x_3^3),e_2,0),\{\,^{(0.2)}a_1,^{(0.5)}a_2,^{(0.5)}a_3\}) \end{array} \right\},$$

and

$$(\widetilde{\mathcal{H}},\mathfrak{Q}) \sqcup_E (\widetilde{\mathcal{K}},\mathfrak{R}) = \left\{ \begin{array}{l} (((x_1^1,x_1^2,x_1^3),e_1,1),\{\,^{(0.5)}a_1,^{(0.6)}a_2,^{(0.3)}a_3\}), \\ (((x_1^1,x_1^2,x_1^3),e_2,1),\{\,^{(0.2)}a_1,^{(0.4)}a_2,^{(0.7)}a_3\}), \\ (((x_1^1,x_1^2,x_1^3),e_3,1),\{\,^{(0.4)}a_1,^{(0.3)}a_2,^{(0.3)}a_3\}), \\ (((x_1^1,x_1^2,x_2^3),e_1,1),\{\,^{(0.4)}a_1,^{(0)}a_2,^{(0.7)}a_3\}), \\ (((x_1^1,x_1^2,x_2^3),e_3,1),\{\,^{(0.2)}a_1,^{(0.6)}a_2,^{(0.3)}a_3\}), \\ (((x_1^1,x_1^2,x_3^3),e_1,1),\{\,^{(0.7)}a_1,^{(0.4)}a_2,^{(0.4)}a_3\}), \\ (((x_1^1,x_1^2,x_3^3),e_2,1),\{\,^{(1)}a_1,^{(0.2)}a_2,^{(0.9)}a_3\}), \\ (((x_1^1,x_1^2,x_3^3),e_3,1),\{\,^{(0.3)}a_1,^{(0.3)}a_2,^{(0.6)}a_3\}), \\ (((x_1^1,x_1^2,x_1^3),e_1,0),\{\,^{(0.4)}a_1,^{(0.2)}a_2,^{(0.5)}a_3\}), \\ (((x_1^1,x_1^2,x_1^3),e_2,0),\{\,^{(0.5)}a_1,^{(0.6)}a_2,^{(0.3)}a_3\}), \\ (((x_1^1,x_1^2,x_1^3),e_3,0),\{\,^{(0.7)}a_1,^{(0.7)}a_2,^{(0.7)}a_3\}), \\ (((x_1^1,x_1^2,x_2^3),e_1,0),\{\,^{(0.5)}a_1,^{(0.6)}a_2,^{(0.5)}a_3\}), \\ (((x_1^1,x_1^2,x_2^3),e_3,0),\{\,^{(0.7)}a_1,^{(0.3)}a_2,^{(0.3)}a_3\}), \\ (((x_1^1,x_1^2,x_3^3),e_1,0),\{\,^{(0.3)}a_1,^{(0.6)}a_2,^{(0.5)}a_3\}), \\ (((x_1^1,x_1^2,x_3^3),e_2,0),\{\,^{(0.2)}a_1,^{(0.5)}a_2,^{(0.5)}a_3\}), \\ (((x_1^1,x_1^2,x_3^3),e_3,0),\{\,^{(0.7)}a_1,^{(0.6)}a_2,^{(0.5)}a_3\}) \end{array} \right\}.$$

Proposition 4.18. *Let $(\widetilde{\mathcal{H}},\mathfrak{Q}),(\widetilde{\mathcal{K}},\mathfrak{R}),(\widetilde{\mathcal{L}},\mathfrak{S}) \in \mathfrak{C}_E\langle A,\mathfrak{P}\rangle$. Then, the following properties are acquired.*

(i): $(\widetilde{\mathcal{H}},\mathfrak{Q})\diamond(\widetilde{\mathcal{K}},\mathfrak{R}) =_E (\widetilde{\mathcal{K}},\mathfrak{R})\diamond(\widetilde{\mathcal{H}},\mathfrak{Q})$ *for each* $\diamond\in\{\Cap_E,\Cup_E\}$.

(ii): $(\widetilde{\mathcal{H}},\mathfrak{Q})\diamond((\widetilde{\mathcal{K}},\mathfrak{R})\diamond(\widetilde{\mathcal{L}},\mathfrak{S})) =_E ((\widetilde{\mathcal{H}},\mathfrak{Q})\diamond(\widetilde{\mathcal{K}},\mathfrak{R}))\diamond(\widetilde{\mathcal{L}},\mathfrak{S})$ *for each* $\diamond\in\{\Cap_E,\Cup_E\}$.

(iii): $(\widetilde{\mathcal{H}},\mathfrak{Q})\diamond((\widetilde{\mathcal{K}},\mathfrak{R})\circ(\widetilde{\mathcal{L}},\mathfrak{S})) =_E ((\widetilde{\mathcal{H}},\mathfrak{Q})\diamond(\widetilde{\mathcal{K}},\mathfrak{R}))\circ((\widetilde{\mathcal{H}},\mathfrak{Q})\diamond(\widetilde{\mathcal{L}},\mathfrak{S}))$ *for each* $\diamond,\circ\in\{\Cap_E,\Cup_E\}$.

(iv): $((\widetilde{\mathcal{H}},\mathfrak{Q})\diamond(\widetilde{\mathcal{K}},\mathfrak{R}))^{r_E} =_E (\widetilde{\mathcal{H}},\mathfrak{Q})^{r_E}\circ(\widetilde{\mathcal{K}},\mathfrak{R})^{r_E}$ *for each* $\diamond,\circ\in\{\Cap_E,\Cup_E\}$ *and* $\diamond\neq\circ$.

Proof. The proofs are straightforward. □

Proposition 4.19. *Let* $(\widetilde{\mathcal{H}},\mathfrak{Q}),(\widetilde{\mathcal{K}},\mathfrak{R}),(\widetilde{\mathcal{L}},\mathfrak{S})\in\mathfrak{C}_E\langle A,\mathfrak{P}\rangle$. *Then, the following properties are acquired.*

(i): $(\widetilde{\mathcal{H}},\mathfrak{Q})\diamond(\widetilde{\mathcal{K}},\mathfrak{R}) =_E (\widetilde{\mathcal{K}},\mathfrak{R})\diamond(\widetilde{\mathcal{H}},\mathfrak{Q})$ *for each* $\diamond\in\{\sqcap_E,\sqcup_E\}$.

(ii): $(\widetilde{\mathcal{H}},\mathfrak{Q})\diamond((\widetilde{\mathcal{K}},\mathfrak{R})\diamond(\widetilde{\mathcal{L}},\mathfrak{S})) =_E ((\widetilde{\mathcal{H}},\mathfrak{Q})\diamond(\widetilde{\mathcal{K}},\mathfrak{R}))\diamond(\widetilde{\mathcal{L}},\mathfrak{S})$ *for each* $\diamond\in\{\sqcap_E,\sqcup_E\}$.

(iii): $(\widetilde{\mathcal{H}},\mathfrak{Q})\diamond((\widetilde{\mathcal{K}},\mathfrak{R})\circ(\widetilde{\mathcal{L}},\mathfrak{S})) =_E ((\widetilde{\mathcal{H}},\mathfrak{Q})\diamond(\widetilde{\mathcal{K}},\mathfrak{R}))\circ((\widetilde{\mathcal{H}},\mathfrak{Q})\diamond(\widetilde{\mathcal{L}},\mathfrak{S}))$ *for each* $\diamond,\circ\in\{\sqcap_E,\sqcup_E\}$.

(iv): $((\widetilde{\mathcal{H}},\mathfrak{Q})\diamond(\widetilde{\mathcal{K}},\mathfrak{R}))^{r_E} =_E (\widetilde{\mathcal{H}},\mathfrak{Q})^{r_E}\circ(\widetilde{\mathcal{K}},\mathfrak{R})^{r_E}$ *for each* $\diamond,\circ\in\{\sqcap_E,\sqcup_E\}$ *and* $\diamond\neq\circ$.

Proof. The proofs are straightforward. □

5. n-ary Fuzzy Hypersoft Expert Sets

In this section, we initiate the theory of n-ary fuzzy hypersoft expert sets including both n-ary fuzzy hypersoft sets and fuzzy hypersoft expert sets.

Throughout this section, $\{A_j : j\in J=\{1,2,...,n\}\}$ is a collection of universal sets such that $A_j\cap A_{j'}=\emptyset$ for each $j,j'\in J=\{1,2,...,n\}$ and $j\neq j'$. $\mathfrak{F}(\mathfrak{A})=\prod_{j\in J}\mathfrak{F}(A_j)=\mathfrak{F}(A_1)\times\mathfrak{F}(A_2)\times...\times\mathfrak{F}(A_n)$ where $\mathfrak{F}(A_j)$ denotes the set of all fuzzy sets in A_j. Also, $X_1,X_2,...,X_m$ are the pairwise disjoint sets of parameters (i.e., $X_i\cap X_{i'}=\emptyset$ for each $i,i'\in I=\{1,2,...,m\}$ and $i\neq i'$), $\mathbf{X}=\prod_{i\in I}X_i=X_1\times X_2\times...\times X_m$, $\mathcal{E}$ is a set of experts, $\mathcal{O}$ is a set of opinions, $\mathfrak{P}=\mathbf{X}\times\mathcal{E}\times\mathcal{O}$ and $\mathfrak{Q}\subseteq\mathfrak{P}$.

Definition 5.1. A pair $(\widetilde{\mathcal{H}}_n,\mathfrak{Q})$ is called an n-ary fuzzy hypersoft expert set over $\mathfrak{A}=\{A_1,A_2,...,A_n\}$, where $\widetilde{\mathcal{H}}_n$ is mapping given by

$$\widetilde{\mathcal{H}}_n:\mathfrak{Q}\to\mathfrak{F}(\mathfrak{A}). \tag{23}$$

Simply, an n-ary fuzzy hypersoft expert set can be given as

$$\begin{aligned}(\widetilde{\mathcal{H}}_n,\mathfrak{Q}) &= \{((\mathbf{x^I},e,o),\widetilde{\mathcal{H}}_n((\mathbf{x^I},e,o))) : (\mathbf{x^I},e,o)\in\mathfrak{Q} \text{ and } \widetilde{\mathcal{H}}_n((\mathbf{x^I},e,o))\in\mathfrak{A}\}\\ &= \left\{\left((\mathbf{x^I},e,o),\begin{pmatrix}\{^{(\mu_{\widetilde{\mathcal{H}}_n((\mathbf{x^I},e,o))}(a^1))}a^1 : a^1\in A_1\},\\ \{^{(\mu_{\widetilde{\mathcal{H}}_n((\mathbf{x^I},e,o))}(a^2))}a^2 : a^2\in A_2\},\\ .\\ .\\ .\\ \{^{(\mu_{\widetilde{\mathcal{H}}_n((\mathbf{x^I},e,o))}(a^m))}a^m : a^m\in A_m\}\end{pmatrix}\right) : (\mathbf{x^I},e,o)\in\mathfrak{Q}\right\},\end{aligned}$$

where $\widetilde{\mathcal{H}}_{n(j)}((\mathbf{x^I}, e, o)) = \{^{(\mu_{\widetilde{\mathcal{H}}_n((\mathbf{x^I},e,o))}(a^j))}a^j : a^j \in A_j\}$ for $j = 1, 2, ..., n$ and it is termed to be an A_j-part of $\widetilde{\mathcal{H}}_n((\mathbf{x^I}, e, o))$.

Especially, if $n = 2, 3, 4$ and 5 then it is called a binary fuzzy hypersoft expert set, ternary fuzzy hypersoft expert set, quaternary fuzzy hypersoft expert set, quinary fuzzy hypersoft expert set, respectively.

Note 8. The set of all n-ary fuzzy hypersoft expert sets over $\mathfrak{A} = \{A_1, A_2, ..., A_n\}$ for $\mathfrak{P}$ is denoted by $\mathfrak{C}_{EN_n}\langle\mathfrak{A}, \mathfrak{P}\rangle$.

Example 5.2. Consider the problem in Examples 3.2 and 4.2. Assume that he/se seeks the opinions of 3 experts with the intention of determining the optimal car(s) for each segment (i.e., A_1, A_2 and A_3), simultaneously. The set of experts is $\mathcal{E} = \{e_1, e_2, e_3\}$ and the set of opinions is $\mathcal{O} = \{o_1 = agree(1),\ o_2 = disagree(0)\}$. For

$$\mathfrak{Q} = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 1), (((x_1^1, x_1^2, x_1^3), e_2, 1), (((x_1^1, x_1^2, x_1^3), e_3, 1), (((x_1^1, x_1^2, x_3^3), e_1, 1), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), (((x_1^1, x_1^2, x_3^3), e_3, 1), (((x_1^1, x_1^2, x_1^3), e_1, 0), (((x_1^1, x_1^2, x_1^3), e_2, 0), \\ (((x_1^1, x_1^2, x_1^3), e_3, 0), (((x_1^1, x_1^2, x_3^3), e_1, 0), (((x_1^1, x_1^2, x_3^3), e_2, 0), (((x_1^1, x_1^2, x_3^3), e_3, 0) \end{array} \right\} \subseteq \mathbf{X} \times \mathcal{E} \times \mathcal{O},$$

it is created the following ternary fuzzy hypersoft expert set.

$$(\widetilde{\mathcal{H}}_3, \mathfrak{Q}) = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 1), (\{^{(0.5)}a_1^1, ^{(0.6)}a_2^1, ^{(0.3)}a_3^1\}, \{^{(0.4)}a_1^2, ^{(0.5)}a_2^2\}, \{^{(0.1)}a_1^3, ^{(0.6)}a_2^3, ^{(0.2)}a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_2, 1), (\{^{(0.2)}a_1^1, ^{(0.4)}a_2^1, ^{(0.7)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.6)}a_2^2\}, \{^{(0.4)}a_1^3, ^{(0.4)}a_2^3, ^{(0.1)}a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_3, 1), (\{^{(0.4)}a_1^1, ^{(0.3)}a_2^1, ^{(0.3)}a_3^1\}, \{^{(0.7)}a_1^2, ^{(0.2)}a_2^2\}, \{^{(0.4)}a_1^3, ^{(0.3)}a_2^3, ^{(0.5)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_1, 1), (\{^{(0.7)}a_1^1, ^{(0.2)}a_2^1, ^{(0.4)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.6)}a_2^2\}, \{^{(0.3)}a_1^3, ^{(0.4)}a_2^3, ^{(1)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), (\{^{(1)}a_1^1, ^{(0.2)}a_2^1, ^{(0.9)}a_3^1\}, \{^{(0.4)}a_1^2, ^{(0.4)}a_2^2\}, \{^{(0.5)}a_1^3, ^{(0.2)}a_2^3, ^{(0.3)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_3, 1), (\{^{(0.3)}a_1^1, ^{(0.3)}a_2^1, ^{(0.6)}a_3^1\}, \{^{(0.5)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0.4)}a_1^3, ^{(0.2)}a_2^3, ^{(0.6)}a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_1, 0), (\{^{(0.4)}a_1^1, ^{(0.2)}a_2^1, ^{(0.5)}a_3^1\}, \{^{(0.6)}a_1^2, ^{(0.2)}a_2^2\}, \{^{(0.1)}a_1^3, ^{(0.2)}a_2^3, ^{(0.3)}a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_2, 0), (\{^{(0.5)}a_1^1, ^{(0.6)}a_2^1, ^{(0.3)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0.4)}a_1^3, ^{(0.5)}a_2^3, ^{(0.1)}a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_3, 0), (\{^{(0.7)}a_1^1, ^{(0.7)}a_2^1, ^{(0.7)}a_3^1\}, \{^{(0.5)}a_1^2, ^{(0.2)}a_2^2\}, \{^{(0)}a_1^3, ^{(0.4)}a_2^3, ^{(0.3)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_1, 0), (\{^{(0.1)}a_1^1, ^{(0.6)}a_2^1, ^{(0.5)}a_3^1\}, \{^{(0.7)}a_1^2, ^{(0.4)}a_2^2\}, \{^{(0.2)}a_1^3, ^{(0.5)}a_2^3, ^{(0.1)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_2, 0), (\{^{(0.1)}a_1^1, ^{(0.5)}a_2^1, ^{(0.3)}a_3^1\}, \{^{(0.2)}a_1^2, ^{(0)}a_2^2\}, \{^{(1)}a_1^3, ^{(0.5)}a_2^3, ^{(0.3)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_3, 0), (\{^{(0.7)}a_1^1, ^{(0.6)}a_2^1, ^{(0.5)}a_3^1\}, \{^{(0.7)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.7)}a_1^3, ^{(0.2)}a_2^3, ^{(0.7)}a_3^3\})) \end{array} \right\}.$$

Definition 5.3. Let $(\widetilde{\mathcal{H}}_n, \mathfrak{Q}) \in \mathfrak{C}_{EN_n}\langle\mathfrak{A}, \mathfrak{P}\rangle$. Then,

- **(a):** $(\widetilde{\mathcal{H}}_n, \mathfrak{Q})^1 = \{(q, \widetilde{\mathcal{H}}_n(q)) : q \in \mathbf{X} \times \mathcal{E} \times \{1\} \subseteq \mathfrak{Q}\}$ is named to be an agree n-ary fuzzy hypersoft expert set over $\mathfrak{A}$.
- **(b):** $(\widetilde{\mathcal{H}}_n, \mathfrak{Q})^0 = \{(q, \widetilde{\mathcal{H}}_n(q)) : q \in \mathbf{X} \times \mathcal{E} \times \{0\} \subseteq \mathfrak{Q}\}$ is named to be a disagree n-ary fuzzy hypersoft expert set over $\mathfrak{A}$.

From the definition, it is clear that $(\widetilde{\mathcal{H}}_n, \mathfrak{Q})^1 \cup (\widetilde{\mathcal{H}}_n, \mathfrak{Q})^0 = (\widetilde{\mathcal{H}}_n, \mathfrak{Q})$.

Example 5.4. We consider the ternary fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}_3, \mathfrak{Q}) \in \mathfrak{C}_{EN_3}\langle\mathfrak{A}, \mathfrak{P}\rangle$ given in Example 5.2. Then, the agree ternary fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})^1$ and the disagree ternary fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})^0$ are respectively

$$(\widetilde{\mathcal{H}}_3, \mathfrak{Q})^1 = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 1), (\{^{(0.5)}a_1^1,^{(0.6)} a_2^1,^{(0.3)} a_3^1\}, \{^{(0.4)}a_1^2,^{(0.5)} a_2^2\}, \{^{(0.1)}a_1^3,^{(0.6)} a_2^3,^{(0.2)} a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_2, 1), (\{^{(0.2)}a_1^1,^{(0.4)} a_2^1,^{(0.7)} a_3^1\}, \{^{(0.3)}a_1^2,^{(0.6)} a_2^2\}, \{^{(0.4)}a_1^3,^{(0.4)} a_2^3,^{(0.1)} a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_3, 1), (\{^{(0.4)}a_1^1,^{(0.3)} a_2^1,^{(0.3)} a_3^1\}, \{^{(0.7)}a_1^2,^{(0.2)} a_2^2\}, \{^{(0.4)}a_1^3,^{(0.3)} a_2^3,^{(0.5)} a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_1, 1), (\{^{(0.7)}a_1^1,^{(0.2)} a_2^1,^{(0.4)} a_3^1\}, \{^{(0.3)}a_1^2,^{(0.6)} a_2^2\}, \{^{(0.3)}a_1^3,^{(0.4)} a_2^3,^{(1)} a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), (\{^{(1)}a_1^1,^{(0.2)} a_2^1,^{(0.9)} a_3^1\}, \{^{(0.4)}a_1^2,^{(0.4)} a_2^2\}, \{^{(0.5)}a_1^3,^{(0.2)} a_2^3,^{(0.3)} a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_3, 1), (\{^{(0.3)}a_1^1,^{(0.3)} a_2^1,^{(0.6)} a_3^1\}, \{^{(0.5)}a_1^2,^{(0.3)} a_2^2\}, \{^{(0.4)}a_1^3,^{(0.2)} a_2^3,^{(0.6)} a_3^3\})) \end{array} \right\},$$

and

$$(\widetilde{\mathcal{H}}_3, \mathfrak{Q})^0 = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 0), (\{^{(0.4)}a_1^1,^{(0.2)} a_2^1,^{(0.5)} a_3^1\}, \{^{(0.6)}a_1^2,^{(0.2)} a_2^2\}, \{^{(0.1)}a_1^3,^{(0.2)} a_2^3,^{(0.3)} a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_2, 0), (\{^{(0.5)}a_1^1,^{(0.6)} a_2^1,^{(0.3)} a_3^1\}, \{^{(0.3)}a_1^2,^{(0.3)} a_2^2\}, \{^{(0.4)}a_1^3,^{(0.5)} a_2^3,^{(0.1)} a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_3, 0), (\{^{(0.7)}a_1^1,^{(0.7)} a_2^1,^{(0.7)} a_3^1\}, \{^{(0.5)}a_1^2,^{(0.2)} a_2^2\}, \{^{(0)}a_1^3,^{(0.4)} a_2^3,^{(0.3)} a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_1, 0), (\{^{(0.1)}a_1^1,^{(0.6)} a_2^1,^{(0.5)} a_3^1\}, \{^{(0.7)}a_1^2,^{(0.4)} a_2^2\}, \{^{(0.2)}a_1^3,^{(0.5)} a_2^3,^{(0.1)} a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_2, 0), (\{^{(0.1)}a_1^1,^{(0.5)} a_2^1,^{(0.3)} a_3^1\}, \{^{(0.2)}a_1^2,^{(0)} a_2^2\}, \{^{(1)}a_1^3,^{(0.5)} a_2^3,^{(0.3)} a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_3, 0), (\{^{(0.7)}a_1^1,^{(0.6)} a_2^1,^{(0.5)} a_3^1\}, \{^{(0.7)}a_1^2,^{(0.7)} a_2^2\}, \{^{(0.7)}a_1^3,^{(0.2)} a_2^3,^{(0.7)} a_3^3\})) \end{array} \right\}.$$

Definition 5.5. Let $(\widetilde{\mathcal{H}}_n, \mathfrak{Q}) \in \mathfrak{C}_{EN_n}\langle \mathfrak{A}, \mathfrak{P} \rangle$.

(a): If $\widetilde{\mathcal{H}}_n((\mathbf{x^I}, e, o)) = (\widehat{\emptyset}, \widehat{\emptyset}, ..., \widehat{\emptyset})$ (i.e., $\widetilde{\mathcal{H}}_{n(j)}((\mathbf{x^I}, e, o)) = \widehat{\emptyset} \quad \forall j \in J$) for each $(\mathbf{x^I}, e, o) \in \mathfrak{Q}$ then it is termed a relative null n-ary fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{\emptyset}_{\mathfrak{Q}}^{EN_n}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is called a null n-ary fuzzy hypersoft expert set and denoted by $\widehat{\emptyset}_{\mathfrak{P}}^{EN_n}$.

(b): If $\widetilde{\mathcal{H}}_n^1((\mathbf{x^I}, e, 1)) = (\widehat{\emptyset}, \widehat{\emptyset}, ..., \widehat{\emptyset})$ (i.e., $\widetilde{\mathcal{H}}_{n(j)}^1((\mathbf{x^I}, e, 1)) = \widehat{\emptyset} \quad \forall j \in J$) for each $(\mathbf{x^I}, e, 1) \in \mathfrak{Q}$ then it is termed to be a relative null agree n-ary fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{\emptyset}_{\mathfrak{Q}}^{EN_n^1}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is called a null agree n-ary fuzzy hypersoft expert set and denoted by $\widehat{\emptyset}_{\mathfrak{P}}^{E_n^1}$.

(c): If $\widetilde{\mathcal{H}}_n^0((\mathbf{x^I}, e, 0)) = (\widehat{\emptyset}, \widehat{\emptyset}, ..., \widehat{\emptyset})$ (i.e., $\widetilde{\mathcal{H}}_{n(j)}^1((\mathbf{x^I}, e, 0)) = \widehat{\emptyset} \quad \forall j \in J$) for each $(\mathbf{x^I}, e, 0) \in \mathfrak{Q}$ then it is named to be a relative null disagree n-ary fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{\emptyset}_{\mathfrak{Q}}^{EN_n^0}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is termed to be a null disagree n-ary fuzzy hypersoft expert set and denoted by $\widehat{\emptyset}_{\mathfrak{P}}^{EN_n^0}$.

(d): If $\widetilde{\mathcal{H}}_n((\mathbf{x^I}, e, o)) = (\widehat{A}_1, \widehat{A}_2, ..., \widehat{A}_n)$ (i.e., $\widetilde{\mathcal{H}}_{n(j)}((\mathbf{x^I}, e, o)) = \widehat{A}_j \quad \forall j \in J$) for each $(\mathbf{x^I}, e, o) \in \mathfrak{Q}$ then it is termed to be a relative whole n-ary fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{A}_{\mathfrak{Q}}^{EN_n}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is called an absolute n-ary fuzzy hypersoft expert set and denoted by $\widehat{A}_{\mathfrak{P}}^{EN_n}$.

(e): If $\widetilde{\mathcal{H}}_n^1((\mathbf{x^I}, e, 1)) = (\widehat{A}_1, \widehat{A}_2, ..., \widehat{A}_n)$ (i.e., $\widetilde{\mathcal{H}}_{n(j)}^1((\mathbf{x^I}, e, 1)) = \widehat{A}_j \quad \forall j \in J$) for each $(\mathbf{x^I}, e, 1) \in \mathfrak{Q}$ then it is called a relative whole agree n-ary fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{A}_{\mathfrak{Q}}^{EN_n^1}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is called an absolute agree n-ary fuzzy hypersoft expert set and denoted by $\widehat{A}_{\mathfrak{P}}^{EN_n^1}$.

(f): If $\widetilde{\mathcal{H}}_n^0((\mathbf{x^I}, e, 0)) = (\widehat{A}_1, \widehat{A}_2, ..., \widehat{A}_n)$ (i.e., $\widetilde{\mathcal{H}}_{n(j)}^0((\mathbf{x^I}, e, 0)) = \widehat{A}_j \quad \forall j \in J$) for each $(\mathbf{x^I}, e, 0) \in \mathfrak{Q}$ then it is named a relative whole disagree n-ary fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$), denoted by $\widehat{A}^{EN^0_n}_{\mathfrak{Q}}$. If $\mathfrak{Q} = \mathfrak{P}$ then it is called an absolute disagree n-ary fuzzy hypersoft expert set and denoted by $\widehat{A}^{EN^0_n}_{\mathfrak{P}}$.

Example 5.6. The following ternary fuzzy hypersoft expert sets are given as examples of relative null disagree ternary fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$) and relative whole agree ternary fuzzy hypersoft expert set (with respect to $\mathfrak{Q}$) over $\mathfrak{A}$.

$$\widehat{\emptyset}^{EN^0_3}_{\mathfrak{Q}} = \left\{ \begin{array}{l} (((x^1_1, x^2_1, x^3_1), e_1, 0), (\{^{(0)}a^1_1, ^{(0)}a^1_2, ^{(0)}a^1_3\}, \{^{(0)}a^2_1, ^{(0)}a^2_2\}, \{^{(0)}a^3_1, ^{(0)}a^3_2, ^{(0)}a^3_3\})), \\ (((x^1_1, x^2_1, x^3_1), e_2, 0), (\{^{(0)}a^1_1, ^{(0)}a^1_2, ^{(0)}a^1_3\}, \{^{(0)}a^2_1, ^{(0)}a^2_2\}, \{^{(0)}a^3_1, ^{(0)}a^3_2, ^{(0)}a^3_3\})), \\ (((x^1_1, x^2_1, x^3_1), e_3, 0), (\{^{(0)}a^1_1, ^{(0)}a^1_2, ^{(0)}a^1_3\}, \{^{(0)}a^2_1, ^{(0)}a^2_2\}, \{^{(0)}a^3_1, ^{(0)}a^3_2, ^{(0)}a^3_3\})), \\ (((x^1_1, x^2_1, x^3_3), e_1, 0), (\{^{(0)}a^1_1, ^{(0)}a^1_2, ^{(0)}a^1_3\}, \{^{(0)}a^2_1, ^{(0)}a^2_2\}, \{^{(0)}a^3_1, ^{(0)}a^3_2, ^{(0)}a^3_3\})), \\ (((x^1_1, x^2_1, x^3_3), e_2, 0), (\{^{(0)}a^1_1, ^{(0)}a^1_2, ^{(0)}a^1_3\}, \{^{(0)}a^2_1, ^{(0)}a^2_2\}, \{^{(0)}a^3_1, ^{(0)}a^3_2, ^{(0)}a^3_3\})), \\ (((x^1_1, x^2_1, x^3_3), e_3, 0), (\{^{(0)}a^1_1, ^{(0)}a^1_2, ^{(0)}a^1_3\}, \{^{(0)}a^2_1, ^{(0)}a^2_2\}, \{^{(0)}a^3_1, ^{(0)}a^3_2, ^{(0)}a^3_3\})) \end{array} \right\}.$$

and

$$\widehat{A}^{EN^1_3}_{\mathfrak{Q}} = \left\{ \begin{array}{l} (((x^1_1, x^2_1, x^3_1), e_1, 1), (\{^{(1)}a^1_1, ^{(1)}a^1_2, ^{(1)}a^1_3\}, \{^{(1)}a^2_1, ^{(1)}a^2_2\}, \{^{(1)}a^3_1, ^{(1)}a^3_2, ^{(1)}a^3_3\})), \\ (((x^1_1, x^2_1, x^3_1), e_2, 1), (\{^{(1)}a^1_1, ^{(1)}a^1_2, ^{(1)}a^1_3\}, \{^{(1)}a^2_1, ^{(1)}a^2_2\}, \{^{(1)}a^3_1, ^{(1)}a^3_2, ^{(1)}a^3_3\})), \\ (((x^1_1, x^2_1, x^3_1), e_3, 1), (\{^{(1)}a^1_1, ^{(1)}a^1_2, ^{(1)}a^1_3\}, \{^{(1)}a^2_1, ^{(1)}a^2_2\}, \{^{(1)}a^3_1, ^{(1)}a^3_2, ^{(1)}a^3_3\})), \\ (((x^1_1, x^2_1, x^3_3), e_1, 1), (\{^{(1)}a^1_1, ^{(1)}a^1_2, ^{(1)}a^1_3\}, \{^{(1)}a^2_1, ^{(1)}a^2_2\}, \{^{(1)}a^3_1, ^{(1)}a^3_2, ^{(1)}a^3_3\})), \\ (((x^1_1, x^2_1, x^3_3), e_2, 1), (\{^{(1)}a^1_1, ^{(1)}a^1_2, ^{(1)}a^1_3\}, \{^{(1)}a^2_1, ^{(1)}a^2_2\}, \{^{(1)}a^3_1, ^{(1)}a^3_2, ^{(1)}a^3_3\})), \\ (((x^1_1, x^2_1, x^3_3), e_3, 1), (\{^{(1)}a^1_1, ^{(1)}a^1_2, ^{(1)}a^1_3\}, \{^{(1)}a^2_1, ^{(1)}a^2_2\}, \{^{(1)}a^3_1, ^{(1)}a^3_2, ^{(1)}a^3_3\})) \end{array} \right\}.$$

Definition 5.7. Let $(\widetilde{\mathcal{H}}_n, \mathfrak{Q}), (\widetilde{\mathcal{K}}_n, \mathfrak{R}) \in \mathfrak{C}_{EN_n}\langle \mathfrak{A}, \mathfrak{P} \rangle$.

(a): $(\widetilde{\mathcal{H}}_n, \mathfrak{Q})$ is called an n-ary fuzzy hypersoft expert subset of $(\widetilde{\mathcal{K}}_n, \mathfrak{R})$, denoted by $(\widetilde{\mathcal{H}}_n, \mathfrak{Q}) \sqsubseteq_{EN_n} (\widetilde{\mathcal{K}}_n, \mathfrak{R})$, if $\mathfrak{Q} \subseteq \mathfrak{R}$ and

$$\widetilde{\mathcal{H}}_{n(j)}((\mathbf{x}^{\mathbf{I}}, e, o)) \subseteq_f \widetilde{\mathcal{K}}_{n(j)}((\mathbf{x}^{\mathbf{I}}, e, o)) \quad \forall j \in J \tag{24}$$

for each $(\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{Q}$.

(b): The n-ary fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}_n, \mathfrak{Q})$ and $(\widetilde{\mathcal{K}}_n, \mathfrak{R})$ are called equal, denoted by $(\widetilde{\mathcal{H}}_n, \mathfrak{Q}) =_{EN_n} (\widetilde{\mathcal{K}}_n, \mathfrak{R})$, if $(\widetilde{\mathcal{H}}_n, \mathfrak{Q}) \sqsubseteq_{EN_n} (\widetilde{\mathcal{K}}_n, \mathfrak{R})$ and $(\widetilde{\mathcal{K}}_n, \mathfrak{R}) \sqsubseteq_{EN_n} (\widetilde{\mathcal{H}}_n, \mathfrak{Q})$.

Example 5.8. Consider $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})$ in Example 5.2 and the $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})^1$ and $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})^0$ Example 5.4. It is clear that $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})^1$ and $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})^0$ are ternary fuzzy hypersoft expert subsets of $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})$. Furthermore, $\widehat{\emptyset}^{EN^0_3}_{\mathfrak{Q}}$ is ternary fuzzy hypersoft expert subset of $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})$.

Definition 5.9. Let $(\widetilde{\mathcal{H}}_n, \mathfrak{Q}) \in \mathfrak{C}_{EN_n}\langle \mathfrak{A}, \mathfrak{P} \rangle$. Then, the relative complement of n-ary fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}_n, \mathfrak{Q})$, denoted by $(\widetilde{\mathcal{H}}_n, \mathfrak{Q})^{r_{EN_n}}$, is defined as

$$(\widetilde{\mathcal{H}}_n, \mathfrak{Q})^{r_{EN_n}} = (\widetilde{\mathcal{H}}^r_n, \mathfrak{Q}), \tag{25}$$

where $\widetilde{\mathcal{H}}^r_{n(j)}((\mathbf{x}^{\mathbf{I}}, e, o))$ is the fuzzy complement of $\widetilde{\mathcal{H}}_{n(j)}((\mathbf{x}^{\mathbf{I}}, e, o))$ $(\forall j \in J)$ for each $(\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{Q}$.

Example 5.10. Consider the ternary fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})$ in Example 5.2. Then, the relative complement of ternary fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})$ is

$$(\widetilde{\mathcal{H}}_3, \mathfrak{Q})^{r_{EN_3}} = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_1^3), e_1, 1), (\{^{(0.5)}a_1^1, ^{(0.4)}a_2^1, ^{(0.7)}a_3^1\}, \{^{(0.6)}a_1^2, ^{(0.5)}a_2^2\}, \{^{(0.9)}a_1^3, ^{(0.4)}a_2^3, ^{(0.8)}a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_2, 1), (\{^{(0.8)}a_1^1, ^{(0.6)}a_2^1, ^{(0.3)}a_3^1\}, \{^{(0.7)}a_1^2, ^{(0.4)}a_2^2\}, \{^{(0.6)}a_1^3, ^{(0.6)}a_2^3, ^{(0.9)}a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_3, 1), (\{^{(0.6)}a_1^1, ^{(0.7)}a_2^1, ^{(0.7)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.8)}a_2^2\}, \{^{(0.6)}a_1^3, ^{(0.7)}a_2^3, ^{(0.5)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_1, 1), (\{^{(0.3)}a_1^1, ^{(0.8)}a_2^1, ^{(0.6)}a_3^1\}, \{^{(0.7)}a_1^2, ^{(0.4)}a_2^2\}, \{^{(0.7)}a_1^3, ^{(0.6)}a_2^3, ^{(0)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_2, 1), (\{^{(1)}a_1^1, ^{(0.8)}a_2^1, ^{(0.1)}a_3^1\}, \{^{(0.6)}a_1^2, ^{(0.6)}a_2^2\}, \{^{(0.5)}a_1^3, ^{(0.8)}a_2^3, ^{(0.7)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_3, 1), (\{^{(0.7)}a_1^1, ^{(0.7)}a_2^1, ^{(0.4)}a_3^1\}, \{^{(0.5)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.6)}a_1^3, ^{(0.8)}a_2^3, ^{(0.4)}a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_1, 0), (\{^{(0.6)}a_1^1, ^{(0.8)}a_2^1, ^{(0.5)}a_3^1\}, \{^{(0.4)}a_1^2, ^{(0.8)}a_2^2\}, \{^{(0.9)}a_1^3, ^{(0.8)}a_2^3, ^{(0.7)}a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_2, 0), (\{^{(0.5)}a_1^1, ^{(0.4)}a_2^1, ^{(0.7)}a_3^1\}, \{^{(0.7)}a_1^2, ^{(0.7)}a_2^2\}, \{^{(0.6)}a_1^3, ^{(0.5)}a_2^3, ^{(0.9)}a_3^3\})), \\ (((x_1^1, x_1^2, x_1^3), e_3, 0), (\{^{(0.3)}a_1^1, ^{(0.3)}a_2^1, ^{(0.3)}a_3^1\}, \{^{(0.5)}a_1^2, ^{(0.8)}a_2^2\}, \{^{(1)}a_1^3, ^{(0.5)}a_2^3, ^{(0.7)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_1, 0), (\{^{(0.9)}a_1^1, ^{(0.4)}a_2^1, ^{(0.5)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.6)}a_2^2\}, \{^{(0.8)}a_1^3, ^{(0.5)}a_2^3, ^{(0.9)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_2, 0), (\{^{(0.9)}a_1^1, ^{(0.5)}a_2^1, ^{(0.7)}a_3^1\}, \{^{(0.8)}a_1^2, ^{(1)}a_2^2\}, \{^{(0)}a_1^3, ^{(0.5)}a_2^3, ^{(0.3)}a_3^3\})), \\ (((x_1^1, x_1^2, x_3^3), e_3, 0), (\{^{(0.3)}a_1^1, ^{(0.4)}a_2^1, ^{(0.5)}a_3^1\}, \{^{(0.3)}a_1^2, ^{(0.3)}a_2^2\}, \{^{(0.3)}a_1^3, ^{(0.8)}a_2^3, ^{(0.3)}a_3^3\})) \end{array} \right\}.$$

Proposition 5.11. *Let $(\widetilde{\mathcal{H}}_n, \mathfrak{Q}) \in \mathfrak{C}_{EN_n}\langle \mathfrak{A}, \mathfrak{P} \rangle$. Then, we have the following.*

- **(i):** $((\widetilde{\mathcal{H}}_n, \mathfrak{Q})^{r_{EN_n}})^{r_{EN_n}} =_{EN_n} (\widetilde{\mathcal{H}}_n, \mathfrak{Q})$.
- **(ii):** $(\widehat{A}_{\mathfrak{Q}}^{EN_n})^{r_{EN}} =_{EN_n} (\widehat{\emptyset}_{\mathfrak{Q}}^{EN_n})$.
- **(iii):** $(\widehat{\emptyset}_{\mathfrak{Q}}^{EN_n})^{r_{EN_n}} =_{EN_n} (\widehat{A}_{\mathfrak{Q}}^{EN_n})$.

Proof. The proofs are straightforward. □

Definition 5.12. Let $(\widetilde{\mathcal{H}}_n, \mathfrak{Q}), (\widetilde{\mathcal{K}}_n, \mathfrak{R}) \in \mathfrak{C}_{EN_n}\langle \mathfrak{A}, \mathfrak{P} \rangle$. Then, the restricted intersection of n-ary fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}_n, \mathfrak{Q})$ and $(\widetilde{\mathcal{K}}_n, \mathfrak{R})$ is denoted and defined by $(\widetilde{\mathcal{L}}_n, \mathfrak{S}) = (\widetilde{\mathcal{H}}_n, \mathfrak{Q}) \Cap_{EN_n} (\widetilde{\mathcal{K}}_n, \mathfrak{R})$ where $\mathfrak{S} = \mathfrak{Q} \cap \mathfrak{R}$ and

$$\widetilde{\mathcal{L}}_{n(j)}((\mathbf{x}^{\mathbf{I}}, e, o)) = \widetilde{\mathcal{H}}_{n(j)}((\mathbf{x}^{\mathbf{I}}, e, o)) \cap_f \widetilde{\mathcal{K}}_{n(j)}((\mathbf{x}^{\mathbf{I}}, e, o)) \quad \forall j \in J \tag{26}$$

for each $(\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{S}$.

Definition 5.13. Let $(\widetilde{\mathcal{H}}_n, \mathfrak{Q}), (\widetilde{\mathcal{K}}_n, \mathfrak{R}) \in \mathfrak{C}_{EN_n}\langle \mathfrak{A}, \mathfrak{P} \rangle$. Then, the extended intersection of n-ary fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}_n, \mathfrak{Q})$ and $(\widetilde{\mathcal{K}}_n, \mathfrak{R})$ is denoted and defined by $(\widetilde{\mathcal{L}}_n, \mathfrak{S}) = (\widetilde{\mathcal{H}}_n, \mathfrak{Q}) \sqcap_{EN_n} (\widetilde{\mathcal{K}}_n, \mathfrak{R})$ where $\mathfrak{S} = \mathfrak{Q} \cup \mathfrak{R}$ and

$$\widetilde{\mathcal{L}}_n((\mathbf{x}^{\mathbf{I}}, e, o)) = \left\{ \begin{array}{l} \widetilde{\mathcal{H}}_n((\mathbf{x}^{\mathbf{I}}, e, o)), \quad \text{if} \quad (\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{Q}, \\ \widetilde{\mathcal{K}}_n((\mathbf{x}^{\mathbf{I}}, e, o)), \quad \text{if} \quad (\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{R}, \\ \widetilde{\mathcal{H}}_n((\mathbf{x}^{\mathbf{I}}, e, o)) \cap_f \widetilde{\mathcal{K}}_n((\mathbf{x}^{\mathbf{I}}, e, o)), \quad \text{if} \quad (\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{Q} \cap \mathfrak{R}, \end{array} \right. \tag{27}$$

for each $(\mathbf{x}^{\mathbf{I}}, e, o) \in \mathfrak{S}$, where Eq. (26) is applied to obtain $\widetilde{\mathcal{H}}_n((\mathbf{x}^{\mathbf{I}}, e, o)) \cap_f \widetilde{\mathcal{K}}_n(\mathbf{x}^{\mathbf{I}})$.

Example 5.14. Consider the ternary fuzzy hypersoft expert set $(\widetilde{\mathcal{H}}_3, \mathfrak{Q})$ in Example 5.2. Also, we suppose that

$$\mathfrak{R} = \left\{ \begin{array}{l} (((x_1^1, x_1^2, x_2^3), e_1, 1), (((x_1^1, x_1^2, x_2^3), e_3, 1), (((x_1^1, x_1^2, x_3^3), e_1, 1), (((x_1^1, x_1^2, x_3^3), e_2, 1), \\ (((x_1^1, x_1^2, x_2^3), e_1, 0), (((x_1^1, x_1^2, x_2^3), e_3, 0), (((x_1^1, x_1^2, x_3^3), e_1, 0), (((x_1^1, x_1^2, x_3^3), e_2, 0) \end{array} \right\} \subseteq \mathbf{X} \times \mathcal{E} \times \mathcal{O},$$

and

$$(\widetilde{\mathcal{K}}_3,\mathfrak{R})=\left\{\begin{array}{l}
(((x_1^1,x_1^2,x_2^3),e_1,1),(\{^{(0.4)}a_1^1,^{(0)}a_2^1,^{(0.7)}a_3^1\},\{^{(0.3)}a_1^2,^{(0.1)}a_2^2\},\{^{(0.6)}a_1^3,^{(0.4)}a_2^3,^{(0.3)}a_3^3\})),\\
(((x_1^1,x_1^2,x_2^3),e_3,1),(\{^{(0.2)}a_1^1,^{(0.6)}a_2^1,^{(0.3)}a_3^1\},\{^{(0.4)}a_1^2,^{(0.2)}a_2^2\},\{^{(0)}a_1^3,^{(0.4)}a_2^3,^{(0.7)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_1,1),(\{^{(0.7)}a_1^1,^{(0.4)}a_2^1,^{(0.4)}a_3^1\},\{^{(0.5)}a_1^2,^{(0.3)}a_2^2\},\{^{(0.3)}a_1^3,^{(0.2)}a_2^3,^{(0.2)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_2,1),(\{^{(0.1)}a_1^1,^{(0.1)}a_2^1,^{(0.5)}a_3^1\},\{^{(0)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.5)}a_1^3,^{(0.2)}a_2^3,^{(0.2)}a_3^3\})),\\
(((x_1^1,x_1^2,x_2^3),e_1,0),(\{^{(0.5)}a_1^1,^{(0.6)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.7)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.3)}a_1^3,^{(0.1)}a_2^3,^{(0.1)}a_3^3\})),\\
(((x_1^1,x_1^2,x_2^3),e_3,0),(\{^{(0.7)}a_1^1,^{(0.3)}a_2^1,^{(0.3)}a_3^1\},\{^{(1)}a_1^2,^{(0.3)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.1)}a_2^3,^{(0)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_1,0),(\{^{(0.3)}a_1^1,^{(0)}a_2^1,^{(0.4)}a_3^1\},\{^{(0.4)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.2)}a_1^3,^{(0.5)}a_2^3,^{(0.6)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_2,0),(\{^{(0.2)}a_1^1,^{(0.3)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.5)}a_1^2,^{(0.6)}a_2^2\},\{^{(0.2)}a_1^3,^{(0.4)}a_2^3,^{(0.5)}a_3^3\}))
\end{array}\right\}.$$

Then, the restricted intersection and extended intersection of ternary fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}_3,\mathfrak{Q})$ and $(\widetilde{\mathcal{K}}_3,\mathfrak{R})$ are respectively

$$(\widetilde{\mathcal{H}}_3,\mathfrak{Q})\Cap_{EN_3}(\widetilde{\mathcal{K}}_3,\mathfrak{R})$$
$$=\left\{\begin{array}{l}
(((x_1^1,x_1^2,x_3^3),e_1,1),(\{^{(0.7)}a_1^1,^{(0.2)}a_2^1,^{(0.4)}a_3^1\},\{^{(0.3)}a_1^2,^{(0.3)}a_2^2\},\{^{(0.3)}a_1^3,^{(0.2)}a_2^3,^{(0.2)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_2,1),(\{^{(0.1)}a_1^1,^{(0.1)}a_2^1,^{(0.5)}a_3^1\},\{^{(0)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.5)}a_1^3,^{(0.2)}a_2^3,^{(0.2)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_1,0),(\{^{(0.1)}a_1^1,^{(0)}a_2^1,^{(0.4)}a_3^1\},\{^{(0.4)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.2)}a_1^3,^{(0.5)}a_2^3,^{(0.1)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_2,0),(\{^{(0.1)}a_1^1,^{(0.3)}a_2^1,^{(0.3)}a_3^1\},\{^{(0.2)}a_1^2,^{(0)}a_2^2\},\{^{(0.2)}a_1^3,^{(0.4)}a_2^3,^{(0.3)}a_3^3\}))
\end{array}\right\},$$

and

$$(\widetilde{\mathcal{H}}_3,\mathfrak{Q})\sqcap_{EN_3}(\widetilde{\widetilde{\mathcal{H}}}_3,\mathfrak{R})$$
$$=\left\{\begin{array}{l}
(((x_1^1,x_1^2,x_1^3),e_1,1),(\{^{(0.5)}a_1^1,^{(0.6)}a_2^1,^{(0.3)}a_3^1\},\{^{(0.4)}a_1^2,^{(0.5)}a_2^2\},\{^{(0.1)}a_1^3,^{(0.6)}a_2^3,^{(0.2)}a_3^3\})),\\
(((x_1^1,x_1^2,x_1^3),e_2,1),(\{^{(0.2)}a_1^1,^{(0.4)}a_2^1,^{(0.7)}a_3^1\},\{^{(0.3)}a_1^2,^{(0.6)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.4)}a_2^3,^{(0.1)}a_3^3\})),\\
(((x_1^1,x_1^2,x_1^3),e_3,1),(\{^{(0.4)}a_1^1,^{(0.3)}a_2^1,^{(0.3)}a_3^1\},\{^{(0.7)}a_1^2,^{(0.2)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.3)}a_2^3,^{(0.5)}a_3^3\})),\\
(((x_1^1,x_1^2,x_2^3),e_1,1),(\{^{(0.4)}a_1^1,^{(0)}a_2^1,^{(0.7)}a_3^1\},\{^{(0.3)}a_1^2,^{(0.1)}a_2^2\},\{^{(0.6)}a_1^3,^{(0.4)}a_2^3,^{(0.3)}a_3^3\})),\\
(((x_1^1,x_1^2,x_2^3),e_3,1),(\{^{(0.2)}a_1^1,^{(0.6)}a_2^1,^{(0.3)}a_3^1\},\{^{(0.4)}a_1^2,^{(0.2)}a_2^2\},\{^{(0)}a_1^3,^{(0.4)}a_2^3,^{(0.7)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_1,1),(\{^{(0.7)}a_1^1,^{(0.2)}a_2^1,^{(0.4)}a_3^1\},\{^{(0.3)}a_1^2,^{(0.3)}a_2^2\},\{^{(0.3)}a_1^3,^{(0.2)}a_2^3,^{(0.2)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_2,1),(\{^{(0.1)}a_1^1,^{(0.1)}a_2^1,^{(0.5)}a_3^1\},\{^{(0)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.5)}a_1^3,^{(0.2)}a_2^3,^{(0.2)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_3,1),(\{^{(0.3)}a_1^1,^{(0.3)}a_2^1,^{(0.6)}a_3^1\},\{^{(0.5)}a_1^2,^{(0.3)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.2)}a_2^3,^{(0.6)}a_3^3\})),\\
(((x_1^1,x_1^2,x_1^3),e_1,0),(\{^{(0.4)}a_1^1,^{(0.2)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.6)}a_1^2,^{(0.2)}a_2^2\},\{^{(0.1)}a_1^3,^{(0.2)}a_2^3,^{(0.3)}a_3^3\})),\\
(((x_1^1,x_1^2,x_1^3),e_2,0),(\{^{(0.5)}a_1^1,^{(0.6)}a_2^1,^{(0.3)}a_3^1\},\{^{(0.3)}a_1^2,^{(0.3)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.5)}a_2^3,^{(0.1)}a_3^3\})),\\
(((x_1^1,x_1^2,x_1^3),e_3,0),(\{^{(0.7)}a_1^1,^{(0.7)}a_2^1,^{(0.7)}a_3^1\},\{^{(0.5)}a_1^2,^{(0.2)}a_2^2\},\{^{(0)}a_1^3,^{(0.4)}a_2^3,^{(0.3)}a_3^3\})),\\
(((x_1^1,x_1^2,x_2^3),e_1,0),(\{^{(0.5)}a_1^1,^{(0.6)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.7)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.3)}a_1^3,^{(0.1)}a_2^3,^{(0.1)}a_3^3\})),\\
(((x_1^1,x_1^2,x_2^3),e_3,0),(\{^{(0.7)}a_1^1,^{(0.3)}a_2^1,^{(0.3)}a_3^1\},\{^{(1)}a_1^2,^{(0.3)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.1)}a_2^3,^{(0)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_1,0),(\{^{(0.1)}a_1^1,^{(0)}a_2^1,^{(0.4)}a_3^1\},\{^{(0.4)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.2)}a_1^3,^{(0.5)}a_2^3,^{(0.1)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_2,0),(\{^{(0.1)}a_1^1,^{(0.3)}a_2^1,^{(0.3)}a_3^1\},\{^{(0.2)}a_1^2,^{(0)}a_2^2\},\{^{(0.2)}a_1^3,^{(0.4)}a_2^3,^{(0.3)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_3,0),(\{^{(0.7)}a_1^1,^{(0.6)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.7)}a_1^2,^{(0.7)}a_2^2\},\{^{(0.7)}a_1^3,^{(0.2)}a_2^3,^{(0.7)}a_3^3\}))
\end{array}\right\}.$$

Definition 5.15. Let $(\widetilde{\mathcal{H}}_n,\mathfrak{Q}),(\widetilde{\mathcal{K}}_n,\mathfrak{R}) \in \mathfrak{C}_{EN_n}\langle\mathfrak{A},\mathfrak{P}\rangle$. Then, the restricted union of n-ary fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}_n,\mathfrak{Q})$ and $(\widetilde{\mathcal{K}}_n,\mathfrak{R})$ is denoted and defined by $(\widetilde{\mathcal{L}}_n,\mathfrak{S}) = (\widetilde{\mathcal{H}}_n,\mathfrak{Q}) \Cup_{EN_n} (\widetilde{\mathcal{K}}_n,\mathfrak{R})$ where $\mathfrak{S} = \mathfrak{Q}\cap\mathfrak{R}$ and

$$\widetilde{\mathcal{L}}_{n(j)}((\mathbf{x}^{\mathbf{I}},e,o)) = \widetilde{\mathcal{H}}_{n(j)}((\mathbf{x}^{\mathbf{I}},e,o)) \cup_f \widetilde{\mathcal{K}}_{n(j)}((\mathbf{x}^{\mathbf{I}},e,o)) \quad \forall j \in J \tag{28}$$

for each $(\mathbf{x}^{\mathbf{I}},e,o) \in \mathfrak{S}$.

Definition 5.16. Let $(\widetilde{\mathcal{H}}_n,\mathfrak{Q}),(\widetilde{\mathcal{K}}_n,\mathfrak{R}) \in \mathfrak{C}_{EN_n}\langle\mathfrak{A},\mathfrak{P}\rangle$. Then, the extended union of n-ary fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}_n,\mathfrak{Q})$ and $(\widetilde{\mathcal{K}}_n,\mathfrak{R})$ is denoted and defined by $(\widetilde{\mathcal{L}}_n,\mathfrak{S}) = (\widetilde{\mathcal{H}}_n,\mathfrak{Q}) \sqcup_{EN_n} (\widetilde{\mathcal{K}}_n,\mathfrak{R})$ where $\mathfrak{S} = \mathfrak{Q}\cup\mathfrak{R}$ and

$$\widetilde{\mathcal{L}}_n((\mathbf{x}^{\mathbf{I}},e,o)) = \begin{cases} \widetilde{\mathcal{H}}_n((\mathbf{x}^{\mathbf{I}},e,o)), & \text{if} \quad (\mathbf{x}^{\mathbf{I}},e,o) \in \mathfrak{Q}, \\ \widetilde{\mathcal{K}}_n((\mathbf{x}^{\mathbf{I}},e,o)), & \text{if} \quad (\mathbf{x}^{\mathbf{I}},e,o) \in \mathfrak{R}, \\ \widetilde{\mathcal{H}}_n((\mathbf{x}^{\mathbf{I}},e,o)) \cup_f \widetilde{\mathcal{K}}_n((\mathbf{x}^{\mathbf{I}},e,o)), & \text{if} \quad (\mathbf{x}^{\mathbf{I}},e,o) \in \mathfrak{Q}\cap\mathfrak{R}, \end{cases} \tag{29}$$

for each $(\mathbf{x}^{\mathbf{I}},e,o) \in \mathfrak{S}$, where Eq. (28) is applied to obtain $\widetilde{\mathcal{H}}_n((\mathbf{x}^{\mathbf{I}},e,o)) \cup_f \widetilde{\mathcal{K}}_n(\mathbf{x}^{\mathbf{I}})$.

Example 5.17. We consider the ternary fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}_3,\mathfrak{Q})$ and $(\widetilde{\mathcal{K}}_3,\mathfrak{R})$ in Examples 5.2 and 5.14, respectively. Then, the restricted union and extended union of ternary fuzzy hypersoft expert sets $(\widetilde{\mathcal{H}}_3,\mathfrak{Q})$ and $(\widetilde{\mathcal{K}}_3,\mathfrak{R})$ are respectively

$$(\widetilde{\mathcal{H}}_3,\mathfrak{Q}) \Cup_{EN_3} (\widetilde{\mathcal{K}}_3,\mathfrak{R}) \\ = \left\{ \begin{array}{l} (((x_1^1,x_1^2,x_3^3),e_1,1),(\{^{(0.7)}a_1^1,^{(0.4)}a_2^1,^{(0.4)}a_3^1\},\{^{(0.5)}a_1^2,^{(0.6)}a_2^2\},\{^{(0.3)}a_1^3,^{(0.4)}a_2^3,^{(1)}a_3^3\})), \\ (((x_1^1,x_1^2,x_3^3),e_2,1),(\{^{(1)}a_1^1,^{(0.2)}a_2^1,^{(0.9)}a_3^1\},\{^{(0.4)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.5)}a_1^3,^{(0.2)}a_2^3,^{(0.3)}a_3^3\})), \\ (((x_1^1,x_1^2,x_3^3),e_1,0),(\{^{(0.3)}a_1^1,^{(0.6)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.7)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.2)}a_1^3,^{(0.5)}a_2^3,^{(0.6)}a_3^3\})), \\ (((x_1^1,x_1^2,x_3^3),e_2,0),(\{^{(0.2)}a_1^1,^{(0.5)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.5)}a_1^2,^{(0.6)}a_2^2\},\{^{(1)}a_1^3,^{(0.5)}a_2^3,^{(0.5)}a_3^3\})) \end{array} \right\},$$

and

$$(\widetilde{\mathcal{H}}_3,\mathfrak{Q})\sqcup_{EN_3}(\widetilde{\mathcal{K}}_3,\mathfrak{R})$$
$$=\left\{\begin{array}{l}
(((x_1^1,x_1^2,x_1^3),e_1,1),(\{^{(0.5)}a_1^1,^{(0.6)}a_2^1,^{(0.3)}a_3^1\},\{^{(0.4)}a_1^2,^{(0.5)}a_2^2\},\{^{(0.1)}a_1^3,^{(0.6)}a_2^3,^{(0.2)}a_3^3\})),\\
(((x_1^1,x_1^2,x_1^3),e_2,1),(\{^{(0.2)}a_1^1,^{(0.4)}a_2^1,^{(0.7)}a_3^1\},\{^{(0.3)}a_1^2,^{(0.6)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.4)}a_2^3,^{(0.1)}a_3^3\})),\\
(((x_1^1,x_1^2,x_1^3),e_3,1),(\{^{(0.4)}a_1^1,^{(0.3)}a_2^1,^{(0.3)}a_3^1\},\{^{(0.7)}a_1^2,^{(0.2)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.3)}a_2^3,^{(0.5)}a_3^3\})),\\
(((x_1^1,x_1^2,x_2^3),e_1,1),(\{^{(0.4)}a_1^1,^{(0)}a_2^1,^{(0.7)}a_3^1\},\{^{(0.3)}a_1^2,^{(0.1)}a_2^2\},\{^{(0.6)}a_1^3,^{(0.4)}a_2^3,^{(0.3)}a_3^3\})),\\
(((x_1^1,x_1^2,x_2^3),e_3,1),(\{^{(0.2)}a_1^1,^{(0.6)}a_2^1,^{(0.3)}a_3^1\},\{^{(0.4)}a_1^2,^{(0.2)}a_2^2\},\{^{(0)}a_1^3,^{(0.4)}a_2^3,^{(0.7)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_1,1),(\{^{(0.7)}a_1^1,^{(0.4)}a_2^1,^{(0.4)}a_3^1\},\{^{(0.5)}a_1^2,^{(0.6)}a_2^2\},\{^{(0.3)}a_1^3,^{(0.4)}a_2^3,^{(1)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_2,1),(\{^{(1)}a_1^1,^{(0.2)}a_2^1,^{(0.9)}a_3^1\},\{^{(0.4)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.5)}a_1^3,^{(0.2)}a_2^3,^{(0.3)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_3,1),(\{^{(0.3)}a_1^1,^{(0.3)}a_2^1,^{(0.6)}a_3^1\},\{^{(0.5)}a_1^2,^{(0.3)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.2)}a_2^3,^{(0.6)}a_3^3\})),\\
(((x_1^1,x_1^2,x_1^3),e_1,0),(\{^{(0.4)}a_1^1,^{(0.2)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.6)}a_1^2,^{(0.2)}a_2^2\},\{^{(0.1)}a_1^3,^{(0.2)}a_2^3,^{(0.3)}a_3^3\})),\\
(((x_1^1,x_1^2,x_1^3),e_2,0),(\{^{(0.5)}a_1^1,^{(0.6)}a_2^1,^{(0.3)}a_3^1\},\{^{(0.3)}a_1^2,^{(0.3)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.5)}a_2^3,^{(0.1)}a_3^3\})),\\
(((x_1^1,x_1^2,x_1^3),e_3,0),(\{^{(0.7)}a_1^1,^{(0.7)}a_2^1,^{(0.7)}a_3^1\},\{^{(0.5)}a_1^2,^{(0.2)}a_2^2\},\{^{(0)}a_1^3,^{(0.4)}a_2^3,^{(0.3)}a_3^3\})),\\
(((x_1^1,x_1^2,x_2^3),e_1,0),(\{^{(0.5)}a_1^1,^{(0.6)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.7)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.3)}a_1^3,^{(0.1)}a_2^3,^{(0.1)}a_3^3\})),\\
(((x_1^1,x_1^2,x_2^3),e_3,0),(\{^{(0.7)}a_1^1,^{(0.3)}a_2^1,^{(0.3)}a_3^1\},\{^{(1)}a_1^2,^{(0.3)}a_2^2\},\{^{(0.4)}a_1^3,^{(0.1)}a_2^3,^{(0)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_1,0),(\{^{(0.3)}a_1^1,^{(0.6)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.7)}a_1^2,^{(0.4)}a_2^2\},\{^{(0.2)}a_1^3,^{(0.5)}a_2^3,^{(0.6)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_2,0),(\{^{(0.2)}a_1^1,^{(0.5)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.5)}a_1^2,^{(0.6)}a_2^2\},\{^{(1)}a_1^3,^{(0.5)}a_2^3,^{(0.5)}a_3^3\})),\\
(((x_1^1,x_1^2,x_3^3),e_3,0),(\{^{(0.7)}a_1^1,^{(0.6)}a_2^1,^{(0.5)}a_3^1\},\{^{(0.7)}a_1^2,^{(0.7)}a_2^2\},\{^{(0.7)}a_1^3,^{(0.2)}a_2^3,^{(0.7)}a_3^3\}))
\end{array}\right\}.$$

Proposition 5.18. *Let* $(\widetilde{\mathcal{H}}_n,\mathfrak{Q}),(\widetilde{\mathcal{K}}_n,\mathfrak{R}),(\widetilde{\mathcal{L}}_n,\mathfrak{S})\in\mathfrak{C}_{EN_n}\langle\mathfrak{A},\mathfrak{P}\rangle$*. Then, the following properties are satisfied.*

- **(i):** $(\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond(\widetilde{\mathcal{K}}_n,\mathfrak{R})=_{EN_n}(\widetilde{\mathcal{K}}_n,\mathfrak{R})\diamond(\widetilde{\mathcal{H}}_n,\mathfrak{Q})$ *for each* $\diamond\in\{\Cap_{EN_n},\Cup_{EN_n}\}$.
- **(ii):** $(\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond((\widetilde{\mathcal{K}}_n,\mathfrak{R})\diamond(\widetilde{\mathcal{L}}_n,\mathfrak{S}))=_{EN_n}((\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond(\widetilde{\mathcal{K}}_n,\mathfrak{R}))\diamond(\widetilde{\mathcal{L}}_n,\mathfrak{S})$ *for each* $\diamond\in\{\Cap_{EN_n},\Cup_{EN_n}\}$.
- **(iii):** $(\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond((\widetilde{\mathcal{H}}_n,\mathfrak{R})\circ(\widetilde{\mathcal{L}}_n,\mathfrak{S}))=_{EN_n}((\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond(\widetilde{\mathcal{K}}_n,\mathfrak{R}))\circ((\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond(\widetilde{\mathcal{L}}_n,\mathfrak{S}))$ *for each* $\diamond,\circ\in\{\Cap_{EN_n},\Cup_{EN_n}\}$.
- **(iv):** $((\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond(\widetilde{\mathcal{K}}_n,\mathfrak{R}))^{r_{EN_n}}=_{EN_n}(\widetilde{\mathcal{H}}_n,\mathfrak{Q})^{r_{EN_n}}\circ(\widetilde{\mathcal{K}}_n,\mathfrak{R})^{r_{EN_n}}$ *for each* $\diamond,\circ\in\{\Cap_{EN_n},\Cup_{EN_n}\}$ *and* $\diamond\neq\circ$.

Proof. The proofs are straightforward. □

Proposition 5.19. *Let* $(\widetilde{\mathcal{H}}_n,\mathfrak{Q}),(\widetilde{\mathcal{K}}_n,\mathfrak{R}),(\widetilde{\mathcal{L}}_n,\mathfrak{S})\in\mathfrak{C}_{EN_n}\langle\mathfrak{A},\mathfrak{P}\rangle$*. Then, the following properties are satisfied.*

- **(i):** $(\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond(\widetilde{\mathcal{K}}_n,\mathfrak{R})=_{EN_n}(\widetilde{\mathcal{K}}_n,\mathfrak{R})\diamond(\widetilde{\mathcal{H}}_n,\mathfrak{Q})$ *for each* $\diamond\in\{\sqcap_{EN_n},\sqcup_{EN_n}\}$.
- **(ii):** $(\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond((\widetilde{\mathcal{K}}_n,\mathfrak{R})\diamond(\widetilde{\mathcal{L}}_n,\mathfrak{S}))=_{EN_n}((\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond(\widetilde{\mathcal{K}}_n,\mathfrak{R}))\diamond(\widetilde{\mathcal{L}}_n,\mathfrak{S})$ *for each* $\diamond\in\{\sqcap_{EN_n},\sqcup_{EN_n}\}$.
- **(iii):** $(\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond((\widetilde{\mathcal{H}}_n,\mathfrak{R})\circ(\widetilde{\mathcal{L}}_n,\mathfrak{S}))=_{EN_n}((\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond(\widetilde{\mathcal{K}}_n,\mathfrak{R}))\circ((\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond(\widetilde{\mathcal{L}}_n,\mathfrak{S}))$ *for each* $\diamond,\circ\in\{\sqcap_{EN_n},\sqcup_{EN_n}\}$.
- **(iv):** $((\widetilde{\mathcal{H}}_n,\mathfrak{Q})\diamond(\widetilde{\mathcal{K}}_n,\mathfrak{R}))^{r_{EN_n}}=_{EN_n}(\widetilde{\mathcal{H}}_n,\mathfrak{Q})^{r_{EN_n}}\circ(\widetilde{\mathcal{K}}_n,\mathfrak{R})^{r_{EN_n}}$ *for each* $\diamond,\circ\in\{\sqcap_{EN_n},\sqcup_{EN_n}\}$ *and* $\diamond\neq\circ$.

Proof. The proofs are straightforward. □

6. An Application of n-ary Fuzzy Hypersoft Expert Sets

In this section, we present a possible application of n-ary fuzzy hypersoft expert set theory in real-life problem.

We construct the following algorithm to determine the optimal choice in each universal set in the n-ary fuzzy hypersoft expert set setting.

Algorithm 1.

(1) Input the n-ary fuzzy hypersoft expert set.
(2) Construct (in the tabular form) the agree n-ary fuzzy hypersoft expert set and the disagree n-ary fuzzy hypersoft expert set.
(3) For the agree n-ary fuzzy hypersoft expert set, calculate $c_k^j = \sum_l (a_k^j)_l$ for each $j \in J$.
(4) For the disagree n-ary fuzzy hypersoft expert set, calculate $d_k^j = \sum_l (a_k^j)_l$ for each $j \in J$.
(5) Compute score value $s_k^j = c_k^j - d_k^j$ for each $j \in J$.
(6) Find γ_j for each $j \in J$ which $s_{\gamma_j} = \max s_k^j$ for each $j \in J$, and then determine the optimal choice a_γ^j in each universal set A_j.

In the tables of agree n-ary fuzzy hypersoft expert set and disagree n-ary fuzzy hypersoft expert set, $(a_k^j)_l$ corresponds the fuzzy value in the l-th row for a_k^j.

Now, we ready to give an application of n-ary fuzzy hypersoft expert set theory in handling real-life problem.

Example 6.1. Suppose that an association wants to determine the best films of the year in the fields of drama, comedy and documentary and to award these films at an award ceremony to be organized by the association. The sets of nominated films of drama, comedy and documentary are $A_1 = \{a_1^1, a_2^1, a_3^1\}$, $A_2 = \{a_1^2, a_2^2\}$ and $A_3 = \{a_1^3, a_2^3, a_3^3\}$, respectively. Also, the association hires the jury members (experts) to determine the best of the films in each category. Assume that $\mathcal{E} = \{e_1, e_2\}$ is a set of jury. The jury should analyze the characteristics or attributes of these films. Therefore, they consider the disjoint parameter sets X_1, X_2 and X_3 based on the story, message and narration of film, respectively. These sets are $X_1 = \{x_1^1 = originality,\ x_2^1 = fiction\}$, $X_2 = \{x_1^2 = ease\ of\ perception,\ x_2^2 = essence\}$, and $X_3 = \{x_1^3 = narrative\ style,\ x_2^3 = audiovisual\ quality\}$, and thereby $\mathbf{X} = X_1 \times X_2 \times X_3$. Following the serious discussion, the jury constructs the following ternary fuzzy hypersoft expert set for $\mathfrak{P} = \mathbf{X} \times \mathcal{E} \times \mathcal{O}$.

$$
(\tilde{\tilde{\mathcal{H}}}, \mathfrak{P}) = \left\{ \begin{array}{l}
(((x_1^1, x_1^2, x_1^3), e_1, 1), (\{{}^{(0.4)}a_1^1, {}^{(0.3)}a_2^1, {}^{(0.2)}a_3^1\}, \{{}^{(0.1)}a_1^2, {}^{(0.7)}a_2^2\}, \{{}^{(0.3)}a_1^3, {}^{(0.4)}a_2^3, {}^{(0.1)}a_3^3\})), \\
(((x_1^1, x_1^2, x_1^3), e_2, 1), (\{{}^{(0.4)}a_1^1, {}^{(0.6)}a_2^1, {}^{(0.4)}a_3^1\}, \{{}^{(0.3)}a_1^2, {}^{(0.2)}a_2^2\}, \{{}^{(0.6)}a_1^3, {}^{(0.8)}a_2^3, {}^{(0.1)}a_3^3\})), \\
(((x_1^1, x_1^2, x_2^3), e_1, 1), (\{{}^{(0.7)}a_1^1, {}^{(0.5)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.2)}a_2^2\}, \{{}^{(0.2)}a_1^3, {}^{(0.1)}a_2^3, {}^{(0.4)}a_3^3\})), \\
(((x_1^1, x_1^2, x_2^3), e_2, 1), (\{{}^{(0.4)}a_1^1, {}^{(0.4)}a_2^1, {}^{(0.2)}a_3^1\}, \{{}^{(0.2)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.3)}a_1^3, {}^{(0.1)}a_2^3, {}^{(0.2)}a_3^3\})), \\
(((x_1^1, x_2^2, x_1^3), e_1, 1), (\{{}^{(0.4)}a_1^1, {}^{(0.2)}a_2^1, {}^{(0.2)}a_3^1\}, \{{}^{(0.6)}a_1^2, {}^{(0.4)}a_2^2\}, \{{}^{(0.3)}a_1^3, {}^{(0.2)}a_2^3, {}^{(0.1)}a_3^3\})), \\
(((x_1^1, x_2^2, x_1^3), e_2, 1), (\{{}^{(0.4)}a_1^1, {}^{(0.2)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.2)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.4)}a_2^3, {}^{(0.1)}a_3^3\})), \\
(((x_1^1, x_2^2, x_2^3), e_1, 1), (\{{}^{(0.1)}a_1^1, {}^{(0.6)}a_2^1, {}^{(0.5)}a_3^1\}, \{{}^{(0.5)}a_1^2, {}^{(0)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.3)}a_2^3, {}^{(0.5)}a_3^3\})), \\
(((x_1^1, x_2^2, x_2^3), e_2, 1), (\{{}^{(0.4)}a_1^1, {}^{(0.7)}a_2^1, {}^{(0.6)}a_3^1\}, \{{}^{(0.8)}a_1^2, {}^{(0.7)}a_2^2\}, \{{}^{(0)}a_1^3, {}^{(0.6)}a_2^3, {}^{(1)}a_3^3\})), \\
(((x_2^1, x_1^2, x_1^3), e_1, 1), (\{{}^{(0.4)}a_1^1, {}^{(0.4)}a_2^1, {}^{(0.2)}a_3^1\}, \{{}^{(0.3)}a_1^2, {}^{(0.6)}a_2^2\}, \{{}^{(0.8)}a_1^3, {}^{(0.4)}a_2^3, {}^{(0.4)}a_3^3\})), \\
(((x_2^1, x_1^2, x_1^3), e_2, 1), (\{{}^{(0.2)}a_1^1, {}^{(0.5)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.2)}a_1^2, {}^{(0.1)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.5)}a_2^3, {}^{(0.3)}a_3^3\})), \\
(((x_2^1, x_1^2, x_2^3), e_1, 1), (\{{}^{(0.5)}a_1^1, {}^{(0.6)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.4)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.2)}a_3^3\})), \\
(((x_2^1, x_1^2, x_2^3), e_2, 1), (\{{}^{(0.3)}a_1^1, {}^{(0.6)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.3)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.2)}a_3^3\})), \\
(((x_2^1, x_2^2, x_1^3), e_1, 1), (\{{}^{(0.1)}a_1^1, {}^{(0.6)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.4)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.2)}a_3^3\})), \\
(((x_2^1, x_2^2, x_1^3), e_2, 1), (\{{}^{(0.7)}a_1^1, {}^{(0.7)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.2)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.2)}a_3^3\})), \\
(((x_2^1, x_2^2, x_2^3), e_1, 1), (\{{}^{(0.7)}a_1^1, {}^{(0.3)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.3)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.8)}a_2^3, {}^{(0.7)}a_3^3\})), \\
(((x_2^1, x_2^2, x_2^3), e_2, 1), (\{{}^{(0.7)}a_1^1, {}^{(0.1)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.2)}a_3^3\})), \\
(((x_1^1, x_1^2, x_1^3), e_1, 0), (\{{}^{(0.5)}a_1^1, {}^{(0.5)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(1)}a_2^2\}, \{{}^{(0.2)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.5)}a_3^3\})), \\
(((x_1^1, x_1^2, x_1^3), e_2, 0), (\{{}^{(0.4)}a_1^1, {}^{(0.6)}a_2^1, {}^{(0.4)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.4)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.4)}a_2^3, {}^{(0.4)}a_3^3\})), \\
(((x_1^1, x_1^2, x_2^3), e_1, 0), (\{{}^{(0.6)}a_1^1, {}^{(0.8)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.2)}a_3^3\})), \\
(((x_1^1, x_1^2, x_2^3), e_2, 0), (\{{}^{(0.9)}a_1^1, {}^{(0.5)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.2)}a_3^3\})), \\
(((x_1^1, x_2^2, x_1^3), e_1, 0), (\{{}^{(0.3)}a_1^1, {}^{(0.3)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.4)}a_3^3\})), \\
(((x_1^1, x_2^2, x_1^3), e_2, 0), (\{{}^{(0.3)}a_1^1, {}^{(0.4)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.6)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.4)}a_3^3\})), \\
(((x_1^1, x_2^2, x_2^3), e_1, 0), (\{{}^{(0.2)}a_1^1, {}^{(0.2)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.2)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.2)}a_3^3\})), \\
(((x_1^1, x_2^2, x_2^3), e_2, 0), (\{{}^{(0.2)}a_1^1, {}^{(0.2)}a_2^1, {}^{(0.4)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.3)}a_3^3\})), \\
(((x_2^1, x_1^2, x_1^3), e_1, 0), (\{{}^{(0.6)}a_1^1, {}^{(0.6)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.5)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.4)}a_3^3\})), \\
(((x_2^1, x_1^2, x_1^3), e_2, 0), (\{{}^{(0.8)}a_1^1, {}^{(0.6)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.2)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.2)}a_3^3\})), \\
(((x_2^1, x_1^2, x_2^3), e_1, 0), (\{{}^{(0.4)}a_1^1, {}^{(0.5)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(1)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.1)}a_3^3\})), \\
(((x_2^1, x_1^2, x_2^3), e_2, 0), (\{{}^{(0.3)}a_1^1, {}^{(0.4)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.5)}a_1^2, {}^{(0.5)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.4)}a_2^3, {}^{(0.2)}a_3^3\})), \\
(((x_2^1, x_2^2, x_1^3), e_1, 0), (\{{}^{(0.5)}a_1^1, {}^{(0.5)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.6)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.3)}a_3^3\})), \\
(((x_2^1, x_2^2, x_1^3), e_2, 0), (\{{}^{(0.4)}a_1^1, {}^{(0.5)}a_2^1, {}^{(0.2)}a_3^1\}, \{{}^{(0.4)}a_1^2, {}^{(0.4)}a_2^2\}, \{{}^{(0.5)}a_1^3, {}^{(0.6)}a_2^3, {}^{(0.4)}a_3^3\})), \\
(((x_2^1, x_2^2, x_2^3), e_1, 0), (\{{}^{(0.2)}a_1^1, {}^{(0.5)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.8)}a_1^2, {}^{(0.3)}a_2^2\}, \{{}^{(0.1)}a_1^3, {}^{(0.7)}a_2^3, {}^{(0.3)}a_3^3\})), \\
(((x_2^1, x_2^2, x_2^3), e_2, 0), (\{{}^{(0.3)}a_1^1, {}^{(0.7)}a_2^1, {}^{(0.3)}a_3^1\}, \{{}^{(0.2)}a_1^2, {}^{(0.6)}a_2^2\}, \{{}^{(0.3)}a_1^3, {}^{(0.7)}a_2^3, {}^{(0.1)}a_3^3\}))
\end{array} \right\}.
$$

The steps of Algorithm 1 may be followed by the association to determine the best of the films in each category.

In Tables 1 and 2, we give the agree ternary fuzzy hypersoft expert set and the disagree ternary fuzzy hypersoft expert set, respectively.

Table 1. Agree ternary fuzzy hypersoft expert set

	a_1^1	a_2^1	a_3^1	a_1^2	a_2^2	a_1^3	a_2^3	a_3^3
$((x_1^1, x_1^2, x_1^3), e_1, 1)$	0.4	0.3	0.2	0.1	0.7	0.3	0.4	0.1
$((x_1^1, x_1^2, x_1^3), e_2, 1)$	0.4	0.6	0.4	0.3	0.2	0.6	0.8	0.1
$((x_1^1, x_1^2, x_2^3), e_1, 1)$	0.7	0.5	0.3	0.4	0.2	0.2	0.1	0.4
$((x_1^1, x_1^2, x_2^3), e_2, 1)$	0.4	0.4	0.2	0.2	0.5	0.3	0.1	0.2
$((x_1^1, x_2^2, x_1^3), e_1, 1)$	0.4	0.2	0.2	0.6	0.4	0.3	0.2	0.1
$((x_1^1, x_2^2, x_1^3), e_2, 1)$	0.4	0.2	0.3	0.2	0.5	0.1	0.4	0.1
$((x_1^1, x_2^2, x_2^3), e_1, 1)$	0.1	0.6	0.5	0.5	0	0.1	0.3	0.5
$((x_1^1, x_2^2, x_2^3), e_2, 1)$	0.4	0.7	0.6	0.8	0.7	0	0.6	1
$((x_2^1, x_1^2, x_1^3), e_1, 1)$	0.4	0.4	0.2	0.3	0.6	0.8	0.4	0.4
$((x_2^1, x_1^2, x_1^3), e_2, 1)$	0.2	0.5	0.3	0.2	0.1	0.1	0.5	0.3
$((x_2^1, x_1^2, x_2^3), e_1, 1)$	0.5	0.6	0.3	0.4	0.5	0.4	0.6	0.2
$((x_2^1, x_1^2, x_2^3), e_2, 1)$	0.3	0.6	0.3	0.3	0.5	0.1	0.6	0.2
$((x_2^1, x_2^2, x_1^3), e_1, 1)$	0.1	0.6	0.3	0.4	0.4	0.1	0.6	0.2
$((x_2^1, x_2^2, x_1^3), e_2, 1)$	0.7	0.7	0.3	0.2	0.5	0.1	0.6	0.2
$((x_2^1, x_2^2, x_2^3), e_1, 1)$	0.7	0.3	0.3	0.4	0.3	0.1	0.8	0.7
$((x_2^1, x_2^2, x_2^3), e_2, 1)$	0.7	0.1	0.3	0.4	0.5	0.1	0.6	0.2

Table 2. Disagree ternary fuzzy hypersoft expert set

	a_1^1	a_2^1	a_3^1	a_1^2	a_2^2	a_1^3	a_2^3	a_3^3
$((x_1^1, x_1^2, x_1^3), e_1, 0)$	0.5	0.5	0.3	0.4	1	0.2	0.6	0.5
$((x_1^1, x_1^2, x_1^3), e_2, 0)$	0.4	0.6	0.4	0.4	0.4	0.1	0.4	0.4
$((x_1^1, x_1^2, x_2^3), e_1, 0)$	0.6	0.8	0.3	0.4	0.5	0.1	0.6	0.2
$((x_1^1, x_1^2, x_2^3), e_2, 0)$	0.9	0.5	0.3	0.4	0.5	0.1	0.6	0.2
$((x_1^1, x_2^2, x_1^3), e_1, 0)$	0.3	0.3	0.3	0.4	0.5	0.1	0.6	0.4
$((x_1^1, x_2^2, x_1^3), e_2, 0)$	0.3	0.4	0.3	0.4	0.6	0.1	0.6	0.4
$((x_1^1, x_2^2, x_2^3), e_1, 0)$	0.2	0.2	0.3	0.4	0.2	0.1	0.6	0.2
$((x_1^1, x_2^2, x_2^3), e_2, 0)$	0.2	0.2	0.4	0.4	0.5	0.1	0.6	0.3
$((x_2^1, x_1^2, x_1^3), e_1, 0)$	0.6	0.6	0.3	0.5	0.5	0.1	0.6	0.4
$((x_2^1, x_1^2, x_1^3), e_2, 0)$	0.8	0.6	0.3	0.4	0.2	0.1	0.6	0.2
$((x_2^1, x_1^2, x_2^3), e_1, 0)$	0.4	0.5	0.3	0.4	1	0.1	0.6	0.1
$((x_2^1, x_1^2, x_2^3), e_2, 0)$	0.3	0.4	0.3	0.5	0.5	0.1	0.4	0.2
$((x_2^1, x_2^2, x_1^3), e_1, 0)$	0.5	0.5	0.3	0.4	0.6	0.1	0.6	0.3
$((x_2^1, x_2^2, x_1^3), e_2, 0)$	0.4	0.5	0.2	0.4	0.4	0.5	0.6	0.4
$((x_2^1, x_2^2, x_2^3), e_1, 0)$	0.2	0.5	0.3	0.8	0.3	0.1	0.7	0.3
$((x_2^1, x_2^2, x_2^3), e_2, 0)$	0.3	0.7	0.3	0.2	0.6	0.3	0.7	0.1

From Tables 1 and Table 2, we have the score values in Table 3.

TABLE 3. Score values s_k^j

	a_1^1	a_2^1	a_3^1	a_1^2	a_2^2	a_1^3	a_2^3	a_3^3
$c_k^j = \sum_l (a_k^j)_l$	$c_1^1 = 6.8$	$c_2^1 = 7.3$	$c_3^1 = 5$	$c_1^2 = 5.7$	$c_2^2 = 6.6$	$c_1^3 = 3.7$	$c_2^3 = 7.6$	$c_3^3 = 5.1$
$d_k^j = \sum_l (a_k^j)_l$	$d_1^1 = 6.9$	$d_2^1 = 7.8$	$d_3^1 = 4.9$	$d_1^2 = 6.8$	$d_2^2 = 8.3$	$c_1^3 = 2.3$	$d_2^3 = 9.4$	$d_3^3 = 4.6$
$s_k^j = c_k^j - d_k^j$	$s_1^1 = -0.1$	$s_2^1 = -0.5$	$s_3^1 = 0.1$	$s_1^2 = -1.1$	$s_2^2 = -1.7$	$s_1^3 = 1.4$	$s_2^3 = -1.8$	$s_3^3 = 0.5$

Since $\max s_{\gamma_1}^1 = s_3^1$ and $\max s_{\gamma_2}^2 = s_1^2$ and $\max s_{\gamma_3}^3 = s_1^3$ from Table 3, the best films of the year in the fields of drama, comedy and documentary are determined as s_3^1, s_1^2 and s_1^3, respectively.

7. Conclusions

In this chapter, the concepts of n-ary fuzzy hypersoft set, fuzzy hypersoft expert set and n-ary fuzzy hypersoft expert set, which are effective mathematical models for dealing with many kinds of uncertainties in the real world, were introduced. Also, the basic operations such as complement, intersection and union of these emerged types of fuzzy hypersoft sets were defined and some of their remarkable properties were discussed. An application was given to illustrate how the n-ary fuzzy hypersoft expert set sets can be useful in solving a problem in real-life. The topics of future research may be developing the n-ary fuzzy hypersoft sets, fuzzy hypersoft expert sets and n-ary fuzzy hypersoft expert sets in theoretical aspects such as describing new operations and characteristic properties in more general frameworks, and also investigating their practical applications in decision making, medical diagnosis and game theory.

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The Application of Neutrosophic Hypersoft Set TOPSIS (NHSS-TOPSIS) in the Selection of Carbon Nano Tube based Field Effective Transistors CNTFETs

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Abstract: Carbon nano tubes (CNT) are the main parts of the electronic devices. Due to high carrier mobility, these devices have very high speed. CNT has a wide range of application in making sensors which can detect different types of diseases, materials, viruses and bacteria. The main problem in the CNT based field effect transistors (CNTFETs) is leakage current. For the reduction of leakage current, high k-gate dielectric are the most suitable materials. The purpose of this research is to find the most suitable high k-gate dielectric for the CNTFETs using Neutrosophic Hypersoft set TOPSIS (NHSS-TOPSIS). Since NHSS-TOPSIS are useful when attributes are more than one and are further bi-furcated. The concepts of TOPSIS has practical applications in computational intelligence, machine learning, image processing, neural networks, medical diagnosis, and decision analysis. A practical application for ranking of alternatives with newly developed NHSS-TOPSIS approach is illustrated by a numerical example for CNTFETs. The validity and superiority of NHSS-TOPSIS with existing approaches is also given with the help of a comparison analysis.

Keywords: Carbon Nano Tube (CNT), Neutrosophic Soft Set (NSS), Hypersoft Set, CNTFETs, K-Dielectrics, TOPSIS, Neutrosophic Hypersoft Set (NHSS), MCDM.

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1. Introduction

Carbon nano tubes (CNT) are long, thin cylinders of carbon. They can be considered as a sheet of graphite rolled into a cylinder. Since its discovery by Iijima in 1991, it has caught the attention of the researchers due to its remarkable structure, physical and chemical properties. CNT has high electron mobility and larger mean free path for carrier transport. The transistors whose structure is based on CNT have larger transconductance [1-5]. Most of the CNT based transistors are made of single walled CNTs. These single walled CNTs have one dimensional structure. Due to this, the carrier movement is limited to a single direction. So the scattering of charge carriers at different angles is prevented. Also, it causes the throw transfer of charge carriers [6-12].

The first CNT based transistor were constructed in 1998 after which a lot of research have been done to improve CNT based transistors. The switching in CNT based field effect transistors (CNTFET) may be arisen at contact or in bulk as compared to conventional bulk switched Si devices. Also, the switching process depends upon CNT diameter, nature of electrodes, geometry of electrodes, device geometry and gate dielectric [13-14]. CNTFETs have many advantages over Si based FETs. CNTFETs may operate in ballistic regime with high k-gate dielectrics which causes the transistor to work at higher speed. They are widely used for making sensors. These sensors are able to detect proteins, DNA sequence, bacteria and toxic materials [15-21].

The performance of CNTFETs is dependent upon source material, drain material and gate dielectric. With the advancement of the electronics, the size (dimension) of electronic devices is becoming smaller day by day. The decrease in size causes a leakage current. The main purpose of high k-gate dielectrics is to reduce leakage current. The high k-gate dielectric which have been used so far with CNTFETs are Hafnium Oxide (HfO_2), Aluminum Oxide (Al_2O_3), Hafnium Silicate ($HfSiO_4$), Zirconia (ZnO_2), Yttrium Oxide (Y_2O_3), Lanthanum Oxide (La_2O_3), Silicon dioxide (SiO_2) and Silicon Nitride (Si_3N_4) [22-31].

Zadeh [32] advanced his significant idea of fuzzy sets in 1965 to deal with various styles of uncertainties and proposed fuzzy sets and this theory was extended by [33] named as Intuitionistic fuzzy number theory. Smarandache [34] initiated the notion of neutrosophic sets which consider indeterminate/uncertain information in today's problems and incorporated not only membership and non-membership grades, but also indeterminacy grades assigned each component of the discourse universe with is limitation that the sum of three independent grades chosen in the interval [0,3]. Later on, fuzzy, intuitionist and neutrosophy theories were extended to fuzzy softset [35,36] intuitionistic soft set [37] and neutrosophic soft set [38]. Smarandache [39] generalized soft set to hyper soft set (HSS's) by changing the function into multi decision function. The HSS's was extended to neutrosophic hypersoft set (NHSS) [40] along with some MCDM techniques like TOPSIS and many applications [41-44]

In this research, eight different k-dielectrics are considered. The motivation of this research is to find which k-dielectric is most suitable alternate of CNTFETs for high power communication. The modelling of this problem shows that attributes are more than one, and are further bi-furcated. Thus, keeping in mind, the set structure of hypersoft sets we decided to apply NHSS-TOPSIS technique to find the optimal choice for k-dielectrics and results are compared with [45].

The remainder of this paper is structured as follows: Firstly, fundamental definitions are given about hypersoft set and neutrosophic hypersoft set theory. After, the step-wise algorithm of NHSS-TOPSIS is present. In section 3, we propose the idea of a case study of CNTFETs and some parameters have been considered. Section 4 provides the comparative analysis of calculation. Finally, in Section 5, a conclusion is outlined with future directions and limitations.

2.Preliminaries

Definition 2.1: Semiconductors [10]

These are the materials which are poorer conductors than metals but better than insulators. For example, Silicon (Si), Germanium (Ge) and Gallium Arsenide.

Definition 2.2: Transistors [6]

It is an electrical device which consist of two PN junctions fabricated on a same single crystal. It has three main parts i.e. emitter, base and collector. Transistors are used to amplify and switch the electronic signals and electrical power.

Definition 2.3: Dielectrics [30]

These are the materials which are poor conductors of electricity. It is an insulator but an effective supporter to electric filed.

Definition 2.4: High k Dielectrics Materials [31]

Dielectric materials which have high value of dielectric constant are called high k dielectric materials. Metal oxides have usually had high dielectric constant. They are usually used in MOSFETs as gate dielectric.

Definition 2.5: Carbon Nano Tube (CNT) [47]

These are tubes made of carbon with diameters typically measured in nanometers range. Carbon nanotube is theoretically distinct as a cylinder fabricated of rolled up grapheme sheet. Carbon nanotubes often refer to single-wall carbon nanotubes (SWCNTs). Single-wall carbon nanotubes are one of the allotropes of carbon, intermediate between fullerene cages and flat graphene. Most of the physical properties of carbon nanotubes derive from graphene.

Definition 2.6: Carbon Nano Tube Field Effective Transistors (CNTFETs) [48]

They are referred as a field-effect transistor in which a single carbon nanotube or an array of carbon nanotubes is used as the channel material instead of bulk silicon which is used in the traditional field-effect transistor. They were first discovered in 1998. CNTFET is a nano scale device that can provide low-power integrated circuits with high performance and high-power density.

Definition 2.7: Neutrosophic Hypersoft Set (NHSS) [39-40]

Let $\mathbb{U}$ be the universal set and $\mathbb{P}(\mathbb{U})$be the power set of $\mathbb{U}$. Consider $l^1, l^2, l^3 \dots l^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $L^1, L^2, L^3 \dots L^n$ with $L^i \cap L^j = \emptyset$, for $i \neq j$ and $i, j \in \{1,2,3 \dots n\}$ and their relation $L^1 \times L^2 \times L^3 \dots L^n = \$$, then the pair $(₣, \$)$ is said to be Neutrosophic Hypersoft set (NHSS) over $\mathbb{U}$ where;

$₣: L^1 \times L^2 \times L^3 \dots L^n \rightarrow \mathbb{P}(\mathbb{U})$and

$₣(L^1 \times L^2 \times L^3 \dots L^n) = \{< x, T(₣(\$)), I(₣(\$)), F(₣(\$)) >, x \in \mathbb{U}\}$ where T is the membership value of truthiness, I is the membership value of indeterminacy and F is the membership value of falsity such that $T, I, F: \mathbb{U} \rightarrow [0,1]$ also $0 \leq T(₣(\$)) + I(₣(\$)) + F(₣(\$)) \leq 3$.

3. Calculations

In this section an algorithm is proposed to solve MCDM problem under neutrosophic environment.

3.1 Algorithm

TOPSIS (Technique for Order Preference by Similarly to Ideal Solution) is a suitable approach to deal with multi-attribute decision making problems.

Consider a multi-attribute decision making problem based on neutrosophic hypersoft sets (NHSSs) in which $\tilde{Ư} = \{\tilde{ư}^1, \tilde{ư}^2, \dots \tilde{ư}^a\}$ be the set of alternatives and $Ļ_1, Ļ_2, \dots Ļ_b$ be the sets of attributes and their corresponding attributive values are respectively the set $Ļ_1^a, Ļ_2^b, \dots Ļ_b^z$ where $a, b, c, \dots z = 1,2, \dots n$. Let $\hat{w}^j$ be the weight of attributes $Ļ_j^z, j = 1,2 \dots b$, where $0 \leq \hat{w}^j \leq 1$ and

$\sum_{j=1}^{b} \hat{w}^{j} = 1$. Suppose that $Đ = (Đ_1, Đ_2, \dots Đ_t)$ be the set of t decision makers and Δ^x be the weight of t decision-makers with $0 \le \Delta^x \le 1$ and $\sum_{x=1}^{t} \Delta^x = 1$. Let $\left[A_{ij}^{x}\right]_{a\times b}$ be the decision matrix where $A_{ij}^{x} = \left(Ţ_{ijk}^{x}, İ_{ijk}^{x}, Ḟ_{ijk}^{x}\right)$, $i = 1,2,3 \dots a, j = 1,2,3, \dots b, k = a, b, c, \dots z$ and $Ţ_{ijk}^{x}, İ_{ijk}^{x}, Ḟ_{ijk}^{x} \in [0,1]$, $0 \le Ţ_{ijk}^{x} + İ_{ijk}^{x} + Ḟ_{ijk}^{x} \le 3$. Utilizing the following steps, the determination strategy for the selection of alternatives can be obtained.

Step 1: Determine the Weight of Decision Makers

Let $\left[A_{ij}^{x}\right]_{a\times b}$ be the decision matrix where it is given as follows:

$$\left[A_{ij}^{x}\right]_{a\times b} = \begin{bmatrix} Ţ_{Ļ_1^a}^{x}(ũ_1), İ_{Ļ_1^a}^{x}(ũ), Ḟ_{Ļ_1^a}^{x}(ũ_1)\; Ţ_{Ļ_2^b}^{x}(ũ_1), İ_{Ļ_2^b}^{x}(ũ_1), Ḟ_{Ļ_2^b}^{x}(ũ_1) \cdots Ţ_{Ļ_b^z}^{x}(ũ), İ_{Ļ_b^z}^{x}(ũ), Ḟ_{Ļ_b^z}^{x}(ũ)\; Ţ_{Ļ_1^a}^{x}(ũ_2), İ_{Ļ_1^a}^{x}(ũ_2), Ḟ_{Ļ_1^a}^{x}(ũ_2)\; Ţ_{Ļ_2^b}^{x}(ũ_2), İ_{Ļ_2^b}^{x}(ũ_2), Ḟ_{Ļ_2^b}^{x}(ũ_2) \\ \cdots Ţ_{Ļ_b^z}^{x}(ũ_2), İ_{Ļ_b^z}^{x}(ũ_2), Ḟ_{Ļ_b^z}^{x}(ũ_2) \;\vdots\;\vdots\;\ddots\;\vdots\; Ţ_{Ļ_1^a}^{x}(ũ_a), İ_{Ļ_1^a}^{x}(ũ_a), Ḟ_{Ļ_1^a}^{x}(ũ_a)\; Ţ_{Ļ_2^b}^{x}(ũ_a), İ_{Ļ_2^b}^{x}(ũ_a), Ḟ_{Ļ}^{x}(ũ_a) \cdots Ţ_{Ļ_b^z}^{x}(ũ_a), İ_{Ļ_b^z}^{x}(ũ_a), Ḟ_{Ļ_b^z}^{x}(ũ_a) \end{bmatrix}$$

To find the ideal matrix, we average all the individual decision matrix A_{ij}^{x} *where* $x = 1,2 \dots t$ with the followings:

$$\left[A_{ij}^{\star`}\right]_{a\times b} =$$

$$\begin{bmatrix} Ţ_{Ļ_1^a}^{\star}(ũ_1), İ_{Ļ_1^a}^{\star}(ũ_1), Ḟ_{Ļ_1^a}^{\star}(ũ_1) & Ţ_{Ļ_2^b}^{\star}(ũ_1), İ_{Ļ_2^b}^{\star}(ũ), Ḟ_{Ļ_2^b}^{\star}(ũ_1) & \cdots & Ţ_{Ļ}^{\star}(u_1), İ_{Ļ_b^z}^{\star}(u_1), Ḟ_{Ļ_b^z}^{\star}(u_1) \\ Ţ_{Ļ_1^a}^{\star}(ũ_2), İ_{Ļ_1^a}^{\star}(ũ_2), Ḟ_{Ļ_1^a}^{\star}(ũ_2) & Ţ_{Ļ}^{\star}(ũ_2), İ_{Ļ_2^b}^{\star}(ũ_2), Ḟ_{Ļ}^{\star}(ũ_2) & \cdots & Ţ_{Ļ_b^z}^{\star}(u_2), İ_{Ļ_b^z}^{\star}(u_2), Ḟ_{Ļ_b^z}^{\star}(u_2) \\ \vdots & \vdots & \ddots & \vdots \\ Ţ_{Ļ}^{\star}(ũ_a), İ_{Ļ_1^a}^{\star}(ũ_a), Ḟ_{Ļ_1^a}^{\star}(u_a) & Ţ_{Ļ_2^b}^{\star}(ũ_a), İ_{Ļ}^{\star}(ũ_a), Ḟ_{Ļ_2^b}^{\star}(ũ_a) & \cdots & Ţ_{Ļ_b^z}^{\star}(ũ_a), İ_{Ļ_b^z}^{\star}(ũ_a), Ḟ_{Ļ_b^z}^{\star}(ũ_a) \end{bmatrix}, \text{ where}$$

$$A_{ij}^{\star} = \left(Ţ_{Ļ_j^k}^{\star}(ũ_i), İ_{Ļ_j^k}^{\star}(ũ), Ḟ_{Ļ_j^k}^{\star}(ũ_i)\right)$$

$$= \left(1 - \prod_{x=1}^{t}\left(1 - Ţ_{Ļ_j^k}^{x}(ũ)\right)^{\frac{1}{t}}, \prod_{x=1}^{t}\left(İ_{Ļ_j^k}^{x}(ũ_i)\right)^{\frac{1}{t}}, \prod_{x=1}^{t}\left(Ḟ_{Ļ_j^k}^{x}(ũ_i)\right)^{\frac{1}{t}}\right)$$

Step 2: Determine the weight of attributes

the elements of A_{ij} in the matrix $\left[A_{ij}\right]_{a\times b}$ is calculated as $\left[A_{ij}\right]_{a\times b} = \left(1 - \prod_{x=1}^{t}\left(1 - Ţ_{Ļ_j^k}^{x}(ũ_i)\right)^{\Delta^x}, \prod_{x=1}^{t}\left(İ_{Ļ_j^k}^{x}(ũ_i)\right)^{\Delta^x}, \prod_{x=1}^{t}\left(Ḟ_{Ļ_j^k}^{x}(ũ_i)\right)^{\Delta^x}\right)$, $i = 1,2,3 \dots a, j = 1,2,3, \dots b$ *and* $x = 1,2, \dots t$.

Step 3: Determine the weight of attributes

In the decision-making procedure, decision-makers may perceive that all attributes are not similarly significant. In this manner, each decision-maker may have their own opinion regarding attribute weights. To acquire the gathering assessment of the picked attributes, all the decision-makers opinions for the importance of each attribute need to be aggregated. For this purpose, weight $\hat{w}^{j}$ of attributes $Ḷ_j^z, j = 1,2 \dots b$ is calculated as $\hat{w}^j = \left(Ț_{Ḷ_j}, İ_{Ḷ_j}, Ḟ_{Ḷ_j}\right) = \left(1 - \prod_{x=1}^{t}\left(1 - Ț_{Ḷ_j}^x\right)^{\Delta^x}, \prod_{x=1}^{t}\left(İ_{Ḷ_j}^x\right)^{\Delta^x}, \prod_{x=1}^{t}\left(Ḟ_{Ḷ_j}^x\right)^{\Delta^x}\right)$.

Step 4: Calculate weighted aggregated decision matrix

After finding the weights of individual attributes, we use the weights to each row of the aggregated decision matrix with $\left[A_{ij}^{\omega}\right]_{a\times b} = \left(Ț_{Ḷ_j^k}^{\omega}(\tilde{ư}_i), İ_{Ḷ}^{\omega}(\tilde{ư}_i), Ḟ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i)\right) = \left((Ț_{Ḷ_j^k}(\tilde{ư}_i).Ț_{Ḷ_j}), \left(İ_{Ḷ_j^k}(\tilde{ư}_i) + İ_{Ḷ_j} - İ_{Ḷ_j^k}(\tilde{ư}_i).İ_{Ḷ_j}\right), \left(Ḟ_{Ḷ_j^k}(\tilde{ư}_i) + Ḟ_{Ḷ_j} - Ḟ_{Ḷ}(\tilde{ư}_i).Ḟ_{Ḷ_j}\right)\right)$. Then, we get a weighted aggregated decision matrix.

Step 5: Determine the ideal solution

In real life we deal with two types of attributes, one is benefit type attributes and the other is cost type attributes. In our MAGDM problem, we also deal with these two types of attributes. Let C_1 be the benefit type attributes and C_2 be the cost type attributes. The neutrosophic hypersoft positive ideal solution is given as $A_j^{\omega^+} = \left(Ț_{Ḷ_j^k}^{\omega^+}(\tilde{ư}_i), İ_{Ḷ_j^k}^{\omega^+}(\tilde{ư}_i), Ḟ_{Ḷ_j^k}^{\omega^+}(\tilde{ư}_i)\right) = \{\left\{Ț_{Ḷ_j^k}^{\omega}(\tilde{ư}_i)\right\}, \left\{İ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i)\right\}, \left\{Ḟ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i)\right\}$ $, j \in C_1$ $\left\{Ț_{Ḷ}^{\omega}(\tilde{ư}_i)\right\}, \left\{İ_{Ḷ}^{\omega}(\tilde{ư}_i)\right\}, \left\{Ḟ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i)\right\}$ $, j \in C_2$. Similarly, the neutrosophic hypersoft negative ideal solution is given as $A_j^{\omega^-} = \left(Ț_{Ḷ_j^k}^{\omega^-}(\tilde{ư}_i), İ_{Ḷ_j^k}^{\omega^-}(\tilde{ư}_i), Ḟ_{Ḷ_j^k}^{\omega^-}(\tilde{ư}_i)\right) =$ $\{\left\{Ț_{Ḷ_j^k}^{\omega}(\tilde{ư}_i)\right\}, \left\{İ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i)\right\}, \left\{Ḟ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i)\right\}$ $, j \in C_1$ $\left\{Ț_{Ḷ_j^k}^{\omega}(\tilde{ư}_i)\right\}, \left\{İ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i)\right\}, \left\{Ḟ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i)\right\}$ $, j \in C_2$

Step 6 Calculate the distances

Now, we should find the Normalized Hamming distance between the alternatives and positive ideal solution with $Đ^{i+}\left(A_{ij}^{\omega}, A_j^{\omega^+}\right) = \frac{1}{3b}\sum_{j=1}^{b} \left(\left|Ț_{Ḷ_j^k}^{\omega}(\tilde{ư}_i) - Ț_{Ḷ_j^k}^{\omega^+}(\tilde{ư}_i)\right| + \left|İ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i) - İ_{Ḷ_j^k}^{\omega^+}(\tilde{ư}_i)\right| + \left|Ḟ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i) - Ḟ_{Ḷ_j^k}^{\omega^+}(\tilde{ư}_i)\right|\right)$. Similarly, we find the Normalized Hamming distance between the alternatives and positive ideal solution as $Đ^{i-}\left(A_{ij}^{\omega}, A_j^{\omega^-}\right) = \frac{1}{3b}\sum_{j=1}^{b} \{\left|Ț_{Ḷ_j^k}^{\omega}(\tilde{ư}_i) - Ț_{Ḷ_j^k}^{\omega^-}(\tilde{ư}_i)\right| + \left|İ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i) - İ_{Ḷ_j^k}^{\omega^-}(\tilde{ư})\right| + \left|Ḟ_{Ḷ_j^k}^{\omega}(\tilde{ư}_i) - Ḟ_{Ḷ_j^k}^{\omega^-}(\tilde{ư}_i)\right|\}$.

Step 7 Calculate the relative closeness coefficient

Relative closeness index is used to rank the alternatives and it is calculated with, $i = 1, \dots, a$,

$$RC^i = \frac{Đ^{i-}\left(A_{ij}^{\omega}, A_j^{\omega^-}\right)}{max^i\left\{Đ^{i-}\left(A_{ij}^{\omega}, A_j^{\omega^-}\right)\right\}} - \frac{Đ^{i+}\left(A_{ij}^{\omega}, A_j^{\omega^+}\right)}{min^i\left\{Đ^{i+}\left(A_{ij}^{\omega}, A_j^{\omega^+}\right)\right\}}. \quad (11)$$

The set of selected alternatives are ranked according to the descending order of relative closeness index.

3.2 Numerical Example:

In this section, we will discuss the case study for the selection of high k-gate dielectric for CNTFETs using mathematical tools. CNTFETs are the main part of electronic devices these days. The problem of the leakage current affects its performance. The high k-gate dielectric can be used to reduce the problem of leakage current. So, the more efficient CNTFETs devices can be constructed alternatives are considered in Table 1.

SR NO.	DIELECTRIC MATERIAL	DIELECTRIC CONSTANT (k)	Band Gap E_g (eV)	Conduction band with respect to Si ΔE_c (eV)	Structure
1	SiO_2	3.9	8.9	3.2	Amorphous
2	Al_2O_3	9.0	8.7	2.8	Amorphous
3	$HfSiO_4$	11	6.5	1.8	Tetra
4	ZrO_2	25	5.8	1.4	Mono, Tetra, Cubic
5	La_2O_3	30	6.0	2.3	Cubic
6	Y_2O_3	15	5.6	2.3	Cubic
7	HfO_2	25	5.7	1.4	Mono, Tetra, Cubic
8	Si_3N_4	7.0	5.1	2.4	Amorphous

TABLE 1: The alternatives with attributive values

Let U be the set of all dielectric materials that can be used at junction interface as gate dielectric, so U= $\{Đ_1, Đ_2, Đ_3, Đ_4, Đ_5, Đ_6, Đ_7, Đ_8\}$

Where $Đ_1 = SiO_2$, $Đ_2 = Al_2O_3$, $Đ_3 = HfSiO_4$, $Đ_4 = ZrO_2$, $Đ_5 = La_2O_3$, $Đ_6 = Y_2O_3$, $Đ_7 = HfO_2$, $Đ_8 = Si_3N_4$

Let us consider the following attributes;

$A_1 =$ Dielectric constant

$A_2 =$ Band Gap E_g (eV)

$A_3 =$ Conduction band with respect to Si ΔE_c (eV)

$A_4 =$ Structure

So $Đ_i$ = Universal set of dielectrics where (i = 1, 2, 3, 4, 5, 6, 7, 8) and A_i = Set of attributes where (i = 1, 2, 3, 4).

$A_1^a = Dielectric\ constant = \{3.9, 9.0, 11, 25, 30, 15, 25, 7\}$
$A_2^b = Band\ Gap\ E_g\ (eV) = \{8.9, 8.7, 6.5, 5.8, 6.0, 5.6, 5.7, 5.1\}$
$A_3^c = Conduction\ band\ with\ respect\ to\ Si\ \Delta E_c\ (eV) = \{3.2, 2.8, 1.8, 1.4, 2.3, 2.3, 1.4, 2.4\}$
$A_4^d = Structure$
$= \{amorphous, amorphous, tetra, (cubic, mono, tetra), cubic, cubic, (cubic, mono, tetra), amorphous\}$

Now, let us define the relation for the function $f : Đ_1 \times Đ_2 \times Đ_3 \times Đ_4 \times Đ_5 \times Đ_6 \times Đ_7 \times Đ_8 \rightarrow P(U)$ as, $f(Đ_1 \times Đ_2 \times Đ_3 \times Đ_4 \times Đ_5 \times Đ_6 \times Đ_7 \times Đ_8) = (\beta = dielectric\ constatn = 30, \Upsilon = Band\ gap = 8.9, \psi = Conduction = 2.3, \dot{\eta} = structure = cubic) =$ $(Đ_4 \times Đ_5 \times Đ_7)$ is the actual sample of the CNT for the CNTFETs. Three patients $\{Đ_4, Đ_5, Đ_7\}$ are selected based on sample. The panel of three doctors (decision-maker) $\{M^1, M^2, M^3\}$ will examine the sample and select the most relevant CNTFETs. These decision-makers give their valuable opinion in the form of neutrosophic number based on their experience and knowledge, and are presented in NHSM, separately, as follows:

$$[M^1]_{3\times4} = \begin{bmatrix} (\beta(0.6,0.5,0.4)) & (\Upsilon(0.2,0.4,0.6)) & (\psi(0.2,0.3,0.2)) & (\dot{\eta}(0.8,0.2,0.2)) \\ (\beta(0.4,0.2,0.2)) & (\Upsilon(1.2,0.4,0.3)) & (\psi(1.0,0.7,0.8)) & (\dot{\eta}(0.2,0.4,0.7)) \\ (\beta(0.7,0.6,0.5)) & (\Upsilon(0.6,0.7,0.19)) & (\psi(0.5,0.4,0.31)) & (\dot{\eta}(0.8,0.21,0.48)) \end{bmatrix}$$

$$[M^2]_{3\times4} = \begin{bmatrix} (\beta(0.1,0.5,0.3)) & (\Upsilon(0.7,0.5,0.4)) & (\psi(0.8,0.3,0.7)) & (\dot{\eta}(0.6,0.4,0.7)) \\ (\beta(0.2,0.4,0.9)) & (\Upsilon(0.3,0.210.0)) & (\psi(1.0,0.1,0.9)) & (\dot{\eta}(0.4,0.1,0.6)) \\ (\beta(0.6,0.5,0.9)) & (\Upsilon(0.4,0.7,0.91)) & (\psi(0.3,0.6,0.41)) & (\dot{\eta}(0.34,0.16,0.19)) \end{bmatrix}$$

$$[M^3]_{3\times4} = \begin{bmatrix} (\beta(0.8,0.2,0.1)) & (\Upsilon(0.6,0.4,0.2)) & (\psi(0.6,0.3,0.5)) & (\dot{\eta}(1.0,0.4,0.5)) \\ (\beta(0.4,0.2,0.4)) & (\Upsilon(3.3,014,0.13)) & (\psi(0.12,0.2,0.44)) & (\dot{\eta}(0.53,0.25,0.4)) \\ (\beta(0.5,0.4,0.6)) & (\Upsilon(0.5,0.6,0.91)) & (\psi(0.3,05.,0.23)) & (\dot{\eta}(0.8,0.1,0.3)) \end{bmatrix}$$

Step 1 Determine the Weights of Decision Makers

To determine the weights of the decision-makers, first, we find the similarity measure between each decision matrix $\{M^1, M^2, M^3\}$ and the ideal matrix $S^\star$ using $S(A_{ij}^x, A_{ij}^\star) = 1 - \frac{1}{3ab}\sum_i^a \sum_j^b \{|Ţ_{Ḷ_j^k}^x(\tilde{u}_i) - Ţ_{Ḷ_j^k}^\star(\tilde{u}_i)| + |İ_{Ḷ_j^k}^x(\tilde{u}_i) - İ_{Ḷ_j^k}^\star(\tilde{u}_i)| + |Ḟ_{Ḷ_j^k}^x(\tilde{u}_i) - Ḟ_{Ḷ_j^k}^\star(\tilde{u}_i)|\}$. So, $S(p_1, p_*) = 0.5641$, $S(p_1, p_*) = 0.1224$, $S(p_1, p_*) = 0.1046$. Now we calculate the weight $\Delta^x\ for\ (x = 1,2,3)$ of each decision-makers using $\Delta^x = \frac{S(A_{ij}^x, A_{ij}^*)}{\sum_{x=1}^t S(A_{ij}^x, A_{ij}^*)}$. We have

$$\Delta^1 = \frac{0.5641}{(0.5641 + 0.1224 + 0.1046)} = 0.7130$$

$$\varDelta^2 = \frac{0.1224}{(0.5641 + 0.1224 + 0.1046)} = 0.1547$$

$$\varDelta^3 = \frac{0.1046}{(0.5641 + 0.1224 + 0.1046)} = 0.1322$$

Step 2 Aggregate Neutrosophic Hypersoft Decision Matrices

Now we construct an aggregated neutrosophic hypersoft decision matrix NHSM, to obtain group decision. We obtain, $[p]_{3\times 4}$ =

$$\begin{bmatrix} (\text{ϐ}(0.116, 0.341, 0.312))\ (\text{'Υ}(0.896.\, 0.341, 0.334))\ (\psi(0.234, 0.241, 0.210))\ (\text{ή}(0.352, 0.112, 0.007)) \\ (\text{ϐ}(0.621, 0.241, 0653))\ (\text{'Υ}(0.342, 0.121, 0.732))\ (\psi(0.234, 0.466, 0.369))\ (\text{ή}(0.251, 0.144, 0.330)) \\ (\text{ϐ}(0.871, 0.636, 0.346))\ (\text{'Υ}(0.212, 0.1111, 0.203))\ (\psi(0.223, 0.761, 0.474))\ (\text{ή}(0.467, 0.831, 0.120)) \end{bmatrix}$$

Step 3 Determine the weight of attributes

Weight $\hat{w}^j$ of attributes $Ḽ_j, j = 1,2 \ldots b$ is calculated using $\hat{w}^j = \left(Ţ_{Ḽ_j}, İ_{Ḽ_j}, Ḟ_{Ḽ_j}\right) = \left(1 - \prod_{x=1}^{t} \left(1 - Ţ_{Ḽ_j}^x\right)^{\varDelta^x}, \prod_{x=1}^{t} \left(İ_{Ḽ_j}^x\right)^{\varDelta^x}, \prod_{x=1}^{t} \left(Ḟ_{Ḽ_j}^x\right)^{\varDelta^x}\right)$.

we get $\hat{w}^1 = (0.7224, 0.6938, 0.2346), \hat{w}^2 = (0.6755, 0.1340, 0.1004), \hat{w}^3 = (0.2821, 0.1269, 0.0992)$

Step 4 Calculate the weighted aggregated decision matrix

After finding the weights of attributes, we apply these weights to each row of aggregated decision matrix using $\left[A_{ij}^{\omega}\right]_{a\times b} = \left(Ţ_{Ḽ_j^k}^{\omega}(\tilde{ư}_i), İ_{Ḽ_j^k}^{\omega}(\tilde{ư}_i), Ḟ_{Ḽ_j^k}^{\omega}(\tilde{ư}_i)\right) = \left((Ţ_{Ḽ_j^k}(\tilde{ư}_i).\, Ţ_{Ḽ_j}), \left(İ_{Ḽ_j^k}(\tilde{ư}_i) + İ_{Ḽ_j} - İ_{Ḽ_j^k}(\tilde{ư}_i).\, İ_{Ḽ_j}\right), \left(Ḟ_{Ḽ_j^k}(\tilde{ư}_i) + Ḟ_{Ḽ_j} - Ḟ_{Ḽ_j^k}(\tilde{ư}_i).\, Ḟ_{Ḽ_j}\right)\right)$. We get a weighted aggregated decision matrix $[S^{\omega}]$ as follows: $[S^{\omega}]_{3\times 4}$ =

$[(\text{ϐ}(0.342, 0.121, 0.732))\ (\text{'Υ}(0.754, 0.466, 0.369))\ (\psi(0.871, 0.636, 0.346))\ (\text{ή}(0.812, 0.1111, 0.203))\ (\text{ϐ}(0.467, 0.83$

Step 5: Determine the ideal solution

Neutrosophic hypersoft positive ideal solution is calculated using S^{ω^+} = $[(\text{ϐ}, (0.23,0.53,013))\ (\text{'Υ}, (0.95, 0.16, 0.62))\ (\psi, (0.75, 0.85, 0.75))\ (\text{ή}, (0.42,0.85,0.13))]$. Similarly, the neutrosophic hypersoft negative ideal solution is given as S^{ω^-} = $[(\text{ϐ}, (0.34, 0.52, 0.77))\ (\text{'Υ}, (0.23, 0.32, 0.21))\ (\psi, (0.86, 0.23, 0.11))\ (\text{ή}, (0.12,0.09,0.03))]$.

Step 6 Calculate the distance measure

Now we find the normalized hamming distance between the alternatives and positive ideal solution using $Đ^{i+}(A_{ij}^{\omega}, A_j^{\omega^+}) = \frac{1}{3b}\sum_{j=1}^{b} \left(\left|Ţ_{Ļ_j^k}^{\omega}(\tilde{ŭ}_i) - Ţ_{Ļ_j^k}^{\omega^+}(\tilde{ŭ}_i)\right| + \left|İ_{Ļ_j^k}^{\omega}(\tilde{ŭ}_i) - İ_{Ļ_j^k}^{\omega^+}(\tilde{ŭ}_i)\right| + \left|Ḟ_{Ļ_j^k}^{\omega}(\tilde{ŭ}) - Ḟ_{Ļ_j^k}^{\omega^+}(\tilde{ŭ})\right|\right)$. We get $Đ^{1+}(S_1^{\omega}, S^{\omega^+}) = 0.342$, $Đ^{2+}(S_2^{\omega}, S^{\omega^+}) = 0.127$, $Đ^{3+}(S_3^{\omega}, S^{\omega^+}) = 0.985$.

Similarly, we will find the normalized hamming distance between the alternatives and negative ideal solution using

$$Đ^{i-}(A_{ij}^{\omega}, A_j^{\omega^-}) = \frac{1}{3b}\sum_{j=1}^{b} \left(\left|Ţ_{Ļ_j^k}^{\omega}(\tilde{ŭ}_i) - Ţ_{Ļ_j^k}^{\omega^-}(\tilde{ŭ}_i)\right| + \left|İ_{Ļ_j^k}^{\omega}(\tilde{ŭ}_i) - İ_{Ļ_j^k}^{\omega^-}(\tilde{ŭ}_i)\right| + \left|Ḟ_{Ļ_j^k}^{\omega}(\tilde{ŭ}_i) - Ḟ_{Ļ_j^k}^{\omega^-}(\tilde{ŭ})\right|\right).$$

We get, $Đ^{1-}(S_1^{\omega}, S^{\omega^-}) = 0.741$, $Đ^{2-}(S_2^{\omega}, S^{\omega^-}) = 0.443$, $Đ^{3-}(S_3^{\omega}, S^{\omega^-}) = 0.332$.

Step 7: Calculate the relative closeness coefficient

Now we will calculate the relative closeness index using $Rp_i = \frac{Đ^{i-}(A_{ij}^{\omega}, A_j^{\omega^-})}{max^i\{Đ^{i-}(A_{ij}^{\omega}, A_j^{\omega^-})\}} - \frac{Đ^{i+}(A_{ij}^{\omega}, A_j^{\omega^+})}{min^i\{Đ^{i+}(A_{ij}^{\omega}, A_j^{\omega^+})\}}$.

We get

$$Rp_1 = \frac{0.4351}{0.4351} - \frac{0.332}{0.345} = -22.40$$

$$Rp_2 = \frac{0.443}{0.4351} - \frac{0.747}{0.345} = -32.50$$

$$Rp_3 = \frac{0.332}{0.4351} - \frac{0.345}{0.345} = -12.50$$

By using NHSS-TOPSIS for neutrosophic hypersoft sets, we can decide that which CNT is good for CNTFETs using the values of relative closeness coefficient in descending order. We rank the selected alternatives as shown in Table. 2 according to the descending order of relative closeness index as $Đ_5 > Đ_4 > Đ_7$. This shows that $Đ_5$ is the alternative which is goof for CNTFETs.

Alternative	$Đ_4$	$Đ_5$	$Đ_7$
Relatives Closeness using NHSS-TOPSIS	-22.40	-32.50	-12.50

Table 2. Relative closeness coefficient measurement and ranking of alternatives

4. Result Discussion

The performance of the CNTFET devices is primarily dependent on gate dielectric, source and drain material. The reduction of size of an electronic device causes leakage current which may be reduced by using high k gate dielectric as gate. The choice of this high k-gate dielectric depends on the properties of materials like dielectric constant, energy band gap etc. La_2O_3 is the best among all the potential candidates as gate dielectric as it has high dielectric constant as compared to all others. The drain current in CNTFET's increases with the relative dielectric constant for a fixed value of the drain voltage. So La_2O_3 gives the highest value of the drain current among all the other candidates for the gate dielectric. Moreover. La_2O_3 has high current on/off ratio and low leakage power which makes it a suitable candidate for CNT based electronic devices [45]. Difference between thermal expansion coefficient of CNT and thermal expansion coefficient of La_2O_3 is very small as compared to difference of other dielectrics [46] and present in Table 3.

Methods	Ranking of alternatives	Best alternatives
Ankita Dixit, Navneet Gupta. [45]	$Đ_5 > Đ_4 > Đ_7$.	$Đ_5$
Ankita Dixit, Navneet Gupta [46]	$Đ_4 > Đ_5 > Đ_7$.	$Đ_5$
The NHSS-TOPSIS	$Đ_5 > Đ_4 > Đ_7$.	$Đ_5$

TABLE 3. Comparison analysis of final ranking with existing methods

5. Conclusions

The concept of neutrosophic hypersoft TOPSIS (NHSS-TOPSIS) is a strong model for MADM. With the development of this technique we robust MADM method for CNTFETs selection by using NHSS-TOPSIS for NHSSs. Meanwhile, a practical application for ranking of alternatives with newly developed MADM approach is illustrated by a numerical example. We computed correlation coefficient and relative closeness by using our method and compared the results with existing methods of [45-46]. The validity and superiority of this method with existing approaches is also given with the help of a comparison analysis. Finally, it is deduced that NHSS-TOPSIS is more efficient, impressive and suitable.

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Conflicts of Interest

The authors declare no conflict of interest.

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Neutrosophic Real Inner Product Spaces

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Abstract: The objective of this work is to define for the first time the concept of inner product in a neutrosophic vector space V(I) and a refined neutrosophic vector space $V(I_1, I_2)$. Also, this paper introduces many interesting properties of neutrosophic real inner products and establishes the theoretical foundations of neutrosophic functional analysis such as neutrosophic normed spaces, refined neutrosophic normed spaces, neutrosophic real inner product spaces, refined neutrosophic inner product spaces, orthogonal basis, neutrosophic and refined neutrosophic Cauchy- Schwartz inequality.

Keywords: Neutrosophic Inner product, refined neutrosophic inner product, neutrosophic normed space, neutrosophic orthogonal basis, neutrosophic functional analysis

1. Introduction

Neutrosophy is a new branch of philosophy founded by Smarandache to study the indeterminacy in all activities and branches of science. Neutrosophic set and its generalizations such as refined neutrosophic set and n-refined neutrosophic set became useful tools in the study of topology [7,8,32], neutrosophic algebraic structures such as modules, groups, matrices, and different kinds of rings [1,3,5,6,9,11,14,15,16,22,23,24,30,33,34], and equations [31]. On the other hand, neutrosophic sets were used widely in applied mathematics and engineering such as optimization, artificial intelligence, health care, decision making , and industry [25,26,27,28,29,35,36,37,40,41]. On the other hand, neutrosophy is very effective and applicative in the study of many applied fields such as probability, statistics, analysis and Diophantine equations [39,42].

The concept of inner product was defined in classical vector spaces as a linear function takes its values in a field such as R or C, which plays an important role in the study of norms, metrics, and functional analysis [12,13].

Agboola et.al presented the concept of weak and strong neutrosophic vector space in [10], and refined neutrosophic vector space in [17,18] from an algebraic view as new generalizations of classical vector spaces. Recently, Smarandache et.al defined n-refined neutrosophic vector space in [20], with many interesting substructures [2]. Sankari et.al have proved that the weak neutrosophic vector space is isomorphic to the direct product of V with itself in [21]. This means that inner products defined over a weak neutrosophic vector space V(I) can be studied easily by taking its isomorphic image to the classical case.

In this work, we aim to extend classical real inner product to the neutrosophic and refined neutrosophic case. We define the concept of inner product over a strong neutrosophic vector space V(I) and over a strong refined neutrosophic vector space V(I_1, I_2) and we study its basic properties, which is considered as a first step in the theory of neutrosophic functional analysis.

The main result of this work is to prove that Cauchy-Schwartz inequality is still true in neutrosophic and refined neutrosophic spaces.

One of the most difficulties is that neutrosophic and refined neutrosophic real numbers has no order relation, so that we define an order relation, so we can study inequalities between such numbers.

Motivation

We regard that there is not a strict definition of inner products in neutrosophic systems based on neutrosophic spaces, thus our motivation is to close this important research gap by defining the basic theoretical concepts of functional analysis based on neutrosophic numbers and spaces.

2. Preliminaries

Definition 2.1 : [10]

Let $(V, +, .)$ be a vector space over the field K, $(V(I), +, .)$ is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Elements of V(I) have the form $x + yI\ ; x, y \in V$, i.e V(I) can be written as $V(I) = V + VI$.

Definition 2.2 : [10]

Let V(I) be a strong neutrosophic vector space over the neutrosophic field K(I) and W(I) be a non empty set of V(I), then W(I) is called a strong neutrosophic subspace if W(I) is itself a strong neutrosophic vector space.

Definition 2.3 : [10]

Let U(I) , W(I) be two strong neutrosophic subspaces of V(I), then we say that V(I) is a direct sum of U(I) and W(I) if and only if for each element x $\in V(I)$, then x can be written uniquely as x = y + z such y$\in U(I)$ and z $\in W(I)$

Definition 2.4 :[10]

Let $v_1, v_2 .. v_s \in V(I) and\ x \in V(I)$, we say that x is a linear combination of $\{ v_i; i = 1..s\}$ if x= $a_1 v_1 + \cdots + a_s v_s$ such $a_i \in K(I)$.

The set $\{ v_i; i = 1..s\}$ is called linearly independent if $a_1 v_1 + \cdots + a_s v_s = 0$ implies $a_i = 0$ for all i.

Theorem 2.5 : [10]

If $\{ v_1, ... v_s \}$ is a basis of V(I) and $f: V(I) \to W(I)$ is a neutrosophic vector space homomorphism, then $\{ f(v_1), ... f(v_s)\}$ is a basis of W(I).

Definition 2.6: [13]

Let V be a vector space over the field R, consider the following function $g: V \times V \to R$, then g is called real inner product if and only if:

(a) $g(a, a) \geq 0, g(a, a) = 0\ if\ and\ only\ if\ a = 0$.

$(b) g(a, b) = g(b, a)$.

(c) $g(ma + nb, c) = mg(a, c) + ng(b, c)$. For all $a, b, c \in V, m, n \in R$.

V is called a real inner product space.

Definition 2.7: [12]

Let V be a real inner product space, the norm of any element $x \in V$ is defined as follows:

$\|x\| = \sqrt{g(x,x)}$.

Definition 2.8: [13]

Let V be any vector space over the field R, the function $\| \quad \|: V \to R$ is called a norm if and only if:

(a) $\|x\| \geq 0, \|m.x\| = |m|.\|x\|$.

$(b)\|x + y\| \leq \|x\| + \|y\| \; for \; all \; x, y \in V and \; m \in R$.

(c) $\|x\| = 0 \; if \; and \; only \; if \; x = 0$.

Theorem 2.9: [12] (Cauchy- Schwartz inequality)

Let V be any real inner product space. Then $|g(x,y)| \leq \|x\|\|y\| for \; all \; x, y \in V$.

3. Neutrosophic inner product spaces

First of all, we shall define a partial order relation ($\leq$) on the neutrosophic field of real numbers $R(I)$.

Definition 3.1: [39]

Let $R(I) = \{a + bI;\ a, b \in R\}$ be the real neutrosophic field, we say that $a + bI \leq c + dI$ if and only if $a \leq c \; and \; a + b \leq c + d$.

Theorem 3.2: [39]

The relation defined in Definition 3.1 is a partial order relation.

Remark 3.3:

According to Theorem 3.2, we are able to define positive neutrosophic real numbers as follows:

$a + bI \geq 0 = 0 + 0.I$ implies that $a \geq 0, a + b \geq 0$.

Absolute value on R(I) can be defined as follows:

$|a + bI| = |a| + I[|a + b| - |a|]$, we can see that $|a + bI| \geq 0$.

Example 3.4:

$x = 2 - I$ is a neutrosophic positive real number, since $2 \geq 0 \; and \; (2 - 1) = 1 \geq 0$.

$2 + I \geq 2$, that is because $2 \geq 2 \; and \; (2 + 1) = 3 \geq (2 + 0) = 2$.

Definition 3.5:

Let V be a vector space over R, V(I) be its corresponding strong neutrosophic vector space over R(I).

Let $f: V(I) \times V(I) \to R(I)$ be a map, we call it a neutrosophic real inner product if it has the following properties:

(a) $f(x,y) = f(y,x)$.

$(b) f(x,x) \geq 0, if \; f(x,x) = 0 \; if \; and \; only \; if \; x = 0$.

(c) $f(ax+by,z)=af(x,z)+bf(y,z)$ for all $x,y,z\in V(I),a,b\in R(I)$.

V(I) is called a neutrosophic real inner product space.

Definition 3.6:

Let V(I) be any neutrosophic real inner product space, consider any two elements $x,y\in V(I)$. We say that $x\perp y$ if and only if $f(x,y)=0$.

Now, we suggest a kind of real neutrosophic inner products which can be derived from any classical inner product on the space V.

Theorem 3.7:

Let V be any inner product space over R, consider $g:V\times V\to R$ as its inner product. Then the corresponding neutrosophic strong vector space $V(I)$ has a neutrosophic real inner product.

Proof:

We define $f:V(I)\times V(I)\to R(I); f(a+bI,c+dI)=g(a,c)+I[g(a+b,c+d)-g(a,c)]$ $for\ all\ a+bI,c+dI\in V(I)$. We prove that f is a neutrosophic inner product.

Let $x=a+bI,y=c+dI,z=m+nI\in V(I), hence\ a,b,c,d,m,n\in V$, let $t=k+sI,l=r+pI\in R(I)$, we have

$f(x,y)=f(a+bI,c+dI)=g(a,c)+I[g(a+b,c+d)-g(a,c)]$=

$g(c,a)+I[g(c+d,a+b)-g(c,a)]$=$f(y,x)$.

$f(x,x)=g(a,a)+I[g(a+b,a+b)-g(a,a)]\geq 0$, that is because $g(a,a)\geq 0\ and\ [g(a+b,a+b)-g(a,a)+g(a,a)]=g(a+b,a+b)\geq 0$.

$f(x,x)=0$ implies $g(a,a)+I[g(a+b,a+b)-g(a,a)]=0$, hence $g(a,a)=0\ and\ g(a+b,a+b)=0$, thus $a=0\ and\ a+b=0$, so that $a=b=0\ and\ x=0$.

Now, we shall compute $tx+ly$.

$tx=ka+I[kb+sa+sb]=ka+I[(k+s)(a+b)-ka], ly=rc+I[rd+pc+pd]=rc+I[(r+p)(c+d)-rc]$, hence

$tx+ly=[ka+rc]+I[(k+s)(a+b)-ka+(r+p)(c+d)-rc]$,

$f(tx+ly,z)=g((ka+rc,m))+I\big[g\big((k+s)(a+b)+(r+p)(c+d),m+n\big)-g\big((ka+rc),m\big)\big]=$

$kg(a,m)+rg(c,m)+I[(k+s)g(a+b,m+n)+(r+p)g(c+d,m+n)-\ kg(a,m)-rg(c,m)]$.

On the other hand we have:

$t.f(x,z) = (k+sI).[g(a,m) + I[g(a+b,m+n) - g(a,m)] = k.g(a,m) + I[k.g(a+b,m+n) - k.g(a,m) + s.g(a,m) + s.g(a+b,m+n) - s.g(a,m)] = k.g(a,m) + I[k.g(a+b,m+n) - k.g(a,m) + s.g(a+b,m+n)]$.

$lf(y,z) = (r+pI).[g(c,m) + I[g(c+d,m+n) - g(c,m)] = r.g(c,m) + I[r.g(c+d,m+n) - r.g(c,m) + p.g(c,m) + p.g(c+d,m+n) - p.g(c,m)] = r.g(c,m) + I[r.g(c+d,m+n) - r.g(c,m) + p.g(c+d,m+n)]$.

Now, we can find that

$f(tx+ly,z) = tf(x,z) + lf(y,z), thus\ f\ is\ a\ neutrosophic\ inner\ product.$

Definition 3.8:

(a) The neutrosophic real inner product introduced in Theorem 3.7 is called the canonical neutrosophic real inner product generated by g.

(b) Let V be any vector space over R, with a classical real inner product g, V(I) be its corresponding neutrosophic strong vector space, let f be the canonical inner product generated by g, the canonical norm of $x = a + bI$ is defined as follows:

$\|x = a + bI\| = \sqrt{f(x,x)}$.

Theorem 3.9:

Let V be any vector space over R, with a classical real inner product g, V(I) be its corresponding neutrosophic strong vector space, let f be the canonical inner product generated by g, we have

(a) $\|x\| = \|a\| + I[\|a+b\| - \|a\|]$ for all $x = a + bI \in V(I)$.

$(b) For\ x = a + bI, y = c + dI,\ x \perp y$ if and only if $a \perp c$, and $a + b \perp c + d$.

(c) $\|a + bI\| = 1$ if and only if $\|a\| = \|a+b\| = 1$.

(d) If $\|a + bI\| = 1$, then $g(a,b) \le 0$.

Proof:

(a) We use definition 3.8 to compute $\|a+bI\|^2 = f(a+bI, a+bI) = g(a,a) + I[g(a+b,a+b) - g(a,a)]=$

$\|a\|^2 + I[\|a+b\|^2 - \|a\|^2]$.

Now, we prove that $\sqrt{f(x,x)} = \|a\| + I[\|a+b\| - \|a\|]$. By easy computing, we find

$[\|a\| + I[\|a+b\| - \|a\|]]^2 = \|a\|^2 + I[\|a+b\|^2 - \|a\|^2] = f(x,x)$, thus

$\|x\| = \|a\| + I[\|a + b\| - \|a\|]$.

(b) $x \perp y$ if and only if $f(x, y) = 0$, hence $g(a, c) + I[g(a + b, c + d) - g(a, c)] = 0$, this implies that $g(a, c) = 0, g(a + b, c + d) = 0$, thus $a \perp c$, and $a + b \perp c + d$.

(c) $\|a + bI\| = 1$ if and only if $\|a\| + I[\|a + b\| - \|a\|] = 1$, hence $\|a\| = 1, \|a + b\| - \|a\| = 0$, thus $\|a + b\| = \|a\| = 1$.

(d) By section (c), we find that $\|a + b\| = \|a\| = 1$, this means $g(a + b, a + b) = g(a, a) = 1$, hence $g(a, a) + 2g(a, b) + g(b, b) = g(a, a), thus\ \|b\|^2 = g(b, b) = -2g(a, b) \geq 0$, *thus* $g(a, b) \leq 0$.

Example 3.10:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I) = R^2(I) = \{(a, b) + (c, d)I; a, b, c, d \in R\}$ is defined as follows:

$f[((a, b) + (c, d)I, (m, n) + (t, s)I)] = g[(a, b), (m, n)] + I[g((a + c, b + d), (m + t, n + s)) - g((a, b), (m, n))] = (a.m + b.n) + I[(a + c).(m + t) + (b + d).(n + s) - a.m - b.n]$, where $a, b, c, d, m, n, s, t \in R$.

(b) Let $x = (1,2) + (1,0)I, y = (-2,1) + (2, -1)I$, we have

$f(x, y) = (1)(-2) + (2)(1) + I[(3)(0) + (2)(0) - (1)(-2) - (2)(1)] = 0, hence\ x \perp y$.

$\|x\| = \|(1,2)\| + I[\|(2,2)\| - \|(1,2)\|] = \sqrt{5} + I[\sqrt{8} - \sqrt{5}]$.

(c) Let $x = (1,0) + (-1,1)I$, we have $\|x\| = \|(1,0)\| + I[\|(0,1)\| - \|(1,0)\|] = 1 + I[1 - 1] = 1$.

We can see that $\|(1,0)\| = \|(1,0) + (-1,1)\| = 1 = \|(1,0)\|$.

Theorem 3.11: (Neutrosophic Cauchy-Schwartz inequality)

Let $x = a + bI, y = c + dI$ any two elements in a strong neutrosophic canonical inner product vector space. Then

$|f(x, y)| \leq \|x\|\|y\|$.

Proof:

We have $|f(x, y)| = |g(a, c)| + I[|g(a + b, c + d)| - |g(a, c)|]$.

$\|x\|\|y\| = \|a\|\|c\| + I[\|a\|\|c + d\| - \|a\|\|c\| + \|a + b\|\|c\| - \|a\|\|c\| + \|a + b\|\|c + d\| - \|c\|\|a + b\| - \|a\|\|c + d\| + \|a\|\|c\|] = \|a\|\|c\| + I[\|a + b\|\|c + d\| - \|a\|\|c\|]$.

By classical Cauchy – Schwartz inequality, we find $|g(a, c)| \leq \|a\|\|c\|$, and $|g(a + b, c + d)| \leq \|a + b\|\|c + d\|$, thus

$|g(a,c)| + I[|g(a+b,c+d)| - |g(a,c)|] \leq \|a\|\|c\| + I[\|a+b\|\|c+d\| - \|a\|\|c\|]$, so that

$|f(x,y)| \leq \|x\|\|y\|$.

Example 3.12:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I) = R^2(I) = \{(a,b) + (c,d)I; a,b,c,d \in R\}$ is defined as follows:

$f\big[\big((a,b) + (c,d)I, (m,n) + (t,s)I\big)\big] = g[(a,b),(m,n)] + I[g\big((a+c,b+d),(m+t,n+s)\big) -$

$g\big((a,b),(m,n)\big)]$= $(a.m + b.n) + I[(a+c).(m+t) + (b+d).(n+s) - a.m - b.n]$, where

$a,b,c,d,m,n,s,t \in R$.

(b) Let $x = (1,1) + (2,-1)I, y = (1,0) + (0,1)I$, we have

$f(x,y) = 1 + I[3-1] = 1 + 2I, |f(x,y)| = 1 + 2I, \|x\| = \sqrt{2} + I\big[3 - \sqrt{2}\big], \|y\| = 1 + I[\sqrt{2} - 1]$.

$\|x\|\|y\| = \sqrt{2} + I\big[2 - \sqrt{2} + 3 - \sqrt{2} + 3\sqrt{2} - 3 - 2 + \sqrt{2}\big] = \sqrt{2} + 2\sqrt{2}I$,

$|f(x,y)| = 1 + 2I \leq \sqrt{2} + 2\sqrt{2}I$. That is because $1 \leq \sqrt{2}, 1 + 2 = 3 \leq \sqrt{2} + 2\sqrt{2}$. (see definition 3.13).

Theorem 3.13:

Let V(I) be a neutrosophic strong real inner product vector space, let $x = a + bI$ be any element in V(I). We have

(a) $\|x\| \geq 0, \|m.x\| = |m|.\|x\|$.

$(b)\|x+y\| \leq \|x\| + \|y\|\ for\ all\ x,y \in V(I) and\ m \in R(I)$.

(c) $\|x\| = 0\ if\ and\ only\ if\ x = 0$.

Proof:

(a) Since$\|x\| = \|a\| + I[\|a+b\| - \|a\|]$, and $\|a\| \geq 0, (\|a+b\| - \|a\|) + \|a\| = \|a+b\| \geq 0$, we get that $\|x\| \geq 0$.

Let $m = c + dI \in R(I); c,d \in R$, we have $m.x = c.a + I[(c+d)(a+b) - c.a]$, hence

$\|m.x\| = \|c.a\| + I[\|(c+d)(a+b) - c.a + c.a\| - \|c.a\|]$=

$|c|\|a\| + I[|(c+d)|\|a+b\| - |c|\|a\|]$=

$[|c| + I[|c+d| - |c|]][\|a\| + I[\|a+b\| - \|a\|]] = |m|.\|x\|$.

(b) Let $x = a + bI, y = c + dI \in V(I); a,b,c,d \in V$, $\|x+y\| = \|(a+c) + (b+d)I\|$=

$\|a+c\| + I[\|a+c+b+d\| - \|a+c\|]$, by regarding classical properties of classical norms, we get

$\|a+c\| \leq \|a\| + \|c\|, \|a+c+b+d\| \leq \|a+b\| + \|c+d\|$, thus

$\|a+c\|+I[\|a+c+b+d\|-\|a+c\|] \leq \|a\|+\|c\|+I[\|a+b\|+\|c+d\|-\|a\|-\|c\|]=\|x\|+\|y\|$.

(c) The proof is trivial and similar to the classical case.

According to the previous theorem, we can define any neutrosophic norm on a strong neutrosophic vector space V(I) as a function $\| \quad \|: V(I) \to R(I)$, where conditions (a), (b), and (c) are true. V(I) is called a strong neutrosophic normed space in this case.

Example 3.14:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I)=R^2(I)=\{(a,b)+(c,d)I; a,b,c,d \in R\}$ is defined as follows:

$f[((a,b)+(c,d)I,(m,n)+(t,s)I)]=g[(a,b),(m,n)]+I[g((a+c,b+d),(m+t,n+s))-g((a,b),(m,n))]=(a.m+b.n)+I[(a+c).(m+t)+(b+d).(n+s)-a.m-b.n]$, where $a,b,c,d,m,n,s,t \in R$.

(b) Let $x=(1,1)+(1,0)I, y=(1,-1)+(0,1)I, m=2+3I$, we have

$x+y=(2,0)+I(1,1), \|x+y\|=\|(2,0)\|+I[\|(3,1)\|-\|(2,0)\|]=2+I[\sqrt{10}-2]$,

$\|x\|=\sqrt{2}+I[\sqrt{5}-\sqrt{2}], \|y\|=\sqrt{2}+I[1-\sqrt{2}]$, it is easy to check that

$\|x+y\| \leq \|x\|+\|y\|$.

(c) $\|m.x\|=\|(2,2)+I[(3,3)+(2,0)+(3,0)\|=\|(2,2)+I(8,3)\|=\sqrt{8}+I[\|(10,5)\|-\sqrt{8}]=\sqrt{8}+I[5\sqrt{5}-\sqrt{8}]$,

$|m|=|2|+I[|3+2|-|2|]=2+3I, \|x\|=\sqrt{2}+I[\sqrt{5}-\sqrt{2}]$, it is easy to see that

$\|m.x\|=|m|.\|x\|$.

It is clear that $R^2(I)$ is a neutrosophic normed space.

Definition 3.15:

Let W be a subspace of V(I), we define the canonical orthogonal complement to be the set

$W^{\perp}=\{x \in V(I); f(x,y)=0 \; for \; all \; y \in W\}$.

Definition 3.16:

Let S be any basis of V(I), we say that S is a canonical orthogonal basis if and only if

$f(x,y)=0 \; for \; all \; x,y \in S$.

Definition 3.17:

Let S be any canonical orthogonal basis of V(I), we say that S is standard if and only if

$\|x\| = 1 \; for \; all \; x \in S$.

Theorem 3.18:

Let W be a subspace of V(I), and $W^{\perp} = \{x \in V(I); f(x,y) = 0 \; for \; all \; y \in W\}$be the canonical orthogonal complement, then $W^{\perp}$ is a strong neutrosophic subspace of V(I).

Proof:

Let x, y be any two elements in $W^{\perp}$, z be any element in W, $m = a + bI$ be any element in R(I), we have

$f(x-y,z) = f(x,z) - f(y,z) = 0 - 0 = 0, thus \; x - y \in W^{\perp}$. On the other hand

$f(m.x,z) = m.f(x,z) = m.0 = 0, thus \; m.x \in W^{\perp}$, hence $W^{\perp}$ is a strong neutrosophic subspace of V(I).

Theorem 3.19:

$W^{\perp^{\perp}} = W$.

Proof:

The proof is similar to the classical case.

Example 3.20:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I) = R^2(I) = \{(a,b) + (c,d)I; a,b,c,d \in R\}$ is defined as follows:

$f\big[\big((a,b) + (c,d)I, (m,n) + (t,s)I\big)\big] = g[(a,b),(m,n)] + I[g\big((a+c,b+d),(m+t,n+s)\big) - g\big((a,b),(m,n)\big)]$= $(a.m + b.n) + I[(a+c).(m+t) + (b+d).(n+s) - a.m - b.n]$, where $a,b,c,d,m,n,s,t \in R$.

(b) $W = \{v = (x,0) + (0,y)I; x,y \in R\}$ is a strong neutrosophic subspace of V(I).

$W^{\perp} = \{w = (t,z) + (k,s)I; t,z,k,s \in R\}; f(v,w) = 0$, this implies

$xt = 0, x(t+k) + y(z+s) - xt = 0, thus \; t = 0$, hence $xk + y(z+s) = 0$ for all $x, y \in R$, thus $k = z + s = 0$, so that $s = -z$ and $W^{\perp} = \{w = (0,z) + (0,-z)I; z \in R\}$.

4. Refined neutrosophic inner product spaces

First of all, we shall define an order relation (≤) on the refined neutrosophic field of real numbers R(I_1, I_2).

Definition 4.1:

Let $\mathrm{R}(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in R\}$ be the real refined neutrosophic field, we say that

$(a, bI_1, cI_2) \leq (x, yI_1, zI_2)$ if and only if $a \leq x$ *and* $a + c \leq x + z, a + b + c \leq x + y + z$.

Theorem 4.2:

The relation defined in Definition 4.1 is an order relation.

Proof:

Let $s = (a, bI_1, cI_2), k = (x, yI_1, zI_2), l = (m, nI_1, tI_2) \in \mathrm{R}(I_1, I_2)$, we have

$s \leq s$ that is because $a \leq a$ *and* $a + b \leq a + b, a + b + c \leq a + b + c$.

Now, suppose that $s \leq k$ *and* $k \leq s$, then $a \leq x, a + c \leq x + z, x \leq a, x + z \leq a + c, a + b + c \leq x + y + z, x + y + z \leq a + b + c$, hence

$a = x, a + c = x + z,\ a + b + c = x + y + z$, which means that $c = z, b = y$ and $s = k$.

Assume that $s \leq k$ *and* $k \leq l$, hence $a \leq x, a + c \leq x + z, a + b + c \leq x + y + z, x \leq m, x + z \leq m + t, x + y + z \leq m + n + t$, this implies that

$a \leq m, a + c \leq m + t, a + b + c \leq m + n + t$, hence $s \leq l$. Thus $\leq$ is an order relation on $\mathrm{R}(I_1, I_2)$.

Remark 4.3:

According to Theorem 4.2, we are able to define positive refined neutrosophic real numbers as follows:

$(a, bI_1, cI_2) \geq 0 = (0, 0I_1, 0I_2)$ implies that $a \geq 0, a + b + c \geq 0, a + c \geq 0$.

Absolute value on R(I) can be defined as follows:

$|(a, bI_1, cI_2)| = (|a|, (|b| - |c|)I_1, (|c| - |a|)I_2)$, we can see that $|(a, bI_1, cI_2)| \geq 0$.

Example 4.4:

$x = (5, -2I_1, -I_2)$ is a refined neutrosophic positive real number, since $5 \geq 0$ *and* $(5 - 1) = 4 \geq 0, (5 - 2 - 1) = 3 \geq 0$.

$(5, -2I_1, -I_2) \geq (2,0,0)$, that is because $5 \geq 2$ *and* $(5 - 1) = 4 \geq (2 + 0) = 2, (5 - 2 - 1) = 2 \geq 2 + 0 + 0 = 2$.

Definition 4.5:

Let V be a vector space over R, $V(I_1, I_2)$ be its corresponding strong refined neutrosophic vector space over $R(I_1, I_2)$. Let f:$V(I_1, I_2) \times V(I_1, I_2) \to R(I_1, I_2)$ be a map, we call it a refined neutrosophic real inner product if it has the following properties:

(a) $f(x, y) = f(y, x)$.

$(b) f(x, x) \geq 0, if\ f(x, x) = 0, then\ x = 0$.

(c) $f(ax + by, z) = af(x, z) + bf(y, z)$ for all $x, y, z \in V(I_1, I_2), a, b \in R(I_1, I_2)$.

$V(I_1, I_2)$ is called a refined neutrosophic real inner product space.

Definition 4.6:

Let $V(I_1, I_2)$ be any refined neutrosophic real inner product space, consider any two elements $x, y \in V(I_1, I_2)$. We say that $x \perp y$ if and only if $f(x, y) = 0$.

Now, we suggest a kind of real refined neutrosophic inner products which can be derived from any classical inner product on the space V.

Theorem 4.7:

Let V be any inner product space over R, consider $g: V \times V \to R$ as its inner product. Then the corresponding refined neutrosophic strong vector space $V(I_1, I_2)$ has a refined neutrosophic real inner product.

Proof:

We define $f: V(I_1, I_2) \times V(I_1, I_2) \to R(I_1, I_2); f\big((a, bI_1, cI_2), (x, yI_1, xI_2)\big) = (g(a, x), I_1[g(a + b + c, x + y + z) - g(a + c, x + z)], I_2[g(a + c, x + z) - g(a, x)])$ $for\ all (a, bI_1, cI_2), (x, yI_1, xI_2) \in V(I_1, I_2)$. We prove that f is a refined neutrosophic inner product.

Let $s = (a, bI_1, cI_2), k = (x, yI_1, zI_2), l = (m, nI_1, tI_2) \in V(I_1, I_2), hence\ a, b, c, x, y, x, t, m, n \in V$, let $i = (e, hI_1, rI_2), j = (p, qI_1, wI_2) \in R(I_1, I_2)$, we have

$f(s, k) = f\big((a, bI_1, cI_2), (x, yI_1, xI_2)\big) = (g(a, x), I_1[g(a + b + c, x + y + z) - g(a + c, x + z)], I_2[g(a + c, x + z) - g(a, x)])$=

$(g(x, a), I_1[g(x + y + z, a + b + c) - g(x + z, a + c)], I_2[g(x + z, a + c) - g(x, a)])$=$f(y, x)$.

$f(s, s) = (g(a, a), I_1[g(a + b + c, a + b + c) - g(a + c, a + c)], I_2[g(a + c, a + c) - g(a, a)]) \geq 0$, that is because $g(a, a) \geq 0 and\ [g(a + c, a + c) - g(a, a) + g(a, a)] = g(a + c, a + c) \geq 0, [g(a + b + c, a + b + c) - g(a + c, a + c) + g(a + c, a + c) - g(a, a) + g(a, a)] = g(a + b + c, a + b + c) \geq 0$.

$f(s,s)=0$ implies $(g(a,a), I_1[g(a+b+c,a+b+c)-g(a+c,a+c)], I_2[g(a+c,a+c)-g(a,a)])=0$, hence $g(a,a)=0$ and $g(a+c,a+c)=0, g(a+b+c,a+b+c)=0$, thus $a=0$ and $a+c=0, a+b+c$, so that $a=b=c=0$ and $s=0$.

Now, we shall compute $is+jk$.

$is=(ea, I_1[eb+ha+hc+hb+rb], I_2[ec+rc+ra])=(ea, I_1[(e+h+r)(a+b+c)-(e+r)(a+c)], I_2[(e+r)(a+c)-ea]), jk=(px, I_1[py+qx+qz+qy+wy], I_2[pz+wz+wx])=(px, I_1[(p+q+r)(x+y+z)-(p+w)(x+z)], I_2[(p+w)(x+z)-px])$, hence

$is+jk=(ea+px, I_1[(e+h+r)(a+b+c)-(e+r)(a+c)+(p+q+r)(x+y+z)-(p+w)(x+z)], I_2[(e+r)(a+c)-ea+(p+w)(x+z)-px])$,

$f(is+jk,l)=(g(ea+px,m), I_1\left[g\big((e+h+r)(a+b+c)+(p+q+w)(x+y+z), m+n+t\big)-g\Big(\big((e+r)(a+c)+(p+w)(x+z)\big), m+t\Big)\right], I_2[g\big((e+r)(a+c)+(p+w)(x+z), m+t\big)-(g(ea+px,m)])=(eg(a,m)+pg(x,m), I_1[(e+h+r)g(a+b+c,m+n+t)+(p+q+w)g(x+y+z,m+n+t)-(e+r)g(a+c,m+t)-(p+w)g(x+z,m+t)], I_2[(e+r)g(a+c,m+t)+(p+w)g(x+z,m+t)-eg(a,m)-pg(x,m)])$.

On the other hand we have:

$i.f(s,l)=(e,hI_1,rI_2).(g(a,m), I_1[g(a+b+c,m+n+t)-g(a+c,m+t)], I_2[g(a+c,m+t)-g(a,m)])=(eg(a,m), I_1[(e+h+r)g(a+b+c,m+n+t)-(e+r)g(a+c,m+t)], I_2[(e+r)g(a+c,m+n)-eg(a,m)])$,

$jf(k,l)=(p,qI_1,wI_2).(g(x,m), I_1[g(x+y+z,m+n+t)-g(x+z,m+t)], I_2[g(x+z,m+t)-g(x,m)])=(pg(x,m), I_1[(p+q+w)g(x+y+z,m+n+t)-(p+w)g(x+z,m+t)], I_2[(p+w)g(x+z,m+n)-pg(x,m)])$.

Now, we can find that

$f(is+jk,l)=if(s.l)+jf(k,l), thus\ f\ is\ a\ refined\ neutrosophic\ inner\ product.$

Definition 4.8:

(a) The refined neutrosophic real inner product introduced in Theorem 3.7 is called the canonical refined neutrosophic real inner product generated by g.

(b) Let V be any vector space over R, with a classical real inner product g, $V(I_1, I_2)$ be its corresponding refined neutrosophic strong vector space, let f be the canonical refined inner product generated by g, the canonical norm of $s = (a, bI_1, cI_2)$ is defined as follows:

$\|s = (a, bI_1, cI_2)\| = \sqrt{f(s,s)}$.

Theorem 4.9:

Let V be any vector space over R, with a classical real inner product g, $V(I_1, I_2)$ be its corresponding refined neutrosophic strong vector space, let f be the canonical refined inner product generated by g, we have

(a) $\|s\| = (\|a\|, I_1[\|a+b+c\| - \|a+c\|], I_2[\|a+c\| - \|a\|])$ for all $s = (a, bI_1, cI_2) \in V(I_1, I_2)$.

$(b) For\ s = (a, bI_1, cI_2), k = (x, yI_1, zI_2)$, $s \perp k$ if and only if $a \perp$x, and $a + c \perp x + z, a + b + c \perp x + y + z$.

(c) $\|(a, bI_1, cI_2)\| = 1$ if and only if $\|a\| = \|a+b+c\| = \|a+c\| = 1$.

Proof:

(a) We compute $\|(a, bI_1, cI_2)\|^2 = f\big((a, bI_1, cI_2), (a, bI_1, cI_2)\big) = (g(a,a), I_1[g(a+b+c, a+b+c) - g(a+c, a+c)], I_2[g(a+c, a+c) - g(a,a)])$=

$(\|a\|^2, I_1[\|a+b+c\|^2 - \|a+c\|^2], I_2[\|a+c\|^2 - \|a\|^2])$.

Now, we prove that $\sqrt{f(s,s)} = (\|a\|, I_1[\|a+b+c\| - \|a+c\|], I_2[\|a+c\| - \|a\|])$. By easy computing, we find

$[(\|a\|, I_1[\|a+b+c\| - \|a+c\|], I_2[\|a+c\| - \|a\|])]^2 = (\|a\|^2, I_1[\|a+b+c\|^2 - \|a+c\|^2], I_2[\|a+c\|^2 - \|a\|^2]) = f(s,s)$, thus

$\|s\| = (\|a\|, I_1[\|a+b+c\| - \|a+c\|], I_2[\|a+c\| - \|a\|])$.

(b) $s \perp k$ if and only if $f(s,k) = 0$, hence $(g(a,x), I_1[g(a+b+c, x+y+z) - g(a+c, x+z)], I_2[g(a+c, x+z) - g(a,x)]) = 0$, this implies that

$g(a,x) = 0, g(a+c, x+z) = 0, g(a+b+c, x+y+z) = 0$, thus $a \perp$x, and $a + c \perp x + z, a + b + c \perp x + y + z$.

(c) $\|(a, bI_1, cI_2)\| = 1$ if and only if $(\|a\|, I_1[\|a+b+c\| - \|a+c\|], I_2[\|a+c\| - \|a\|]) = (1,0,0)$, hence $\|a\| = 1, \|a+c\| - \|a\| = 0$, thus $\|a+c\| = \|a\| = 1, \|a+b+c\| - \|a+c\|, thus\ \|a+b+c\| = \|a+c\| = 1$.

Example 4.10:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I_1, I_2) = R^2(I_1, I_2) = \{((a,b),(c,d)I_1,(m,n)I_2); a,b,c,d,m,n \in R\}$ is defined as follows:

$f[(((a,b),(c,d)I_1,(m,n)I_2),((x,y),(z,t)I_1,(k,s)I_2))] = (g[(a,b),(x,y)], I_1\big[g\big((a+c+m,b+d+n),(x+z+k,y+t+s)\big) - g\big((a+m,b+n),(x+k,y+s)\big)\big], I_2\left[g\big((a+m,b+n),(x+k,y+s)\big) - g[(a,b),(x,y)]\right]) = (ax+by, I_1[(a+c+m)(x+z+k)+(b+d+n)(y+t+s)-(a+m)(x+k)-(b+n)(y+s)], I_2[(a+m)(x+k)+(b+n)(y+s)-ax-by])$, where

$a,b,c,d,m,n,x,y,z,t,k,s \in R$.

(b) Let $x = ((1,1),(2,-1)I_1,(0,0)I_2), y = ((-1,1),(1,-1)I_1,(0,0)I_2)$, we have

$f(x,y) = \big((1)(-1)+(1)(1), I_1[(3)(0)+(0)(6)-(1)(-1)-(1)(1)], I_2[(1)(-1)+(1)(1)-(1)(-1)-(1)(1)]\big) = (0,0,0), hence\ x \perp y$.

$\|x\| = (\|(1,1)\|, I_1[\|(1,1)+(2,-1)+(0,0)\| - \|(1,1)+(0,0)\|], I_2[\|(1,1)+(0,0)\| - \|(1,1)\|]) = (\sqrt{2}, I_1\big[3-\sqrt{2}\big], I_2[0])$.

Theorem 4.11: (Refined neutrosophic Cauchy-Schwartz inequality)

Let $x = (a, bI_1, cI_2), y = (m, nI_1, tI_2)$ any two elements in a refined strong neutrosophic canonical inner product vector space. Then

$|f(x,y)| \le \|x\|\|y\|$.

Proof:

We have $|f(x,y)| = (|g(a,m)|, I_1[|g(a+b+c,m+n+t)| - |g(a+c,m+t)|], I_2[|g(a+c,m+t) - |g(a,m)|])$.

$\|x\|\|y\| = (\|a\|\|m\|, I_1[\|a+b+c\|\|m+n+t\| - \|a+c\|\|m+t\|], I_2[\|a+c\|\|m+t\| - \|a\|\|m\|])$.

By classical Cauchy – Schwartz inequality, we find $|g(a,m)| \le \|a\|\|m\|$, and

$|g(a+b+c,m+n+t)| \le \|a+b+c\|\|m+n+t\|$, $|g(a+c,m+t)| \le \|a+c\|\|m+t\|$ thus

$(|g(a,m)|, I_1[|g(a+b+c,m+n+t)| - |g(a+c,m+t)|], I_2[|g(a+c,m+t)| - |g(a,m)|]) \le (\|a\|\|m\|, I_1[\|a+b+c\|\|m+n+t\| - \|a+c\|\|m+t\|], I_2[\|a+c\|\|m+t\| - \|a\|\|m\|])$, so that

$|f(x,y)| \le \|x\|\|y\|$.

Example 4.12:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I_1,I_2)=R^2(I_1,I_2)=\{((a,b),(c,d)I_1,(m,n)I_2);a,b,c,d,m,n\in R\}$ is defined as follows:

$f[(((a,b),(c,d)I_1,(m,n)I_2),((x,y),(z,t)I_1,(k,s)I_2))]=(g[(a,b),(x,y)],I_1\big[g\big((a+c+m,b+d+n),(x+z+k,y+t+s)\big)-g\big((a+m,b+n),(x+k,y+s)\big)\big],I_2\Big[g\big((a+m,b+n),(x+k,y+s)\big)-g[(a,b),(x,y)]\Big])=(ax+by,I_1[(a+c+m)(x+z+k)+(b+d+n)(y+t+s)-(a+m)(x+k)-(b+n)(y+s)],I_2[(a+m)(x+k)+(b+n)(y+s)-ax-by])$, where

$a,b,c,d,m,n,x,y,z,t,k,s\in R$.

(b) Let $x=((1,1),(2,-1)I_1,(0,2)I_2),y=((-1,1),(3,1)I_1,(-2,4)I_2)$, we have

$f(x,y)=(0,I_1[0+8+3-15],I_2[-3+15-0])=(0,I_1[-4],I_2[12]),|f(x,y)|=(0,-8I_1,12I_2),\|x\|=\big(\sqrt{2},I_1\big[\sqrt{14}-\sqrt{10}\big],I_2\big[\sqrt{10}-\sqrt{2}\big]\big),\|y\|=(\sqrt{2},I_1\big[6-\sqrt{34}\big],I_2[\sqrt{34}-\sqrt{2}])$.

$\|x\|\|y\|=\big(2,I_1\big[6\sqrt{14}-\sqrt{340}\big)\big],I_2[\sqrt{340}-2]\big)$,

$|f(x,y)|=(0,-8I_1,12I_2)\le\big(2,I_1\big[6\sqrt{14}-\sqrt{340}\big)\big],I_2[\sqrt{340}-2]\big)$. That is because $0\le 2,0+12\le 2+\big(\sqrt{340}-2\big),0-8+12=4\le\big(\sqrt{2}+6-\sqrt{34}+\sqrt{34}-\sqrt{2}\big)=6$.

Theorem 4.13:

Let $\mathrm{V}(I_1,I_2)$ be a refined neutrosophic strong real inner product vector space, let $x=(a,bI_1,cI_2)$ be any element in $\mathrm{V}(I_1,I_2)$. We have

(a) $\|x\|\ge 0,\|m.x\|=|m|.\|x\|$.

$(b)\|x+y\|\le\|x\|+\|y\|\ for\ all\ x,y\in V(I) and\ m\in R(I)$.

(c) $\|x\|=0\ if\ and\ only\ if\ x=0$.

Proof:

(a) Since $\|x\|=(\|a\|,I_1[\|a+b+c\|-\|a+c\|],I_2[\|a+c\|-\|a\|])$, and $\|a\|\ge 0,(\|a+c\|-\|a\|)+\|a\|=\|a+c\|\ge 0, and\ (\|a+b+c\|-\|a+c\|)+(\|a+c\|-\|a\|)+(\|a\|)=\|a+b+c\|\ge 0$, we get that $\|x\|\ge 0$.

Let $m=(n,pI_1,qI_2)\in R(I_1,I_2);n,p,q\in R$, we have $m.x=(n.a,I_1[(n+p+q)(a+b+c)-(n+q)(a+c)],I_2[(n+q)(a+c)-n.a])$, hence

$\|m.x\|=(\|\mathrm{n.a}\|,I_1[\|(n+p+q)(a+b+c)\|-\|(n+q)(a+c)\|],I_2[\|(n+q)(a+c)\|-\|n.a\|])=[(|n|,(|n+p+q|-|n+q|)I_1,(|n+q|-|n|)I_2)][(\|a\|,(\|a+b+c\|-\|a+c\|)I_1,(\|a+c\|-\|a\|)I_2)]=|m|.\|x\|$.

(b) Let $x=(a,bI_1,cI_2)),y=(m,nI_1,tI_2)\in \mathrm{V}(I_1,I_2);a,b,c,m,n,t\in V,\ \|x+y\|=\|(a+m,I_1[b+n],I_2[c+t])\|=$

$(\|a+m\|,I_1[\|a+m+b+n+c+t\|-\|a+m+c+t\|],I_2[\|a+m+c+t\|-\|a+m\|])$, by regarding classical properties of classical norms, we get

$\|a+m\|\le\|a\|+\|m\|,\|a+m+c+t\|\le\|a+c\|+\|m+t\|,\|a+m+b+n+c+t\|\le\|a+b+c\|+\|m+n+t\|$, thus

$(\|a+m\|,I_1[\|a+m+b+n+c+t\|-\|a+m+c+t\|],I_2[\|a+m+c+t\|-\|a+m\|])\le(\|a\|+\|m\|,I_1[\|a+b+c\|+\|m+n+t\|-\|a+c\|-\|m+t\|],I_2[\|a+c\|+\|m+t\|-\|a\|-\|m\|])=\|x\|+\|y\|$.

(c) The proof is trivial and similar to the classical case.

According to the previous theorem, we can define any neutrosophic norm on a strong neutrosophic vector space $\mathrm{V}(I_1,I_2)$ as a function $\|\ \|:\mathrm{V}(I_1,I_2)\to \mathrm{R}(I_1,I_2)$, where conditions (a), (b), and (c) are true. $\mathrm{V}(I_1,I_2)$ is called a strong neutrosophic normed space in this case.

Example 4.14:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I_1,I_2)=R^2(I_1,I_2)=\{((a,b),(c,d)I_1,(m,n)I_2);a,b,c,d,m,n\in R\}$ is defined as follows:

$f[(((a,b),(c,d)I_1,(m,n)I_2),((x,y),(z,t)I_1,(k,s)I_2))]=(g[(a,b),(x,y)],I_1\big[g\big((a+c+m,b+d+n),(x+z+k,y+t+s)\big)-g\big((a+m,b+n),(x+k,y+s)\big)\big],I_2\Big[g\big((a+m,b+n),(x+k,y+s)\big)-g[(a,b),(x,y)]\Big])=(ax+by,I_1[(a+c+m)(x+z+k)+(b+d+n)(y+t+s)-(a+m)(x+k)-(b+n)(y+s)],I_2[(a+m)(x+k)+(b+n)(y+s)-ax-by])$, where

$a,b,c,d,m,n,x,y,z,t,k,s\in R$.

(b) Let $x=\big((1,1),(1,0)I_1,(-1,2)I_2\big),y=\big((1,-1),(0,1)I_1,(-2,1)I_2\big),m=(2,3I_1,I_2)$, we have

$x+y=\big((2,0),I_1(1,1),I_2(-3,3)\big),\|x+y\|=(\|(2,0)\|,I_1[\|(0,4)\|-\|(-1,3)\|],I_2[\|(0,4)\|-\|(2,0)\|])=(2,I_1[4-\sqrt{10}],I_2[\sqrt{10}-2])$,

$\|x\|=\big(\sqrt{2},I_1[\sqrt{10}-3],I_2[3-\sqrt{2}]\big),\|y\|=(\sqrt{2},I_1[\sqrt{2}-1],I_2[1-\sqrt{2}])$, it is easy to check that

$\|x+y\|\le\|x\|+\|y\|$.

(c) $\|m.x\| = \|((2,2), I_1[((3,3) + (2,0) + (3,0) + (1,0) + (-3,6), I_2[(1,1) + (-2,4) + (-1,2)])\| =$

$\|((2,2), I_1(6,9), I_2(-2,7))\| = (\sqrt{8}, I_1[\|(6,18)\| - \|(0,9)\|], I_2[\|(0,9)\| - \|(2,2)\|]) = (\sqrt{8}, I_1[\sqrt{36+324} -$

$\sqrt{81}], I_2[9 - \sqrt{8}]) = (\sqrt{8}, (6\sqrt{10} - 9)I_1, (9 - \sqrt{8})I_2),$

$|m| = (|2|, I_1[|3+2+1| - |2+1|], I_2[|1+2| - |2|]) = (2, 3I_1, I_2), \|x\| = (\sqrt{2}, I_1[\sqrt{10} - 3], I_2[3 - \sqrt{2}]),$

it is easy to see that

$\|m.x\| = |m|.\|x\|.$

It is clear that $R^2(I_1, I_2)$ is a neutrosophic normed space.

Definition 4.15:

Let W be a subspace of $V(I_1, I_2)$, we define the canonical orthogonal complement to be the set

$W^{\perp} = \{x \in V(I_1, I_2); f(x,y) = 0 \; for \; all \; y \in W\}.$

Definition 4.16:

Let S be any basis of $V(I_1, I_2)$, we say that S is a canonical orthogonal basis if and only if

$f(x,y) = 0 \; for \; all \; x, y \in S.$

Definition 4.17:

Let S be any canonical orthogonal basis of $V(I_1, I_2)$, we say that S is standard if and only if

$\|x\| = 1 \; for \; all \; x \in S.$

Theorem 4.18:

Let W be a subspace of $V(I_1, I_2)$, and $W^{\perp} = \{x \in V(I_1, I_2); f(x,y) = (0,0,0) \; for \; all \; y \in W\}$ be the canonical orthogonal complement, then $W^{\perp}$ is a strong refined neutrosophic subspace of V(I).

Proof:

Let x, y be any two elements in $W^{\perp}$, z be any element in W, m=(a, bI_1, cI_2) be any element in $R(I_1, I_2)$, we have

$f(x-y,z) = f(x,z) - f(y,z) = (0,0,0) - (0,0,0) = (0,0,0), thus \; x - y \in W^{\perp}$. On the other hand

$f(m.x,z) = m.f(x,z) = m.(0,0,0) = (0,0,0), thus \; m.x \in W^{\perp}$, hence $W^{\perp}$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$.

Theorem 4.19:

$W^{\perp^{\perp}} = W.$

Proof:

The proof is similar to the classical case.

Example 4.20:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I_1,I_2) = R^2(I_1,I_2) = \{((a,b),(c,d)I_1,(m,n)I_2); a,b,c,d,m,n \in R\}$ is defined as follows:

$f[(((a,b),(c,d)I_1,(m,n)I_2),((x,y),(z,t)I_1,(k,s)I_2))] = (g[(a,b),(x,y)], I_1\big[g\big((a+c+m,b+d+n),(x+z+k,y+t+s)\big) - g\big((a+m,b+n),(x+k,y+s)\big)\big], I_2\Big[g\big((a+m,b+n),(x+k,y+s)\big) - g[(a,b),(x,y)]\Big])$= $(ax+by, I_1[(a+c+m)(x+z+k)+(b+d+n)(y+t+s)-(a+m)(x+k)-(b+n)(y+s)], I_2[(a+m)(x+k)+(b+n)(y+s)-ax-by])$, where $a,b,c,d,m,n,x,y,z,t,k,s \in R$.

(b) $W = \{v = ((x,0),(0,y)I_1,(0,0)I_2); x,y \in R\}$ is a strong neutrosophic subspace of $V(I_1,I_2)$.

$W^{\perp} = \big\{w = \big((t,z),(k,s)I_1,(p,q)I_2\big); t,z,k,s,p,q \in R\big\}; f(v,w) = (0,0,0)$, this implies

$xt = 0, x(t+k+p)+y(z+s+q)-x(t+p)+y(z+q) = 0, x(t+p)+y(z+q)-xt = 0, thus\ t = 0$, hence $x(t+k+p)+y(z+s+q) = 0, x(t+p)+y(z+q) = 0$ for all $x,y \in R$, thus

$z+q = t+p = 0, and\ t+k+p = z+s+q = 0$, so that $q = -z, t = -p = 0, k = s = 0$ and $W^{\perp} = \big\{w = \big((0,-q),(0,0)I_1,(0,q)I_2\big); p,q \in R\big\}$.

4. Conclusions

In this article, we have defined the concept of real inner product over a strong neutrosophic vector space V(I) and strong refined neutrosophic space $V(I_1,I_2)$, as well as neutrosophic and refined neutrosophic normed space. Many interesting properties were studied and proved, especially neutrosophic and refined neutrosophic Cauchy- Schwartz inequality, where we have proved that it is still correct in neutrosophic spaces.

This work opens a wide door to study neutrosophic functional analysis, neutrosophic orthogonal standard basis, and neutrosophic matrices and isometrics in the future, especially inequalities, since we have determined a strong partial ordering relation between neutrosophic numbers.

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Multicriteria group decision making based on neutrosophic analytic hierarchy process: Suggested modifications

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Abstract

To avoid any conflict toward a previous work [1], we clarify certain parts that need explanation and suggest some modifications that will enhance the performance of the suggested algorithm. We enhance the algorithm by making it work in any environment and under various conditions without the occurrence of any warnings. Moreover, we suggest various future directions that will help researchers in the application of neutrosophic analytic hierarchy process. Given that scientific research is always renewed and developed, we always strive to reach the best methods and solutions. Thus, we present this study as a good guide for researchers in their future works.

Keywords: Neutrosophic Set; Analytic Hierarchy Process; Triangular Neutrosophic Number.

1. Introduction

The analytic hierarchy process (AHP) is one of the most important multicriteria decision-making techniques [1-2]; thus, it is applied in various fields. However, as traditional AHP fails to consider imprecise and incomplete information, it needs to be developed. For example, Saaty's AHP produces rank reversal; thus, in 2015, Smarandache proposed a new procedure called "alpha-discounting method for multicriteria decision making" [4-7].

In view of the important role of neutrosophic theory in various fields and applications, we are the first to present the analytic hierarch process in the neutrosophic environment [1]. Generally, scientific research is always evolving, and new discoveries are made every day, which might change the usual rules or methods. Thus, we must present the latest developments to guide researchers toward the right path.

In this study, we present an accurate version of the score function, which was presented in [1] and has never been presented in the literature. We also present some modifications of the proposed method that researchers can use in their future works.

The remaining parts of this paper are organized as follows. In Section 2, we present a suggested modification of the score function and neutrosophic scaling of AHP. Section 3 discusses the managerial implications and benefits of the suggested modifications. Section 4 presents the conclusion and future work suggestions.

2. Suggested Modifications

In this section, some modifications of the presented score function in [1] are introduced, and a new neutrosophic scaling for the comparison matrices of AHP is presented.

2.1 Modification of the Existing Score Function

In Section 3 "Methodology, " especially in Step 4, if we have a single-value triangular neutrosophic number $\tilde{a} = \left((a_1, a_2, a_3); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\right)$, then the score function for converting it to its crisp value is as follows:

$$S(\tilde{a}) = \frac{a_1+a_2+a_3}{9} * \left(2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}}\right). \quad (1)$$

The accuracy function is

$$A(\tilde{a}) = \frac{a_1+a_2+a_3}{9} * (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} + \beta_{\tilde{a}}). \quad (2)$$

2.2 Modification of the Illustrative Example

By solving the presented example in [1] for evaluating job applicants, the neutrosophic pairwise comparison matrix of the criteria, which is presented in Table 4 in [1], is exactly as presented in Table 1 in the present work.

Table 1. Neutrosophic pairwise comparison matrix of criteria

	Presentable	**Years of experience**	**Age**
Presentable	$\tilde{1}$	$\tilde{2}$	$\tilde{6}$
Years of experience	$\tilde{2}^{-1}$	$\tilde{1}$	$\tilde{7}$
Age	$\tilde{6}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}$

Notes: $\tilde{1} = (0.5,1,3); (0.5,0.2,0.3)$, $\tilde{2} = (0,2,3); (0.3,0.7,0.7)$, $\tilde{6} = (3,6,12); (0.1,0.3,0.5)$, $\tilde{7} = (2,7,15); (0.4,0.4,0.5)$.

By using a modified score function (i.e., Eq. (1)), we obtain the same data as in Table 5 in [1]. For the modified neutrosophic pairwise comparisons of the applicants according to a presentable criterion, which was presented in Table 7 in [2], $\tilde{1} = (0.5,1,3); (0.5,0.2,0.3)$, $\tilde{2} = (0,2,3); (0.6,0.2,0.3)$, $\tilde{3} = (0,3,9); (0.3,0.5,0.6)$, $\tilde{4} = (2,4,6); (0.2,0.5,0.6)$, $\tilde{5} = (3,5,15); (0.4,0.5,0.6)$, $\tilde{6} = (0,6,12); (0.4,0.5,0.6)$, $\tilde{7} = (2,7,11); (0.1,0.2,0.5)$, and $\tilde{9} = (4,9,20); (0.2,0.5,0.6)$.

By applying the modified score function using the suggested steps in [1] and correcting the typographical errors to be $\tilde{1} = (0.5,1,3); (0.5,0.2,0.3)$, $\tilde{2} = (0,2,3); (0.6,0.2,0.3)$, $\tilde{3} = (0,3,9); (0.3,0.5,0.6)$, $\tilde{4} = (2,4,6); (0.2,0.5,0.6)$, $\tilde{5} = (3,5,15); (0.4,0.5,0.6)$, $\tilde{6} = (0,6,12); (0.4,0.5,0.6)$, $\tilde{7} = (2,7,11); (0.1,0.2,0.5)$, $\tilde{9} = (4,9,20); (0.2,0.5,0.6)$, we set Tables 7 and 8 in [1] as Tables 2 and 3 here, respectively.

Table 2. Neutrosophic pairwise comparison matrix of alternatives regarding presentable criterion

	A1	**A2**	**A3**	**A4**	**A5**
A1	$\tilde{1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{9}$
A2		$\tilde{1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{7}$
A3			$\tilde{1}$	$\tilde{4}$	$\tilde{9}$
A4				$\tilde{1}$	$\tilde{5}$
A5					$\tilde{1}$

Table 3. Crisp pairwise comparison matrix of alternatives regarding presentable criterion

	A1	A2	A3	A4	A5
A1	1	1	1.60	1	4.03
A2	1	1	1	1.60	3.11
A3	0.62	1	1	1.46	4.03
A4	1	0.62	0.68	1	3.32
A5	0.25	0.32	0.25	0.30	1

The weights of the alternatives are as follows: $A1 = 0.26, A2 = 0.24, A3 = 0.23, A4 = 0.19,$ and $A5 = 0.06$.
Moreover, Table 13 in [1] is set as Table 4 here.

Table 4. Neutrosophic pairwise comparison matrix of alternatives regarding age

	A1	A2	A3	A4	A5
A1	$\tilde{1}$	$\tilde{3}$	$\tilde{7}$	$\tilde{6}$	$\tilde{7}$
A2		$\tilde{1}$	$\tilde{4}$	$\tilde{7}$	$\tilde{5}$
A3			$\tilde{1}$	$\tilde{3}$	$\tilde{6}$
A4				$\tilde{1}$	$\tilde{9}$
A5					$\tilde{1}$

The weights of the alternatives regarding to age are as follows: $A1 = 0.36, A2 = 0.26, A3 = 0.16, A4 = 0.14,$ and $A5 = 0.07$.

Furthermore, all the elements in the comparison matrix are positive, and the upper value of the triangular neutrosophic number is greater than zero.

2.3 Modification of the Methodology

This subsection presents a modified approach for solving neutrosophic AHP. Table 5 presents a new ranking scale for the alternatives and criteria.

The steps for solving the neutrosophic AHP are as follows.

Step 1. Same as in [1].

Step 2. Same as in [1]. However, for constructing the neutrosophic pairwise comparison matrix, use the scale presented in Table 5.

Steps 3 and 4. Same as in [1]. However, for converting the neutrosophic pairwise comparison matrix, use Eq. (1) instead of Eq. (4) in [1].

Steps 5 and 6. Same as in [1].

Table 5. Linguistic variables for ranking the alternatives and criteria for neutrosophic AHP

Neutrosophic scale of Saaty	Linguistic terms	Lower, median, and upper values of the triangular number	Degree of certainty of expert opinion
$\tilde{1}$	Equally important	$\langle(1,1,1)\rangle$	Absolutely uncertain (0, 0, 1)
$\tilde{3}$	Slightly important	$\langle(2,3,4)\rangle$	Uncertain (0.25 ,0.75, 0.75)
$\tilde{5}$	Strongly important	$\langle(4,5,6)\rangle$	Slightly certain (0.45, 0.60, 0.60)
$\tilde{7}$	Very strongly important	$\langle(6,7,8)\rangle$	Median certainty (0.50, 0.50, 0.50)
$\tilde{9}$	Absolutely important	$\langle(9,9,9)\rangle$	Certain (0.75, 0.20, 0.20)
$\tilde{2}$	Sporadic values among two close scales	$\langle(1,2,3)\rangle$	Strongly certain (0.85, 0.15, 0.15)
$\tilde{4}$		$\langle(3,4,5)\rangle$	Very strongly certain (0.90, 0.10, 0.10)
$\tilde{6}$		$\langle(5,6,7)\rangle$	Absolutely certain (1.00, 0.00, 0.0)
$\tilde{8}$		$\langle(7,8,9)\rangle$	

2.4 Illustrative Example

For illustrating how the suggested method works, let us solve a simple example.

If we need to purchase an MP3 player and *i* have three criteria for buying, namely, storage, availability, and color [3], then we have four available alternatives A, B, C, and D. We want to evaluate the four available alternatives to select the best one.
The hierarchy for evaluating the available alternatives of MP3 players is shown in Fig. 1.

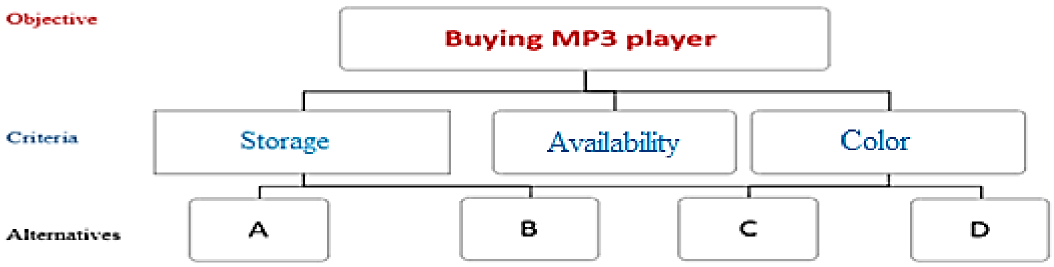

Fig. 1. Hierarchy tree for evaluating various types of MP3 player

Table 6 shows the neutrosophic pairwise comparison matrix of criteria using the suggested scale.

Table 6. Neutrosophic pairwise comparison matrix of criteria

	Storage	**Availability**	**Color**
Storage	$\langle(1,1,1);\ (1,0,0)\rangle$	$\langle(2,3,4);\ (0.85,0.15,0.15)\rangle$	$\langle(4,5,6);\ (0.9,0.1,0.1)\rangle$
Availability		$\langle(1,1,1);\ (1,0,0)\rangle$	$\langle(2,3,4);\ (0.85,0.15,0.15)\rangle$
Color			$\langle(1,1,1);\ (1,0,0)\rangle$

By using Eq. (1), the crisp form of the neutrosophic pairwise comparison matrix of the criteria are shown in Table 7.

Table 7. Crisp pairwise comparison matrix of criteria

	Storage	Availability	Color
Storage	1	2.55	4.5
Availability	0.39	1	2.55
Color	0.22	0.39	1

The weights for the criteria are as follows: weight of storage = 0.61, weight of availability = 0.27, and weight of color = 0.12.

Table 8 shows the neutrosophic pairwise comparison matrix of the alternatives regarding storage using the suggested scale.

Table 8. Neutrosophic pairwise comparison matrix of alternatives regarding storage criterion

Storage	A	B	C	D
A	$\langle(1,1,1);\ (1,0,0)\rangle$	$\langle(4,5,6);\ (0.9,0.1,0.1)\rangle$	$\langle(4,5,6);\ (0.9,0.1,0.1)\rangle$	$\langle(4,5,6);\ (0.9,0.1,0.1)\rangle$
B		$\langle(1,1,1);\ (1,0,0)\rangle$	$\langle(2,3,4);\ (0.85,0.15,0.15)\rangle$	$\langle(4,5,6);\ (0.9,0.1,0.1)\rangle$
C			$\langle(1,1,1);\ (1,0,0)\rangle$	$\langle(4,5,6);\ (0.9,0.1,0.1)\rangle$
D				$\langle(1,1,1);\ (1,0,0)\rangle$

By using Eq. (1), the crisp form of the neutrosophic pairwise comparison matrix of the alternatives regarding storage criterion is shown in Table 9.

Table 9. Crisp pairwise comparison matrix of alternatives regarding storage criterion

Storage	A	B	C	D
A	1	4.5	4.5	4.5
B	0.22	1	2.55	4.5
C	0.22	0.39	1	4.5
D	0.22	0.22	0.22	1

The weights for the alternatives are as follows: weight of A = 0.55, weight of B = 0.23, weight of C = 0.16, and weight of D = 0.07.

Table 10 presents the neutrosophic pairwise comparison matrix of the alternatives regarding availability using the suggested scale.

Table 10. Neutrosophic pairwise comparison matrix of alternatives regarding availability criterion

Availability	A	B	C	D
A	$\langle(1,1,1);\ (1,0,0)\rangle$	$\langle(7,8,9);\ (0.85,0.15,0.15)\rangle$	$\langle(7,8,9);\ (0.85,0.15,0.15)\rangle$	$\langle(7,8,9);\ (0.85,0.15,0.15)\rangle$
B		$\langle(1,1,1);\ (1,0,0)\rangle$	$\langle(2,3,4);\ (0.85,0.15,0.15)\rangle$	$\langle(2,3,4);\ (0.85,0.15,0.15)\rangle$
C			$\langle(1,1,1);\ (1,0,0)\rangle$	$\langle(2,3,4);\ (0.85,0.15,0.15)\rangle$
D				$\langle(1,1,1);\ (1,0,0)\rangle$

By using Eq. (1), the crisp form of the neutrosophic pairwise comparison matrix of the alternatives regarding availability criterion is shown in Table 11.

Table 11. Crisp pairwise comparison matrix of alternatives regarding availability criterion

Availability	A	B	C	D
A	1	6.8	6.8	6.8
B	0.15	1	2.55	2.55
C	0.15	0.39	1	2.55
D	0.15	0.39	0.39	1

The weights for the alternatives are as follows: weight of A = 0.661, weight of B = 0.163, weight of C = 0.109, and weight of D = 0.065.

The neutrosophic pairwise comparison matrix of the alternatives regarding color using the suggested scale is presented in Table 12.

Table 12. Neutrosophic pairwise comparison matrix of alternatives regarding color criterion

Color	A	B	C	D
A	$\langle(1,1,1);\ (1,0,0)\rangle$	$\langle(1,2,3);\ (1,0,0)\rangle$	$\langle(3,4,5);\ (1,0,0)\rangle$	$\langle(7,8,9);\ (1,0,0)\rangle$
B		$\langle(1,1,1);\ (1,0,0)\rangle$	$\langle(1,2,3);\ (1,0,0)\rangle$	$\langle(3,4,5);\ (1,0,0)\rangle$
C			$\langle(1,1,1);\ (1,0,0)\rangle$	$\langle(2,3,4);\ (1,0,0)\rangle$
D				$\langle(1,1,1);\ (1,0,0)\rangle$

By using Eq. (1), the crisp form of the neutrosophic pairwise comparison matrix of the alternatives regarding color criterion is shown in Table 13.

Table 13. Crisp pairwise comparison matrix of alternatives regarding color criterion

Color	A	B	C	D
A	1	2	4	8
B		1	2	4
C			1	3
D				1

The weights for the alternatives are as follows: weight of A = 0.529, weight of B = 0.264, weight of C = 0.147, and weight of D = 0.06.

Then, the relative scores for the alternatives are as follows:

$$\begin{bmatrix} 0.55 & 0.66 & 0.53 \\ 0.23 & 0.16 & 0.26 \\ 0.16 & 0.11 & 0.15 \\ 0.07 & 0.06 & 0.06 \end{bmatrix} \times \begin{bmatrix} 0.61 \\ 0.27 \\ 0.12 \end{bmatrix} = \begin{bmatrix} 0.57 \\ 0.21 \\ 0.14 \\ 0.07 \end{bmatrix}.$$

Findings show that Alternative A is the best one.

3. Managerial Implications

Selecting suitable alternatives requires a ranking method that usually contains several selection scopes. Habitually, there exist several conflicting criteria that makes the selection process difficult. The suggested neutrosophic AHP displays its applicability to handle vague and imprecise information, which exists usually in reality. Then, we can reach robust decisions by using the suggested method. The suggested neutrosophic AHP has the same benefits with the classical AHP besides the following advantages: offers user with a richer structural framework than the classical, fuzzy, and intuitionistic fuzzy AHP; defines the preference judgment values of the decision maker efficiently; and considers three degrees, namely, membership, indeterminacy, and non-membership degrees, which simulate natural human thinking. Generally, the suggested method in this study can be extended to diverse decisions related to other problems. The proposed method can be utilized as a reference guide for researchers to produce precise decisions about any problem in any organization. Governments can also use the proposed method to make precise decisions about any social, economic, and environmental problems.

4. Conclusions and Future Directions

Clarifications and modifications of the suggested score function and method for neutrosophic AHP are illustrated here to avoid any conflict among researchers and help them in future application of neutrosophic AHP in various fields. By using the suggested score function and the suggested scale for neutrosophic AHP, researchers can overcome various problems that they may face in the future application of neutrosophic AHP.

In the future, we recommend researchers to use the proposed scale for rating criteria and alternatives of neutrosophic AHP and use the presented score function in various case studies for its benefits and applicability. Moreover, we recommend researchers to propose novel methods to enhance the degree of consistency instead of repeating the exercise in cases of obtaining inconsistent comparison matrices.

Conflict of interest

We do not have any commercial or associative interest that signifies a conflict of interest in submitting our work.

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Novel Properties of Edge Irregular Single Valued Neutrosophic Graphs

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Abstract. In this paper, some types of edge irregular single valued neutrosophic graphs such as neighbourly edge totally irregular single valued neutrosophic graphs, strongly edge irregular single valued neutrosophic graphs and strongly edge totally irregular single valued neutrosophic graphs are introduced. A comparative study between neighbourly edge irregular single valued neutrosophic graphs and neighbourly edge totally irregular single valued neutrosophic graphs is done. Likewise some properties of them are studied. Finally, we have given some interesting results about edge irregular single valued neutrosophic graphs that are very useful in computer science and networks.

Keywords: Edge irregular SVNG; Neighbourly edge irregular SVNG; Neighbourly edge totally irregular SVNG; Strongly edge irregular SVNG; Strongly edge totally irregular SVNG.

1. **Introduction**

In 1736, Euler first introduced the concept of graph theory. In the history of mathematics, the solution given by Euler of the well known Konigsberg bridge problem is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of combinatorics. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as logic, geometry, algebra, topology, analysis, number theory, information theory, artificial intelligence, operations research, optimization, neural networks, planning, computer science and etc [10–12, 14].

Fuzzy set theory, introduced by Zadeh in 1965, is a mathematical tool for handling uncertainties like vagueness, ambiguity and imprecision in linguistic variables [41]. Research on theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its application. Fuzzy set theory has emerged as a potential area of interdisciplinary research and fuzzy graph theory is of recent interest.

In 1983, Atanassov [3,4] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [41]. Atanassov added a new component(which determines the degree of non-membership) in the definition of fuzzy set. The concept of neutrosophic set was introduced by F. Smarandache [31, 32] which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. The theory is generalization of classical sets and fuzzy sets and is applied in decision making problems, control theory, medicines, topology and in many more real life problems.

In 1975, Rosenfeld [25] introduced the concept of fuzzy graphs. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Later on, Bhattacharya gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [13].

Later, Broumi et al. [7] presented the concept of single valued neutrosophic graphs by combining the single valued neutrosophic set theory and the graph theory, and defined different types of single valued neutrosophic graphs (SVNG).

In the literature, many extensions of fuzzy graphs and their properties have been deeply studied by several researchers, such as intuitionistic fuzzy graphs, interval valued fuzzy graphs, interval valued intuitionistic fuzzy graphs, bipolar fuzzy graphs and etc [1, 2, 5, 6, 8, 9, 15, 21–24, 26, 30, 33–40].

Nagoorgani and Radha [17, 18] introduced the concept of regular fuzzy graphs and defined degree of a vertex in fuzzy graphs. Nagoorgani and Latha [16] introduced the concept of irregular fuzzy graphs, neighbourly irregular fuzzy graphs and highly irregular fuzzy graphs in 2008. Nandhini and Nandhini introduced the concept of strongly irregular fuzzy graphs and discussed about its properties [19].

Radha and Kumaravel [20] introduced the concept of edge degree, total edge degree in fuzzy graph and edge regular fuzzy graphs and discussed about the degree of an edge in some fuzzy graphs. Santhi Maheswari and Sekar introduced the concept of edge irregular fuzzy graphs and edge totally irregular fuzzy graphs and discussed about its properties [27]. Also, Santhi Maheswari and Sekar introduced the concept of neighbourly edge irregular fuzzy graphs, neighbourly edge totally irregular fuzzy graphs, strongly edge irregular fuzzy graphs and strongly edge totally irregular fuzzy graphs and discussed about its properties [28, 29].

This is the background to introduce neighbourly edge irregular single valued neutrosophic graphs, neighbourly edge totally irregular single valued neutrosophic graphs, strongly edge irregular single valued neutrosophic graphs, strongly edge totally irregular single valued neutrosophic graphs and discussed some of their properties. Also neighbourly edge irregularity and strongly edge irregularity on some single valued neutrosophic graphs whose underlying crisp graphs are a path, a cycle and a star are studied.

2. Preliminaries

We present some known definitions and results for ready reference to go through the work presented in this paper.

Definition 2.1. *A graph is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of edges of G^*. A graph G^* is finite if its vertex set and edge set are finite.*

Definition 2.2. *The degree $d_{G^*}(v)$ of a vertex v in G^* or simply $d(v)$ is the number of edges of G^* incident with vertex v.*

Definition 2.3. *A Fuzzy graph denoted by $G : (\sigma, \mu)$ on the graph $G^* : (V, E)$ is a pair of functions (σ, μ) where $\sigma : V \longrightarrow [0, 1]$ is a fuzzy subset of a non empty set V and $\mu : E \longrightarrow [0, 1]$ is a symmetric fuzzy relation on σ such that for all u and v in V the relation $\mu(u, v) = \mu(uv) \leq \min[\sigma(u), \sigma(v)]$ is satisfied.*

Definition 2.4. *An intuitionistic fuzzy graph (IFG) is of the form $G : (\sigma, \mu)$ where $\sigma = (\sigma_1, \sigma_2)$ and $\mu = (\mu_1, \mu_2)$ such that*
(1) The functions $\sigma_1 : V \longrightarrow [0, 1]$ and $\sigma_2 : V \longrightarrow [0, 1]$ denote the degree of membership and nonmembership of the element $u \in V$, respectively, and $0 \leq \sigma_1(u) + \sigma_2(u) \leq 1$ for every $u \in V$;
(2) The functions $\mu_1 : V \times V \longrightarrow [0, 1]$ and $\mu_2 : V \times V \longrightarrow [0, 1]$ are the degree of membership and nonmembership of the edge $uv \in E$, respectively, such that $\mu_1(uv) \leq \min[\sigma_1(u), \sigma_1(v)]$ and $\mu_2(uv) \geq \max[\sigma_2(u), \sigma_2(v)]$ and $0 \leq \mu_1(uv) + \mu_2(uv) \leq 1$ for every uv in E.

Definition 2.5. *A single valued neutrosophic graph (SVNG) is of the form $G : (A, B)$ where $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ such that*
(1) The functions $T_A : V \longrightarrow [0, 1]$, $I_A : V \longrightarrow [0, 1]$ and $F_A : V \longrightarrow [0, 1]$ denote the degree of truth-membership, the degree of indeterminacy-membership and the degree of falsity-membership of the element $u \in V$, respectively, and $0 \leq T_A(u) + I_A(u) + F_A(u) \leq 3$ for every $u \in V$;
(2) The functions $T_B : V \times V \longrightarrow [0, 1], I_B : V \times V \longrightarrow [0, 1]$ and $F_B : V \times V \longrightarrow [0, 1]$ are the degree of truth-membership, the degree of indeterminacy-membership and the degree of falsity-membership of the edge $uv \in E$, respectively, such that $T_B(uv) \leq \min[T_A(u), T_A(v)]$, $I_B(uv) \geq$

$\max[I_A(u), I_A(v)]$ *and* $F_B(uv) \geq \max[F_A(u), F_A(v)]$ *and* $0 \leq T_B(uv) + I_B(uv) + F_B(uv) \leq 3$ *for every* uv *in* E.

Definition 2.6. *Let* $G : (A, B)$ *be a SVNG on* $G^* : (V, E)$. *Then the degree of a vertex* u *is defined as* $d_G(u) = (d_{T_A}(u), d_{I_A}(u), d_{F_A}(u))$ *where* $d_{T_A}(u) = \Sigma_{v\neq u} T_B(uv)$, $d_{I_A}(u) = \Sigma_{v\neq u} I_B(uv)$ *and* $d_{F_A}(u) = \Sigma_{v\neq u} F_B(uv)$.

Definition 2.7. *Let* $G : (A, B)$ *be a SVNG on* $G^* : (V, E)$. *Then the total degree of a vertex* u *is defined by* $td_G(u) = (t(d_{T_A}(u), td_{I_A}(u), td_{F_A}(u))$ *where* $td_{T_A}(u) = \Sigma_{v\neq u} T_B(uv) + T_A(u)$, $td_{I_A}(u) = \Sigma_{v\neq u} I_B(uv) + I_A(u)$ *and* $td_{F_A}(u) = \Sigma_{v\neq u} F_B(uv) + F_A(u)$.

Definition 2.8. *Let* $G : (A, B)$ *be a SVNG on* $G^* : (V, E)$. *Then:*
(1) G *is irregular, if there is a vertex which is adjacent to vertices with distinct degrees.*
(2) G *is totally irregular, if there is a vertex which is adjacent to vertices with distinct total degrees.*

Definition 2.9. *Let* $G : (A, B)$ *be a connected single valued neutrosophic graph on* $G^* : (V, E)$. *Then:*
(1) G *is said to be a neighbourly irregular single valued neutrosophic graph if every pair of adjacent vertices have distinct degrees.*
(2) G *is said to be a neighbourly totally single valued neutrosophic fuzzy graph if every pair of adjacent vertices have distinct total degrees.*
(3) G *is said to be a strongly irregular single valued neutrosophic graph if every pair of vertices have distinct degrees.*
(4) G *is said to be a strongly totally irregular single valued neutrosophic graph if every pair of vertices have distinct total degrees.*
(5) G *is said to be a highly irregular single valued neutrosophic graph if every vertex in* G *is adjacent to the vertices having distinct degrees.*
(6) G *is said to be a highly totally irregular single valued neutrosophic graph if every vertex in* G *is adjacent to the vertices having distinct total degrees.*

Definition 2.10. *Let* $G : (A, B)$ *be a SVNG on* $G^* : (V, E)$. *The degree of an edge* uv *is defined as* $d_G(uv) = (d_{T_B}(uv), d_{I_B}(uv), d_{F_B}(uv))$ *where* $d_{T_B}(uv) = d_{T_A}(u) + d_{T_A}(v) - 2T_B(uv)$, $d_{I_B}(uv) = d_{I_A}(u) + d_{I_A}(v) - 2I_B(uv)$ *and* $d_{F_B}(uv) = d_{F_A}(u) + d_{F_A}(v) - 2F_B(uv)$.

Definition 2.11. *Let* $G : (A, B)$ *be a SVNG on* $G^* : (V, E)$. *The total degree of an edge* uv *is defined as* $td_G(uv) = (td_{T_B}(uv), td_{I_B}(uv), td_{F_B}(uv))$ *where* $td_{T_B}(uv) = d_{T_A}(u) + d_{T_A}(v) - T_B(uv) = d_{T_B}(uv) + T_B(uv)$, $td_{I_B}(uv) = d_{I_A}(u) + d_{I_A}(v) - I_B(uv) = d_{I_B}(uv) + I_B(uv)$ *and* $td_{F_B}(uv) = d_{F_A}(u) + d_{F_A}(v) - F_B(uv) = d_{F_B}(uv) + F_B(uv)$.

3. Neighbourly edge irregular single valued neutrosophic graphs and neighbourly edge totally irregular single valued neutrosophic graphs

In this section, neighbourly edge irregular single valued neutrosophic graphs and neighbourly edge totally irregular single valued neutrosophic graphs are introduced.

Definition 3.1. *Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$. Then G is said to be:*
(1) A neighbourly edge irregular single valued neutrosophic graph if every pair of adjacent edges have distinct degrees.
(2) A neighbourly edge totally irregular single valued neutrosophic graph if every pair of adjacent edges have distinct total degrees.

Example 3.2. *Graph which is both neighbourly edge irregular single valued neutrosophic graph and neighbourly edge totally irregular single valued neutrosophic graph.*
Consider $G^* : (V, E)$ where $V = \{u, v, w, x\}$ and $E = \{uv, vw, wx, xu\}$.

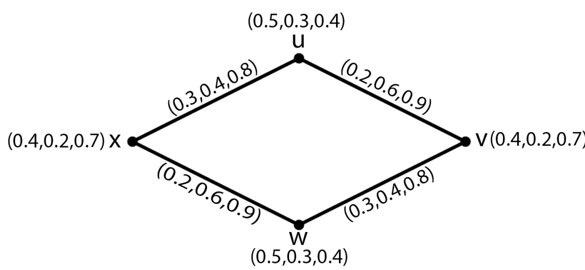

Figure 1. Both neighbourly edge irregular SVNG and neighbourly edge totally irregular SVNG.

From Figure 1,
$d_G(u) = d_G(v) = d_G(w) = d_G(x) = (0.5, 1.0, 1.7)$.
Degrees of the edges are calculated as follows
$d_G(uv) = d_G(wx) = (0.6, 0.8, 1.6), d_G(vw) = d_G(xu) = (0.4, 1.2, 1.8)$.
It is noted that every pair of adjacent edges have distinct degrees. Hence G is a neighbourly edge irregular single valued neutrosophic graph.
Total degrees of the edges are calculated as follows
$td_G(uv) = td_G(wx) = (0.8, 1.4, 2.5), td_G(vw) = td_G(xu) = (0.7, 1.6, 2.6)$.
It is observed that every pair of adjacent edges having distinct total degrees. So, G is a neighbourly edge totally irregular single valued neutrosophic graph.
Hence G is both neighbourly edge irregular single valued neutrosophic graph and neighbourly edge totally irregular single valued neutrosophic graph.

Example 3.3. *Neighbourly edge irregular single valued neutrosophic graph do not need to be neighbourly edge totally irregular single valued neutrosophic graph.*
Consider $G : (A, B)$ be a single valued neutrosophic graph such that $G^* : (V, E)$ is a star on four vertices.

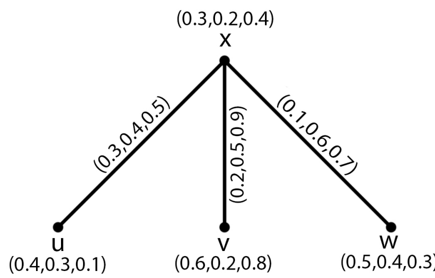

Figure 2. Neighbourly edge irregular SVNG, not neighbourly edge totally irregular SVNG.

From Figure 2,
$d_G(u) = (0.3, 0.4, 0.5), d_G(v) = (0.2, 0.5, 0.9), d_G(w) = (0.1, 0.6, 0.7), d_G(x) = (0.6, 1.5, 2.1);$
$d_G(ux) = (0.3, 1.1, 1.6), d_G(vx) = (0.4, 1.0, 1.2), d_G(wx) = (0.5, 0.9, 1.4);$
$td_G(ux) = td_G(vx) = td_G(wx) = (0.6, 1.5, 2.1).$
Here, $d_G(ux) \neq d_G(vx) \neq d_G(wx)$. Hence G is a neighbourly edge irregular single valued neutrosophic graph. But G is not a neighbourly edge totally irregular single valued neutrosophic graph, since all edges have same total degrees.

Example 3.4. *Neighbourly edge totally irregular single valued neutrosophic graphs don't need to be neighbourly edge irregular single valued neutrosophic graphs. Following shows this subject:*
Consider $G : (A, B)$ be a single valued neutrosophic graph such that $G^* : (V, E)$ is a path on four vertices.

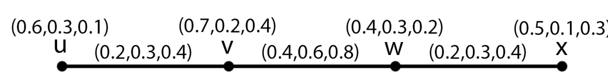

Figure 3. Neighbourly edge totally irregular single valued neutrosophic graph, not neighbourly edge irregular single valued neutrosophic graph

From Figure 3,
$d_G(u) = d_G(x) = (0.2, 0.3, 0.4), d_G(v) = d_G(w) = (0.6, 0.9, 1.2);$
$d_G(uv) = d_G(vw) = d_G(wx) = (0.4, 0.6, 0.8);$
$td_G(uv) = td_G(wx) = (0.6, 0.9, 1.2), td_G(vw) = (0.8, 1.2, 1.6).$
Here, $d_G(uv) = d_G(vw) = d_G(wx)$. Hence G is not a neighbourly edge irregular single valued

neutrosophic graph. But G is a neighbourly edge totally irregular single valued neutrosophic graph, since $td_G(uv) \neq td_G(vw)$ and $td_G(vw) \neq td_G(wx)$.

Theorem 3.5. *Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$ and $B : (T_B, I_B, F_B)$ is a constant function. Then G is a neighbourly edge irregular single valued neutrosophic graph, if and only if G is a neighbourly edge totally irregular single valued neutrosophic graph.*

Proof: Assume that $B : (T_B, I_B, F_B)$ is a constant function, let $B(uv) = C$, for all uv in E, where $C = (C_T, C_I, C_F)$ is constant.
Let uv and vw be pair of adjacent edges in E, then we have
$d_G(uv) \neq d_G(vw)$
$\iff d_G(uv) + C \neq d_G(vw) + C$
$\iff (d_{T_B}(uv), d_{I_B}(uv), d_{F_B}(uv)) + (C_T, C_I, C_F) \neq (d_{T_B}(vw), d_{I_B}(vw), d_{F_B}(vw)) + (C_T, C_I, C_F)$
$\iff (d_{T_B}(uv) + C_T, d_{I_B}(uv) + C_I, d_{F_B}(uv) + C_F) \neq (d_{T_B}(vw) + C_T, d_{I_B}(vw) + C_I, d_{F_B}(vw) + C_F)$
$\iff (d_{T_B}(uv) + T_B(uv), d_{I_B}(uv) + I_B(uv), d_{F_B}(uv) + F_B(uv)) \neq (d_{T_B}(vw) + T_B(vw), d_{I_B}(vw) + I_B(vw), d_{F_B}(vw) + F_B(vw))$
$\iff (td_{T_B}(uv), td_{I_B}(uv), td_{F_B}(uv)) \neq (td_{T_B}(vw), td_{I_B}(vw), td_{F_B}(vw))$
$\iff td_G(uv) \neq td_G(vw)$.
Therefore every pair of adjacent edges have distinct degrees if and only if have distinct total degrees. Hence G is a neighbourly edge irregular single valued neutrosophic graph if and only if G is a neighbourly edge totally irregular single valued neutrosophic graph.

□

Remark 3.6. *Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$. If G is both neighbourly edge irregular single valued neutrosophic graph and neighbourly edge totally irregular single valued neutrosophic graph, Then B don't need to be a constant function.*

Example 3.7. *Consider $G : (A, B)$ be a single valued neutrosophic graph such that $G^* : (V, E)$ is a path on four vertices.*

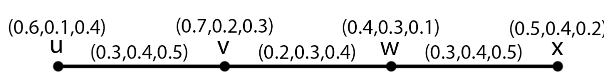

Figure 4. B is not a constant function.

From Figure 4,
$d_G(u) = d_G(x) = (0.3, 0.4, 0.5), d_G(v) = d_G(w) = (0.5, 0.7, 0.9)$;
$d_G(uv) = d_G(wx) = (0.2, 0.3, 0.4), d_G(vw) = (0.6, 0.8, 1.0)$;

$td_G(uv) = td_G(wx) = (0.5, 0.7, 0.9), td_G(vw) = (0.8, 1.1, 1.4)$.
Here, $d_G(uv) \neq d_G(vw)$ and $d_G(vw) \neq d_G(wx)$. Hence G is a neighbourly edge irregular single valued neutrosophic graph. Also, $td_G(uv) \neq td_G(vw)$ and $td_G(vw) \neq td_G(wx)$. Hence G is a neighbourly edge totally irregular single valued neutrosophic graph. But B is not constant function.

Theorem 3.8. *Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$ and $B : (T_B, I_B, F_B)$ is a constant function. If G is a strongly irregular single valued neutrosophic graph, then G is a neighbourly edge irregular single valued neutrosophic graph.*

Proof: Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$. Assume that $B : (T_B, I_B, F_B)$ is a constant function, let $B(uv) = C$, for all uv in E, where $C = (C_T, C_I, C_F)$ is constant.
Let uv and vw be any two adjacent edges in G. Let us suppose that G is a strongly irregular single valued neutrosophic graph. Then every pair of vertices in G having distinct degrees, and hence
$d_G(u) \neq d_G(v) \neq d_G(w)$
$\Rightarrow (d_{T_A}(u), d_{I_A}(u), d_{F_A}(u)) \neq (d_{T_A}(v), d_{I_A}(v), d_{F_A}(v)) \neq (d_{T_A}(w), d_{I_A}(w), d_{F_A}(w))$
$\Rightarrow (d_{T_A}(u), d_{I_A}(u), d_{F_A}(u)) + (d_{T_A}(v), d_{I_A}(v), d_{F_A}(v)) - 2(C_T, C_I, C_F) \neq (d_{T_A}(v), d_{I_A}(v), d_{F_A}(v)) + (d_{T_A}(w), d_{I_A}(w), d_{F_A}(w)) - 2(C_T, C_I, C_F)$
$\Rightarrow (d_{T_A}(u)+d_{T_A}(v)-2C_T, d_{I_A}(u)+d_{I_A}(v)-2C_I, d_{F_A}(u)+d_{F_A}(v))-2C_F) \neq (d_{T_A}(v)+d_{T_A}(w)-2C_T, d_{I_A}(v) + d_{I_A}(w) - 2C_I, d_{F_A}(v)) + d_{F_A}(w) - 2C_F)$
$\Rightarrow (d_{T_A}(u) + d_{T_A}(v) - 2T_B(uv), d_{I_A}(u) + d_{I_A}(v) - 2I_B(uv), d_{F_A}(u) + d_{F_A}(v)) - 2F_B(uv)) \neq (d_{T_A}(v) + d_{T_A}(w) - 2T_B(vw), d_{I_A}(v) + d_{I_A}(w) - 2I_B(vw), d_{F_A}(v)) + d_{F_A}(w) - 2F_B(vw))$
$\Rightarrow (d_{T_B}(uv), d_{I_B}(uv), d_{F_B}(uv)) \neq (d_{T_B}(vw), d_{I_B}(vw), d_{F_B}(vw))$
$\Rightarrow d_G(uv) \neq d_G(vw)$.
Therefore every pair of adjacent edges have distinct degrees, hence G is a neighbourly edge irregular single valued neutrosophic graph.

□

Similar to the above theorem can be considered the following theorem:

Theorem 3.9. *Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$ and $B : (T_B, I_B, F_B)$ is a constant function. If G is a strongly irregular single valued neutrosophic graph, then G is a neighbourly edge totally irregular single valued neutrosophic graph.*

Remark 3.10. *Converse of the above theorems don't need to be true.*

Example 3.11. *Consider $G : (A, B)$ be a single valued neutrosophic graph such that $G^* : (V, E)$ is a path on four vertices.*

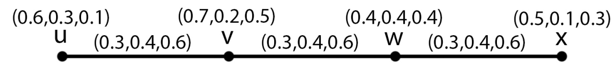

Figure 5. Both neighbourly edge irregular SVNG and neighbourly edge totally irregular SVNG, not strongly irregular SVNG

From Figure 5,

$d_G(u) = d_G(x) = (0.3, 0.4, 0.6), d_G(v) = d_G(w) = (0.6, 0.8, 1.2)$. *Here, G is not a strongly irregular single valued neutrosophic graph.*

$d_G(uv) = d_G(wx) = (0.3, 0.4, 0.6), d_G(vw) = (0.6, 0.8, 1.2)$;

$td_G(uv) = td_G(vw) = td_G(wx) = (0.6, 0.8, 1.2)$.

It is noted that $d_G(uv) \neq d_G(vw)$ and $d_G(vw) \neq d_G(wx)$. And also, $td_G(uv) \neq td_G(vw)$ and $td_G(vw) \neq td_G(wx)$. Hence G is both neighbourly edge irregular single valued neutrosophic graph and neighbourly edge totally irregular single valued neutrosophic graph. But G is not a strongly irregular single valued neutrosophic graph.

Theorem 3.12. *Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$ and $B : (T_B, I_B, F_B)$ is a constant function. Then G is a highly irregular single valued neutrosophic graph if and only if G is a neighbourly edge irregular single valued neutrosophic graph.*

Proof: Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$. Assume that $B : (T_B, I_B, F_B)$ is a constant function, let $B(uv) = C$, for all uv in E, where $C = (C_T, C_I, C_F)$ is constant.

Let uv and vw be any two adjacent edges in G. Then we have

$d_G(u) \neq d_G(w)$

$\iff (d_{T_A}(u), d_{I_A}(u), d_{F_A}(u)) \neq (d_{T_A}(w), d_{I_A}(w), d_{F_A}(w))$

$\iff (d_{T_A}(u), d_{I_A}(u), d_{F_A}(u)) + (d_{T_A}(v), d_{I_A}(v), d_{F_A}(v)) - 2(C_T, C_I, C_F) \neq (d_{T_A}(v), d_{I_A}(v), d_{F_A}(v)) + (d_{T_A}(w), d_{I_A}(w), d_{F_A}(w)) - 2(C_T, C_I, C_F)$

$\iff (d_{T_A}(u) + d_{T_A}(v) - 2C_T, d_{I_A}(u) + d_{I_A}(v) - 2C_I, d_{F_A}(u) + d_{F_A}(v)) - 2C_F) \neq (d_{T_A}(v) + d_{T_A}(w) - 2C_T, d_{I_A}(v) + d_{I_A}(w) - 2C_I, d_{F_A}(v)) + d_{F_A}(w) - 2C_F)$

$\iff (d_{T_A}(u) + d_{T_A}(v) - 2T_B(uv), d_{I_A}(u) + d_{I_A}(v) - 2I_B(uv), d_{F_A}(u) + d_{F_A}(v)) - 2F_B(uv)) \neq (d_{T_A}(v) + d_{T_A}(w) - 2T_B(vw), d_{I_A}(v) + d_{I_A}(w) - 2I_B(vw), d_{F_A}(v)) + d_{F_A}(w) - 2F_B(vw))$

$\iff (d_{T_B}(uv), d_{I_B}(uv), d_{F_B}(uv)) \neq (d_{T_B}(vw), d_{I_B}(vw), d_{F_B}(vw))$

$\iff d_G(uv) \neq d_G(vw)$.

Therefore every pair of adjacent edges have distinct degrees, if and only if every vertex adjacent to the vertices having distinct degrees. Hence G is a highly irregular single valued neutrosophic graph, if and only if G is a neighbourly edge irregular single valued neutrosophic graph.

□

Theorem 3.13. *Let $G:(A,B)$ be a connected single valued neutrosophic graph on $G^*:(V,E)$ and $B:(T_B,I_B,F_B)$ is a constant function. Then G is highly irregular single valued neutrosophic graph if and only if G is neighbourly edge totally irregular single valued neutrosophic graph.*

proof: Proof is similar to Theorem 3.12.

□

Theorem 3.14. *Let $G:(A,B)$ be a single valued neutrosophic graph on $G^*:(V,E)$, a star $K_{1,n}$. Then G is a totally edge regular single valued neutrosophic graph. Also, if the degrees of truth-membership, indeterminacy-membership and falsity-membership of no two edges are same, then G is a neighbourly edge irregular single valued neutrosophic graph.*

proof: Let $v_1,v_2,v_3,...,v_n$ be the vertices adjacent to the vertex x. Let $e_1,e_2,e_3,...,e_n$ be the edges of a star G^* in that order having the degrees of truth-membership $p_1,p_2,p_3,...,p_n$, the degrees of indeterminacy-membership $q_1,q_2,q_3,...,q_n$ and the degrees of falsity-membership $r_1,r_2,r_3,...,r_n$

that $0 \le p_i+q_i+r_i \le 3$ for every $1 \le i \le n$. Then,

$td_G(e_i) = (td_{T_B}(e_i), td_{I_B}(e_i), td_{F_B}(e_i)) = (d_{T_B}(e_i)+T_B(e_i), d_{I_B}(e_i)+I_B(e_i), d_{F_B}(e_i)+F_B(e_i)) = ((p_1+p_2+p_3+...+p_n)-p_i+p_i, (q_1+q_2+q_3+...+q_n)-q_i+q_i, (r_1+r_2+r_3+...+r_n)-r_i+r_i) = (p_1+p_2+p_3+...+p_n, q_1+q_2+q_3+...+q_n, r_1+r_2+r_3+...+r_n)$.

All edges e_i , $(1 \le i \le n)$, having same total degrees. Hence G is a totally edge regular single valued neutrosophic graph. Now, if $p_i \ne p_j$, $q_i \ne q_j$ and $r_i \ne r_j$ for every $1 \le i,j \le n$, then we have

$d_G(e_i) = (d_{T_B}(e_i), d_{I_B}(e_i), d_{F_B}(e_i)) = (d_{T_A}(x)+d_{T_A}(v_i)-2T_B(xv_i), d_{I_A}(x)+d_{I_A}(v_i)-2I_B(xv_i), d_{F_A}(x)+d_{F_A}(v_i)-2F_B(xv_i)) = ((p_1+p_2+p_3+...+p_n)+p_i-2p_i, (q_1+q_2+q_3+...+q_n)+q_i-2q_i, (r_1+r_2+r_3+...+r_n)+r_i-2r_i) = ((p_1+p_2+p_3+...+p_n)-p_i, (q_1+q_2+q_3+...+q_n)-q_i, (r_1+r_2+r_3+...+r_n)-r_i)$ for every $1 \le i \le n$.

Therefore, all edges e_i ,$(1 \le i \le n)$, having distinct degrees. Hence G is a neighbourly edge irregular single valued neutrosophic graph.

□

Theorem 3.15. *Let $G:(A,B)$ be a single valued neutrosophic graph such that $G^*:(V,E)$ is a path on $2m (m>1)$ vertices. If the degrees of truth-membership, indeterminacy-membership* *and falsity-membership of the edges e_i, $i = 1, 3, 5, ..., 2m-1$, are p_1 , q_1 and r_1,respectively, and the degrees of truth-membership, indeterminacy-membership and falsity-membership of the edges e_i, $i = 2, 4, 6, ..., 2m-2$, are p_2 , q_2 and r_2, respectively, such that $p_1 \neq p_2$, $p_2 \neq 2p_1$, $q_1 \neq q_2$, $q_2 \neq 2q_1$, $r_1 \neq r_2$ and $r_2 \neq 2r_1$, then G is both neighbourly edge irregular single valued neutrosophic graph and neighbourly edge totally irregular single valued neutrosophic graph.*

proof: Let $G : (A, B)$ be a single valued neutrosophic graph on $G^* : (V, E)$, a path on $2m(m > 1)$ vertices. Let $e_1, e_2, e_3, ..., e_{2m-1}$ be the edges of path G^*. If the alternate edges have the same degrees of truth-membership, indeterminacy-membership and falsity-membership, such that

$$B(e_i) = (T_B(e_i), I_B(e_i), F_B(e_i)) = \begin{cases} (p_1, q_1, r_1) \ if \ i \ is \ odd \\ (p_2, q_2, r_2) \ if \ i \ is \ even. \end{cases}$$

where $0 \leq p_i + q_i + r_i \leq 3$ and $p_1 \neq p_2$, $p_2 \neq 2p_1$, $q_1 \neq q_2$, $q_2 \neq 2q_1$, $r_1 \neq r_2$ and $r_2 \neq 2r_1$, then

$d_G(e_1) = ((p_1) + (p_1 + p_2) - 2p_1, (q_1) + (q_1 + q_2) - 2q_1, (r_1) + (r_1 + r_2) - 2r_1) = (p_2, q_2, r_2)$

for $i = 3, 5, 7, ..., 2m-3$;

$d_G(e_i) = ((p_1 + p_2) + (p_1 + p_2) - 2p_1, (q_1 + q_2) + (q_1 + q_2) - 2q_1, (r_1 + r_2) + (r_1 + r_2) - 2r_1) = (2p_2, 2q_2, 2r_2)$

for $i = 2, 4, 6, ..., 2m-2$;

$d_G(e_i) = ((p_1 + p_2) + (p_1 + p_2) - 2p_2, (q_1 + q_2) + (q_1 + q_2) - 2q_2, (r_1 + r_2) + (r_1 + r_2) - 2r_2) = (2p_1, 2q_1, 2r_1)$

$d_G(e_{2m-1}) = ((p_1 + p_2) + (p_1) - 2p_1, (q_1 + q_2) + (q_1) - 2q_1, (r_1 + r_2) + (r_1) - 2r_1) = (p_2, q_2, r_2)$.

We observe that the adjacent edges have distinct degrees. Hence G is a neighbourly edge irregular single valued neutrosophic graph. Also we have

$td_G(e_1) = (p_1 + p_2, q_1 + q_2, r_1 + r_2)$

$td_G(e_i) = (2p_1 + p_2, 2q_1 + q_2, 2r_1 + r_2)$ for $i = 2, 4, 6, ..., 2m-2$

$td_G(e_i) = (p_1 + 2p_2, q_1 + 2q_2, r_1 + 2r_2)$ for $i = 3, 5, 7, ..., 2m-3$

$td_G(e_{2m-1}) = (p_1 + p_2, q_1 + q_2, r_1 + r_2)$.

Therefore the adjacent edges have distinct total degrees, hence G is a neighbourly edge totally irregular single valued neutrosophic graph.

□

Theorem 3.16. *Let $G : (A, B)$ be a single valued neutrosophic graph such that $G^* : (V, E)$ is an even cycle of length $2m$. If the alternate edges have the same degrees of truth-membership,* *the same degrees of indeterminacy-membership and the same degrees of falsity-membership , then G is both neighbourly edge irregular single valued neutrosophic graph and neighbourly edge totally irregular single valued neutrosophic graph.*

proof: Let $G : (A, B)$ be a single valued neutrosophic fuzzy graph on $G^* : (V, E)$, an even cycle of length $2m$. Let $e_1, e_2, e_3, ..., e_{2m}$ be the edges of cycle G^*. If the alternate edges have the same degrees of truth-membership, the same degrees of indeterminacy-membership and the same degrees of falsity-membership, such that

$$B(e_i) = (T_B(e_i), I_B(e_i), I_B(e_i)) = \begin{cases} (p_1, q_1, r_1) \; if \; i \; is \; odd \\ (p_2, q_2, r_2) \; if \; i \; is \; even. \end{cases}$$

where $0 \leq p_i + q_i + q_i \leq 3$ and $p_1 \neq p_2$, $q_1 \neq q_2$ and $r_1 \neq r_2$,then
for $i = 1, 3, 5, 7, ..., 2m - 1$;
$d_G(e_i) = ((p_1 + p_2) + (p_1 + p_2) - 2p_1, (q_1 + q_2) + (q_1 + q_2) - 2q_1, (r_1 + r_2) + (r_1 + r_2) - 2r_1) = (2p_2, 2q_2, 2r_2)$
for $i = 2, 4, 6, ..., 2m$;
$d_G(e_i) = ((p_1 + p_2) + (p_1 + p_2) - 2p_2, (q_1 + q_2) + (q_1 + q_2) - 2q_2, (r_1 + r_2) + (r_1 + r_2) - 2r_2) = (2p_1, 2q_1, 2r_1)$

We observe that the adjacent edges have distinct degrees. Hence G is a neighbourly edge irregular single valued neutrosophic graphs. Also we have
$td_G(e_i) = (p_1 + 2p_2, q_1 + 2q_2, r_1 + 2r_2)$ for $i = 1, 3, 5, 7, ..., 2m - 1$
$td_G(e_i) = (2p_1 + p_2, 2q_1 + q_2, 2r_1 + r_2)$ for $i = 2, 4, 6, ..., 2m$.

Therefore the adjacent edges have distinct total degrees, hence G is a neighbourly edge totally irregular single valued neutrosophic graph.

□

Theorem 3.17. *Let $G : (A, B)$ be a single valued neutrosophic graph on $G^* : (V, E)$, a cycle on $m(m \geq 4)$ vertices. If the degrees of truth-membership, indeterminacy-membership and falsity-membership of the edges $e_1, e_2, e_3, ..., e_m$ are $p_1, p_2, p_3, ..., p_m$ such that $p_1 < p_2 < p_3 < ... < p_m$, $q_1, q_2, q_3, ..., q_m$ such that $q_1 > q_2 > q_3 > ... > q_m$ and $r_1, r_2, r_3, ..., r_m$ such that $r_1 > r_2 > r_3 > ... > r_m$, respectively, then G is both neighbourly edge irregular single valued neutrosophic graph and neighbourly edge totally irregular single valued neutrosophic graph.*

proof: Let $G : (A, B)$ be a single valued neutrosophic graph on $G^* : (V, E)$, a cycle on $m(m \geq 4)$ vertices. Let $e_1, e_2, e_3, ..., e_m$ be the edges of cycle G^* in that order. Let degrees of truth-membership, indeterminacy-membership and falsity-membership of the edges

$e_1, e_2, e_3, ..., e_m$ are $p_1, p_2, p_3, ..., p_m$ such that $p_1 < p_2 < p_3 < ... < p_m$, $q_1, q_2, q_3, ..., q_m$ such that $q_1 > q_2 > q_3 > ... > q_m$ and $r_1, r_2, r_3, ..., r_m$ such that $r_1 > r_2 > r_3 > ... > r_m$, respectively, then

$d_G(v_1) = (p_1 + p_m, q_1 + q_m, r_1 + r_m)$

$d_G(v_i) = (p_{i-1} + p_i, q_{i-1} + q_i, r_{i-1} + r_i)$ for $i = 2, 3, 4, 5, ..., m$

$d_G(e_1) = (p_2 + p_m, q_2 + q_m, r_2 + r_m)$

$d_G(e_i) = (p_{i-1} + p_{i+1}, q_{i-1} + q_{i+1}, r_{i-1} + r_{i+1})$ for $i = 2, 3, 4, 5, ..., m - 1$

$d_G(e_m) = (p_1 + p_{m-1}, q_1 + q_{m-1}, r_1 + r_{m-1})$.

We observe that the adjacent edges have distinct degrees. Hence G is a neighbourly edge irregular single valued neutrosophic graph.

$td_G(e_1) = (p_1 + p_2 + p_m, q_1 + q_2 + q_m, r_1 + r_2 + r_m)$

$td_G(e_i) = (p_{i-1} + p_i + p_{i+1}, q_{i-1} + q_i + q_{i+1}, r_{i-1} + r_i + r_{i+1})$ for $i = 2, 3, 4, 5, ..., m - 1$

$td_G(e_m) = (p_1 + p_{m-1} + p_m, q_1 + q_{m-1} + q_m, r_1 + r_{m-1} + r_m)$.

We note that the adjacent edges have distinct total degrees. Hence G is a neighbourly edge totally irregular single valued neutrosophic graph.

□

4. Strongly edge irregular single valued neutrosophic graphs and strongly edge totally irregular single valued neutrosophic graphs

Now, In this section, we study strongly edge irregular single valued neutrosophic graphs and strongly edge totally irregular single valued neutrosophic graphs.

Definition 4.1. *Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$. Then G is said to be:*
(1)A strongly edge irregular single valued neutrosophic graph if every pair of edges having distinct degrees (or no two edges have same degree). (2) A strongly edge totally irregular single valued neutrosophic graph if every pair of edges having distinct total degrees (or no two edges have same total degree).

Example 4.2. *Graph which is both strongly edge irregular single valued neutrosophic graph and strongly edge totally irregular single valued neutrosophic graph.*
Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$ which is a cycle of length five.

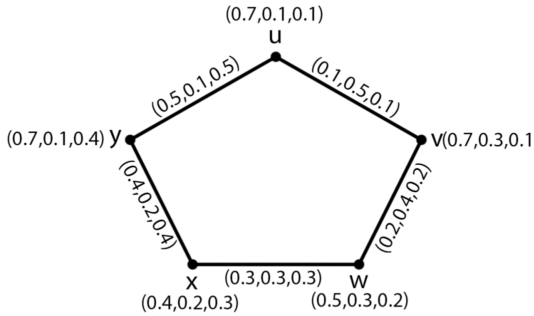

Figure 6. Both strongly edge irregular SVNG and strongly edge totally irregular SVNG.

From Figure 6,
$d_G(u) = (0.6, 0.6, 0.6), d_G(v) = (0.3, 0.9, 0.3), d_G(w) = (0.5, 0.7, 0.5), d_G(x) = (0.7, 0.5, 0.7),$
$d_G(y) = (0.9, 0.3, 0.9).$
Degrees of the edges are calculated as follows
$d_G(uv) = (0.7, 0.5, 0.7), d_G(vw) = (0.4, 0.8, 0.4), d_G(wx) = (0.6, 0.6, 0.6),$
$d_G(xy) = (0.8, 0.4, 0.8), d_G(yu) = (0.5, 0.7, 0.5).$
It is noted that every pair of edges having distinct degrees. Hence G is a strongly edge irregular single valued neutrosophic graph.
Total degrees of the edges are calculated as follows
$td_G(uv) = (0.8, 1.0, 0.8), td_G(vw) = (0.6, 1.2, 0.6), td_G(wx) = (0.9, 0.9, 0.9),$
$td_G(xy) = (1.2, 0.6, 1.2), td_G(yu) = (1.0, 0.8, 1.0).$
It is observed that every pair of edges having distinct total degrees. So, G is a strongly edge totally irregular single valued neutrosophic graph.
Hence G is both strongly edge irregular single valued neutrosophic graph and strongly edge totally irregular single valued neutrosophic graph.

Example 4.3. *Strongly edge irregular single valued neutrosophic graph need not be strongly edge totally irregular single valued neutrosophic graph.*

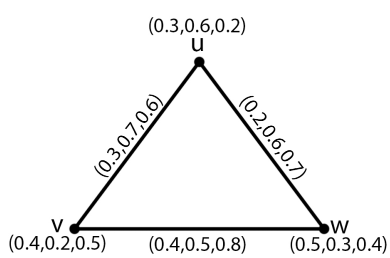

Figure 7. Strongly edge irregular SVNG, not strongly edge totally irregular SVNG

Consider $G : (A, B)$ be a single valued neutrosophic graph such that $G^* : (V, E)$, a cycle of length three.
From Figure 7,
$d_G(u) = (0.5, 1.3, 1.3), d_G(v) = (0.7, 1.2, 1.4), d_G(w) = (0.6, 1.1, 1.5)$;
$d_G(uv) = (0.6, 1.1, 1.5), d_G(vw) = (0.5, 1.3, 1.3), d_G(wu) = (0.7, 1.2, 1.4)$;
$td_G(uv) = td_G(wx) = td_G(wu) = (0.9, 1.8, 2.1)$.
noted that G is strongly edge irregular single valued neutrosophic graph, since every pair of edges having distinct degrees. Also, G is not strongly edge totally irregular single valued neutrosophic graph, since all the edges having same total degree. Hence strongly edge irregular single valued neutrosophic graph need not be strongly edge totally irregular single valued neutrosophic graph.

Example 4.4. *Strongly edge totally irregular single valued neutrosophic graphs need not be strongly edge irregular single valued neutrosophic graphs.*
Consider $G : (A, B)$ be a single valued neutrosophic graph such that $G^* : (V, E)$, a cycle of length four.

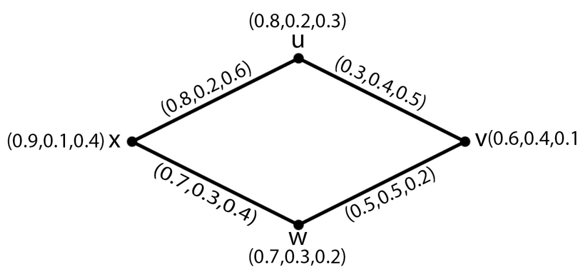

Figure 8. Strongly edge totally irregular SVNG, not strongly edge irregular SVNG.

From Figure 8,
$d_G(u) = (1.1, 0.6, 1.1), d_G(v) = (0.8, 0.9, 0.7), d_G(w) = (1.2, 0.8, 0.6), d_G(x) = (1.5, 0.5, 1.0)$;
$d_G(uv) = d_G(wx) = (1.3, 0.7, 0.8), d_G(vw) = d_G(xu) = (1.0, 0.7, 0.9)$;
$td_G(uv) = (1.6, 1.1, 1.3), td_G(vw) = (1.5, 1.2, 1.1), td_G(wx) = (2.0, 1.0, 1.2)$,
$d_G(xu) = (1.9, 0.8, 1.5)$.
It is noted that $d_G(uv) = d_G(wx)$. Hence G is not strongly edge irregular single valued neutrosophic graph.
But G is strongly edge totally irregular single valued neutrosophic graph, since $td_G(uv) \neq td_G(vw) \neq td_G(wx) \neq td_G(xu)$.
Hence strongly edge totally irregular single valued neutrosophic graph need not be strongly edge irregular single valued neutrosophic graph.

Theorem 4.5. *Let $G:(A,B)$ be a connected single valued neutrosophic graph on $G^*:(V,E)$ and $B:(T_B,I_B,F_B)$ is a constant function. Then G is a strongly edge irregular single valued neutrosophic graph, if and only if G is a strongly edge totally irregular single valued neutrosophic graph.*

proof: Assume that $B:(T_B,I_B,F_B)$ is a constant function, let $B(uv)=C$, for all uv in E, where $C=(C_T,C_I,C_F)$ is constant.

Let uv and xy be any pair of edges in E. Then we have

$d_G(uv)\neq d_G(xy)$

$\Longleftrightarrow d_G(uv)+C\neq d_G(xy)+C$

$\Longleftrightarrow (d_{T_B}(uv),d_{I_B}(uv),d_{F_B}(uv))+(C_T,C_I,C_F)\neq(d_{T_B}(xy),d_{I_B}(xy),d_{F_B}(xy))+(C_T,C_I,C_F)$

$\Longleftrightarrow (d_{T_B}(uv)+C_T,d_{I_B}(uv)+C_I,d_{F_B}(uv)+C_F)\neq(d_{T_B}(xy)+C_T,d_{I_B}(xy)+C_I,d_{F_B}(xy)+C_F)$

$\Longleftrightarrow (d_{T_B}(uv)+T_B(uv),d_{I_B}(uv)+I_B(uv),d_{F_B}(uv)+F_B(uv))\neq(d_{T_B}(xy)+T_B(xy),d_{I_B}(xy)+I_B(xy),d_{F_B}(xy)+F_B(xy))$

$\Longleftrightarrow (td_{T_B}(uv),td_{I_B}(uv),td_{F_B}(uv))\neq(td_{T_B}(xy),td_{I_B}(xy),td_{F_B}(xy))$

$\Longleftrightarrow td_G(uv)\neq td_G(xy)$

Therefore every pair of edges have distinct degrees if and only if have distinct total degrees. Hence G is strongly edge irregular single valued neutrosophic graph if and only if G is a strongly edge totally irregular single valued neutrosophic graph.

□

Remark 4.6. *Let $G:(A,B)$ be a connected single valued neutrosophic graph on $G^*:(V,E)$. If G is both strongly edge irregular single valued neutrosophic graph and strongly edge totally irregular single valued neutrosophic graph, Then B need not be a constant function.*

Example 4.7. *Consider $G:(A,B)$ be a single valued neutrosophic graph such that $G^*:(V,E)$ is a cycle of length five.*

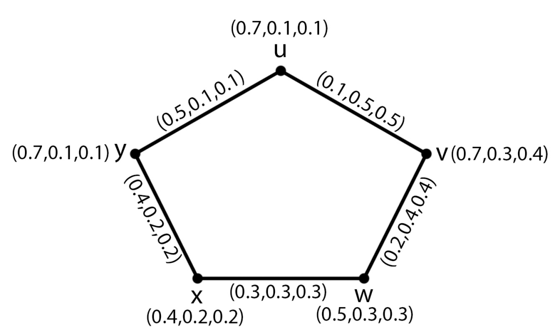

Figure 9. B is not a constant function.

From Figure 9,
$d_G(u) = (0.6, 0.6, 0.6), d_G(v) = (0.3, 0.9, 0.9), d_G(w) = (0.5, 0.7, 0.7), d_G(x) = (0.7, 0.5, 0.5)$,
$d_G(y) = (0.9, 0.3, 0.3)$.
Also, $d_G(uv) = (0.7, 0.5, 0.5), d_G(vw) = (0.4, 0.8, 0.8), d_G(wx) = (0.6, 0.6, 0.6)$,
$d_G(xy) = (0.8, 0.4, 0.4), d_G(yu) = (0.5, 0.7, 0.7)$.
It is noted that every pair of edges in G having distinct degrees. Hence G is a strongly edge irregular single valued neutrosophic graph.
Also, $td_G(uv) = (0.8, 1.0, 1.0), td_G(vw) = (0.6, 1.2, 1.2), td_G(wx) = (0.9, 0.9, 0.9)$,
$td_G(xy) = (1.2, 0.6, 0.6), td_G(yu) = (1.0, 0.8, 0.8)$.
Note that every pair of edges in G having distinct total degrees. Hence G is a strongly edge totally irregular single valued neutrosophic graph. Therefore G is both strongly edge irregular single valued neutrosophic graph and strongly edge totally irregular single valued neutrosophic graph. But μ is not a constant function.

Theorem 4.8. *Let $G : (A, B)$ be a single valued neutrosophic graph on $G^* : (V, E)$. If G is a strongly edge irregular single valued neutrosophic graph, then G is a neighbourly edge irregular single valued neutrosophic graph.*

proof: Let $G : (A, B)$ be a single valued neutrosophic graph on $G^* : (V, E)$. Let us assume that G is a strongly edge irregular single valued neutrosophic graph, then every pair of edges in G have distinct degrees. So every pair of adjacent edges have distinct degrees. Hence G is a neighbourly edge irregular single valued neutrosophic graph.

□

Theorem 4.9. *Let $G : (A, B)$ be a single valued neutrosophic graph on $G^* : (V, E)$. If G is a strongly edge totally irregular single valued neutrosophic graph, then G is a neighbourly edge totally irregular single valued neutrosophic graph.*

proof: Let $G : (A, B)$ be a single valued neutrosophic graph on $G^* : (V, E)$. Let us assume that G is a strongly edge totally irregular single valued neutrosophic graph, then every pair of edges in G have distinct total degrees. So every pair of adjacent edges have distinct total degrees. Hence G is a neighbourly edge totally irregular single valued neutrosophic graph.

□

Remark 4.10. *Converse of the above Theorems 4.8 and 4.9 need not be true.*

Example 4.11. *Consider $G : (A, B)$ be a fuzzy graph such that $G^* : (V, E)$ is a path on four vertices.*

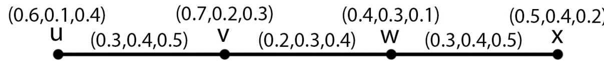

Figure 10. Neighbourly edge irregular SVNG, not strongly edge irregular SVNG; Neighbourly edge totally irregular SVNG, not strongly edge totally irregular SVNG.

From Figure 10,
$d_G(u) = (0.3, 0.4, 0.5), d_G(v) = (0.5, 0.7, 0.9), d_G(w) = (0.5, 0.7, 0.9),$
$d_G(x) = (0.3, 0.4, 0.5);$
$d_G(uv) = d_G(wx) = (0.2, 0.3, 0.4), d_G(vw) = (0.6, 0.8, 1.0);$
$td_G(uv) = td_G(wx) = (0.5, 0.7, 0.9), td_G(vw) = (0.8, 1.1, 1.4).$
Here, $d_G(uv) \neq d_G(vw)$ and $d_G(vw) \neq d_G(wx)$. Hence G is a neighbourly edge irregular single valued neutrosophic graph. But G is not a strongly edge irregular single valued neutrosophic graph, since $d_G(uv) \neq d_G(wx)$. Also, note that $td_G(uv) \neq td_G(vw)$ and $td_G(vw) \neq td_G(wx)$. Hence G is a neighbourly edge totally irregular single valued neutrosophic graph. But G is not a strongly edge totally irregular single valued neutrosophic graph, since $td_G(uv) \neq td_G(wx)$.

Theorem 4.12. *Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$ and $B : (T_B, I_B, F_B)$ is a constant function. If G is a strongly edge irregular single valued neutrosophic graph, then G is an irregular single valued neutrosophic graph.*

proof: Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$. Assume that $B : (T_B, I_B, F_B)$ is a constant function, let $B(uv) = C$, for all uv in E, where $C = (C_T, C_I, C_F)$ is constant.
Let us Suppose that G is a strongly edge irregular single valued neutrosophic graph. Then every pair of edges having distinct degrees. Let uv and vw be adjacent edges in G having distinct degrees, and hence
$d_G(uv) \neq d_G(vw)$
$\Rightarrow (d_{T_B}(uv), d_{I_B}(uv), d_{F_B}(uv)) \neq (d_{T_B}(vw), d_{I_B}(vw), d_{F_B}(vw))$
$\Rightarrow (d_{T_A}(u) + d_{T_A}(v) - 2T_B(uv), d_{I_A}(u) + d_{I_A}(v) - 2I_B(uv), d_{F_A}(u) + d_{F_A}(v)) - 2F_B(uv)) \neq (d_{T_A}(v) + d_{T_A}(w) - 2T_B(vw), d_{I_A}(v) + d_{I_A}(w) - 2I_B(vw), d_{F_A}(v)) + d_{F_A}(w) - 2F_B(vw))$
$\Rightarrow (d_{T_A}(u)+d_{T_A}(v)-2C_T, d_{I_A}(u)+d_{I_A}(v)-2C_I, d_{F_A}(u)+d_{F_A}(v))-2C_F) \neq (d_{T_A}(v)+d_{T_A}(w)-2C_T, d_{I_A}(v) + d_{I_A}(w) - 2C_I, d_{F_A}(v)) + d_{F_A}(w) - 2C_F)$
$\Rightarrow (d_{T_A}(u) + d_{T_A}(v), d_{I_A}(u) + d_{I_A}(v), d_{F_A}(u) + d_{F_A}(v)) - 2(C_T, C_I, C_F) \neq (d_{T_A}(v) + d_{T_A}(w), d_{I_A}(v) + d_{I_A}(w), d_{F_A}(v)) + d_{F_A}(w)) - 2(C_T, C_I, C_F)$
$\Rightarrow (d_{T_A}(u) + d_{T_A}(v), d_{I_A}(u) + d_{I_A}(v), d_{F_A}(u) + d_{F_A}(v)) \neq (d_{T_A}(v) + d_{T_A}(w), d_{I_A}(v) + d_{I_A}(w), d_{F_A}(v)) + d_{F_A}(w))$
$\Rightarrow (d_{T_A}(u), d_{I_A}(u), d_{F_A}(u)) + (d_{T_A}(v), d_{I_A}(v), d_{F_A}(v)) \neq (d_{T_A}(v), d_{I_A}(v), d_{F_A}(v)) + (d_{T_A}(w), d_{I_A}(w), d_{F_A}(w))$

$\Rightarrow d_G(u) + d_G(v) \neq d_G(v) + d_G(w)$
$\Rightarrow d_G(u) \neq d_G(w)$
So there exists a vertex v which is adjacent to vertices u and w having distinct degrees. Hence G is an irregular single valued neutrosophic graph.

□

Theorem 4.13. *Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$ and $B : (T_B, I_B, F_B)$ is a constant function. If G is a strongly edge totally irregular single valued neutrosophic graph, then G is an irregular single valued neutrosophic graph.*

proof: Proof is similar to the above Theorem 4.12.

□

Remark 4.14. *Converse of the above Theorems 4.12 and 4.13 need not be true.*

Example 4.15. *Consider $G : (A, B)$ be a single valued neutrosophic graph such that $G^* : (V, E)$ is a path on four vertices.*

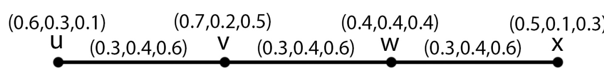

Figure 11. Irregular SVNG, not strongly edge irregular SVNG and not strongly edge totally irregular SVNG.

From Figure 11,
$d_G(u) = d_G(x) = (0.3, 0.4, 0.6), d_G(v) = d_G(w) = (0.6, 0.8, 1.2)$. *Here, G is an irregular single valued neutrosophic graph.*
Also, $d_G(uv) = d_G(wx) = (0.3, 0.4, 0.6), d_G(vw) = (0.6, 0.8, 1.2)$;
$td_G(uv) = td_G(wx) = (0.6, 0.8, 1.2), td_G(vw) = (0.9, 1.2, 1.8)$.
It is noted that $d_G(uv) = d_G(wx)$. Hence G is not a strongly edge irregular single valued neutrosophic graph. Also, $td_G(uv) = td_G(wx)$. Hence G is not a strongly edge totally irregular single valued neutrosophic graph.

Theorem 4.16. *Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$ and $B : (T_B, I_B, F_B)$ is a constant function. If G is a strongly edge irregular single valued neutrosophic graph, Then G is a highly irregular single valued neutrosophic graph.*

proof: Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$. Assume that $B : (T_B, I_B, F_B)$ is a constant function, let $B(uv) = C$ for all uv in E, where $C = (C_T, C_I, C_F)$ is constant.
Let v be any vertex adjacent with u, w and x. Then uv, vw and vx are adjacent edges in G.

Let us suppose that G is strongly edge irregular single valued neutrosophic graph. Then every pair of edges in G have distinct degrees. So every pair of adjacent edges in G have distinct degrees. Hence $d_G(uv) \neq d_G(vw) \neq d_G(vx)$

$\Rightarrow (d_{T_B}(uv), d_{I_B}(uv), d_{F_B}(uv)) \neq (d_{T_B}(vw), d_{I_B}(vw), d_{F_B}(vw)) \neq (d_{T_B}(vx), d_{I_B}(vx), d_{F_B}(vx))$

$\Rightarrow (d_{T_A}(u) + d_{T_A}(v) - 2T_B(uv), d_{I_A}(u) + d_{I_A}(v) - 2I_B(uv), d_{F_A}(u) + d_{F_A}(v)) - 2F_B(uv)) \neq (d_{T_A}(v) + d_{T_A}(w) - 2T_B(vw), d_{I_A}(v) + d_{I_A}(w) - 2I_B(vw), d_{F_A}(v)) + d_{F_A}(w) - 2F_B(vw)) \neq (d_{T_A}(v) + d_{T_A}(x) - 2T_B(vx), d_{I_A}(v) + d_{I_A}(x) - 2I_B(vx), d_{F_A}(v)) + d_{F_A}(x) - 2F_B(vx))$

$\Rightarrow (d_{T_A}(u)+d_{T_A}(v)-2C_T, d_{I_A}(u)+d_{I_A}(v)-2C_I, d_{F_A}(u)+d_{F_A}(v))-2C_F) \neq (d_{T_A}(v)+d_{T_A}(w)-2C_T, d_{I_A}(v)+d_{I_A}(w)-2C_I, d_{F_A}(v))+d_{F_A}(w)-2C_F) \neq (d_{T_A}(v)+d_{T_A}(x)-2C_T, d_{I_A}(v)+d_{I_A}(x)-2C_I, d_{F_A}(v)) + d_{F_A}(x) - 2C_F)$

$\Rightarrow (d_{T_A}(u) + d_{T_A}(v), d_{I_A}(u) + d_{I_A}(v), d_{F_A}(u) + d_{F_A}(v)) \neq (d_{T_A}(v) + d_{T_A}(w), d_{I_A}(v) + d_{I_A}(w), d_{F_A}(v)) + d_{F_A}(w)) \neq (d_{T_A}(v) + d_{T_A}(x), d_{I_A}(v) + d_{I_A}(x), d_{F_A}(v)) + d_{F_A}(x))$

$\Rightarrow (d_{T_A}(u), d_{I_A}(u), d_{F_A}(u)) + (d_{T_A}(v), d_{I_A}(v), d_{F_A}(v)) \neq (d_{T_A}(v), d_{I_A}(v), d_{F_A}(v)) + (d_{T_A}(w), d_{I_A}(w), d_{F_A}(w)) \neq (d_{T_A}(v), d_{I_A}(v), d_{F_A}(v)) + (d_{T_A}(x), d_{I_A}(x), d_{F_A}(x))$

$\Rightarrow (d_{T_A}(u), d_{I_A}(u), d_{F_A}(u)) \neq (d_{T_A}(w), d_{I_A}(w), d_{F_A}(w)) \neq (d_{T_A}(x), d_{I_A}(x), d_{F_A}(x))$

$\Rightarrow d_G(u) \neq d_G(w) \neq d_G(x)$

Therefore the vertex v is adjacent to the vertices with distinct degrees. Hence G is a highly irregular single valued neutrosophic graph.

□

Theorem 4.17. *Let $G : (A, B)$ be a connected single valued neutrosophic graph on $G^* : (V, E)$ and $B : (T_B, I_B, F_B)$ is a constant function. If G is a strongly edge totally irregular single valued neutrosophic graph, Then G is a highly irregular single valued neutrosophic graph.*

proof: Proof is similar to the above Theorem 4.16.

□

Remark 4.18. *Converse of the above Theorems 4.16 and 4.17 need not be true.*

Example 4.19. *Consider $G : (A, B)$ be a single valued neutrosophic graph such that $G^* : (V, E)$ is a path on four vertices.*

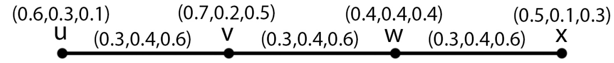

Figure 12. highly irregular SVNG, not strongly edge irregular SVNG and not strongly edge totally irregular SVNG

From Figure 12,

$d_G(u) = d_G(x) = (0.3, 0.4, 0.6), d_G(v) = d_G(w) = (0.6, 0.8, 1.2)$. *Here, G is a highly irregular*

single valued neutrosophic graph.
Note that $d_G(uv) = d_G(wx) = (0.3, 0.4, 0.6), d_G(vw) = (0.6, 0.8, 1.2)$. *So, G is not a strongly edge irregular single valued neutrosophic graph.*
Also, $td_G(uv) = td_G(wx) = (0.6, 0.8, 1.2), td_G(vw) = (0.9, 1.2, 1.8)$. *So, G is not a strongly edge totally irregular single valued neutrosophic graph.*

Theorem 4.20. *Let* $G : (A, B)$ *be a single valued neutrosophic graph on* $G^* : (V, E)$*, a star* $K_{1,n}$*. Then G is a totally edge regular single valued neutrosophic graph. Also, if the degrees of truth-membership, indeterminacy-membership and falsity-membership of no two edges are same, then G is a strongly edge irregular single valued neutrosophic graph.*

proof: Proof is similar to Theorem 3.14.

□

Theorem 4.21. *Let* $G : (A, B)$ *be a single valued neutrosophic graph such that* $G^* : (V, E)$ *is a path on* $2m(m > 1)$ *vertices.If the degrees of truth-membership, indeterminacy-membership and falsity-membership of the edges* $e_1, e_2, e_3, ..., e_{2m-1}$ *are* $p_1, p_2, p_3, ..., p_{2m-1}$ *such that* $p_1 < p_2 < p_3 < ... < p_{2m-1}$, $q_1, q_2, q_3, ..., q_{2m-1}$ *such that* $q_1 > q_2 > q_3 > ... > q_{2m-1}$ *and* $r_1, r_2, r_3, ..., r_{2m-1}$ *such that* $r_1 > r_2 > r_3 > ... > r_{2m-1}$*, respectively, then G is both strongly edge irregular single valued neutrosophic graph and strongly edge totally irregular single valued neutrosophic graph.*

proof: Let $G : (A, B)$ be a single valued neutrosophic graph on $G^* : (V, E)$, a path on $2m(m > 1)$ vertices. Let $e_1, e_2, e_3, ..., e_{2m-1}$ be the edges of path G^* in that order. Let degrees of truth-membership, indeterminacy-membership and falsity-membership of the edges $e_1, e_2, e_3, ..., e_{2m-1}$ are $p_1, p_2, p_3, ..., p_{2m-1}$ such that $p_1 < p_2 < p_3 < ... < p_{2m-1}$, $q_1, q_2, q_3, ..., q_{2m-1}$ such that $q_1 > q_2 > q_3 > ... > q_{2m-1}$ and $r_1, r_2, r_3, ..., r_{2m-1}$ such that $r_1 > r_2 > r_3 > ... > r_{2m-1}$, respectively, then
$d_G(v_1) = (p_1, q_1, r_1)$
$d_G(v_i) = (p_{i-1} + p_i, q_{i-1} + q_i, r_{i-1} + r_i)$ for $i = 2, 3, 4, 5, ..., 2m - 1$
$d_G(v_m) = (p_{2m-1}, q_{2m-1}, r_{2m-1})$
$d_G(e_1) = (p_2, q_2, r_2)$
$d_G(e_i) = (p_{i-1} + p_{i+1}, q_{i-1} + q_{i+1}, r_{i-1} + r_{i+1})$ for $i = 2, 3, 4, 5, ..., 2m - 2$
$d_G(e_{2m-1}) = (p_{2m-2}, q_{2m-2}, r_{2m-2})$
We observe that any pair of edges have distinct degrees. Hence G is a strongly edge irregular single valued neutrosophic graph. Also we have
$td_G(e_1) = (p_1 + p_2, q_1 + q_2, r_1 + r_2)$
$td_G(e_i) = (p_{i-1} + p_i + p_{i+1}, q_{i-1} + q_i + q_{i+1}, r_{i-1} + r_i + r_{i+1})$ for $i = 2, 3, 4, 5, ..., 2m - 2$

$td_G(e_{2m-1}) = (p_{2m-2} + p_{2m-1}, q_{2m-2} + q_{2m-1}, r_{2m-2} + r_{2m-1})$

Therefore any pair of edges have distinct total degrees, hence G is a strongly edge totally irregular single valued neutrosophic graph.

□

Theorem 4.22. *Let $G : (A, B)$ be a single valued neutrosophic graph such that $G^* : (V, E)$ is a cycle on $m(m \geq 4)$ vertices. If the degrees of truth-membership, indeterminacy-membership and falsity-membership of the edges $e_1, e_2, e_3, ..., e_m$ are $p_1, p_2, p_3, ..., p_m$ such that $p_1 < p_2 < p_3 < ... < p_m$, $q_1, q_2, q_3, ..., q_m$ such that $q_1 > q_2 > q_3 > ... > q_m$ and $r_1, r_2, r_3, ..., r_m$ such that $r_1 > r_2 > r_3 > ... > r_m$, respectively, then G is both strongly edge irregular single valued neutrosophic graph and strongly edge totally irregular single valued neutrosophic graph.*

proof: Let $G : (A, B)$ be a single valued neutrosophic graph on $G^* : (V, E)$, a cycle on $m(m \geq 4)$ vertices. Let $e_1, e_2, e_3, ..., e_m$ be the edges of cycle G^* in that order. Let degrees of truth-membership, indeterminacy-membership and falsity-membership of the edges $e_1, e_2, e_3, ..., e_m$ are $p_1, p_2, p_3, ..., p_m$ such that $p_1 < p_2 < p_3 < ... < p_m$, $q_1, q_2, q_3, ..., q_m$ such that $q_1 > q_2 > q_3 > ... > q_m$ and $r_1, r_2, r_3, ..., r_m$ such that $r_1 > r_2 > r_3 > ... > r_m$, respectively,then

$d_G(v_1) = (p_1 + p_m, q_1 + q_m, r_1 + r_m)$

$d_G(v_i) = (p_{i-1} + p_i, q_{i-1} + q_i, r_{i-1} + r_i)$ for $i = 2, 3, 4, 5, ..., m$

$d_G(e_1) = (p_2 + p_m, q_2 + q_m, r_2 + r_m)$

$d_G(e_i) = (p_{i-1} + p_{i+1}, q_{i-1} + q_{i+1}, r_{i-1} + r_{i+1})$ for $i = 2, 3, 4, 5, ..., m - 1$

$d_G(e_m) = (p_1 + p_{m-1}, q_1 + q_{m-1}, r_1 + r_{m-1})$

We observe that any pair of edges have distinct degrees. Hence G is a strongly edge irregular single valued neutrosophic graph.

$td_G(e_1) = (p_1 + p_2 + p_m, q_1 + q_2 + q_m, r_1 + r_2 + r_m)$

$td_G(e_i) = (p_{i-1} + p_i + p_{i+1}, q_{i-1} + q_i + q_{i+1}, r_{i-1} + r_i + r_{i+1})$ for $i = 2, 3, 4, 5, ..., m - 1$

$td_G(e_m) = (p_1 + p_{m-1} + p_m, q_1 + q_{m-1} + q_m, r_1 + r_{m-1} + r_m)$

We note that any pair of edges have distinct total degrees. Hence G is a strongly edge totally irregular single valued neutrosophic graph.

□

5. Conclusion

It is well known that graphs are among the most ubiquitous models of both natural and human-made structures. They can be used to model many types of relations and process dynamics in computer science, physical, biological and social systems. In general graphs theory has a wide range of applications in diverse fields. In this paper, we introduced some types of edge irregular single valued neutrosophic graphs and properties of them. A comparative study between neighbourly edge irregular single valued neutrosophic graphs and neighbourly edge totally irregular single valued neutrosophic graphs and also between strongly edge irregular single valued neutrosophic graphs and strongly edge totally irregular single valued neutrosophic graphs did. Also some properties of neighbourly edge irregular single valued neutrosophic graphs and strongly edge irregular single valued neutrosophic graphs studied, and they examined for neighbourly edge totally irregular single valued neutrosophic graphs and strongly edge totally irregular single valued neutrosophic graphs.

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Plithogenic Probability & Statistics are generalizations of MultiVariate Probability & Statistics

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Abstract: In this paper we exemplify the types of Plithogenic Probability and respectively Plithogenic Statistics. Several applications are given.

The Plithogenic Probability of an event to occur is composed from the chances that the event occurs with respect to all random variables (parameters) that determine it. Each such a variable is described by a Probability Distribution (Density) Function, which may be a classical, (T,I,F)-neutrosophic, I-neutrosophic, (T,F)-intuitionistic fuzzy, (T,N,F)-picture fuzzy, (T,N,F)-spherical fuzzy, or (other fuzzy extension) distribution function.

The Plithogenic Probability is a generalization of the classical MultiVariate Probability.

The analysis of the events described by the plithogenic probability is the Plithogenic Statistics.

Keywords: MultiVariate Probability; MultiVariate Statistics; Plithogenic Probability; Plithogenic Refined Probability; Plithogenic Statistics; Plithogenic Refined Statistics; Neutrosophic Data; Neutrosophic Sample, Neutrosophic Population, Neutrosophic Random Variables; Plithogenic Variate Data

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1. Introduction

The Plithogeny, as generalization of Dialectics and Neutrosophy, and then its applications to Plithogenic Set/Logic/Probability/Statistics (as generalization of fuzzy, intuitionistic fuzzy, neutrosophic set/logic/probability/statistics) [1, 2] were introduced by Smarandache in 2017.

Plithogeny is the genesis or origination, creation, formation, development, and evolution of new entities from dynamics and organic fusions of contradictory and/or neutrals and/or non-contradictory multiple old entities.

Plithogenic means what is pertaining to plithogeny. Etymologically, plitho-geny comes from: (Gr.) πλήθος (plithos) = crowd, many while -geny < (Gr.) -γενιά (-geniá) = generation, the production of something

2. Neutrosophic (or Indeterminate) Data

Neutrosophic (or Indeterminate) Data is a vague, unclear, incomplete, partially unknown, conflicting indeterminate data. The neutrosophic data can be metrical, or categorical, or both. Plithogenic Variate Data summarizes the associations (or inter-relationships) between Neutrosophic variables. While Neutrosophic Variable is a variable (or function, operator), that deals with neutrosophic data) either in its arguments or in its values, or in both. The problem to solve may have many dimensions, therefore multiple measurements and observations are needed since there are many sides to the problem, not only one. Neutrosophic variables may be: dependent; independent; or partially dependent, partially independent, and partially indeterminate.

(i.e. unknown if dependent or independent). The data's attributes (features, functions etc.) are investigated by survey-based techniques within the frame of Neutrosophic Conjoint Analysis (which includes the choice based conjoint and the adaptive choice-based conjoint.) Indeterminacy may occur at the level of attributes as well. We may thus deal with neutrosophic (indeterminate, unclear, partially known etc.) attributes [3-14].

3. Classical MultiVariate Analysis vs. Plithogenic Variate Analysis

The **Classical MultiVariate Analysis** (MVA) studies a system, which is characterized by many variables, or one may call it a system-of-systems. The variables, i.e. the subsystems, and the system as a whole are also classical (i.e. they do not deal with indeterminacy). Many classical measurements are needed, and the classical relations between variables to be determined. This system-of-systems is generally represented by a surrogate approximate model.

The **Plithogenic Variate Analysis** (PVA) is an extension of of the classical MultiVariate Analysis, where indeterminate data or procedures, that are called neutrosophic data and respectively neutrosophic procedures, are allowed. Therefore PVA deals with neutrosophic/ indeterminate variables, neutrosophic/indeterminate subsystems, and neutrosophic/indeterminate system-of-systems as a whole.

Therefore the Plithogenic Variate Analysis studies a neutrosophic/indeterminate system as a whole, characterized by many neutrosophic/indeterminate variables (i.e. neutrosophic/indeterminate sub-systems), and many neutrosophic/indeterminate relationships. Hence many neutrosophic measurements and observations are needed.

The Plithogenic Variate Analysis requires complex computations, hence it is more complicated than the Classical MultiVariate Analysis due to the neutrosophic (indeterminate) data it deals with; nonetheless the PVA better reflects our world, giving results nearer to real-life situation. With the dramatic increase of computers power this complexity is overcome.

The Plithogenic Variate Analysis elucidates each attribute of the data, using various methods, such as: regression/factor/cluster/path/discriminant/latent (trait or profile)/multilevel analysis / structural equation/recursive partition/redundancy/ constrained correspondence/ artificial neural networks, multidimensional scaling, and so on.

The **Plithogenic UniVariate Analysis** (PUVA) comprises the procedures for analysis of neutrosophic/indeterminate data that contains only one neutrosophic/indeterminate variable.

4. Plithogenic Probability

The Plithogenic Probability of an event to occur is composed from the chances that the event occurs with respect to all random variables (parameters) that determine it.

The Plithogenic Probability, based on Plithogenic Variate Analysis, is a multi-dimensional probability ("plitho" means "many", synonym with "multi"). We may say that it is a probability of sub-probabilities, where each sub-probability describes the behavior of one variable. We assume that the event we study is produced by one or more variables.

Each variable is represented by a **Probability Distribution (Density) Function** (PDF).

5. Subclasses of Plithogenic Probability are:

(i) If all PDFs are classical, then we have a classical **MultiVariate Probability**.

(ii) If all PDFs are in the neutrosophic style, i.e. of the form (T, I, F),
where T is the chance that the event occurs,
I is the indeterminate-chance of the event to occur or not,
and F is the chance that the event do not occur, with T, I, F $\in [0, 1]$, $0 \leq T+I+F \leq 3$,
then we have a **Plithogenic Neutrosophic Probability**.

(iii) If all PDFs are indeterminate functions (i.e. functions that have indeterminate data in their arguments, or in their values, or in both), then we have a **Plithogenic Indeterminate Probability**.

(iv) If all PDFs are Intuitionistic Fuzzy in the form of (T, F),
where T is the chance that the event occurs,
and F is the chance that the event do not occur, with T, F $\in$ [0, 1], $0 \leq T + F \leq 1$,
then we have a **Plithogenic Intuitionistic Fuzzy Probability**.

(v) If all PDFs are in the Picture Fuzzy Set style, i.e. of the form (T, N, F),
where T is the chance that the event occurs,
N is the neutral-chance of the event to occur or not,
and F is the chance that the event do not occur, with T, N, F$\in$ [0, 1], $0 \leq T+N+F \leq 1$,
then we have a **Plithogenic Picture Fuzzy Probability**.

(vi) If all PDFs are in the Spherical Fuzzy Set style, i.e. of the form (T, H, F),
where T is the chance that the event occurs,
H is the neutral-chance of the event to occur or not,
and F is the chance that the event do not occur, with T,H,F $\in$ [0, 1], $0 \leq T^2+H^2+F^2 \leq 1$,
then we have a **Plithogenic Spherical Fuzzy Probability**.

(vii) In general, if all PDFs are in any (fuzzy-extension set) style,
then we have a **Plithogenic (*fuzzy-extension*) Probability**.

(viii) If some PDFs are in one of the above styles, while others are in different styles,
then we have a **Plithogenic Hybrid Probability**.

6. Plithogenic Refined Probability

The most general form of probability is **Plithogenic Refined Probability**, when the components of T (Truth = Occurrence), I (Indeterminate-Occurrence), and F (Falsehood-NonOccurrence) are refined/split into sub-components: $T_1, T_2, \ldots, T_p$ (sub-truths = sub-occurrences) and $I_1, I_2, \ldots, I_r$ (sub-indeterminate-occurrences), and $F_1, F_2, \ldots, F_s$ (sub-falsehoods = sub-nonoccurrences), where p, r, s $\geq$ 0 are integers, and $p + r + s \geq 1$.

All the above sub-classes of plithogenic probability may be refined this way.

7. Convergence from MultiVariate to UniVariate Analysis

In order to be able to make a decision, we need to convert from Plithogenic (*MultiVariate*) Probability and Statistics to Plithogenic *UniVariate* Probability and Statistics. Actually we need to fusion (combine) all variables and obtain a single cumulative variable.

The **Classical Probability Space** is complete, i.e. all possible event that may occur are known.

For example, let's consider a soccer game between teams A and B. The classical probability space is CPS = {A wins, tie game, B wins}.

The **Neutrosophic Probability Space** is in general incomplete, i.e. not all possible events are known, and there also are events that are only partially known. In our world, most real probability spaces are neutrosophic.

Example. Considering the same soccer game, the neutrosophic probability space NPS = {A wins, tie game, B wins, interrupted game, etc.}, "interrupted" means that due to some unexpected weather conditions, or to a surprising terrorist attack on the stadium, etc. the game is interrupted and rescheduled (this has happened in our world many times).

Let's assume an event E in a given (classical or neutrosophic) probability space is determined by $n \geq 2$ variables $v_1, v_2, \ldots, v_n$, and we denote it as $E(v_1, v_2, \ldots, v_n)$. The multi-variate probability of the event E to occur, denoted by MVP(E), depends on many probabilities, i.e. on the probability that the event E occurs with respect to variable v_1, denoted by $P_1(E(v_1))$, on the probability that the event E occurs with respect to variable v_2, denoted by $P_2(E(v_2))$, and so on.
Therefore, $MVP(E(v_1, v_2, \ldots, v_n)) = (P_1(E(v_1)), P_2(E(v_2)), \ldots, P_n(E(v_n)))$.
The variables $v_1, v_2, \ldots, v_n$, and the probabilities $P_1, P_2, \ldots, P_n$, may be classical, or having some degree of indeterminacy.

In order *to convert from multi-probability to uni-probability*, we apply various logical operators (conjunctions, disjunctions, negations, implications, etc. and their combinations, depending on the application to do and on the expert) on the multi-probability.

Such applications are presented towards the end of the paper.

7. Plithogenic Statistics

Plithogenic Statistics (PS) encompasses the analysis and observations of the events studied by the Plithogenic Probability.

Plithogenic Statistics is a generalization of classical MultiVariate Statistics, and it is a simultaneous analysis of many outcome neutrosophic/indeterminate variables, and it as well is a multi-indeterminate statistics.

8. Subclasses of Plithogenic Statistics are:

- **MultiVariate Statistics**
- **Plithogenic Neutrosophic Statistics**
- **Plithogenic Indeterminate Statistics**
- **Plithogenic Intuitionistic Fuzzy Statistics**
- **Plithogenic Picture Fuzzy Statistics**
- **Plithogenic Spherical Fuzzy Statistics**
- and in general: **Plithogenic (*fuzzy-extension*) Statistics**
- and **Plithogenic Hybrid Statistics**.

9. Plithogenic Refined Statistics are, similarly, the most general form of statistics that studies the analysis and observations of the events described by the Plithogenic Refined Probability.

10. Applications of Plithogenic Probability

We retour our 2017 example [1] and pass it through all sub-classes of Plithogenic Probability.

In the Spring 2021 semester, at The University of New Mexico, United States, in a program of Electrical Engineering, Jenifer needs to pass four courses in order to graduate at the end of the semester: two courses of Mathematics (Second Order Differential Equations, and Stochastic Analysis), and two courses of Mechanics (Fluid Mechanics and Solid Mechanics). What is the Plithogenic Probability that Jenifer will graduate?

Her chances of graduating are estimated by the university's advisors.

There are four variables (courses), v_1, v_2, v_3, v_4 respectively, that generate four probability distributions. We consider the discrete probability distribution functions.
[For the continuous ones, it will be similar.]

10.1. Classical MultiVariate Probability (CMVP)

The advisers have estimated that CMVP(Jenifer) = (0.5, 0.6, 0.8, 0.4),

which means that Jenifer has 50% chance to pass the Second Order Differential Equations class, 60% chance of passing the Stochastic Analysis class (both as part of Mathematics), and 80% chance of passing the Fluid Mechanics class, and 40% chance of passing the Solid Mechanics class (both as part of Mechanics).

Since she has to pass all four classes, Jenifer's chance of graduating is min{0.5, 0.6, 0.8, 0.4} = 0.4 or 40% chance.

10.2 Plithogenic Neutrosophic Probability

PNP(Jenifer) = ((0.5, 0.9, 0.2), (0.6, 0.7, 0.4); (0.8, 0.2, 0.1), (0.4, 0.3, 0.5)),

which similarly means that :

Jenifer's chance to pass the Second Order Differential Equations class is 50%, and the indeterminate chance is 90%, and the chance to fail it is 20%. Similarly for the other three classes.

In conclusion: (min{0.5, 0.6, 0.8, 0.4}, max{0.9, 0.7, 0.2, 0.3}, max{0.2, 0.4, 0.1, 0.5}) = (0.4, 0.9, 0.5).

10.3. Plithogenic Indeterminate Probability

PIP(Jenifer) = ([0.4, 0.5], 0.2 or 0.4, 0.1 or unknown)
which unclear information, i.e. chance of graduating is between [40%, 50%], 20% or 40% is indeterminate-chance of graduating, and chance of not graduating is 10% or unknown (i.e. the advisors were not able to estimate it well).

10.4. Plithogenic Intuitionistic Fuzzy Probability (PIFP) provides more information

PIFP(Jenifer) = ((0.5, 0.2), (0.6, 0.4); (0.8, 0.1), (0.4, 0.5)),

which means that Jenifer has 50% chance to pass the Second Order Differential Equations class, and 20% chance to fail it. And similarly for the other three classes.

In conclusion: ((min{0.5, 0.6, 0.8, 0.4), max{0.2, 0.4, 0.1, 0.5}) = (0.4, 0.5).

10.5. Plithogenic Picture Fuzzy Probability (PPFP) brings even more information

PPFP(Jenifer) = ((0.5,0.1, 0.2), (0.6, 0.0, 0.4); (0.8, 0.1, 0.1), (0.4, 0.0, 0.5)),

which means that Jenifer's chance to pass the Second Order Differential Equations class is 50%, and 10% neutral chance, and 20% chance to fail it. Similarly for the other three classes.

In conclusion: PSFP(Jenifer) = ((min{0.5, 0.6, 0.8, 0.4), max{0.1, 0.0, 0.1, 0.0}, max{0.2, 0.4, 0.1, 0.5}) = (0.4, 0.1, 0.5).

10.6. Plithogenic Spherical Fuzzy Probability (PSFP) enlarges the value spectrum of the previous one

PSFP(Jenifer) = ((0.5, 0.3, 0.2), (0.6, 0.5, 0.4); (0.8, 0.3, 0.1), (0.4, 0.6, 0.5)),

with the same meaning as the previous one.

In conclusion: PSFP(Jenifer) = ((min{0.5, 0.6, 0.8, 0.4), max{0.3, 0.5, 0.3, 0.6}, max{0.2, 0.4, 0.1, 0.5}) = (0.4, 0.6, 0.5).

10.7. Plithogenic Hybrid Probability (PHP)

PHP(Jenifer) = (0.5; (0.7, 0.1, 0.4); (0.1, 0.2); 0.4 or 0.3),

which means that Jenifer has 50% chance to pass the Second Order Differential Equations class; 70% chance of passing and 10% indeterminate-chance and 40% chance of failing the Stochastic Analysis class (both as part of Mathematics); and 10% chance of passing and 20% of failing the Fluid Mechanics class; and 40% or 30% chance of passing the Solid Mechanics class (both as part of Mechanics).

We have mixed herein: the fuzzy, neutrosophic, intuitionistic fuzzy, and indeterminate above cases.

Or PHP(Jenifer) = ((0.5, 0.0, 0.0); (0.7, 0.1, 0.4); (0.1, 0.7, 0.2); (0.4 or 0.3, 0.0, 0.0)), since for the intuitionistic fuzzy the hesitancy is : $1 - 0.1 - 0.2 = 0.7$.

In conclusion: PHP(Jenifer) = ((min{0.5, 0.7, 0.1, 0.4 or 0.3), max{0.0, 0.1, 0.7, 0.0}, max{0.0, 0.4, 0.2, 0.0}) = (0.1, 0.7, 0.4).

10.8. Plithogenic Refined Probability (PRP)

Let's assume that for each class, Jenifer has to pass an oral test and a written test. Therefore T,I,F are refined/split into:

T_1(oral test), T_2(written test);

I_1(oral test), I_2(written test);

F1(oral test), F2(written test).
Then, we may have, as an example:
PRP(Jenifer) = (((0.5, 0.6), (0.4, 0.7), (0.1, 0.2)),
((0.6, 0.8), (0.0, 0.7), (0.3, 0.4)),
((0.8, 0.8), (0.1, 0.2), (0.1, 0.0)),
((0.3, 0.7), (0.2, 0.3), (0.5, 0.4))),
which means :
with respect to the first class,
Jenifer's chance to pass the oral test is 50% and the written test is 60%;
indeterminate-chance to pass the oral test is 40% and the written test is 70%;
and chance not to pass the oral test is 10% and the written test 20%.
Similarly for the other classes.

11. Converging/Transforming from MultiVariate to UniVariate Analysis

11.1. From Classical MultiVariate Probability (CMVP) to UniVariate Probability

(i) Since Jenifer has to pass *all* four classes, we use the conjunction operator: $v_1 \wedge v_2 \wedge v_3 \wedge v_4$. In this case it is fuzzy conjunction (t-norm).

Therefore, Jenifer's chance of graduating is CMVP(Jenifer) = min{0.5, 0.6, 0.8, 0.4} = 0.4, or 40% chance.

(ii) Let's change the example and assume that for Jenifer to graduate she needs to pass at least one class among the four. Now we use the disjunction operator: $v_1 \vee v_2 \vee v_3 \vee v_4$ or fuzzy disjunction (t-conorm). Therefore, Jenifer's chance of graduating is CMVP(Jenifer) = max{0.5, 0.6, 0.8, 0.4} = 0.8, or 80% chance.

(iii) Let's change again the example and assume that for Jenifer to graduate she needs to pass at least one class of Mathematics and at least one class of Mechanics. Then, we use a mixture of conjunctions and disjunctions: CMVP(Jenifer) = $(v_1 \vee v_2) \wedge (v_3 \vee v_4)$ = min{max{v_1, v_2}, max{v_3, v_4}} = min{max{0.5, 0.6}, max{0.8, 0.4}} = min{0.6, 0.8} = 0.6, or 60% chance.

11.2 From Plithogenic Neutrosophic Probability (PNP) to UniVariate Neutrosophic Probability

In conclusion: PNP(Jenifer) = ((min{0.5, 0.6, 0.8, 0.4), max{0.9, 0.7, 0.2, 0.3}, max{0.2, 0.4, 0.1, 0.5}) = (0.4, 0.9, 0.5).

11.3. This Plithogenic Indeterminate Probability happens to be a UniVariate Indeterminate Probability

Therefore, no converting (or transformation) needed.
PIP(Jenifer) = ([0.4, 0.5], 0.2 or 0.4, 0.1 or unknown)
with unclear information, i.e. chance of graduating is between [40%, 50%], 20% or 40% indeterminate-chance of graduating, and chance of not graduating is (10% or unknown – i.e. the advisors were not able to estimate it well).

11.4. From Plithogenic Intuitionistic Fuzzy Probability (PIFP) to UniVariate Intuitionistic Fuzzy Probability

In conclusion: PIPF(Jenifer) = ((min{0.5, 0.6, 0.8, 0.4), max{0.2, 0.4, 0.1, 0.5}) = (0.4, 0.5), which means that chance of graduating is 40%, and chance of not graduating 50%.

11.5. From Plithogenic Picture Fuzzy Probability (PPFP) to UniVariate Picture Fuzzy Probability

In conclusion: PSFP(Jenifer) = ((min{0.5, 0.6, 0.8, 0.4), max{0.1, 0.0, 0.1, 0.0}, max{0.2, 0.4, 0.1, 0.5}) = (0.4, 0.1, 0.5), or 40% chance to graduate, 10% neutral chance, and 50% chance not to graduate.

11.6. From Plithogenic Spherical Fuzzy Probability (PSFP) to UniVariate Spherical Fuzzy Probability

In conclusion: PSFP(Jenifer) = ((min{0.5, 0.6, 0.8, 0.4), max{0.3, 0.5, 0.3, 0.6}, max{0.2, 0.4, 0.1, 0.5}) = (0.4, 0.6, 0.5), or 40% chance to graduate, 60% hesitant chance, and 50% chance not to graduate.

11.7. From Plithogenic Hybrid Probability (PHP) to UniVariate Hybrid Probability

In conclusion: PHP(Jenifer) = ((min{0.5, 0.7, 0.1, 0.4 or 0.3), max{0.0, 0.1, 0.7, 0.0}, max{0.0, 0.4, 0.2, 0.0}) = (0.1, 0.7, 0.4).

10.8. From Plithogenic Refined Probability (PRP) to UniVariate Refined Probability ???

Taking min of T1's, min T2's, and max of I1's, max of I2's, and max of F1's, F2's, one gets:
PRP(Jenifer) = ((0.3, 0.6), (0.4, 0.7), (0.5, 0.4)).
Whence, Jenifer's chance, with respect to all classes,
to pass the oral test is 30% and the written test is 60%;
indeterminate-chance to pass the oral test is 40% and the written test is 70%;
and chance not to pass the oral test is 50% and the written test 40%.
But, with respect to graduation, we use again the fuzzy conjunction:
PRP(Jenifer) = (min{0.3, 0.6}, max{0.4, 0.7}, max{0.5, 0.4}) = (0.3, 0.7, 0.5), or
Jenifer's chance to graduate is 30%, indeterminate-chance of graduating 70%, and 50% chance not to graduate.

11. Corresponding Applications of Plithogenic Statistics

A prospective is made on the university student population, that was enrolled this semester, in order to determine the chance of the average students to graduate.

Let's take a random sample of the university's student population in order to investigate what's the chance of graduating for an enrolled average student.

By *inference statistics*, we estimate the population's average student to be similar to the sample's average student.

We may have a **classical random sample**, i.e. the sample size is known and all sample individuals belong 100% of the population - i.e. the individuals are full-time students; or a **neutrosophic random sample** {i.e. the sample size may be unknown or only approximately known), and some or all individuals may only partially belonging to the population (for example part-time students), or may have taken some extra classes above the norm.

Even the university's student population is a **neutrosophic population**, since the number of students changes almost continuously (some students drop, others enroll earlier or later), and not all students are 100% enrolled: there are full-time, part-time, and even over-time (i.e. students enrolled in more than the required full time number or credit hour classes).

In a **classical population**, the population size is known, and all population individuals belong 100% to the population.

Let T = truth, with T belongs to [0,1], be the chance to graduate, I = indeterminate, with I belongs to [0,1], be the indeterminate-chance to graduate, and F = falsehood, with F belongs to [0,1], be the chance not to graduate, where $0 \le T + I + F \le 3$.

Let's assume, the classical or neutrosophic random sample has the size $n \ge 2$, and a student S_j, $1 \le j \le n$, has the plithogenic neutrosophic probabilty of (graduating, indeterminate-graduating, not graduating), respectively (T_j, I_j, F_j), with all T_j, I_j, F_j belong to [0, 1], $0 \le T_j + I_j + F_j \le 3$.

Make the average of all sample students, assuming the sample size is $n \ge 2$,

$$\frac{1}{n}\sum_{j=1}^{n}(T_j, I_j, F_j) = (\frac{1}{n}\sum_{j=1}^{n}T_j, \frac{1}{n}\sum_{j=1}^{n}I_j, \frac{1}{n}\sum_{j=1}^{n}F_j) .$$

For the Plithogenic Refined Neutrosophic Probabilities, the average is a straight-forward extension. Let the student S_j, $1 \le j \le n$, have the Plithogenic Refined Neutrosophic Probability:

PRNP(S_j) = $(T_1(j), T_2(j), ..., T_p(j); I_1(j), I_2(j), ..., I_r(j); F_1(j), F_2(j), ..., F_s(j))$

where p, r, s ≥ 0 are integers, and p + r + s ≥ 1.

The refined neutrosophic sub-components with index 0, such as $T_0(j), I_0(j), F_0(j)$, if any, are discarded.

All refined neutrosophic sub-components

$T_k(j), 1 \le k \le p, I_l(j), 1 \le l \le r, F_m(j), 1 \le m \le s$, are single-valued in [0, 1].

Then, the average of PRNPs of the sample students is:

$$\frac{1}{n}\sum_{j=1}^{n} \text{PRNP}\left(\text{S}_j\right) = \frac{1}{n}\sum_{j=1}^{n}(T_1(j), T_2(j), ..., T_p(j); I_1(j), I_2(j), ..., I_r(j); F_1(j), F_2(j), ..., F_s(j))$$

=

$$(\frac{1}{n}\sum_{j=1}^{n}T_1(j), \frac{1}{n}\sum_{j=1}^{n}T_2(j), ..., \frac{1}{n}\sum_{j=1}^{n}T_p(j); \frac{1}{n}\sum_{j=1}^{n}I_1(j), \frac{1}{n}\sum_{j=1}^{n}I_2(j), ..., \frac{1}{n}\sum_{j=1}^{n}I_r(j); \frac{1}{n}\sum_{j=1}^{n}F_1(j), \frac{1}{n}\sum_{j=1}^{n}F_2(j), ..., \frac{1}{n}\sum_{j=1}^{n}F_s(j))$$

And we get the sample average students' plithogenic refined probability to (graduate, indeterminate graduate, not graduate).

For the cases when one or two among T, I, F are missing, we simply discard them.

An average student is not among the best, not among the worst.

Let's consider Jenifer is an average student, whose plithogenic probabilities have been obtained after sampling and computing the average of plithogenic probabilities of all its students - since we have already her data.

Considering the inference statistics, we simply substitute Jenifer by average student.

We consider the simplest case when T,I,F are single-valued neutrosophic (SVN) components in [0,1]. But the cases when T,I,F are hesitant-valued (finite discrete subsets of [0,1]) neutrosophic HVN components, or interval-valued included in [0,1] neutrosophic (IVN) component, or in general subset-valued included in [0,1] neutrosophic (SVN) components.

10.1. Classical MultiVariate Statistics (CMVS)

Since the average student has to pass all four classes, his chance of graduating is min{0.5, 0.6, 0.8, 0.4} = 0.4 or 40% chance. In conclusion: CMVS(average student) = 0.4.

10.2. Plithogenic Neutrosophic Statistics (PNS)

In conclusion: PNP(average student) = ((min{0.5, 0.6, 0.8, 0.4), max{0.9, 0.7, 0.2, 0.3}, max{0.2, 0.4, 0.1, 0.5}) = ((0.4, 0.9, 0.5)).

An average student has 40% chance to graduate, 90% indeterminate chance of graduating, and 50% chance not to graduate.

10.3. Plithogenic Indeterminate Statistics

An average student has (40% or 50%) chance of graduating, 90% indeterminate chance of graduating, and (50% or unknown) chance of not graduating.

10.4. Plithogenic Intuitionistic Fuzzy Statistics

An average student has 40% chance to graduate and 50% chance not to graduate.

10.5. Plithogenic Picture Fuzzy Statistics

An average student has 40% chance to graduate, 10% indeterminate chance to graduate, and 50% chance not to graduate.

10.6. Plithogenic Spherical Fuzzy Statistics

An average student has 40% chance to graduate, 60% indeterminate chance of graduating, and 50% chance not to graduate.

10.7. Plithogenic Hybrid Statistics

An average student has 40% chance to graduate, 60% indeterminate chance of graduating, and 50% chance not to graduate.

10.8. Plithogenic Refined Statistics

An average student's chance to graduate is 30%, indeterminate-chance of graduating 70%, and 50% chance not to graduate.

11. Conclusion

We have presented in this paper many types of Plithogenic Probability and corresponding Plithogenic Statistics, together with some application.

The Plithogenic Probability of an event to occur is composed from the chances that the event occurs with respect to all random variables (parameters) that determine it. Each such a variable is described by a Probability Distribution (Density) Function, which may be a classical, (T,I,F)-neutrosophic, I-neutrosophic, (T,F)-intuitionistic fuzzy, (T,N,F)-picture fuzzy, (T,N,F)-spherical fuzzy, or (other fuzzy extension) distribution function.

The Plithogenic Probability is a generalization of the classical MultiVariate Probability.

The analysis of the events described by the plithogenic probability constitutes the Plithogenic Statistics.

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The neutrosophic integrals and integration methods

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Abstract: the purpose of this article is to study the neutrosophic integrals and integration methods, Where the method of integration by substitution is defined, and theorems have been proven useful for facilitating the calculation of integration for some neutrosophic functions from that contain indeterminacy. Also, neutrosophic trigonometric identities are defined, in addition to studying all cases of the integrating products of neutrosophic trigonometric functions. Where detailed examples were given to clarify each case.

Keywords: neutrosophic indefinite integral; substitution method; indeterminacy; neutrosophic trigonometric functions.

1. Introduction

As an alternative to the existing logics, Smarandache proposed the Neutrosophic Logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of neutrosophy is a new branch of philosophy introduced by Smarandache [3-13]. He presented the definition of the standard form of neutrosophic real number and conditions for the division of two neutrosophic real numbers to exist, he defined the standard form of neutrosophic complex number, and found root index $n \geq 2$ of a neutrosophic real and complex number [2-4], studying the concept of the Neutrosophic probability [3-5], the Neutrosophic statistics [4][6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1-8]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [9]. Edalatpanah proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side represented with triangular neutrosophic numbers [10]. Chakraborty used pentagonal neutrosophic number in networking problem, and Shortest Path Problem [11-12]. Y.Alhasan studied the concepts of neutrosophic complex numbers and the general exponential form of a neutrosophic complex [7][14]. On the other hand, M.Abdel-Basset presented study in the science of neutrosophic about an approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number [15]. Also, neutrosophic sets played an important role in applied science such as health care, industry, and optimization [16-17-18-19]. Recently, there are increasing efforts to study the neutrosophic generalized structures and spaces such as refined neutrosophic modules, spaces, equations, and rings [21-22-23].

Integration is important in human life, and one of its most important applications is the calculation of area, size and arc length. In our reality we find things that cannot be precisely defined, and that contain an indeterminacy part. This is the reason for studying neutrosophic integration and methods of its integration in this paper.

Paper consists of 5 sections. In 1th section, provides an introduction, in which neutrosophic science review has given. In 2th section, some definitions and examples of neutrosophic real number neutrosophic indefinite integral and are discussed. The 3th section frames the neutrosophic integration by substitution method, and their employment in finding integrals that include roots, logarithms, exponents, or a function with its derivative (between them is the process of multiplication or division), and the study of the related theories. The 4th section introduces the integrating products of neutrosophic trigonometric function in three states and neutrosophic trigonometric identities, as they have been used in finding some types of neutrosophic trigonometric integrals. In 5th section, a conclusion to the paper is given.

2. Preliminaries

2.1. Neutrosophic Real Number [4]

Suppose that w is a neutrosophic number, then it takes the following standard form: $w = a + bI$ where a, b are real coefficients, and I represent indeterminacy, such $0.I = 0$ and $I^n = I$, for all positive integers n.

2.2. Division of neutrosophic real numbers [4]

Suppose that w_1, w_2 are two neutrosophic numbers, where

$$w_1 = a_1 + b_1 I \ , \qquad w_2 = a_2 + b_2 I$$

To find $(a_1 + b_1 I) \div (a_2 + b_2 I)$, we can write:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} \equiv x + yI$$

where x and y are real unknowns.

$$a_1 + b_1 I \equiv (a_2 + b_2 I)(x + yI)$$

$$a_1 + b_1 I \equiv a_2 x + (b_2 x + a_2 y + b_2 y)I$$

by identifying the coefficients, we get

$$a_1 = a_2 x$$

$$b_1 = b_2 x + (a_2 + b_{2)} y$$

We obtain unique one solution only, provided that:

$$\begin{vmatrix} a_2 & 0 \\ b_2 & a_2 + b_2 \end{vmatrix} \neq 0 \ \Rightarrow \ a_2(a_2 + b_2) \neq 0$$

Hence: $a_2 \neq 0$ and $a_2 \neq -b_2$ are the conditions for the division of two neutrosophic real numbers to exist.

Then:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} . I$$

2.3. Neutrosophic Indefinite Integral [14]

We just extend the classical definition of anti-derivative. The neutrosophic antiderivative of neutrosophic function $f(x)$ is the neutrosophic function $F(x)$ such that $F'(x) = f(x)$.

Example2.4.1:

$Let\ \ f: R \to R \cup \{I\},\ f(x) = -3x^2 + (4x-5)I.$

Then:

$$F(x) = \int[-3x^2 + (4x-5)I]dx$$
$$= -x^3 + (2x^2 - 5x)I + C$$

where C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while I = indeterminacy).

Example2.4.2: (Refined Indeterminacy).

$Let\ g: \mathbb{R} \to \mathbb{R} \cup \{I_1\} \cup \{I_2\} \cup \{I_3\}$, where I_1, I_2, and I_3 are types of sub indeterminacies,

$$g(x) = 7x - 2I_1 + x^2 I_2 + 4x^3 I_3$$

Then:

$$F(x) = \int[7x - 2I_1 + x^2 I_2 + 4x^3 I_3]dx$$
$$= \frac{7x^2}{2} - 2xI_1 + \frac{x^3}{3}I_{\ 2} + x^4 I_3 + a + bI$$

where a and b are real constants.

3. Neutrosophic integration by substitution method

Definition3.1

$Let\ \ f: D_f \subseteq R \to R_f \cup \{I\}$, to evaluate $\int f(x)dx$

Put: $x = g(u) \quad \Rightarrow \quad dx = \acute{g}(u)du$

By substitution, we get:

$$\int f(x)dx = \int f(u)\acute{g}(u)du$$

then we can directly integral it.

Theorme3.1:

If $\int f(x, I)dx = \varphi(x, I)\ \ then,$

$$\int f\big((a+bI)x + c + dI\big)\, dx = \left(\frac{1}{a} - \frac{b}{a(a+b)}I\right)\varphi\big((a+bI)x + c + dI\big) + C$$

where C is an indeterminate real constant, $a \neq 0\ , a \neq -b\ and\ b, c, d$ are real numbers, while I = indeterminacy.

Proof:

Put: $(ba + I)x + c + dI = u \qquad \Rightarrow (ax + bI)dx = du$

$$\Rightarrow dx = \frac{1}{a+bI}\ du$$

$$\Rightarrow dx = \left(\frac{1}{a} - \frac{b}{a(a+b)}I\right)du$$

$$\int f\big((ba + I)x + c + dI\big)\, dx = \int f(u)\left(\frac{1}{a} - \frac{b}{a(a+b)}I\right)du$$

$$= \left(\frac{1}{a} - \frac{b}{a(a+b)}I\right)\varphi(u) + C$$

back to the variable x, we get:

$$\int f\big((a+bI)x+c+dI\big)\,dx = \left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)\varphi\big((a+bI)x+c+dI\big)+C$$

3.1. We can prove each of the following, using the previous theorem:

1) $\int ((a+bI)x+c+dI)^n dx = \left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)\frac{((a+bI)x+c+dI)^{n+1}}{n+1}+C$

2) $\int \frac{1}{(a+bI)x+c+dI}dx = \left(\frac{1}{a}-\frac{b}{a(a+b)}I\right) ln|(a+bI)x+c+dI| + \boldsymbol{C}$

3) $\int e^{(a+bI)x+c+dI}\,dx = \left(\frac{1}{a}-\frac{b}{a(a+b)}I\right) e^{(a+bI)x+c+dI} + C$

4) $\int \frac{1}{\sqrt{(a+bI)x+c+dI}}dx = 2\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)\sqrt{(a+bI)x+c+dI} + \boldsymbol{C}$

5) $\int cos((a+bI)x+c+dI)\,dx = \left(\frac{1}{a}-\frac{b}{a(a+b)}I\right) cos((a+bI)x+c+dI) + \boldsymbol{C}$

6) $\int sin((a+bI)x+c+dI)\,dx = -\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right) sin((a+bI)x+c+dI) + \boldsymbol{C}$

7) $\int sec^2((a+bI)x+c+dI)\,dx = \left(\frac{1}{a}-\frac{b}{a(a+b)}I\right) tan((a+bI)x+c+dI) + \boldsymbol{C}$

8) $\int csc^2((a+bI)x+c+dI)\,dx = -\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right) cot((a+bI)x+c+dI) + \boldsymbol{C}$

9) $\int sec((a+bI)x+c+dI)\,tan((a+bI)x+c+dI)\,dx$
$= \left(\frac{1}{a}-\frac{b}{a(a+b)}I\right) sec((a+bI)x+c+dI) + \boldsymbol{C}$

10) $\int csc((a+bI)x+c+dI)\,cot((a+bI)x+c+dI)\,dx$
$= -\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right) csc((a+bI)x+c+dI) + \boldsymbol{C}$

where C is an indeterminate real constant, $a \neq 0, a \neq -b$ *and* b, c, d are real numbers, while I = indeterminacy.

Example3.1.1

1) $\int ((3-5I)x+4)^7 dx = \left(\frac{1}{3}-\frac{5}{6}I\right)\frac{((3-5I)x+4)^8}{8} + \boldsymbol{C}$

2) $\int \frac{1}{(6+5I)x-7I}dx = \left(\frac{1}{6}-\frac{5}{66}I\right) ln|(6+5I)x-7I| + \boldsymbol{C}$

3) $\int \boldsymbol{e}^{(2+I)x-3}\,dx = \left(\frac{1}{2}-\frac{1}{6}I\right)\boldsymbol{e}^{(2+I)x-3} + \boldsymbol{C}$

4) $\int cos\big((1+4I)x+I\big)\,dx = \left(1-\frac{4}{5}I\right) cos\big((1+4I)x+I\big) + \boldsymbol{C}$

5) $\int sec^2\big((6+5I)x-7I\big)\,dx = \left(\frac{1}{6}-\frac{5}{66}I\right) tan\big((6+5I)x-7I\big)+\boldsymbol{C}$

6) $\int csc\big((2+I)x-3\big)\,cot\big((2+I)x-3\big)\,dx = -\left(\frac{1}{2}-\frac{1}{6}I\right) csc\big((2+I)x-3\big)+\boldsymbol{C}$

7) $\int \frac{1}{\sqrt{(2+5I)x+3I}}\,dx = 2\left(\frac{1}{2}-\frac{5}{14}I\right)\sqrt{(2+5I)x+3I}+\boldsymbol{C} = (1-5I)\sqrt{(2+5I)x+3I}+\boldsymbol{C}$

Theorme3.2:
Let $f: D_f \subseteq R \to R_f \cup \{I\}$ then:

$$\int \frac{\acute{f}(x,I)}{f(x,I)}\,dx = ln|f(x,I)| + C$$

where C is an indeterminate real constant (i.e. constant of the form $a+bI$, where a,b are real numbers, while I = indeterminacy).

Proof:

Put: $f(x,I)=u \quad \Rightarrow \acute{f}(x,I)dx = du$

$$\Rightarrow dx = \frac{1}{\acute{f}(x,I)}\,du$$

$$\Rightarrow dx = \frac{1}{\acute{u}}\,du$$

$$\int \frac{\acute{f}(x,I)}{f(x,I)}\,dx = \int \frac{\acute{u}}{u}\frac{1}{\acute{u}}\,du = \int \frac{1}{u}\,du = ln|u| + C$$

back to the variable $f(x,I)$, we get:

$$\int \frac{\acute{f}(x,I)}{f(x,I)}\,dx = ln|f(x,I)| + C$$

Example3.1.2

1) $\int \frac{(1+2I)x^3}{(1+2I)x^4+5I}\,dx = \frac{1}{4} ln|(1+2I)x^4+5I| + C$

2) $\int \frac{(2+I)e^{(2+I)x-3}-2I}{e^{(2+I)x-3}-2xI}\,dx = ln\left|e^{(2+I)x-3}-2xI\right| + C$

3) $\int \tan(1+7I)x\,dx = \int \frac{\sin(1+7I)x}{\cos(1+7I)x}\,dx = \left(-1+\frac{7}{8}I\right) ln|\cos(1+7I)x| + C$

4) $\int \frac{1}{1+\tan(1+2I)x}\,dx = \int \frac{1}{1+\frac{\sin(1+2I)}{\cos(1+2I)}}\,dx = \frac{1}{2}\int \frac{2\cos(1+2I)x}{\cos(1+2I)x+\sin(1+2I)x}\,dx$

$$= \frac{1}{2}\int \frac{\cos(1+2I)x+\sin(1+2I)x+\cos(1+2I)x-\sin(1+2I)x}{\cos(1+2I)+\sin(1+2I)x}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos(1+2I)x-\sin(1+2I)\,x}{\cos(1+2I)x+\sin(1+2I)x}\,dx$$

$$= \frac{1}{2}x + \left(\frac{1}{2} - \frac{1}{3}I\right) ln|\cos(1+2I)x + \sin(1+2I)x| + C$$

Theorme3.3:
Let $f: D_f \subseteq R \to R_f \cup \{I\}$, then:

$$\int \frac{\acute{f}(x,I)}{\sqrt{f(x,I)}} dx = 2\sqrt{f(x,I)} + C$$

where C is an indeterminate real constant (i.e. constant of the form $a+bI$, where a,b are real numbers, while I = indeterminacy).

Proof:

Put: $f(x,I) = u \quad \Rightarrow \acute{f}(x,I)dx = du$

$$\Rightarrow dx = \frac{1}{\acute{f}(x,I)} du$$

$$\Rightarrow dx = \frac{1}{\acute{u}} du$$

$$\int \frac{\acute{f}(x,I)}{\sqrt{f(x,I)}} dx = \int \frac{\acute{u}}{\sqrt{u}} \frac{1}{\acute{u}} du = \int \frac{1}{\sqrt{u}} du = 2\sqrt{u} + C$$

back to the variable $f(x,I)$, we get:

$$\int \frac{\acute{f}(x,I)}{\sqrt{f(x,I)}} dx = 2\sqrt{f(x,I)} + C$$

Example3.1.3

1) $\int \frac{-(2+5I)x + 4I}{\sqrt{(2+5I)x^2 - 8xI}} dx = \frac{-1}{2}\sqrt{(2+5I)x^2 - 8xI} + C$

2) $\int \frac{3x^2}{\sqrt{(2+5I)x^3 + 3I}} dx = \left(1 - \frac{5}{7}I\right)\sqrt{(2+5I)x^3 + 3I} + C$

Theorme3.4:
$f: D_f \subseteq R \to R_f \cup \{I\}$, then:

$$\int [f(x,I)]^n \acute{f}(x)\, dx = \frac{[f(x,I)]^{n+1}}{n+1} + C$$

Where n is any rational number. C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a,b are real numbers, while I = indeterminacy).

Proof:

Put: $f(x,I) = u \quad \Rightarrow \acute{f}(x,I)dx = du$

$$\Rightarrow dx = \frac{1}{\acute{f}(x,I)} du$$

$$\Rightarrow dx = \frac{1}{\acute{u}} du$$

$$\int [f(x,I)]^n \acute{f}(x,I) dx = \int u^n \acute{u} \frac{1}{\acute{u}} du = \int u^n du = \frac{u^{n+1}}{n+1} + C$$

back to the variable $f(x,I)$, we get:

$$\int [f(x,I)]^n f\acute{(}x)dx = \frac{[f(x,I)]^{n+1}}{n+1} + C$$

Example3.1.4

1) $\int x^4[(2+5I)x^5]^{12}dx = \left(\frac{1}{10} - \frac{1}{20}I\right)\frac{[(2+5I)x^5]^{13}}{13} + C$

2) $\int \frac{1}{\sqrt{(4-2I)x}}\left(\sqrt{(4-2I)x}\right)^{10} dx = \left(\frac{1}{2} + \frac{1}{2}I\right)\frac{\left(\sqrt{(4-2I)x}\right)^{11}}{11} + C$

4. Integrating products of neutrosophic trigonometric function:

I. $\int \sin^m(a+bI)x \cos^n(a+bI)x\ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following two cases:

- Case $n\ is$ odd:
  - Split of $\cos(a+bI)x$
  - Apply $\cos^2(a+bI)x = 1 - \sin^2(a+bI)x$
  - We substitution $u = \sin(a+bI)x$
- Case $m\ is$ odd:
  - Split of $\sin(a+bI)x$
  - Apply $\sin^2(a+bI)x = 1 - \cos^2(a+bI)x$
  - We substitution $u = \cos(a+bI)x$

Example4.1

Find: $\int \sin^2(3+7I)x \cos^3(3+7I)x\ dx$

Solution:

$$\int \sin^2(3+7I)x \cos^3(3+7I)x\ dx = \int \sin^2(3+7I)x \cos^2(3+7I)x\ \ \cos(3+7I)x\ dx$$

$$= \int \sin^2(3+7I)x\,(1 - \sin^2(3+7I)x)\ \ \cos(3+7I)x\ dx$$

$$= \int (\sin^2(3+7I)x - \sin^4(3+7I)x)\ \ \cos(3+7I)x\ dx$$

By substitution:

$$u = \sin(3+7I)x \qquad \Rightarrow \frac{1}{3+7I}\,du = \cos(3+7I)x\ dx$$

$$\Rightarrow \int (\sin^2(3+7I)x - \sin^4(3+7I)x)\ \ \cos(3+7I)x\ dx = \frac{1}{3+7I}\int (\mathrm{u}^2 - \mathrm{u}^4)\,du$$

$$= \left(\frac{1}{3} - \frac{7}{30}I\right)\left(\frac{\mathrm{u}^3}{3} - \frac{\mathrm{u}^5}{5}\right) + C = \left(\frac{1}{3} - \frac{7}{30}I\right)\left(\frac{\sin^3(3+7I)x}{3} - \frac{\sin^5(3+7I)x}{5}\right) + C$$

II. $\int \tan^m(a+bI)x \sec^n(a+bI)x \ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following cases:

- Case n *is* even:
  - Split of $\sec^2(a+bI)x$
  - Apply $\sec^2(a+bI)x = 1+\tan^2(a+bI)x$
  - We substitution $u=\tan(a+bI)x$

- Case m *is* odd:
  - Split of $\sec(a+bI)x \tan(a+bI)x$
  - Apply $\tan^2(a+bI)x = \text{ses}^2(a+bI)x - 1$
  - We substitution $u=\text{ses}(a+bI)x$

- Case m even and n odd:
  - Apply $\tan^2(a+bI)x = \text{ses}^2(a+bI)x - 1$
  - We substitution $u=\text{ses}(a+bI)x$ or $u=\tan(a+bI)x$, depending on the case.

Example4.2

Find: $\int \tan^2(2-5I)x \ \sec^4(2-5I)x \ dx$

Solution:

$$n = 4 \ \text{(even)}$$

$$\int \tan^2(2-5I)x \ \sec^4(2-5I)x \ dx$$

$$= \int \tan^2(2-5I)x \ \sec^2(2-5I)x \ \ \sec^2(2-5I)x \ dx$$

$$= \int (\tan^2(2-5I)x + \tan^4(2-5I)x) \ \ \sec^2(2-5I)x \ dx$$

By substitution:

$$u = \tan(2-5I)x \qquad \Rightarrow \frac{1}{2-5I} du = \sec^2(2-5I)x \ dx$$

$$\Rightarrow \int (\tan^2(2-5I)x + \tan^4(2-5I)x) \ \ \sec^2(2-5I)x \ dx = \frac{1}{2-5I}\int (\mathrm{u}^2+\mathrm{u}^4)\, du$$

$$= \left(\frac{1}{2} - \frac{5}{6}I\right)\left(\frac{\mathrm{u}^3}{3} - \frac{\mathrm{u}^5}{5}\right) + C = \left(\frac{1}{2} - \frac{5}{6}I\right)\left(\frac{\tan^3(3+7I)x}{3} - \frac{\tan^5(3+7I)x}{5}\right) + C$$

Example4.3

Find: $\int \tan^3(6+I)x \ \sec^3(6+I)x \ dx$

Solution:

$$m = 4 \ \text{(odd)}$$

$$\int \tan^3(6+I)x \ \sec^3(6+I)x \ dx = \int \tan^2(6+I)x \ \sec^2(6+I)x \ \sec(6+\mathrm{I})x \tan(6+I)x\, dx$$

$$= \int (\sec^4(6+I)x - \sec^2(6+I)x) \ \sec(6+\mathrm{I})x \tan(6+I)x\, dx$$

By substitution:

$$u = \sec(6+\mathrm{I})x \qquad \Rightarrow \frac{1}{6+\mathrm{I}}\,du = \sec(6+\mathrm{I})x\tan(6+I)x\ dx$$

$$\Rightarrow \int (\sec^4(6+I)x - \sec^2(6+I)x)\ \sec(6+\mathrm{I})x\tan(6+I)x\,dx = \frac{1}{6+\mathrm{I}}\int (\mathrm{u}^4+\mathrm{u}^2)\,du$$

$$= \left(\frac{1}{6}-\frac{1}{42}I\right)\left(\frac{\mathrm{u}^3}{3}-\frac{\mathrm{u}^5}{5}\right)+C = \left(\frac{1}{6}-\frac{1}{42}I\right)\left(\frac{\sec^5(3+7I)x}{5}-\frac{\sec^3(3+7I)x}{3}\right)+C$$

III. $\int \cot^m(a+bI)x\csc^n(a+bI)x\ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following cases:

- Case n *is* even:
  - Split of $\csc^2(a+bI)x$
  - Apply $\csc^2(a+bI)x = 1+\cot^2(a+bI)x$
  - We substitution $u = \cot(a+bI)x$
- Case m *is* odd:
  - Split of $\csc(a+bI)x\cot(a+bI)x$
  - Apply $\cot^2(a+bI)x = \csc^2(a+bI)x - 1$
  - We substitution $u = \csc(a+bI)x$
- Case m even and n odd:
  - Apply $\cot^2(a+bI)x = \csc^2(a+bI)x - 1$
  - We substitution $u = \csc(a+bI)x$ or $u = \cot(a+bI)x$, depending on the case.

Example4.4

Find: $\int \tan^2(2-5I)x\ \sec^4(2-5I)x\ dx$

Solution:

$$n = 4\ \text{(even)}$$

$$\int \tan^2(2-5I)x\ \sec^4(2-5I)x\ dx$$

$$= \int \tan^2(2-5I)x\ \sec^2(2-5I)x\ \ \sec^2(2-5I)x\ dx$$

$$= \int (\tan^2(2-5I)x + \tan^4(2-5I)x)\ \ \sec^2(2-5I)x\ dx$$

By substitution:

$$u = \tan(2-5I)x \qquad \Rightarrow \frac{1}{2-5I}\,du = \sec^2(2-5I)x\ dx$$

$$\Rightarrow \int (\tan^2(2-5I)x + \tan^4(2-5I)x)\ \ \sec^2(2-5I)x\ dx = \frac{1}{2-5I}\int (\mathrm{u}^2+\mathrm{u}^4)\,du$$

$$= \left(\frac{1}{2}-\frac{5}{6}I\right)\left(\frac{\mathrm{u}^3}{3}-\frac{\mathrm{u}^5}{5}\right)+C = \left(\frac{1}{2}-\frac{5}{6}I\right)\left(\frac{\tan^3(3+7I)x}{3}-\frac{\tan^5(3+7I)x}{5}\right)+C$$

Example4.5

Find: $\int \tan^3(6+I)x\ \sec^3(6+I)x\ dx$

Solution:

$$m = 3\ \text{(odd)}$$

$$\int \tan^3(6+I)x\ \sec^3(6+I)x\ dx = \int \tan^2(6+I)x\ \sec^2(6+I)x\ \sec(6+\mathrm{I})x \tan(6+I)x\, dx$$

$$= \int (\sec^4(6+I)x - \sec^2(6+I)x)\ \sec(6+\mathrm{I})x \tan(6+I)x\, dx$$

By substitution:

$$u = \sec(6+\mathrm{I})x \qquad \Rightarrow \frac{1}{6+\mathrm{I}}\, du = \sec(6+\mathrm{I})x \tan(6+I)x\ dx$$

$$\Rightarrow \int (\sec^4(6+I)x - \sec^2(6+I)x)\ \sec(6+\mathrm{I})x \tan(6+I)x\, dx = \frac{1}{6+\mathrm{I}}\int (\mathrm{u}^4 + \mathrm{u}^2)\, du$$

$$= \left(\frac{1}{6} - \frac{1}{42}I\right)\left(\frac{\mathrm{u}^3}{3} - \frac{\mathrm{u}^5}{5}\right) + C = \left(\frac{1}{6} - \frac{1}{42}I\right)\left(\frac{\sec^5(3+7I)x}{5} - \frac{\sec^3(3+7I)x}{3}\right) + C$$

Example4.6

Find: $\int \cot^4(1+4I)x\ \csc^4(1+4I)x\ dx$

Solution:

$$n = 4\ \text{(even)}$$

$$\int \sqrt{\cot(1+4I)x}\ \csc^4(1+4I)x\ dx$$

$$= \int \cot^{1/2}(1+4I)x\ \csc^2(1+4I)x \csc^2(1+4I)x\ dx$$

$$= \int \left(\cot^{1/2}(1+4I)x + \cot^{3/2}(1+4I)x\right) \csc^2(1+4I)x\ dx$$

By substitution:

$$u = \cot(1+4I)x \qquad \Rightarrow \frac{1}{1+4I}\, du = \csc^2(1+4I)x\ dx$$

$$\Rightarrow \int \left(\cot^{1/2}(1+4I)x + \cot^{3/2}(1+4I)x\right) \csc^2(1+4I)x\ dx$$

$$= \frac{1}{1+4I}\int \left(\mathrm{u}^{1/2} + \mathrm{u}^{3/2}\right) du$$

$$= (1-I)\left(\frac{2\ \mathrm{u}^{3/2}}{3} - \frac{2\ \mathrm{u}^{5/2}}{5}\right) + C = (1-I)\left(\frac{2\ \cot^{3/2}(1+4I)x}{3} - \frac{2\ \cot^{5/2}(1+4I)x}{5}\right) + C$$

4.1Neutrosophic trigonometric identities:

1) $sin(a+bI)x \cos(c+dI)\, x = \frac{1}{2}\, [\sin(a+bI+c+dI) + \sin(a+bI-c-dI)]$

2) $cos(a+bI)x \sin(c+dI)\, x = \frac{1}{2}\, [\sin(a+bI+c+dI) - \sin(a+bI-c-dI)]$

3) $cos(a+bI)x \cos(c+dI)\, x = \frac{1}{2}\, [\cos(a+bI+c+dI) + \cos(a+bI-c-dI)]$

4) $sin(a+bI)x \sin(c+dI)\, x = \frac{-1}{2}\, [\cos(a+bI+c+dI) - \cos(a+bI-c-dI)]$

Where $a \neq c\ (not\ zero)\ and\ b, d$ are real numbers, while I = indeterminacy.

Example4.1.1

Find: 1) $\int sin(7+3I)x\cos(6+3I)\,x\,dx = \int \frac{1}{2}\,[sin(13+6I)x+\sin x]\;dx$

$$=\frac{1}{2}\left[-\left(\frac{1}{13}-\frac{6}{247}I\right)cos(7+3I)x-\cos x\right]+C$$

2) $\int cos(2-I)x\cos(3+4I)\,x\,dx = \int \frac{1}{2}\,[cos(5+3I)x+\cos(-1-5I)\,x]\;dx$

$$=\int \frac{1}{2}\,[cos(5+3I)x+\cos(1+5I)\,x]\;dx$$

$$=\frac{1}{2}\left[\left(\frac{1}{5}-\frac{3}{40}I\right)sin(5+3I)x+\left(1-\frac{5}{6}I\right)sin(1+5I)x\right]+C$$

3) $\int sin(2+I)x\,sin(1+3I)\,x\,dx = \int \frac{-1}{2}\,[\cos(3+4I)x-\cos\,(1-2\mathrm{I})x]\;dx$

$$=\frac{1}{2}\left[\left(\frac{1}{3}-\frac{4}{21}I\right)sin(3+4I)x-(1-2\mathrm{I})sin(1-2\mathrm{I})x\right]+C$$

5. Conclusions

The integral is very important in our life, and is used especially for example in calculating areas whose shape is not familiar. This led us to study neutrosophic integrals for some neutrosophic functions from that contain indeterminacy. Where the method of integration by substitution and the neutrosophic trigonometric identities are defined, in addition to studying cases of the integrating products of neutrosophic trigonometric functions. This paper is considered an introduction to the applications in neutrosophic integrals.

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