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The neutrosophic integrals and integration methods

Yaser Ahmad Alhasan

Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, KSA; y.alhasan@psau.edu.sa

Abstract: the purpose of this article is to study the neutrosophic integrals and integration methods, Where the method of integration by substitution is defined, and theorems have been proven useful for facilitating the calculation of integration for some neutrosophic functions from that contain indeterminacy. Also, neutrosophic trigonometric identities are defined, in addition to studying all cases of the integrating products of neutrosophic trigonometric functions. Where detailed examples were given to clarify each case.

Keywords: neutrosophic indefinite integral; substitution method; indeterminacy; neutrosophic trigonometric functions.

1. Introduction

As an alternative to the existing logics, Smarandache proposed the Neutrosophic Logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of neutrosophy is a new branch of philosophy introduced by Smarandache [3-13]. He presented the definition of the standard form of neutrosophic real number and conditions for the division of two neutrosophic real numbers to exist, he defined the standard form of neutrosophic complex number, and found root index $n \geq 2$ of a neutrosophic real and complex number [2-4], studying the concept of the Neutrosophic probability [3-5], the Neutrosophic statistics [4][6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1-8]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [9]. Edalatpanah proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side represented with triangular neutrosophic numbers [10]. Chakraborty used pentagonal neutrosophic number in networking problem, and Shortest Path Problem [11-12]. Y.Alhasan studied the concepts of neutrosophic complex numbers and the general exponential form of a neutrosophic complex [7][14]. On the other hand, M.Abdel-Basset presented study in the science of neutrosophic about an approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number [15]. Also, neutrosophic sets played an important role in applied science such as health care, industry, and optimization [16-17-18-19]. Recently, there are increasing efforts to study the neutrosophic generalized structures and spaces such as refined neutrosophic modules, spaces, equations, and rings [21-22-23].

Integration is important in human life, and one of its most important applications is the calculation of area, size and arc length. In our reality we find things that cannot be precisely defined, and that contain an indeterminacy part. This is the reason for studying neutrosophic integration and methods of its integration in this paper.

Paper consists of 5 sections. In 1th section, provides an introduction, in which neutrosophic science review has given. In 2th section, some definitions and examples of neutrosophic real number neutrosophic indefinite integral and are discussed. The 3th section frames the neutrosophic integration by substitution method, and their employment in finding integrals that include roots, logarithms, exponents, or a function with its derivative (between them is the process of multiplication or division), and the study of the related theories. The 4th section introduces the integrating products of neutrosophic trigonometric function in three states and neutrosophic trigonometric identities, as they have been used in finding some types of neutrosophic trigonometric integrals. In 5th section, a conclusion to the paper is given.

2. Preliminaries

2.1. Neutrosophic Real Number [4]

Suppose that w is a neutrosophic number, then it takes the following standard form: $w = a + bI$ where a, b are real coefficients, and I represent indeterminacy, such $0.\mathrm{I} = 0$ and $\mathrm{I}^n = \mathrm{I}$, for all positive integers n.

2.2. Division of neutrosophic real numbers [4]

Suppose that w_1, w_2 are two neutrosophic numbers, where

$$w_1 = a_1 + b_1 I \ , \qquad w_2 = a_2 + b_2 I$$

To find $(a_1 + b_1 I) \div (a_2 + b_2 I)$, we can write:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} \equiv x + yI$$

where x and y are real unknowns.

$$a_1 + b_1 I \equiv (a_2 + b_2 I)(x + yI)$$

$$a_1 + b_1 I \equiv a_2 x + (b_2 x + a_2 y + b_2 y) I$$

by identifying the coefficients, we get

$$a_1 = a_2 x$$

$$b_1 = b_2 x + (a_2 + b_{2)} y$$

We obtain unique one solution only, provided that:

$$\begin{vmatrix} a_2 & 0 \\ b_2 & a_2 + b_2 \end{vmatrix} \neq 0 \quad \Rightarrow \quad a_2(a_2 + b_2) \neq 0$$

Hence: $a_2 \neq 0$ and $a_2 \neq -b_2$ are the conditions for the division of two neutrosophic real numbers to exist.

Then:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} . I$$

2.3. Neutrosophic Indefinite Integral [14]

We just extend the classical definition of anti-derivative. The neutrosophic antiderivative of neutrosophic function $f(x)$ is the neutrosophic function $F(x)$ such that $F'(x) = f(x)$.

Example2.4.1:

Let $f: R \to R \cup \{I\}$, $f(x) = -3x^2 + (4x-5)I$.

Then:

$$F(x) = \int [-3x^2 + (4x-5)I]dx$$
$$= -x^3 + (2x^2 - 5x)I + C$$

where C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while I = indeterminacy).

Example2.4.2: (Refined Indeterminacy).

Let $g: \mathbb{R} \to \mathbb{R} \cup \{I_1\} \cup \{I_2\} \cup \{I_3\}$, where I_1, I_2, and I_3 are types of sub indeterminacies,

$$g(x) = 7x - 2I_1 + x^2 I_2 + 4x^3 I_3$$

Then:

$$F(x) = \int [7x - 2I_1 + x^2 I_2 + 4x^3 I_3]dx$$
$$= \frac{7x^2}{2} - 2xI_1 + \frac{x^3}{3} I_2 + x^4 I_3 + a + bI$$

where a and b are real constants.

3. Neutrosophic integration by substitution method

Definition3.1

Let $f: D_f \subseteq R \to R_f \cup \{I\}$, to evaluate $\int f(x)dx$

Put: $x = g(u) \quad \Rightarrow \quad dx = \acute{g}(u)du$

By substitution, we get:

$$\int f(x)dx = \int f(u)\acute{g}(u)du$$

then we can directly integral it.

Theorme3.1:

If $\int f(x,I)dx = \varphi(x,I)$ *then*,

$$\int f\big((a+bI)x + c + dI\big)\, dx = \left(\frac{1}{a} - \frac{b}{a(a+b)} I\right) \varphi\big((a+bI)x + c + dI\big) + C$$

where C is an indeterminate real constant, $a \neq 0\,, a \neq -b$ *and* b, c, d are real numbers, while I = indeterminacy.

Proof:

Put: $(ba + I)x + c + dI = u \qquad \Rightarrow (ax + bI)dx = du$

$$\Rightarrow dx = \frac{1}{a+bI}\, du$$

$$\Rightarrow dx = \left(\frac{1}{a} - \frac{b}{a(a+b)} I\right) du$$

$$\int f\big((ba + I)x + c + dI\big)\, dx = \int f(u) \left(\frac{1}{a} - \frac{b}{a(a+b)} I\right) du$$

$$= \left(\frac{1}{a} - \frac{b}{a(a+b)} I\right) \varphi(u) + C$$

back to the variable x, we get:

$$\int f\big((a+bI)x+c+dI\big)\,dx=\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)\varphi\big((a+bI)x+c+dI\big)+C$$

3.1. We can prove each of the following, using the previous theorem:

1) $\int ((a+bI)x+c+dI)^n dx=\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)\frac{((a+bI)x+c+dI)^{n+1}}{n+1}+C$

2) $\int \frac{1}{(a+bI)x+c+dI}dx=\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)ln|(a+bI)x+c+dI|+\boldsymbol{C}$

3) $\int e^{(a+bI)x+c+dI}\,dx=\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)e^{(a+bI)x+c+dI}+C$

4) $\int \frac{1}{\sqrt{(a+bI)x+c+dI}}dx=2\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)\sqrt{(a+bI)x+c+dI}+\boldsymbol{C}$

5) $\int cos((a+bI)x+c+dI)\,dx=\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)cos((a+bI)x+c+dI)+\boldsymbol{C}$

6) $\int sin((a+bI)x+c+dI)\,dx=-\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)sin((a+bI)x+c+dI)+\boldsymbol{C}$

7) $\int sec^2((a+bI)x+c+dI)\,dx=\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)tan((a+bI)x+c+dI)+\boldsymbol{C}$

8) $\int csc^2((a+bI)x+c+dI)\,dx=-\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)cot((a+bI)x+c+dI)+\boldsymbol{C}$

9) $\int sec((a+bI)x+c+dI)\,tan((a+bI)x+c+dI)\,dx$
$=\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)sec((a+bI)x+c+dI)+\boldsymbol{C}$

10) $\int csc((a+bI)x+c+dI)\,cot((a+bI)x+c+dI)\,dx$
$=-\left(\frac{1}{a}-\frac{b}{a(a+b)}I\right)csc((a+bI)x+c+dI)+\boldsymbol{C}$

where C is an indeterminate real constant, $a\neq 0, a\neq -b$ and b,c,d are real numbers, while $I=$ indeterminacy.

Example3.1.1

1) $\int ((3-5I)x+4)^7dx=\left(\frac{1}{3}-\frac{5}{6}I\right)\frac{((3-5I)x+4)^8}{8}+\boldsymbol{C}$

2) $\int \frac{1}{(6+5I)x-7I}dx=\left(\frac{1}{6}-\frac{5}{66}I\right)ln|(6+5I)x-7I|+\boldsymbol{C}$

3) $\int \boldsymbol{e}^{(2+I)x-3}\,dx=\left(\frac{1}{2}-\frac{1}{6}I\right)\boldsymbol{e}^{(2+I)x-3}+\boldsymbol{C}$

4) $\int cos\big((1+4I)x+I\big)\,dx=\left(1-\frac{4}{5}I\right)cos\big((1+4I)x+I\big)+\boldsymbol{C}$

5) $\int sec^2\big((6+5I)x-7I\big)\,dx=\left(\frac{1}{6}-\frac{5}{66}I\right)tan\big((6+5I)x-7I\big)+\boldsymbol{C}$

6) $\int csc\big((2+I)x-3\big)\,cot\big((2+I)x-3\big)\,dx=-\left(\frac{1}{2}-\frac{1}{6}I\right)csc\big((2+I)x-3\big)+\boldsymbol{C}$

7) $\int \frac{1}{\sqrt{(2+5I)x+3I}}dx=2\left(\frac{1}{2}-\frac{5}{14}I\right)\sqrt{(2+5I)x+3I}+\boldsymbol{C}=(1-5I)\sqrt{(2+5I)x+3I}+\boldsymbol{C}$

Theorme3.2:
Let $f\colon D_f\subseteq R\to R_f\cup\{I\}$ then:

$$\int\frac{\acute{f}(x,I)}{f(x,I)}dx=ln|f(x,I)|+C$$

where C is an indeterminate real constant (i.e. constant of the form $a+bI$, where a,b are real numbers, while I = indeterminacy).

Proof:

Put: $f(x,I)=u \quad \Rightarrow \acute{f}(x,I)dx=du$

$$\Rightarrow dx=\frac{1}{\acute{f}(x,I)}\,du$$

$$\Rightarrow dx=\frac{1}{\acute{u}}\,du$$

$$\int\frac{\acute{f}(x,I)}{f(x,I)}dx=\int\frac{\acute{u}}{u}\frac{1}{\acute{u}}du=\int\frac{1}{u}du=ln|u|+C$$

back to the variable $f(x,I)$, we get:

$$\int\frac{\acute{f}(x,I)}{f(x,I)}dx=ln|f(x,I)|+C$$

Example3.1.2

1) $\int\frac{(1+2I)x^3}{(1+2I)x^4+5I}dx=\frac{1}{4}ln|(1+2I)x^4+5I|+C$

2) $\int\frac{(2+I)e^{(2+I)x-3}-2I}{e^{(2+I)x-3}-2xI}dx=ln\left|e^{(2+I)x-3}-2xI\right|+C$

3) $\int\tan(1+7I)x\,dx=\int\frac{\sin(1+7I)x}{\cos(1+7I)x}dx=\left(-1+\frac{7}{8}I\right)ln|\cos(1+7I)x|+C$

4) $\int\frac{1}{1+\tan(1+2I)x}dx=\int\frac{1}{1+\frac{\sin(1+2I)}{\cos(1+2I)}}dx=\frac{1}{2}\int\frac{2\cos(1+2I)x}{\cos(1+2I)x+\sin(1+2I)x}dx$

$=\frac{1}{2}\int\frac{\cos(1+2I)x+\sin(1+2I)x+\cos(1+2I)x-\sin(1+2I)x}{\cos(1+2I)+\sin(1+2I)x}dx$

$=\frac{1}{2}\int dx+\frac{1}{2}\int\frac{\cos(1+2I)x-\sin(1+2I)\,x}{\cos(1+2I)x+\sin(1+2I)x}dx$

$$=\frac{1}{2}x+\left(\frac{1}{2}-\frac{1}{3}I\right)ln|\cos(1+2I)x+\sin(1+2I)x|+C$$

Theorme3.3:
Let $f: D_f \subseteq R \to R_f \cup \{I\}$, then:

$$\int \frac{\acute{f}(x,I)}{\sqrt{f(x,I)}}dx = 2\sqrt{f(x,I)}+C$$

where C is an indeterminate real constant (i.e. constant of the form $a+bI$, where a,b are real numbers, while I = indeterminacy).

Proof:

Put: $f(x,I)=u \quad \Rightarrow \acute{f}(x,I)dx = du$

$$\Rightarrow dx = \frac{1}{\acute{f}(x,I)}\,du$$

$$\Rightarrow dx = \frac{1}{\acute{u}}\,du$$

$$\int \frac{\acute{f}(x,I)}{\sqrt{f(x,I)}}dx = \int \frac{\acute{u}}{\sqrt{u}}\frac{1}{\acute{u}}du = \int \frac{1}{\sqrt{u}}du = 2\sqrt{u}+C$$

back to the variable $f(x,I)$, we get:

$$\int \frac{\acute{f}(x,I)}{\sqrt{f(x,I)}}dx = 2\sqrt{f(x,I)}+C$$

Example3.1.3

1) $$\int \frac{-(2+5I)x+4I}{\sqrt{(2+5I)x^2-8xI}}dx = \frac{-1}{2}\sqrt{(2+5I)x^2-8xI}+C$$

2) $$\int \frac{3x^2}{\sqrt{(2+5I)x^3+3I}}dx = \left(1-\frac{5}{7}I\right)\sqrt{(2+5I)x^3+3I}+C$$

Theorme3.4:
$f: D_f \subseteq R \to R_f \cup \{I\}$, then:

$$\int [f(x,I)]^n \acute{f}(x)\,dx = \frac{[f(x,I)]^{n+1}}{n+1}+C$$

Where n is any rational number. C is an indeterminate real constant (i.e. constant of the form $a+bI$, where a,b are real numbers, while I = indeterminacy).

Proof:

Put: $f(x,I)=u \quad \Rightarrow \acute{f}(x,I)dx = du$

$$\Rightarrow dx = \frac{1}{\acute{f}(x,I)}\,du$$

$$\Rightarrow dx = \frac{1}{\acute{u}}\,du$$

$$\int [f(x,I)]^n \acute{f}(x,I)dx = \int u^n \acute{u}\frac{1}{\acute{u}}du = \int u^n du = \frac{u^{n+1}}{n+1}+C$$

back to the variable $f(x,I)$, we get:

$$\int [f(x,I)]^n f\acute{(}x)dx = \frac{[f(x,I)]^{n+1}}{n+1} + C$$

Example3.1.4

1) $\int x^4[(2+5I)x^5]^{12}dx = \left(\frac{1}{10} - \frac{1}{20}I\right)\frac{[(2+5I)x^5]^{13}}{13} + C$

2) $\int \frac{1}{\sqrt{(4-2I)x}}\left(\sqrt{(4-2I)x}\right)^{10} dx = \left(\frac{1}{2} + \frac{1}{2}I\right)\frac{\left(\sqrt{(4-2I)x}\right)^{11}}{11} + C$

4. Integrating products of neutrosophic trigonometric function:

I. $\int \sin^m(a+bI)x \cos^n(a+bI)x \; dx$, where m and n are positive integers.
To find this integral, we can distinguish the following two cases:

- Case $n \; is$ odd:
  - Split of $\cos(a+bI)x$
  - Apply $\cos^2(a+bI)x = 1 - \sin^2(a+bI)x$
  - We substitution $u = \sin(a+bI)x$
- Case $m \; is$ odd:
  - Split of $\sin(a+bI)x$
  - Apply $\sin^2(a+bI)x = 1 - \cos^2(a+bI)x$
  - We substitution $u = \cos(a+bI)x$

Example4.1

Find: $\int \sin^2(3+7I)x \cos^3(3+7I)x \; dx$

Solution:

$$\int \sin^2(3+7I)x \cos^3(3+7I)x \; dx = \int \sin^2(3+7I)x \cos^2(3+7I)x \;\; \cos(3+7I)x \; dx$$

$$= \int \sin^2(3+7I)x \,(1 - \sin^2(3+7I)x) \;\; \cos(3+7I)x \; dx$$

$$= \int (\sin^2(3+7I)x - \sin^4(3+7I)x) \;\; \cos(3+7I)x \; dx$$

By substitution:

$$u = \sin(3+7I)x \qquad \Rightarrow \frac{1}{3+7I}\, du = \cos(3+7I)x \; dx$$

$$\Rightarrow \int (\sin^2(3+7I)x - \sin^4(3+7I)x) \;\; \cos(3+7I)x \; dx = \frac{1}{3+7I}\int (\mathrm{u}^2 - \mathrm{u}^4)\, du$$

$$= \left(\frac{1}{3} - \frac{7}{30}I\right)\left(\frac{\mathrm{u}^3}{3} - \frac{\mathrm{u}^5}{5}\right) + C = \left(\frac{1}{3} - \frac{7}{30}I\right)\left(\frac{\sin^3(3+7I)x}{3} - \frac{\sin^5(3+7I)x}{5}\right) + C$$

II. $\int \tan^m(a+bI)x \sec^n(a+bI)x \ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following cases:

- Case n *is* even:
  - Split of $\sec^2(a+bI)x$
  - Apply $\sec^2(a+bI)x = 1+\tan^2(a+bI)x$
  - We substitution $u=\tan(a+bI)x$

- Case m *is* odd:
  - Split of $\sec(a+bI)x \tan(a+bI)x$
  - Apply $\tan^2(a+bI)x = \text{ses}^2(a+bI)x - 1$
  - We substitution $u=\text{ses}(a+bI)x$

- Case m even and n odd:
  - Apply $\tan^2(a+bI)x = \text{ses}^2(a+bI)x - 1$
  - We substitution $u=\text{ses}(a+bI)x$ or $u=\tan(a+bI)x$, depending on the case.

Example4.2

Find: $\int \tan^2(2-5I)x \ \sec^4(2-5I)x \ dx$

Solution:

$$n = 4 \ \text{(even)}$$

$$\int \tan^2(2-5I)x \ \sec^4(2-5I)x \ dx$$

$$= \int \tan^2(2-5I)x \ \sec^2(2-5I)x \ \ \sec^2(2-5I)x \ dx$$

$$= \int (\tan^2(2-5I)x + \tan^4(2-5I)x) \ \ \sec^2(2-5I)x \ dx$$

By substitution:

$$u = \tan(2-5I)x \qquad \Rightarrow \frac{1}{2-5I}\,du = \sec^2(2-5I)x \ dx$$

$$\Rightarrow \int (\tan^2(2-5I)x + \tan^4(2-5I)x) \ \ \sec^2(2-5I)x \ dx = \frac{1}{2-5I}\int (\mathrm{u}^2+\mathrm{u}^4)\,du$$

$$= \left(\frac{1}{2}-\frac{5}{6}I\right)\left(\frac{\mathrm{u}^3}{3}-\frac{\mathrm{u}^5}{5}\right)+C = \left(\frac{1}{2}-\frac{5}{6}I\right)\left(\frac{\tan^3(3+7I)x}{3}-\frac{\tan^5(3+7I)x}{5}\right)+C$$

Example4.3

Find: $\int \tan^3(6+I)x \ \sec^3(6+I)x \ dx$

Solution:

$$m = 4 \ \text{(odd)}$$

$$\int \tan^3(6+I)x \ \sec^3(6+I)x \ dx = \int \tan^2(6+I)x \ \sec^2(6+I)x \ \sec(6+\mathrm{I})x \tan(6+I)x\,dx$$

$$= \int (\sec^4(6+I)x - \sec^2(6+I)x) \ \sec(6+\mathrm{I})x \tan(6+I)x\,dx$$

By substitution:

$$u = \sec(6+\mathrm{I})x \qquad \Rightarrow \frac{1}{6+\mathrm{I}}\, du = \sec(6+\mathrm{I})x \tan(6+I)x \ dx$$

$$\Rightarrow \int (\sec^4(6+I)x - \sec^2(6+I)x) \ \sec(6+\mathrm{I})x \tan(6+I)x \, dx = \frac{1}{6+\mathrm{I}} \int (\mathrm{u}^4 + \mathrm{u}^2)\, du$$

$$= \left(\frac{1}{6} - \frac{1}{42} I\right)\left(\frac{\mathrm{u}^3}{3} - \frac{\mathrm{u}^5}{5}\right) + C = \left(\frac{1}{6} - \frac{1}{42} I\right)\left(\frac{\sec^5(3+7I)x}{5} - \frac{\sec^3(3+7I)x}{3}\right) + C$$

III. $\int \cot^m(a+bI)x \csc^n(a+bI)x \ dx$, where m and n are positive integers.
To find this integral, we can distinguish the following cases:

- Case n *is* even:
  - Split of $\csc^2(a+bI)x$
  - Apply $\csc^2(a+bI)x = 1 + \cot^2(a+bI)x$
  - We substitution $u = \cot(a+bI)x$
- Case m *is* odd:
  - Split of $\csc(a+bI)x \cot(a+bI)x$
  - Apply $\cot^2(a+bI)x = \csc^2(a+bI)x - 1$
  - We substitution $u = \csc(a+bI)x$
- Case m even and n odd:
  - Apply $\cot^2(a+bI)x = \csc^2(a+bI)x - 1$
  - We substitution $u = \csc(a+bI)x$ or $u = \cot(a+bI)x$, depending on the case.

Example4.4

Find: $\int \tan^2(2-5I)x \ \sec^4(2-5I)x \ dx$

Solution:

$$n = 4 \ \text{(even)}$$

$$\int \tan^2(2-5I)x \ \sec^4(2-5I)x \ dx$$

$$= \int \tan^2(2-5I)x \ \sec^2(2-5I)x \ \ \sec^2(2-5I)x \ dx$$

$$= \int (\tan^2(2-5I)x + \tan^4(2-5I)x) \ \ \sec^2(2-5I)x \ dx$$

By substitution:

$$u = \tan(2-5I)x \qquad \Rightarrow \frac{1}{2-5I}\, du = \sec^2(2-5I)x \ dx$$

$$\Rightarrow \int (\tan^2(2-5I)x + \tan^4(2-5I)x) \ \ \sec^2(2-5I)x \ dx = \frac{1}{2-5I}\int (\mathrm{u}^2 + \mathrm{u}^4)\, du$$

$$= \left(\frac{1}{2} - \frac{5}{6} I\right)\left(\frac{\mathrm{u}^3}{3} - \frac{\mathrm{u}^5}{5}\right) + C = \left(\frac{1}{2} - \frac{5}{6} I\right)\left(\frac{\tan^3(3+7I)x}{3} - \frac{\tan^5(3+7I)x}{5}\right) + C$$

Example4.5

Find: $\int \tan^3(6+I)x \ \sec^3(6+I)x \ dx$

Solution:

$$m = 3 \ \text{(odd)}$$

$$\int \tan^3(6+I)x\ \sec^3(6+I)x\ dx = \int \tan^2(6+I)x\ \sec^2(6+I)x\ \sec(6+\mathrm{I})x\tan(6+I)x\,dx$$

$$= \int (\sec^4(6+I)x - \sec^2(6+I)x)\ \sec(6+\mathrm{I})x\tan(6+I)x\,dx$$

By substitution:

$$u = \sec(6+\mathrm{I})x \qquad \Rightarrow \frac{1}{6+\mathrm{I}}\,du = \sec(6+\mathrm{I})x\tan(6+I)x\ dx$$

$$\Rightarrow \int (\sec^4(6+I)x - \sec^2(6+I)x)\ \sec(6+\mathrm{I})x\tan(6+I)x\,dx = \frac{1}{6+\mathrm{I}}\int (\mathrm{u}^4+\mathrm{u}^2)\,du$$

$$= \left(\frac{1}{6}-\frac{1}{42}I\right)\left(\frac{\mathrm{u}^3}{3}-\frac{\mathrm{u}^5}{5}\right)+C = \left(\frac{1}{6}-\frac{1}{42}I\right)\left(\frac{\sec^5(3+7I)x}{5}-\frac{\sec^3(3+7I)x}{3}\right)+C$$

Example4.6

Find: $\int \cot^4(1+4I)x\ \csc^4(1+4I)x\ dx$

Solution:

$$n = 4\ \text{(even)}$$

$$\int \sqrt{\cot(1+4I)x}\ \csc^4(1+4I)x\ dx$$

$$= \int \cot^{1/2}(1+4I)x\ \csc^2(1+4I)x\csc^2(1+4I)x\ dx$$

$$= \int \left(\cot^{1/2}(1+4I)x + \cot^{3/2}(1+4I)x\right)\csc^2(1+4I)x\ dx$$

By substitution:

$$u = \cot(1+4I)x \qquad \Rightarrow \frac{1}{1+4I}\,du = \csc^2(1+4I)x\ dx$$

$$\Rightarrow \int \left(\cot^{1/2}(1+4I)x + \cot^{3/2}(1+4I)x\right)\csc^2(1+4I)x\ dx$$

$$= \frac{1}{1+4I}\int \left(\mathrm{u}^{1/2}+\mathrm{u}^{3/2}\right)du$$

$$= (1-I)\left(\frac{2\,\mathrm{u}^{3/2}}{3}-\frac{2\,\mathrm{u}^{5/2}}{5}\right)+C = (1-I)\left(\frac{2\ \cot^{3/2}(1+4I)x}{3}-\frac{2\ \cot^{5/2}(1+4I)x}{5}\right)+C$$

4.1Neutrosophic trigonometric identities:

1) $sin(a+bI)x\cos(c+dI)\,x = \frac{1}{2}\,[\sin(a+bI+c+dI)+\sin(a+bI-c-dI)]$

2) $cos(a+bI)x\sin(c+dI)\,x = \frac{1}{2}\,[\sin(a+bI+c+dI)-\sin(a+bI-c-dI)]$

3) $cos(a+bI)x\cos(c+dI)\,x = \frac{1}{2}\,[\cos(a+bI+c+dI)+\cos(a+bI-c-dI)]$

4) $sin(a+bI)x\sin(c+dI)\,x = \frac{-1}{2}\,[\cos(a+bI+c+dI)-\cos(a+bI-c-dI)]$

Where $a \neq c\ (not\ zero)\ and\ b, d$ are real numbers, while I = indeterminacy.

Example4.1.1

Find: 1) $\int sin(7+3I)x\cos(6+3I)\,x\,dx = \int \frac{1}{2}\,[sin(13+6I)x + \sin x]\;dx$

$$= \frac{1}{2}\left[-\left(\frac{1}{13} - \frac{6}{247}I\right)cos(7+3I)x - \cos x\right] + C$$

2) $\int cos(2-I)x\cos(3+4I)\,x\,dx = \int \frac{1}{2}\,[cos(5+3I)x + \cos(-1-5I)\,x]\;dx$

$$= \int \frac{1}{2}\,[cos(5+3I)x + \cos(1+5I)\,x]\;dx$$

$$= \frac{1}{2}\left[\left(\frac{1}{5} - \frac{3}{40}I\right)sin(5+3I)x + \left(1 - \frac{5}{6}I\right)sin(1+5I)x\right] + C$$

3) $\int sin(2+I)x\,sin(1+3I)\,x\,dx = \int \frac{-1}{2}\,[\cos(3+4I)x - \cos\,(1-2\mathrm{I})x]\;dx$

$$= \frac{1}{2}\left[\left(\frac{1}{3} - \frac{4}{21}I\right)sin(3+4I)x - (1-2\mathrm{I})sin(1-2\mathrm{I})x\right] + C$$

5. Conclusions

The integral is very important in our life, and is used especially for example in calculating areas whose shape is not familiar. This led us to study neutrosophic integrals for some neutrosophic functions from that contain indeterminacy. Where the method of integration by substitution and the neutrosophic trigonometric identities are defined, in addition to studying cases of the integrating products of neutrosophic trigonometric functions. This paper is considered an introduction to the applications in neutrosophic integrals.

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