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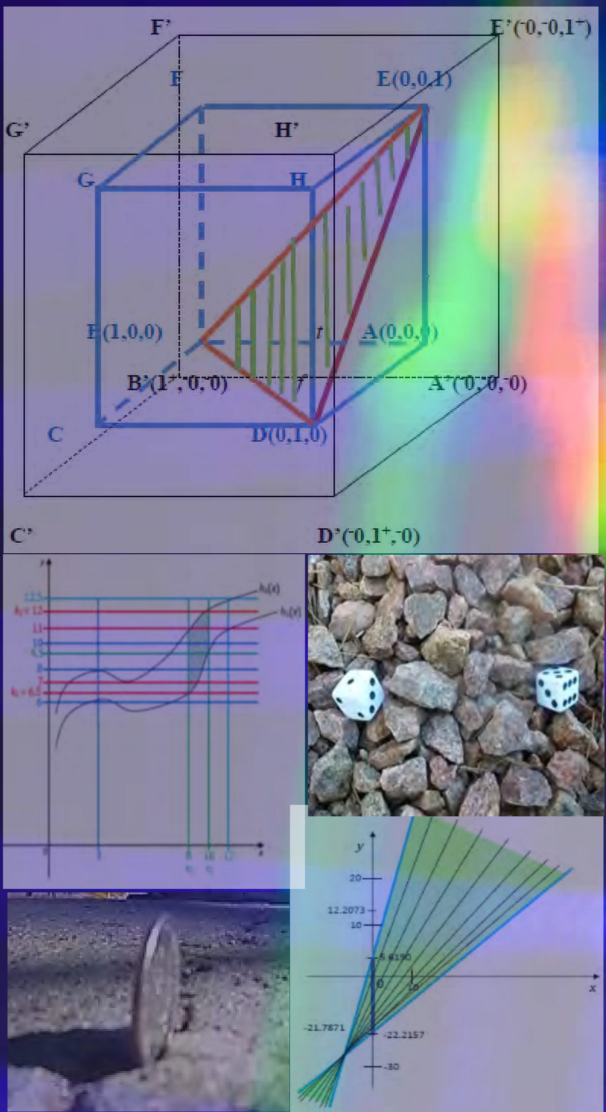

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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March 20, 2019

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Univ New Mexico, Gallup Campus

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A hybrid Model Using MCDM Methods and Bipolar Neutrosophic Sets for Select Optimal Wind Turbine: Case Study in Egypt

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Abstract: The wind turbine selection problem is important for countries under change of climate and global warming. The importance wind turbine has increased due to toward countries used the renewable energy. The information of selection wind turbines is often vague and imprecise. Therefore, this paper develops a methodology for wind turbines selection problem based on neutrosophic information. Bipolar neutrosophic sets (BNSs) is a very common tool for performing potentially uncertain information provided by experts and decision makers. So, the BNSs is a useful for dealing with uncertain complex situations. The wind turbine is contain the different and conflict criteria. Thus, the concept of multi-criteria decision making (MCDM) is used. This paper used MCDM method for selection wind turbine problem. First. Used the entropy weight to calculate the weights of criteria. Then the Weighted sum method (WSM), visekriterijumsko kompromisno rangiranje (VIKOR), Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS), Evaluation based on Distance from Average Solution (EDAS) are used to select best turbine. The case study in Egypt is provided. The comparative analysis is done to test the reliability of the proposed methodology. Finally the sensitivity analysis is performed.

Keywords: Wind Turbine Selection, MCDM, Entropy Weight, TOPSIS, EDAS, WSM, VICKOR

1. Introduction

Every day the global warming and change of climate are increased in the world. Consequence this, the awareness of the world are increased toward saving the ecosphere and going to use the fossil fuel [1]. The countries that depend on the energy from fossil fuel are converting to a renewable energy. In recent years, a new resources of energy is explored due to diminution of fossil fuel. There are many sources of renewable energy for instance, wind, wave, solar and others.

To beat the global warming, the wind energy is introduced as one of many plans[2]. Every day the value of wind energy is increasing so, several states and countries are gain money by using the wind power [3]. By the sun and unbalance abhorring of the land and sea with variance of pressure the wind energy is produced. In recent years, with quick growth, the substitute of the traditional energy systems is the wind energy[4]. The most vital parts of theses energy system are wind turbines. The energy of electricity is produced in wind turbines by converting energy of motion of wind. So, choice the wind turbines is a critical work and must be precise for long term processes.

Many countries is seeking to build the wind farms due to have many advantages like creating many jobs through increasing the attract investment by deployment the economic, the security of energy is increased, the quality of air is enhanced, the emissions of co_2 is reduced, the dependence on the using imported fuel is decreased and the prices of power will stable. There are three costs are incurred by wind farms to produce electricity. These costs are include: capital costs that contain the building power plant costs, the costs of running that contain the costs of operations and maintenance of the wind farm and the costs of financing that include the costs of running and constructing the wind farm. The cost of capital is very great. The choice best wind turbine is a high weight as a wind turbine cost make up the mainstream of the total cost for wind farm project. The selection an appropriate wind turbine that include many of problems such as effective and efficient wind farm development, maximum energy output and efficient wind farm design. So in this paper take into considerations these factors.

In the previous studies, the researchers are proposed many of techniques for selection wind turbines problem for instance heuristics, Meta heuristics and models of probability [5]. Though, these approaches have many confines and disadvantages[6]. The decision model has limitations, one of these limitations it is simple due to has one criterion[7]. Though, the problem of choice wind turbines has several different conflict criteria[8]. So, the multi criteria decision making (MCDM) is the best solution to this problem. The methods of MCDM is a preferable with numerous criteria of wind turbines and each criterion is conflict with other[9].

The criteria of wind turbines find in many units and scale. But must put all criteria in one unit with less magnitude[6]. MCDM approaches are used with the fuzzy theory to overcome this difficulties[10]. Using the fuzzy theory with the truth and false value[11]. But the fuzzy has limitations that not take into considerations the indeterminacy value although fuzzy sets has many generations such as intuitionistic fuzzy sets and hesitant fuzzy sets [12]. To overcome these limitations the neutrosophic set is presented. Neutrosophic sets is generalization of fuzzy sets and introduced by. Florentin Smarandache [13, 14]. Neutrosophic sets is used in many fields like industry, healthcare and others [15]. It has truth, false and indeterminacy value. In this work use the MCDM methods with neutrosophic numbers to select the best wind turbines.

To select best wind turbines, needs a regular approaches due to this selection is a complex and difficult but it is vital and essential to wind farms. So, needs in this work to evolve approaches and methods to this problem to aid Egypt to build a new wind farm in government red sea and introduce best wind turbines for designing.

In this work, the criteria is collected from the literature and the weights of criteria is computed by entropy weight method[16]. The entropy weight method is not used in previous research with wind turbines. Experts and decision makers build the decision matrix between criteria and alternative by using linguistic term of neutrosophic number.

To rank the wind turbines the MCDM methods are proposed. In this paper proposed WSM, VIKOR, TOPSIS and EDAS methods with the bipolar neutrosophic numbers (BNNs) to select best alternative (turbine). The WSM is the simplest additive weighted method. It is most commonly used MCDM methods. It used in this paper to rank the wind turbine. The VIKOR method is a commonly MCDM method. It used to solve the problems of decision making with different and conflicting criteria. this method is used to rank the wind turbines. The TOPSIS method is a common MCDM methods. It is used to select best alternatives. This method solve the MCDM problems in different areas. It used in this paper to rank the wind turbines. The EDAS method is an effective and efficient to solve the problems with conflicting criteria. It used to rank the wind turbines.

With this kind of problem these four methods are not used before with other. So in this work integrate the entropy weight, WSM, VIKOR, TOPSIS and EDAS with the BNNs as an innovation to select best wind turbines to help the government of Egypt to build a new wind farm in the government red sea. This a MCDM model is used to rank the wind turbine by taking into account different criteria and turbines.

The rest of this paper was organized as follow: The literature review is presented in section 2. Section 3 presented the methodology of this paper. The case study is presented in section 4. The comparative analysis is performed in section 5. In section 6 the sensitivity analysis. Finally the conclusions of this study is presented in section 7.

2. Review of Literature

The position and importance of wind turbines is increased due to the several number of needs and usage of wind energy. Researcher have many works in technical structure and design the wind turbines due to it is the vital part to produce the wind energy [17]. Although, the works in selection wind turbines problem are relatively insufficient [18-20]

Rosales et al. compare wind turbines based on the energy cost using two variables hub height and total efficiency due to number of non-experts choose the wind turbines based on the commercial offers. The main drawbacks in their work dataset that signifies only a subclass of the total population of commercialized horizontal axis wind turbines [21]. Sedaghata et al. discuss a new strategy for the wind turbines selection problem. They depend on three variables the capacity, annual production of energy and electricity cost. The main results found that wind turbines with lower rated power will reduce the cost of electricity and wind turbines with greater rated power will produce greater capacity and annual production of energy. The main drawbacks of their study not used many of criteria they depend only three criteria [22] .

The selection wind turbine problem is contain the uncertainty information. So proposed the fuzzy theory to deal with uncertainty. Pang et al. proposed in their study fuzzy theory to overcome the uncertainty and vague information [23]. But the fuzzy theory has limitations. The main limitations of fuzzy theory not deal with indeterminacy value. So, the neutrosophic sets is proposed in this study to overcome the uncertainty information. The main advantage of neutrosophic sets that deal with the indeterminacy value. It has three value truth, indeterminacy and false [24]. The neutrosophic sets has many generalizations like Bipolar Neutrosophic Sets (BNSs). Abdel-Basset et al. proposed the BNSs for professional selection problem [25]. Broumi et al. proposed the BNSs for shortest path problem. [26] Based on this, no previous study used the BNSs for selection wind turbine problem. So in this paper proposed the BNSs for overcome the uncertainty information in selection wind turbine problem. Using concept the MCDM for dealing with different and conflict criteria.

The studies in wind turbines selections using MCDM methods is relatively few[9]. The analytical hierarchy process (AHP) approach is the commonly used in wind turbines selection problem[3, 20]. The AHP method has many advantage as build the pairwise comparison and check the consistency to test consistent the opinions of the decision makers. Also it has disadvantage as biased pairwise and complexity. In this study used the entropy weight method to calculate the weights of criteria. It is not used before in the previous study with the selection wind turbines problem. But used into another fields. Wang et al. used the entropy weight method with the Pythagorean fuzzy for valuation the express quality of service. The main limitation in their study that not into consideration the indeterminacy value [27]. Zeng et al. used the entropy weight method to sustainable supplier selection with single value neutrosophic sets [28]. Xiao et al. used the entropy weight method with fuzzy theory for assessment the urban taxi-carpooling matching schemes [29]. So in this study used the entropy weight method to calculate the weights of criteria due to has many advantage as deal with uncertainty, compute the degree of confusion and less entropy value can produce more of information.

There are many MCDM methods to calculate the best alternatives (wind turbines). WSM is one of the simplest and mostly widely used MCDM methods. Rehman and Khan used the WSM for selection best wind turbine. They used five criteria and eighteen turbines. They used the C++ program to perform simulation [1]. Yörükoğlu and Aydın used the MULTIMOORA method to select wind turbines[17].

VIKOR method is used to solve decision making problems with conflict and different units of criteria. The main advantage from this method that focus on the basic information as result this, reduce the computational complexity [30]. VIKOR method is not used in previous selection wind turbine problem. Abdel-Basset et al. used the VIKOR method for assessment the performance financial of manufacturing industries [31]. Li et al. used the VIKOR method for selection machine tool [32]. Krishankumar et al. used the VIKOR method for problem of personnel selection [33].

TOPSIS method is a common MCDM methods. It is used for calculate the best alternatives. It is used for solving MCDM problems in several areas. The main concept of TOPSIS is that the highest alternative rank should have the lower distance from the positive ideal solution [34]. The TOPSIS method is used in wind turbine selection problem. Supciller et al. used the TOPSIS method for determine the best wind turbine with case study in Turkey. They used the single value neutrosophic set with twenty one criteria [24]. Ahmet et al. used the AHP-TOPSIS to with hesitant fuzzy for assessment wind turbines. The main limitation sin their study that is not take into considerations the indeterminacy value [3].

EDAS method is also a MCDM methods. It is used for solving decision making problems and determine the best alternatives. It is easy and useful for applying to different conflicting criteria. The main rule for this method that is the best alternative is computed by shortness distance from the average solutions [34]. Supciller et al. used the EDAS method to select best turbine for a case study in Turkey [24]. Kahraman et al. used the EDAS method with the Intuitionistic fuzzy for selection solid waste disposal site problem [35].

So in this work discuss many of criteria that conflict with others for wind turbines selection problem. Used the entropy weight method to calculate the weights of criteria for the first time in this problem. Used the WSM, VIKOR, TOPSIS and EDAS to select best turbine. The VIKOR method is used the first time in selection wind turbine problem.

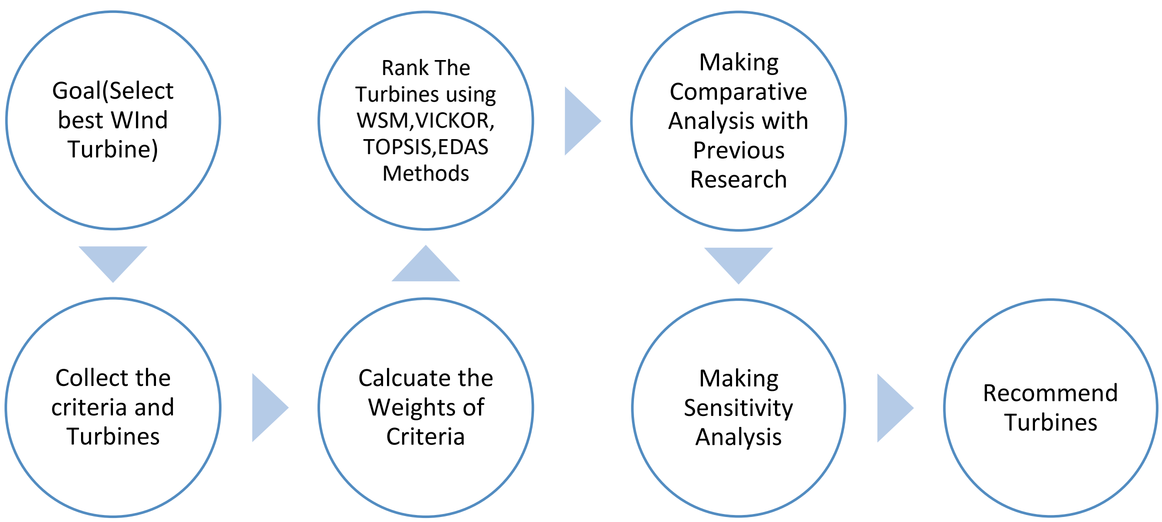

Fig 1. The framework for this study

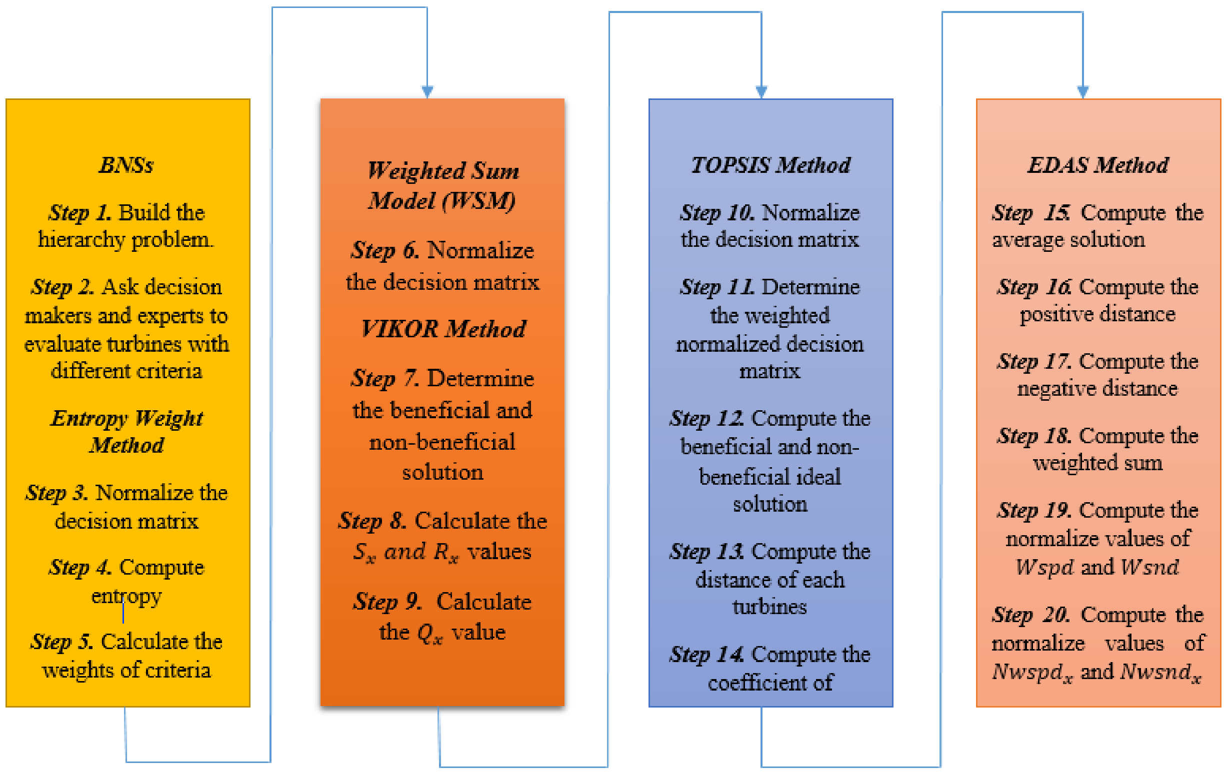

Fig 2. The methodology for this paper

3. Methodology

This paper introduced the integrate BNSs with a MCDM entropy weight method for selection best wind turbine to build a new farm in Egypt. The entropy weight method is used to determine the weights of all criteria. Then used the WSM, VICKOR, TOPSIS and EDAS to rank the wind turbines. Then the best wind turbine is recommended. Fig 1. Show the framework for this study. .Fig 2. Show the methodology for this study. The steps of methodology is presented as follow:

3.1 Bipolar Neutrosophic Sets (BNSs)

In this sub section, suggested linguistic information of BNNs and the functions of score, accuracy and certainty. Bipolar Neutrosophic sets are suggested to solve the MCDM problems. BNNs are consist from Truth (T^+, T^-), Indeterminacy (I^+, I^-) and False (F^+, F^-) where T^+, I^+, F^+ [0,1] are positive and $T^-, I^-, F^- \rightarrow$ [0,1] are negative. Table 1 show the Linguistic variable and scale of BNNs. Where Very perfect (linguistic term) is the highest value and very Bad (linguistic term) is the lowest value. The score, accuracy and certainty functions are shown in the following Eqs. (1, 2, 3)[25]:

$$\tilde{R}(\widetilde{C_1}) = (T_1^+ + 1 - I_1^+ - F_1^+ + 1 + T_1^- - I_1^- - F_1^-)/6 \quad (1)$$

$$\tilde{C}(\widetilde{C_1}) = (T_1^+ - F_1^+ + T_1^- - F_1^-) \quad (2)$$

$$\tilde{E}(\widetilde{C_1}) = (T_1^+ + F_1^-) \quad (3)$$

The steps of BNSs is presented as follow:

Step 1. Build the hierarchy problem.

The main goal form this study that select best wind turbine. Then collect the main and sub criteria, where u refers to the criteria ($u = 1, 2, 3, 4, \dots \dots x$) and x refers to number of criteria. Then determine wind turbines (Alternatives), where v refers to turbines ($v = 1,2,3,, \dots \dots y$) and y refers to number of turbines.

Step 2. Ask decision makers and experts to evaluate turbines with different criteria.

Building the decision matrix between criteria turbines with the opinions of experts by using scale of BNNs in Table 1 by Eq. (4). Then Deneutrosophic the BNNs by Eq. (1) to obtain one value instead of six value. Then aggregate the decision matrix of opinions experts into one matrix by Eqs (5, 6).

$$\mathrm{P^D} = \begin{bmatrix} \mathrm{P_{11}^D} & \cdots & \mathrm{P_{1u}^D} \\ \vdots & \ddots & \vdots \\ \mathrm{P_{v1}^D} & \cdots & \mathrm{P_{vu}^D} \end{bmatrix} \quad (4)$$

Where, D indicates to number of experts.

$$P_{xy} = \frac{\sum_{D=1}^{D} P_{uv}}{D} \quad (5)$$

$$\mathrm{P} = \begin{bmatrix} \mathrm{P_{11}} & \cdots & \mathrm{P_{1x}} \\ \vdots & \ddots & \vdots \\ \mathrm{P_{y1}} & \cdots & \mathrm{P_{vu}} \end{bmatrix} \quad (6)$$

3.2 *Proposed The MCDM Methods*

The following steps for entropy, WSM, VIKOR, TOPSIS and EDAS methods.

3.2.1 *Entropy Weight Method*

Entropy weight method is used to determine the weights of criteria. The following steps show the entropy weight[36]:

Step 3. Normalize the decision matrix

Start with the decision matrix with aggregated the opinion of experts. Then normalize the aggregation decision matrix using Eq. (7).

$$N_{xy} = \frac{\mathrm{P_{xy}}}{\sum_{y=1}^{v} \mathrm{P_{xy}}} \tag{7}$$

Step 4. Compute entropy

The entropy is computed by the multiply the ln of normalized decision matrix by normalized decision matrix then compute the summation of it. Finally multiply this summation by the negative L by using Eq. (8):

$$O_x = -\mathrm{L} \sum_{y=1}^{v} N_{xy} \ln N_{xy} \tag{8}$$

Where L = $1/\ln(y)$

Step 5. Calculate the weights of criteria using Eq. (9)

$$W_x = \frac{1-O_x}{\sum_{y=1}^{u}(1-O_x)} \tag{9}$$

3.2.2 *Weighted Sum Model (WSM)*

Step 6. Normalize the decision matrix***[36]***

Start with the aggregation decision matrix and multiply each weight by the value of decision matrix and then obtain the normalization matrix by using Eq. (10). Then ranking the turbines descanting according to normalize value

$$Z_x = \sum_{y=1}^{u} W_x P_{xy} \tag{10}$$

3.2.3 VIseKriterijumska Optimizacija I Kompromisno Resenje *(VIKOR)*

VIKOR method is used to rank turbines with different conflict criteria. The following steps of VIKOR method [36].

Step 7. Determine the beneficial-ideal solution (B^{+}) and non-beneficial-ideal solution (B^{-}) using Eqs. (11, 12)

$$B_x^{+} = \max_x P_{xy} \text{ for Positive criteria and } B_x^{+} = \min_x P_{xy} \text{ for negative criteria} \quad (11)$$

$$B_x^{-} = \min_x P_{xy} \text{ for Positive criteria and } B_x^{-} = \max_x P_{xy} \text{ for negative criteria} \quad (12)$$

Step 8. Calculate the S_x *and* R_x values using Eqs. (13, 14)

$$\mathrm{S_x} = \sum_{y=1}^{v}(W_y * \frac{B_x^{+} - P_{xy}}{B_x^{+} - B_x^{-}}) \quad (13)$$

$$\mathrm{R_x} = \max_y(W_y * \frac{B_x^{+} - P_{xy}}{B_x^{+} - B_x^{-}}) \quad (14)$$

Step 9. Calculate the Q_x value using Eq. (15). Then rank the turbines ascending to value of Q_x.

$$Q_x = h\left(\frac{\mathrm{S_x} - \min_x \mathrm{S_x}}{\max_x \mathrm{S_x} - \min_x \mathrm{S_x}}\right) + (1-h)\left(\frac{\mathrm{R_x} - \min_x \mathrm{R_x}}{\max_x \mathrm{R_x} - \min_x \mathrm{R_x}}\right) \quad (15)$$

Value of h refers to highest group utility of strategy weight and (1-h) refers to individual regret of weight. Usually, the value of h is equal to 0.5 and the value of h can be range from 0 to 1.

4.2.3 Technique for Order Performance by Similarity to Ideal Solution (TOPSIS)

The steps of TOPSIS method is presented as follow[36]:

Step 10. Normalize the decision matrix

Start with the aggregation decision matrix between criteria and turbines. Then normalize the decision matrix using Eq. (16)

$$N_{xy} = \frac{P_{xy}}{\sqrt{\sum_{y=1}^{v} P_{xy}^2}} \quad (16)$$

Step 11. Determine the weighted normalized decision matrix

Multiply the weights of criteria by the normalize decision matrix to calculate the weighted normalized decision matrix using Eq. (17).

$$I_{xy} = N_{xy} W_y \quad (17)$$

Step 12. Compute the beneficial ideal solution (f^{+}) and non-beneficial ideal solution (f^{-}) using Eqs. (18, 19)

$$f_x^{+} = \max_x P_{xy} \text{ for Positive criteria and } f_x^{+} = \min_x P_{xy} \text{ for negative criteria} \quad (18)$$

$$f_x^{-} = \min_x P_{xy} \text{ for Positive criteria and } f_x^{-} = \max_x P_{xy} \text{ for negative criteria} \quad (19)$$

Step 13. Compute the distance of each turbines from beneficial and non-beneficial ideal solution by using Eqs. (20, 21)

$A_y^+ = \sum_x^u (I_{xy} - f_x^+)^2$ for positive criteria (20)

$A_y^- = \sum_x^u (I_{xy} - f_x^-)^2$ for cost criteria (21)

Step 14. Compute the coefficient of closeness

From the distance of each turbine, calculate the value of closeness coefficient using Eq. (22). Then rank turbine according the descending order of value closeness coefficient.

$$\mathrm{G_y} = \frac{A_y^-}{A_y^+ + A_y^-} \quad (22)$$

4.2.4 Evaluation based on Distance from Average Solution (EDAS)

The steps of EDAS method is presented as follow[24]:

Step 15. Compute the average solution

Start with the aggregation decision matrix. Then compute the average solution by divide the value of decision matrix by the number of turbines using Eq. (23)

$$vg_{\mathrm{y}} = \frac{\sum_{x=1}^{b} p_{xy}}{b} \quad (23)$$

Step 16. From the average solution compute the positive distance for positive and cost criteria using Eqs. (24,25)

$$\mathrm{Pos}_{xy}^+ = \frac{\max(0,(p_{xy} - vg_{\mathrm{y}}))}{vg_{\mathrm{y}}} \quad \text{For positive criteria} \quad (24)$$

$$\mathrm{Pos}_{xy}^- = \frac{\max(0,(vg_{\mathrm{y}} - p_{xy}))}{vg_{\mathrm{y}}} \quad \text{For cost criteria} \quad (25)$$

Step 17. From the average solution compute the negative distance for positive and cost criteria using Eqs. (26,27)

$$\mathrm{Neg}_{xy}^+ = \frac{\max(0,(vg_{\mathrm{y}} - p_{xy}))}{vg_{\mathrm{y}}} \quad \text{For positive criteria} \quad (26)$$

$$\mathrm{Neg}_{xy}^- = \frac{\max(0,(p_{xy} - vg_{\mathrm{y}}))}{vg_{\mathrm{y}}} \quad \text{For cost criteria} \quad (27)$$

Step 18. Compute the weighted sum of positive distance

From the positive distance for positive and negative criteria multiply the weight of criteria by the positive distance and compute the sum of this multiplication using Eqs. (28, 29)

$$Wspd_x = \sum_{y=1}^{v} W_y \, \mathrm{Pos_{xy}} \quad (28)$$

$$Wsnd_x = \sum_{y=1}^{v} W_y \, \mathrm{Neg_{xy}} \quad (29)$$

Step 19. Compute the normalize values of $Wspd$ and $Wsnd$ using Eqs. (30,31)

$$Nwspd_x = \frac{Wspd_x}{\max(Wspd_x)} \quad (30)$$

$$Nwsnd_x = 1 - \frac{Wsnd_x}{\max(Wsnd_x)} \quad (31)$$

Step 20. Compute the normalize values of $Nwspd_x$ and $Nwsnd_x$

After compute the value of nor_x rank turbines according descending order of value nor_x using Eq. (32)

$$nor_x = 0.5 * (Nwspd_x + Nwsnd_x) \quad (32)$$

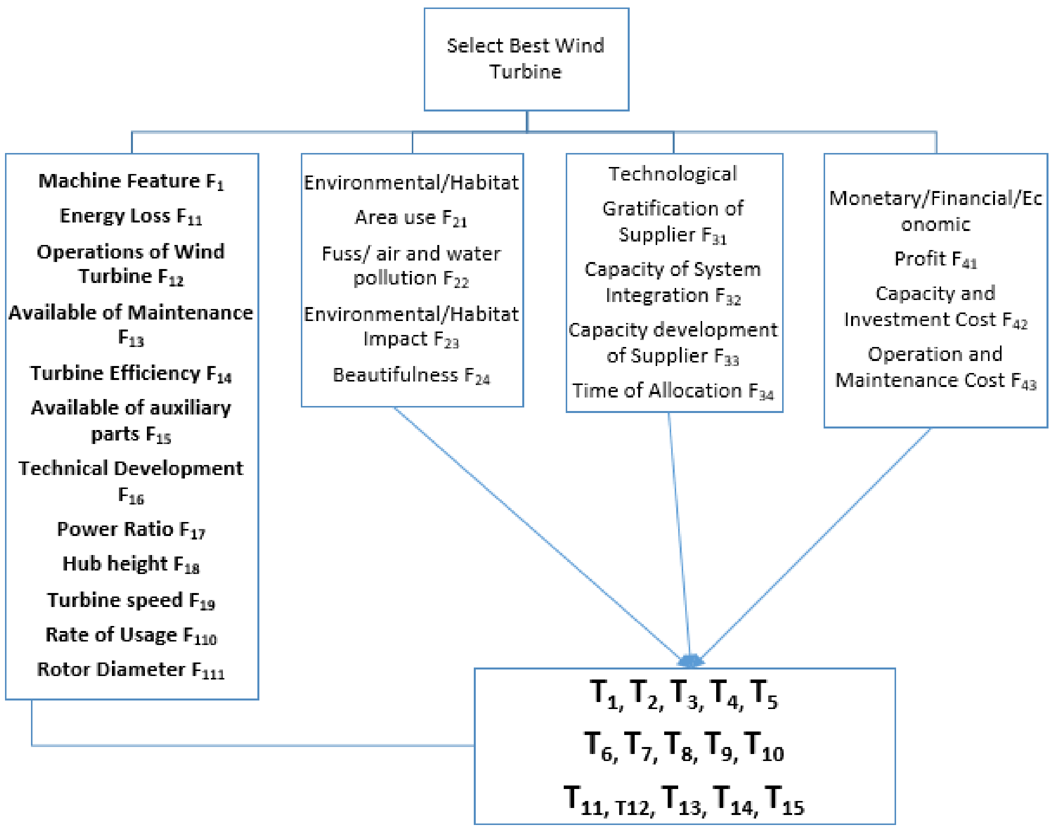

Fig 3. The structure between criteria, sub criteria and turbines

4. Case Study

Egypt vision in 2030 depend on decreasing using the fossil fuels and increasing using the renewable energy. One of the important renewable energy is wind turbine. The choice the wind turbines is the important issue. So in this study choice the best wind turbine to help Egypt vision to build different wind farms.

Start this study by collect a collection of decision makers and experts. This collection includes of four people working in companies of renewable energy in Egypt. Two of this group working as a manger on renewable energy. The other two experts working as mechanical engineering. Two of them have a PHD degree in engineering and others have a master degree in engineering. The all of these experts and decision makers have a weighty degree in expertise. All these decision makers have the same weight of degree. Making interview with these decision makers for recognizing the criteria and alternatives with their opinions.

Making into considerations the different types of wind turbines. The fifteen alternatives (wind turbines) are selected. T_1 V164-9.5MW, T_2 SG 8.0-167 DD, T_3 GW154 6.7MW, T_4 Senvion 6.2M152, T_5 GE Haliade 150-6MW, T_6 Ming Yang SCD 6.0, T_7 Doosan WindS500,T_8 Hitachi HTW5.2-136, T_9 H151-5.0MW, T_{10} AD 5-135, T_{11} E-126 7.580, T_{12} Haliade-X, T_{13} SG 11.0-193 DD Flex, T_{14} D10000-185, T_{15} V164-10.0. The criteria and sub criteria are identified and collected based on the survey of literature. The opinions of decision makers and experts is presented based on the BNSs. Fig 3 show the criteria, sub criteria and alternatives for this study. The criteria is divided to positive and negative (cost) criteria. The F_{11}, F_{21}, F_{22}, F_{23}, F_{34}, F_{42}, F_{43} criteria are negative and the rest of criteria are positive.

The entropy weight method is used to compute the weights of criteria. Then used the WSM, VIKOR, TOPSIS and EDAS methods are used to rank the turbines (alternatives).

4.1 Computing the weights of criteria by entropy weight method

The group of decision makers and experts assets the criteria to compute the importance of the criteria by entropy weight method. First the linguistic term is introduced to four decision makers to build the decision matrix. Then, replace the linguistic term by BNNs in Table 1. The opinions of four experts is used to build the decision matrix by using Eq. (4). Then, convert the BNNs into the crisp value (one value instead of six value of BNNs) by using Eq. (1). Hence, have the four decision matrix so, need to aggregate it into one matrix by using Eqs. (5,6) in Table 2.

The steps of entropy weight method is applied in next stage. Start with the aggregated decision matrix between the criteria and turbines (alternatives). First normalize the aggregated decision matrix by using

Eq. (7). Then, compute the entropy by using Eq. (8). Finally the weights of main and sub criteria is computed by using Eq. (8). In Table 3.the ranking and weights of main and sub criteria.

The results of entropy weight show the importance of the criteria and sub criteria between other. The Machine feature (F_1) is the highest important main criteria equal 0.42302 then Technological (F_3) is after machine feature with value 0.19949 then, importance of Monetary criteria (F_4) is lower than technological criteria, then the Habitat criteria (F_2) is the lowest criteria in four main criteria.

The results of sub criteria show that the operation and maintenance cost (F_{43}) with value 0.08431 is the highest weight in sub criteria and the technical development (F_{16}) with value 0.02093 is the lowest weight in sub criteria.

Table 1. Scale of BNSs.

Linguistic term	**BNNs**	
Linguistic Variable	**T_1^+,I_1^+,F_1^+,**	**T_1^-,I_1^-,F_1^-**
Very Bad	<0.1,0.9,0.8,	-0.8,-0.3,-0.1>
Bad	<0.3,0.5,0.7,	-0.6,-0.4,-0.4>
Medium	<0.45,0.45,0.5,	-0.45,-0.5.-0.45>
Perfect	<0.7,0.3,0.4,	-0.3,-0.6,-0.8>
Very Perfect	<0.9,0.1,0.2,	-0.2,-0.7,-0.9>

Table 2. The aggregated decision matrix between criteria and turbines (alternatives)

Criteria/turbines	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}	F_{19}	F_{110}	F_{111}
T_1	0.683	0.500	0.383	0.683	0.683	0.833	0.500	0.683	0.500	0.500	0.683
T_2	0.833	0.833	0.683	0.500	0.683	0.833	0.383	0.683	0.683	0.500	0.833
T_3	0.683	0.383	0.833	0.683	0.500	0.833	0.383	0.833	0.683	0.683	0.833
T_4	0.383	0.683	0.833	0.383	0.683	0.683	0.167	0.833	0.833	0.833	0.833
T_5	0.683	0.833	0.383	0.683	0.383	0.683	0.683	0.683	0.833	0.833	0.683
T_6	0.833	0.683	0.383	0.383	0.383	0.500	0.383	0.383	0.683	0.683	0.383
T_7	0.833	0.833	0.167	0.683	0.833	0.683	0.167	0.833	0.383	0.833	0.167
T_8	0.833	0.383	0.683	0.383	0.833	0.383	0.683	0.683	0.683	0.383	0.383
T_9	0.833	0.167	0.683	0.683	0.383	0.833	0.833	0.683	0.500	0.167	0.833
T_{10}	0.683	0.383	0.383	0.683	0.683	0.833	0.383	0.383	0.500	0.683	0.683
T_{11}	0.833	0.683	0.683	0.833	0.383	0.683	0.167	0.500	0.683	0.833	0.383
T_{12}	0.833	0.683	0.383	0.683	0.833	0.383	0.383	0.500	0.833	0.383	0.500
T_{13}	0.683	0.833	0.167	0.383	0.683	0.683	0.833	0.833	0.683	0.383	0.500
T_{14}	0.683	0.383	0.683	0.833	0.383	0.167	0.683	0.683	0.683	0.683	0.683
T_{15}	0.683	0.383	0.683	0.383	0.683	0.833	0.383	0.833	0.833	0.683	0.683
Criteria/turbines	F_{21}	F_{22}	F_{23}	F_{24}	F_{31}	F_{32}	F_{33}	F_{34}	F_{41}	F_{42}	F_{43}
T_1	0.500	0.500	0.683	0.500	0.833	0.683	0.500	0.383	0.683	0.500	0.500
T_2	0.500	0.383	0.833	0.500	0.683	0.833	0.383	0.500	0.683	0.683	0.383
T_3	0.683	0.383	0.683	0.683	0.383	0.833	0.383	0.167	0.833	0.683	0.383

T_4	0.383	0.833	0.683	0.683	0.167	0.683	0.167	0.683	0.833	0.383	0.167
T_5	0.167	0.683	0.383	0.833	0.383	0.383	0.683	0.833	0.500	0.167	0.167
T_6	0.383	0.383	0.383	0.833	0.683	0.383	0.683	0.683	0.500	0.383	0.683
T_7	0.683	0.833	0.167	0.833	0.683	0.167	0.683	0.683	0.833	0.683	0.683
T_8	0.683	0.383	0.167	0.683	0.500	0.383	0.833	0.500	0.383	0.833	0.833
T_9	0.683	0.683	0.683	0.683	0.500	0.683	0.833	0.500	0.167	0.833	0.833
T_{10}	0.833	0.383	0.683	0.500	0.683	0.683	0.500	0.683	0.683	0.500	0.833
T_{11}	0.383	0.833	0.500	0.500	0.683	0.833	0.500	0.683	0.683	0.500	0.383
T_{12}	0.383	0.683	0.500	0.383	0.167	0.833	0.500	0.833	0.833	0.683	0.383
T_{13}	0.500	0.383	0.683	0.167	0.383	0.683	0.683	0.383	0.500	0.683	0.167
T_{14}	0.500	0.833	0.833	0.383	0.500	0.683	0.683	0.167	0.500	0.833	0.167
T_{15}	0.683	0.683	0.683	0.683	0.683	0.167	0.833	0.500	0.683	0.683	0.683

Table 3. Final weights and ranking for the main and sub-criteria.

Main criteria	Weights	Rank	Sub-criteria	Weights	Rank
Machine Feature F_1	0.423017	1	Energy Loss F_{11}	0.021954	1
			Operations of Wind Turbine F_{12}	0.049721	8
			Available of Maintenance F_{13}	0.030774	21
			Turbine Efficiency F_{14}	0.042039	15
			Available of auxiliary parts F_{15}	0.031543	20
			Technical Development F_{16}	0.020927	22
			Power Ratio F_{17}	0.052603	6
			Hub height F_{18}	0.041303	16
			Turbine speed F_{19}	0.039383	17
			Rate of Usage F_{110}	0.049055	9
			Rotor Diameter F_{111}	0.043716	14
Environmental/Habitat F_2	0.18805	4	Area use F_{21}	0.048228	10
			Fuss/ air and water pollution F_{22}	0.03899	19
			Environmental/Habitat Impact F_{23}	0.053389	5
			Beautifulness F_{24}	0.047438	11
Technological F_3	0.199494	2	Gratification of Supplier F_{31}	0.062704	3
			Capacity of System Integration F_{32}	0.039025	18
			Capacity development of Supplier F_{33}	0.051427	7
			Time of Allocation F_{34}	0.046337	12
Monetary F_4	0.189444	3	Profit F_{41}	0.06014	4
			Capacity and Investment Cost F_{42}	0.044995	13
			Operation and Maintenance Cost F_{43}	0.084309	2

4.2 Rank Turbines

The wind turbines is ranked by the SWM, VIKOR, TOPSIS, EDAS methods. First Apply the WSM method.

The WSM is applied to rank wind turbines. Start with the aggregated decision matrix in Table 2. Then applied Eq. (10) to obtain final rank by multiply the weights of criteria by the value of aggregated decision matrix. The rank wind turbines by WSM method is presented in Table 4.

The results of WSM method show that T_9 is the highest rank with value 0.6126 and T_6 is the lowest rank with value 0.50064.

Table 4. The rank of turbines by WSM method

Turbines/Rank	Values	Rank	Total Points
T_1	0.594988	T_9	12
T_2	0.564966	T_7	8
T_3	0.580817	T_{10}	10
T_4	0.538728	T_1	3
T_5	0.555143	T_{15}	6
T_6	0.500635	T_3	1
T_7	0.603511	T_{13}	14
T_8	0.557474	T_2	7
T_9	0.612693	T_8	15
T_{10}	0.600935	T_5	13
T_{11}	0.549546	T_{11}	5
T_{12}	0.541628	T_{12}	4
T_{13}	0.571428	T_4	9
T_{14}	0.502519	T_{14}	2
T_{15}	0.592822	T_6	11

The second method (VIKOR) is applied to rank the turbines. First start with the aggregated decision matrix in Table 2. Then compute the beneficial-ideal solution (B^+) and non-beneficial-ideal solution (B^-) for positive and negative criteria by using Eqs. (11,12). Then the value of S_x is computed by using Eq. (13). Then compute the value of R_x by using Eq. (14). Finally applying Eq. (15) to compute the value of Q_x. Based on this, the rank of turbines is ordered ascending by value of Q_x. Table 5 presented the values of S_x, R_x, Q_x and ranking of turbines.

The results from applying the VIKOR method show that the T_2 is the highest rank with value 0.12725 and the T_9 is the lowest rank with value 1.

Table 5. The values of S_x, R_x, Q_x and rank of turbines by VIKOR method

Turbines/Rank	S_x	R_x	Q_x	Rank	Total Points
T_1	0.54934	0.066592	0.698403	T_2	4
T_2	0.40505	0.047397	0.127254	T_5	15

T_3	0.451533	0.04898	0.250675	T_4	11
T_4	0.353708	0.062704	0.215514	T_7	13
T_5	0.426901	0.046337	0.161515	T_3	14
T_6	0.443595	0.073311	0.553539	T_{14}	7
T_7	0.454352	0.047653	0.239425	T_{13}	12
T_8	0.489558	0.07881	0.727372	T_{11}	3
T_9	0.580291	0.084309	1	T_6	1
T_{10}	0.533824	0.07209	0.736572	T_{12}	2
T_{11}	0.515349	0.052603	0.4392	T_{15}	8
T_{12}	0.513857	0.062704	0.568914	T_1	6
T_{13}	0.46983	0.053389	0.349105	T_8	9
T_{14}	0.461978	0.051427	0.305942	T_{10}	10
T_{15}	0.548904	0.066592	0.697443	T_9	5

The third method (TOPSIS) is applied to rank turbines. Start with the combined decision matrix in Table 2. Then compute the normalized decision matrix by using Eq. (16). From the normalized decision matrix the Eq. (17) is applied to compute the weighted normalized decision matrix. Then compute the value of beneficial-ideal solution and non-beneficial-ideal solution for positive and negative criteria by using Eqs. (18,19). Then Applying Eqs. (20,21) to compute the distance of each turbine from beneficial and non-beneficial for positive and negative criteria. Finally Applying Eq. (22) for the compute the value of coefficient closeness G_y. The rank of turbines is computed descending by the value of G_y. In Table 6 the values of $A_y^+, A_y^- G_y$ and rank of turbines is presented.

The results of TOPSIS method show that the T_2 is the highest rank with value 0.6248 and T_9 is the lowest rank with value 0.4185.

Table 6. The values of $A_y^+, A_y^- G_y$ rank of turbines by TOPSIS method

Turbines/Rank	A_y^+	A_y^-	G_y	Rank	Total Points
T_1	0.031261	0.027524	0.468217	T_2	4
T_2	0.021399	0.035644	0.624863	T_4	15
T_3	0.023555	0.032831	0.582246	T_5	12
T_4	0.025083	0.037006	0.596013	T_3	14
T_5	0.023556	0.034485	0.594151	T_{14}	13
T_6	0.029447	0.031434	0.516321	T_7	8
T_7	0.026516	0.031088	0.539692	T_{11}	10
T_8	0.03366	0.027441	0.449111	T_6	2
T_9	0.036287	0.026118	0.418525	T_{13}	1
T_{10}	0.031895	0.028337	0.470463	T_{12}	5
T_{11}	0.028071	0.030586	0.521442	T_{10}	9

T_{12}	0.02889	0.028215	0.494095	T_1	6
T_{13}	0.02753	0.029085	0.513738	T_{15}	7
T_{14}	0.025255	0.033744	0.571948	T_8	11
T_{15}	0.030396	0.025705	0.458189	T_9	3

The fourth method (EDAS) is applied to obtain the rank of turbines. First start with aggregated decision matrix in Table 2. Then compute the average solution by using Eq. (23). Then compute the positive distance for positive and negative criteria by using Eqs. (24,25). Then compute the negative distance for positive and negative criteria by using Eqs. (26,27). Then compute the weighted sum of positive distance and negative distance by using Eqs. (28,29). Then compute the normalize value for weighted sum of positive ($NWSPd_x$) and negative distance ($NWSnd_x$) by using Eqs. (30,31) in Table 7. Finally compute the normalized value (Nor_x) for ($NWSPd_x, NWSnd_x$) by using Eqs. (32,33) in Table 7. The final rank is computed based on descending value of Nor_x in Table 7.

The results of EDAS method show that the T_4 is the highest rank with value 0.612422 and the T_3 is the lowest rank with value 0.4435.

Table 7. The values of $NWSPd_x, NWSnd_x, Nor_x$ and rank of turbines by EDAS method

Turbines/Rank	$NWSPd_x$	$NWSnd_x$	Nor_x	Rank	Total Points
T_1	0.44604	0.778805	0.612422	T_4	8
T_2	0.788992	0.445451	0.617222	T_9	10
T_3	0.510227	0.376734	0.44348	T_8	1
T_4	1	0.559512	0.779756	T_6	15
T_5	0.707808	0.501764	0.604786	T_{11}	7
T_6	0.70301	0.660142	0.681576	T_2	12
T_7	0.584757	0.534367	0.559562	T_{10}	5
T_8	0.607156	0.816269	0.711713	T_1	13
T_9	0.454275	1	0.727137	T_5	14
T_{10}	0.463	0.763534	0.613267	T_{14}	9
T_{11}	0.525765	0.716991	0.621378	T_7	11
T_{12}	0.402231	0.611706	0.506969	T_{12}	4
T_{13}	0.485332	0.508127	0.49673	T_{13}	3
T_{14}	0.599665	0.532189	0.565927	T_{15}	6
T_{15}	0.270496	0.62297	0.446733	T_3	2

Finally is this section make combination rank for four methods by total points. The concept of total points is applied as if the T_1 is the highest rank take 15 points and lowest rank take 1 points and so on. Table 8. Show the combined rank of four methods[36].

The results of combined four method show that the T_2 is the highest rank with highest total points and T_{12} is the lowest rank with lowest total points

Table 8. The combined rank of four methods.

Turbines/Rank	Total Points	Rank
T_1	28	T_2
T_2	48	T_4
T_3	34	T_7
T_4	45	T_5
T_5	40	T_3
T_6	28	T_{11}
T_7	41	T_9
T_8	25	T_{10}
T_9	31	T_{14}
T_{10}	29	T_1
T_{11}	33	T_6
T_{12}	20	T_{13}
T_{13}	28	T_8
T_{14}	29	T_{15}
T_{15}	21	T_{12}

5. Comparative analysis

In this section making the comparative analysis to test the reliability of this proposed methodology. Making two comparative analysis with SVNSs and Hesitant Fuzzy sets as follow:

5.1 Comparison by Single Valued Neutrosophic Sets

Aliye Ayca Supciller and Fatih Toprak[24] used SWARA, TOPSIS and EDAS methods to select best wind turbines. The SWARA method is used to calculate the weights of criteria. So, make comparison between SWARA and entropy weight method (method in this study).

The results of SWARA show that the F_1 = 0.4029, F_2 = 0.12241, F_3 = 0.30441, F_4 = 0.17069. Table 9. Show the weights of main criteria and Table 10. Show the weights of sub criteria by the entropy weight and

SWARA method. Results show that, in main criteria the highest weight by SWARA method is F_1 and the lowest weight is F_2, the highest weight by entropy weight method is F_1 and the lowest weight is F_2. In sub criteria the highest weights by SWARA is F_{16} and lowest weights is F_6 and highest weight by entropy is F_{22} and the lowest weight is F_6.

In ranking the turbine, make comparison between SVNSs TOPSIS and EDAS with BNSs TOPSIS, WSM, VIKOR and EDAS methods. By using the weights of SWARA and entropy weight methods the turbines is ranked. Table 11. Show the ranking by comparison study. Results show that, In SVNSs TOPSIS the T_2 is the highest rank and T_9 is the lowest rank. In SVNSs EDAS method, T_4 the highest rank and T_{13} is the lowest rank. In BNSs WSM method T_9 is the highest rank and T_6 is the lowest rank. In BNSs TOPSIS the highest rank is T_2 and the lowest rank is T_9. In BNSs VIKOR the T_2 is the highest rank and T_9 is the lowest rank. In BNSs T_4 is the highest rank and T_3 is the lowest rank.

Table 9. The weights of main criteria by entropy and SWARA methods.

Criteria/Rank	SWARA	Rank by SWARA method	Entropy weight	Rank by the entropy weight
F_1	0.40249	F_1	0.423017	F_1
F_2	0.12241	F_3	0.188045	F_3
F_3	0.30441	F_4	0.199494	F_4
F_4	0.17069	F_2	0.189444	F_2

Table 10. The rank weights of sub criteria by SWARA and entropy weight methods

Criteria/Rank	SWARA	Rank of SWARA	Entropy weight	Rank of entropy weight
F_{11}	0.002027	F_{16}	0.021954	F_{22}
F_{12}	0.098079	F_7	0.049721	F_{16}
F_{13}	0.003187	F_{22}	0.030774	F_{20}
F_{14}	0.022589	F_2	0.042039	F_{14}
F_{15}	0.005242	F_{14}	0.031543	F_7
F_{16}	0.001137	F_{10}	0.020927	F_{18}
F_{17}	0.150387	F_{18}	0.052603	F_2
F_{18}	0.012434	F_{20}	0.041303	F_{10}
F_{19}	0.009173	F_{11}	0.039383	F_{12}
F_{10}	0.061087	F_{12}	0.049055	F_{15}
F_{11}	0.037147	F_{19}	0.043716	F_{19}
F_{21}	0.031047	F_4	0.048228	F_{21}
F_{22}	0.004184	F_{21}	0.03899	F_{11}
F_{23}	0.077627	F_{17}	0.053389	F_4
F_{24}	0.009553	F_8	0.047438	F_8
F_{31}	0.200746	F_{15}	0.062704	F_9
F_{32}	0.014744	F_9	0.039025	F_{17}

F_{33}	0.059922	F_5	0.051427	F_{13}
F_{34}	0.028996	F_{13}	0.046337	F_5
F_{41}	0.038642	F_3	0.06014	F_3
F_{42}	0.018698	F_1	0.044995	F_1
F_{43}	0.113353	F_6	0.084309	F_6

Table 11. The rank of turbines by this study methods and SVNSs TOPSIS and EDAS methods.

Turbines/Rank	SVNSs TOPSIS	SVNSs EDAS	BNSs WSM	BNSs TOPSIS	BNSs VIKOR	BNSs EDAS
T_1	T_2	T_4	T_9	T_2	T_2	T_4
T_2	T_4	T_{10}	T_7	T_4	T_5	T_9
T_3	T_5	T_8	T_{10}	T_5	T_4	T_8
T_4	T_{14}	T_9	T_1	T_3	T_7	T_6
T_5	T_3	T_{11}	T_{15}	T_{14}	T_3	T_{11}
T_6	T_{11}	T_6	T_3	T_7	T_{14}	T_2
T_7	T_7	T_{12}	T_{13}	T_{11}	T_{13}	T_{10}
T_8	T_6	T_1	T_2	T_6	T_{11}	T_1
T_9	T_{13}	T_5	T_8	T_{13}	T_6	T_5
T_{10}	T_{12}	T_2	T_5	T_{12}	T_{12}	T_{14}
T_{11}	T_1	T_{14}	T_{11}	T_{10}	T_{15}	T_7
T_{12}	T_{10}	T_7	T_{12}	T_1	T_1	T_{12}
T_{13}	T_8	T_{15}	T_4	T_{15}	T_8	T_{13}
T_{14}	T_{15}	T_3	T_{14}	T_8	T_{10}	T_{15}
T_{15}	T_9	T_{13}	T_6	T_9	T_9	T_3

5.2 Comparison by Hesitant Fuzzy AHP and TOPSIS[3]

Making a comparison between Hesitant Fuzzy AHP-TOPSIS with this study. First Applying the AHP method to calculate the weights of main and sub criteria. Table 12. Show the comparison weights between AHP and entropy weight method. The results of comparison weight of main criteria show that, the highest weight by AHP method is F_1 and F_2 is the lowest weight. In entropy weight, the F_1 is the highest weight and F_2 is the lowest weight. The weights of sub criteria is computed and ranked in Table 13. In AHP method the F_{20} is the highest weigh in sub criteria and F_{15} is the lowest weight. In entropy weight the F_{22} is the highest weight and F_6 is the lowest weight.

After comparison with the weights of criteria, the turbines is ranked. Comparison by the Hesitant Fuzzy TOPSIS and BNSs WSM, TOPSIS, VIKOR and EDAS. The Rank of turbines is computed in Table 14. In Hesitant Fuzzy TOPSIS show that T_4 is the highest rank and T_9 is the lowest rank.

Table 12. The weights of main criteria by entropy weight and AHP methods.

Criteria/Rank	AHP	Rank by AHP method	Entropy weight	Rank by the entropy weight
F_1	0.355425	F_1	0.423017	F_1
F_2	0.131329	F_3	0.188045	F_3
F_3	0.270759	F_4	0.199494	F_4
F_4	0.242487	F_2	0.189444	F_2

Table 13. The rank weights of sub criteria by AHP and entropy weight methods

Criteria/Rank	AHP	Rank of AHP	Entropy weight	Rank of entropy weight
F_{11}	0.04962	F_{20}	0.021954	F_{22}
F_{12}	0.045957	F_{16}	0.049721	F_{16}
F_{13}	0.035905	F_{19}	0.030774	F_{20}
F_{14}	0.035779	F_{21}	0.042039	F_{14}
F_{15}	0.030804	F_{18}	0.031543	F_7
F_{16}	0.030648	F_{17}	0.020927	F_{18}
F_{17}	0.029155	F_{22}	0.052603	F_2
F_{18}	0.026058	F_1	0.041303	F_{10}
F_{19}	0.024442	F_{12}	0.039383	F_{12}
F_{10}	0.023972	F_2	0.049055	F_{15}
F_{11}	0.023085	F_3	0.043716	F_{19}
F_{21}	0.048331	F_4	0.048228	F_{21}
F_{22}	0.029932	F_{14}	0.03899	F_{11}
F_{23}	0.031933	F_5	0.053389	F_4
F_{24}	0.021132	F_6	0.047438	F_8
F_{31}	0.099028	F_{13}	0.062704	F_9
F_{32}	0.053956	F_7	0.039025	F_{17}
F_{33}	0.056504	F_8	0.051427	F_{13}
F_{34}	0.06127	F_9	0.046337	F_5
F_{41}	0.131909	F_{10}	0.06014	F_3
F_{42}	0.057006	F_{11}	0.044995	F_1
F_{43}	0.053572	F_{15}	0.084309	F_6

Table 14. The rank of turbines by this study methods and Hesitant Fuzzy TOPSIS

Turbines/Rank	Hesitant Fuzzy TOPSIS	BNSs WSM	BNSs TOPSIS	BNSs VIKOR	BNSs EDAS
T_1	T_4	T_9	T_2	T_2	T_4
T_2	T_2	T_7	T_4	T_5	T_9
T_3	T_5	T_{10}	T_5	T_4	T_8
T_4	T_7	T_1	T_3	T_7	T_6
T_5	T_3	T_{15}	T_{14}	T_3	T_{11}
T_6	T_{11}	T_3	T_7	T_{14}	T_2
T_7	T_{14}	T_{13}	T_{11}	T_{13}	T_{10}
T_8	T_{12}	T_2	T_6	T_{11}	T_1
T_9	T_{13}	T_8	T_{13}	T_6	T_5
T_{10}	T_{15}	T_5	T_{12}	T_{12}	T_{14}

T_{11}	T_6	T_{11}	T_{10}	T_{15}	T_7
T_{12}	T_{10}	T_{12}	T_1	T_1	T_{12}
T_{13}	T_1	T_4	T_{15}	T_8	T_{13}
T_{14}	T_8	T_{14}	T_8	T_{10}	T_{15}
T_{15}	T_9	T_6	T_9	T_9	T_3

6. **Sensitivity analysis**

The change criteria weights can affect rank. So needs to change weights of criteria to assess the rank of turbines. In this paper proposed five cases weights changes in Table 15[36]. In case 1 proposed equally weights important for four main criteria. The next cases based on the machine feature, environmental, technological and monetary criteria. The weights of criteria in these cases obtained by divide the weight of criteria by number of criteria (four criteria). Table 16 show the rank of turbines under different cases and methods.

In WSM method, In case 1, the T_9 is the highest turbine rank and T_{14} is the lowest turbine rank. In case 2, T_9 is the highest turbine rank and T_6 is the lowest turbine rank. In case 3, T_9 is the highest turbine rank and T_6 is the lowest turbine rank. In case 4, T_9 is the highest turbine rank and T_{14} is the lowest turbine rank. In case 5, T_{10} is the highest turbine rank and T_{14} is the lowest turbine rank.

In VIKOR method, In case 1, the T_5 is the highest turbine rank and T_9 is the lowest turbine rank. In case 2, T_4 is the highest turbine rank and T_9 is the lowest turbine rank. In case 3, T_6 is the highest turbine rank and T_1 is the lowest turbine rank. In case 4, T_{13} is the highest turbine rank and T_{12} is the lowest turbine rank. In case 5, T_4 is the highest turbine rank and T_5 is the lowest turbine rank.

In TOPSIS method, In case 1, the T_5 is the highest turbine rank and T_9 is the lowest turbine rank. In case 2, T_2 is the highest turbine rank and T_9 is the lowest turbine rank. In case 3, T_5 is the highest turbine rank and T_1 is the lowest turbine rank. In case 4, T_{13} is the highest turbine rank and T_{12} is the lowest turbine rank. In case 5, T_4 is the highest turbine rank and T_9 is the lowest turbine rank.

In EDAS method, In case 1, the T_4 is the highest turbine rank and T_{15} is the lowest turbine rank. In case 2, T_4 is the highest turbine rank and T_3 is the lowest turbine rank. In case 3, T_6 is the highest turbine rank and T_{15} is the lowest turbine rank. In case 4, T_4 is the highest turbine rank and T_3 is the lowest turbine rank. In case 5, T_4 is the highest turbine rank and T_{12} is the lowest turbine rank.

Due to the MCDM methods have different rank results. So, proposed the combination method to aggregate the turbines rank. If there are h alternative, the highest rank takes h points and second rank takes h-1 points, third rank takes h-2 points and so on. The turbines is the highest points takes the best turbines[36]. Table 17 show the combination rank.

Table 15. The five case of change weight.

Turbines/Rank	Machine Feature	Environmental/Habitat	Technological	Monetary
Case 1 Equal important	0.25	0.25	0.25	0.25
Case 2 Machine Feature	0.5	0.1667	0.1667	0.1667
Case 3 Environmental/Habitat	0.1667	0.5	0.1667	0.1667
Case 4 Technological	0.1667	0.1667	0.5	0.1667
Case 5 Monetary	0.1667	0.1667	0.1667	0.5

Table 16. The rank of turbines by five cases of weights.

WSM					VIKOR					TOPSIS					EDAS				
Case 1	Case 2	Case 3	Case 4	Case 5	Case 1	Case 2	Case 3	Case 4	Case 5	Case 1	Case 2	Case 3	Case 4	Case 5	Case 1	Case 2	Case 3	Case 4	Case 5
T9	T9	T9	T9	T10	T5	T4	T6	T13	T4	T5	T2	T5	T13	T4	T4	T4	T6	T4	T4
T10	T7	T15	T10	T9	T2	T2	T5	T3	T5	T2	T4	T6	T3	T5	T9	T9	T4	T6	T9
T15	T10	T7	T1	T15	T4	T5	T4	T6	T3	T4	T5	T4	T6	T3	T6	T8	T9	T8	T8
T7	T1	T10	T15	T1	T3	T3	T2	T7	T14	T3	T3	T7	T7	T14	T8	T6	T1	T9	T6
T1	T15	T3	T3	T7	T7	T13	T11	T8	T2	T14	T14	T2	T1	T2	T5	T10	T8	T5	T14
T3	T3	T1	T7	T11	T14	T7	T14	T9	T12	T11	T7	T11	T10	T11	T1	T11	T7	T11	T5
T13	T2	T5	T13	T3	T11	T12	T7	T1	T11	T7	T13	T8	T2	T12	T11	T2	T5	T1	T2
T5	T13	T13	T5	T8	T13	T14	T8	T5	T7	T6	T11	T14	T9	T7	T2	T1	T11	T14	T11
T8	T8	T8	T11	T13	T6	T6	T15	T2	T13	T13	T12	T10	T15	T10	T14	T5	T12	T12	T10
T11	T5	T11	T8	T2	T12	T11	T10	T15	T6	T12	T6	T9	T8	T13	T10	T7	T2	T2	T1
T2	T11	T4	T2	T5	T15	T1	T12	T10	T10	T10	T1	T12	T14	T6	T7	T14	T13	T10	T3
T12	T12	T12	T12	T12	T10	T15	T3	T14	T15	T15	T10	T15	T5	T15	T13	T12	T14	T13	T13
T4	T4	T2	T4	T6	T1	T8	T13	T4	T1	T1	T8	T13	T4	T1	T12	T13	T10	T7	T15

T6	T14	T14	T6	T4	T8	T10	T9	T11	T8	T8	T15	T3	T11	T8	T3	T15	T3	T15	T7
T14	T6	T6	T14	T14	T9	T9	T1	T12	T9	T9	T9	T1	T12	T9	T15	T3	T15	T3	T12

Table 17. The combination rank for five case weights.

WSM	VIKOR	TOPSIS	EDAS
T9	T5	T5	T4
T10	T4	T2	T9
T15	T2	T4	T6
T7	T3	T3	T8
T1	T7	T7	T5
T3	T6	T14	T1
T13	T13	T6	T11
T5	T14	T11	T2
T8	T11	T13	T14
T11	T12	T10	T10
T2	T8	T12	T7
T12	T15	T1	T12
T4	T10	T8	T13
T6	T1	T15	T3
T14	T9	T9	T15

7. Conclusions

Many countries go toward using the renewable energy instead of using fossil fuel in recent years. The wind energy is a source of renewable energy. So, increasing the importance of selection the best wind turbine. In this paper discuss the selection best wind turbine for Egypt to build a new farm in the government red sea. First the criteria is collected from the literature review. The opinions of experts and decision makers are collected. The twenty two sub criteria and four main criteria is collected. The fifteen turbines were determined. The weights of criteria is computed by the entropy weight method. The turbines were ranked by the WSM, VIKOR, TOPSIS and EDAS methods with bipolar neutrosophic sets. Base on the results show that the T_2 is the highest rank and T_{12} is the lowest rank.

The future work can apply another MCDM methods for this problem.

Conflict of interest
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This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] Rehman, S. and S.A. Khan, *Multi-criteria wind turbine selection using weighted sum approach.* International Journal of Advanced Computer Science and Applications, 2017. **8**(6): p. 128-132.

[2] Gamboa, G. and G. Munda, *The problem of windfarm location: A social multi-criteria evaluation framework.* Energy policy, 2007. **35**(3): p. 1564-1583.

[3] Beskese, A., et al., *Wind turbine evaluation using the hesitant fuzzy AHP-TOPSIS method with a case in Turkey.* Journal of Intelligent & Fuzzy Systems, 2020. **38**(1): p. 997-1011.

[4] Lee, A.H., H.H. Chen, and H.-Y. Kang, *Multi-criteria decision making on strategic selection of wind farms.* Renewable Energy, 2009. **34**(1): p. 120-126.

[5] Neto, J.X.V., et al., *Wind turbine blade geometry design based on multi-objective optimization using metaheuristics.* Energy, 2018. **162**: p. 645-658.

[6] Rehman, S. and S.A. Khan, *Fuzzy logic based multi-criteria wind turbine selection strategy—A case study of Qassim, Saudi Arabia.* Energies, 2016. **9**(11): p. 872.

[7] Kaya, T. and C. Kahraman, *Multicriteria renewable energy planning using an integrated fuzzy VIKOR & AHP methodology: The case of Istanbul.* Energy, 2010. **35**(6): p. 2517-2527.

[8] Rehman, S. and S.A. Khan, *Goal Programming-Based Two-Tier Multi-Criteria Decision-Making Approach for Wind Turbine Selection.* Applied Artificial Intelligence, 2019. **33**(1): p. 27-53.

[9] Shirgholami, Z., S.N. Zangeneh, and M. Bortolini, *Decision system to support the practitioners in the wind farm design: A case study for Iran mainland.* Sustainable Energy Technologies and Assessments, 2016. **16**: p. 1-10.

[10] Çolak, M. and İ. Kaya, *Prioritization of renewable energy alternatives by using an integrated fuzzy MCDM model: A real case application for Turkey.* Renewable and sustainable energy reviews, 2017. **80**: p. 840-853.

[11] Qu, F., et al., *Wind turbine fault detection based on expanded linguistic terms and rules using non-singleton fuzzy logic.* Applied Energy, 2020. **262**: p. 114469.

[12] Madi, E.N., J.M. Garibaldi, and C. Wagner. *An exploration of issues and limitations in current methods of TOPSIS and fuzzy TOPSIS*. in *2016 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*. 2016. IEEE.

[13] Smarandache, F., *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: neutrsophic logic. Neutrosophy, neutrosophic set, neutrosophic probability*. 2005: Infinite Study.

[14] Faizi, S., et al., *Decision making with uncertainty using hesitant fuzzy sets.* International Journal of Fuzzy Systems, 2018. **20**(1): p. 93-103.

[15] Junaid, M., et al., *A neutrosophic ahp and topsis framework for supply chain risk assessment in automotive industry of Pakistan.* Sustainability, 2020. **12**(1): p. 154.

[16] Cui, Y., et al., *Water resources carrying capacity evaluation and diagnosis based on set pair analysis and improved the entropy weight method.* Entropy, 2018. **20**(5): p. 359.

[17] Yörükoğlu, M. and S. Aydın, *Wind turbine selection by using MULTIMOORA method.* Energy Systems, 2020: p. 1-14.

[18] Kongprasit, S., J. Waewsak, and T. Chaichana, *Wind turbine and local acceptance in Southern Thailand.* Energy Procedia, 2017. **138**: p. 380-385.

[19] Sarja, J. and V. Halonen, *Wind turbine selection criteria: A customer perspective.* Journal of Energy and Power Engineering, 2013. **7**(9): p. 1795.

[20] Şağbanşua, L. and F. Balo, *Multi-criteria decision making for 1.5 MW wind turbine selection.* Procedia computer science, 2017. **111**: p. 413-419.

[21] Arias-Rosales, A. and G. Osorio-Gómez, *Wind turbine selection method based on the statistical analysis of nominal specifications for estimating the cost of energy.* Applied Energy, 2018. **228**: p. 980-998.

[22] Sedaghat, A., et al., *A new strategy for wind turbine selection using optimization based on rated wind speed.* Energy Procedia, 2019. **160**: p. 582-589.

[23] Pang, N., et al., *Selection of Wind Turbine Based on Fuzzy Analytic Network Process: A Case Study in China.* Sustainability, 2021. **13**(4): p. 1792.

[24] Supciller, A.A. and F. Toprak, *Selection of wind turbines with multi-criteria decision making techniques involving neutrosophic numbers: A case from Turkey.* Energy, 2020. **207**: p. 118237.
[25] Abdel-Basset, M., et al., *A bipolar neutrosophic multi criteria decision making framework for professional selection.* Applied Sciences, 2020. **10**(4): p. 1202.
[26] Broumi, S., et al. *Shortest path problem under bipolar neutrosphic setting*. in *Applied Mechanics and Materials*. 2017. Trans Tech Publ.
[27] Wang, L., H. Garg, and N. Li, *Pythagorean fuzzy interactive Hamacher power aggregation operators for assessment of express service quality with entropy weight.* Soft Computing, 2021. **25**(2): p. 973-993.
[28] Zeng, S., et al., *A multi-criteria sustainable supplier selection framework based on neutrosophic fuzzy data and entropy weighting.* Sustainable Development, 2020. **28**(5): p. 1431-1440.
[29] Xiao, Q., et al., *Evaluation of urban taxi-carpooling matching schemes based on entropy weight fuzzy matter-element.* Applied Soft Computing, 2019. **81**: p. 105493.
[30] Manna, S., T.M. Basu, and S.K. Mondal, *A soft set based VIKOR approach for some decision-making problems under complex neutrosophic environment.* Engineering Applications of Artificial Intelligence, 2020. **89**: p. 103432.
[31] Abdel-Basset, M., et al., *An integrated plithogenic MCDM approach for financial performance evaluation of manufacturing industries.* Risk Management, 2020. **22**(3): p. 192-218.
[32] Li, H., et al., *A novel hybrid MCDM model for machine tool selection using fuzzy DEMATEL, entropy weighting and later defuzzification VIKOR.* Applied Soft Computing, 2020. **91**: p. 106207.
[33] Krishankumar, R., et al., *A novel extension to VIKOR method under intuitionistic fuzzy context for solving personnel selection problem.* Soft Computing, 2020. **24**(2): p. 1063-1081.
[34] Keshavarz Ghorabaee, M., et al., *Multi-criteria inventory classification using a new method of evaluation based on distance from average solution (EDAS).* Informatica, 2015. **26**(3): p. 435-451.
[35] Kahraman, C., et al., *Intuitionistic fuzzy EDAS method: an application to solid waste disposal site selection.* Journal of Environmental Engineering and Landscape Management, 2017. **25**(1): p. 1-12.
[36] Lee, H.-C. and C.-T. Chang, *Comparative analysis of MCDM methods for ranking renewable energy sources in Taiwan.* Renewable and Sustainable Energy Reviews, 2018. **92**: p. 883-896.

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Graphical Representation of Type-2 Neutrosophic sets

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Abstract: Neutrosophic set is the universality of the fuzzy and intuitionistic fuzzy sets. If the value of grade of membership contains uncertainty then that problems or situations can be dealt by type-2 fuzzy and intuitionistic fuzzy sets. It is not possible to work in enigmatic and uncertain situations and indeterminate situations as well. This present study introduces, graphical representation of type-2 neutrosophic set (T2NS) to deal the level of uncertainty in truth, indeterminate and false part of the information from footprint of uncertainty (FOU). This graphical representation helps as a learning strategy of type-2 neutrosophic sets. Also discussed the advantage of T2NS.

Keywords: Type 2 neutrosophic number; graphical representation; score and accuracy function

1. Introduction

Fuzzy set (FS) [32] is an extension of the conventional set where the elements have membership degrees. It encounters the uncertainty, partial truth and vagueness of each and every element in the set. FS is also called as type-1 FS. Since it has crisp membership values take from [0, 1], it is unable to deal more uncertainties generally exist in the real world problems. To sort out this issue type-2 fuzzy sets have been introduced to deal more uncertainties as its membership values itself a fuzzy number with a unique dimension called footprint of uncertainty (FOU). FOU tells about the level of uncertainty of the problem by giving more degrees of freedom and it is the fundamental difference between the type-1 and type-2 FSs.

Structure of the rule is same in these two types but the output is different. An expert can decide the membership value in type-1 fuzzy exactly where for type-2 fuzzy it is not instead expert will provide interval based membership values. Defuzzification is the method for getting crisp outputs from type-2 fuzzy sets and gives the flexibilities in decision-making. In this way, type-2 fuzzy sets have the capability of dealing uncertainty in high level by providing missing components, which is very useful in decision-making.

Atanassov[5] stated that intuitionistic fuzzy sets (IFSs) are the generalization of FSs by adding degree of non-membership of the elements in the set. It provides the theoretical support to handle the hesitation information provided by the people in judging questions. In addition, it looks more precisely to uncertainty analysis and afford the chance of having precise model using existing observation and intelligence. Both these types are soft methods and hence directed to soft computing and approximate reasoning [23]. Type-2 IFSs more successful by enhancing the ability of dealing with more uncertainties as T2FSs with extra component non-membership. T2FSs and T2IFSs have a broad range of practical applications.

Impreciseness in real world problems has become an advantageous modeling field for FSs and its generalization. Many efforts have been made to employ the approach of these sets for reducing the impreciseness from such problems [24]. Dubois et al.[15]inferred that some of the experts have disputed that demanding precision in grades of membership of the elements of the sets may sound sarcastic. Though it is a challenged one automatically, it leads to interval valued fuzzy sets. In the fields of engineering, economics intervals were used to produce the values of quantities due to uncertainty. Bustince et al.[13] mentioned that, all the above mentioned types are only because of uncertainty of human knowledge representation.

Though FSs and IFSs scope with the dealing of uncertainty in real world problems, they are unable to deal indeterminacy of the information or data. Hence, neutrosophic set introduced by Smarandache [27]. Then its special cases like single valued and interval valued neutrosophic sets have been introduced to deal more uncertainty of the problem by dealing more indeterminacy of the information and data provided by the experts or people according to the questionnaire provided [7,33].

The remaining part of the paper is organized as follows. In section 2, review of literature has been given. In section 3, basic concepts of type-2 neutrosophic set have been presented. In section 4, numerical validation is done for the concepts using type-2 neutrosophic sets. In section 5, graphical representation of type-2 neutrosophic set is presented. In section 6, advantages of T2NS are presented. In section 7, we conclude the work with the future direction.

2-Review of Literature

Atanassov [2] introduced the concept of intuitionistic fuzzy sets. Atanassov [4] described the theory of intuitionistic fuzzy concept. Atanassov [4] introduced the geometric interpretation, discrete norm, graph concepts under IFS environment. Zhao and Xia [33] proposed the concept of IFS under type-2 setting. Coung et al [14] defined some operations of IFS. Jana [21] proposed arithmetic operations on type-2 IFS and used the proposed concepts in transportation problem. Singh and Garg [29] proposed distance measure between IFSs and applied in a decision-making problem. Anusuya and Sathya [2] solved shortest path problem (SPP) using the complement of a type-2 fuzzy number. Anusuya and Sathya [1] proposed a new approach for solving SPP under type-2 fuzzy environment. Lee and Lee [25] solved SPP using type-2 fuzzy weighted graph. Basset et al [26] proposed a novel methodology for supplier selection using TOPSIS approach.

Kumar and Pandey [24] made a discussion Qwitching Type-2 fuzzy sets and IFS in an application of medical diagnosis. Khatibi and Montazer [23] analyzed the performance of medical pattern recognition using IFS and FS. Dubois et al[15]discussed about the difficulties of using the terminologies of FS theory. Bustince et al [13] presented a wider view on the relationship of interval type-2 fuzzy sets and interval valued FSs.

Smarandache [27] introduced the concept of neutrosophy and its probability, logic and set. Smarandache [28] introduced neutrosophic theory, its logic and set to solve the problem with indeterminacy. Wang et al [30] introduced single valued neutrosophic set (SVNS) as the special case of neutrosophic set (NS). Wang et al [31] introduced interval valued NSs and applied in the field computing technology. Broumi and Smarandache [6] proposed cosine similarity measure of interval valued NSs. Broumi et al [11] introduced interval valued neutrosophic sets and its operations.

Broumi et al [8] solved minimum spanning tree problem under interval valued bipolar neutrosophic setting. Broumi et al [9] solve SPP using single valued neutrosophic graphs. Broumi et al[10]analyzed SPP using trapezoidal NS. Broumi et al [11] introduced N-valued interval NSs and applied in medical diagnosis problem. Nagarajan et al [17] have done edge detection on DICOM Image using triangular norms under type-2 fuzzy. Nagarajan et al [18] have done image extraction using the concept of type-2 fuzzy.

Nagarajan et al. [19] introduced interval type2 fuzzy logic washing machine. Nagarajan et al [20], proposed fuzzy optimization techniques based on hidden Markov model using interval type-2 fuzzy parameters. Broumi et al [12] solved SPP using triangular and trapezoidal interval NSs.

Nagarajan et al [16] analyzed traffic control management using interval type-2 FSs and interval neutrosophic sets and their aggregation operators. Also proposed a new score function for interval neutrosophic numbers. Karaaslan and Hunu [22] introduced type-2 single valued neutrosophic sets and solved multicriteria group decision-making problem based on TOPSIS method under type-2 single valued neutrosophic environment.

Though many concepts and types have been introduced, type-2 neutrosophic set with truth, indeterminacy and falsity components as the subparts for all the three components truth, indeterminacy and falsity is yet to be studied. Hence the scope and aim of this paper.

3-Preliminaries

We introduce several basic concepts of T2NN and operations on T2NN.

Definition of type 2 neutrosophic number [26]

Let Z be the limited universe of discourse and $F[0,1]$ be the set of all triangular neutrosophic numbers on $F[0,1]$. A type 2 neutrosophic number set (T2NNS) $\widetilde{U}$ in Z is represented by $\widetilde{U}=\{\langle z, \widetilde{T}_{\widetilde{U}}(z), \tilde{I}_{\widetilde{U}}(z), \widetilde{F}_{\widetilde{U}}(z) \mid z \in Z\rangle\}$, where $\widetilde{T}_{\widetilde{U}}(z): Z \to F[0,1], \tilde{I}_{\widetilde{U}}(z): Z \to F[0,1], \widetilde{F}_{\widetilde{U}}(z): Z \to F[0,1]$. A type 2 neutrosophic number set (T2NNS) $\widetilde{T}_{\widetilde{A}}(z) = \left(T_{T_{\widetilde{U}}}(z), T_{I_{\widetilde{U}}}(z), T_{F_{\widetilde{U}}}(z)\right)$, $\tilde{I}_{\widetilde{U}}(z) = \left(I_{T_{\widetilde{U}}}(z), I_{I_{\widetilde{U}}}(z), I_{F_{\widetilde{U}}}(z)\right)$, $\widetilde{F}_{\widetilde{U}}(z) = \left(F_{T_{\widetilde{U}}}(z), F_{I_{\widetilde{U}}}(z), F_{F_{\widetilde{U}}}(z)\right)$, respectively, denote the truth, indeterminacy, and falsity memberships of z in $\widetilde{U}$ and for every $z \in Z$: $0 \leq \widetilde{T}_{\widetilde{U}}(z)^3 + \tilde{I}_{\widetilde{U}}(z)^3 + \widetilde{F}_{\widetilde{U}}(z)^3 \leq 3$; for convenience, we consider that

$\widetilde{U} = \left\langle \left(T_{T_{\widetilde{U}}}(z), T_{I_{\widetilde{U}}}(z), T_{F_{\widetilde{U}}}(z)\right), \left(I_{T_{\widetilde{U}}}(z), I_{I_{\widetilde{U}}}(z), I_{F_{\widetilde{U}}}(z)\right), \left(F_{T_{\widetilde{U}}}(z), F_{I_{\widetilde{U}}}(z), F_{F_{\widetilde{U}}}(z)\right) \right\rangle$ as a type 2 neutrosophic number.

Definition 2[26]

Suppose $\widetilde{U}_1 = \left\langle \left(T_{T_{\widetilde{U}_1}}(z), T_{I_{\widetilde{U}_1}}(z), T_{F_{\widetilde{U}_1}}(z)\right), \left(I_{T_{\widetilde{U}_1}}(z), I_{I_{\widetilde{U}_1}}(z), I_{F_{\widetilde{U}_1}}(z)\right), \left(F_{T_{\widetilde{U}_1}}(z), F_{I_{\widetilde{U}_1}}(z), F_{F_{\widetilde{U}_1}}(z)\right) \right\rangle$ and $\widetilde{U}_2 = \left\langle \left(T_{T_{\widetilde{U}_2}}(z), T_{I_{\widetilde{U}_2}}(z), T_{F_{\widetilde{U}_2}}(z)\right), \left(I_{T_{\widetilde{U}_2}}(z), I_{I_{\widetilde{U}_2}}(z), I_{F_{\widetilde{U}_2}}(z)\right), \left(F_{T_{\widetilde{U}_2}}(z), F_{I_{\widetilde{U}_2}}(z), F_{F_{\widetilde{U}_2}}(z)\right) \right\rangle$ are two

T2NNS in the set real numbers. Then the procedures are defined as follows:

$$\tilde{U}_1 \oplus \tilde{U}_2 = \left\langle \begin{array}{c} \left(\begin{array}{c} \left(T_{T_{\tilde{U}_1}}(z) + T_{T_{\tilde{U}_2}}(z) - T_{T_{\tilde{U}_1}}(z).T_{T_{\tilde{U}_2}}(z)\right), \left(T_{I_{\tilde{U}_1}}(z) + T_{I_{\tilde{U}_2}}(z) - T_{I_{\tilde{U}_1}}(z).T_{I_{\tilde{U}_2}}(z)\right), \\ \left(T_{F_{\tilde{U}_1}}(z) + T_{F_{\tilde{U}_2}}(z) - T_{F_{\tilde{U}_1}}(z).T_{F_{\tilde{U}_2}}(z)\right) \end{array} \right), \\ \left(I_{T_{\tilde{U}_1}}(z).I_{T_{\tilde{U}_2}}(z), I_{I_{\tilde{U}_1}}(z).I_{I_{\tilde{U}_2}}(z), I_{F_{\tilde{U}_1}}(z).I_{F_{\tilde{U}_2}}(z)\right), \\ \left(F_{T_{\tilde{U}_1}}(z).F_{T_{\tilde{U}_2}}(z), F_{I_{\tilde{U}_1}}(z).F_{I_{\tilde{U}_2}}(z), F_{F_{\tilde{U}_1}}(z).F_{F_{\tilde{U}_2}}(z)\right) \end{array} \right\rangle \quad (1)$$

$$\tilde{U}_1 \otimes \tilde{U}_2 = \left\langle \begin{array}{c} \left(\left(T_{T_{\tilde{U}_1}}(z).T_{T_{\tilde{U}_2}}(z), T_{I_{\tilde{U}_1}}(z).T_{I_{\tilde{U}_2}}(z), T_{F_{\tilde{U}_1}}(z).T_{F_{\tilde{U}_2}}(z)\right)\right), \\ \left(\begin{array}{c} \left(I_{T_{\tilde{U}_1}}(z) + I_{T_{\tilde{U}_2}}(z) - I_{T_{\tilde{U}_1}}(z).I_{T_{\tilde{U}_2}}(z)\right), \left(I_{I_{\tilde{U}_1}}(z) + I_{I_{\tilde{U}_2}}(z) - I_{I_{\tilde{U}_1}}(z).I_{I_{\tilde{U}_2}}(z)\right), \\ \left(I_{F_{\tilde{U}_1}}(z) + I_{F_{\tilde{U}_2}}(z) - I_{F_{\tilde{U}_1}}(z).I_{F_{\tilde{U}_2}}(z)\right) \end{array} \right) \\ \left(\begin{array}{c} \left(F_{T_{\tilde{U}_1}}(z) + F_{T_{\tilde{U}_2}}(z) - F_{T_{\tilde{U}_1}}(z).F_{T_{\tilde{U}_2}}(z)\right), \left(F_{I_{\tilde{U}_1}}(z) + F_{I_{\tilde{U}_2}}(z) - F_{I_{\tilde{U}_1}}(z).F_{I_{\tilde{U}_2}}(z)\right), \\ \left(F_{F_{\tilde{U}_1}}(z) + F_{F_{\tilde{U}_2}}(z) - F_{F_{\tilde{U}_1}}(z).F_{F_{\tilde{U}_2}}(z)\right) \end{array} \right) \end{array} \right\rangle$$

(2)

$$\delta\tilde{U} = \left\langle \begin{array}{c} \left(1-(1-T_{T_{\tilde{U}_1}}(z))^{\delta}, 1-(1-T_{I_{\tilde{U}_1}}(z))^{\delta}, 1-(1-T_{F_{\tilde{U}_1}}(z))^{\delta}\right), \\ \left(\left(I_{T_{\tilde{U}_1}}(z)\right)^{\delta}, \left(I_{I_{\tilde{U}_1}}(z)\right)^{\delta}, \left(I_{F_{\tilde{U}_1}}(z)\right)^{\delta}\right), \\ \left(\left(F_{T_{\tilde{U}_1}}(z)\right)^{\delta}, \left(F_{I_{\tilde{U}_1}}(z)\right)^{\delta}, \left(F_{F_{\tilde{U}_1}}(z)\right)^{\delta}\right) \end{array} \right\rangle \quad \text{for} \quad \delta > 0$$

(3)

$$\widetilde{U}^{\delta} = \left\langle \begin{matrix} \left(\left(T_{T_{\widetilde{U}_1}}(z)\right)^{\delta}, \left(T_{I_{\widetilde{U}_1}}(z)\right)^{\delta}, \left(T_{F_{\widetilde{U}_1}}(z)\right)^{\delta}\right), \\ \left(1-\left(1-I_{T_{\widetilde{U}_1}}(z)\right)^{\delta}, \quad 1-\left(1-I_{I_{\widetilde{U}_1}}(z)\right)^{\delta}, 1-\left(1-I_{F_{\widetilde{U}_1}}(z)\right)^{\delta}\right), \\ \left(1-\left(1-F_{T_{\widetilde{U}_1}}(z)\right)^{\delta}, \quad 1-\left(1-F_{I_{\widetilde{U}_1}}(z)\right)^{\delta}, 1-\left(1-F_{F_{\widetilde{U}_1}}(z)\right)^{\delta}\right) \end{matrix} \right\rangle$$

for $\delta > 0$

(4)

The procedures defined in definition 2 satisfy the following properties:

$\widetilde{U}_1 \oplus \widetilde{U}_2 = \widetilde{U}_2 \oplus \widetilde{U}_1$, $\widetilde{U}_1 \otimes \widetilde{U}_2 = \widetilde{U}_2 \otimes \widetilde{U}_1$;

$\delta(\widetilde{U}_1 \oplus \widetilde{U}_2) = \delta\widetilde{U}_1 \oplus \delta\widetilde{U}_2$, $(\widetilde{U}_1 \otimes \widetilde{U}_2)^{\delta} = \widetilde{U}_1^{\ \delta} \otimes \widetilde{U}_2^{\ \delta}$ for $\delta > 0$, and

$\delta_1\widetilde{U}_1 \oplus \delta_2\widetilde{U}_1 = (\delta_1 + \delta_2)\widetilde{U}_1$, $\widetilde{U}_1^{\ \delta_1} \oplus \widetilde{U}_1^{\ \delta_2} = \widetilde{U}_1^{\ (\delta_1 + \delta_2)}$ for $\delta_1, \delta_2 > 0$.

Definition 3 [26]

Suppose that $\widetilde{U}_1 = \left\langle \left(T_{T_{\widetilde{U}_1}}(z), T_{I_{\widetilde{U}_1}}(z), T_{F_{\widetilde{U}_1}}(z)\right), \left(I_{T_{\widetilde{U}_1}}(z), I_{I_{\widetilde{U}_1}}(z), I_{F_{\widetilde{U}_1}}(z)\right), \left(F_{T_{\widetilde{U}_1}}(z), F_{I_{\widetilde{U}_1}}(z), F_{F_{\widetilde{U}_1}}(z)\right) \right\rangle$

are T2NNS in the set of real numbers, the score function $S(\widetilde{U}_1)$ of $\widetilde{U}_1$ is defined as follows:

$$S(\widetilde{U}_1) = \frac{1}{12}\left\langle 8 + \left(T_{T_{\widetilde{U}_1}}(z) + 2\left(T_{I_{\widetilde{U}_1}}(z)\right) + T_{F_{\widetilde{U}_1}}(z)\right) - \left(I_{T_{\widetilde{U}_1}}(z) + 2\left(I_{I_{\widetilde{U}_1}}(z)\right) + I_{F_{\widetilde{U}_1}}(z)\right) - \left(F_{T_{\widetilde{U}_1}}(z) + 2\left(F_{I_{\widetilde{U}_1}}(z)\right) + F_{F_{\widetilde{U}_1}}(z)\right)\right\rangle \quad (5)$$

$$A(\widetilde{U}_1) = \frac{1}{4}\left\langle \left(T_{T_{\widetilde{U}_1}}(z) + 2\left(T_{I_{\widetilde{U}_1}}(z)\right) + T_{F_{\widetilde{U}_1}}(z)\right) - \left(F_{T_{\widetilde{U}_1}}(z) + 2\left(F_{I_{\widetilde{U}_1}}(z)\right) + F_{F_{\widetilde{U}_1}}(z)\right)\right\rangle$$

(6)

Definition 4 [26].

Suppose that $\widetilde{U}_1 = \left\langle \left(T_{T_{\widetilde{U}_1}}(z), T_{I_{\widetilde{U}_1}}(z), T_{F_{\widetilde{U}_1}}(z)\right), \left(I_{T_{\widetilde{U}_1}}(z), I_{I_{\widetilde{U}_1}}(z), I_{F_{\widetilde{U}_1}}(z)\right), \left(F_{T_{\widetilde{U}_1}}(z), F_{I_{\widetilde{U}_1}}(z), F_{F_{\widetilde{U}_1}}(z)\right) \right\rangle$

and $\widetilde{U}_2 = \left\langle \left(T_{T_{\widetilde{U}_2}}(z), T_{I_{\widetilde{U}_2}}(z), T_{F_{\widetilde{U}_2}}(z)\right), \left(I_{T_{\widetilde{U}_2}}(z), I_{I_{\widetilde{U}_2}}(z), I_{F_{\widetilde{U}_2}}(z)\right), \left(F_{T_{\widetilde{U}_2}}(z), F_{I_{\widetilde{U}_2}}(z), F_{F_{\widetilde{U}_2}}(z)\right) \right\rangle$ are two

T2NNS in the set of real numbers. Suppose that $S(\widetilde{U}_i)$ and $A(\widetilde{U}_i)$ are the score and accuracy functions of T2NNS $\widetilde{U}_i$(i = 1, 2), then the order relations are defined as follows:

If $\tilde{S}(\widetilde{U}_1) > \tilde{S}(\widetilde{U}_2)$, then $\widetilde{U}_1$ is greater than $\widetilde{U}_2$, that is $\widetilde{U}_1$ is superior to $\widetilde{U}_2$, denoted by $\widetilde{U}_1 > \widetilde{U}_2$;

If $\tilde{S}(\widetilde{U}_1) = \tilde{S}(\widetilde{U}_2)$, $\widetilde{A}(\widetilde{U}_1) > \widetilde{A}(\widetilde{U}_2)$)then $\widetilde{U}_1$ is superior than $\widetilde{U}_2$, that is $\widetilde{U}_1$ is superior to $\widetilde{U}_2$, denoted by $\widetilde{U}_1 > \widetilde{U}_2$;

If $\tilde{S}(\widetilde{U}_1) = \tilde{S}(\widetilde{U}_2)$, $\widetilde{A}(\widetilde{U}_1) = \widetilde{A}(\widetilde{U}_2)$) then $\widetilde{U}_1$ is equal to $\widetilde{U}_2$, that is $\widetilde{U}_1$ is indifferent to $\widetilde{U}_2$, denoted by $\widetilde{U}_1 = \widetilde{U}_2$;

4-Numreical examples

In this section, numerical examples are given for validating the concepts.

Example 1. Consider two T2NNS in the group of real numbers:

$$\widetilde{U}_1 = \left\langle \left(T_{T_{\widetilde{U}_1}}(z), T_{I_{\widetilde{U}_1}}(z), T_{F_{\widetilde{U}_1}}(z)\right), \left(I_{T_{\widetilde{U}_1}}(z), I_{I_{\widetilde{U}_1}}(z), I_{F_{\widetilde{U}_1}}(z)\right), \left(F_{T_{\widetilde{U}_1}}(z), F_{I_{\widetilde{U}_1}}(z), F_{F_{\widetilde{U}_1}}(z)\right) \right\rangle \text{ and}$$

$$\widetilde{U}_2 = \left\langle \left(T_{T_{\widetilde{U}_2}}(z), T_{I_{\widetilde{U}_2}}(z), T_{F_{\widetilde{U}_2}}(z)\right), \left(I_{T_{\widetilde{U}_2}}(z), I_{I_{\widetilde{U}_2}}(z), I_{F_{\widetilde{U}_2}}(z)\right), \left(F_{T_{\widetilde{U}_2}}(z), F_{I_{\widetilde{U}_2}}(z), F_{F_{\widetilde{U}_2}}(z)\right) \right\rangle$$

$\widetilde{U}_1 = \langle (0.65, 0.70, 0.75), (0.20, 0.15, 0.30), (0.15, 0.20, 0.10) \rangle$, $\widetilde{U}_2 = \langle (0.45, 0.40, 0.55), (0.35, 0.45, 0.30), (0.25, 0.35, 0.40) \rangle$.

From score function, we get the following outcomes:

Score value of $\tilde{S}(\widetilde{U}_1) = \big(8 + (2.8 - 0.8 - .065)\big)/12 = 0.78$, and $\tilde{S}(\widetilde{U}_2) = \big(8 + (1.8 - 1.55 - 1.35)\big)/12 = 0.58$;

Accuracy value of $A(\widetilde{U}_1) = (2.8 - 0.65)/4 = 0.54$, and $A(\widetilde{U}_2) = (1.8 - 1.35)/4 = 0.11$; it's obvious that $A_1 > A_2$.

Example 2. Consider two T2NNS in the set of real numbers: $\widetilde{U}_1 = \langle (0.50, 0.20, 0.35), (0.30, 0.45, 0.30), (0.10, 0.25, 0.35) \rangle$, $\widetilde{U}_2 = \langle (0.15, 0.60, 0.20), (0.35, 0.20, 0.30), (0.45, 0.35, 0.20) \rangle$.From Eqs. (5) and (6), we obtain the following results:

Score value of $\tilde{S}(\tilde{U}_1) = (8+(1.25-1.5-0.95))/12 = 0.57$, and $\tilde{S}(\tilde{U}_2) =$

$(8+(1.55-1.05-1.35))/12 = 0.60$;

Accuracy value of $A(\tilde{U}_1) = (1.25-0.95)/4 = 0.075$, and $A(\tilde{U}_2) = (1.55-1.35)/12 = 0.05$;

it's obvious that $A_2 > A_1$.

5-Graphical Representation of T2NS

Here, graphical representation of type-2 neutrosophic set is introduced (Figure 1). The method of analyzing numerical data is called graphical representation. It exposes the relation between data and the concept in a diagram. Here we present the graphical representation of type-2 neutrosophic sets which is useful to exhibit the relation of truth, indeterminacy and falsity of the data and concept. This representation is a learning system of T2NS. Here footprint of uncertainty (FOU) for truth, indeterminacy and falsity represents the level of uncertainty exist.

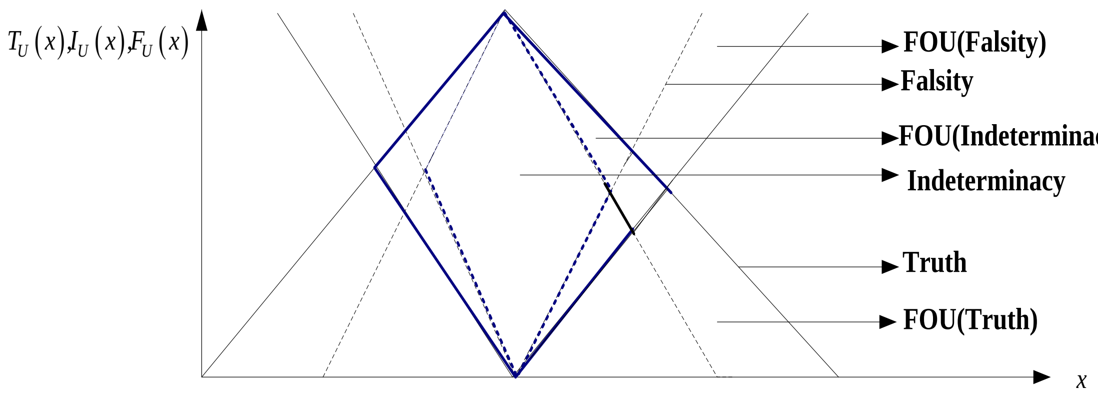

Figure 1. Graphical Representation of Type-2 Neutrosophic Set (T2NS)

z((T_T, T_I, T_F), (I_T, I_I, I_F), (F_T, F_I, F_F)) is a Type-2 Neutrosophic Number, which means that each neutrosophic component T, I, and F is split into its truth, indeterminacy, and falsehood subparts. The procedure of splitting may be executed recurrently, as many times as needed, obtaining a general Type-n Neutrosophic Number, for any integer $\eta \geq 2$.

6- Advantage of Type-2 neutrosophic set

Problems entailing linguistic variables and uncertainty can be deal with efficiently by type-2 fuzzy sets and type-2 intuitionistic fuzzy sets. But modeling the problems which involve incompatible or inconsistant information is very challenging one by these sets. Also, in a neutrosophic set, the membership functions of the three functions namely truth, indeterminacy and falsity are not uncertain. So it is not able to deal with the information which is of the form of word and sentences in artificial languages called linguistic variables as this variable reduces the overall computational complexity of any real world problem. Since FOU represents the level of uncertainty exist, T2NS has more capability of reducing uncertainty and indeterminacy of the data in real world problems than other sets. Also, in T2FS, truth, indeterminacy and falsity membership functions are independent of each other and they may be can considered as fuzzy sets and therefore assigning different linguistic variables is possible. Hence, the advantage of T2NS.

7-Conclusion

Neutrosophic logic and sets are the one, which deals uncertainty of the real world problems in an optimized way due its unique capability of handling indeterminacy of the problem. Since type-2 neutrosophic logic can deal more uncertainties using primary and secondary membership functions shortest path problem can be solved in with accurate result. In this paper, graphical representation of T2NS has been introduced and the advantage T2NS has been discussed. In future, this work may be extended to different sets.

References

[1] V. Anusuya, and R. Sathya, New Approach for Solving Type-2 Fuzzy Shortest Path Problem. Annals of Pure and Applied Mathematics, 2014; N 8(1),pp. 83-92.

[2] V. Anusuya, and R. Sathya, Shortest path with complement of type-2 fuzzy number. Malaya Journal of Matematik, 2013; N(1),pp.71–76.

[3] K.T. Atanassov, Intuitionistic Fuzzy Sets, Springer-Verlag, Heidelberg, 1999.

[4] K.T. Atanassov, On Intuitionistic Fuzzy Sets Theory, Springer, Berlin, Heidelberg, 2012.

[5] K.T. Atanassov, Intuitionistic Fuzzy Sets. Fuzzy Sets and Systems, 2006; N20, pp.87-96.

[6] S. Broumi, and F. Smarandache, Cosine similarity measure of interval valued neutrosophic sets. Infinite Study, 2014; DOI : 10.5281/nenodo.30151.

[7] S.Broumi, F.Smarandache, M. Talea and A. Bakali, Operations on Interval Valued Neutrosophic Graphs, chapter in book- New Trends in Neutrosophic Theory and Applications- Florentin Smarandache and Surpati Pramanik (Editors), 2016, pp. 231-254. ISBN 978-1-59973-498-9

[8] S.Broumi, A.Bakali, M.Talea, F. Smarandache, and R. Verma, Computing Minimum Spanning Tree in Interval Valued Bipolar Neutrosophic Environment. International Journal of Modeling and Optimization, 2017;N7(5), pp.300-304.

[9] S.Broumi, A.Bakali, M. Talea, and F. Smarandache, Shortest Path Problem on Single Valued Neutrosophic Graphs. 2017 International Symposium on Networks, Computers and Communications (ISNCC).

[10] S.Broumi, A.Bakali, M. Talea and F. Smarandache, Shortest Path Problem under Trapezoidal Neutrosophic Information, Computing Conference 2017; pp. 142-148.

[11] S.Broumi, I.Deli, and F. Smarandache, N-valued interval neutrosophic sets and their application in medical diagnosis. Critical Review. Center for Mathematics of Uncertainty, Creighton University,Omaha, NE, USA, 2015;10, pp.45-69

[12] S. Broumi, D.Nagarajan, A. Bakali, M.Talea, F. Smarandache, and M. Lathamaheswari, The shortest path problem in interval valued trapezoidal and triangular neutrosophic environment. Complex and Intelligent Systems, 2019. http://doi.org/10.1007/s40747-019-0092-5.

[13] H.Bustince, J. Fernandez, H. Hagras, F.Herrera, M. Pagola, and Barrenechea, Interval Type-2 Fuzzy Sets are generalization of Interval-Valued Fuzzy Sets: Towards a Wider view on their relationship. IEEE Transactions on Fuzzy Systems, 2014 doi: 10.1109/TFUZZ.2014.2362149.

[14] B.C.Coung, T.H. Anh, and B.D.Hai, , Some Operations on Type-2 Intuitionistic Fuzzy sets, Journal of Computer Science and Cybernetics, 2012; 28,pp.274-283.

[15] D. Dubois, S. Gottwald, P. Hajaek, J. Kacprzyk, and H. Prade, Terminological difficulties in fuzzy set theory. Fuzzy Sets and Systems,2005; N (156), pp.485-491.

[16] D. Nagarajan, M. Lathamaheswari, S. Broumi, and J. Kavikumar, A new perspective on traffic control management using triangular interval type-2 fuzzy sets and interval neutrosophic sets. Operations Research perspectives, 2019, Article in Press. https://doi.org/10.1016/j.orp.2019.100099

[17] D. Nagarajan, M. Lathamaheswari, R. Sujatha, and J. Kavikumar, Edge Detection on DICOM Image using Triangular Norms in Type-2 Fuzzy. International Journal of Advanced Computer Science and Applications, 9, 2018 ;(11), pp.462-475.

[18] D.Nagarajan, M.Lathamaheswari, J. Kavikumar, and Hamzha. A Type-2 Fuzzy in Image Extraction for DICOM Image, International Journal of Advanced Computer Science and Applications.,2018; 9(12), pp.352- 362.

[19] D. Nagarajan, M. Lathamaheswari, J. Kavikumar, and E. Deenadayalan, Interval Type-2 Fuzzy Logic Washing Machine. International Journal of Fuzzy Logic and Intelligent Systems, 2019; 9(14), pp.223-233.

[20] D. Nagarajan, M. Lathamaheswari, J. Kavikumar, and N. Kumaresan, Fuzzy Optimization Techniques by Hidden Markov Model with Interval Type-2 Fuzzy Parameters. International Journal of Fuzzy Systems, 2019. https://doi.org/10.1007/s40815-019-00738-3.

[21] D. K. Jana, Novel Arithmetic Operations on Type-2 Intuitionistic Fuzzy and its application to transportation problem, Pacific Science Review A: Natural Science and Engineering, 2016;18, pp178-189.

[22] F. Karaaslan, F. Hunu, Type-2 single-valued neutrosophic sets and their applications in multicriteria group decision making based on TOPSIS method. Journal of Ambient Intelligence and Humanized Computing, 2020. https://doi.org/10.1007/s12652-020-01686-9.

[23] Khatibi, V and G.A. Montazer, Intuitionistic Fuzzy set vs. fuzzy set application in medical pattern recognition. Artificial Intelligence in Medicine, 2009; N (47), pp.43-52.

[24] K. Kumar, and D. Pandey, Discussion on the Qwitching Type-2 Fuzzy Sets and Intuitionistic Fuzzy Sets: An Application in Medical Diagnosis. Journal of Information and Optimization Sciences, 2015; N 39(2), pp.427-444.

[25] S. Lee, and K.H. Lee, Shortest Path Problem in a Type-2 Fuzzy Weighted Graph, Journal of Korean Institute of Intelligent Systems. 2001; 11(6), pp.528-531

[26] M. Abdel-Basset, M. Saleh, A. Gamal, F. Smarandache, An Approach of TOPSIS Technique for Developing Supplier Selection with Group Decision Making under Type-2 Neutrosophic Number, Applied Soft Computing Journal, 2019; 77, pp.438–452.
[27] F.Smarandache, Neutrosophy. Neutrosophic Probability, Set, and Logic.ProQuest Information & Learning, Ann Arbor, Michigan, USA,1998, 105 p.
[28] F. Smarandache, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosphy, Neutrosophic Logic, Set, Probability and Statistics University of New Mexico, Gallup, NM 87301, USA.
[29] S. Singh, and H. Garg, Distance measures between type-2 intuitionistic fuzzy sets and their applications to MCDM process, Application Intelligence, 2017; 46, pp.788-799.
[30] H. Wang, F.Smarandache, Y. Zhang, and R. Sunderraman, Single valued Neutrosophic Sets. Multisspace and Multistructure, 2010; N4, pp.410-413.
[31] H. Wang, F. Smarandache, R. Sunderraman, and Y.Q. Zhang, Interval neutrosophic sets and logic: theory and applications in computing: Theory and applications in computing, Infinite Study; 2005.
[32] L.A. Zadeh, Fuzzy sets. Information and control, 1965; N8(3), pp.338-53.
[33] T. Zhao, and J. Xia, Type-2 intuitionistic fuzzy sets. Control Theory Applications, 2012; N29, pp.1215–1222.

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PESTEL Analysis to Identify Key Barriers to Smart Cities Development in India

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Abstract: The development of smart cities has been gaining attention not only in India but across borders. The development of these cities is supposed to bring India as a smart India in the global market. With all odds in favor; these projects are not gaining their expected on-time results. This has motivated us to think that there are loopholes that are putting hindrance to smart cities development. The aim of this study is to identify and prioritize mathematically the key barriers using the neutrosophic PESTEL analysis technique. An extensive literature survey of the problem provides a lot of factors which are categorized in six main factors such as social, political, legal, ethical and technological factors. Present work using neutrosophic PESTEL analysis finds that social and political factors with 93% and 83% are the key barriers to the development of smart cities in India. Other factors such as Technological and economic factors come at second position securing percentage 75% and 60% respectively. Environmental and legal factors come at last securing 49% and 43% respectively. The research's main focus is to identify and prioritize quantitatively the most important barriers which come into smart cities development in India. This research in many ways would aid the Government agencies and policymakers to prioritize the key barriers at an early stage so that the development may take place as expected and get completed within the stipulated time frame.

Keywords: Smart Cities, Fuzzy Logic, Fuzzy Cognitive Maps, Neutrosophic Logic, Neutrosophic Cognitive Maps, PESTEL Analysis, Machine Learning.

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1. Introduction

The development of smart cities is gaining much attention all over the globe in the last 20 years [1-3]. A smart city in this context can be defined as a city with technological advancements and modernized territory. These cities are capable of dealing with issues like social, economic and technical in such a way that these could lead to superior infrastructure and services [4-8]. With the advent of information & technology together with the policies of the Indian government, these projects seem to get completed in near future. In India people are heading from rural to urbanization at faster rate. This rate is expected to expand the cities to 600 million by 2030. A study by [9] has predicted that at least 200 million people would move from rural areas to urban areas within 15 years from now. According to United Nations Population Fund (UNFPA) [10], a substantial share of population will move from rural to urban areas by 2050. This movement of population is not normal as it would lead to population nearly equal to existing populations of some prominent counties like the United Kingdom (UK), Germany and France taken together. This shift from rural areas to urban areas is how ever slow in India. According to census 2011 this is only 31.5% of total population in India. The reason behind this may be insufficient government policies together with managing the urban dynamics.

The issue of development of smart cities in India is undertaken as an initiative to improve the quality of life and providing basic necessities to its people [11-12]. However, these cities need a totally different perspective for its development in India. The need of the hour is faster development of these cities but it is not achieved despite so much of planning and initiatives. The development of these cities generate a lot of problems which can be physical problems like pollution, resource management, traffic, digitization of data and many more [13-14]. There is also lack of strategic planning pertaining to smart cities development [15]. The projects related to smart cities in India are not getting completed at an expected time despite lot of resources and planning. Moreover these projects are taking longer to get completed without serving the objectives of the project. These loopholes in carrying out these projects have motivated us to identify barriers to the development of smart cities in India. Also, the need of these cities are getting increased at faster rate than its development in India. These reasons serve as primary cause for taking current research at present time.

The current work is grouped in following sections, Section 2 explains well the materials and methods required to understand the work carried out in present research. Section 3 of Results exposes the application of Neutrosophic PESTEL analysis in identifying the key barriers in the development of smart cities in India. This section shows how the situation is modelled in current work using various methods together with giving detailed description of results obtained. Section 4 concludes the paper with more emphasis on future work.

2. Material and Methods

Now, let us understand neutrosophy which is further combined with PESTEL analysis in the present work. Neutrosophic theory was proposed by Florentin Smarandache [18] which is popular theory for the treatment of uncertainties. It helps in generalizing crisp fuzzy sets and theories by introducing some concepts such as neutrosophic sets and neutrosophic logic [18]. When PESTEL technique is used together with neutrosophic logic it becomes neutrosophic PESTEL analysis as used in [19]. The study carried out in [19] is related to food industry where authors have tried to identify factors that affect Food Industry using this technique. The current study uses the technique of PESTEL analysis in determining the key barriers for smart cities development in India. The technique based on neutrosophic cognitive maps permits us to analyze specific topics mathematically using neutrosophic sets and systems. The approach is based on expressive technique with a quantitative strategy. Since this PESTEL analysis is combined with neutrosophy therefore it is required to understand certain concepts which are needed in order to carry out the mathematical work. For this let understand the concept of neutrosophic logic presented by Florentin Smarandache [20]. The latest developments could easily be referred from the work done in [40-44].

Definition1. Let $N = \{(T, I, F): T, I, F \in (0,1)\}$ be a neutrosophic set. Let $m: P \to N$ be a propositional relation into N, i.e., for every $p \in P$ there is association with a value in N, as mentioned in Equation 1,expressing that p is F% false, I% indeterminate and T% true.

$$m(p) = (T, I, F) \qquad (1)$$

Hence, the generalization of fuzzy logic is termed as neutrosophic logic, based on the notion of neutrosophy [18] [21]

Definition2. A Neutrosophic matrix is a matrix= $[a_{ij}]_{ij}$ where $i = 1,2,3, \dots \dots .., m$ and $j = 1,2,3, \dots \dots .., n$ such that each $a_{ij} \in K(I)$ where $K(I)$ is a neutrosophic ring [22] Now let understand this neutrosophic matrix by an example. Suppose each element of matrix is represented by $a + bI$ where a and b are real numbers and I is a factor of indeterminacy.

For Example:

$$\begin{pmatrix} -1 & I & 5I \\ I & 4 & 7 \end{pmatrix} \begin{pmatrix} I & 9I & 6 \\ 0 & I & 0 \\ -4 & 7 & 5 \end{pmatrix} = \begin{pmatrix} -21I & 27I & -6+25I \\ -28+I & 49+13I & 35+6I \end{pmatrix}$$

Definition 3. A graph is called a neutrosophic graph if there exists an indeterminate node or an indeterminate edge where $a_{ij} = 0$ it means there is no connection between nodes i and j, $a_{ij} = 1$ means there is a connection between nodes i and j and $a_{ij} = I$ means that connection is indeterminate (unknown).

Definition 4. A Neutrosophic Cognitive Map is a directed graph [23] with nodes as events or policies and causalities or relationship as determinate &indeterminate edges. The determinate edge or unbroken edge between two concepts shows that the relationship is certain and known whereas on the other hand the indeterminate or dotted edge between the concepts shows that the relationship is not certain or may be unknown. The below graph shows an example of neutrosophic graph in which the edge between node v_4 and v_1 is termed as determinate and edge between v_1 and v_5 is termed as indeterminate. This is represented in Figure 1.

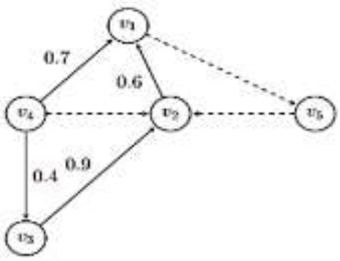

Figure 2 Example of Neutrosophic Cognitive Map

In order to include an indeterminate framework in the PESTEL analysis Neutrosophic Cognitive Maps (NCMs) are used extensively in this research. Neutrosophic Cognitive Maps (NCMs) are regarded as generalization of Fuzzy Cognitive Maps (FCMs). Fuzzy Cognitive Maps with their possible applications are well explained in [24]. The nodes in FCMs represent events or variables which are modelled to ascertain the possible relationship among them. The arcs among the nodes shows the relationship among nodes which could be positive or negative. These relationships are termed as causal where '+1' indicates the positivity and '-1' indicates the negativity of relation. Though FCMs are very much effective in modelling any situation but it lacks on certain grounds like it could not model uncertain, indeterminate and not known relations. To the rescue Neutrosophic Cognitive Maps (NCMs) are introduced in [18] [22] [31] are a way different from fuzzy cognitive maps (FCMs). FCMs do not include the notion of indeterminacy in them which is always present is neutrosophic cognitive mapping making it more efficient and accurate.

Below shown framework in Figure 2 presents a way to analyze factors for identifying and characterizing barriers to smart cities development in India with a model called PESTEL.

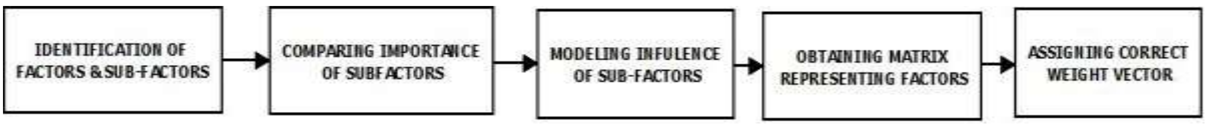

Figure 3 Framework for obtaining characteristics in every factor being analyzed by PESTEL model

The analysis using PESTEL has gained popularity since its mention in [17]. The term PESTEL was coined in the book titled "Exploring Corporate Strategy" by Johnson and Scholes. PESTEL analysis is a technique that strategically tries to identify the external environment that influences the factors such as political, economic, social, technological, environmental and legal. The factors obtained are later integrated corresponding to analysis of PESTEL and then modelled using Neutrosophic Cognitive Maps (NCMs). This modelling provides for quantitative analysis of characteristics under consideration which correspond to analysis of factors. Further a neutrosophic adjacency matrix is formulated. By taking into consideration the adjacency matrix and their absolute weights the static analysis is applied. When it is talked about static analysis in neutrosophy or neutrosophic cognitive maps [26] the special types of numbers called neutrosophic numbers like "$a + bI$" are taken in considerations where "a" and "b" are real numbers and "I" is called indetermination [27]. This not only deals with neutrosiphication but also de- neutrosiphication. This is well proposed in [28] by Salmeron and Smarandache where $I \in (0,1)$ and takes the value which is minimum or maximum. For static analysis of factors some measures are the need of the hour. The measures which are used extensively in proposed model are described below by equations 2-6. These measures are based on the absolute values of adjacency matrix [25]. These measures are used in further calculations for PESTEL analysis. Now let understand each one by one:

Definition 5. Out-degree in any graph represents the strength of outgoing relationship of a variable. It is described as sum of all elements of a row in corresponding neutrosophic adjacency matrix. It is represented in equation numbered 2.

$$\text{Outdegree (node)} = \sum_{i=1}^{n} |c_{ij}|, \tag{2}$$

Definition 6. In-degree in any graph represents the incoming relations from a variable. It is described as sum of all column elements in corresponding neutrosophic adjacency matrix. It is represented in equation numbered 3.

$$\text{Indegree (node)} = \sum_{i=1}^{n} |c_{ij}|, \tag{3}$$

Definition 7. Total degree or total centrality can be expressed as the sum of in-degree and out-degree. Equation 4 gives its mathematical notation.

$$\text{Total Degree} = \text{Indegree (node)} + \text{Outdegree (node)}, \tag{4}$$

Finally using equations 5 and 6 the averages of extreme values is calculated which is mostly used to obtain single value used in calculation [29]. This value is used in our case study for recognition of the features or characteristics.

$$\partial([a, b]) = \frac{a+b}{2}, \tag{5}$$

Then,

$$A > B \leftrightarrow \frac{a+b}{2} > \frac{c+d}{2}, \tag{6}$$

3. Results and Discussion

Initially the barriers are identified by doing extensive literature survey and taking experts opinion in this regard [16]. The experts are from academia, industry and public-private organizations. This study takes into account all types of barriers to smart cities development in India; which are mentioned in the related literature. The number of sub-factors under each category is same in order to maintain the homogeneity of calculation. Later these are grouped in six categories namely social, political, legal, ethical and technological for ease of carrying out effective analysis. These sub-factors and factors when summarized can be illustrated as follows:

- *Political Factors:* These factors are crucial to be considered is this regard since all policies and financial aid is being issued from government agencies. These sub-factors within this could be lack of trust between governed and the government together with lack of developing a common information system model as given in [33-34].

- *Economic Factors:* These factors are of utmost importance since all companies whether national, international directly get affected by these factors. The sub-factors within this may be the cost of high infrastructure together with training & skill development as mentioned in [35].
- *Social Factors:* The present work deals with the barriers to smart cities development in India. This has direct association with demographic changes which are happening all around the globe [30]. Social factors includes sub-factors such as lack of citizen's participation together with low consciousness of the community.
- *Technological Factors:* It is also an important factor that should be taken into consideration since this factor has direct implication on the quality of services which are required for development of smart cities. The sub factors here are lack of technological knowledge, privacy and security issues as mentioned in [36-37].
- *Environmental Factors:* The sustainability consideration is one of factor that is always taken in mind when such projects are undertaken. The sub-factors under Environmental factors are lacking ecological view in behavior and lack of sustainability considerations [33] [38].
- *Legal & Ethical Factors:* These factors mainly include cultural issues and issues in openness of data [35] [39]. Figure 3 below gives the grouping of sub-factors in six different main factors.

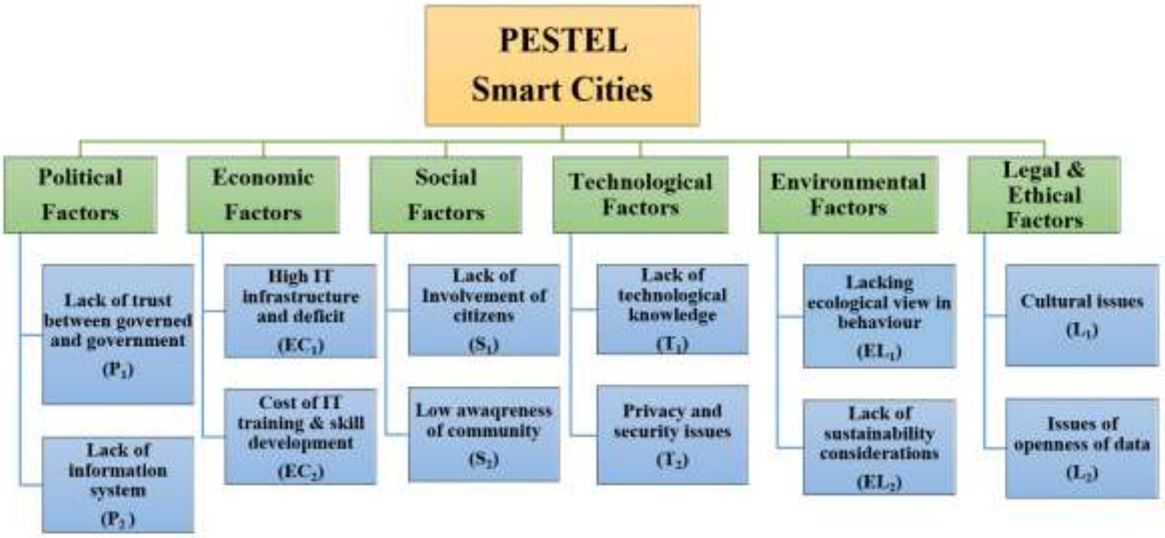

Figure 4 PESTEL hierarchical model for identifying key barriers to smart cities development in India.

Since, all above mentioned factors which are analyzed using PESTEL technique are always linguistic. Therefore in order to analyses them, these factors need to be quantified so to obtain higher interpretability. In order to quantify all the linguistic terms the technique of neutrosophic cognitive maps [31] is used. Now taking in account all the factors which are being considered in this study for analyzing the barriers to smart cities development in India, the NCM is formed. Later this cognitive mapping is used to form the neutrosophic adjacency matrix which forms the basis of all our further calculations. This adjacency matrix is represented in Table 1 below.

Table 1 Neutrosophic Adjacency Matrix

Variables	P1	P2	EC1	EC2	S1	S2	T1	T2	EL1	EL2	L1	L2
P1	0	0	0	-0.37	0	0	0	0	0	0	0	0
P2	0	0	0	0	0	0	0.31	0	0	0	0	0
EC1	0	0	0	0	0	0	0	0	0	0	0	0
EC2	0	0	0	0	0	0	0	0.37	0	0	0	0
S1	0.49	I	0	0	0	0	0.37	0	0	0	0	0
S2	0	0	0	0	I	0	0	0	0	0	0	0

T1	0	0	0	0	0	0	0	0	0	0	0	0
T2	0	0	0	0.46	0	0	0	0	0	0	0	0
EL1	0	0	0	0	0	0	0	0	0.31	0	0	0
EL2	0	0	0	0	0	0	0	0	0	0	0	0
L1	0	0	0	0	0	0	0	0	0	0.37	0	0
L2	0	0	0	0	0	0	0	0	0	0	0	0.25

Now for static analysis of factors the measures of centrality are calculated. The calculation which is required for calculating the measures of centrality is done using equations 2, 3 and 4. These measures of centrality are based on in-degree and out-degree measures. Table 2 below shows the measures of centrality.

Table 2 Measures of centrality, out-degree, and in-degree

Node	**Id**	**Od**	**Total degree**
P1	0.49	0.37	0.86
P2	I	0.31	I+0.31
EC1	0	0	0
EC2	0.83	0.37	1.20
S1	I	I+0.86	2I+0.86
S2	0	I	I
T1	0.68	0	0.68
T2	0.37	0.46	0.83
EL1	0.31	0.31	0.62
EL2	0.37	0	0.37
L1	0	0.37	0.37
L2	0.25	0.25	0.50

3.1 Results

The results are obtained after the process of de-neutrosophication. This may be referred from Salmeron and Smarandache's work in [25]. *'I'* is replaced by minimum and maximum values in the range *(0, 1)*. The results which are obtained in Table 2 above are being converted to intervals if it contains a value which is indeterminate i.e. *'I'*. Further the de-neutrosiphication function which is represented using symbol ∂ is applied according to equation number 5. The value of ∂ forms the basis for setting up orders of preferences of the barriers which put hindrance to the smart cities development in India. It is represented in Table 3.

Table 3 Total degree, de-neutrosophicated and ordinal number of every variable

Variable	**Final Value**	**∂ (Vi)**	**Order of Preference**
P1	0.86	0.86	3
P2	[0.31, 1.31]	0.81	5
EC1	0	0	10
EC2	1.20	1.20	2
S1	[0.86, 1.86]	1.36	1
S2	[0, 1]	0.50	8
T1	0.68	0.68	6
T2	0.83	0.83	4
EL1	0.62	0.62	7

EL2	0.37	0.37	9
L1	0.37	0.37	9
L2	0.50	0.50	8

According to neutrosophic PESTEL analysis the results obtained are as follows$S1 > EC2 > P1 > T2 > P2 > T1 > EL1 > L2 > L1 > EC1$. The symbol '>' indicates the preference of the factor over the other. This shows the comparison among sub-factors indicating at ground level the importance of each sub-factor. But if comparison is needed taking in consideration all the factors the average of sub-factors corresponding to particular factor is taken. Later this average is converted in percentage and the results are shown using following pie chart in Figure 4.

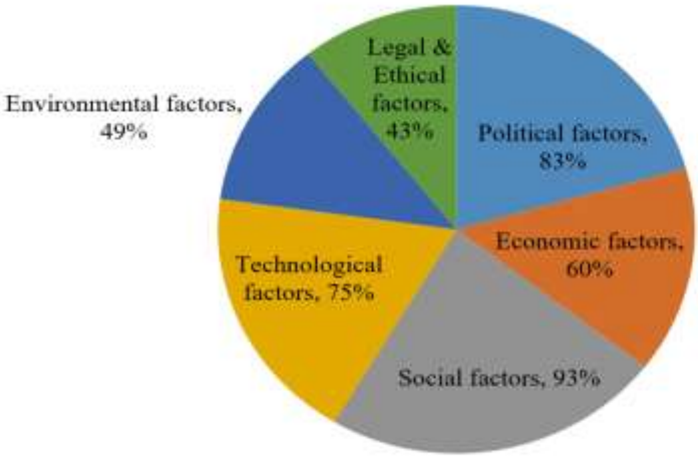

Figure 5 Neutrosophic PESTEL Analysis by Factors

Now overall PESTEL analysis of the factors is done. The results obtained indicate that social factors and political factors are the key barriers to the smart cities development in India contributing for 93% and 83% respectively. Technological and economic factors come at second position contributing 75% and 60% respectively. Environmental and legal factors come at last securing 49% and 43% respectively.

4. Conclusion

The present work seeks to find out the key barriers linked to smart cities development in India using neutrosophic PESTEL technique. The present work contributes to existing research in the related field by providing more realistic modelling of the situation using Neutrosophy which as per our knowledge is not applied in this regard earlier. This work is the need of the hour since there are many projects of the Government, for the development of smart cities, but these are not attaining their expected on-time results due to some uncertain reasons. In the present work, a comprehensive literature review through various literature surveys has disclosed various factors and sub-factors putting hindrance in such projects. Since PESTEL analysis deals with political, economic, social, technological, environmental and legal factors, these sub- factors are grouped in six key barriers. These barriers are modelled using neutrosophic cognitive maps for quantitative analysis and later neutrosophic adjacency matrix is formulated. Further, a comprehensive static analysis is done in order to identify and prioritize the key barriers. The findings of current research are mentioned below:

- The social and political factors contributing 93% and 83% respectively are the key barriers to the development of smart cities in India.

- Technological and economic factors come at second position contributing 75% and 60% respectively as a barrier.
- Environmental and legal factors come at last securing 49% and 43% respectively.

The present works also demonstrate that neutrosophic PESTEL analysis can be applied to more complex problems. Future work in this regard includes implementing and designing machine learning algorithms for carrying out the simulation using neutrosophic theories. Earlier proposed algorithms in machine learning for PESTEL analysis might be combined with neutrosophic approaches so that the output obtained could be validated with more optimized results.

References:

[1] Albino, V., Berardi, U., & Dangelico, R. M. Smart cities: Definitions, dimensions, performance, and initiatives. Journal ofUrban Technology, 22(1), 3–21. (2015).
[2] Koo, C., Ricci, F., Cobanoglu, C., & Okumus, F. Special issue on smart, connected hospitality and tourism. Information Systems Frontiers, 19(4), 699–703. (2017).
[3] Mori, K., & Christodoulou, A. Review of sustainability indices and indicators: Towards a new City sustainability index (CSI). Environmental Impact Assessment Review, 32(1), 94–106. (2012).
[4] Bakıcı, T.,Almirall, E.,&Wareham, J. Asmart city initiative: The case of Barcelona. Journal of the Knowledge Economy, 4(2), 135–148. (2013).
[5] Cruz-Jesus, F., Oliveira, T., Bacao, F., &Irani, Z. Assessing the pattern between economic and digital development of countries. Information Systems Frontiers, 19(4), 835–854. (2017).
[6] Washburn, D., Sindhu, U., Balaouras, S., Dines, R. A., Hayes, N. M., &Nelson, L. E. Helping CIOs understand BSmart City^ initiatives: Defining the smart city, its drivers, and the role of the CIO. Cambridge: Forrester Research. (2010).
[7] Zygiaris, S. Smart city reference model: Assisting planners to conceptualize the building of smart city innovation ecosystems. Journal of the Knowledge Economy, 4(2), 217–231. (2013).
[8] Chatterjee, S., & Kar, A. K. Alignment of IT authority and citizens of proposed smart cities in India: System security and privacy perspective. Global Journal of Flexible Systems Management, 19(1), 95–107. (2018b).
[9] Mckinsey Report. Combating the challenges of urbanization in emerging markets: Lessons from India. Accessed from https://www.mckinsey.com/industries/capital-projects-and-infrastructure/ourinsights/combating-the-challenges-of-urbanization-in-emergingmarkets-lessonsfromIndiaon. 22th April 2018.
[10] UNFPA. State of World population: Unleashing the potential of urban growth. United Nations Population Fund (UNFPA), New York, 2008.
[11] OCED. "The path to happiness lies in good health and a good job, the better life index shows". The organisation for economic cooperationand development (OECD), Paris, France (2015).
[12] http://www.oecd.org/newsroom/the-path-tohappiness-lies-in-goodhealth-and-a-good-job-the-better-life-index-shows.htm
[13] The Indian Express. "Quality of life: Which is the best Indian city to live in?" (2016). http://indianexpress.com/article/india/indianews-india/where-to-live-in-india-which-city-offersthe bestquality-of-living-2891060/
[14] Chourabi, H., Nam, T.,Walker, S., Gil-Garcia, J. R.,Mellouli, S., Nahon, K., Pardo, T. A. & Scholl, H. J. Understanding smart cities: An integrative framework. 45th Hawaii international conference on system science, 2289–2297. (2012).
[15] Chauhan, S., Agarwal, N., & Kar, A. K. Addressing big data challenges in smart cities: A systematic literature review. info,18(4), 73–90. (2016).
[16] Angelidou, M. "Smart cities: A conjuncture of four forces". Cities, 47, 95–106. (2016).
[17] Rana, Nripendra P., et al. "Barriers to the development of smart cities in Indian context." Information Systems Frontiers 21.3: 503-525. (2019).

[18] Gassner, M. (2008), PESTEL- Strategy against the mastery of external risks (PESTEL–Strategie zur Beherrschung ex-terner Risiken)(In German), . Düsseldorf: Symposion. Available in http://www.symposion.de/kapitel33630101_WERK7001009.html, Consulted on March 21, 2018.
[19] Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability: Neutrsophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability: Infinite Study. (2005)
[20] Parraga Alava, R., Muñoz Murillo, J., Barre Zambrano, R., Zambrano Vélez, M. I., and Leyva Vázquez, M. Y. PEST Analysis Based on Neutrosophic Cognitive Maps: A Case Study for Food Industry. Neutrosophic Sets and Systems, 21, 84-92. (2018).
[21] Wang, H., Smarandache, F., Sunderraman, R., and Zhang, Y. Q. Interval Neutrosophic Sets and Logic:Theory and Applications in Computing: Theory and Applications in Computing : Hexis. (2005).
[22] Menendez Vera, P. J., Menendez Delgado, C. F., Pena Gonzalez, M., and Leyva Vazquez, M. Marketing skills as determinants that underpin the competitiveness of the rice industry in Yaguachi canton. (2005) Application of SVN numbers to the prioritization of strategies, Neutrosophic Sets and Systems, 13, 70-7
[23] Kandasamy, W.V. and F. Smarandache. Fuzzy Neutrosophic Models for Social Scientists. Education Publisher Inc. (2015).
[24] Srinivas and Shekar. Cognitive Maps. (1997). https://web.itu.edu.tr/topcuil/ya/MDM02xCognitiveMaps.pdf
[25] Axelrod, R.M.Structure of decision: The cognitive maps of political elites: Princeton, NJ, Princeton University Press. (1976).
[26] Salmerona, J.L. and F. Smarandache. Redesigning Decision Matrix Method with an indeterminacy-based inference process. Multispace and Multistructure. Neutrosophic Transdisciplinarity (100 Collected Papers of Sciences), 4: p. 151. (2010)
[27] Bello Lara, R., González Espinosa, S., Martín Ravelo, A., Leyva Vázquez M. Y. Model of static analysis in fuzzy graphs based on compound indicators of centrality (Modelo para el análisis estáticoen grafos difusos basado en indicado-res compuestos de centralidad)(In Spanish). Revista Cubana de Ciencias Informáticas. 9(2), 52-65. (2015).
[28] Smarandache, F. Refined literal indeterminacy and the multiplication law of sub indeterminacies. Neutrosophic Sets and Systems, 9, 58-63. (2015).
[29] Salmerona, J.L. and F. Smarandache. Redesigning Decision Matrix Method with an indeterminacy-based inference process. Multispace and Multistructure. Neutrosophic Transdisciplinarity (100 Collected Papers of Sciences), 4: p. 151. (2010).
[30] Merigó, J. New extensions to the OWA operators and its application in decision making, PhD Thesis, Department of Business Administration, University of Barcelona, Spain. (2008).
[31] Yüksel, İ. Developing a multi-criteria decision making model for PESTEL analysis. International Journal of Busi-ness and Management, 7(24), 52-66. (2012)
[32] Zafar, Aasim, and Mohd Anas Wajid. "Neutrosophic Cognitive Maps for Situation Analysis." Neutrosophic Sets & Systems 29. (2019).
[33] Navas, J.E.; Guerras, L.A. The Management of the Enterprise: Theory and Applications (La Dirección Estratégica de la Empresa. Teoría y Aplicaciones)(In Spanish), Thompson Civitas. (2007).
[34] Kogan, N. and Lee, K.J. Exploratory research on success factors and challenges of smart city projects. MSc Thesis, Kyung Hee University, Seoul, South Korea. (2014).
[35] Letaifa, S. B. How to strategize smart cities: Revealing the SMART model. Journal of Business Research, 68(7), 1414–1419. (2015).
[36] Monzon, A. Smart cities concept and challenges: Bases for the assessment of smart city projects. International conference on smart cities and green ICT systems, 17–31. (2015).
[37] Scuotto, V., Ferraris, A., & Bresciani, S. Internet of things: Applications and challenges in smart cities: A case study of IBM smart city projects. Business Process Management Journal, 22(2), 357–367. (2016).
[38] Elmaghraby, A. S., & Losavio, M. M. Cyber security challenges in smart cities: Safety, security and privacy. Journal of Advanced Research, 5(4), 491–497. (2014).
[39] Yoon, A. How smart cities enable urban sustainability. (2015). https://www.triplepundit.com/2015/08/smart-citiesenable-urban-sustainability/

[40] Nam, T. and Pardo, T. A. Conceptualizing smart city with dimensions of technology, people, and institutions. 12th Annual International Digital Government Research Conference: Digital Government Innovation in Challenging Times, 282-291. (2011).
[41] Abdel-Basset, Mohamed, M. Saleh, Abduallah Gamal, and Florentin Smarandache. "An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number." Applied Soft Computing 77 (2019): 438-452.
[42] Abdel-Baset, Mohamed, Victor Chang, Abduallah Gamal, and Florentin Smarandache. "An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field." Computers in Industry 106 (2019): 94-110.
[43] Abdel-Baset, Mohamed, Victor Chang, and Abduallah Gamal. "Evaluation of the green supply chain management practices: A novel neutrosophic approach." Computers in Industry 108 (2019): 210-220.
[44] Abdel-Basset, Mohamed, Abduallah Gamal, Ripon K. Chakrabortty, and Michael Ryan. "Development of a hybrid multi-criteria decision-making approach for sustainability evaluation of bioenergy production technologies: A case study." Journal of Cleaner Production 290 (2021): 125805.
[45] Abdel-Basset, Mohamed, Abduallah Gamal, Ripon K. Chakrabortty, and Michael J. Ryan. "Evaluation approach for sustainable renewable energy systems under uncertain environment: A case study." Renewable Energy 168 (2021): 1073-1095.

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Some New Structures in Neutrosophic Metric Spaces

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Abstract: Neutrosophic sets deals with inconsistent, indeterminate and imprecise datas. The concept of Neutrosophic Metric Space (NMS) uses the idea of continuous t- norm and continuous t - conorm in intuitionistic fuzzy metric spaces. In this paper, we introduce the definition of subcompatible maps of types (J-1 and J-2). We extend the structure of weak non-Archimedian with the help of subcompatible maps of types (J-1 and J-2) in NMS. Finally, we obtain common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

Keywords: Weak non-Archimedean, NMS, Compatible map, Sub compatible, Subcompatible maps of types (J-1) and (J-2).

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1. **Introduction**

Fuzzy set was presented by Zadeh [22] as a class of elements with a grade of membership. Kramosil and Michalek [8] defined new notion called Fuzzy Metric Space (FMS). Later, many authors have examined the concept of fuzzy metric in various aspects. In 2013, Muthuraj and Pandiselvi [17] introduced the concept of compatible mappings of type (P-1) andtype (P-2) in generalized fuzzy metric spaces and obtains common fixed point theorems are obtained forcompatible maps of type (P-1) and type (P- 2). Since then, many authors have obtained fixed point results in fuzzy metric space using these compatible notions.

Atanassov [1] introduced and studied the notion of intuitionistic fuzzy set by generalizing the notion of fuzzy set. Park [9] defined the notion of intuitionistic fuzzy metric space as a generalization of fuzzy metric space. In 1998, Smarandache [14-16] characterized the new concept called neutrosophic logic and neutrosophic set and explored many results in it. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. Baset et al. [2] Explored the neutrosophic applications in dif and only iferent fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making.

In 2019, Kirisci et al [9] defined NMS as a generalization of IFMS and brings about fixed point theorems in complete NMS. Erduran et.al.[13] introduced the concept of weak non-Archimedean intuitionistic fuzzy metric space and proved a common fixed point theorem for a pair of generalized (φ, Ψ) – contractive mappings. Later Jeyaraman at el [19,20] proved Fixed point results in non-Archimedean generalized intuitionistic fuzzy metric spaces. In 2020, Sowndrarajan Jeyaraman and Florentin Smarandache [18] proved some fixed point results for contraction theorems in neutrosophic metric spaces.

In this paper, we introduce the definition of sub compatible maps and sub compatible maps of types (J-1) and (J-2) in weak non-Archimedean NMS and give some examples and relationship between these definitions. We extend the structure of weak non-Archimedian with the help of subcompatible maps of types (J-1 and J-2) in NMS. Thereafter, we prove common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

2. **Preliminaries**

Definition: 2.1

A binary operation * : [0, 1] x [0, 1] → [0, 1] is a continuous t-norm [CTN] if it satisfies the following conditions :

(i) * is commutative and associative,

(ii) * is continuous,

(iii) $\varepsilon_1 * 1 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,

(iv) $\varepsilon_1 * \varepsilon_2 \leq \varepsilon_3 * \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, 1]$.

Definition: 2.2

A binary operation ⋄ : [0, 1] x [0, 1] → [0, 1] is a continuous t-conorm [CTC] if it satisfies the following conditions:

(i) ⋄ is commutative and associative,

(ii) ⋄ is continuous,

(iii) $\varepsilon_1 \diamond 0 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,

(iv) $\varepsilon_1 \diamond \varepsilon_2 \leq \varepsilon_3 \diamond \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4 \in [0, 1]$.

Definition: 2.3

A 6-tuple $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is said to be an NMS (shortly NMS), if Σ is an arbitrary non empty set, $*$ is a neutrosophic CTN, $\diamond$ is a neutrosophic CTC and Ξ, Θ *and* Υ are neutrosophic on $\Sigma^3 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\zeta, \eta, \delta, \omega \in \Sigma$, $\lambda \in \mathbb{R}^+$.

1. $0 \leq \Xi(\zeta, \eta, \delta, \lambda) \leq 1$; $0 \leq \Theta(\zeta, \eta, \delta, \lambda) \leq 1$; $0 \leq \Upsilon(\zeta, \eta, \delta, \lambda) \leq 1$;
2. $\Xi(\zeta, \eta, \delta, \lambda) + \Theta(\zeta, \eta, \delta, \lambda) + \Upsilon(\zeta, \eta, \delta, \lambda) \leq 3$;
3. $\Xi(\zeta, \eta, \delta, \lambda) = 1$ if and only if $\zeta = \eta = \delta$;
4. $\Xi(\zeta, \eta, \delta, \lambda) = \Xi(\rho(\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
5. $\Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, \mu) \leq \Xi(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;

6. $\Xi(\zeta, \eta, \delta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous ;
7. $\lim_{\lambda\to\infty} \Xi(\zeta, \eta, \delta, \lambda) = 1$ for all $\lambda > 0$;
8. $\Theta(\zeta, \eta, \delta, \lambda) = 0$ if and only if $\zeta = \eta = \delta$;
9. $\Theta(\zeta, \eta, \delta, \lambda) = \Theta(\rho(\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
10. $\Theta(\zeta, \eta, \omega, \lambda) \diamond \Theta(\omega, \delta, \delta, \mu) \geq \Theta(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
11. $\Theta(\zeta, \eta, \delta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous;
12. $\lim_{\lambda\to\infty} \Theta(\zeta, \eta, \delta, \lambda) = 0$ for all $\lambda > 0$;
13. $\Upsilon(\zeta, \eta, \delta, \lambda) = 0$ if and only if $\zeta = \eta = \delta$;
14. $\Upsilon(\zeta, \eta, \delta, \lambda) = \Upsilon(\rho(\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
15. $\Upsilon(\zeta, \eta, \omega, \lambda) \diamond \Upsilon(\omega, \delta, \delta, \mu) \geq \Upsilon(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
16. $\Upsilon(\zeta, \eta, \delta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous;
17. $\lim_{\lambda\to\infty} \Upsilon(\zeta, \eta, \delta, \lambda) = 0$ for all $\lambda > 0$;
18. If $\lambda > 0$ then $\Xi(\zeta, \eta, \delta, \lambda) = 0$; $\Theta(\zeta, \eta, \delta, \lambda) = 1$; $\Upsilon(\zeta, \eta, \delta, \lambda) = 1$.

Then, (Ξ, Θ, Υ) is called an NMS on Σ. The functions Ξ, Θ *and* Υ denote degree of closedness, neturalness and non-closedness between ζ, η *and* δ with respect to λ respectively.

Example: 2.4

Let (Σ, D) be a metric space. Define $\omega * \tau = \min\{\omega, \tau\}$ and $\omega \diamond \tau = \max\{\omega, \tau\}$ and $\Xi, \Theta, \Upsilon : \Sigma^3 \times \mathbb{R}^+ \to [0, 1]$ defined by, we define $\Xi(\zeta, \eta, \delta, \lambda) = \frac{\lambda}{\lambda + D(\zeta,\eta,\delta)}$; $\Theta(\zeta, \eta, \delta, \lambda) = \frac{D(\zeta,\eta,\delta)}{\lambda + D(\zeta,\eta,\delta)}$; $\Upsilon(\zeta, \eta, \delta, \lambda) = \frac{D(\zeta,\eta,\delta)}{\lambda}$ for all $\zeta, \eta, \delta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is called NMS induced by a metric D the standard neutrosophic metric.

Remark: 2.5

In NMS$\Xi(\zeta, \eta, \delta, \lambda, .)$ is non-decreasing, $\Theta(\zeta, \eta, \delta, .)$ is non-increasing and $\Upsilon(\zeta, \eta, \delta, .)$ is decreasing for all $\zeta, \eta, \delta \in \Sigma$.

In the above definition, if the triangular inequality (v), (x) and (xv) are replaced by the following:

$$\Xi(\zeta, \eta, \delta, \max\{\lambda, \mu\}) \geq \Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, \mu),$$
$$\Theta(\zeta, \eta, \delta, \min\{\lambda, \mu\}) \leq \Theta(\zeta, \eta, \omega, \lambda) \diamond \Theta(\omega, \delta, \delta, \mu),$$
$$\Upsilon(\zeta, \eta, \delta, \min\{\lambda, \mu\}) \leq \Upsilon(\zeta, \eta, \omega, \lambda) \diamond \Upsilon(\omega, \delta, \delta, \mu)$$

or equivalently

$$\Xi(\zeta, \eta, \delta, \lambda) \geq \Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, \lambda),$$
$$\Theta(\zeta, \eta, \delta, \lambda) \leq \Theta(\zeta, \eta, \omega, \lambda) \diamond \Theta(\omega, \delta, \delta, \lambda),$$
$$\Upsilon(\zeta, \eta, \delta, \lambda) \leq \Upsilon(\zeta, \eta, \omega, \lambda) \diamond \Upsilon(\omega, \delta, \delta, \lambda).$$

Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is called non-Archimedean NMS. It is easy to check that the triangle inequality (NA) implies (5), (10) and (15), that is, every non-Archimedean NMS is itself an NMS.

Example:2.6

Let Σ be a non-empty set with at least two elements. Define $\Xi(\zeta, \eta, \delta, \lambda)$ by: If we define the neutrosophic set $(\Sigma, \Xi, \Theta, \Upsilon)$ by $\Xi(\zeta, \zeta, \zeta, \lambda) = 1$, $\Theta(\zeta, \zeta, \zeta, \lambda) = 0$ and $\Upsilon(\zeta, \zeta, \zeta, \lambda) = 0$ for all $\zeta \in \Sigma$ and $\lambda > 0$, and $\Xi(\zeta, \eta, \delta, \lambda) = 0$, $\Theta(\zeta, \eta, \delta, \lambda) = 1$ and $\Upsilon(\zeta, \eta, \delta, \lambda) = 1$, for $\zeta \neq \eta \neq \delta$ and $0 < \lambda \leq 1$, and $\Xi(\zeta, \eta, \delta, \lambda) = 1$, $\Theta(\zeta, \eta, \delta, \lambda) = 0$ and $\Upsilon(\zeta, \eta, \delta, \lambda) = 0$, for $\zeta \neq \eta \neq \delta$ and $\lambda > 1$. Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a non-Archimedean NMS with arbitrary $*$ is a neutrosophic CTN, $\diamond$ is a neutrosophic CTC. Clearly $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is also an NMS.

Definition:2.7

In Definition 2.3, if the triangular inequality (v), (x) and (xv) are replaced by the following:

$\Xi(\zeta,\eta,\delta,\lambda) \geq \max\{\Xi(\zeta,\eta,\omega,\lambda) * \Xi(\omega,\delta,\delta,,\lambda/2), \Xi(\zeta,\eta,\omega,\lambda/2) * \Xi(\omega,\delta,\delta,\lambda)\}$,

$\Theta(\zeta,\eta,\delta,\lambda) \leq \min\{\Theta(\zeta,\eta,\omega,\lambda) \diamond \Theta(\omega,\delta,\delta,,\lambda/2), \Theta(\zeta,\eta,\omega,\lambda/2) \diamond \Theta(\omega,\delta,\delta,\lambda)\}$,

$\Upsilon(\zeta,\eta,\delta,\lambda) \leq \min\{\Upsilon(\zeta,\eta,\omega,\lambda) \diamond \Upsilon(\omega,\delta,\delta,,\lambda/2), \Upsilon(\zeta,\eta,\omega,\lambda/2) \diamond \Upsilon(\omega,\delta,\delta,\lambda)\}$,

for all $\Xi,\Theta,\Upsilon \in \Sigma$ and $\lambda > 0$, then $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$is said to be a Weak Non- Archimedean (WNA) NMS.

Obviously, every non-Archimedean NMS is itself a weak non-Archimedean NMS.

The inequality (WNA) does not imply that $\Xi(\zeta,\eta,\delta,\lambda,.)$ is non-decreasing , $\Theta(\zeta,\eta,\delta,.)$ is non-increasing and $\Upsilon(\zeta,\eta,\delta,.)$ is decreasing. Thus, a weak non-Archimedean NMS is not necessarily an NMS.

Example: 2.8

Let $\Sigma=[0,\infty)$ and define $\Xi(\zeta,\eta,\delta,\lambda)$; $\Theta(\zeta,\eta,\delta,\lambda)$ and $\Upsilon(\zeta,\eta,\delta,\lambda)$ by

$$\Xi(\zeta,\eta,\delta,\lambda) = \begin{cases} 1, & \zeta=\eta=\delta \\ \frac{\lambda}{\lambda+1}, & \zeta \neq \eta \neq \delta \end{cases},$$

$$\Theta(\zeta,\eta,\delta,\lambda) = \begin{cases} 0, & \zeta=\eta=\delta \\ \frac{1}{\lambda+1}, & \zeta \neq \eta \neq \delta \end{cases},$$

$$\Upsilon(\zeta,\eta,\delta,\lambda) = \begin{cases} 0, & \zeta=\eta=\delta \\ \lambda+1, & \zeta \neq \eta \neq \delta \end{cases},$$

for all $\lambda > 0$. $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega+\tau-\omega\tau\}$ for every $\omega, \tau \in [0,1]$.

Definition: 2.9

Let Γ and Ω be maps from an NMS $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0,$$

for all $\lambda>0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition: 2.10

Let Γ and Ω be self mappings of an NMS $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$. Then the mappings are said to be compatible of type (J-1), if

$$\lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0,$$

for all $\lambda>0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition: 2.11

Let Γ and Ω be self mappings of an NMS $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$. Then the mappings are said to be compatible of type (J-2), if

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and}$$

$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$

for all $\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition:2.12

Let Γ and Ω be maps from an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ into itself. The maps Γ and Ω are said to be Occasionally Weakly Compatible (OWC) if and only if there is a point $\zeta \in \Sigma$ which is a coincidence point of Γ and Ωat which Γ and Ω commute i.e., there is a point $\zeta \in \Sigma$ such that $\Gamma\zeta = \Omega\zeta$ and $\Gamma\Omega\zeta = \Omega\Gamma\zeta$.

Definition:2.13

Let Γ and Ω be maps from an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$. The maps Γ and Ω are said to be reciprocally continuous if $\lim_{n\to\infty} \Gamma\Omega\zeta_n = \Gamma\zeta$, $\lim_{n\to\infty} \Omega\Gamma\zeta_n, = \Omega\zeta$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

3. Types Of Subcompatible Maps In Weak Non-Archimedean NMS.

Definition:3.1

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subsequently continuous if there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfy $\lim_{n\to\infty} \Gamma\mathcal{S}\zeta_n = \Gamma\zeta$, $\lim_{n\to\infty} \Omega\Gamma\zeta_n, = \Omega\zeta$.

Clearly, if Γ and $\Omega$are continuous or reciprocally continuous, then they are subsequentially continuous, but converse is not true in general.

Example: 3.2

Let $\Sigma = [0, \infty)$ and define, for all $\lambda > 0$, $\Xi(\zeta, \eta, \delta, \lambda)$; $\Theta(\zeta, \eta, \delta, \lambda)$ and $\Upsilon(\zeta, \eta, \delta, \lambda)$ by

$$\Xi(\zeta, \eta, \delta, \lambda) = \begin{cases} 1, & \zeta = \eta = \delta, \\ \frac{\lambda}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Theta(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \frac{1}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Upsilon(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \lambda+1, & \zeta \neq \eta \neq \delta. \end{cases}$$

Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$. Define Γ and Ωas follows:

$$\Gamma\zeta = \begin{cases} 2, & \zeta < 3 \\ \zeta, & \zeta \geq 3 \end{cases}, \quad \Omega\zeta = \begin{cases} 2\zeta - 4, & \zeta \leq 3, \\ 3, & \zeta > 3. \end{cases}$$

Clearly Γ and Ω are discontinuous at $\zeta = 3$. Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 3 - \frac{1}{n}$ for n = 1, 2 …, then $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 2$, $2 \in \Sigma$ and $\lim_{n\to\infty} \Gamma\Omega\zeta_n = 2 = \Gamma(2)$, $\lim_{n\to\infty} \Omega\Gamma\zeta_n = 0 = \Omega(2)$. Therefore, Γ and Ω are subsequentially continuous. Now, let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 3 + \frac{1}{n}$ for n = 1,2,…, then $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 3$, $3 \in \Sigma$ and $\lim_{n\to\infty} \Omega\Gamma\zeta_n = 3 \neq 2 = \Omega(3)$. Hence Γ and Ω are not reciprocally continuous.

Definition: 3.3

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subcompatible if and only if there exist a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfies

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0.$$

It is easy to see that two owc maps are subcompatible, however the converse is not true in general. It is also interesting to see the following one-way implication:

Commuting ⇒ Weakly commuting ⇒ Compatibility ⇒ Weak compatibility ⇒ OWC⇒ Sub compatibility.

Definition:3.4

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$be a weak non-Archimedean NMS. Self- maps Γ and Ωon Σ are said to be subcompatible of type (J-1) if there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfies

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Xi\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and,}$$
$$\lim_{n\to\infty} \Upsilon\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Clearly, if Γ and Ω are compatible of type (J-1), then they are subcompatible of type (J-1), but converse is not true in general.

Example: 3.5

Let $\Sigma = [0, \infty)$. Define $\Xi(\zeta, \eta, \delta, \lambda)$; $\Theta(\zeta, \eta, \delta, \lambda)$ and $\Upsilon(\zeta, \eta, \delta, \lambda)$ by
$\Xi(\zeta, \eta, \delta, \lambda) = \frac{\lambda}{\lambda+|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}$, $\Theta(\zeta, \eta, \delta, \lambda) = \frac{|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}{\lambda+|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}$ and $\Upsilon(\zeta, \eta, \delta, \lambda) = \frac{|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}{\lambda}$ for all $\lambda > 0$. Then, $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$.

Define Γ and Ω as follows:

$$\Gamma x = \begin{cases} \zeta^2 + 1, & \zeta < 1 \\ 2\zeta - 1, & \zeta \geq 1 \end{cases}, \quad \Omega\zeta = \begin{cases} \zeta + 1, & \zeta < 1 \\ 3\zeta - 2, & \zeta \geq 1 \end{cases}.$$

Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 1 + \frac{1}{n}$, for n = 1, 2…, then $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 1$, $1 \in \Sigma$ and

$$\Gamma\Omega\zeta_n = \Gamma\left(1 + \frac{3}{n}\right) = 2\left(1 + \frac{3}{n}\right) - 1 = 1 + \left(\frac{6}{n}\right),$$
$$\Omega\Gamma\zeta_n = \Omega\left(1 + \frac{2}{n}\right) = 3\left(1 + \frac{2}{n}\right) - 2 = 1 + \left(\frac{6}{n}\right),$$

$$\Gamma\Gamma\zeta_n = \Gamma\left(1+\frac{2}{n}\right) = 2\left(1+\frac{2}{n}\right)\text{-}1 = 1+\left(\frac{4}{n}\right),$$
$$\Omega\Omega\zeta_n = \Omega\left(1+\frac{3}{n}\right) = 3\left(1+\frac{3}{n}\right)\text{-}2 = 1+\left(\frac{9}{n}\right).$$

Therefore,

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

And,

$$\lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

That is, Γ and Ω are subcompatible of type (J-1) but if we consider a sequence $\zeta_n = 1\text{-}\frac{1}{n}$ for n = 1,2,…, then $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 2$, 2 ∈ Σ and

$$\Gamma\Omega\zeta_n = \Gamma\left(2-\frac{1}{n}\right) = 2\left(2-\frac{1}{n}\right)\text{-}1 = 3\text{-}\left(\frac{2}{n}\right),\ \Omega\Gamma\zeta_n = \Omega\left(\left(1-\frac{1}{n}\right)^2+1\right) = 3\left(\left(1-\frac{1}{n}\right)^2+1\right)\text{-}2,$$
$$\Gamma\Gamma\zeta_n = \Gamma\left(\left(1-\frac{1}{n}\right)^2+1\right) = \Gamma\left(1-\frac{2}{n}+\frac{1}{n^2}\right) = \left(1-\frac{2}{n}+\frac{1}{n^2}\right)^2+1,$$
$$\Omega\Omega\zeta_n = \Omega\left(2-\frac{1}{n}\right) = 3\left(2-\frac{1}{n}\right)\text{-}2 = 4\text{ - }\left(\frac{3}{n}\right).$$

Therefore,

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0,$$
$$\lim_{n\to\infty} \Xi\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \neq 1,$$
$$\lim_{n\to\infty} \Theta\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \neq 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \neq 0.$$

That is, Γ and Ω are not compatible of type (J-1).

Definition: 3.6

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subcompatible of type (J-1) if and only if there exist a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfies

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

Clearly, if Γ and Ω are compatible of type (J-2), then they are subcompatible of type (J-2), but converse is not true in general.

Example: 3.7

Let Σ = [0, ∞) and define $\Xi\,(\zeta, \eta, \delta, \lambda)$; $\Theta\,(\zeta, \eta, \delta, \lambda)$ and $\Upsilon\,(\zeta, \eta, \delta, \lambda)$ by

$$\Xi\,(\zeta, \eta, \delta, \lambda) = \begin{cases} 1, & \zeta = \eta = \delta, \\ \frac{\lambda}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Theta\ (\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \frac{1}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Upsilon\ (\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \lambda + 1, & \zeta \neq \eta \neq \delta. \end{cases}$$

Then, $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$. Define Γ and Ωas follows:

$$\Gamma\zeta = \zeta^2,\ \ \Omega\zeta = \begin{cases} \zeta + 2, & \zeta \in [0,4] \cup (5, \infty) \\ \zeta + 12, & \zeta \in (4,5]\ . \end{cases}$$

Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 2+\frac{1}{n}$ for n = 1,2…, then $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = 4$, and $\Gamma\Gamma\zeta_n = \Gamma\left(\left(2+\frac{1}{n}\right)^2\right) = \left(2+\frac{1}{n}\right)^4$, $\Omega\Omega\zeta_n = \Omega\left(4+\frac{1}{n}\right) = 4+\frac{1}{n} + 12 = 16 + \frac{1}{n}$.

Therefore,

$$\lim_{n\to\infty} \Xi\ (\Gamma\Gamma_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\ (\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\ (\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

That is, Γ and Ω are subcompatible of type (J-2) but if we consider a sequence $\zeta_n = 2-\frac{1}{n}$ for n = 1, 2,…, then $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = 4$ and $\Gamma\Gamma\zeta_n = \Gamma\left(\left(2-\frac{1}{n}\right)^2\right) = \left(2-\frac{1}{n}\right)^4$, $\Omega\Omega\zeta_n = \Omega\left(4-\frac{1}{n}\right) = 4-\frac{1}{n} + 2 = 6 - \frac{1}{n}$.

Therefore,

$$\lim_{n\to\infty} \Xi\ (\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 1,$$
$$\lim_{n\to\infty} \Theta\ (\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\ (\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0.$$

That is, Γ and Ω are not compatible of type (J-2).

Preposition: 3.8

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS and Γ, Ω: Σ → Σ are subsequentially continuous mappings. Γ and Ω are subcompatible maps if and only if they are not subcompatible of type (J-1).

Proof:

Suppose Γ and Ω are subcompatible, then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\lim_{n\to\infty} \Xi\ (\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 0.$$

Since Γ and Ωare subsequentially continuous, we have

$\lim_{n\to\infty}\Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty}\Gamma\Gamma\zeta_n$, $\lim_{n\to\infty}\Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty}\Omega\Omega\zeta_n$.

Thus, from the inequality (WNA), for all $\lambda > 0$,

$$\Xi\ (\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq \Xi\ (\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) * \Xi\ (\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Theta(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Upsilon\ (\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$

and it follows that

$$\Xi\ (\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,$$

$$\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \le 0 \diamond 0 = 0,$$
$$\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \le 0 \diamond 0 = 0.$$

That is, for all $\lambda > 0$,

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

By the same way,

$$\lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Consequently, Γ and Ω are subcompatible of type (J-1).

Conversely, suppose that Γ and Ω are subcompatible of type (J-1), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1, \lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0 \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0 \text{ and } \lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n\to\infty} \Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty} \Gamma\Gamma\zeta_n, \quad \lim_{n\to\infty} \Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty} \Omega\Omega\zeta_n.$$

Now, from the inequality (WNA), for all $\lambda > 0$,

$$\Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \ge \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) * \Xi(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \le \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \diamond \Theta(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \le \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \diamond \Upsilon(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$

and, it follows that,

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \ge 1 * 1 = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \le 0 \diamond 0 = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \le 0 \diamond 0 = 0,$$

which implies that

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0.$$

Therefore, Γ and Ω are subcompatible.

Preposition: 3.9

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS and $\Gamma, \Omega: \Sigma \to \Sigma$ are subsequentially continuous mappings. Γ and Ω are subcompatible maps if and only if they are not subcompatible of type (J-2).

Proof:

Suppose Γ and Ω are subcompatible, then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \delta$, $\delta \in \Sigma$ and satisfy

$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1$, $\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$, and $\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$.

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n\to\infty} \Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty} \Gamma\Gamma\zeta_n,\quad \lim_{n\to\infty} \Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty} \Omega\Omega\zeta_n.$$

Thus, from the inequality (WNA),

$$\begin{aligned}
\Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\geq \Xi\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) * \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2)\\
&\geq \Xi\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) * \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2) *\\
&\qquad \Xi(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/4),\\
\Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq \Theta\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2)\\
&\leq \Theta\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2) \diamond\\
&\qquad \Theta\,(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/4) \text{ and}\\
\Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq \Upsilon\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2)\\
&\leq \Upsilon\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2) \diamond\\
&\qquad \Upsilon\,(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/4),
\end{aligned}$$

for all $\lambda > 0$, and, it follows that, for all $\lambda > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,
\end{aligned}$$

which implies that,

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.
\end{aligned}$$

Consequently, Γ and Ω are subcompatible of type (J-2). Conversely, suppose that Γ and Ω are subcompatible of type (J-2), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.
\end{aligned}$$

Now, from the inequality (WNA), we have

$$\begin{aligned}
\Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\geq \Xi\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) * \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\geq \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) * \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\qquad * \Xi\,(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/4),\\
\Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\leq \Theta\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Theta(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\leq \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\qquad \diamond \Theta\,(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/4) \text{ and}\\
\Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\leq \Upsilon\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\leq \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\qquad \diamond \Upsilon\,(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/4),
\end{aligned}$$

and, it follows that, for all $\lambda > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \geq 1 * 1 * 1 = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \leq 0 \diamond 0 \diamond 0 = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \leq 0 \diamond 0 \diamond 0 = 0,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0.$$

Therefore, Γ and Ω are subcompatible.

Preposition: 3.10

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS and $\Gamma, \Omega: \Sigma \to \Sigma$ are subsequentially continuous mappings. Γ and Ω are subcompatible maps of type (J-1) if and only if they are subcompatible of type (J-2).

Proof:

Suppose Γ and Ωare subcompatible of type (J-1), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfy

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and,}$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Xi\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and,}$$
$$\lim_{n\to\infty} \Upsilon\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n\to\infty} \Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty} \Gamma\Gamma\zeta_n,\ \lim_{n\to\infty} \Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty} \Omega\Omega\zeta_n.$$

Thus, from the inequality (WNA),

$$\Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq \Xi\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) * \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Theta\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Upsilon\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$

and, it follows that

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$

which implies that

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

Therefore, Γ and Ωare subcompatible of type (J-2).

Conversely, suppose that Γ and Ω are subcompatible of type (J-2), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

Now, from the inequality (WNA), we have

$$\Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq \Xi(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) * \Xi(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Theta(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Theta(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Upsilon(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$

and, it follows that

$$\lim_{n\to\infty}(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,$$
$$\lim_{n\to\infty}\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$
$$\lim_{n\to\infty}\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$

which implies that, for all $\lambda > 0$,

$$\lim_{n\to\infty}(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

By the same way, we obtain that

$$\lim_{n\to\infty}(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Therefore, Γ and Ω are subcompatible of type (J-1).

4. Main Theorems

Theorem: 4.1

Let Γ, Λ, Ω and H be self-maps of a weak non-Archimedean NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ and let the pairs (Γ, Ω) and (Λ, H) are subcompatible maps of type(J-1) and subsequentially continuous.

$$\Xi(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \geq \psi(\min\{\Xi(\Omega\zeta, H\eta, H\eta, \lambda), \Xi(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi(\Lambda\eta, H\eta, H\eta, \lambda), \tfrac{1}{2}[\Xi(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \quad (4.1.1)$$
$$\Theta(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \phi(\max\{\Theta(\Omega\zeta, H\eta, H\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta(\Lambda\eta, H\eta, H\eta, \lambda), \tfrac{1}{2}[\Theta(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \quad (4.1.2)$$
$$\Upsilon(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \varphi(\max\{\Upsilon(\Omega\zeta, H\eta, H\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon(\Lambda\eta, H\eta, H\eta, \lambda), \tfrac{1}{2}[\Upsilon(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \quad (4.1.3)$$

for all $\zeta, \eta \in \Sigma$, $\lambda > 0$, where $\psi, \phi, \varphi : [0,1] \to [0,1]$ are continuous functions such that $\psi(s) > s$, $\phi(s) < s$ and $\varphi(s) < s$ for each $s \in (0,1)$. Then Γ, Λ, Ω and H have a unique common fixed point in Σ.

Proof

Since the pairs (Γ, Ω) and (Λ, H) are subcompatible maps of type (J-1) and subsequentially continuous, then there exist two sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \delta$, $\delta \in \Sigma$ and satisfy

$$\lim_{n\to\infty}\Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = \Xi(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = \Theta(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = \Upsilon(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 0,$$
$$\lim_{n\to\infty}\Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = \Xi(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = \Theta(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = \Upsilon(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 0.$$

$\lim_{n\to\infty}\Lambda\zeta_n = \lim_{n\to\infty}H\zeta_n = \omega$, $\omega \in \Sigma$, and

$$\lim_{n\to\infty}\Xi(\Lambda H\eta_n, HH\eta_n, HH\eta_n, \lambda) = \Xi(\Lambda\omega, H\omega, H\omega, \lambda) = 1,$$

$\lim_{n\to\infty} \Theta$ (ΛHη$_n$, HHη$_n$, HHη$_n$, λ) = Θ (Λω, Hω, Hω, λ) = 0,
$\lim_{n\to\infty} \Upsilon$ (ΛHη$_n$, HHη$_n$, HHη$_n$, λ) = ϒ(Λω, Hω, Hω, λ) = 0,
$\lim_{n\to\infty} \Xi$ (HΛη$_n$, ΛΛη$_n$, ΛΛη$_n$, λ) = Ξ(Hω, Λω, Λω, λ) = 1,
$\lim_{n\to\infty} \Theta$ (HΛη$_n$, ΛΛη$_n$, ΛΛη$_n$, λ) = Θ (Hω, Λω, Λω, λ) = 0,
$\lim_{n\to\infty} \Upsilon$ (HΛη$_n$, ΛΛη$_n$, ΛΛη$_n$, λ) = ϒ (Hω, Λω, Λω, λ) = 0.

Therefore, Γδ = Ωδ and Λω = Hω, that is δ is a coincidence point of Γ and Ω, ω is a coincidence point of Λ and H.Now, we prove that δ = ω. By using (3.1) for ζ = ζ$_n$ and η = η$_n$, we get
Ξ(Γζ$_n$, Λη$_n$, Λη$_n$, λ) ≥ ψ (min {Ξ(Ωζ$_n$, Hη$_n$, Hη$_n$, λ), Ξ(Γζ$_n$, Ωζ$_n$, Ωζ$_n$, λ), Ξ (Λη$_n$, Hη$_n$, Hη$_n$, λ), $\frac{1}{2}$[Ξ (Λη$_n$, Ωζ$_n$, Ωζ$_n$, λ) + Ξ (Γζ$_n$, Hη$_n$, Hη$_n$, λ)]}),
Θ (Γζ$_n$, Λη$_n$, Λη$_n$, λ) ≤ ϕ (max {Θ (Ωζ$_n$, Hη$_n$, Hη$_n$, λ) , Θ(Γζ$_n$, Ωζ$_n$, Ωζ$_n$, λ), Θ (Λη$_n$, Hη$_n$, Hη$_n$, λ), $\frac{1}{2}$[Θ (Λη$_n$, Ωζ$_n$, Ωζ$_n$, λ) + Θ (Γζ$_n$, Hη$_n$, Hη$_n$, λ)]}).
ϒ(Γζ$_n$, Λη$_n$, Λη$_n$, λ) ≤ φ(max {ϒ(Ωζ$_n$, Hη$_n$, Hη$_n$, λ) , ϒ(Γζ$_n$, Ωζ$_n$, Ωζ$_n$, λ), ϒ (Λη$_n$, Hη$_n$, Hη$_n$, λ), $\frac{1}{2}$[ϒ (Λη$_n$, Ωζ$_n$, Ωζ$_n$, λ) + ϒ (Γζ$_n$, Hη$_n$, Hη$_n$, λ)]}).

Taking the limit n → ∞, we have
Ξ (δ, ω, ω, λ) ≥ ψ (min { Ξ (δ, ω, ω, λ), Ξ (δ, δ, δ, λ) , Ξ (ω, ω, ω, λ), $\frac{1}{2}$ [Ξ (ω, δ, δ, λ) + Ξ (δ, ω, ω, λ)]}),
Θ (δ, ω, ω, λ) ≤ ϕ (max { Θ(δ, ω, ω, λ), Θ (δ, δ, δ, λ) , Θ (ω, ω, ω, λ), $\frac{1}{2}$ [Θ (ω, δ, δ, λ) + Θ (δ, ω, ω, λ)]}),
ϒ(δ, ω, ω, λ) ≤ φ (max { ϒ(δ, ω, ω, λ), ϒ(δ, δ, δ, λ), ϒ(ω, ω, ω, λ), $\frac{1}{2}$ [ϒ(ω, δ, δ, λ) + ϒ(δ, ω, ω, λ)]}),
that is,

Ξ (δ., ω, ω, λ) ≥ ψ (Ξ (δ, ω, ω, λ)) > Ξ (δ, ω, ω, λ),
Θ (δ., ω, ω, λ) ≤ ϕ (Θ (δ, ω, ω, λ)) < Θ (δ, ω, ω, λ),
ϒ (δ., ω, ω, λ) ≤ φ (ϒ (δ, ω, ω, λ)) < ϒ (δ, ω, ω, λ),

which yield δ = ω.

Again using (3.1) for ζ = δ and η = η$_n$, we obtain
Ξ (Γδ, Λη$_n$, Λη$_n$, λ) ≥ ψ (min {Ξ (Ωδ, Hη$_n$, Hη$_n$, λ), Ξ (Γδ, Ωδ, Ωδ, λ), Ξ (Λη$_n$, Hη$_n$, Hη$_n$, λ), $\frac{1}{2}$[Ξ (Λη$_n$, Ωδ, Ωδ, λ) + Ξ (Γδ, Hη$_n$, Hη$_n$, λ)]}),
Θ {Γδ, Λη$_n$, Λη$_n$, λ) ≤ ϕ (max {Θ (Ωδ, Hη$_n$, Hη$_n$, λ), Θ (Γδ, Ωδ, Ωδ, λ), Θ (Λη$_n$, Hη$_n$, Hη$_n$, λ), $\frac{1}{2}$[Θ (Λη$_n$, Ωδ, Ωδ, λ) + Θ (Γδ, Hη$_n$, Hη$_n$, λ)]}).
ϒ{Γδ, Λη$_n$, Λη$_n$, λ) ≤ φ(max {ϒ(Ωδ, Hη$_n$, Hη$_n$, λ), ϒ(Γδ, Ωδ, Ωδ, λ), ϒ(Λη$_n$, Hη$_n$, Hη$_n$, λ), $\frac{1}{2}$[ϒ(Λη$_n$, Ωδ, Ωδ, λ) + ϒ (Γδ, Hη$_n$, Hη$_n$, λ)]}).

Taking the limit as n → ∞, we have,
Ξ(Γδ, ω, ω, λ) ≥ ψ (min {Ξ(Ωδ, ω, ω, λ), Ξ(Γδ, Ωδ, Ωδ, λ), Ξ (ω, ω, ω, λ), $\frac{1}{2}$[Ξ (ω, Ωδ, Ωδ, λ) + Ξ (Γδ, ω, ω, λ)]}),
Θ(Γδ, ω, ω, λ) ≤ ϕ (max {Θ(Ωδ, ω, ω, λ), Θ(Γδ, Ωδ, Ωδ, λ), Θ (ω, ω, ω, λ), $\frac{1}{2}$[Θ (ω, Ωδ, Ωδ, λ) + Θ (Γδ, ω, ω, λ)]}),
ϒ(Γδ, ω, ω, λ) ≤ φ (max {ϒ(Ωδ, ω, ω, λ), ϒ(Γδ, Ωδ, Ωδ, λ), ϒ (ω, ω, ω, λ), $\frac{1}{2}$[ϒ (ω, Ωδ, Ωδ, λ) + ϒ (Γδ, ω, ω, λ)]}).

That is,

Ξ(Γδ, ω, ω, λ) ≥ ψ (Ξ(Γδ, ω, ω, λ)) > Ξ(Γδ, ω, ω, λ),
Θ(Γδ, ω, ω, λ) ≤ ϕ (Θ(Γδ, ω, ω, λ)) < Θ(Γδ, ω, ω, λ),
ϒ(Γδ, ω, ω, λ) ≤ φ (ϒ(Γδ, ω, ω, λ)) < ϒ(Γδ, ω, ω, λ).

which yield Γδ = ω = δ.

Therefore δ = ω is a common fixed point of Γ, Λ, Ω and H.

For uniqueness, suppose that there exist another fixed point u of Γ, Λ, Ω and H.

Then from (3.1), we have

$$\Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda) \geq \psi(\min \{\Xi(\Omega\delta, Hu, Hu, \lambda), \Xi(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Xi(\Lambda u, Hu, Hu, \lambda), \frac{1}{2}[\Xi(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Xi(\Gamma\delta, Hu, Hu, \lambda)]\})$$
$$= \psi(\min \{\Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 1, \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \frac{1}{2}[\Xi(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\})$$
$$= \psi(\Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda)$$
$$> \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda),$$

$$\Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda) \leq \phi(\max \{\Theta(\Omega\delta, Hu, Hu, \lambda), \Theta(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Theta(\Lambda u, Hu, Hu, \lambda), \frac{1}{2}[\Theta(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Theta(\Gamma\delta, Hu, Hu, \lambda)]\})$$
$$= \phi(\max \{\Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 0, \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \frac{1}{2}[\Theta(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\})$$
$$= \phi(\Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda)$$
$$< \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda),$$

$$\Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda) \leq \varphi(\max \{\Upsilon(\Omega\delta, Hu, Hu, \lambda), \Upsilon(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Upsilon(\Lambda u, Hu, Hu, \lambda), \frac{1}{2}[\Upsilon(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Upsilon(\Gamma\delta, Hu, Hu, \lambda)]\})$$
$$= \varphi(\max \{\Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 0, \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \frac{1}{2}[\Upsilon(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\})$$
$$= \varphi(\Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda)$$
$$< \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda),$$

which yield δ = u. Therefore, uniqueness follows.

If we put Ω = H in Theorem 3.1, we get the following result.

Corollary: 4.2

Let Γ, Λ, and Ω be self-maps of a weak non-Archimedean NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ and let the pairs (Γ, Ω) and (Λ, Ω) are subcompatible maps of type (J-1) and subsequentially continuous. If

$$\Xi(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \geq \psi(\min \{\Xi(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Xi(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Xi(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \quad (4.2.1)$$

$$\Theta(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \phi(\max \{\Theta(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Theta(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \quad (4.2.2)$$

$$\Upsilon(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \varphi(\max \{\Upsilon(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Upsilon(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \quad (4.2.3)$$

for all ζ, η ∈ Σ, $\lambda > 0$, where $\psi, \phi, \varphi : [0,1] \to [0,1]$ are continuous functions such that $\psi(s) > s$, φ(s) < s and $\varphi(s) < s$ for each s ∈ (0,1). Then Γ, Λ and Ω have a unique common fixed point in Σ.

If we put Γ = Λ and Ω = H in Theorem 4.1, we get the following result.

Corollary: 4.3

Let Γ and Ω be self-maps of a weak non-Archimedean NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ and let the pairs (Γ, Ω) is subcompatible maps of type (J-1) and subsequentially continuous. If

$$\Xi(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \geq \psi\,(\min\{\Xi(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Xi(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Xi(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.1)$$

$$\Theta(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \leq \phi\,(\max\{\Theta(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Theta(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.2)$$

$$\Upsilon(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \leq \varphi\,(\max\{\Upsilon(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Upsilon(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.3)$$

for all $\zeta, \eta \in \Sigma$, $\lambda > 0$, where $\psi, \phi, \varphi : [0,1] \rightarrow [0,1]$ are continuous functions such that $\psi(s) > s$, $\varphi(s) < s$ and $\varphi(s) < s$ for each $s \in (0,1)$. Then Γ and Ω have a unique common fixed point in Σ.

5. Conclusion

In this work, we obtained new structure of weak non-Archimedian with the help of subcompatible maps of types (J-1) and (J-2) in NMS. Also, we proved common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

References:

[1] Abdel-Basset, M., et al. (2019). "An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number." Applied Soft Computing 77: 438-452.

[2] Abdel-Basset, M., et al. (2019). "A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection." Journal of medical systems 43(2): 38.

[3] Abdel-Basset, M., Gamal, A., Chakrabortty, R.K., Ryan, M., 2021. A new hybrid multi-criteria decision-making approach for location selection of sustainable offshore wind energy stations: A case study. J. Clean. Prod. 280. https://doi.org/10.1016/j.jclepro.2020.124462

[4] Abdel-Basset, Mohamed, Gamal, A., Chakrabortty, R.K., Ryan, M.J., 2020a. Evaluation of sustainable hydrogen production options using an advanced hybrid MCDM approach: A case study. Int. J. Hydrogen Energy. https://doi.org/https://doi.org/10.1016/j.ijhydene.2020.10.232.

[5] Abdel-Basset, M, Mohamed, R, Sallam, K, and Elhoseny, M, A novel decision-making model for sustainable supply chain maintenance under uncertainty environment. Journal of Cleaner Production, 122324, (2020).

[6] Alaca, C, Altun, I, and Turkoglu, D, On compatible mappings of type (I) and (II) in intuitionistic fuzzy metric spaces, Comm. Korean Math. Soci., 23(3)(2008), 427– 446.

[7] Atanassov, K, Intuitionistic fuzzy sets, Fuzzy, Sets and Systems 20, 87- 96, (1986).

[8] George. A, Veeramani. P , On some results in fuzzy metric spaces, Fuzzy sets and Systems, 395 – 399, (1994).

[9] Grabiec. M, Fixed Points in Fuzzy Metric Spaces, Fuzzy Sets and Systems, 27
a. 385 – 389, (1988).

[10] Jeyaraman, M, Poovaragavan, D, Fixed Point Results in Non- Archimedean Generalized Fuzzy Metric Spaces, International Journal of Pure and Applied Mathematics, 119(9), 213-222, 2018.

[11] Jeyaraman, M, Suganthi, M, Sowndrarajan, S, Fixed point results in non-Archimedean generalized intuitionistic fuzzy metric spaces, Notes on Intuitionistic Fuzzy Sets,25(4), 48-58, 2019.

[12] Jungck, G, Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9, 771 – 779, (1986),

[13] Jungck, G, Murthy, P.P, and Cho, Y.J, Compatible mappings of type (A) and common fixed points, Math. Japan., 38(2), 381 – 390, (1993).

[14] Kramosil, I, and Michalek, J, Fuzzy metric and statistical metric space, Kybernetika, 11, 326 – 334, (1975).

[15] Kirisci, M, Simsek, N, Neutrosophic metric spaces, Mathematical Sciences (2020) 14, 241–248.

[16] Muthuraj. R and Pandiselvi.R, Common fixed point theorems for compatible mappings of type (P-1) and type (P-2) in M-fuzzy metric spaces, Arab journal of Mathematics, and Mathematical Sciences, 1-10, (2013).

[17] Park. J. H, Intuitionistic fuzzy metric spaces, Chaos Solit. Fract., 22(2004), 1039 - 1046.

[18] Pathak. H. K, Cho. Y. J, Kang. S. M and Madharia. B, Compatible mappings oftype (C) and common fixed point theorems of Gregus type, Demon. Math.,1(3), 499 – 518, (1998)

[19] Singh. M. R, and Singh, Y. M, Compatible mappings of type (E) and common fixed point theorems of Meir-Keeler type, Internat. J. Math. Sci. & Eng. Appl., 1 (2), 299 – 315, (2007).

[20] Smarandache, F. A Unifying Field in Logics, Neutrosophy: Neutrosophic Probability, Set and Logic; American Research Press: Reheboth, MA, USA, 1998.

[21] Smarandache, F. Neutrosophy: Neutrosophic Probability, Set and Logic; ProQuest Information and Learning: Ann Arbor, MI, USA, 1998; pp. 105.

[22] Smarandache, F. Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Inter J Pure Appl Math. Vol 24, pp 287-297, 2005.

[23] Sola Erduran.F , Yildiz. C Kutukcu .S, A Common fixed point theorem in weak non-Archimedean intuitionistic fuzzy metric spaces, International journal of Open Compt. Math., 7, 1.1 – 1.7, (2014).

[24] Sowndrarajan, S, Jeyaraman, M, and Florentin Smarandache, Fixed Point Results for Contraction Theorems in Neutrosophic Metric Spaces, Neutrosophic Sets and Systems.,36, 308-310, 2020.

[25] Wadei F. Al-Omeri, Saeid Jafari and Florentin Smarandache, Neutrosophic Fixed Point Theorems and Cone Metric Spaces, Neutrosophic Sets and Systems, Volume 31, pp 250-264, 2020.

[26] Zadeh. L. A, Fuzzy sets, Infor. and Control.,8338 – 353, (1965).

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Neutrosophic Pythagorean Soft Set With T and F as Dependent Neutrosophic Components

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Abstract: The aim of this paper is to introduce the new concept of Neutrosophic Pythagorean soft set with T and F as dependent components and have also discussed some of its properties.

Keywords: Neutrosophic set, Neutrosophic pythagorean set, Neutrosophic soft set, Neutrosophic pythagorean soft set.

1. Introduction

The fuzzy set was introduced by Zadeh [19] in 1965. In 1968, Chang [4] defined the concept of fuzzy topological space and generalized some basic notions of topology. Intuitionistic fuzzy set was introduced by Atanassov [2,3] in 1983.The concept of Neutrosophic set was introduced by F. Smarandache which is a mathematical tool for handling problems involving imprecise, indeterminancy and inconsistent data.
In 2018 Smarandache [16] generalized the gvlSoft Set to the Hyper Soft Set by transforming the classical uni-argument function F into a multi-argument function:
In 2016, F. Smarandache [13] introduced for the first time the degree of dependence between the components of fuzzy set and neutrosophic sets. The main idea of Neutrosophic sets is to characterize each value statement in a 3D – Neutrosophic space, where each dimension of the space represents respectively the truth membership, falsity membership and the indeterminacy, when two components T and F are dependent and I is independent then T+I+F≤ 2.
Pabitra kumar Maji had combined the Neutrosophic set with soft sets and introduced a new mathematical model Neutrosophic soft set. I.Arockiarani [2] introduced the new concept of fuzzy neutrosophic soft set. Yager introduced pythagorean fuzzy sets. R. Jhansi [6] introduced the concept of Pythagorean Neutrosophic set with T and F as dependent components.
In this we have to introduce the concept of neutrosophic pythagorean soft set with truth membership and false membership as dependent components and the indeterminacy as independent component and establish some of its properties.

2.Preliminaries

Definition:2.1[13]

Let U be a universe. A Neutrosophic set A on U can be defined as follows:

$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in U\}$

Where $T_A, I_A, F_A: U \to [0,1]$ *and* $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$

Here, $T_A(x)$ is the degree of membership, $I_A(x)$ is the degree of indeterminancy and $F_A(x)$ is the degree of non-membership.

Definition:2.2[6]

Let U be a universe. A Pythagorean neutrosophic set with T and F are dependent neutrosophic components A on U is an object of the form

$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in U\}$

Where $T_A, I_A, F_A: U \to [0,1]$ *and* $0 \leq (T_A(x))^2 + (I_A(x))^2 + (F_A(x))^2 \leq 2$

Here, $T_A(x)$ is the degree of membership, $I_A(x)$ is the degree of inderminancy and $F_A(x)$ is the degree of non-membership.

*He*re, $T_A(x)$ and $F_A(x)$are dependent components and $I_A(x)$ is an independent component.

Definition:2.3[2]

Let U be the initial universe set and E be set of parameters. Consider a non-empty set A on E, Let P(U) denote the set of all neutrosophic sets of U. The collection (F, A) is termed to be neutrosophic soft set over U, where F is a mapping given by F: A → P(U).

3.Neutrosophic Pythagorean Soft Set (NPSS or NPS Set)

Definition:3.1

Let U be the initial universe set and E be set of parameters. Consider a non-empty set A on E, Let P(U) denote the set of all neutrosophic pythagorean sets of U. The collection (F, A) is termed to be neutrosophic pythagorean soft set over U, where F is a mapping given by F: A → P(U).

Definition:3.2

A neutrosophic pythagorean soft set A is contained in another neutrosophic pythagorean soft set B (i.e) A$\subseteq B$ if $T_A(x) \leq T_B(x), I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x)$

Definition:3.3

The complement of a neutrosophic pythagorean soft set (F, A) Denoted by $(F, A)^c$ and is defined as

F^c(x)= $\{< x, F_A(x), 1 - I_A(x), T_A(x) >: x \in U\}$

Definition:3.4

Let U be a non-empty set, A = $< x, T_A(x), I_A(x), F_A(x) >$ and

B = $< x, T_B(x), I_B(x), F_B(x) >$ are neutrosophic pythagorean soft (NPS) sets. Then

A ∪ B = $< x, max(T_A(x), T_B(x)), max(I_A(x), I_B(x)), min(F_A(x), F_B(x)) >$

A ∩ B = $< x, min(T_A(x), T_B(x)), min(I_A(x), I_B(x)), max(F_A(x), F_B(x)) >$

Definition:3.5

A neutrosophic pythagorean soft set (F, A) over the universe U is said to be empty neutrosophic pythagorean soft set with respect to the parameter A if $T_{F(e)} = 0, I_{F(e)} = 0, F_{F(e)}$=1,$\forall x \in U, \forall e \in A$. It is denoted by 0_N

Definition:3.6

A neutrosophic pythagorean soft set (F, A) over the universe U is said to be universe neutrosophic pythagorean soft set with respect to the parameter A if $T_{F(e)} = 1, I_{F(e)} = 1, F_{F(e)}$=0,$\forall x \in U, \forall e \in A$. It is denoted by 1_N

Remark: $0_N^c = 1_N$ and $1_N^c = 0_N$

Definition:3.7

Let A and B be two neutrosophic pythagorean soft sets then A\B may be defined as

A\B = $< x, min(T_A(x), F_B(x)), min(I_A(x), 1 - I_B(x)), max(F_A(x), T_B(x)) >$

Definition:3.8

FE is called neutrosophic pythagorean soft set over U if F(e) = 1_N for any $e \in E$. We denote it by U_E

F_E is called relative null neutrosophic pythagorean soft set over U if F(e) = 0_N for any $e \in E$. We denote it by $\emptyset_E$

Obviously $\emptyset_E = U_E^c$ and $U_E = \emptyset_E^c$

Definition:3.9

The complement of a neutrosophic pythagorean soft set (F, A) can also be defined as

$(F, A)^c = U_E \backslash F(e)$ for all $e \in A$.

Note: We denote U_E by U in the proofs of proposition.

Definition:3.10

If (F, A) and (G, B) be two neutrosophic pythagorean soft set then "(F, A) AND (G, B)" is a denoted by (F, A) ∧(G, B) and is defined by (F, A)∧ (G, B) = (H, A ×B)

where H (a, b) = F(a) ∩ G(b) $\forall a \in A$ and $\forall b \in B$, where ∩ is the operation intersection of NPS set.

Definition:3.11

If (F, A) and (G, B) be two neutrosophic pythagorean soft set then "(F, A) OR (G, B)" is a denoted by (F, A) ∨(G, B) and is defined by (F, A)∨ (G, B) = (K, A ×B)

where K (a, b) = F(a) ∪ G(b) $\forall a \in A$ and $\forall b \in B$, where ∪ is the operation union of NPS set.

Theorem :3.12

Let (F, A) and (G, B) be NPS set in $NPSS(U)_A$ .Then the following are true.

(i) (F, A)⊆ (G, A) iff (F, A) ∩ (G, A) = (F, A)

(ii) (F, A)⊆ (G, A) iff (F, A) ∪ (G, A) = (F, A)

Proof:

(i)Suppose that (F, A)⊆ (G, A), then F(e) ⊆G(e) for all e ∈ A. Let (F, A) ∩ (G, A) = (H, A).

Since H(e) = F(e) ∩ G(e) =F(e) for all e ∈ A, by definition (H, A) = (F, A).

Consider (F, A) $\cap$ (G, A) = (F, A). Let (F, A) $\cap$ (G, A) = (H, A). Since H(e) = F(e) $\cap$ G(e) =F(e) for all e $\in$ A, we know that F(e) $\subseteq$ G(e) for all e $\in$ A. Hence (F, A)$\subseteq$ (G, A).

(ii)The proof is similar to (i).

Theorem :3.13

Let (F, A), (G, A), (H, A), and (S, A) be NPS set in $NPSS(U)_A$ .Then the following are true.

(i) If (F, A) $\cap$ (G, A) =$\emptyset_A$, then (F, A)$\subseteq$ (G, A)^c

(ii) If (F, A)$\subseteq$ (G, A) and (G, A)$\subseteq$ (H, A) then (F, A)$\subseteq$ (H, A)

(iii) If (F, A)$\subseteq$ (G, A) and (H, A)$\subseteq$ (S, A) then (F, A) $\cap$ (H, A) $\subseteq$ (G, A) $\cap$ (S, A)

(iv) (F, A) $\subseteq$ (G, A) iff (G, A)^c $\subseteq$(F, A)^c

Proof:

(i)Suppose that (F, A) $\cap$ (G, A) =$\emptyset_A$.Then F(e)$\cap$ G(e) = $\emptyset$. So, F(e) $\subseteq$ U\G(e)= $G^c(e)$ for all e $\in$ A. therefore we have (F, A)$\subseteq$ (G, A)^c

Proof of (ii) and (iii) are obvious.

(iv) (F, A)$\subseteq$ (G, A)$\Leftrightarrow$ F(e) $\subseteq$ G(e) for all e $\in$ A.

$\Leftrightarrow$ (G(e))^c $\subseteq$ (F(e))^c for all e $\in$ A.

$\Leftrightarrow$(G, A)^c $\subseteq$(F, A)^c

Definition:3.14

Let I be an arbitrary index $\{(F_i, A)\}_{i\in I}$ be a subfamily of NPSS(U)$_A$.

(i)The union of these NPSS is the NPSS (H, A) where H(e) = $\cup_{i\in I} F_i\ (e)$ for each e $\in A$.

We write $\cup_{i\in I}(F_i\,, A)$= (H, A)

(ii)The intersection of these NPSS is the NPSS (M, A) where M(e) = $\cap_{i\in I}\ F_i(e)$ for each e $\in A$.

We write $\cap_{i\in I}\,(F_i, A)$ = (M, A)

Theorem:3.15

Let I be an arbitrary index set and $\{(F_i, A)\}_{i\in I}$ be a subfamily of NPSS(U)A. Then

(i) $(\bigcup_{i\in I}(F_i, A))^C = \bigcap_{i\in I}(F_i, A)^C$

(ii) $(\bigcap_{i\in I}(F_i, A))^C = \bigcup_{i\in I}(F_i, A)^C$

Proof:

(i) $(\bigcup_{i\in I}(F_i, A))^C$ = (H, A) ^{C}, By definition $H^C(e) = U_E \backslash H(e) = U_E \backslash \bigcup_{i\in I} F_i(e) = \bigcap_{i\in I}(U_E \backslash F_i(e))$

for all e ∈ A. On the other hand, $(\bigcap_{i\in I}(F_i, A))^C$(K, A).

By definition, $K(e) = \bigcap_{i\in I} F_i^C(e) = \bigcap_{i\in I}(U - F_i(e))$ for all e ∈ A.

(ii) It is obvious.

Note: We denote $\emptyset_E$ *by* $\emptyset$ and U_E *by* U.

Theorem:3.16

(i) $(\emptyset, A)^c$ = (U, A)

(ii) $(U, A)^c = (\emptyset, A)$

Proof:

Let (∅, A) = (F, A)

Then ∀e ∈ A,

$F(e) = \{< x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x) >: x \in U\}$

$= \{(x,0,0,1): x \in U\}$

Now, $(\emptyset, A)^c = (F, A)^c$

Then ∀e ∈ A,

$(F(e))^c = \{< x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x) >: x \in U\}^c$

$= \{< x, F_{F(e)}(x), 1 - I_{F(e)}(x), T_{F(e)}(x) >: x \in U\}$

$=\{(x1,1,0): x \in U\}$ = U

Thus $(\emptyset, A)^c$ = (U, A)

(i) Proof is similar to (i)

Theorem:3.17

(i) (F, A)∪ (∅, A) = (F, A)

(ii) (F, A)∪ (U, A) = (U, A)

Proof:

(i) (F, A) = {e, $(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)): x \in U\}$ ∀e ∈ A

(∅, A) = {e, (x,0,0,1): x ∈ U} ∀e ∈ A

(F, A)∪ (∅, A) = {e, $(x, max(T_{F(e)}(x),0), max(I_{F(e)}(x),0), min(F_{F(e)}(x),1)): x \in U\}$ ∀e ∈ A

= {e, $(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)): x \in U\}$ ∀e ∈ A

= (F, A)

(ii) Proof is similar to (i).

Theorem:3.18

(i) (F, A)∩ (∅, A) = (∅, A)

(ii) (F, A)∩ (U, A) = (F, A)

Proof:

(i) (F, A) = {e, $(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)): x \in U\}$ ∀e ∈ A

(∅, A) = {e, (x,0,0,1): x ∈ U} ∀e ∈ A

(F, A)∩ (∅, A) = {e, $(x, min(T_{F(e)}(x),0), min(I_{F(e)}(x),0), max(F_{F(e)}(x),1)): x \in U\}$ ∀e ∈ A

= {e, $, (x, 0,0,1): x \in U\}$ ∀e ∈ A

= (∅, A)

(ii) Proof is similar to (i).

Theorem:3.19

(i) (F, A)∪ (∅, B) = (F, A) iff B⊆A

(ii) (F, A)∪ (*U*, B) = (U, A) iff A⊆ *B*

Proof:

(i) We have for (F, A)

F(e) = $\{(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)) : x \in U\}$ ∀e ∈ A

Also let (∅, B) = (G, B) then

G(e) = {(x, 0,0,1): x ∈ U} ∀e ∈ B

Let (F, A)∪ (∅, B) = (F, A)∪ (*G*, B) = (H, C) where C = A ∪B and for all $e \in C$

H(e) may be defined as

$$\begin{cases} \{(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)) : x \in U\} \; if \; e \in A - B \\ \{(x, T_{G(e)}(x), I_{G(e)}(x), F_{G(e)}(x)) : x \in U\} \; if \; e \in B - A \\ \{(x, max(T_{F(e)}(x), T_{G(e)}(x)), max(I_{F(e)}(x), I_{G(e)}(x)), min(F_{F(e)}(x), F_{G(e)}(x)) : x \in U\} \; if \; e \in A \cap B \end{cases}$$

$$= \begin{cases} \{(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)) : x \in U\} \; if \; e \in A - B \\ \{(x, 0,0,1) : x \in U\} \; if \; e \in B - A \\ \{(x, max(T_{F(e)}(x), 0), max(I_{F(e)}(x), 0), min(F_{F(e)}(x), 1) : x \in U\} \; if \; e \in A \cap B \end{cases}$$

$$= \begin{cases} \{(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)) : x \in U\} \; if \; e \in A - B \\ \{(x, 0,0,1) : x \in U\} \; if \; e \in B - A \\ \{(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)) : x \in U\} \; if \; e \in A \cap B \end{cases}$$

Let B⊆A

$$\text{Then H (e)} = \begin{cases} \{(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)) : x \in U\} \; if \; e \in A - B \\ \{(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)) : x \in U\} \; if \; e \in A \cap B \end{cases}$$

= F(e) $\forall e \in A$

Conversely Let $(F, A) \cup (\emptyset, B) = (F, A)$

Then A = A ∪ B⟹ $B \subseteq A$

(ii) Proof is similar to (i)

Theorem:3.20

(i) (F, A)∩ (∅, B) = (∅, A∩B)

(ii) (F, A)∩ (U, B) = (F, A∩B)

Proof:

(i) We have for (F, A)

F(e) = $\{(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)): x \in U\}$ ∀e ∈ A

Also let (∅, B) = (G, B) then

G(e) = {(x, 0,0,1): x ∈ U} ∀e ∈ B

Let (F, A) ∩ (∅,B) = (F, A) ∩(G, B) = (H, C) where C = A ∩B and ∀e ∈ C

H(e) = $\{(x, min(T_{F(e)}(x), T_{G(e)}(x)), min(I_{F(e)}(x), I_{G(e)}(x)), max(F_{F(e)}(x), F_{G(e)}(x)): x \in U\}$

= $\{ (x, min(T_{F(e)}(x), 0), min(I_{F(e)}(x), 0), max(F_{F(e)}(x), 1)): x \in U\}$

= $\{(x, 0,0,1): x \in U\}$

= (G, B) = (∅, B)

Thus (F, A)∩ (∅, B) = (∅, B) = (∅, A∩B)

(ii) Proof is similar to (i).

Theorem:3.21

(i) ((F, A)∪ (G, B))^{C} ⊆ (F, A)^{C} ∪ (G, Bj)^{C}

(ii) (F, A)^{C}∩ (G, B)^{C} ⊆((F, A)∩ (G, B))^{C}

Proof:

Let (F, A)∪(G, B) = (H, C) Where C = A∪B and∀ e∈ C

H(e) may be defined as

$$\begin{cases} \{(x, T_{F(e)}(x), I_{F(e)}(x), F_{F(e)}(x)): x \in U\} \text{ if } e \in A - B \\ \{(x, T_{G(e)}(x), I_{G(e)}(x), F_{G(e)}(x)): x \in U\} \text{ if } e \in B - A \\ \{(x, max(T_{F(e)}(x), T_{G(e)}(x)), max(I_{F(e)}(x), I_{G(e)}(x)), min(F_{F(e)}(x), F_{G(e)}(x)): x \in U\} \text{ if } e \in A \cap B \end{cases}$$

Thus (F,A)∪(G,B))^{C} = (H,C)^{C} Where C = A∪B and∀ e∈ C

$$(H(e))^{C} = \begin{cases} (F(e))^{C} \text{ if } e \in A - B \\ (G(e))^{C} \text{ if } e \in B - A \\ (F(e) \cup G(e))^{C} \text{ if } e \in A \cap B \end{cases}$$

=

$$\begin{cases} \{(x, F_{F(e)}(x), 1 - I_{F(e)}(x), T_{F(e)}(x)) : x \in U\} \text{ } if\ e \in A - B \\ \{(x, F_{G(e)}(x), 1 - I_{G(e)}(x), T_{G(e)}(x)) : x \in U\} \text{ } if\ e \in B - A \\ \{(x, min(F_{F(e)}(x), F_{G(e)}(x)), 1 - max(I_{F(e)}(x), I_{G(e)}(x)), max(T_{F(e)}(x), T_{G(e)}(x)) : x \in U\} \text{ } if e \in A \cap B \end{cases}$$

Again (F, A) ^{C} ∪ (G, B) ^{C} = (I, J) say J = A∪B and ∀ e∈ J

$$I(e) = \begin{cases} (F(e))^C \ if\ e \in A - B \\ (G(e))^C \ if\ e \in B - A \\ (F(e) \cup G(e))^C if\ e \in A \cap B \end{cases}$$

=

$$\begin{cases} \{(x, F_{F(e)}(x), 1 - I_{F(e)}(x), T_{F(e)}(x)) : x \in U\} \text{ } if\ e \in A - B \\ \{(x, F_{G(e)}(x), 1 - I_{G(e)}(x), T_{G(e)}(x)) : x \in U\} \text{ } if\ e \in B - A \\ \{(x, min(F_{F(e)}(x), F_{G(e)}(x)), 1 - max(I_{F(e)}(x), I_{G(e)}(x)), max(T_{F(e)}(x), T_{G(e)}(x)) : x \in U\} \text{ } if e \in A \cap B \end{cases}$$

So, C⊆ J ∀ e∈ J, $(H(e))^C$ ⊆I(e)

Thus (F, A)∪(G, B)) ^{C} ⊆ (F, A) ^{C} ∪ (G, B) ^{C}

(ii) Let (F, A)∩(G, B) = (H, C) Where C = A∩B and∀ e∈ C

H(e) = F(e)∩G(e)

$$=\{(x, min(T_{F(e)}(x), T_{G(e)}(x)), min(I_{F(e)}(x), I_{G(e)}(x)), max(F_{F(e)}(x), F_{G(e)}(x))\}$$

Thus ((F, A)∩(G, B)) ^{C} = (H, C^C Where C = A∩B and∀ e∈ C

$$(H(e))^C = \{(x, min(T_{F(e)}(x), T_{G(e)}(x)), min(I_{F(e)}(x), I_{G(e)}(x)), max(F_{F(e)}(x), F_{G(e)}(x))\}^C$$

$$=\{(x, max(F_{F(e)}(x), F_{G(e)}(x)), 1 - min(I_{F(e)}(x), I_{G(e)}(x)), min(T_{F(e)}(x), T_{G(e)}(x))\}$$

Again (F, A) ^{C}∩ (G, B) ^{C} = (I, J) say where J= A∩ B and ∀ e∈ J

I(e) = (F(e)) ^{C}∩ (G(e)) ^{C}

$$=\{(x, min(F_{F(e)}(x), F_{G(e)}(x)), min(1 - I_{F(e)}(x), 1 - I_{G(e)}(x)), max(T_{F(e)}(x), T_{G(e)}(x))\}$$

We see that C = J and ∀ e∈ J, I(e) ⊆(H(e)) ^{C}

Thus (F, A) ^{C} ∩ (G, B) ^{C} ⊆((F, A)∩ (G, B)) ^{C}

Theorem :3.22

Let (F, A) and (G, A) are two neutrosophic pythagorean soft sets over the same universe U. We have the following

(i) ((F, A)∪ (G, A)) ^{C} = (F, A) ^{C} ∩ (G, A) ^{C}

(ii) ((F, A)∩ (G, A)) ^{C} = (F, A) ^{C}∪ (G, A) ^{C}

Proof:

(i) Let (F, A) ∪ (G, A) = (H, A) ∀ e∈ A

H(e) = F(e)∪G(e)

$$=\{(x, max(T_{F(e)}(x), T_{G(e)}(x)), max(I_{F(e)}(x), I_{G(e)}(x)), min(F_{F(e)}(x), F_{G(e)}(x))\}$$

Thus (F, A) ∪ (G, A))^C = (H, A)^C ∀ e∈ A

(H(e)) ^{C} = (F(e)∪G(e)) ^{C}

$$=\{(x, max(T_{F(e)}(x), T_{G(e)}(x)), max(I_{F(e)}(x), I_{G(e)}(x)), min(F_{F(e)}(x), F_{G(e)}(x))\}^C$$

$$=\{(x, min(F_{F(e)}(x), F_{G(e)}(x)), 1 - max(I_{F(e)}(x), I_{G(e)}(x)), min(T_{F(e)}(x), T_{G(e)}(x))$$

Again (F, A) ^{C}∩ (G, A) ^{C} = (I, A) where∀ $e \in A$

I(e) = (F(e)) ^{C}∩ (G(e)) ^{C}

$$=\{(x, min(F_{F(e)}(x), F_{G(e)}(x)), min(1 - I_{F(e)}(x), 1 - I_{G(e)}(x)), max(T_{F(e)}(x), T_{G(e)}(x))\}$$

$$=\{(x, min(F_{F(e)}(x), F_{G(e)}(x)), 1 - max(I_{F(e)}(x), I_{G(e)}(x)), max(T_{F(e)}(x), T_{G(e)}(x))$$

Thus ((F, A)∪ (G, A)) ^{C} = (F, A) ^{C} ∩ (G, A) ^{C}

(ii) Let (F, A) ∩ (G, A) = (H, A) ∀ e∈ A

H(e) = F(e)∩G(e)

$$=\{(x, min(T_{F(e)}(x), T_{G(e)}(x)), min(I_{F(e)}(x), I_{G(e)}(x)), max(F_{F(e)}(x), F_{G(e)}(x))\} \ \forall \ e \in A$$

Thus (F, A) ∩ (G, A)) ^{C} = (H, A) ^{C}

(H(e)) ^{C} = (F(e)∩G(e)) ^{C}

$= \{(x, min(T_{F(e)}(x), T_{G(e)}(x)), min(I_{F(e)}(x), I_{G(e)}(x)), max(F_{F(e)}(x), F_{G(e)}(x))\ \}^C$ =

$\{(x, max(F_{F(e)}(x), F_{G(e)}(x)), 1 - min(I_{F(e)}; (x), I_{G(e)}(x)), min(T_{F(e)}(x), T_{G(e)}(x))\} \forall\ e \in\ A$

Again (F, A) ^{C}∪ (G, A) ^{C} = (I, A) where$\forall\ e \in\ A$

I(e) = (F(e)) ^{C}∪ (G(e)) ^{C}

$= \{(x, max(F_{F(e)}(x), F_{G(e)}(x)), max(1 - I_{F(e)}(x), 1 - I_{G(e)}(x)), min(T_{F(e)}(x), T_{G(e)}(x))\}$

$= \{(x, max(F_{F(e)}(x), F_{G(e)}(x)), 1 - min(I_{F(e)}(x), I_{G(e)}(x)), min(T_{F(e)}(x), T_{G(e)}(x))$

Thus ((F, A)∩ (G, A)) ^{C} = (F, A) ^{C} ∪ (G, A) ^{C}

Theorem:3.23

Let (F, A) and (G, A) are two neutrosophic pythagorean soft sets over the same universe U. We have the following

(i) ((F, A)∧ (G, A)) ^{C} = (F, A) ^{C} ∨ (G, A) ^{C}

(ii) ((F, A)∨ (G, A)) ^{C} = (F, A) ^{C}∧ (G, A) ^{C}

Proof:

Let (F, A) ∧ (G, B) = (H, A ×B) where H (a, b) =F(a) ∩G(b) $\forall\ a \in\ A$ and $\forall\ b \in\ B$ where ∩ is the operation intersection of NPSS.

Thus H (a, b) = F(a) ∩G(b)

$= \{(x, min(T_{F(a)}(x), T_{G(b)}(x)), min(I_{F(a)}(x), I_{G(b)}(x)), max(F_{F(a)}(x), F_{G(b)}(x))\}$

((F, A) ∧ (G, B)) ^{C} = (H, A ×B) ^{C} $\forall(\ a, b) \in\ A \times B$

Thus (H (a, b)) ^{C}=$\{(x, min(T_{F(a)}(x), T_{G(b)}(x)), min(I_{F(a)}(x), I_{G(b)}(x)), max(F_{F(a)}(x), F_{G(b)}(x))\}^C$

=

$\{(x, max(F_{F(a)}(x), F_{G(b)}(x)), 1 - min(I_{F(a)}(x), I_{G(b)}(x)), min(T_{F(a)}(x), T_{G(b)}(x))\}$

Let (F, A) ^{C} ∨ (G, A) ^{C} = (R, A ×B) where R (a, b) =(F(a)) ^{C} ∪ (G(b)) ^{C} $\forall\ a \in\ A$ and $\forall\ b \in\ B$ where ∪ is the operation union of NPSS.

R (a, b) = $\{(x, max(F_{F(a)}(x), F_{G(b)}(x)), max(1 - I_{F(a)}(x), 1 - I_{F(b)}(x)), min(T_{F(a)}(x), T_{G(b)}(x))\}$

$=\{(x, max(F_{F(a)}(x), F_{G(b)}(x)), 1 - min(I_{F(a)}(x), I_{G(b)}(x)), min(T_{F(a)}(x), T_{G(b)}(x))\}$

Hence ((F, A)$\wedge$ (G, A)) ^{C} = (F, A) ^{C} $\vee$ (G, A) ^{C}

Similarly, we can prove (ii)

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Conclusion

In this paper, I have defined the concept of neutrosophic pythagorean soft sets with dependent components by combining the concept of neutrosophic pythagorean set and neutrosophic set. Then we have discussed the properties of union, intersection and complement of neutrosophic pythagorean soft set. This may helpful in future study of generalized neutrosophic pythagorean soft set in neutrosophic pythagorean soft topological spaces. This may lead to the new properties of separation axioms in neutrosophic pythagorean soft topological space.

References

[1] I. Arockiarani, R. Dhavaseelan, S. Jafari, M. Parimala, On some notations and functions in neutrosophic topological spaces, Neutrosophic sets and systems

[2] I. Arockiarani, I.R. Sumathi and J, Martina Jency, Fuzzy neutrosophic soft topological spaces, IJMA-4[10], oct-2013.

[3] K. Atanassov, Intuitionistic fuzzy sets, in V. Sgurev, ed., vii ITKRS Session, Sofia (June 1983 central Sci. and Techn. Library, Bulg. Academy of Sciences (1983)).

[4] M. Bora, T.J. Neog and D.K. Sut, "A study on some operations of fuzzy soft sets" International Journal of Mathematics Trends and Technology- Volume3 Issue- 2- 2012.

[5] M. Irfan Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, "On some new operations in soft set theory", Comput. Math Appl.57 (2009) 1547-1553.

[6] R, Jhansi, K. Mohana and Florentin Smarandache, Correlation measure for pythagorean Neutrosophic sets with T and F as dependent neutrosophic components

[7] D. Molodtsov, Soft set Theory - First Results, Comput.Math.Appl. 37 (1999)19-31.

[8] P.K. Maji, R. Biswas ans A. R. Roy, "Fuzzy soft sets", Journal of Fuzzy Mathematics, Vol. 9, no.3, pp – 589-602, 2001

[9] P. K. Maji, R. Biswas ans A. R. Roy, "Intuitionistic Fuzzy soft sets", The journal of fuzzy Mathematics, Vol. 9, (3) (2001), 677 – 692.

[10] Pabitra Kumar Maji, Neutrosophic soft set, Annals of Fuzzy Mathematics and Informatics, Volume 5, No.1, (2013).,157-168.

[11] A.A. Salama and S.A. Al – Bloumi, Neutrosophic Set and Neutrosophic topological spaces, IOSR Journal of Math., Vol. (3) ISSUE4 (2012), 31 – 35

[12] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011)1786 – 1799.

[13] F. Smarandache, Degree of Dependence and independence of the sub components of fuzzy set and neutrosophic set, Neutrosophic sets and systems, vol 11,2016 95-97

[14] F. Smarandache, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability and Statistics University of New Mexico, Gallup, NM 87301, USA (2002).

[15] F. Smarandache, Neutrosophic set, A generialization of the intuituionistics fuzzy sets, Inter. J. Pure Appl. Math., 24 (2005), 287 – 297.

[16] F. Smarandache: Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set, Neutrosophic Sets and Systems, vol. 22, 2018, pp. 168-170.

[17] B. Tanay and M.B. Kandemir, Topological structure of fuzzy soft sets, Comput.Math.Appl. 61 (2001),2952 – 2957.

[18] Xindong Peng, Yong Yang, Some results for pythagorean fuzzy sets, International Journal of Intelligent systems,30(2015).1133-1160.

[19] L. A. Zadeh, Fuzzy Sets, Inform and Control 8(1965) 338 – 353.

[20] Zhaowen Li, Rongchen cui, On the topological structure of intuitionistic fuzzy soft sets, Annals of Fuzzy Mathematics and Informatics, Volume 5, No.1, (2013),229-239.

[21] Zorlutuna, M. Akdag, W. K. Min, S. Atmaca, "Remarks on soft topological spaces", Annals of Fuzzy Mathematics and Informatics, Volume3, No.2, (2012), pp.171-185.

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A Robust Machine Learning Algorithm for Cosmic Galaxy Images Classification Using Neutrosophic Score Features.

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Abstract: The development of galaxy images classification automated schemes is necessary to identify, classify, and study the evolution and formation of galaxies in our universe as it is one of the main challenges faced by astronomers today. Scientists can also build a deeper understanding of galaxies evolution and formation by classifying them into various classes. This paper proposed a robust novel hybrid automated intelligent algorithm based on neutrosophic techniques (NTs) and machine learning techniques for classifying the galaxy morphological astronomical images into various types of galaxies images (Hubble types) based on its features into three main classes; Elliptical, Spiral and Irregular. A nine classifiers performance was assessed based on the machine learning (ML) techniques by using a combination of a sets of morphic features (MFs); obtained from image analysis and principal component analysis (PCA) features. The results indicated that; the classifier which called, multilayer perceptron (MLP) gives the better results for the features set consisting of nine MFs and 24 PCs features among all tested cases; Mean squared error (MSE) = 0.0021; Normalized mean squared error (NMSE)= 0.0371; Correlation coefficient (r) = 0.9889, and the Error = 0.7751 with an accuracy 99.2249 %. Then, to improve the system efficiency; the neutrosophic techniques were applied in combination with the classifier that gave the best results in the previous step on the same extracted features to get a three robust component namely; membership, indeterminacy and non-membership components to fed to the neural network. The results showed that; the combination between the NTs and MLP classifier for (MFs with 4PCs) gives the best results; MSE = 0.0001; NMSE = 0.0009; r = 0.9997, and Error = 0.4212 with an accuracy about 99.5788 % in total for all chosen sets of features. The results showed the high performance of the proposed method comparing with other methods. The experimental results are performed based on a sample from (EFIGI) catalog.

Keywords: Galaxy classification, image processing, Multi-Layer Perceptron, Neutrosophic Techniques, Machine Learning Techniques.

1. Introduction

Galaxies are the areas where hydrogen transforms into luminous stars and celestial bodies that are gravitationally bound, they have different sizes, colors and shapes, it consisting of dust, gas and billions of nuclear-powered stars that also contain most chemical elements [1]. One of the most important problems for astronomers is the galaxies classification, as it can provide significant information about the evolution and origin of the universe. It is becoming an essential trouble due to the fact of astrophysicists often use vast information databases either to check present theories, or to structure a new inference to clarify the physical processes that control nature of the universe, galaxies and the star formation [2]. The morphological classification of galaxies is a system used to classify galaxies based on their appearance and their external structure by astronomers. The system created by Sir Edwin Hubble in 1936 is the most frequent classification scheme; as he divides galaxies into three main classes in accordance of their visual appearance, namely; (i) Elliptical galaxies: are without features objects, smooth, denoted by letter E followed by integer n (E0, E3, E5, and E7); (ii) Spiral galaxies: have structures like the disk and are distinguished by a flat disk contains stars in the central bulge and on the spiral arms. The usual spirals are indicated by letter S and located at the upper half (S0, Sa, Sb, Sc, and Sd). The lower half of spiral galaxies denoted by SB (SBa, SBb, and SBc) knowen as barred spiral galaxies; and (iii) Irregular galaxies: that has very irregular shapes without galactic bulges or the spiral arms of spiral galaxies are indicated by letter I (Im, and Ibm) as shown in Figure 1 [3]. This scheme is also referred to as the "Hubble Tuning Fork". This classification updated by another classification scheme called De Vaucouleurs scheme in 1959, to obtain the Revised Hubble System (RHS). In 1958 Morgan proposed further scheme and in 1960 Van Den Berghin proposed another one. NASA also provided a universal classification, called the revised morphological types [4].

Wide catalogs have been used by astronomers over recent decades to research the fundamental physics of the universe and test theories [5]. One of the most successful modern data collection projects in astronomy is Sloan Digital Sky Survey (SDSS), which uses a dedicated 2.5-m wide-angle optical telescope for spectroscopic surveys and multi-filter imaging, its data collection began in 2000 and in its final data release, it covered most of the sky area [6].

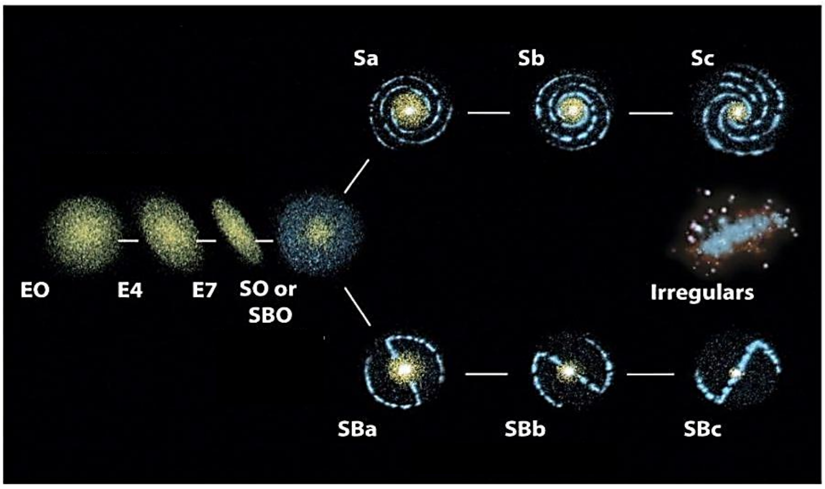


Figure 1. Hubble Galaxy Classification Scheme (Hubble 1936) [3].

Galaxies used to be manually classified into classes based on their visual characteristics in the past, however as the present-day sky surveys include images of millions of galaxies, it is found that implementing classification algorithms can assist in solving this issue. Advances in algorithms and computational tools have begun to allow galaxy morphology automated analysis by analyzing its internal structure in the past few years. Other attempts have been made to implement the artificial neural networks using the raw pixel data as well as the image-extracted features as inputs to the artificial neural networks. To solve classical artificial intelligence issues, several deep learning algorithms have been used. All of these algorithms are a branch of machine learning techniques.

In recent years, many machine learning algorithms have played a significant role in finding solutions with classification problems, such as; Abd Elfattah et al. [7], presented Artificial Neural Network algorithms together with features extraction algorithms based on invariant moment. They have used self-organized feature maps SOFMs and time lag recurrent network TLRNs. The results showed that using SOFMs classifier have better results about 98.49 % in combination with invariant moments for feature extraction than using TLRNs classifier in combination with invariant moments about 97.29 %. Ferrari et al. [8], proposed linear discriminant analysis (LDA) to automatically classify galaxies images by measured the parameters of galaxy morphology, including asymmetry, concentration, the Gini coefficient, smoothness, entropy, spirality and moment with accuracy of more than 90%. Dieleman et al. [9], developed a rotation invariant convolutional neural network (CNN) algorithm. The achieved accuracy was approximately 99%.

Polsterer et al. [10], proposed an unsupervised method, called parallelized rotation/flipping invariant kohonen (PINK) maps to classify a number of galaxy images by extracting a set of features. This approach is focused on an enhancement of self-organizing maps with invariant similarity measure flipping and intensive rotation. Also, it is used an environment of a multi-core CPU/GPU. Selim et al. [6], presented a supervised machine learning algorithm for classifying galaxy images from the Zsoltfrei catalog based on Non-Negative matrix factorization method. The accuracy was about approximately 93% compared to other manually classified methods. Selim et al. [4], presented a new supervised machine learning algorithm for classifying galaxy images automatically from the EFIGI catalog based on the nonnegative matrix factorization method. The algorithm was used a dataset of 700 images (a large dataset) and 110 images (a small dataset). The results showed that the achieved accuracy was about 92% for large dataset and 93% for small dataset.

Aniyan et al. [11], developed a convolutional neural network (CNN) approach and 3 binary classifiers, using images from the Fanaroff–Riley (FRI and FRII) class and bent-tailed radio galaxies. The results showed that the precision is at 91%, 75% and 95% for FRI, FRII and bent-tailed radio galaxies classes, while the recall is at 91% and at 79% for each (FRI, FRIIs) and the bent-tailed class respectively. Khalifa et al. [12], proposed a CNN and a SoftMax classifier model using a sample from the EFIGI catalogue. The accuracy result was about 97.272%. Khalifa et al. [13], developed a deep convolutional neural network algorithm for galaxy images classification using EFIGI catalogue with an accuracy about 97.772%. Abd Elaziz et al. [1], proposed a new algorithm depending on the machine learning techniques using feature selection method called, artificial bee colony based on gegenbauer orthogonal moments using a sample from the EFIGI catalog. The achieved accuracy was about 94.63%. Zhu et al. [5], proposed a Residual Networks (ResNets) variant along with convolutional neural networks (CNNs) for classification of galaxy morphology. The achieved accuracy was about 95.2083%.

It is clear that prior researchers obtained successful results to some degree from the above researches. By using a new approach in the current work, the performance rate is very high by using a combination between neutrosophic techniques and machine learning based classifiers. This paper's contributions are summarized as follows:

- Building a novel hybrid automated algorithm to classify galaxies images in an efficient manner.
- Through these proposed tools, the galaxy classification process would decrease the system's complexity while achieving higher efficiencies needed to classify astronomical objects.

- Astronomers will be gaining a better understanding about the evolution of galaxies and to test many theories about the universe. In addition, they will be able to make use of the addition of this information to these catalogues of galaxies.
- Several research students will be able to use these methods in discovering new astronomical objects and use it in their projects.
- There can additionally be extension to these projects through classifying the galaxies morphology into extra than three classes.
- The excited people can add more objects of astronomical for classification, such as nebulae, stars, etc.

The paper is organized as follows: Section 2 presents the basic concepts of galaxy classification techniques. Section 3 presents the proposed approach for galaxies classification and its phases. Section 4 presents the experimental result and discussion. Finally, section 5 addresses conclusion (last section).

2. Galaxy Classification Techniques

The main objective of classification is to measure the structural characteristics of the galaxies or to distinguish between different types of these galaxies. The challenge is to create a robust automated intelligent algorithm that generates a high-performance classification which more efficient, simple, fast rating and produces higher results. There are different types of classifiers based on artificial neural networks (ANNs), nine of them were used and evaluated based on a set of selected features. We also utilized a mix between two different techniques, neutrosophic and machine learning techniques as described in the next section.

2.1 Multilayer Perceptron (MLP)

MLP is a feed forward network with layers usually trained on fixed back propagation. These networks have used into a myriad of applications that require classification of fixed patterns. Its advantages, is that any continuous and logical function can be represented as long as hidden units are suitable with using the appropriate activation function. In addition, it is simple in use, and it can round whatever output or input map [14]. MLP includes; input layer, more than a hidden layer that able to generate more efficient results and the output layer. The approach for updating weights known as back propagation in ANNs to gain the accuracy, it is aims to produce results with the least number of errors. The error is only seen at the output layer and that error is disseminated back again to preceding layers of neural network. Finally, the new weights are up to date and repetition followed again. Since the error size is large at the output layer, the same ratio of error is propagated again to the preceding layer [15].

2.2 Modular Neural Network (MNN)

MNN are a special category of MLP. It using multiple parallel MLPs, these networks process inputs and then reassemble the results. This helps to build inside the topology a certain structure, which enhances job specialization in each subunit (sub-module). Modular networks have no full connection between their layers, unlike MLP standard networks. Therefore, for the same size network, fewer weights are needed. This helps to decrease the number of necessary training models and accelerate training times. There are lots of methods to divide MLP in modules. It is not clear how the modular structure is best structured using the data [16].

2.3 Generalized Feed-Forward (GFF)

GFF networks are a generalization for MLP so that connections can leap over one or extra layers. MLP could in theory, solve any issue by the generalized forward feeding networks. However, in practice, GFF networks addresses the issue more effectively. So, MLP takes more training cycles of hundreds of times than the GFF network comprising the same number of processing components [17].

2.4 Principal Component Analysis (PCA) Network

PCA networks combine supervised and unsupervised learning within the same structure. It is an unsupervised linear procedure finds major components and a set of uncorrelated features from the input. The nonlinear classification of these components is supervised by MLP [18].

2.5 Jordan/Elman Network (JEN)

JEN expand multi-layered perception of context units, which are processing elements which remember previous operation. Network context units offer the ability to extract the time information from the data. The output of the network is copied by the Jordan network, while the first hidden processing elements actions, are transmitted to the contextual units in the Elman network. There are also networks that feed the final hidden layer to the context units and the input [19].

2.6 Self-Organized Maps (SOMs)

SOMs networks convert the arbitrary dimension of the input into a topological (neighborhood preservation) restriction of a one- or two-dimensional discrete map. The main benefit of this network is the clustering provided by the SOM, which uses a self-organizing mechanism to reduce the input space to representative features. Then the input space basic structure is preserved, while the space dimensions are reduced. The feature maps are calculated by using unsupervised learning called Kohonen method. SOM output may be used as inputs to a supervised neural network used in classification like the network of MLP [20].

2.7 Radial Basis Function (RBF) Networks

RBF are a non-linear and hybrid networks that normally including one hidden layer of processing elements. Instead of the sigmoidal functions used by MLPs, the RBF layer using Gaussian transfer functions. The learning process of these networks is far quicker than MLPs. Gaussian's widths and centers are set by rules of non-supervised learning, and supervised learning is carried out on the output layer. All the network weights may be analytically determined if the net of generalized regression (GRNN) probabilistic (PNN) is selected. During this case, by definition, the cluster centers number and the model's number is equivalent, and the same variance is given to all of them [21].

2.8 Recurrent Networks (RNs)

RNs are the latest technology in classifying the time pattern of non-linear time series, identification of the system and prediction. Recurrent networks actually have two types: (i) partially recurrent net begin with a fully recurrent network, then it adding a feed forward connection which bypasses the recurrent portion and treating it effectively as a state memory and (ii) fully recurrent net that feedback the hidden layer to themselves. RNs may have a memory with an unlimited depth, and therefore they are finding the relationships both by the instantaneous space of input and through time [22].

2.9 Time Lagged Recurrent Networks (TLRNs)

TLRN are a multilayered perceptron networks with systems of memory of short-term. In real-world, many data consist of information in its time structure, that is, how data adjustments over time. However, most neural networks are fixed classifiers in purely. Time lagged recurrent networks are the latest technology in classifying the time pattern of non-linear time series, identification of the system and prediction [23].

3. The Proposed Architecture

The proposed architecture framework for classification of galaxy images is presented in details in Figure 2. It includes four basic phases, (i) Preprocessing phase, followed by (ii) Feature extraction phase, (iii) Neutrosophic techniques phase, then (iv) Machine learning and Classification phase. In this section, these four phases are defined in details with the involved steps, the features and the characteristics of each phase.

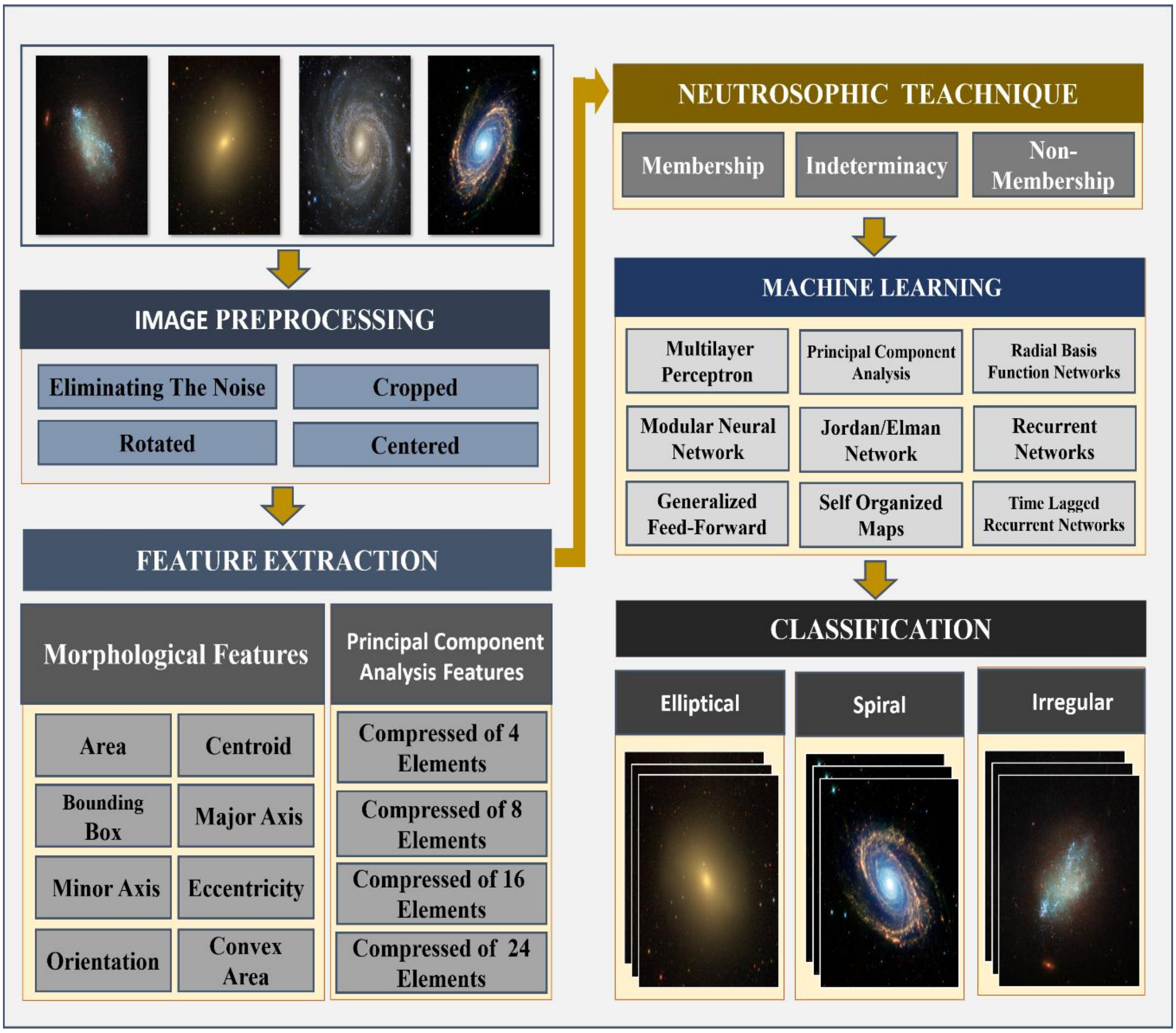


Figure 2. Visualizes The Proposed Architecture Galaxy Classification Framework.

3.1 Image Preprocessing Phase

Before extracting any features from the images, the pre-processing phase is necessary to enhance the images and also before applying the machine learning algorithm. Also, the dataset that collected is often of various sizes, colors, noise, positioning, etc. The image processing goal is to create fixed images, all of which are equal in color, noise elimination and size before feeding the images to the neural network [24].

3.2 Feature extraction phase

The extraction of features leads to some shift in the original features by performing transformations and combinations to create other more effective features by extracting the features from the entered data to improve the accuracy of learning models. This stage decreases data dimensions by removing redundant data and thus maintaining the most prominent features, as it improving the speed of training and inference. In generally, the features extraction is used to indicate the creation of linear sets of continuous features with a fine discriminatory force among categories [25]. In this paper, nine sets of morphic features (MFs) in addition to a set of principal components (PCs); (4, 8, 16 and 24) were extracted from the galaxy images, to get the efficient and the most salient features from the chosen database images.

3.2.1 Principal Component Analysis (PCA) Features

PCA is a statistical second order method which converts a number of associated variables into a smaller number of unassociated variables known as PCs. In general, principal component analysis is used to minimize the data set dimensions with maintaining as much information as possible. The data can be represented through a few base vectors instead of using all the covariance matrix PCs [26, 27]. In this paper, PCs were extracted from the whole training set; in which every image was expressed as a row vector. After that, every image coefficient was transformed into a collection of features and this is the new galaxy information representation.

3.2.2 Morphological Features (MFs)

MFs are based on the visual characteristics of the galaxy such as: (i) Minor axis: the minor ellipse axis length in pixels, which has the same natural central second moment as the area, (ii) Major axis: the major ellipse axis length in pixels, which has the same natural central second moment as the area; (iii) Orientation: expresses areas which can be described as having one dimension locally, in terms of edges or lines for example; (iv) Area: the real pixel count in the area; (v) Bounding box: shows the square with the smallest scale in which all points are located inside; (vi) Eccentricity: represents the degree to which the image of the galaxy deviates from being circular; (vii) Centroid: represents all straight lines intersection that divide the image into two equal-moment parts informally around the line, which is the average of all image points; (viii) Extrema: expresses the minimum value or the maximum value which takes in a point either in its entirety within the function domain; or within a particular neighborhood; and (ix) Convex area: represents the line segment that connects any two points within the shape, which are fully contained in the figure.

3.3 Neutrosophic Techniques (NTs) Phase

NTs are the techniques, which uses neutrosophic sets and the principles of neutrosophic logic for the classification. NTs includes a neutrosophic simple rule-based method such as: if X and Y then Z, to solve the problem rather than trying to model a mathematical similar system of fuzzy approach [28]. The architecture of the neutrosophic inference classification system using fuzzy approach is based on the fuzzy inference method principles of Mamdani [29]. The neutrosophic classification system block diagram is illustrated in Figure 3. The values of the neutrosophic components T, I and F are independent of each other. Thus, three components were constructed using MATLAB's fuzzy logic toolbox: one for the truth component of neutrosophic, the second for the component of indeterminacy and the third for the component of falsity. A correlation is drawn between the membership functions of these components in order to capture the input and output truthfulness, indeterminacy and falsity, although the operation of neutrosophic components are independent from each other [30].

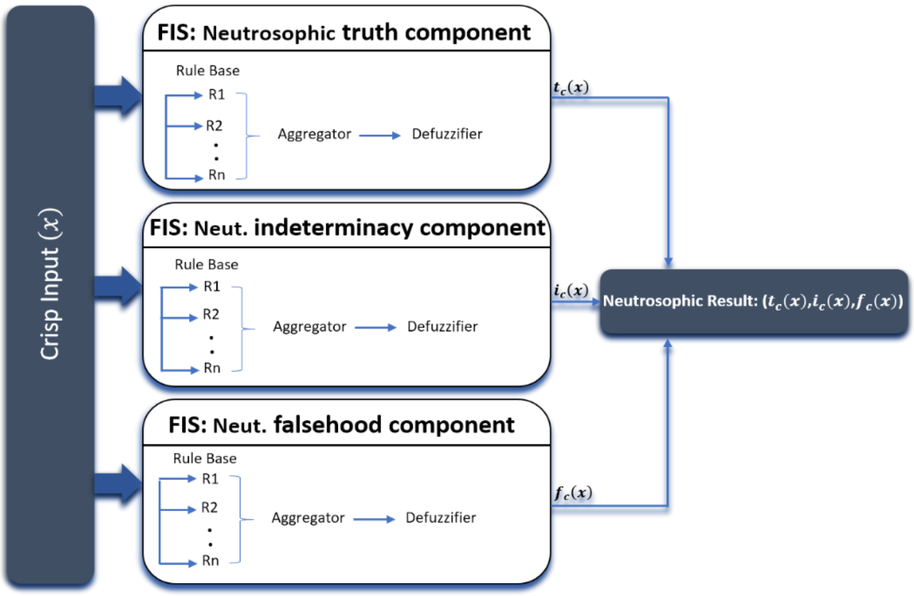


Figure 3. Block Diagram for The Neutrosophic Components [30].

The NRCS is a Neutrosophic Rule-based Classification System where neutrosophic logic is used as a method to represent various types of knowledge about the existing issue, and also to model relationships and interactions between their variables that exist [31]. Figure 4, illustrates the general structure of NRCSs.

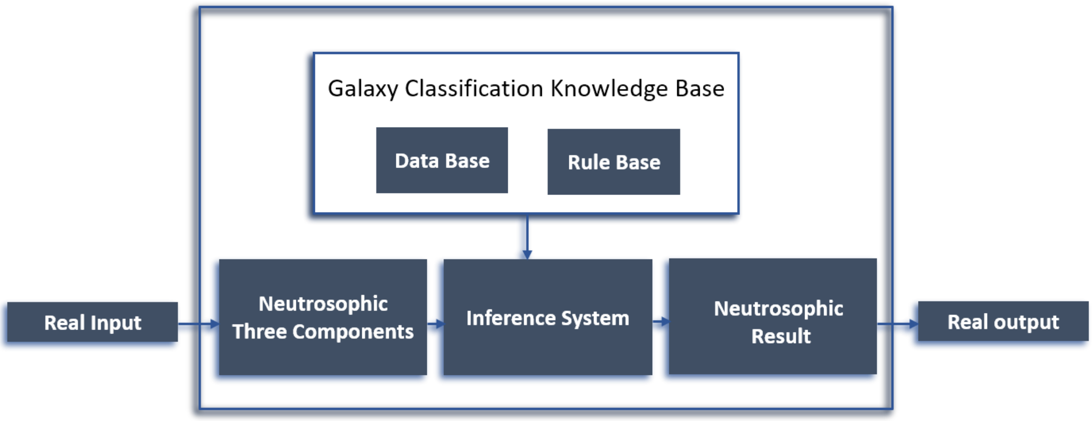


Figure 4. Basic Structure of a Neutrosophic Rule-Based Classifier System [31].

Let U be a universal set while W is a collection of bright pixels units, where W is a sub-set in U. The neutrosophic pixels sets (PNS) of images are defined by, three degrees; T, I and F. The degree of membership can be defined as T, the degree of indeterminacy can be defined as I, and the degree of non-membership can be defined as F. A pixel P in the image is defined as $P\ (T, I, F)$ which belonging to W with true in bright pixels (t%), indeterminate (i%) and false (f%), where t differs in T and i differs in I and f differs in F. The $p\ (i, j)$ pixel in domain of the image turns into:

$$NDP_{NS}(i,j) = \{T(i,j), I(i,j), F(i,j)\} \tag{1}$$

Where $NDP_{Ns}(i,j)$ is the neutrosophic domain for image pixels, $T(i,j)$ belongs to white group, $I(i,j)$ belongs to indeterminate group and $F(i,j)$ belongs to non-white group. Which can be described as [32]:

$$P_{NS}(i,j) = \{T(i,j), I(i,j), F(i,j)\} \tag{2}$$

$$T(i,j) = \frac{\overline{g(i,j)} - \bar{g}_{min}}{\bar{g}_{max} - \bar{g}_{min}} \tag{3}$$

$$I(i,j) = 1 - \frac{H_o(i,j) - H_o}{H_{o_{max}} - H_{o_{min}}} \tag{4}$$

$$F(i,j) = 1 - T(i,j) \tag{5}$$

$$H_o(i,j) = abs\big(g(i,j)\big) - \overline{g(i,j)} \tag{6}$$

Where $\overline{g(i,j)}$ is the local mean value of window size pixels, and $Ho(i,j)$ can be described as the homogeneity value of T at(i,j), which represented with using the absolute value of the various between the intensity $g(i,j)$ and the local mean value $\overline{g(i,j)}$.

The neutrosophic set fundamental concepts are presented by Smarandache in [33, 34] and Salama et al. in [35-38]. In 2014, Salama et al. [39] implemented and designed an object-oriented programming [OOP] to deal with the operations of the neutrosophic data.

After extracting the features using MFs and PCA techniques, the neutrosophic techniques were used to extract three important and efficient components for every morphic and principal component feature which correlated with the most important variables for the algorithm of classification. The steps in this stage are given below, as shown in Figure 5:

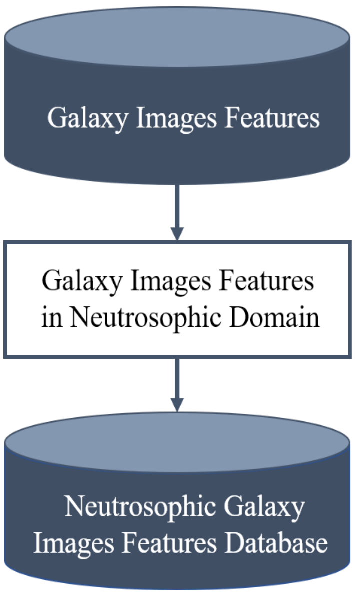


Figure 5. Neutrosophic Galaxy Images Architecture.

- Firstly, we extracted the most efficient and outstanding features to represent each image content within the database.
- Secondly, using the neutrosophic techniques, the data features have been converted from classic mode that helps with the classification process and therefore, we can make a better choice and arrange all alternatives according to three functions namely; membership, indeterminacy, and non-membership components.
- Hence, the database of neutrosophic features of galaxy images was obtained.
- finally, the three components were used to fed to the neural network to get more efficient and accurate result.

3.4 Machine learning and Classification Phase

In the machine learning and classification phase; Firstly, nine classifiers are used based on the ML techniques to conduct the learning method from the extracted features without using NTs as follows, Multilayer Perceptron (MLP), Modular Neural Network (MNN), Generalized Feed-Forward (GFF), Principal Component Analysis (PCA) Network, Jordan/Elman Network (JEN), Self-Organized Maps (SOMs), Radial Basis Function (RBF) Networks, Recurrent Networks (RNs), and Time Lagged Recurrent Networks (TLRNs). Algorithm 1, illustrates pseudo-code for the first proposed algorithm.

Algorithm 1: First Proposed Galaxy Classification Algorithm Pseudo Code

Begin

1) **Converting** every image from the RGB to gray scale image.
2) **Defining** the galaxy limits considering it as a nebulous object.
3) **Locating** the exact position of the galaxy by the position of the center of the spheroid bulge.
4) **Choosing** a suitable window size for eliminating the noise through property of preserving the edges.
5) **Measuring** galaxy intensity region and building a complete map for the background to subtract it from the initial image.
6) S**ubtracting** outside objects and bright stars from outside the galaxy bulge.
7) **Rotating** each galaxy image into a horizontal position.
8) **Cropping** the galaxy body from the image.
9) **Centering** the galaxy body to seem uniform.
10) **Extracting** the visual characteristics of each galaxy (Morphological features).
11) **Computing** the input data dimensions.
12) **Assuming** a galaxy image vector set (n= 255); every (n) vector has the identical length (K).
13) **Making** an n^2dimensional vector for every image.
14) **Establishing** the PCA analysis input by arranging galaxy images in a big matrix.
15) **Creating** the n dimensional matrix for every galaxy images type.
16) **Applying** the PCA algorithm, to transform these vectors to a new vector for every case of the galaxies images vectors.
17) **Computing** the mean for every type of the galaxy vector, then subtract it from every data value.
18) **Computing** the input data matrix covariance matrix (C).
19) **Calculating** the covariance matrix by choosing the largest eigenvalues (K) and the eigenvector, where (K << N).
20) **Obtaining** the new feature vector of eigenvectors of principal components.
21) **Extracting** the final data computed, i.e., the new data set.
22) I**ntroducing** the new extracted features to the ANNs based classifiers to conduct the learning Method.
23) **Obtaining** the classified galaxy images into three types of galaxies**.**

End

Then, after seeing the results we using a combination between neutrosophic techniques and machine learning based classifier including the best classifier which gave the best results in the first algorithm classification phase and it was the multilayer perceptron (MLP) classifier. After extracting the neutrosophic three components (membership, indeterminacy and non-membership) for each principal component and morphic feature in order to get more efficient and accurate result, the three components were fed to the MLP classifier, and then compared with other methods, which indicated that our method has performed better in classifying the images of galaxies to one of three types from the obtained database. Algorithm 2, illustrates pseudo-Code for the second proposed algorithm. While Figure 6 represents the MLP classifier structure model, with an illustration of the number of input layers, hidden layers, and the output layers that were used for small set of images.

Algorithm 2: Second Proposed Galaxy Classification Algorithm Pseudo Code

Begin

1) **Extracting** the neutrosophic three components for each principal component and morphic feature.
2) **Introducing** the extracted three components of features to the ANNs classifier to conduct the learning Method.
3) **Obtaining** the classified galaxy images into three types of galaxies with highly efficient results**.**

End

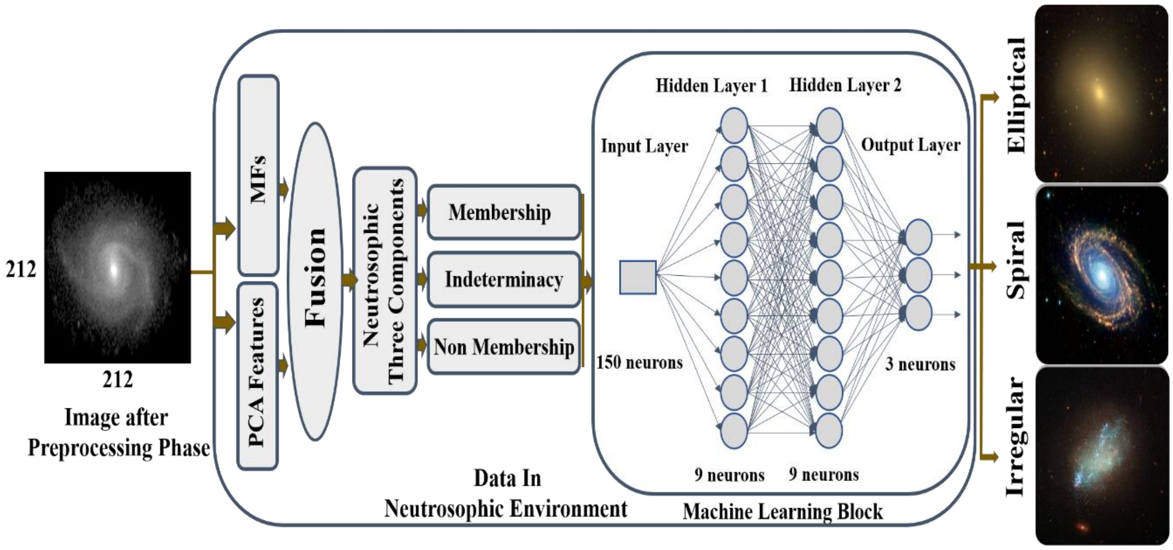


Figure 6. The MLP Classifier Structure Model

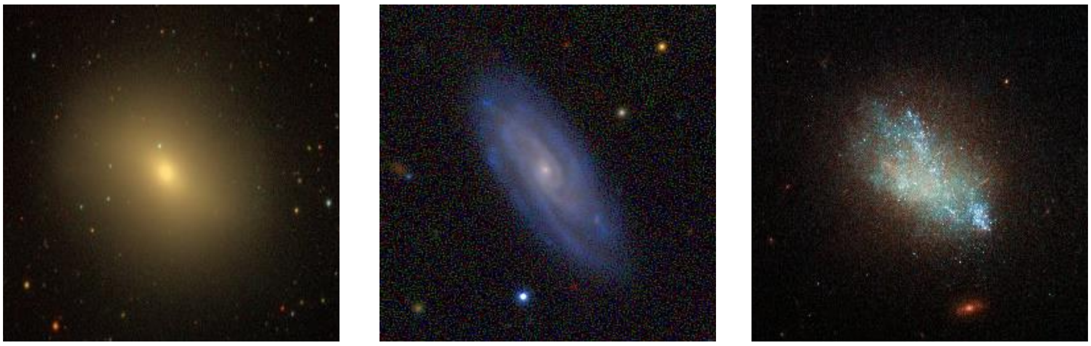
4. Experimental Results and Discussion

4.1 Data set Collection

In this research, the used images were collected from the EFIGI catalog [40], the database contains a sample of all kinds of Hubble galaxies. EFIGI database are usually used in astronomical study as a benchmark, as the images are distinguished by good quality and high resolution. The catalog integrates data from standard catalogs and surveys such as (NASA Extragalactic Database, Value-Added Galaxy Catalogue, HyperLeda, Sloan Digital Sky Survey, and the Principal Galaxy Catalogue) [12]. Also, types of Hubble galaxies, such as the Elliptical, Spiral, and Irregular galaxies, were chosen according to the images captured availability. The images sizes differ in height and width. The galaxies types and the number of training and testing images for every type of galaxy are shown in Table (1), while Figure 7 displays three samples from the galaxy images forms.

Table 1. Number of galaxy type images in each set (data, training, and testing).

Galaxy Type	Data Set	Training Set	Testing Set
Elliptical	150	105	45
Spiral	130	100	30
Irregular	110	85	25



(a) (b) (c)

Figure 7. Samples of EFIGI Catalog (a) Elliptical type, (b) Spiral type, (c) Irregular type

4.2 Performance measures

Performance measures can be characterized as a logical and mathematical structure used for calculating how near the actual results are to what was projected or predicted. The performance measures are used to compare the predictions of the trained model with the actual data from the set of test data in machine learning regression experiments. These metrics results can directly affect the process of decision-making for choosing the types of machine learning algorithms [41]. In order to assess the efficiency of the classification approach, five various performance indices metrics have been used: (i) Mean-Square Error (MSE); (ii) Normalized Mean Square Error (NMSE); (iii) Correlation Coefficient (R); (iv) Error Percentage; and (v) Accuracy. They are described as follows:

4.2.1 Mean squared error (MSE)

MSE calculates the average of the squares of the errors, i.e., the average squared difference among the values included in the estimator and the calculated quantity's actual values. MSE is a function of risk, as it Corresponding to the squared error loss' expected value. If Pij is n predictions vector, and T_i is the real values vector. MSE is calculated for the predictor as within the equation (7) [2]:

$$MSE = \frac{\sum_{J=0}^{n}(P_{ij} - T_j)^2}{n} \quad (7)$$

where Pij is the predicted value outside of n sample cases or fitness cases measured through case i for fitness case j, while the Tj is the goal value of fitness case j.

4.2.2 Normalized mean squared error (NMSE)

NMSE is estimated the total deviations as described in the equation (8) between expected and measured values [2]:

$$NMSE = \sum_{J=0}^{n} \left(P_{ij} - T_j\right)^2 (n \times P \times T), \tag{8}$$

$$P = \sum\nolimits_{i=1}^{n} P_{ij}/n \ and \ P = \sum\nolimits_{j=1}^{n} T_j/n$$

where Pij is the predicted value outside of n sample cases or fitness cases measured through case i for fitness case j, while the Tj is the goal value of fitness case j.

4.2.3 Correlation coefficient (r)

r is the quantity which gives for the original image, the quality of the suitable least squares. The correlation coefficient is given as in equation (9) for two data sets x, y as follow [42]:

$$r = \frac{cov(x.y)}{\sigma x \times \sigma y} \tag{9}$$

where $cov(x.y)$ is the covariance of x and y, while σx and σy are the image (x and y) standard deviation.

4.2.4 Error percentage (Error %)

The percentage of error is determined by subtract the value that accepted from the value that calculated [42].

4.2.5 Accuracy

Accuracy is a qualitative performance characteristic, expressing the closeness of agreement between a measurement result and the value of the measurand. y_i is the corresponding true value of the predicted value and $\hat{y}_i$ is the predicted value of the $i-th$ sample. Then, the fraction of right predictions over $n_{samples}$ is defined as [5]:

$$Accuracy(y_i, \hat{y}_i) = \frac{1}{n_{samples}} \sum_{i=0}^{n_{samples}-1} 1(\hat{y}_i - y_i) \tag{10}$$

4.3 Results and discussion

The proposed system was trained using 390 images, with two hidden layers, and 1000 epochs, also was implemented by using (MATLAB 2017b) software package with a specific CPU and run in 64bit windows support environment. All tests were performed using a server with core i5 Intel processor 2.60 GHz, and Ram with 4.00 GB.

In the image preprocessing phase, each image converted from the color image that contains three matrices to a gray scale image containing one matrix in order for the system to be fast and more efficient, every image of galaxy is converted as follows: **(i) *Eliminating the image noise*** by removing outside objects and bright stars by converting image to black and white (binarize image) and isolating the perimeter of the biggest object for more accurate images, **(ii) *Rotating*** every image into a horizontal position in order to have higher performance for the classifier**, (iii)** ***Cropping*** the galaxy body to delete those irrelevant elements from the background and selecting the image bright part, and **(iv) *Centering*** the galaxy object to seem uniform for more efficient extraction of features. ***(v) Resizing* the image** to a standard size (212 × 212) for mnimizing the origenal data dimentions. As seen in figure 8.

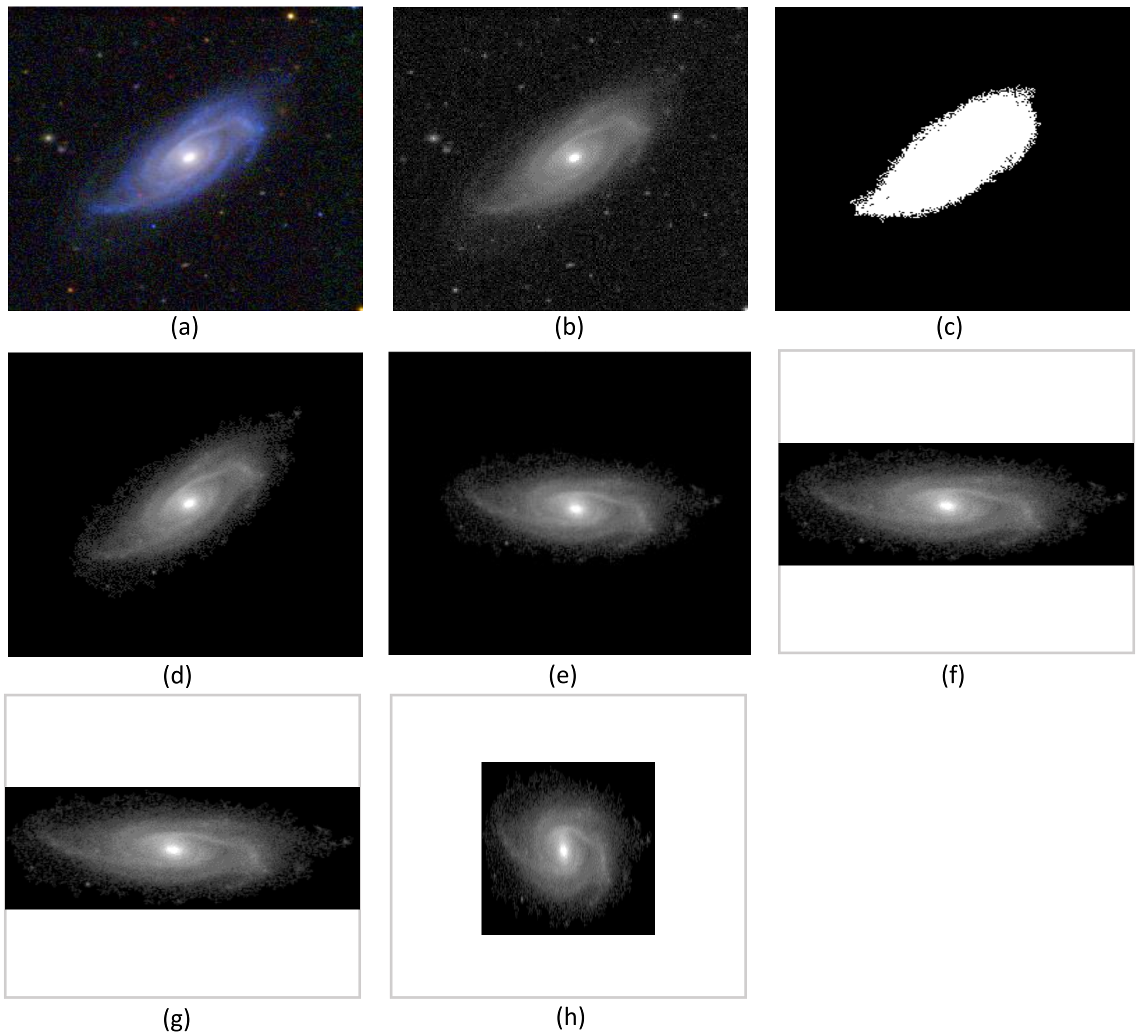


Figure 8. Spiral Galaxy (a) Raw Image, (b) Gray Scale Image, (c) Binarize Image, (d) Image without Noise, (e) Rotated Image, (f) Cropped Image, (g) Centered Image and (h) Resized Image.

The images were transformed using 4, 8, 16 and 24 element PC vector bases in the dataset as five sets of features have been organized: (i) MFs only; (ii) MFs along with 4 PCs, (iii) MFs along with 8 PCs, (iv) MFs along with 16 PCs, and (v) MFs along with 24 PCs. There is no need to extract even more PCs because the findings indicate no apparent difference with the usage of PCs greater than 24 PC. The all-tested classifiers performance, was assessed using various performance measures; MSE, NMSE, r, Error, and Accuracy. Figure 8. to Figure 12. shows the results of the nine ANN classifiers ratings discussed. The results showed the following; the MLP-based classifier using MFs only; gives lower result from all selected features sets about; MSE = 0.1611; NMSE = 0.6011; R = 0.6131, and the Error = 4.7111 with an accuracy 95.2889% but when using the MFs with 24 PCs, the classifier performance give higher result from all chosen features sets about; MSE = 0.0021; NMSE= 0.0371; r = 0.9889, and the Error = 0.7751 with achieving an accuracy of 99.2249 % (figure 9. a., b.).

The MNN-based classifier using MFs only gives lower result from all selected features sets about; MSE = 0.2051; NMSE = 0.6991; r = 0.5901, and the Error = 5.8110 with an accuracy 94.189 % but when using the MFs with 24 PCs, the classifier performance gives higher result from all chosen features sets about; MSE = 0.1113; NMSE = 0.2605; r =0.7926, and the Error = 2.1432 with an accuracy 97.8568 % (figure 9. c., d.).

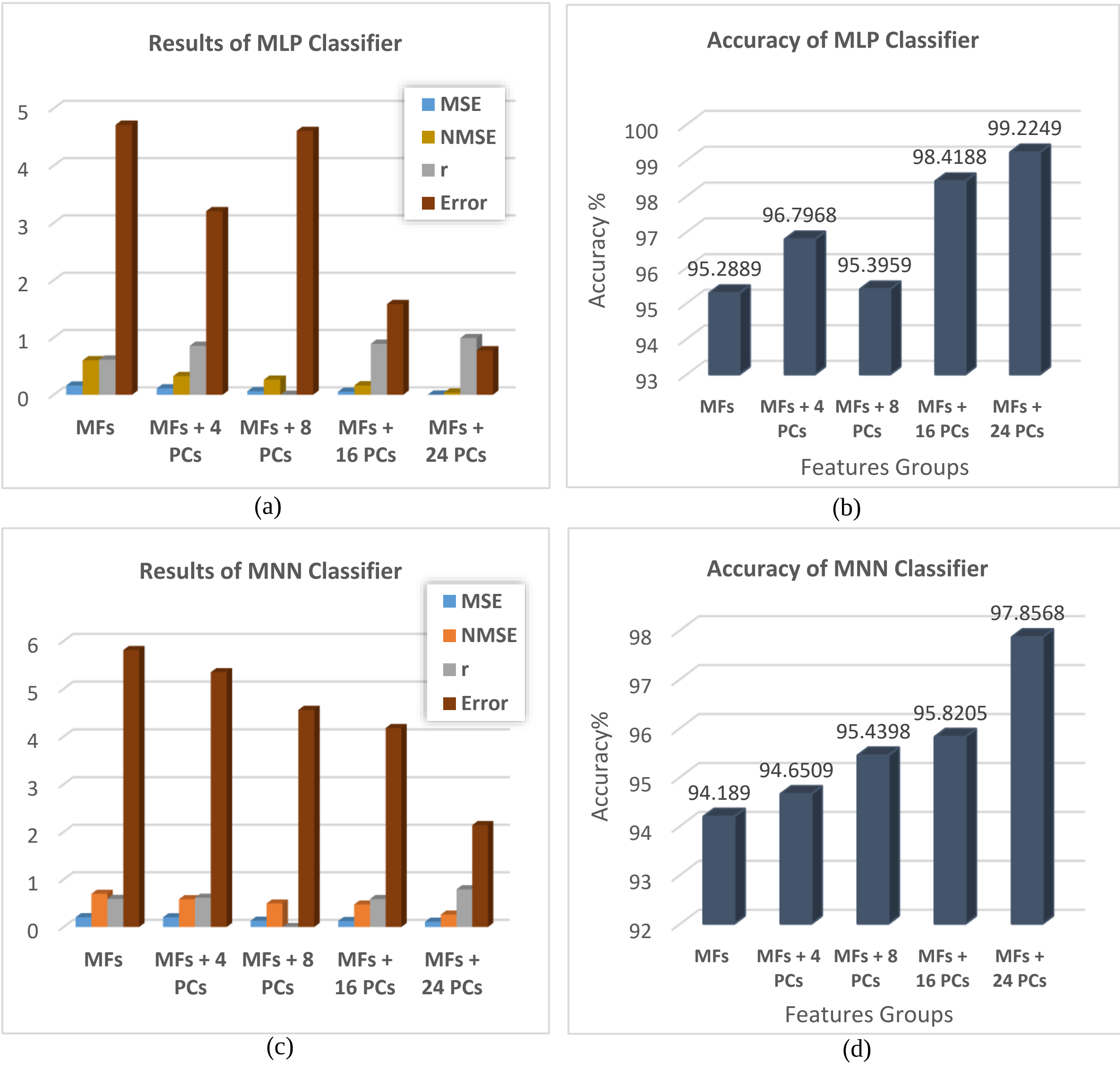


Figure 9. MLP Classification Results; (a) MSE, NMSE, r and Error, (b) Accuracy, and MNN Classification Results; (c) MSE, NMSE, r and Error, (d) Accuracy.

The GFF-based classifier using MFs only gives lower result from all selected features sets about; MSE = 0.1551; NMSE = 0.5791; r = 0.7031, and the Error = 4.6011 with an accuracy 95.3989 % but when using the MFs with 24 PCs, the classifier performance gives higher result from all chosen features sets about; MSE = 0.0119; NMSE = 0.0459; r = 0.9817, and the Error = 1.0211 with an accuracy 98.9789% (figure 10. a., b.).

The PCA-based classifier using MFs with 24 PCs gives lower result from all selected features sets about; MSE = 0.1640; NMSE = 0.6413; r = 0.3897, and the Error = 6.4533 with an accuracy 93.5467 % but when using the MFs with 4 PCs, the classifier performance gives higher result from all chosen features sets about; MSE = 0.1635; NMSE = 0.6412; r = 0.6246, and the Error = 5.4536 with an accuracy 94.5464% (figure 10. c., d.).

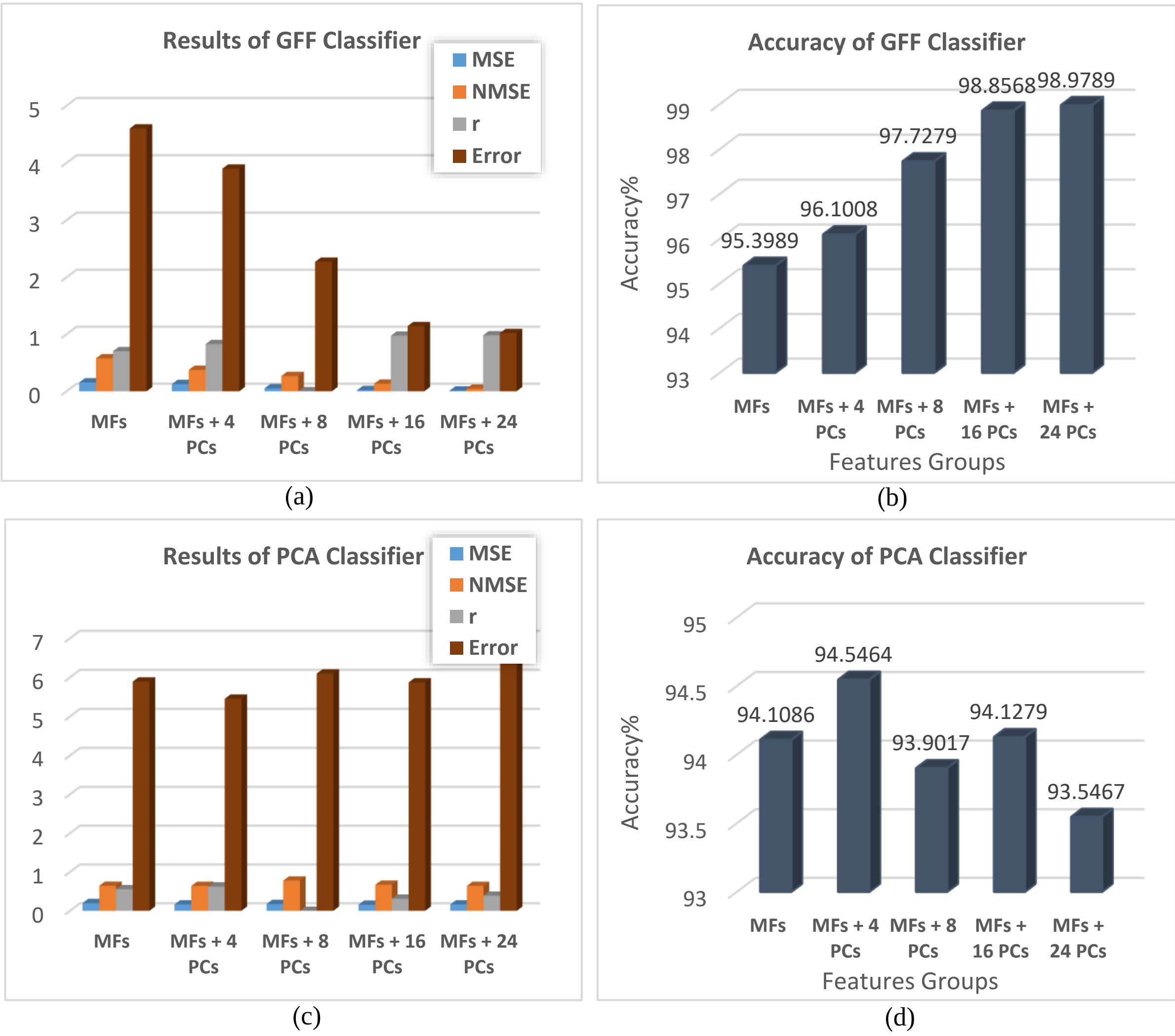


Figure 10. GFF Classification Results; (a) MSE, NMSE, r and Error, (b) Accuracy, and PCA Classification Results; (c) MSE, NMSE, r and Error, (d) Accuracy.

The JEN-based classifier using MFs only gives lower result from all selected features sets about; MSE = 0.1131; NMSE = 0.3582; r = 0.8523, and the Error = 3.7112 with an accuracy 96.2888 % but when using the MFs with 24 PCs, the classifier performance gives higher result from all chosen features sets about; MSE = 0.0136; NMSE = 0.0205; r=0.9872, and the Error = 0.9664 with an accuracy 99.0336% (figure 11. a., b.).

The SOMs-based classifier using MFs only gives lower result from all selected features sets about; MSE = 0.1235; NMSE = 0.5383; r = 0.6830, and the Error = 4.6557 with an accuracy 95.3443 % but when using the MFs with 16 PCs, the classifier performance gives higher result from all chosen features sets about; MSE = 0.1367; NMSE = 0.3541; r = 0.6247, and the Error = 3.1382 with an accuracy 96.8618 % (figure 11. c., d.).

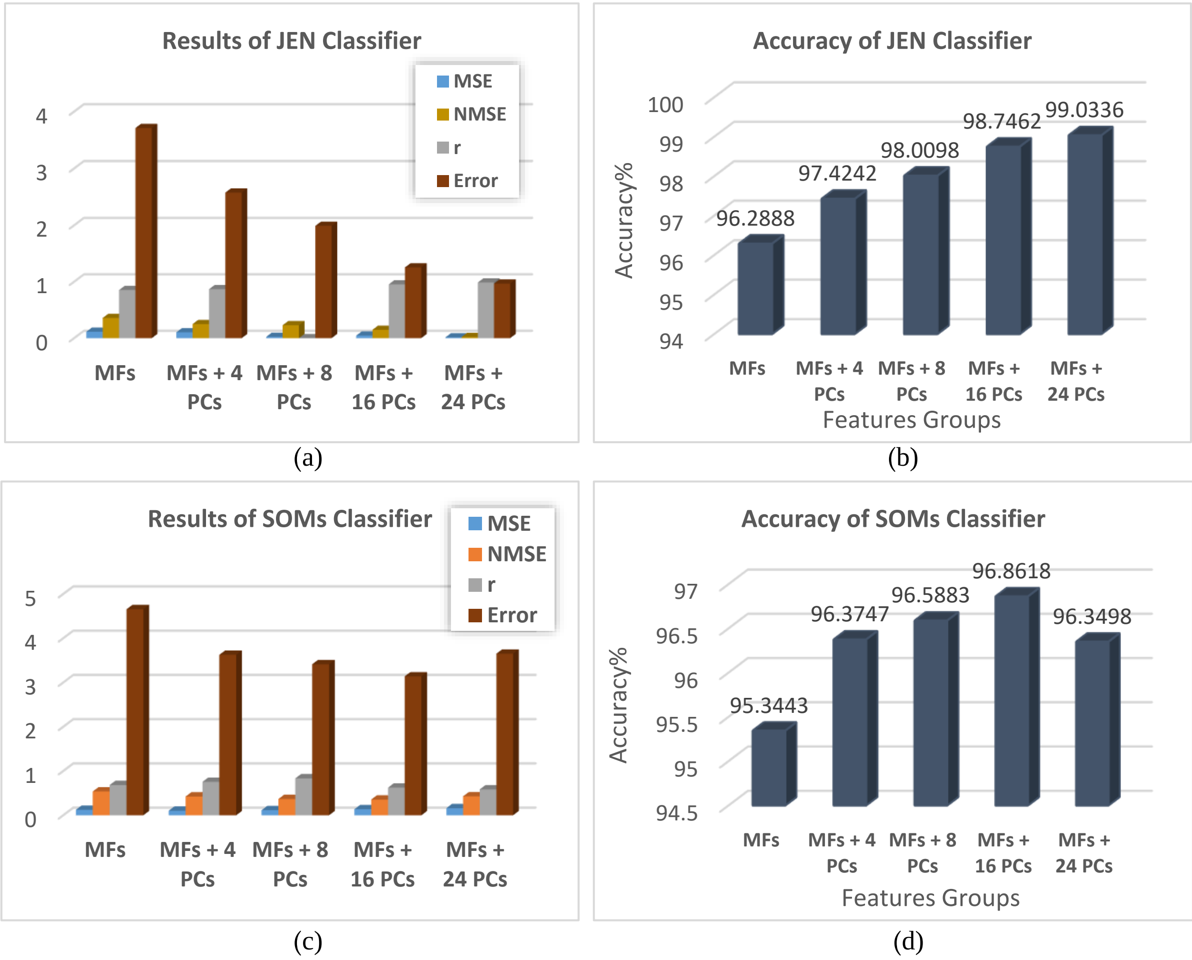


Figure 11. JEN Classification Results; (a) MSE, NMSE, r and Error, (b) Accuracy, and SOMs Classification Results; (c) MSE, NMSE, r and Error, (d) Accuracy.

The RBF-based classifier using MFs with 24 PCs gives lower result from all selected features sets about; MSE = 0.2632; NMSE = 0.8682; r = 0.1233, and the Error = 9.2390 with an accuracy 90.761% but when using the MFs only, the classifier performance gives higher result from all chosen features sets about; MSE = 0.1234; NMSE = 0.5381; r= 0.6832, and the Error = 4.6578 (figure 12. a., b.) with an accuracy 95.3422%.

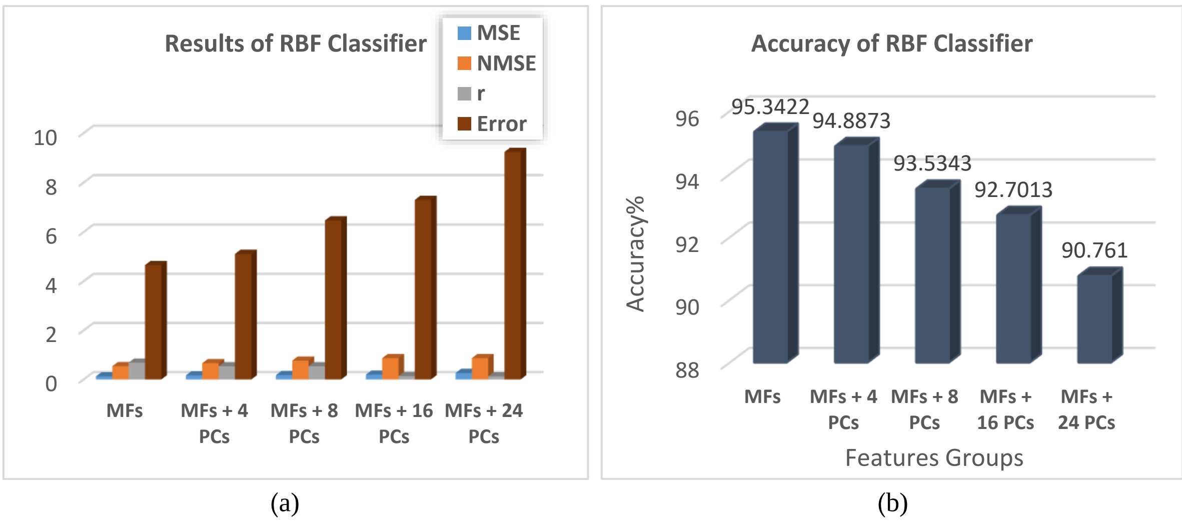


Figure 12. RBF Classification Results; (a) MSE, NMSE, r and Error, (b) Accuracy.

The RN-based classifier using MFs with 4 PCs gives lower result from all selected features sets about; MSE = 0.1998; NMSE = 0.6539; r = 0.5235, and the Error = 5.3830 with an accuracy 94.617% but when using the MFs with 24 PCs, the classifier performance gives higher result from all chosen features sets about; MSE = 0.1509; NMSE = 0.3397; r = 0.8257, and the Error = 2.4287 with an accuracy 97.5713% (figure 13. a., b.).

The TLRN-based classifier using MFs with 8 PCs gives lower result from all selected features sets about; MSE = 0.1999; NMSE = 0.5739; r = 0.5920, and the Error = 7.0739 with an accuracy 92.9261 % but when using the MFs with 16 PCs, the classifier performance gives higher result from all chosen features sets about; MSE = 0.0321; NMSE = 0.156; r = 0.9201, and the Error = 1.986 with an accuracy 98.014% (figure 13. c., d.).

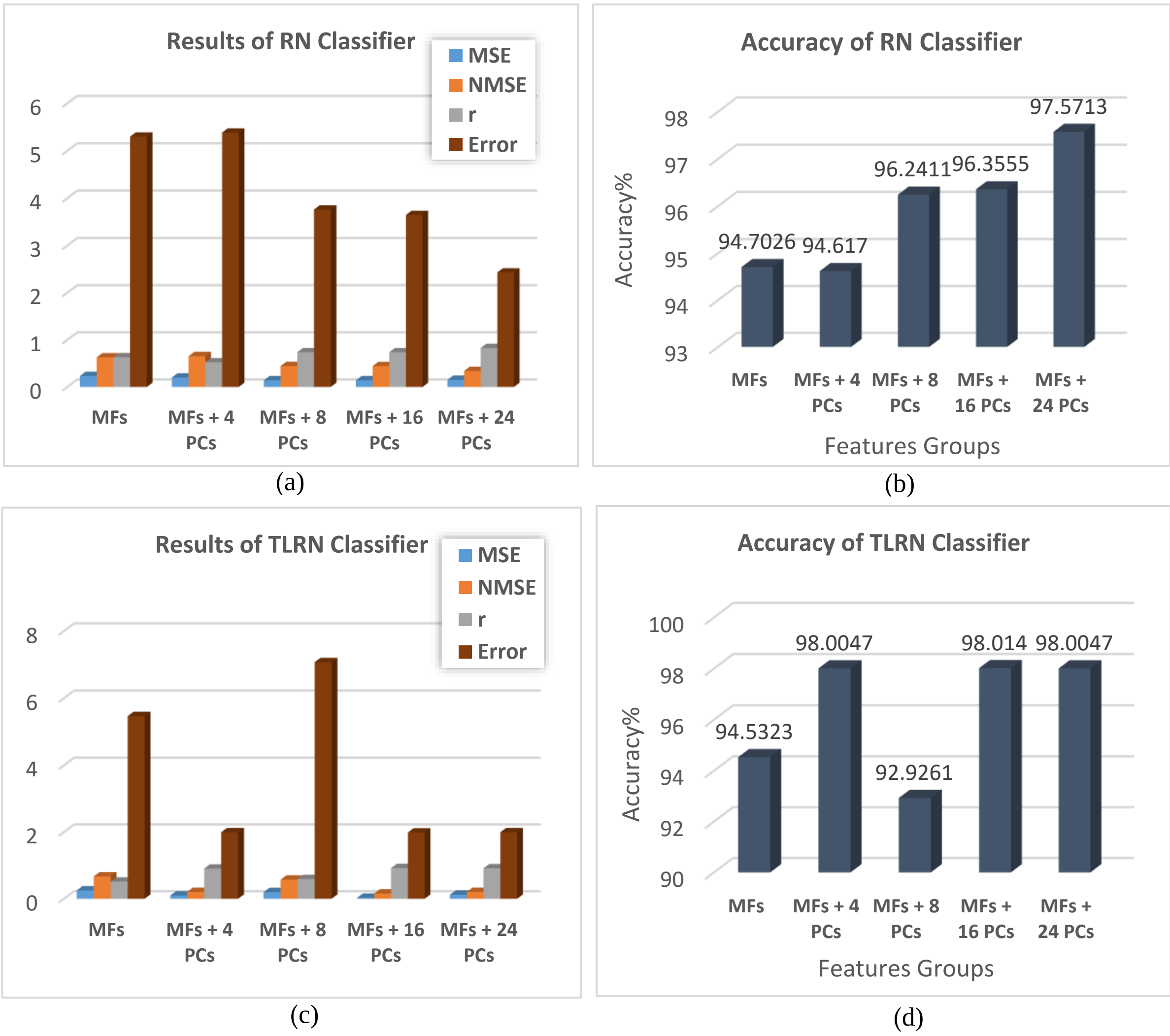


Figure 13. RN Classification Results; (a) MSE, NMSE, r and Error, (b) Accuracy, and TLRN Classification Results; (c) MSE, NMSE, r and Error, (d) Accuracy.

According to the above results, it has been found that the MLP-based classifier gives the best results for all sets of chosen features among all tested ML classifiers while, the RBF-based classifier gives the worst results.

Then, we applied the NTs in combination with the best classifier that give the best results through all tested ML classifiers (MLP). The NTs were applied on the chosen features to get a three efficient component. Figure 14. shows the neutrosophic components (NCs) graph; membership, indeterminacy, and non-membership for the MFs along with PCs features in the neutrosophic environment.

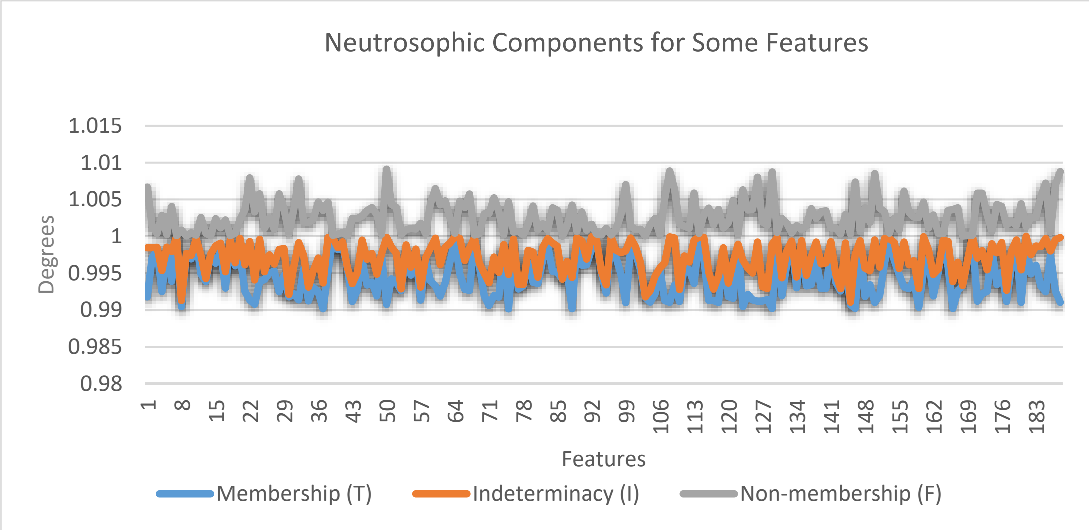


Figure 14. Neutrosophic Three Components for Some Features.

From above results, it has been found that, the MLP classifier of (MFs with 24 PCs) features gives the best results without using NTs through all tested cases for all sets of chosen features but when applied the NTs on the all tested cases for all sets of features and fed to the neural network, the results showed that; using NTs of (MFs with 4PCs) in combination with classifier of MLP gives better result than using the classifier of MLP with the other sets of features; MSE = 0.0001; NMSE = 0.0009; r = 0.9997, and Error = 0.4212 with total accuracy 99.5788 % as summarized in Table (2). The proposed system advantage is depending on a small features number that have been tested for classification, i.e., only MFs with 4PCs, and this reduces system complexity and saves time while getting higher efficiency in the classification process. This indicates, that a small set of features is enough to classify images of galaxy using the NTs. Additionally, there is no need to extract even more 4 PCs with using NTs because the results indicate no obvious difference with the use of PCs more than 4pc.

Table 2. Performance measures* of the proposed algorithm.

Performance Measures	NCs for MFs	NCs for MFs + 4 PCs	NCs for MFs + 8 PCs	NCs for MFs + 16 PCs	NCs for MFs + 24 PCs
MSE	0.0015	0.0001	0.0012	0.0014	0.0021
NMSE	0.0112	0.0009	0.0011	0.0013	0.0032
r	0.8984	0.9997	0.8988	0.9956	0.9950
Error	2.6875	0.4212	1.1743	1.4889	0.4257
Accuracy %	97.3125	99.5788	98.8257	98.5111	99.5743

* NCs: neutrosophic components, MFs: morphological features; PCs: principal components; MSE: Mean squared error; NMSE: normalized mean squared error; and r: correlation coefficient

Figure 15. represents the comparative results between the proposed system and other related works, which shows that the proposed method has high performance results for classifying galaxy images outperforms those of other related works.

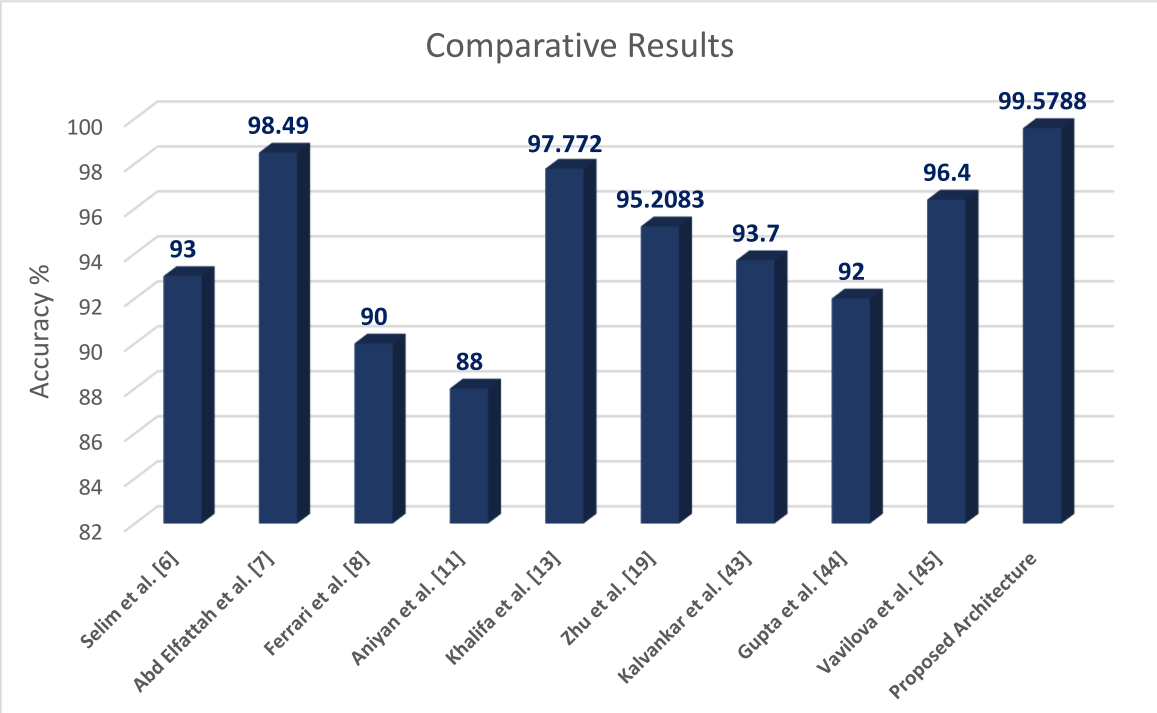


Figure 15. Comparative Accuracy Results for Galaxies Classification Comparing with Other Related Works.

The proposed algorithm achieved accuracy with 99.5788% at the testing process. Comparing with other related works, the proposed system improved the accuracy by 6.5788%, 1.0888%, 9.5788%, 11.5788%, 1.8068%, 4.3705%, 5.8788%, 7.5788%, and 3.1788% comparing with [6], [7], [8], [11], [13], [19], [43], [44], and [45], respectively. Overall, the proposed architecture provides high efficiency in addition has a high level of reliability and achieves advanced performance.

5. Conclusions

Modern sky surveys like, upcoming Large Synoptic Survey Telescope (LSST), Dark Energy Surveys (DES), and COSMOS surveys continue to generate more data, so, the classification of galaxies is one of the most important research topics and studies over the years. The main concern of research was the Hubble classification, as it allowed galaxies to be classified into one of three types based on their morphological features: Elliptical, Spiral, and Irregular. In this paper, a novel automated intelligent system for galaxy images classification into various galaxies types, which combines neural networks and their variants, machine learning algorithms and neutrosophic techniques to build more intelligent classification system was introduced. The obtained results showed that the use of MFs along with 4PCs for feature extraction in combination with NTs and the classifier of MLP gives better results compared to other methods as the testing accuracy was about 99.5788% in total and the performance measures; MSE = 0.0001; NMSE = 0.0009; r = 0.9997, and Error = 0.4212. In addition, we found that a small set of features is enough to classify images of galaxy and this necessary for reducing system complexity and saving time while getting higher efficiency in the process of classification. The results also illustrate the challenge of using the classification system on the irregular galaxies category. Since the category includes galaxies with no definite form that do not follow the criteria of any of the other categories, the category within the set of features is difficult to classify and less diagnostic. Therefore, it is confused with other objects frequently. In upcoming surveys; our algorithm can be applied to a large-scale galaxy classification.

References

1. Abd Elaziz, M., Hosny, K.M. and Selim, I. M. (2019). Galaxies image classification using artificial bee colony based on orthogonal Gegenbauer moments. Soft Computing, 23(19), 9573-9583.
2. Abd Elfattah, M., Elbendary, N., Elminir, H.K., El-Soud, M.A.A. and Hassanien, A.E. (2014). Galaxies image classification using empirical mode decomposition and machine learning techniques. In 2014 International Conference on Engineering and Technology (ICET), IEEE, 1-5.
3. Sharma, P. and Baral, A. (2018). Galaxy Classification Using Neural Networks: A Review. In 2018 International Conference on Audio, Language and Image Processing (ICALIP), IEEE, 179-183.
4. Selim, I.M. and Abd El Aziz, M. (2017). Automated morphological classification of galaxies based on projection gradient nonnegative matrix factorization algorithm. Experimental Astronomy, 43(2), 131-144.
5. Zhu, X.P., Dai, J.M., Bian, C.J., Chen, Y., Chen, S. and Hu, C. (2019). Galaxy morphology classification with deep convolutional neural networks. Astrophysics and Space Science, 364(4), 55.
6. Selim, I., Keshk, A.E. and El Shourbugy, B.M. (2016). Galaxy image classification using non-negative matrix factorization. Int. J. Comput. Appl, 137(5), 4-8.
7. Abd Elfattah, M., El-Bendary, N., Elsoud, M.A.A., Hassanien, A.E. and Tolba, M.F. (2013). An intelligent approach for galaxies images classification. In 13th International Conference on Hybrid Intelligent Systems (HIS 2013), IEEE, 167-172.
8. Ferrari, F., de Carvalho, R.R. and Trevisan, M. (2015). Morfometryka—a new way of establishing morphological classification of galaxies. The Astrophysical Journal, 814(1), 55.
9. Dieleman, S., Willett, K.W. and Dambre, J. (2015). Rotation-invariant convolutional neural networks for galaxy morphology prediction. Monthly notices of the royal astronomical society, 450(2), 1441-1459.
10. Polsterer, K.L., Gieseke, F. and Igel, C. (2015). Automatic galaxy classification via machine learning techniques: Parallelized rotation/flipping INvariant Kohonen maps (PINK). Astronomical data analysis software and systems XXIV (ADASS XXIV), vol 495, p 81.
11. Aniyan, A.K. and Thorat, K. (2017). Classifying radio galaxies with the convolutional neural network. The Astrophysical Journal Supplement Series, 230(2), 20.
12. Khalifa, N.E.M., Taha, M.H.N., Hassanien, A.E. and Selim, I.M. (2017). Deep galaxy: Classification of galaxies based on deep convolutional neural networks. arXiv preprint arXiv:1709.02245.
13. Khalifa, N.E., Taha, M.H., Hassanien, A.E. and Selim, I. (2018). Deep galaxy V2: Robust deep convolutional neural networks for galaxy morphology classifications. In 2018 International Conference on Computing Sciences and Engineering (ICCSE), IEEE, 1-6.
14. Mohamed, M.A. and Atta, M.M. (2010). Automated classification of galaxies using transformed domain features. Int. J. Comput. Sci. Network Security, 10, pp.86-91.
15. Biswas, M. and Adlak, R. (2018). Classification of Galaxy Morphologies using Artificial Neural Network. In 2018 4th International Conference for Convergence in Technology (I2CT), IEEE, 1-4.
16. Lobato, Justo, Pablo Cañizares, Manuel A. Rodrigo, Ciprian-George Piuleac, Silvia Curteanu, and José J. Linares. "Direct and inverse neural networks modelling applied to study the influence of the gas diffusion layer properties on PBI-based PEM fuel cells." international journal of hydrogen energy 35, no. 15 (2010): 7889-7897.

17. Abdel-Azim, Mohamed, A. I. Abdel-Fatah, and Mohamed Awad. "Performance analysis of artificial neural network intrusion detection systems." In 2009 International Conference on Electrical and Electronics Engineering-ELECO 2009, pp. II-385. IEEE, 2009. .
18. Hsieh, W.W., 2001. Nonlinear principal component analysis by neural networks. Tellus A, 53(5), 599-615.
19. Dhule, V., Pathare, S., Sadaphale, R. and Tekade, A., Comparison of ANN Networks Using Parks-Hilbert Approach.
20. Yin, H., 2008. The self-organizing maps: background, theories, extensions and applications. In Computational intelligence: A compendium (pp. 715-762). Springer, Berlin, Heidelberg.
21. Markopoulos, A.P., Georgiopoulos, S. and Manolakos, D.E. (2016). On the use of back propagation and radial basis function neural networks in surface roughness prediction. Journal of Industrial Engineering International, 12(3), pp.389-400.
22. Atiya, A.F. and Parlos, A.G. (2000). New results on recurrent network training: unifying the algorithms and accelerating convergence. IEEE transactions on neural networks, 11(3), pp.697-709.
23. Liang, Y. and Kelemen, A., (2009). Time lagged recurrent neural network for temporal gene expression classification. International Journal of Computational Intelligence in Bioinformatics and Systems Biology, 1(1), 86-99.
24. Bastanfard, A. and Amirkhani, D. (2019). Automatic Classification of Galaxies Based on SVM. In 2019 9th International Conference on Computer and Knowledge Engineering (ICCKE), IEEE, pp. 32-39.
25. Khalid, S., Khalil, T. and Nasreen, S. (2014). A survey of feature selection and feature extraction techniques in machine learning. In 2014 Science and Information Conference (pp. 372-378). IEEE.
26. Cateni, S., Vannucci, M., Vannocci, M. and Colla, V. (2012). Variable selection and feature extraction through artificial intelligence techniques. Multivariate Analysis in Management, Engineering and the Science, pp.103-118.
27. Mankar, V.R. and Ghatol, A.A. (2009). Design of Adaptive Filter Using Jordan/Elman Neural Network in a Typical EMG Signal Noise Removal. Advances in artificial neural systems Vol. 2009, p. 9.
28. Ansari, A.Q., Biswas, R. and Aggarwal, S. (2013). Neutrosophic classifier: An extension of fuzzy classifer. Applied soft computing, 13(1), 563-573.
29. Mohammed, M. N., Syamsudin, H., Al-Zubaidi, S., AKS, R. R., Yusuf, E. (2020). Novel COVID-19 detection and diagnosis system using IOT based smart helmet. International Journal of Psychosocial Rehabilitation, 24 (7), 2296- 2303.
30. Yasser, I., Twakol, A., El-Khalek, A. A., Samrah, A. and Salama, A.A. (2020). COVID-X: Novel Health-Fog Framework Based on Neutrosophic Classifier for Confrontation Covid-19. Neutrosophic Sets and Systems, 35(1), p.1.
31. Fang, Y., Zhang, H., Xie, J., Lin, M., Ying, L., Pang, P., and Ji, W. (2020). Sensitivity of chest CT for COVID-19: comparison to RT-PCR. Radiology, p.200432.
32. Salama, A. A., Smarandache, F., Eisa, M. (2014). Introduction to image processing via neutrosophic techniques. Neutrosophic Sets and Systems, vol. 5, 59-64.
33. Smarandache, F. (2002). Neutrosophy and neutrosophic logic. In First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico, Gallup, NM, Vol. 87301, 338-353.
34. Smarandache, F. (1999). A unifying field in Logics: Neutrosophic Logic. In Philosophy, American Research Press, 1-141.
35. ELwahsh, H., Gamala, M., Salama, A.A. & El-Henawy, I.M. (2017). Modeling Neutrosophic Data by Self-Organizing Feature Map: MANETs Data Case Study. Procdica Computer science, 121, 152-159.

36. Salama, A.A., Eisa, M., ElGhawalby, H. and Fawzy, A.E. (2019). A New Approach in Content-Based Image Retrieval Neutrosophic Domain. In Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets, Springer, Cham., vol 369 , 361-369.
37. Salama, A.A. and Smarandache, F. (2015). Neutrosophic crisp set theory. Infinite Study. Education Publishing 1313 Chesapeake, Avenue, Columbus, Ohio 43212.
38. Salama, A.A., Abdelfattah, M., El-Ghareeb, H.A. and Manie, A.M. (2014). Design and implementation of neutrosophic data operations using object-oriented programming. International Journal of Computer Application, 5(4), 163-175.
39. Salama, A.A., El-Ghareeb, H. A., Manie, A. M., and Smarandache, F. (2014). Introduction to Develop Some Software Programs for Dealing with Neutrosophic Sets. Neutrosophic Sets and Systems, 3, 51-52.
40. De Lapparent, V., Baillard, A. and Bertin, E. (2011). The EFIGI catalogue of 4458 nearby galaxies with morphology-II. Statistical properties along the Hubble sequence. Astronomy & Astrophysics, 532, p.A75.
41. Botchkarev, A. (2019). A NEW TYPOLOGY DESIGN OF PERFORMANCE METRICS TO MEASURE ERRORS IN MACHINE LEARNING REGRESSION ALGORITHMS. Interdisciplinary Journal of Information, Knowledge & Management, 14.
42. Elfattah, M.A., El-Bendary, N., Abou Elsoud, M.A., Platoš, J. and Hassanien, A.E. (2014). Principal component analysis neural network hybrid Classification Approach for Galaxies Images. In Innovations in Bio-inspired Computing and Applications, Springer, Cham, 225-237.
43. Kalvankar, S., Pandit, H. and Parwate, P., (2020). Galaxy Morphology Classification using EfficientNet Architectures. arXiv preprint arXiv:2008.13611.
44. Gupta, R., Srijith, P.K. and Desai, S., (2020). Galaxy Morphology Classification using Neural Ordinary Differential Equations. arXiv preprint arXiv:2012.07735.
45. Vavilova, I.B., Dobrycheva, D.V., Vasylenko, M.Y., Elyiv, A.A., Melnyk, O.V. and Khramtsov, V., (2021). Machine learning technique for morphological classification of galaxies from the SDSS. I. Photometry-based approach.

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Analysis of Covid-19 via Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps

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Abstract: As far world history is concerned, human being faced many problems like world war, terrorist attack, bomb blasters, natural disasters and so on. But all these problems are visible and were able to come across in few months or years. Nowadays situation is entirely different from the past world history, because all the worlds are fighting with an invisible enemy called Novel Corono virus disease (Covid-19). If we compare with other diseases like Swine flu, Ebola virus this disease is very infectious and the death rate is also severe all around the world. The covid-19 pandemic has affected everyone both physically, mentally and changed our life style. Almost all the countries are suffering from this disease and the countries are struggling to control from this epidemic. Researchers in all the disciplines are exploring ways to control this disease and trying to find a vaccine. So this is a correct situation to mathematically analyze the disease covid-19 like symptoms, spreading way and precaution method using Fuzzy cognitive maps and Neutrosophic cognitive maps. Both the methods work on expert's opinion. Since medical field involves uncertainty and indeterminacy, we have chosen fuzzy set and neutrosophic sets for our study.

Keywords: Fuzzy cognitive maps, Neutrosophic cognitive maps, Covid-19, Symptoms, Prediction.

1. Introduction

Each and every human in the universe is looking back to the ever seen killing disease covid-19. All the peoples are living with fear because of severity in spreading and increasing in death rate. This virus was first detected in Wuhan, one of the cities in China in late December 2019 and rapidly spread in more than 100 countries within few months. It has been declared as a Pandemic by WHO in March 2020. As a precaution, all the countries have declared lock down, working class have been working from home, people are advised to maintain social distance in the public place, continuous hand washing using sanitizer after returning from public place, compulsory wearing mask, avoiding unnecessary outing, online classes for students and so on. But still control of this disease is a challenge. So far there is no prescribed medicine to cure this disease. Various researches is actively going on to study the nature of the virus and to find a vaccine.

On the other hand, all the countries are experiencing an economic downturn because of continuous lockdown. Small merchants, farmers, building contractors and migrant workers are having a hard time. After hundred years, the world is facing a very big disaster because of disease. So we have to put more effort to save from this destruction. As a part of this we have to first create awareness about this disease to the society. To create awareness in the sense this study helps to analysis this disease using fuzzy technique and an extension of fuzzy, called neutrosophic theory. The question is whether this fuzzy techniques is suitable for our study?. The answer is the crisp value 'yes'. Since, for any kind of prediction of (new deadly) diseases or to identify, in the beginning stage there is no crisp answer in the medical field. i.e we cannot say 'yes' or 'no' for the disease during the first examination of the patient. The physician is always in the fuzzy state, because of the complexity of the human body, common or similar symptoms for many disease. We can also say it is indeterminant (extension of fuzzy) to identify the disease for the physician. The doctor wants many parameters like patients medical history, laboratory results, physical examination of the patient body and so on to diagnose the disease. So in comparison to the crisp set, fuzzy set gives gradual membership and also some concepts in the medical term are indeterminant, so it is suitable to use Fuzzy Cognitive Maps (FCMs) and Neutrosophic Cognitive Maps (NCMs) for our study.

This work is constructed as follows: The motivation and background is given in section-2. Necessary basic concepts needed for this study is discussed in section-3. In section-4 analysis of Covid-19 using FCMs and NCMs is presented and the conclusion of the proposed work is given in section-4.

2. Motivation and Background

Decision making in the medical field involves uncertainty and indeterminancy and this motivate us to apply FCMs and NCMs in this work. In the literature FCMs and NCMs are widely applied in many fields, including medical. We will review few here. To study the uncertainty, the concept of fuzzy set was introduced by Zadeh [40]. In 1965, Bart Kosko [20] introduced FCMs as a combination of fuzzy logic and cognitive maps. To analysis the indeterminacy Florentin Smarandache and Vasantha Kandasamy [35] proposed the new technique called NCMs as an extension of FCMS. As a new approach, for diagnosing process of meniscus injury Antigoni et al.[5] used evolutionary type of FCMs called Dynamic Fuzzy Cognitive Knowledge Networks. In another study to identify pulmonary disease Evangelia et al. [14] applied time dependent FCMs. Gaurav[16] combined two techniques namely NCMs and Genetic algorithm for medical diagnosis.

Neutrosophic environment is more suitable to handle indeterminate situation in medical diagnosis. So, we chosen NCMs for the proposed study. Neutrosophic sets are applied for the analysis of medical imaging. For instance, Abdel-Basset et al.[1] used Plithogenic set as a generalization of neutrosophic set for the identification of Covid-19 using primary symptoms and CT scans. Neutrosophic sets are widely applied in various fields for decision making [2, 25, 26, 27].

For decision making in medical field, Albert William et al.[3] applied FCMs to identify the symptoms of breast cancer. For diagnosing Rheumatoid Arthristis (RAs) Chitra et al. [11] applied Gene selection and Dynamic Neutrosophic Cognitive Map with Bat Algorithm (DNCM-BA). Deepika et al. [12] used neutrosophic sets for medical image identification. For medical diagnosis research, Mumtuz Ali et al. [24] applied algebraic neutrosophic measures, Chao Zhang et al. [8] used single valued neutrosophic probabilistic rough multisets, Masooma et al.[22] proposed m-polar neutrosophic topology and shawkat [34] introduces n-valued refined neutrosophic soft sets. Nowadays most of the people in the globe are affected by diabetes. To identify risk factors caused by diabetes AshrafulAlam [6] used FCMs and Muhammad Aslam et al. [23] applied neutrosophic statistic. Nivetha [29] used decagonal linguistic neutrosophic FCMs to analysis the risk factors of life style disease. In another study, Vasantha Kandasamy et al. [35] applied FCMs by taking concepts as symptoms and disease. For medical decision support, FCMs are applied in [39,40].

FCM architecture is proposed for obstetrics by Chrysostomos [10]. In another application Amirkani at el. [4] Studied taxonomy, methods and applications of FCMs in medicine. For tracking urinary infection Douli et al. [13] applied FCMs using 25 clinical and 13 diagnosis concepts. As a new application Neil et al. [28] introduces FCMs in nursing research. Combining with non-linear Hebbian learning algorithm FCMs can be used for prediction of stroke by Khodadadi et al. [19]. In another study, Papageorgiou et al. [31] applied the concept of FCMs to analysis the risk factor caused by familial breast cancer. To analysis the symptoms of migraine Merlyn Margaret [21] used Induced FCMs.

For medical diagnosis Innocent et al. [18] applied various fuzzy methods like clustering, fuzzy set aggregation and type-2 fuzzy sets. For the treatment of fuzzy disease, fuzzy prototypes used by

Ruben et al. [32]. In another study for decision making in medical diagnosis Palash et al.[30] used an advanced distance measures on intuitionistic fuzzy sets. To diagnose the rare disease , applications in medicine using fuzzy logic and fuzzy logic inference is given in [9,15,36,38]. Type-2 fuzzy sets is used to diagnosis the common disease by Besime et al.[7]. Sundaresan et al. [37] proposed fuzzy membership matrix to identify the different treatment stages in medical diagnosis. Hamidi et al. [17] used the application of Neutro-BCK algebra for the study in covid-19 among country wise.

From this review of literature FCMs and NCMs are widely applied in many medical field and this is another motivation for the current work.

3. Basic things needed for the study

3.1 Definition

A fuzzy weighted directed graph involving concepts like policies, events etc as nodes and the link connecting them represent the casual relationship between the concepts is a FCM. If the nodes of the FCM are fuzzy sets then the nodes are fuzzy nodes. NCM is differ from FCM only when the relation between the concepts is indeterminant and it is denoted by $'I'$.

3.2 Definition

An FCMs is said to be simple if the edge weights are taken from the set$\{-\mathbf{1},\mathbf{0},\mathbf{1}\}$ and for simple NCMs it is from $\{-\mathbf{1},\mathbf{0},\mathbf{1},\boldsymbol{I}\}$.

3.3Definition

Let $(\boldsymbol{C_1},\boldsymbol{C_2},\dots,\boldsymbol{C_n})$ be the concepts of the FCMs. Using this concept the directed graph is drawn with edge weight $\boldsymbol{e_{ij}} \in \{-\mathbf{1},\mathbf{0},\mathbf{1}\}$. Here $\boldsymbol{e_{ij}} = \mathbf{1}$ means positive casuality between the concepts. In otherwayincrease(or decrease) to the corresponding increase(or decrease) in the other and vice-versa for $\boldsymbol{e_{ij}} = -\mathbf{1}$. If the concepts has no relation indicates$\boldsymbol{e_{ij}} = \mathbf{0}$. Define the adjacency matrix $\boldsymbol{E} = (\boldsymbol{e_{ij}})$ where $\boldsymbol{e_{ij}}$ is the weight of the corresponding edge $'\boldsymbol{C_i}\boldsymbol{C_j}'$. This adjacency matrix also known as the connection matrix of the FCMs. Similarly for NCMs using the same concepts of FCMs we can draw the directed graph with edge weight $\boldsymbol{e_{ij}} \in \{-\mathbf{1},\mathbf{0},\mathbf{1},\boldsymbol{I}\}$. Here the adjacency matrix is denoted by $\boldsymbol{N}(\boldsymbol{E}) = (\boldsymbol{e_{ij}})$. If $\boldsymbol{e_{ij}} = '\boldsymbol{I}'$means the relation between the concepts is indeterminate and it is denoted by dotted line in the directed graph. The adjacency matrix $\boldsymbol{N}(\boldsymbol{E})$ is called the neutrosophic adjacency matrix of the NCMs.

3.4 Definition

Let $(C_1, C_2, \dots, C_n)$ be the concepts of the FCMs(NCMs).The instantaneous state vector $A = (a_1, a_2, \dots, a_n)$ where $a_i \in \{0, 1, I\}$ represent the ON-OFF-INDETERMINATE position of the node. If $a_i = 0$ means OFF, $a_i = 1$ means ON and $a_i = I$ means indeterminate for $i = 1, 2, 3, .., n.$

3.5 Definition

If the edges form a directed cycle then FCMs (NCMs) is said to be cyclic, otherwise acyclic. An FCMs (NCMs) with cycles is said to have a feedback. If the FCMs (NCMs) has a feedback, then the FCMs (NCMs) is called a dynamical system.

3.6 Definition

Let $C_1C_2, C_2C_3, \dots C_mC_n$(for $m \neq n$) be a cycle. We will say that the dynamical system goes round and round, if the concept $'C_i'$ is ON and if the causality passes through the edges of the cycle and again causes $'C_i'$. This is true for any $'C_i'$ for $i = 1, 2, 3, \dots, n.$ The hidden pattern is the equilibrium state of the dynamical system. If the equilibrium state is a unique state vector, then it is fixed point or limit cycle. This is applicable for both FCMs and NCMs.

3.1.1 The pseudo code for the proposed method

1. Collect the concepts or nodes for the covid-19 problem.

2. Construct the directed graph, neutrosophic directed graph and the corresponding adjacency matrix E and $N(E)$through experts (Doctors) opinion.

3. Take any concept $C_i(i = 1, 2, 3, \dots, n)$ in ON state.

4. To find the hidden pattern of $C_i(i = 1, 2, 3, \dots, n)$, the instaneous input vector $A_1 = (a_1, a_2, \dots, a_n)$ is defined by assigning $a_1 = 1$for $i = 1$ and other $a_i's = 0$ if the concept C_1 is switch ON and similarly for other concepts.

5. Multiply A_1 with E and $N(E)$, we get another row vector namely $(b_1, b_2, \dots, b_n)$. Here the new operation is introduced called threshold operation and it is denoted by the symbol $' \rightarrow '.$ This operation is done by putting b_i=1 to the corresponding ON state concept $C_i(i = 1, 2, 3, \dots, n)$ and 0 for remaining b_i. After this updation we will get another vector called A_2.

6. Multiply A_2 with E and $N(E)$ and repeat the same procedure to reach the fixed point. Similarly we follow the same procedure to find the hidden pattern and indeterminacy for all the concepts of the disease covid-19. Both FCMs and NCMs will function mainly on expert's opinion. To avoid biasness, it is necessary to consider more than one experts.

4. Reasons and Secure from Novel Coronavirus (covid-19) using FCMs and NCMs

This work concentrates on reasons, transmission mode and precaution method of the coronavirus (Covid-19) using FCMs and NCMs. The different concepts considered for this analysis is identified and is given in the Table-1.

Table-1 Concepts considered for the proposed work

Concepts	Explanation
C_1	Fever with cold, cough and difficulty in breathing
C_2	No symptoms
C_3	Maintaining social distance, wearing mask and continuous hand wash.
C_4	High blood pressure, diabetes, tuberculosis, cancer patient, elder people who are violating C_3.
C_5	Travelling history
C_6	Possibility of Covid-19.
C_7	High risk factor for getting Covid-19.
C_8	Prevention measures from Covid-19.

We are taken the above eight main concepts for this study. First we work on FCMs .In Figure 1 we give the directed graph and the connection square matrix E according to first experts opinion.

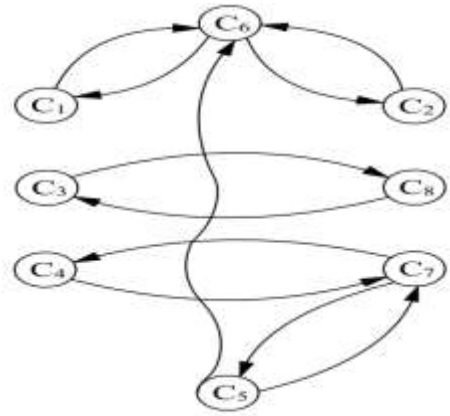

Figure-1 Directed graph given by the first expert for the analysis of covid-19.

The connection matrix $\boldsymbol{E}$ is given by

$$E = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad (1)$$

Case-1 First we consider the concept C_6 i.e the possibility of covid-19. Take $A_1 = (0,0,0,0,0,1,0,0)$ the effect of A_1 on E is given by

$$A_1E = (1,1,0,0,0,0,0,0)$$

$$\to (1,1,0,0,0,1,0,0)$$

$$= A_2. \quad (2)$$

$$A_2E = (1,1,0,0,0,2,0,0)$$

$$\to (1,1,0,0,0,1,0,0)$$

$$= A_3. \quad (3)$$

Here $A_2 = A_3$.

Case-2 Next we consider the concept C_7 i.e High risk for getting covid-19. Take $A_1 = (0,0,0,0,0,1,0,0)$ the effect of A_1 on E is given by

$$A_1E = (0,0,0,1,1,0,0,0)$$

$$\rightarrow (0,0,0,1,1,0,1,0)$$

$$= A_2. \quad (4)$$

$$A_2E = (0,0,0,1,1,1,2,0)$$

$$\rightarrow (0,0,0,1,1,1,1,0)$$

$$= A_3. \quad (5)$$

$$A_3E = (1,1,0,1,1,1,2,0)$$

$$\rightarrow (1,1,0,1,1,1,1,0)$$

$$= A_4. \quad (6)$$

$$A_4E = (1,1,0,1,1,3,2,0)$$

$$\rightarrow (1,1,0,1,1,1,1,0)$$

$$= A_5. \quad (7)$$

Here $A_4 = A_5$.

Case-3 Now we consider the concept C_3 i.e Maintaining social distance,wearing mask and continuous hand wash i.e Take $A_1 = (0,0,1,0,0,0,0,0)$ the effect of A_1 on E is given by

$$A_1E = (0,0,0,0,0,0,0,1)$$

$$\rightarrow (0,0,1,0,0,0,0,1)$$

$$= A_2. \quad (8)$$

$$A_2E = (0,0,1,0,0,0,0,1)$$

$$\rightarrow (0,0,1,0,0,0,0,1)$$

$$= A_3. \quad (9)$$

Here $A_2 = A_3$.

Case-4 Finally we consider the concept C_8.i.e the prevention method from covid-19. Take $A_1 = (0,0,0,0,0,0,0,1)$ the effect of A_1 on E is given by

$$A_1E = (0,0,1,0,0,0,0,0)$$

$$\rightarrow (0,0,1,0,0,0,0,1)$$

$$= A_2 \quad (10)$$

$$A_2E = (0,0,1,0,0,0,0,1)$$

$$\rightarrow (0,0,1,0,0,0,0,1)$$

$$= A_3 \quad (11)$$

Here $A_2 = A_3$

Now the first expert is allow to give the answers concerning the indeterminacy of the concepts. The corresponding neutrosophic graph is given in Figure-2.

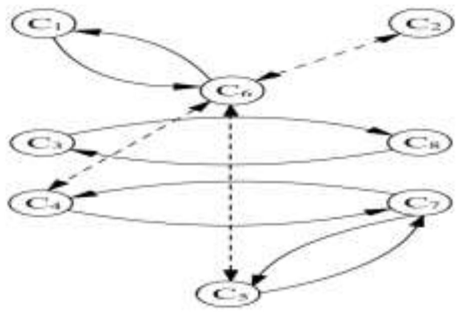

Figure-2 Neutrosophic directed graph given by the first expert for the analysis of covid-19.

The neutrosophic adjacency matrix $N(\boldsymbol{E})$ is given by

$$N(E) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & I & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 1 & 0 \\ 1 & I & 0 & I & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad (12)$$

Case-1First we consider the concept C_6 i.e the possibility of covid-19. Take $A_1 = (0,0,0,0,0,1,0,0)$ the effect of A_1 on E is given by

$$A_1N(E) = (1, I, 0, I, I, 0,0,0)$$

$$\rightarrow (1, I, 0, I, I, 1,0,0)$$

$$= A_2 \quad (13)$$

$$\mathrm{A}_2 N(E) = (1, I, 0, I, I, 1 + 3I^2, 2I, 0)$$

$$\rightarrow (1, I, 0, I, I, 1, 2I, 0).$$

$$= A_3. \quad (14)$$

$$A_3 N(E) = (1, I, 0, 3I, 2I, 1 + 3I^2, 2I, 0)$$

$$\rightarrow (1, I, 0, I, I, 1, 2I, 0).$$

$$= A_4. \quad (15)$$

Here $A_3 = A_4$.

Next we construct the FCMs based on the second expert with the same set of attributes. we give the directed graph in Figure-3 and the connection square matrix **E** according to second experts opinion.

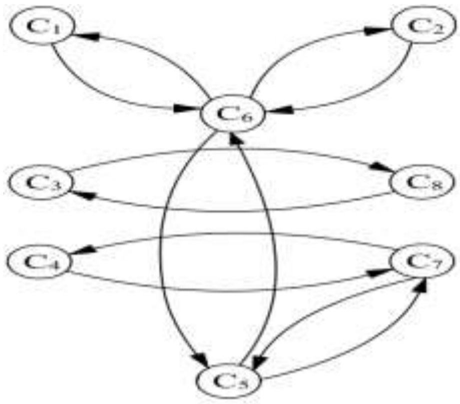

Figure-3 Directed graph given by the second expert for the analysis of covid-19.

The corresponding connection matrix $\boldsymbol{E}$ is given by

$$E = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad (16)$$

Case-1 we consider the concept C_6 i.e the possibility of covid-19. Take $A_1 = (0,0,0,0,0,1,0,0)$ the effect of A_1 on E is given by

$$A_1E = (1,1,0,0,1,0,0,0)$$

$$\rightarrow (1,1,0,0,1,1,0,0)$$

$$=A_2 \quad (17)$$

$$A_2E = (1,1,0,0,1,3,1,0)$$

$$\rightarrow (1,1,0,0,1,1,1,0)$$

$$= A_3. \quad (18)$$

$$A_3E = (1,1,0,1,2,3,1,0)$$

$$\rightarrow (1,1,0,1,1,1,1,0)$$

$$= A_4. \quad (19)$$

$$A_4E = (1,1,0,1,2,3,2,0)$$

$$\rightarrow (1,1,0,1,1,1,1,0)$$

$$= A_5. \quad (20)$$

Here $A_4 = A_5$.

Now the second expert is allow to give the options concerning the indeterminacy of the concepts. The corresponding neutrosophic graph is given in Figure-4.

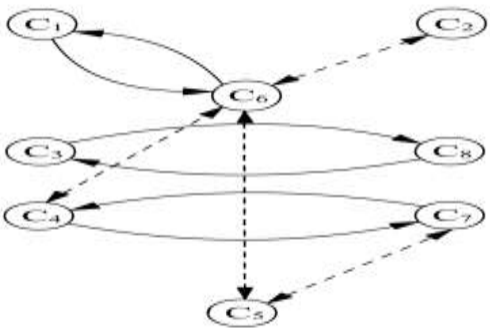

Figure-4 Neutrosophic directed graph given by the second expert for the analysis of covid-19.

The neutrosophic adjacency matrix $N(\boldsymbol{E})$ is given by

$C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5 \quad C_6 \quad C_7 \quad C_8$

$$N(E_1)=\begin{matrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{matrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & I & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 1 & 0 \\ 1 & I & 0 & I & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & I & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

Case-1 Again we consider the concept C_6 i.e the possibility of covid-19. Take $A_1 = (0,0,0,0,0,1,0,0)$ the effect of A_1 on E is given by

$$A_1 N(E) = (1, I, 0, I, I, 0,0,0)$$

$$\to (1, I, 0, I, I, 1,0,0)$$

$$= A_2 \tag{22}$$

$$A_2 N(E) = (1, I, 0,2I, I, 1,2I, 0)$$

$$\to (1, I, 0,2I, I, 1,2I, 0).$$

$$= A_3. \tag{23}$$

$$A_3 N(E) = (1, I, 0,3I, 3I, 1,3I, 0)$$

$$\to (1, I, 0,3I, 3I, 1,3I, 0)$$

$$= A_4. \tag{24}$$

Here $A_3 = A_4$.

5. Conclusion

FCMs and NCMs play a very important role in medical field because it involves uncertainty and indeterminacy. This study uses both the techniques for the analysis of covid-19 and we reached many important solutions. First, the results for the various attributes for this covid-19 based on FCMs is discussed.

According to first expert from case-1, the possibility of covid-19 mainly because of fever with cold, cough and difficulty in breathing and no symptoms. From case-2, persons having fever with cold, cough and difficulty in breathing, without any symptoms, persons having disease mentioned in C_4 violating the precaution methods, travelling from countries to countries are all have to get possibility of getting and high risk of getting this disease. Maintaining social distance, wearing mask and continuous hand washing are the main precaution method and vice-versa for this disease from case-3 and case-4. From the second expert is concern, except the concepts C_3 and C_8 all are main reasons for the possibilities of this disease. Next, we see the indeterminant factor regarding this disease using NCMs.

As far NCMs is concern we are getting the same fixed points for both the experts. Persons having no symptoms, high blood pressure, diabetes, tuberculosis, cancer patients, older people who are not following any precaution method mentioned in C_3,travelling history persons are indeterminant factors for the physician to decide for the possibility and the high risk of getting of this covid-19. The results of this study is suitably matches the current world situation of this disease and we have to strictly follow the precautions mentioned in C_3 to prevent from this invisible disease. The proposed study mathematically analyze the disease using fuzzy and neutrosophic techniques, and it is very useful to all to know the root cause of the epidemic as well as the procedures to be followed to protect from this disease. In future this research can be extended using various fuzzy techniques.

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References

[1] Abdel-Basset,M.; Mohamed,R.; Elhoseny,M. A model for the effective COVID-19 identification in uncertainty environment using primary symptoms and CT scans. *Health Informatics Journal*, **2020**, pp 1-18.

[2] Abdel-Basset,M.; Nabeeh,N.A.; El-Ghareeb,H.A.; Abelfetouh,A. Utilizing Neutrosophic Theory to Solve Transition Difficulties in IoT-based Enterprises. *Enterprise Information Systems*, **2019**, 14, pp 1-21.

[3] Albert William,M.; Victor Devadoss.A.; Janet Sheeba.A. A Study on the Symptoms of Breast Cancer using Fuzzy Cognitive Maps. *International Journal of Engineering Research & Technology*, **2012**, 1, pp 1-4.

[4] Amirkani,A.; Papageorgiou,E.I.; Mohseni,A.; Mosavi,M.R. A Review of Fuzzy Cognitive Maps in Medicine: Taxonomy, Methods, and Applications. *Computer Methods and Programs in Biomedicine*, **2017**, 142, pp 129-145.

[5] Antigoni.P.Anninou.; Peter,P.Groumpos.; Panagiotis Poulios.; Ioannis.Gkliatis. A New Approach of Dynamic Fuzzy Cognitive Knowledge Networks in Modelling Diagnosing Process of Meniscus Injury. *International Federation of Automatic Control* (IFAC papers online), **2017**, 50-1, pp 5861-5866.

[6] Ashraful Alam,Md. Fuzzy Cognitive Maps Approach to Identify Risk Factors of Diabetes. *Journal of Physical Sciences*, **2017**,22, pp13-21.

[7] Besime,E.; Abiyev,R.H. Diagnosis of Common Diseases Using Type-2 Fuzzy System. *ICMLSC 2019: Proceedings of the 3rd International Conference on Machine Learning and Soft Computing,* **2019**, pp 239–243.

[8] Chao,Z.; Deyu,L.; Said Broumi and ArunKumar,S. Medical Diagnosis based on Single Valued Neutrosophic Probabilistic Rough Multisets over Two Universes. *Symmetry*, **2018**, 10(6), pp 1-16.

[9] Choudhury,B.S. ;Dhara,P.S.; Saha,P. An application of fuzzy logic on importing medicines.*International Journal of Healthcare Management*, **2019**, pp 1-7.

[10] Chrysostomos,D.S.; Voula,C.G. Fuzzy Cognitive Maps Structure for Medical Decision Support Systems. *Studies in Fuzziness and Soft Computing*, **2008**, pp 1-5.

[11] Chitra,B.; Nedunchelian,R. Gene selection and dynamic neutrosophic cognitive map with bat algorithm (DNCM-BA) for diagnose of rheumatoid arthritis (RAs). *International Journal of Engineering & Technology,***2018**, 7, pp 242-250.

[12] Deepika,K. ; Savita,G.; Sukhwinder, S. Applications of Neutrosophic Sets in Medical Image Denoising and Segmentation. *New Trends in Neutrosophic Theory and Applications,* **2016**, pp *257-275.*

[13] Douali,N.; Papageorgiou,E.I.; Roo, J.D.; Cools,H.; Jaulent,M.C. Clinical Decision Support System based on Fuzzy Cognitive Maps. *Journal of Computer Science & Systems Biology*, **2015**, 8(1), pp 112-120.

[14] Evangelia,B.; Chrysostomos,D.; George,M.; Voula,C,G. Time Dependent Fuzzy Cognitive Maps for Medical Diagnosis.*Artificial Intelligence: Methods and Applications*, **2014**, 8445, pp 544-554.

[15] Greeda,J.; Mageswari,A,; Nithya.R.A. Study on Fuzzy Logic and its Applications in Medicine. *International Journal of Pure and Applied Mathematics*, **2018**, 119(16), pp 1515-1525.

[16] Gaurav.; Megha Kumar.; Kanika,B.; Swati,A. Hybrid Model for Medical Diagnosis using Neutrosophic Cognitive Maps with Genetic Algorithms. *IEEE International Conference on Fuzzy Systems*(FUZZ-IEEE), **2015**, pp 1-7.

[17] Hamidi,M.; Smarandache.F. Neutro-BCK-Algebra. *International journal of Neutrosophic Science,* **2020,** 8(2), pp 110-117.

[18] Innocent,P.R.; John,R.I.; Garibald,G.M. Fuzzy Methods for Medical Diagnosis. *Applied Artificial Intelligence,* **2005**, 19, pp 69-98.

[19] Khodadadi,M., ; Shayanfar,H. ; Maghooli,K. ; Hooshang Mazinan,A. Prediction of Stroke Probability Occurrence Based on Fuzzy Cognitive Maps. AUTOMATIKA, **2019**, 4, pp 385-392.

[20] Kosko, B. Fuzzy Cognitive Maps. *International Journal of Man Machine studies*, **1986**, 24, pp 65–75.

[21] Merlyn Margaret,H.; Lilly Merline,W.A. Study on the Symptoms of Migraine using Induced Fuzzy Cognitive Maps. AIP Conference Proceedings, **2019**, 2112, pp 1-5.

[22] Masooma,R.H.; Muhammad,R.; Smarandache.F. m-Polar Neutrosophic Topology with Applications to Multi-criteria Decision-Making in Medical Diagnosis and Clustering Analysis. *International Journal of Fuzzy Systems*, **2020**, 22(1), pp 273-292.

[23] Muhammad,A.; Osama H. A.; Rehan,K. New Diagnosis Test under the Neutrosophic Statistics: An Application to Diabetic Patients. *BioMed Research International*, **2020**, 7, pp 1-7**.**

[24] Mumtaz ,A. ; Nguyen,V.M. ; Le, H.S. A Neutrosophic Recommender System for Medical Diagnosis Based on Algebraic Neutrosophic Measures. *Applied Soft Computing*, **2018**, 71,pp 1054-1071.

[25] Nabeeh,N.A.; Abdel-Basset,M.; Soliman,G. A model for evaluating credit rating and its impact on sustainability performance, *Journal of Cleaner Production*, **2021**, 280, pp 1-16.

[26] Nabeeh,N.A.; Smarandache,F.; Abdel-Basset,M.; El-Ghareeb,H.A.; Abelfetouh,A. An Integrated Approach and its Application to Personal Selection: A New Trend in Brain Processing and Analysis. *IEEE Access*. **2019,** 7, pp 29734-29744.

[27] Nabeeh,N.A.; Abdel-Basset,M.; El-Ghareeb,H.A.; Abelfetouh,A. Neutrosophic Multi-Criteria Decision Making Approach for IoT-Based Entreprises. *IEEE Access*, **2019**,7, pp 59559-59574.

[28] Neil, A.; Hilah,S. Fuzzy cognitive mapping: An Old Tool with New Uses in Nursing Research. Journal of Advanced Nursing, **2019**, 75(12), pp 1-8.

[29] Nivetha,M.;Aleeswari,A. ;Lilly Merline,W.Risk Factors of Lifestyle Diseases – Analysis by Decagonal Linguistic Neutrosophic Fuzzy Cognitive Map, *Materials Today: Proceedings*, **2020**, 24(3), pp 1939-1943.

[30] Palash,D.;Soumendra, G.Fuzzy Decision Making in Medical Diagnosis Using an Advanced Distance Measure on Intuitionistic Fuzzy Sets. *The Open Cybernetics & Systemics Journal,* **2018,** 12, pp 136-149.

[31] Papageorgiou,E.I.;Jayashree,S.; Akila,K.; Nikolaos,P.A. Risk Management Model for Familial Breast Cancer: A New Application using Fuzzy Cognitive Map method**.** *Computer methods and programs in biomedicine***, 2015,** 122(2), pp 1-13**.**

[32] Ruben,R.C.;Olivas,J.A. ;Romero,F.P.;Francisco,A.G.;Jesus,S.G.An Application of Fuzzy Prototypes to theDiagnosis and Treatment of Fuzzy Diseases. *International Journal of Intelligent Systems,* **2016**, pp 1-17.

[33] Shaista,H.; Muhammad,A. Medical Decision Support Systems Based on Fuzzy Cognitive Maps. *International Journal of Biomathematics*, **2019**, 12, pp 1-34.

[34] Shawkat,A.; Ayman,A.H. n-Valued Refined Neutrosophic Soft Sets and their Applications in Decision Making Problems and Medical Diagnosis. *Journal of Artificial Intelligence and Soft Computing Research***,** **2018**, 8(1), pp 79-86.

[35] Smarandache, F.; Vasantha kandasamy W.B, "Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps", Xiquan, Phoenix (2003).

[36] Sunny,T.; JatinderSingh,B. Medical Applications on Fuzzy Logic Inference System: A Review. International *Journal of Advanced Networking and Applications*, **2019**, 10(4), pp 3944-3950.

[37] Sundaresan,T.; Sheeja,G. ;Govindarajan,A. Different Treatment Stages in Medical Diagnosis using Fuzzy Membership Matrix. *IOP Conference. Series: Journal of Physics*, **2018**, 1000, pp 1-7.

[38] Tatiana,K.; Maka,K.; Karaman,P. Fuzzy Logic in Diagnostics of Rare Diseases. *Fuzziness and Medicine*, STUDFUZZ, **2013**, 302, pp. 379–399.

[39] Voula ,C.G.; Chrysostomos,D.S. Fuzzy Cognitive Maps for Medical Decision Support – A Paradigm from Obstetrics. *32nd Annual International Conference of the IEEE EMBS Buenos Aires*, Argentina, **2010**, pp 1174-1177.

[40] Zadeh ,L.A. Fuzzy Sets. *Information and Control*, **2011**, 8, pp139-146.

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Neutrosophic Theory and Its Application in Various Queueing Models: Case Studies

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Abstract: Queueing theory is an important technique to study and evaluate the performance of system. Queueing theory is applied in many applications such as logistics, finance, emergency services and project management, etc. In this research we apply neutrosophic philosophy in queueing theory. We deal with several queue models such as M/M/1 queue, M/M/s queue and M/M/1/b queue. We illustrate, solve, and find the performance measures of M/M/1, M/M/s, and M/M/1/b crisp queue models via examples with exact arrival rate and service rate. Queueing models affect by many factors such as arrival rate, service rate, number of servers, etc. These factors are not constantly expressed by accurate times; hence we express the parameters of queueing system by the neutrosophic. We express arriving rates and serving rates by neutrosophic values. We also illustrate, solve, and find the performance measures of NM/NM/1, NM/NM/s, and NM/NM/1/b neutrosophic queue models via examples. We concluded that the performance measures of neutrosophic queue models is more accurate than crisp queue models.

Keywords: Neutrosophic Set, Queueing Theory, Poisson Process, Exponential Distribution.

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1. Introduction

In 1909 Erlang developed queueing theory for modeling waiting lines and developing effectual systems that decrease waiting times of customers and makes it conceivable to serve more customers and growth profits of organizations.

In classical queueing theory the statement of well determined knowledge of queueing system's parameters such as arrival, service and departure rate are important and it is often imprecise in reality [1,2].

For dealing with problems of classical queueing theory, many researchers presented queueing theory in fuzzy environment to deal with uncertainty in parameters of queueing systems as in [3,4].

Since fuzzy and intuitionistic fuzzy theories does not represent reality efficiently and fails to simulate human thinking, Smarandache in 1995 presented neutrosophic logic. Neutrosophic logic is a generalization of fuzzy and intuitionistic fuzzy logic [5,6,7,8]. Neutrosophic logic able to deal with indeterminacy of data besides considering truth and falsity degrees. So, presenting queueing theory in neutrosophic environment makes decisions more competent [3,9,10,11,12,13,14,16].

In Neutrosophic set truth, indeterminacy, and falsity degrees are real values ranges from] $0^{-}, \ 1^{+}$[with no restriction on the sum. For simplifying application of neutrosophic set in real cases, a single valued neutrosophic set is presented [15].

Now we can say that neutrosophic queueing theory has imprecise values of parameters. For example, let λ which is the arrival rate in the form $\lambda_N = \lambda + I$ and μ which is the service rate in the form $\mu_N = \mu + I$, where I determines the indeterminant part of the given values.

In this research we show the important role of neutrosophic theory [9,10] to deal with vague parameters of some queueing models that is: (NM/NM/1) :(FCFS/∞/∞) queue, (NM/NM/s) :(FCFS/∞/∞) queue and (NM/NM/1) :(FCFS/∞/b) queue, and we prove the applicability and superiority of neutrosophic performance via solving various examples.

The remaining parts of this research consist of the following: In Section 2, we briefly discussed the queueing theory preliminaries. Section 3 discusses the fundamental steps of the neutrosophic queueing theory. In section 4, real case studies are solved for showing important role of neutrosophic in queueing theory. Section 5 presents the conclusion, findings and offers future work suggestions.

2. Queueing Theory Preliminaries

In this section the major preliminaries and concepts of single server waiting line model and multi-server waiting line model are presented.

The structure of waiting line systems presented in Figure 1.

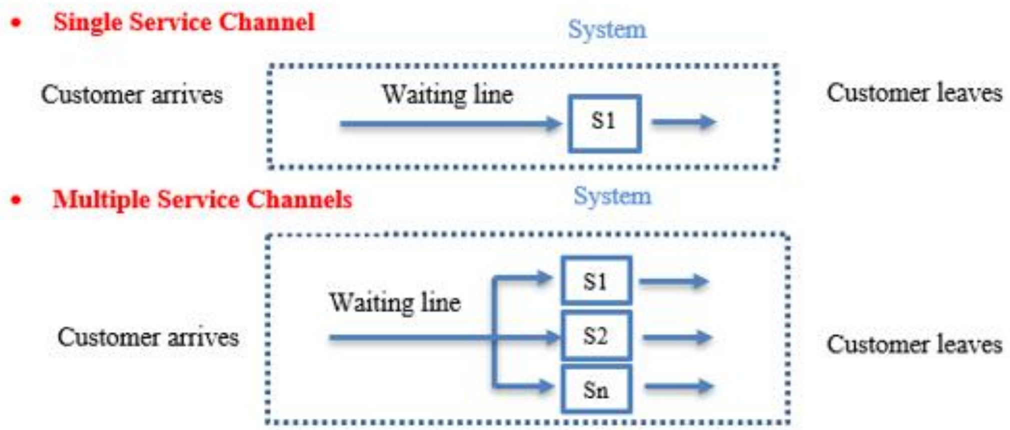

Fig.1. The structures **of** waiting line system

2.1 Single Server Waiting Line Model

2.1.1 (M/M/1) :(FCFS/∞/∞) [3,16]

The waiting line model considers the elementary in server, queue, and stage. There assumptions on this model are as follows:

1. The customers do not leave the queue and their population is infinite.

2. The arrival of customer are specified by a Poisson distribution with a mean arrival rate λ, so the time between the arrival of consecutive customers is specified by an exponential distribution with an average of $1/\lambda$.

3. The service rate of customer is specified by a Poisson distribution with a mean service rate of μ , so the service time of customer is defined by an exponential distribution with an average of $1/\mu$.

4. The customers are served according to first-come, first-served.

We can calculate the operating features of a waiting line system using the following formulas:

$\lambda =$ mean arrival rate of customers

$\mu =$ mean service rate

$\rho = \frac{\lambda}{\mu}$ = average utilization of system (1)

$L_S = \frac{\lambda}{\mu - \lambda} =$ average number of customers in the system (2)

$L_Q = \rho\, L$= average number of customers waiting in line (3)

$W_S = \frac{1}{\mu - \lambda}$ = average time customers spent in the system (4)

$W_Q = \rho \;\; W_S$= average time customers spent waiting in line (5)

$P_n = (1 - P)P^n =$ the probability that n customers are in the service system at a given time (6)

2.1.2 (M/M/1) :(FCFS/∞/b) [3, 16]

The interarrival times and serving times are specified in this model according to exponential distribution, there is one server for customers. The customers are served according to FCFS policy, the calling source is infinite and system size is finite by b including the one being served.

The performance measures of the system are as follows:

$\mathrm{P(k)} = \frac{\rho^{k}(1-\rho)}{(1-\rho^{b+1})}$ (7)

$L_Q = \frac{\rho^2[\,1-b\rho^{b-1}+(b-1)\rho^b\,]}{(1-\rho)(1-\rho^{b+1})}$ (8)

$L_s = L_Q + Eff\rho \;\; ; \;\; Eff\rho = \frac{Eff\,\lambda}{\mu} \quad , Eff\,\lambda = \lambda(1 - p(b))$ (9)

$W_Q = \frac{L_Q}{Eff\,\lambda}$ (10)

$W_S = \frac{L_S}{Eff\,\lambda}$ (11)

2.2 Multi-Server Waiting Line Model (M/M/s) :(FCFS/∞/∞) [3,16]

The assumptions on this model are described as follows:

1. The customers come from a single line.

2. The customers are served by the first server available.

3. There are s identical servers, the service time of each server is specified by exponential distribution and the mean service time is expressed by $1/\mu$. We can describe the operating features using the following formulas:

$s =$ the number of servers in the system

$\rho = \frac{\lambda}{s\mu}$ = the average utilization of system (12)

$$p_0 = \left[\sum_{n=0}^{s-1} \frac{\left(\lambda/\mu\right)^n}{n!} + \frac{\left(\lambda/\mu\right)^s}{s!}\left(\frac{1}{1-\rho}\right)\right]^{-1} \tag{13}$$

= the probability that no customers are in the system

$L_Q = \frac{p_0\left(\lambda/\mu\right)^s \rho}{s!\,(1-\rho)^2}$ = the average number of customers waiting in line (14)

$W_Q = \frac{L_Q}{\lambda} =$ the average time spent waiting in line (15)

$W_S = W_Q + \frac{1}{\mu} =$ the average time spent in the system, including service (16)

$L_S = \lambda$ W = the average number of customers in the service system (17)

$$P_n = \begin{cases} \frac{\left(\lambda/\mu\right)^n}{n!} p_0 & for\ n \le s \\ \frac{\left(\lambda/\mu\right)^n}{s!s^{n-s}} p_0 & for\ n > s \end{cases} \tag{18}$$

= the probability that n customers are in the system at a given time

3. Neutrosophic Queueing Theory Preliminaries

In this section the major preliminaries and concepts of neutrosophic queues are presented.

3.1 Neutrosophic Queue

Neutrosophic queue is a queueing system in which queueing parameters such as average rate of customers entering the queueing system (λ), and average rate of customers being served (μ) are neutrosophic numbers [3,16].

In neutrosophic queueing λ is denoted by $\lambda_N = [\lambda^L, \lambda^U]$and μ is denoted by $\mu_N = [\mu^L, \mu^U]$. Then, the neutrosophic traffic intensity if we have s servers is denoted by

$$\rho_N = \frac{\lambda_N}{s\mu_N} = \frac{[\lambda^L, \lambda^U]}{s[\mu^L, \mu^U]}. \quad (19)$$

3.2 Arithmetic Operations of Interval Values

Let $[c_1, d_1], [c_2, d_2]$ be two Intervals where $c_1, c_2, d_1, d_2 \in \mathbb{R}$ and for practical cases set $c_1 > 0, c_2 > 0, d_1 > 0, d_2 > 0$ then:

$$[c_1, d_1] + [c_2, d_2] = [c_1 + c_2, d_1 + d_2] \quad (20)$$

$$[c_1, d_1] - [c_2, d_2] = [c_1 - d_2, d_1 - c_2] \quad (21)$$

$$[c_1, d_1] * [c_2, d_2] = [c_1 c_2, d_1 d_2] \quad (22)$$

$$[c_1, d_1]/[c_2, d_2] = [c_1, d_1] * \frac{1}{[c_2, d_2]} = \left[\frac{c_1}{d_2}, \frac{d_1}{c_2}\right] \quad (23)$$

3.3 (NM/NM/1) :(FCFS/∞/∞) Queue [16]

After replacing crisp parameters by neutrosophic parameters, the neutrosophic probability that arriving customer will find k customers in queue are as follows:

$$NP(k) = (1 - \rho_N){\rho_N}^k; k = 0,1, \ldots$$

$$NP(k) = \left(1 - \left[\frac{\lambda^L}{\mu^U}, \frac{\lambda^u}{\mu^L}\right]\right)\left[\frac{\lambda^L}{\mu^u}, \frac{\lambda^U}{\mu^L}\right]^k; k = 0,1, \ldots$$

$$NP(k) = \left[1 - \frac{\lambda^U}{\mu^L}, 1 - \frac{\lambda^L}{\mu^U}\right]\left[\left(\frac{\lambda^L}{\mu^U}\right)^k, \left(\frac{\lambda^U}{\mu^L}\right)^k\right]; k = 0,1, \ldots$$

$$NP(k) = \left[\left(1 - \frac{\lambda^U}{\mu^L}\right)\left(\frac{\lambda^L}{\mu^U}\right)^k, \left(1 - \frac{\lambda^L}{\mu^U}\right)\left(\frac{\lambda^U}{\mu^L}\right)^k\right]; k = 0,1, \ldots \quad (24)$$

The performance measures of the system are as follows:

- Neutrosophic expected number of customers in system:

  $NL_s= \rho_N/(1-\rho_N)$, then

$$NL_s = \left[\frac{\frac{\lambda^L}{\mu^U}}{1-\frac{\lambda^L}{\mu^U}}, \frac{\frac{\lambda^u}{\mu^L}}{1-\frac{\lambda^U}{\mu^L}}\right] \quad (25)$$

- Neutrosophic expected number of customers in queue:

  $NL_Q= {\rho_N}^2/(1-\rho_N)$, then

$$NL_Q = \left[\frac{\left(\frac{\lambda^L}{\mu^U}\right)^2}{1-\frac{\lambda^L}{\mu^U}}, \frac{\left(\frac{\lambda^U}{\mu^L}\right)^2}{1-\frac{\lambda^U}{\mu^L}}\right] \quad (26)$$

- Neutrosophic expected waiting time in system:

  $\mathrm{N}W_s = \frac{1}{\mu_N - \lambda_N}$, then

$$NW_s = \left[\frac{1}{\mu^U-\lambda^L}, \frac{1}{\mu^L-\lambda^U}\right] \quad (27)$$

- Neutrosophic expected waiting time in queue:

  $NW_Q = \rho_N/(\mu_N - \lambda_N)$, then

$$NW_Q = \left[\frac{\frac{\lambda^L}{\mu^U}}{\mu^U-\lambda^L}, \frac{\frac{\lambda^U}{\mu^L}}{\mu^L-\lambda^U}\right] \quad (28)$$

3.4 (NM/NM/s) :(FCFS/∞/∞) Queue [16]

NM/NM/s queue is the same as NM/NM/1 queue except that in NM/NM/s customers are being served by s parallel homogeneous servers according to FCFS policy.

The neutrosophic probability that K customers in the queue will be:

$$NP(k)=\begin{cases}\frac{(s\rho_N)^k}{k!}NP(0); k<s\\ \frac{{\rho_N}^k s^s}{s!}NP(0); k\geq s\end{cases}$$

$$NP(k)=\begin{cases}\frac{\left(s\left[\frac{\lambda^L}{\mu^U},\frac{\lambda^U}{\mu^L}\right]\right)^k}{k!}NP(0); k<s\\ \frac{\left[\frac{\lambda^L}{\mu^l},\frac{\lambda^U}{\mu^L}\right]^k s^s}{s!}NP(0); k\geq s\end{cases} \tag{29}$$

Where: $NP(0)=\left(\sum_{n=0}^{s-1}\frac{(s\rho_N)^n}{n!}+\frac{(s\rho_N)^s}{s!}\cdot\frac{1}{1-\rho_N}\right)^{-1}$

$$NP(0)=\left(\sum_{n=0}^{s-1}\frac{\left(s\left[\frac{\lambda^L}{\mu},\frac{\lambda^u}{\mu^L}\right]\right)^n}{n!}+\frac{\left(s\left[\frac{\lambda^L}{\mu^u},\frac{\lambda^u}{\mu^L}\right]\right)^s}{s!}\cdot\frac{1}{1-\left[\frac{\lambda^L}{\mu^U},\frac{\lambda^U}{\mu^L}\right]}\right)^{-1} \text{ OR}$$

$$NP(0)=\left[\sum_{n=0}^{s-1}\frac{\left({}^{\lambda_N}/_{\mu_N}\right)^n}{n!}+\frac{\left({}^{\lambda_N}/_{\mu_N}\right)^s}{s!}\left(\frac{1}{1-\rho_N}\right)\right]^{-1} \tag{30}$$

The neutrosophic performance measures will be as follows:

- The average number of customers waiting in line:

$$NL_Q=\frac{NP(0)\left(\frac{\lambda_N}{\mu_N}\right)^S\rho_N}{S!(1-\rho_N)^2} \tag{31}$$

- The average waiting time of customer in line:

$$NW_Q=\frac{NL_Q}{\lambda_N} \tag{32}$$

- The average waiting time of customer in the system:

$$NW_S=NW_Q+\frac{1}{\mu_N} \tag{33}$$

- The average number of customers in the system:

$$NL_s=\ \lambda_N NW_S \tag{34}$$

3.5 (NM/NM/1) :(FCFS/∞/b) Queue [16]

In this model the interarrival times and serving times are specified according to neutrosophic exponential distribution, there is one server for customers. The customers are served according to FCFS policy, the calling source is infinite and system size is finite by b including the one being served. The neutrosophic probability that K customer in the queue will be:

$$NP(k) = \frac{\rho_N{}^k(1-\rho_N)}{(1-\rho_N{}^{b+1})}; k = 0..b$$

$$NP(k) = \frac{\left[\frac{\lambda^L}{\mu^U}, \frac{\lambda^U}{\mu^L}\right]^k \left(1-\left[\frac{\lambda^L}{\mu^L}, \frac{\lambda^U}{\mu^L}\right]\right)}{\left(1-\left[\frac{\lambda^L}{\mu^U}, \frac{\lambda^U}{\mu^L}\right]^{b+1}\right)}; k = 0..b \quad (35)$$

The performance measures will be then as follows:

- The average number of customers waiting in line:

$$NL_Q = \frac{\rho_N{}^2[1-b\rho_N{}^{b-1}+(b-1)\rho_N{}^b]}{(1-\rho_N)(1-\rho_N{}^{b+1})}$$, then

$$NL_Q = \frac{\left[\frac{\lambda^L}{\mu^U}, \frac{\lambda^U}{\mu^L}\right]^2 \left[1-b\left[\frac{\lambda^L}{\mu^U}, \frac{\lambda^U}{\mu^L}\right]^{b-1} + (b-1)\left[\frac{\lambda^L}{\mu^U}, \frac{\lambda^v}{\mu^L}\right]^b\right]}{\left[1-\frac{\lambda^U}{\mu^L}, 1-\frac{\lambda^L}{\mu^U}\right]\left(1-\left[\frac{\lambda^L}{\mu^U}, \frac{\lambda^U}{\mu^L}\right]^{b+1}\right)} \quad (36)$$

- The average number of customers in the system:

$$NL_S = NL_Q + Eff\rho_N, \quad (37)$$

$$Eff\ \ \rho_N = \frac{Eff\lambda_N}{\mu_N}, Eff\lambda_N = \lambda_N(1-NP(b)) \quad (38)$$

$$Eff\ \ \lambda_N = [\lambda^L, \lambda^U](1-NP(b)) \quad (39)$$

- The average waiting time of customer in line:

$$NW_Q = \frac{1}{Eff\lambda_N} NL_Q \quad (40)$$

- The average waiting time of customer in the system:

$$NW_S = \frac{1}{Eff\lambda_N} NL_S \quad (41)$$

The methodology for solving neutrosophic queueing models presented in Figure 2.

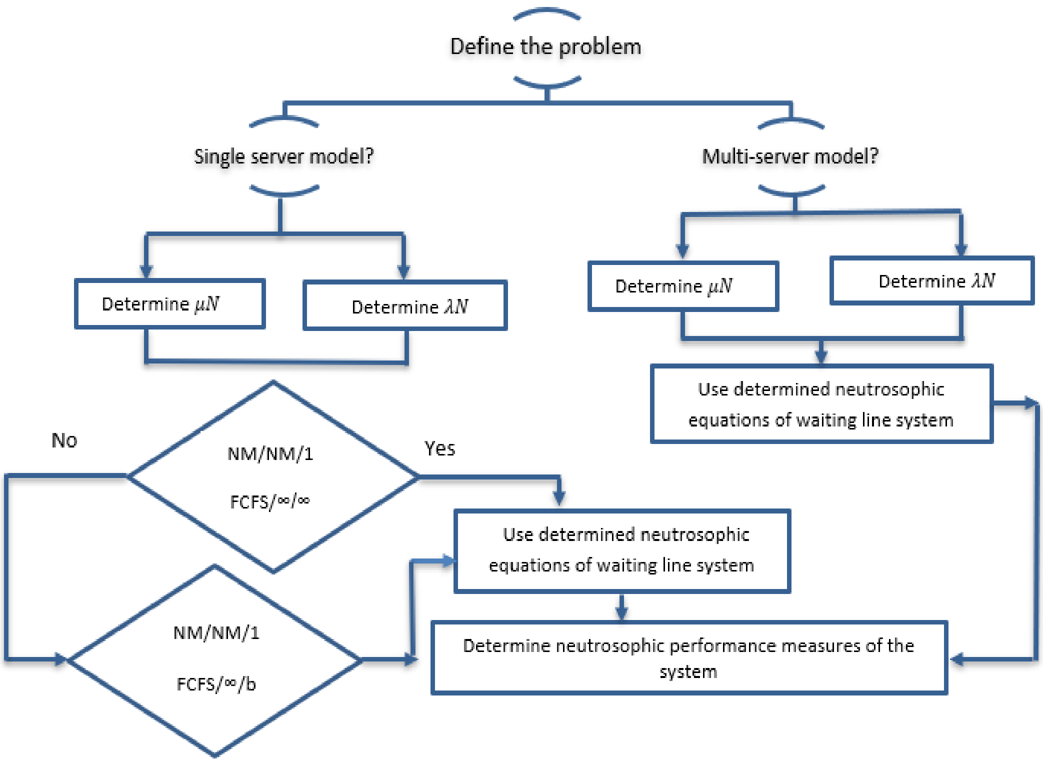

Fig.2. The methodology for solving neutrosophic queueing models

4. Case Studies

In this section various case studies on crisp and neutrosophic queues are presented and solved.

4.1 Example on ((M/M/1) :(FCFS/∞/∞) Crisp Queue Model

The computer lab at State University helps the students by help desk. The students stand in front of the desk to wait for help. Students are served according to priority rule first-come, first-served. Students arrive according to Poisson process with a mean arrival rate 15 students per hour. Students are served by service rate exponentially distributed with an average 20 students per hour. Find the performance measures of the system.

(a) The average utilization of the system

(b) The average number of students in the system

(c) The average number of students waiting in queue

(d) The average waiting time in the system

(e) The average waiting time in queue

Crisp solution

(a) By using Eq. (1), the average utilization is as follows: $\rho = \frac{15}{20}$ = 0.75, or 75%.

(b) By using Eq.(2), the average number of students in the system is as follows: L_s = $\frac{15}{20-15} = 3$ students

(c) By using Eq.(3), the average number of students waiting in queue: $L_Q = 0.75 \times 3$ =2.25 students

(d) By using Eq.(4), the average waiting time in the system: W_s = $\frac{1}{20-15}$= 0.2 hours, or 12 minutes

(e) By using Eq.(5), the average waiting time in queue: W_Q = 0.75 × 0.2 = 0.15 hours, or 9 minutes

4.2 Example on (NM/NM/1) :(FCFS/∞/∞) Neutrosophic Queue Model

The computer lab at State University helps the students by help desk. The students stand in front of the desk to wait for help. Students are served according to priority rule first-come, first-served. Students arrive according to Poisson process with a mean arrival rate between 14 and 16 students per hour. Students are served by service rate exponentially distributed with an average 19 and 21 students per hour. Find the performance measures of the system.

(a) The average utilization of the system

(b) The average number of students in the system

(c) The average number of students waiting in queue

(d) The average waiting time in the system

(e) The average waiting time in queue

Neutrosophic solution

$\lambda_N = [\,14{,}16]$ students per hour.
$\mu_N = [19{,}21]$ students per hour.

a) Average utilization: $\rho_N = \frac{\lambda_N}{\mu_N} = \frac{[\,14,16]}{[19,21]}$ = [0.66, 0.84]. We can say the efficiency of the system ranges between 0.66 and 0.84 and 0.75 (crisp value) ∈ [0.66,0.84].

b) By using Eq.(25), the average number of students in the system: $NL_s = \frac{[0.66\,,0.84]}{(1-[0.66\,,0.84])} = \frac{[0.66\,,0.84]}{[0.16\,,0.34]} = [1.94\,,5.25]$. Which means that expected number of students in system ranges between 1.94 and 5.25 and 3 (crisp value) ∈ [1.94 , 5.25].

c) By using Eq. (26), the average number of students in queue: $NL_Q = \frac{[0.66,0.84]^2}{(1-[0.66,0.84])} = \frac{[0.4356,0.7056]}{[0.16,0.34]} =$ [1.28, 4.41]. Which means that expected number of students in queue ranges between 1.28 and 4.41 and 2.25 (crisp value) $\in$ [1.28 , 4.41].

d) By using Eq. (27), the average waiting time in the system: $NW_s = \frac{1}{[3,7]}$ = [0.14,0.33].

Which means that mean waiting time in system ranges between 8.4 mins and 19.8 mins and 12 mins (crisp value) $\in$ [8.4 , 19.8].

e) By using Eq. (28), the average waiting time a student in queue: $NW_Q = \frac{[0.66,0.84]}{[3,7]} =$ [0.09, 0.28]. Which means that mean waiting time in queue ranges between 5.4 mins and 16.8 mins and 9 mins (crisp value) $\in$ [5.4 , 16.8].

4.3 Example on ((M/M/s) :(FCFS/∞/∞) Crisp Queue Model

State University has intended to maximize the number of assignments. Instead of a single person working at the help desk, the university planned to have three servers. The students will arrive at a rate of 45 per hour, according to a poison distribution. The service rate for each of the three servers is 18 students per hour with exponential service times. Find the following performance measures of the system.

(a) The average utilization of the help desk

(b) The average number of students in the queue

(c) The average waiting time in the queue

(d) The average waiting time in the system

(e) The average number of students in the system

Crisp solution

(a) Average utilization: $\rho = \frac{\lambda}{s\mu} = \frac{45}{3\times 18} = $ = 0.833, or 83.3%

(b) The average number of students in the queue:
Firstly, we find the probability that there are no students in the system using Eq. (13) as follows:

$$p_0 = \left[\frac{(45/18)^0}{0!} + \frac{(45/18)^1}{1!} + \frac{(45/18)^2}{2!} + \left(\frac{(45/18)^3}{3!}\left(\frac{1}{1-0.833}\right)\right)\right]^{-1}$$

$= \frac{1}{22.215}$ = 0.045, or 4.5% of having no students in the system

By using Eq. (14), the average number of students in the queue is as follows:

$$L_Q = \frac{0.045(45/18)^3 \times 0.833}{3! \times (1-0.833)^2} = \frac{0.5857}{0.1673} = 3.5 \text{ students}$$

(c) By using Eq. (15), the average waiting time in the queue: $W_q = \frac{3.5}{45} =$ 0.078 hour, or 4.68 minutes

(d) By using Eq. (16), the average waiting time in the system: W_s =0.078 + $\frac{1}{18}$ = 0.134 hour, or 8.04 minutes

(e) By using Eq. (17), the average number of students in the system: $L_s = 45(0.134) = 6.03$ students

4.4 Example on (NM/NM/s) :(FCFS/∞/∞) Neutrosophic Queue Model

State University has intended to maximize the number of assignments. Instead of a single person working at the help desk, the university planned to have three servers. The students will arrive at a rate of [44, 46] students per hour, according to Poisson distribution. The service rate for each of the three servers is [17,19] students per hour with exponential service times. Find the following performance measures of the system.

(a) The average utilization of the help desk

(b) The average number of students in the queue

(c) The average waiting time in the queue

(d) The average waiting time in the system

(e) The average number of students in the system

Neutrosophic solution

$\lambda_N = [\,44{,}46]$ students per hour.

$\mu_N = [17{,}19]$ students per hour.

a) Average utilization: $\rho_N = \frac{\lambda_N}{s\mu_N} = \frac{[\,44,46]}{3[17,19]} = \frac{[\,44,46]}{[51,57]}$ = [0.77,0.90].We can say that the efficiency of the system ranges between .0.77 and 0.9 and 0.83 (crisp value) ∈ [0.77 , 0.90].

b) The average number of students in the queue:

Firstly, we find the probability that there are no students in the system using Eq. (30) as follows:

$$NP(0) = \left[\frac{([2.3,2.7])^0}{0!} + \frac{([2.3,2.7])^1}{1!} + \frac{([2.3,2.7])^2}{2!} + \left(\frac{([2.3,2.7])^3}{3!}\left(\frac{1}{1-[0.77,0.9]}\right)\right)\right]^{-1}$$

$=\left[[5.9,7.3] + \left(\frac{[12.16,19.8]}{6}\left(\frac{1}{[0.1,0.23]}\right)\right)\right]^{-1}$

$=[[5.9\ ,7.3] + ([2.02\ ,3.3][4.3\ ,10])]^{-1}$

= [0.024, 0.068] and we can say that the probability that we will find no student in the system ranges between .0.0.024 and 0.068 and 0.045 (crisp value) ∈ [0.024, 0.68].

By using Eq. (31), the average number of students waiting in line is as follows:

$NL_Q = \frac{[0.024,0.068]\ ([2.3,2.7])^3[0.77,0.9]}{3!\ ([0.1,0.23])^2}$ = [0.69, 20].

Which means that expected number of students in queue ranges between 0.69 and 20 and 3.5 (crisp value) ∈ [0.069, 20].

(c) By using Eq. (32), the average waiting time in the queue is as follows:

$NW_Q = \frac{[0.69,\ 20]}{[44\ ,46]} = [0.015\ ,0.45]$ hour $= [0.9\ ,27]$ minutes which means that mean waiting time in queue ranges between 0.9 mins and 27 mins and 4.68 (crisp value) ∈ $[0.9\ ,27]$minutes.

(d) By using Eq. (33), the average waiting time in the system is as follows:

NW_S =[0.015,0.45] + $\frac{1}{[17,19]}$ = [0.06,0.5] hour = [3.6,30] minutes which means that mean waiting time in system ranges between 3.6 mins and 30 mins and 8.04 (crisp value) ∈ [3.6,30] minutes.

(e) By using Eq. (34), the average number of students in the system is as follows:

NL_s= [44,46] [0.06,0.5] = [2.64,23] students, which means that expected number of students in system ranges between 2.64 and 23 and 6.03 (crisp value) ∈ [2.64,23].

4.5 Example on (M/M/1) :(FCFS/∞/b) Crisp Queue Model

The packets of wireless access gateway arrive at a mean rate of 125 packets per second, they are buffered until they can be transmitted. The gateway takes 500 seconds to transmit a packet. The gateway currently has 13 places (including the packet being transmitted) and packets that arrive when the buffer is full are lost. Find the probability that a new packet is going to be lost, then find the performance measures of the system.

Crisp solution

By using Eq. (1), $\rho = \frac{125}{500} = 0.25$

Then, by using Eq. (7) the probability that a new packet is going to be lost is as follows:

$$\mathrm{P(k)} = \frac{(0.25)^{13}(0.75)}{1-(0.25)^{14}} = 1.12 \times 10^{-8}$$

The performance measures of the system are as follows:

(a) By using Eq. (8), the average number of packets waiting in the queue is as follows:

$$L_Q = \frac{(0.25)^2[\,1-13(0.25)^{12} + 12\,(0.25)^{13}]}{0.75\,[1-(0.25)^{14}]} = 0.0834$$

(b) By using Eq. (9), the average number of packets in the system is as follows:

$$Eff\,\lambda = 125(1-1.12\times 10^{-8}) = 124.9$$

$$Eff\rho = \frac{124.9}{500} = 0.249$$

Hence $L_s = 0.0834 + 0.249 = 0.3324$

(a) By using Eq. (10), the average waiting time in the queue is as follows:

$W_Q = \frac{0.0834}{124.9}$ =0.00066 seconds

(b) By using Eq. (11), the average waiting time in the system is as follows:

$W_S = \frac{0.3324}{124.9} = 0.00266$ seconds

4.6 Example on (NM/NM/1) :(FCFS/∞/b) Neutrosophic Queue Model

The packets of wireless access gateway arrive at a mean rate of $[124,126]$ packets per second, and they are buffered until they can be transmitted. The gateway takes $[499,501]$ seconds to transmit a packet. The gateway currently has 13 places (including the packet being transmitted) and packets that arrive when the buffer is full are lost. Calculate the probability that a new packet is going to be lost. Find the performance measures of the system.

Neutrosophic solution

$\lambda_N = [\ 124{,}126]$ packets per second.

$\mu_N = [499{,}501]$ packets per second.

$\rho_N = \frac{\lambda_N}{\mu_N} = \frac{[\ 124{,}126]}{[499{,}501]} = [0.247{,}0.252]$ and 0.25 (crisp value) $\in [0.247\ , 0.252]$.

By using Eq. (35), the probability that a new packet is going to be lost is as follows:

$\mathrm{NP(k)} = \frac{[0.247\ ,0.252]^{13}(1-[0.247\ ,0.252])}{(1-[0.247\ ,0.252]^{14})} = [95 \times 10^{-10}, 124 \times 10^{-10}]$

and 1.12×10^{-8} (crisp value) $\in [95 \times 10^{-10}, 124 \times 10^{-10}]$

The performance measures of the system are as follows:

(a) By using Eq. (36), the average number of packets waiting in line is as follows:

$$NL_Q = \frac{[0.247\ ,0.252]^2[\ 1-13[0.247\ ,0.252]^{12}+ 12\ [0.247\ ,0.252]^{13}]}{(1-[0.247\ ,0.252])(1-[0.247\ ,0.252]^{14})} = \frac{[0.060{,}0.0629]}{[0.747{,}0.752]} = [0.079{,}0.084]$$

Means that average number of waiting packets will be between 0.079 and 0.084 and 0.0834 (crisp value) $\in [0.079{,}0.084]$.

(b) By using Eq. (37), Eq. (38), and Eq. (39), the average number of packets in the system is as follows:

$$Eff\ \lambda_N = [124{,}126][1 - (95 \times 10^{-10}, 124 \times 10^{-10})] =$$

$$[124{,}126][0.99999998{,}0.99999999] = [123.9{,}125.9]$$

$$Eff\rho = \frac{[123.9{,}125.9]}{[499{,}501]} = [0.2473{,}0.2523]$$

Hence, $NL_s = [0.079{,}0.084] + [0.2473{,}0.2523] = [0.3263{,}0.3363]$

Means that average number of packets in the system will be between 0.3263 and 0.3363 and 0.3324 (crisp value) $\in$ [0.3263,0.3363].

(c) By using Eq. (40), the average waiting time in the queue is as follows:

$NW_Q = \frac{[0.079{,}0.084]}{[123.9{,}125.9]}$ = [0.00062,0.00067] seconds and 0.00066 (crisp value) $\in$[0.00062,0.00067].

(d) By using Eq. (41), the average waiting time in the system is as follows:

$$NW_S = \frac{[0.3263,0.3363]}{[123.9,125.9]} = [0.0025,0.0027] \text{ seconds and } 0.0026 \text{ (crisp value) } \in [0.0025,0.0027].$$

5. Conclusions and Future Directions

We concluded that the neutrosophic queueing theory is better than the crisp queueing theory when we deal with imprecise data. We have presented three types of queues in neutrosophic environment: (NM/NM/1) :(FCFS/∞/∞) queue, (NM/NM/s) :(FCFS/∞/∞) queue and (NM/NM/1) :(FCFS/∞/b) queue. We evaluate the neutrosophic performance measures for three queueing models according to crisp and neutrosophic queueing models. Neutrosophic queueing models gives better results than crisp queueing models.

In the future we can study other types of queueing systems in neutrosophic environment. We can also use triangular and trapezoidal neutrosophic numbers in various queueing theory models. Also, various types of neutrosophic sets such as single, interval and bipolar neutrosophic sets will apply in our future research in queueing theory.

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References

1. F Shortle J.; M Thompson J.; Gross D.; M Harris C., Fundamentals of Queueing Theory, 5th ed.; Wiley: United States of America, 2018; pp. 1–27, 77–96.
2. Nick T. Thomopoulos; Fundamentals of Queuing Systems, 1st ed.; Springer: United States of America, 2012; pp. 19–26, 41–48.
3. Mohamed Bisher Zeina, Khudr Al-Kridi and Mohammed Taher Anan, New Approach to FM/FM/1 Queue's Performance Measures, Journal of King Abdulaziz University: "Science", Vol.30, No. 1., 2017.
4. Chen, S.P., Eur. J., Parametric Nonlinear Programming for Analyzing Fuzzy Queues with Infinite Capacity, Oper. Res., Vol. 157, 2009.
5. Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. American Research Press, Rehoboth, NM, 1999.
6. Smarandache, F, Neutrosophic set a generalization of the intuitionistic fuzzy sets. Inter. J. Pure Appl. Math., 24, 287 – 297, 2005.
7. Smarandache. F, Introduction to Neutrosophic Measure, Integral, Probability. Sitech Education publisher, 2015.
8. Smarandache, F, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico, Gallup, NM 87301, USA,2002.
9. Rafif Alhabib, Moustafa Mzher Ranna, Haitham Farah, A.A. Salama, Some Neutrosophic Probability Distributions. Neutrosophic Sets and Systems, Vol. 22, 2018.
10. Salama.A.A, Smarandache. F, Neutrosophic Crisp Set Theory. Education Publishing, Columbus, 2015.
11. Patro.S.K, Smarandache. F. the Neutrosophic Statistical Distribution, More Problems, More Solutions. Neutrosophic Sets and Systems, Vol. 12, 2016.
12. Smarandache. F, Neutrosophical statistics, Sitech & Education publishing, 2014.
13. Smarandache, F. & Pramanik, S. (Eds). (2018). New trends in neutrosophic theory and applications, Vol.2. Brussels: Pons Editions.
14. Smarandache, F. & Pramanik, S. (Eds). (2016). New trends in neutrosophic theory and applications. Brussels: Pons Editions.

15. Abdel-Basset, M., Mohamed, M., & Smarandache, F. (2018). An extension of neutrosophic AHP–SWOT analysis for strategic planning and decision-making. Symmetry, 10(4), 116.
16. Zeina, M. B. (2020). Neutrosophic M/M/1, M/M/c, M/M/1/b Queueing Systems. Infinite Study.

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Sustainable Energy Production's Spatial Determination Framework, Based on Multi Criteria Decision Making and Geographic Information System Under Neutrosophic Environment: A Case Study in Egypt

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Abstract: The development of wind energy projects (WEP) have been encouraged, since the last decade. Therefore, WEP grows exponentially, which makes wind energy the trend of energy production for many countries. The success of wind energy project relies on the choice of the appropriate site for wind power plant, often decided by the application of Multi Criteria Decision Making (MCDM). The MCDM methodologies for location selection have a range of shortcomings: (1) the incomplete use of knowledge, (2) the lack of evidence in the decision-making process; and (3) the problem of ignoring the interaction between parameters. This paper presents a new framework for the location selection of wind power stations, based on the incorporating of geographic information system (GIS) and analytical network process (ANP) through neutrosophic environment to cover MCDM's shortcoming. First, an assessment model is built for wind farm site selection. Then, in the specialist committee decision, the bipolar neutrosophic set is used to express missing knowledge. In addition, we take the relationship problem into account by collecting the opinions of experts. Finally, the GIS is used to determine the wind farm potential zones. The suggested framework for the identification of wind farm sites is validated by the use of a case study from Egypt.

Keywords: WEP, neutrosophic, MCDM, GIS

1. Introduction

Electricity consumption is directly growing with time in accordance: urban, technical development, civilians, and agricultural expansion. Energy production is depended mainly on fossil fuels, which: is decreased by time (unsustainable), as well as the high-cost extraction, directly reflected in consumers, and environment pollution effects.

The electricity power importance and its resources, led to the increased interest in alternative and renewable energy resources. Wind is one of the sustainable power resources. Wind power be provided as ample oil fuel, contributes the preservation of the environment, as well as facilitate development in remote area. Wind Energy Location (WEL) is one of the most important factors of wind power production projects, WPL is cornerstone of wind power efficiency and generation cost, as well as to the environmental impacts. Therefore, WEL determination is a vital issue that must be analyzed in depth in order to be effective technically, economically, environmentally, and society. WEL is affected by many factors, these factors must be carefully and systematically identified for making a decision of the holistic approach. Because of the difficulty of trade-off among the alternative available factors and criteria has been the focus of using decision support tools. this paper adopted Multi-Criteria Decision Making (MCDM) approach for WEL determination.

MCDM is one of the operational research sub-disciplines that specifically assesses different competing criteria in decision making1[1]. Although the decision-making preferences must be used to classify the solutions, there is no uniquely appropriate solutions to such problems. Better informed decision-making is assisted by proper structuring and consistent consideration of various parameters for complex problems. MCDM methods demonstrated success in the assessment process in several problem-solving domains.

While the MCDM methods offer an efficient basis for the selection of the ideal location for renewable energy plant with contradictory and multiple criteria, the decision to choose a WEL still has several restrictions. One of challenges is the general uncertainty of determining the selection as the decision takes place before the wind farm is set up, so due to the complexities and location-specific variables, it is often difficult to exactly predict or evaluate correct assessment details. In addition, the reported opinions of experts appear to be uncertain to a large extent, and the level of satisfaction cannot be calculated in an accurate way. Therefore, in an incomplete and imperfect knowledge atmosphere, the site selection decision is made.

The analytic network process (ANP) is one of the best ways to solve dependency and feedback issues between criteria and sub-criteria in decision-making problems under the assumption that they are independent or show self-relation. As there are several complicated interdependencies among the criteria used, there's many ambiguous (non-deterministic) sub-criteria and their connections, the bipolar neutrosophic set-Analytic Network process (BNS -ANP) appears to be an effective tool for determining the best wind farm locations.

There are many factors involved in the wind farm site-selection process, such as social-economic, spatial, ecological and environmental considerations. The Multi-Criteria Decision-Making Approach (MCDM) is efficient in solving dynamic and contradictory multi-layer problems (e.g., benefits, drawbacks, costs, rewards) and is ideal for providing graded decision alternatives to site selection [2]. On the other hand, the Geographic Information System (GIS) instrument, as a powerful method for gathering." preserving, handling, measuring, evaluating, manipulating and mapping geographic information, could play a critical role in the possible evaluation and site selection of wind resources on the basis of its capacity to provide indicator databases and visualized map [3-5].

The integration of MCDM and GIS has also been broadly applicable to site selection analysis. Example studies cover onshore wind farm site selection [3, 6-8]. And Various MCDM techniques are possible to account for the complexity of decision-making under uncertain circumstances and imprecise, especially in the wind farm site selection field. For example, the integration of GIS and the weighted linear combination (WLC) technique was investigated by Gorsevski et al [9] to produce the suitability index of each site under the map layer for Northwest Ohio onshore wind farms.

Sánchez-Lozano et al. [10] First removed unsuitable areas on the basis of relevant legal limitations and consideration of such criteria, and then identified ideal locations for power generation facilities in the Spanish region of Murcia using the ELECTRE-TRI system based on GIS. S. Ali et al. [11] suggested a combined approach to GIS and MCDM to identify the best location for the placement of wind farms. G. Villacreses et al. [2] introduced a GIS with MCDM techniques to determine the optimal site for the construction of wind farms in Ecuador, selecting as the most appropriate location in the Andean zone of Ecuador. Diez-Rodrı́guez et al. [12] developed a methodology for future use in strategic environmental assessment through the application of a technical Group-Spatial Decision Support system (GSDSS) that incorporates information and methods of collective intelligence, complexity theory and geo-prospective.

In order to deal with onshore wind farm site selection, The Analytic Hierarchy Process (AHP) and GIS were combined by S. Ali et al. [3] to classify the ideal sites in Songkhla Province, Thailand for utility-scale onshore wind farms. Gigović et al. [13] developed a model based on the combination of GIS, Decision Making Trial and Assessment Laboratory (DEMATEL), ANP, and Multi-Attributive Boundary Approximation Area Comparison (MABAC), to decide the sites for the construction of wind farms in the province of Vojvodina, Serbia. a fuzzy TOPSIS and Complex Proportional Assessment (COPRAS) model was proposed to select appropriate wind farm locations by Dhiman and Deb [14].

To deal with uncertainty and imprecision Zadeh first proposed the concept of fuzzy sets (FSs) and intuitionistic fuzzy sets (IFs) [15] and [16] respectively. In view of the fact that uncertainties are correlated with the weight determination of the proposed evaluation indicators and their scores relevant to all candidate locations, the fixed values are not adequate to characterize the characteristics of the indicators. As a result, uncertain MCDM approaches have appeared in the field of site selection for wind farms. For example, Ayodele et al. [17] suggested a type-2 fuzzy AHP GIS-based model to decide the appropriate wind farms in Nigeria, where fuzzy sets were used to describe the inconsistency, vagueness and uncertainties of the decision-making process. Y. Wu et al. [18] Firstly, used intuitionist fuzzy numbers and fuzzy measures to represent the intuitive preferences of the experts and to rate the degrees of importance between criteria. Finally, the acceptability of alternate locations for the wind farm project in China was assessed. In addition, in the context of Southeastern Spain [6], the Southeastern Corridor of Pakistan [19] and Vietnam [20], fuzzy AHP and fuzzy TOPSIS have also been shown to be successful in sustainable site selection for onshore wind farms.

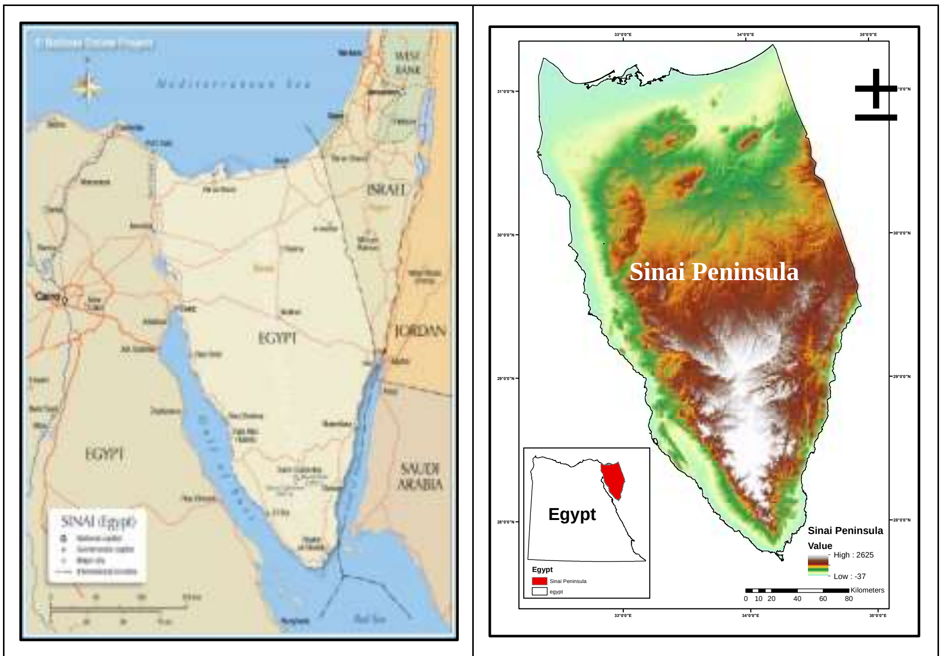
Fuzzy focuses only on the membership function (degree of truth) and does not take into account the degree of non-membership (degree of falsehood) and indeterminacy, so fail to represent indeterminacy and uncertainty. Smarandache [21] subsequently developed the neutrosophic set concept, which can deal with indeterminacy. Compared to the fuzzy set and the intuitionistic fuzzy sets, which are unable to deal with indeterminacy effectively. Neutrosophic set (NS) is the generalization of (FSs) and (IFs). numerous types of MCDM approaches are incorporate by neutrosophic set. Neutrosophic sets have many benefits when compared with (FS) and (IFs). Consequently, it is extensively studied by many researchers [22-26].

This paper presents an assessment model for wind farm location selections based on bipolar neutrosophic set (BNS) that can handle vagueness, indeterminacy and improve reliability. BNS is applied with ANP method and GIS to add to the field of wind power station literature. After that, an empirical case study has been considered to illustrate the applicability of this proposed approach.

The remainder of this paper is planned as follows: Section 2 describes the study area. Section 3 describes the bipolar neutrosophic numbers background theory. Section 4 describes Materials and methods. Section 5 presents results and discussion, followed by Section 6 which contains concluding remarks.

2. Study Area

Sinai is a 61,000 km2 triangular peninsula in northeastern Egypt that connects the vast continental land masses of Africa and Asia between latitudes 27° 43' and 31° 19' North and longitudes 32° 19' and 34° 54' East. The peninsula is located between the gulfs of Aqaba and Suez and is bounded to the north by the Mediterranean Sea as shown in Fig. (la). It is split into two administrative regions, with north Sinai covering approximately 27,564 km^2 and south Sinai covering approximately 31,272. Km^2. The Peninsula also covers portions of three governorates; namely Ismailia, Suez, and Port Saied Governorates. Desert plains, sand dunes and sea shores, plateaus and mountainous areas are included in the geographical geography Digital Elevation Model (DEM) of the Sinai Peninsula is shown in Fig. (1b). With a shoreline reaching 205 km, the Mediterranean Sea borders the Peninsula from the north.



(a) (b)

Figure 1. (a) Administrative Boundary, (b) Digital Elevation Model of the Sinai Peninsula.

3. Bipolar Neutrosophic Set (BNS)

Bipolarity is described as the human mind's propensity to reason and make decisions based on positive and negative consequences. Positive statements express what is probable, satisfactory, permissible, expected, or considered suitable. Negative statements, on the other hand, convey what is impossible, forbidden, or rejected [27]. In this section, some important definitions of bipolar neutrosophic numbers (BNNs) are introduced [28].

Definition 3.1 A BNS A in X is defined as an object of the form $A = \{\langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)\rangle : x \in X\}$ where $T^+, I^+, F^+ : X \to [1,0]$ and $T^-, I^-, F^- : X \to [-1,0]$. The positive membership degree $T^+(x), I^+(x), F^+(x)$ represent the truth membership, the indeterminacy membership, and the falsity membership of $x \in A$, respectively. And the negative membership degree $T^-(x), I^-(x), F^-(x)$ represent the truth membership, the indeterminacy membership, and the falsity membership of $x \in A$.

Definition 3.2 Suppose that $\tilde{a}_1 = \langle T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^-\rangle$ and $\tilde{a}_2 = \langle T_2^+, I_2^+, F_2^+, T_2^-, I_2^-, F_2^-\rangle$ be tow Bipolar Neutrosophic Numbers. Then, there are the following operational rules:

$$\lambda\tilde{a}_1 = \langle 1-(1-T_1^+)^\lambda, (I_1^+)^\lambda, (F_1^+)^\lambda, -(-T_1^-)^\lambda, -(-I_1^-)^\lambda, -\left(1-\left(1-(-F_1^-)\right)^\lambda\right)\rangle \quad (1)$$

$$\tilde{a}_1{}^\lambda = \langle (T_1^+)^\lambda, 1-(1-I_1^+)^\lambda, 1-(1-F_1^+)^\lambda, -\left(1-\left(1-(-T_1^-)\right)^\lambda\right), -(-I_1^-)^\lambda, -(-F_1^-)^\lambda\rangle \quad (2)$$

$$\tilde{a}_1 + \tilde{a}_2 = \langle T_1^+ + T_2^+ - T_1^+T_2^+, I_1^+I_2^+, F_1^+F_2^+, -T_1^-T_2^-, -(-I_1^- - I_2^- - I_1^-I_2^-), -(-F_1^- - F_2^- - F_1^-F_2^-)\rangle \quad (3)$$

$$\tilde{a}_1.\tilde{a}_2 = \langle T_1^+ + T_2^+, I_1^+ + I_2^+ - I_1^+I_2^+, F_1^+ + F_2^+ - F_1^+F_2^+, -(-T_1^- - T_2^- - T_1^-T_2^-), -I_1^-I_2^-, -F_1^-F_2^-\rangle \; Where\ \lambda > 0 \quad (4)$$

Definition 3.3 Suppose that $\tilde{a}_1 = \langle T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^-\rangle$ be a Bipolar Neutrosophic Number. Then, the score function $S(\tilde{a}_1)$, accuracy function $A(\tilde{a}_1)$ and certainty function $C(\tilde{a}_1)$ of a Bipolar Neutrosophic Number can be defined as follows:

$$S(\tilde{a}_1) = (T_1^+ + 1 - I_1^+ + 1 - F_1^+ + 1 + T_1^- - I_1^- - F_1^-)/6 \quad (5)$$

$$A(\tilde{a}_1) = T_1^+ - F_1^+ + T_1^- - T_1^- \quad (6)$$

$$C(\tilde{a}_1) = T_1^+ - F_1^- \quad (7)$$

4. Materials and Methods

This section describes the proposed framework and the used data sets with its resources. The framework is an integration among BNS, ANP, and GIS (BAG).

4.1 Data Set

Table (1) summarizes the researcher's data set that were collected from numerous resources including governmental agencies, open sources, and related literature. GIS and remote sensing technology have been used in combination to process, Integrate, and analyze spatial data. The software used for this study are ArcGIS 10.3 and Global Mapper v17.1 to make them usable in the wind farm site selection model. The weights of the criteria were generated using the Bipolar neutrosophic set (BNS) and Analytic Network Process (ANP), the mathematical model implemented in Microsoft Excel.

Table 1. Data Sources Used in the Study.

Format	Data Set	Source
Raster	Digital Elevation Model.	United States Geological Survey Earth Explorer.
	Wind Speeds and Directions.	National Authority for Remote Sensing and Space Sciences, Egyptian Metrological Authority, The Global Wind Atlas, NASA Power Data Access Viewer.
	Land Cover.	Food and Agriculture Organization AFRICOVER Data
	Birds Flyway.	Bird Life International
Vector	Roads, Urban Areas, Water Surfaces, Airports, Power Lines.	Egyptian Survey Authority
	Protected Areas.	Egyptian Environmental Affairs Agency

4.2 BAG Framework Description

BAG utilizes GIS capabilities in geospatial data management and MCDM versatility to merge accurate data (e.g., slope, land usage, elevation, etc.) with value-based data (e.g., specialists views, standards, surveys, etc.) in a neutrosophic framework for the selection of suitable locations for wind farms. the BAG framework is comprising the following stages as shown in Fig.(2) .

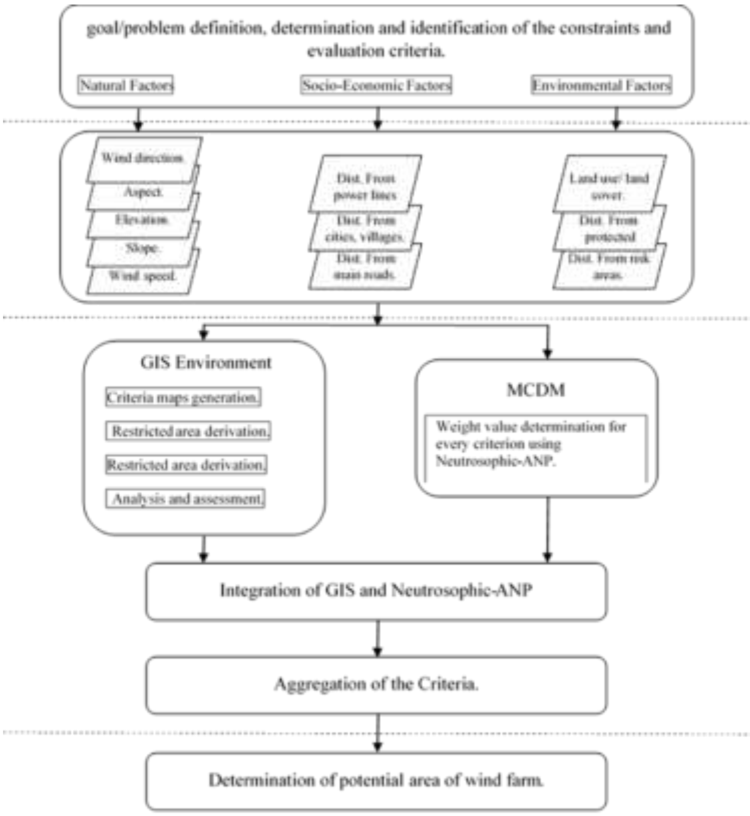


Figure 2. BAG Framework.

Stage 1: preliminary study, Data acquisition and Pre-Processing

This stage involves definition of goal/problem, determination and identification of the constraints and evaluation criteria, and analysis of generally suitable sites.

Stage 2: Restricted area identification

Due to residential areas, water bodies, natural reserves or protected areas, it is deemed impractical to install such a system in such an environment. The definition of that area helps the definition of the area of usable zones for the construction of a wind farm system to be eliminated. First, certain areas are excluded which, due to factual factors and legal requirements, may be deemed to be unsuitable for locating wind farms. Buffer zones, i.e., minimal lengths, across these regions are also excluded in some cases under Egyptian legislation.

The procedure of exclusion is applied in ArcGIS. The BUFFER tool is used to build a buffer zone around a specified type of field. In a next step all feature datasets are transformed into a raster dataset. then, Based on Boolean logic, the criteria are assigned a true or false value by the IS NULL and CON tools. All restricted areas are marked as false and therefore obtain a value score of 0. After that, "multiply' all restrictions. Finally, the exclusion area map will show the technically available maximum land for wind energy development in the study area.

Stage 3: Criteria Standardization

Although each criteria attribute has its measuring scale, standardization is used to perform transformation of attributes into a common suitability. that produces transformed attributes in a common reference rate scale. For example, the criteria attributes for each sub-model were transformed from the original values to a common suitability scale ranged from 1-10 (10 means more favorable, and zero means unsuitable pixels).

Stage 4: Analysis, and assessment

After exclusionary areas were identified and excluded from the all area of Study area, the potential suitable area for wind farm construction is the remainder area. This potentially suitable area must be evaluated to select the preferred sites. In this study, we used ArcGIS spatial analyst which provides affluence set of spatial analysis and modeling tools and functions for both raster and vector data. The analytical capabilities of Spatial Analyst facilitate spatial manipulation and generate data based on spatial analysis and displaying the results of spatial analysis. Here are described the GIS analytical procedures that have been applied individually or used in sequence within ArcGIS to evaluate the initial suitability for wind farm construction:

1. Euclidian distance analysis: Euclidian distance tool describes each cells relationship to a source based on the straight-line distance. The output of this tool is raster map .
2. Reclassify analysis: Provide a variety of methods that allow you to reclassify or change input cells to alternative values.

Stage 5: Bipolar Neutrosophic ANP application

In main nine steps, Bipolar Neutrosophic ANP can be summed up as follows :

Step 1. Model Builder: Building a model and transforming an issue into a network structure concept. There must be an accessible transformation of a problem into a logical structure, such as a network. The problem is transformed into a network system at this step, where all aspects can contact with each other.

Step 2. Experts Determination: A process to select a committee of experts including scholars and professionals in relevant fields such as social sciences, energy, environmental protection and economy. It is important to take into account the diverse perspectives of experts based on their background and areas of expertise.

Step 3. Linguistic Evaluation: Experts suggest their linguistic expressions for assessing the relative importance of criteria.

Step 4. BNS Transformation: Transforms the linguistic expressions to Bipolar neutrosophic numbers. For criteria weights, the linguistic expressions are as shown in Table (2).

Table 2. Bipolar Neutrosophic Scale for Comparison Matrix [28].

Linguistic Expressions	**Bipolar Neutrosophic Numbers Scale** $\langle \boldsymbol{T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)} \rangle$
Absolutely Influential (AI)	$\langle (0.9,0.1,0.1), (-0.4,-0.8,-0.9) \rangle$
Very Highly Influential (VHI)	$\langle (0.8,0.5,0.5), (-0.3,-0.8,-0.8) \rangle$
Equally Influential (EI)	$\langle (0.5,0.5,0.5), (-0.5,-0.5,-0.5) \rangle$
Influential (I)	$\langle (0.4,0.2,0.7), (-0.5,-0.2,-0.1) \rangle$
Almost Influential (ALI)	$\langle (0.1,0.8,0.7), (-0.9,-0.2,-0.1) \rangle$

Step 5. Deneutrosophication: Determine the score value of linguistic terms for each factor, Using the Eq. (5) for converting bipolar neutrosophic numbers into crisp values.

Step 6. Pair-wise Comparisons Constructions: Constructing a pair-wise relation of all the decision-making variables and estimate the criteria priority . Decision elements for each group are compared pairwise, equivalent to the pair-wise comparison conducted in AHP. Groups themselves are also evaluated on the basis of their position and influence on the achievement of the goals and on the interdependencies between each group's criteria. Through the eigenvector, the impact of criteria on each other can be presented .

Step 7. Generate a Super Matrix: In order to achieve overall objectives in an interconnected environment, Vectors of internal importance must be inserted into unique columns of the matrix which is called the super matrix. It is essentially a partition matrix that displays the relations among two groups in a system. The hierarchy's super matrix can be defined as:

$$W_h = \begin{bmatrix} 0 & 0 & 0 \\ W_{21} & 0 & 0 \\ 0 & W_{32} & I \end{bmatrix} \tag{8}$$

Where in this super matrix, $W_{21}$is a vector that demonstrates the impacts of the target on criteria, W_{32} demonstrates the impacts of criteria on alternatives, and I represents the unit matrix. If the parameters for inner relations are used, the hierarchy model will be transformed to network model. Criteria interactions are by inserting W_{22} into the W_h super matrix to be the W_n matrix.

$$W_n = \begin{bmatrix} 0 & 0 & 0 \\ W_{21} & W_{22} & 0 \\ 0 & W_{32} & I \end{bmatrix} \tag{9}$$

Step 8. Constructing the weighted super matrix: This matrix is known as the initial super matrix. For obtaining the unweighted super matrix the inner priorities vectors, matrices and elements replaced in the initial super matrix. By multiplying the unweighted super matrix values in the group matrix, the weighted super matrix is obtained. Then, Using Eq. (10) in the final stage for calculating the limited super matrix.

$$\lim_{k\to\infty} W^k \tag{10}$$

Step 9. Choosing the right choice: In the limited super matrix, the alternatives final weight obtaining from the alternative's column. An alternative is regarded to be the right choice when becoming the greatest weight in this matrix. In the proposed technique, Bipolar neutrosophic ANP can be applied for determining the weights of the criteria. After that, the weights of the criteria can be used in ARCGIS to determine alternatives.

Stage 5: Aggregation of the Criteria:

It is important to aggregate the criteria after calculating of the clusters/criteria weights. WLC is used in the requirements aggregation process. Each standardized criteria map (each cell within each map) is multiplied by the weight of its criteria and the results are then summed. To integrate the assessment (factors) criteria as per the WLC process, the following mathematical expression was used:

$$S = \sum W_i X_i \tag{11}$$

Where S is suitability, W_i is the normalized value of the weight of factor i, and X_i is the criterion score of factor i.

In the next stage, the required locations need to be segregated by removing the cells from the suitability map with the highest values for showing the position of wind farms. By integrating the arithmetic operations and queries in the GIS application, the cells are filtered then identifying wind farm installation sites.

5. Results and Discussions

In accordance with recent developments and political developments in Egypt over the past few years, and in line with the trend of the State in promoting the use of renewable energies in most industrial, agricultural, tourism and other applications, nevertheless the issue of selecting wind farm site still prominent. Decision making process on choosing the best site is a big issue for MCDM. In this research, the solution to the problem has been achieved in an environment of ambiguity (fuzziness) and uncertainty by merging the Bipolar Neutrosophic, ANP, and GIS in the following steps :

Step 1: preliminary study, Data acquisition and Pre-Processing

In this research, we used a data-set that included climatic, topographic, hydrologic, and geological factor. Based on several literatures, case studies concerning wind farm site selection and local conditions, different criteria were reviewed and eleven criteria were selected to evaluate the suitable sites for wind farms, criteria have been classified into three main groups because groups play an important role in the ANP method; natural, environmental and socio-economic factors. These were the most important criteria for selecting suitable sites.

1. Natural factors

Includes wind speed, Elevation, Slope, Aspect direction, and wind direction. Wind speed is a critical factor to generate wind turbine's electricity. To order to produce wind energy, wind speed above certain rates is vital [7]. The height has an impact on the technical capability of installing a wind-turbine and maximizes construction and maintenance costs, the high-altitude sites (above 1500 m.a.sl) or near cliffs are usually not appropriate for installing wind turbines [29-31]. Sloping grounds are considered to be less suitable for wind turbine improvement, which increases the cost of building and maintaining turbines dramatically [7, 32]. Terrain location should be taken into account, as the ideal factor. Aspect relative to the direction of the wind [33], and wind turbines are located through the prevailing wind direction to be effective.

2. Environmental factors

Include Proximity to airports, distance to environmental interest areas; and land cover/land use of ground surface. The distance between airports and wind turbines affects the safety of flights, therefore, the location of the for airports factor should be taken into account. Moreover, Wind turbines may interfere with radio transmissions, radar and microwave signals due to their heights hence the need to site them away from airports [34]. when deciding where turbines should be installed, the wind turbine effect on environmental interest areas (protected areas, bird migration flyway) should be taken into account [35,36]. Moreover, the possibility of floods happening near wind farms during the winter should be taking into account as a crucial factor affects the functionality of the turbines, and in order to prevent damage to the turbine components, wind turbine fins are lowered and disconnected. And all the mechanical parts of wind power turbines have to be kept away from the water. One of the most important factors for energy investments is land use/land cover. Wind farms should be installed in the area in which they negligibly interfere with existing land use outside protected areas, artificial surfaces, wetlands, aquatic and forest areas [33].

3. Socio-economic factors

Include Proximity to power grid, Proximity to cities, distance to roads. In order to reduce the costs associated with the construction of wind farms and to reduce electric transport costs generated in the national energy distribution system, wind farms should be located in the vicinity of the current transmission grids [33]. One of the key technical considerations, therefore, is the need to shorten the distance between wind-turbines -as the source of renewable energy- and the existing national energy network. The wind farm must be located far from the cities and villages to achieve the protection and lower noise interference [33]. Distance to roads has an impact on the expenses of installing and maintaining wind turbines, but due to safety reasons, the location of wind turbines should be properly positioned at a set distance from roads and railways [33].

After that, all maps taken as GIS layers for the whole area of Sinai Peninsula and projected into WGS_1984_UTM_Zone_36N of the Universal Transverse Mercator System (UTM) of projected coordinates. Then all vector data sets were converted to raster data set. Clip or mask the data set with study area boundary, and ensuring that all cell size equal 30×30.

Step 2: Identification and Exclusion of restricted areas.

Table (3) shows the exclusionary criteria and buffer zones for potential wind farms. Based on a predefined criterion, the restrictive method uses the Boolean logic approach to define the possibility of locating a wind farm. Logical math tools represent the right conditions as 1 for the area with a probability of being a wind farm location and false conditions as 0 for an area with an impediment for wind farm locating.

Table 3. The List of Exclusionary Criteria and Corresponding Buffer Distance.

Criteria	Exclusionary Criteria	Buffer Zones
Natural	Elevation	>2000 m
	Slope	>15%

	Wind Speed	<5
Scio-Economic	Roads	0-500 m
	Power Lines	0-500 m
	Urban Areas	0-2500 m
Environmental	Land Cover / Land Use	Water Bodies, Urban Areas.
	Protected Areas	0-2000 m

Step 3: Criteria standardization to a common scale.

For our research, we used the simplest formula for linear standardization which is called the maximum score procedure. The formula divides each raw criterion value by the maximum criterion value as shown in Eq. (12).

$$x'_{ij} = \frac{x_{ij}}{x_j^{max}} \quad (12)$$

Where x'_{ij} is the standardized score for the i^{th} decision alternative and the j^{th} criterion, x_{ij} is the raw data value, and x_j^{max} is the maximum score for the j^{th} criterion.

Step 4: Analysis, and assessment.

Euclidian distance function (multiple buffers) in ArcGIS Spatial Analyst was used to calculate the distance from transmission power lines; urban areas; roads; and protected areas. Then, reclassify analysis function in ArcGIS Spatial Analyst was used to reclassify the study area into classes. The complete classification has been presented in Table (4). Fig. (3) shows an example for the reclassified maps.

Table 4. Criteria Suitability Classes.

Suitability Rating	Classes	Slope	Elevation	Wind Speed	D.F. Roads
3	Most Suitable	0 - 2.5	0 - 50	10.8 - 16.2	0 - 2627
2	Suitable	2.5 - 5	50 - 100	7.6 - 10.8	2627 - 7342
1	Less Suitable	5 - 15	100 - 600	4.4 - 7.6	7342 - 30981
0	Not Suitable	> 15	> 600	2.4 - 4.4	30981 - 58219
Suitability Rating	**Classes**	**D.F. Power Lines**	**D.F. Urban Areas**	**Land Cover / Land Use**	**Protected Areas**
3	Most Suitable	0 - 6557	0 - 4922	Bare Land	> 2000 m
2	Suitable	6557 - 15953	4922 - 12639	-	-
1	Less Suitable	15953 - 48713	12639 - 43710	-	-
0	Not Suitable	48713 - 76364	43710 - 73454	Sabkha	-

Step 5: Constructing the structure of the problem .

The general criteria and sub-criteria for selections are mentioned in Table (5). Fig. (2) presented a schematic diagram of the problem.

Step 6: Determine a committee of decision makers.

Step 7: Use linguistic variables to express the opinion of specialists Using the scales mentioned previously in Table (2).

Step 8: Determine the inner-relationship among the sub-criteria, as in Table (6).

Table 5. Criteria for Wind Farm Selection.

Criteria	Sub-Criteria
Natural (c_1)	Slope (c_{11})
	Wind Direction (c_{12})
	Wind Speed (c_{13})
	Elevation (c_{14})
	Aspect (c_{15})
Scio-Economic (c_2)	D.F. Roads (c_{21})
	D.F. Power Lines (c_{22})
	D.F. Urban Areas (c_{23})
Environmental (c_3)	Land Cover / Land Use (c_{31})
	Protected Areas (c_{32})

Table 6. Sub-criteria Dependencies.

Sub-Criteria	Rely on	Sub-Criteria	Rely on
c_{11}	(c_{12}, c_{22}, c_{31})	c_{21}	$(c_{22}, c_{23}, c_{31}, c_{32})$
c_{12}	(c_{11}, c_{21}, c_{32})	c_{22}	(c_{11}, c_{13}, c_{15})
c_{13}	$(c_{12}, c_{21}, c_{22}, c_{23})$	c_{23}	$(c_{11}, c_{13}, c_{15}, c_{21})$
c_{14}	(c_{13}, c_{21}, c_{32})	c_{31}	(c_{21}, c_{23}, c_{32})
c_{15}	(c_{11}, c_{13}, c_{22})	c_{32}	(c_{14}, c_{21}, c_{31})

Step 9: constructing the pairwise comparison matrix between the main criteria as follows :

- Construct W_{21} as presented in Table (7).
- Replace the linguistic scale by Bipolar Neutrosophic numbers by using Table (2).
- De-neutrosophication of the Bipolar neutrosophic numbers to crisp values as presented in table (8) using Eq. (5).
- Check the consistency by computing the CR of the comparison matrices with less or equal 0.1.
- Calculated the interdependences for sub-criteria as Demonstrated in Tables (9-18).
- Constructed the W_{22} matrix as presented in Table (19).
- Constructed the weight matrix and calculate the weight of criteria using $W_{criteria} = W_{21} \times W_{22}$, as shown in Table (19).

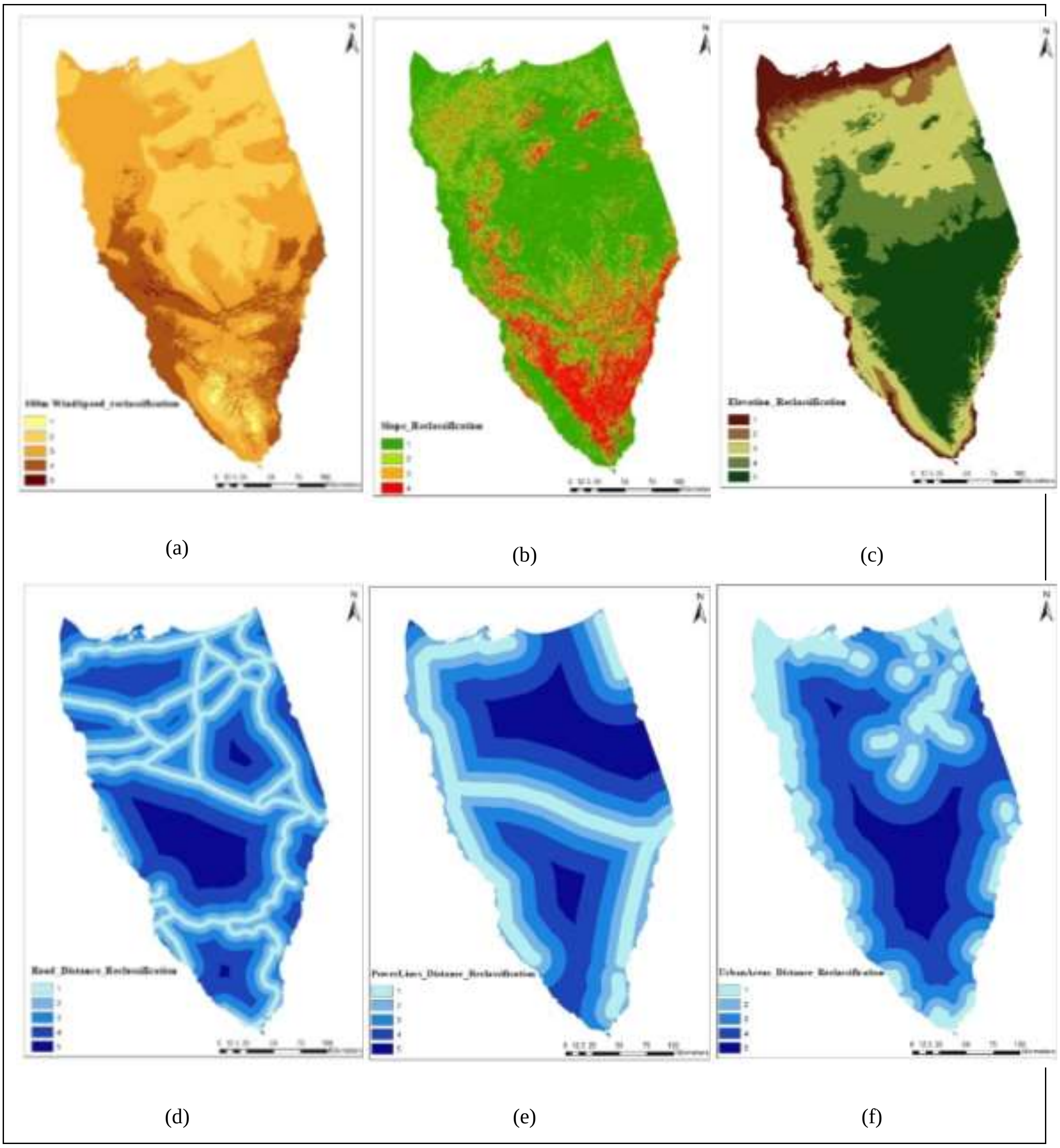


Figure 3. Reclassified Factors Maps for: (a) Wind Speeds; (b) Slope; (c) Elevation; (d) Roads; (e) Power Lines; (f) Urban Areas.

Table 7. Pairwise Comparison for W_{21}.

W_{21}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{21}	c_{22}	c_{23}	c_{31}	c_{32}
c_{11}	EI	1/I	ALI	1/AI	AI	AI	1/ALI	VHI	1/EI	EI
c_{12}	I	EI	1/VHI	1/ALI	1/VHI	ALI	1/AI	ALI	VHI	1/VHI
c_{13}	1/ALI	VHI	EI	AI	1/AI	AI	EI	AI	EI	ALI
c_{14}	AI	ALI	1/AI	EI	ALI	ALI	AI	1/ALI	1/ALI	ALI

c_{15}	1/AI	VHI	AI	1/ALI	EI	VHI	AI	1/I	1/AI	1/AI
c_{21}	1/AI	1/ALI	1/AI	1/ALI	1/VHI	EI	1/AI	EI	AI	VHI
c_{22}	ALI	AI	1/EI	1/AI	1/AI	AI	EI	AI	1/EI	ALI
c_{23}	1/VHI	1/ALI	1/AI	ALI	I	1/EI	1/AI	EI	AI	1/AI
c_{31}	EI	1/VHI	1/EI	ALI	AI	1/AI	EI	1/AI	EI	ALI
c_{32}	1/EI	VHI	1/ALI	1/ALI	AI	1/VHI	1/ALI	AI	1/ALI	EI

Table 8. W_{21} De-neutrosophication Matrix.

W_{21}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{21}	c_{22}	c_{23}	c_{31}	c_{32}	W_{21}Weight
c_{11}	0.5	2.609	0.167	1.2	0.833	0.833	6	0.683	2	0.5	0.096
c_{12}	0.383	0.5	1.463	6	1.463	0.167	1.2	0.167	0.683	1.463	0.088
c_{13}	6	0.683	0.5	0.833	1.2	0.833	0.5	0.833	0.5	0.167	0.088
c_{14}	0.833	0.167	1.2	0.5	0.167	0.167	0.833	6	6	0.167	0.098
c_{15}	1.2	0.683	0.833	6	0.5	0.683	0.833	2.609	1.2	1.2	0.100
c_{21}	1.2	6	1.2	6	1.463	0.5	1.2	0.5	0.833	0.683	0.115
c_{22}	0.167	0.833	2	1.2	1.2	0.833	0.5	0.833	2	0.167	0.066
c_{23}	1.463	6	1.2	0.167	0.383	2	1.2	0.5	0.833	1.2	0.109
c_{31}	0.5	1.463	2	0.167	0.833	1.2	0.5	1.2	0.5	0.167	0.063
c_{32}	2	0.683	6	6	0.833	1.463	6	0.833	6	0.5	0.176

Table 9. Interdependencies Matrix of Factor C_{11}.

C_{11}	C_{12}	C_{22}	C_{31}	W_{22}
C_{12}	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	⟨0.1,0.8,0.7, −0.9, −0.2, −0.1⟩	1/⟨0.9,0.1,0.1, −0.4, −0.8, −0.9⟩	0.217
C_{22}	1/⟨0.1,0.8,0.7, −0.9, −0.2, −0.1⟩	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	⟨0.8,0.5,0.5, −0.3, −0.8, −0.8⟩	0.447
C_{31}	⟨0.9,0.1,0.1, −0.4, −0.8, −0.9⟩	1/⟨0.8,0.5,0.5, −0.3, −0.8, −0.8⟩	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	0.337

Table 10. Interdependencies Matrix of Factor C_{12}.

C_{12}	C_{11}	C_{21}	C_{32}	W_{22}
C_{11}	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	⟨0.9,0.1,0.1, −0.4, −0.8, −0.9⟩	1/⟨0.1,0.8,0.7, −0.9, −0.2, −0.1⟩	0.458
C_{21}	1/⟨0.9,0.1,0.1, −0.4, −0.8, −0.9⟩	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	0.288
C_{32}	⟨0.1,0.8,0.7, −0.9, −0.2, −0.1⟩	1/⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	0.254

Table 11. Interdependencies Matrix of Factor C_{13}.

C_{13}	C_{12}	C_{21}	C_{22}	C_{23}	W_{22}
C_{12}	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	⟨0.8,0.5,0.5, −0.3, −0.8, −0.8⟩	1/⟨0.9,0.1,0.1, −0.4, −0.8, −0.9⟩	0.162
C_{21}	1/⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	⟨1/0.4,0.2,0.7, −0.5, −0.2, −0.1⟩	⟨0.4,0.2,0.7, −0.5, −0.2, −0.1⟩	0.312
C_{22}	1/⟨0.8,0.5,0.5, −0.3, −0.8, −0.8⟩	⟨0.4,0.2,0.7, −0.5, −0.2, −0.1⟩	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	1/⟨0.4,0.2,0.7, −0.5, −0.2, −0.1⟩	0.269
C_{23}	⟨0.9,0.1,0.1, −0.4, −0.8, −0.9⟩	1/⟨0.4,0.2,0.7, −0.5, −0.2, −0.1⟩	⟨0.4,0.2,0.7, −0.5, −0.2, −0.1⟩	⟨0.5,0.5,0.5, −0.5, −0.5, −0.5⟩	0.256

Table 12. Interdependencies Matrix of Factor C_{14}.

C_{14}	C_{13}	C_{21}	C_{32}	W_{22}
C_{13}	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.1,0.8,0.7,-0.9,-0.2,-0.1\rangle$	$\langle 0.9,0.1,0.1,-0.4,-0.8,-0.9\rangle$	0.467
C_{21}	$\langle 0.1,0.8,0.7,-0.9,-0.2,-0.1\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	0.227
C_{32}	$1/\langle 0.9,0.1,0.1,-0.4,-0.8,-0.9\rangle$	$\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	0.306

Table 13. Interdependencies Matrix of Factor C_{15}.

C_{15}	C_{11}	C_{13}	C_{22}	W_{22}
C_{11}	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.4,0.2,0.7,-0.5,-0.2,-0.1\rangle$	$\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	0.386
C_{13}	$\langle 0.4,0.2,0.7,-0.5,-0.2,-0.1\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	0.283
C_{22}	$1/\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	0.331

Table 14. Interdependencies Matrix of Factor C_{21}.

C_{21}	C_{22}	C_{23}	C_{31}	C_{32}	W_{22}
C_{22}	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$1/\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	0.182
C_{23}	$1/\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.4,0.2,0.7,-0.5,-0.2,-0.1\rangle$	$\langle 0.4,0.2,0.7,-0.5,-0.2,-0.1\rangle$	0.313
C_{31}	$1/\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$\langle 0.4,0.2,0.7,-0.5,-0.2,-0.1\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	0.247
C_{32}	$\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$1/\langle 0.4,0.2,0.7,-0.5,-0.2,-0.1\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	0.258

Table 15. Interdependencies Matrix of Factor C_{22}.

$\mathbf{C_{22}}$	C_{11}	C_{13}	$\mathbf{C_{15}}$	W_{22}
C_{11}	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$\langle 0.1,0.8,0.7,-0.9,-0.2,-0.1\rangle$	$1/\langle 0.9,0.1,0.1,-0.4,-0.8,-0.9\rangle$	0.217
C_{13}	$1/\langle 0.1,0.8,0.7,-0.9,-0.2,-0.1\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	0.447
$\mathbf{C_{15}}$	$\langle 0.9,0.1,0.1,-0.4,-0.8,-0.9\rangle$	$1/\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	0.337

Table 16. Interdependencies Matrix of Factor C_{23}.

$\mathbf{C_{23}}$	C_{11}	C_{13}	$\mathbf{C_{15}}$	$\mathbf{C_{21}}$	W_{22}
C_{11}	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$1/\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	0.182
C_{13}	$1/\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.4,0.2,0.7,-0.5,-0.2,-0.1\rangle$	$\langle 0.4,0.2,0.7,-0.5,-0.2,-0.1\rangle$	0.313
$\mathbf{C_{15}}$	$1/\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$\langle 0.4,0.2,0.7,-0.5,-0.2,-0.1\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	0.247
$\mathbf{C_{21}}$	$\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$1/\langle 0.4,0.2,0.7,-0.5,-0.2,-0.1\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	0.258

Table 17. Interdependencies Matrix of Factor C_{31}.

C_{31}	$\mathbf{C_{21}}$	C_{23}	C_{32}	W_{22}
$\mathbf{C_{21}}$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.1,0.8,0.7,-0.9,-0.2,-0.1\rangle$	$\langle 0.9,0.1,0.1,-0.4,-0.8,-0.9\rangle$	0.467
C_{23}	$\langle 0.1,0.8,0.7,-0.9,-0.2,-0.1\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	0.227
C_{32}	$1/\langle 0.9,0.1,0.1,-0.4,-0.8,-0.9\rangle$	$\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	0.306

Table 18. Interdependencies Matrix of Factor C_{32}.

C_{32}	C_{14}	C_{21}	C_{31}	W_{22}
C_{14}	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.1,0.8,0.7,-0.9,-0.2,-0.1\rangle$	$\langle 0.9,0.1,0.1,-0.4,-0.8,-0.9\rangle$	0.467
C_{21}	$\langle 0.1,0.8,0.7,-0.9,-0.2,-0.1\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	$1/\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	0.227
C_{31}	$1/\langle 0.9,0.1,0.1,-0.4,-0.8,-0.9\rangle$	$\langle 0.8,0.5,0.5,-0.3,-0.8,-0.8\rangle$	$\langle 0.5,0.5,0.5,-0.5,-0.5,-0.5\rangle$	0.306

Table 19. ANP Final Weight for Criteria.

W_{22}	W_{11}	W_{12}	W_{13}	W_{14}	W_{15}	W_{21}	W_{22}	W_{23}	W_{31}	W_{32}	W_{21}	Total Criteria Weight ($W_{criteria}$)
W_{11}	0	0.458	0	0	0.386	0	0.217	0.182	0	0	0.096	0.113
W_{12}	0.217	0	0.162	0	0	0	0	0	0	0	0.088	0.035
W_{13}	0	0	0	0.467	0.283	0	0.447	0.313	0	0	0.088	0.138
W_{14}	0	0	0	0	0	0	0	0	0	0.467	0.098	0.082
W_{15}	0	0	0	0	0	0	0.337	0.247	0	0	0.100	0.049
W_{21}	0	0.288	0.312	0.227	0	0	0	0.258	0.467	0.227	0.115	0.173
W_{22}	0.447	0	0.269	0	0.331	0.182	0	0	0	0	0.066	0.121
W_{23}	0	0	0.256	0	0	0.313	0	0	0.227	0	0.109	0.073
W_{31}	0.337	0	0	0	0	0.247	0	0	0	0.306	0.063	0.115
W_{32}	0	0.254	0	0.306	0	0.258	0	0	0.306	0	0.176	0.101

Step 10: Aggregation of the Criteria:

The weighted overlay in ArcGIS was used to combine the different geospatial layers for the modelling criteria. The study area's final suitability scores were calculated by reclassifying the weighted overlay scores into four classes, with the areas corresponding to the exclusionary areas being graded as "not-suitable". As seen in fig. (4).

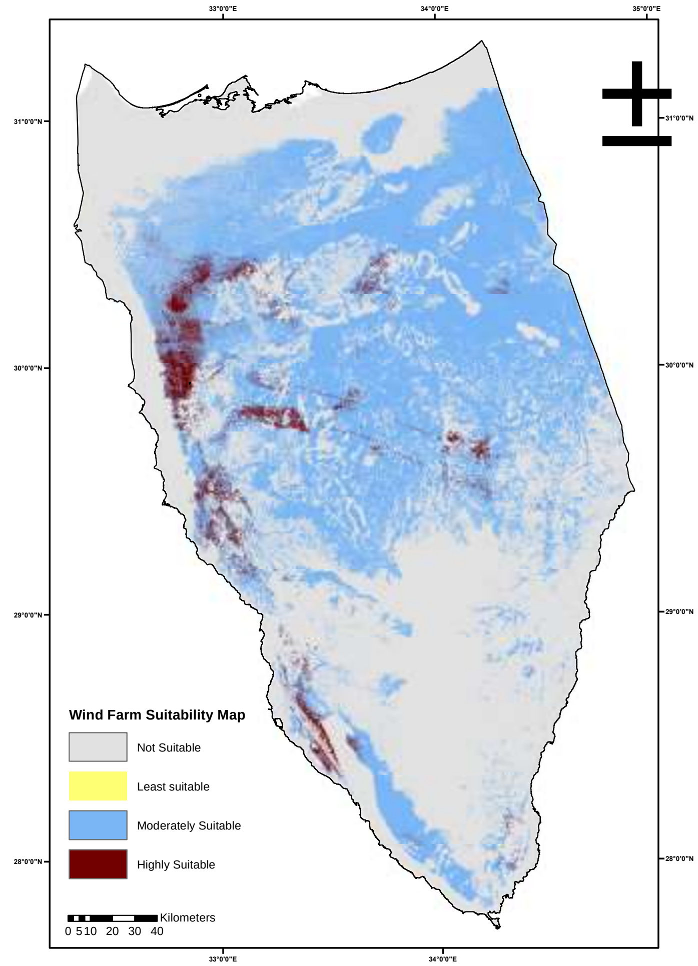


Figure 4. Suitability Maps of the Wind Farms.

6. Conclusion

This paper introduces a new model for mapping potential wind energy zones in Sinai Peninsula in Egypt that combines remote sensing data and a spatial decision support model. we use bipolar neutrosophic numbers to explain the values of attributes to accommodate the shortage of judgement knowledge .

The selection of suitable sites for wind farms in Sinai Peninsula is based on a number of interrelated factors of geography, climate and land use-land cover. For studying such factors, remote sensing (ASTER) and GIS techniques were used and a Spatial Multicriteria Decision Making (SMDM) model was designed.

The creation of a spatial decision model resulted from the incorporation of interpreted data obtained from a series of layers regarding natural and environmental characteristics, as well as Scio-economic. The research resulted in a suitability index map with various suitable zones for grid-connected wind power plant construction.

It is concluded that Spatial Multicriteria Decision Making model managed to solve the site selection problem and fulfill the objective of the study. It considered the most effective criteria, i.e., natural, environmental and Scio-economic, and their relative importance in the decision making. In addition, to accommodate missing details, the bipolar neutrosophic set is included in the specialist committee judgement. Such decisions support tool studied need more attention from both researchers and decision makers.

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References

[1] A. Kumar, B. Sah, A. R. Singh, Y. Deng, X. He, P. Kumar and R. C. Bansal, "A review of multi criteria decision making (MCDM) towards sustainable renewable energy development," *Renewable and Sustainable Energy Reviews,* vol. 69, p. 596–609, 2017.

[2] G. Villacreses, G. Gaona, J. Martínez-Gómez and D. J. Jijón, "Wind farms suitability location using geographical information system (GIS), based on multi-criteria decision making (MCDM) methods: The case of continental Ecuador," *Renewable energy,* vol. 109, p. 275–286, 2017.

[3] S. Ali, J. Taweekun, K. Techato, J. Waewsak and S. Gyawali, "GIS based site suitability assessment for wind and solar farms in Songkhla, Thailand," *Renewable Energy,* vol. 132, p. 1360–1372, 2019.

[4] S. M. Bina, S. Jalilinasrabady, H. Fujii and H. Farabi-Asl, "A comprehensive approach for wind power plant potential assessment, application to northwestern Iran," *Energy,* vol. 164, p. 344–358, 2018.

[5] L. Castro-Santos, G. P. Garcia, T. Simões and A. Estanqueiro, "Planning of the installation of offshore renewable energies: A GIS approach of the Portuguese roadmap," *Renewable Energy,* vol. 132, p. 1251–1262, 2019.

[6] J. M. Sánchez-Lozano, M. S. García-Cascales and M. T. Lamata, "GIS-based onshore wind farm site selection using Fuzzy Multi-Criteria Decision Making methods. Evaluating the case of Southeastern Spain," *Applied Energy,* vol. 171, p. 86–102, 2016.

[7] T. Höfer, Y. Sunak, H. Siddique and R. Madlener, "Wind farm siting using a spatial Analytic Hierarchy Process approach: A case study of the Städteregion Aachen," *Applied energy,* vol. 163, p. 222–243, 2016.

[8] S. Değirmenci, F. Bingöl and S. C. Sofuoglu, "MCDM analysis of wind energy in Turkey: decision making based on environmental impact," *Environmental Science and Pollution Research,* vol. 25, p. 19753–19766, 2018.

[9] P. V. Gorsevski, S. C. Cathcart, G. Mirzaei, M. M. Jamali, X. Ye and E. Gomezdelcampo, "A group-based spatial decision support system for wind farm site selection in Northwest Ohio," *Energy Policy,* vol. 55, p. 374–385, 2013.

[10] J. M. Sánchez-Lozano, M. S. García-Cascales and M. T. Lamata, "Identification and selection of potential sites for onshore wind farms development in Region of Murcia, Spain," *Energy,* vol. 73, p. 311–324, 2014.

[11] S. Ali, S.-M. Lee and C.-M. Jang, "Determination of the most optimal on-shore wind farm site location using a GIS-MCDM methodology: Evaluating the case of South Korea," *Energies,* vol. 10, p. 2072, 2017.

[12] J. J. Diez-Rodríguez, T. B. Fischer and S. Di Zio, "Introducing a group spatial decision support system for use in strategic environmental assessment of onshore wind farm development in Mexico," *Journal of Cleaner Production,* vol. 220, p. 1239–1254, 2019.

[13] L. Gigović, D. Pamučar, D. Božanić and S. Ljubojević, "Application of the GIS-DANP-MABAC multi-criteria model for selecting the location of wind farms: A case study of Vojvodina, Serbia," *Renewable energy,* vol. 103, p. 501–521, 2017.

[14] H. S. Dhiman and D. Deb, "Fuzzy TOPSIS and fuzzy COPRAS based multi-criteria decision making for hybrid wind farms," *Energy,* vol. 202, p. 117755, 2020.

[15] L. A. Zadeh, "Fuzzy sets," *Information and Control,* vol. 8, pp. 338-353, 1965.

[16] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems,* vol. 20, pp. 87-96, 1986.

[17] T. R. Ayodele, A. S. O. Ogunjuyigbe, O. Odigie and J. L. Munda, "A multi-criteria GIS based model for wind farm site selection using interval type-2 fuzzy analytic hierarchy process: The case study of Nigeria," *Applied Energy,* vol. 228, p. 1853–1869, 2018.

[18] Y. Wu, S. Geng, H. Xu and H. Zhang, "Study of decision framework of wind farm project plan selection under intuitionistic fuzzy set and fuzzy measure environment," *Energy Conversion and Management,* vol. 87, p. 274–284, 2014.

[19] Y. A. Solangi, Q. Tan, M. W. A. Khan, N. H. Mirjat and I. Ahmed, "The selection of wind power project location in the Southeastern Corridor of Pakistan: A factor analysis, AHP, and fuzzy-TOPSIS application," *Energies,* vol. 11, p. 1940, 2018.

[20] C.-N. Wang, Y.-F. Huang, Y.-C. Chai, V. T. Nguyen and others, "A multi-criteria decision making (MCDM) for renewable energy plants location selection in Vietnam under a fuzzy environment," *Applied Sciences,* vol. 8, p. 2069, 2018.

[21] F. Smarandache, "A unifying field in Logics: Neutrosophic Logic.," in *Philosophy,* American Research Press, 1999, p. 1–141.

[22] A. Kharal, "A neutrosophic multi-criteria decision making method," *New Mathematics and Natural Computation,* vol. 10, p. 143–162, 2014.

[23] P. Liu and Y. Wang, "Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean," *Neural Computing and Applications,* vol. 25, p. 2001–2010, 2014.

[24] J.-j. Peng, J.-q. Wang, J. Wang, H.-y. Zhang and X.-h. Chen, "Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems," *International journal of systems science,* vol. 47, p. 2342–2358, 2016.

[25] J. Ye, "Single valued neutrosophic cross-entropy for multicriteria decision making problems," *Applied Mathematical Modelling,* vol. 38, p. 1170–1175, 2014.

[26] J. Ye, "Trapezoidal neutrosophic set and its application to multiple attribute decision-making," *Neural Computing and Applications,* vol. 26, p. 1157–1166, 2015.

[27] P. Bosc and O. Pivert, "On a fuzzy bipolar relational algebra," *Information Sciences,* vol. 219, p. 1–16, 2013.

[28] M. Abdel-Basset, A. Gamal, L. H. Son, F. Smarandache and others, "A bipolar neutrosophic multi criteria decision making framework for professional selection," Applied Sciences, vol. 10, p. 1202, 2020.

[29] D. Latinopoulos and K. Kechagia, "A GIS-based multi-criteria evaluation for wind farm site selection. A regional scale application in Greece," *Renewable Energy,* vol. 78, p. 550–560, 2015.

[30] K. B. Atici, A. B. Simsek, A. Ulucan and M. U. Tosun, "A GIS-based Multiple Criteria Decision Analysis approach for wind power plant site selection," *Utilities Policy,* vol. 37, p. 86–96, 2015.

[31] V. Gass, J. Schmidt, F. Strauss and E. Schmid, "Assessing the economic wind power potential in Austria," *Energy policy,* vol. 53, p. 323–330, 2013.

[32] Y. Noorollahi, H. Yousefi and M. Mohammadi, "Multi-criteria decision support system for wind farm site selection using GIS," *Sustainable Energy Technologies and Assessments,* vol. 13, p. 38–50, 2016.

[33] R. Van Haaren and V. Fthenakis, "GIS-based wind farm site selection using spatial multi-criteria analysis (SMCA): Evaluating the case for New York State," *Renewable and sustainable energy reviews,* vol. 15, p. 3332–3340, 2011.

[34] D. Pamučar, L. Gigović, Z. Bajić and M. Janošević, "Location selection for wind farms using GIS multi-criteria hybrid model: An approach based on fuzzy and rough numbers," *Sustainability,* vol. 9, p. 1315, 2017.

[35] P. E. Baffoe and D. Sarpong, "Selecting suitable sites for wind energy development in Ghana," *Ghana Mining Journal,* vol. 16, p. 8–20, 2016.

[36] D. G. Vagiona and M. Kamilakis, "Sustainable site selection for offshore wind farms in the South Aegean—Greece," *Sustainability,* vol. 10, p. 749, 2018.

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Voluntary Risk disclosure Assessment in The Corporate Board Structure under uncertainty: A Case Study of Saudi Arabian Companies

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Abstract: The global crisis of financial and corporate scandals of governance have run to calls for well voluntary risk disclosure. Firms limit this disclosure of voluntary by the proprietary cost theory to evade the risk of opposing actions. So, this paper investigates and assessment voluntary risk disclosure in the corporate board structure. These voluntary risk disclosures contain the conflict and multiple criteria. So the concept of multi-criteria decision-making (MCDM) is used to overcome this problem. The neutrosophic sets are used to deal with uncertain information. This paper used the Analytic Network Process (ANP) and Decision-making trial and evaluation laboratory (DEMATEL) methods to assess voluntary risk disclosure. The ANP is used to calculate the weights of criteria. DEMATEL method is used to assess and show the impact of voluntary risk disclosure. This research uses Saudi Arabian companies as a case study.

Keywords: ANP; DEMATEL; SVNSs; Neutrosophic Sets; voluntary risk disclosure.

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1. Introduction

Mandatory disclosure is known as the corporates are needed to disclosure minimum level of information according to standards of accepted account. Mandatory disclosure includes related information about the company's performance and results of financial reporting[1]. To make an economic and financial decision, attaining the appropriate information is very important for many stakeholders[2, 3]. Annually in companies, the financial reporting is published as a source of information for internal and externals stakeholders. Financial reporting consider as a tool of communication for moving information of non-financial and financial to attention stakeholders[3].

Corporates have become more attention to possible risks that can affect the performance of systems and sustainability in the last years due to the global crisis of finance [4]. By the review of the literature, the stakeholders obtain a little information about risks that might affect corporates. So, the investors, shareholders, and stakeholders are pressure corporates to obtain and disclose risks to help them to lessen the uncertainty in decisions and to do better management in potential risks[5]. This needs more and more information than the standards generally accepted. This is known as voluntary disclosure, which means reporting information of financial and non-financial related operations of corporates, this gives more information and explanations beyond the framework set by regulations. Eng and Mak state that "voluntary disclosure is measured by the amount and detail of non-mandatory information that is contained in the management discussion and analysis in the annual report"[6]. While mandatory disclosure regulations ensure access to basic information, voluntary disclosures should be augmented by companies[7]. The level of voluntary disclosure depends on the attitude of board members towards voluntary disclosure and the benefits and costs involved[8]. So, the voluntary disclosure information helps the users and producers for the development of the accounting standards and policies[2]. Mandatory voluntary risk disclosure includes risk information disclosed by companies as specified in the International Financial Reporting Standards (IFRS) and Saudi GAAP. Voluntary disclosure of risk is any other risk information that appears in the narrative sections of the annual corporate reports. Both of these risk types are measured by the number of risk information sentences used in the accounting literature.

Reporting of risks is important in corporate disclosure practices due to offers information and details that related to corporate investment options [9]. The previous studies found deficiencies in the disclosure of risk and vague in corporate annual reports[10]. The voluntary risk disclosure is known as "the inclusion of information about managers' estimates, judgments, reliance on market-based accounting policies such as impairment, derivative hedging, financial instruments, and fair value as well as the disclosure of concentrated operations, non-financial information about corporations' plans, recruiting strategy, and other operational, economic, political and financial risks"[11]. So, voluntary risk disclosure is important for helping corporates to overcome uncertainties.

The capital market of Saudi Arabian is still in the stage of development with efforts to enhance its performance compared to the global capital market[12]. Moreover, though it is one of the major global oil sources, the Saudi government is investing heavily to diversify the economy by incorporating other industries including the tourism and entertainment sectors[13]. This will attract investors in local and global companies. So, the Saudi Government needs to include that corporates disclose enough information about their performance, risk vague and uncertainty. Hence, voluntary risk disclosure becomes more significant for the stability and profitability of Saudi corporates.

There are several theories that have been employed by researchers to examine how corporate board structure might influence the performance of companies. The current research employs agency theory to study the correlation between corporate board structure and level of voluntary risk disclosure in the Saudi context. Agency theory has been used by many studies to link corporate governance and voluntary risk disclosure. This theory posits that a corporate comprises of the agent and the principal. Agency theory can reduce agency loss. Agency theory advocates for the frequency of board meetings to characterize an active board of directors. Boards of directors that meet frequently are likely to result in risk reporting. Agency theory suggests that autonomous directors have no management role in the corporate, hence information concealment is minimized. Agency theory provides that presence of autonomous directors yields quality financial reports that are factual hence credible. Agency theory suggests that the characterization of corporate boards in terms of age, size, autonomy, and diversity does impact on the practice of voluntary risk disclosure

The voluntary risk disclosure more uncertain and vague information. So, this study proposed the neutrosophic sets to overcome this problem. The neutrosophic sets are generalized from fuzzy sets. Fuzzy sets cannot deal perfectly with uncertainty because not take into consideration the indeterminacy value[14]. This study proposed the single-valued neutrosophic sets (SVNSs). It is a subset of neutrosophic sets. It includes the Truth, Indeterminacy, False values (T,I,F)[15].

This kind of information includes multiple conflict criteria. So, proposed the concept of MCDM for overcoming it[16]. The MCDM method is used for assessing voluntary risk disclosure. The ANP method is used to obtaining the weights of criteria[17]. The DEMETAL is used to show impact and assessment the voluntary risk disclosure[18].

The main contributions in this work, assessment and show the impact of voluntary risk disclosure by using the neutrosophic sets to overcome uncertainty information, which not used in previous research, the ANP and DEMATAL are used as an MCDM method for assessing the voluntary risk disclosure not used in previous research and proposed a case study in Saudi Arabian companies.

The rest in this paper is organized as follow: section 2 present the review of the literature. Section 3 introduces the methodology. The case study is presented in section 4. The analysis of VRD is presented in section 5. Finally, section 5 introduces the conclusions of this work.

2. Review of Literature

Voluntary risk disclosure is an important process for corporates due to the decrease issue of inconsistent information. The benefits of voluntary risk disclosure that help in relieve issues between director's boards and stakeholders. Can decrease problems by enough information disclosing risks and uncertainties, hence the investors can acquire more and more confidence in corporate due to symmetry and consistent information[19].

Elshandidy & Neri study the impact of corporate governance on voluntary risk disclosure practices in the UK and Italy and also study the influence of those practices on market fluidity[20]. The results have many influences on organizers and investors in both the UK and Italy. Al-Maghzom et al. scout corporate governance and the demographic feature of top management teams as the determinants of voluntary risk disclosure practices in listed banks [10]. They make a case study in all Saudi Arabian banks from 2009 to 2013. The results of their study show that outer ownership, gender, audit committee meetings, profitability, the board size, and volume are primary determinants of voluntary risk disclosure practices in Saudi listed banks. Al-Janadi et al. measure and contrast the standard of voluntary disclosure practices in Saudi Arabia and the UAE by using a modified voluntary disclosure index[21]. Their results found that the level of voluntary disclosure is low and decreases for most of the items of social and ecological information. Al-Maghzom & Abdullah address the current hole in the disclosure literature by investigating voluntary risk disclosure in a developing economy (Saudi Arabia)[22]. Habbash et al. determine the voluntary disclosure level in Saudi Arabia and identify the main drivers of voluntary disclosure in Saudi Arabia[23].

The neutrosophic sets are used to overcome the uncertainty in voluntary risk disclosure. Karabašević et al. used the neutrosophic sets to select the e-commerce development strategies[24]. Dung et al. use the interval neutrosophic sets for personnel selection[25]. Broumi et al. SVNSs to shortest path problem[26]. Akram et al used the SVNSs for the physician selection problem[27]. The MCDM is used in this paper to deal with conflict criteria. ANP and DEMATEL are MCDM methods. Yang et al. used the ANP method for calculating the weights of criteria for a novel cluster weighted[17]. Abdel-Baset et al. used the ANP method for achieving sustainable supplier selection[28]. The DEMATEL method is used to determine the degrees of impact of these criteria. Han & Deng are used the fuzzy DEMATEL method to identify the critical success factors[29]. Abdel-Basset et al. used the neutrosophic with DEMATEL method for developing supplier selection criteria[30]. Mao et al. used the DEMATEL method handling dependent evidence [31].

The review of the literature found that no study used the ANP and DEMATEL method for voluntary risk disclosure and no study used the neutrosophic sets with this kind of problem. So, this study proposed the hybrid ANP and DEMATEL method for impact and assessment of the risk closure in companies for Saudi Arabian.

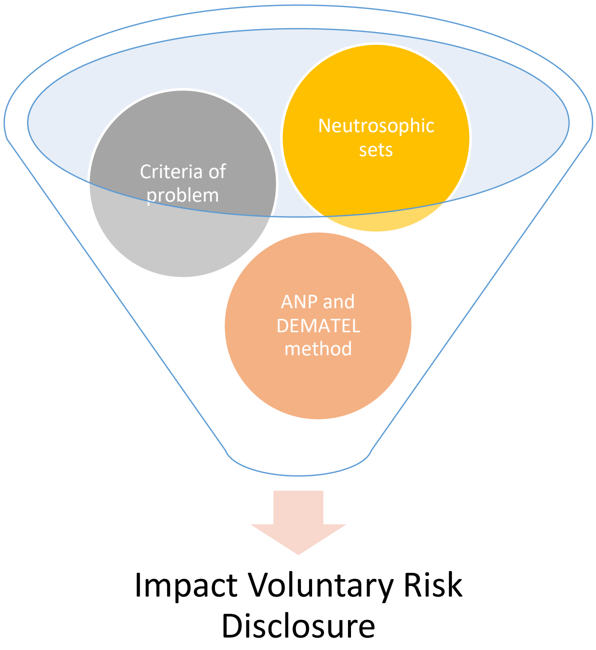

Fig 1. Research Methodology for this paper

3. Methodology

This methodology integrates the ANP and DEMATEL MCDM method under a neutrosophic environment for voluntary risk disclosure. This methodology has two-stage. In the first stage proposed the SVNSs and ANP method. ANP is used to calculate the weights of criteria. The second stage is used to show the impact and assessment of the voluntary risk disclosure. Fig 1. Show the research methodology.

3.1 First Stage ANP Method

ANP is a common MCDM method. It is modified on the AHP method. The main benefits of ANP consider dependency between elements of the problem. ANP is used to calculate the weights of criteria.

Words are described semantic better than numbers. This paper used the SVNSs as a linguistic variable. Table 1. present the single-valued neutrosophic numbers (SVNNs) and linguistic variables[15]. The steps of the ANP method are organized as follows [28]: Fig 2. Show the steps of the ANP method.

Step 1: Select a group of decision-makers and experts

Step 2: Collect the criteria from the review of the literature.

Step 3: Build the structure of the problem.

Step 4: Build the pairwise comparison matrix between criteria by using Eq. (1)

$$P^T = \begin{bmatrix} P_{11}^T & \cdots & P_{1b}^T \\ \vdots & \ddots & \vdots \\ P_{a1}^T & \cdots & P_{ab}^T \end{bmatrix} \quad (1)$$

Where T presents the decision-makers.

Step 5: Obtaining the crisp value.

After building the pairwise comparison matrix need to convert the three values (T,I,F) with one value by applying the score function by using Eq. (2)

$$S(P_{kl}^T) = \frac{2 + T_{kl}^T - I_{kl}^T - F_{kl}^T}{3} \quad (2)$$

Where, $T_{kl}^T - I_{kl}^T - F_{kl}^T$ presents truth, indeterminacy, and falsity of the SVNSs.

Step 6: Combine the pairwise comparison matrix

After obtaining the crisp value (one value) need to combine the opinions of decision-makers into one value by using Eq. (3).

$$P_{kl} = \frac{\sum_{T=1}^{T} P_{kl}}{T} \quad (3)$$

Step 7: Build the combined pairwise comparison matrix.

After combined the opinions of decision-makers build the combined matrix by using Eq. (4)

$$P = \begin{bmatrix} P_{11} & \cdots & P_{1l} \\ \vdots & \ddots & \vdots \\ P_{k1} & \cdots & P_{kl} \end{bmatrix} \quad (4)$$

Step 8: Calculate the weights of criteria

The weights of criteria are computed by computing the eigenvector which will be used in the building of the supermatrix of interdependences.

Step 9: Compute the weights of sub-criteria.

The weights of sub-criteria are computed by multiplying the weights of the interdependences matrix by the weights of local weight which was obtained by comparison matrix of opinions decision-makers.

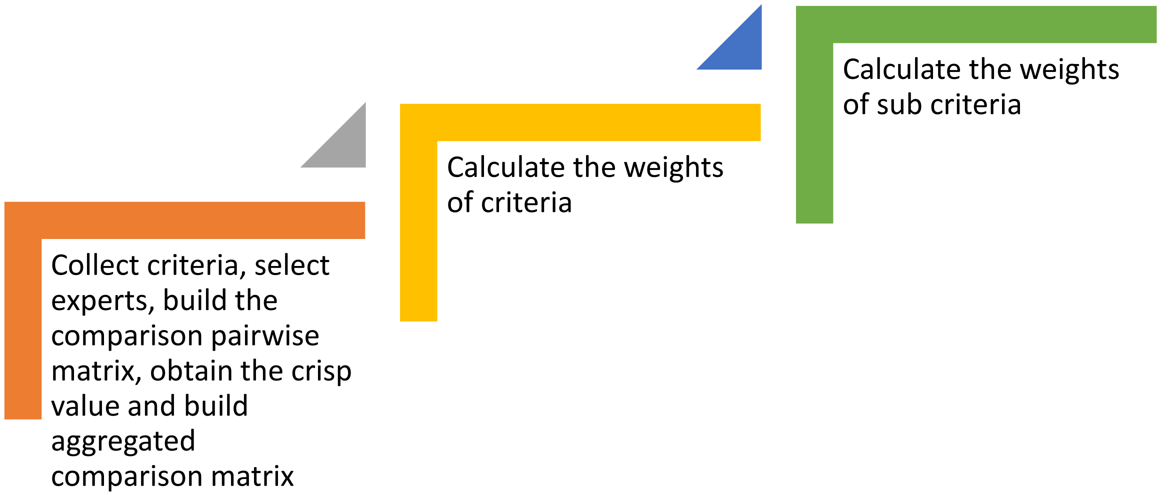

Fig 2. The steps of the ANP method

3.2. The Second Stage DEMATEL Method

The DEMATEL method is used for assessing voluntary risk disclosure and shows the impact of criteria. DEMATEL is an MCDM method. It is used to solve complex problems. The following steps of the DEMATAL method as follows [30]:

Step 10: Build the direct relation matrix.

By using the combined pairwise matrix in step 7 the direct relation matrix is built.

Step 11: Normalize the direct relation matrix.

The normalized direct relation matrix is computed by using Eqs. (5,6)

$$N = \frac{1}{\max_{1 \le a \le b} \sum_{b=1}^{l} P_{kl}} \quad (5)$$

$$M = N \text{ x } P \quad (6)$$

Step 12: Calculate the total relation matrix.

The total relation matrix is computed by using software Matlab to attain the identity matrix by using Eq. (7).

$$R = M(I - M)^{-1} \quad (7)$$

Step 13: Attaining the sum of rows and columns.

The sum of rows and columns is obtained as X and Y respectively. Then calculate X+Y and X-Y

Step 14: Drawing the cause and effect diagram.

The cause diagram presents in Horizontal the value of X+Y and Vertically X-Y.

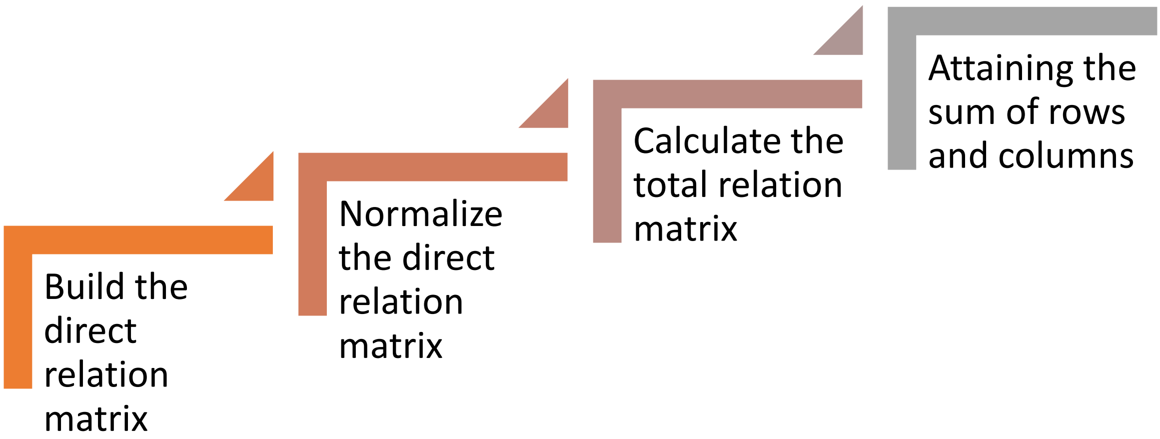

Fig 3. Show the steps of the DEMATEL method

Fig 4. The criteria of this study

4. Case Study

This study is made on companies of Saudi Arabian. The hybrid model was applied to voluntary risk disclosure. The target population of the study was all companies listed on the Saudi Stock Exchange, called Tadawul. Given the aim of this research, the sample included financial and non-financial companies. Financial institutions included banks, whereas non-financial institutions included all other listed companies. Since there are only 11 listed banks, all of them were selected and included 30 in the study. Also, out of 160 listed non-financial companies, 14 non-financial companies were randomly selected.

This study used the three decision-makers and experts to assess the criteria by their opinions. The criteria are determined from the review of the literature. The seven criteria are used in this research. Fig 4. Show the main criteria of this work. The five factors are proposed to show impact of board composition in the (voluntary risk disclosure) VRD. The five factor include: Gender F_1, Independent directors F_2, Board qualification F_3, Audit Committee meetings F_4, Board size F_5.

First by using the linguistic term in Table 1. the pairwise comparison matrix is built by opinions of experts by using Eq. (1). Then replace the linguistic term with SVNNs. Then convert the SVNNs into crisp value by score function by using Eq. (2). Then aggregate the crisp value to obtain one value instead of three values of three matrices by using Eq. (3). Then build the combined pairwise comparison matrix by using Eq. (4) to obtain one matrix. Table 2. Show the combined pairwise comparison matrix from main criteria. Tables 3 to 9 show the interdependency matrix for the main criteria. Table 10 shows the comparative for impact criteria. Fig 5. Show the weights of criteria. The weights of the criteria show that C_7 Operational Risk is the highest risk by the ANP method and C_1 Reputation Risk is the lowest risk. The rank risks in Table 11.

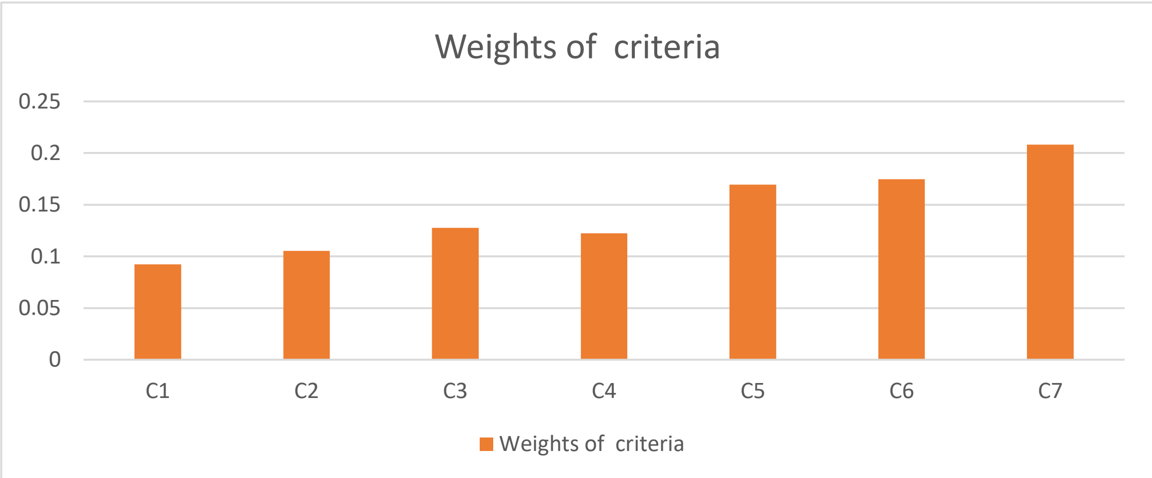

Fig 5. The weights of the criteria

Table 1. SVNSs scale.

Linguistic Term	SVNNs
Very Immoral	<0.25,0.7,0.7>
Immoral	<0.35,0.6,0.6>
Medium	<0.45,0.5,0.45>

Honest	<0.75,0.35,0.25>
Very Honest	<0.85,0.2,0.2>

Table 2. The combined pairwise comparison of criteria and local weight.

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	Local Weight
C_1	0.5	0.783367	0.527767	0.672233	0.750033	0.494433	0.494433	0.087967
C_2	1.281388	0.5	0.6389	0.7167	0.494433	0.672233	0.750033	0.103278
C_3	2.147428	1.742882	0.5	0.8167	0.494433	0.750033	0.3833	0.12385
C_4	1.281388	1.22444	1.22444	0.5	0.750033	0.527767	0.750033	0.121667
C_5	2.608923	2.204376	2.204376	1.338336	0.5	0.672233	0.783367	0.184447
C_6	1.281388	1.338336	1.338336	2.147428	1.685934	0.5	0.783367	0.179213
C_7	2.204376	1.338336	2.608923	1.338336	1.281388	1.281388	0.5	0.199578

Table 3. Interdependency matrix of the criteria related to C_1 Reputation Risk.

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	Local Weight
C_1	0.5	0.8167	0.605567	0.6389	0.750033	0.605567	0.3833	0.085857
C_2	1.22444	0.5	0.672233	0.605567	0.750033	0.494433	0.7167	0.100001
C_3	1.79983	1.685934	0.5	0.783367	0.6389	0.750033	0.672233	0.127074
C_4	2.147428	1.395284	1.281388	0.5	0.672233	0.6389	0.8167	0.139085
C_5	1.338336	1.742882	1.742882	1.685934	0.5	0.672233	0.7167	0.159205
C_6	1.395284	2.204376	1.338336	1.742882	1.685934	0.5	0.3833	0.171669
C_7	2.608923	1.395284	1.685934	1.22444	1.395284	2.608923	0.5	0.217109

Table 4. Interdependency matrix of the criteria related to C_2 Compliance Risk.

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	Local Weight
C_1	0.5	0.6389	0.6389	0.6389	0.6389	0.8167	0.8167	0.097589
C_2	1.742882	0.5	0.6389	0.6389	0.6389	0.750033	0.7167	0.107704
C_3	1.742882	1.742882	0.5	0.8167	0.6389	0.3833	0.527767	0.112032

C_4	1.281388	1.22444	1.22444	0.5	0.3833	0.750033	0.527767	0.10576
C_5	1.79983	1.742882	1.742882	2.608923	0.5	0.672233	0.7167	0.170402
C_6	2.147428	2.608923	2.608923	1.338336	1.685934	0.5	0.3833	0.193487
C_7	1.22444	1.395284	2.147428	2.147428	1.395284	2.608923	0.5	0.213027

Table 5. Interdependency matrix of the criteria related to C_3 Commodity.

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	Local Weight
C_1	0.5	0.7167	0.527767	0.6389	0.527767	0.605567	0.527767	0.084499
C_2	1.395284	0.5	0.750033	0.527767	0.750033	0.783367	0.7167	0.1109
C_3	2.147428	1.338336	0.5	0.8167	0.6389	0.527767	0.527767	0.118158
C_4	1.685934	1.22444	1.22444	0.5	0.494433	0.494433	0.672233	0.11665
C_5	1.79983	1.742882	1.742882	2.204376	0.5	0.8167	0.7167	0.173688
C_6	1.685934	2.147428	2.147428	2.204376	1.22444	0.5	0.8167	0.197791
C_7	2.147428	1.395284	2.147428	1.685934	1.395284	1.22444	0.5	0.198313

Table 6. Interdependency matrix of the criteria related to C_4 Sustainability Risk.

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	Local Weight
C_1	0.5	0.750033	0.672233	0.750033	0.672233	0.605567	0.527767	0.093228
C_2	1.338336	0.5	0.750033	0.6389	0.6389	0.783367	0.7167	0.109671
C_3	1.685934	1.338336	0.5	0.783367	0.6389	0.605567	0.783367	0.122658
C_4	1.685934	1.281388	1.281388	0.5	0.494433	0.494433	0.672233	0.120819
C_5	2.204376	1.742882	1.742882	2.204376	0.5	0.783367	0.7167	0.180828
C_6	1.338336	1.79983	1.79983	2.204376	1.281388	0.5	0.8167	0.18619
C_7	2.147428	1.395284	1.281388	1.685934	1.395284	1.22444	0.5	0.186606

Table 7. Interdependency matrix of the main criteria related to C_5 Technological Risk.

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	Local Weight
C_1	0.5	0.750033	0.6389	0.6389	0.783367	0.527767	0.527767	0.0899

C_2	1.338336	0.5	0.527767	0.6389	0.6389	0.783367	0.7167	0.104511
C_3	1.742882	2.147428	0.5	0.783367	0.6389	0.6389	0.6389	0.130527
C_4	1.685934	1.281388	1.281388	0.5	0.3833	0.605567	0.6389	0.118516
C_5	1.685934	1.742882	1.742882	2.608923	0.5	0.783367	0.7167	0.177868
C_6	1.281388	1.742882	1.742882	1.79983	1.281388	0.5	0.527767	0.165805
C_7	2.147428	1.395284	1.742882	1.742882	1.395284	2.147428	0.5	0.212873

Table 8. Interdependency matrix of the main criteria related to C_6 Strategic Risk.

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	Local Weight
C_1	0.5	0.8167	0.605567	0.6389	0.750033	0.605567	0.672233	0.097709
C_2	1.22444	0.5	0.672233	0.6389	0.750033	0.494433	0.7167	0.103149
C_3	1.79983	1.685934	0.5	0.783367	0.6389	0.750033	0.783367	0.136629
C_4	1.685934	1.281388	1.281388	0.5	0.672233	0.6389	0.527767	0.12808
C_5	1.338336	1.742882	1.742882	1.685934	0.5	0.783367	0.605567	0.162535
C_6	1.338336	1.338336	1.338336	1.742882	1.281388	0.5	0.672233	0.161323
C_7	1.685934	1.395284	1.281388	2.147428	1.79983	1.685934	0.5	0.210576

Table 9. Interdependency matrix of the main criteria related to C_7 Operational Risk.

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	Local Weight
C_1	0.5	0.8167	0.605567	0.6389	0.750033	0.605567	0.527767	0.094102
C_2	1.22444	0.5	0.672233	0.6389	0.750033	0.494433	0.7167	0.103519
C_3	1.79983	1.685934	0.5	0.783367	0.6389	0.750033	0.672233	0.133783
C_4	1.685934	1.281388	1.281388	0.5	0.672233	0.6389	0.494433	0.126593
C_5	1.338336	1.742882	1.742882	1.685934	0.5	0.783367	0.605567	0.162214
C_6	1.338336	1.338336	1.338336	1.742882	1.281388	0.5	0.8167	0.165547
C_7	2.147428	1.395284	1.685934	2.204376	1.79983	1.22444	0.5	0.214242

Table 10. The comparative impact of seven criteria and rank of risks.

Criteria	Weights of criteria	Rank
C_1	0.092312	7
C_2	0.105421	6
C_3	0.127567	4
C_4	0.122383	5
C_5	0.169425	3
C_6	0.174767	2
C_7	0.208124	1

The results of the DEMATEL method discuss as follow. First, build the direct relation matrix in step 7. Then applying Eqs. (5,6) for normalizing the direct relation matrix. Table 11. Present the normalized matrix for the criteria. Then use the Matlab code for obtaining the total relation matrix in Table 12. Then the sum of rows and columns in Table 13. The results of the cause diagram show that C_3 Commodity risk had the greatest impact and Operational risk C_7 had a lesser impact. Fig 6. Show the cause diagram.

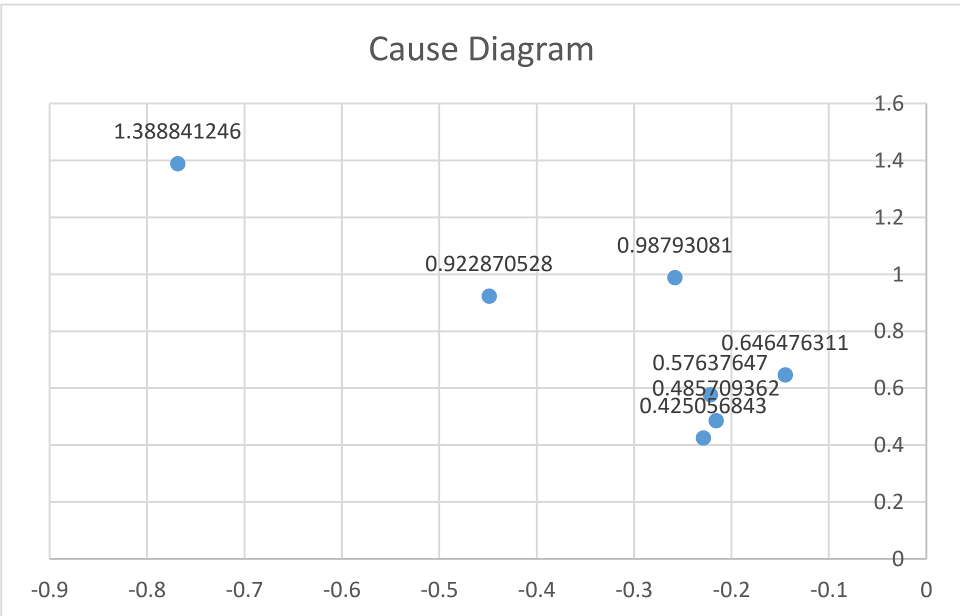

Fig 6. The cause diagram by DEMATEL method.

Table 11. The normalized decision matrix for main criteria.

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7
C_1	0.052476	0.082215	0.05539	0.070552	0.078717	0.051891	0.051891
C_2	0.134483	0.052476	0.067053	0.075218	0.051891	0.070552	0.078717
C_3	0.225375	0.182917	0.052476	0.085714	0.051891	0.078717	0.040228
C_4	0.134483	0.128506	0.128506	0.052476	0.078717	0.05539	0.078717
C_5	0.273809	0.231352	0.231352	0.14046	0.052476	0.070552	0.082215
C_6	0.134483	0.14046	0.14046	0.225375	0.176941	0.052476	0.082215
C_7	0.231352	0.14046	0.273809	0.14046	0.134483	0.134483	0.052476

Table 12. The total relation matrix.

Main Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7
C_1	-0.02384	0.0216	-0.00471	0.022175	0.041601	0.019845	0.021454
C_2	0.053234	-0.01624	-0.00057	0.017843	0.003389	0.032477	0.044792
C_3	0.131494	0.105464	-0.02012	0.016379	-0.00687	0.032037	-0.00757
	0.026911	0.043409	0.051409	-0.01504	0.026146	0.007276	0.037166
C_5	0.114421	0.102734	0.125921	0.04172	-0.03044	-0.00443	0.015003
C_6	-0.02795	0.007524	0.016442	0.133228	0.10441	-0.01425	0.017613
C_7	0.052674	-0.01144	0.154788	0.026163	0.045809	0.059633	-0.01752

Table 13. The sum of rows and columns.

Main Criteria	X	Y	X-Y	X+Y
C_1	0.098121	0.326936	-0.22882	0.425057
C_2	0.13493	0.350779	-0.21585	0.485709
C_3	0.25081	0.395666	-0.14486	0.646476
C_4	0.177274	0.399102	-0.22183	0.576376
C_5	0.36493	0.623001	-0.25807	0.987931
C_6	0.237016	0.685854	-0.44884	0.922871

C_7	0.310103	1.078739	-0.76864	1.388841

The results of board composition show the weights of six factors. The Board size F_5 is the highest weights in the five factors with value 0.245594, then the Audit Committee meetings F_4 with value 0.210658, then Independent directors F_3 = 0.21379, then Board qualification F_2= 0.17068, then the lowest factor is Gender F_1=0.15277. Table 14. Show the decision matrix between criteria and others. Table 15 show the weights and rank of six factors.

With the DEMATEL method show that the F_5 board size has high impact on VRD and the F_1 Gender has lowest impact on VRD. Table 16. Show the sum or rows and columns and rank of impact on VRD. Fig 7. Show the cause diagram of board composition.

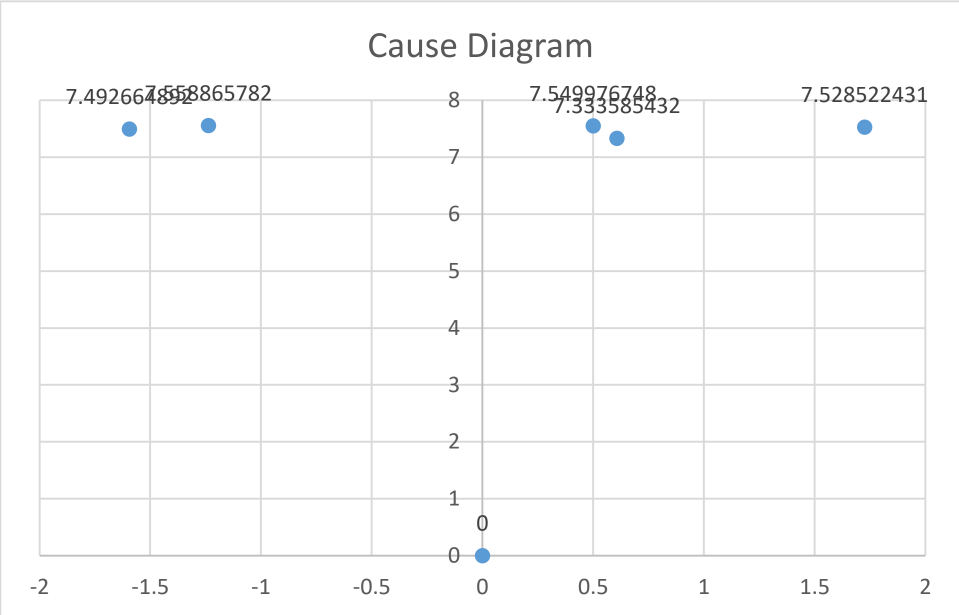

Fig 7. Cause Diagram of Six Factors of board composition.

Table 14. The combined pairwise comparison of six factors.

Criteria	F_1	F_2	F_3	F_4	F_5
F_1	0.5	0.783367	0.750033	0.8167	0.7167
F_2	1.281388	0.5	0.6389	0.7167	0.783367
F_3	1.338336	1.742882	0.5	0.8167	0.750033
F_4	1.281388	1.22444	1.22444	0.5	0.750033
F_5	1.281388	1.338336	1.338336	1.338336	0.5

Table 15. The weights and rank of six factor of board composition on VRD.

Six Factors	Weights of Six Factors	Rank
F_1	0.159277	5
F_2	0.17068	4
F_3	0.21379	2
F_4	0.210658	3
F_5	0.245594	1

Table 16. The sum of rows and columns of board composition for six factor.

Six Factors	X	Y	X-Y	X+Y
F_1	2.949194	4.543471	-1.59428	7.492665
F_2	3.160792	4.398073	-1.23728	7.558866
F_3	4.024743	3.525234	0.499509	7.549977
F_4	3.97023	3.363355	0.606875	7.333585
F_5	4.626848	2.901674	1.725174	7.528522

5. Analysis of VRD

The dependent variable for the study was voluntary risk disclosure. A disclosure index, which is coded as VRD, was developed based on a seven criteria. The regression equation that was used to test the hypothesis was of the form:

$$VRD = \beta_0 + \beta_1 BSIZE + \beta_2 BQUAL + \beta_3 INDEP + \beta_4 ACMEET + \beta_5 GENDER + \beta_6 IFRS + e$$

Where

VRD: Voluntary risk disclosure,

BSIZE: Board size,

BQUAL: Board qualification

INDEP: Independent directors

ACMEET: Audit committee meetings

GENDER: Number of females on the board

IFRS: International Financial Reporting Standards

The outputs of the multiple regression analysis indicate the regression model was statistically significant: F (5, 74) 3.542 and p < 0.05. The R2 was 0.193, which means that 19.3% of the variance in the level of voluntary risk disclosure was explained by the five independent variables. Independent variables has an impact on the voluntary risk disclosure practices by the Saudi listed corporates.

6. Conclusions

This paper study the voluntary risk disclosure in companies of Saudi Arabian. This paper proposes seven criteria. The neutrosophic sets are used to deal with uncertainty. The MCDM method is used in this paper like ANP and DEMATEL methods. ANP is used to calculate the weights of main and sub-criteria. DEMATEL method is used to assess and show the impact of voluntary risk disclosure. The main results show that the Operational risks are the highest impact and reputation risk is the lowest impact. Future work can use other MCDM methods like TOPSIS and VIKOR.

References

[1] Bhasin, M.L., R. Makarov, and N. Orazalin, *Determinants of voluntary disclosure in the banking sector: An empirical study.* International Journal of Contemporary Business Studies, 2012. **3**(3).

[2] Alhazaimeh, A., R. Palaniappan, and M. Almsafir, *The impact of corporate governance and ownership structure on voluntary disclosure in annual reports among listed Jordanian companies.* Procedia-Social and Behavioral Sciences, 2014. **129**: p. 341-348.

[3] Beattie, V., B. McInnes, and S. Fearnley. *A methodology for analysing and evaluating narratives in annual reports: a comprehensive descriptive profile and metrics for disclosure quality attributes*. in *Accounting forum*. 2004. Elsevier.

[4] Brown, P., W. Beekes, and P. Verhoeven, *Corporate governance, accounting and finance: A review.* Accounting & finance, 2011. **51**(1): p. 96-172.

[5] Neifar, S. and A. Jarboui, *Corporate governance and operational risk voluntary disclosure: Evidence from Islamic banks.* Research in International Business and Finance, 2018. **46**: p. 43-54.

[6] Eng, L.L. and Y.T. Mak, *Corporate governance and voluntary disclosure.* Journal of accounting and public policy, 2003. **22**(4): p. 325-345.

[7] Ho, S.S. and K.S. Wong, *A study of the relationship between corporate governance structures and the extent of voluntary disclosure.* Journal of International Accounting, Auditing and Taxation, 2001. **10**(2): p. 139-156.

[8] Chau, G. and S.J. Gray, *Family ownership, board independence and voluntary disclosure: Evidence from Hong Kong.* Journal of International Accounting, Auditing and Taxation, 2010. **19**(2): p. 93-109.

[9] Abraham, S. and P. Cox, *Analysing the determinants of narrative risk information in UK FTSE 100 annual reports.* The British Accounting Review, 2007. **39**(3): p. 227-248.

[10] Al-Maghzom, A., K. Hussainey, and D.A. Aly, *Corporate governance and risk disclosure: Evidence from Saudi Arabia.* Corporate Ownership and Control Journal, 2016. **13**(2).

[11] Hassan, M.K., *UAE corporations-specific characteristics and level of risk disclosure.* Managerial Auditing Journal, 2009.

[12] Moshashai, D., A.M. Leber, and J.D. Savage, *Saudi Arabia plans for its economic future: Vision 2030, the National Transformation Plan and Saudi fiscal reform.* British Journal of Middle Eastern Studies, 2020. **47**(3): p. 381-401.

[13] Nurunnabi, M., *Transformation from an oil-based economy to a knowledge-based economy in Saudi Arabia: the direction of Saudi vision 2030.* Journal of the Knowledge Economy, 2017. **8**(2): p. 536-564.

[14] Chinnadurai, V. and M. Sindhu, *A Novel Approach for Pairwise Separation Axioms on Bi-Soft Topology Using Neutrosophic Sets and An Output Validation in Real Life Application.* Neutrosophic Sets and Systems, 2020. **35**: p. 435-463.

[15] Wang, H., et al., *Single valued neutrosophic sets*. 2010: Infinite study.

[16] Tsaur, S.-H., T.-Y. Chang, and C.-H. Yen, *The evaluation of airline service quality by fuzzy MCDM.* Tourism management, 2002. **23**(2): p. 107-115.

[17] Yang, J.L. and G.-H. Tzeng, *An integrated MCDM technique combined with DEMATEL for a novel cluster-weighted with ANP method.* Expert Systems with Applications, 2011. **38**(3): p. 1417-1424.

[18] Tzeng, G.-H., C.-H. Chiang, and C.-W. Li, *Evaluating intertwined effects in e-learning programs: A novel hybrid MCDM model based on factor analysis and DEMATEL.* Expert systems with Applications, 2007. **32**(4): p. 1028-1044.

[19] Elshandidy, T., et al., *Risk reporting: A review of the literature and implications for future research.* Journal of Accounting Literature, 2018. **40**: p. 54-82.

[20] Elshandidy, T. and L. Neri, *Corporate governance, risk disclosure practices, and market liquidity: Comparative evidence from the UK and I taly.* Corporate Governance: An International Review, 2015. **23**(4): p. 331-356.

[21] Al-Janadi, Y., R.A. Rahman, and N.H. Omar, *The level of voluntary disclosure practices among public listed companies in Saudi Arabia and the UAE: Using a modified voluntary disclosure index.* International Journal of disclosure and Governance, 2012. **9**(2): p. 181-201.

[22] Al-Maghzom, A., *The determinants and consequences of risk disclosure in Saudi banks*. 2016, University of Gloucestershire.

[23] Habbash, M., K. Hussainey, and A.E. Awad, *The determinants of voluntary disclosure in Saudi Arabia: an empirical study.* International Journal of Accounting, Auditing and Performance Evaluation, 2016. **12**(3): p. 213-236.

[24] Karabašević, D., et al., *A Novel Extension of the TOPSIS Method Adapted for the Use of Single-Valued Neutrosophic Sets and Hamming Distance for E-Commerce Development Strategies Selection.* Symmetry, 2020. **12**(8): p. 1263.

[25] Dung, V., et al., *TOPSIS approach using interval neutrosophic sets for personnel selection*. 2018: Infinite Study.

[26] Broumi, S., et al. *Shortest path problem on single valued neutrosophic graphs*. in *2017 international symposium on networks, computers and communications (ISNCC)*. 2017. IEEE.

[27] Sun, R., J. Hu, and X. Chen, *Novel single-valued neutrosophic decision-making approaches based on prospect theory and their applications in physician selection.* Soft Computing, 2019. **23**(1): p. 211-225.

[28] Abdel-Baset, M., et al., *An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field.* Computers in Industry, 2019. **106**: p. 94-110.

[29] Han, Y. and Y. Deng, *An enhanced fuzzy evidential DEMATEL method with its application to identify critical success factors.* Soft computing, 2018. **22**(15): p. 5073-5090.

[30] Abdel-Basset, M., et al., *A hybrid approach of neutrosophic sets and DEMATEL method for developing supplier selection criteria.* Design Automation for Embedded Systems, 2018. **22**(3): p. 257-278.

[31] Mao, S., et al., *A hybrid DEMATEL-FRACTAL method of handling dependent evidences.* Engineering Applications of Artificial Intelligence, 2020. **91**: p. 103543.

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Properties of Productional NeutroOrderedSemigroups

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Abstract. The introducing of NeutroAlgebra by Smarandache opened the door for researchers to define many related new concepts. NeutroOrderedAlgebra was one of these new related definitions. The aim of this paper is to study productional NeutroOrderedSemigroup. In this regard, we firstly present many examples and study subsets of productional NeutroOrderedSemigroups. Then, we find sufficient conditions for the productional NeutroSemigroup to be a NeutroOrderedSemigroup. Finally, we find sufficient conditions for subsets of the productional NeutroOrderedSemigroup to be NeutroOrderedSubSemigroups, NeutroOrderedIdeals, and NeutroOrderedFilters.

Keywords: NeutroSemigroup, NeutrosOrderedSemigroup, NeutroOrderedIdeal, NeutroOrderedFilter, Productional NeutroOrderedSemigroup.

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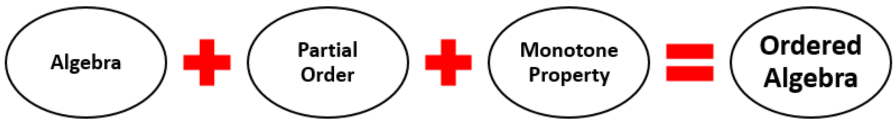
1. **Introduction**

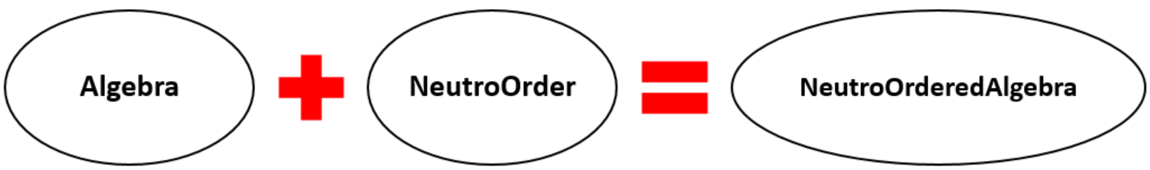
Smarandache [1–3] introduced NeutroAlgebra as a generalization of the known Algebra. It is known that in an Algebra, operations are well defined and axioms are always true whereas for NeutroAlgebra, operations and axioms are partially true, partially indeterminate, and partially false. The latter is considered as an extension of Partial Algebra where operations and axioms are partially true and partially false. Many researchers worked on special types of NeutroAlgebras by applying them to different types of algebraic structures such as semigroups, groups, rings, BE-Algebras, CI-Algebras, BCK-Algebras, etc. For more details about NeutroStructures, the reader may see [4–8]. l order on it that satisfies the monotone property, we get an Ordered Algebra (as illustrated in Figure 1). And starting with a partial order on a

Figure 1. Ordered Algebra



NeutroAlgebra, we get a NeutroStructure. The latter if it satisfies the conditions of **NeutroOrder**, it becomes a NeutroOrderedAlgebra (as illustrated in Figure 2). In [9], the authors

Figure 2. NeutroOrderedAlgebra



defined NeutroOrderedAlgebra and applied it to semigroups by studying NeutroOrderedSemigroups and their subsets such as NeutrosOrderedSubSemigroups, NeutroOrderedIdeals, and NeutroOrderedFilters.

Our paper is concerned about Cartesian product of NeutroOrderedSemigroups and the remainder part of it is as follows: In Section 2, we present some definitions and examples related to NeutroOrderedSemigroups. In Section 3, we define productional NeutroOrderedSemigroup and find sufficient conditions for the Cartesian product of NeutroSemigroups and semigroups to be NeutroOrderedSemigroups. Finally in Section 4, we find sufficient conditions for subsets of the productional NeutroOrderedSemigroup to be NeutroOrderedSubSemigroups, NeutroOrderedIdeals, and NeutroOrderedFilters.

2. NeutroOrderedSemigroups

In this section, we present some definitions and examples about NeutroOrderedSemigroups, introduced and studied by the authors in [9], that are used throughout the paper.

Definition 2.1. [10] Let $(S, \cdot)$ be a semigroup ("$\cdot$" is an associative and a binary closed operation) and "$\leq$" a partial order on S. Then $(S, \cdot, \leq)$ is an *ordered semigroup* if for every $x \leq y \in S$, $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all $z \in S$.

Definition 2.2. [10] Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq M \subseteq S$. Then

(1) M is an *ordered subsemigroup* of S if $(M, \cdot, \leq)$ is an ordered semigroup and $(x] \subseteq M$ for all $x \in M$. i.e., if $y \leq x$ then $y \in M$.
(2) M is an *ordered left ideal* of S if M is an ordered subsemigroup of S and for all $x \in M$, $r \in S$, we have $rx \in M$.
(3) M is an *ordered right ideal* of S if M is an ordered subsemigroup of S and for all $x \in M$, $r \in S$, we have $xr \in M$.
(4) M is an *ordered ideal* of S if M is both: an ordered left ideal of S and an ordered right ideal of S.
(5) M is an *ordered filter* of S if $(M, \cdot)$ is a semigroup and for all $x, y \in S$ with $x \cdot y \in M$, we have $x, y \in M$ and $[y) \subseteq M$ for all $y \in M$. i.e., if $y \in M$ with $y \leq x$ then $x \in M$.

For more details about semigroup theory and ordered algebraic structures, we refer to [10, 11].

Definition 2.3. [2] Let A be any non-empty set and "$\cdot$" be an operation on A. Then "$\cdot$" is called a *NeutroOperation* on A if the following conditions hold.

(1) There exist $x, y \in A$ with $x \cdot y \in A$. (This condition is called degree of truth, "T".)
(2) There exist $x, y \in A$ with $x \cdot y \notin A$. (This condition is called degree of falsity, "F".)
(3) There exist $x, y \in A$ with $x \cdot y$ is indeterminate in A. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical binary closed operation, and from $(0, 0, 1)$ that represents the AntiOperation.

Definition 2.4. [2] Let A be any non-empty set and "$\cdot$" be an operation on A. Then "$\cdot$" is called a *NeutroAssociative* on A if there exist $x, y, z, a, b, c, e, f, g \in A$ satisfying the following conditions.

(1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$; (This condition is called degree of truth, "T".)
(2) $a \cdot (b \cdot c) \neq (a \cdot b) \cdot c$; (This condition is called degree of falsity, "F".)
(3) $e \cdot (f \cdot g)$ is indeterminate or $(e \cdot f) \cdot g$ is indeterminate or we can not find if $e \cdot (f \cdot g)$ and $(e \cdot f) \cdot g$ are equal. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical associative axiom, and from $(0, 0, 1)$ that represents the AntiAssociativeAxiom.

Definition 2.5. [2] Let A be any non-empty set and "$\cdot$" be an operation on A. Then $(A, \cdot)$ is called a *NeutroSemigroup* if "$\cdot$" is either a NeutroOperation or NeutroAssociative.

Definition 2.6. [9] Let $(S, \cdot)$ be a NeutroSemigroup and "$\leq$" be a partial order (reflexive, anti-symmetric, and transitive) on S. Then $(S, \cdot, \leq)$ is a *NeutroOrderedSemigroup* if the following conditions hold.

(1) There exist $x \leq y \in S$ with $x \neq y$ such that $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all $z \in S$. (This condition is called degree of truth, "T".)

(2) There exist $x \leq y \in S$ and $z \in S$ such that $z \cdot x \nleq z \cdot y$ or $x \cdot z \nleq y \cdot z$. (This condition is called degree of falsity, "F".)

(3) There exist $x \leq y \in S$ and $z \in S$ such that $z \cdot x$ or $z \cdot y$ or $x \cdot z$ or $y \cdot z$ are indeterminate, or the relation between $z \cdot x$ and $z \cdot y$, or the relation between $x \cdot z$ and $y \cdot z$ are indeterminate. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical Ordered Semigroup, and from $(0, 0, 1)$ that represents the AntiOrderedSemigroup.

Definition 2.7. [9] Let $(S, \cdot, \leq)$ be a NeutroOrderedSemigroup . If "$\leq$" is a total order on A then A is called *NeutroTotalOrderedSemigroup*.

Example 2.8. [9] Let $S_1 = \{s, a, m\}$ and $(S_1, \cdot_1)$ be defined by the following table.

$\cdot_1$	s	a	m
s	s	m	s
a	m	a	m
m	m	m	m

By defining the total order

$$\leq_1 = \{(m, m), (m, s), (m, a), (s, s), (s, a), (a, a)\}$$

on S_1, we get that $(S_1, \cdot_1, \leq_1)$ is a NeutroTotalOrderedSemigroup.

Example 2.9. Let $S_2 = \{0, 1, 2, 3\}$ and $(S_2, \cdot_2')$ be defined by the following table.

$\cdot_2'$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	3	2
3	0	1	3	2

By defining the partial order

$$\leq_2' = \{(0, 0), (0, 1), (0, 2), (1, 1), (2, 2), (3, 3)\}$$

on S_2, we get that $(S_2, \cdot_2', \leq_2')$ is a NeutroOrderedSemigroup.

Example 2.10. [9] Let $S_3 = \{0, 1, 2, 3, 4\}$ and $(S_3, \cdot_3)$ be defined by the following table.

$\cdot_3$	0	1	2	3	4
0	0	0	0	3	0
1	0	1	2	1	1
2	0	4	2	3	3
3	0	4	2	3	3
4	0	0	0	4	0

By defining the partial order

$$\leq_3 = \{(0, 0), (0, 1), (0, 3), (0, 4), (1, 1), (1, 3), (1, 4), (2, 2), (3, 3), (3, 4), (4, 4)\}$$

on S_3, we get that $(S_3, \cdot_3, \leq_3)$ is a NeutroOrderedSemigroup.

Example 2.11. Let $\mathbb{Z}$ be the set of integers and define "$\star$" on $\mathbb{Z}$ as follows: $x \star y = xy - 2$ for all $x, y \in \mathbb{Z}$. We define the partial order "$\leq_\star$" on $\mathbb{Z}$ as $-2 \leq_\star x$ for all $x \in \mathbb{Z}$ and for $a, b \geq -2$, $a \leq_\star b$ is equivalent to $a \leq b$ and for $a, b < -2$, $a \leq_\star b$ is equivalent to $a \geq b$. In this way, we get $-2 \leq_\star -1 \leq_\star 0 \leq_\star 1 \leq_\star \ldots$ and $-2 \leq_\star -3 \leq_\star -4 \leq_\star \ldots$. Then $(\mathbb{Z}, \star, \leq_\star)$ is a NeutroOrderedSemigroup.

Definition 2.12. [9] Let $(S, \cdot, \leq)$ be a NeutroOrderedSemigroup and $\emptyset \neq M \subseteq S$. Then

(1) M is a *NeutroOrderedSubSemigroup* of S if $(M, \cdot, \leq)$ is a NeutroOrderedSemigroup and there exist $x \in M$ with $(x] = \{y \in S : y \leq x\} \subseteq M$.

(2) M is a *NeutroOrderedLeftIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $r \cdot x \in M$ for all $r \in S$.

(3) M is a *NeutroOrderedRightIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $x \cdot r \in M$ for all $r \in S$

(4) M is a *NeutroOrderedIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $r \cdot x \in M$ and $x \cdot r \in M$ for all $r \in S$.

(5) M is a *NeutroOrderedFilter* of S if $(M, \cdot, \leq)$ is a NeutroOrderedSemigroup and there exists $x \in S$ such that for all $y, z \in S$ with $x \cdot y \in M$ and $z \cdot x \in M$, we have $y, z \in M$ and there exists $y \in M$ $[y) = \{x \in S : y \leq x\} \subseteq M$.

Definition 2.13. [9] Let $(A, \star, \leq_A)$ and $(B, \circledast, \leq_B)$ be NeutroOrderedSemigroups and $\phi : A \to B$ be a function. Then

(1) ϕ is called *NeutroOrderedHomomorphism* if $\phi(x \star y) = \phi(x) \circledast \phi(y)$ for some $x, y \in A$ and there exist $a \leq_A b \in A$ with $a \neq b$ such that $\phi(a) \leq_B \phi(b)$.

(2) ϕ is called *NeutroOrderedIsomomorphism* if ϕ is a bijective NeutroOrderedHomomorphism.

(3) ϕ is called *NeutroOrderedStrongHomomorphism* if $\phi(x \star y) = \phi(x) \circledast \phi(y)$ for all $x, y \in A$ and $a \leq_A b \in A$ is equivalent to $\phi(a) \leq_B \phi(b) \in B$.

(4) ϕ is called *NeutroOrderedStrongIsomomorphism* if ϕ is a bijective NeutroOrdered-StrongHomomorphism.

Example 2.14. Let $(S_3, \cdot_3, \leq_3)$ be the NeutroOrderedSemigroup presented in Example 2.10. Then $I = \{0, 1, 2\}$ is both: a NeutroOrderedLefttIdeal and a NeutroOrderedRightIdeal of S_3.

Example 2.15. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NeutroOrderedSemigroup presented in Example 2.11. Then $I = \{-2, -1, 0, 1, -2, -3, -4, \ldots\}$ is a NeutroOrderedIdeal of $\mathbb{Z}$.

Example 2.16. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NeutroOrderedSemigroup presented in Example 2.11. Then $F = \{-2, -1, 0, 1, 2, 3, 4, \ldots\}$ is a NeutroOrderedFilter of $\mathbb{Z}$.

3. Productional NeutroOrderedSemigroups

Let $(A_\alpha, \leq_\alpha)$ be a partial ordered set for all $\alpha \in \Gamma$. We define "$\leq$" on $\prod_{\alpha \in \Gamma} A_\alpha$ as follows: For all $(x_\alpha), (y_\alpha) \in \prod_{\alpha \in \Gamma} A_\alpha$,

$$(x_\alpha) \leq (y_\alpha) \Longleftrightarrow x_\alpha \leq_\alpha y_\alpha \text{ for all } \alpha \in \Gamma.$$

One can easily see that $(\prod_{\alpha \in \Gamma} A_\alpha, \leq)$ is a partial ordered set.

Let A_α be any non-empty set for all $\alpha \in \Gamma$ and "$\cdot_\alpha$" be an operation on A_α. We define "$\cdot$" on $\prod_{\alpha \in \Gamma} A_\alpha$ as follows: For all $(x_\alpha), (y_\alpha) \in \prod_{\alpha \in \Gamma} A_\alpha$, $(x_\alpha) \cdot (y_\alpha) = (x_\alpha \cdot_\alpha y_\alpha)$.

Throughout the paper, we write NOS instead of NeutroOrderedSemigroup.

Theorem 3.1. *Let $(G_1, \leq_1), (G_2, \leq_2)$ be partially ordered sets with operations $\cdot_1, \cdot_2$ respectively. Then $(G_1 \times G_2, \cdot, \leq)$ is an NOS if one of the following statements is true.*

(1) *G_1 and G_2 are NeutroSemigroups with at least one of them is an NOS.*

(2) *One of G_1, G_2 is an NOS and the other is a semigroup.*

Proof. Without loss of generality, let G_1 be an NOS. We prove 1. and 2. is done similarly. We have three cases for "$\cdot_1$" and "$\cdot_2$": Case "$\cdot_1$" is a NeutroOperation, Case "$\cdot_2$" is a NeutroOperation, and Case "$\cdot_1$" and "$\cdot_2$" are NeutroAssociative.

Case "$\cdot_1$" is a NeutroOperation. There exist $x_1, y_1, a_1, b_1 \in G_1$ such that $x_1 \cdot_1 y_1 \in G_1$ and $a_1 \cdot_1 b_1 \notin G_1$ or $x_1 \cdot_1 y_1$ is indeterminate in G_1. Since G_2 is a NeutroSemigroup, it follows that there exist $x_2, y_2 \in G_2 \neq \emptyset$ such that $x_2 \cdot_2 y_2 \in G_2$ or $x_2 \cdot_2 y_2$ is indeterminate in G_2 (If no such elements exist then G_2 will be an AntiSemigroup.). Then $(x_1, x_2) \cdot (y_1, y_2) \in G_1 \times G_2$ and $(a_1, x_2) \cdot (b_1, y_2) \notin G_1 \times G_2$ or $(x_1, x_2) \cdot (y_1, y_2)$ is indeterminate in $G_1 \times G_2$. Thus "$\cdot$" is a NeutroOperation.

Case "$\cdot_2$" is a NeutroOperation. This case can be done in a similar way to Case "$\cdot_1$" is a

NeutroOperation.

Case "$\cdot_1$" and "$\cdot_2$" are NeutroAssociative. There exist $x_1, y_1, z_1, a_1, b_1, c_1 \in G_1$ and $x_2, y_2, z_2, a_2, b_2, c_2 \in G_2$ such that

$$x_1 \cdot_1 (y_1 \cdot_1 z_1) = (x_1 \cdot_1 y_1) \cdot_1 z_1 \ , \ a_1 \cdot_1 (b_1 \cdot_1 c_1) \neq (a_1 \cdot_1 b_1) \cdot_1 c_1,$$

$$x_2 \cdot_2 (y_2 \cdot_2 z_2) = (x_2 \cdot_2 y_2) \cdot_2 z_2 \ , \text{ and } a_2 \cdot_2 (b_2 \cdot_2 c_2) \neq (a_2 \cdot_2 b_2) \cdot_2 c_2.$$

The latter implies that

$$(x_1, x_2) \cdot ((y_1, y_2) \cdot (z_1, z_2)) = ((x_1, x_2) \cdot (y_1, y_2)) \cdot (z_1, z_2)$$

and

$$(a_1, a_2) \cdot ((b_1, b_2) \cdot (c_1, c_2)) = ((a_1, a_2) \cdot (b_1, b_2)) \cdot (c_1, c_2).$$

Thus, "$\cdot$" is NeutroAssociative.

Having "$\leq_1$" a NeutroOrder on G_1 implies that:

(1) There exist $x \leq_1 y \in G_1$ with $x \neq y$ such that $z \cdot_1 x \leq_1 z \cdot_1 y$ and $x \cdot_1 z \leq_1 y \cdot_1 z$ for all $z \in G_1$.

(2) There exist $x \leq_1 y \in G_1$ and $z \in G_1$ such that $z \cdot_1 x \nleq z \cdot_1 y$ or $x \cdot_1 z \nleq y \cdot_1 z$.

(3) There exist $x \leq_1 y \in G_1$ and $z \in G_1$ such that $z \cdot_1 x$ or $z \cdot_1 y$ or $x \cdot_1 z$ or $y \cdot_1 z$ are indeterminate, or the relation between $z \cdot_1 x$ and $z \cdot_1 y$, or the relation between $x \cdot_1 z$ and $y \cdot_1 z$ are indeterminate.

Where (T, I, F) is different from $(1, 0, 0)$ and from $(0, 0, 1)$.

Having $b \leq_2 b$ for all $b \in G_2$ implies that:

By (1), we get that there exist $(x, b) \leq (y, b) \in G_1 \times G_2$ with $(x, b) \neq (y, b)$. For all $(z, a) \in G_1 \times G_2$, we have either $a \cdot_2 b \in G_2$ or $a \cdot_2 b \notin G_2$ or $a \cdot_2 b$ is indeterminate in G_2. Similarly for $b \cdot_2 a$. The latter implies that $(z, a) \cdot (x, b) \leq (z, a) \cdot (y, b)$ and $(x, b) \cdot (z, a) \leq (y, b) \cdot (z, a)$ or $(z, a) \cdot (x, b) \leq (z, a) \cdot (y, b)$ is indeterminate in $G_1 \times G_2$ or $(x, b) \cdot (z, a) \leq (y, b) \cdot (z, a)$ is indeterminate in $G_1 \times G_2$.

By (2), we get that there exist $(x, b) \leq (y, b) \in G_1 \times G_2$ and $(z, a) \in G_1 \times G_2$ such that $(z, a) \cdot (x, b) \nleq (z, a) \cdot (y, b)$ or $(x, b) \cdot (z, a) \nleq (y, b) \cdot (z, a)$ or $(z, a) \cdot (x, b) \leq (z, a) \cdot (y, b)$ is indeterminate in $G_1 \times G_2$ or $(x, b) \cdot (z, a) \leq (y, b) \cdot (z, a)$ is indeterminate in $G_1 \times G_2$.

By (3), we get that there exist $(x, b) \leq (y, b) \in G_1 \times G_2$ and $(z, a) \in G_1 \times G_2$ such that $(z, a) \cdot (x, b) \leq (z, a) \cdot (y, b)$ is indeterminate in $G_1 \times G_2$ or $(x, b) \cdot (z, a) \leq (y, b) \cdot (z, a)$ is indeterminate in $G_1 \times G_2$ or $(z, a) \cdot (x, b)$ is indeterminate in $G_1 \times G_2$ or $(x, b) \cdot (z, a)$ is indeterminate in $G_1 \times G_2$. Therefore, $(G_1 \times G_2, \cdot, \leq)$ is an NOS. □

Theorem 3.1 implies that $G_1 \times G_2$ is an NOS if either G_1, G_2 are both NOS, G_1 is an NOS and G_2 is a NeutroSemigroup, G_1 is an NOS and G_2 is a semigroup (or odered semigroup),

G_1 is a NeutroSemigroup and G_2 is an NOS, or G_1 is a semigroup (or ordered semigroup) and G_2 is an NOS.

We present a generalization of Theorem 3.1.

Theorem 3.2. *Let $(G_\alpha, \leq_\alpha)$ be a partially ordered set with operation "$\cdot_\alpha$" for all $\alpha \in \Gamma$. Then $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq)$ is an NOS if there exist $\alpha_0 \in \Gamma$ such that $(G_{\alpha_0}, \cdot_{\alpha_0}, \leq_{\alpha_0})$ is an NOS and $(G_\alpha, \cdot_\alpha)$ is a semigroup or NeutroSemigroup for all $\alpha \in \Gamma - \{\alpha_0\}$.*

Notation 1. *Let $(G_\alpha, \leq_\alpha)$ be a partially ordered set with operation "$\cdot_\alpha$" for all $\alpha \in \Gamma$. If $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq)$ is an NOS then we call it the **productional NOS**.*

Proposition 3.3. *Let $(G_1, \cdot_1, \leq_1)$ and $(G_2, \cdot_2, \leq_2)$ be NeutroTotalOrderedSemigroups with $|G_1|, |G_2| \geq 2$. Then $(G_1 \times G_2, \cdot, \leq)$ is not a NeutroTotalOrderedSemigroup.*

Proof. Since $(G_1, \cdot_1, \leq_1)$ and $(G_2, \cdot_2, \leq_2)$ are NeutroTotalOrderedSemigroups with $|G_1| \geq 2$ and $|G_2| \geq 2$, it follows that there exist $a \leq_1 b \in G_1$, $c \leq_2 d \in G_2$ with $a \neq b$ and $c \neq d$. One can easily see that $(a, d) \nleq (b, c) \in G_1 \times G_2$ and $(b, c) \nleq (a, d) \in G_1 \times G_2$. Therefore, $(G_1 \times G_2, \cdot, \leq)$ is not a NeutroTotalOrderedSemigroup. □

Corollary 3.4. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be NeutroTotalOrderedSemigroups for all $\alpha \in \Gamma$ with $|G_{\alpha_0}|, |G_{\alpha_1}| \geq 2$ for $\alpha_0 \neq \alpha_1 \in \Gamma$. Then $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq)$ is not a NeutroTotalOrderedSemigroup.*

Proof. The proof follows from Proposition 3.3. □

Example 3.5. Let $S_1 = \{s, a, m\}$, $(S_1, \cdot_1, \leq_1)$ be the NOS presented in Example 2.8, and "$\leq_1'$" be the trivial order on S_1. Theorem 3.1 asserts that Cartesian product $(S_1 \times S_1, \cdot, \leq)$ resulting from $(S_1, \cdot_1, \leq_1)$ and $(S_1, \cdot_1, \leq_1')$ is an NOS of order 9.

Example 3.6. Let $S_1 = \{s, a, m\}$, $(S_1, \cdot_1, \leq_1)$ be the NOS presented in Example 2.8, and $(\mathbb{R}, \cdot_s, \leq_u)$ be the semigroup of real numbers under standard multiplication and usual order. Theorem 3.1 asserts that Cartesian product $(\mathbb{R} \times S_1, \cdot, \leq)$ is an NOS of infinite order.

Example 3.7. Let $S_1 = \{s, a, m\}$ and $(S_1, \cdot_1, \leq_1)$ be the NOS presented in Example 2.8. Theorem 3.2 asserts that $(S_1 \times S_1 \times S_1, \cdot, \leq)$ is an NOS of order 27. Moreover, by means of Proposition 3.3, $(S_1 \times S_1 \times S_1, \cdot, \leq)$ is not a NeutroTotalOrderedSemigroup.

Example 3.8. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NOS presented in Example 2.11 and $(\mathbb{Z}_n, \odot, \leq_t)$ be the semigroup under standard multiplication of integers modulo n and "$\leq_t$" is defined as follows. For all $\overline{x}, \overline{y} \in \mathbb{Z}_n$ with $0 \leq x, y \leq n-1$,

$$\overline{x} \leq_t \overline{y} \Longleftrightarrow x \leq y \in \mathbb{Z}.$$

Then $(\mathbb{Z}_n \times \mathbb{Z}, \cdot, \leq)$ is an NOS.

Proposition 3.9. *Let $(G_\alpha, \leq_\alpha)$ be a partially ordered set with operation "$\cdot_\alpha$" for all $\alpha \in \Gamma$ and $(G_{\alpha_0}, \cdot_{\alpha_0}, \leq_{\alpha_0})$ be an NOS for some $\alpha_0 \in \Gamma$. Then $\phi : (\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq) \to G_{\alpha_0}$ with $\phi((x_\alpha)) = x_{\alpha_0}$ is a NeutroOrderedHomomorphism.*

Proof. The proof is straightforward. □

Remark 3.10. If $|\Gamma| \geq 2$ and there exist $\alpha \neq \alpha_0 \in \Gamma$ with $|G_\alpha| \geq 2$ then the NeutroOrdered-Homomorphism ϕ in Proposition 3.9 is not a NeutroOrderedIsomorphism.

Remark 3.11. If $|\Gamma| \geq 2$ and there exist $\alpha \neq \alpha_0 \in \Gamma$ with $|G_\alpha| \geq 2$ then $G_{\alpha_0} \ncong_s \prod_{\alpha\in\Gamma} G_\alpha$. This is clear as there exist no bijective function from G_{α_0} to $\prod_{\alpha\in\Gamma} G_\alpha$.

Proposition 3.12. *There are infinite non-isomorphic NOS.*

Proof. Let $(G, \cdot_G, \leq_G)$ be an NOS with $|G| \geq 2$, $\Gamma \subseteq \mathbb{R}$, and $|\Gamma| \geq 2$. Theorem 3.2 asserts that $(\prod_{\alpha\in\Gamma} G, \cdot, \leq)$ is an NOS for every $\Gamma \subseteq \mathbb{R}$. For all $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}$ with $|\Gamma_1| \neq |\Gamma_2|$, Remark 3.11 asserts that $\prod_{\alpha\in\Gamma_1} G \ncong_s \prod_{\alpha\in\Gamma_2} G$. Therefore, there are infinite non-isomorphic NOS. □

Example 3.13. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NOS presented in Example 2.11. Then for every $n \in \mathbb{N}$, we have $(\prod_{i=1}^{n} \mathbb{Z}, \cdot, \leq)$ is an NOS. Moreover, we have infinite such non-isomorphic NOS.

Theorem 3.14. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ and $(G'_\alpha, \cdot'_\alpha, \leq'_\alpha)$ be NOS for all $\alpha \in \Gamma$. Then the following statements hold.*

(1) *If there is a NeutroOrderedHomomorphism from G_α to G'_α for all $\alpha \in \Gamma$ then there is a NeutroOrderedHomomorphism from $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq)$ to $(\prod_{\alpha\in\Gamma} G'_\alpha, \cdot', \leq')$.*
(2) *If there is a NeutroOrderedStrongHomomorphism from G_α to G'_α for all $\alpha \in \Gamma$ then there is a NeutroOrderedStrongHomomorphism from $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq)$ to $(\prod_{\alpha\in\Gamma} G'_\alpha, \cdot', \leq')$.*
(3) *If $G_\alpha \cong G'_\alpha$ for all $\alpha \in \Gamma$ then $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq) \cong (\prod_{\alpha\in\Gamma} G'_\alpha, \cdot', \leq')$.*
(4) *If $G_\alpha \cong_s G'_\alpha$ for all $\alpha \in \Gamma$ then $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq) \cong_s (\prod_{\alpha\in\Gamma} G'_\alpha, \cdot', \leq')$.*

Proof. We prove 1. and the proof of 2., 3., and 4. are done similarly. Let $\phi_\alpha : G_\alpha \to G'_\alpha$ be a NeutroOrderedHomomorphism and define $\phi : \prod_{\alpha\in\Gamma} G_\alpha \to \prod_{\alpha\in\Gamma} G'_\alpha$ as follows: For all $(x_\alpha) \in \prod_{\alpha\in\Gamma} G_\alpha$,

$$\phi((x_\alpha)) = (\phi_\alpha(x_\alpha)).$$

one can easily see that ϕ is a NeutroOrderedHomomorphism. □

4. Subsets of productional NeutroOrderedSemigroups

In this section, we find some sufficient conditions for subsets of the productional NOS to be NeutroOrderedSubSemigroups, NeutroOrderedIdeals, and NeutroOrderedFilters. Moreover, we present some related examples.

Proposition 4.1. *Let $(A_\alpha, \leq_\alpha)$ be a partial ordered set for all $\alpha \in \Gamma$ and $(x_\alpha) \in \prod_{\alpha\in\Gamma} A_\alpha$. Then $((x_\alpha)] = \prod_{\alpha\in\Gamma}(x_\alpha]$.*

Proof. Let $(y_\alpha) \in ((x_\alpha)]$. Then $(y_\alpha) \leq (x_\alpha)$. The latter implies that $y_\alpha \leq_\alpha x_\alpha$ for all $\alpha \in \Gamma$ and hence, $y_\alpha \in (x_\alpha]$ for all $\alpha \in \Gamma$. We get now that $(y_\alpha) \in \prod_{\alpha\in\Gamma}(x_\alpha]$. Thus, $((x_\alpha)] \subseteq \prod_{\alpha\in\Gamma}(x_\alpha]$. Similarly, we can prove that $\prod_{\alpha\in\Gamma}(x_\alpha] \subseteq ((x_\alpha)]$. □

Proposition 4.2. *Let $(A_\alpha, \leq_\alpha)$ be a partial ordered set for all $\alpha \in \Gamma$ and $(x_\alpha) \in \prod_{\alpha\in\Gamma} A_\alpha$. Then $[(x_\alpha)) = \prod_{\alpha\in\Gamma}[x_\alpha)$.*

Proof. The proof is similar to that of Proposition 4.1. □

Theorem 4.3. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If S_α is a NeutroOrderedSubSemigroup of G_α for all $\alpha \in \Gamma$ then $\prod_{\alpha\in\Gamma} S_\alpha$ is a NeutroOrderedSubSemigroup of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. For all $\alpha \in \Gamma$, we have S_α an NOS (as it is NeutroOrderedSubSemigroup of G_α). Theorem 3.2 asserts that $\prod_{\alpha\in\Gamma} S_\alpha$ is an NOS. Since S_α is a NeutroOrderedSubSemigroup of G_α for every $\alpha \in \Gamma$, it follows that for every $\alpha \in \Gamma$ there exist $x_\alpha \in S_\alpha$ with $(x_\alpha] \subseteq S_\alpha$. Using Proposition 4.1, we get that there exist $(x_\alpha) \in \prod_{\alpha\in\Gamma} S_\alpha$ such that $((x_\alpha)] = \prod_{\alpha\in\Gamma}(x_\alpha] \subseteq \prod_{\alpha\in\Gamma} S_\alpha$. Therefore, $\prod_{\alpha\in\Gamma} S_\alpha$ is a NeutroOrderedSubSemigroup of $\prod_{\alpha\in\Gamma} G_\alpha$. □

Corollary 4.4. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If there exists $\alpha_0 \in \Gamma$ such that S_{α_0} is a NeutroOrderedSubSemigroup of G_{α_0} then $\prod_{\alpha\in\Gamma,\alpha<\alpha_0} G_\alpha \times S_{\alpha_0} \times \prod_{\alpha\in\Gamma,\alpha>\alpha_0}$ is a NeutroOrderedSubSemigroup of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof follows from Theorem 4.3 and having G_α a NeutroOrderedSubSemigroup of itself. □

Theorem 4.5. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If I_α is a NeutroOrderedLeftIdeal of G_α for all $\alpha \in \Gamma$ then $\prod_{\alpha\in\Gamma} I_\alpha$ is a NeutroOrderedLeftIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. Having every NeutroOrderedLeftIdeal a NeutroOrderedSubSemigroup and that I_α is a NeutroOrderedLeftIdeal of G_α for all $\alpha \in \Gamma$ implies, by means of Theorem 4.3, that $\prod_{\alpha\in\Gamma} I_\alpha$ is a NeutroOrderedSubSemigroup of $\prod_{\alpha\in\Gamma} G_\alpha$. Since I_α is a NeutroOrderedLeftIdeal of G_α for all $\alpha \in \Gamma$, it follows that for every $\alpha \in \Gamma$ there exist $x_\alpha \in I_\alpha$ such that $r_\alpha \cdot_\alpha x_\alpha \in I_\alpha$ for all $r_\alpha \in G_\alpha$. The latter implies that there exist $(x_\alpha) \in \prod_{\alpha\in\Gamma} I_\alpha$ such that $(r_\alpha)\cdot(x_\alpha) = (r_\alpha \cdot_\alpha x_\alpha) \in \prod_{\alpha\in\Gamma} I_\alpha$ for all $(r_\alpha) \in \prod_{\alpha\in\Gamma} G_\alpha$. Therefore, $\prod_{\alpha\in\Gamma} I_\alpha$ is a NeutroOrderedLeftIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$. □

Corollary 4.6. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If there exists $\alpha_0 \in \Gamma$ such that I_{α_0} is a NeutroOrderedLeftIdeal of G_{α_0} and for $\alpha \neq \alpha_0$ there exist $x_\alpha \in G_\alpha$ such that $r_\alpha \cdot_\alpha x_\alpha \in G_\alpha$ for all $r_\alpha \in G_\alpha$ then $\prod_{\alpha\in\Gamma,\alpha<\alpha_0} G_\alpha \times I_{\alpha_0} \times \prod_{\alpha\in\Gamma,\alpha>\alpha_0}$ is a NeutroOrderedLeftIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof follows from Theorem 4.5 and having G_α a NeutroOrderedLeftIdeal of itself. □

Theorem 4.7. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If I_α is a NeutroOrderedRightIdeal of G_α for all $\alpha \in \Gamma$ then $\prod_{\alpha\in\Gamma} I_\alpha$ is a NeutroOrderedRightIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof is similar to that of Theorem 4.5. □

Corollary 4.8. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If there exists $\alpha_0 \in \Gamma$ such that I_{α_0} is a NeutroOrderedRightIdeal of G_{α_0} and for $\alpha \neq \alpha_0$ there exist $x_\alpha \in G_\alpha$ such that $x_\alpha \cdot_\alpha r_\alpha \in G_\alpha$ for all $r_\alpha \in G_\alpha$ then $\prod_{\alpha\in\Gamma,\alpha<\alpha_0} G_\alpha \times I_{\alpha_0} \times \prod_{\alpha\in\Gamma,\alpha>\alpha_0}$ is a NeutroOrderedRightIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof follows from Theorem 4.7 and having G_α a NeutroOrderedRightIdeal of itself. □

Theorem 4.9. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If I_α is a NeutroOrderedIdeal of G_α for all $\alpha \in \Gamma$ then $\prod_{\alpha\in\Gamma} S_\alpha$ is a NeutroOrderedIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof is similar to that of Theorem 4.5. □

Corollary 4.10. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If there exists $\alpha_0 \in \Gamma$ such that I_{α_0} is a NeutroOrderedIdeal of G_{α_0} and for $\alpha \neq \alpha_0$ there exist $x_\alpha \in G_\alpha$ such that $r_\alpha \cdot_\alpha x_\alpha, x_\alpha \cdot_\alpha r_\alpha \in G_\alpha$ for all $r_\alpha \in G_\alpha$ then $\prod_{\alpha\in\Gamma,\alpha<\alpha_0} G_\alpha \times I_{\alpha_0} \times \prod_{\alpha\in\Gamma,\alpha>\alpha_0}$ is a NeutroOrderedIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof follows from Theorem 4.9 and having G_α a NeutroOrderedIdeal of itself. □

Example 4.11. Let $(S_3, \cdot_3, \leq_3)$ be the NeutroOrderedSemigroup presented in Example 2.10. Example 2.14 asserts that $I = \{0,1,2\}$ is both: a NeutroOrderedLefttIdeal and a NeutroOrderedRightIdeal of S_3. Theorem 4.5 and Theorem 4.7 imply that $I \times I = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$ is both: a NeutroOrderedLefttIdeal and a NeutroOrderedRightIdeal of $S_3 \times S_3$. Moreover, $I \times S_3$ and $S_3 \times I$ are both: NeutroOrderedLefttIdeals and NeutroOrderedRightIdeals of $S_3 \times S_3$.

Example 4.12. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NeutroOrderedSemigroup presented in Example 2.11. Example 2.15 asserts that $I = \{-2,-1,0,1,-2,-3,-4,\ldots\}$ is a NeutroOrderedIdeal of $\mathbb{Z}$. Theorem 4.9 asserts that $I \times I \times I$ is NeutroOrderedIdeal of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Theorem 4.13. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If F_α is a NeutroOrderedFilter of G_α for all $\alpha \in \Gamma$ then $\prod_{\alpha\in\Gamma} F_\alpha$ is a NeutroOrderedFilter of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. For all $\alpha \in \Gamma$, we have F_α an NOS (as it is NeutroOrderedFilter of G_α). Theorem 3.2 asserts that $\prod_{\alpha\in\Gamma} S_\alpha$ is an NOS. Having F_α is a NeutroOrderedFilter of G_α for all $\alpha \in \Gamma$ implies that for every $\alpha \in \Gamma$ there exist $x_\alpha \in F_\alpha$ such that for all $y_\alpha, z_\alpha \in F_\alpha$, $x_\alpha \cdot_\alpha y_\alpha \in F_\alpha$ and $z_\alpha \cdot_\alpha x_\alpha \in F_\alpha$ imply that $y_\alpha, z_\alpha \in F_\alpha$. We get now that there exist $(x_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$ such that for all $(y_\alpha),(z_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$, $(x_\alpha)\cdot(y_\alpha) = (x_\alpha \cdot_\alpha y_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$ and $(z_\alpha)\cdot(x_\alpha) = (z_\alpha \cdot_\alpha x_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$ imply that $(y_\alpha),(z_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$. Since F_α is a NeutroOrderedFilter of G_α for every $\alpha \in \Gamma$, it follows that for every $\alpha \in \Gamma$ there exist $x_\alpha \in F_\alpha$ with $[x_\alpha) \subseteq F_\alpha$. Using Proposition 4.2, we get that there exist $(x_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$ such that $[(x_\alpha)) = \prod_{\alpha\in\Gamma}[x_\alpha) \subseteq \prod_{\alpha\in\Gamma} F_\alpha$. Therefore, $\prod_{\alpha\in\Gamma} F_\alpha$ is a NeutroOrderedFilter of $\prod_{\alpha\in\Gamma} G_\alpha$. □

Corollary 4.14. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If there exists $\alpha_0 \in \Gamma$ such that F_{α_0} is a NeutroOrderedFilter of G_{α_0} then $\prod_{\alpha\in\Gamma,\alpha<\alpha_0} G_\alpha \times F_{\alpha_0} \times \prod_{\alpha\in\Gamma,\alpha>\alpha_0}$ is a NeutroOrderedFilter of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof follows from Theorem 4.13 and having G_α a NeutroOrderedFilter of itself. □

Example 4.15. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NeutroOrderedSemigroup presented in Example 2.11. Example 2.16 asserts that $F = \{-2,-1,0,1,2,3,4,\ldots\}$ is a NeutroOrderedFilter of $\mathbb{Z}$. Theorem 4.13 implies that $F \times F \times F \times F$ is a NeutroOrderedFilter of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Moreover, $\mathbb{Z} \times \mathbb{Z} \times F \times \mathbb{Z}$ is a NeutroOrderedFilter of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

5. Conclusion

The class of NeutroAlgebras is very large. This paper considered NeutroOrderedSemigroups (introduced by the authors in [9]) as a subclass of NeutroAlgebras. Results related to productional NOS and its subsets were investigated and some examples were elaborated.

For future work, it will be interesting to investigate the following.

(1) Find necessary conditions for the productional NeutroSemigroup to be NeutroOrderedSemigroup.
(2) Check the possibility of introducing the quotient NeutroOrderedSemigroup and investigate its properties.
(3) Study other types of productional NetroOrderedStructures.

References

[1] F. Smarandache, Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures, in Advances of Standard and Nonstandard Neutrosophic Theories, Pons Publishing House Brussels, Belgium, Ch. 6, pp. 240-265, 2019; http://fs.unm.edu/AdvancesOfStandardAndNonstandard.pdf

[2] F. Smarandache, Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited), Neutrosophic Sets and Systems, Vol. 31, pp. 1-16, 2020. DOI: 10.5281/zenodo.3638232, http://fs.unm.edu/NSS/NeutroAlgebraic-AntiAlgebraic-Structures.pdf

[3] F. Smarandache, NeutroAlgebra is a Generalization of Partial Algebra, International Journal of Neutrosophic Science (IJNS), Vol. 2, No. 1, pp. 08-17, 2020, http://fs.unm.edu/NeutroAlgebra.pdf.

[4] A.A.A. Agboola, Introduction to NeutroGroups, International Journal of Neutrosophic Science(IJNS), Vol. 6, 2020, pp. 41-47.

[5] A.A.A. Agboola, Introduction to NeutroRings, International Journal of Neutrosophic Science (IJNS), Vol. 7, 2020, pp. 62-73.

[6] M. Hamidi, F. Smarandache,Neutro-BCK-Algebra, International Journal of Neutrosophic Science (IJNS), Vol. 8, 2020, pp. 110-117.

[7] F. Smarandache, A. Rezaei, H.S. Kim, A New Trend to Extensions of CI- algebras, International Journal of Neutrosophic Science(IJNS) Vol. 5, No. 1 , pp. 8-15, 2020; DOI: 10.5281/zenodo.3788124

[8] A. Rezaei, F. Smarandache, On Neutro-BE-algebras and Anti-BE-algebras, International Journal of Neutrosophic Science(IJNS), Vol. 4, 2020, pp. 8-15.

[9] M. Al-Tahan, F. Smarandache, B. Davvaz, NeutroOrderedAlgebra: Applications to Semigroups, Neutrosophic Sets and Systems, Vol. 39, pp. 133-147, 2021.

[10] L. Fuchs, Partially Ordered Algebraic Systems, Int. Ser. of Monographs on Pure and Appl. Math. 28, Pergamon Press, Oxford, 1963.

[11] A.H. Clifford, G.B. Preston, The Algebraic Theory of Semigroups, Vol 1, Amer. Math. Soc., Math. Surveys 7, Providence, Rhode Island, 1977.

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Hypersoft Expert Set With Application in Decision Making for Recruitment Process

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Abstract. Many researchers have created some models based on soft set, to solve problems in decision making and medical diagnosis, but most of these models deal only with one expert. This causes a problem with the users, especially with those who use questionnaires in their work and studies. Therefore we present a new model i.e. Hypersoft Expert Set which not only addresses this limitation of soft-like models by emphasizing the opinion of all experts but also resolves the inadequacy of soft set for disjoint attribute-valued sets corresponding to distinct attributes. In this study, the existing concept of soft expert set is generalized to hypersoft expert set which is more flexible and useful. Some fundamental properties (i.e. subset, not set and equal set), results (i.e. commutative, associative, distributive and D' Morgan Laws) and set-theoretic operations (i.e. complement, union intersection AND, and OR) are discussed. An algorithm is proposed to solve decision-making problems and applied to recruitment process for hiring "right person for the right job".

Keywords: Soft Set; Soft Expert Set; Hypersoft Set; Hypersoft Expert Set.

1. Introduction

Soft set presented by Molodtsov [1] is considered as a new parameterized family of subsets of the universe of discourse, which addresses the inadequacy of fuzzy-like structures for parameterization tools. It has helped the researcher (experts) to solve efficiently the decision-making problems involving some sort of uncertainty. Many researchers [2]- [13]studied and broadened this concept and applied to different fields. The gluing concept of soft set with expert system initiated by Alkhazaleh et al. [15] to emphasize the due status of the opinions of all experts regarding taking any decision in decision-making system. Al-Quran et al. [16] proposed neutrosophic vague soft expert set theory, Alkhazaleh et al. [17] characterized fuzzy soft expert set and its application. Bashir et al. [18, 19] presented possibility fuzzy soft expert set and fuzzy parameterized soft expert set. Sahin et al. [20] investigated neutrosophic soft expert sets. Alhazaymeh et al. [21, 22] studied mapping on generalized vague soft expert set and generalized vague soft expert set. Alhazaymeh et al. [23] explained the application of generalized vague soft expert set in decision making. Hassan et al. [24] reviewed Q-neutrosophic soft expert set and its application in decision making. Uluay et al. [25] studied generalized neutrosophic soft expert set for multiple-criteria decision-making. Al-Qudah et al. [26] explained bipolar fuzzy soft expert set and its application in decision making. Al-Qudah et al. [27] investigated complex multi-fuzzy soft expert set and its application. Al-Quran et al. [28] presented the complex neutrosophic soft expert set and its application in decision making. Pramanik et al. [29] studied the topsis for single valued neutrosophic soft expert set based multi-attribute decision making problems. Abu Qamar et al. [30] investigated the generalized Q-neutrosophic soft expert set for decision under uncertainty. Adam et al. [31] characterized the multi Q-fuzzy soft expert set and its application. Ulucay et al. [32] presented the time-neutrosophic soft expert sets and its decision making problem. Al-Quran et al. [33] studied fuzzy parameterised single valued neutrosophic soft expert set theory and its application in decision making. Hazaymeh et al. [34] researched generalized fuzzy soft expert set.

There are many real life scenarios when we are to deal with disjoint attribute-valued set for distinct attributes. In 2018, Smarandache [35] addressed this inadequacy of soft with the development of new structure hypersoft set by replacing single attribute-valued function to multi-attribute valued function. In 2020, Saeed et al. [36, 37] extended the concept and discussed the fundamentals of hypersoft set such as hypersoft subset, complement, not hypersoft set, aggregation operators along with hypersoft set relation, sub relation,complement relation, function, matrices and operations on hypersoft matrices. In the same year, Mujahid et al. [38] discussed hypersoft points in different fuzzy-like envorinments. In 2020, Rahman et al. [39] defined complex hypersoft set and developed the hybrids of hypersoft set with complex fuzzy set, complex intuitionistic fuzzy set and complex neutrosophic set respectively. They also discussed their fundamentals i.e. subset, equal sets, null set, absolute set etc. and theoretic operations i.e. complement, union, intersection etc. In 2020, Rahman et al. [40] conceptualized convexity cum concavity on hypersoft set and presented its pictorial versions with illustrative examples.

Dealing with disjoint attribute-valued sets is of great importance and it is vital for sensible decisions in decision-making techniques. Results will be varied and be considered inclined and odd on ignoring such kind of sets. Therefore, it is the need of the literature to adequate the exiting literature of soft and expert set for multi-attribute function. Having motivation from [15] and [35–38], new notions of hypersoft expert set are developed and an application is discussed in decision making through proposed method. The pattern of rest of the paper is: section 2 reviews the notions of soft sets, soft expert set, hypersoft set and relevant definitions used in the proposed work. Section 3, presents notions of hypersoft expert set with properties. Section 4, demonstrates an application of this concept in a decision-making problem. Section 5, concludes the paper.

1.1. *Motivation*

The novelty of hypersoft expert set (HSE-set) is as:

- it is the extension of soft set and soft expert set,
- it tackles all the hindrances of soft set and soft expert set for dealing with further partitions of attributes into attribute-valued sets,
- it facilitates the decision-makers to have decisions for uncertain scenarios without encountering with any inclined situation.

2. **Preliminaries**

In this section, some basic definitions and terms regarding the main study, are presented from the literature.

Definition 2.1. [1]
Let $P(\Omega)$ denote power set of Ω(universe of discourse) and F be a collection of parameters defining Ω. A *soft set* Ψ_M is defined by mapping

$$\Psi_M : F \to P(\Omega)$$

Definition 2.2. [3]
The union of two soft sets (Ψ_1, A_1) and (Ψ_2, A_2) over Ω is the soft set (Ψ_3, A_3) ; $A_3 \doteq A_1 \cup A_2$, and $\forall\ \xi \in A_3$,

$$\Psi_3(\xi) = \begin{cases} \Psi_1(\xi) & ; \xi \in A_1 - A_2 \\ \Psi_2(\xi) & ; \xi \in A_2 - A_1 \\ \Psi_1(\xi) \cup \Psi_2(\xi) & ; \xi \in A_1 \cap A_2 \end{cases}$$

Definition 2.3. [14]
The extended intersection of two soft sets (Ψ_1, A_1) and (Ψ_2, A_2) with Ω is the soft set (Ψ_3, A_3) while $A_3 \doteq A_1 \cup A_2$, ; $\xi \in A_3$,

$$\Psi_3(\xi) = \begin{cases} \Psi_1(\xi) & ; \xi \in A_1 - A_2 \\ \Psi_2(\xi) & ; \xi \in A_2 - A_1 \\ \Psi_1(\xi) \cup \Psi_2(\xi) & ; \xi \in A_1 \cap A_2 \end{cases}$$

Definition 2.4. [15]
Assume that Y be a set of specialists (operators) and $\ddot{O}$ be a set of conclusions, $T = F \times Y \times \ddot{O}$ with $S \subseteq T$ where Ω denotes the universe , F a set of parameters.
A pair(Φ_H, S) is known as a *soft expert set* over Ω, where Ψ_H is a mapping given by

$$\Phi_H : S \to P(\Omega)$$

Definition 2.5. [15]
A $(\Phi_1, S) \subseteq (\Phi_2, P)$ over Ω, if
(i) $S \subseteq P$,
(ii) $\forall\ \alpha \in P, \Phi_2(\alpha) \subseteq \Phi_1(\alpha)$.
While (Φ_2, P) is known as a *soft expert superset* of (Φ_1, S).

Definition 2.6. [19]
Let $h_1, h_2, h_3, ..., h_m$, for $m \geq 1$, be m distinct attributes, whose corresponding attribute values are respectively the sets $H_1, H_2, H_3, ..., H_m$, with $H_i \cap H_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., m\}$. Then the pair (Ψ, G), where $G = H_1 \times H_2 \times H_3 \times . ... \times H_m$ and $\Psi : G \to P(\Omega)$ is called a *hypersoft Set* over Ω.

3. Hypersoft Expert set (HSE-Set)

In this section, the fundamentals of hypersoft expert set are established and its basic properties, laws and operations are generalized

Definition 3.1. Hypersoft Expert set (HSE-Set)
A pair(Ψ, S) is known as a *hypersoft expert set* over Ω, where

$$\Psi : S \to P(\Omega)$$

where

- $S \subseteq T = G \times D \times C$
- $G = G_1 \times G_2 \times G_3 \times .... \times G_n$ where $G_1, G_2, G_3, ..., G_n$ are disjoint attributive sets corresponding to n distinct attributes $g_1, g_2, g_3, ..., g_n$
- D be a set of specialists (operators)
- C be a set of conclusions

For simplicity, $C = \{0 = disagree, 1 = agree\}$.

Example 3.2. Suppose that an organization manufactured modern kinds of its brands and intends to proceed the assessment of certain specialists about concerning these products. Let $\Omega = \{v_1, v_2, v_3, v_4\}$ be a set of products and

$G_1 = \{g_{11}, g_{12}\}$

$G_2 = \{g_{21}, g_{22}\}$

$G_3 = \{g_{31}, g_{32}\}$

be disjoint attributive sets for distinct attributes g_1= simple to use, g_2= nature, g_3= modest. Now $G = G_1 \times G_2 \times G_3$

$$G = \left\{ \begin{array}{l} a_1 = (g_{11}, g_{21}, g_{31}), a_2 = (g_{11}, g_{21}, g_{32}), a_3 = (g_{11}, g_{22}, g_{31}), a_4 = (g_{11}, g_{22}, g_{32}), \\ a_5 = (g_{12}, g_{21}, g_{31}), a_6 = (g_{12}, g_{21}, g_{32}), a_7 = (g_{12}, g_{22}, g_{31}), a_8 = (g_{12}, g_{22}, g_{32}) \end{array} \right\}$$

Now $T = G \times D \times C$

$$T = \left\{ \begin{array}{l} (a_1, s, 0), (a_1, s, 1), (a_1, t, 0), (a_1, t, 1), (a_1, u, 0), (a_1, u, 1), \\ (a_2, s, 0), (a_2, s, 1), (a_2, t, 0), (a_2, t, 1), (a_2, u, 0), (a_2, u, 1), \\ (a_3, s, 0), (a_3, s, 1), (a_3, t, 0), (a_3, t, 1), (a_3, u, 0), (a_3, u, 1), \\ (a_4, s, 0), (a_4, s, 1), (a_4, t, 0), (a_4, t, 1), (a_4, u, 0), (a_4, u, 1), \\ (a_5, s, 0), (a_5, s, 1), (a_5, t, 0), (a_5, t, 1), (a_5, u, 0), (a_5, u, 1), \\ (a_6, s, 0), (a_6, s, 1), (a_6, t, 0), (a_6, t, 1), (a_6, u, 0), (a_6, u, 1), \\ (a_7, s, 0), (a_7, s, 1), (a_7, t, 0), (a_7, t, 1), (a_7, u, 0), (a_7, u, 1), \\ (a_8, s, 0), (a_8, s, 1), (a_8, t, 0), (a_8, t, 1), (a_8, u, 0), (a_8, u, 1) \end{array} \right\}$$

let

$$S = \left\{ \begin{array}{l} (a_1, s, 0), (a_1, s, 1), (a_1, t, 0), (a_1, t, 1), (a_1, u, 0), (a_1, u, 1), \\ (a_3, s, 0), (a_3, s, 1), (a_3, t, 0), (a_3, t, 1), (a_3, u, 0), (a_3, u, 1), \\ (a_5, s, 0), (a_5, s, 1), (a_5, t, 0), (a_5, t, 1), (a_5, u, 0), (a_5, u, 1), \end{array} \right\}$$

be a subset of T and $D = \{s, t, u\}$ be a set of specialists.

Assume that the organization has appropriated a survey to three specialists to settle the choices on the organization's products, and we get the accompanying:

$\Psi_1 = \Psi(a_1, s, 1) = \{v_1, v_2, v_4\}$,

$\Psi_2 = \Psi(a_1, t, 1) = \{v_3, v_4\}$,

$\Psi_3 = \Psi(a_1, u, 1) = \{v_3, v_4\}$,

$\Psi_4 = \Psi(a_3, s, 1) = \{v_4\}$,

$\Psi_5 = \Psi(a_3, t, 1) = \{v_1, v_3\}$,

$\Psi_6 = \Psi(a_3, u, 1) = \{v_1, v_2, v_4\}$,

$\Psi_7 = \Psi(a_5, s, 1) = \{v_3, v_4\}$,

$\Psi_8 = \Psi(a_5, t, 1) = \{v_1, v_2\}$,
$\Psi_9 = \Psi(a_5, u, 1) = \{v_4\}$,
$\Psi_{10} = \Psi(a_1, s, 0) = \{v_3\}$,
$\Psi_{11} = \Psi(a_1, t, 0) = \{v_2, v_3\}$,
$\Psi_{12} = \Psi(a_1, u, 0) = \{v_1, v_2\}$,
$\Psi_{13} = \Psi(a_3, s, 0) = \{v_1, v_2, v_3\}$,
$\Psi_{14} = \Psi(a_3, t, 0) = \{v_2, v_4\}$,
$\Psi_{15} = \Psi(a_3, u, 0) = \{v_3\}$,
$\Psi_{16} = \Psi(a_5, s, 0) = \{v_1, v_2\}$,
$\Psi_{17} = \Psi(a_5, t, 0) = \{v_3, v_4\}$,
$\Psi_{18} = \Psi(a_5, u, 0) = \{v_1, v_2, v_3\}$,
The hypersoft expert set is

$$(\Psi, \mathrm{S}) = \left\{ \begin{array}{l} ((a_1, s, 1), \{v_1, v_2, v_4\}), ((a_1, t, 1), \{v_1, v_4\}), ((a_1, u, 1), \{v_3, v_4\}), \\ ((a_3, s, 1), \{v_4\}), ((a_3, t, 1), \{v_1, v_3\}), ((a_3, u, 1), \{v_1, v_2, v_4\}), \\ ((a_5, s, 1), \{v_3, v_4\}), ((a_5, t, 1), \{v_1, v_2\}), ((a_5, u, 1), \{v_4\}), \\ ((a_1, s, 0), \{v_3\}), ((a_1, t, 0), \{v_2, v_3\}), ((a_1, u, 0), \{v_1, v_2\}), \\ ((a_3, s, 0), \{v_1, v_2, v_3\}), ((a_3, t, 0), \{v_2, v_4\}), ((a_3, u, 0), \{v_3\}) \\ ((a_5, s, 0), \{v_1, v_2\}), ((a_5, t, 0), \{v_3, v_4\}), ((a_5, u, 0), \{v_1, v_2, v_3\}) \end{array} \right\}$$

Note that in this example the first specialist, s, "agrees" that the "simple to use" products are v_1, v_2, and v_4. The subsequent specialist t, "agrees" that the "simple to use" products are v_1 and v_4, and the third specialist, u, "agrees" that the "simple to use" products are v_3 and v_4. See here every one of specialists "agree" that product v_4 is "anything but simple to use."

Definition 3.3. Hypersoft Expert subset
A hypersoft expert set (Ψ_1, S) is said to be hypersoft expert subset of (Ψ_2, R) over Ω, if
(i) $S \subseteq R$,
(ii) $\forall\ \alpha \in S, \Psi_1(\alpha) \subseteq \Psi_2(\alpha)$.
and denoted by $(\Psi_1, S) \subseteq (\Psi_2, R)$. Similarly (Ψ_2, R) is said to be *hypersoft expert superset* of (Ψ_1, S).

Example 3.4. Considering Example 3.2, Suppose

$$A_1 = \Big\{ (a_1, s, 1), (a_3, s, 0), (a_1, t, 1), (a_3, t, 1), (a_3, t, 0), (a_1, u, 0), (a_3, u, 1) \Big\}$$

$$A_2 = \Big\{ (a_1, s, 1), (a_3, s, 0), (a_3, s, 1), (a_1, t, 1), (a_3, t, 1), (a_5, t, 0), (a_3, t, 0), (a_1, u, 0), (a_3, u, 1), (a_5, u, 1) \Big\}$$

It is clear that $A_1 \subset A_2$. Suppose (Ψ_1, A_1) and (Ψ_2, A_2) be defined as following

$$(\Psi_1, \mathrm{A}_1) = \left\{ \begin{array}{l} ((a_1, s, 1), \{v_1, v_2\}), ((a_1, t, 1), \{v_1\}), \\ ((a_3, t, 1), \{v_1, v_3\}), ((a_3, u, 1), \{v_1, v_2\}), \\ ((a_1, u, 0), \{v_1\}), ((a_3, s, 0), \{v_1, v_2\}), \\ ((a_3, t, 0), \{v_2, v_4\}) \end{array} \right\}$$

$$(\Psi_2, \mathrm{A}_2) = \left\{ \begin{array}{l} ((a_1, s, 1), \{v_1, v_2, v_4\}), ((a_1, t, 1), \{v_1, v_4\}), \\ ((a_3, s, 1), \{v_4\}), ((a_3, t, 1), \{v_1, v_3\}), \\ ((a_5, u, 1), \{v_4\}), ((a_3, u, 1), \{v_1, v_2, v_4\}), \\ ((a_1, u, 0), \{v_1, v_2\}), ((a_5, t, 0), \{v_3, v_4\}), \\ ((a_3, s, 0), \{v_1, v_2, v_3\}), ((a_3, t, 0), \{v_2, v_4\}) \end{array} \right\}$$

which implies that $(\Psi_1, A_1) \subseteq (\Psi_2, A_2)$.

Definition 3.5. Two hypersoft expert sets (Ψ_1, A_1) and (Ψ_2, A_2) over Ω are said to be equal if (Ψ_1, A_1) is a hypersoft expert subset of (Ψ_2, A_2) and (Ψ_2, A_2) is a hypersoft expert subset of (Ψ_1, A_1).

Definition 3.6. Let G be a set as defined in definition 3.1 and D , a set of experts. The NOT set of $T = G \times D \times C$ denoted by $\sim T$, is defined by $\sim T = \{(\sim g_i, d_j, c_k) \forall i, j, k\}$ where $\sim g_i$ is not g_i.

Definition 3.7. The complement of a hypersoft expert set (Ψ, S), denoted by $(\Psi, S)^c$, is defined by $(\Psi, S)^c = (\Psi^c, \sim S)$ while $\Psi^c : \sim S \to P(\Omega)$ is a mapping given by $\Psi^c(\beta) = \Omega - \Psi(\sim \beta)$, where $\beta \in \sim S$.

Example 3.8. Taking complement of hypersoft expert set determined in 3.2, we have

$$(\Psi, \mathrm{S})^c = \left\{ \begin{array}{l} ((\sim a_1, s, 1), \{v_3\}), ((\sim a_1, t, 1), \{v_2, v_3\}), ((\sim a_1, u, 1), \{v_1, v_2\}), \\ ((\sim a_3, s, 1), \{v_1, v_2, v_3\}), ((\sim a_3, t, 1), \{v_2, v_4\}), ((\sim a_3, u, 1), \{v_1, v_2, v_4\}), \\ ((\sim a_5, s, 1), \{v_1, v_2\}), ((\sim a_5, t, 1), \{v_3, v_4\}), ((\sim a_5, u, 1), \{v_1, v_2, v_3\}), \\ ((\sim a_1, s, 0), \{v_1, v_2, v_4\}), ((\sim a_1, t, 0), \{v_1, v_4\}), ((\sim a_1, u, 0), \{v_1, v_2\}), \\ ((\sim a_3, s, 0), \{v_4\}), ((\sim a_3, t, 0), \{v_1, v_3\}), ((\sim (a_3, u, 0), \{v_3\}) \\ ((\sim a_5, s, 0), \{v_3, v_4\}), ((\sim a_5, t, 0), \{v_1, v_3\},), ((\sim a_5, u, 0), \{v_4\}) \end{array} \right\}$$

Definition 3.9. An agree-hypersoft expert set $(\Psi, S)_{ag}$ over Ω, is a hypersoft expert subset of (Ψ, S) and is characterized as
$(\Psi, S)_{ag} = \{\Psi_{ag}(\beta) : \beta \in G \times D \times \{1\}\}$.

Example 3.10. Finding agree-hypersoft expert set determined in 3.2, we get

$$(\Psi, \mathrm{S}) = \left\{ \begin{array}{l} ((a_1, s, 1), \{v_1, v_2, v_4\}), ((a_1, t, 1), \{v_1, v_4\}), ((a_1, u, 1), \{v_3, v_4\}), \\ ((a_3, s, 1), \{v_4\}), ((a_3, t, 1), \{v_1, v_3\}), ((a_3, u, 1), \{v_1, v_2, v_4\}), \\ ((a_5, s, 1), \{v_3, v_4\}), ((a_5, t, 1), \{v_1, v_2\}), ((a_5, u, 1), \{v_4\}) \end{array} \right\}$$

Definition 3.11. A disagree-hypersoft expert set $(\Psi, S)_{dag}$ over Ω, is a hypersoft expert subset of (Ψ, S) and is characterized as
$(\Psi, S)_{dag} = \{\Psi_{dag}(\beta) : \beta \in G \times D \times \{0\}\}$.

Example 3.12. Getting disagree-hypersoft expert set determined in 3.2,

$$(\Psi, \mathrm{S}) = \left\{ \begin{array}{l} ((a_1, s, 0), \{v_3\},), ((a_1, t, 0), \{v_2, v_3\}), ((a_1, u, 0), \{v_1, v_2\}), \\ ((a_3, s, 0), \{v_1, v_2, v_3\}), ((a_3, t, 0), \{v_2, v_4\}), ((a_3, u, 0), \{v_3\}) \\ ((a_5, s, 0), \{v_1, v_2\}), ((a_5, t, 0), \{v_3, v_4\}), ((a_5, u, 0), \{v_1, v_2, v_3\}) \end{array} \right\}$$

Proposition 3.13. *If (Ψ, S) is a hypersoft expert set over Ω, then*

(1). $((\Psi, S)^c)^c = (\Psi, S)$
(2). $(\Psi, S)^c_{ag} = (\Psi, S)_{dag}$
(3). $(\Psi, S)^c_{dag} = (\Psi, S)_{ag}$

Definition 3.14. The union of (Ψ_1, S) and (Ψ_2, R) over Ω is (Ψ_3, L) with $L = S \cup R$, defined as

$$\Psi_3(\beta) = \left\{ \begin{array}{cl} S(\beta) & ; \beta \in S - R \\ R(\beta) & ; \beta \in R - S \\ S(\beta) \cup R(\beta) & ; \beta \in S \cap R \end{array} \right.$$

Example 3.15. Taking into consideration the concept of example 3.2, consider the following two sets

$$A_1 = \left\{ (a_1, s, 1), (a_3, s, 0), (a_3, s, 1), (a_1, t, 1), (a_3, t, 1), (a_5, t, 0), (a_3, t, 0), (a_1, u, 0), (a_3, u, 1), (a_5, u, 1) \right\}$$

$$A_2 = \left\{ (a_1, s, 1), (a_3, s, 0), (a_3, s, 1), (a_1, t, 1), (a_3, t, 1), (a_5, t, 0), (a_3, t, 0), (a_1, u, 0), (a_3, u, 1) \right\}$$

Suppose (Ψ_1, A_1) and (Ψ_2, A_2) over Ω are two hypersoft expert sets such that

$$(\Psi_1, \mathrm{A}_1) = \left\{ \begin{array}{l} ((a_1, s, 1), \{v_1, v_2\}), ((a_1, t, 1), \{v_1\}), ((a_3, t, 1), \{v_1, v_3\}), \\ ((a_3, u, 1), \{v_1, v_2\}), ((a_1, u, 0), \{v_1\}), ((a_3, s, 0), \{v_1, v_2\}), \\ ((a_3, t, 0), \{v_2, v_4\}) \end{array} \right\}$$

$$(\Psi_2, \mathrm{A}_2) = \left\{ \begin{array}{l} ((a_1, s, 1), \{v_1, v_2, v_4\}), ((a_1, t, 1), \{v_1, v_4\}), \\ ((a_3, s, 1), \{v_4\}), ((a_3, t, 1), \{v_1, v_3\}), \\ ((a_5, u, 1), \{v_4\}), (((a_3, u, 1), \{v_1, v_2, v_4\}) \\ ((a_1, u, 0), \{v_1, v_2\}), ((a_5, t, 0), \{v_3, v_4\}), \\ ((a_3, s, 0), \{v_1, v_2, v_3\}), ((a_3, t, 0), \{v_2, v_4\}) \end{array} \right\}$$

Then $(\Psi_1, A_1) \cup (\Psi_2, A_2) = (\Psi_3, A_3)$

$$(\Psi_3, \mathrm{A}_3) = \left\{ \begin{array}{l} ((a_1, s, 1), \{v_1, v_2, v_4\}), ((a_1, t, 1), \{v_1, v_4\}), \\ ((a_3, s, 1), \{v_4\}), ((a_3, t, 1), \{v_1, v_3\}), \\ ((a_5, u, 1), \{v_4\}), ((a_3, u, 1), \{v_1, v_2, v_4\}), \\ ((a_1, u, 0), \{v_1, v_2\}), ((a_5, t, 0), \{v_3, v_4\}), \\ ((a_3, s, 0), \{v_1, v_2, v_3\}), ((a_3, t, 0), \{v_2, v_4\}) \end{array} \right\}$$

Proposition 3.16. *If* (Ψ_1, A_1),(Ψ_2, A_2) *and* (Ψ_3, A_3) *are three hypersoft expert sets over* Ω*, then*

(1). $(\Psi_1, A_1) \cup (\Psi_2, A_2) = (\Psi_2, A_2) \cup (\Psi_1, A_1)$
(2). $((\Psi_1, A_1) \cup (\Psi_2, A_2)) \cup (\Psi_3, A_3) = (\Psi_1, A_1) \cup ((\Psi_2, A_2) \cup (\Psi_3, A_3))$

Definition 3.17. The intersection of (Ψ_1, S) and (Ψ_2, R) over Ω is (Ψ_3, L) with $L = S \cap R$, defined as

$$\Psi_3(\beta) = \begin{cases} S(\beta) & ; \beta \in S - R \\ R(\beta) & ; \beta \in R - S \\ S(\beta) \cap R(\beta) & ; \beta \in S \cap R \end{cases}$$

Example 3.18. Taking into consideration the concept of example 3.2, consider the following two sets

$$A_1 = \left\{ (a_1, s, 1), (a_3, s, 0), (a_3, s, 1), (a_1, t, 1), (a_3, t, 1), (a_5, t, 0), (a_3, t, 0), (a_1, u, 0), (a_3, u, 1), (a_5, u, 1) \right\}$$

$$A_2 = \left\{ (a_1, s, 1), (a_3, s, 0), (a_3, s, 1), (a_1, t, 1), (a_3, t, 1), (a_5, t, 0), (a_3, t, 0), (a_1, u, 0), (a_3, u, 1) \right\}$$

Suppose (Ψ_1, A_1) and (Ψ_2, A_2) over Ω are two hypersoft expert sets such that

$$(\Psi_1, \mathrm{A}_1) = \left\{ \begin{array}{l} ((a_1, s, 1), \{v_1, v_2\}), ((a_1, t, 1), \{v_1\}), ((a_3, t, 1), \{v_1, v_3\}), \\ ((a_3, u, 1), \{v_1, v_2\}), ((a_1, u, 0), \{v_1\}), ((a_3, s, 0), \{v_1, v_2\}), \\ ((a_3, t, 0), \{v_2, v_4\}) \end{array} \right\}$$

$$(\Psi_2, \mathrm{A}_2) = \begin{Bmatrix} ((a_1, s, 1), \{v_1, v_2, v_4\}), ((a_1, t, 1), \{v_1, v_4\}), \\ ((a_3, s, 1), \{v_4\}), ((a_3, t, 1), \{v_1, v_3\},), \\ ((a_5, u, 1), \{v_4\}), (((a_3, u, 1), \{v_1, v_2, v_4\}) \\ ((a_1, u, 0), \{v_1, v_2\}), ((a_5, t, 0), \{v_3, v_4\}), \\ ((a_3, s, 0), \{v_1, v_2, v_3\}), ((a_3, t, 0), \{v_2, v_4\}) \end{Bmatrix}$$

Then $(\Psi_1, A_1) \cap (\Psi_2, A_2) = (\Psi_3, A_3)$

$$(\Psi_1, \mathrm{A}_1) = \begin{Bmatrix} ((a_1, s, 1), \{v_1, v_2\}), ((a_1, t, 1), \{v_1\}), \\ ((a_3, t, 1), \{v_1, v_3\}), ((a_3, u, 1), \{v_1, v_2\}), \\ ((a_1, u, 0), \{v_1\}), ((a_3, s, 0), \{v_1, v_2\}), \\ ((a_3, t, 0), \{v_2, v_4\}) \end{Bmatrix}$$

Proposition 3.19. *If (Ψ_1, A_1),(Ψ_2, A_2) and (Ψ_3, A_3) are three hypersoft expert sets over Ω, then*

(1). $(\Psi_1, A_1) \cap (\Psi_2, A_2) = (\Psi_2, A_2) \cap (\Psi_1, A_1)$
(2). $((\Psi_1, A_1) \cap (\Psi_2, A_2)) \cap (\Psi_3, A_3) = (\Psi_1, A_1) \cap ((\Psi_2, A_2) \cap (\Psi_3, A_3))$

Proposition 3.20. *If (Ψ_1, A_1),(Ψ_2, A_2) and (Ψ_3, A_3) are three hypersoft expert sets over Ω, then*

(1). $(\Psi_1, A_1) \cup ((\Psi_2, A_2) \cap (\Psi_3, A_3)) = ((\Psi_1, A_1) \cup ((\Psi_2, A_2)) \cap ((\Psi_1, A_1) \cup (\Psi_3, A_3))$
(2). $(\Psi_1, A_1) \cap ((\Psi_2, A_2) \cup (\Psi_3, A_3)) = ((\Psi_1, A_1) \cap ((\Psi_2, A_2)) \cup ((\Psi_1, A_1) \cap (\Psi_3, A_3))$

Definition 3.21. If (Ψ_1, A_1) and (Ψ_2, A_2) are two hypersoft expert sets over Ω then (Ψ_1, A_1) AND (Ψ_2, A_2) denoted by $(\Psi_1, A_1) \wedge (\Psi_2, A_2)$ is defined by
$(\Psi_1, A_1) \wedge (\Psi_2, A_2) = (\Psi_3, A_1 \times A_2)$,
while $\Psi_3(\beta, \gamma) = \Psi_1(\beta) \cap \Psi_2(\gamma), \forall (\beta, \gamma) \in A_1 \times A_2$.

Example 3.22. Taking into consideration the concept of example 3.2, let two sets

$$A_1 = \Big\{ (a_1, s, 1), (a_1, t, 1), (a_3, s, 1), (a_3, s, 0) \Big\}$$

$$A_2 = \Big\{ (a_1, s, 1), (a_3, s, 0), (a_3, s, 1) \Big\}$$

Suppose (Ψ_1, A_1) and (Ψ_2, A_2) over Ω are two hypersoft expert sets such that

$$(\Psi_1, \mathrm{A}_1) = \left\{ \begin{array}{l} ((a_1, s, 1), \{v_1, v_2\}), ((a_1, t, 1), \{v_1\}), \\ ((a_3, s, 1), \{v_4\}), ((a_3, s, 0), \{v_1, v_2\}), \end{array} \right\}$$

$$(\Psi_2, \mathrm{A}_2) = \left\{ ((a_1, s, 1), \{v_1, v_2, v_4\}), ((a_3, s, 0), \{v_1, v_2, v_3\}), \right\}$$

Then $(\Psi_1, A_1) \wedge (\Psi_2, A_2) = (\Psi_3, A_1 \times A_2)$,

$$(\Psi_3, \mathrm{A}_1 \times \mathrm{A}_2) = \left\{ \begin{array}{l} (((a_1, s, 1), (a_1, s, 1)), \{v_1, v_2\}), (((a_1, s, 1), (a_3, s, 0)), \{v_1, v \\ \\ \\ \end{array} \right.$$

Proposition 3.25. *If* (Ψ_1, A_1),(Ψ_2, A_2) *and* (Ψ_3, A_3) *are three hypersoft expert sets over* Ω, *then*

(1). $((\Psi_1, A_1) \wedge (\Psi_2, A_2))^c = ((\Psi_1, A_1))^c \vee ((\Psi_2, A_2))^c$

(2). $((\Psi_1, A_1) \vee (\Psi_2, A_2))^c = ((\Psi_1, A_1))^c \wedge ((\Psi_2, A_2))^c$

Proposition 3.26. *If* (Ψ_1, A_1),(Ψ_2, A_2) *and* (Ψ_3, A_3) *are three hypersoft expert sets over* Ω, *then*

(1). $((\Psi_1, A_1) \wedge (\Psi_2, A_2)) \wedge (\Psi_3, A_3) = (\Psi_1, A_1) \wedge ((\Psi_2, A_2) \wedge (\Psi_3, A_3))$

(2). $((\Psi_1, A_1) \vee (\Psi_2, A_2)) \vee (\Psi_3, A_3) = (\Psi_1, A_1) \vee ((\Psi_2, A_2) \vee (\Psi_3, A_3))$

(3). $(\Psi_1, A_1) \vee ((\Psi_2, A_2) \wedge (\Psi_3, A_3) = ((\Psi_1, A_1) \vee ((\Psi_2, A_2)) \wedge ((\Psi_1, A_1) \vee (\Psi_3, A_3))$

(4). $(\Psi_1, A_1) \wedge ((\Psi_2, A_2) \vee (\Psi_3, A_3)) = ((\Psi_1, A_1) \wedge ((\Psi_2, A_2)) \vee ((\Psi_1, A_1) \wedge (\Psi_3, A_3))$

4. An Applications to Hypersoft expert set

In this section, an application of hypersoft expert set theory in a decision making problem, is presented.

Statement of the problem

A manufacturing company advertises a "job opportunity" to fill its a vacant position. Its main slogan is "the right person for the right post". Eight applications received from the suitable candidates and company wants to complete this hiring process through through the selection board of some experts with some prescribed attributes.

Proposed Algorithm

The following algorithm may be followed by the company to fill the position.

(1). Construct hypersoft soft expert set (Ψ, K),

(2). Determine an agree-hypersoft expert set and a aisagree-hypersoft expert set,

(3). Compute $d_i = \sum_i c_{ij}$ for agree-hypersoft expert set,

(4). Determine $f_i = \sum_i c_{ij}$ for disagree-hypersoft expert set,

(5). Determine $g_j = d_j - f_j$ for agree-hypersoft expert set,

(6). Compute n, for which $p_n = \max p_j$ for agree-hypersoft expert set,

Step-1

Let eight candidates form the universe of discourse $\Omega = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$ and X $= \{E_1, E_2, E_3\}$ be a set of experts (committee members) for this recruitment process. The following are the attribute-valued sets for prescribed attributes:

$H_1 = Qualification = \{h_1, h_2\}$

$H_2 = Experience = \{h_3, h_4\}$

$H_3 = ComputerKnowledge = \{h_5, h_6\}$

$H_4 = Confidence = \{h_7, h_8\}$

$H_5 = Skills = \{h_9, h_{10}\}$

and then

$H = H_1 \times H_2 \times H_3 \times H_4 \times H_5$

$$H = \left\{ \begin{array}{l} (h_1,h_3,h_5,h_7,h_9),(h_1,h_3,h_5,h_7,h_{10}),(h_1,h_3,h_5,h_8,h_9),(h_1,h_3,h_5,h_8,h_{10}),(h_1,h_3,h_6,h_7,h_9), \\ (h_1,h_3,h_6,h_7,h_{10}),(h_1,h_3,h_6,h_8,h_9),(h_1,h_3,h_6,h_8,h_{10}),(h_1,h_4,h_5,h_7,h_9),(h_1,h_4,h_5,h_7,h_{10}), \\ (h_1,h_4,h_5,h_8,h_9),(h_1,h_4,h_5,h_8,h_{10}),(h_1,h_4,h_6,h_7,h_9),(h_1,h_4,h_6,h_7,h_{10}),(h_1,h_4,h_6,h_8,h_9), \\ (h_1,h_4,h_6,h_8,h_{10}),(h_2,h_3,h_5,h_7,h_9),(h_2,h_3,h_5,h_7,h_{10}),(h_2,h_3,h_5,h_8,h_9),(h_2,h_3,h_5,h_8,h_{10}), \\ (h_2,h_3,h_6,h_7,h_9),(h_2,h_3,h_6,h_7,h_{10}),(h_2,h_3,h_6,h_8,h_9),(h_2,h_3,h_6,h_8,h_{10}),(h_2,h_4,h_5,h_7,h_9), \\ (h_2,h_4,h_5,h_7,h_{10}),(h_2,h_4,h_5,h_8,h_9),(h_2,h_4,h_5,h_8,h_{10}),(h_2,h_4,h_6,h_7,h_9),(h_2,h_4,h_6,h_7,h_{10}), \\ (h_2,h_4,h_6,h_8,h_9),(h_2,h_4,h_6,h_8,h_{10}) \end{array} \right\}$$

and now take $K \subseteq H$ as

$K = \{a_1 = (h_1,h_3,h_5,h_7,h_9), a_2 = (h_1,h_3,h_6,h_7,h_{10}), a_3 = (h_1,h_4,h_6,h_8,h_9), a_4 = (h_2,h_3,h_6,h_8,h_9), a_5 = (h_2,h_4,h_6,h_7,h_{10})\}$

and

$$(\Psi, \mathrm{K}) = \left\{ \begin{array}{l} ((a_1,E_1,1) = \{c_1,c_2,c_4,c_7,c_8\}),((a_1,E_2,1) = \{c_1,c_4,c_5,c_8\}), \\ ((a_1,E_3,1) = \{c_1,c_3,c_4,c_5,c_6,c_7,c_8\}), \\ ((a_2,E_1,1) = \{c_3,c_5,c_8\}),((a_2,E_2,1) = \{c_1,c_3,c_4,c_5,c_6,c_8\}), \\ ((a_2,E_3,1) = \{c_1,c_2,c_4,c_7,c_8\}), \\ ((a_3,E_1,1) = \{c_3,c_4,c_5,c_7\}),((a_3,E_2,1) = \{c_1,c_2,c_5,c_8\}), \\ ((a_3,E_3,1) = \{c_1,c_7,c_8\}), \\ ((a_4,E_1,1) = \{c_1,c_7,c_8\}),((a_4,E_2,1) = \{c_5,c_1,c_4,c_8\}), \\ ((a_4,E_3,1) = \{c_1,c_6,c_7,c_8\}), \\ ((a_5,E_1,1) = \{c_1,c_3,c_4,c_5,c_7,c_8\}),((a_5,E_2,1) = \{c_1,c_4,c_5,c_8\}), \\ ((a_5,E_3,1) = \{c_1,c_3,c_4,c_5,c_7,c_8\}), \\ ((a_1,E_1,0) = \{c_3,c_5,c_6\}),((a_1,E_2,0) = \{c_2,c_3,c_6,c_7\}), \\ ((a_1,E_3,0) = \{c_2,c_5\}), \\ ((a_2,E_1,0) = \{c_1,c_2,c_4,c_5,c_6,c_7\}),((a_2,E_2,0) = \{c_2,c_7\}), \\ ((a_2,E_3,0) = \{c_2,c_3,c_4,c_5,c_6\}), \\ ((a_3,E_1,0) = \{c_1,c_2,c_6,c_8\}),((a_3,E_2,0) = \{c_3,c_4,c_6,c_7\}), \\ ((a_3,E_3,0) = \{c_2,c_3,c_4,c_5,c_7\}), \\ ((a_4,E_1,0) = \{c_2,c_3,c_3,c_4,c_5,c_7\}),((a_4,E_2,0) = \{c_2,c_3,c_6,c_7\}), \\ ((a_4,E_3,0) = \{c_2,c_3,c_4,c_5\}), \\ ((a_5,E_1,0) = \{c_4,c_6,c_7\}),((a_5,E_2,0) = \{c_2,c_3,c_6,c_7\}), \\ ((a_5,E_3,0) = \{c_2,c_4,c_6\}) \end{array} \right\}$$

is a hypersoft expert set.

Step-2

Table 1 presents an agree-hypersoft expert set and table 2 presents a disagree-hypersoft expert

Table 1. Agree-hypersoft expert set

v	c_1	c_2	c_3	c_4	c_5	c_6	c_8	c_7
(a_1, E_1)	✓	✓	×	✓	×	×	✓	✓
(a_2, E_1)	×	×	✓	×	✓	×	×	✓
(a_3, E_1)	×	×	✓	✓	✓	×	✓	×
(a_4, E_1)	✓	×	×	×	✓	×	✓	✓
(a_5, E_1)	✓	✓	✓	×	✓	×	×	✓
(a_1, E_2)	✓	×	×	✓	×	×	×	✓
(a_2, E_2)	✓	×	✓	✓	✓	✓	×	✓
(a_3, E_2)	×	×	✓	✓	✓	×	✓	×
(a_4, E_2)	✓	×	×	✓	✓	×	×	✓
(a_5, E_2)	✓	×	×	✓	✓	×	×	✓
(a_1, E_3)	✓	×	✓	✓	×	✓	✓	✓
(a_2, E_3)	✓	✓	×	✓	×	×	✓	✓
(a_3, E_3)	✓	×	×	×	×	×	✓	✓
(a_4, E_3)	✓	×	×	×	×	✓	✓	✓
(a_5, E_3)	✓	×	✓	×	✓	×	✓	✓
$d_j = \sum_i c_{ij}$	$d_1 = 12$	$d_2 = 3$	$d_3 = 7$	$d_4 = 9$	$d_5 = 9$	$d_6 = 3$	$d_7 = 9$	$d_8 = 13$

set respectively, such that if $c_i \in F_1(\beta)$ then $c_{ij} = \checkmark = 1$ otherwise $c_{ij} = \times = 0$, and if

$$c_i \in F_0(\beta)$$

then $c_{ij} = \checkmark = 1$ otherwise $c_{ij} = \times = 0$ where c_{ij} are the entries in tables 1 and 2.

Step-(3-6)

Table 3 presents $d_i = \sum_i c_{ij}$ for agree-hypersoft expert set, $f_i = \sum_i c_{ij}$ for disagree-hypersoft expert set, $g_j = d_j - f_j$ for agree-hypersoft expert set, and n, for which $p_n =$ max p_j for agree-hypersoft expert set.

Decision

As g_8 is maximum, so candidate c_8 is preferred to be selected for the said post. Then max g_8, so the committee will choose candidate 8 for the job.

5. Conclusions

Insufficiency of soft set and expert set for multi-attribute function (attribute-valued sets) is addressed with the development and characterization of novel hybrid structure i.e. hypersoft expert set, in this study. Moreover

(1) The fundamentals of hypersoft expert set (HSE-Set) are established and the basic properties of HSE-Set like subset, superset, equal sets, not set, agree HSE-Set and disagree HSE-Set are described with examples.

Table 2. Disagree-hypersoft expert set

V	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
(a_1, E_1)	×	×	✓	×	×	✓	×	×
(a_2, E_1)	✓	✓	×	✓	×	✓	✓	×
(a_3, E_1)	✓	✓	×	×	×	✓	×	✓
(a_4, E_1)	×	✓	✓	✓	✓	✓	×	×
(a_5, E_1)	×	×	×	✓	×	✓	✓	×
(a_1, E_2)	×	✓	✓	×	×	✓	✓	×
(a_2, E_2)	×	✓	×	×	×	×	✓	×
(a_3, E_2)	✓	✓	×	×	×	✓	×	✓
(a_4, E_2)	×	✓	✓	×	×	✓	✓	×
(a_5, E_2)	×	✓	✓	×	×	✓	✓	×
(a_1, E_3)	×	✓	×	×	✓	×	×	×
(a_2, E_3)	×	×	✓	×	✓	✓	×	×
(a_3, E_3)	×	✓	✓	✓	✓	✓	×	×
(a_4, E_3)	×	✓	✓	✓	✓	×	×	×
(a_5, E_3)	×	✓	×	✓	×	✓	×	×
$f_i = \sum_i c_{ij}$	$f_1 = 3$	$f_2 = 12$	$f_3 = 8$	$f_4 = 6$	$f_5 = 5$	$f_6 = 12$	$f_7 = 6$	$f_8 = 2$

Table 3. Optimal

$d_i = \sum_i c_{ij}$	$f_i = \sum_i c_{ij}$	$g_j = d_j - f_j$
$d_1 = 12$	$f_1 = 3$	$g_1 = 9$
$d_2 = 3$	$f_2 = 12$	$g_2 = -9$
$d_3 = 7$	$f_3 = 8$	$g_3 = -1$
$d_4 = 9$	$f_4 = 6$	$g_4 = 3$
$d_5 = 9$	$f_5 = 5$	$g_5 = 4$
$d_6 = 3$	$f_6 = 12$	$g_6 = -9$
$d_7 = 9$	$f_7 = 6$	$g_7 = 3$
$d_8 = 13$	$f_8 = 2$	$g_8 = 11$

(2) The essential set-theoretic operations on HSE-Set like complement, union, intersection, OR and AND operations are established and some laws such as commutative, associative and De Morgan are presented with suitable examples.

(3) A decision-making application regarding recruitment process is presented with the help of proposed algorithm.

(4) A daily life based example is discussed for the understanding of decision making process.

(5) Future work may include the extension of the presented work for other hypersoft-like hybrids i.e. fuzzy, intuitionistic fuzzy, interval-valued fuzzy, pythagorean fuzzy etc.

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References

1. Molodtsov, D. Soft set theoryfirst results, *Computers and Mathematics with Applications.* **1999,** 37(4-5), 19-31.
2. Chen, D., Tsang, E. C. C., Yeung, D. S. and Wang, X. The parameterization reduction of soft sets and its applications, *Computers and Mathematics with Applications.*, **2005**, 49(5-6), 757-763.
3. Maji, P. K., Roy, A. R. and Biswas, R. Soft set theory, *Computers and Mathematics with Applications.*, **2003**, 45(4-5), 555-562.
4. Maji, P. K., Roy, A. R. and Biswas, R. An application of soft sets in a decision making problem, *Computers and Mathematics with Applications.* **2002**, 44(8-9), 1077-1083.
5. Alkhazaleh, S., and A. R. Salleh, and Hassan, N. Soft multisets theory, *Applied Mathematical Sciences.* **2011** 5(72), 3561-3573.
6. Alkhazaleh, S., and A. R. Salleh, and Hassan, N. Possibility fuzzy soft set, *Advances in Decision Sciences*, **2011**.
7. Alkhazaleh, S., and A. R. Salleh, and Hassan, N. Fuzzy parameterized interval-valued fuzzy soft set, *Applied Mathematical Sciences.* **2011**, 5(67), 3335-3346.
8. Abdel-Basset, M., Saleh, M., Gamal, A., Smarandache, F. An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. *Applied Soft Computing*,**2019**, 77, 438-452.
9. Abdel-Baset, M., Chang, V., Gamal, A., Smarandache, F. An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field. *Computers in Industry*, **2019**, 106, 94-110.
10. Abdel-Baset, M., Chang, V., Gamal, A. Evaluation of the green supply chain management practices: A novel neutrosophic approach. *Computers in Industry*, **2019**, 108, 210-220.
11. Abdel-Basset, M., Gamal, A., Manogaran, G., Long, H. V. A novel group decision making model based on neutrosophic sets for heart disease diagnosis. Multimedia Tools and Applications, **2019**, 1-26.
12. Abdel-Basset, Mohamed, Gamal, A., Son, L.H., Smarandache, F., A bipolar neutrosophic multi criteria decision making framework for professional selection. *Appl. Sci.*, **2019**, 10.
13. Abdel-Basset, M., Manogaran, G., Gamal, A., Chang, V., A Novel Intelligent Medical Decision Support Model Based on Soft Computing and IoT. *IEEE Internet Things J.*, **2019**, 7.
14. Ali, M. I., Feng, F., Liu, X., Min, W. K. and Shabir, M., On some new operations in soft set theory, *Computers and Mathematics with Applications.* **2009**, 57(9), 1547-1553.
15. Alkhazaleh, S., and A. R. Salleh, Soft Expert Sets. Adv. Decis. Sci., 757868-1, 2011.
16. Al-Quran, A. and Hassan, N. Neutrosophic vague soft expert set theory. *Journal of Intelligent and Fuzzy Systems.* **2016**, 30(6), 3691-3702.
17. Alkhazaleh, S., and Salleh, A. R. Fuzzy soft expert set and its application. *Applied Mathematics*, **2014**.
18. Bashir, M., and Salleh, A. R. Possibility fuzzy soft expert set. *Open Journal of Applied Sciences*, **2012**, 12, 208-211.
19. Bashir, M., and Salleh, A. R. Fuzzy parameterized soft expert set. In Abstract and Applied Analysis, **2012**. *Hindawi.*
20. Sahin, M., Alkhazaleh, S., and Ulucay, V. Neutrosophic soft expert sets. *Applied Mathematics*, **2015**, 6(1), 116.
21. Alhazaymeh, K., and Hassan, N. Mapping on generalized vague soft expert set. *International Journal of Pure and Applied Mathematics*, **2014**, 93(3), 369-376.

22. Alhazaymeh, K., and Hassan, N. Generalized vague soft expert set. *International Journal of Pure and Applied Mathematics*,**2014**, 93(3), 351-360.
23. Alhazaymeh, K., and Hassan, N. Application of generalized vague soft expert set in decision making. *International Journal of Pure and Applied Mathematics*, **2014**, 93(3), 361-367.
24. Hassan, N., Uluay, V., and ahin, M. Q-neutrosophic soft expert set and its application in decision making. *International Journal of Fuzzy System Applications* (IJFSA), **2018**, 7(4), 37-61.
25. Uluay, V., ahin, M., and Hassan, N. Generalized neutrosophic soft expert set for multiple-criteria decision-making. *Symmetry*, **2018** 10(10), 437.
26. Al-Qudah, Y., and Hassan, N. Bipolar fuzzy soft expert set and its application in decision making. *International Journal of Applied Decision Sciences*, **2017**, 10(2), 175-191.
27. Al-Qudah, Y., and Hassan, N. Complex multi-fuzzy soft expert set and its application. Int. J. Math. *Comput. Sci*, **2017**, 14(1), 149-176.
28. Al-Quran, A., and Hassan, N. The complex neutrosophic soft expert set and its application in decision making. *Journal of Intelligent and Fuzzy Systems*, **2018**, 34(1), 569-582.
29. Pramanik, S., Dey, P. P., and Giri, B. C. TOPSIS for single valued neutrosophic soft expert set based multi-attribute decision making problems. *Neutrosophic Sets and Systems*, **2015**, 10, 88-95.
30. Abu Qamar, M., and Hassan, N. Generalized Q-neutrosophic soft expert set for decision under uncertainty. *Symmetry*, **2018**, 10(11), 621.
31. Adam, F., and Hassan, N. Multi Q-fuzzy soft expert set and its application. *Journal of Intelligent and Fuzzy Systems*, **2016**, 30(2), 943-950.
32. Ulucay, V., Sahin, M., and Olgun, N. Time-neutrosophic soft expert sets and its decision making problem. *Infinite Study*. **2018**.
33. Al-Quran, A., and Hassan, N. Fuzzy parameterised single valued neutrosophic soft expert set theory and its application in decision making. *International Journal of Applied Decision Science*, **2016**, 9(2), 212-227.
34. Hazaymeh, A. A., Abdullah, I. B., Balkhi, Z. T., and Ibrahim, R. I. Generalized fuzzy soft expert set. *Journal of Applied Mathematics*, **2012**.
35. Smarandache, F., Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set, *Neutrosophic Sets Syst.* **2018**, 22, 168-170.
36. Saeed, M.; Ahsan, M.; Siddique, M. K., and Ahmad, M. R., A Study of The Fundamentals of Hypersoft Set Theory, *International Journal of Scientific and Engineering Research* **2020**, 11.
37. Saeed, M., Rahman, A. U., Ahsan, M. and Smarandache, F. An Inclusive Study on Fundamentals of Hypersoft Set. Theory and Application of Hypersoft Set, 1.
38. Abbas, M.; Murtaza, G.; and Smarandache, F., Basic operations on hypersoft sets and hypersoft point, *Neutrosophic Sets Syst.* **2020**, 35, 407-421.
39. Rahman, A.U., Saeed, M., Smarandache, F., and Ahmad, M.R., Development of Hybrids of Hypersoft Set with Complex Fuzzy Set, Complex Intuitionistic Fuzzy set and Complex Neutrosophic Set, *Neutrosophic Sets and Syst.* **2020**, 38, 335-354.
40. Rahman, A.U., Saeed, M., and Smarandache, F., Convex and Concave Hypersoft Sets with Some Properties, *Neutrosophic Sets Syst.* **2020**, 38, 497-508.

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Common Fixed Point Results in Neutrosophic Metric Spaces

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Abstract. In this manuscript, We investigate new outcomes in the field of neutrosophic metric space due to Kirisci and Simsek. We analyse weakly commuting and R-weakly commuting in the setting of neutrosophic metric space and prove some fixed point results. We develop the results to obtain common fixed point theorem in neutrosophic version. We validate our results by suitable examples.

Keywords: Fixed point; Neutrosophic metric Space; Banach contraction; Weakly commuting; R-Weakly commuting.

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1. Introduction

Fuzzy Sets(FSs) was presented by Zadeh [22] as a class of elements with a grade of membership. Kramosil and Michalek [10] defined new notion called Fuzzy Metric Space (FMS). George and Veeramani [6] redefined the concept of FMS with the assistance of triangular norms. Afterward, numerous researchers have analyzed the characteristics of FMS and proved many fixed point results. FMS has wide range of applications in applied science fields such as fixed point theory, decision making, medical imaging and signal processing. In 1986, Atanassov [1] defined Intuitionistic Fuzzy Sets(IFSs) by adding non - membership to FSs. Park [15] defined Intuitionistic Fuzzy Metric Space (IFMS) from the concept of IFSs and given some fixed point results. Fixed point theorems related to FMS and IFMS given by Alaca et al [2] and numerous researchers [5, 12, 17]. In 1998, Smarandache [20] characterized the new idea called neutrosophic set. In general the notion of fuzzy set and IFS deal with degree of membership

and non - membership respectively. Neutrosophic set is a generalized state of Fuzzy and Intuitionistic Fuzzy Set by incorporating degree of indeterminacy. In addition, several researchers contributed significantly to develop the neutrosophic theory. Recently, Baset et al. [3, 4] explored the neutrosophic applications in different fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making, financial performance and evaluation of manufacturing industries. In 2019, Kirisci et al [11] defined neutrosophic metric space as a generalization of IFMS and brings about fixed point theorems in complete neutrosophic metric space. In 2020, Sowndrarajan and Jeyaraman et al [21] proved some fixed point results in neutrosophic metric spaces.
In this paper, we define the concept of weakly commuting and R-weakly commuting mappings in the setting of neutrosophic metric space and prove common fixed point theorems with the help of Pant's theorem. [14].

2. Preliminaries

Definition 2.1 [20] Let Σ be a non-empty fixed set. A Neutrosophic Set (NS for short) N in Σ is an object having the form $N = \{\langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle : a \in \Sigma\}$ where the functions $\xi_N(a), \varrho_N(a)$ and $\nu_N(a)$ represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element a $\in$ N to the set Σ.
A neutrosophic set $N = \{\langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle : a \in \Sigma\}$ is expressed as an ordered triple $N = \langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle$ in Σ. In NS, there is no restriction on $(\xi_N(a), \varrho_N(a), \nu_N(a))$ other than they are subsets of $]^{-}0, 1^{+}[$.
Triangular Norms (TNs) were initiated by menger. Triangular Co norms(TCs) knowns as dual operations of triangular norms.

Definition 2.2 [7] A binary operation $\star : [0, 1] \times [0, 1] \to [0, 1]$ is called continuous t - norm (CTN) if it satisfies the following conditions;
For all $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in [0,\ 1]$
(i) $\zeta_1 \star 1 = \zeta_1$;
(ii) If $\zeta_1 \le \zeta_3$ and $\zeta_2 \le \zeta_4$ then $\zeta_1 \star \zeta_2 \le \zeta_3 \star \zeta_4$;
(iii) $\star$ is continuous;
(iv) $\star$ is commutative and associative.

Definition 2.3 [7] A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is called continuous t - co norm (CTC) if it satisfies the following conditions;
For all $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in [0,\ 1]$
(i) $\zeta_1 \diamond 0 = \zeta_1$;

(ii) If $\zeta_1 \leq \zeta_3$ and $\zeta_2 \leq \zeta_4$ then $\zeta_1 \diamond \zeta_2 \leq \zeta_3 \diamond \zeta_4$;

(iii) $\diamond$ is continuous;

(iv) $\diamond$ is commutative and associative.

Remark 2.4 [11] From the definitions of CTN and CTC, we note that if we take $0 < \zeta_1, \zeta_2 < 1$ for $\zeta_1 < \zeta_2$ then there exist $0 < \zeta_3, \zeta_4 < 1$ such that $\zeta_1 \star \zeta_3 \geq \zeta_2$ and $\zeta_1 \geq \zeta_2 \diamond \zeta_4$. Further we choose $\zeta_5 \in (0,1)$ then there exists $\zeta_6, \zeta_7 \in (0,1)$ such that $\zeta_6 \star \zeta_6 \geq \zeta_5$ and $\zeta_7 \diamond \zeta_7 \leq \zeta_5$.

3. Neutrosophic Metric Spaces

In this section, we apply neutrosophic theory to generalize the intuitionistic fuzzy metric space. we also discuss some properties and examples in it.

Definition 3.1 [21] A 6 - tuple $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$is called Neutrosophic Metric Space(NMS), if Σ is an arbitrary non empty set, $\star$ is a neutrosophic CTN and $\diamond$ is a neutrosophic CTC and Ξ, Θ, Υ are neutrosophic sets on $\Sigma^2 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\zeta, \eta, \omega \in \Sigma, \lambda \in \mathbb{R}^+$

(i) $0 \leq \Xi(\zeta,\eta,\lambda) \leq 1;\ 0 \leq \Theta(\zeta,\eta,\lambda) \leq 1;\ 0 \leq \Upsilon(\zeta,\eta,\lambda) \leq 1;$

(ii) $\Xi(\zeta,\eta,\lambda) + \Theta(\zeta,\eta,\lambda) + \Upsilon(\zeta,\eta,\lambda) \leq 3;$

(iii) $\Xi(\zeta,\eta,\lambda) = 1$ if and only if $\zeta = \eta$;

(iv) $\Xi(\zeta,\eta,\lambda) = \Xi(\eta,\zeta,\lambda)$ for $\lambda > 0$;

(v) $\Xi(\zeta,\eta,\lambda) \star \Xi(\eta,\zeta,\mu) \leq \Xi(\zeta,\omega,\lambda+\mu)$, for all $\lambda, \mu > 0$;

(vi) $\Xi(\zeta,\eta,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous ;

(vii) $lim_{\lambda\to\infty}\Xi(\zeta,\eta,\lambda) = 1$ for all $\lambda > 0$;

(viii) $\Theta(\zeta,\eta,\lambda) = 0$ if and only if $\zeta = \eta$;

(ix) $\Theta(\zeta,\eta,\lambda) = \Theta(\eta,\zeta,\lambda)$ for $\lambda > 0$;

(x) $\Theta(\zeta,\eta,\lambda) \diamond \Theta(\zeta,\omega,\mu) \geq \Theta(\zeta,\omega,\lambda+\mu)$, for all $\lambda, \mu > 0$;

(xi) $\Theta(\zeta,\eta,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous ;

(xii) $lim_{\lambda\to\infty}\Theta(\zeta,\eta,\lambda) = 0$ for all $\lambda > 0$;

(xiii) $\Upsilon(\zeta,\eta,\lambda) = 0$ if and only if $\zeta = \eta$;

(xiv) $\Upsilon(\zeta,\eta,\lambda) = \Upsilon(\eta,\zeta,\lambda)$ for $\lambda > 0$;

(xv) $\Upsilon(\zeta,\eta,\lambda) \diamond \Upsilon(\zeta,\omega,\mu) \geq \Upsilon(\zeta,\omega,\lambda+\mu)$, for all $\lambda, \mu > 0$;

(xvi) $\Upsilon(\zeta,\eta,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous ;

(xvii) $lim_{\lambda\to\infty}\Upsilon(\zeta,\eta,\lambda) = 0$ for all $\lambda > 0$;

(xviii) If $\lambda > 0$ then $\Xi(\zeta,\eta,\lambda) = 0, \Theta(\zeta,\eta,\lambda) = 1, \Upsilon(\zeta,\eta,\lambda) = 1$.

Then (Ξ, Θ, Υ) is called Neutrosophic Metric on Σ. The functons Ξ, Θ and Υ denote degree of closedness, neturalness and non - closedness between ζ and η with respect to λ respectively.

Example 3.2 [21] Let (Σ, d) be a metric space. Define $\zeta\star\eta = min\{\zeta, \eta\}$ and $\zeta\diamond\eta = max\{\zeta, \eta\}$, and $\Xi, \Theta, \Upsilon : \Sigma^2 \times \mathbb{R}^+ \to [0, 1]$ defined by

$$\Xi(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}; \quad \Theta(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)}; \quad \Upsilon(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda}$$

for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is called neutrosophic metric space induced by a metric d the standard neutrosophic metric.

Remark 3.3 [11] In neutrosophic metric space Ξ is non - decreasing , Θ is a non - increasing, Υ is decreasing function for all $\zeta, \eta \in \Sigma$.

Definition 3.4 Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be neutrosophic metric space . Then

(a) $\{\zeta_n\}$ in Σ is converging to a point $\zeta \in \Sigma$ if for each $\lambda > 0$

$$lim_{n\to\infty}\Xi(\zeta_n, \zeta, \lambda) = 1; \quad lim_{n\to\infty}\Theta(\zeta_n, \zeta, \lambda) = 0; \quad lim_{n\to\infty}\Upsilon(\zeta_n, \zeta, \lambda) = 0.$$

(b) ζ_n in Σ is called a Cauchy if for each $\epsilon > 0$ and $\lambda > 0$ there exist $N \in \mathbb{N}$ such that $\Xi(\zeta_{n+p}, \zeta_n, \lambda) = 1$; $\Theta(\zeta_{n+p}, \zeta_n, \lambda) = 0$; $\Upsilon(\zeta_{n+p}, \zeta_n, \lambda) = 0$.

(c) $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is said to be complete neutrosophic metric space if every Cauchy sequence is convergence in it.

4. **Main Results**

In this section, we present some interesting concepts such as weakly commuting and R-weakly commuting as an extensive work from Banach's contraction principle with suitable examples.

Theorem 4.1 Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be a complete neutrosophic metric space. Let $\varphi, \varrho : \Sigma \to \Sigma$ be functions satisfying the following conditions:

(i) $\varphi(\Sigma) \subseteq \varrho(\Sigma)$;

(ii) ϱ is continuous;

(iii) there exists $0 \le k \le 1$ such that, for all $\zeta, \eta, \omega \in \Sigma$

$$\Xi(\varphi(\zeta), \varphi(\eta), k\lambda) \ge \Xi(\varrho(\zeta), \varrho(\eta), \lambda),$$

$$\Theta(\varphi(\zeta), \varphi(\eta), k\lambda) \le \Theta(\varrho(\zeta), \varrho(\eta), \lambda),$$

$$\Upsilon(\varphi(\zeta), \varphi(\eta), k\lambda) \le \Upsilon(\varrho(\zeta), \varrho(\eta), \lambda).$$

Then ϱ and φ have a unique common unique fixed point in Σ provided ϱ and φ commute on Σ.

Proof: Let $\zeta_0 \in \Sigma$, from (i) we can get ζ_1 such that $\varrho(\zeta_1) = \varphi(\zeta_0)$. By mathematical induction, we define ζ_n in Σ such that $\varrho(\zeta_n) = \varphi(\zeta_{n-1})$. Again by induction

$$\Xi(\varrho(\zeta_n),\varrho(\zeta_{n+1}),\lambda)=\Xi(\varphi(\zeta_{n-1}),\varphi(\zeta_n),\lambda)\quad\geq\Xi(\varrho(\zeta_{n-1}),\varrho(\zeta_n),\frac{\lambda}{k})\cdots\quad\geq\Xi(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{k^n}),$$
$$\Theta(\varrho(\zeta_n),\varrho(\zeta_{n+1}),\lambda)=\Theta(\varphi(\zeta_{n-1}),\varphi(\zeta_n),\lambda)\quad\leq\Theta(\varrho(\zeta_{n-1}),\varrho(\zeta_n),\frac{\lambda}{k})\cdots\quad\leq\Theta(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{k^n}),$$
$$\Upsilon(\varrho(\zeta_n),\varrho(\zeta_{n+1}),\lambda)=\Upsilon(\varphi(\zeta_{n-1}),\varphi(\zeta_n),\lambda)\quad\leq\Upsilon(\varrho(\zeta_{n-1}),\varrho(\zeta_n),\frac{\lambda}{k})\cdots\quad\leq\Upsilon(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{k^n}),$$

for all $n > 0$ and $\lambda > 0$. Thus, for any non-negative integer p, we have

$$\begin{aligned}\Xi(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)&\geq\Xi\varrho(\zeta_n),\varrho(\zeta_{n+1}),\frac{\lambda}{k})\star\cdots^{(p-times)}\cdots\star\Xi(\varrho(\zeta_{n+p-1}),\varrho(\zeta_{n+p}),\frac{\lambda}{k})\\&\geq\Xi(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})\star\cdots^{(p-times)}\cdots\star\Xi(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^{n+p-1}}),\\\Theta(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)&\leq\Theta(\varrho(\zeta_n),\varrho(\zeta_{n+1}),\frac{\lambda}{k})\diamond\cdots^{(p-times)}\cdots\diamond\Theta(\varrho(\zeta_{n+p-1}),\varrho(\zeta_{n+p}),\frac{\lambda}{k})\\&\leq\Theta(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})\diamond\cdots^{(p-times)}\cdots\diamond\Theta(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^{n+p-1}}),\\\Upsilon(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)&\leq\Upsilon(\varrho(\zeta_n),\varrho(\zeta_{n+1}),\frac{\lambda}{k})\diamond\cdots^{(p-times)}\cdots\diamond\Upsilon(\varrho(\zeta_{n+p-1}),\varrho(\zeta_{n+p}),\frac{\lambda}{k})\\&\leq\Upsilon(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})\diamond\cdots^{(p-times)}\cdots\diamond\Upsilon(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^{n+p-1}}).\end{aligned}$$

by conditions (vii), (xii) and (xvii) of definition (3.1), we get

$$lim_{n\to\infty}\Xi(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})=1,$$
$$lim_{n\to\infty}\Theta(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})=0,$$
$$lim_{n\to\infty}\Upsilon(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})=0.$$

It follows that

$$lim_{n\to\infty}\Xi(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)\geq 1\star\cdots^{(p-times)}\cdots\star 1=1,$$
$$lim_{n\to\infty}\Theta(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)\leq 0\diamond\cdots^{(p-times)}\cdots\diamond 0=0,$$
$$lim_{n\to\infty}\Upsilon(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)\leq 0\diamond\cdots^{(p-times)}\cdots\diamond 0=0.$$

Since Σ is complete NMS, $\{\varrho(\zeta_n)\}$ is a Cauchy sequence that converges to a point η and $\varphi(\zeta_{n-1}) = \varrho(\zeta_n)$ converges to the same point η. From (iii), it is shown that continuity of ϱ implies continuity of φ. Hence, $\{\varphi(\varrho(\zeta_n))\}$ converges to $\varphi(\eta)$. However, ϱ and φ are commute on Σ, $\varphi(\varrho(\zeta_n)) = \varrho(\varphi(\zeta_n))$ and so $\varrho(\varphi(\zeta_n))$ converges to $\varrho(\eta)$. Thus $\varrho(\eta) = \varphi(\eta)$, which implies $\varrho(\varrho(\eta)) = \varrho(\varphi(\eta))$. Thus, we get

$$\begin{aligned}
\Xi(\varphi(\eta),\varphi(\varphi(\eta)),\lambda) &\geq \Xi(\varrho(\eta),\varrho(\varphi(\eta)),\frac{\lambda}{k})\\
&\geq \Xi(\varphi(\eta),\varphi(\varphi(\eta)),\frac{\lambda}{k})\\
&\geq \cdots\\
&\geq \Xi(\varphi(\eta),\varphi(\varphi(\eta)),\frac{\lambda}{k^n}),\\
\Theta(\varphi(\eta),\varphi(\varphi(\eta)),\lambda) &\leq \Theta(\varrho(\eta),\varrho(\varphi(\eta)),\frac{\lambda}{k})\\
&\leq \Theta(\varphi(\eta),\varphi(\varphi(\eta)),\frac{\lambda}{k})\\
&\leq \cdots\\
&\leq \Theta(\varphi(\eta),\varphi(\varphi(\eta)),\frac{\lambda}{k^n}),\\
\Upsilon(\varphi(\eta),\varphi(\varphi(\eta)),\lambda) &\leq \Upsilon(\varrho(\eta),\varrho(\varphi(\eta)),\frac{\lambda}{k})\\
&\leq \Upsilon(\varphi(\eta),\varphi(\varphi(\eta)),\frac{\lambda}{k})\\
&\leq \cdots\\
&\leq \Upsilon(\varphi(\eta)),\varphi(\varphi(\eta)),\frac{\lambda}{k^n}).
\end{aligned}$$

Therefore from the definition of (3.1), it follows that $\varphi(\eta) = \varphi(\varphi(\eta))$. Thus $\varphi(\eta) = \varphi(\varphi(\eta)) = \varrho(\varphi(\eta))$. Hence $\varphi(\eta)$ is a common fixed point of the mappings ϱ and φ.

To prove uniqueness, let us assume η and ω are two fixed points of ϱ and φ, then

$$\begin{aligned}
&1 \geq \Xi(\zeta,\omega,\lambda) = \Xi(\varphi(\eta),\varphi(\omega),\lambda) \geq \Xi(\varrho(\zeta),\varrho(\omega),\frac{\lambda}{k})\\
&\quad = \Xi(\eta,\omega,\frac{\lambda}{k}) \geq \cdots \geq \Xi(\zeta,\omega,\frac{\lambda}{k^n}),\\
&0 \leq \Theta(\zeta,\omega,\lambda) = \Theta(\varphi(\eta),\varphi(\omega),\lambda) \leq \Theta(\varrho(\zeta),\varrho(\omega),\frac{\lambda}{k})\\
&\quad = \Theta(\eta,\omega,\frac{\lambda}{k}) \leq \cdots \leq \Theta(\zeta,\omega,\frac{\lambda}{k^n}),\\
&0 \leq \Upsilon(\zeta,\omega,\lambda) = \Upsilon(\varphi(\eta),\varphi(\omega),\lambda) \leq \Upsilon(\varrho(\zeta),\varrho(\omega),\frac{\lambda}{k})\\
&\quad = \Upsilon(\eta,\omega,\frac{\lambda}{k}) \leq \cdots \leq \Upsilon(\zeta,\omega,\frac{\lambda}{k^n}).
\end{aligned}$$

From the definition(3.1), we get

$$\lim_{n\to\infty}\Xi(\eta,\omega,\frac{\lambda}{k^n}) = 1, \quad \lim_{n\to\infty}\Theta(\eta,\omega,\frac{\lambda}{k^n}) = 0, \quad \lim_{n\to\infty}\Upsilon(\eta,\omega,\frac{\lambda}{k^n}) = 0.$$

It follows that

$$1 \geq \Xi(\eta,\omega,\lambda) \geq 1,\ 0 \leq \Theta(\eta,\omega,\lambda) \leq 0,\ 0 \leq \Upsilon(\eta,\omega,\lambda) \leq 0,$$

which states that $\eta = \omega$. Hence, we obtain a unique common fixed point of both φ and ϱ.

Example 4.2 Let $\Sigma = \{\frac{1}{n}; n \in \mathbb{N}\}$ with the standard metric $d(\zeta,\eta) = |\zeta - \eta|$. For all $\zeta, \eta \in \Sigma$ and $\lambda \in [0,\infty)$, define

$$\Xi(\zeta,\eta,\lambda) = \begin{cases} 0, & if \quad \lambda = 0 \\ \frac{\lambda}{\lambda + d(\zeta,\eta)}, & if \quad \lambda > 0 \end{cases}$$

$$\Theta(\zeta,\eta,\lambda) = \begin{cases} 1 & if \quad \lambda = 0 \\ \frac{d(\zeta,\eta)}{k\lambda + d(\zeta,\eta)} & if \quad k > 0,\ \lambda > 0 \end{cases}$$

$$\Upsilon(\zeta,\eta,\lambda) = \frac{d(\zeta,\eta)}{\lambda} \quad if \quad \lambda > 0.$$

for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is called complete neutrosophic metric space on Σ, Here $\star$ is defined by $\zeta \star \eta = \zeta\eta$ and $\diamond$ is defined as $\zeta \diamond \eta = min\{1,\ \zeta + \eta\}$. Define $\varphi(\zeta) = \frac{\zeta}{9}; \varrho(\zeta) = \frac{\zeta}{3}$. Clearly $\varphi(\Sigma) \subseteq \varrho(\Sigma)$, Also for $k = \frac{1}{3}$, we get

$$\Xi(\varphi(\zeta), \varphi(\eta), \frac{\lambda}{3}) = \frac{\frac{\lambda}{3}}{\frac{\lambda}{3} + d(\varphi(\zeta), \varphi(\eta))} \geq \frac{\lambda}{\lambda + \frac{d(\zeta,\eta)}{3}} = \Xi(\varrho(\zeta), \varrho(\eta), \lambda),$$

Similarly, we get

$$\Theta(\varphi(\zeta), \varphi(\eta), \frac{\lambda}{3}) \leq \Theta(\varrho(\zeta)\varrho(\eta), \lambda),$$
$$\Upsilon(\varphi(\zeta), \varphi(\eta), \frac{\lambda}{3}) \leq \Upsilon(\varrho(\zeta), \varrho(\eta), \lambda).$$

Hence the conditions in Theorem (4.1) are satisfied and so ϱ and φ have common fixed point 0.

Definition 4.3 [18] Let ϱ and φ be two self mappings from neutrosophic metric space $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ into itself. The mappings ϱ and φ is called weakly commuting if for all $\zeta \in \Sigma$

$$\Xi(\varrho\varphi(\zeta)) \geq \Xi(\varphi\varrho(\zeta)), \quad \Theta(\varrho\varphi(\zeta)) \leq \Theta(\varphi\varrho(\zeta)), \quad \Upsilon(\varrho\varphi(\zeta)) \leq \Upsilon(\varphi\varrho(\zeta)).$$

Definition 4.4 Let ϱ and φ be two self mappings from neutrosophic metric space $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ into itself. The mappings ϱ and φ is called R-weakly commuting if there exist a positive real number R such that for all $\zeta \in \Sigma$.

$$\Xi(\varrho\varphi(\zeta), \varphi\varrho(\zeta), \lambda) \geq \Xi(\varphi(\zeta), \varrho(\zeta), \frac{\lambda}{R}),$$
$$\Theta(\varrho\varphi(\zeta), \varphi\varrho(\zeta), \lambda) \leq \Theta(\varphi(\zeta), \varrho(\zeta), \frac{\lambda}{R}),$$
$$\Upsilon(\varrho\varphi(\zeta), \varphi\varrho(\zeta), \lambda) \leq \Upsilon(\varphi(\zeta), \varrho(\zeta), \frac{\lambda}{R}).$$

Remark 4.5 In Neutrosophic metric spaces, Weak commutativity implies R-weak commutativity, but weak commutativity can be derived from R-weakly commuting only when $R \leq 1$.

Example 4.6 Let $\Sigma = \mathbb{R}$ be set of all real numbers. $\star$ and $\diamond$ defined by $a \star b = ab, \quad a \diamond b = min\{1, a+b\}$, define $d(\zeta,\eta) = |\zeta - \eta|$

$$\Xi(\zeta,\eta,\lambda) = \left(exp\left(\frac{d(\zeta,\eta)}{\lambda}\right)\right)^{-1},$$

$$\Theta(\zeta,\eta,\lambda) = \frac{exp\left(\frac{d(\zeta,\eta)}{\lambda}\right) - 1}{exp\left(\frac{d(\zeta,\eta)}{\lambda}\right)},$$

$$\Upsilon(\zeta,\eta,\lambda) = exp\left(\frac{d(\zeta,\eta)}{\lambda}\right).$$

for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is a neutrosophic metric space. We define $\varrho(\zeta) = 2\zeta - 1$ and $g(\zeta) = \zeta^2$. Then , we have

$$\Xi(\varrho\varphi(\zeta), \varphi\varrho(\eta), \lambda) = \left(exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right)\right)^{-1},$$

$$\Theta(\varrho\varphi(\zeta), \varphi\varrho(\eta), \lambda) = \frac{exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right) - 1}{exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right)},$$

$$\Upsilon(\varrho\varphi(\zeta), \varphi\varrho(\eta), \lambda) = exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right).$$

Also, we have,

$$\Xi(\varrho(\zeta), \varphi(\eta), \frac{\lambda}{2}) = \left(exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right)\right)^{-1},$$

$$\Theta(\varrho(\zeta), \varphi(\eta), \frac{\lambda}{2}) = \frac{exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right) - 1}{exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right)},$$

$$\Upsilon(\varrho(\zeta), \varphi(\eta), \frac{\lambda}{2}) = exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right).$$

Therefore, the self mappings ϱ and φ are R-weakly commuting only for R = 2, but converse is not true since the exponential function is non-decreasing.
Now, we define R-weakly commuting on Σ and prove the neutrosophic version of Pant's theorem.

Definition 4.7 Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is a neutrosophic metric space and ϱ and φ be R-weakly commuting self-mappings of Σ satisfying the following condition:

$$\Xi(\varrho(\zeta), \varrho(\eta), \lambda) \geq r\ \Xi(\varphi(\zeta), \varphi(\eta), \lambda)$$

$$\Theta(\varrho(\zeta), \varrho(\eta), \lambda) \leq r\ \Theta(\varphi(\zeta), \varphi(\eta), \lambda)$$

$$\Upsilon(\varrho(\zeta), \varrho(\eta), \lambda) \leq r\ \Upsilon(\varphi(\zeta), \varphi(\eta), \lambda)$$

for all $\zeta, \eta \in \Sigma$. where $r : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $r(\lambda) < \lambda$ for all $\lambda > 0$. By hypothesis of theorem, ϱ and φ have a unique common fixed point in Σ.

Now, we prove the neutrosophic version of Pant's theorem.

Theorem 4.8 Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is a complete neutrosophic metric space and ϱ *and* φ be R-weakly commuting self-mappings of Σ satisfying the following condition:

(i) $\varphi(\Sigma) \subseteq \varrho(\Sigma)$;

(ii) ϱ or φ is continuous;

(iii) There exists $0 \le \lambda \le 1$ such that, for all $\zeta, \eta, \omega \in \Sigma$

$$\Xi(\varrho(\zeta), \varrho(\eta), \lambda) \ge \gamma(\Xi(\varphi(\zeta), \varphi(\eta), \lambda)),$$
$$\Theta(\varrho(\zeta), \varrho(\eta), \lambda) \le \gamma'(\Theta(\varphi(\zeta), \varphi(\eta), \lambda)),$$
$$\Upsilon(\varrho(\zeta), \varrho(\eta), \lambda) \le \gamma''(\Upsilon(\varphi(\zeta), \varphi(\eta), \lambda)),$$

where γ, γ' and $\gamma'' : [0\ 1] \to [0\ 1]$ are continuous function such that $\gamma(\lambda) > \lambda$, $\gamma'(\lambda) < \lambda$ and $\gamma''(\lambda) < \lambda$.

(iv) If the sequence $\{\zeta_n\}$ and $\{\eta_n\}$ in Σ are such that, for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$, $lim_{n\to\infty}\zeta_n = \zeta$ and $lim_{n\to\infty}\eta_n = \eta$ implies,

$$lim_{n\to\infty}\Xi(\zeta_n, \eta_n, \lambda) = \Xi(\zeta, \eta, \lambda),$$
$$lim_{n\to\infty}\Theta(\zeta_n, \eta_n, \lambda) = \Theta(\zeta, \eta, \lambda),$$
$$lim_{n\to\infty}\Upsilon(\zeta_n, \eta_n, \lambda) = \Upsilon(\zeta, \eta, \lambda).$$

Then ϱ and φ have a unique common unique fixed point in Σ.

Proof. Let ζ_0 be an arbitrary point in Σ. By the condition (i), Let $\zeta_1 \in \Sigma$ such that $\varrho(\zeta_0) = \varphi(\zeta_1)$. So we choose ζ_{n+1} such that $\varrho(\zeta_n) = \varphi(\zeta_{n+1})$ for all $n \ge 0$. Then, for all $\lambda > 0$,

$$\begin{aligned}
\Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda) &\ge \gamma(\Xi(\varphi(\zeta_n), \varphi(\zeta_{n+1}), \lambda)) \\
&= \gamma(\Xi(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda) \\
&> \Xi(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda) \qquad (4.8.1) \\
\Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda) &\le \gamma'(\Theta(\varphi(\zeta_n), \varphi(\zeta_{n+1}), \lambda)) \\
&= \gamma'(\Theta(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda) \\
&< \Theta(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda) \qquad (4.8.2) \\
\Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda) &\le \gamma''(\Upsilon(\varphi(\zeta_n), \varphi(\zeta_{n+1}), \lambda)) \\
&= \gamma''(\Upsilon(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda) \\
&< \Upsilon(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda) \qquad (4.8.3)
\end{aligned}$$

since $\gamma(\lambda) > \lambda$, $\gamma'(\lambda) < \lambda$ and $\gamma''(\lambda) < \lambda$ for all $0 < \lambda < 1$. Thus

$\{\Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda)\}$ is an increasing sequence of positive real numbers in [0, 1] and $\{\Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda)\}$, $\{\Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda)\}$ is a decreasing sequence of positive real number in [0, 1]. Therefore, they converge to the limits $S \leq 1$, $S^{'} < 0$ and $S^{''} < 0$, respectively.
Now, we claim that $S = 1$, $S^{'} = 0$ and $S^{''} = 0$. For, let $S < 1$. Letting $n \to \infty$ in (4.8.1), we have $S \geq \gamma(S) > S$, which is a contradiction and so $S = 1$. Similarly, let $S^{'} > 0$ and $S^{''} > 0$. Letting $n \to \infty$ in (4.8.2) and (4.8.3), we have $S^{'} \geq \gamma(S^{'}) > S^{'}$ and $S^{''} \geq \gamma(S^{''}) > S^{''}$. which is a contradiction and so $S^{'} = 0$ and $S^{''} = 0$.
Now for any positive integer p and $\lambda > 0$, we get

$$\begin{aligned}
\Xi(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \lambda) &\geq \Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \star \cdots \star \Xi(\varrho(\zeta_{n+p-1}), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) \\
&\geq \Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \star \cdots \star \Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}), \\
\Theta(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \lambda) &\leq \Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \diamond \cdots \diamond \Theta(\varrho(\zeta_{n+p-1}), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) \\
&\leq \Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \diamond \cdots \diamond \Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}), \\
\Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \lambda) &\leq \Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \diamond \cdots \diamond \Upsilon(\varrho(\zeta_{n+p-1}), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) \\
&\leq \Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \diamond \cdots \diamond \Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}).
\end{aligned}$$

Since, we have

$$\begin{aligned}
lim_{n\to\infty}\Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) &= 1, \\
lim_{n\to\infty}\Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) &= 0, \\
lim_{n\to\infty}\Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) &= 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
lim_{n\to\infty}\Xi(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) &\geq 1 \star \cdots \star 1 \geq 1, \\
lim_{n\to\infty}\Theta(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) &\leq 0 \diamond \cdots \diamond \leq 0, \\
lim_{n\to\infty}\Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) &\leq 0 \diamond \cdots \diamond \leq 0.
\end{aligned}$$

Thus, by definition (3.4), $\{\varrho(\zeta_n)\}$ is a Cauchy sequence and by the completeness of Σ, $\{\varrho(\zeta_n)\}$converges to a point $\omega \in \Sigma$. Also, $\{\varphi(\zeta_n)\}$ converges to the point ω.
Suppose that, by (ii) the mapping ϱ is continuous. Then $lim_{n\to\infty}\varrho\varrho(\zeta_n) = \varrho(\omega)$ and $lim_{n\to\infty}\varrho\varphi(\zeta_n) = \varrho(\omega)$. Further, since ϱ and φ are R-weakly commuting, we have

$$\Xi(\varrho\varphi(\zeta_n), \varphi\varrho(\zeta_n), \lambda) \geq \Xi(\varrho(\zeta_n), \varphi(\zeta_n), \frac{\lambda}{R}),$$
$$\Theta(\varphi\varrho(\zeta_n), \varrho\varphi(\zeta_n), \lambda) \leq \Theta(\varrho(\zeta_n), \varphi(\zeta_n), \frac{\lambda}{R}),$$
$$\Upsilon(\varphi\varrho(\zeta_n), \varrho\varphi(\zeta_n), \lambda) \leq \Upsilon(\varrho(\zeta_n), \varphi(\zeta_n), \frac{\lambda}{R}).$$

Letting $n \to \infty$ by the definition of NMS, we have $lim_{n\to\infty}\varphi\varrho(\zeta_n) = \varrho(\omega.)$
Now, we show that $\omega = \varrho(\omega)$. Suppose $\omega \neq \varrho(\omega)$. Then there exists $\lambda > 0$ such that

$$\Xi(\omega, \varrho(\omega), \lambda) < 1, \quad \Theta(\omega, \varrho(\omega), \lambda) > 1, \quad \Upsilon(\omega, \varrho(\omega), \lambda) > 1.$$

By (iii), we have

$$\Xi(\varrho(\zeta_n), \varrho\varrho(\zeta_n), \lambda) \geq \gamma(\Xi(\varphi(\zeta_n), \varphi\varrho(\zeta_n), \lambda)),$$
$$\Theta(\varrho(\zeta_n), \varrho\varrho(\zeta_n), \lambda) \leq \gamma^{'}(\Theta(\varphi(\zeta_n), \varphi\varrho(\zeta_n), \lambda)),$$
$$\Upsilon(\varrho(\zeta_n), \varrho\varrho(\zeta_n), \lambda) \leq \gamma^{''}(\Upsilon(\varphi(\zeta_n), \varphi\varrho(\zeta_n), \lambda)).$$

Letting $n \to \infty$ in the above inequalities, we get

$$\Xi(\omega, \varrho(\omega), \lambda) \geq \gamma(\Xi(\omega, \varrho(\omega), \lambda)) > \Xi(\omega, \varrho(\omega), \lambda),$$
$$\Theta(\omega, \varrho(\omega), \lambda) \leq \gamma^{'}(\Theta(\omega, \varrho(\omega), \lambda)) < \Theta(\omega, \varrho(\omega), \lambda),$$
$$\Upsilon(\omega, \varrho(\omega), \lambda) \leq \gamma^{''}(\Upsilon(\omega, \varrho(\omega), \lambda)) < \Upsilon(\omega, \varrho(\omega), \lambda).$$

Which are contradiction. Therefore, $\omega = \varrho(\omega)$. By condition (i), we can find a point $\omega_1 \in \Sigma$ such that $\omega = \varrho(\omega) = \varphi(\omega_1)$. Now, it follows that,

$$\Xi(\varrho\varrho(\zeta_n), \varrho(\zeta_1), \lambda) \geq \gamma(\Xi(\varphi\varrho(\zeta_n), \varphi(\zeta_1), \lambda)),$$
$$\Theta(\varrho\varrho(\zeta_n), \varrho(\zeta_1), \lambda) \leq \gamma^{'}(\Theta(\varphi\varrho(\zeta_n), \varphi(\zeta_1), \lambda)),$$
$$\Upsilon(\varrho\varrho(\zeta_n), \varrho(\zeta_1), \lambda) \leq \gamma^{''}(\Upsilon(\varphi\varrho(\zeta_n), \varphi(\zeta_1), \lambda)).$$

Letting $n \to \infty$ in the above inequalities, we have

$$\Xi(\varrho(\omega), \varrho(\omega_1), \lambda) \geq \gamma(\Xi(\varrho(\omega), \varphi(\omega_1), \lambda)) = 1,$$
$$\Theta(\varrho(\omega), \varrho(\omega_1), \lambda) \leq \gamma^{'}(\Theta(\varrho(\omega), \varphi(\omega_1), \lambda)) = 0,$$
$$\Upsilon(\varrho(\omega), \varrho(\omega_1), \lambda) \leq \gamma^{''}(\Upsilon(\varrho(\omega), \varphi(\omega_1), \lambda)) = 0.$$

which implies that $\varrho(\omega) = \varrho(\omega_1)$ since $\gamma(\lambda) = 1$, $\gamma^{'}(\lambda) = 0$ and $\gamma^{''}(\lambda) = 0$ for $\lambda = 1$. So, we get $\omega = \varrho(\omega) = \varrho(\omega_1) = \varphi(\omega_1)$. For any $\lambda > 0$,

$$\Xi(\varrho(\zeta), \varphi(\zeta), \lambda) = \Xi(\varrho\varphi(\zeta_1), \varphi\varrho(\zeta_1), \lambda)) \geq \Xi(\varrho(\zeta_1), \varphi(\zeta)1), \frac{\lambda}{R}) = 1,$$
$$\Theta(\varrho(\zeta), \varphi(\zeta), \lambda) = \Theta(\varrho\varphi(\zeta_1), \varphi\varrho(\zeta_1), \lambda)) \leq \Theta(\varrho(\zeta_1), \varphi(\zeta)1), \frac{\lambda}{R}) = 0,$$
$$\Upsilon(\varrho(\zeta), \varphi(\zeta), \lambda) = \Upsilon(\varrho\varphi(\zeta_1), \varphi\varrho(\zeta_1), \lambda)) \leq \Upsilon(\varrho(\zeta_1), \varphi(\zeta)1), \frac{\lambda}{R}) = 0.$$

Which again implies that $\varrho(\omega) = \varphi(\omega)$. Hence ω is a common fixed point of ϱ and φ. Next, we prove the uniqueness, let η ($\eta \neq \omega$) be another common fixed point of ϱ and φ. Then there exists $\lambda > 0$ such that,$\Xi(\omega,\eta,\lambda) < 1, \Theta(\omega,\eta,\lambda) > 0, \Upsilon(\omega,\eta,\lambda) > 0$ and

$$\Xi(\omega,\eta,\lambda) = \Xi(\varrho(\zeta),\varphi(\zeta),\lambda) \geq \gamma(\Xi(\varphi(\omega),\varphi(\eta),\lambda)) = \gamma(\Xi(\omega,\eta,\lambda) > \Xi(\omega,\eta,\lambda),$$
$$\Theta(\omega,\eta,\lambda) = \Theta(\varrho(\zeta),\varphi(\zeta),\lambda) \leq \gamma^{'}(\Theta(\varphi(\omega),\varphi(\eta),\lambda)) = \gamma^{'}(\Theta(\omega,\eta,\lambda) < \Theta(\omega,\eta,\lambda),$$
$$\Upsilon(\omega,\eta,\lambda) = \Upsilon(\varrho(\zeta),\varphi(\zeta),\lambda) \leq \gamma^{''}(\Upsilon(\varphi(\omega),\varphi(\eta),\lambda)) = \gamma^{''}(\Upsilon(\omega,\eta,\lambda) < \Upsilon(\omega,\eta,\lambda).$$

Which is a contradiction. Since $\gamma(\lambda) > \lambda$, $\gamma^{'}(\lambda) < \lambda$ and $\gamma^{''}(\lambda) < \lambda$ for any $0 < \lambda < 1$. Therefore $\eta = \omega$. Hence η is the only common fixed point of ϱ and φ. Hence Proved.

Now, we prove an example to validate the above theorem.

Example 4.9 Let $\Sigma = \{\frac{1}{n}; n \in \mathbb{N}\} \cup \{0\}$ with metric d defined by $d(\zeta,\eta) = |\zeta - \eta|$. For all $\zeta,\eta \in \Sigma$ and $\lambda \in (0,\infty)$, define

$$\Xi(\zeta,\eta,\lambda) = \frac{\lambda}{\lambda + |\zeta - \eta|}; \quad \Theta(\zeta,\eta,\lambda) = \frac{|\zeta - \eta|}{\lambda + |\zeta - \eta|}; \quad \Upsilon(\zeta,\eta,\lambda) = \frac{|\zeta - \eta|}{\lambda}$$

Clearly $(\Sigma,\Xi,\Theta,\Upsilon,\star,\diamond)$ is a complete neutrosophic metric space on Σ. Here $\star$ is defined by $\zeta \star \eta = \zeta\eta$ and $\diamond$ is defined as $\zeta \diamond \eta = min\{1,\ \zeta + \eta\}$.

Define

$$\varrho(\zeta) = 1, \quad \varphi(\zeta) = \begin{cases} 1, & if \quad \zeta \quad is\ a\ rational\ number \\ 0, & if \quad \zeta \quad is\ an\ irrational\ number. \end{cases}$$

It is evident that $\varrho \subset \varphi$, also ϱ is continuous and φ is discontinuous. Define a function $\gamma : [0,1] \to [0,1]$ by $\gamma(\lambda) = \sqrt{\lambda}$ for any $0 < \lambda < 1$ and $\gamma(\lambda) = 1\ for\ \lambda = 1, \gamma^{'} : [0,1] \to [0,1]$ by $\gamma^{'}(\lambda) = \lambda^2$ for any $0 < \lambda < 1$ and $\gamma^{'}(\lambda) = 0$, for $\lambda = 0$. Next, we define $\gamma^{''} : [0,1] \to [0,1]$ by $\gamma^{''}(\lambda) = \lambda^2$ for any $0 < \lambda < 1$ and $\gamma^{''}(\lambda) = 0$, for $\lambda = 0$. Then $\gamma(\lambda) > \lambda \quad \gamma^{'}(\lambda) < \lambda \quad \gamma^{''}(\lambda) < \lambda$ for any $0 < \lambda < 1$, we have

$$\Xi(\varrho(\zeta),\varrho(\eta),\lambda) \geq \gamma\Xi(\varphi(\zeta),\varphi(\eta),\lambda),$$
$$\Theta(\varrho(\zeta),\varrho(\eta),\lambda) \leq \gamma^{'}\Theta(\varphi(\zeta),\varphi(\eta),\lambda),$$
$$\Upsilon(\varrho(\zeta),\varrho(\eta),\lambda) \leq \gamma^{''}\Upsilon(\varphi(\zeta),\varphi(\eta),\lambda).$$

for all $\zeta,\eta \in \Sigma$. Also ϱ *and* φ are R- weakly commuting. Thus, all the conditions of Theorem (4.8) are satisfied and so ϱ and φ have 1 as a common fixed point.

Conclusion: In this manuscript, we explored new results in the notion of neutrosophic metric spaces (NMS) due to Kirisci, Simsek. We first formulated the definition of weakly commuting and R-weakly commuting mappings in NMS and proved the neutrosophic version of Pant's theorem. Also, we have given some examples to validate our results.

References

[1] Atanassov K, Intuitionistic Fuzzy Sets.Fuzzy Sets Syst 1986; Vol 20, pp 87-96.

[2] Alaca, C., Turkoglu D., Yildiz C. Fixed points in intuitionistic fuzzy metric spaces, Chaos Solitons and Fractals 2006, Vol 29,pp 1073-1078.

[3] Basset, M., Gamal, A., Son, L. H., and Smarandache, F. (2020). A Bipolar Neutrosophic Multi Criteria Decision Making Framework for Professional Selection. Applied Sciences, 10,4, 2020.

[4] Basset, M., Mohamed, R., Zaied, A. E. N. H., Gamal, A., and Smarandache, F. (2020). Solving the supply chain problem using the best-worst method based on a novel Plithogenic model. In Optimization Theory Based on Neutrosophic and Plithogenic Sets Academic Press, pp 1-19, 2020.

[5] Coker. D, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88 (1997), 81-99.

[6] George A, Veeramani P. On some results in fuzzy metric spaces. Fuzzy Sets Syst 1994; Vol 64, pp 395-399.

[7] Schweizer B and Sklar A, Statistical metric spaces, Pacific J. Math. 10, 314-334,1960.

[8] Jeyaraman. M, Sowndrarajan. S and Poovaragavan. D., Some Fixed Point Theorem for Generalized $(\Psi - \Phi)$ - Contractive Mappings in Strong M-Fuzzy Metric Spaces, International Journal of Pure and Applied Mathematics., 119 (12) pp 3119-3131, 2018.

[9] Jungck. G, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986), 771-774.

[10] Kramosil O, Michalek J. Fuzzy metric and statistical metric spaces. Kybernetika 1975, Vol 11, pp 326-334.

[11] Kirisci, M., Simsek, N. Neutrosophic metric spaces. Math Sci 14, 241248 (2020).

[12] Kubiak T, Common fixed point theorems of pairwise commuting maps, Math. Nachr. 118 (1984), 123-127.

[13] Menger, K.M. Statistical metrics, Proc Nat Acad Sci,1942, Vol 28, pp 535537.

[14] Pant R.P, Common fixed points of noncommuting mappings, J. Math. Anal. Appl. 188 (1994), 436-440.

[15] Park J.H. Intuitionistic fuzzy metric spaces. Chaos, Solitons and Fractals 2004, Vol 22, pp 1039-1046.

[16] Pathak H.K., Y. J. Cho and S. M. Kang, Remarks on R-weakly commuting mappings and common fixed point theorems, Bull. Korean Math. Soc. 34 (1997), 247-257.

[17] Rhoades.E and Sessa. S, Common fixed point theorems for three mappings under a weak commutativity condition, Indian J. Pure Appl. Math. 17 (1986), 47-57.

[18] Sessa. S, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. 32 (1982), No.32, 149-153.

[19] Smarandache, F. A Unifying Field in Logics, Neutrosophy: Neutrosophic Probability, Set and Logic; American Research Press: Reheboth, MA, USA, 1998.

[20] Smarandache, F. Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Inter J Pure Appl Math. Vol 24, pp 287297, 2005.

[21] Sowndrarajan. S, Jeyaraman. M, Florentin Smarandache., Fixed Point theorems in neutrosophic metric spaces, Neutrosophic Sets and Systems, Volume 36, pp 251-268, 2020.

[22] Zadeh LA. Fuzy sets. Inform Control 1965, Vol 8, pp 338353.

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Neutrosophic $\mathcal{N}-$structures on Sheffer stroke Hilbert algebras

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Abstract. In this study, a neutrosophic $\mathcal{N}-$subalgebra and a level set of a neutrosophic $\mathcal{N}-$structure are defined on Sheffer stroke Hilbert algebras. By determining a subalgebra on Sheffer stroke Hilbert algebras, it is proved that the level set of neutrosophic $\mathcal{N}-$subalgebras on this algebra is its subalgebra and vice versa. It is stated that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke Hilbert algebra forms a complete distributive lattice. Finally, a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is described and some of properties are given. Also, it is shown that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse is generally not valid.

Keywords: Sheffer stroke (Hilbert algebra); ideal; neutrosophic $\mathcal{N}-$subalgebra; neutrosophic $\mathcal{N}-$ideal.

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1. Introduction

The Sheffer operation (or, Sheffer stroke) was originally introduced by H. M. Sheffer [29]. Because Sheffer stroke, which is also called NAND operator, is one of the two operators that can be used by itself without any other logical operators, to construct a logical system, any axiom of the system is restated by only this operation.. Thus, it is easy to control some properties of the new constructed system. Since the axioms of Boolean algebra, which is an algebraic counterpart of the well-known classical propositional calculi, can be written by only using the Sheffer operation [21], it causes that the Sheffer stroke is applied to many algebraic structures such as orthoimplication algebras [1], ortholattices [8], Sheffer stroke non-associative MV-algebras [9] and its filters [24], Sheffer stroke BL-algebras and (fuzzy) filters [25], Sheffer stroke UP-algebras [26] and Sheffer stroke BG-algebras [27]. Besides, Hilbert algebras, which were introduced by Henkin and Skolem [12], are algebraic parts of the propositional logic including the implication operator and the constant element 1 [28]. Also,these algebras are dual to positive implicative BCK-algebras [10], [13, 14]. Specially, Busneag and Diego widely studied on Hilbert algebras and the related notions [4–6] and [11]. Recently, Oner et al. presented Hilbert algebras with Sheffer operation and its (fuzzy) filters [22]- [23].

On the other side, Atanassov introduced the degree of nonmembership(or falsehood (f)) and intuitionistic fuzzy sets [2] which are generalizations of fuzzy sets [33] with the degree of membership (or truth (t)). Then Smarandache introduced the degree of indeteminacy/neutrality and neutrosophic sets which are generalizations of intuitionistic fuzzy sets with the degrees of membership and nonmembership [30,31]. In a sence, there exist three functions called membership (t), indeteminacy (i) and nonmembership (f) functions in neutrosophic sets. Particularly, Jun et al. applied neutrospohic sets to BCK/BCI-algebras and semigroups [3, 7, 15–20, 32, 34].

We give general definitions and notions of Sheffer stroke Hilbert algebras, $\mathcal{N}-$functions and neutrosophic $\mathcal{N}-$structures defined by these functions on a nonempty universe X. Then a neutrosophic $\mathcal{N}-$subalgebra and a $(\alpha, \beta, \gamma)-$level set are defined by means of $\mathcal{N}-$functions on Sheffer stroke Hilbert algebras. After describing a subalgebra of Sheffer stroke Hilbert algebras, we show that the $(\alpha, \beta, \gamma)-$level set of a neutrosophic $\mathcal{N}-$subalgebra defined by its $\mathcal{N}-$functions on this algebra is its subalgebra and the inverse is also valid. Also, it is proved that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke Hilbert algebra forms a complete distributive lattice. Some properties of neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke Hilbert algebra are investigated. Moreover, a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is defined by means of $\mathcal{N}-$functions and it is demonstrated that $\mathcal{N}-$functions which define a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra are order-preserving. It is stated that $(\alpha, \beta, \gamma)-$level set of a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is its ideal and the inverse holds. Besides, some features of a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra are presented and it is shown that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse is not valid in general. Finally, new subsets of a Sheffer stroke Hilbert algebra are determined by $\mathcal{N}-$functions on the algebra and it is shown that these subsets are ideals of a Sheffer stroke Hilbert algebra for its neutrosophic $\mathcal{N}-$ideal. However, the validity of the inverse is satisfied under the special conditions.

2. **Preliminaries**

In this section, basic definitions and notions about Sheffer stroke Hilbert algebras and neutrosophic $\mathcal{N}-$structures.

Definition 2.1. [8] Let $\mathcal{H} = \langle H, | \rangle$ be a groupoid. The operation $|$ is said to be a *Sheffer stroke operation* if it satisfies the following conditions:

$(S1)$ $x|y = y|x$,

$(S2)$ $(x|x)|(x|y) = x$,

$(S3)$ $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,

$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Definition 2.2. [22] A Sheffer stroke Hilbert algebra is a structure $\langle H, |\rangle$ of type (2), in which H is a non-empty set and $|$ is a Sheffer stroke operation on H such that the following identities are satisfied for all $x, y, z \in H$:

(SHa_1) $(x|((y|(z|z))|(y|(z|z))))|(((x|(y|y))|((x|(z|z))|(x|(z|z))))|((x|(y|y))|((x|(z|z))|(x|(z|z))))) = x|(x|x)$,

(SHa_2) If $x|(y|y) = y|(x|x) = x|(x|x)$ then $x = y$.

Lemma 2.3. *[22] Let $\langle H, |\rangle$ be a Sheffer Stroke Hilbert algebra. Then the following identities hold for all $x \in H$:*

(i) $x|(x|x) = 1$,

(ii) $x|(1|1) = 1$,

(iii) $1|(x|x) = x$.

Lemma 2.4. *[22] Let $\langle H, |\rangle$ be a Sheffer stroke Hilbert algebra. Then the relation $x \leq y$ iff $x|(y|y) = 1$ is a partial order on H, that will be called natural ordering on H. With respect to this ordering, 1 is the largest element of H.*

If a Sheffer stroke Hilbert algebra $\langle H, |\rangle$ has the least element 0, then a unary operation * can be defined by $x^* = x|(0|0)$, for all x in H [22].

Lemma 2.5. *[22] Let $\langle H, |\rangle$ be a Sheffer stroke Hilbert algebra with 0. Then the followings hold, for all $x \in H$*

(i) $0|0 = 1$ *and* $1|1 = 0$,

(ii) $1^* = 0$ *and* $0^* = 1$,

(iii) $x|1 = x|x$,

(iv) $x^* = x|x$,

(v) $x|0 = 1$,

(vi) $(x^*)^* = x$,

(vii) $x|x^* = 1$.

Definition 2.6. [22] A non-empty subset I of H is called an ideal if

$(SSHI1)$ $0 \in I$,

$(SSHI2)$ $(x|(y|y))|(x|(y|y)) \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Theorem 2.7. *[22] Let I be a subset of H such that $0 \in I$. Then I is an ideal of H if and only if $x \leq y$ and $y \in I$ imply $x \in I$ for all $x \in H$.*

Definition 2.8. [15] $\mathcal{F}(X, [-1, 0])$ denotes the collection of functions from a set X to $[-1, 0]$ and a element of $\mathcal{F}(X, [-1, 0])$ is called a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}-$function on X). An $\mathcal{N}-$structure refers to an ordered pair (X, f) of X and $\mathcal{N}-$function f on X.

Definition 2.9. [20] A neutrosophic $\mathcal{N}-$structure over a nonempty universe X is defined by

$$X_N := \frac{X}{(T_N, I_N, F_N)} = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X\},$$

where T_N, I_N and F_N are $\mathcal{N}-$function on X, called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively.

Every neutrosophic $\mathcal{N}-$structure X_N over X satisfies the condition

$$(\forall x \in X)(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0).$$

Definition 2.10. [16] Let X_N be a neutrosophic $\mathcal{N}-$structure on a set X and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$T_N^{\alpha} := \{x \in X : T_N(x) \leq \alpha\},$$

$$I_N^{\beta} := \{x \in X : I_N(x) \geq \beta\}$$

and

$$F_N^{\gamma} := \{x \in X : F_N(x) \leq \gamma\}.$$

The set

$$X_N(\alpha, \beta, \gamma) := \{x \in X : T_N(x) \leq \alpha, I_N(x) \geq \beta \ and \ T_N(x) \leq \gamma\}$$

is called the $(\alpha, \beta, \gamma)-$level set of X_N. Moreover, $X_N(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$.

Consider sets

$$X_N^{w_t} := \{x \in X : T_N(x) \leq T_N(w_t),$$

$$X_N^{w_i} := \{x \in X : I_N(x) \geq I_N(w_i)$$

and

$$X_N^{w_f} := \{x \in X : F_N(x) \leq F_N(w_f),$$

for any $w_t, w_i, w_f \in X$. Obviously, $w_t \in X_N^{w_t}, w_i \in X_N^{w_i}$ and $w_f \in X_N^{w_f}$ [16].

3. Neutrosophic $\mathcal{N}-$structures

In this section, we present neutrosophic $\mathcal{N}-$subalgebras and neutrosophic $\mathcal{N}-$ideals on Sheffer stroke Hilbert algebras. Unless otherwise specified, H states a Sheffer stroke Hilbert algebra.

Definition 3.1. A neutrosophic $\mathcal{N}-$subalgebra H_N on a Sheffer stroke Hilbert algebra H is called a neutrosophic $\mathcal{N}-$structure of H satisfying the conditions

$$T_N((x(y|y))|(x(y|y))) \leq \bigvee\{T_N(x), T_N(y)\},$$

$$I_N((x(y|y))|(x(y|y))) \geq \bigwedge\{I_N(x), I_N(y)\}$$

and

$$F_N((x(y|y))|(x(y|y))) \leq \bigvee\{F_N(x), F_N(y)\},$$

for all $x, y \in H$.

Example 3.2. Consider a Sheffer stroke Hilbert algebra $\langle H, |\rangle$, where the set $H = \{0, p, q, 1\}$ and the Sheffer operation $|$ on H has the Cayley table as below [22]:

TABLE 1

$\vert$	1	p	q	0
1	0	q	p	1
p	q	q	1	1
q	p	1	p	1
0	1	1	1	1

A neutrosophic $\mathcal{N}-$structure $H_N = \{\frac{0}{(-0.81, -0.13, -0.47)}, \frac{p}{(-0.69, -0.32, -0.35)}, \frac{q}{(-0.69, -0.32, -0.35)}, \frac{1}{(-0.56, -0.99, -0.42)}\}$ on H is a neutrosophic $\mathcal{N}-$subalgebra of H.

Definition 3.3. Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. For the sets

$$T_N^\alpha := \{x \in H : T_N(x) \leq \alpha\},$$

$$I_N^\beta := \{x \in H : I_N(x) \geq \beta\}$$

and

$$F_N^\gamma := \{x \in H : F_N(x) \leq \gamma\},$$

the set

$$H_N(\alpha, \beta, \gamma) := \{x \in H : T_N(x) \leq \alpha, I_N(x) \geq \beta \text{ and } F_N(x) \leq \gamma\}$$

is called the $(\alpha, \beta, \gamma)-$level set of H_N. Also, $H_N(\alpha, \beta, \gamma) = T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$.

Definition 3.4. A nonempty subset G of a Sheffer stroke Hilbert algebra H is called a subalgebra of H if $(x|(y|y))|(x|(y|y)) \in G$, for all $x, y \in G$.

Example 3.5. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. Then $\{0, 1\}$ is a subalgebra of H.

Theorem 3.6. *Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If H_N is a neutrosophic $\mathcal{N}-$subalgebra of H, then the nonempty $(\alpha, \beta, \gamma)-$level set of H_N is a subalgebra of H.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$subalgebra of H and x, y be any elements of $H_N(\alpha, \beta, \gamma)$. Then $T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma$ and $T_N(y) \leq \alpha, I_N(y) \geq \beta, F_N(y) \leq \gamma$. Thus, it is obtained that

$$T_N((x(y|y))|(x(y|y))) \leq \bigvee\{T_N(x), T_N(y)\} \leq \alpha,$$
$$I_N((x(y|y))|(x(y|y))) \geq \bigwedge\{I_N(x), I_N(y)\} \geq \beta$$

and

$$F_N((x(y|y))|(x(y|y))) \leq \bigvee\{F_N(x), F_N(y)\} \leq \gamma,$$

for all $x, y \in H$. So, $(x|(y|y))|(x|(y|y)) \in H_N(\alpha, \beta, \gamma)$ which means that $H_N(\alpha, \beta, \gamma)$ is a subalgebra of H. □

Theorem 3.7. *Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} be subalgebras of H, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then H_N is a neutrosophic $\mathcal{N}-$subalgebra of H.*

Proof. Let $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} be subalgebras of H, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Suppose that x and y be any elements of H such that $a = T_N((x(y|y))|(x(y|y))) > \bigvee\{T_N(x), T_N(y)\} = b$. Then $b < \alpha_1 < a$ where $\alpha_1 = \frac{1}{2}(a + b) \in [-1, 0)$. Thus, $x, y \in T_N^{\alpha_1}$ but $(x(y|y))|(x(y|y)) \notin T_N^{\alpha_1}$ which is a contradiction. Hence, $T_N((x(y|y))|(x(y|y))) \leq \bigvee\{T_N(x), T_N(y)\}$, for all $x, y \in H$.

Assume that x and y be any elements of H such that $u = I_N((x(y|y))|(x(y|y))) < \bigwedge\{I_N(x), I_N(y)\} = v$. Then $u < \beta_1 < v$ in which $\beta_1 = \frac{1}{2}(u + v) \in [-1, 0)$. So, $x, y \in I_N^{\beta_1}$ while $(x(y|y))|(x(y|y)) \notin I_N^{\beta_1}$ which is a contradiction. Thus, $I_N((x(y|y))|(x(y|y))) \geq \bigwedge\{I_N(x), I_N(y)\}$, for all $x, y \in H$.

Suppose that x and y be any elements of H such that $m = F_N((x(y|y))|(x(y|y))) > \bigvee\{F_N(x), F_N(y)\} = n$. Then $n < \gamma_1 < m$ where $\gamma_1 = \frac{1}{2}(m + n) \in [-1, 0)$. Hence, $x, y \in F_N^{\gamma_1}$ but $(x(y|y))|(x(y|y)) \notin F_N^{\gamma_1}$ which is a contradiction. Therefore, $F_N((x(y|y))|(x(y|y))) \leq \bigvee\{F_N(x), F_N(y)\}$, for all $x, y \in H$.

Thereby, H_N is a neutrosophic $\mathcal{N}-$subalgebra of H. □

Theorem 3.8. *Let $\{H_{N_i} : i \in \mathbb{N}\}$ be a family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke Hilbert algebra H. Then $\{H_{N_i} : i \in \mathbb{N}\}$ forms a complete distributive lattice.*

Proof. Let G be a nonempty subset of $\{H_{N_i} : i \in \mathbb{N}\}$. Since H_{N_i} is a neutrosophic $\mathcal{N}-$subalgebra of H, for all $H_{N_i} \in G$, it satisfies

$$T_N((x(y|y))|(x(y|y))) \leq \bigvee\{T_N(x), T_N(y)\},$$

$$I_N((x(y|y))|(x(y|y))) \geq \bigwedge\{I_N(x), I_N(y)\}$$

and

$$F_N((x(y|y))|(x(y|y))) \leq \bigvee\{F_N(x), F_N(y)\},$$

for all $x, y \in H$. Then $\bigcap G$ satisfies these inequalities, which means that $\bigcap G$ is a neutrosophic $\mathcal{N}-$subalgebra of H.

Let P be a family of all neutrosophic $\mathcal{N}-$subalgebras of H containing $\bigcup\{H_{N_i} : i \in \mathbb{N}\}$. Then $\bigcap P$ is a neutrosophic $\mathcal{N}-$subalgebra of H.

If $\bigwedge_{i\in\mathbb{N}} H_{N_i} = \bigcap_{i\in\mathbb{N}} H_{N_i}$ and $\bigvee_{i\in\mathbb{N}} H_{N_i} = \bigcap P$, then $(\{H_{N_i} : i \in \mathbb{N}\}, \bigvee, \bigwedge)$ is a complete lattice. Also, it is distibutive by the definitions of $\bigvee$ and $\bigwedge$. □

Proposition 3.9. *If a neutrosophic $\mathcal{N}-$structure H_N on a Sheffer stroke Hilbert algebra H is a neutrosophic $\mathcal{N}-$subalgebra of H, then $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$, for all $x \in H$.*

Proof. By substituting $[y := 0]$ in the inequalities in Definition 3.1, we have from Lemma 2.3 (i) and Lemma 2.5 (i) that

$$T_N(0) = T_N(1|1) = T_N((x(x|x))|(x(x|x))) \leq \bigvee\{T_N(x), T_N(x)\} = T_N(x),$$

$$I_N(0) = I_N(1|1) = I_N((x(x|x))|(x(x|x))) \geq \bigwedge\{I_N(x), I_N(x)\} = I_N(x)$$

and

$$F_N(0) = F_N(1|1) = F_N((x(x|x))|(x(x|x))) \leq \bigvee\{F_N(x), F_N(x)\} = F_N(x),$$

for all $x \in H$. □

The inverse of Proposition 3.9 is generally not true.

Example 3.10. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$H_N = \{\frac{0}{(-1, 0, -1)}, \frac{p}{(-0.2, -0.2, -0.2)}, \frac{q}{(-0.3, -0.3, -0.3)}, \frac{1}{(-0.4, -0.4, -0.4)}\}$$

on H is not a neutrosophic $\mathcal{N}-$subalgebra of H since

$$T_N((1(q|q))|(1(q|q))) = T_N(p) = -0.2 > -0.3 = \bigvee\{-0.3, -0.4\} = \bigvee\{T_N(1), T_N(q)\}.$$

Lemma 3.11. *Let H_N be a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke Hilbert algebra H. If there exists a sequence $\{a_n\}$ in H such that $\lim_{n\longrightarrow\infty} T_N(a_n) = -1 = \lim_{n\longrightarrow\infty} F_N(a_n)$ and $\lim_{n\longrightarrow\infty} I_N(a_n) = 0$, then $T_N(0) = -1 = F_N(0)$ and $I_N(0) = 0$.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke Hilbert algebra H. Assume that there exists a sequence $\{a_n\}$ in H such that $\lim_{n\longrightarrow\infty} T_N(a_n) = -1 = \lim_{n\longrightarrow\infty} F_N(a_n)$ and $\lim_{n\longrightarrow\infty} I_N(a_n) = 0$. Since $T_N(0) \leq T_N(a_n)$, $I_N(0) \geq I_N(a_n)$ and $F_N(0) \leq F_N(a_n)$, for every $n \in \mathbb{Z}^+$ from Proposition 3.9, it follows that

$$-1 = \lim_{n\longrightarrow\infty} -1 \leq \lim_{n\longrightarrow\infty} T_N(0) = T_N(0) \leq \lim_{n\longrightarrow\infty} T_N(a_n) = -1,$$

$$0 = \lim_{n\longrightarrow\infty} 0 \geq \lim_{n\longrightarrow\infty} I_N(0) = I_N(0) \geq \lim_{n\longrightarrow\infty} I_N(a_n) = 0$$

and

$$-1 = \lim_{n\longrightarrow\infty} -1 \leq \lim_{n\longrightarrow\infty} F_N(0) = F_N(0) \leq \lim_{n\longrightarrow\infty} F_N(a_n) = -1.$$

Hence, $T_N(0) = -1 = F_N(0)$ and $I_N(0) = 0$. □

Proposition 3.12. *Every neutrosophic $\mathcal{N}-$subalgebra H_N of a Sheffer stroke Hilbert algebra H satisfies*

$$T_N((x(y|y))|(x(y|y))) \leq T_N(y),$$

$$I_N((x(y|y))|(x(y|y))) \geq I_N(y)$$

and

$$F_N((x(y|y))|(x(y|y))) \leq F_N(y),$$

for all $x, y \in H$ if and only if T_N, I_N and F_N are constant.

Proof. (⇒) Since

$$\begin{aligned} T_N(x) &= T_N((x|x)|(x|x)) \\ &= T_N((1|((x|x)|(x|x)))|(1|((x|x)|(x|x)))) \\ &= T_N((x|1)|(x|1)) \\ &= T_N((x|(0|0))|(x|(0|0))) \\ &\leq T_N(0), \end{aligned}$$

and similarly, $I_N(0) \leq I_N(x)$, $F_N(x) \leq F_N(0)$ from (S1), (S2), Lemma 2.3 (iii) and Lemma 2.5 (i), we have from Proposition 3.9 that $T_N(x) = T_N(0)$, $I_N(x) = I_N(0)$ and $F_N(x) = F_N(0)$, for all $x \in X$.

(⇐) It is obvious by the fact that H_N is a neutrosophic $\mathcal{N}-$subalgebra of H and T_N, I_N and F_N are constant. □

Definition 3.13. A neutrosophic $\mathcal{N}-$structure H_N on H is called a neutrosophic $\mathcal{N}-$ideal of H if

$$T_N(0) \leq T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\},$$

$$I_N(0) \geq I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\}$$

and

$$F_N(0) \leq F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\},$$

for all $x, y \in H$.

Example 3.14. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$H_N = \{\frac{0}{(-1,0,-0.21)}, \frac{p}{(-1,0,-0.21)}, \frac{q}{(-0.71,-0.55,-0.11)}, \frac{1}{(-0.71,-0.55,-0.11)}\}$$

on H is a neutrosophic $\mathcal{N}-$ideal of H.

Proposition 3.15. *Let H_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra H. Then $x \leq y$ implies $T_N(x) \leq T_N(y)$, $I_N(x) \geq I_N(y)$ and $F_N(x) \leq F_N(y)$, for all $x, y \in H$.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra H and $x \leq y$. Then $x|(y|y) = 1$ from Lemma 2.4, and so, $(x|(y|y))|(x|(y|y)) = 1|1 = 0$ from Lemma 2.5 (i). Thus,

$$T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} = \bigvee\{T_N(0), T_N(y)\} = T_N(y)$$

$$I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} = \bigwedge\{I_N(0), I_N(y)\} = I_N(y)$$

and

$$F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} = \bigvee\{F_N(0), F_N(y)\} = F_N(y).$$

□

The inverse of Proposition 3.15 does not hold in general.

Example 3.16. Consider the neutrosophic $\mathcal{N}-$ideal of H in Example 3.14. Then $p \nleq q$ when $T_N(p) = -1 \leq -0.71 = T_N(q)$.

Lemma 3.17. *Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If H_N is a neutrosophic $\mathcal{N}-$ideal of H, then the nonempty set $H_N(\alpha, \beta, \gamma)$ is an ideal of H.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra H and $H_N(\alpha,\beta,\gamma) \neq \emptyset$, for any $\alpha,\beta,\gamma \in [-1,0]$ with $-3 \leq \alpha+\beta+\gamma \leq 0$. Since $T_N(0) \leq T_N(x) \leq \alpha, I_N(0) \geq I_N(x) \geq \beta$ and $F_N(0) \leq F_N(x) \leq \gamma$, for any $x \in H_N(\alpha,\beta,\gamma)$, we have $0 \in H_N(\alpha,\beta,\gamma)$. Let $(x|(y|y))|(x|(y|y)), y \in H_N(\alpha,\beta,\gamma)$. Then $T_N((x|(y|y))|(x|(y|y))) \leq \alpha$, $I_N((x|(y|y))|(x|(y|y))) \geq \beta$, $F_N((x|(y|y))|(x|(y|y))) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. Since

$$T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} \leq \bigvee\{\alpha,\alpha\} = \alpha,$$

$$I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \geq \bigwedge\{\beta,\beta\} = \beta$$

and

$$F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \leq \bigvee\{\gamma,\gamma\} = \gamma,$$

for all $x, y \in H$, we get $x \in H_N(\alpha,\beta,\gamma)$ which means that $H_N(\alpha,\beta,\gamma)$ is an ideal of H. □

Lemma 3.18. *Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and $T_N^\alpha, I_N^\beta, F_N^\gamma$ be ideals of H, for all $\alpha,\beta,\gamma \in [-1,0]$ with $-3 \leq \alpha+\beta+\gamma \leq 0$. Then H_N is a neutrosophic $\mathcal{N}-$ideal of H.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and $T_N^\alpha, I_N^\beta, F_N^\gamma$ be ideals of H, for all $\alpha,\beta,\gamma \in [-1,0]$ with $-3 \leq \alpha+\beta+\gamma \leq 0$. Suppose that x_0, y_0 and z_0 be any elements of H such that $T_N(0) > T_N(x_0)$, $I_N(0) < I_N(y_0)$ and $F_N(0) > F_N(z_0)$. If $\alpha = \frac{1}{2}(T_N(0)+T_N(x_0))$, $\beta = \frac{1}{2}(I_N(0)+I_N(y_0))$ and $\gamma = \frac{1}{2}(F_N(0)+F_N(z_0))$, for $\alpha,\beta,\gamma \in [-1,0)$, then $T_N(0) > \alpha > T_N(x_0)$, $I_N(0) < \beta < I_N(y_0)$ and $F_N(0) > \gamma > F_N(z_0)$ which imply that $0 \notin T_N^\alpha, 0 \notin I_N^\beta$ and $0 \notin F_N^\gamma$, respectively. This contradicts with (SSHI1). So, $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$, for all $x \in H$. Assume that x_1, x_2, x_3, y_1, y_2 and y_3 be any elements of H such that

$$a_1 = T_N(x_1) > \bigvee\{T_N((x_1|(y_1|y_1))|(x_1|(y_1|y_1))), T_N(y_1)\} = b_1,$$

$$a_2 = I_N(x_2) < \bigwedge\{I_N((x_2|(y_2|y_2))|(x_2|(y_2|y_2))), I_N(y_2)\} = b_2$$

and

$$a_3 = F_N(x_3) > \bigvee\{F_N((x_3|(y_3|y_3))|(x_3|(y_3|y_3))), F_N(y_3)\} = b_3.$$

If $\alpha' = \frac{1}{2}(a_1+b_1)$, $\beta' = \frac{1}{2}(a_2+b_2)$ and $\gamma' = \frac{1}{2}(a_3+b_3)$, then $b_1 < \alpha' < a_1$, $a_2 < \beta' < b_2$ and $b_3 < \gamma' < a_3$. Thus, $(x_1|(y_1|y_1))|(x_1|(y_1|y_1)), y_1 \in T_N^{\alpha'}$, $(x_2|(y_2|y_2))|(x_2|(y_2|y_2)), y_2 \in I_N^{\beta'}$ and $(x_3|(y_3|y_3))|(x_3|(y_3|y_3)), y_3 \in F_N^{\gamma'}$, and so, $x_1 \in T_N^{\alpha'}$, $x_2 \in I_N^{\beta'}$ and $x_3 \in F_N^{\gamma'}$ which contradicts with the assumption. Therefore,

$$T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\},$$

$$I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\}$$

and

$$F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\},$$

for all $x, y \in H$. Thereby, H_N is a neutrosophic $\mathcal{N}-$ideal of H. □

Lemma 3.19. *Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H. Then H_N is a neutrosophic $\mathcal{N}-$ideal of H if and only if $(x|(y|y))|(x|(y|y)) \leq z$ implies*

$$T_N(x) \leq \bigvee\{T_N(y), T_N(z)\},$$

$$I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\}$$

and

$$F_N(x) \leq \bigvee\{F_N(y), F_N(z)\},$$

for all $x, y, z \in H$.

Proof. (⇒) Let H_N be a neutrosophic $\mathcal{N}-$ideal of H and $(x|(y|y))|(x|(y|y)) \leq z$. Then $(((x|(y|y))|(x|(y|y)))|(z|z))|(((x|(y|y))|(x|(y|y)))|(z|z)) = 1|1 = 0$ from Lemma 2.4 and Lemma 2.5 (i). Since

$$\begin{aligned} T_N((x|(y|y))|(x|(y|y))) &\leq \bigvee\{T_N((((x|(y|y))|(x|(y|y)))|(z|z))| \\ &\qquad (((x|(y|y))|(x|(y|y)))|(z|z))), T_N(z)\} \\ &= \bigvee\{T_N(0), T_N(z)\} \\ &= T_N(z), \end{aligned}$$

$$\begin{aligned} I_N((x|(y|y))|(x|(y|y))) &\geq \bigwedge\{I_N((((x|(y|y))|(x|(y|y)))|(z|z))| \\ &\qquad (((x|(y|y))|(x|(y|y)))|(z|z))), I_N(z)\} \\ &= \bigwedge\{I_N(0), I_N(z)\} \\ &= I_N(z) \end{aligned}$$

and

$$\begin{aligned} F_N((x|(y|y))|(x|(y|y))) &\leq \bigvee\{F_N((((x|(y|y))|(x|(y|y)))|(z|z))| \\ &\qquad (((x|(y|y))|(x|(y|y)))|(z|z))), F_N(z)\} \\ &= \bigvee\{F_N(0), F_N(z)\} \\ &= F_N(z), \end{aligned}$$

we have

$$T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} \leq \bigvee\{T_N(y), T_N(z)\},$$

$$I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \geq \bigwedge\{I_N(y), I_N(z)\}$$

and

$$F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \leq \bigvee\{F_N(y), F_N(z)\},$$

for all $x, y, z \in H$.

($\Leftarrow$) Let H_N be a neutrosophic $\mathcal{N}-$structure on H such that $(x|(y|y))|(x|(y|y)) \leq z$ implies

$$T_N(x) \leq \bigvee\{T_N(y), T_N(z)\},$$

$$I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\}$$

and

$$F_N(x) \leq \bigvee\{F_N(y), F_N(z)\},$$

for all $x, y, z \in H$. Since $(0|(x|x))|(0|(x|x)) = ((x|x)|(1|1))|((x|x)|(1|1)) = 1|1 = 0 \leq z$ from (S1), Lemma 2.3 (ii) and Lemma 2.5 (i), we get $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$, for all $x \in H$. Since

$$((x|(x|(y|y)))|(x|(x|(y|y))))|(y|y) = (x|(y|y))|((x|(y|y))|(x|(y|y))) = 1$$

from (S1), (S3) and Lemma 2.3 (i), it follows from Lemma 2.4 that $(x|(x|(y|y)))|(x|(x|(y|y))) \leq y$. Since $(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y)))))|(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))) = (x|(x|(y|y)))|(x|(x|(y|y))) \leq y$ from (S2), it is obtained that

$$T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\},$$

$$I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\}$$

and

$$F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\},$$

for all $x, y, z \in H$. Thus, H_N is a neutrosophic $\mathcal{N}-$ideal of H. $\square$

Theorem 3.20. *Every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra H is a neutrosophic $\mathcal{N}-$subalgebra of H.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$ideal of H. Then it follows from (S1), (S3), Lemma 2.3 (i)-(ii), Lemma 2.5 (i) and Definition 3.13 that

$$\begin{aligned} T_N((x|(y|y))|(x|(y|y))) &\leq \bigvee\{T_N((((x|(y|y))|(x|(y|y)))|(x|x))| \\ &\quad (((x|(y|y))|(x|(y|y)))|(x|x))), T_N(x)\} \\ &= \bigvee\{T_N(((y|y)|((x|(x|x))|(x|(x|x))))| \\ &\quad ((y|y)|((x|(x|x))|(x|(x|x))))), T_N(x)\} \\ &= \bigvee\{T_N(((y|y)|(1|1))|((y|y)|(1|1))), T_N(x)\} \\ &= \bigvee\{T_N(1|1), T_N(x)\} \\ &= \bigvee\{T_N(0), T_N(x)\} \\ &= T_N(x) \\ &\leq \bigvee\{T_N(x), T_N(y)\}, \end{aligned}$$

and similarly,

$$I_N((x|(y|y))|(x|(y|y))) \geq \bigwedge\{I_N(x), I_N(y)\},$$

$$F_N((x|(y|y))|(x|(y|y))) \leq \bigvee\{F_N(x), F_N(y)\},$$

for all $x, y \in H$. Hence, H_N is a neutrosophic $\mathcal{N}-$subalgebra of H. □

The inverse of Theorem 3.20 is mostly not true.

Example 3.21. The neutrosophic $\mathcal{N}-$subalgebra H_N of H in Example 3.2 is not a neutrosophic $\mathcal{N}-$ideal of H since $T_N(1) = -0.56 > -0.69 = T_N(p) = \bigvee\{T_N(p), T_N(q)\} = \bigvee\{T_N((1|(q|q))|(1|(q|q))), T_N(q)\}$.

Definition 3.22. Let H be a Sheffer stroke Hilbert algebra. We define

$$H_N^{x_t} := \{x \in H : T_N(x) \leq T_N(x_t)\},$$

$$H_N^{x_i} := \{x \in H : I_N(x) \geq I_N(x_i)\}$$

and

$$H_N^{x_f} := \{x \in H : F_N(x) \leq F_N(x_f)\},$$

for all $x_t, x_i, x_f \in H$. Obviously, $x_t \in H_N^{x_t}, x_i \in H_N^{x_i}$ and $x_f \in H_N^{x_f}$.

Example 3.23. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. Let $T_N(0) = -0.11, T_N(p) = -0.14, T_N(q) = -0.17, T_N(1) = -0.2, I_N(0) = -0.12, I_N(p) = -0.15$, $I_N(q) = -0.13, I_N(1) = -0.21, F_N(0) = -0.22, F_N(p) = -0.19, F_N(q) = -0.2, F_N(1) = -0.23, x_t = 1, x_i = p$ and $x_f = q$. Then

$$H_N^{x_t} = \{x \in H : T_N(x) \leq T_N(1)\} = \{1\},$$

$$H_N^{x_i} = \{x \in H : I_N(x) \geq I_N(p)\} = \{0, p, q\}$$

and

$$H_N^{x_f} = \{x \in H : F_N(x) \leq F_N(q)\} = \{0, q, 1\}.$$

Theorem 3.24. *Let x_t, x_i and x_f be any elements of a Sheffer stroke Hilbert algebra H. If H_N is a neutrosophic $\mathcal{N}-$ideal of H, then $H_N^{x_t}, H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra H. Since $T_N(0) \leq T_N(x_t)$, $I_N(0) \geq I_N(x_i)$ and $F_N(0) \leq F_N(x_f)$, for any $x_t, x_i, x_f \in H$, it follows that $0 \in H_N^{x_t}$, $0 \in H_N^{x_i}$ and $0 \in H_N^{x_f}$. Let $(x_1|(y_1|y_1))|(x_1|(y_1|y_1)), y_1 \in H_N^{x_t}$, $(x_2|(y_2|y_2))|(x_2|(y_2|y_2))$, $y_2 \in H_N^{x_i}$ and $(x_3|(y_3|y_3))|(x_3|(y_3|y_3))$, $y_3 \in H_N^{x_f}$. Then $T_N((x_1|(y_1|y_1))|(x_1|(y_1|y_1))) \leq T_N(x_t)$, $T_N(y_1) \leq T_N(x_t)$, $I_N((x_2|(y_2|y_2))|(x_2|(y_2|y_2))) \geq I_N(x_i)$, $I_N(y_2) \geq I_N(x_i)$ and $F_N((x_3|(y_3|y_3))|(x_3|(y_3|y_3))) \leq F_N(x_f)$, $F_N(y_3) \leq F_N(x_f)$. Since

$$T_N(x_1) \leq \bigvee\{T_N((x_1|(y_1|y_1))|(x_1|(y_1|y_1))), T_N(y_1)\} \leq T_N(x_t),$$

$$I_N(x_2) \geq \bigwedge\{I_N((x_2|(y_2|y_2))|(x_2|(y_2|y_2))), I_N(y_2)\} \geq I_N(x_i)$$

and

$$F_N(x_3) \leq \bigvee\{F_N((x_3|(y_3|y_3))|(x_3|(y_3|y_3))), F_N(y_3)\} \leq F_N(x_f),$$

we get $x_1 \in H_N^{x_t}$, $x_2 \in H_N^{x_i}$ and $x_3 \in H_N^{x_f}$. Therefore, $H_N^{x_t}$, $H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H. □

Example 3.25. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. For a neutrosophic $\mathcal{N}-$ideal

$$H_N = \{\frac{0}{(-0.69,-0.1,-0.41)}, \frac{p}{(-0.57,-0.27,-0.38)}, \frac{q}{(-0.69,-0.1,-0.41)}, \frac{1}{(-0.57,-0.27,-0.38)}\}$$

of H, $x_t = p$ and $x_i = x_f = q \in H$, the subsets

$$H_N^{x_t} = \{x \in H : T_N(x) \leq T_N(p)\} = H,$$

$$H_N^{x_i} = \{x \in H : I_N(x) \geq I_N(q)\} = \{0, q\}$$

and

$$H_N^{x_f} = \{x \in H : F_N(x) \leq F_N(q)\} = \{0, q\}$$

of H are ideals of H.

Theorem 3.26. *Let x_t, x_i and x_f be any elements of a Sheffer stroke Hilbert algebra H and H_N be a neutrosophic $\mathcal{N}-$structure on H.*

(1) *If $H_N^{x_t}, H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H, then the following condition is satisfied:*

$$T_N(x) \geq \bigvee\{T_N((y|(z|z))(y|(z|z))), T_N(z)\} \Rightarrow T_N(x) \geq T_N(y),$$

$$I_N(x) \leq \bigwedge\{I_N((y|(z|z))(y|(z|z))), I_N(z)\} \Rightarrow I_N(x) \leq I_N(y) \quad \textit{and} \tag{1}$$

$$F_N(x) \geq \bigvee\{F_N((y|(z|z))(y|(z|z))), F_N(z)\} \Rightarrow F_N(x) \geq F_N(y),$$

for all $x, y, z \in H$.

(2) *If H_N satisfies the condition (1) and*

$$T_N(0) \leq T_N(x), \quad I_N(0) \geq I_N(x) \quad \textit{and} \quad F_N(0) \leq F_N(x), \quad \textit{for all } x \in H, \tag{2}$$

then $H_N^{x_t}, H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H, for all $x_t \in T_N^{-1}$, $x_i \in I_N^{-1}$ and $x_f \in F_N^{-1}$.

Proof. Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H.

(1) $H_N^{x_t}, H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H, for any $x_t, x_i, x_f \in H$, and x, y, z be any elements of H such that $T_N(x) \geq \bigvee\{T_N((y|(z|z))(y|(z|z))), T_N(z)\}$, $I_N(x) \leq \bigvee\{I_N((y|(z|z))(y|(z|z))), I_N(z)\}$ and $F_N(x) \geq \bigvee\{F_N((y|(z|z))(y|(z|z))), F_N(z)\}$. Then $(y|(z|z))(y|(z|z))), z \in H_N^{x_t} \cap H_N^{x_i} \cap H_N^{x_f}$ where $x_t = x_i = x_f = x$. So, it is obtained from

(SSHI2) that $y \in H_N^{x_t} \cap H_N^{x_i} \cap H_N^{x_f}$ where $x_t = x_i = x_f = x$. Thus, $T_N(x) \geq T_N(y)$, $I_N(x) \leq I_N(y)$ and $F_N(x) \geq F_N(y)$.

(2) Let $x_t \in T_N^{-1}$, $x_i \in I_N^{-1}$ and $x_f \in F_N^{-1}$ and H_N be a neutrosophic $\mathcal{N}-$structure on H satisfying the conditions (1) and (2). Then $0 \in H_N^{x_t}$, $0 \in H_N^{x_i}$ and $0 \in H_N^{x_f}$ from the condition (2). Let $(x_1|(y_1|y_1))|(x_1|(y_1|y_1)), y_1 \in H_N^{x_t}$, $(x_2|(y_2|y_2))|(x_2|(y_2|y_2)), y_2 \in H_N^{x_i}$ and $(x_3|(y_3|y_3))|(x_3|(y_3|y_3)), y_3 \in H_N^{x_f}$. Thus, $T_N((x_1|(y_1|y_1))|(x_1|(y_1|y_1))) \leq T_N(x_t)$, $T_N(y_1) \leq T_N(x_t)$, $I_N((x_2|(y_2|y_2))|(x_2|(y_2|y_2))) \geq I_N(x_i)$, $I_N(y_2) \geq I_N(x_i)$ and $F_N((x_3|(y_3|y_3))|(x_3|(y_3|y_3))) \leq F_N(x_f)$, $F_N(y_3) \leq F_N(x_f)$. Since

$$\bigvee\{T_N((x_1|(y_1|y_1))|(x_1|(y_1|y_1))), T_N(y_1)\} \leq T_N(x_t),$$

$$\bigwedge\{I_N((x_2|(y_2|y_2))|(x_2|(y_2|y_2))), I_N(y_2)\} \geq I_N(x_i)$$

and

$$\bigvee\{F_N((x_3|(y_3|y_3))|(x_3|(y_3|y_3))), F_N(y_3)\} \leq F_N(x_f),$$

we have from the condition (1) that $T_N(x_1) \leq T_N(x_t)$, $I_N(x_2) \geq I_N(x_i)$ and $F_N(x_3) \leq F_N(x_f)$ which imply that $x_1 \in H_N^{x_t}$, $x_2 \in H_N^{x_i}$ and $x_3 \in H_N^{x_f}$. Thereby, $H_N^{x_t}$, $H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H. □

Example 3.27. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. Let $T_N(0) = T_N(q) = -0.997, T_N(p) = T_N(1) = 0, I_N(0) = I_N(q) = -0.08, I_N(p) = I_N(1) = -1, F_N(0) = F_N(q) = -0.8, F_N(p) = F_N(1) = -0.7$. Then the ideals $H_N^{x_t} = \{0, q\}, H_N^{x_i} = \{0\}$ and $H_N^{x_f} = H$ of H satisfy the condition (1), for $x_t = q, x_i = 0$ and $x_f = p \in H$.

Also, let

$$H_N = \{\frac{0}{(-0.7, -0.13, -0.6)}, \frac{p}{(-0.7, -0.13, -0.6)}, \frac{q}{(-0.41, -0.87, -0.52)}, \frac{1}{(-0.41, -0.87, -0.52)}\}$$

be a neutrosophic $\mathcal{N}-$structure on H satisfyinh the conditions (1) and (2). For $x_t = p, x_i = 1$ and $x_f = q \in H$, the subsets

$$H_N^{x_t} = \{x \in H : T_N(x) \leq T_N(p)\} = \{0, p\},$$

$$H_N^{x_i} = \{x \in H : I_N(x) \geq I_N(1)\} = H$$

and

$$H_N^{x_f} = \{x \in H : F_N(x) \leq F_N(q)\} = H$$

of H are ideals of H.

4. Conclusion

In this study, we have studied neutrosophic $\mathcal{N}-$structures defined by $\mathcal{N}-$functions on Sheffer stroke Hilbert algebras. By giving basic definitions and notions about Sheffer stroke Hilbert algebras and neutrosophic $\mathcal{N}-$structures defined by $\mathcal{N}-$functions on a nonempty universe X, a neutrosophic $\mathcal{N}-$subalgebra and a $(\alpha,\beta,\gamma)-$level set of a neutrosophic $\mathcal{N}-$structure are described by $\mathcal{N}-$functions on Sheffer stroke Hilbert algebras. It is proved that the $(\alpha,\beta,\gamma)-$level set of a neutrosophic $\mathcal{N}-$subalgebra defined by the $\mathcal{N}-$functions on this algebra is its subalgebra and also the inverse is valid. We show that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke Hilbert algebra forms a complete distributive lattice. Besides, it is demonstrated that every neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke Hilbert algebra satisfies $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$, for all $x \in H$ but a neutrosophic $\mathcal{N}-$structure of a Sheffer stroke Hilbert algebra satisfying the property is mostly not its neutrosophic $\mathcal{N}-$subalgebra. Also, it is comprehensively examined the statement which $\mathcal{N}-$functions defining a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke Hilbert algebra are constant. After describing a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra by means of $\mathcal{N}-$functions, we demonstrate that $\mathcal{N}-$functions defining a neutrosophic $\mathcal{N}-$ideal of the algebra are order-preserving whereas the inverse does not hold in general. In fact, $(\alpha,\beta,\gamma)-$level set of a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is its ideal and vice versa. we present that a lemma is equivalent to the definition of the neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra, and that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse does not usually hold. Moreover, new three subsets $H_N^{x_t}, H_N^{x_i}$ and $H_N^{x_f}$ of a Sheffer stroke Hilbert algebra are described by $\mathcal{N}-$functions and certain elements x_t, x_i and x_f of the algebra. It is proved that these subsets are ideals of a Sheffer stroke Hilbert algebra for its neutrosophic $\mathcal{N}-$ideal defined by the $\mathcal{N}-$functions. A neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra is generally not the $\mathcal{N}-$ideal in the case which these subsets are its ideals.

In the future works, we wish to study on plithogenic sets of Sheffer stroke Hilbert algebras and neutrosophic structures of other Sheffer stroke algebras.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Abbott, J. C. Implicational algebras. Bulletin Mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie 1967, 11(1), pp. 3-23.
2. Atanassov, K. T. Intuitionistic fuzzy sets. Fuzzy Sets Syst. 1986, 20, pp. 87-96.
3. Borumand Saeid, A.; Jun, Y. B. Neutrosophic subalgebras of BCK/BCI-algebras based on neutrosophic points. Ann. Fuzzy Math. Inform. 2017, 14, pp. 87-97.
4. Busneag, D. A note on deductive systems of a Hilbert algebra. Kobe J. Math. 1985, 2, pp. 29-35.

5. Busneag, D. Hilbert algebras of fractions and maximal Hilbert algebras of quotients. Kobe J. Math. 1988, 5, pp. 161-172.
6. Busneag, D. Hertz algebras of fractions and maximal Hertz algebras of quotients. Math. Japon. 1993, 39, pp. 461-469.
7. Borzooei, R. A.; Zhang, X.; Smarandache, F.; Jun, Y. B. Commutative generalized neutrosophic ideals in BCK-algebras. Symmetry 2018, 10(350).
8. Chajda, I. Sheffer operation in ortholattices. Acta Universitatis Palackianae Olomucensis, Facultas Rerum Naturalium. Mathematica 2005, 44(1), pp. 19-23.
9. Chajda, I.; Halaš, R.; Länger, H. Operations and structures derived from non-associative MV-algebras. Soft Computing 2019, 23(12), pp. 3935-3944.
10. Cornish, W. H. On positive implicative BCK-algebras. Mathematics Seminar Notes 1980, 8, pp. 455-468.
11. Diego, A. Sur les algebras de Hilbert. Gauthier-Villars: Paris, Collection de Logique Math. 1966; Série A, No. 21, 56 pages.
12. Henkin, L. An algebraic characterization of quantifiers. Fundam. Math. 1950, 37, pp. 63-74.
13. Iorgulescu, A. Algebras of logic as BCK algebras. Editura ASE: Bucharest, 2008.
14. Iséki, K.; Tanaka, S. An introduction to the theory of BCK-algebras. Math. Japonica 1978, 23(1), pp. 1-26.
15. Jun, Y. B.; Lee, K. J.; Song, S. Z. $\mathcal{N}-$ideals of BCK/BCI-algebras. J. Chungcheong Math. Soc. 2009, 22, pp. 417-437.
16. Jun, Y. B.; Smarandache, F.; Bordbar, H. Neutrosophic $\mathcal{N}-$structures applied to BCK/BCI-algebras. Information 2017, 8(128), pp. 1-12.
17. Jun, Y. B. Neutrosophic subalgebras of several types in BCK/BCI-algebras. Ann. Fuzzy Math. Inform. 2017, 14, pp. 75-86.
18. Jun, Y. B.; Kim, S. J.; Smarandache, F. Interval neutrosophic sets with applications in BCK/BCI-algebras. Axioms 2018, 7(23), pp. 1-13.
19. Jun, Y. B.; Smarandache, F.; Song, S. Z.; Khan, M., Neutrosophic positive implicative $\mathcal{N}-$ideals in BCK-algebras. Axioms 2018, 7(3), pp. 1-13.
20. Khan, M.; Anis, S.; Smarandache, F.; Jun, Y. B. Neutrosophic $\mathcal{N}-$structures and their applications in semigroups. Infinite Study 2017.
21. McCune, W.; Veroff, R.; Fitelson, B.,; Harris, K.; Feist, A.; Wos, L. Short single axioms for Boolean algebra. Journal of Automated Reasoning 2002, 29(1), pp. 1-16 (2002).
22. Oner, T.; Katican, T.; Borumand Saeid, A. Relation between Sheffer Stroke and Hilbert Algebras. Categories and General Algebraic Structures with Applications 2021, 14(1), pp. 245-268.
23. Oner, T.; Katican T.; Borumand Saeid, A. Fuzzy filters of Sheffer stroke Hilbert algebras. Journal of Intelligent and Fuzzy Systems 2021, 40(1), pp. 759-772.
24. Oner, T.; Katican, T.; Borumand Saeid, A.; Terziler, M. Filters of strong Sheffer stroke non-associative MV-algebras. Analele Stiintifice ale Universitatii Ovidius Constanta 2021, 29(1), pp. 143-164.
25. Oner, T.; Katican T.; Borumand Saeid, A. (Fuzzy) filters of Sheffer stroke BL-algebras. Kragujevac Journal of Mathematics 2023, 47(1), pp. 39-55.
26. Oner, T.; Katican T.; Borumand Saeid, A. On Sheffer stroke UP-algebras. Discussione Mathematicae General Algebra and Applications (in press).
27. Oner, T.; Kalkan, T.; Kircali Gursoy, N. Sheffer stroke BG-algebras. International Journal of Maps in Mathematics 2021, 4(1), pp. 27-30.
28. Rasiowa, H. An algebraic approach to non-classical logics. Studies in Logic and the Foundations of Mathematics: North-Holland and PWN, 1974; Volume 78.
29. Sheffer, H. M. A set of five independent postulates for Boolean algebras, with application to logical constants. Transactions of the American Mathematical Society 1913, 14(4), pp. 481-488.

30. Smarandache, F. A unifying field in logic. Neutrosophy: neutrosophic probability, set and logic. American Research Press: Rehoboth, NM, USA, 1999.
31. Smarandache, F. Neutrosophic set-a generalization of the intuitionistic fuzzy set. Int. J. Pure Appl. Math. 2005, 24, pp. 287-297.
32. Song, S. Z.; Smarandache, F.; Jun, Y. B. Neutrosophic commutative $\mathcal{N}-$ideals in BCK-algebras. Information 2017, 8(130).
33. Zadeh L. A. Fuzzy sets. Inf. Control 1965, 8, pp. 338-353.
34. Öztürk, M. A.; Jun, Y. B. Neutrosophic ideals in BCK/BCI-algebras based on neutrosophic points. J. Int. Math. Virtual Inst. 2018, 8, pp. 1-17.

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Some New Type of Lacunary Statistically Convergent Sequences In Neutrosophic Normed Space

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Abstract. The idea of statistical convergence was introduced by Fast [H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244] afterwards studied by many authors. In [J.A. Fridy, C. Orhan, Lacunary statistical convergence, Pacific J. Math. 160 (1993) 43–51],Fridy and Orhan proposed the concept of lacunary statistical convergence. In present paper, we introduce lacunary statistically convergent in neutrosophic normed space (brifly, NNS). We define the concept of lacunary statistical Cauchy sequence in NNS and derive the relation between statistical completness and completeness in NNS . We give some basic properties of these concepts.

Keywords: NNS, t-norm,t-conorm, Statistical convergence,Lacunary statistical convergence,Lacunary statistical Cauchy and Lacunary statistical completeness.

1. **Introduction**

Zadeh [6] was the first who introduced the theory of fuzzy sets. It has a very influential impact on all the scientific fields and is quite necessary for the real- life situations. Atanassov [9] generalized the concepts of fuzzy sets, known as intuitionistic fuzzy sets. One of the dominant problems in fuzzy topology is to obtain an appropriate hypothesis of the fuzzy metric space. Moreover, Park [2] used George and Veeramani's [1] thought of applying t-norm and t-conorm to fuzzy metric spaces for defining intuitionistic fuzzy metric spaces and studying its fundamental features. After a while, Smarandache [10] introduced the notion of neutrosophic sets (NS), which is a different kind of the notion of the classical set theory.by adding an intermediate membership function. This set is a formal setting trying to measure the truth, indeterminacy, and falsehood. Smarandache [16] further studied the differences between intuitionistic fuzzy set and neutrosophic set and the corresponding relations between these two sets. The basic differences are as follows:

(i) Neutrosophic set can distinguish between relative truth = 1 and absolute truth = 1^+. This has application in philosophy. For this reason, the unitary standard interval $[0,1]$ used in intuitionistic fuzzy set has been extended to the unitary non-standard interval $]^-0,1^+[$ in neutrosophic set.

(ii) In neutrosophic set, there is no condition on $\mathcal{T}$(truth), $\mathcal{H}$(indeterminacy) and $\mathcal{F}$(falsehood) other than they are subsets of $]^-0,1^+[$, therefore:

$$^-0 \leq \inf \mathcal{T} + \inf \mathcal{H} + \inf \mathcal{F} \leq \sup \mathcal{T} + \sup \mathcal{H} + \sup \mathcal{F} \leq 3^+.$$

(iii) In neutrosophic set, the components T(truth), H(indeterminacy) and F(falsehood) can also be non-standard subsets included in the unitary non-standard interval $]^-0,1^+[$, not only standard subsets, included in the unitary standard interval $[0,1]$ as in intuitionistic fuzzy logic.

Neutrosophic sets are more effective and flexible because it handles, besides independent components, also partially dependent and partially independent components, while intuitionistic fuzzy sets cannot deal with these. Further, Smarandache [17–19] investigated neutroalgebra which is generalization of partial algebra , neutroalgebraic structures and antialgebraic structures. Moreover, Bera and Mahapatra [11] defined neutrosophic soft linear spaces (NSLSs). In [11] neutrosophic norm, Cauchy sequence in NSNLS, the convexity of NSNLS, metric in NSNLS were studied. There has been much progress in the study of neutrosophic theory in different fields by various authors.

Fast [13] proposed the concept of statistical convergence and later on studied by many researchers. Friday and Orhan [15] have investigated the theory of lacunary statistical convergence. Later on, the concepts of statistical convergence of double sequences have been analyzed in IFNS by Mursaleen and Mohiuddin [12]. Quite recently, Kirisci and Simsek [7] introduced the notion of neutrosophic normed space and statistical convergence. Since neutrosophic normed space is a natural generalization of IFNS and statistical convergence.

In the present paper we will study lacunary statistical convergence and lacunary statistical Cauchy in neutrosophic normed space. we will the study the concept of statistical completeness which would provide a ordinary framework to study the completeness of neutrosophic normed space. We outline the present work as follows. In Section 2, we recall some basic definitions related to the neutrosophic normed space.In Section 3, in this paper we porposed lacunary statistical convergence in NNS and prove our main results. Finally, Section 4 is devoted to introduce a recent concept, i.e. (lacunary) statistical completeness and find its relation with completeness of NNS.

2. Preliminaries

Throughtout this article, $\mathbb{N}$ will denote the set of natural numbers. Using definitions of continuous t-norm and continuous t-conorm (see [14]), Kirisci and Simsek [7] proposed the notion of NNS which is defined as follows:

Definition 2.1. [14] Given an operation $\star : [0,1] \times [0,1] \longrightarrow [0,1]$ then it is called continuous t-norm if it satisfies the following conditions:
(a) $\star$ is associative and commutative,
(b) $\star$ is continuous,
(c) $c \star 1 = c$ for all $c \in [0,1]$,
(d) $c \star d \leq f \star g$ whenever $c \leq f$ and $d \leq g$ for each $c, d, f, g \in [0,1]$.

Definition 2.2. [14] Given an operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ then it is called continuous t-conorm if it satisfies the following conditions:
(a) $\diamond$ is associative and commutative,
(b) $\diamond$ is continuous,
(c) $c \diamond 0 = c$ for all $c \in [0,1]$,
(d) $c \diamond d \leq f \diamond g$ whenever $c \leq f$ and $d \leq g$ for each $c, d, f, g \in [0,1]$.

From above definitions, we note that if we choose $0 < e_1, e_2 < 1$ with $e_1 > e_2$, then there exist $0 < e_3, e_4 < 1$ such that $e_1 * e_3 \geq e_2, e_1 \geq e_4 \diamond e_2$. Further, if we choose $e_5 \in (0,1)$, then there exist $e_6, e_7 \in (0,1)$ such that $e_6 * e_6 \geq e_5$ and $e_7 \diamond e_7 \leq e_5$.

Definition 2.3. [5] The intuitionistic fuzzy set A which is a subset of non-empty set X is an ordered triplet defined by

$$A = \{< x, \mathcal{T}(x),\ \mathcal{F}(x) >: x \in X\},$$

where $\mathcal{T}(x), \mathcal{F}(x) : X \to [0,1]$ represent the degree of membership and degree of nonmembership respectively in such a way that

$$0 \leq \mathcal{T}(x) + \mathcal{F}(x) \leq 1$$

Also, $1 - \mathcal{T}(x) - \mathcal{F}(x)$ is called degree of hesitancy. The intuitionistic fuzzy components $\mathcal{T}(x), \mathcal{F}(x)$ and degree of hesitancy are dependent on each other.

Definition 2.4. [10] Let A be a subset of non-empty set X. Then,

$$A_{NS} = \{< x, \mathcal{T}(x), \mathcal{I}(x), \mathcal{F}(x) >: x \in X\},$$

where $\mathcal{T}(x), \mathcal{I}(x), \mathcal{F}(x) : X \to [0,1]$ represent the degree of truth-membership, degree of indeterminacy-membership, and degree of false-nonmembership respectively in such a way that

$$0 \leq \mathcal{T}(x) + \mathcal{I}(x) + \mathcal{F}(x) \leq 3.$$

The neutrosophic components $\mathcal{T}(x), \mathcal{I}(x)$ and $\mathcal{F}(x)$ are independent of each other.

Definition 2.5. [20, 21] The complement of an interval neutrosophic set P is denoted by P_{-} and is defined by

$\mathcal{T}_{P_{-}}(x) = \mathcal{F}_P(x)$;

$\inf \mathcal{H}_{P_{-}}(x) = 1 - \sup \mathcal{H}_P(x);\ \sup \mathcal{H}_{P_{-}}(x) = 1 - \inf \mathcal{H}_P(x)$;

$\mathcal{F}_{P_{-}}(x) = \mathcal{T}_P(x);\ \forall\ x \in X$.

Definition 2.6. [8] Let P and R be two neutrosophic sets in a non-empty set X.Then,

(a) $P \subset R \iff \mathcal{T}_P(x) \leq \mathcal{T}_R(x), \mathcal{H}_P(x) \leq \mathcal{H}_R(x), \mathcal{F}_P(x) \geq \mathcal{F}_R(x)\ \forall\ x \in X$

(b) $P = R \iff \mathcal{T}_P(x) = \mathcal{T}_R(x), \mathcal{H}_P(x) = \mathcal{H}_R(x), \mathcal{F}_P(x) = \mathcal{F}_R(x)\ \forall\ x \in X$

(c) $P \cap R = \{\langle x, \min(\mathcal{T}_P(x), \mathcal{T}_R(x)), \min(\mathcal{H}_P(x), \mathcal{H}_R(x)), \min(\mathcal{F}_P(x), \mathcal{F}_R(x))\rangle \mid x \in X\}$

(d) $P \cup R = \{\langle x, \max(\mathcal{T}_P(x), \mathcal{T}_R(x)), \max(\mathcal{H}_P(x), \mathcal{H}_R(x)), \max(\mathcal{F}_P(x), \mathcal{F}_R(x))\rangle \mid x \in X\}$

(e) $P^c = \{\langle x, \mathcal{F}_P(x), 1 - \mathcal{H}_R(x)), \mathcal{T}_P(x)\rangle \mid x \in X\}$

(f) $P \backslash R = \{\langle x, \mathcal{T}_P(x) \min \mathcal{F}_R(x), \mathcal{H}_P(x) \min 1 - \mathcal{H}_R(x), \mathcal{F}_P(x) \max \mathcal{T}_R(x)\rangle \mid x \in X\}$.

Definition 2.7. [7] Let U be a vector space and $\mathcal{N} = \{< v, \mathcal{T}(v), \mathcal{H}(v), \mathcal{F}(v) >: v \in U\}$ be a normed space(NS) such that $\mathcal{T}(v), \mathcal{H}(v), \mathcal{F}(v) : U \times \mathbb{R}^+ \to [0, 1]$. Let $*$ and $\diamond$ be continuous t-norm and t-conorm respectively. If the subsequent conditions holds, then the four-tuple $(U, \mathcal{N}, *, \diamond)$ is called neutrosophic normed space (NNS): For all $v, w \in U$ and $\eta, s > 0$ and for each $a \neq 0$,

(1) $0 \leq \mathcal{T}(u, \eta) \leq 1$, $0 \leq \mathcal{H}(v, \eta) \leq 1$, $0 \leq \mathcal{F}(v, \eta) \leq 1$

(2) $\mathcal{T}(v, \eta) + \mathcal{H}(v, \eta) + \mathcal{F}(v, \eta) \leq 3, (\eta \in \mathbb{R}^+)$

(3) $\mathcal{T}(v, t\eta) = 1$ (for $\eta > 0$) if and only if $v = 0$,

(4) $\mathcal{T}(cv, \eta) = \mathcal{T}(v, \frac{\eta}{|c|})$ for each $c \neq 0$,

(5) $\mathcal{T}(v, \eta) * \mathcal{T}(w, s) \leq \mathcal{T}(v + w, \eta + s)$,

(6) $\mathcal{T}(v, .)$ is continuous non-decreasing function,

(7) $\lim_{\eta\to\infty} \mathcal{T}(v,\eta) = 1$

(8) $\mathcal{H}(v,\eta) = 0$ (for $\eta > 0$) if and only if $v = 0$,

(9) $\mathcal{H}(cv,\eta) = \mathcal{H}(v, \frac{\eta}{|c|})$ for each $c \neq 0$,

(10) $\mathcal{H}(v,\eta) * \mathcal{H}(v,s) \geq \mathcal{H}(v+w, \eta+s)$,

(11) $\mathcal{H}(v,.)$ is continuous non-increasing function,

(12) $\lim_{\eta\to\infty} \mathcal{H}(v,\eta) = 1$

(13) $\mathcal{F}(v,\eta) = 0$ (for $\eta > 0$) if and only if $v = 0$,

(14) $\mathcal{F}(cv,\eta) = \mathcal{F}(v, \frac{\eta}{|c|})$ for each $c \neq 0$,

(15) $\mathcal{F}(v,\eta) * \mathcal{F}(v,\mathcal{T}) \geq \mathcal{F}(v+w, \eta+s)$,

(16) $\mathcal{F}(v,.)$ is continuous non-increasing function,

(17) $\lim_{\eta\to\infty} \mathcal{F}(v,\eta) = 1$

(18) If $\eta \leq 0$, then $\mathcal{T}(v,\eta) = 0, \mathcal{H}(v,\eta) = 1$ and $\mathcal{F}(v,\eta) = 1$.

In this case, $\mathcal{N} = (\mathcal{T}, \mathcal{H}, \mathcal{F})$ is said to be neutrosophic norm (NN).

Example 2.8. [7] Let $(U, \|.\|)$ be a normed space. Given the operations $*$ and $\diamond$ in such a way that: $v * w = vw$, $v \diamond w = v + w - vw$. For $\eta > \|v\|$ and $\eta > 0$

$$\mathcal{T}(v,\eta) = \frac{\eta}{\eta + \|v\|}, \mathcal{H}(v,\eta) = \frac{\|v\|}{\eta + \|v\|} \text{ and } \mathcal{F}(v,\eta) = \frac{\|v\|}{\eta} \tag{1}$$

for all $v, w \in U$. If we take $\eta \leq \|v\|$, then $\mathcal{T}(v,\eta) = 0, \mathcal{H}(v,\eta) = 1$ and $\mathcal{F}(v,\eta) = 1$. Then $(U, \mathcal{N}, *, \diamond)$ is NNS in such a way that $\mathcal{N} : U \times \mathbb{R}^+ \to [0,1]$.

Example 2.9. Let $(U = \mathbb{R}, \|.\|)$ be a normed space where $\|x\| = |x|\ \forall\ x \in \mathbb{R}$. Give the operations $*$ and $\diamond$ in such a way that: $v * w = \min\{v, w\}$ and $v \diamond w = \max\{v, w\}.\forall\ v, w\ \in [0,1]$ and Define,

$$\mathcal{T}(v,\eta) = \frac{\eta}{\eta + k\|v\|}, \mathcal{H}(v,t) = \frac{k\|v\|}{\eta + \|v\|} \text{ and } \mathcal{F}(v,\eta) = \frac{k\|v\|}{\eta} \tag{2}$$

where $k > 0$ Then $A = \{(v,\eta), \mathcal{T}(v,\eta), \mathcal{H}(v,\eta), \mathcal{F}(v,\eta) : (v,\eta) \in U \times \mathbb{R}^+\}$ is a NNS on U

Definition 2.10. [7] Let $(U, \mathcal{N}, *, \diamond)$ be a NNS. Then, the sequence (a_n) is said to be convergent to $\xi \in X$ with respect to the NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$ if for each $\epsilon,\ \eta > 0$, there exists $N \in \mathbb{N}$, in such a manner that

$$\mathcal{T}(a_n - \xi, \eta) > 1 - \epsilon, \mathcal{H}(a_n - \xi, \eta) < \epsilon \text{ and } \mathcal{F}(a_n - \xi, \eta) < \epsilon \tag{3}$$

for all $n \geq N$, i.e.,

$$\lim_{n\to\infty} \mathcal{T}(a_n - \xi, \eta) = 1,\ \lim_{n\to\infty} \mathcal{H}(a_n - \xi, \eta) = 0 \text{ and } \lim_{n\to\infty} \mathcal{F}(a_n - \xi, \eta) = 0.$$

In such case, we denote $\mathcal{N} - \lim a_n = \xi$.

Definition 2.11. [7] Let $(U, \mathcal{N}, *, \diamond)$ be a NNS. Then, the sequence (u_n) is known as Cauchy sequence with respect to the NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$ if for each $\epsilon,\ \eta > 0$, there exists $N \in \mathbb{N}$, in such a manner that

$$\mathcal{T}(a_n - a_m, \eta) > 1 - \epsilon, \mathcal{H}(a_n - a_m, \eta) < \epsilon \text{ and } \mathcal{F}(a_n - a_m, \eta) < \epsilon \tag{4}$$

for all $n, m \geq N$.

Definition 2.12. [15]

A lacunary sequence is an increasing integer sequence $\theta = \{n_r\}$ such that $n_0 = 0$ and $h_r = n_r - n_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (n_{r-1}, n_r]$ and the ratio $\frac{n_r}{n_{r-1}}$ will be abbreviated as q_r. Let $K \subseteq \mathbb{N}$. The number

$$\delta_\theta(K) = \frac{1}{h_r} \mid \{n \in I_r : n \in K\} \mid$$

is called θ-density of K, provided the limit exists.

Definition 2.13. [15] Let θ be a lacunary sequence. A sequence $a = \{a_n\}$ of numbers is said to be lacunary statistically convergent (briefly S_θ- convergent) to the number ξ if for every $\epsilon > 0$, the set $K(\epsilon)$ has θ-density zero, where

$$K(\epsilon) = \{a \in I_r : |a_n - \xi| \geq \epsilon\}.$$

In this case we write $S_\theta - \lim a = \xi$.

3. **Lacunary Statistical convergence in NNS**

In this section, we introduce lacunary statistical convergence in NNS. First, we define the subsequent definition

Definition 3.1. Let $(U, \mathcal{N}, *, \diamond)$ be NNS. A sequence (a_n) is called Lacunary statistical convergent with respect to NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$, if there exist $\xi \in U$ such that, the set

$$\Big\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \epsilon \text{ or } \mathcal{H}(a_n - \xi, \eta) \geq \epsilon, \mathcal{F}(a_n - \xi, \eta) \geq \epsilon\Big\}$$

has density zero, for every $\epsilon,\ \eta > 0$ or equivalently,

$$\lim_{n} \frac{1}{n} |\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \epsilon \text{ or } \mathcal{H}(a_n - \xi, \eta) \geq \epsilon, \mathcal{F}(a_n - \xi, \eta) \geq \epsilon\}| = 0.$$

We write $\mathcal{N}_\theta - \lim a = \xi$.

Using the above definition and properties of θ-density, we have the subsequent lemma.

Lemma 3.2. *Let $(U, \mathcal{N}, *, \diamond)$ be a NNS and θ be a lacunary sequence . Then, for each $\epsilon,\ \eta > 0$, the subsequent statements are equivalent:*

(1) $\mathcal{N}_\theta - \lim x = \xi$

(2) $\delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, t) \leq 1 - \epsilon, \mathcal{H}(a_n - \xi, \eta) \geq \epsilon, \mathcal{F}(a_n - \xi, \eta) \geq \epsilon\}\Big) = 0$

(3) $\delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) > 1 - \epsilon, \mathcal{H}(a_n - \xi, \eta) < \epsilon \textit{ and } \mathcal{F}(a_n - \xi, \eta) < \epsilon\}\Big) = 1.$

(4) $\delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) > 1 - \epsilon\}\Big) = \delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{H}(a_n - \xi, \eta) < \epsilon\}\Big) = \delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{F}(a_n - \xi, \eta) < \epsilon\}\Big) = 1$

(5) $\mathcal{N}_\theta - \lim \mathcal{T}(a_n - \xi, \eta) = 1$, $\mathcal{N}_\theta - \lim \mathcal{H}(a_n - \xi, \eta) = 0$ *and* $\mathcal{N}_\theta - \lim \mathcal{F}(a_n - \xi, \eta) = 0$.

Theorem 3.3. *Let θ be a lacunary sequence and $(U, \mathcal{N}, *, \diamond)$ be a NNS. If a sequence $a = (a_n)$ is lacunary statistically convergent with respect to NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$ then $\mathcal{N}_\theta -$ limit is unique.*

Proof. Consider, $\mathcal{N}_\theta - \lim a = \xi_1$, $\mathcal{N}_\theta - \lim a = \xi_2$ and $\xi_1 \neq \xi_2$. Given $\epsilon > 0$, $\alpha > 0$ and $(1 - \alpha) * (1 - \alpha) > 1 - \epsilon$ and $\alpha \diamond \alpha < \epsilon$ Then, for any $\eta > 0$, define the following sets as:

$$W_{\mathcal{T},1}(\alpha, \eta) = \{k \in \mathbb{N} : \mathcal{T}(a_n - \xi_1, \frac{\eta}{2}) \leq 1 - \alpha\}$$

$$W_{\mathcal{T},2}(\alpha, \eta) = \{n \in \mathbb{N} : \mathcal{T}(a_n - \xi_2, \frac{\eta}{2}) \leq 1 - \alpha\}$$

$$W_{\mathcal{H},1}(\alpha, \eta) = \{n \in \mathbb{N} : \mathcal{H}(a_n - \xi_1, \frac{\eta}{2}) \geq \alpha\}$$

$$W_{\mathcal{H},2}(\alpha, \eta) = \{n \in \mathbb{N} : \mathcal{H}(a_n - \xi_2, \frac{\eta}{2}) \geq \alpha\}$$

$$W_{\mathcal{F},1}(\alpha, \eta) = \{n \in \mathbb{N} : \mathcal{F}(a_n - \xi_1, \frac{\eta}{2}) \geq \alpha\}$$

$$W_{\mathcal{F},2}(\alpha,\eta) = \{n \in \mathbb{N} : \mathcal{F}(a_n - \xi_2, \frac{t}{2}) \geq \alpha\}$$

Since $\mathcal{N}_\theta - \lim a = \xi_1$, then using Lemma 3.2, for every $\eta > 0$, we have

$$\delta_\theta(W_{\mathcal{T},1}(\epsilon,\eta)) = \delta_\theta(W_{\mathcal{H},1}(\epsilon,\eta)) = \delta_\theta(W_{\mathcal{F},1}(\epsilon,\eta)) = 0 \tag{5}$$

Furthermore, using $\mathcal{N}_\theta - \lim a = \xi_2$, for all $\eta > 0$, we get

$$\delta_\theta(W_{\mathcal{T},2}(\epsilon,\eta)) = \delta_\theta(W_{\mathcal{H},2}(\epsilon,\eta)) = \delta_\theta(W_{\mathcal{F},2}(\epsilon,\eta)) = 0. \tag{6}$$

Now let

$$W(\epsilon,\eta) = \{(W_{\mathcal{T},1}(\epsilon,\eta) \cup W_{\mathcal{T},1}(\epsilon,\eta))\} \cap \{(W_{\mathcal{H},1}(\epsilon,\eta) \cup W_{\mathcal{H},1}(\epsilon,\eta))\} \cap \{(W_{\mathcal{F},1}(\epsilon,\eta) \cup W_{\mathcal{F},1}(\epsilon,\eta))\}$$

Then observe that $\delta_\theta(W(\epsilon,\eta)) = 0$ which implies $\delta_\theta(\mathbb{N}\backslash W(\epsilon,\eta)) = 1$ if $k \in \mathbb{N}\backslash W(\epsilon,\eta)$, then we have three possible cases.

(a) $(\{n \in \mathbb{N}\backslash W_{\mathcal{T},1}(\epsilon,\eta) \cup W_{\mathcal{T},1}(\epsilon,\eta)\})$

(b) $(\{n \in \mathbb{N}\backslash W_{\mathcal{H},1}(\epsilon,\eta) \cup W_{\mathcal{H},1}(\epsilon,\eta)\})$

(c) $(\{n \in \mathbb{N}\backslash W_{\mathcal{F},1}(\epsilon,\eta) \cup W_{\mathcal{F},1}(\epsilon,\eta)\})$.

Therefore, one obtain
$\mathcal{T}(\xi_1 - \xi_2, \eta) = \mathcal{T}(a_n - \xi_1, \frac{\eta}{2}) * \mathcal{T}(a_n - \xi_2, \frac{\eta}{2}) > (1-\alpha) * (1-\alpha)$.
Since $(1-\alpha) * (1-\alpha) > 1-\epsilon$.
It follows that $\mathcal{T}(\xi_1 - \xi_2, \eta) > 1-\epsilon$.
Since $\epsilon > 0$ was arbitrary, we get $\mathcal{T}(\xi_1 - \xi_2, \eta) = 1$ for all $\eta > 0$, which gives $\xi_1 = \xi_2$.
Contrarily (b), if $n \in \mathbb{N}\backslash W_{\mathcal{H},1}(\epsilon,\eta) \cup W_{\mathcal{H},1}(\epsilon,\eta)$. Then,
$\mathcal{H}(\xi_1 - \xi_2, \eta) \leq \mathcal{H}(a_n - \xi_1, \frac{\eta}{2}) \diamond \mathcal{H}(a_n - \xi_2, \frac{\eta}{2}) < \alpha \diamond \alpha$
Now, utilizing the fact that $\alpha \diamond \alpha < \epsilon$, it can be easily seen that
$\mathcal{H}(\xi_1 - \xi_2, \eta) < \epsilon$.
So, $\mathcal{H}(\xi_1 - \xi_2, \eta) = 0$ for all $\eta > 0$, implies $\xi_1 = \xi_2$.
and if $n \in \mathbb{N}\backslash W_{\mathcal{F},1}(\epsilon,\eta) \cup W_{\mathcal{F},1}(\epsilon,\eta)$.Then
$\mathcal{F}(\xi_1 - \xi_2, \eta) \leq \mathcal{F}(a_n - \xi_1, \frac{\eta}{2}) \diamond \mathcal{F}(a_n - \xi_2, \frac{\eta}{2}) < \alpha \diamond \alpha$
Since $\alpha \diamond \alpha < \epsilon$, it follows that.
$\mathcal{F}(\xi_1 - \xi_2, \eta) < \epsilon$. we have
$\mathcal{F}(\xi_1 - \xi_2, \eta) = 0$ for all $\eta > 0$, which implies $\xi_1 = \xi_2$.
Therefore, in all cases, we conclude that $\mathcal{N}_\theta-$ limit is unique.

□

Theorem 3.4. *Let θ be a lacunary sequence and $(U, \mathcal{N}, *, \diamond)$ be an NNS. If $\mathcal{N} - \lim a = \xi$ then $\mathcal{N}_\theta - \lim a = \xi$. but converse need not be true.*

Proof. Let $\lim a = \xi$. Then for each $\epsilon,\ \eta > 0$, there is a number $n_0 \in \mathbb{N}$ such that $\mathcal{T}(a_n - \xi, \eta) > 1 - \epsilon$, $\mathcal{H}(a_n - \xi, \eta) < \epsilon$ and $\mathcal{F}(a_n - \xi, \eta) < \epsilon$ for all $n \geq n_0$.
Hence, the set

$$\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \epsilon \text{ or } \mathcal{H}(a_n - \xi, \eta) \geq \epsilon, \mathcal{T}(a_n - \xi, \eta) \geq \epsilon\} \tag{7}$$

has at most finitely many terms. Since each and every finite subset of $\mathbb{N}$ has density zero and hence

$$\delta_\theta(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \epsilon \text{ or } \diamond H(a_n - \xi, \eta) \geq \epsilon, \mathcal{F}(a_n - \xi, \eta) \geq \epsilon\}) = 0. \tag{8}$$

Therefore, $\mathcal{N}_\theta - \lim a = \xi$. □

For converse, we construct the following example:

Example 3.5. Let $(U, \|.\|)$ be a NS. Consider $U = \mathbb{R}$ and for all $v, w \in [0, 1]$, define $v * w = vw$ and $v \diamond w = \min\{v + w, 1\}$. Take

$$\mathcal{T}(v, \eta) = \frac{\eta}{\eta + \|v\|}, \mathcal{H}(v, \eta) = \frac{\|v\|}{\eta + \|v\|}, \mathcal{F}(v, t) = \frac{\|v\|}{\eta}$$

for all $\eta > 0$. Then $(U, \mathcal{N}, *, \diamond)$ be NNS. Define a sequence $a = (a_n)$ by,

$$a_n = \begin{cases} n, & \text{if } n_r - [\sqrt{h_r}] + 1 \leq n \leq n_r, r \in \mathbb{N} \\ 0, & otherwise. \end{cases} \tag{9}$$

Consider

$$K_r(\epsilon, \eta) = \{n \leq I_r : \mathcal{T}(a_n, \eta) \leq 1 - \epsilon \text{ or } \mathcal{H}(a_n, \eta) \geq \epsilon, \mathcal{F}(a_n, \eta) \geq \epsilon\}$$

for every $\epsilon \in (0, 1)$ and for any $\eta > 0$. Then, we have

$$\begin{aligned} K_r(\epsilon, \eta) &= \{n \leq I_r : \frac{\eta}{\eta + \|v\|} \leq 1 - \epsilon \text{ or } \frac{\|v\|}{\eta + \|v\|} \geq \epsilon, \frac{\|v\|}{\eta} \geq \epsilon\} \\ &= \{n \leq I_r : \|v\| \geq \frac{\eta\epsilon}{1 - \epsilon} \text{ or } \|v\| \geq \eta\epsilon\} \\ &\subseteq \{n \leq I_r : a_n = n\} \end{aligned}$$

Thus,

$$\frac{1}{h_r} \mid \{n \in I_r : n \in K_r(\epsilon, \eta)\} \mid \leq \frac{\sqrt{h_r}}{h_r} \to 0 \text{ as } r \to \infty.$$

Therefore,

$$\mathcal{N}_\theta - \lim_k a_n = 0.$$

But the sequence $a = \{a_n\}$ is not convergent to 0.

Theorem 3.6. *Let $(U, \mathcal{N}, *, \diamond)$ be an NNS. Then for any lacunary sequence θ, $\mathcal{N}_\theta - \lim a_n = \xi$ iff there exists a increasing index sequence $K = \{k_1 < k_2 < \dots\} \subseteq \mathbb{N}$ while $\delta_\theta(N) = 1$, then $\mathcal{N} - \lim_j a_{n_j} = \xi$.*

Proof. Let $\mathcal{N}_\theta - \lim a_n = \xi$. For any $\eta > 0$ and $\alpha = 1, 2, 3, \dots$

$$W(\alpha, \eta) = \big\{n \leq k : \mathcal{T}(a_n - \xi, \eta) > 1 - \frac{1}{\alpha} \text{ and } \mathcal{H}(a_n - \xi, \eta) < \frac{1}{\alpha},\ \mathcal{F}(a_n - \xi, \eta) < \frac{1}{\alpha}\big\}$$

and

$$Q(\alpha, \eta) = \big\{n \leq k : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \frac{1}{\alpha} \text{ or } \mathcal{H}(a_n - \xi, \eta) \geq \frac{1}{\alpha},\ \mathcal{F}(a_n - \xi, \eta) \geq \frac{1}{\alpha}\big\}$$

Then, $\delta_\theta(Q(\alpha, \eta)) = 0$, since $\mathcal{N}_\theta - \lim a_n = \xi$. Further, for $\eta > 0$ and $\alpha = 1, 2, 3, \dots$

$$W(\alpha, \eta) \supset W(\alpha + 1, \eta)$$

and so,

$$\delta_\theta(W(\alpha, \eta)) = 1. \tag{10}$$

Now, we imply that for $n \in W(\alpha, \eta), \lim a_n = \xi$. Assume that $\lim a_n \neq \xi$ for some $n \in W(\alpha, \eta)$. Then, there is $\beta > 0$ and a +ve integer N such that $\mathcal{T}(a_n - \xi, \eta) \leq 1 - \beta$ or $\mathcal{H}(a_n - \xi, \eta) \geq \beta, \mathcal{F}(a_n - \xi, \eta) \geq \beta$ for all $n \geq N$. Let $\mathcal{T}(a_n - \xi, \eta) > 1 - \beta$ or $\mathcal{H}(a_n - \xi, \eta) < \beta, \mathcal{F}(a_n - \xi, \eta) < \beta$ for all $n > N$. Hence

$$\lim_k \frac{1}{k} \big|\{n \leq N : \mathcal{T}(a_n - \xi, \eta) > 1 - \beta \text{ and } \mathcal{H}(a_n - \xi, \eta) < \beta, \mathcal{F}(a_n - \xi, \eta) < \beta\}\big| = 0.$$

Since $\beta > \frac{1}{\alpha}$, we obtain $\delta_\theta(W(\alpha, \eta)) = 0$, which contradicts equation (10). that's why, $\mathcal{N}_\theta - \lim a_n = \xi$.

Conversely, assume that there exists an increasing index sequence $K = \{k_1 < k_2 < \dots\} \subseteq \mathbb{N}$ while $\delta_\theta(K) = 1$, then $\lim_k x_{n_k} = \xi$ i.e., there exists a $K \in \mathbb{N}$ such that $\mathcal{T}(a_n - \xi, \eta) > 1 - \alpha, \mathcal{H}(a_n - \xi, \eta) < \alpha, \mathcal{F}(a_n - \xi, \eta) < \alpha$ for every $\alpha > 0$ and $\eta > 0$. In that case

$$Q_\theta(\alpha, \eta) = \big\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \eta) \leq 1 - \alpha \text{ and } \mathcal{H}(a_n - \xi, \eta) \geq \alpha, \mathcal{F}(a_n - \xi, \eta) \geq \alpha\big\}$$

$$\subseteq \mathbb{N} - \{k_{N+1}, k_{N+2}, \dots\}.$$

Therefore $\delta_\theta(Q(\alpha, \eta)) \leq 1 - 1 = 0$. Hence $\mathcal{N}_\theta - \lim a_n = \xi$. □

4. Lacunary statistical Completeness in NNS

Definition 4.1. Let $(U,\mathcal{N},*,\diamond)$ be an NNS and θ be any lacunary sequence. The sequence (a_n) is called Lacunary statistically Cauchy with respect to Neutrosophic norm(NN) in NNS U, if there exists $M = M(\epsilon)$, for every $\epsilon > 0$ and $\eta > 0$ such that, the set

$$\delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{T}(a_n - a_M, \eta) \leq 1 - \epsilon \text{ or } \mathcal{H}(a_n - a_M, \eta) \geq \epsilon, \mathcal{F}(a_n - a_M, \eta) \geq \epsilon\}\Big) = 0.$$

Theorem 4.2. *Let $(U,\mathcal{N},*,\diamond)$ be an NNS and θ be any lacunary sequence. If a sequence $\{a_n\}$ is $\mathcal{N}_\theta$-statistically convergent, then it is $\mathcal{N}_\theta$-statistically Cauchy with respect to the NN $(\mathcal{T},\mathcal{H},\mathcal{F})$.*

Proof. Let a sequence $a = \{a_n\}$ is a lacunary statistically convergent in NNS U. We obtained $(1-\epsilon)*(1-\epsilon) > 1-\alpha$ and $\epsilon \diamond \epsilon < \alpha$ for a given $\epsilon > 0$ and choose $\alpha > 0$. Then, we get

$$\begin{aligned}\delta_\theta\left(W(\epsilon,\eta)\right) = \delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \frac{\eta}{2}) \leq 1-\epsilon \text{ or }\\ \mathcal{H}(a_n - \xi, \frac{\eta}{2}) \geq \epsilon, \mathcal{F}(a_n - \xi, \frac{\eta}{2}) \geq \epsilon\}\Big) = 0.\end{aligned} \tag{11}$$

and so

$$\begin{aligned}\delta_\theta\Big(W^c(\epsilon,\eta)\Big) = \delta_\theta\Big(\{n \in \mathbb{N} : \mathcal{T}(a_n - \xi, \frac{\eta}{2}) > 1-\epsilon \text{ or }\\ \mathcal{H}(a_n - \xi, \frac{\eta}{2}) < \epsilon, \mathcal{F}(a_n - \xi, \frac{\eta}{2}) < \epsilon\}\Big) = 1\end{aligned}$$

for $\eta > 0$. Let $p \in W^c(\epsilon,\eta)$ then

$$\mathcal{T}(a_n - \xi, \frac{\eta}{2}) > 1-\epsilon \text{ and } \mathcal{H}(a_n - \xi, \frac{\eta}{2}) < \epsilon, \mathcal{F}(a_n - \xi, \frac{\eta}{2}) < \epsilon.$$

Let

$$Q(\epsilon,\eta) = \{n \in \mathbb{N} : \mathcal{T}(a_n - a_m, \eta) \leq 1-\alpha \text{ or } \mathcal{H}(a_n - a_m, \eta) \geq \alpha, \mathcal{F}(a_n - a_m, \eta) \geq \alpha\}$$

Now, we have to show that $Q(\epsilon,\eta) \subset W(\epsilon,\eta)$. Let $q \in Q(\epsilon,\eta) \setminus W(\epsilon,\eta)$. Then

$$\mathcal{T}(a_q - a_n, \eta) \leq 1-\alpha \text{ and } \mathcal{T}(a_q - \xi, \frac{\eta}{2}) > 1-\epsilon,$$

in particular $\mathcal{T}(a_q - \xi, \frac{\eta}{2}) > 1-\epsilon$. At the same time,

$$1-\alpha \geq \mathcal{T}(a_q - a_n, \eta) \geq \mathcal{T}(u_q - \xi, \frac{\eta}{2}) * \mathcal{T}(a_n - \xi, \frac{\eta}{2}) > (1-\epsilon)*(1-\epsilon) > 1-\alpha$$

which is impossible. Moreover,

$$\mathcal{H}(a_q - a_n, \eta) \geq \alpha \text{ and } \mathcal{H}(a_q - \xi, \frac{\eta}{2}) < \alpha$$

in a similar way, $\mathcal{H}(a_n - \xi, \frac{\eta}{2}) < \epsilon$. Then,

$$\alpha \leq \mathcal{H}(a_q - a_n, \eta) \leq \mathcal{H}(a_q - \xi, \frac{\eta}{2}) \diamond \mathcal{H}(a_n - \xi, \frac{\eta}{2}) < \epsilon \diamond \epsilon < \alpha$$

which is impossible. Similarly,

$$\mathcal{F}(a_q - a_n, \eta) \geq \alpha \text{ and } \mathcal{F}(a_q - \xi, \frac{\eta}{2}) < \alpha$$

in particular $\mathcal{F}(a_n - \xi, \frac{t}{2}) < \epsilon$. Then,

$$\alpha \leq \mathcal{F}(a_q - a_n, \eta) \leq \mathcal{F}(a_q - \xi, \frac{\eta}{2}) \diamond \mathcal{F}(a_n - \xi, \frac{\eta}{2}) < \epsilon \diamond \epsilon < \alpha$$

which is impossible. suppose we consider, $Q(\epsilon, \eta) \subset W(\epsilon, t)$.
Then, by (11) $\delta_\theta\Big(Q(\epsilon, \eta)\Big) = 0$. Hence, sequence (a_n) is $\mathcal{N}_\theta$-Statistical Cauchy with respect to NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$. □

Definition 4.3. The NNS $(U, \mathcal{N}, \star, \diamond)$ is called statistically $(\mathcal{N}_\theta)$ complete, if every statistically $(\mathcal{N}_\theta)$,respectively) Cauchy sequence with respect to NN $(U, \mathcal{N}, \star, \diamond)$ is statistically $\mathcal{N}_\theta$, respectively) convergent with respect to NN $(\mathcal{T}, \mathcal{H}, \mathcal{F})$.

Theorem 4.4. *Let $(U, \mathcal{N}, *, \diamond)$ be a NNS and θ be any lacunary sequence. Then every sequence $a = (a_n)$ in U is $\mathcal{N}_\theta-$complete but not complete in general.*

Proof. Let (a_n) be $\mathcal{N}_\theta-$statistical Cauchy but not $\mathcal{N}_\theta-$statistical convergent in NNS. Choose $\alpha > 0$. We get $(1-\epsilon) * (1-\epsilon) > 1 - \alpha$ and $\epsilon \diamond \epsilon < \alpha$, for a given $\epsilon > 0$ and $\eta > 0$. Since (a_n) is not $\mathcal{N}_\theta$- statistical convergent in NNS.

$$\mathcal{T}(a_n - a_M, \eta) \geq \mathcal{T}(a_n - \xi, \frac{\eta}{2}) * \mathcal{T}(a_M - \xi, \frac{\eta}{2}) > (1-\epsilon) * (1-\epsilon) > 1 - \alpha$$

$$\mathcal{H}(a_n - a_M, \eta) \leq \mathcal{H}(a_n - \xi, \frac{\eta}{2}) \diamond \mathcal{H}(a_M - \xi, \frac{\eta}{2}) < \epsilon \diamond \epsilon < \alpha$$

$$\mathcal{F}(a_n - a_M, \eta) \leq \mathcal{F}(u_n - \xi, \frac{\eta}{2}) \diamond \mathcal{F}(a_n - \xi, \frac{\eta}{2}) < \epsilon \diamond \epsilon < \alpha$$

For,

$$W(\epsilon, \alpha) = \{n \in \mathbb{N}, B_{a_n - a_M}(\epsilon) \leq 1 - \alpha\}$$

Since $\delta_\theta\Big(W^C(\epsilon, \alpha)\Big) = 0$ and so $\delta_\theta\Big(W(\epsilon, \alpha)\Big) = 1$ which is in disagreement, since (a_n) was $\mathcal{N}_\theta-$ statistical Cauchy in NNS. So that (a_n) must be $\mathcal{N}_\theta-$statistical convergent in NNS. consequently, entire NNS is $\mathcal{N}_\theta-$statistically complete. □

Example 4.5. [3] Consider, $U = (0, 1]$ and

$$\mathcal{T}(u, \eta) = \frac{\eta}{\eta + |v|}, \mathcal{H}(v, \eta) = \frac{|v|}{\eta + |v|}, \mathcal{F}(u, \eta) = \frac{|v|}{\eta}$$

for all $v \in U$. Then $(U, \mathcal{N}, *, \diamond, \wedge, \vee)$ where min $= \wedge$ and max $= \vee$ is NNS but it's incomplete, since the sequence $(\frac{1}{m})$ is Cauchy with respect to $(\mathcal{T}, \mathcal{H}, \mathcal{F})$ but not convergent with regarding to the present $(\mathcal{T}, \mathcal{H}, \mathcal{F})$.

5. Conclusions

Since every standard norm defines an neutrosophic norm, our results are more general than the corresponding results in [4].The statistical convergence is a generalization of the usual convergence. Furthermore, definition provides a new techniques to investigate the completeness in the sense of statistical convergence. These are illustrated by suitable examples. Their related properties and structural characteristics have been discussed.

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References

1. George, A., and P. Veeramani. 1997. On some results of analysis for fuzzy metric spaces. Fuzzy Sets and Systems 90: 365–368.
2. J.H. Park, Intuitionistic fuzzy metric spaces,*Chaos Solitons Fractals.*, **22** (2004), 1039–1046.
3. R. Saadati, J.H. Park, On the intuitionistic fuzzy topological spaces, Chaos Solitons Fractals 27 (2006) 331-344.
4. J.A. Fridy, C. Orhan, Lacunary statistical convergence, Pacific J. Math.160 (1993) 43-51.
5. K. Atanassov, Intuitionistic fuzzy sets, fuzzy sets Syst., 20(1986): 87-96
6. L. A. Zadeh, Fuzzy sets, Inform Control (1965); 8:338-53
7. M. Kirişci, N. Şimşek,Neutrosophic normed spaces and statistical convergence, *The Journal of Analysis*, (2020) 1-15
8. Smarandache, F. (1999). A unifying field in logics, Neutrosophic Probability, Set and Logic, Rehoboth: American Research Press.
9. K. Atanassov, Intuitionistic fuzzy sets, fuzzy sets Syst., 20(1986): 87-96
10. F. Smarandache, Neutrosophic set-a generalization of the intuitionistic fuzzy set, International journal of pure and applied mathematics, 24(3) **2005** pages 287.
11. T. Bera, N. K. Mahapatra, On neutrosophic soft linear spaces, Fuzzy Information and Engineering, 9(3) **2017**, 299-324.
12. M. Mursaleen, S.A. Mohiuddine, Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, Chaos, Solitons & Fractals, Vol. 41(5) 2414-2421 **2009.**
13. Fast H (1951) Sur la convergence statistique. *Colloq. Math.* (2): 241-244
14. K. Menger, Statistical metrics, Proceedings of the National Academy of Sciences of the United States of America, *Vol.*,**28(12)** (1942), 535.
15. J.A. Fridy, C. Orhan, Lacunary statistical convergence,*Pacific J. Math.* 160 **1993** 43-51.
16. F. Smarandache, Neutrosophic Set is a Generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set, Spherical Fuzzy Set, and q-Rung Orthopair Fuzzy Set, while Neutrosophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision (revisited), *Journal of New Theory,* **29 2019**, 1-31.
17. F. Smarandache, NeutroAlgebra is a Generalization of Partial Algebra, *International Journal of Neutrosophic Science (IJNS),* Volume 2, **2020** .
18. F. Smarandache, NeutroAlgebraic Structures and AntiAlgebraic Structures, in Advances of Standard and Nonstandard Neutrosophic Theories , *Pons Publishing House Brussels, Belgium.*, **Chapter No. 6**, pages 240 - 265 , 2019.

19. F. Smarandache, NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited), *Neutrosophic Sets and Systems*, **vol. 31, pp 1-16,** (2020).
20. Florentin Smarandache, A Unifying Field in Logics: Neutrosophic Logic, Multiple-Valued Logic / An International Journal, Vol. 8, No. 3, 385-438, 2002, http://fs.unm.edu /eBook-Neutrosophics6.pdf
21. Florentin Smarandache, Neutrosophy. Neutrosophic Probability, Set, and Logic. ProQuest Information and Learning, Ann Arbor, Michigan, USA, 105 p., 1998, 2000, 2002, 2005, 2006.
22. V. A. Khan, H. Fatima, M. Daud. Khan, Space of neutrosophic lambda- statistical convergence sequence and their properties, Vol. 2021 (23), jmcs **2021,** 1-9. http://dx.doi.org/10.22436/jmcs.023.01.01
23. V. A. Khan, M. Ahmad, H. Fatima, M. F. Khan, On some results in intuitionistic fuzzy ideal convergence double sequence spaces Advances in Difference Equations, Vol. 2019(1), **2019** 1-10.

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A Generalized Neutrosophic Metric Space and Coupled Coincidence Point Results

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Abstract. This work introduces the notion of J-Neutrosophic Metric Space using the concept of Neutrosophic Sets. We analyze and extend some coupled coincidence point results on θ_{JN}-coupled and compatible mappings $\mathcal{G} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ where $\mathcal{G}$ has the mixed $\mathfrak{h}$-monotone property. We support the proposed result with suitable example.

Keywords: J-Neutrosophic Metric Space; JN-Compatible; JN-Mixed $\mathfrak{h}$-Monotone Property; θ_{JN}-Coupled Mappings; JN-Coupled Coincidence Point

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1. Introduction

The classical set theory [3, 8] evolves through various extensions as it laid a foundation for modern mathematics. The especially notable extension is the concept of fuzzy sets [9, 22] which introduced graded identity in set theory. This significant feature of assigning graded membership polarized the researchers to come out with numerous analysis and applications over kinds of fuzzy metric spaces.

In 1983, Atanassov [1] made an excitement with the introduction of Intutionstic Fuzzy Sets by adding the idea of nonmembership grade to fuzzy set theory. Since then numerous work have been done to bring out new results and to extend existing concepts over intuitionistic fuzzy setting.

In the year 1995, Florentin Smarandache [17–19] introduced Neutrosophy, an extension of intuitionistic fuzzy set, which claims that between an idea and its opposite, there is a continuum-power spectrum of neutralities. As neutrosophy adds neutralities to intuitionistic fuzzy sets, it inspired the research community and the field is currently growing fruitfully with so many investigations, analysis, computing techniques and applications [4–7, 11, 13, 15, 20].

In the meanwhile, Mustafa and Sims [14] defined the following generalized metric space.

Definition 1.1. [14]Let $\mathcal{A}$ be a nonempty set. $G : \mathcal{A}^3 \to (-\infty, +\infty)$ is called a Generalized Metric (Shortly, G-metric) on $\mathcal{A}$ if for all $\mu, \rho, \upsilon, \gamma \in \mathcal{A}$,

(G1) $G(\mu, \mu, \rho) > 0$ if $\mu \neq \rho$,

(G2) $G(\mu, \rho, \upsilon) = 0$ if and only if $\mu = \rho = \upsilon$

(G3) $G(\mu, \mu, \rho) \leq G(\mu, \rho, \upsilon)$ if $\rho \neq \upsilon$

(G4) $G(\mu, \rho, \upsilon)$ is symmetry in all three variables.

(G5) $G(\mu, \rho, \upsilon) \leq G(\mu, \gamma, \gamma) + G(\gamma, \rho, \upsilon)$

The pair $(\mathcal{A}, G)$ is called generalized metric space.

This space was then used by Sun and Yang [21] to bring out the notion of generalized fuzzy metric space. Mohiuddine and Alotaibi [12] used it to introduce intuitionistic fuzzy metric space. As a consequence, numerous terms and definitions are introduced along with related results in these settings. Notable among them is the concept of common fixed point, coupled coincidence point and mixed $\mathfrak{h}$-monotone property that are given by Bhaskar and Lakshmikantham [2] and Lakshmikantham and Ciric [10].

Definition 1.2. [2]Let $\mathcal{A}$ be a set with partial order $\leq$. The mapping $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is said to have the mixed monotone property if the following conditions hold.

(i) $\mu_1, \mu_2 \in \mathcal{A}$, $\mu_1 \leq \mu_2$ implies $\mathcal{G}(\mu_1, \rho) \leq \mathcal{G}(\mu_2, \rho)$ for all $\rho \in \mathcal{A}$;

(ii) $\rho_1, \rho_2 \in \mathcal{A}$, $\rho_1 \leq \rho_2$ implies $\mathcal{G}(\mu, \rho_1) \leq \mathcal{G}(\mu, \rho_2)$ for all $\mu \in \mathcal{A}$.

Definition 1.3. [2]$(\mu, \rho) \in \mathcal{A} \times \mathcal{A}$ is a coupled fixed point of $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ if $\mathcal{G}(\mu, \rho) = \mu$ and $\mathcal{G}(\rho, \mu) = \rho$.

Definition 1.4. [10]Let $\mathcal{A}$ be a set with partial order $\leq$. The mappings $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ have mixed $\mathfrak{h}$-monotone property if the following conditions hold.

(i) $\mu_1, \mu_2 \in \mathcal{A}$, $\mathfrak{h}(\mu_1) \leq \mathfrak{h}(\mu_2)$ implies $\mathcal{G}(\mu_1, \rho) \leq \mathcal{G}(\mu_2, \rho)$ for all $\rho \in \mathcal{A}$;

(ii) $\rho_1, \rho_2 \in \mathcal{A}$, $\mathfrak{h}(\rho_1) \leq \mathfrak{h}(\rho_2)$ implies $\mathcal{G}(\mu, \rho_1) \leq \mathcal{G}(\mu, \rho_2)$ for all $\mu \in \mathcal{A}$.

Definition 1.5. [10]$(\mu, \rho) \in \mathcal{A} \times \mathcal{A}$ is a coupled coincidence point of $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ if $\mathcal{G}(\mu, \rho) = \mathfrak{h}(\mu)$ and $\mathcal{G}(\rho, \mu) = \mathfrak{h}(\rho)$.

In this scenario, we present here the notion of J-Neutrosophy Metric Space. We propose coincidence point results for compatible, coupled mappings that are having a kind of mixed monotone property in the newly defined space with a partial order.

2. J-Neutrosophic Metric Space

Let us start with the definitions of following binary operations which will be the main frame in defining the J-Neutrosophic Metric Space.

Definition 2.1. [16] A binary operation $\odot : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm (Shortly, CTN) if
(i) $\odot$ is commutative, associative and continuous,
(ii) $t \odot 1 = t$ for all $t \in [0,1]$
(iii) $t \odot s \leq u \odot v$ whenever $t \leq u$ and $s \leq v$, and $s, t, u, v \in [0,1]$.

Definition 2.2. [16] A binary operation $\oslash : [0,1] \times [0,1]$ is a continuous t-conorm (Shortly, CTCN) if
(i) $\oslash$ is commutative, associative and continuous
(ii) $t \oslash 0 = t$ for all $t \in [0,1]$
(iii) $t \oslash s \leq u \oslash v$ whenever $t \leq u$ and $s \leq v$, and $s, t, u, v \in [0,1]$.

The following definition defines the new space, namely, J-Neutrosophic Metric Space.

Definition 2.3. Consider a nonempty set $\mathcal{A}$, a CTN $\odot$, two CTCNs $\oslash$, $\otimes$ and fuzzy sets J, S, F on $\mathcal{A}^3 \times [0,1]$. A 7-tuple $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is called a J-Neutrosophic Metric Space (Shortly, JNMS) if for each $\mu, \rho, \upsilon, \gamma \in \mathcal{A}$ and $t > 0$,

(JN1) $J(\mu, \rho, \upsilon, t) + S(\mu, \rho, \upsilon, t) + F(\mu, \rho, \mu, t) \leq 3$,
(JN2) $J(\mu, \rho, \upsilon, t) > 0$,
(JN3) $J(\mu, \rho, \upsilon, t)$ is symmetry in μ, ρ and υ,
(JN4) $J(\mu, \mu, \rho, t) \geq J(\mu, \rho, \upsilon, t)$ if $\rho \neq \upsilon$,
(JN5) $J(\mu, \rho, \upsilon, t) = 1$ if and only if $\mu = \rho = \upsilon$,
(JN6) $J(\mu, \rho, \upsilon, t + s) \geq J(\mu, \gamma, \gamma, t) \odot J(\gamma, \rho, \upsilon, s)$,
(JN7) $J(\mu, \rho, \upsilon, t)$ is continuous with respect to t,
(JN8) J is nondecreasing on $[0, +\infty]$,
$\lim_{q \to +\infty} J(\mu, \rho, \upsilon) = 1$, $\lim_{q \to 0} J(\mu, \rho, \upsilon) = 0$,
(JN9) $S(\mu, \rho, \upsilon, t) < 1$,
(JN10) $S(\mu, \rho, \upsilon, t)$ is symmetry in μ, ρ and υ,
(JN11) $S(\mu, \mu, \rho, t) \leq S(\mu, \rho, \upsilon, t)$ if $\rho \neq \upsilon$,
(JN12) $S(\mu, \rho, \upsilon, t) = 0$ if and only if $\mu = \rho = \upsilon$,
(JN13) $S(\mu, \rho, \upsilon, t + s) \leq S(\mu, \gamma, \gamma, t) \oslash S(\gamma, \rho, \upsilon, s)$,

(JN14) $S(\mu, \rho, \upsilon, t)$ is continuous with respect to t,

(JN15) S is nonincreasing on $[0, +\infty]$

$\lim_{q \to +\infty} S(\mu, \rho, \upsilon) = 0$, $\lim_{q \to 0} S(\mu, \rho, \upsilon) = 1$

(JN16) $F(\mu, \rho, \upsilon, t) < 1$,

(JN17) $F(\mu, \rho, \upsilon, t)$ is symmetry in μ, ρ and υ,

(JN18) $F(\mu, \mu, \rho, t) \leq F(\mu, \rho, \upsilon, t)$ if $\rho \neq \upsilon$,

(JN19) $F(\mu, \rho, \upsilon, t) = 0$ if and only if $\mu = \rho = \upsilon$,

(JN20) $F(\mu, \rho, \upsilon, t + s) \leq F(\mu, \gamma, \gamma, t) \otimes F(\gamma, \rho, \upsilon, s)$,

(JN21) $F(\mu, \rho, \upsilon, t)$ is continuous with respect to t.

(JN22) F is nonincreasing on $[0, +\infty]$,

$\lim_{q \to +\infty} F(\mu, \rho, \upsilon) = 0$, $\lim_{q \to 0} F(\mu, \rho, \upsilon) = 1$.

The triplet (J, S, F) is called J-Neutrosophic Metric on $\mathcal{A}$.

Remark 2.4. $J(\mu, \rho, \upsilon, t)$, $S(\mu, \rho, \upsilon, t)$ and $F(\mu, \rho, \upsilon, t)$ represent, respectively, the degree of nearness, the degree of non-nearness and the degree of indeterminacy between μ, ρ and υ with respect to t.

Example 2.5. Let $(\mathcal{A}, G)$ be a generalized metric space. Define the fuzzy sets J, S, F by

$$J(\mu, \rho, \upsilon, t) = \frac{t}{t + G(\mu, \rho, \upsilon)},$$

$$S(\mu, \rho, \upsilon, t) = \frac{G(\mu, \rho, \upsilon)}{t + G(\mu, \rho, \upsilon)} \textit{ and}$$

$$F(\mu, \rho, \upsilon, t) = \frac{G(\mu, \rho, \upsilon)}{t} \text{ for all } \mu, \rho, \upsilon \in \mathcal{A}$$

Define $\odot$, $\oslash$ and $\otimes$ by $a \odot b = ab$, $a \oslash b = min\{a + b, 1\}$ and $a \otimes b = max\{a, b\}$. Then $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is a JNMS.

Definition 2.6. Consider a JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$. Let $\mu \in \mathcal{A}$, $r \in (0, 1)$ and $t > 0$. The JN-open ball $B(\mu, r, t)$ with centre at μ and radius r is defined by

$$B(\mu, r, t) = \{\rho \in \mathcal{A} : J(\mu, \mu, \rho) > 1 - r,\ S(\mu, \mu, \rho) < r,\ F(\mu, \mu, \rho) < r\}.$$

Remark 2.7. The above definition leads to the following facts.

(1) Every JN-open ball is an open set.

(2) Every JNMS is Hausdorff.

It follows from the above remark that the collection $\{B(\mu, r, t) : \mu \in \mathcal{A},\ r \in (0, 1),\ t > 0\}$ forms a base for the JN-metric topology on $\mathcal{A}$ and this topology coincides with the metric topology arising from the generalized metric.

Definition 2.8. A sequence $\{\mu_q\}$ in a JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is JN-Convergent to x if it converges to x in the JN-metric topology.

Remark 2.9. If $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is a JNMS, $\{\mu_{\mathrm{q}}\}$ is a sequence in $\mathcal{A}$ and $\mu \in \mathcal{A}$, then the following are equivalent:

(1) $\{\mu_{\mathrm{q}}\}$ is JN-convergent to μ.
(2) $d_G(\mu_{\mathrm{q}}, \mu) \longrightarrow 0$ as $n \longrightarrow +\infty$.
(3) $J(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu, t) \longrightarrow 1$, $S(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu, t) \longrightarrow 0$, $F(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu, t) \longrightarrow 0$ as $n \longrightarrow +\infty$.
(4) $J(\mu_{\mathrm{q}}, \mu, \mu, t) \longrightarrow 1$, $S(\mu_{\mathrm{q}}, \mu, \mu, t) \longrightarrow 0$, $F(\mu_{\mathrm{q}}, \mu, \mu, t) \longrightarrow 0$ as $n \longrightarrow +\infty$.
(5) $J(\mu_{\mathrm{q}}, \mu_m, \mu, t) \longrightarrow 1$, $S(\mu_{\mathrm{q}}, \mu_m, \mu, t) \longrightarrow 0$, $F(\mu_{\mathrm{q}}, \mu_m, \mu, t) \longrightarrow 0$ as $n \longrightarrow +\infty$.

Definition 2.10. A sequence $\{\mu_{\mathrm{q}}\}$ in a JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is JN-Cauchy if for every $\epsilon > 0$ and $t > 0$ there exists $N \in \mathbb{N}$ such that $J(\mu_{\mathrm{q}}, \mu_m, \mu_l, t) > 1 - \epsilon$, $S(\mu_{\mathrm{q}}, \mu_m, \mu_l, t) < \epsilon$, $F(\mu_{\mathrm{q}}, \mu_m, \mu_l, t) < \epsilon$ for all $n, m, l \in \mathbb{N}$. A JNMS is said to be JN-Complete if every JN-Cauchy sequence is JN-convergent.

Definition 2.11. A JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is said to be regular if the following conditions hold:

(i) If a nondecreasing sequence μ_{q} in $\mathcal{A}$ JN-converges to μ, then $\mu_{\mathrm{q}} \leq \mu$ for all n.
(ii) If a nonincreasing sequence μ_{q} in $\mathcal{A}$ JN-converges to μ, then $\mu_{\mathrm{q}} \geq \mu$ for all n.

Definition 2.12. Let $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ be a JNMS. The mappings $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \longrightarrow \mathcal{A}$ are said to be JN-Compatible if for all $t > 0$,

$$\lim_{\mathrm{q}\to\infty} J(\mathfrak{h}\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}}), \mathfrak{h}\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}}), \mathcal{G}(\mathfrak{h}x_{\mathrm{q}}, \mathfrak{h}\rho_{\mathrm{q}}), t) = 1,$$
$$\lim_{\mathrm{q}\to\infty} S(\mathfrak{h}\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}}), \mathfrak{h}\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}}), \mathcal{G}(\mathfrak{h}x_{\mathrm{q}}, \mathfrak{h}\rho_{\mathrm{q}}), t) = 0,$$
$$\lim_{\mathrm{q}\to\infty} F(\mathfrak{h}\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}}), \mathfrak{h}\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}}), \mathcal{G}(\mathfrak{h}x_{\mathrm{q}}, \mathfrak{h}\rho_{\mathrm{q}}), t) = 0,$$
$$\lim_{\mathrm{q}\to\infty} J(\mathfrak{h}\mathcal{G}(\rho_{\mathrm{q}}, \mu_{\mathrm{q}}), \mathfrak{h}\mathcal{G}(\rho_{\mathrm{q}}, \mu_{\mathrm{q}}), \mathcal{G}(\mathfrak{h}y_{\mathrm{q}}, \mathfrak{h}\mu_{\mathrm{q}}), t) = 1,$$
$$\lim_{\mathrm{q}\to\infty} S(\mathfrak{h}\mathcal{G}(\rho_{\mathrm{q}}, \mu_{\mathrm{q}}), \mathfrak{h}\mathcal{G}(\rho_{\mathrm{q}}, \mu_{\mathrm{q}}), \mathcal{G}(\mathfrak{h}y_{\mathrm{q}}, \mathfrak{h}\mu_{\mathrm{q}}), t) = 0 \text{ and}$$
$$\lim_{\mathrm{q}\to+\infty} F(\mathfrak{h}\mathcal{G}(\rho_{\mathrm{q}}, \mu_{\mathrm{q}}), \mathfrak{h}\mathcal{G}(\rho_{\mathrm{q}}, \mu_{\mathrm{q}}), \mathcal{G}(\mathfrak{h}y_{\mathrm{q}}, \mathfrak{h}\mu_{\mathrm{q}}), t) = 0,$$

where $\{\mu_{\mathrm{q}}\}$ and $\{\rho_{\mathrm{q}}\}$ are sequences in $\mathcal{A}$ such that $\lim_{\mathrm{q}\to+\infty} \mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}}) = \lim_{\mathrm{q}\to+\infty} \mathfrak{h}\mu_{\mathrm{q}} = \mu$ and $\lim_{\mathrm{q}\to+\infty} \mathcal{G}(\rho_{\mathrm{q}}, \mu_{\mathrm{q}}) = \lim_{\mathrm{q}\to+\infty} \mathfrak{h}\rho_{\mathrm{q}} = \rho$ for some $\mu, \rho \in \mathcal{A}$.

To continue the work, we need to define the family Θ of strictly increasing, upper semi-continuous functions $\theta : [0, +\infty) \to [0, \infty)$ in which $\theta(0) = \{0\}, \theta(t) < t$ and $\sum_{\mathrm{q}=1}^{+\infty} \theta^n(t) < +\infty$ for all $t > 0$.

Definition 2.13. Let $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ be a JNMS. The mapping $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is said to be self $\theta_{JN}-$coupled if there exists $\theta \in \Theta$ such that

$$\begin{aligned} J(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &\geq \Big\{J(\mu,\mu,\gamma,t)\odot J(\mu,\mu,\mathcal{G}(\mu,\rho),t)\\ &\odot J(\gamma,\gamma,\mathcal{G}(\gamma,\sigma),t)\Big\},\\ S(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &\leq \Big\{S(\mu,\mu,\gamma,t)\oslash S(\mu,\mu,\mathcal{G}(\mu,\rho),t)\\ &\oslash S(\gamma,\gamma,\mathcal{G}(\gamma,\sigma),t),\\ F(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &\leq \Big\{F(\mu,\mu,\gamma,t)\otimes F(\mu,\mu,\mathcal{G}(\mu,\rho),t)\\ &\otimes F(\gamma,\gamma,\mathcal{G}(\gamma,\sigma),t)\Big\}, \end{aligned}$$

for all $\mu, \rho, \gamma, \sigma \in \mathcal{A}$ and $t > 0$ with $\mu \leq \gamma$, $\rho \geq \sigma$ or $\mu \geq \gamma$, $\rho \leq \sigma$.

Definition 2.14. Let $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ be a JNMS. The mappings $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \longrightarrow \mathcal{A}$ are said to be $\theta_{JN}-$coupled if there exists $\theta \in \Theta$ such that

$$\begin{aligned} J(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &\geq \Big\{J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),u,t)\odot J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),F(\mu,\rho),t)\\ &\odot J(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),F(\gamma,\sigma),t)\Big\},\\ S(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &\leq \Big\{S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\gamma,t)\oslash S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t)\\ &\otimes S(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\},\\ F(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &\leq \Big\{F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\gamma,t)\otimes F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t)\\ &\oslash F(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\}, \end{aligned}$$

for all $\mu, \rho, \gamma, \sigma \in \mathcal{A}$ and $t > 0$ with $\mathfrak{h}(\mu) \leq \mathfrak{h}(\gamma)$, $\mathfrak{h}(\rho) \geq \mathfrak{h}(\sigma)$ or $\mathfrak{h}(\mu) \geq \mathfrak{h}(\gamma)$, $\mathfrak{h}(\rho) \leq \mathfrak{h}(\sigma)$.

Lemma 2.15. *Let $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ be a JNMS and*

$$\Lambda_\kappa(\mu,\rho,\upsilon) = inf\{t > 0 : J(\mu,\rho,\upsilon,t) > 1-\kappa, S(\mu,\rho,\upsilon,t) < \kappa, F(\mu,\rho,\upsilon,t) < \kappa\}$$

for all $\mu, \rho, \upsilon \in \mathcal{A}$, $\kappa \in$(0,1] and $t > 0$. Then for each $\kappa \in$(0,1], there exists $\mu \in$(0,1] such that $\Lambda_\kappa(\mu_1,\mu_1,\mu_{\mathrm{q}}) \leq \sum_{\mathrm{q}=1}^{n-1} \Lambda_\mu(\mu_{\mathrm{q}},\mu_{\mathrm{q}},\mu_{\mathrm{q}+1})$

Proof. For $\kappa \in$(0,1], choose $\mu \in$(0,1] such that $\odot^{(n-1)}(1-\mu) > 1-\kappa$, $\oslash^{(n-1)}\mu < \kappa$, $\otimes^{(n-1)}\mu < \kappa$. Let $\epsilon > 0$. Then

$$\begin{aligned} &J(\mu_1,\mu_1,\mu_{\mathrm{q}},\Lambda_\mu(\mu_{\mathrm{q}},\mu_{\mathrm{q}},\mu_{\mathrm{q}+1}) + (n-1)\epsilon)\\ &\geq J(\mu_1,\mu_1,\mu_2,\Lambda_\mu(\mu_1,\mu_1,\mu_2)+\epsilon)\odot\cdots\odot J(\mu_{\mathrm{q}-1},\mu_{\mathrm{q}-1},\mu_{\mathrm{q}},\Lambda_\mu(\mu_{\mathrm{q}-1},\mu_{\mathrm{q}-1},\mu_{\mathrm{q}})+\epsilon)\\ &> \odot^{(n-1)}(1-\mu)\\ &> 1-\kappa \end{aligned}$$

$$
\begin{aligned}
&S(\mu_1, \mu_1, \mu_{\mathrm{q}}, \Lambda_\mu(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}) + (n-1)\epsilon) \\
&\leq S(\mu_1, \mu_1, \mu_2, \Lambda_\mu(\mu_1, \mu_1, \mu_2) + \epsilon) \oslash \cdots \oslash S(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}, \Lambda_\mu(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}) + \epsilon) \\
&< \oslash^{(n-1)}\mu \\
&< \kappa
\end{aligned}
$$

$$
\begin{aligned}
&F(\mu_1, \mu_1, \mu_{\mathrm{q}}, \Lambda_\mu(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}) + (n-1)\epsilon) \\
&\leq S(\mu_1, \mu_1, \mu_2, \Lambda_\mu(\mu_1, \mu_1, \mu_2) + \epsilon) \otimes \cdots \otimes S(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}, \Lambda_\mu(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}) + \epsilon) \\
&< \otimes^{(n-1)}\mu \\
&< \kappa
\end{aligned}
$$

Therefore, we have $\Lambda_\mu(\mu_1, \mu_1, \mu_{\mathrm{q}}) \leq \Lambda_\mu(\mu_1, \mu_1, \mu_2) + \cdots + \Lambda_\mu(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}) + (n-1)\epsilon$. As $\epsilon > 0$ is arbitrary, $\Lambda_\mu(\mu_1, \mu_1, \mu_{\mathrm{q}}) \leq \Lambda_\mu(\mu_1, \mu_1, \mu_2) + \cdots + \Lambda_\mu(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}})$. □

Lemma 2.16. *A sequence $\{\mu_{\mathrm{q}}\}$ in a JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is JN-Cauchy if for some $\theta \in \Theta$,*

$$
\begin{aligned}
J(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, \theta(t)) &\geq J(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}, t) \odot J(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, t), \\
S(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, \theta(t)) &\leq S(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}, t) \oslash S(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, t), \\
F(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, \theta(t)) &\leq F(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}, t) \otimes F(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, t),
\end{aligned}
$$

for all $t > 0$.

Proof. Denote $\Lambda_\mu(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}})$ by a_{q}. For given $\epsilon > 0$ and each a_{q}, we can find $\mathsf{m}_{\mathrm{q}} > a_{\mathrm{q}}$ such that $\theta(\mathsf{m}_{\mathrm{q}}) < \theta(a_{\mathrm{q}}) + \epsilon$. Now,

$$
\begin{aligned}
J(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, \mathsf{m}_{\mathrm{q}}) &> 1 - \kappa, \\
S(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, \mathsf{m}_{\mathrm{q}}) &< \kappa, \\
F(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, \mathsf{m}_{\mathrm{q}}) &< \kappa.
\end{aligned}
$$

Take $M_{\mathrm{q}} = max\{\mathsf{m}_{\mathrm{q}}, \mathsf{m}_{\mathrm{q}+1}\}$, then

$$
\begin{aligned}
J(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, \theta(M_{\mathrm{q}})) &\geq J(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}, M_{\mathrm{q}}) \odot J(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, M_{\mathrm{q}}) \\
&\geq J(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}, \mathsf{m}_{\mathrm{q}}) \odot J(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, \mathsf{m}_{\mathrm{q}+1}) \\
&> 1 - \kappa.
\end{aligned}
$$

$$
\begin{aligned}
S(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, \theta(M_{\mathrm{q}})) &\leq S(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}, M_{\mathrm{q}}) \oslash S(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, M_{\mathrm{q}}) \\
&\leq S(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}, \mathsf{m}_{\mathrm{q}}) \oslash S(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}, \mathsf{m}_{\mathrm{q}+1}) \\
&< \kappa.
\end{aligned}
$$

$$\begin{aligned} F(\mu_q,\mu_q,\mu_{q+1},\theta(M_q)) &\leq F(\mu_{q-1},\mu_{q-1},\mu_q,M_q \otimes F(\mu_q,\mu_q,\mu_{q+1},M_q) \\ &\leq F(\mu_{q-1},\mu_{q-1},\mu_q,\mathsf{m}_q) \otimes F(\mu_q,\mu_q,\mu_{q+1},\mathsf{m}_{q+1}) \\ &< \kappa. \end{aligned}$$

From Lemma 2.15,

$$\Lambda_\kappa(\mu_q,\mu_q,\mu_{q+1}) \leq \theta(M_q) = max\{\theta(\mathsf{m}_q),\theta(\mathsf{m}_{q+1})\} \leq max\{\theta(a_q),\theta(a_{q+1})\} + \epsilon.$$

From the choice of ϵ, $a_{q+1} \leq max\{\theta(a_q),\theta(a_{q+1})\}$. Hence, for $\epsilon > 0$, we can find n_0 for which $J(\mu_q,\mu_q,\mu_m,\epsilon) > 1-\kappa$, $S(\mu_q,\mu_q,\mu_m,\epsilon) < \kappa$ and $F(\mu_q,\mu_q,\mu_m,\epsilon) < \kappa$ for all $n,m \geq n_0$. Therefore $\{\mu_q\}$ is a JN-Cauchy sequence. □

3. Main Results

The following theorem exhibits the existence of coupled coincidence point of two continuous, compatible functions $\mathcal{G} : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ where $\mathcal{G}$ has the mixed $\mathfrak{h}$-monotone property.

Theorem 3.1. *Consider a complete JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ where $\mathcal{A}$ is a partially ordered set. Consider the mappings $\mathcal{G} : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ where*

(a) $\mathcal{G}(\mathcal{A}\times\mathcal{A}) \subseteq \mathfrak{h}(\mathcal{A})$,
(b) *$\mathcal{G}$ and $\mathfrak{h}$ are continuous,*
(c) *$\mathcal{G}$ and $\mathfrak{h}$ are compatible,*
(d) *$\mathcal{G}$ has mixed $\mathfrak{h}$-monotone property,*
(e) *$\mathcal{G}$ and $\mathfrak{h}$ are $\theta_{JN}-$coupled for some $\theta \in \Theta$.*

If there exist $\mu_0,\rho_0 \in \mathcal{A}$ for which $\mathfrak{h}(\mu_0) \leq \mathcal{G}(\mu_0,\rho_0)$ and $\mathfrak{h}(\rho_0) \geq \mathcal{G}(\rho_0,\mu_0)$, then $\mathcal{G}$ and $\mathfrak{h}$ have a coupled coincidence point.

Proof. Define sequences $\{\mu_q\}$ and $\{\rho_q\}$ by $\mathfrak{h}(\mu_{q+1}) = \mathcal{G}(\mu_q,\rho_q)$ and $\mathfrak{h}(\rho_{q+1}) = \mathcal{G}(\rho_q,\mu_q)$, $n \geq 0$. Let us prove by induction that

$$\mathfrak{h}(\mu_q) \leq \mathfrak{h}(\mu_{q+1}),\ \mathfrak{h}(\rho_q) \geq \mathfrak{h}(\rho_{q+1}) \text{ for all } n \geq 0. \tag{1}$$

The choice of μ_0,ρ_0 gives that $\mathfrak{h}(\mu_0) \leq \mathfrak{h}(\mu_1)$, $\mathfrak{h}(\rho_0) \geq \mathfrak{h}(\rho_1)$. Suppose (1) is true for $n = m$. Then by the mixed $\mathfrak{h}$-monotone property of $\mathcal{G}$, we have that

$$\mathfrak{h}(\mu_{q+1}) = \mathcal{G}(\mu_q,\rho_q) \leq \mathcal{G}(\mu_{q+1},\rho_q),\ \mathfrak{h}(\rho_{q+1}) = \mathcal{G}(\rho_q,\mu_q) \geq \mathcal{G}(\rho_{q+1},\mu_q)$$

which gives that

$$\mathfrak{h}(\mu_{q+2}) = \mathcal{G}(\mu_{q+1},\rho_{q+1}) \geq \mathcal{G}(\mu_{q+1},\rho_q),\ \mathfrak{h}(\rho_{q+2}) = \mathcal{G}(\rho_{q+1},\mu_{q+1}) \leq \mathcal{G}(\rho_{q+1},\mu_q).$$

Hence $\mathfrak{h}(\mu_{q+1}) \leq \mathfrak{h}(\mu_{q+2})$, $\mathfrak{h}(\rho_{q+1}) \geq \mathfrak{h}(\rho_{q+2})$ and 1 follows.

If $\mu = \mu_{q-1}, \rho = \rho_{q-1}, \gamma = \mu_q, \sigma = \rho_q$, then from (e) and (1), we have that

$$J(\mathcal{G}(\mu_{q-1}, \rho_{q-1}), \mathcal{G}(\mu_{q-1}, \rho_{q-1}), \mathcal{G}(\mu_q, \rho_q), \theta(t)) \geq \Big\{ J(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \odot J(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathcal{G}(\mu_{q-1}, \rho_{q-1}), t) \odot J(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathcal{G}(\mu_q, \rho_q), t) \Big\},$$

$$S(\mathcal{G}(\mu_{q-1}, \rho_{q-1}), \mathcal{G}(\mu_{q-1}, \rho_{q-1}), \mathcal{G}(\mu_q, \rho_q), \theta(t)) \leq \Big\{ S(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \oslash S(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathcal{G}(\mu_{q-1}, \rho_{q-1}), t) \oslash S(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathcal{G}(\mu_q, \rho_q), t) \Big\},$$

$$F(\mathcal{G}(\mu_{q-1}, \rho_{q-1}), \mathcal{G}(\mu_{q-1}, \rho_{q-1}), \mathcal{G}(\mu_q, \rho_q), \theta(t)) \leq \Big\{ J(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \otimes F(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), F(\mu_{q-1}, \rho_{q-1}), t) \otimes F(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathcal{G}(\mu_q, \rho_q), t) \Big\}.$$

These inequalities imply that

$$\begin{aligned} J(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathfrak{h}(\mu_{q+1}), \theta(t)) &\geq \Big\{ J(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \odot J(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \\ &\quad \odot J(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathfrak{h}(\mu_{q+1}), t) \Big\} \\ &= J(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \odot J(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathfrak{h}(\mu_{q+1}), t). \end{aligned}$$

$$\begin{aligned} S(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathfrak{h}(\mu_{q+1}), \theta(t)) &\leq \Big\{ S(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \oslash S(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \\ &\quad \oslash S(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathfrak{h}(\mu_{q+1}), t) \Big\} \\ &= S(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \oslash S(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathfrak{h}(\mu_{q+1}), t). \end{aligned}$$

$$\begin{aligned} F(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathfrak{h}(\mu_{q+1}), \theta(t)) &\leq \Big\{ F(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \otimes F(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \\ &\quad \otimes F(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathfrak{h}(\mu_{q+1}), t) \Big\} \\ &= F(\mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_{q-1}), \mathfrak{h}(\mu_q), t) \otimes F(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu_q), \mathfrak{h}(\mu_{q+1}), t). \end{aligned}$$

By lemma 2.16, $\{\mathfrak{h}(\mu_q)\}$ is JN-Cauchy.

If $x = \rho_q, y = \mu_q, u = \rho_{q-1}, v = \mu_{q-1}$, (1) gives that

$$\begin{aligned} J(\mathcal{G}(\rho_{q-1}, \mu_{q-1}), \mathcal{G}(\rho_{q-1}, \mu_{q-1}), \mathcal{G}(\rho_q, \mu_q), \theta(t)) \geq \Big\{ & J(\mathfrak{h}(\rho_{q-1}), \mathfrak{h}(\rho_{q-1}), \mathfrak{h}(\rho_q), t) \\ & \odot J(\mathfrak{h}(\rho_{q-1}), \mathfrak{h}(\rho_{q-1}), \mathcal{G}(\rho_{q-1}, \mu_{q-1}), t) \\ & \odot J(\mathfrak{h}(\rho_q), \mathfrak{h}(\rho_q), \mathcal{G}(\rho_q, \mu_q), t) \Big\}. \end{aligned}$$

$$\begin{aligned} S(\mathcal{G}(\rho_{q-1}, \mu_{q-1}), \mathcal{G}(\rho_{q-1}, \mu_{q-1}), \mathcal{G}(\rho_q, \mu_q), \theta(t)) \leq \Big\{ & S(\mathfrak{h}(\rho_{q-1}), \mathfrak{h}(\rho_{q-1}), \mathfrak{h}(\rho_q), t) \\ & \oslash S(\mathfrak{h}(\rho_{q-1}), \mathfrak{h}(\rho_{q-1}), \mathcal{G}(\rho_{q-1}, \mu_{q-1}), t) \\ & \oslash S(\mathfrak{h}(\rho_q), \mathfrak{h}(\rho_q), \mathcal{G}(\rho_q, \mu_q), t) \Big\}. \end{aligned}$$

$$F(\mathcal{G}(\rho_{\mathrm{q}-1},\mu_{\mathrm{q}-1}),\mathcal{G}(\rho_{\mathrm{q}-1},\mu_{\mathrm{q}-1}),\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}}),\theta(t)) \leq \Big\{F(\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}}),t) \otimes F(\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathcal{G}(\rho_{\mathrm{q}-1},\mu_{\mathrm{q}-1}),t) \otimes F(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}}),\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}}),t)\Big\}.$$

These inequalities give that

$$\begin{aligned} J(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}+1}),\theta(t)) &\geq \Big\{J(\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}}),t) \odot J(\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}}),t) \\ &\odot J(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}+1}),t)\Big\} \\ &= J(\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}}),t) \odot J(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}+1}),t). \end{aligned}$$

$$\begin{aligned} S(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}+1}),\theta(t)) &\leq \Big\{S(\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}}),t) \oslash S(\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}}),t) \\ &\oslash S(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}+1}),t)\Big\} \\ &= S(\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}}),t) \oslash S(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}+1}),t). \end{aligned}$$

$$\begin{aligned} F(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}+1}),\theta(t)) &\leq \Big\{F(\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}}),t) \otimes F(\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}}),t) \\ &\otimes F(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}+1}),t)\Big\} \\ &= F(\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}-1}),\mathfrak{h}(\rho_{\mathrm{q}}),t) \otimes F(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}+1}),t). \end{aligned}$$

By lemma 2.16 $\{\mathfrak{h}(\mu_{\mathrm{q}})\}$ is JN-Cauchy.
The completeness of $\mathcal{A}$ gives $\mu,\rho \in \mathcal{A}$ such that

$$\lim_{\mathrm{q}\to+\infty} \mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}}) = \lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\mu_{\mathrm{q}}) = \mu,\ \lim_{\mathrm{q}\to+\infty} \mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}}) = \lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\rho_{\mathrm{q}}) = \rho.$$

$\mathcal{G}$ and $\mathfrak{h}$ are JN-compatible. Hence, for all $t > 0$, we have that

$$\begin{aligned} &\lim_{\mathrm{q}\to+\infty} J(\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\mu_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}})),t) = 1, \\ &\lim_{\mathrm{q}\to+\infty} S(\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\mu_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}})),t) = 0, \\ &\lim_{\mathrm{q}\to+\infty} F(\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\mu_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}})),t) = 0, \\ &\lim_{\mathrm{q}\to+\infty} J(\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\mu_{\mathrm{q}})),t) = 1, \\ &\lim_{\mathrm{q}\to+\infty} S(\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\mu_{\mathrm{q}})),t) = 0, \\ &\lim_{\mathrm{q}\to+\infty} F(\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\mu_{\mathrm{q}})),t) = 0. \end{aligned}$$

Since $\mathcal{G}$ and $\mathfrak{h}$ are continuous, we have, for all $t > 0$, that

$$J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu, \rho), t) = 1,\ J(\mathfrak{h}(\rho), \mathfrak{h}(\rho), \mathcal{G}(\rho, \mu), t) = 1$$
$$S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu, \rho), t) = 0,\ S(\mathfrak{h}(\rho), \mathfrak{h}(\rho), \mathcal{G}(\rho, \mu), t) = 0$$
$$F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu, \rho), t) = 0,\ F(\mathfrak{h}(\rho), \mathfrak{h}(\rho), \mathcal{G}(\rho, \mu), t) = 0.$$

Thus we can conclude that $\mathcal{G}(\mu, \rho) = \mathfrak{h}(\mu)$ and $\mathcal{G}(\rho, \mu) = \mathfrak{h}(\rho)$. □

In theorem 3.1, if we take $\mathcal{A}$ to be regular, then $\mathcal{G}$ need not be continuous to get the results. The next theorem proves the same.

Theorem 3.2. *Consider a complete JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ where $\mathcal{A}$ is regular and partially ordered. Consider the mappings $\mathcal{G} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ where*

(a) $\mathcal{G}(\mathcal{A} \times \mathcal{A}) \subseteq \mathfrak{h}(\mathcal{A})$,
(b) $\mathfrak{h}$ *is continuous,*
(c) $\mathcal{G}$ *and* $\mathfrak{h}$ *are compatible,*
(d) $\mathcal{G}$ *and* $\mathfrak{h}$ *are* $\theta_{JN} -$ *coupled,*
(e) $\mathcal{G}$ *and* $\mathfrak{h}$ *are* $\theta_{JN}-$*coupled for some* $\theta \in \Theta$.

If there exist $\mu_0, \rho_0 \in \mathcal{A}$ for which $\mathfrak{h}(\mu_0) \leq \mathcal{G}(\mu_0, \rho_0)$ and $\mathfrak{h}(\rho_0) \geq F(\rho_0, \mu_0)$, then $\mathcal{G}$ and $\mathfrak{h}$ have a coupled coincidence point.

Proof. Since $\mathcal{A}$ is regular, $\mathfrak{h}(\mu_{\mathrm{q}}) \leq \mu$ and $\mathfrak{h}(\rho_{\mathrm{q}}) \geq \rho$, where $\mu_{\mathrm{q}}) \to \mu$, $\rho_{\mathrm{q}}) \to \rho$ as $q \to +\infty$. As $\mathcal{G}$ and $\mathfrak{h}$ are compatible and $\mathfrak{h}$ is continuous,

$$\lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}})) = \mathfrak{h}(\mu) = \lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}})) = \lim_{\mathrm{q}\to+\infty} \mathcal{G}(\mathfrak{h}(\mu_{\mathrm{q}}), \mathfrak{h}(\rho_{\mathrm{q}})),$$
$$\lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\mathfrak{h}(\rho_{\mathrm{q}})) = \mathfrak{h}(\rho) = \lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}}, \mu_{\mathrm{q}})) = \lim_{\mathrm{q}\to+\infty} \mathcal{G}(\mathfrak{h}(\rho_{\mathrm{q}}), \mathfrak{h}(\mu_{\mathrm{q}})).$$

For all $0 \leq k < 1$, we have that

$$\begin{aligned}
J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu, \rho), \theta(t)) \geq \Big\{ & J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \theta(t) - \theta(kt)) \\
& \odot J(\mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathcal{G}(\mu, \rho), \theta(kt)) \Big\} \\
S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), F(\mu, \rho), \theta(t)) \leq \Big\{ & S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \theta(t) - \theta(kt)) \\
& \oslash S(\mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathcal{G}(\mu, \rho), \theta(kt)) \Big\} \\
F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu, \rho), \theta(t)) \leq \Big\{ & F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \theta(t) - \theta(kt)) \\
& \otimes F(\mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathcal{G}(\mu, \rho), \theta(kt)) \Big\}.
\end{aligned}$$

Letting $n \to +\infty$ in the above inequalities, we get that

$$\begin{aligned}
J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho).\theta(t)) &\geq \lim_{q\to+\infty}\Big\{J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\mathfrak{h}(\mu_{q+1}),\theta(t)-\theta(kt)) \\
&\odot J(\mathfrak{h}(\mathfrak{h}(\mu_{q+1}),\mathfrak{h}(\mathfrak{h}(\mu_{q+1}),\mathcal{G}(\mu,\rho),\theta(kt))\Big\} \\
&\geq \lim_{q\to+\infty} J(\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),\mathcal{G}(\mu,\rho),\theta(kt)), \\
S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho).\theta(t)) &\leq \lim_{q\to+\infty}\Big\{S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\mathfrak{h}(\mu_{q+1}),\theta(t)-\theta(kt)) \\
&\oslash S(\mathfrak{h}(\mathfrak{h}(\mu_{q+1}),\mathfrak{h}(\mathfrak{h}(\mu_{q+1}),\mathcal{G}(\mu,\rho),\theta(kt))\Big\} \\
&\leq \lim_{q\to+\infty} S(\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),F(\mu,\rho),\theta(kt)), \\
F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho).\theta(t)) &\leq \lim_{q\to+\infty}\Big\{J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\mathfrak{h}(\mu_{q+1}),\theta(t)-\theta(kt)) \\
&\otimes F(\mathfrak{h}(\mathfrak{h}(\mu_{q+1}),\mathfrak{h}(\mathfrak{h}(\mu_{q+1}),\mathcal{G}(\mu,\rho),\theta(kt))\Big\} \\
&\leq \lim_{q\to+\infty} F(\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),\mathcal{G}(\mu,\rho),\theta(kt)).
\end{aligned}$$

Since $\mathcal{G}$ and $\mathfrak{h}$ are $\theta_{JN}-$coupled, from the above inequalities, we obtain that

$$\begin{aligned}
J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho).\theta(t)) &\geq \Big\{J(\mathfrak{h}(\mathfrak{h}(\mu_q),\mathfrak{h}(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu),kt)\odot J(\mathfrak{h}(\mathfrak{h}(\mu_q),\mathfrak{h}(\mathfrak{h}(\mu_q),\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),kt) \\
&\odot J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),kt)\Big\} \\
&\geq J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),kt), \\
S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho).\theta(t)) &\geq \Big\{S(\mathfrak{h}(\mathfrak{h}(\mu_q),\mathfrak{h}(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu),kt)\oslash S(\mathfrak{h}(\mathfrak{h}(\mu_q),\mathfrak{h}(\mathfrak{h}(\mu_q),\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),kt) \\
&\oslash S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),kt)\Big\} \\
&\geq S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),kt), \\
F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho).\theta(t)) &\geq \Big\{F(\mathfrak{h}(\mathfrak{h}(\mu_q),\mathfrak{h}(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu),kt)\otimes F(\mathfrak{h}(\mathfrak{h}(\mu_q),\mathfrak{h}(\mathfrak{h}(\mu_q),\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),kt) \\
&\otimes F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),kt)\Big\} \\
&\geq F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),kt)
\end{aligned}$$

Allowing k tending to 1, we obtain that $\mathcal{G}(\mu,\rho)=\mathfrak{h}(\mu)$. In a similar way, we can obtain that $\mathcal{G}(\rho,\mu)=\mathfrak{h}(\rho)$. □

If we take $\mathfrak{h}$ to be the identity mapping in the above theorems, then it leads to the following corollary.

Corollary 3.3. *Consider a complete JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ where $\mathcal{A}$ is a partially ordered set. Let $\mathcal{G}:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ and $\mathfrak{h}:\mathcal{A}\to\mathcal{A}$. Assume that*

(a) *either $\mathcal{A}$ is regular or $\mathcal{G}$ is continuous,*

(b) *$\mathfrak{h}$ is continuous,*

(c) *$\mathcal{G}$ has mixed monotone property,*

(d) *$\mathcal{G}$ is self $\theta_{JN}-$coupled for some $\theta \in \Theta$.*

If there exist $\mu_0, \rho_0 \in \mathcal{A}$ for which $\mathfrak{h}(\mu_0) \leq \mathcal{G}(\mu_0, \rho_0)$ and $\mathfrak{h}(\rho_0) \geq \mathcal{G}(\rho_0, \mu_0)$, then $\mathcal{G}$ has a coupled fixed point.

Example 3.4. Consider the JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ as in Example 2.5 where $\mathcal{A} = [0, 1]$ is with natural ordering and $G(\mu, \rho, \upsilon) =\mid \mu - \rho \mid + \mid \rho - \upsilon \mid + \mid \upsilon - \mu \mid$ for all $\mu, \rho, \in \mathcal{A}$. Let $\theta(t) = \frac{2t}{8}$, for $t \in [0, +\infty)$. Let us consider the functions $\mathcal{G} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{G}(\mu, \rho) = \begin{cases} \frac{\mu^3 - 2\rho^3}{8}, & \text{if } \mu \geq \rho, \\ 0 & \text{otherwise,} \end{cases}$$
$$\mathfrak{h}(\mu) = \mu^3.$$

Consider sequences $\{\mu_{\rm q}\}$ and $\{\rho_{\rm q}\}$ in $\mathcal{A}$ such that

$$\lim_{{\rm q}\to+\infty} \mathcal{G}(\mu_{\rm q}, \rho_{\rm q}) = \lim_{{\rm q}\to+\infty} \mathfrak{h}(\mu_{\rm q}) \text{ and } \lim_{{\rm q}\to+\infty} \mathcal{G}(\rho_{\rm q}, \mu_{\rm q}) = \lim_{{\rm q}\to+\infty} \mathfrak{h}(\rho_{\rm q}).$$

It is then obvious that all these limit values must be zero. Let us show that $\mathcal{G}$ and $\mathfrak{h}$ are compatible.

$$\begin{aligned} J(\mathfrak{h}(\mathcal{G}(\mu_{\rm q}, \rho_{\rm q})), \mathfrak{h}(\mathcal{G}(\mu_{\rm q}, \rho_{\rm q})), \mathcal{G}(\mathfrak{h}(\mu_{\rm q}), \mathfrak{h}(\rho_{\rm q})), t) &= J(\mathfrak{h}(\frac{\mu_{\rm q}^3 - 2\rho_{\rm q}^3}{8})), \mathfrak{h}(\frac{\mu_{\rm q}^3 - 2\rho_{\rm q}^3}{8}), \mathcal{G}(\mu_{\rm q}^3, \rho_{\rm q}^3), t) \\ &= J((\frac{\mu_{\rm q}^3 - 2\rho_{\rm q}^3}{8})^3, (\frac{\mu_{\rm q}^3 - 2\rho_{\rm q}^3}{8})^3, \frac{\mu_{\rm q}^9 - 2\rho_{\rm q}^9}{8}, t) \\ &= \frac{t}{t + 2 \mid (\frac{\mu_{\rm q}^3 - 2\rho_{\rm q}^3}{8})^3 - \frac{\mu_{\rm q}^9 - 2\rho_{\rm q}^9}{8} \mid} \\ &\to 1 \text{ as } n \to +\infty. \end{aligned}$$

$$\begin{aligned} S(\mathfrak{h}(\mathcal{G}(\mu_{\rm q}, \rho_{\rm q})), \mathfrak{h}(\mathcal{G}(\mu_{\rm q}, \rho_{\rm q})), \mathcal{G}(\mathfrak{h}(\mu_{\rm q}), \mathfrak{h}(\rho_{\rm q})), t) &= S(\mathfrak{h}(\frac{\mu_{\rm q}^3 - 2\rho_{\rm q}^3}{8})), \mathfrak{h}(\frac{\mu_{\rm q}^3 - 2\rho_{\rm q}^3}{8}), \mathcal{G}(\mu_{\rm q}^3, \rho_{\rm q}^3), t) \\ &= S((\frac{\mu_{\rm q}^3 - 2\rho_{\rm q}^3}{8})^3, (\frac{\mu_{\rm q}^3 - 2\rho_{\rm q}^3}{8})^3, \frac{\mu_{\rm q}^9 - 2\rho_{\rm q}^9}{8}, t) \\ &= \frac{2 \mid (\frac{\mu_{\rm q}^3 - 2\rho_{\rm q}^3}{8})^3 - \frac{\mu_{\rm q}^9 - 2\rho_{\rm q}^9}{8} \mid}{t + 2 \mid (\frac{\mu_{\rm q}^3 - 2\rho_{\rm q}^3}{8})^3 - \frac{\mu_{\rm q}^9 - 2\rho_{\rm q}^9}{8} \mid} \\ &\to 0 \text{ as } n \to +\infty. \end{aligned}$$

$$\begin{aligned}F(\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\mu_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}})),t) &= F(\mathfrak{h}(\frac{\mu_{\mathrm{q}}^3-2\rho_{\mathrm{q}}^3}{8})),\mathfrak{h}(\frac{\mu_{\mathrm{q}}^3-2\rho_{\mathrm{q}}^3}{8}),\mathcal{G}(\mu_{\mathrm{q}}^3,\rho_{\mathrm{q}}^3),t)\\ &= F((\frac{\mu_{\mathrm{q}}^3-2\rho_{\mathrm{q}}^3}{8})^3,(\frac{\mu_{\mathrm{q}}^3-2\rho_{\mathrm{q}}^3}{8})^3,\frac{\mu_{\mathrm{q}}^9-2\rho_{\mathrm{q}}^9}{8},t)\\ &= \frac{2\mid(\frac{\mu_{\mathrm{q}}^3-2\rho_{\mathrm{q}}^3}{8})^3-\frac{\mu_{\mathrm{q}}^9-2\rho_{\mathrm{q}}^9}{8}\mid}{t}\\ &\to 0 \text{ as } n\to+\infty.\end{aligned}$$

In a similar way, we can deduce that

$$\begin{aligned}J(\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\mu_{\mathrm{q}})),t)&\to 1,\\ S(\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\mu_{\mathrm{q}})),t)&\to 0,\\ F(\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\mu_{\mathrm{q}})),t)&\to 0.\end{aligned}$$

Therefore $\mathcal{G}$ and $\mathfrak{h}$ are compatible. Take $\mu_0 = 0, \rho_0 = \alpha and \mu, \rho,$ in $\mathcal{A}$ such that $\mathfrak{h}(\mu_0) = \mathcal{G}(\mu_0,\rho_0)$, $\mathfrak{h}(\rho_0) = \mathcal{G}(\rho_0,\mu_0)$ and $\mathfrak{h}(\mu) \leq \mathfrak{h}(\gamma), \mathfrak{h}(\rho) \geq \mathfrak{h}(\sigma)$. Let us consider the following cases to verify 3.1(e).
case(i) $\mu \geq \rho$, $\gamma \geq \sigma$.

$$\begin{aligned}J(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &= J(\frac{\mu^3-2\rho^3}{8},\frac{\mu^3-2\rho^3}{8},\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8})\\ &= \frac{t}{t+\mid(\mu^3-2\rho^3)-(\gamma^3-2\sigma^3)\mid}\\ &\geq \frac{t}{t+2\mid\gamma^3-\frac{\gamma^3-2\sigma^3}{8}\mid}\\ &\geq J(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\\ &\geq \Big\{J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)\odot J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t)\\ &\qquad \odot J(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\}\end{aligned}$$

$$\begin{aligned}
S(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),F(\gamma,\sigma),\theta(t)) &= S(\frac{\mu^3-2\rho^3}{8},\frac{\mu^3-2\rho^3}{8},\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8}) \\
&= \frac{|(\mu^3-2\rho^3)-(\gamma^3-2\sigma^3)|}{t+|(\mu^3-2\rho^3)-(\gamma^3-2\sigma^3)|} \\
&\leq \frac{2\mid\gamma^3-\frac{\gamma^3-2\sigma^3}{8}\mid}{t+2\mid\gamma^3-\frac{\gamma^3-2\sigma^3}{8}\mid} \\
&\leq S(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t) \\
&\leq \Big\{S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)\oslash S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t) \\
&\quad\oslash S(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\} \\
F(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &= F(\frac{\mu^3-2\rho^3}{8},\frac{\mu^3-2\rho^ho3}{8},\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8}) \\
&= \frac{|(\mu^3-2\rho^3)-(\gamma^3-2\sigma^3)|}{t} \\
&\leq \frac{2\mid\gamma^3-\frac{\gamma^3-2\sigma^3}{8}\mid}{t} \\
&\leq F(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t) \\
&\leq \Big\{F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)\otimes F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t) \\
&\quad\otimes F(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\}
\end{aligned}$$

case(ii) $\mu<\rho$, $\gamma\geq\sigma$.

$$\begin{aligned}
J(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &= J(0,0,\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8}) \\
&= \frac{t}{t+\mid\frac{\gamma^3-2\sigma^3}{8}\mid}\geq\frac{t}{t+8\mid\mu^3-\gamma^3\mid} \\
&\geq J(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t) \\
&\geq \Big\{J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)\odot J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t) \\
&\quad\odot J(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\} \\
S(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &= S(0,0,\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8}) \\
&= \frac{\mid\frac{\gamma^3-2\sigma^3}{8}\mid}{t+\mid\frac{\gamma^3-2\sigma^3}{8}\mid} \\
&\leq \frac{8\mid\mu^3-\gamma^3\mid}{t+8\mid\mu^3-\gamma^3\mid} \\
&\leq S(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t) \\
&\leq \Big\{S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)\oslash S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t) \\
&\quad\oslash S(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\}
\end{aligned}$$

$$\begin{aligned} F(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &= F(0,0,\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8}) \\ &= \frac{\mid \frac{\gamma^3-2\sigma^3}{8} \mid}{t} \\ &\leq \frac{8 \mid \mu^3-\gamma^3 \mid}{t} \\ &\leq F(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t) \\ &\leq \Big\{ F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t) \\ &\qquad \otimes F(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\} \end{aligned}$$

case(iii) $\mu < \rho$, $\gamma < \sigma$.
This case is obvious, since we have that

$$J(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) = 1,$$
$$S(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) = 0,$$
$$F(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) = 0.$$

We have thus shown that $\mathcal{G}$ and $\mathfrak{h}$ fit into the theorem 3.1. Therefore $\mathcal{G}$ and $\mathfrak{h}$ must have a coupled coincidence point and it is the point $(0,0)$.

4. **Conclusion**

This work built a generalized neutrosophic metric space, called J-Neutrosophic metric space, based on the concept of neutrosophy. We proved coupled coincidence point results for JN-compatible mappings satisfying certain conditions. As the space introduced here considers the indeterminacy along with the degree of nearness and the degree of non-nearness and generalizes the ideas of intuitionistic sets, fuzzy sets, classical sets, paraconsistent sets and dialetheist sets, this work has the scope of further extension and analysis.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Attanssov, K., Intuitionistic fuzzy sets. Fuzzy sets and systems 1986, 20, 87-96.
2. Bhaskar, T.G.; Lakshmikantam, V., Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. 2006, TMA 65, 1379-1397.
3. Cantor, G., Ueber eine Eigenschaft des Begriffes aller reelen algenraishen Zahlen. Journal fiir die reine und angewandte Mathematic 1874, 77, 258-262.
4. Das, R., Smarandache, F., Tripathy, B.C., Neutrosophic fuzzy matrices and some algebraic operations. Neutrosophic Sets and Systems, 2020, 32,401-409.
5. Das, R.; Tripathy, B.C., Neutrosophic multi-set topological space. Neutrosophic Sets and Systems, 2020, 35, 142-152.

6. Das, S.; Das, R.; Tripathy, B.C., Multi-criteria group decision making model using single-valued neutrosophic set. LogForum, 16(3), 421-429.
7. Eser; Derya, A new edge detection approach via neutrosophy based on maximum norm entropy, Expert Systems with Applications, 115(2019), 499-511.
8. Johnson Philip, A history of set theory, Prindle, Weber & Schmidt, 1972, ISBN 0-87150-154-6.
9. Klaua, D., Uber einen Ansatz zur mehrwertigen Mengenlehre. Monatsb, Deutsh Akad. Wiss. Berlin 1965, 7, 859-876.
10. Lakshmikantham, V.; Ciric, L., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Analysis: Theory, Methods and Appl. 2009, 70(12),4341-4349.
11. Leyva-Vázquez, M.; Smarandache, F.; Ricardo, J.E., Artificial intelligence: challenges, perspectives and neutrosophy role (Master Conference). Dilemas Contemporáneos: Educación, Política y Valore 2018, 6(Special).
12. Mohiuddine, S.A.; Alotaibi, A., Coupled coincidence point theorems for compatible mappings in partially ordered intuitionistic generalized fuzzy metric spaces. Fixed point theory and applications 2013, 2013(1), 265.
13. Mukherjee, A.; Das, R., Neutrosophic bipolar vague soft set and its application to decision making problems. Neutrosophic Sets and Systems, 2020, 32,410-424.
14. Mustafa, Z.; Sims, B., A new approach to generalized metric spaces. J. Nonlinear convex Anal. 2006, 7(2), 289-297.
15. Rashno, E.; Akbari, A.; Nasersharif, B., A Convoloutional Neural Network model based on Neutrosophy for Noisy Speech Recognition. Infinite Study 2019.
16. Schweizer, B.; Sklar, A., Statistical metric spaces. Pac. J. Math. 1960, 10, 313-334.
17. Smarandache, F., Neutrosophy / Neutrosophic Probability, Set, and Logic. American Research Press, 1998.
18. Smarandache, F., Neutrosophy, a new Branch of Philosophy. Infinite Study, 2002.
19. Smarandache, F., Neutrosophic set, a generalization of intuitionistic fuzzy sets. Int. J. Pure Appl. Math. 2005, 24, 287-297.
20. Sowndrarajan, S.; Jeyaraman, M.; Smarandache, F., Fixed Point Results for Contraction Theorems in Neutrosophic Metric Spaces. Neutrosophic Sets and Systems 2020, 36, 308-318.
21. Sun, G.; Yang, K.; Generalized fuzzy metric spaces with properties. Res. J. Appl. Sci. 2010, 2, 673-678.
22. Zadeh, L.A., Fuzzy sets. Inform and control, 1965, 8, 338-353.

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Neutrosophic $\mathcal{N}$-Topological Ordered Space

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Abstract. This research article presents a new concept, "Neutrosophic $\mathcal{N}$-topological ordered space". Also we define some of the separation axioms, weakly neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space and Neutrosophic $\mathcal{N}\varsigma$-regularly ordered space in Neutrosophic $\mathcal{N}$-topological ordered space. Besides giving some of the innovative properties of these spaces.

Keywords: Neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space, Neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space, Weakly neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space, Almost Neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space and Neutrosophic $\mathcal{N}\varsigma$-regularly ordered space.

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1. **Introduction**

L.A. Zadeh introduced the concept of fuzzy sets [14]. The theory of fuzzy topological spaces was developed by Chang [3] . The study of intutionistic fuzzy set was established by Atanassov [1] in 1983. In [4], the another notion called intutionistic fuzzy topological space was found by Coker. F. Smarandache originated the concepts of neutrosophy and neutrosophic set ([12], [13]). The concept of neutrosophic crisp set and neutrosophic crisp topological space were introduced by A.A. Salama and S.A. Alblowi [11]. Leopoldo Nachbin [9] initiated the study of topological ordered spaces in 1965. Lellis Thivagar et al. [6] have proposed the concept of N-topological space. Recently we found the new concept called $\mathcal{N}$-topological ordered spaces [5]. In this paper, we investigate the concept called Neutrosophic $\mathcal{N}$-topological Ordered Space. And also, we establish some of the Separation Axioms and its characterizations.

2. Preliminaries

Definition 2.1. [8] Let X be a non-empty set, $\tau_1, \tau_2, ..., \tau_N$ be N-arbitrary topologies defined on X and let the collection $N\tau$ be defined by

$N\tau = \{S \subseteq X : S = (\cup_{i=1}^{N} A_i) \bigcup (\cap_{i=1}^{N} B_i), A_i, B_i \in \tau_i\}$

satisfying the following axioms:

(i) $X, \emptyset \in N\tau$.

(ii) $\bigcup_{i=1}^{\infty} S_i \in N\tau$ for all $S_i \in N\tau$.

(iii) $\bigcap_{i=1}^{n} S_i \in N\tau$ for all $S_i \in N\tau$.

Then the pair $(X, N\tau)$ is called a *N-topological space* on X and the elements of the collection $N\tau$ are known as *$N\tau$-open sets* on X. A subset A of X is called *$N\tau$-closed* on X if the complement of A is $N\tau$-open on X. The set of all $N\tau$-open sets on X and the set of all $N\tau$-closed sets on X are respectively, denoted by $N\tau O(X)$ and $N\tau C(X)$.

Definition 2.2. [5] An $\mathcal{N}$-topological Space $(X, \mathcal{N}\tau)$ equipped with a partial order relation $\leq$ (that is, *Reflexive*, *Transitive* and *Antisymmetric*) is called an $\mathcal{N}$-topological Ordered Space $(X, \mathcal{N}\tau, \leq)$.

Definition 2.3. [12] Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ where $\mu_A(x), \sigma_A(x)$, $\gamma_A(x)$ which represents the degree of membership function, the degree of indeterminacy and the degree of non-membership function respectively of each element $x \in X$ to the set A. Also $^{-}0 \leq \mu_A(x) + \sigma_A(x) + \gamma_A(x) \leq 3^{+}$ for all $x \in X$.

Remark 2.4. [12, 13] (1) A neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ can be identified to an ordered triple set $\langle \mu_A, \sigma_A, \gamma_A\rangle$ in $]0^{-}, 1^{+}[$ on X.

(2) For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \sigma_A, \gamma_A\rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$

Definition 2.5. [10] Let $\{A_i, i \in J\}$ be an arbitrary family of neutrosophic sets in X. Then

(a) $\cap A_i = \{\langle x, \wedge\mu_{A_i}(x), \wedge\sigma_{A_i}(x), \vee\gamma_{A_i}(x)\rangle : x \in X\}$;

(b) $\cup A_i = \{\langle x, \vee\mu_{A_i}(x), \vee\sigma_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle : x \in X\}$

Definition 2.6. [10]

$0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$

Definition 2.7. [6] A neutrosophic N-topology on a non-empty set X is a family $N_n\tau$ of neutrosophic sets in X satisfying the following axioms:

(i) $0_N, 1_N \in N_n\tau$

(ii) $\cup_{i=1}^{\infty} A_i \in N_n\tau$ for all $A_i \in N_n\tau$

(iii) $\cap_{i=1}^{n} A_i \in N_n\tau$ for all $A_i \in N_n\tau$.

Then the pair $(X, N_n\tau)$ is called neutrosophic N-topological space and each neutrosophic set in $N_n\tau$ is called neutrosophic $N_n\tau$-open set. The complement of neutrosophic $N_n\tau$-open set is called neutrosophic $N_n\tau$-closed set.

Definition 2.8. [6] Let $(X, N_n\tau)$ be a neutrosophic N-topological space on X and A be a neutrosophic set on X, then $N_nint(A)$ and $N_ncl(A)$ are respectively defined as

(i) $N_nint(A) = \cup\{G\colon G \subseteq A$ and G is a $N_n\tau - opensetinX\}$

(ii) $N_ncl(A) = \cap\{F\colon A \subseteq F$ and F is a $N_n\tau - closedsetinX\}$

Definition 2.9. [10] A neutrosophic set A = $\langle x, \mu_A, \sigma_A, \gamma_A\rangle$ in a neutrosophic topological space (X,T) is said to be a neutrosophic neighbourhood of a neutrosophic point $x_{r,t,s} \in X$, if there exists a neutrosophic open set B = $\langle x, \mu_B, \sigma_B, \gamma_B\rangle$ with $x_{r,t,s} \subseteq B \subseteq A$.

Notation 1. *[10] We denote neutrosophic neighbourhood A of a in X by neutrosophic neighbourhood A of a neutrosophic point $a_{r,t,s}$ for $a \in X$*

Definition 2.10. [10] A neutrosophic set $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ in a partially ordered set $(X, \leq)$ is said to be

(i) an increasing neutrosophic set if $x \leq y$ implies $A(x) \subseteq A(y)$. That is, $\mu_A(x) \leq \mu_A(y), \sigma_A(x) \leq \sigma_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$.

(ii) a decreasing neutrosophic set if $x \leq y$ implies $A(x) \supseteq A(y)$. That is, $\mu_A(x) \geq \mu_A(y), \sigma_A(x) \geq \sigma_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$.

Definition 2.11. A neutrosophic set A is called neutrosophic $\mathcal{N}\varsigma$-clopen set if it is both neutrosophic $\mathcal{N}\varsigma$-open set and neutrosophic $\mathcal{N}\varsigma$-closed set.

3. Neutrosophic $\mathcal{N}$-topological Ordered Space

In this paper, we define the notation of Neutrosophic $\mathcal{N}$-Topological Space as Neutrosophic $\mathcal{N}$-TS, partial order relation as por and also Neutrosophic $\mathcal{N}$-topological Ordered Space as Neutrosophic $\mathcal{N}$-TOS. We found some results of Neutrosophic $\mathcal{N}$-topological Ordered Spaces like Neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space, Neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space, weakly Neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space, almost Neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space and Neutrosophic $\mathcal{N}\varsigma$-T_3-ordered space.

Definition 3.1. A neutrosophic $\mathcal{N}$-TS $(X, \mathcal{N}_n\varsigma)$ equipped with a por $\leq$ is called Neutrosophic $\mathcal{N}$-TOS $(X, \mathcal{N}_n\varsigma, \leq)$.

Definition 3.2. For every $u, v \in X$ such that $u \nleq v$ (i.e., u is not related to v) in X, if there exists a decreasing neutrosophic $\mathcal{N}\varsigma$-open set G containing v such that $u \notin G$, then neutrosophic $\mathcal{N}$-TOS $(X, \mathcal{N}_n\varsigma, \leq)$ is called *upper* neutrosophic $\mathcal{N}\varsigma$-T_1-*ordered* space.

Definition 3.3. For every $u, v \in X$ such that $u \nleq v$ (i.e., u is not related to v) in X, if there exists an increasing neutrosophic $\mathcal{N}\varsigma$-open set H containing u such that $v \notin H$, then neutrosophic $\mathcal{N}$-TOS $(X, \mathcal{N}_n\varsigma, \leq)$ is called *lower* neutrosophic $\mathcal{N}\varsigma$-T_1-*ordered* space.

Definition 3.4. $(X, \mathcal{N}_n\varsigma, \leq)$ is said to be neutrosophic $\mathcal{N}\varsigma$-T_1-*ordered space* if it is both lower and upper neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space.

Example 3.5. Let $X = \{a, b, c\}$ with a por $\leq$. For $\mathcal{N} = 2$, let the neutrosophic sets be $U = \{x, (0.2, 0.2, 0.4), \quad (0.3, 0.3, 0.1), (0.6, 0.6, 0.2)\}$ and $V = \{x, (0.4, 0.4, 0.4), (0.4, 0.4, 0.3), (0.4, 0.4, 0.3)\}$. Then $U \cup V = \{(x, (0.4, 0.4, 0.4), (0.4, 0.4, 0.3), (0.4, 0.4, 0.3))\}$ and $U \cap V = \{(x, (0.2, 0.2, 0.4), (0.3, 0.3, 0.1), (0.6, 0.6, 0.2))\}$. Considering $\varsigma_1 = \{0_N, 1_N, U\}$ and $\varsigma_2 = \{0_N, 1_N, V\}$, then $2\varsigma O(X) = \{0_N, 1_N, U, V, U \cap V, U \cup V\}$ which is a neutrosophic bitopology on X. Then $(X, 2_n\varsigma, \leq)$ is a neutrosophic bi-topological ordered space. Let $a_{(0.15,0.2,0.4)}$ and $b_{(0.15,0.15,0.25)}$ be any two neutrosophic points on X. For $a_{(0.15,0.2,0.4)} \nleq b_{(0.15,0.15,0.25)}$, there exists an increasing neutrosophic 2ς-neighbourhood U of $a_{(0.15,0.2,0.4)}$ such that U is not a neutrosophic 2ς-neighbourhood of $b_{(0.15,0.15,0.25)}$. Therefore, $(X, 2_n\varsigma, \leq)$ is a lower neutrosophic 2ς-T_1-ordered space. Similarly, we can do for upper neutrosophic 2ς-T_1-ordered space. For $\mathcal{N} = 3$, define the neutrosophic sets $U = \{x, (0.3, 0.3, 0.5), (0.5, 0.5, 0.3), (0.7, 0.7, 0.2)\}$, $V = \{x, (0.6, 0.6, 0.5), (0.6, 0.6, 0.5), (0.6, 0.6, 0.5)\}$. Then $U \cup V = \{(x, (0.6, 0.6, 0.5), (0.6, 0.6, 0.5), (0.6, 0.6, 0.5))\}$ and $U \cap V = \{(x, (0.3, 0.3, 0.5), (0.5, 0.5, 0.3), (0.7, 0.7, 0.2))\}$. Considering $\varsigma_1 = \{0_N, 1_N, U\}$, $\varsigma_2 = \{0_N, 1_N, V\}$ and $\varsigma_3 = \{0_N, 1_N\}$, then $3\varsigma O(X) = \{0_N, 1_N, U, V, U \cap V, U \cup V\}$ which is a neutrosophic tritopology on X. Then $(X, 3_n\varsigma, \leq)$ is neutrosophic tri-topological ordered space. Let $a_{(0.25,0.3,0.5)}, b_{(0.25,0.25,0.35)} \in X$ such that $a_{(0.25,0.3,0.5)} \nleq b_{(0.25,0.25,0.35)}$. Then there exists an increasing neutrosophic 3ς-neighbourhood U of $a_{(0.25,0.3,0.5)}$ such that U is not a neutrosophic 3ς-neighbourhood of $b_{(0.25,0.25,0.35)}$. Therefore, $(X, 3_n\varsigma, \leq)$ is a lower neutrosophic 3ς-T_1-ordered space. Similarly, we can do for upper neutrosophic 3ς-T_1-ordered space.

Theorem 3.6. *For a neutrosophic $\mathcal{N}$-TOS $(X, \mathcal{N}_n\varsigma, \leq)$, the following are equivalent:*

(i) X is a lower(respectively upper) neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space.

(ii) For each $u, v \in X$ such that $u \nleq v$, there exists an increasing(respectively decreasing) neutrosophic $\mathcal{N}\varsigma$-open set $G = \langle x, \mu_G, \sigma_G, \gamma_G \rangle$ containing u(respectively v) such that $r \nleq v$(respectively $u \nleq r$) for all $r \in G$.

Proof. Now we prove the theorem only for lower neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space.

(i) $\Rightarrow$ (ii): Let $u \nleq v$. By hypothesis, there exists an increasing neutrosophic $\mathcal{N}\varsigma$-open set G containing u such that $v \notin G$. If $r \in G$ and $r \leq v$, then $v \in G$, a contradiction. Therefore, $r \nleq v$ for all $r \in G$.

(ii) $\Rightarrow$ (i): Let $u, v \in X$ such that $u \nleq v$. Therefore there exists an increasing neutrosophic $\mathcal{N}\varsigma$-open set G containing u such that $r \nleq v$ for all $r \in G$. Then $i(G)$ is an increasing neutrosophic $\mathcal{N}\varsigma$-open set of u such that $v \notin i(G)$. This implies that X is a lower neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space. Similar proof holds for upper neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space. □

Theorem 3.7. *If $(X, \mathcal{N}_n\varsigma, \leq)$ is a lower(respectively upper) neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space and $\mathcal{N}_n\varsigma \subseteq \mathcal{N}_n\varsigma^*$, then $(X, \mathcal{N}_n\varsigma^*, \leq)$ is a lower(respectively upper) neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space.*

Proof. Let $(X, \mathcal{N}_n\varsigma, \leq)$ be a lower neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space. Then if $u, v \in X$ such that $u \nleq v$, there exists an increasing neutrosophic $\mathcal{N}\varsigma$-open set $U = \langle x, \mu_U, \sigma_U, \gamma_U \rangle$ of u such that U is not a neutropsophic $\mathcal{N}\varsigma$-open set of v. Since $\mathcal{N}_n\varsigma \subseteq \mathcal{N}_n\varsigma^*$, therefore if $u, v \in X$ such that $u \nleq v$, there exists an increasing neutrosophic $\mathcal{N}\varsigma^*$-open set U^* of u such that U^* is not a neutrosophic $\mathcal{N}\varsigma^*$-open set of v. Thus $(X, \mathcal{N}_n\varsigma^*, \leq)$ is a lower neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space. Similarly, we can prove for upper neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space. □

Definition 3.8. For each pair of elements $u \nleq v$ in X, there exists neutrosophic $\mathcal{N}\varsigma$-open sets $G = \langle x, \mu_G, \sigma_G, \gamma_G \rangle$ and $H = \langle x, \mu_H, \sigma_H, \gamma_H \rangle$ such that G is an increasing neutrosophic $\mathcal{N}\varsigma$-neighbourhood of u, H is a decreasing neutrosophic $\mathcal{N}\varsigma$-neighbourhood of v and $G \cap H = 0_N$, then $(X, \mathcal{N}_n\varsigma, \leq)$ is defined to be neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space.

Theorem 3.9. *For a neutrosophic $\mathcal{N}$-TOS $(X, \mathcal{N}_n\varsigma, \leq)$, the following are equivalent:*

(i) X is a neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space.

(ii) For each pair $u, v \in X$ such that $u \nleq v$, there exists neutrosophic $\mathcal{N}\varsigma$-open sets $G = \langle x, \mu_G, \sigma_G, \gamma_G \rangle$ and $H = \langle x, \mu_H, \sigma_H, \gamma_H \rangle$ such that $u \in G$, $v \in H$ and $s \in G$, $t \in H$ together imply that $s \nleq t$.

(iii) The graph of the partial order of X is a neutrosophic $\mathcal{N}\varsigma^*$-closed where $\mathcal{N}\varsigma^*$ is the product topology for $X \times X$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i): Let $u, v \in X$ with $u \nleq v$, there exists neutrosophic $\mathcal{N}\varsigma$-open sets G and H satisfying the properties in (ii). Since $i(G)$ is an increasing neutrosophic $\mathcal{N}\varsigma$-open set and $d(H)$ is a decreasing neutrosophic $\mathcal{N}\varsigma$-open set, we have $i(G) \cap d(H) = 0_N$. Suppose if $w \in i(G) \cap d(H)$, there exists $s \in G$ such that $s \leq w$ and there exists $t \in H$ such that $w \leq t$. Then $s \leq t$, a contradiction. Therefore $i(G) \cap d(H) = 0_N$. Hence X is neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space.

(i) $\Rightarrow$ (iii): Let G be the graph of the partial order of X and $(s, t) \in \mathcal{N}_n\varsigma^*$-$cl(G)$ and $(s, t) \notin G$. Then $s \nleq t$ and therefore there exists an increasing neutrosophic $\mathcal{N}\varsigma$-open set A of

s and a decreasing neutrosophic $\mathcal{N}\varsigma$-open set B of t such that $A \cap B = 0_N$. $A \times B$ being a neutrosophic $\mathcal{N}\varsigma^*$-open set of (s,t), $(A \times B) \cap G = 0_N$. Thus $(s,t) \in A \times B$. It follows that $(s,s) \in A$ which implies $s \leq t$. Since A is an increasing neutrosophic $\mathcal{N}\varsigma$-open set, $t \in A$. Then $A \cap B \neq 0_N$, a contradiction. Therefore, $(s,t) \notin \mathcal{N}_n\varsigma^*$-$cl(G)$ and consequently, G is neutrosophic $\mathcal{N}\varsigma^*$-closed.

(iii) $\Rightarrow$ (i): Suppose $s \nleq t$. Then $(s,s) \notin G$ where G is the graph of the partial order of X. Since G is neutrosophic $\mathcal{N}\varsigma^*$-closed, there exists neutrosophic $\mathcal{N}\varsigma^*$-open sets S and T such that $(s,t) \in S \times T$ and $(S \times T) \cap G = 0_N$. Let $S^* = i(S)$ and $T^* = d(T)$. Then S^* is an increasing neutrosophic $\mathcal{N}\varsigma$-open set of s, T^* is a decreasing neutrosophic $\mathcal{N}\varsigma$-open set of t. Also $S^* \cap T^* = 0_N$, because suppose if $r \in S^* \cap T^*$, then there exists $p \in S, q \in T$ such that $p \leq r \leq q$ which implies $p \leq q$. So $(p,q) \in (S \times T) \cap G$, a contradiction. Therefore, $S^* \cap T^*$ must be empty. Hence X is neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space. □

Theorem 3.10. *A neutrosophic $\mathcal{N}$-TOS $(X, \mathcal{N}_n\varsigma, \leq)$ is a neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space if and only if for each $r \in X$, there exists an increasing(respectively decreasing) neutrosophic $\mathcal{N}\varsigma$-clopen subset of X containing r.*

Proof. If X is neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space and let $H \subseteq X$, then H is the required increasing (respectively decreasing) neutrosophic $\mathcal{N}\varsigma$-clopen subset of X for all $r \in X$. Conversely, let us assume $r \nleq s$ in X. By hypothesis, there exists an increasing(respectively decreasing) neutrosophic $\mathcal{N}\varsigma$-clopen subset H in X containing r. If $s \in H$, then there is nothing to prove. If $s \notin H$, then $X \setminus H$ is a decreasing neutrosophic $\mathcal{N}\varsigma$-clopen subset of X containing s. Also $H \cap X \setminus H = \emptyset$. Hence $(X, \mathcal{N}_n\varsigma, \leq)$ is a neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space.

□

4. Weakly Neutrosophic $\mathcal{N}\varsigma$-T_2-Ordered and Almost Neutrosophic $\mathcal{N}\varsigma$-T_2-Ordered Space

Definition 4.1. A neutrosophic $\mathcal{N}$-TOS is said to be weakly neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space if for given $v < u$(that is $v \leq u$ and $v \neq u$), there exists neutrosophic $\mathcal{N}\varsigma$-open sets $G = \langle x, \mu_G, \sigma_G, \gamma_G \rangle$ and $H = \langle x, \mu_H, \sigma_H, \gamma_H \rangle$ containing u and v respectively such that $r \in G$ and $s \in H$ together imply that $s < r$.

Definition 4.2. A neutrosophic $\mathcal{N}$-TOS is said to be an almost neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space if for given $u \parallel v$, there exists neutrosophic $\mathcal{N}\varsigma$-open sets $G = \langle x, \mu_G, \sigma_G, \gamma_G \rangle$ and $H = \langle x, \mu_H, \sigma_H, \gamma_H \rangle$ containing u and v respectively such that $r \in G$ and $s \in H$ together imply that $r \parallel s$.

Theorem 4.3. *A neutrosophic $\mathcal{N}$-TOS $(X, \mathcal{N}_n\varsigma, \leq)$ is a neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space if and only if it is weakly neutrosophic $\mathcal{N}\varsigma$-T_2-ordered and almost neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space.*

Proof. Let $(X, \mathcal{N}_n\varsigma, \leq)$ be a neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space. Then it is weakly neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space. Let $u \parallel v$. Then $u \nleq v$ and $v \nleq u$. Since X is neutrosophic $\mathcal{N}\varsigma$-T_2-ordered and $u \nleq v$, then there exists neutrosophic $\mathcal{N}\varsigma$-open sets G and H containing u and v respectively such that $r \in G$ and $s \in H$ together imply that $r \nleq s$. Since $v \nleq u$, there exists neutrosophic $\mathcal{N}\varsigma$-open sets H^* of $v$and G^* of u such that $s \in H^*$ and $r \in G^*$ together imply that $s \nleq r$. Thus $G \cap G^*$ is a neutrosophic $\mathcal{N}\varsigma$-open set containing u and $H \cap H^*$ is a neutrosophic $\mathcal{N}\varsigma$-open set containing v such that $r \in G \cap G^*$, $s \in H \cap H^*$ together imply that $r \parallel s$. Hence X is almost neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space.

Conversely, if $u \nleq v$, then either $v < u$ or $v \nleq u$. If $v < u$ and since X is weakly neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space, then there exists neutrosophic $\mathcal{N}\varsigma$-open sets G and H containing u and v respectively such that $r \in G$, $s \in H$ implies that $s < r$, that is $r \nleq s$. If $v \nleq u$, then obviously $u \parallel v$. And since X is almost neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space, for given $u \parallel v$, there exists neutrosophic $\mathcal{N}\varsigma$-open sets G^* and H^* containing u and v respectively such that $r \in G^*$ and $s \in H^*$ together imply that $r \parallel s$. Therefore $(X, \mathcal{N}_n\varsigma, \leq)$ is a neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space. □

5. Neutrosophic $\mathcal{N}\varsigma$-Regularly Ordered Space

Definition 5.1. Let $(X, \mathcal{N}_n\varsigma, \leq)$ be a neutrosophic $\mathcal{N}$-TOS. If for each decreasing(respectively increasing) neutrosophic $\mathcal{N}\varsigma$-closed subset W in X and for each $s \notin W$, there exists a neutrosophic $\mathcal{N}\varsigma$-neighbourhood G of s and a neutrosophic $\mathcal{N}\varsigma$-neighbourhood H of W such that G is increasing(respectively decreasing), H is decreasing(respectively increasing) and $G \cap H = 0_N$, then $(X, \mathcal{N}_n\varsigma, \leq)$ is said to be lower(respectively upper) neutrosophic $\mathcal{N}\varsigma$-regularly ordered space.

Definition 5.2. $(X, \mathcal{N}_n\varsigma, \leq)$ is said to be neutrosophic $\mathcal{N}\varsigma$-regularly ordered space if it is both lower and upper neutrosophic $\mathcal{N}\varsigma$-regularly ordered space.

Definition 5.3. A neutrosophic $\mathcal{N}\varsigma$-T_1-ordered neutrosophic $\mathcal{N}\varsigma$-regularly ordered space is called $\mathcal{N}\varsigma$-T_3-ordered space.

Theorem 5.4. *Every neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space, lower or upper neutrosophic $\mathcal{N}\varsigma$-regularly ordered space is neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space.*

Proof. Let X be a neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space, lower neutrosophic $\mathcal{N}\varsigma$-regularly ordered space and let $u \nleq v$. Since X is neutrosophic $\mathcal{N}\varsigma$-T_1-ordered space, $[\leftarrow, v]$ is neutrosophic $\mathcal{N}\varsigma$-closed. Also $[\leftarrow, v]$ is a decreasing neutrosophic set. Since $u \notin [\leftarrow, v]$, there exists an increasing neutrosophic $\mathcal{N}\varsigma$-neighbourhood G of u and a decreasing neutrosophic $\mathcal{N}\varsigma$-neighbourhood H of $[\leftarrow, v]$ such that $G \cap H = 0_N$. Since $v \in [\leftarrow, v] \subseteq H$, X is a neutrosophic $\mathcal{N}\varsigma$-T_2-ordered space. □

6. Conclusions

In this paper, we defined a new concept "Neutrosophic $\mathcal{N}$-Topological Ordered Spaces". some characterisitics of separation axioms $\mathcal{N}\varsigma$-T_i-ordered space $(i = 0, 1, 2, 3)$ dealing with neutrosophic were studied here. In our future work, we deal with neutrosophic $\mathcal{N}\varsigma$-T_i-ordered space (i=4,5) and its characteristics in Neutrosophic $\mathcal{N}$-Topological Ordered Spaces.

References

1. Atnassov K.T., Intutionistic fuzzy sets, Fuzzy Sets and Systems 1986, 20, 87-96.
2. Balaji G., Ganesh G., Rajesh N.; Separation Axioms in Bitopological Ordered Spaces, International Journal of Pure and Applied Mathematics 2017, Vol.114(2), 221-229.
3. Chang C.L., Fuzzy topological spaces, Journal of Mathematical Analysis and Applications 1968, 24, 182-190.
4. Coker D., An introduction to intutionistic fuzzy topological spaces, Fuzzy Sets and Systems 1997, 88, 81-89.
5. Firthous Fatima S. and Durgaselvi A., $\mathcal{N}$-Topological Ordered Spaces, AIP Conference Proceedings 2020, 2261, 030022(1-5).
6. Lellis Thivagar M., Jafari S., Antonysamy V. and Sutha Devi V., The ingenuity of Neutrosophic Topology via N-Topology, Neutrosophic Sets and Systems 2019, 19, 1.
7. Lellis Thivagar M., Ramesh V., Arockia Dasan M.; Computation of Separation Axioms in N-topology, Journal of Ultra Scientist of Physical Sciences 2017, Vol.29(2), 77-82.
8. Lellis Thivagar M., Ramesh V., Arockia Dasan M.; On new structure of N-topology, Cogent Mathematics (Taylor and Francise) 2016, Vol.3, 1204104.
9. Leopoldo Nachbin, Topology and order, D.Van Nostrand Inc., Princeton, New Jersey, 1965.
10. Narmada Devi R., Dhavaseelan R., Jafari S., On Separation axioms in an Ordered Neutrosophic Bitopological Space, Neutrosophic Sets and Systems 2017, 18, 1.
11. Salama A.A. and Alblowi S.A., Neutrosophic Set and Neutrosophic Topological Spaces, IOSR Journal of Mathematics 2012, Vol.3, Issue 4, 31-35.
12. Smarandache F., Neutrosophy and neutrosophic logic. In: F. Smarandache (Ed.), Neutrosophy, neutrosophic logic, set, probability and statistics. Proceedings of the International Conference, University of New Mexico, Gallup, NM 87301,USA (2002).
13. Smarandache F., A unifying field in logics: Neutrosophy and neutrosophic logic. Neutrosophy, neutrosophic set, probability. American Research Press, Rehoboth, NM, 1999.
14. Zadeh L.A., Fuzzy Sets, Information and Control 1965, 8(3), 338-353.

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Complex neutrosophic N-soft sets: A new model with applications

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Abstract: In this paper we establish the notion of complex single-valued neutrosophic N-soft set. It improves the traits of three general models, namely, single-valued neutrosophic sets, single-valued neutrosophic soft sets and single-valued neutrosophic N-soft sets, in such way that it makes two dimensional ambiguous information and parameterized grading evaluation compatible. We explain the modeling abilities of complex single-valued neutrosophic N-soft sets and investigate some of their fundamental properties. Moreover, the intended approach hinges on rational attributes to support the choice of the most suitable solution. The proposed method is explicated through an example from the islamic banking industry. We also perform a comparative analysis with respect to the neutrosophic TOPSIS method.

Keywords: Complex single-valued neutrosophic set, N-soft set, TOPSIS method, MAGDM.

1 Introduction

A fascinating research article by Smarandache [29] has attracted the attention of many researchers since 1998. Neutrosophic sets ($\mathcal{N}Ss$) had been born that year. They are based on formal logic that contemplates the nature, origin, and scope of objectivities with their relations for numerous intellectual spectra. The neutrosophic theory comprises probability, set theory, logics, and statistics. As such it copes with real life events characterized by degree of satisfaction, dissatisfaction and indeterminacy. It is therefore acknowledged to provide a generalization of both classic set, fuzzy set, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and Pythagorean fuzzy sets [38, 17, 33]. Neutrosophic-inspired sets are classified into many subclasses like interval-valued neutrosophic sets, single-valued neutrosophic sets ($SV\mathcal{N}Ss$), and the subclass known as simplified neutrosophic set. The $SV\mathcal{N}Ss$ were introduced by Wang and Smarandache [31, 30]. They can be characterized by three real valued functions whose values are taken from the unit closed interval $[0, 1]$, therefore it is more convenient and applicable in many areas of science and engineering. After Wang and Smarandache, the single-valued neutrosophic environment has been scrutinized extensively. For example, Ye [34] provided a correlation coefficient between $SV\mathcal{N}Ss$ which became a useful tool for decision making, and Akram and Luqman [6] illustrated the concept of $SV\mathcal{N}Ss$ with the flavor of hypergraphs.

Another breakthrough was Ramot et al. [26] who extended the 1-dimensional fuzzy perspective [38] to 2-dimensional phenomena. The resulting model was called complex fuzzy sets. This new perspective prompted many authors to adapt existing models to the complex spirit. Thus complex intuitionistic fuzzy sets [15] and complex Pythagorean fuzzy sets [37], which are precisely related to multi-attribute decision making (MAGDM) phenomena, were soon developed.

The two aforementioned expressions of vagueness were made compatible by Ali and Smarandache [13]. These authors put forward the notion of complex neutrosophic set under the influence of both neutrosophic sets [29] and complex fuzzy sets [26].

In MAGDM problems, the opinions of people are not invariably expressed through binary evaluations. It is often easier to bring up decisions using non-binary evaluations, specifically in the case of qualitative information such as the perceived performance of banking industry, people's morality, hospital assistance, etc. Hence, Fatimah et al. [21] firstly presented N-soft sets and applied them on decision making methods based on non-binary evaluations. N-soft sets extended the scope of soft sets [25] whose foundation is that any alternative can be characterized by a selected list of attributes. Many real examples were given [11, 21]. Stimulated from the novel concept of N-soft set, Akram et al. [5] solved decision making problems using the hybrid combination of fuzzy set with N-soft set that improves the performance of fuzzy soft sets [10]. Further, Akram et al. [9] presented the novel idea of intuitionistic fuzzy N-soft sets ($IFNS_fSs$), Pythagorean fuzzy N-soft sets ($PFNS_fSs$) have been introduced by Zhang [39] in 2020, and recently the multi-fuzzy N-soft set model has been presented alongside its applications to decision-making [22]. This proves that N-soft sets are a trendy topic and that the model is amenable to hybridization from many standpoints including rough set theory [11] and hesitancy [4] in addition to the ideas discussed above.

The theoretical models called neutrosophic soft sets ($\mathcal{N}S_fSs$) and single-valued neutrosophic soft sets ($SV\mathcal{N}S_fSs$) were put forward by Maji [40] and Jana et al. [23], respectively. The parametrized nature of the attributes that characterizes soft set theory is combined with neutrosophic information and the possibilities of these new models are discussed in detail. Ashraf and Butt [16] and Riaz et al. [27] first established a theoretical model for neutrosophic N-soft sets ($\mathcal{N}NS_fSs$). They made applications to business and the medical field supported by the TOPSIS method, respectively. Moreover, Sahin et al. [28] used the framework of ($SV\mathcal{N}S_fSs$) for the development of a TOPSIS method which helped to find the most suitable supplier for a production industry. In 2015, Ye [36] introduced single-valued neutrosophic linguistic numbers ($SV\mathcal{N}LNs$) as an extension of intuitionistic linguistic numbers and further set theoretical description for single-valued neutrosophic linguistic-TOPSIS method. More recently, Akram et al. [7, 8] have presented new decision making methods.

In this manuscript we present a quite general model known as complex single-valued neutrosophic N-soft set ($CSV\mathcal{N}S_fS$). It describes the possibility that the parameterized nature of the universe may be complex single-valued neutrosophic, which comprises functions for satisfaction degree, hesitancy degree and dissatisfaction degree whose values are taken from the complex unit circle. The hesitancy degree and ordered grades endow the $CSV\mathcal{N}S_fS$ with excellent qualities, so much so that this model dominates over the existing $C\mathcal{N}Ss$, $\mathcal{N}NS_fSs$ and $SV\mathcal{N}Ss$.

The motivation for this paper depends upon the following elements:

1. The $\mathcal{N}NS_fSs$ and $IFNS_fSs$ have the ability to express situations including an indeterminacy part with ordered grades, but they are not designed to deal with two dimensional ambiguity in the parametric information.
2. Moreover, $SV\mathcal{N}Ss$ and $C\mathcal{N}Ss$ can tackle the hesitancy degree in human judgment with periodic terms, but they cannot assist us in the decision making problems based on non-binary evaluations or ranking systems.
3. These limitations encouraged us to present the idea of $CSV\mathcal{N}S_fS$ which competently handles the phase term of 2-dimensional problems with ordered grades, indeterminacy, hesitancy and incomplete figures in their decisions.

The practical contribution of this article is the formalization of the $CSV\mathcal{N}S_fS$-TOPSIS technique for solving MAGDM problems that require the use of $CSV\mathcal{N}S_f$ information. For this purpose, we define some basic notions and the $CSV\mathcal{N}S_fSSs$ and $CSV\mathcal{N}S_f$ averaging and geometric operators. These operators allow us to combine the decisions according to the performance of the alternatives and the weightage of the relevant attributes and experts. We also define score and accuracy function sof $CSV\mathcal{N}S_fNs$ for the sake of $CSV\mathcal{N}S_f$-PIS and $CSV\mathcal{N}S_f$-NIS. Finally, we can sort out the alternatives using a revised closeness index whose values are totally based upon the normalized Euclidean distance.

The authenticity of the presented technique is verified by a numerical example that concerns the monitoring performance of the Islamic banking industry on the basis of the CAMELS rating system. Moreover, a comparison of the proposed model with the $SV\mathcal{N}$-TOPSIS method substantiates the accuracy and reliability of the results and of our novel technique. For further useful notions related to N-soft sets not discussed in the paper, the readers are referred to [1, 2, 12]

The arrangement of this paper is as follows. Section 2 contains some basic definitions related to the proposed model. In Section 3 we describe the main features of the presented theory with some operations and properties. Section 4 presents the score function, accuracy function and some aggregation operators related to $CSV\mathcal{N}NS_fNs$. Section 5, gives a brief description for the $CSV\mathcal{N}NS_f$-TOPSIS method with a specific algorithm. Section 6, models a MAGDM problem and applies the proposed technique to find a solution. Section 7 comprises the comparison analysis with the $CSV\mathcal{N}$-TOPSIS method. In Section 8, we come to the conclusion with some ideas for future research works.

2 Preliminaries

Definition 1. *[29] A neutrosophic set ($\mathcal{N}S$) Ψ on a universe of discourse $\mathbb{U}$ has the form:*

$$\Psi = \langle u, \mathbb{T}_\Psi(u), \mathbb{I}_\Psi(u), \mathbb{F}_\Psi(u) : u \in \mathbb{U}\rangle,$$

where, $\mathbb{T}_\Psi(u)$, $\mathbb{I}_\Psi(u)$ and $\mathbb{F}_\Psi(u)$ are degree of satisfaction, degree of indeterminacy and degree of dissatisfaction, respectively, belongs to non-standard interval $]^-0, 1^+[$, for every $u \in \mathbb{U}$.

Definition 2. *[31] A single-valued neutrosophic set ($SV\mathcal{N}S$) Ψ on a universe of discourse $\mathbb{U}$ has the form*

$$\Psi = \langle u, \mathbb{T}_\Psi(u), \mathbb{I}_\Psi(u), \mathbb{F}_\Psi(u) : u \in \mathbb{U}\rangle,$$

where $\mathbb{T}_\Psi(u), \mathbb{I}_\Psi(u), \mathbb{F}_\Psi(u) : U \to [0,1]$ are the degree of truthness, degree of hesitancy and degree of falsity, respectively, without any condition on the sum of $\mathbb{T}_\Psi(u)$, $\mathbb{I}_\Psi(u)$ and $\mathbb{F}_\Psi(u)$ for all $u \in \mathbb{U}$. The triplet $(\mathbb{T}_\Psi, \mathbb{I}_\Psi, \mathbb{F}_\Psi)$ is called single-valued neutrosophic number ($SV\mathcal{N}N$).

Definition 3. *[13] A complex single-valued neutrosophic set ($CSV\mathcal{N}S$) Ψ, on the universe $\mathbb{U}$ is defined as:*

$$\Psi = \langle u, \mathbb{T}_\Psi(u), \mathbb{I}_\Psi(u), \mathbb{F}_\Psi(u) : u \in \mathbb{U}\rangle,$$

where $\mathbb{T}_\Psi(u) = p_\Psi(u)e^{i2\pi t_\Psi(u)}$, $\mathbb{I}_\Psi(u) = q_\Psi(u)e^{i2\pi\omega_\Psi(u)}$ and $\mathbb{F}_\Psi(u) = r_\Psi(u)e^{i2\pi f_\Psi(u)}$, denote the degree of truthness, degree of hesitancy and degree of falsity, respectively, without any conditions on the sum of amplitude terms $p_\Psi(u)$, $q_\Psi(u)$, $r_\Psi(u) : U \to [0,1]$ or the phase terms $t_\Psi(u)$, $\omega_\Psi(u)$, $f_\Psi(u) : U \to [0,1]$ for all $u \in \mathbb{U}$. The triplet $(p_\Psi(u)e^{i2\pi t_\Psi(u)}, q_\Psi(u)e^{i2\pi\omega_\Psi(u)}, r_\Psi(u)e^{i2\pi f_\Psi(u)})$ is called complex single-valued neutrosophic number ($CSV\mathcal{N}N$).

Definition 4. *[25] Let $\mathbb{U}$ be a non-empty set and K be a set of parameters and $Y \subseteq K$. A soft set S_fS over $\mathbb{U}$ is a pair (Φ, Y), where $\Phi : K \to P(\mathbb{U})$ is a set-valued function defined as:*

$$(\Phi, Y) = \{\langle y_w, \Phi(y_w)\rangle | y_w \in Y, \Phi(y_w) \in P(\mathbb{U})\}.$$

Definition 5. *Let $\mathbb{U}$ be a non-empty set and K be a set of parameters and $Y \subseteq K$. A complex single-valued neutrosophic soft set $CSV\mathcal{N}S_fS$ over $\mathbb{U}$ is a pair (Φ, Y), where $\Phi : K \to \mathbb{P}(CSV\mathcal{N}S)$ is a set-valued function defined as:*

$$\begin{aligned}(\Phi, Y) &= \{\langle y_w, \Phi(y_w)\rangle | y_w \in Y, \Phi(y_w) \in \mathbb{P}(CSV\mathcal{N}S)\}\\ &= \{\langle y_w, (u_s, (\mathbb{T}_{ws}, \mathbb{I}_{ws}, \mathbb{F}_{ws}))\rangle\}\\ &= \{\langle y_w, (u_s, (p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}}))\rangle\},\end{aligned}$$

where $\mathbb{P}(CSV\mathcal{N}S)$ is the collection of all subsets of $CSV\mathcal{N}Ss$ over the non-empty set $\mathbb{U}$ and $p_{ws}, t_{ws}, q_{ws}, \omega_{ws}, r_{ws}, f_{ws} \in [0, 1]$.

Definition 6. *[21]Let $\mathbb{U}$ be a non-empty set and K be a set of parameters and $Y \subseteq K$. Let $H = \{0, 1, 2, \ldots, N-1\}$ be a set of ordered grades with $N \in \{2, 3, \ldots\}$. A triple (Φ, Y, N) is called N-soft set (NS_fS) over $\mathbb{U}$ if Φ is a mapping define as $\Phi : Y \to 2^{\mathbb{U}\times H}$, that is there exist a unique pair $(u_s, h^s_w) \in \mathbb{U} \times H$ such that $(u_s, h^s_w) \in \Phi(y_w)$, where $u_s \in \mathbb{U}, h^s_w \in H$.*

3 Complex single-valued neutrosophic N-soft sets

Definition 7. *Let $\mathbb{U}$ be a non-empty set and K be a set of parameters with $Y \subseteq K$. Let $H = \{0, 1, 2, \ldots, N-1\}$ be a set of ordered grades with $N \in \{2, 3, \ldots\}$. A triple (Φ_Ψ, Y, N) is called a complex single-valued neutrosophic N-soft set $(CSV\mathcal{N}NS_fS)$ on Y, if (Φ, Y, N) is an NS_fS on $\mathbb{U}$, and $\Phi_\Psi : Y \to 2^{\mathbb{U}\times H} \times \mathbb{CSVNN}$ is a mapping, which is defined as:*

$$\begin{aligned}\Phi_\Psi(y_w) &= \{\langle(\Phi(y_w), \Psi(y_w))\rangle : y_w \in Y\},\\ &= \{\langle((u_s, h^s_w), (\mathbb{T}_{ws}, \mathbb{I}_{ws}, \mathbb{F}_{ws}))\rangle\},\\ &= \{\langle((u_s, h^s_w), (p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}}))\rangle\},\end{aligned}$$

where $\Phi : Y \to 2^{\mathbb{U}\times H}$, $\Psi : Y \to \mathbb{CSVNN}$, and $\mathbb{CSVNN}$ denotes the collection of all complex single-valued neutrosophic numbers of $\mathbb{U}$, h^s_w denotes the rank of parameter for the alternative y_w and $p_{ws}, t_{ws}, q_{ws}, \omega_{ws}, r_{ws}, f_{ws} \in [0, 1]$, with no conditions on their sum.

Example 1. *Let $\mathbb{U} = \{\mathbb{U}_1 =$ Emirates, $\mathbb{U}_2 =$ Eithad Airways, $\mathbb{U}_3 =$ Turkish airlines, $\mathbb{U}_4 =$ Flynas $\}$ be the set of airlines from Pakistan to Turkey and $Y = \{Y_1 =$ Price , $Y_2 =$ Entertainment, $Y_3 =$ luxuries, $Y_4 =$ Safety $\}$ be the characteristics which are experienced by the passengers and then passengers assigned ratings to these airlines. These ratings are aggregated by the experts and form a 6-soft set given Table 1, where*

0 means 'very Bad'

1 means 'Bad'

2 means 'Ok'

3 means 'Good'

4 means 'Great'

5 means 'Excellent'

Table 1: 6-soft set evaluated by experts

$Y/\mathbb{U}$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_3$	$\mathbb{U}_4$
Y_1	3	5	0	1
Y_2	1	4	2	0
Y_3	2	1	4	3
Y_4	5	0	1	2

For handling the alternatives with fuzziness property related to parameters, we need $CSV\mathcal{N}NS_fSs$. Therefore, authorities defined grading criteria, given in Table 2, for the evaluation of airlines under the environment of $CSV\mathcal{N}NS_fSs$, where Table 2 is evaluated from the following criteria:

$$\begin{aligned}&when\ h^s_w = 0, && -4.00 \le S(\Psi) < -3.30,\\ &when\ h^s_w = 1, && -3.30 \le S(\Psi) < -2.20,\\ &when\ h^s_w = 2, && -2.20 \le S(\Psi) < -1.00,\\ &when\ h^s_w = 3, && -1.00 \le S(\Psi) < 0.20,\\ &when\ h^s_w = 4, && 0.20 \le S(\Psi) < 1.20,\\ &when\ h^s_w = 5, && 1.20 \le S(\Psi) \le 2.000.\end{aligned}$$

Table 2: Grading criteria for $CSV\mathcal{N}6SS$

h_z^w/J	degree of truthness		degree of indeterminacy		degree of falsity	
grades	p_w	$2\pi t_w$	q_w	$2\pi\omega_w$	r_w	$2\pi f_w$
$h_w^s = 0$	$[0.00, 0.15)$	$[0.0, 0.3\pi)$	$(0.90, 1.00]$	$[1.8\pi, 2.0\pi]$	$(0.90, 1.00]$	$[1.8\pi, 2.0\pi]$
$h_w^s = 1$	$[0.15, 0.30)$	$[0.3\pi, 0.6\pi)$	$(0.70, 0.90]$	$[1.4\pi, 1.8\pi)$	$(0.70, 0.90]$	$[1.4\pi, 1.8\pi)$
$h_w^s = 2$	$[0.30, 0.50)$	$[0.6\pi, 1.0\pi)$	$(0.50, 0.70]$	$[1.0\pi, 1.4\pi)$	$(0.50, 0.70]$	$[1.0\pi, 1.4\pi)$
$h_w^s = 3$	$[0.50, 0.70)$	$[1.0\pi, 1.4\pi)$	$(0.30, 0.50]$	$[0.6\pi, 1.0\pi)$	$(0.30, 0.50]$	$[0.6\pi, 1.0\pi)$
$h_w^s = 4$	$[0.70, 0.90)$	$[1.4\pi, 1.8\pi)$	$(0.15, 0.30]$	$[0.3\pi, 0.6\pi)$	$(0.15, 0.30]$	$[0.3\pi, 0.6\pi)$
$h_w^s = 5$	$[0.90, 1.00]$	$[1.8\pi, 2\pi]$	$[0.00, 0.15]$	$[0.0, 0.3\pi)$	$[0.00, 0.15]$	$[0.0, 0.3\pi)$

Using the prescribed information, the $CSV\mathcal{N}6S_fS$, shown in 3, is defined as:

$$\begin{aligned}\Phi_\Psi(Y_1) &= \{((\mathbb{U}_1,3),(0.60e^{i1.26\pi},0.35e^{i0.68\pi},0.4e^{i0.84\pi})),((\mathbb{U}_2,5),(0.95e^{i1.92\pi},0.05e^{i0.12\pi},0.12e^{i0.26\pi})),\\ &\quad ((\mathbb{U}_3,0),(0.06e^{i0.14\pi},0.95e^{i1.92\pi},0.97e^{i1.96\pi})),((\mathbb{U}_4,1),(0.24e^{i0.50\pi},0.86e^{i1.70\pi},0.87e^{i1.72\pi}))\},\end{aligned}$$

$$\begin{aligned}\Phi_\Psi(Y_2) &= \{((\mathbb{U}_1,1),(0.17e^{i0.40\pi},0.75e^{i1.48\pi},0.81e^{i1.66\pi})),((\mathbb{U}_2,4),(0.81e^{i1.66\pi},0.22e^{i0.42\pi},0.25e^{i0.48\pi})),\\ &\quad ((\mathbb{U}_3,2),(0.36e^{i0.74\pi},0.58e^{i1.18\pi},0.54e^{i1.10\pi})),((\mathbb{U}_4,0),(0.08e^{i0.20\pi},0.96e^{i1.94\pi},0.98e^{1.98\pi}))\},\end{aligned}$$

$$\begin{aligned}\Phi_\Psi(Y_3) &= \{((\mathbb{U}_1,2),(0.32e^{i0.70\pi},0.55e^{i1.12\pi},0.52e^{i1.06\pi})),((\mathbb{U}_2,1),(0.2e^{i0.42\pi},0.76e^{i1.54\pi},0.78e^{i1.58\pi})),\\ &\quad ((\mathbb{U}_3,4),(0.75e^{i1.52\pi},0.17e^{i0.36\pi},0.20e^{i0.38\pi})),((\mathbb{U}_4,3),(0.69e^{i1.36\pi},0.41e^{i0.84\pi},0.48e^{0.94\pi}))\},\end{aligned}$$

$$\begin{aligned}\Phi_\Psi(Y_4) &= \{((\mathbb{U}_1,5),(0.98e^{i1.94\pi},0.01e^{i0.04\pi},0.1e^{0.24i\pi})),((\mathbb{U}_2,0),(0.03e^{i0.10\pi},0.91e^{i1.84\pi},0.93e^{1.88i\pi})),\\ &\quad ((\mathbb{U}_3,1),(0.21e^{i0.46\pi},0.79e^{i1.60\pi},0.83e^{i1.68\pi})),((\mathbb{U}_4,2),(0.38e^{i0.78\pi},0.59e^{i1.20\pi},0.55e^{i1.12\pi}))\}.\end{aligned}$$

Table 3: The $CSV\mathcal{N}6S_fS$ $(\Phi_\Psi, Y, 6)$

$(\Phi_\Psi, Y, 6)$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_3$	$\mathbb{U}_4$
Y_1	$(3,(0.60e^{i1.26\pi},0.35e^{i0.68\pi},0.40e^{i0.84\pi}))$	$(5,(0.95e^{i1.92\pi},0.05e^{i0.12\pi},0.12e^{i0.26\pi}))$	$(0,(0.06e^{i0.14\pi},0.95e^{i1.92\pi},0.97e^{i1.96\pi}))$	$(1,(0.24e^{i0.50\pi},0.86e^{i1.70\pi},0.87e^{i1.72\pi}))$
Y_2	$(1,(0.17e^{i0.40\pi},0.75e^{i1.48\pi},0.81e^{i1.66\pi}))$	$(4,(0.81e^{i1.66\pi},0.22e^{i0.42\pi},0.25e^{i0.48\pi}))$	$(2,(0.36e^{i0.74\pi},0.58e^{i1.18\pi},0.54e^{i1.10\pi}))$	$(0,(0.08e^{i0.20\pi},0.96e^{i1.94\pi},0.98e^{1.98\pi}))$
Y_3	$(2,(0.32e^{i0.70\pi},0.55e^{i1.12\pi},0.52e^{i1.06\pi}))$	$(1,(0.20e^{i0.42\pi},0.76e^{i1.54\pi},0.78e^{i1.58\pi}))$	$(4,(0.75e^{i1.52\pi},0.17e^{i0.36\pi},0.20e^{i0.38\pi}))$	$(3,(0.69e^{i1.36\pi},0.41e^{i0.84\pi},0.48e^{0.94\pi}))$
Y_4	$(5,(0.98e^{i1.94\pi},0.01e^{i0.04\pi},0.10e^{0.24i\pi}))$	$(0,(0.03e^{i0.10\pi},0.91e^{i1.84\pi},0.93e^{1.88i\pi}))$	$(1,(0.21e^{i0.46\pi},0.79e^{i1.60\pi},0.83e^{i1.68\pi}))$	$(2,(0.38e^{i0.78\pi},0.59e^{i1.20\pi},0.55e^{i1.12\pi}))$

Definition 8. *A $CSV\mathcal{N}S_fS(\Phi_\Psi, Y, N)$ over a non-empty set $\mathbb{U}$ is said to be efficient where (Φ, Y, N) is an NS_fS, if $\Phi_\Psi(y_w) = \langle(u_s, N-1), 1, 0, 0\rangle$ for some $y_w \in Y, u_s \in \mathbb{U}$.*

Example 2. *Let $(\Phi_\Psi, Y, 6)$ be $CSV\mathcal{N}6S_fS$, as in Example 1. From Table 3, it is clear that Example 1 is not efficient.*

Definition 9. *Let (Φ_Ψ, Y, N_1) and (χ_A, C, N_2) be two $CSV\mathcal{N}S_fSs$ on a universe of discourse $\mathbb{U}$. Then, they are said to be equal if and only if $\Phi = \chi$, $\Psi = A$, $Y = C$ and $N_1 = N_2$.*

Definition 10. *Let (Φ_Ψ, Y, N) be a $CSV\mathcal{N}S_fS$ on $\mathbb{U}$. The weak complement of $CSV\mathcal{N}S_fS$ is defined as the weak complement of the N-soft set (Φ, Y, N), that is, any N-soft set such that $\Phi^c(y_w) \cap \Phi(y_w) = \emptyset$ for all $y_w \in Y$. The weak complement of $CSV\mathcal{N}S_fS$ of (Φ_Ψ, Y, N) is represented as (Φ^c_Ψ, Y, N).*

Example 3. *Let $(\Phi_\Psi, Y, 6)$ be $CSV\mathcal{N}6S_fS$, as in Example 1. The weak complement (Φ^c_Ψ, Y, N) is given in Table 4.*

Table 4: A weak complement of the $CSV\mathcal{N}6S_fS$ $(\Phi_\Psi, Y, 6)$

$(\Phi^c_\Psi, Y, 6)$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_3$	$\mathbb{U}_4$
Y_1	$(5,(0.60e^{i1.26\pi},0.35e^{i0.68\pi},0.40e^{i0.84\pi}))$	$(4,(0.95e^{i1.92\pi},0.05e^{i0.12\pi},0.12e^{i0.26\pi}))$	$(1,(0.06e^{i0.14\pi},0.95e^{i1.92\pi},0.97e^{i1.96\pi}))$	$(3,(0.24e^{i0.50\pi},0.86e^{i1.70\pi},0.87e^{i1.72\pi}))$
Y_2	$(4,(0.17e^{i0.40\pi},0.75e^{i1.48\pi},0.81e^{i1.66\pi}))$	$(1,(0.81e^{i1.66\pi},0.22e^{i0.42\pi},0.25e^{i0.48\pi}))$	$(3,(0.36e^{i0.74\pi},0.58e^{i1.18\pi},0.54e^{i1.10\pi}))$	$(5,(0.08e^{i0.20\pi},0.96e^{i1.94\pi},0.98e^{1.98\pi}))$
Y_3	$(4,(0.32e^{i0.70\pi},0.55e^{i1.12\pi},0.52e^{i1.06\pi}))$	$(3,(0.20e^{i0.42\pi},0.76e^{i1.54\pi},0.78e^{i1.58\pi}))$	$(0,(0.75e^{i1.52\pi},0.17e^{i0.36\pi},0.20e^{i0.38\pi}))$	$(5,(0.69e^{i1.36\pi},0.41e^{i0.84\pi},0.48e^{0.94\pi}))$
Y_4	$(0,(0.98e^{i1.94\pi},0.01e^{i0.04\pi},0.10e^{0.24i\pi}))$	$(2,(0.03e^{i0.10\pi},0.91e^{i1.84\pi},0.93e^{1.88i\pi}))$	$(3,(0.21e^{i0.46\pi},0.79e^{i1.60\pi},0.83e^{i1.68\pi}))$	$(3,(0.38e^{i0.78\pi},0.59e^{i1.20\pi},0.55e^{i1.12\pi}))$

Table 6: The complex single-valued neutrosophic complement (Φ_{Φ^c}, Y, N) of the $CSV\mathcal{N}6S_fS$

$(\Phi_{\Psi^c}, Y, 6)$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_3$	$\mathbb{U}_4$
Y_1	$(3,(0.40e^{i0.84\pi},0.65e^{i1.32\pi},0.60e^{i1.26\pi}))$	$(5,(0.12e^{i0.26\pi},0.95e^{i1.88\pi},0.95e^{i1.92\pi}))$	$(0,(0.97e^{i1.96\pi},1.05e^{i1.92\pi},0.06e^{i0.14\pi}))$	$(1,(0.24e^{i0.50\pi},0.14e^{i0.24\pi},0.87e^{i1.72\pi}))$
Y_2	$(1,(0.81e^{i1.66\pi},0.25e^{i0.52\pi},0.17e^{i0.40\pi}))$	$(4,(0.25e^{i0.48\pi},0.88e^{i1.58\pi},0.81e^{i1.66\pi}))$	$(2,(0.54e^{i1.10\pi},0.42e^{i0.08\pi},0.36e^{i0.74\pi}))$	$(0,(0.98e^{1.98\pi},0.04e^{i0.06\pi},0.08e^{i0.20\pi}))$
Y_3	$(2,(0.52e^{i1.06\pi},0.45e^{i0.88\pi},0.32e^{i0.70\pi}))$	$(1,(0.78e^{i1.58\pi},0.24e^{i0.46\pi},0.20e^{i0.42\pi}))$	$(4,(0.20e^{i0.38\pi},0.83e^{i1.64\pi},0.75e^{i1.52\pi}))$	$(3,(0.48e^{0.94\pi},0.59e^{i1.16\pi},0.69e^{i1.36\pi}))$
Y_4	$(5,(0.10e^{0.24i\pi},0.09e^{i1.96\pi},0.98e^{i1.94\pi}))$	$(0,(0.93e^{1.88i\pi},0.09e^{i0.16\pi},0.03e^{i0.10\pi}))$	$(1,(0.83e^{i1.68\pi},0.21e^{i0.40\pi},0.21e^{i0.46\pi}))$	$(2,(0.55e^{i1.12\pi},0.41e^{i0.80\pi},0.38e^{i0.78\pi}))$

Definition 11. *Let (Φ_Ψ, Y, N) be a $CSV\mathcal{N}NS_fS$ on $\mathbb{U}$. The Strong complement of $CSV\mathcal{N}NS_fS$, denoted as (Φ'_Ψ, Y, N), is defined as:*

$$\Phi'(y_w) = \begin{cases} h^s_w - 1, & \text{if } h^s_w = (N-1) - h^s_w, \\ (N-1) - h^s_w, & \text{otherwise,} \end{cases}$$

for all $y_w \in Y$ and $u_s \in \mathbb{U}$, satisfying the condition $(\Phi_\Psi, Y, N) \cap (\Phi'_\Psi, Y, N) = \emptyset$.

Example 4. *Let $(\Phi_\Psi, Y, 6)$ be $CSV\mathcal{N}6S_fS$, then the strong complement $(\Phi'_\Psi, Y, 6)$ of Example 1 is given in Table 5 such that $(\Phi_\Psi, Y, 6) \cap (\Phi'_\Psi, Y, 6) = \emptyset$.*

Table 5: Strong complement of $(\Phi_\Psi, Y, 6)$

$(\Phi_\Psi, Y, 6)$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_3$	$\mathbb{U}_4$
Y_1	$(2,(0.60e^{i1.26\pi},0.35e^{i0.68\pi},0.40e^{i0.84\pi}))$	$(0,(0.95e^{i1.92\pi},0.05e^{i0.12\pi},0.12e^{i0.26\pi}))$	$(5,(0.06e^{i0.14\pi},0.95e^{i1.92\pi},0.97e^{i1.96\pi}))$	$(4,(0.24e^{i0.50\pi},0.86e^{i1.70\pi},0.87e^{i1.72\pi}))$
Y_2	$(4,(0.17e^{i0.40\pi},0.75e^{i1.48\pi},0.81e^{i1.66\pi}))$	$(1,(0.81e^{i1.66\pi},0.22e^{i0.42\pi},0.25e^{i0.48\pi}))$	$(3,(0.36e^{i0.74\pi},0.58e^{i1.18\pi},0.54e^{i1.10\pi}))$	$(5,(0.08e^{i0.20\pi},0.96e^{i1.94\pi},0.98e^{i1.98\pi}))$
Y_3	$(3,(0.32e^{i0.70\pi},0.55e^{i1.12\pi},0.52e^{i1.06\pi}))$	$(4,(0.20e^{i0.42\pi},0.76e^{i1.54\pi},0.78e^{i1.58\pi}))$	$(1,(0.75e^{i1.52\pi},0.17e^{i0.36\pi},0.20e^{i0.38\pi}))$	$(2,(0.69e^{i1.36\pi},0.41e^{i0.84\pi},0.48e^{0.94\pi}))$
Y_4	$(0,(0.98e^{i1.94\pi},0.01e^{i0.04\pi},0.10e^{0.24i\pi}))$	$(5,(0.03e^{i0.10\pi},0.91e^{i1.84\pi},0.93e^{1.88i\pi}))$	$(4,(0.21e^{i0.46\pi},0.79e^{i1.60\pi},0.83e^{i1.68\pi}))$	$(3,(0.38e^{i0.78\pi},0.59e^{i1.20\pi},0.55e^{i1.12\pi}))$

Proposition 12. *A strong complement of $CSV\mathcal{N}NS_fS$ is also a weak complement but week complement may or may not be strong complement.*

Proof. The proof is straight forward from the definitions of strong complement and weak complement. □

Definition 13. *Let (Φ_Ψ, Y, N) be a $CSV\mathcal{N}S_fS$ on $\mathbb{U}$. The complex single-valued neutrosophic complement of $CSV\mathcal{N}S_fS$ is denoted as (Φ_{Ψ^c}, Y, N) and is defined as*

$$\Phi_{\Psi^c}(y_w) = \langle (u_s, h^s_w, (\mathbb{F}_{ws}, 1 - \mathbb{I}_{ws}, \mathbb{T}_{ws})) \rangle = \langle (u_s, h^s_w, (r_{ws}e^{i2\pi f_{ws}}, (1 - q_{ws})e^{i2\pi(1-\omega_{ws})}, p_{ws}e^{i2\pi t_{ws}})) \rangle.$$

Example 5. *Let $(\Phi_\Psi, Y, 6)$ be $CSV\mathcal{N}6S_fS$, as in Example 1. The complex single-valued neutrosophic complement (Φ_{Ψ^c}, Y, N), is given in Table 6.*

Definition 14. *Let (Φ_Ψ, Y, N) be a $CSV\mathcal{N}S_fS$ on $\mathbb{U}$. $(F^c_{J^c}, Z, N)$ is referred to as a weak complex single-valued neutrosophic complement of $((\Phi_\Psi)^c, Y, N)$ if and only if (Φ^c_Ψ, Y, N) is a weak complement and (Φ_{Ψ^c}, Y, N) is a complex single-valued neutrosophic complement of (Φ_Ψ, Y, N).*

Example 6. *Let $(\Phi_\Psi, Y, 6)$ be $CSV\mathcal{N}6S_fS$, as in Example 1. The weak complex single-valued neutrosophic complement $(\Phi^c_{\Phi^c}, Y, N)$, is given in Table 7.*

Table 7: The weak complex single-valued neutrosophic complement $(\Phi^c_{\Psi^c}, Y, 6)$ of the $CSV\mathcal{N}6S_fS$

$(\Phi^c_{\Psi^c}, Y, 6)$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_3$	$\mathbb{U}_4$
Y_1	$(5,(0.40e^{i0.84\pi},0.65e^{i1.32\pi},0.60e^{i1.26\pi}))$	$(4,(0.12e^{i0.26\pi},0.95e^{i1.88\pi},0.95e^{i1.92\pi}))$	$(1,(0.97e^{i1.96\pi},1.05e^{i1.92\pi},0.06e^{i0.14\pi}))$	$(3,(0.24e^{i0.50\pi},0.14e^{i0.24\pi},0.87e^{i1.72\pi}))$
Y_2	$(4,(0.81e^{i1.66\pi},0.25e^{i0.52\pi},0.17e^{i0.40\pi}))$	$(1,(0.25e^{i0.48\pi},0.88e^{i1.58\pi},0.81e^{i1.66\pi}))$	$(3,(0.54e^{i1.10\pi},0.42e^{i0.08\pi},0.36e^{i0.74\pi}))$	$(5,(0.98e^{1.98\pi},0.04e^{i0.06\pi},0.08e^{i0.20\pi}))$
Y_3	$(4,(0.52e^{i1.06\pi},0.45e^{i0.88\pi},0.32e^{i0.70\pi}))$	$(3,(0.78e^{i1.58\pi},0.24e^{i0.46\pi},0.20e^{i0.42\pi}))$	$(0,(0.20e^{i0.38\pi},0.83e^{i1.64\pi},0.75e^{i1.52\pi}))$	$(5,(0.48e^{0.94\pi},0.59e^{i1.16\pi},0.69e^{i1.36\pi}))$
Y_4	$(0,(0.10e^{0.24i\pi},0.09e^{i1.96\pi},0.98e^{i1.94\pi}))$	$(2,(0.93e^{1.88i\pi},0.09e^{i0.16\pi},0.03e^{i0.10\pi}))$	$(3,(0.83e^{i1.68\pi},0.21e^{i0.40\pi},0.21e^{i0.46\pi}))$	$(3,(0.55e^{i1.12\pi},0.41e^{i0.80\pi},0.38e^{i0.78\pi}))$

Definition 15. *Let (Φ_Ψ, Y, N) be a $CSV\mathcal{N}NS_fS$ on $\mathbb{U}$, then the strong complex single-valued neutrosophic complement $((\Phi_\Psi)', Y, N)$ is defined as a strong complement (Φ'_Ψ, Y, N) and a complex single-valued neutrosophic complement (Φ_{Ψ^c}, Y, N) of (Φ_Ψ, Y, N), defined as:*

$\Phi'_{\Psi^c}(y_w) =$

$$\begin{cases} (h^s_w - 1, (r_{ws}e^{i2\pi f_{ws}}, (1-q_{ws})e^{i2\pi(1-\omega_{ws})}, p_{ws}e^{i2\pi t_{ws}})) & \text{if } h^s_w = (N-1) - h^s_w, \\ ((N-1) - h^s_w, (r_{ws}e^{i2\pi f_{ws}}, (1-q_{ws})e^{i2\pi(1-\omega_{ws})}, p_{ws}e^{i2\pi t_{ws}})) & \text{otherwise,} \end{cases}$$

for all $y_w \in Y$ and $u_s \in \mathbb{U}$.

Example 7. *Let* $(\Phi_\Psi, Y, 6)$ *be* $CSV\mathcal{N}6S_fS$ *on* $\mathbb{U}$, *then the strong single-valued neutrosophic complement* (Φ'_{Ψ^c}, Y, N), *of* $(\Phi_\Psi, Y, 6)$ *arranged in Table 3, is calculated in Table 8.*

Table 8: Strong single-valued neutrosophic complement of $(\Phi_\Psi, Y, 6)$

$(\Phi'_{\Psi^c}, Y, 6)$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_3$	$\mathbb{U}_4$
Y_1	$(2, (0.40e^{i0.84\pi}, 0.65e^{i1.32\pi}, 0.60e^{i1.26\pi}))$	$(0, (0.12e^{i0.26\pi}, 0.95e^{i1.88\pi}, 0.95e^{i1.92\pi}))$	$(5, (0.97e^{i1.96\pi}, 1.05e^{i1.92\pi}, 0.06e^{i0.14\pi}))$	$(4, (0.24e^{i0.50\pi}, 0.14e^{i0.24\pi}, 0.87e^{i1.72\pi}))$
Y_2	$(4, (0.81e^{i1.66\pi}, 0.25e^{i0.52\pi}, 0.17e^{i0.40\pi}))$	$(1, (0.25e^{i0.48\pi}, 0.88e^{i1.58\pi}, 0.81e^{i1.66\pi}))$	$(3, (0.54e^{i1.10\pi}, 0.42e^{i0.08\pi}, 0.36e^{i0.74\pi}))$	$(5, (0.98e^{1.98\pi}, 0.04e^{i0.06\pi}, 0.08e^{i0.20\pi}))$
Y_3	$(3, (0.52e^{i1.06\pi}, 0.45e^{i0.88\pi}, 0.32e^{i0.70\pi}))$	$(4, (0.78e^{i1.58\pi}, 0.24e^{i0.46\pi}, 0.20e^{i0.42\pi}))$	$(1, (0.20e^{i0.38\pi}, 0.83e^{i1.64\pi}, 0.75e^{i1.52\pi}))$	$(2, (0.48e^{0.94\pi}, 0.59e^{i1.16\pi}, 0.69e^{i1.36\pi}))$
Y_4	$(0, (0.10e^{0.24i\pi}, 0.09e^{i1.96\pi}, 0.98e^{i1.94\pi}))$	$(5, (0.93e^{1.88i\pi}, 0.09e^{i0.16\pi}, 0.03e^{i0.10\pi}))$	$(4, (0.83e^{i1.68\pi}, 0.21e^{i0.40\pi}, 0.21e^{i0.46\pi}))$	$(3, (0.55e^{i1.12\pi}, 0.41e^{i0.80\pi}, 0.38e^{i0.78\pi}))$

Proposition 16. *Let* $((\Phi_\Psi)^c, Y, N)$ *and* $((\Phi_\Psi)', Y, N)$ *be weak and strong complex single-valued neutrosophic complement of* $CSV\mathcal{N}NS_fS$ (Φ_Ψ, Y, N), *then*

1 $((\Phi^c)^c_\Psi, Y, N) \neq (\Phi_\Psi, Y, N)$,

2 $(((\Phi_\Psi)^c)^c, Y, N) \neq (\Phi_\Psi, Y, N)$,

3 $((\Phi')'_\Psi, Y, N) \left\{ \begin{array}{ll} = (\Phi_\Psi, Y, N) & \text{if } N \text{ is even} \\ \neq (\Phi_\Psi, Y, N) & \text{if } N \text{ is odd.} \end{array} \right\}.$

4 $([(\Phi_\Psi)']', Y, N) \left\{ \begin{array}{ll} = (\Phi_\Psi, Y, N) & \text{if } N \text{ is even} \\ \neq (\Phi_\Psi, Y, N) & \text{if } N \text{ is odd.} \end{array} \right\}.$

Proof. The proof is straight forward from the definitions. □

Definition 17. *Let* $\mathbb{U}$ *be a non-empty set and* (Φ_Ψ, Y, N_1) *and* (χ_A, C, N_2) *be* $CSV\mathcal{N}_1S_fS$ *and* $CSV\mathcal{N}_2S_fS$ *on* $\mathbb{U}$, *respectively, their restricted intersection is defined as* $(L_M, G, O) = (\Phi_\Psi, Y, N_1)\hat{\cap}(\chi_A, C, N_2)$, *with* $L_M = \Phi_\Psi \hat{\cap} \chi_A$, $G = Y \cap C$, $O = \min(N_1, N_2)$, *i.e.,* $\forall x_w \in G$, $u_s \in \mathbb{U}$ *we have*

$$\begin{aligned} & L_M(x_w) = \langle (h^s_w, (\mathbb{T}_{ws}, \mathbb{I}_{ws}, \mathbb{F}_{ws})) \rangle, \\ = \; & \langle (\min(h^{1s}_w, h^{2s}_w), \min(\mathbb{T}^1_{ws}, \mathbb{T}^2_{ws}), \max(\mathbb{I}^1_{ws}, \mathbb{I}^2_{ws}), \max(\mathbb{F}^1_{ws}, \mathbb{F}^2_{ws})) \rangle, \\ = \; & \langle (\min(h^{1s}_w, h^{2s}_w), \min(p^1_{ws}, p^2_{ws}) e^{i2\pi \min(t^1_{ws}, t^2_{ws})}, \max(q^1_{ws}, q^2_{ws}) e^{i2\pi \max(\omega^1_{ws}, \omega^2_{ws})}, \max(r^1_{ws}, r^2_{ws}) e^{i2\pi \max(f^1_{ws}, f^2_{ws})}) \rangle, \end{aligned}$$

where $(h^{1s}_w, (\mathbb{T}^1_{ws}, \mathbb{I}^1_{ws}, \mathbb{F}^1_{ws})) = (h^{1s}_w, (p^1_{ws} e^{i2\pi t^1_{ws}}, q^1_{ws} e^{i2\pi \omega^1_{ws}}, r^1_{ws} e^{i2\pi f^1_{ws}})) \in \Phi_\Psi$ *and* $(h^{2s}_w, (\mathbb{T}^2_{ws}, \mathbb{I}^2_{ws}, \mathbb{F}^2_{ws}))$ $= (h^{2s}_w, (p^2_{ws} e^{i2\pi t^2_{ws}}, q^2_{ws} e^{i2\pi \omega^2_{ws}}, r^2_{ws} e^{i2\pi f^2_{ws}})) \in \chi_A$.

Table 9: The $CSF5S_fS(\chi_A, C, 5)$

	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_5$
Y_1	$(0, (0.12e^{i0.23\pi}, 0.91e^{i1.84\pi}, 0.96e^{i1.96\pi}))$	$(1, (0.21e^{i0.42\pi}, 0.77e^{i1.50\pi}, 0.82e^{i1.66\pi}))$	$(0, (0.05e^{i0.14\pi}, 0.87e^{i1.72\pi}, 0.88e^{i1.80\pi}))$
Y_2	$(2, (0.42e^{i0.82\pi}, 0.51e^{i1.04\pi}, 0.56e^{i1.10\pi}))$	$(4, (0.88e^{i1.78\pi}, 0.09e^{i0.16\pi}, 0.06e^{i0.10\pi}))$	$(4, (0.90e^{i1.84\pi}, 0.11e^{i0.20\pi}, 0.14e^{i0.26\pi}))$
Y_3	$(3, (0.81e^{i1.64\pi}, 0.17e^{i0.36\pi}, 0.19e^{i0.40\pi}))$	$(3, (0.83e^{i1.68\pi}, 0.27e^{i0.56\pi}, 0.30e^{i0.58\pi}))$	$(1, (0.26e^{i0.48\pi}, 0.72e^{i1.42\pi}, 0.75e^{i1.52\pi}))$
Y_4	$(4, (0.95e^{i1.80\pi}, 0.012e^{i0.02\pi}, 0.10e^{0.22i\pi}))$	$(3, (0.70e^{i1.42\pi}, 0.26e^{i0.54\pi}, 0.31e^{0.64i\pi}))$	$(2, (0.49e^{i1.02\pi}, 0.61e^{i1.24\pi}, 0.59e^{i1.16\pi}))$

Example 8. *The restricted intersection* (L_M, G, O) *of* $(\Phi_\Psi, Y, 6)$ *and* $(\chi_A, C, 5)$, *given in Table 3 and Table 9, arranged in 10.*

Table 10: The restricted intersection $(L_M, G, 5)$

$(L_M, G, 56)$	$\mathbb{U}_1$	$\mathbb{U}_2$
Y_1	$(0, (0.12e^{i0.23\pi}, 0.91e^{i1.84\pi}, 0.96e^{i1.96\pi}))$	$(1, (0.21e^{i0.42\pi}, 0.77e^{i1.50\pi}, 0.82e^{i1.66\pi}))$
Y_2	$(1, (0.17e^{i0.40\pi}, 0.75e^{i1.48\pi}, 0.81e^{i1.66\pi}))$	$(4, (0.81e^{i1.66\pi}, 0.22e^{i0.42\pi}, 0.25e^{i0.48\pi}))$
Y_3	$(2, (0.32e^{i0.70\pi}, 0.55e^{i1.12\pi}, 0.52e^{i1.06\pi}))$	$(1, (0.20e^{i0.42\pi}, 0.76e^{i1.54\pi}, 0.78e^{i1.58\pi}))$
Y_4	$(4, (0.95e^{i1.80\pi}, 0.012e^{i0.02\pi}, 0.10e^{0.22i\pi}))$	$(0, (0.03e^{i0.10\pi}, 0.91e^{i1.84\pi}, 0.93e^{1.88i\pi}))$

Definition 18. *Let* (Φ_Ψ, Y, N_1) *and* (χ_A, C, N_2) *be* $CSV\mathcal{N}_1S_fS$ *and* $CSV\mathcal{N}_2S_fS$ *on* $\mathbb{U}$, *respectively, their extended intersection is defined as* $(\mathcal{D}_Q, T, \mathfrak{S}) = (\Phi_\Psi, Y, N_1)\check{\cap}(\chi_A, C, N_2)$, *with* $\mathcal{D}_Q = \Phi_\Psi \check{\cap} \chi_A$, $T = Y \cup C$, $\mathfrak{S} = \max(N_1, N_2)$, *that is,* $\forall x_w \in T$ *and* $u_s \in \mathbb{U}$, *we have*

$$\mathcal{D}_Q(x_w) = \begin{cases} (h^{1s}_w, (\mathbb{T}^1_{ws}, \mathbb{I}^1_{ws}, \mathbb{F}^1_{ws})), & \text{if } x_w \in Y - C, \\ (h^{2s}_w, (\mathbb{T}^2_{ws}, \mathbb{I}^2_{ws}, \mathbb{F}^2_{ws})), & \text{if } x_w \in C - Y, \\ \Big(\min(h^{1s}_w, h^{2s}_w), \min(p^1_{ws}, p^2_{ws}) e^{i2\pi \min(t^1_{ws}, t^2_{ws})}, \max(q^1_{ws}, q^2_{ws}) e^{i2\pi \max(\omega^1_{ws}, \omega^2_{ws})}, \max(r^1_{ws}, r^2_{ws}) e^{i2\pi \max(f^1_{ws}, f^2_{ws})} \Big), & \text{if } x_w \in C \cap Y \end{cases}$$

where $(h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)) = (h_w^{1s}, (p_{ws}^1 e^{i2\pi t_{ws}^1}, q_{ws}^1 e^{i2\pi\omega_{ws}^1}, r_{ws}^1 e^{i2\pi f_{ws}^1})) \in \Phi_\Psi$ *and* $(h_w^{2s}, (\mathbb{T}_{ws}^2, \mathbb{I}_{ws}^2, \mathbb{F}_{ws}^2)) = (h_w^{2s}$ $, (p_{ws}^2 e^{i2\pi t_{ws}^2}, q_{ws}^2 e^{i2\pi\omega_{ws}^2}, r_{ws}^2 e^{i2\pi f_{ws}^2})) \in \chi_A$.

Example 9. *The extended intersection* $(\mathcal{D}_Q, T, 6)$ *of* $(\Phi_\Psi, Y, 6)$ *and* $(\chi_A, C, 5)$, *given in Table 3 and Table 9, arranged in 11.*

Table 11: The extended intersection$(\mathcal{D}_Q, T, \mathfrak{S})$

$(\mathcal{D}_Q, T, \mathfrak{S})$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_3$	$\mathbb{U}_4$
Y_1	$(0, (0.12e^{i0.23\pi}, 0.91e^{i1.84\pi}, 0.96e^{i1.96\pi}))$	$(1, (0.21e^{i0.42\pi}, 0.77e^{i1.50\pi}, 0.82e^{i1.66\pi}))$	$(0, (0.06e^{i0.14\pi}, 0.95e^{i1.92\pi}, 0.97e^{i1.96\pi}))$	$(1, (0.24e^{i0.50\pi}, 0.86e^{i1.70\pi}, 0.87e^{i1.72\pi}))$
Y_2	$(1, (0.17e^{i0.40\pi}, 0.75e^{i1.48\pi}, 0.81e^{i1.66\pi}))$	$(4, (0.81e^{i1.66\pi}, 0.22e^{i0.42\pi}, 0.25e^{i0.48\pi}))$	$(2, (0.36e^{i0.74\pi}, 0.58e^{i1.18\pi}, 0.54e^{i1.10\pi}))$	$(0, (0.08e^{i0.20\pi}, 0.96e^{i1.94\pi}, 0.98e^{1.98\pi}))$
Y_3	$(2, (0.32e^{i0.70\pi}, 0.55e^{i1.12\pi}, 0.52e^{i1.06\pi}))$	$(1, (0.20e^{i0.42\pi}, 0.76e^{i1.54\pi}, 0.78e^{i1.58\pi}))$	$(4, (0.75e^{i1.52\pi}, 0.17e^{i0.36\pi}, 0.20e^{i0.38\pi}))$	$(3, (0.69e^{i1.36\pi}, 0.41e^{i0.84\pi}, 0.48e^{0.94\pi}))$
Y_4	$(4, (0.95e^{i1.80\pi}, 0.012e^{i0.02\pi}, 0.10e^{0.22i\pi}))$	$(0, (0.03e^{i0.10\pi}, 0.91e^{i1.84\pi}, 0.93e^{1.88i\pi}))$	$(1, (0.21e^{i0.46\pi}, 0.79e^{i1.60\pi}, 0.83e^{i1.68\pi}))$	$(2, (0.38e^{i0.78\pi}, 0.59e^{i1.20\pi}, 0.55e^{i1.12\pi}))$
	$\mathbb{U}_5$			
Y_1	$(0, (0.05e^{i0.14\pi}, 0.87e^{i1.72\pi}, 0.88e^{i1.80\pi}))$			
Y_2	$(4, (0.90e^{i1.84\pi}, 0.11e^{i0.20\pi}, 0.14e^{i0.26\pi}))$			
Y_3	$(1, (0.26e^{i0.48\pi}, 0.72e^{i1.42\pi}, 0.75e^{i1.52\pi}))$			
Y_4	$(2, (0.49e^{i1.02\pi}, 0.61e^{i1.24\pi}, 0.59e^{i1.16\pi}))$			

Definition 19. *Let* $\mathbb{U}$ *be a non-empty set and* (Φ_Ψ, Y, N_1) *and* (χ_A, C, N_2) *be* $CSV\mathcal{N}_1S_fS$ *and* $CSV\mathcal{N}_2S_fS$ *on* $\mathbb{U}$, *respectively, their restricted union is defined as* $(\mathbb{L}_\mathbb{M}, \mathfrak{G}, \mathfrak{O}) = (\Phi_\Psi, Y, N_1)\hat{\cup}(\chi_A, C, N_2)$, *with* $\mathbb{L}_\mathbb{M} = \Phi_\Psi\hat{\cup}\chi_A$, $\mathfrak{G} = Y \cap C$, $\mathfrak{O} = \max(N_1, N_2)$, *i.e.,* $\forall x_w \in \mathfrak{G}$, $u_s \in \mathbb{U}$ *we have*

$$\begin{aligned}\mathbb{L}_\mathbb{M}(x_w) &= \langle (h_w^s, (\mathbb{T}_{ws}, \mathbb{I}_{ws}, \mathbb{F}_{ws}))\rangle, \\ &= \langle (\min(h_w^{1s}, h_w^{2s}), \min(\mathbb{T}_{ws}^1, \mathbb{T}_{ws}^2), \max(\mathbb{I}_{ws}^1, \mathbb{I}_{ws}^2), \max(\mathbb{F}_{ws}^1, \mathbb{F}_{ws}^2))\rangle, \\ &= \langle (\max(h_w^{1s}, h_w^{2s}), \max(p_{ws}^1, p_{ws}^2)e^{i2\pi\max(t_{ws}^1, t_{ws}^2)}, \min(q_{ws}^1, q_{ws}^2)e^{i2\pi\min(\omega_{ws}^1, \omega_{ws}^2)}, \min(r_{ws}^1, r_{ws}^2)e^{i2\pi\min(f_{ws}^1, f_{ws}^2)})\rangle,\end{aligned}$$

where $(h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)) = (h_w^{1s}, (p_{ws}^1 e^{i2\pi t_{ws}^1}, q_{ws}^1 e^{i2\pi\omega_{ws}^1}, r_{ws}^1 e^{i2\pi f_{ws}^1})) \in \Phi_\Psi$ *and* $(h_w^{2s}, (\mathbb{T}_{ws}^2, \mathbb{I}_{ws}^2, \mathbb{F}_{ws}^2))$ $= (h_w^{2s}, (p_{ws}^2 e^{i2\pi t_{ws}^2}, q_{ws}^2 e^{i2\pi\omega_{ws}^2}, r_{ws}^2 e^{i2\pi f_{ws}^2})) \in \chi_A$.

Example 10. *The restricted union* (L_M, G, O) *of* $(\Phi_\Psi, Y, 6)$ *and* $(\chi_A, C, 5)$, *given in Table 3 and Table 9, arranged in 12.*

Table 12: Restricted union $(\mathbb{L}_\mathbb{M}, \mathfrak{G}, \mathfrak{O})$

$(\mathbb{L}_\mathbb{M}, \mathfrak{G}, \mathfrak{O})$	$\mathbb{U}_1$	$\mathbb{U}_2$
Y_1	$(3, (0.60e^{i1.26\pi}, 0.35e^{i0.68\pi}, 0.40e^{i0.84\pi}))$	$(5, (0.95e^{i1.92\pi}, 0.05e^{i0.12\pi}, 0.12e^{i0.26\pi}))$
Y_2	$(2, (0.42e^{i0.82\pi}, 0.51e^{i1.04\pi}, 0.56e^{i1.10\pi}))$	$(4, (0.88e^{i1.78\pi}, 0.09e^{i0.16\pi}, 0.06e^{i0.10\pi}))$
Y_3	$(3, (0.81e^{i1.64\pi}, 0.17e^{i0.36\pi}, 0.19e^{i0.40\pi}))$	$(3, (0.83e^{i1.68\pi}, 0.27e^{i0.56\pi}, 0.30e^{i0.58\pi}))$
Y_4	$(5, (0.98e^{i1.94\pi}, 0.01e^{i0.04\pi}, 0.10e^{0.24i\pi}))$	$(3, (0.70e^{i1.42\pi}, 0.26e^{i0.54\pi}, 0.31e^{0.64i\pi}))$

Definition 20. *Let* (Φ_Ψ, Y, N_1) *and* (χ_A, C, N_2) *be* $CSV\mathcal{N}_1S_fS$ *and* $CSV\mathcal{N}_2S_fS$ *on* $\mathbb{U}$, *respectively, their extended union is defined as* $(\mathcal{P}_\mathcal{Q}, \mathcal{T}, \mathfrak{B}) = (\Phi_\Psi, Y, N_1)\breve{\cup}(\chi_A, C, N_2)$, *with* $\mathcal{P}_\mathcal{Q} = \Phi_\Psi\breve{\cup}\chi_A$, $\mathcal{T} = Y \cup C$, $\mathfrak{B} = \max(N_1, N_2)$, *that is,* $\forall x_w \in T$ *and* $u_s \in \mathbb{U}$, *we have*

$$\mathcal{P}_\mathcal{Q}(x_w) = \begin{cases} (h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)), & \text{if } x_w \in Y - C, \\ (h_w^{2s}, (\mathbb{T}_{ws}^2, \mathbb{I}_{ws}^2, \mathbb{F}_{ws}^2)), & \text{if } x_w \in C - Y, \\ \Big(\max(h_w^{1s}, h_w^{2s}), \max(p_{ws}^1, p_{ws}^2)e^{i2\pi\max(t_{ws}^1, t_{ws}^2)}, \min(q_{ws}^1, q_{ws}^2)e^{i2\pi\min(\omega_{ws}^1, \omega_{ws}^2)}, \min(r_{ws}^1, r_{ws}^2)e^{i2\pi\min(f_{ws}^1, f_{ws}^2)}\Big), & \text{if } x_w \in C \cap Y \end{cases}$$

where $(h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)) = (h_w^{1s}, (p_{ws}^1 e^{i2\pi t_{ws}^1}, q_{ws}^1 e^{i2\pi\omega_{ws}^1}, r_{ws}^1 e^{i2\pi f_{ws}^1})) \in \Phi_\Psi$ *and* $(h_w^{2s}, (\mathbb{T}_{ws}^2, \mathbb{I}_{ws}^2, \mathbb{F}_{ws}^2)) = (h_w^{2s}$ $, (p_{ws}^2 e^{i2\pi t_{ws}^2}, q_{ws}^2 e^{i2\pi\omega_{ws}^2}, r_{ws}^2 e^{i2\pi f_{ws}^2})) \in \chi_A$.

Example 11. *The extended union* (L_M, G, O) *of* $(\Phi_\Psi, Y, 6)$ *and* $(\chi_A, C, 5)$, *given in Table 3 and Table 9, arranged in 13.*

Table 13: Extended union $(\mathcal{P}_{\mathcal{Q}}, \mathcal{T}, \mathfrak{B})$

$(\mathcal{P}_{\mathcal{Q}}, \mathcal{T}, \mathfrak{B})$	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_3$	$\mathbb{U}_4$
Y_1	$(3, (0.60e^{i1.26\pi}, 0.35e^{i0.68\pi}, 0.40e^{i0.84\pi}))$	$(5, (0.95e^{i1.92\pi}, 0.05e^{i0.12\pi}, 0.12e^{i0.26\pi}))$	$(0, (0.06e^{i0.14\pi}, 0.95e^{i1.92\pi}, 0.97e^{i1.96\pi}))$	$(1, (0.24e^{i0.50\pi}, 0.86e^{i1.70\pi}, 0.87e^{i1.72\pi}))$
Y_2	$(2, (0.42e^{i0.82\pi}, 0.51e^{i1.04\pi}, 0.56e^{i1.10\pi}))$	$(4, (0.88e^{i1.78\pi}, 0.09e^{i0.16\pi}, 0.06e^{i0.10\pi}))$	$(2, (0.36e^{i0.74\pi}, 0.58e^{i1.18\pi}, 0.54e^{i1.10\pi}))$	$(0, (0.08e^{i0.20\pi}, 0.96e^{i1.94\pi}, 0.98e^{1.98\pi}))$
Y_3	$(3, (0.81e^{i1.64\pi}, 0.17e^{i0.36\pi}, 0.19e^{i0.40\pi}))$	$(3, (0.83e^{i1.68\pi}, 0.27e^{i0.56\pi}, 0.30e^{i0.58\pi}))$	$(4, (0.75e^{i1.52\pi}, 0.17e^{i0.36\pi}, 0.20e^{i0.38\pi}))$	$(3, (0.69e^{i1.36\pi}, 0.41e^{i0.84\pi}, 0.48e^{0.94\pi}))$
Y_4	$(5, (0.98e^{i1.94\pi}, 0.01e^{i0.04\pi}, 0.10e^{0.24i\pi}))$	$(3, (0.70e^{i1.42\pi}, 0.26e^{i0.54\pi}, 0.31e^{0.64i\pi}))$	$(1, (0.21e^{i0.46\pi}, 0.79e^{i1.60\pi}, 0.83e^{i1.68\pi}))$	$(2, (0.38e^{i0.78\pi}, 0.59e^{i1.20\pi}, 0.55e^{i1.12\pi}))$
	$\mathbb{U}_5$			
Y_1	$(0, (0.05e^{i0.14\pi}, 0.87e^{i1.72\pi}, 0.88e^{i1.80\pi}))$			
Y_2	$(4, (0.90e^{i1.84\pi}, 0.11e^{i0.20\pi}, 0.14e^{i0.26\pi}))$			
Y_3	$(1, (0.26e^{i0.48\pi}, 0.72e^{i1.42\pi}, 0.75e^{i1.52\pi}))$			
Y_4	$(2, (0.49e^{i1.02\pi}, 0.61e^{i1.24\pi}, 0.59e^{i1.16\pi}))$			

Now we discuss some properties and their proofs.

Theorem 21. *Let (Φ_Ψ, Y, N_1) be a $CSV\mathcal{N}NS_fS$ over a non-empty set $\mathbb{U}$. Then,*

1 $(\Phi_\Psi, Y, N_1)\check{\cap}(\Phi_\Psi, Y, N_1) = (\Phi_\Psi, Y, N_1)$

2 $(\Phi_\Psi, Y, N_1)\hat{\cap}(\Phi_\Psi, Y, N_1) = (\Phi_\Psi, Y, N_1)$

3 $(\Phi_\Psi, Y, N_1)\check{\cup}(\Phi_\Psi, Y, N_1) = (\Phi_\Psi, Y, N_1)$

4 $(\Phi_\Psi, Y, N_1)\hat{\cup}(\Phi_\Psi, Y, N_1) = (\Phi_\Psi, Y, N_1)$

Proof. 1.

$$R.H.S = (\Phi_\Psi, Y, N_1)\check{\cap}(\Phi_\Psi, Y, N_1), \tag{1}$$

where the extended intersection of two $CSV\mathcal{N}NS_fSs$ is calculated as:

$$(\mathcal{D}_Q, T, \mathfrak{S}) = (\Phi_\Psi, Y, N_1)\check{\cap}(\Phi_\Psi, Y, N_1), \tag{2}$$

with $T = Y \cup Y$, $\mathfrak{S} = \max(N_1, N_1)$ and

$$\mathcal{D}_Q(x_w) = \begin{cases} (h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)), & \text{if } x_w \in Y - Y, \\ (h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)), & \text{if } x_w \in Y - Y, \\ (\min(h_w^{1s}, h_w^{1s}), (\min(\mathbb{T}_{ws}^1, \mathbb{T}_{ws}^1), \max(\mathbb{I}_{ws}^1, \mathbb{I}_{ws}^1), \max(\mathbb{F}_{ws}^1, \mathbb{F}_{ws}^1))), & \text{if } x_w \in Y \cap Y. \end{cases}$$

Case 1 : If $x_w \in Y - Y = \emptyset$,

$$\mathcal{D}_Q(x_w) = \Phi_\Psi(x_w). \tag{3}$$

Case 2 : If $x_w \in Y - Y = \emptyset$,

$$\mathcal{D}_Q(x_w) = \Phi_\Psi(x_w). \tag{4}$$

Case 3 : If $x_w \in Y \cap Y = Y$,

$$\begin{aligned} \mathcal{D}_Q(x_w) &= (\min(h_w^{1s}, h_w^{1s}), (\min(\mathbb{T}_{ws}^1, \mathbb{T}_{ws}^1), \max(\mathbb{I}_{ws}^1, \mathbb{I}_{ws}^1), \max(\mathbb{F}_{ws}^1, \mathbb{F}_{ws}^1))), \\ &= (h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)), \\ &= \Phi_\Psi(x_w). \end{aligned} \tag{5}$$

From Equations 2, 3, 4 and 5, $(\mathcal{D}_Q, T, \mathfrak{S}) = (\Phi_\Psi, Y, N_1)$ and further Eq.1 implies $(\Phi_\Psi, Y, N_1)\check{\cap}(\Phi_\Psi, Y, N_1) = (\Phi_\Psi, Y, N_1)$.

2.

$$R.H.S = (\Phi_\Psi, Y, N_1)\hat{\cap}(\Phi_\Psi, Y, N_1), \tag{6}$$

where the restricted intersection of two $CSV\mathcal{N}NS_fSs$ is calculated as:

$$(L_M, G, O) = (\Phi_\Psi, Y, N_1)\hat{\cap}(\Phi_\Psi, Y, N_1), \tag{7}$$

with $G = Y \cap Y = Y$, $O = \min(N_1, N_1) = N_1$ and

$$\begin{aligned} L_M(x_w) &= (\min(h_w^{1s}, h_w^{1s}), (\min(\mathbb{T}_{ws}^1, \mathbb{T}_{ws}^1), \max(\mathbb{I}_{ws}^1, \mathbb{I}_{ws}^1), \max(\mathbb{F}_{ws}^1, \mathbb{F}_{ws}^1))), \\ &= (h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)), \\ &= \Phi_\Psi(x_w), \end{aligned} \tag{8}$$

clearly, from Equations 6, 7 and 8, we get the required result.

3.

$$R.H.S = (\Phi_\Psi, Y, N_1)\check{\cup}(\Phi_\Psi, Y, N_1), \tag{9}$$

where the extended union of two $CSV\mathcal{N}NS_fSs$ is calculated as:

$$(\mathcal{P}_{\mathcal{Q}}, \mathcal{T}, \mathfrak{B}) = (\Phi_\Psi, Y, N_1)\check{\cup}(\Phi_\Psi, Y, N_1), \tag{10}$$

with $\mathcal{T} = Y \cup Y$, $\mathfrak{B} = \max(N_1, N_1)$ and

$$\mathcal{P}_{\mathcal{Q}}(x_w) = \begin{cases} (h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)), & \text{if } x_w \in Y - Y, \\ (h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)), & \text{if } x_w \in Y - Y, \\ (\max(h_w^{1s}, h_w^{1s}), (\max(\mathbb{T}_{ws}^1, \mathbb{T}_{ws}^1), \min(\mathbb{I}_{ws}^1, \mathbb{I}_{ws}^1), \min(\mathbb{F}_{ws}^1, \mathbb{F}_{ws}^1))), & \text{if } x_w \in Y \cap Y. \end{cases}$$

Case 1 : If $x_w \in Y - Y = \emptyset$,

$$\mathcal{P}_{\mathcal{Q}}(x_w) = \Phi_\Psi(x_w). \tag{11}$$

Case 2 : If $x_w \in Y - Y = \emptyset$,

$$\mathcal{P}_{\mathcal{Q}}(x_w) = \Phi_\Psi(x_w). \tag{12}$$

Case 3 : If $x_w \in Y \cap Y = Y$,

$$\begin{aligned} \mathcal{P}_{\mathcal{Q}}(x_w) &= (\max(h_w^{1s}, h_w^{1s}), (\max(\mathbb{T}_{ws}^1, \mathbb{T}_{ws}^1), \min(\mathbb{I}_{ws}^1, \mathbb{I}_{ws}^1), \min(\mathbb{F}_{ws}^1, \mathbb{F}_{ws}^1))), \\ &= (h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)), \\ &= \Phi_\Psi(x_w). \end{aligned} \tag{13}$$

From Equations 9, 10, 11, 12 and 13, we get $(\Phi_\Psi, Y, N_1)\check{\cup}(\Phi_\Psi, Y, N_1) = (\Phi_\Psi, Y, N_1)$.

4.
$$R.H.S = (\Phi_\Psi, Y, N_1)\hat{\cup}(\Phi_\Psi, Y, N_1), \tag{14}$$

where the restricted union of two $CSV\mathcal{N}NS_fSs$ is calculated as:

$$(\mathbb{L}_{\mathbb{M}}, \mathfrak{G}, \mathfrak{D}) = (\Phi_\Psi, Y, N_1)\hat{\cup}(\Phi_\Psi, Y, N_1), \tag{15}$$

with $\mathfrak{G} = Y \cap Y = Y$, $\mathfrak{D} = \max(N_1, N_1) = N_1$ and

$$\begin{aligned} \mathbb{L}_{\mathbb{M}}(x_w) &= (\max(h_w^{1s}, h_w^{1s}), (\max(\mathbb{T}_{ws}^1, \mathbb{T}_{ws}^1), \min(\mathbb{I}_{ws}^1, \mathbb{I}_{ws}^1), \min(\mathbb{F}_{ws}^1, \mathbb{F}_{ws}^1))), \\ &= (h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)), \\ &= \Phi_\Psi(x_w), \end{aligned} \tag{16}$$

clearly, from Equations 14, 15 and 16, we get the required result.

□

Theorem 22. *Let (Φ_Ψ, Y, N_1) and (χ_A, C, N_2) be $CSV\mathcal{N}N_1S_fS$ and $CSV\mathcal{N}N_2S_fS$, respectively, over the same universe $\mathbb{U}$, then the absorption properties hold:*

1. $((\Phi_\Psi, Y, N_1)\check{\cup}(\chi_A, C, N_2))\hat{\cap}(\Phi_\Psi, E, N_1) = (\Phi_\Psi, Y, N_1)$
2. $(\Phi_\Psi, Y, N_1)\check{\cup}((\chi_A, C, N_2)\hat{\cap}(\Phi_\Psi, E, N_1)) = (\Phi_\Psi, Y, N_1)$
3. $((\Phi_\Psi, Y, N_1)\hat{\cap}(\chi_A, C, N_2))\check{\cup}(\Phi_\Psi, E, N_1) = (\Phi_\Psi, Y, N_1)$
4. $(\Phi_\Psi, Y, N_1)\hat{\cap}((\chi_A, C, N_2)\check{\cup}(\Phi_\Psi, E, N_1)) = (\Phi_\Psi, Y, N_1)$

Proof. 1. Let the extended union of $CSV\mathcal{N}N_1S_fS$ (Φ_Ψ, Y, N_1) and $CSV\mathcal{N}N_2S_fS$ (χ_A, C, N_2), be

$$(\mathcal{P}_{\mathcal{Q}}, \mathcal{T}, \mathfrak{B}) = (\Phi_\Psi, Y, N_1)\check{\cup}(\chi_A, C, N_2),$$

with $\mathcal{T} = Y \cup C$, $\mathfrak{B} = \max(N_1, N_2)$ and
$\mathcal{P}_{\mathcal{Q}}(x_w) = (h_w^s, (\mathbb{T}_{ws}, \mathbb{I}_{ws}, \mathbb{F}_{ws})) =$

$$\begin{cases} (h_w^{1s}, (\mathbb{T}_{ws}^1, \mathbb{I}_{ws}^1, \mathbb{F}_{ws}^1)), & \text{if } x_w \in Y - C, \\ (h_w^{2s}, (\mathbb{T}_{ws}^2, \mathbb{I}_{ws}^2, \mathbb{F}_{ws}^2)), & \text{if } x_w \in C - Y, \\ (\max(h_w^{1s}, h_w^{2s}), (\max(\mathbb{T}_{ws}^1, \mathbb{T}_{ws}^2), \min(\mathbb{I}_{ws}^1, \mathbb{I}_{ws}^2), \min(\mathbb{F}_{ws}^1, \mathbb{F}_{ws}^2))), & \text{if } x_w \in Y \cap C. \end{cases} \tag{17}$$

Now, consider the restricted intersection of $(\mathcal{P}_{\mathcal{Q}}, \mathcal{T}, \mathfrak{B})$ and (Φ_Ψ, Y, N_1), that is defined as

$$(L_M, G, O) = (\mathcal{P}_{\mathcal{Q}}, \mathcal{T}, \mathfrak{B})\hat{\cap}(\Phi_\Psi, Y, N_1),$$

with $G = \mathcal{T} \cap Y$, $O = \min(\mathfrak{B}, N_1) = N_1$ and

$$L_M(x_w) = (\min(h_w^s, h_w^{1s}), (\min(\mathbb{T}_{ws}, \mathbb{T}_{ws}^1), \max(\mathbb{I}_{ws}, \mathbb{I}_{ws}^1), \max(\mathbb{F}_{ws}, \mathbb{F}_{ws}^1))), \tag{18}$$

for all $x_w \in G = Y \cap C$, so that $x_w \in W$, $x_w \in C$. If $x_w \in W$, then there are three cases.

Case 1: if $x_w \in Y - C$, using Equations 17 and 18, we get,

$$\begin{aligned} L_M(x_w) &= (\min(h_w^{1s}, h_w^{1s}), (\min(\mathbb{T}_{ws}^1, \mathbb{T}_{ws}^1), \max(\mathbb{I}_{ws}^1, \mathbb{I}_{ws}^1), \max(\mathbb{F}_{ws}^1, \mathbb{F}_{ws}^1))) \\ &= (h_w^{1s}, \mathbb{T}_{ws}^1, \mathbb{F}_{ws}^1) \\ &= \Phi_\Psi(x_w) \end{aligned} \tag{19}$$

Case 2: if $x_w \in C - Y$, since $x_w \in G = Y \cap C$ implies $x_w \in Y$, therefore, this case is omitted.

Case 3: if $x_w \in C \cap Y$, using Equations 17 and 18, we get,

$$\begin{aligned} L_M(x_w) &= (\min(\max(h_w^{1s}, h_w^{2s}), h_w^{1s}), (\min(\max(\mathbb{T}_{ws}^1, \mathbb{T}_{ws}^2), \mathbb{T}_{ws}^1), \max(\min(\mathbb{I}_{ws}^1, \mathbb{T}_{ws}^2), \mathbb{I}_{ws}^1), \max(\min(\mathbb{F}_{ws}^1, \mathbb{T}_{ws}^2), \mathbb{F}_{ws}^1))) \\ &= (h_w^{1s}, \mathbb{T}_{ws}^1, \mathbb{F}_{ws}^1) \\ &= \Phi_\Psi(x_w) \end{aligned} \tag{20}$$

Thus from Equations 19 and 20, we get $((\Phi_\Psi, E, N_1)\breve{\cup}(\chi_A, C, N_2))\hat{\cap}(\Phi_\Psi, E, N_1) = (\Phi_\Psi, E, N_1)$.

2. proofs of 2, 3 and 4 are same as above.

□

Theorem 23. *Let (Φ_Ψ, Y, N_1), (χ_A, C, N_2) and $(\Upsilon_\kappa, \varrho, N_3)$ be any three $CSV\mathcal{N}N_1S_fS$, $CSV\mathcal{N}N_2S_fS$, and $CSV\mathcal{N}N_3S_fS$, and over the same universe $\mathbb{U}$, then the following properties hold:*

1 $(\Phi_\Psi, Y, N_1)\breve{\cup}(\chi_A, C, N_2) = (\chi_A, C, N_2)\breve{\cup}(\Phi_\Psi, Y, N_1)$,
2 $(\Phi_\Psi, Y, N_1)\hat{\cup}(\chi_A, C, N_2) = (\chi_A, C, N_2)\hat{\cup}(\Phi_\Psi, Y, N_1)$,
3 $(\Phi_\Psi, Y, N_1)\breve{\cap}(\chi_A, C, N_2) = (\chi_A, C, N_2)\breve{\cap}(\Phi_\Psi, Y, N_1)$,
4 $(\Phi_\Psi, Y, N_1)\hat{\cap}(\chi_A, C, N_2) = (\chi_A, C, N_2)\hat{\cap}(\Phi_\Psi, Y, N_1)$,
5 $((\Phi_\Psi, Y, N_1)\breve{\cup}(\chi_A, C, N_2))\breve{\cup}(\Upsilon_\kappa, \varrho, N_3) = (\Phi_\Psi, Y, N_1)\breve{\cup}(((\chi_A, C, N_2))\breve{\cup}(\Upsilon_\kappa, \varrho, N_3))$,
6 $((\Phi_\Psi, Y, N_1)\hat{\cup}(\chi_A, C, N_2))\hat{\cup}(\Upsilon_\kappa, \varrho, N_3) = (\Phi_\Psi, Y, N_1)\hat{\cup}(((\chi_A, C, N_2))\hat{\cup}(\Upsilon_\kappa, \varrho, N_3))$,
7 $((\Phi_\Psi, Y, N_1)\breve{\cap}(\chi_A, C, N_2))\breve{\cap}(\Upsilon_\kappa, \varrho, N_3) = (\Phi_\Psi, Y, N_1)\breve{\cap}(((\chi_A, C, N_2))\breve{\cap}(\Upsilon_\kappa, \varrho, N_3))$,
8 $((\Phi_\Psi, Y, N_1)\hat{\cap}(\chi_A, C, N_2))\hat{\cap}(\Upsilon_\kappa, \varrho, N_3) = (\Phi_\Psi, Y, N_1)\hat{\cap}(((\chi_A, C, N_2))\hat{\cap}(\Upsilon_\kappa, \varrho, N_3))$,
9 $(\Phi_\Psi, Y, N_1)\breve{\cup}((\chi_A, C, N_2)\hat{\cap}(\Upsilon_\kappa, \varrho, N_3)) = ((\Phi_\Psi, Y, N_1)\breve{\cup}(\chi_A, C, N_2))\hat{\cap}((\Phi_\Psi, Y, N_1)\breve{\cup}(\Upsilon_\kappa, \varrho, N_3))$,
10 $(\Phi_\Psi, Y, N_1)\breve{\cap}((\chi_A, C, N_2)\hat{\cup}(\Upsilon_\kappa, \varrho, N_3)) = ((\Phi_\Psi, Y, N_1)\breve{\cap}(\chi_A, C, N_2))\hat{\cup}((\Phi_\Psi, Y, N_1)\breve{\cup}(\Upsilon_\kappa, \varrho, N_3))$,
11 $(\Phi_\Psi, Y, N_1)\hat{\cup}((\chi_A, C, N_2)\breve{\cap}(\Upsilon_\kappa, \varrho, N_3)) = ((\Phi_\Psi, Y, N_1)\hat{\cup}(\chi_A, C, N_2))\breve{\cap}((\Phi_\Psi, Y, N_1)\hat{\cup}(\Upsilon_\kappa, \varrho, N_3))$,
12 $(\Phi_\Psi, Y, N_1)\hat{\cap}((\chi_A, C, N_2)\breve{\cup}(\Upsilon_\kappa, \varrho, N_3)) = ((\Phi_\Psi, Y, N_1)\hat{\cap}(\chi_A, C, N_2))\breve{\cup}((\Phi_\Psi, Y, N_1)\hat{\cap}(\Upsilon_\kappa, \varrho, N_3))$.

4 Complex single-valued neutrosophic N-soft number

Definition 24. *Let $\Phi_\Psi(y_w) = ((u_s, h_w^s), (p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}}))$ be a $CSV\mathcal{N}NS_fS$. Then the complex single-valued neutrosophic N-soft number $(CSV\mathcal{N}NS_fN)$ is defined as:*

$$\alpha_{ws} = (h_w^s, p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}}),$$

Definition 25. *Consider a $CCSV\mathcal{N}NS_fN$ $\alpha_{ws} = (h_w^s, p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}})$. The score function $S(\alpha_{ws})$ is:*

$$S(\alpha_{ws}) = \frac{h_w^s}{N-1} + (p_{ws} - q_{ws} - r_{ws}) + [t_{ws} - \omega_{ws} - f_{ws}], \tag{21}$$

where $S(\alpha_{ws}) \in [-4, 3]$. The accuracy function $A(\alpha_{ws})$ is:

$$A(\alpha_{ws}) = \frac{h_w^s}{N-1} + (p_{ws} + q_{ws} + r_{ws}) + [t_{ws} + \omega_{ws} + f_{ws}] \tag{22}$$

where $A(\alpha_{ws}) \in [0, 7]$, respectively.

Definition 26. *Let $\alpha_{ws} = (h_w^s, p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}})$ and $\alpha_{ls} = (h_l^s, p_{ls}e^{i2\pi t_{ls}}, q_{ls}e^{i2\pi\omega_{ls}}, r_{ls}e^{i2\pi f_{ls}})$ be two $CSV\mathcal{N}NS_fNs$*

1. *If* $S_{\alpha_{ws}} < S_{\alpha_{ls}}$*, then* $\alpha_{ws} \prec \alpha_{ls}$ *(*α_{ws} *is inferior to* α_{ls}*),*
2. *If* $S_{\alpha_{ws}} = S_{\alpha_{ls}}$*, then*

   i $A_{\alpha_{ws}} < A_{\alpha_{ls}}$*, then* $\alpha_{ws} \prec \alpha_{ls}$ *(*α_{ws} *is inferior to* α_{ls}*),*

   ii $A_{\alpha_{ws}} = A_{\alpha_{ls}}$*, then* $\alpha_{ws} \sim \alpha_{ls}$ *(*α_{ws} *is equivalent to* α_{ls}*).*

Definition 27. *Let* $\alpha_{ws} = (h^s_w, p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}})$ *and* $\alpha_{ls} = (h^s_l, p_{ls}e^{i2\pi t_{ls}}, q_{ls}e^{i2\pi\omega_{ls}}, r_{ls}e^{i2\pi f_{ls}})$ *be two* $CSV\mathcal{N}NS_fNs$ *and* $\beta > 0$*. Some operation for* $CSV\mathcal{N}NS_fNs$ *are*

$$\beta\alpha_{ws} = \Big(h^s_w, [1-(1-p_{ws})^\beta]e^{i2\pi[1-(1-t_{ws})^\beta]}, q^\beta_{ws}e^{i2\pi\omega^\beta_{ws}}, r^\beta_{ws}e^{i2\pi f^\beta_{ws}}\Big),$$
$$\alpha^\beta_{ws} = \Big(h^s_w, p^\beta_{ws}e^{i2\pi t^\beta_{ws}}, [1-(1-q_{ws})^\beta]e^{i2\pi[1-(1-\omega_{ws})^\beta]}, [1-(1-r_{ws})^\beta]e^{i2\pi[1-(1-f_{ws})^\beta]}\Big),$$
$$\alpha_{ws}\oplus\alpha_{ls} = \Big(\max(h^s_w,h^s_l), (p_{ws}+p_{ls}-p_{ws}p_{ls})e^{i2\pi(t_{ws}+t_{ls}-t_{ws}t_{ls})}, (q_{ws}q_{ls})e^{i2\pi(\omega_{ws}\omega_{ls})}, (r_{ws}r_{ls})e^{i2\pi(f_{ws}f_{ls})}\Big),$$
$$\alpha_{ws}\otimes\alpha_{ls} = \Big(\min(h^s_w,h^s_l), (p_{ws}p_{ls})e^{i2\pi(t_{ws}t_{ls})}, (q_{ws}+q_{ls}-q_{ws}q_{ls})e^{i2\pi(\omega_{ws}+\omega_{ls}-\omega_{ws}\omega_{ls})}, (r_{ws}+r_{ls}-r_{ws}r_{ls})e^{i2\pi(f_{ws}+f_{ls}-f_{ws}f_{ls})}\Big).$$

Definition 28. *Let* $\alpha_{ws} = (h^s_w, p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}})$ *and* $\alpha_{ls} = (h^s_l, p_{ls}e^{i2\pi t_{ls}}, q_{ls}e^{i2\pi\omega_{ls}}, r_{ls}e^{i2\pi f_{ls}})$ *be two* $CSV\mathcal{N}NS_fNs$ *and* $\beta > 0$*, then the following properties hold:*

1. $\alpha_{ws}\oplus\alpha_{ls} = \alpha_{ls}\oplus\alpha_{ws}$,
2. $\alpha_{ws}\otimes\alpha_{ls} = \alpha_{ls}\otimes\alpha_{ws}$,
3. $\beta\alpha_{ws}\oplus\beta\alpha_{ls} = \beta(\alpha_{ls}\oplus\alpha_{ws}), \beta > 0$,
4. $\beta_1\alpha_{ws}\oplus\beta_1\alpha_{ws} = (\beta_1+\beta_2)\alpha_{ws}, \beta_1, \beta_2 > 0$,
5. $\alpha^\beta_{ws}\otimes\alpha^\beta_{ls} = (\alpha_{ls}\otimes\alpha_{ws})^\beta, \beta > 0$,
6. $\alpha^{\beta_1}_{ws}\otimes\alpha^{\beta_1}_{ws} = \alpha^{(\beta_1+\beta_2)}_{ws}. \beta_1, \beta_2 > 0.$

Proof. 1.

$$\begin{aligned}\alpha_{ws}\oplus\alpha_{ls} &= \Big(\max(h^s_w,h^s_l), (p_{ws}+p_{ls}-p_{ws}p_{ls})e^{i2\pi(t_{ws}+t_{ls}-t_{ws}t_{ls})}, (q_{ws}q_{ls})e^{i2\pi(\omega_{ws}\omega_{ls})}, (r_{ws}r_{ls})e^{i2\pi(f_{ws}f_{ls})}\Big),\\ &= \Big(\max(h^s_l,h^s_w), (p_{ls}+p_{ws}-p_{ls}p_{ws})e^{i2\pi(t_{ls}+t_{ws}-t_{ls}t_{ws})}, (q_{ls}q_{ws})e^{i2\pi(\omega_{ls}\omega_{ws})}, (r_{ls}r_{ws})e^{i2\pi(f_{ls}f_{ws})}\Big),\\ &= \alpha_{ls}\oplus\alpha_{ws}.\end{aligned}$$

2.

$$\begin{aligned}\alpha_{ws}\otimes\alpha_{ls} &= \Big(\min(h^s_w,h^s_l), (p_{ws}p_{ls})e^{i2\pi(t_{ws}t_{ls})}, (q_{ws}+q_{ls}-q_{ws}q_{ls})e^{i2\pi(\omega_{ws}+\omega_{ls}-\omega_{ws}\omega_{ls})}, (r_{ws}+r_{ls}-r_{ws}r_{ls})e^{i2\pi(f_{ws}+f_{ls}-f_{ws}f_{ls})}\Big)\\ &= \Big(\min(h^s_l,h^s_w), (p_{ls}p_{ws})e^{i2\pi(t_{ls}t_{ws})}, (q_{ls}+q_{ws}-q_{ls}q_{ws})e^{i2\pi(\omega_{ls}+\omega_{ws}-\omega_{ls}\omega_{ws})}, (r_{ls}+r_{ws}-r_{ls}r_{ws})e^{i2\pi(f_{ls}+f_{ws}-f_{ls}f_{ws})}\Big)\\ &= \alpha_{ls}\otimes\alpha_{ws}.\end{aligned}$$

3.

$$\begin{aligned}\beta\alpha_{ws}\oplus\beta\alpha_{ls} &= \Big(h^s_w, [1-(1-p_{ws})^\beta]e^{i2\pi[1-(1-t_{ws})^\beta]}, q^\beta_{ws}e^{i2\pi\omega^\beta_{ws}}, r^\beta_{ws}e^{i2\pi f^\beta_{ws}}) \oplus (h^s_l, [1-(1-p_{ls})^\beta]e^{i2\pi[1-(1-t_{ls})^\beta]},\\ &\quad q^\beta_{ls}e^{i2\pi\omega^\beta_{ls}}, r^\beta_{ls}e^{i2\pi f^\beta_{ls}}\Big)\\ &= \Big(\max(h^s_w,h^s_l), ([1-(1-p_{ws})^\beta]+[1-(1-p_{ls})^\beta]-[1-(1-p_{ws})^\beta][1-(1-p_{ls})^\beta])\\ &\quad e^{i2\pi([1-(1-t_{ws})^\beta]+[1-(1-t_{ls})^\beta]-[1-(1-t_{ws})^\beta][1-(1-t_{ls})^\beta])}, (q^\beta_{ws}q^\beta_{ls})e^{i2\pi(\omega^\beta_{ws}\omega^\beta_{ls})}, (r^\beta_{ws}r^\beta_{ls})e^{i2\pi(q^\beta_{ws}q^\beta_{ls})}\Big)\\ &= \Big(\max(h^s_w,h^s_l), [1-(1-p_{ws}+p_{ls}-p_{ws}p_{ls})^\beta]e^{i2\pi[1-(1-t_{ws}+t_{ls}-t_{ws}t_{ls})^\beta]}, (q_{ws}q_{ls})^\beta e^{i2\pi(\omega_{ws}\omega_{ls})^\beta},\\ &\quad (r_{ws}r_{ls})^\beta e^{i2\pi(f_{ws}f_{ls})^\beta}\Big)\\ &= \beta\Big(\max(h^s_w,h^s_l), (p_{ws}+p_{ls}-p_{ws}p_{ls})e^{2\pi(p_{ws}+p_{ls}-p_{ws}p_{ls})}, (q_{ws}q_{ls})e^{2\pi(\omega_{ws}\omega_{ls})}, (r_{ws}r_{ls})e^{2\pi(f_{ws}f_{ls})}\Big)\\ &= \beta\left(\alpha_{ws}\oplus\alpha_{ls}\right).\end{aligned}$$

Similarly, we can prove 4, 5 and 6.

□

Definition 29. *Let* $\alpha_{ws} = (h^s_w, p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}})$ $(w = 1, 2, \ldots, k)$ *be a collection of* $CSV\mathcal{N}NS_fNs$ *and* ν_w *be the weight vectors of* α_{ws} *with* $\nu_w > 0$ *and* $\sum_{w=1}^{k} \nu_w = 1$. *The complex single-valued neutrosophic* N*-soft weighted average operator* $(CSV\mathcal{N}NS_fWA)$ *is a mapping* $CSV\mathcal{N}NS_fWA : \mathcal{J}^k \to \mathcal{J}$, *where* $\mathcal{J}$ *is the set of* $CSV\mathcal{N}NS_fNs$, *defined as follows:*

$$CSV\mathcal{N}NS_fWA(\alpha_{1s}, \alpha_{2s}, \ldots, \alpha_{ls}) = (\nu_1\alpha_{1s} \oplus \nu_2\alpha_{2s} \oplus \ldots \oplus \nu_k\alpha_{ks})$$
$$= \Big(\max_{w=1}^{k}(h^s_w), [1 - \Pi^k_{w=1}(1 - p_{ws})^{\nu_w}]e^{i2\pi[1-\,\Pi^k_{w=1}(1\,-\,t_{ws})^{\nu_w}]}, [\Pi^k_{w=1}(q_{ws})^{\nu_w}]e^{i2\pi[\Pi^k_{w=1}(\omega_{ws})^{\nu_w}]}, [\Pi^k_{w=1}(r_{ws})^{\nu_w}]e^{i2\pi[\Pi^k_{w=1}(f_{ws})^{\nu_w}]}\Big).$$

Definition 30. *Let* $\alpha_{ws} = (h^s_w, p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}})$ $(w = 1, 2, \ldots, k)$ *be a collection of* $CSV\mathcal{N}NS_fNs$ *and* ν_w *be the weight vectors of* α_{ws} *with* $\nu_w > 0$ *and* $\sum_{w=1}^{k} \nu_w = 1$. *The complex single-valued neutrosophic* N*-soft ordered weighted average operator* $(CSV\mathcal{N}NS_fOWA)$ *is a mapping* $CSV\mathcal{N}NS_fOWA : \mathcal{J}^k \to \mathcal{J}$, *where* $\mathcal{J}$ *is the set of* $CSV\mathcal{N}NS_fNs$, *defined as follows:*

$$\begin{aligned}
&CSV\mathcal{N}NS_fOWA(\alpha_{1s}, \alpha_{2s}, \ldots, \alpha_{ls})\\
&= \big(\nu_1\alpha_{\varrho(1s)} \oplus \nu_2\alpha_{\varrho(2s)} \oplus \ldots \oplus \nu_k\alpha_{\varrho(ks)}\big)\\
&= \Big(\max_{w=1}^{k}(h^s_\varrho(w)), [1 - \Pi^k_{w=1}(1 - p_{\varrho(ws)})^{\nu_w}]e^{i2\pi[1-\,\Pi^k_{w=1}(1\,-\,t_{\varrho(ws)})^{\nu_w}]}, [\Pi^k_{w=1}(q_{\varrho(ws)})^{\nu_w}]e^{i2\pi[\Pi^k_{w=1}(\omega_{\varrho(ws)})^{\nu_w}]},\\
&[\Pi^k_{w=1}(r_{\varrho(ws)})^{\nu_w}]e^{i2\pi[\Pi^k_{w=1}(f_{\varrho(ws)})^{\nu_w}]}\Big).
\end{aligned}$$

where, $\varrho(ws)$ *is a permutation ordered by* $\alpha_{\varrho(ws)} \geq \alpha_{\phi(vs)}$, *for all* $w < v$, $(w, v = 1, 2, \ldots, k)$ *and* $(s = 1, 2, \ldots, t)$.

Definition 31. *Let* $\alpha_{ws} = (h^s_w, p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}})$ $(i = 1, 2, \ldots, l)$ *be a collection of* $CSV\mathcal{N}NS_fNs$ *and* ν_w *be the weight vectors of* α_{ws} *with* $\nu_w > 0$ *and* $\sum_{w=1}^{k} \nu_w = 1$. *The single-valued neutrosophic* N*-soft weighted geometric operator* $(CSV\mathcal{N}NS_fWG)$ *is a mapping* $CSV\mathcal{N}NS_fWG : \mathcal{J}^k \to \mathcal{J}$, *where* $\mathcal{J}$ *is the set of* $CSV\mathcal{N}NS_fNs$, *defined as follows:*

$$CSV\mathcal{N}NS_fWG(\alpha_{1s}, \alpha_{2s}, \ldots, \alpha_{ks}) = (\alpha_{1s}^{\nu_1} \otimes \alpha_{2s}^{\nu_2} \otimes \ldots \otimes \alpha_{ks}^{\nu_k})$$
$$= \Big(\min_{w=1}^{k}(h^s_w), [\Pi^k_{w=1}(p_{ws})^{\nu_w}]e^{[\Pi^k_{w=1}(t_{ws})^{\nu_w}]}, [1 - \Pi^k_{w=1}(1 - q_{ws})^{\nu_w}]e^{[1-\,\Pi^k_{w=1}(1\,-\,\omega_{ws})^{\nu_w}]}, [1 - \Pi^k_{w=1}(1 - r_{ws})^{\nu_w}]e^{[\Pi^k_{w=1}(1\,-\,f_{ws})^{\nu_w}]}\Big).$$

Definition 32. *Let* $\alpha_{ws} = (h^s_w, p_{ws}e^{i2\pi t_{ws}}, q_{ws}e^{i2\pi\omega_{ws}}, r_{ws}e^{i2\pi f_{ws}})$ $(i = 1, 2, \ldots, l)$ *be a collection of* $CSV\mathcal{N}NS_fNs$ *and* ν_w *be the weight vectors of* α_{ws} *with* $\nu_w > 0$ *and* $\sum_{w=1}^{k} \nu_w = 1$. *The single-valued neutrosophic* N*-soft ordered weighted geometric operator* $(CSV\mathcal{N}NS_fOWG)$ *is a mapping* $CSV\mathcal{N}NS_fOWG : \mathcal{J}^K \to \mathcal{J}$, *where* $\mathcal{J}$ *is the set of* $CSV\mathcal{N}NS_fNs$, *defined as follows:*

$$\begin{aligned}
&CSV\mathcal{N}NS_fOWG(\alpha_{1s}, \alpha_{2s}, \ldots, \alpha_{ks})\\
&= \big(\alpha_{\varrho(1s)^{\nu_1}} \otimes \alpha_{\varrho(2s)^{\nu_2}} \otimes \ldots \otimes \alpha_{\varrho(ks)^{\nu_k}}\big)\\
&= \Big(\min_{i=1}^{l}(h^s_\varrho(w)), [\Pi^k_{w=1}(p_{\varrho(ws)})^{\nu_w}]e^{i2\pi[\Pi^k_{w=1}(t_{\varrho(ws)})^{\nu_w}]}, [1 - \Pi^k_{w=1}(1 - q_{\varrho(ws)})^{\nu_w}]e^{i2\pi[1-\,\Pi^k_{w=1}(1\,-\,\omega_{\varrho(ws)})^{\nu_w}]},\\
&[1 - \Pi^k_{w=1}(1 - r_{\varrho(ws)})^{\nu_w}]e^{i2\pi[1-\,\Pi^k_{w=1}(1\,-\,q_{\varrho(ws)})^{\nu_w}]}\Big),
\end{aligned}$$

where, $\varrho(ws)$ *is a permutation ordered by* $\alpha_{\varrho(ws)} \geq \alpha_{\phi(vs)}$, *for all* $w < v$, $(w, v = 1, 2, \ldots, k)$ *and* $(s = 1, 2, \ldots, t)$.

5 Complex single-valued neutrosophic N-soft TOPSIS method

In this section, we developed methodology for TOPSIS method under the framework of $CSV\mathcal{N}NS_fSs$ for solving multi-attribute group decision making (MAGDM) problem. For the optimal solution of the MADM problem, TOPSIS method specifically used ideal solutions of that problem. Consider a MAGDM problem with $\mathbb{U} = \{\mathbb{U}_1, \mathbb{U}_2, \mathbb{U}_3, \ldots, \mathbb{U}_t\}$ and $Y = \{Y_1, Y_2, Y_3, \ldots, Y_k\}$ be the set of alternative and attributes decided by the experts $\widetilde{Z}_1, \widetilde{Z}_2, \widetilde{Z}_3, \ldots, \widetilde{Z}_f$, where the experts weight vector for this MAGDM problem is $\nu = (\nu_1, \nu_2, \nu_3, \ldots, \nu_k)^T$. The procedure for $CSV\mathcal{N}NS_f$-TOPSIS method is as follows:

5.1 Organizing the complex single-valued neutrosophic N-soft decision matrix

After studied the MADM problem properly, decision makers use rating system for assigning rank to each alternative, parallel to each semantic term, relative to the attributes that indeed form a NS_fS. Further, decision making panel associate $CSV\mathcal{N}NS_fN$ corresponding to each rank (ordered grade) by defining grading criteria related to the aptitude of the MADM problem. Therefore, a complex single-valued neutrosophic N-soft decision matrix $(CSV\mathcal{N}NS_fDM)$ $\mathbb{H} = (\mathbb{H}_{ws}^{(j)})_{(s\times w)}$ is organized as follow:

$$\mathbb{H}^{(j)} = \begin{pmatrix} (h_1^{1\,(j)}, \mathbb{T}_{11}^{(j)}, \mathbb{I}_{11}^{(j)}, \mathbb{F}_{11}^{(j)}) & (h_2^{1\,(j)}, \mathbb{T}_{12}^{(j)}, \mathbb{I}_{12}^{(j)}, \mathbb{F}_{12}^{(j)}) & \dots & (h_k^{1\,(j)}, \mathbb{T}_{1k}^{(j)}, \mathbb{I}_{1k}^{(j)}, \mathbb{F}_{1k}^{(j)}) \\ (h_1^{2\,(j)}, \mathbb{T}_{21}^{(j)}, \mathbb{I}_{21}^{(j)}, \mathbb{F}_{21}^{(j)}) & (h_2^{2\,(j)}, \mathbb{T}_{22}^{(j)}, \mathbb{I}_{22}^{(j)}, \mathbb{F}_{22}^{(j)}) & \dots & (h_m^{2\,(j)}, \mathbb{T}_{2k}^{(j)}, \mathbb{I}_{2k}^{(j)}, \mathbb{F}_{2k}^{(j)}) \\ \vdots & \vdots & \ddots & \vdots \\ (h_1^{t\,(j)}, \mathbb{T}_{t1}^{(j)}, \mathbb{I}_{t1}^{(j)}, \mathbb{F}_{t1}^{(j)}) & (h_2^{t\,(j)}, \mathbb{T}_{t2}^{(j)}, \mathbb{I}_{t2}^{(j)}, \mathbb{F}_{t2}^{(j)}) & \dots & (h_k^{t\,(j)}, \mathbb{T}_{tk}^{(j)}, \mathbb{I}_{tk}^{(j)}, \mathbb{F}_{tk}^{(j)}) \end{pmatrix},$$

where, $\mathbb{H}_{ws}^{(j)} = ((h_i^j)^{(j)}, \mathbb{T}_{ws}^{(j)}, \mathbb{I}_{ws}^{(j)}, \mathbb{F}_{ws}^{(j)}) = (h_w^s, p_{ws}^{(j)} e^{i2\pi t_{ws}^{(j)}}, q_{ws}^{(j)} e^{i2\pi\omega_{ws}^{(j)}}, r_{ws}^{(j)} e^{i2\pi f_{ws}^{(j)}})$, $s = \{1, 2, 3, \dots, t\}$, $j = \{1, 2, 3, \dots, f\}$, and $w = \{1, 2, 3, \dots, k\}$.

5.2 Aggregated complex single-valued neutrosophic N-soft decision matrix

As the decision makers (experts) are not equally weighted in MAGDM problems, therefore by utilizing the weightage of each expert decided by the panel we cumulate the decision of all experts and get aggregated complex single-valued neutrosophic N-soft decision matrix $(ACSV\mathcal{N}NS_fDM)$. The $CSV\mathcal{N}NS_fWA$ operator or $CSV\mathcal{N}NS_fWG$ operator are precisely used to commulate the $CSV\mathcal{N}NS_fDM$ $(\mathcal{H})$ as follows:

$$\begin{aligned} \mathcal{H}_{ws} &= CSV\mathcal{N}NS_fWA(\mathbb{H}_{ws}^{(1)}, \mathbb{H}_{ws}^{(2)}, \dots, \mathbb{H}_{ws}^{(f)}); \\ (OR) &= CSV\mathcal{N}NS_fWG(\mathbb{H}_{ws}^{(1)}, \mathbb{H}_{ws}^{(2)}, \dots, \mathbb{H}_{ws}^{(f)}); \end{aligned}$$

where, $\mathcal{H}_{ws} = (h_1^1, \mathbb{T}_{ws}, \mathbb{I}_{ws}, \mathbb{F}_{ws}) = (h_w^s, p_{ws} e^{i2\pi t_{ws}}, q_{ws} e^{i2\pi\omega_{ws}}, r_{ws} e^{i2\pi f_{ws}})$.
The $ACSV\mathcal{N}N_fSDM$ denoted as:

$$\mathcal{H} = \begin{pmatrix} (h_1^1, \mathbb{T}_{11}, \mathbb{I}_{11}, \mathbb{F}_{11}) & (h_2^1, \mathbb{T}_{12}, \mathbb{I}_{12}, \mathbb{F}_{12}) & \dots & (h_k^1, \mathbb{T}_{1k}, \mathbb{I}_{1k}, \mathbb{F}_{1k}) \\ (h_1^2, \mathbb{T}_{21}, \mathbb{I}_{21}, \mathbb{F}_{21}) & (h_2^2, \mathbb{T}_{22}, \mathbb{I}_{22}, \mathbb{F}_{22}) & \dots & (h_k^2, \mathbb{T}_{2k}, \mathbb{I}_{2k}, \mathbb{F}_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ (h_1^s, \mathbb{T}_{s1}, \mathbb{I}_{s1}, \mathbb{F}_{s1}) & (h_2^q, \mathbb{T}_{s2}, \mathbb{I}_{s2}, \mathbb{F}_{s2}) & \dots & (h_k^s, \mathbb{T}_{sk}, \mathbb{I}_{sk}, \mathbb{F}_{sk}) \end{pmatrix}.$$

5.3 Weights for parameters

To highlight the influence of the parameters in the MAGDM problem, experts judged each parameter and assign grades as the weight of the parameter. Further, $CSV\mathcal{N}NS_fNs$ are associated to each grade using the grading criteria finalized by the panel. Let $\theta_w^{(j)} = (h_w^{(j)}, \mathbb{T}_w^{(j)}, \mathbb{I}_w^{(j)}, \mathbb{F}_w^{(j)})$ be the weight of wth parameter given by the jth expert in the MAGDM problem. Let $\theta = (\theta_1, \theta_2, \dots, \theta_k)^T = (h_w, \mathbb{T}_w, \mathbb{I}_w, \mathbb{F}_w)$ be the weight vector of attributes that is summarized, by $CSV\mathcal{N}NS_fWA$ operator or $CSV\mathcal{N}NS_fWG$ operator, as follows:

$$\begin{aligned} \theta_w &= CSV\mathcal{N}NS_fWA(\theta_1^{(j)}, \theta_2^{(j)}, \dots, \theta_k^{(j)}); \\ (OR) &= CSV\mathcal{N}NS_fWG(\theta_1^{(j)}, \theta_2^{(j)}, \dots, \theta_k^{(j)}). \end{aligned}$$

where, $\theta_w = (h_1, \mathbb{T}_w, \mathbb{I}_w, \mathbb{F}_w) = (h_w, p_w e^{i2\pi t_w}, q_w e^{i2\pi\omega_w}, r_w e^{i2\pi f_w})$.

5.4 Aggregated weighted complex single-valued neutrosophic N-soft decision matrix

The $ACSV\mathcal{N}N_fSDM$ $\mathcal{H}$ is used within the weight vector $(\theta_1, \theta_2, \dots, \theta_k)^T$ of parameter for the formulation of aggregated weighted single-valued neutrosophic N-soft decision matrix $(AWCSV\mathcal{N}NS_fDM)$. The calculations for are performed as follows:

$$\begin{aligned} \bar{H}_{ws} &= \mathcal{H}_{ws} \otimes \theta_w \\ &= (\min((h_w^s), h_w), (\mathbb{T}_{ws}\mathbb{T}_w), (\mathbb{I}_{ws} + \mathbb{I}_w - \mathbb{I}_{ws}\mathbb{I}_w), (\mathbb{F}_{ws} + \mathbb{F}_i - \mathbb{F}_{ws}\mathbb{F}_W)) \\ &= \left(\min(h_w^s, h_w), p_{ws}p_w e^{i2\pi t_{ws} t_w}, (q_{ws} + q_w - q_{ws}q_w) e^{i2\pi[\omega_{ws} + \omega_{ws} - \omega_{ws}\omega_w]}, (r_{ws} + r_w - r_{ws}r_w) e^{i2\pi[f_{ws} + f_{ws} - f_{ws} f_w]} \right) \\ &= (\bar{h}_w^s, \bar{\mathbb{T}}_{ws}, \bar{\mathbb{I}}_{ws}, \bar{\mathbb{F}}_{ws}) \\ &= (\bar{h}_w^s, \bar{p}_{ws} e^{i2\pi\bar{t}_{ws}}, \bar{q}_{ws} e^{i2\pi\bar{\omega}_{ws}}, \bar{r}_{ws} e^{i2\pi\bar{f}_{ws}}). \end{aligned}$$

The $AWCSV\mathcal{N}NS_fDM$ is:

$$\bar{H}_{ws} = \begin{pmatrix} (\bar{h}_1^1, \bar{\mathbb{T}}_{11}, \bar{\mathbb{I}}_{11}, \bar{\mathbb{F}}_{11}) & (\bar{h}_2^1, \bar{\mathbb{T}}_{12}, \bar{\mathbb{I}}_{12}, \bar{\mathbb{F}}_{12}) & \dots & (\bar{h}_k^1, \bar{\mathbb{T}}_{1k}, \bar{\mathbb{I}}_{1k}, \bar{\mathbb{F}}_{1k}) \\ (\bar{h}_1^2, \bar{\mathbb{T}}_{21}, \bar{\mathbb{I}}_{21}, \bar{\mathbb{F}}_{21}) & (\bar{h}_2^2, \bar{\mathbb{T}}_{22}, \bar{\mathbb{I}}_{22}, \bar{\mathbb{F}}_{22}) & \dots & (\bar{h}_k^2, \bar{\mathbb{T}}_{2k}, \bar{\mathbb{I}}_{2k}, \bar{\mathbb{F}}_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ (\bar{h}_1^s, \bar{\mathbb{T}}_{s1}, \bar{\mathbb{I}}_{s1}, \bar{\mathbb{F}}_{s1}) & (\bar{h}_2^s, \bar{\mathbb{T}}_{s2}, \bar{\mathbb{I}}_{s2}, \bar{\mathbb{F}}_{s2}) & \dots & (\bar{h}_k^q, \bar{\mathbb{T}}_{sk}, \bar{\mathbb{I}}_{sk}, \bar{\mathbb{F}}_{sk}) \end{pmatrix}.$$

5.5 Complex single-valued neutrosophic N-soft ideal solutions

Let $\mathbb{BT}$ be the collection of benefit-type criteria and $\mathbb{CT}$ be the collection of cost-type criteria opted from the number of parameters, keeping in view the expertise of the given problem. Using these collection we are able to evaluate the complex single-valued neutrosophic positive ideal solution $CSV\mathcal{N}N_fS$-PIS and complex single-valued neutrosophic N-soft negative ideal solution $CSV\mathcal{N}NS_f$-NIS of the MAGDM problem. The $CSV\mathcal{N}NS_f$-PIS, related to the parameter Y_w, is defined as:

$$\bar{H}_w^{PIS} = \begin{cases} \max\limits_{j=1}^{s} \bar{H}_{ws}, & \text{if } Y_w \in \mathbb{BT}, \\ \min\limits_{j=1}^{s} \bar{H}_{ws}, & \text{if } Y_w \in \mathbb{CT}, \end{cases}$$

and the $CSV\mathcal{N}NS_f$-NIS is defined as:

$$\bar{H}_w^{NIS} = \begin{cases} \max\limits_{j=1}^{s} \bar{H}_{ws}, & \text{if } Y_w \in \mathbb{CT}, \\ \min\limits_{j=1}^{s} \bar{H}_{ws}, & \text{if } Y_w \in \mathbb{BT}. \end{cases}$$

The $CSV\mathcal{N}NS_f\text{-}PIS$ and $CSV\mathcal{N}NS_f\text{-}NIS$ are denoted as: $\bar{H}_w^{PIS} = (\dot{h}_w, \dot{p}_w e^{i2\pi \dot{t}_w}, \dot{q}_w e^{i2\pi\dot{\omega}_w}, \dot{r}_w e^{i2\pi \dot{f}_w})$., and $\bar{H}_w^{NIS} = (\ddot{h}_w, \ddot{p}_w e^{i2\pi \ddot{t}_w}, \ddot{q}_w e^{i2\pi\ddot{\omega}_w}, \ddot{r}_w e^{i2\pi \ddot{f}_w})$, respectively.

5.6 Formulation of normalized Euclidean distance

For evaluating the alternatives distance from the ideal solution, we can used similarity measures or distance measure. Moreover, from distance measures we used the normalized Euclidean distance. The normalized Euclidean distance of any of the alternative $\mathbb{U}_s$ from the $CSV\mathcal{N}NS_f\text{-}PIS$ is defined as:

$$d(\bar{H}_w^{PIS}, \mathbb{U}_s) = \Big(\frac{1}{7w}\sum_{w=1}^{k}[(\frac{\dot{h}_w}{N-1}) - (\frac{\bar{h}_w^s}{N-1})^2 + (\dot{p}_w - \bar{p}_{ws})^2 + (\dot{q}_w - \bar{q}_{ws})^2 + (\dot{r}_w - \bar{r}_{ws})^2 + (\dot{t}_w - \bar{t}_{ws})^2 + (\dot{\omega}_w - \bar{\omega}_{ws})^2 + (\dot{f}_w - \bar{f}_{ws})^2]\Big) \tag{23}$$

The normalized Euclidean distance between the $CSV\mathcal{N}NS_f$-NIS and any of the alternative $\mathbb{U}_s$, can be evaluated as follows:

$$d(\bar{H}_w^{NIS}, \mathbb{U}_s) = \Big(\frac{1}{7w}\sum_{w=1}^{k}[(\frac{\ddot{h}_w}{N-1}) - (\frac{\bar{h}_w^s}{N-1})^2 + (\ddot{p}_w - \bar{p}_{ws})^2 + (\ddot{q}_w - \bar{q}_{ws})^2 + (\ddot{r}_w - \bar{r}_{ws})^2 + (\ddot{t}_w - \bar{t}_{ws})^2 + (\ddot{\omega}_w - \bar{\omega}_{ws})^2 + (\ddot{f}_w - \bar{f}_{ws})^2]\Big) \tag{24}$$

5.7 Revised closeness index

In TOPSIS method, at last we left with two values related to the alternative that prescribed the distance of that particular alternative from $CSV\mathcal{N}NS_f$-PIS and $CSV\mathcal{N}NS_f$-NIS. Therefore, revised closeness index is utilized for the choice of right solution. The revised closeness index $\Lambda(\mathbb{U}_s)$ is calculated as:

$$\Lambda(\mathbb{U}_s) = \frac{d(\bar{H}_w^{PIS}, \mathbb{U}_s)}{\min\limits_s d(\bar{H}_w^{PIS}, \mathbb{U}_s)} - \frac{d(\bar{H}_w^{NIS}, \mathbb{U}_s)}{\max\limits_s d(\bar{H}_w^{NIS}, \mathbb{U}_s)}, \tag{25}$$

where, $s = 1, 2, \dots, t$.

5.8 Identify dominant alternative

For the evaluation of dominant alternative with respect to their performance in MAGDM problem, revised closeness index related to each alternative arranged in ascending order. So that the alternative with least revised closeness index will be the required one.
For solving a MAGDM problem, the Algorithm 1 is given as:

Algorithm 1: Steps to deal MAGDM problem by $CSV\mathbb{N}N_f$-TOPSIS method

1. **Input**:
   $\mathbb{U}$: Set of alternatives,
   Y : Set of attributes,
   ν : Weight vector for experts $\widetilde{Z}_j$,
   $NS_fS : (\Phi_\Psi, Y, N)$ with $H = \{0, 1, 2, 3, \ldots, N-1\}$, $N \in \{1, 2, 3, \ldots\}$,
2. Construct the $CSV\mathbb{N}NS_fDM$ $\mathbb{H}^{(j)}$, using the input data.
3. Evaluate the $ACSV\mathbb{N}NS_fDM$ as follows:

$$\mathcal{H}_{ws} = \Big(\max_{j=1}^{f}(h^s_w)^{(j)}, [1-\Pi_{j=1}^{f}(1-p^{(j)}_{ws})^{\nu_w}]e^{i2\pi[1-\Pi_{j=1}^{f}(1-t^{(j)}_{ws})^{\nu_w}]}, [\Pi_{j=1}^{f}(q^{(j)}_{ws})^{\nu_w}]e^{i2\pi[\Pi_{j=1}^{f}(\omega^{(j)}_{ws})^{\nu_w}]}, [\Pi_{j=1}^{f}(r^{(j)}_{ws})^{\nu_w}]e^{i2\pi[\Pi_{j=1}^{f}(f^{(j)}_{ws})^{\nu_w}]}\Big).$$

4. Calculating the weight vector $\theta = (\theta_1, \theta_2, \ldots, \theta_k)^T$ for parameters as:

$$\theta_w = \Big(\max_{j=1}^{f}(h_w)^{(j)}, [1-\Pi_{j=1}^{f}(1-p^{(j)}_{w})^{\nu_w}]e^{i2\pi[1-\Pi_{j=1}^{f}(1-t^{(j)}_{w})^{\nu_w}]}, [\Pi_{j=1}^{f}(q^{(j)}_{w})^{\nu_w}]e^{i2\pi[\Pi_{j=1}^{f}(\omega^{(j)}_{w})^{\nu_w}]}, [\Pi_{j=1}^{f}(r^{(j)}_{w})^{\nu_w}]e^{i2\pi[\Pi_{j=1}^{f}(f^{(j)}_{w})^{\nu_w}]}\Big).$$

5. Compute the $AWCSV\mathbb{N}NS_fDM$ using $ACSV\mathbb{N}NS_fDM$ and the weight vector of attributes θ_w, as follows:

$$\bar{H}_{ws} = \left(\min(h^s_w, h_w), p_{ws}p_w e^{i2\pi t_{ws}t_w}, (q_{ws}+q_w-q_{ws}q_w)e^{i2\pi[\omega_{ws}+\omega_{ws}-\omega_{ws}\omega_w]}, (r_{ws}+r_w-r_{ws}r_w)e^{i2\pi[f_{ws}+f_{ws}-f_{ws}f_w]}\right).$$

6. Evaluate the $CSV\mathbb{N}NS_f$ PIS and $CSV\mathbb{N}NS_f$ NIS.
7. Evaluate the normalized Euclidean distance $d(\bar{H}^{PIS}_w, \mathbb{U}_s)$ and $d(\bar{H}^{NIS}_w, \mathbb{U}_s)$
8. Evaluate the revised closeness index $\Lambda(\mathbb{U}_s)$.
9. Arranged revised closeness index in ascending order.

Output: Choose the alternative with minimum revised closeness index.

6 Application

In this section, we solve a MAGDM problem using $CSV\mathcal{N}NS_f - TOPSIS$ method for analyzing the performance of Islamic banks in Pakistan with CAMELS rating system.

6.1 Monitoring performance of Islamic banking industry on the basis of CAMELS rating system.

The banks are more closely monitored other than any field of economy because of their constitution and important role in the economy of the country. Analyzing the banking system create more assurance and reliability in making both short and long term decisions, that in return give on to healthier business in the country. In banking industry, one of the flourishing institute is Islamic banking that follow the rules of Islamic Shariah and promote the Islamic principles to the transaction of financial banking. The evaluation of financial performance of Islamic banking in Pakistan using the CAMELS model and TOPSIS method is necessary for higher level of efficiency that further help to set a benchmark for the country. In this MAGDM problem, following Islamic banks are considered as alternatives:

$\mathbb{U}_1$: Bank Albarka(BA)

$\mathbb{U}_2$: Bank Islamic (BIL)

$\mathbb{U}_3$: Dubai Islamic Bank (DIB)

$\mathbb{U}_4$: Muslim Commercial Bank (MCB)

$\mathbb{U}_5$: Meezan Bank (MBL)

For this MAGDM problem, decision making panel consists of three experts $\widetilde{Z}_1, \widetilde{Z}_2, \widetilde{Z}_3$ that collected data from the official websites of the banks according to the CAMELS model. CAMELS model is generally apply to analyze the performance of the banks on the basis of five different attributes described as follow:

Y_1 : **C**apital adequacy: Experts rank the capital adequacy by checking the factors of growth plan and capacity to control financial risk and loan.

Y_2 : **A**sset quality: In this attribute the banking stability is measure whenever the bank faced loss of values of the assets.

Y_3 : **M**anagement: Experts rate this attribute by measuring the efficiency of banks while dealing with daily activities.

Y_4 : **E**arning capacity: This attribute includes the existing assets, earnings and growth of the banks, as well as to remain competitive in economy.

Y_5 : **L**iquidity: This attribute examine on the basis of the availability of adequate funds by converting assets into the cash.

We solve this MAGDM problem by following the $CSV\mathcal{N}NS_f$-TOPSIS method.

Step 1: According to these attributes each expert model 5-soft set in Table14 where

0 means 'Bad'

1 means 'Ok'

2 means 'Good'

3 means 'Great'

4 means 'Excellent'

Table 14: Initial rating by decision making experts

$Parameters$	$Alternatives$	$\widetilde{Z}_1$	$\widetilde{Z}_2$,	$\widetilde{Z}_3$
Y_1	$\mathbb{U}_1$	$****\ =4$	$***\ =3$	$**\ =2$
	$\mathbb{U}_2$	$****\ =4$	$*\ =1$	$***\ =3$
	$\mathbb{U}_3$	$***\ =3$	$**\ =2$	$*\ =1$
	$\mathbb{U}_4$	$***\ =3$	$*\ =1$	$***\ =3$
	$\mathbb{U}_5$	$****\ =4$	$\bullet\ =0$	$*\ =1$
Y_2	$\mathbb{U}_1$	$***\ =3$	$**\ =2$	$***\ =3$
	$\mathbb{U}_2$	$***\ =3$	$****\ =4$	$****\ =4$
	$\mathbb{U}_3$	$****\ =4$	$\bullet\ =0$	$**\ =2$
	$\mathbb{U}_4$	$****\ =4$	$***\ =3$	$****\ =4$
	$\mathbb{U}_5$	$****\ =4$	$*\ =1$	$**\ =2$
Y_3	$\mathbb{U}_1$	$\bullet\ =0$	$*\ =1$	$**\ =2$
	$\mathbb{U}_2$	$****\ =4$	$****\ =4$	$***\ =3$
	$\mathbb{U}_3$	$\bullet\ =0$	$**\ =2$	$*\ =1$
	$\mathbb{U}_4$	$\bullet\ =0$	$***\ =3$	$****\ =4$
	$\mathbb{U}_5$	$\bullet\ =0$	$\bullet\ =0$	$*\ =1$
Y_4	$\mathbb{U}_1$	$\bullet\ =0$	$*\ =1$	$\bullet\ =0$
	$\mathbb{U}_2$	$\bullet\ =0$	$***\ =3$	$**\ =2$
	$\mathbb{U}_3$	$*\ =1$	$**\ =2$	$***\ =3$
	$\mathbb{U}_4$	$****\ =4$	$****\ =4$	$***\ =3$
	$\mathbb{U}_5$	$\bullet\ =0$	$\bullet\ =0$	$*\ =1$
Y_5	$\mathbb{U}_1$	$**\ =2$	$\bullet\ =0$	$*\ =1$
	$\mathbb{U}_2$	$***\ =3$	$**\ =2$	$*\ =1$
	$\mathbb{U}_3$	$***\ =3$	$****\ =4$	$**\ =2$
	$\mathbb{U}_4$	$****\ =4$	$***\ =3$	$**\ =2$
	$\mathbb{U}_5$	$\bullet\ =0$	$*\ =1$	$\bullet\ =0$

To assign $CSV\mathcal{N}NS_fS$ to each rank in Table 14, experts defined grading criteria given in Table 15 and Tables 16, 17, 18 representing the decision of the experts $\widetilde{Z}_1, \widetilde{Z}_2, \widetilde{Z}_3$, respectively.

Table 15: Grading criteria for $CSV\mathcal{N}5SS$

h_z^w/J	degree of truthness		degree of indeterminacy		degree of falsity	
grades	p_w	$2\pi t_w$	q_w	$2\pi\omega_w$	r_w	$2\pi f_w$
$h_w^s = 0$	$[0.00, 0.15)$	$[0.0, 0.3\pi)$	$(0.85, 1.00]$	$(1.7\pi, 2.0\pi]$	$(0.85, 1.00]$	$(1.7\pi, 2.0\pi]$
$h_w^s = 1$	$[0.15, 0.35)$	$[0.3\pi, 0.7\pi)$	$(0.65, 0.85]$	$(1.3\pi, 1.7\pi]$	$(0.65, 0.85]$	$(1.3\pi, 1.7\pi]$
$h_w^s = 2$	$[0.35, 0.65)$	$[0.7\pi, 1.3\pi)$	$(0.35, 0.65]$	$(0.7\pi, 1.3\pi]$	$(0.35, 0.65]$	$(0.7\pi, 1.3\pi]$
$h_w^s = 3$	$[0.65, 0.85)$	$[1.3\pi, 1.7\pi)$	$(0.15, 0.35]$	$(0.3\pi, 0.7\pi]$	$(0.15, 0.35]$	$(0.3\pi, 0.7\pi]$
$h_w^s = 4$	$[0.85, 1.00)$	$[1.7\pi, 2.0\pi)$	$[0.00, 0.15)$	$[0.0, 0.3\pi)$	$[0.00, 0.15)$	$(0.0, 0.3\pi]$

Table 16: $CSV\mathcal{N}DM$ related to expert $\widetilde{Z}_1$,

	Y_1	Y_2	Y_3	Y_4
$\mathbb{U}_1$	$(4, (0.86e^{i1.76\pi}, 0.08e^{i0.14\pi}, 0.07e^{i0.12\pi}))$	$(3, (0.71e^{i1.46\pi}, 0.31e^{i0.64\pi}, 0.29e^{i0.60\pi}))$	$(0, (0.11e^{i0.26\pi}, 0.91e^{i1.84\pi}, 0.93e^{i1.88\pi}))$	$(0, (0.12e^{i0.28\pi}, 0.87e^{i1.72\pi}, 0.86e^{i1.74\pi}))$
$\mathbb{U}_2$	$(4, (0.87e^{i1.78\pi}, 0.09e^{i0.16\pi}, 0.08e^{i0.14\pi}))$	$(3, (0.66e^{i1.36\pi}, 0.27e^{i0.56\pi}, 0.31e^{i0.60\pi}))$	$(4, (0.89e^{i1.74\pi}, 0.04e^{i0.10\pi}, 0.11e^{i0.24\pi}))$	$(0, (0.13e^{i0.28\pi}, 0.87e^{i1.72\pi}, 0.86e^{1.74\pi}))$
$\mathbb{U}_3$	$(3, (0.69e^{i1.42\pi}, 0.19e^{i0.40\pi}, 0.22e^{i0.46\pi}))$	$(4, (0.88e^{i1.72\pi}, 0.06e^{i0.14\pi}, 0.10e^{i0.18\pi}))$	$(0, (0.14e^{i0.26\pi}, 0.88e^{i1.74\pi}, 0.89e^{i1.76\pi}))$	$(1, (0.34e^{i0.64\pi}, 0.66e^{i1.32\pi}, 0.67e^{1.36\pi}))$
$\mathbb{U}_4$	$(3, (0.82e^{i1.78\pi}, 0.18e^{i0.38\pi}, 0.21e^{0.44i\pi}))$	$(4, (0.91e^{i1.86\pi}, 0.02e^{i0.02\pi}, 0.03e^{0.08i\pi}))$	$(0, (0.13e^{i0.28\pi}, 0.88e^{i1.74\pi}, 0.86e^{i1.74\pi}))$	$(4, (0.93e^{i1.90\pi}, 0.04e^{i0.06\pi}, 0.01e^{i0.04\pi}))$
$\mathbb{U}_5$	$(4, (0.87e^{i1.78\pi}, 0.13e^{i0.28\pi}, 0.12e^{0.26i\pi}))$	$(4, (0.90e^{i1.84\pi}, 0.07e^{i0.12\pi}, 0.10e^{0.22i\pi}))$	$(0, (0.02e^{i0.08\pi}, 0.95e^{i1.72\pi}, 0.97e^{i1.78\pi}))$	$(0, (0.03e^{i0.02\pi}, 0.96e^{i1.90\pi}, 0.98e^{i1.70\pi}))$

	Y_5
$\mathbb{U}_1$	$(2, (0.61e^{i1.18\pi}, 0.41e^{i0.84\pi}, 0.43e^{i0.88\pi}))$
$\mathbb{U}_2$	$(3, (0.67e^{i1.38\pi}, 0.25e^{i0.48\pi}, 0.23e^{i0.44\pi}))$
$\mathbb{U}_3$	$(3, (0.71e^{i1.44\pi}, 0.24e^{i0.50\pi}, 0.27e^{i0.52\pi}))$
$\mathbb{U}_4$	$(4, (0.96e^{i1.94\pi}, 0.05e^{i0.08\pi}, 0.03e^{i0.04\pi}))$
$\mathbb{U}_5$	$(0, (0.05e^{i0.06\pi}, 0.95e^{i1.84\pi}, 0.94e^{1.86i\pi}))$

Table 17: $CSV\mathcal{N}DM$ related to expert $\widetilde{Z}_2$,

	Y_1	Y_2	Y_3	Y_4
$\mathbb{U}_1$	$(3, (0.72e^{i1.46\pi}, 0.32e^{i0.66\pi}, 0.66e^{i0.68\pi}))$	$(2, (0.41e^{i0.86\pi}, 0.51e^{i1.04\pi}, 0.61e^{i1.24\pi}))$	$(1, (0.16e^{i0.36\pi}, 0.69e^{i1.40\pi}, 0.72e^{i1.46\pi}))$	$(1, (0.17e^{i0.28\pi}, 0.75e^{i1.52\pi}, 0.77e^{i1.56\pi}))$
$\mathbb{U}_2$	$(1, (0.19e^{i0.42\pi}, 0.72e^{i1.46\pi}, 0.75e^{i1.52\pi}))$	$(4, (0.93e^{i1.82\pi}, 0.12e^{i0.26\pi}, 0.13e^{i0.28\pi}))$	$(4, (0.88e^{i1.74\pi}, 0.08e^{i0.18\pi}, 0.10e^{i0.22\pi}))$	$(3, (0.73e^{i0.75\pi}, 0.23e^{i0.48\pi}, 0.20e^{0.38\pi}))$
$\mathbb{U}_3$	$(2, (0.45e^{i0.94\pi}, 0.46e^{i0.94\pi}, 0.56e^{i1.04\pi}))$	$(0, (0.09e^{i0.14\pi}, 0.87e^{i1.76\pi}, 0.86e^{i1.74\pi}))$	$(2, (0.58e^{i1.20\pi}, 0.37e^{i0.74\pi}, 0.39e^{i0.80\pi}))$	$(2, (0.59e^{i1.22\pi}, 0.53e^{i1.08\pi}, 0.44e^{0.86\pi}))$
$\mathbb{U}_4$	$(1, (0.32e^{i0.68\pi}, 0.67e^{i1.38\pi}, 0.69e^{1.36i\pi}))$	$(3, (0.84e^{i1.66\pi}, 0.16e^{i0.34\pi}, 0.17e^{0.36i\pi}))$	$(3, (0.83e^{i1.62\pi}, 0.18e^{i0.38\pi}, 0.19e^{i0.40\pi}))$	$(4, (0.98e^{i1.98\pi}, 0.10e^{i0.16\pi}, 0.01e^{i0.04\pi}))$
$\mathbb{U}_5$	$(0, (0.11e^{i0.26\pi}, 0.90e^{i1.82\pi}, 0.91e^{1.84i\pi}))$	$(1, (0.22e^{i0.46\pi}, 0.81e^{i1.64\pi}, 0.84e^{1.66i\pi}))$	$(0, (0.08e^{i0.20\pi}, 0.91e^{i1.80\pi}, 0.92e^{i1.82\pi}))$	$(0, (0.07e^{i0.18\pi}, 0.87e^{i1.72\pi}, 0.88e^{i1.74\pi}))$

	Y_5
$\mathbb{U}_1$	$(0, (0.06e^{i0.08\pi}, 0.91e^{i1.84\pi}, 0.92e^{i1.86\pi}))$
$\mathbb{U}_2$	$(2, (0.64e^{i1.26\pi}, 0.36e^{i0.74\pi}, 0.37e^{i0.76\pi}))$
$\mathbb{U}_3$	$(4, (0.92e^{i1.82\pi}, 0.05e^{i0.08\pi}, 0.12e^{i0.22\pi}))$
$\mathbb{U}_4$	$(3, (0.81e^{i1.02\pi}, 0.20e^{i0.42\pi}, 0.19e^{i0.26\pi}))$
$\mathbb{U}_5$	$(1, (0.23e^{i0.48\pi}, 0.83e^{i1.68\pi}, 0.82e^{1.66i\pi}))$

Table 18: $CSV\mathcal{N}DM$ related to expert $\widetilde{Z}_3$,

	Y_1	Y_2	Y_3	Y_4
$\mathbb{U}_1$	$(2, (0.62e^{i1.20\pi}, 0.36e^{i1.74\pi}, 0.39e^{i0.80\pi}))$	$(3, (0.70e^{i1.36\pi}, 0.26e^{i1.50\pi}, 0.28e^{i1.58\pi}))$	$(2, (0.59e^{1.22\pi}, 0.43^{i0.88\pi}, 0.42e^{i0.86\pi}))$	$(0, (0.86e^{i1.74\pi}, 0.02e^{i0.02\pi}, 0.03e^{i0.04\pi}))$
$\mathbb{U}_2$	$(3, (0.81e^{i1.66\pi}, 0.20e^{i0.28\pi}, 0.18e^{i0.28\pi}))$	$(4, (0.95e^{i1.88\pi}, 0.05e^{i0.08\pi}, 0.07e^{i0.16\pi}))$	$(3, (0.80e^{i1.64\pi}, 0.21e^{i0.40\pi}, 0.22e^{i0.46\pi}))$	$(2, (0.62e^{i1.28\pi}, 0.36e^{i0.74\pi}, 0.38e^{0.78\pi}))$
$\mathbb{U}_3$	$(1, (0.31e^{i0.66\pi}, 0.68e^{i1.38\pi}, 0.69e^{i1.40\pi}))$	$(2, (0.60e^{i1.22\pi}, 0.41e^{i0.80\pi}, 0.42e^{i0.84\pi}))$	$(1, (0.29e^{i0.62\pi}, 0.70e^{i1.42\pi}, 0.72e^{i1.46\pi}))$	$(3, (0.79e^{i1.62\pi}, 0.23e^{i0.44\pi}, 0.20e^{0.42\pi}))$
$\mathbb{U}_4$	$(3, (0.84e^{i1.72\pi}, 0.17e^{i0.32\pi}, 0.16e^{0.34i\pi}))$	$(4, (0.96e^{i1.96\pi}, 0.03e^{i0.08\pi}, 0.02e^{0.06i\pi}))$	$(4, (0.98e^{i1.93\pi}, 0.04e^{i0.06\pi}, 0.03e^{i0.04\pi}))$	$(3, (0.82e^{i1.68\pi}, 0.18e^{i0.34\pi}, 0.19e^{i0.36\pi}))$
$\mathbb{U}_5$	$(1, (0.27e^{i0.38\pi}, 0.74e^{i1.46\pi}, 0.73e^{1.50i\pi}))$	$(2, (0.57e^{i1.10\pi}, 0.45e^{i0.92\pi}, 0.47e^{0.96i\pi}))$	$(1, (0.25e^{i0.34\pi}, 0.76e^{i1.54\pi}, 0.78e^{i1.58\pi}))$	$(1, (0.23e^{i0.50\pi}, 0.79e^{i1.60\pi}, 0.81e^{i1.64\pi}))$

	Y_5
$\mathbb{U}_1$	$(1, (0.31e^{i0.64\pi}, 0.69e^{i1.36\pi}, 0.68e^{i1.38\pi}))$
$\mathbb{U}_2$	$(1, (0.34e^{i0.64\pi}, 0.66e^{i1.34\pi}, 0.67e^{i1.38\pi}))$
$\mathbb{U}_3$	$(2, (0.61e^{i1.26\pi}, 0.39e^{i0.80\pi}, 0.40e^{i0.82\pi}))$
$\mathbb{U}_4$	$(2, (0.63e^{i1.22\pi}, 0.38e^{i0.74\pi}, 0.37e^{i0.72\pi}))$
$\mathbb{U}_5$	$(0, (0.30e^{i0.58\pi}, 0.95e^{i1.92\pi}, 0.96e^{1.94i\pi}))$

Step 2: The decision of all experts cumulated using the $CSV\mathcal{N}NS_fWA$ operator with $\nu = (0.33, 0.40, 0.27)^T$ be the weight vector for the experts so that we get $ACSV\mathcal{N}NS_fDM$ summarized in Table 19.

Table 19: Aggregated complex single-valued neutrosophic N-soft decision matrix

	Y_1	Y_2	Y_3
$\mathbb{U}_1$	$(4,(0.0680e^{i0.1440\pi},0.9139e^{i1.8191\pi},0.9096e^{i1.8092\pi}))$	$(3,(0.0430e^{i0.0920\pi},0.9591e^{i1.9208\pi},0.9568e^{i1.91598\pi}))$	$(2,(0.0041e^{i0.0098\pi},0.9966e^{i1.9940\pi},0.9974e^{i1.9956\pi}))$
$\mathbb{U}_2$	$(4,(0.0710e^{i0.1520\pi},0.9177e^{i1.8278\pi},0.9139e^{i1.81914\pi}))$	$(4,(0.0380e^{i0.95569\pi},0.9544e^{i1.9114\pi},0.9591e^{i1.9159\pi}))$	$(4,(0.0760e^{i0.1400\pi},0.8916e^{i1.7974\pi},0.92434e^{i1.8544\pi}))$
$\mathbb{U}_3$	$(3,(0.0040e^{i0.0860\pi},0.9425e^{i1.8885\pi},0.9474e^{i1.8979\pi}))$	$(4,(0.0720e^{i0.1360\pi},0.9045e^{i1.8192\pi},0.9212e^{i1.8355\pi}))$	$(2,(0.0053e^{i0.0098\pi},0.9954e^{i1.9900\pi},0.9958e^{i1.9909\pi}))$
$\mathbb{U}_4$	$(3,(0.0590e^{i0.1260\pi},0.9407e^{i1.8850\pi},0.9458e^{i1.8949\pi}))$	$(4,(0.0820e^{i0.0180\pi},0.8698e^{i1.6972\pi},0.8825e^{i1.7832\pi}))$	$(4,(0.0050e^{i0.0108\pi},0.9954e^{i1.9901\pi},0.9946e^{i1.9901\pi}))$
$\mathbb{U}_5$	$(4,(0.070e^{i0.1520\pi},0.9323e^{i1.8597\pi},0.9272e^{i1.8597\pi}))$	$(4,(0.0780e^{i0.1720\pi},0.9096e^{i1.8092\pi},0.9212e^{i1.8486\pi}))$	$(1,(0.0007e^{i0.0028\pi},0.9982e^{i1.9970\pi},0.9989e^{i1.9986\pi}))$

	Y_4	Y_5
$\mathbb{U}_1$	$(1,(0.0045e^{i0.0108\pi},0.9977e^{i1.9948\pi},0.9970e^{i1.9932\pi}))$	$(2,(0.0330e^{i0.0624\pi},0.9687e^{i1.9391\pi},0.9704e^{i1.9423\pi}))$
$\mathbb{U}_2$	$(3,(0.0050e^{i0.0106\pi},0.9950e^{i1.9892\pi},0.9946e^{i1.9900\pi}))$	$(2,(0.0387e^{i0.0817\pi},0.9517e^{i1.9008\pi},0.9489e^{i1.8949\pi}))$
$\mathbb{U}_3$	$(3,(0.0146e^{i0.0272\pi},0.9853e^{i1.9716\pi},0.9858e^{i1.9726\pi}))$	$(4,(0.0432e^{i0.0887\pi},0.9517e^{i1.9008\pi},0.9544e^{i1.9062\pi}))$
$\mathbb{U}_4$	$(4,(0.0904e^{i0.2024\pi},0.8916e^{i1.7650\pi},0.8486e^{i1.7397\pi}))$	$(4,(0.1084e^{i0.2349\pi},0.8987e^{i1.7832\pi},0.8825e^{i1.7832\pi}))$
$\mathbb{U}_5$	$(1,(0.0010e^{i0.0007\pi},0.9985e^{i1.9962\pi},0.9992e^{i1.9978\pi}))$	$(1,(0.0182e^{i0.00216\pi},0.9982e^{i1.9955\pi},0.9978e^{i1.9948\pi}))$

Step 3: In CAMELS model each attribute has its own worth and value that continuously change as the time passing out, therefore experts rank them and then assigned $CSV\mathcal{N}NS_fNs$ accordingly. We summarized the weights of the experts related to the attributes, are arranged in Table 20, using the $CSV\mathcal{N}NS_fWA$ operator and get the weight vector θ, given as:

$$\chi = \begin{pmatrix} (2,(0.0079e^{i0.0168\pi},0.9893e^{i1.9794\pi},0.9902e^{1.9814i\pi})) \\ (4,(0.0387e^{i0.0794\pi},0.9388e^{i1.8814\pi},0.9425e^{1.8884i\pi})) \\ (4,(0.0820e^{i0.1720\pi},0.9298e^{i1.8544\pi},0.9243e^{1.8424i\pi})) \\ (3,(0.0408e^{i0.0804\pi},0.9458e^{i1.8948\pi},0.9489e^{1.9008i\pi})) \\ (3,(0.0180e^{i0.0372\pi},0.9642e^{i1.9304\pi},0.9842e^{1.9672i\pi})) \end{pmatrix}$$

Table 20: Weights for attributes from experts

	$\widetilde{Z}_1$	$\widetilde{Z}_2$	$\widetilde{Z}_3$
Y_1	$(1,(0.20e^{i0.42\pi},0.74e^{i1.50\pi},0.76e^{i1.54\pi}))$	$(2,(0.42e^{i0.86\pi},0.38e^{i0.778\pi},0.62e^{i1.22\pi}))$	$(0,(0.09e^{i0.24\pi},0.92e^{i1.86\pi},0.95e^{i1.88\pi}))$
Y_2	$(3,(0.67e^{i1.36\pi},0.17e^{i0.36\pi},0.19e^{i0.40\pi}))$	$(4,(0.93e^{i1.88\pi},0.09e^{i0.16\pi},0.14e^{i0.26\pi}))$	$(1,(0.18e^{i0.38\pi},0.70e^{i1.42\pi},0.72e^{i1.46\pi}))$
Y_3	$(4,(0.91e^{i1.84\pi},0.13e^{i0.24\pi},0.11e^{i0.20\pi}))$	$(1,(0.16e^{i0.34\pi},0.66e^{i1.37\pi},0.68e^{i1.38\pi}))$	$(2,(0.44e^{i0.90\pi},0.40e^{i0.82\pi},0.60e^{i1.18\pi}))$
Y_4	$(3,(0.69e^{i1.40\pi},0.21e^{i0.44\pi},0.23e^{i0.48\pi}))$	$(3,(0.71e^{i1.42\pi},0.25e^{i0.52\pi},0.27e^{i0.56\pi}))$	$(3,(0.75e^{i1.53\pi},0.31e^{i0.64\pi},0.33e^{i0.68\pi}))$
Y_5	$(2,(0.40e^{i0.82\pi},0.36e^{i0.74\pi},0.64e^{i1.26\pi}))$	$(3,(0.73e^{i1.48\pi},0.29e^{i0.60\pi},0.30e^{i0.62\pi}))$	$(3,(0.77e^{i1.56\pi},0.31e^{i0.60\pi},0.26e^{i0.50\pi}))$

Step 4: The weight vector θ and $ACSV\mathcal{N}NS_fDM$ are encapsulated using the $CSV\mathcal{N}NS_fWG$ operator into $AWCSV\mathcal{N}NS_fDM$, compile in Table 21.

Table 21: Aggregated weighted complex single-valued neutrosophic N-soft decision matrix

	Y_1	Y_2	Y_3
$\mathbb{U}_1$	$(2,(0.00053e^{i0.00120\pi},0.99900e^{i1.99812\pi},0.99911e^{i1.99822\pi}))$	$(3,(0.00016e^{i0.00036\pi},0.99749e^{i1.99528\pi},0.99751e^{i1.9953\pi}))$	$(2,(0.00032e^{i0.00084\pi},0.99976e^{i1.99956\pi},0.99980e^{i1.99964\pi}))$
$\mathbb{U}_2$	$(2,(0.00055e^{i0.00126\pi},0.99912e^{i1.99822\pi},0.99916e^{i1.99832\pi}))$	$(4,(0.00014e^{i0.00032\pi},0.99720e^{i1.99474\pi},0.99764e^{i1.99530\pi}))$	$(4,(0.00062e^{i0.00120\pi},0.99239e^{i1.98524\pi},0.99427e^{i1.98852\pi}))$
$\mathbb{U}_3$	$(2,(0.00031e^{i0.00072\pi},0.99938e^{i1.99884\pi},0.99948e^{i1.99906\pi}))$	$(4,(0.00028e^{i0.00052\pi},0.99416e^{i1.98868\pi},0.99546e^{i1.9908\pi}))$	$(2,(0.00043e^{i0.00084\pi},0.99968e^{i1.99926\pi},0.99968e^{i1.99928\pi}))$
$\mathbb{U}_4$	$(2,(0.00046e^{i0.00104\pi},0.99936e^{i1.99880\pi},0.99946e^{i1.99902\pi}))$	$(4,(0.00032e^{i0.00072\pi},0.99203e^{i1.98928\pi},0.99324e^{i1.9879\pi}))$	$(4,(0.00041e^{i0.00092\pi},0.99968e^{i1.99926\pi},0.99959e^{i1.99920\pi}))$
$\mathbb{U}_5$	$(2,(0.00055e^{i0.00128\pi},0.99927e^{i1.99856\pi},0.99928e^{i1.99868\pi}))$	$(4,(0.00030e^{i0.00068\pi},0.994466e^{i1.98868\pi},0.99546e^{i1.9916\pi}))$	$(1,(0.00057e^{i0.00024\pi},0.99987e^{i1.99978\pi},0.99990e^{i1.99980\pi}))$

	Y_4	Y_5
$\mathbb{U}_1$	$(1,(0.00018e^{i0.00044\pi},0.99987e^{i1.99972\pi},0.99984e^{i1.99964\pi}))$	$(2,(0.00059e^{i0.00116\pi},0.99880e^{i1.99786\pi},0.99953e^{i1.99904\pi}))$
$\mathbb{U}_2$	$(3,(0.00020e^{i0.00044\pi},0.99973e^{i1.99942\pi},0.99972e^{i1.99948\pi}))$	$(3,(0.00069e^{i0.00152\pi},0.99827e^{i1.99652\pi},0.99919e^{i1.99826\pi}))$
$\mathbb{U}_3$	$(3,(0.00059e^{i0.00114\pi},0.99920e^{i1.99850\pi},0.99927e^{i1.99986\pi}))$	$(3,(0.00077e^{i0.00164\pi},0.99827e^{i1.99654\pi},0.99927e^{i1.9984\pi}))$
$\mathbb{U}_4$	$(3,(0.00368e^{i0.00850\pi},0.99412e^{i1.98760\pi},0.99220e^{i1.98680\pi}))$	$(3,(0.00195e^{i0.00436\pi},0.99637e^{i1.99242\pi},0.99814e^{i1.99644\pi}))$
$\mathbb{U}_5$	$(1,(0.00004e^{i0.00002\pi},0.99990e^{i1.99980\pi},1.00000e^{i1.99980\pi}))$	$(1,(0.00032e^{i0.00068\pi},0.99990e^{i1.99980\pi},0.99990e^{i2.00000\pi}))$

Step 5 The groundwork of the TOPSIS method that differentiate it from others is to evaluate the PIS and NIS that help to find out optimal solution using the tool of distance measure. The criteria evaluated for this MAGDM problem based on CAMELS model and all are related to benefit-type criteria. Therefore, the $CSV\mathcal{N}NS_f$-PIS and $CSV\mathcal{N}NS_f$-NIS, taking into account the nature of the attributes, are arranged in Table 22.

Table 22: $CSV\mathcal{N}NS_f$-PIS and $CSV\mathcal{N}NS_f$-NIS

$\mathbb{U}_s$	$\tilde{H}_w^{PIS}$	$\tilde{H}_w^{NIS}$
$\mathbb{U}_1$	$(2,(0.00055e^{i0.00126\pi},0.99912e^{i1.99822\pi},0.99916e^{i1.99832\pi}))$	$(2,(0.00031e^{i0.00072\pi},0.99938e^{i1.99880\pi},0.99948e^{i1.99906\pi}))$
$\mathbb{U}_2$	$(4,(0.00032e^{i0.00072\pi},0.99203e^{i1.98928\pi},0.99324e^{i1.98790\pi}))$	$(3,(0.00016e^{i0.00036\pi},0.99749e^{i1.99528\pi},0.99751e^{i1.99530\pi}))$
$\mathbb{U}_3$	$(4,(0.00062e^{i0.00120\pi},0.99239e^{i0.03400\pi},0.99427e^{i1.98852\pi}))$	$(1,(0.00570e^{i0.00024\pi},0.99987e^{i1.99970\pi},0.99989e^{1.99980\pi}))$
$\mathbb{U}_4$	$(3,(0.00368e^{i0.00850\pi},0.99412e^{i1.98760\pi},0.99220e^{1.98680\pi}))$	$(1,(0.00004e^{i0.00002\pi},0.99990e^{i1.99980\pi},1.00000e^{1.99980\pi}))$
$\mathbb{U}_5$	$(2,(0.00195e^{i0.00436\pi},0.99637e^{i1.99242\pi},0.99814e^{i1.99644\pi}))$	$(1,(0.00032e^{i0.00068\pi},0.99990e^{i1.99980\pi},0.99990e^{i2.0000\pi}))$

Step 6 For distance measure, normalized Euclidean distance is used that precisely evaluate the distance between the alternatives and the ideal solutions, simultaneously. Table 23 describe the distance of each alternative from $CSV\mathcal{N}NS_f$-PIS and $CSV\mathcal{N}NS_f$-NIS, respectively.

Table 23: Distance measures of alternatives from ideal solution

$\mathbb{U}_s$	$d(\bar{H}_w^{PIS}, \mathbb{U}_s)$	$d(\bar{H}_w^{NIS}, \mathbb{U}_s)$
$\mathbb{U}_1$	0.133746	0.059764
$\mathbb{U}_2$	0.005061	0.179298
$\mathbb{U}_3$	0.084647	0.13363
$\mathbb{U}_4$	0.003998	0.1793085
$\mathbb{U}_5$	0.174320	0.042260

Step 7: Revised closeness index is used for ranking the alternatives having the properties of closeness and far-away from the ideal solution at a time. The numeric values of revised closeness index calculated in Table **??**

Table 24: Index of alternatives

$\mathbb{U}_s$	$\Lambda(\mathbb{U}_s)$
$\mathbb{U}_1$	33.1199
$\mathbb{U}_2$	0.26594
$\mathbb{U}_3$	20.4343
$\mathbb{U}_4$	0.00000
$\mathbb{U}_5$	43.3661

Step 8: Clearly, from the values of revised closeness index we can easily highlight the bank with best performance that is actually the $\mathbb{U}_4 = MCB$ opting as best performer in Pakistan, where, the ascending order of the values of revised closeness index describe the ranks of the banks on the basis of the CAMELS model and TOPSIS method, shown in Table 25

Table 25: Ranking of alternatives

Alternative	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_3$	$\mathbb{U}_4$	$\mathbb{U}_5$
Ranking	4	2	3	1	5

7 Comparison

To prove the versatility of the $CSV\mathcal{N}NS_f$-TOPSIS method we compare the proposed method with $SV\mathcal{N}$-TOPSIS method [28] by solving the describe MAGDM problem of "Monitoring performance of Islamic banking industry on the basis of CAMELS rating syste" by SVN-TOPSIS method[28]. The evaluation of the problem by SVN-TOPSIS method [28] is as follows:

Step 1 For the implication of SVN-TOPSIS method on the proposed MAGDM problem we have to exclude the grading part as well as reduce the periodic terms to zero in the $CSV\mathcal{N}NS_fN$, so that experts $\widetilde{Z}_1, \widetilde{Z}_2, \widetilde{Z}_3$ assigned SVNs to each rank given in Tables 26, 27 and 28, respectively.

Table 26: $SV\mathcal{N}DM$ related to expert $\widetilde{Z}_1$,

	Y_1	Y_2	Y_3	Y_4	Y_5
$\mathbb{U}_1$	$(0.86, 0.08, 0.07)$	$(0.71, 0.31, 0.29)$	$(0.11, 0.91, 0.93)$	$(0.12, 0.87, 0.86)$	$(0.61, 0.41, 0.43)$
$\mathbb{U}_2$	$(0.87, 0.09, 0.08)$	$(0.66, 0.27, 0.31)$	$(0.89, 0.04, 0.11)$	$(0.13, 0.87, 0.86)$	$(0.67, 0.25, 0.23)$
$\mathbb{U}_3$	$(0.69, 0.19, 0.22)$	$(0.88, 0.06, 0.10)$	$(0.14, 0.88, 0.89)$	$(0.34, 0.66, 0.67)$	$(0.71, 0.24e, 0.27)$
$\mathbb{U}_4$	$(0.82, 0.18, 0.21)$	$(0.91, 0.02, 0.03)$	$(0.13, 0.88, 0.86)$	$(0.93, 0.04, 0.01)$	$(0.96, 0.05, 0.03)$
$\mathbb{U}_5$	$(0.87, 0.13, 0.12)$	$(0.90, 0.07, 0.10)$	$(0.02, 0.95, 0.97)$	$(0.03, 0.96, 0.98)$	$(0.05, 0.95, 0.94)$

Table 27: $SV\mathcal{N}DM$ related to expert $\widetilde{Z}_2$,

	Y_1	Y_2	Y_3	Y_4	
$\mathbb{U}_1$	$(0.72, 0.32, 0.66)$	$(0.41, 0.51, 0.61)$	$(0.16, 0.69, 0.72)$	$(0.17, 0.75, 0.77)$	$(0.06, 0.91, 0.92)$
$\mathbb{U}_2$	$(0.19, 0.72, 0.75)$	$(0.93, 0.12, 0.13)$	$(0.88, 0.08, 0.10)$	$(0.73, 0.23, 0.20)$	$(0.64, 0.36, 0.37)$
$\mathbb{U}_3$	$(0.45, 0.46, 0.56)$	$(0.09, 0.87, 0.86)$	$(0.58, 0.37, 0.39)$	$(0.59, 0.53, 0.44)$	$(0.92, 0.05, 0.12)$
$\mathbb{U}_4$	$(0.32, 0.67, 0.69)$	$(0.84, 0.16, 0.17)$	$(0.83, 0.18, 0.19)$	$(0.98, 0.10, 0.01)$	$(0.81, 0.20, 0.19)$
$\mathbb{U}_5$	$(0.11, 0.90, 0.91)$	$(0.22, 0.81, 0.84)$	$(0.08, 0.91, 0.92)$	$(0.07, 0.87, 0.88)$	$(0.23, 0.83, 0.82)$

Table 28: $SV\mathcal{N}DM$ related to expert $\widetilde{Z}_3$,

	Y_1	Y_2	Y_3	Y_4	Y_5
$\mathbb{U}_1$	$(0.62, 0.36, 0.39)$	$(0.70, 0.26, 0.28)$	$(0.59, 0.43, 0.42)$	$(0.86, 0.02, 0.03)$	$(0.31, 0.69, 0.68)$
$\mathbb{U}_2$	$(0.81, 0.20, 0.18)$	$(0.95, 0.05, 0.07)$	$(0.80, 0.21, 0.22)$	$(0.62, 0.36, 0.38)$	$(0.34, 0.66, 0.67)$
$\mathbb{U}_3$	$(0.31, 0.68, 0.69)$	$(0.60, 0.41, 0.42)$	$(0.29, 0.70, 0.72)$	$(0.79, 0.23, 0.20)$	$(0.61, 0.39, 0.40)$
$\mathbb{U}_4$	$(0.84, 0.17, 0.16)$	$(0.96, 0.03, 0.02)$	$(0.98, 0.04, 0.03)$	$(0.82, 0.18, 0.19)$	$(0.63, 0.38, 0.37)$
$\mathbb{U}_5$	$(0.27, 0.74, 0.73)$	$(0.57, 0.45, 0.47)$	$(0.25, 0.76, 0.78)$	$(0.23, 0.79, 0.81)$	$(0.30, 0.95, 0.96)$

Step 2 The weights of experts $\nu = (0.33, 0.40, 0.27)^T$ and averaging operator [28], we can cumulate the aggregated single-valued neutrosophic decision matrix $(ASV\mathcal{N}DM)$, as follows:

$$\mathcal{H}_{ws} = \Big([1 - \Pi_{j=1}^{f}(1 - p_{ws}^{(j)})^{\nu_w}], [\Pi_{j=1}^{f}(q_{ws}^{(j)})^{\nu_w}], [\Pi_{j=1}^{f}(r_{ws}^{(j)})^{\nu_w}]\Big).$$

The $ASVNDM$ is arranged in Table 29.

Table 29: $ASV\mathcal{N}DM$

	Y_1	Y_2	Y_3	Y_4	Y_5
$\mathbb{U}_1$	$(0.0680, 0.9139, 0.9096)$	$(0.0430, 0.9591, 0.9568)$	$(0.0041, 0.9966, 0.9974)$	$(0.0045, 0.9977, 0.9970)$	$(0.0330, 0.9687, 0.9704)$
$\mathbb{U}_2$	$(0.0710, 0.9177, 0.9139)$	$(0.0380, 0.9544, 0.9591)$	$(0.0760, 0.8916, 0.92434)$	$(0.0050, 0.9950, 0.9946)$	$(0.0387, 0.9517, 0.9489)$
$\mathbb{U}_3$	$(0.0040, 0.9425, 0.9474)$	$(0.0720, 0.9045, 0.9212)$	$(0.0053, 0.9954, 0.9958)$	$(0.0146, 0.9853, 0.9858)$	$(0.0432, 0.9517, 0.9544)$
$\mathbb{U}_4$	$(0.0590, 0.9407, 0.9458)$	$(0.0820, 0.8698, 0.8825)$	$(0.0050, 0.9954, 0.9946)$	$(0.0904, 0.8916, 0.8486)$	$(0.1084, 0.8987, 0.8825)$
$\mathbb{U}_5$	$(0.070, 0.9323, 0.9272)$	$(0.0780, 0.9096, 0.9212)$	$(0.0007, 0.9982, 0.9989)$	$(0.0010, 0.9985, 0.9992)$	$(0.0182, 0.9982, 0.9978)$

Step 3 The weights for attributes are calculated, by summarizing the experts opinion about the nature of attributes given in Table 30, as follows:

$$\theta_w = \Big([1 - \Pi_{j=1}^{f}(1 - p_w^{(j)})^{\nu_w}], [\Pi_{j=1}^{f}(q_w^{(j)})^{\nu_w}], [\Pi_{j=1}^{f}(r_w^{(j)})^{\nu_w}]\Big).$$

Thus we have,

$$\theta = \begin{pmatrix} (0.0079, 0.9893, 0.9902) \\ (0.0387, 0.9388, 0.9425) \\ (0.0820, 0.9298, 0.9243) \\ (0.0408, 0.9458, 0.9489) \\ (0.0180, 0.9642, 0.9842) \end{pmatrix}.$$

Table 30: Weights for attributes from experts

	$\widetilde{Z}_1$	$\widetilde{Z}_2$	$\widetilde{Z}_3$
Y_1	$(0.20, 0.74, 0.76)$	$(0.42, 0.38, 0.62)$	$(0.09, 0.92, 0.95)$
Y_2	$(0.67, 0.17, 0.19)$	$(0.93, 0.09, 0.14)$	$(0.18, 0.70, 0.72)$
Y_3	$(0.91, 0.13, 0.11)$	$(0.16, 0.66, 0.68)$	$(0.44, 0.40, 0.60)$
Y_4	$(0.69, 0.21, 0.23)$	$(0.71, 0.25, 0.27)$	$(0.75, 0.31, 0.33)$
Y_5	$(0.40, 0.36, 0.64)$	$(0.73, 0.29, 0.30)$	$(0.77, 0.31, 0.26)$

Step 4 The aggregated weighted single-valued neutrosophic decision matrix($AWSV\mathcal{N}DM$), shown in Table 31, calculated as:

$$\bar{H}_{ws} = \Big(p_{ws}p_w, (q_{ws} + q_w - q_{ws}q_w), (r_{ws} + r_w - r_{ws}r_w)\Big).$$

Table 31: $AWSV\mathcal{N}DM$

	Y_1	Y_2	Y_3	Y_4	Y_5
$\mathbb{U}_1$	$(0.00053, 0.99900, 0.99911)$	$(0.00016, 0.99749, 0.99751)$	$(0.00032, 0.99976, 0.99980)$	$(0.00018, 0.99987, 0.99984)$	$(0.00059, 0.99880, 0.99953)$
$\mathbb{U}_2$	$(0.00055, 0.99912, 0.99916)$	$(0.00014, 0.99720, 0.99764)$	$(0.00062, 0.99239, 0.99427)$	$(0.00020, 0.99973, 0.99972)$	$(0.00069, 0.99827, 0.99919)$
$\mathbb{U}_3$	$(0.00031, 0.99938, 0.99948)$	$(0.00028, 0.99416, 0.99546)$	$(0.00043, 0.99968, 0.99968)$	$(0.00059, 0.99920, 0.99927)$	$(0.00077, 0.99827, 0.99927)$
$\mathbb{U}_4$	$(0.00046, 0.99936, 0.99946)$	$(0.00032, 0.99203, 0.99324)$	$(0.00041, 0.99968, 0.99959)$	$(0.00368, 0.99412, 0.99220)$	$(0.00195, 0.99637, 0.99814)$
$\mathbb{U}_5$	$(0.00055, 0.99927, 0.99928)$	$(0.00030, 0.994466, 0.99546)$	$(0.00057, 0.99987, 0.99990)$	$(0.00004, 0.99990, 1.00000)$	$(0.00032, 0.99990, 0.99990)$

Step 5 Keeping in view the nature of data, Equation 26 and 27 is used for the evaluation of the single-valued neutrosophic positive ideal solution and negative ideal solution arranged in Table 32.

$$\bar{H}_w^{PIS} = \begin{cases} (\max_s \bar{\mathbb{T}}_{ws}, \min_s \bar{\mathbb{I}}_{ws}, \min_s \bar{\mathbb{F}}_{ws}), & \text{if } Y_w \in \mathbb{BT}, \\ (\min_s \bar{\mathbb{T}}_{ws}, \max_s \bar{\mathbb{I}}_{ws}, \max_s \bar{\mathbb{F}}_{ws}), & \text{if } Y_w \in \mathbb{CT}, \end{cases} \tag{26}$$

and

$$\bar{H}_w^{NIS} = \begin{cases} (\min_s \bar{\mathbb{T}}_{ws}, \max_s \bar{\mathbb{I}}_{ws}, \max_s \bar{\mathbb{F}}_{ws}), & \text{if } Y_w \in \mathbb{BT}, \\ (\max_s \bar{\mathbb{T}}_{ws}, \min_s \bar{\mathbb{I}}_{ws}, \min_s \bar{\mathbb{F}}_{ws}), & \text{if } Y_w \in \mathbb{CT}, \end{cases} \tag{27}$$

Table 32: $SV\mathcal{N}$-PIS and $SV\mathcal{N}$-NIS

$\mathbb{U}_1$	$(0.00055, 0.99900, 0.99911)$	$(0.00031, 0.99938, 0.99948)$
$\mathbb{U}_2$	$(0.00032, 0.99203, 0.99324)$	$(0.00014, 0.99749, 0.99764)$
$\mathbb{U}_3$	$(0.00062, 0.99239, 0.99427)$	$(0.00033, 0.99987, 0.99990)$
$\mathbb{U}_4$	$(0.00368, 0.99412, 0.99220)$	$(0.00004, 0.99990, 1.00000)$
$\mathbb{U}_5$	$(0.00195, 0.99637, 0.99814)$	$(0.00032, 0.99990, 0.99990)$

Step 6 To measure distance of alternatives from PIS and NIS, Euclidean distance used. The calculated values are given in Table 33

Table 33: Distance measures of alternatives from ideal solution

$\mathbb{U}_s$	$d(\bar{H}_w^{PIS}, \mathbb{U}_s)$	$d(\bar{H}_w^{NIS}, \mathbb{U}_s)$
$\mathbb{U}_1$	0.00935	0.00078
$\mathbb{U}_2$	0.00762	0.00660
$\mathbb{U}_3$	0.00810	0.00260
$\mathbb{U}_4$	0.00500	0.00763
$\mathbb{U}_5$	0.00890	0.00210

Step 7 The revised closeness index calculated using Equation 28 , is tabulated in Table 34 and the ranks evaluated through the index values are arranged in Table 35 in descending order, according to which $\mathbb{U}_4$ is the best performer.

$$\Lambda(\mathbb{U}_s) = \frac{d(\bar{H}_w^{NIS}, \mathbb{U}_s)}{d(\bar{H}_w^{PIS}, \mathbb{U}_s) + d(\bar{H}_w^{NIS}, \mathbb{U}_s)}, \tag{28}$$

where, $s = 1, 2, \ldots, k$.

Table 34: Revised closeness index of each alternative

Alternative	$\Lambda(\mathbb{U}_s)$
$\mathbb{U}_1$	0.0769
$\mathbb{U}_2$	0.4641
$\mathbb{U}_3$	0.2429
$\mathbb{U}_4$	0.6041
$\mathbb{U}_5$	0.1900

Table 35: Ranking in single-valued neutrosophic environment

Alternative	$\mathbb{U}_1$	$\mathbb{U}_2$	$\mathbb{U}_3$	$\mathbb{U}_4$	$\mathbb{U}_5$
Ranking	5	2	3	1	1

7.1 Discussion

1. The comparison of the $CSV\mathcal{N}NS_f$-TOPSIS method with the existing SVN-TOPSIS method have same findings for the Islamic bank as best performer in Pakistan but the consequences relevant to the ranks of other banks have no analogy given in Table 36.

Table 36: Comparison

Model	Ranks	Best Performer
$SV\mathcal{N}$-TOPSIS [28]	$\mathbb{U}_4 > \mathbb{U}_2 > \mathbb{U}_3 > \mathbb{U}_5 > \mathbb{U}_1$	$\mathbb{U}_4$
$CSV\mathcal{N}NS_f$-TOPSIS(Proposed)	$\mathbb{U}_4 > \mathbb{U}_2 > \mathbb{U}_3 > \mathbb{U}_1 > \mathbb{U}_5$	$\mathbb{U}_4$

2. The expertise of the presented methodology $CSV\mathcal{N}NS_f$-TOPSIS method to manipulate the indeterminacy degree and two dimensional information in the MAGDM problems by using the frame of $CSV\mathcal{N}NS_fSs$.
3. The presented methodology of $CSV\mathcal{N}NS_f$-TOPSIS method has potential to operate the problems of $IFNS_fSs$, being the generalization of the $IFSs$.
4. The presented model has proficiency to overcome the latest problems characterized by parameterized ordered evaluation system but the existing methods have no grip on such problems.
5. By employing $N = 2$ and periodic terms equal to zero, we switch from $CSV\mathcal{N}NNS_f$ environment to single-valued environment so that the $CSV\mathcal{N}NS_f$-TOPSIS method could sensibly handled the daily life problems under single-valued environment.

8 Conclusion

In this paper we have merged the idea of single-valued neutrosophic set with N-soft sets, and in doing so, we have initiated the idea of $CSV\mathcal{N}NS_fSs$. These sets combine the 2-dimensional single-valued neutrosophic nature of the attributes with parameterized ordered grades which demonstrates their superiority over FNS_fS, $IFNS_fS$ and $\mathcal{N}NS_fS$. A MAGDM model of TOPSIS method is extended to handle the real life problems under the frame of $CSV\mathcal{N}NS_fSs$ in which the ordered grades are assigned to each alternative as initial evaluation that are further characterized by $CSV\mathcal{N}NS_fNs$. The PIS and NIS in $CSV\mathcal{N}NS_f$-TOPSIS method have been determined by the score function which has been further employed to quantify the distance measures and the closeness index that sort the alternatives from highest to lowest rank. An example from the banking industry and the comparison with single-valued neutrosophic TOPSIS method have clarified the accuracy and superiority of the presented technique. The new model and method pioneer a promising avenue for research in the decision making arena that we have only hinted at in this paper. Moreover, the proposed $CSV\mathcal{N}NS_f$-TOPSIS method does not evaluate the relative importance of the normalized Euclidean distances. Therefore we will work for the extension of the VIKOR method under a $CSV\mathcal{N}NS_f$ environment, which might be more credible and trustworthy.

Data availability: No data were used to support this study.
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References

[1] M. Akram, G. Ali, J. C. R. Alcantud, *Hybrid multi-attribute decision-making model based on (m, N)-soft rough sets*, Journal of Intelligent and Fuzzy Systems, **36**(6)(2019), 6325-6342.

[2] M. Akram, G. Ali, J. C. R. Alcantud and F. Fatimah, *Parameter reductions in N-soft sets and their applications in decision-making*, Expert Systems, (2020), e12601.

[3] M. Akram, A. Adeel, and J. C. R. Alcantud, *Fuzzy N-soft sets: A novel model with applications*, Journal of Intelligent and Fuzzy Systems, **35**(4)(2018), 4757-4771.

[4] M. Akram, A. Adeel, and J. C. R. Alcantud, *Hesitant fuzzy N-soft sets: A new model with applications in decision-making*, Journal of Intelligent and Fuzzy Systems, **36**(6)(2018), 6113-6127.

[5] M. Akram, A. Adeel and J.C.R. Alcantud, *Group decision-making methods based on hesitant N- soft sets*, Expert System with Applications, **115**(2019), 95-105.

[6] M. Akram, A. Luqman, *Certain network models using single-valued neutrosophic directed hypergraphs*, Journal of Intelligent and Fuzzy Systems, **33**(1)(2017), 575-588.

[7] M. Akram, F. Waseem and A. N. Al-Kenani, *A hybrid decision-making approach under complex Pythagorean fuzzy N-soft sets*, International Journal of Computational Intelligence Systems, 2021, https://doi.org/10.2991/ijcis.d.210331.002.

[8] M. Akram, M. Shabir, A.N. Al Kenani and J. C R Alcantud, *Hybrid decision making frameworks under complex spherical fuzzy N-soft sets*, Journal of Mathematics, vol. 2021, Article ID 5563215, 46 pages, 2021. https://doi.org/10.1155/2021/5563215.

[9] M. Akram, G. Ali, and J. C. R. Alcantud, *New decision-making hybrid model: Intuitionistic fuzzy N-soft rough sets*, Soft Computing, **23**(20)(2019), 9853-9868.

[10] J.C.R. Alcantud *A novel algorithm for fuzzy soft set based decision making from multiobserver input parameter data set*, Information Fusion, **29**(2016), 142-148.

[11] J. C. R. Alcantud, F. Feng and R.R. Yager, *An N-soft set approach to rough sets*, IEEE Transactions on Fuzzy Systems, **28**(2020), 2996–3007.

[12] G. Ali and M. Akram, Decision-making method based on fuzzy N-soft expert sets, Arabian Journal for Science and Engineering, **45**(12)(2020), 10381-10400.

[13] M. Ali and F. Smarandache, *Complex neutrosophic set*, Neural Computing and Applications, **27**(01)(2017), 1-18.

[14] A. Alkouri and A. Salleh, *Complex intuitionistic fuzzy sets*, 2nd international conference on fundamental and applied sciences, **1482**(2012), 464-470.

[15] A.M. Alkouri, A.R. Salleh, *Complex intuitionistic fuzzy sets*, AIP Confrence Proceedings, **1482**(1)(2012).

[16] A. Ashraf and M. A. Butt, *Extension of TOPSIS method under single-valued neutrosophic N-soft environment*, Neutrosophic sets and systems, **41** (2021), 286-303.

[17] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1)(1986), 87-96.

[18] F. Chiclana, F. Herrera and E. Herrera-Viedma, *The ordered weighted geometric operator, Properties and application*, In: Proc of 8th International Conference on Information Processing and Management of Uncertainty in Knowledgebased Systems, Madrid, (2000), 985-991.

[19] C.T. Chen, *Extension of TOPSIS for group decision-making under fuzzy environment*, Fuzzy Sets and Systems, **114**(1)(2000), 1-9.

[20] B.C. Cuong and V. Kreinovich, *Picture fuzzy sets-a new concept for computational intelligence problems*, In 2013 Third World Congress on Information and Communication Technologies (WICT 2013) (2013), 1-6.

[21] F. Fatimah, D. Rosadi, R.B.F. Hakim and J.C.R. Alcantud, *N-soft sets and their decision-making algorithms*, Soft Computing, **22**(12)(2018), 3829-3842.

[22] F. Fatimah and J.C.R. Alcantud, *The multi-fuzzy N-soft set and its applications to decision-making*, Neural Computing and Applications, 2021, forthcoming.

[23] C. Jana and M. Pal, *A robust single-valued neutrosophic soft aggregation operators in multi-criteria decision making*, Symmetry, **11**(2019), 110.

[24] P. K. Maji, *Neutrosophic soft set*, Annal of Fuzzy Mathematics and Informatics, **5**(1)(2013), 157-168.

[25] D.A. Molodtsov, *Soft set theory-first results*, Computers and Mathematics with Applications, **37**(4-5)(1999), 19-31.

[26] D. Ramot, R. Milo, M. Friedman and A. Kandel, *Complex fuzzy sets*, IEEE Transactions on Fuzzy Systems, **10**(2)(2002), 171-186.

[27] M. Riaz, K. Naeem, I. Zareef and D. Afzal, *Neutrosophic N-soft set with TOPSIS method for multiple attribute decision making*, Neutrosophic sets and systems, **32** (2020), 1-24.

[28] R. Sahin and M. Yigider, *A multi-criteria neutrosophic group decision making metod based TOPSIS for supplier selection*, Applied Mathematics and Informatin Sciences, **10**(5)(2016), 1843-1852.

[29] F. Smarandache, *Neutrosophy: Neutrosophic probabillity, set and logic* , American Research Press, Rehoboth, USA, (1998).

[30] F.Smarandache, *A unifying field in logics., Neutrosophic probability, set and logic*, Rehoboth: American Research Press, (1999).

[31] H. Wang, F. Smarandache, Y. Q. Zhang and R. Sundarraman, *Single valued neutrosophic set* , Multispace and Multistructure, **4**(2011), 410-413.

[32] Z. Xu, *Intuitionistic fuzzy aggregation operators*, IEEE Transaction on Fuzzy Systems, **15**(6)(2007), 1179-1187.

[33] R.R. Yager, *Pythagorean membership grades in multicriteria decision making*, IEEE Transaction on Fuzzy Systems, **22**(4)(2013), 958-965.

[34] J. Ye, *Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment*, International Journal of General Systems, **42**(4)(2013), 386-394.

[35] J. Ye, *Single-valued neutrosophic minimum spanning tree and its clustering method*, Journal of Intelligent Systems, **23**(3)(2014), 311-324.

[36] J. Ye, *An extended TOPSIS method for multiple attribute decision making based on single valued neutrosophic linguistic numbers*, Journal of Intelligent and fuzzy Systems, **28**(1)(2015), 247-255.

[37] K. Ullah, T. Mahmood, Z. Ali and N. Jan (2020)*On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition*. Complex and Intellegent System **6**(2020), 15-27.

[38] L.A. Zadeh, *Fuzzy sets*, Information and Control, **8**(3)(1965), 338-356.

[39] H. Zhang, D. Jia-Hua and C. Yan, *Multi-attribute group decision-making methods based on Pythagorean fuzzy N-soft sets*, IEEE Access, **8**(2020), 62298-62309.

[40] G.T. Zhang, S, Dillon, K.Y. Cai, J. Ma and J. Lu, *Operation properties and δ-equalities of complex fuzzy sets*, International Journal of Approximate Reasoning, **50**(2009), 1227-1249.

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Medical diagnosis based on single-valued neutrosophic information

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Abstract:

Women with heart disease during pregnancy are at higher risk, which can harm the fetus. This risk can be reduced if we diagnose and treat it early. The decision-making system is very helpful in such situations. Many clinical decision-making systems have been proposed, but they are too complicated for medical experts to understand and adapt. Here, we develop a new neutrosophic model for early diagnosis and explain it using explainable artificial techniques. Our model is taking eight symptoms and signs as inputs and determines the diagnosis, type of treatment, and prognosis. Age, obesity, smoking, family pathological history, personal pathological history, electrocardiogram, ultrasound, and functional class are the inputs of this model. Six diagnoses can be made- obstruction at existing, obstruction at entry, rhythm disorder, conduction disorders, congenital diseases, genetic diseases. The types of treatments are- pregnancy interruption, diuretic treatment, anti-arrhythmic treatment, treatment with beta-blockers and anticoagulants treatment. The prognosis is- eutectic delivery, dystocic delivery, the child with complications, child without complications, mother with complications, and mother without complications. The main parts of this system are neutrosophication, knowledge base, inference engine, de-neutrosophication, and explainability. To present the entire execution of the proposed system, we design an algorithm and compute its time complexity to demonstrate the working of the entire system. We compared the results of different methods to gain confidence in our model.

Keywords: Neutrosophic sets, decision-making, heart diseases, algorithm, explainable articial intelligence.

1 Introduction

In medical, computer-aided medical diagnosing applications facilitates doctors to take decisions swiftly. Many models and applications have been designed for this purpose but the major drawback of such models is their complexity and a lot of mathematical work. Their complex models make it difficult for doctors to adopt. Explainable artificial intelligence (XAI) is an approach that makes such a model understandable for doctors and they feel comfortable adopting them. XAI also helps doctors to check the accuracy of the decisions as well.

The medical data is very much sensitive and contains a lot of ambiguities because each doctor has his own opinion, and using these opinions it becomes difficult to take one exact decision. In such environments, fuzzy logic plays an important role to make human-like decisions among multiple decisions. The concept of fuzzy sets was introduced by Zadeh in 1965 after that many extensions of fuzzy sets were proposed, intuitionistic fuzzy is also one of them which works with membership and non-membership [1]. But the restriction on sum of membership and non-membership restricts the selection of membership and nonmembership values.

In 1995, Smarandache introduced a novel branch of philosophy known as neutrosophy to eliminate this issue [2]. Neutrosophy is the essence of the neutrosophic set (NS) and neutrosophic logic (NL). NS concurrently considers true membership, falsity membership, and indeterminacy membership, which are more effective and consistent as compare to fuzzy systems and intuitionistic fuzzy systems. The single-valued neutrosophic set (SVNS) is an extension of the NS [3]-[4]. There are many applications of fuzzy and its extensions are discussed in literature, some of them are [5]-[11].

Abdel-Basset et al. proposed a novel neutrosophic multi-criteria decision-making (MCDM) model that used the neutrosophic analytical network process (ANP), and the TOPSIS method for deciding on the election of an appropriate candidate for job [12]. They using this MCDM technique selection of Chief executive officer (CEO) vacancy. The proposed MCDM procedure combined quantitative and qualitative information for personnel selection.

In 2019, Hashmi et al. proposed a new concept of m-polar neutrosophic set (MPNS) and topological structure on m-polar neutrosophic set by fusing polar fuzzy set (MPFS) and neutrosophic set [13]. They proposed a score function for the estimate of m-polar neutrosophic numbers (MPNNs). m-polar neutrosophic topology is established and defined the interior, exterior, and frontier for m-polar neutrosophic sets (MPNs).

In 2020, Pamucar et al. introduced a neutrosophic decision-making model for the selection of supplier [14]. Their target was to decrease the risk and disruptions to the supply chain and to preserve the stability of the supply-chain system. Also, the unpredictable situations in a supply chain force decisionmakers and authorities to choose a fuzzybased evaluation platform to guarantee safe and secure outcomes. They proposed a new weight aggregator that uses a pairwise comparison.

In 2020, Pamucar et al. evaluated and prioritized the energy storage technology alternatives (methods) by considering technical cost, and environmental and social criteria. They proposed a hybrid trapezoidal neutrosophic fuzzy numbers based Dombi weighted geometric averaging operator and MultiAtributive Ideal-Real Comparative Analysis (MAIRCA) model [15]. They employed a case study in Romania is carried out to illustrate the applicability of the proposed model.

In 2020, Chakraborty et al. suggested the notion of cylindrical neutrosophic single-valued numbers from a different perspective and aspects to obtained its true result [16]. This concept is based on score and accuracy functions that are used to convert fuzzy numbers into a crisp number to make decisions on different problems. This technique was applied to real-life examples like networking and obtain the result that it is a better choice for decision making instead of neutrosophic numbers when falsity and indeterminacy function are dependent. The technique introduced in this paper is helpful in engineering and science-related field for diagnosis purposes.

In 2020, Aslan et al. presented the notion of neutrosociology [17]. It is a more effective way to find out uncertainties in social theories as it is impossible to find out with classical maths. The researchers used similarity measures of single-valued neutrosophic numbers for sociology-related decision-making problems. By using neutrosophic numbers and sets related to other social theories, new similarity measures introduced, and checked the appropriateness of these formulas.

In 2020, Tan et al. proposed a multi-attribute decision-making method for the decision-making problems where the attribute weight is unknown. Using information of entropy evaluation is performed [18]. They defined new formulas of single-valued neutrosophic similarities and single-valued neutrosophic entropy. Moreover, the connection between them is also discussed. Karabasevic et al. introduced a unique type of the TOPSIS method applicable for the use of single-valued neutrosophic sets [19]. The motivations of this study are described below:

1. Women with heart disease during pregnancy are at higher risk, which can have a negative effect on the fetus.
2. Medical data may contain unclear information. Using such data may lead to a wrong diagnosis if not managed efficiently.
3. Modern decision-making systems are very much complicated, and due to non-transparency, it becomes difficult for medical professionals to adopt them.

Our contributions to this research are as follows:

1. We developed a novel decision-making model to facilitate doctors to early predict diagnosis, type of treatment, and prognosis.
2. Single-valued neutrosophic sets are used for decision-making because they are very close to human reasoning. SVNS focuses on the degree of truth, the degree of indeterminacy, and the degree of falsity simultaneously. Also, there is no restriction on its sum of membership and non-membership.
3. We combine the theory of explainable AI to form our system adaptable for medical experts. The degree of explanation is estimated using causability. Explainable AI and causability AI systems support building the trust of medical experts.
4. We have designed an algorithm to explain the entire functioning of the model, as well as to measure its time complexity.
5. We compared our results with the rest of the techniques.

The rest of this article has been designed subsequently: Section 2 concisely reconsiders essential concepts of neutrosophic sets, explainable AI, and causability measures. Section 3 reviews the explainable neutrosophic clinical decision-making systems for the treatment of pregnant women with heart diseases. Section 4 offers a case study to present the effectiveness of the system. Section 5 matches the values of the proposed model with other theories. Section 6 ends this article and presents possible future research directions.

2 Preliminaries

This part recalls some of the preparatory notions that require to be read to completely benefit from this study.

2.1. Explainability [20]: In artificial intelligence (AI), explainability is the extent to which the internal mechanism of the algorithm can be explained in humans terms. Two explainable models have been found in the literature; the post-hoc explainability, and ante-hoc explaining models. The scope of the post-hoc model is local, in which it explains only the specific component of the algorithm, not the complete system. Its examples are LIME and BETA models. Whereas, the ante-hoc system are interpretable by design. Its examples are linear regression, decision trees, and fuzzy inference systems.

2.2. Causability [20]: Causability measures the quality of explainable. This means that the provided explanation is how much effective and understandable to humans.

2.3. Single-valued neutrosophic set (SVNS) [3]: Let A be a sample space. A SVNS B on a non-empty set A is described by a truth membership function $T_B : A \to [0,1]$, indeterminacy membership function $I_B : A \to [0,1]$ and a falsity membership function $F_B : A \to [0,1]$. Thus, $B = \{< w, T_B(w), I_B(w), F_B(w) > | w \in A\}$. There is no restraint on the sum of $T_B(w)$, $I_B(w)$ and $F_B(w)$ for all $w \in A$.

2.4. Neutrosophic logic [21]: Neutrosophic logic (NL) was bestowed by Smarandache as an extension of fuzzy logic, intuitionistic logic, and para-consistent logic. This logic contains three basic parts, that is, truth membership, indeterminacy membership, and falsity membership.

2.5. Single-valued neutrosophic number [22]: Let V be a SVNN which is defined as $V = ([(u_1, v_1, w_1, x_1); \rho], [(u_2, v_2, w_2, x_2); \sigma], [(u_3, v_3, w_3, x_3); \omega])$ where $\rho, \sigma, \omega \in [0,1]$, the truth membership function $(\mu_V) : \mathbb{R} \to [0, \rho]$, indeterminacy membership function $(\nu_V) : \mathbb{R} \to [\sigma, 1]$, and falsity membership function $(\lambda_V) : \mathbb{R} \to [\omega, 1]$ are represented as follows:

$$\mu_V(w) = \begin{cases} \mu_{Al}(w), & \text{if } u_1 \leq w \leq v_1, \\ \rho, & \text{if } v_1 \leq w \leq w_1, \\ \mu_{Au}(w), & \text{if } w_1 \leq w \leq x_1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_V(w) = \begin{cases} \nu_{Al}(w), & \text{if } u_2 \leq w \leq v_2, \\ \sigma, & \text{if } v_2 \leq w \leq w_2, \\ \nu_{Au}(w), & \text{if } w_2 \leq w \leq x_2, \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_V(w) = \begin{cases} \lambda_{Al}(w), & \text{if } u_3 \leq w \leq v_3, \\ \omega, & \text{if } v_3 \leq w \leq w_3, \\ \lambda_{Au}(w), & \text{if } w_3 \leq w \leq x_3, \\ 1, & \text{otherwise.} \end{cases}$$

2.6. Operations of SVNS [3]: Let A be a space of points and V_1 and V_2 are the two SVNSs and V_3 contains their result.

Intersection: The intersection of V_1 and V_2 is represented as follows:

$$T_{V_3}(w) = \min(T_{V_1}(w), T_{V_2}(w)),$$
$$I_{V_3}(w) = \max(I_{V_1}(w), I_{V_2}(w)),$$
$$F_{V_3}(w) = \max(F_{V_1}(w), F_{V_2}(w)),$$

for all w in A.

Union: The union of V_1 and V_2 is represented as follows:

$$T_{V_3}(w) = \max(T_{V_1}(w), T_{V_2}(w)),$$
$$I_{V_3}(w) = \min(I_{V_1}(w), I_{V_2}(w)),$$
$$F_{V_3}(w) = \min(F_{V_1}(w), F_{V_2}(w)),$$

for all w in A.

2.7. Major factors :

There are following major factors that can increase the risks during pregnancy.

Age (S1): The length of time that a person has lived or a thing has existed.

Obesity (S2): Obesity is a complicated condition including an unnecessary volume of body fat.

Smoking (S3): Smoking is a habit of gasping smoke of tobacco or a drug.

Family pathological history (S4): Pathology includes investigating the cause of sickness, how it occurs, the impact of the sickness on cells, and the consequence of the sickness. If this sickness comes from family, then it is called family pathological history.

Personal pathological history (S6): If the sufferer itself has an own record of sickness.

Electrocardiogram (S7): An electrocardiogram shows the electrical signals in your heart.

Ultrasound (S8): An ultrasound scan is a medical examination that employs high-frequency sound waves to take live pictures from the inside of your body.

Functional class (S9) [23]: The World Health Organisation (WHO) functional class system was designed to determine the rigor of somebodys manifestations and how they influence day-to-day actions.

2.8 Diagnosis:

Obstruction at exit (OEX) [24]: It is a procedure used to deliver babies who have airway compression due to certain blockage.

Obstruction at entry (OEN) [24]: In pregnancy obstruction at entry is rare and is most generally created by adhesions from past abdominal surgery.

Rhythm disorders (RD) [25]: A biological cycle present in the human body gets upset when the sleep-wake cycle dis-coordinate with the environment and hinder a daily routine.

Conduction disorder (CD) [26]: The heart relies on electrical signals that originate the heartbeat in rhythm, when certain signals obstruct it results in conduction disorder.

Congenital disease (CD) [27]: A medical condition present in babies by birth occurs during the fetal stage of development or is acquired from parents or produced by environmental factors. It is also known as birth defects.

Genetic disease (GD) [27]: A change in DNA sequence from normal sequence results in Genetic disorder. It can be produced in the whole body or a particular part of the body by a mutation in one gene, mutation in multiple genes, or change in the sequence of genes.

2.9 Treatments:

Pregnancy interruption (PI) [28]: Discontinued Pregnancy is recognized as Pregnancy interruption. It can either be done artificially called abortion or naturally due to fetal aberration.

Diuretic treatment (DT) [29]: These are medications cause a net impairment of sodium and water in urine. It is also called water pills.

Anti-arrhythmic treatment (AAT) [29]: These medications are used when the heart rate goes fast or have an extra heart beat. This condition is called tachycardia. This pill helps to restore the regular beat of the heart.

Treatment with beta-blockers (TBB): These medicines are used to decrease high blood pressure by opening up the nerves and arteries to recover blood flow.

Anticoagulants treatment (ACT) [29]: This medicine is utilized to block blood clots and also stopping them from growing big. It also decreases the risks of heart attack and strokes.

Eutectic delivery (ED) [30]: Delivery performs mixing certain drugs.

Dystocic delivery (PD) [30]: Slow cervix dilation during delivery.

Mother with complication (MC): Any health problems a mother ought during or before pregnancy such as blood pressure, anemia, infections, etc.

The following matrices show the relation connecting symptoms and diagnosis. treatment, and prognosis. The following matrices help to examine the correctness of the nal diagnosis, treatment, and prognosis.

$$
\begin{array}{c} \\ A \\ OB \\ TAB \\ FPH \\ PPH \\ ECG \\ ECO \\ FC \end{array}
\begin{array}{c}
\begin{array}{cccccc} OEX & OEN & RD & CDS & CD & GD \end{array} \\
\begin{pmatrix}
(0.9,0.2,0.3) & (0.2,0.2,0.4) & (0.8,0.3,0.2) & (0.9,0.2,0.3) & (0.9,0.1,0.3) & (0.9,0.2,0.3) \\
(0.2,0.6,0.4) & (0.3,0.5,0.6) & (0.9,0.2,0.3) & (0.8,0.2,0.1) & (0.9,0.2,0.2) & (0.8,0.2,0.3) \\
(0.9,0.1,0.3) & (0.8,0.1,0.2) & (0.9,0.2,0.3) & (0.9,0.2,0.3) & (0.8,0.1,0.2) & (0.2,0.1,0.4) \\
(0.2,0.2,0.3) & (0.2,0.4,0.1) & (0.2,0.1,0.1) & (0.3,0.6,0.5) & (0.2,0.8,0.1) & (0.8,0.2,0.3) \\
(0.9,0.1,0.2) & (0.8,0.2,0.3) & (0.9,0.2,0.3) & (0.9,0.2,0.2) & (0.2,0.5,0.5) & (0.9,0.2,0.3) \\
(0.8,0.2,0.3) & (0.7,0.2,0.3) & (0.2,0.2,0.4) & (0.2,0.5,0.6) & (0.9,0.2,0.2) & (0.3,0.8,0.6) \\
(0.7,0.2,0.2) & (0.8,0.2,0.3) & (0.2,0.8,0.5) & (0.9,0.2,0.2) & (0.8,0.2,0.3) & (0.2,0.1,0.5) \\
(0.9,0.2,0.1) & (0.9,0.2,0.3) & (0.9,0.2,0.3) & (0.9,0.1,0.3) & (0.9,0.2,0.2) & (0.8,0.2,0.3)
\end{pmatrix}
\end{array},
$$

$$
\begin{array}{c} \\ A \\ OB \\ TAB \\ FPH \\ PPH \\ ECG \\ ECO \\ FC \end{array}
\begin{array}{c}
\begin{array}{ccccc} PI & DT & AAT & TBB & ACT \end{array} \\
\begin{pmatrix}
(0.9,0.2,0.3) & (0.2,0.1,0.5) & (0.9,0.4,0.3) & (0.8,0.2,0.3) & (0.7,0.1,0.1) \\
(0.8,0.4,0.2) & (0.2,0.1,0.5) & (0.7,0.3,0.3) & (0.9,0.2,0.3) & (0.9,0.2,0.3) \\
(0.2,0.2,0.5) & (0.2,0.3,0.4) & (0.2,0.1,0.5) & (0.9,0.2,0.3) & (0.9,0.2,0.3) \\
(0.2,0.1,0.5) & (0.3,0.1,0.5) & (0.2,0.5,0.5) & (0.2,0.4,0.5) & (0.7,0.2,0.3) \\
(0.9,0.2,0.3) & (0.9,0.2,0.3) & (0.2,0.1,0.5) & (0.8,0.4,0.2) & (0.9,0.2,0.3) \\
(0.2,0.4,0.1) & (0.2,0.1,0.2) & (0.8,0.2,0.3) & (0.7,0.2,0.3) & (0.9,0.2,0.3) \\
(0.8,0.1,0.3) & (0.9,0.2,0.3) & (0.9,0.2,0.3) & (0.2,0.1,0.3) & (0.1,0.1,0.2) \\
(0.7,0.2,0.3) & (0.9,0.2,0.3) & (0.8,0.2,0.3) & (0.9,0.2,0.3) & (0.9,0.2,0.1)
\end{pmatrix}
\end{array},
$$

$$
\begin{array}{c} \\ A \\ OB \\ TAB \\ FPH \\ PPH \\ ECG \\ ECO \\ FC \end{array}
\begin{array}{c}
\begin{array}{cccccc} ED & PD & CHC & NOCHC & MC & NOMC \end{array} \\
\begin{pmatrix}
(0.1,0.1,0.2) & (0.9,0.2,0.3) & (0.1,0.1,0.2) & (0.1,0.3,0.2) & (0.7,0.2,0.3) & (0.7,0.2,0.2) \\
(0.9,0.2,0.3) & (0.9,0.2,0.3) & (0.3,0.1,0.2) & (0.1,0.1,0.2) & (0.9,0.2,0.3) & (0.9,0.1,0.2) \\
(0.1,0.1,0.2) & (0.3,0.1,0.2) & (0.8,0.2,0.3) & (0.3,0.1,0.2) & (0.9,0.2,0.3) & (0.1,0.1,0.2) \\
(0.1,0.1,0.2) & (0.2,0.1,0.2) & (0.9,0.2,0.2) & (0.9,0.2,0.3) & (0.1,0.1,0.2) & (0.7,0.2,0.3) \\
(0.9,0.2,0.4) & (0.9,0.2,0.3) & (0.9,0.2,0.3) & (0.7,0.2,0.3) & (0.8,0.2,0.3) & (0.9,0.2,0.3) \\
(0.9,0.2,0.2) & (0.1,0.1,0.2) & (0.3,0.1,0.2) & (0.1,0.1,0.2) & (0.1,0.1,0.2) & (0.3,0.1,0.2) \\
(0.7,0.2,0.3) & (0.9,0.2,0.3) & (0.7,0.2,0.3) & (0.9,0.2,0.3) & (0.1,0.1,0.2) & (0.8,0.2,0.3) \\
(0.7,0.2,0.3) & (0.9,0.2,0.3) & (0.8,0.2,0.2) & (0.9,0.2,0.1) & (0.9,0.5,0.3) & (0.9,0.2,0.3)
\end{pmatrix}
\end{array}.
$$

3 Explainable single-valued neutrosophic medical decision-making system for the treatment of pregnant women with cardiac diseases

Let's understand the complete working of the decision-making system for the treatment of pregnant women with cardiac problems. We use SVNN for diagnosing, treatment and prognosis. To make this system simple and transparent, we used XAI methods. These methods help to explain the entire working of a particular module. The causality helps to determine the effectiveness of the explanation. Also, We devise an algorithm and calculates its time complexity.

3.1 Basic structure of explainable single-valued neutrosophic medical decision-making system

The proposed system consists of five main parts. The first part is neutrosophication which contains truth membership, indeterminacy memberships, and falsity membership of each input variable. Section parts is a knowledge base that contains all possible rules of the systems. The third part is the inference engine which contains active decision-making rules. The fourth part is de-neutrosophication which contains truth memberships, indeterminacy memberships, and falsity memberships of the output variable. The fifth part is explainability which is integrated with each part of the systems to explain its working. Figure 1 presents the block diagram of the prescribed system.

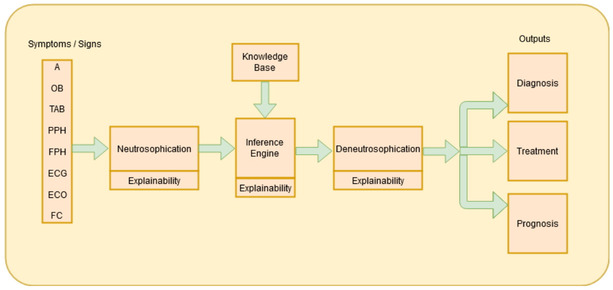

Figure 1: Block Diagram

3.2 Algorithm

The algorithm of the introduced model is as follows:

Algorithm 1 Steps to diagnosis, determine type of treatment, and prognosis by applying single-valued neutrosophic logic.

1: **Inputs:** Take inputs values from user: Age (A), obesity (OB), smoking (TAB), family pathological history (FPH), personal pathological history (PPH), electrocardiogram (ECG), ultrasound (ECO), and functional class (FC).
2: Define truth membership functions, indeterminacy membership functions, and falsity membership functions of each input and output variables.
3: Use these functions to determine degree of truth, degree of indeterminacy, and degree of falsity of each input variable against provided input values in step 1. This process is called neutrosophication process.
4: Defines rules of the system and determine firing strength of each rule using following formulas:

$$T(w) = \min(\mu_A(w), \mu_{OB}(w), \mu_{TAB}(w), \mu_{FPH}(w), \mu_{PPH}(w), \mu_{ECG}(w), \mu_{ECO}(w), \mu_{FC}(w)).$$
$$I(w) = \max(\nu_A(w), \nu_{OB}(w), \nu_{TAB}(w), \nu_{FPH}(w), \nu_{PPH}(w), \nu_{ECG}(w), \nu_{ECO}(w), \nu_{FC}(w)).$$
$$F(w) = \max(\lambda_A(w), \lambda_{OB}(w), \lambda_{TAB}(w), \lambda_{FPH}(w), \lambda_{PPH}(w), \lambda_{ECG}(w)\lambda_{ECO}(w), \lambda_{FC}(w)).$$

5: Use following de-neutrosophication formula to determine the values of each diagnosis, type of treatment, and prognosis. [22]:

$$V = (p + 2q + r + s + 2t + u + v + 2w + x)/12,$$

where, (p, q, r) are three points of truth membership, (s, t, u) are three points of indeterminacy membership, and (v, w, x) are three points of falsity membership.
6: Determine the final values of diagnosis, treatment, and prognosis.
7: Obtained the highest values among all values.
8: **Output:** The maximum value will be the final decision of diagnosis, treatment, and prognosis.

Neutrosophication

		Time Complexity
(i).	Start	
(ii).	Create three linked lists to store values of truth MFs, indeterminacy MFs, and falsity MFs. DF, IF, FF= null;	1
(iii).	Create pointers ptr1= DF; ptr2=IF; ptr3=FF	1
(iv).	for(n=1 to numbers of functions)	n
(v).	newDF.data=truth value of nth MF	n-1
(vi).	ptr1.link=newDF	n-1
(vii).	newIF.data=indeterminacy value of nth MF	n-1

(viii). ptr2.link=newIF n-1

(ix). newFF.data=falsity value of nth MF n-1

(x). ptr3.link=newFF n-1

(xi). end for

(xii). End Time complexity= $O(n)$

Inference Engine

(i). Start

(ii). Create three linked lists to store values of rules
Truth, Indeterminacy, Falsity= null; 1

(iii). Create pointers
ptr1= Truth; ptr2=Indeterminacy; ptr3=Falsity 1

(iv). for(n=1 to numbers of rules) n

(v). newTruth=minimum of truth MFs n-1

(vi). ptr1.link=newTruth n-1

(vii). newInd=maximum of indeterminacy MFs n-1

(viii). ptr2.link=newInd n-1

(ix). newFalsity= maximum falsity MFs n-1

(x). ptr3.ink=newFalsity n-1

(xi). end for

(xii). for(n=1 to numbers of rules) n

(xiii). Truth-value= min from truth values n-1

(xiv). Indeterminacy-value=max if indeterminacy values n-1

(xv). Falsity-value= max of falsity values n-1

(xvi). end for

(xvii). End Time complexity= $O(n)$

Defuzzification

(i). Start

(ii). for(n=1 to numbers of diagnosis) n

(iii). for(k=1 to numbers of output functions) n(n-1)

(iv). W1[n][k]=(p+2q+r+s+2t+u+v+2w+x)/12 (n-1)(n-2)

(v). V1[n]=take max from W1[n][k] (n-1)(n-2)

(vi). end for

(vii). end for

(viii). for(n=1 to number of treatments) n

(ix). for(k=1 to number of output functions) n(n-1)

(x). W2[n][k]=(p+2q+r+s+2t+u+v+2w+x)/12 (n-1)(n-2)

(xi). V2[n]=take max from W2[n][k] (n-1)(n-2)

(xii). end for

```
(xiii).    end for
(xiv).     for(n=1 to number of prognosis)                              n
(xv).         for(k=1 to number of output functions)                    n(n-1)
(xvi).            W3[n][k]=(p+2q+r+s+2t+u+v+2w+x)/12                    (n-1)(n-2)
(xvii).           V3[n]=take max from W3[n][k]                          (n-1)(n-2)
(xviii).      end for
(xix).     end for
(xx).      for(n=1 to number of diagnosis)                              n
(xxi).         Diagnosis=take max from V1[n]                            n-1
(xxii).    end for
(xxiii).   for(n=1 to no. of treatment)                                 n
(xxiv).        Treatment=take max from V2[n]                            n-1
(xxv).     end for
(xxvi).    for(n=1 to no. of prognosis)                                 n
(xxvii).       Prognosis=take max from V3[n]                            n-1
(xxviii).  end for
(xxix). End                                          Time complexity= O(n^2)
```

Let's compute the total time taken by this algorithm. Line 1 takes constant time. Neutrosophication is O(n) time process, therefore, line 2 takes O(n) time. The inference engine is O(n) time process, therefore, line 3 takes O(n) time. The de-neutrosophication process takes $O(n^2)$ time, so, line 4 takes $O(n^2)$. Lines 5, line 6, and line 7 take constant time. Hence, the overall time complexity of Algorithm 1 is $O(n^2)$.

3.3 Working of explainable neutrosophic clinical decision-making system for pregnant women with heart diseases

Our model is taking eight symptoms as inputs and computes the values of diagnosis, treatment, and prognosis. Table 1 shows the scale of each variable.

Table 1: Scale of input variables

Sr. no.	Symptoms	Scale
S1	Age-A (year/s)	0-100
S2	Obesity-OB	0-100
S3	Smoking-TAB	1/2
S4	Family pathological history-FPH	1/2
S5	Personal pathological history-PPH	1/2
S6	Electrocardiogram-ECG	0-100
S7	Ultrasound-ECO	0-100
S8	Functional class-FC	0-100

The scale is divided into three types of MFs. The plot of each symptom is depicted in Figure 2, Figure 3, Figure 4, Figure 5, Figure 6, Figure 7, Figure 8, Figure 9, Figure 10, Figure 11, Figure 12, Figure 13, and Figure 14.

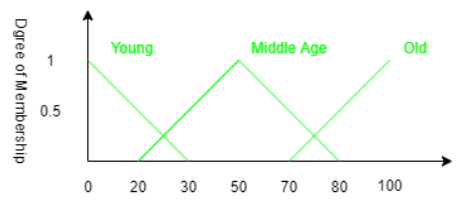

Figure 2: Age-truth MFs

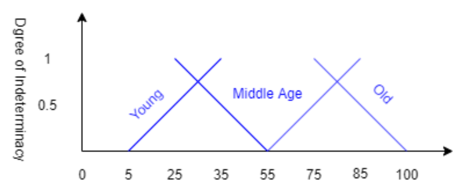

Figure 3: Age-indeterminacy MFs

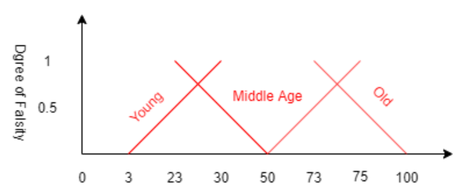

Figure 4: Age-falsity MFs

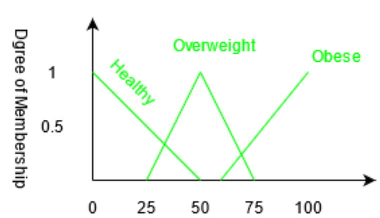

Figure 5: Obesity-truth MFs

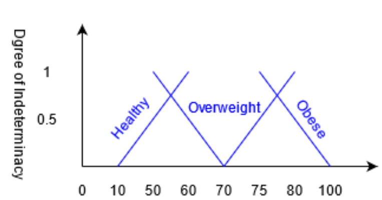

Figure 6: Obesity-indeterminacy MFs

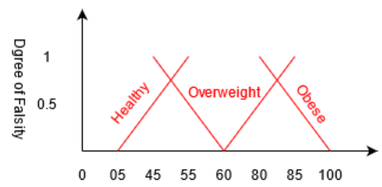

Figure 7: Obesity- falsity MFs

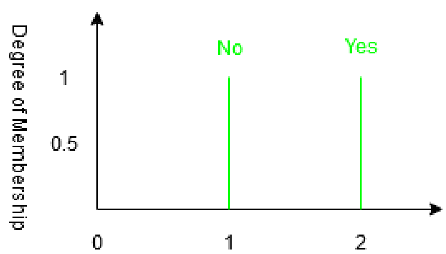

Figure 8: Smoking, personal and family pathological history- truth MFs

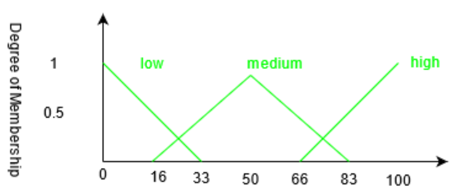

Figure 9: Electrocardiogram-truth MFs

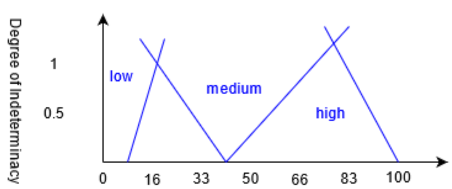

Figure 10: Electrocardiogram-indeterminacy MFs

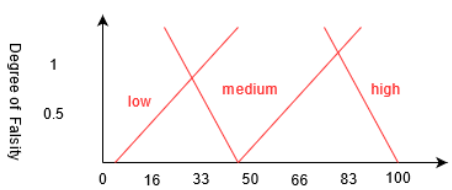

Figure 11: Electrocardiogram-falsity MFs

Figure 12: Function class-truth MFs

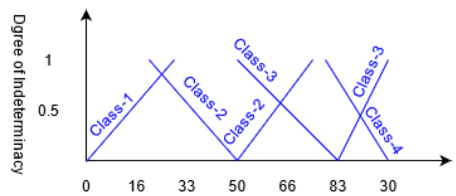

Figure 13: Function class- indeterminacy MFs

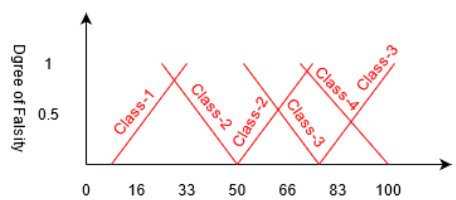

Figure 14: Function class-falsity MFs

The mathematical equations of truth MFs, indeterminacy MFs, and falsity MFs of age are as follows:

$$\mu_{young}(w) = \begin{cases} \dfrac{30-w}{30}, & \text{if } w \in [0-30], \\ 0, & \text{otherwise.} \end{cases} \qquad \mu_{old}(w) = \begin{cases} \dfrac{w-70}{30}, & \text{if } w \in [70-100], \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{middle-age}(w) = \begin{cases} \dfrac{w-20}{30}, & \text{if } w \in [20-50], \\ \dfrac{80-w}{30}, & \text{if } w \in [50-80], \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{young}(w) = \begin{cases} \dfrac{w-5}{30}, & \text{if } w \in [5-35], \\ 1, & \text{otherwise.} \end{cases} \qquad \nu_{old}(w) = \begin{cases} \dfrac{100-w}{25}, & \text{if } w \in [75-100], \\ 1, & \text{otherwise.} \end{cases}$$

$$\nu_{middle-age}(w) = \begin{cases} \dfrac{55-w}{30}, & \text{if } w \in [25-55], \\ \dfrac{w-55}{30}, & \text{if } w \in [55-85], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{young}(w) = \begin{cases} \dfrac{w-3}{27}, & \text{if } w \in [3-30], \\ 1, & \text{otherwise.} \end{cases} \qquad \lambda_{old}(w) = \begin{cases} \dfrac{100-w}{27}, & \text{if } w \in [73-100], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{middle-age}(w) = \begin{cases} \dfrac{50-w}{27}, & \text{if } w \in [23-50], \\ \dfrac{w-50}{25}, & \text{if } w \in [50-75], \\ 1, & \text{otherwise.} \end{cases}$$

The mathematical equations of truth MFs, indeterminacy MFs, and falsity MFs of obesity are as follows:

$$\mu_{healthy}(w)=\begin{cases}\frac{50-w}{50-0}, & \text{if } w\in[0-50],\\ 0, & \text{otherwise.}\end{cases}$$

$$\mu_{obese}(w)=\begin{cases}\frac{w-65}{35}, & \text{if } w\in[65-100],\\ 0, & \text{otherwise.}\end{cases}$$

$$\mu_{over-weight}(w)=\begin{cases}\frac{w-25}{25}, & \text{if } w\in[25-50],\\ \frac{75-w}{25}, & \text{if } w\in[50-75],\\ 0, & \text{otherwise.}\end{cases}$$

$$\nu_{healthy}(w)=\begin{cases}\frac{w-10}{50}, & \text{if } w\in[10-60],\\ 1, & \text{otherwise.}\end{cases}$$

$$\nu_{obese}(w)=\begin{cases}\frac{100-w}{25}, & \text{if } w\in[75-100],\\ 1, & \text{otherwise.}\end{cases}$$

$$\nu_{over-weight}(w)=\begin{cases}\frac{70-w}{20}, & \text{if } w\in[50-70],\\ \frac{w-70}{20}, & \text{if } w\in[70-90],\\ 1, & \text{otherwise.}\end{cases}$$

$$\lambda_{healthy}(w)=\begin{cases}\frac{w-5}{50}, & \text{if } w\in[5-55],\\ 1, & \text{otherwise.}\end{cases}$$

$$\lambda_{obese}(w)=\begin{cases}\frac{100-w}{20}, & \text{if } w\in[80-100],\\ 1, & \text{otherwise.}\end{cases}$$

$$\lambda_{over-weight}(w)=\begin{cases}\frac{60-w}{15}, & \text{if } w\in[45-60],\\ \frac{w-60}{25}, & \text{if } w\in[60-85],\\ 1, & \text{otherwise.}\end{cases}$$

The mathematical equations of truth MFs, indeterminacy MFs, and falsity MFs of smoking, personal pathological history and family pathological history are as follows:

$\mu_{yes}(w)=1.$
$\nu_{yes}(w)=0.$
$\lambda_{yes}(w)=0.$

$\mu_{no}(w)=1.$
$\nu_{no}(w)=0.$
$\lambda_{no}(w)=0.$

The mathematical form of degree of membership, degree of indeterminacy, and degree of falsity of electrocardiogram are as follows:

$$\mu_{low}(w)=\begin{cases}\frac{33.33-w}{33.33-0}, & \text{if } w\in[0-33.33],\\ 0, & \text{otherwise.}\end{cases}$$

$$\mu_{high}(w)=\begin{cases}\frac{w-43.33}{23.33}, & \text{if } w\in[43.33-66.67],\\ 0, & \text{otherwise.}\end{cases}$$

$$\mu_{medium}(w)=\begin{cases}\frac{w-16.67}{16.67}, & \text{if } w\in[16.67-33.33],\\ \frac{50-w}{16.67}, & \text{if } w\in[33.33-50],\\ 0, & \text{otherwise.}\end{cases}$$

$$\nu_{low}(w)=\begin{cases}\frac{w-6.67}{33.33}, & \text{if } w\in[6.67-40],\\ 1, & \text{otherwise.}\end{cases}$$

$$\nu_{high}(w)=\begin{cases}\frac{66.67-w}{16.67}, & \text{if } w\in[50-66.67],\\ 1, & \text{otherwise.}\end{cases}$$

$$\nu_{medium}(w)=\begin{cases}\frac{46.67-w}{13.33}, & \text{if } w\in[33.33-46.67],\\ \frac{w-46.67}{13.33}, & \text{if } w\in[46.67-60],\\ 1, & \text{otherwise.}\end{cases}$$

$$\lambda_{low}(w)=\begin{cases}\frac{w-3.33}{33.33}, & \text{if } w\in[3.33-36.67],\\ 1, & \text{otherwise.}\end{cases}$$

$$\lambda_{high}(w)=\begin{cases}\frac{66.67-w}{13.33}, & \text{if } w\in[53.33-66.67],\\ 1, & \text{otherwise.}\end{cases}$$

$$\lambda_{medium}(w)=\begin{cases}\frac{40-w}{10}, & \text{if } w\in[30-40],\\ \frac{w-40}{16.67}, & \text{if } w\in[40-56.67],\\ 1, & \text{otherwise.}\end{cases}$$

The mathematical form of degree of membership, degree of indeterminacy, and degree of falsity of ultrasound are as follows:

$$\mu_{low}(w) = \begin{cases} \dfrac{33.33 - w}{33.33 - 0}, & \text{if } w \in [0 - 33.33], \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{high}(w) = \begin{cases} \dfrac{w - 43.33}{23.33}, & \text{if } w \in [43.33 - 66.67], \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{medium}(w) = \begin{cases} \dfrac{w - 16.67}{16.67}, & \text{if } w \in [16.67 - 33.33], \\ \dfrac{50 - w}{16.67}, & \text{if } w \in [33.33 - 50], \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{low}(w) = \begin{cases} \dfrac{w - 6.67}{33.33}, & \text{if } w \in [6.67 - 40], \\ 1, & \text{otherwise.} \end{cases}$$

$$\nu_{high}(w) = \begin{cases} \dfrac{66.67 - w}{16.67}, & \text{if } w \in [50 - 66.67], \\ 1, & \text{otherwise.} \end{cases}$$

$$\nu_{medium}(w) = \begin{cases} \dfrac{46.67 - w}{13.33}, & \text{if } w \in [33.33 - 46.67], \\ \dfrac{w - 46.67}{13.33}, & \text{if } w \in [46.67 - 60], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{low}(w) = \begin{cases} \dfrac{w - 3.33}{33.33}, & \text{if } w \in [3.33 - 36.67], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{high}(w) = \begin{cases} \dfrac{66.67 - w}{13.33}, & \text{if } w \in [53.33 - 66.67], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{medium}(w) = \begin{cases} \dfrac{40 - w}{10}, & \text{if } w \in [30 - 40], \\ \dfrac{w - 40}{16.67}, & \text{if } w \in [40 - 56.67], \\ 1, & \text{otherwise.} \end{cases}$$

The mathematical equations of truth MFs, indeterminacy MFs, and falsity MFs functional class are as follows:

$$\mu_{class-1}(w) = \begin{cases} \dfrac{33.33 - w}{33.33}, & \text{if } w \in [0 - 33.33], \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{class-3}(w) = \begin{cases} \dfrac{w - 33.33}{33.33}, & \text{if } w \in [33.33 - 66.67], \\ \dfrac{100 - w}{33.33}, & \text{if } w \in [66.67 - 100], \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{class-2}(w) = \begin{cases} \dfrac{w - 33.33}{33.33}, & \text{if } w \in [0 - 33.33], \\ \dfrac{66.67 - w}{33.33}, & \text{if } w \in [33.33 - 66.67], \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{class-4}(w) = \begin{cases} \dfrac{w - 66.67}{33.33}, & \text{if } w \in [66.67 - 100], \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{class-1}(w) = \begin{cases} \dfrac{w - 0}{26.67}, & \text{if } w \in [0 - 26.67], \\ 1, & \text{otherwise.} \end{cases}$$

$$\nu_{class-3}(w) = \begin{cases} \dfrac{83.33 - w}{43.33}, & \text{if } w \in [40 - 83.33], \\ \dfrac{w - 83.33}{16.67}, & \text{if } w \in [83.33 - 100], \\ 1, & \text{otherwise.} \end{cases}$$

$$\nu_{class-2}(w) = \begin{cases} \dfrac{50 - w}{30}, & \text{if } w \in [20 - 50], \\ \dfrac{w - 50}{26.67}, & \text{if } w \in [50 - 76.67], \\ 1, & \text{otherwise.} \end{cases}$$

$$\nu_{class-4}(w) = \begin{cases} \dfrac{100 - w}{20}, & \text{if } w \in [80 - 100], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{class-1}(w) = \begin{cases} \dfrac{w - 10}{23.33}, & \text{if } w \in [10 - 33.33], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{class-3}(w) = \begin{cases} \dfrac{76.67 - w}{36.67}, & \text{if } w \in [40 - 76.67], \\ \dfrac{w - 76.67}{23.33}, & \text{if } w \in [76.67 - 100], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{class-2}(w) = \begin{cases} \dfrac{50 - w}{20}, & \text{if } w \in [30 - 50], \\ \dfrac{w - 50}{23.33}, & \text{if } w \in [50 - 73.33], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{class-4}(w) = \begin{cases} \dfrac{100 - w}{30}, & \text{if } w \in [70 - 100], \\ 1, & \text{otherwise.} \end{cases}$$

3.3.1 Ante-hoc explanation

In this section, we dene mathematical equations of truth MFs, indeterminacy MFs, and falsity MFs. These equations help to convert crisp input into linguistics values. We also draw the plots of each input variable. The value of these functions extends between 0 to 1. The value of each linguistic variable is the output of the neutrosophication module.

3.4 Inference engine

The inference engine is the main part of the system which contains the firing strength of all active rules. There are 330 rules in the system, some of them are written here:

R1−**IF** (age=middle-age, obesity=obese, smoking=yes, personal pathological history=yes, family pathological history=no, electrocardiogram/heartbeat=low, ultrasound=no-complications, functional class=class-3) **THEN** (obstruction at exist=medium, obstruction at entry=low, rhythm disorder=high, conduction disorders=medium, congenital diseases=medium, genetic diseases=medium) **AND** (pregnancy interruption=low, diuretic treatment=low, anti-arrhythmic treatment=low, treatment with beta blockers=high and anticoagulants treatment=medium) **AND** (eutectic delivery=low, dystocic delivery=low, child with complications=medium, child without complications=low, mother with complications=high, mother without complications=medium).

R2−**IF** (age=middle-age, obesity=overweight, smoking=yes, personal pathological history=no, family pathological history=yes, electrocardiogram/heartbeat=low, ultrasound=complications, functional class=class-2) **THEN** (obstruction at exist=low, obstruction at entry=low, rhythm disorder=low, conduction disorders=low, congenital diseases=medium, genetic diseases=high) **AND** (pregnancy interruption=low, diuretic treatment=low, anti-arrhythmic treatment=low, treatment with beta blockers=medium and anticoagulants treatment=high) **AND** (eutectic delivery=low, dystocic delivery=low, child with complications=high, child without complications=low, mother with complications=medium, mother without complications=low).

R3 − **IF** (age=old, obesity=obese, smoking=no, personal pathological history=no, family pathological history=yes, electrocardiogram/heartbeat=high, ultrasound=complications functional class=class-4) **THEN** (obstruction at exist=high, obstruction at entry=medium, rhythm disorder=low, conduction disorders=low, congenital diseases=low, genetic diseases=low) **AND** (pregnancy interruption=low, diuretic treatment=low, anti-arrhythmic treatment=high, treatment with beta blockers=medium and anticoagulants treatment=low) **AND** (eutectic delivery=low, dystocic delivery=high, child with complications=medium, child without complications=low, mother with complications=medium, mother without complications=low).

R4 − **IF** (age=young, obesity=healthy, smoking=yes, personal pathological history=yes, family pathological history=no, electrocardiogram/heartbeat=high, ultrasound=complications, functional class=class-3) **THEN** (obstruction at exist=medium, obstruction at entry=high, rhythm disorder=low, conduction disorders=low, congenital diseases=low, genetic diseases=low) **AND** (pregnancy interruption=medium, diuretic treatment=high, anti-arrhythmic treatment=low, treatment with beta blockers=low and anticoagulants treatment=low) **AND** (eutectic delivery=low, dystocic delivery=low, child with complications=high, child without complications=low, mother with complications=medium, mother without complications=low).

3.4.1 Ante-hoc Explanation

The knowledgebase is a vital part of our system. It contains all possible rules which are highlighted against the provided inputs. These rules are IF-THEN statements, which describe how to associate inputs with desired outputs. This module receives input from the neutrosophication module and the knowledge base, it is determined that which rules are currently triggered, and their firing strength is computed in the inference engine. To compute the firing strengths of rules, we take the minimum value out of all values of truth memberships, the maximum value of all indeterminacy membership functions, and the maximum value of all falsity membership functions.

3.5 De-Neutrosophication

The last phase of our system is de-neutrosophication We used de-neutrosophication formula to covert linguistic values into crisp values discussed in [22].

$$V=(p+2q+r+s+2t+u+v+2w+x)/12,$$

where, (p,q,r) are the points of truth MF, (s,t,u) are the points of indeterminacy MF, and (v,w,x) are the points of falsity MFs. The plots of truth MFs, indeterminacy MFs, and falsity MFs for output parameter are depicted in Figure 15, Figure 16, and Figure 17.

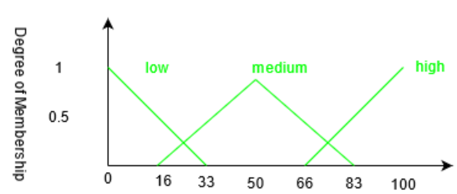

Figure 15: Diagnosis, treatment, prognosis-truth MFs

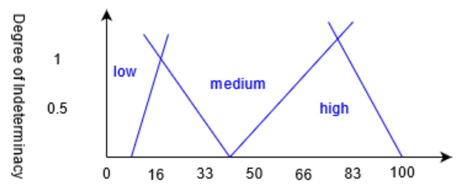

Figure 16: Diagnosis, treatment, prognosis- indeterminacy MFs

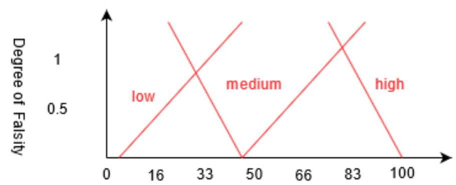

Figure 17: Diagnosis, treatment, prognosis- falsity MFs

The mathematical equation of truth MFs, indeterminacy MFs, and falsity MFs of outputs are as follows:

$$\mu_{low}(w) = \begin{cases} \dfrac{33.33 - w}{33.33 - 0}, & \text{if } w \in [0 - 33.33], \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{high}(w) = \begin{cases} \dfrac{w - 43.33}{23.33}, & \text{if } w \in [43.33 - 66.67], \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{medium}(w) = \begin{cases} \dfrac{w - 16.67}{16.67}, & \text{if } w \in [16.67 - 33.33], \\ \dfrac{50 - w}{16.67}, & \text{if } w \in [33.33 - 50], \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{low}(w) = \begin{cases} \dfrac{w - 6.67}{33.33}, & \text{if } w \in [6.67 - 40], \\ 1, & \text{otherwise.} \end{cases}$$

$$\nu_{high}(w) = \begin{cases} \dfrac{66.67 - w}{16.67}, & \text{if } w \in [50 - 66.67], \\ 1, & \text{otherwise.} \end{cases}$$

$$\nu_{medium}(w) = \begin{cases} \dfrac{46.67 - w}{13.33}, & \text{if } w \in [33.33 - 46.67], \\ \dfrac{w - 46.67}{13.33}, & \text{if } w \in [46.67 - 60], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{low}(w) = \begin{cases} \dfrac{w - 3.33}{33.33}, & \text{if } w \in [3.33 - 36.67], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{high}(w) = \begin{cases} \dfrac{66.67 - w}{13.33}, & \text{if } w \in [53.33 - 66.67], \\ 1, & \text{otherwise.} \end{cases}$$

$$\lambda_{medium}(w) = \begin{cases} \dfrac{40 - w}{10}, & \text{if } w \in [30 - 40], \\ \dfrac{w - 40}{16.67}, & \text{if } w \in [40 - 56.67], \\ 1, & \text{otherwise.} \end{cases}$$

3.5.1 Ante-hoc Explanation

De-neutrosophication is the last phase of our model. The input of part is the weight of active rules. It contains truth MFs, indeterminacy MFs, and falsity MFs of each output variable. It converts linguistic values to crisp values by using de-neutrosophication formulas. Its crisp values help to determine diagnosis, treatment, and prognosis.

4 Case study

This section demonstrate a numerical example of our system to explain its entire working to the readers. For this purpose, consider an input: (age, obesity, smoking, personal pathological history, family pathological history, electrocardiogram/heart-beat, ultrasound, functional class)=(40, 75, 2, 2, 1, 16.67, 16.6, 76.67). Lets see the execution of each module of the proposed system.

4.1 Neutrosophication

The first step is neutrosophication. We obtained the following linguistic values against the provided inputs.

Age(40)= $(\mu_{yng}, \mu_{m-a}, \mu_{ol})$=(0,0.67,0), $(\nu_{yng}, \nu_{m-a}, \nu_{ol})$=(1,0.5,1), ($\lambda_{yng}$, λ_{m-a}, λ_{ol})=(1,0.37,1).

Obesity(75)= $(\mu_{healthy}, \mu_{over-weight}, \mu obese)$ =(1,1,0.29), $(\nu_{healthy}, \nu_{over-weight}, \nu_{obese})$ =(1,0.25,1), $(\lambda_{male}, \lambda_{female})$ =(1,0.6,1).

Smoking(2)= (μ_y, μ_n) =(1,0), (ν_y, ν_n) =(0,1), (λ_y, λ_n) =(0,1).

Personal pathological history(2)= (μ_y, μ_n) =(1,0), (ν_y, ν_n) =(0,1), (λ_y, λ_n) =(0,1).

Family pathological history(1)= (μ_y, μ_n) =(0,1), (ν_y, ν_n) =(1,0), (λ_y, λ_n) =(1,0).

Electrocardiogram(16.67)= $(\mu_{lo}, \mu_{med}, \mu_{hi})$ =(0.5,0,0), $(\nu_{lo}, \nu_{med}, \nu_{hi})$ =(0.3,1,1), $(\lambda_{lo}, \lambda_{med}, \lambda_{hi})$ =(0.4,1,1).

Ultrasound(16.67)= $(\mu_{lo}, \mu_{med}, \mu hi)$ =(0.5,0,0), $(\nu_{lo}, \nu_{med}, \nu_{hi})$ =(0.3,1,1), $(\lambda_{lo}, \lambda_{med}, \lambda_{hi})$ =(0.4,1,1).

Function class(76.67)= $(\mu_{c1}, \mu_{c2}, \mu c3), \mu c4$ =(0,0,0.7,0.3), $(\nu_{c1}, \nu_{c2}, \nu_{c3}, \nu_{c4})$ =(1,1,0.15,1), $(\lambda_{c1}, \lambda_{c2}, \lambda_{c3}, \lambda_{c4})$ =(1,1,1,0.78).

4.1.1 Explanation

In this module, we take exemplary values of inputs to understand the complete working of the proposed model. We passed the input values from truth MFs, indeterminacy MFs, and falsity MFs of each input and find out the degree of each function. The degree of each function can lies between 0 to 1.

4.2 Inference Engine

The next phase is the inference engine. Let's pass linguistic values to the inference engine to get the active rules. **R1** − **IF** (age=middle-age, obesity=over-weight, smoking=yes, personal pathological history=yes, family pathological history=no, electrocardiogram/heart-beat=low, ultrasound=no-complications, functional class=class-3) **THEN** (obstruction at exist=medium, obstruction at entry=high, rhythm disorder=low, conduction disorders=low, congenital diseases=low, genetic diseases=low) **AND** (pregnancy interruption=medium, diuretic treatment=high, anti-arrhythmic treatment=low, treatment with beta blockers=low and anticoagulants treatment=low) **AND** (eutectic delivery=low, dystocic delivery=low, child with complications=high, child without complications=low, mother with complications=medium, mother without complications=low).

R2−**IF** (age=middle-age, obesity=over-weight, smoking=yes, personal pathological history=yes, family pathological history=no, electrocardiogram/heart-beat=low, ultrasound=no-complications, functional class=class-4) **THEN** (obstruction at exist=medium, obstruction at entry=medium, rhythm disorder=medium, conduction disorders=low, congenital diseases=low, genetic diseases=low) **AND** (pregnancy interruption=medium, diuretic treatment=high, anti-arrhythmic treatment=low, treatment with beta blockers=low and anticoagulants treatment=low) **AND** (eutectic delivery=low, dystocic delivery=low, child with complications=medium, child without complications=low, mother with complications=low, mother without complications=low).

R3−**IF** (age=middle-age, obesity=obese, smoking=yes, personal pathological history=yes, family pathological history=no, electrocardiogram/heart-beat=low, ultrasound=no-complications, functional class=class-3) **THEN** (obstruction at exist=low, obstruction at entry=high, rhythm disorder=low, conduction disorders=low, congenital diseases=low, genetic diseases=low) **AND** (pregnancy interruption=low, diuretic treatment=high, anti-arrhythmic treatment=low, treatment with beta blockers=low and anticoagulants treatment=low) **AND** (eutectic delivery=low, dystocic delivery=low, child with complications=high, child without complications=low, mother with complications=medium, mother without complications=low).

R4−**IF** (age=middle-age, obesity=obese, smoking=yes, personal pathological history=yes, family pathological history=no, electrocardiogram/heart-beat=low, ultrasound=no-complications, functional class=class-4) **THEN** (obstruction at exist=medium, obstruction at entry=high, rhythm disorder=low, conduction disorders=medium, congenital diseases=low, genetic diseases=low) **AND** (pregnancy interruption=medium, diuretic treatment=medium, anti-arrhythmic treatment=low, treatment with beta blockers=low and anticoagulants treatment=low) **AND** (eutectic delivery=low, dystocic delivery=low, child with complications=high, child without complications=low, mother with complications=medium, mother without complications=low).

The output generated from inference engine is shown in Table 2. The abbreviations used in the table are as follows: MA=middle-age, OB=obese, OW=over-weight, Y=yes, N=no, L=low, C1=class-1, C2-class-2, C3=class-3, and C4=class-4.

Table 2: Active rules in inference engine

Input	R60	R61	R62	R63
A	MA(0.67, 0.5 ,0.37)	MA (0.67, 0.5 ,0.37)	MA (0.67, 0.5, 0.37)	MA(0.67, 0.5 ,0.37)
OB	OW(1, 0.25, 0.6)	OW(1, 0.25, 0.6)	OB(0.29, 0.25, 0.6)	OB(0.29, 0.25, 0.6)
TAB	Y(1, 0 ,0)	Y(1, 0, 0)	Y(1, 0, 0)	Y(1, 0, 0)
FPH	N(1, 0, 0)	N(1 ,0, 0)	N(1, 0, 0)	N(1 ,0 ,0)
PPH	Y(1 ,0 ,0)	Y (1, 0, 0)	Y(1, 0, 0)	Y(1 ,0 ,0)
ECG	L(0.5, 0.3, 0.4)	L (0.5, 0.3 ,0.4)	L(0.5, 0.3 ,0.4)	L(0.5, 0.3 ,0.4)
ECO	L (0.5, 0.3 ,0.4)	L(0.5, 0.3, 0.4)	L (0.5, 0.3, 0.4)	L(0.5, 0.3 ,0.4)
FC	C3(0.7, 0.15, 1)	C4(0.3, 1 ,0.78)	C3(0.7, 0.15, 1)	C4(0.3, 1, 0.78)
(min,max,max)	(0.5 ,0.5, 1)	(0.3, 1, 0.78)	(0.29, 0.5, 1)	(0.29, 1, 0.78)

4.2.1 Explanation

The linguistic values obtained from the neutrosophication section are passed to the inference engine. Inference engine gets all rules from the knowledge base and against these linguistic values determines the active rules and computes their firing strengths. In our example, R60, R61, R62, and R63 are fired.

4.3 De-neutrosophication

The last phase of the proposed model is de-neutrosophication to get the final findings. Let's see the de-neutrosophication process and determines the final findings.

OEX:

$$low = \frac{0 + 2 \times 0 + 1 + 0.2 + 2 \times 0.2 + 1.2 + 0.1 + 2 \times 0.1 + 1.1}{12},$$

$low = 0.341.$

$$medium = \frac{0.5 + 2 \times 1.5 + 2.5 + 0.4 + 2 \times 1.2 + 2.4 + 1.6 + 2 \times 1.4 + 2.6}{12},$$

$medium = 1.57.$

OEN:

$$high = \frac{2 + 2 \times 3 + 3 + 1.9 + 2 \times 3 + 3 + 2.2 + 2 \times 3 + 3}{12},$$

$high = 2.8.$

$$medium = \frac{0.5 + 2 \times 1.5 + 2.5 + 0.4 + 2 \times 1.2 + 2.4 + 1.6 + 2 \times 1.4 + 2.6}{12},$$

$medium = 1.57.$

RD:

$$low = \frac{0 + 2 \times 0 + 1 + 0.2 + 2 \times 0.2 + 1.2 + 0.1 + 2 \times 0.1 + 1.1}{12},$$

$low = 0.341.$

$$medium = \frac{0.5 + 2 \times 1.5 + 2.5 + 0.4 + 2 \times 1.2 + 2.4 + 1.6 + 2 \times 1.4 + 2.6}{12},$$

$medium = 1.57.$

CDS:

$$low = \frac{0+2\times 0+1+0.2+2\times 0.2+1.2+0.1+2\times 0.1+1.1}{12},$$

$low = 0.341.$

CD:

$$low = \frac{0+2\times 0+1+0.2+2\times 0.2+1.2+0.1+2\times 0.1+1.1}{12},$$

$low = 0.341.$

$$medium = \frac{0.5+2\times 1.5+2.5+0.4+2\times 1.2+2.4+1.6+2\times 1.4+2.6}{12},$$

$medium = 1.57.$

GD

$$low = \frac{0+2\times 0+1+0.2+2\times 0.2+1.2+0.1+2\times 0.1+1.1}{12},$$

$low = 0.341.$

The maximum value is of OEN. Now we will perform de-neutrosophication for treatment using de-neutrosophication method proposed in [22]:

PI:

$$low = \frac{0+2\times 0+1+0.2+2\times 0.2+1.2+0.1+2\times 0.1+1.1}{12},$$

$low = 0.341.$

$$medium = \frac{0.5+2\times 1.5+2.5+0.4+2\times 1.2+2.4+1.6+2\times 1.4+2.6}{12},$$

$medium = 1.57.$

DT

$$medium = \frac{0.5+2\times 1.5+2.5+0.4+2\times 1.2+2.4+1.6+2\times 1.4+2.6}{12},$$

$medium = 1.57.$

$$high = \frac{2+2\times 3+3+1.9+2\times 3+3+2.2+2\times 3+3}{12},$$

$high = 2.8.$

AAT:

$$low = \frac{0+2\times 0+1+0.2+2\times 0.2+1.2+0.1+2\times 0.1+1.1}{12},$$

$low = 0.341.$

TBB:

$$low = \frac{0+2\times 0+1+0.2+2\times 0.2+1.2+0.1+2\times 0.1+1.1}{12},$$

$low = 0.341.$

ACT:

$$low = \frac{0+2\times 0+1+0.2+2\times 0.2+1.2+0.1+2\times 0.1+1.1}{12},$$

$low = 0.341.$

The maximum value is of DT. Now we will perform de-neutrosophication for prognosis using de-neutrosophication method proposed in [22]:

ED:

$$low = \frac{0 + 2 \times 0 + 1 + 0.2 + 2 \times 0.2 + 1.2 + 0.1 + 2 \times 0.1 + 1.1}{12},$$

$low = 0.341.$

PD:

$$low = \frac{0 + 2 \times 0 + 1 + 0.2 + 2 \times 0.2 + 1.2 + 0.1 + 2 \times 0.1 + 1.1}{12},$$

$low = 0.341.$

CHC

$$high = \frac{2 + 2 \times 3 + 3 + 1.9 + 2 \times 3 + 3 + 2.2 + 2 \times 3 + 3}{12},$$

$$medium = \frac{0.5 + 2 \times 1.5 + 2.5 + 0.4 + 2 \times 1.2 + 2.4 + 1.6 + 2 \times 1.4 + 2.6}{12},$$

$medium = 1.57.$

$high = 2.8.$

NOCHC:

$$low = \frac{0 + 2 \times 0 + 1 + 0.2 + 2 \times 0.2 + 1.2 + 0.1 + 2 \times 0.1 + 1.1}{12},$$

$low = 0.341.$

MC:

$$low = \frac{0 + 2 \times 0 + 1 + 0.2 + 2 \times 0.2 + 1.2 + 0.1 + 2 \times 0.1 + 1.1}{12},$$

$low = 0.341.$

$$medium = \frac{0.5 + 2 \times 1.5 + 2.5 + 0.4 + 2 \times 1.2 + 2.4 + 1.6 + 2 \times 1.4 + 2.6}{12},$$

$medium = 1.57.$

NOMC

$$low = \frac{0 + 2 \times 0 + 1 + 0.2 + 2 \times 0.2 + 1.2 + 0.1 + 2 \times 0.1 + 1.1}{12},$$

$low = 0.341.$

The maximum value is of CHC.

4.3.1 Explanation

The final step of the proposed model is de-neutrosophication. Here, we used a de-neutrosophication formula to obtain the value of each diagnosis, treatment, and prognosis. The among all values of diagnosis, we pick the maximum value which is considered as the final decision about the diagnosis. The same procedure is adopted for treatment and prognosis.

4.4 Three-layered causal hierarchy

Now we will determine the quality of our explanation using Peral et al method [31]-[32]:

Level 1: Association- How many considered symptoms and signs relate to the particularized diagnosis, treatment, and prognosis? This inquiry is investigated by medical specialists and decided that all the specified symptoms and signs are almost associated with diagnosis, treatment, and prognosis.
Level 2: Intervention- What will happen if the doctor adopted the recommended method will the patient get diagnosed earlier? According to the specialists, the recommended model encourages doctors to diagnose, treatment, and prognosis pregnant women at the earliest.
Level 3: Counterfactuals- Was the diagnosis, treatment, and prognosis that influences the specified symptoms? After analyzing the outcomes of diagnosis, treatment, and prognosis with symptoms, it is inferred that diagnosis, treatment, and prognosis cause most of the signs. The diagnosis, treatment, and prognosis are firmly linked with specified symptoms.

5 Comparison Analysis

This segment presents a contrastive examination of the outcomes achieved from our model and with the present decision-making methods using different data sets [22], [6]-[7]. In the research, various schemes of decision-making are presented. Here we analyzed fuzzy soft sets and fuzzy cognitive maps to analyze the recommended model. We have considered eighteen data sets for testing purposes. The outcomes achieved by these techniques are quite alike to the final findings as we obtained from our method. All approaches recognized the similar diagnosis, treatment, and prognosis. The terminal values achieved by these models are reviewed in Figure 18, Figure 19, and Figure 20.

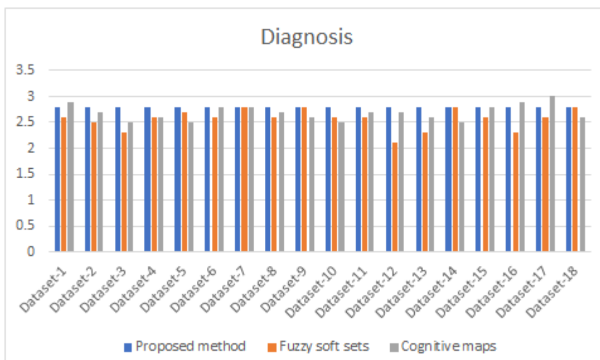

Figure 18: Diagnosis-Comparison Analysis

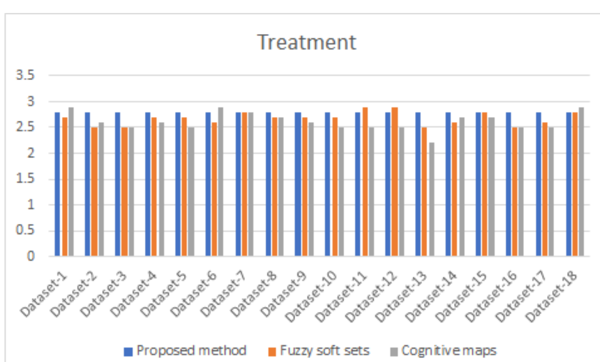

Figure 19: Treatment-Comparison Analysis

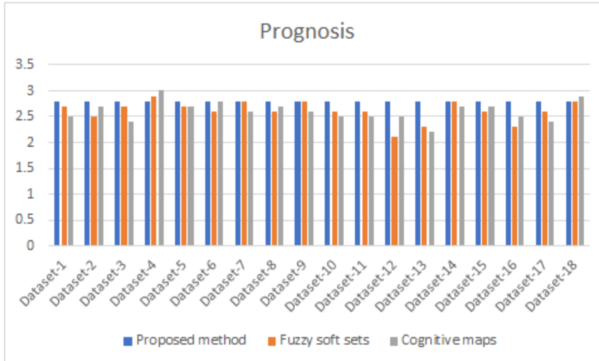

Figure 20: Prognosis-Comparison analysis

After comparing, we conclude that the proposed system is the best alternative to the existing model which offers explainability part as well.

6 Conclusion and future directions

This study presented a novel explainable single-valued neutrosophic decision-making model for the treatment of pregnant women with cardiac diseases. To make this system more effective and understandable to medical experts, we used XAI techniques and measures the quality of explanation as well. The principal contributions and advantages of our research are as follows:

1. Our methods help medical specialists to early diagnosis, identify the type of treatment, and prognosis so that prudent actions can be brought timely. The system considered eight parameters as inputs and computes the value of each diagnosis, type of treatment, and prognosis. The proposed system consists of five main parts, neutrosophication, knowledge base, inference engine, de-neutrosophication, and explainability.
2. In this study ante-hoc explanation is used to make the systems more understandable to the medical experts, and its quality is also measured.
3. To demonstrate the working of the system we devise an algorithm and computed its time complexity as well.
4. Also, a comparative analysis is performed to get the precision of the system and concluded that all decision-making methods highlighted the same diagnosis, treatment, and prognosis.

The recommended model can be applied in many other problems where we need to decide uncertain situations. Examples of such problems are diagnosis and treatment of cancers and other diseases, irrigation in agriculture, any industrial decision-making problems, and many more.

References

[1] Zadeh, L.A., *Fuzzy sets*, Information and Control, **8**(3)(1965), 338-353.

[2] Smarandache, F., *A unifying field in logics. neutrosophy: neutrosophic probability, set and logic*, (1999), 1-141.

[3] Akram, M., *Single-valued neutrosophic graphs*, Infosys Science Foundation Series in Mathematical Sciences, Springer, (2018).

[4] Wang, H., Smarandache, F., Zhang, Y., and Sunderraman, R., *Single-valued neutrosophic sets*, Multispace Multistruct, **4**(2010), 410-413.

[5] Habib, S., and Akram, M., and Ashraf, A., *Fuzzy climate decision support systems for tomatoes in high tunnels*, International Journal of Fuzzy Systems, **19**(3)(2017), 751-775.

[6] Habib, S., and Akram, M., *Diagnostic methods and risk analysis based on fuzzy soft information*, International Journal of Biomathematics, **11**(8)(2018), 1850096.

[7] Habib, S., and Akram, M., *Medical decision support systems based on fuzzy cognitive maps*, International Journal of Biomathematics, **12**(6)(2019), 1950069.

[8] Habib, S., us Salam, W., Butt, M. A., Akram, M., and Smarandache, F., *A neutrosophic clinical decision-making system for cardiovascular diseases risk analysis*, Journal of Intelligent & Fuzzy Systems, (Preprint)(2020), 1-23.

[9] Ye, J., *Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses*, Artificial intelligence in medicine, **63**(3)(2015), 171-179.

[10] Singh, P. K., *Medical diagnoses using three-way fuzzy concept lattice and their Euclidean distance*, Computational and Applied Mathematics, **37**(3)(2018), 3282-3306.

[11] Singh, P. K., *Three-way fuzzy concept lattice representation using neutrosophic set*, International Journal of Machine Learning and Cybernetics, **8**(1)(2017), 69-79.

[12] Abdel-Basset, M., Gamal, A., Son, L. H., and Smarandache, F., *A bipolar neutrosophic multi criteria decision making framework for professional selection*, Applied Sciences, **10**(4)(2020), 1202.

[13] Hashmi, M. R., Riaz, M., and Smarandache, F., *m-Polar neutrosophic topology with applications to multi-criteria decision-making in medical diagnosis and clustering analysis*, International Journal of Fuzzy Systems, **22**(1)(2019), 273-292.

[14] Pamucar, D., Yazdani, M., Obradovic, R., Kumar, A., and TorresJimenez, M., *A novel fuzzy hybrid neutrosophic decisionmaking approach for the resilient supplier selection problem*, International Journal of Intelligent Systems, **35**(12)(2020), 1934-1986.

[15] Pamucar, D., Deveci, M., Schitea, D., Erikin, L., Iordache, M., and Iordache, I., *Developing a novel fuzzy neutrosophic numbers based decision making analysis for prioritizing the energy storage technologies*, International Journal of Hydrogen Energy, **45**(43)(2020), 23027-23047.

[16] Chakraborty, A., Mondal, S. P., Alam, S., and Mahata, A., *Cylindrical neutrosophic single-valued number and its application in networking problem, multi-criterion group decision-making problem and graph theory*, CAAI Transactions on Intelligence Technology, **5**(2)(2020), 68-77.

[17] Aslan, C., Kargn, A., and ahin, M., *Neutrosophic modeling of talcott parsonss action and decision-making applications for it*, Symmetry, **12**(7)(2020), 1166.

[18] Tan, R. P., *Decision-making method based on new entropy and refined single-valued neutrosophic sets and its application in typhoon disaster assessment* Applied Intelligence, (2020), 1-25.

[19] Karabasevic, D., Stanujkic, D., Zavadskas, E. K., Stanimirovic, P., Popovic, G., Predic, B., and Ulutas, A., *A novel extension of the TOPSIS method adapted for the use of single-valued neutrosophic sets and hamming distance for e-commerce development strategies selection*, Symmetry, *12*(8)(2020), 1263.

[20] Holzinger, A., Langs, G., Denk, H., Zatloukal, K., and Mller, H., *Causability and explainability of artificial intelligence in medicine*, Wiley Interdisciplinary Reviews: Data Mining and Knowledge Discovery, **9**(4)(2019), p. e1312.

[21] Smarandache, F., *A unifying field in Logics: neutrosophic logic philosophy*, American Research Press, (1999), 1-141.

[22] Chakraborty, A., Mondal, S.P., Ahmadian, A., Senu, N., Alam, S., and Salahshour, S. *Different forms of triangular neutrosophic numbers, de-neutrosophication techniques, and their applications*, Symmetry, **10**(8)(2018), 327.

[23] https://www.phaeurope.org/about-ph/classification-and-who-functional-class/.

[24] Daimon, A., Terai, Y., Nagayasu, Y., Okamoto, A., Sano, T., Suzuki, Y., and Ohmichi, M, . *A case of intestinal obstruction in pregnancy diagnosed by MRI and treated by intravenous hyperalimentation*, Case reports in obstetrics and gynecology, 2016.

[25] https://www.mayoclinic.org/diseases-conditions/heart-arrhythmia/symptoms-causes/syc-20350668.

[26] https://www.nhlbi.nih.gov/health-topics/conduction-disorders

[27] https://www.spine-health.com/glossary/congenital-disorder

[28] https://www.health.harvard.edu/medical-tests-and-procedures/abortion-termination-of-pregnancy-a-to-z.

[29] https://www.mayoclinic.org/diseases-conditions/high-blood-pressure/in-depth/diuretics/art-20048129.

[30] Udigwe, G. O., Eleje, G. U., Ihekwoaba, E. C., Udegbunam, O. I., Egeonu, R. O., and Okwuosa, A. O., *Acute intestinal obstruction complicating abdominal pregnancy: conservative management and successful outcome*, Case reports in obstetrics and gynecology, 2016.

[31] Pearl, J., *Theoretical impediments to machine learning with seven sparks from the causal revolution*, (2018), arXiv preprint arXiv:1801.04016.

[32] Pearl, J., and Mackenzie, D., *The book of why: the new science of cause and effect*, Basic Books, (2018).

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Studying health and inclusive education: sentiment analysis using neutrosophy as a research tool

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Abstract: The coronavirus (COVID-19), from its propagation and virulence, has constituted the second global pandemic of the XXI century, claiming humanity, lives worldwide, in high scales, being Ecuador one of the affected countries and Guayas Canton with the highest morbidity and mortality, reasons why it has generated a social distancing as preventive measures. Present work aimed at the survey and intervention of health and educational problems manifested by parents and students with disability in regular education in District 09D17 of Milagro. For the first time, neutrosophic sets have been used to analyze interviews as a qualitative research tool. This paper is the first step of research that points out the uncertainties arising in qualitative data analysis. Among its main achievements are the change of behavior of the intervened families towards healthier lifestyles in the area of nutrition, psycho-pedagogical and social care, preparation for the life of these students, as well as the level of organization of students and researchers, teamwork, the use of communicational and digital tools to reach an improvement in the quality of life of these children.

Keywords: Transtheoretical model, Health education, Virtual education, Families of students with disability, adaptive behavior. Neutrosophy, SVN, sentiment analysis, neutrosophic research method.

1. Introduction

At a global level, the current society is going through a health crisis known to all: COVID-19, in which health systems face an arduous task as a result of the high morbimortality resulting from the pandemic, therefore, a group of actions are taken in most countries, imposing restrictions in general that contribute to reducing the transmission of the virus, and among these, it begins to experiment in the educational field a different social learning modality, being virtual education the hope of improving quality of life of the population[1].

In this current context, parents of families and students with Special Educational Needs (SEN in Spanish) [2]associated or not to disability (Intellectual and developmental, Physical, Sensory, Mental, constitute a population group of high vulnerability, from chronic diseases generated first by the ageing of parents, along with unfavorable lifestyles and then those acquired in their children as part of the manifest disabilities, which reveal difficulties in the quality of health.

In addition to this analysis, difficulties in accessing information are observed, which causes digital exclusion[3] in the educational field, the lack of knowledge of support systems for the education of their children, family isolation, chaos, translated into hopelessness, in this new role that they must assume from their homes towards their children and access to health services. In this sense, the educational inclusion of these students in virtual mode becomes a severe problem for the family and the educational institution because they try to support them from their place. However, it requires a personalized approach, which does not consistently achieve its purpose, as this feeling is reflected with high prevalence in social networks; on the other hand, it was seen in statistics of the Milagros canton the increase of health problems in this population mentioned above.

With these elements, the group of researchers and students of the Special Education career decide to conduct a study that leads to improving the health and quality of inclusive education of these students, in conditions of isolation of the population, through training to parents, to create practical attitudes and try to raise awareness in this vulnerable group the incorporation of healthier lifestyles that includes: hygiene, nutrition, sports and creative activities, relating social, environmental and economic factors by using health education, through the Transtheoretical model and based on neutrosophic research method.

2. Neutrosophy in sentiment analysis basic concepts

Neutrosophy is a mathematical theory developed by Romanian Scholar Florentin Smarandache to deal with indetermination[4]. It has been the base for developing new methods to deal with indeterminate and inconsistent information as neutrosophic sets neutrosophic logic and, especially, in decision-making problems [5]. The truth value in the neutrosophic set is as follow[6]:

Let be $\boldsymbol{N} = \{(\boldsymbol{T}, \boldsymbol{I}, \boldsymbol{F}): \boldsymbol{T}, \boldsymbol{I}, \boldsymbol{F} \subseteq [\mathbf{0}, \mathbf{1}]\}\boldsymbol{n}$, be a neutrosophic evaluation of a mapping of a group of formulas propositional to $\boldsymbol{N}$, and for each sentence $\boldsymbol{p}$:

$$\boldsymbol{v}(\boldsymbol{p}) = (\boldsymbol{T}, \boldsymbol{I}, \boldsymbol{F}) \qquad (1)$$

To facilitate the practical application in real-world problems[7], the use of Single-Value neutrosophic Sets (SVNS) was proposed, through which it is likely to use linguistic terms to obtain greater interpretability of the results[8].

Let X be a universe of discourse, an SVNS A over X has the following form[9]:

$$\boldsymbol{A} = \{\langle \boldsymbol{x}, \boldsymbol{u_a}(\boldsymbol{x}), \boldsymbol{r_a}(\boldsymbol{x}), \boldsymbol{v_a}(\boldsymbol{x})\rangle : \boldsymbol{x} \in \boldsymbol{X}\} \qquad (2)$$

Where

$$\boldsymbol{u_a}(\boldsymbol{x}): \boldsymbol{X} \to [\mathbf{0}, \mathbf{1}], \boldsymbol{r_a}(\boldsymbol{x}): \boldsymbol{X} \to [\mathbf{0}, \mathbf{1}]\ \boldsymbol{y}\ \boldsymbol{v_a}(\boldsymbol{x}): \boldsymbol{X} \to [\mathbf{0}, \mathbf{1}]$$

with

$$\mathbf{0} \leq \boldsymbol{u_a}(\boldsymbol{x}), \boldsymbol{r_a}(\boldsymbol{x}), \boldsymbol{v_a}(\boldsymbol{x}) \leq \mathbf{3}, \forall\ \boldsymbol{x} \in \boldsymbol{X}$$

The intervals $\boldsymbol{u_a}(\boldsymbol{x}), \boldsymbol{r_a}(\boldsymbol{x})$ **y** $\boldsymbol{v_a}(\boldsymbol{x})$ denote the memberships related to true, indeterminate and false from x in A, respectively[10]. For convenience reasons, a Single Value Neutrosophic Number (SVN) is expressed as A = (a, b, c), where a, b, c ∈ [0.1] and 0 ≤ a + b + c ≤ 3.

Let A = (a, b, c) be a single valued neutrosophic number, a score function S related to a single valued neutrosophic value, based on the truth-membership degree, indeterminacy-membership degree and falsity membership degree is defined by[11]:

$$s(V_i) = 2 + T_i - F_i - I_i \qquad (4)$$

The score function for single-valued neutrosophic sets is proposed to make the distinction between numbers.

In social sciences, a primary research methodology such as one-on-one interviews constitutes a widely used technique to derive meaningful insights and draw broad conclusions[12]. Once transcribed, these interviews help in providing qualitative analyses. However, such analyses are subjective and draw heavily from the unconscious biases of the authors or researchers. That apart, the learning from every new interview diminishes at a high rate and is not an efficient use of the researchers' valuable time. Neutrosophy, which features the concept of indeterminacy, has not been widely used in sentiment analysis of interviews. Neutrosophic sets are used for sentiment analysis of interviews as a qualitative research tool[13]. This study is the first step of research that points out the uncertainties in the discursive analysis[14].

2. Materials and Methods (proposed work with more details)

The population was represented by 197 families of students with Special Educational Needs to be Associated with disability of District 09D17 Milagro, according to a database provided by the Inclusion Support Unit (UDAI), of which 150 families were surveyed, as part of the Hermeneutic interpretative analysis in the 1st stage known as Structuring and coding of the research, to obtain the guiding categories of the study (Health in inclusive education), which in turn facilitated the emergence of the sensitizing categories, to delimit the object of study and establish the comprehensive analysis of the connections revealed and interpret its results in the research.

With this sample, two instruments were applied at the beginning and in the first evaluative cut to parents about their family member with SEN associated or not to disability: a closed interview that contemplated: General data of their child (age, sex, weight, height, body mass index (BMI), problems of families and their participation in the education of their children), the second which measure the development of adaptive behaviour skills of students with Special Educational Needs (SEN), including indicators structured in the following categories: communication, hygiene and health, home life, social skills, use of community services, self-direction, self-care and safety, leisure and free time, and functional academic skills, under the leadership of researchers and students of the Special Education career.

The neutrosophic research method was used for sentiment analysis on interviews[15]. The Neutrosophic Research Method is a generalization of Hegel's dialectic. It suggests that scientific and humanistic research will advance via studying not only the opposite ideas but the neutral ideas related to them as well in order to have a broad picture of the whole problem to **solve**[16].

3. Results

A pipeline using Orange Data mining[17] to analyze sentiment in interviews was developed. The sentiment analysis component predicts sentiment for each document in a corpus.

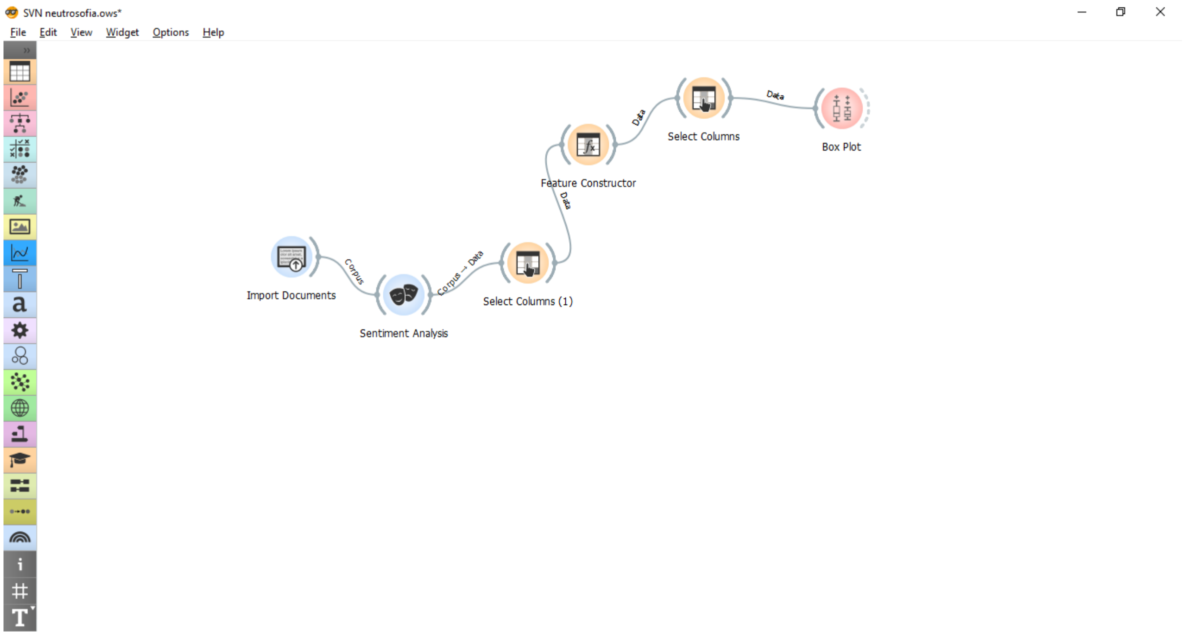

Figure 1. *Orange pipeline*

A group of 5 interviews were used, and as sentiment analysis, the VADER (Valence Aware Dictionary for Sentiment Reasoning text sentiment analysis) model is used [18]. VADER is sensitive to both polarity (positive/negative) and intensity (strength) of emotion. For convenience, a Single Value Neutrosophic Number (SVNS) in sentiment analysis is expressed as A = (pos, net, neg), where pos, net, and neg positive are positive, neutral and negative composite scores, respectively(Table 1).

Table 1. SVN number associated in interviews.

Case	SVN number
Case1	(0.052,0.909,0.04)
Case 2	(0.336,0.622,0.042)
Case 3	(0.044,0.814,0.142)
Case 4	(0,1,0)
Case 5	(0.075,0.746,0.179)

Scores of every interview were calculated using a feature contractor component

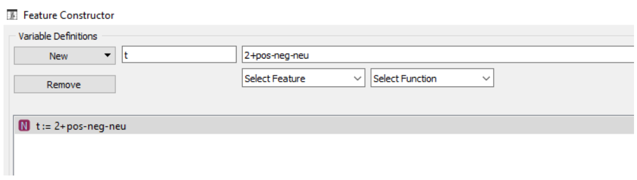

Figure 2. *Score calculation with* feature contractor component

The calculation results using Eq 4 as de-neutrification method [19] are shown in Figure 3.

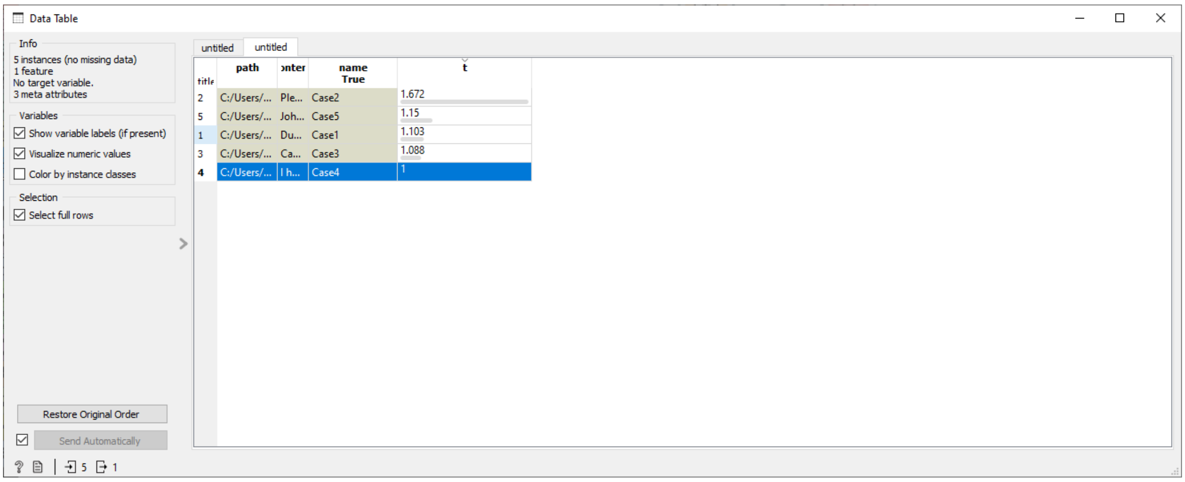

Figure 3. *Score calculation*

The methods show and sentiment score for every interview, that score function allow to rank single-valued neutrosophic numbers and gives a single numerical value. The researcher exploited a written interview transcript rich with observations and insights and quantified it using neutrosophy in conjunction with other research methods.

The highest scores are evidenced in the sensitizing categories Adaptive behaviour skills represented by leisure and free time with 12.43%, Social skills with 11.64% and main problems of families with 11.51%, while the relative values with the lowest scores are Health conditions with 1.35%, and Special Educational Needs to be associated or not with disability with 1.65%, which is coherent with the current reality, taking into account the deficiencies in the preparation of the family in the educational intervention of their children.

Within the adaptive behavioral skills of students with SEN associated or not to a disability, those indicators corresponding to leisure and free time skills and the development of social skills stand out because these areas in the initial state were depressed, constituting a causality of family conflicts, so they have been worked intensively, Although radical changes in the final state are not appreciated from their condition, both have evidenced reversed improvements in the other sensitizing categories studied, including the strengthening of family communication within and towards other families, a fundamental element of impact in terms of results.

Another of the statistical tools applied was the calculation of the emergency index[19], and upon studying the results, it was found that the highest incidence was due to the following:

· Main problems of families in the initial stage: indicators such as the following prevailed: parents' lack of knowledge of the developmental condition of their children and the support systems they need, generating disorientation, apathy, feeling of exclusion in the educational context, helplessness in educating their children (stress maintained in 100%, malnutrition due to obesity nine students (Morbid in 8 and 1 student Moderate), epilepsy, emotional disturbances, low self-esteem, insecurity in the work they do, low tolerance to the frustration of mistakes in parents and children, rejection of school, sadness, grief for the loss of family members with the pandemic, on the other hand, shows various disabilities with great diversity in terms of levels of development (age).

Although intellectual and developmental disabilities prevailed with more than 20% and adolescence as a complex stage within the development, although childhood was represented, all this determined the design of the training system and the innovative and Educommunication activities articulated in the educational process in the initial stage.

In the final state, encouraging results were observed in the change of behavior of the families and the degree of training that they have acquired in carrying out activities with their children, especially those related to the management of technological and Educommunication tools, degree of acceptance towards their children, greater tolerance and resilience in the current historical context and consequently less frustration.

· Involvement of parents in their children's education: in this category, the evidence shows that, in the initial state, family actions were insufficient towards the education of their children, hostility, covert rejection, domestic violence (verbal and physical) prevailed, translated into the need to deepen in areas of conflicts and how to achieve peaceful coexistence in their family, evasion of the role that corresponded to them in the education of their children, resistance to change, no involvement in the educational processes of their children, therefore their participation as a family is focused on feeding, protecting, caring for their children and the formation of some values, as part of their cultural function, the rest was characterized by complaints towards the state and absence of activities that propitiate support systems in each of the students within the educational process, primarily those support actions directed to the use of virtual technological tools.

In the final state, each of the families has evolved according to their condition and reality, from each workshop given, coupled with various communication tools used, along with the specialized accompaniment of specialists in Pediatric Medicine, Psychology, Special Pedagogy, Technology, Sports in the various activities articulated to the workshops applied in each module with the following topics: 1st Module: Main problems of families and students with SEN associated with disability, 2nd Module: Support systems for educational inclusion and the 3rd Module: Curricular adaptations in virtual education and the family, together with the work carried out in a personalized way by researchers and students of the Special Education career in the pedagogical reinforcement applied in the 3rd module.

· The development of Adaptive Behavioral Skills in the initial state: were contemplated in the type: Communication, Hygiene and health, Homelife, Social skills, Use of community services, Self-direction, Self-care and safety, leisure and free time, Functional academic skills, each of them, were structured in an essential group of evaluative indicators, articulated at the time of evaluation to three categories (always, sometimes and never), this type of evaluation allowed catching the slightest result, a fundamental element that focused on observing the evolution of the process optimistically.

In the final stage of the study, the educational practice and the statistical tools presented above reveal the participation of each of these skills in 100% of the schoolchildren studied in more or less development, with those of the Communicative, Hygiene and health, Social skills, Self-care, Leisure and free time type standing out above the others due to their complexity and importance in the adaptation to the social environment, which was strengthened by the increased work of the family as the main protagonist of this historical context.

An essential element that stood out is the guiding role played by the reflections of each of the workshops applied by the specialists, systematically uploaded to the Blog, elaborated as an Educomunicasional tool for this project, the research group and the career of Special Education and how parents have been involved spontaneously in all this work, characterized by dialogue and active participation, high self-esteem of families who already consider themselves part of this educational process, which has led to the transformation of the modes of action and the improvement of healthier lifestyles.

4. Results

It is possible to evidence precisely the total dependence that exists between the guiding category (Health in educational inclusion) and the initial and final state of the sensitizing categories, in particular: the states of health, problematic of families before the role of educational care of their children with SEN associated or not to disability and the development of adaptive behaviour skills in preparation for the life of these schoolchildren as described by [20].

It is appropriate to highlight how social and emotional learning has gained greater prominence in the results of this study and demonstrates the postulates of the Social Learning Theory [21], pointing to the importance of this learning according to the guiding principles: Attention, retention, reproduction and motivation of the knowledge learned through observation and imitation of the closest people of these students, tinged by the emotional learning that is performed by identifying the expressions, and emotions that this knowledge brings to their routine life.

The digital inclusion of this population group (students with SEN associated with disability and their families) contributes significantly to the improvement of inclusive education and constitutes a gateway to work within the family, which should continue to be studied.

Its limitations remain to continue generating dynamics that systematically strengthen these aspects mentioned above, even outside this pandemic stage.

.This study is the first step of research that points out the uncertainties solving in discursive analysis. Results show the practical applicability of the proposal ease of use and interpretation by experts

Conclusions (

1. The *Transtheoretical model applied in health education* in the study has been represented by the interactivity, negotiation and active participation of each of the acting groups based on a relationship of respect, which has been the cornerstone in the action of the project, strengthening family and social communication in general and their change of healthier lifestyles.
2. The social and emotional learning facilitated by parents strengthens the development of adaptive behavior skills, generating active and coherent participation of these students, strengthening their inclusive education.
3. The continuity of training for parents should be continued, not with the intention that they assume the role of a specialist, but rather taking advantage of their role as a family so that they can incorporate new tools that stimulate their actions, sensitization towards working with their children in an assertive family environment, permeated with love, respect and understanding.
4. This study is the first step of research that points out the uncertainties in discursive analysis using a qualitative research approach in line with the Smarandache proposal. Datamining tool Orange was adapted to the neutrosophic environment.
5. Future work will concentrate on multi-refined neutrosophic set (MRNS) in interview sentiment analysis.

References

[1] UNESCO, "COVID-19 Impact on Education," *UNESCO Inst. Stat. data*, 2020.

[2] K. Asbury, L. Fox, E. Deniz, A. Code, and U. Toseeb, "How is COVID-19 Affecting the Mental Health of

Children with Special Educational Needs and Disabilities and Their Families?," *J. Autism Dev. Disord.*, 2020, doi: 10.1007/s10803-020-04577-2.

[3] J. GIL-QUINTANA and A. CANO-ALFARO, "Digital inclusion: A challenge for school organization, planning and teaching," *Rev. Mediterr. Comun.*, 2020, doi: 10.14198/MEDCOM2020.11.1.6.

[4] F. Smarandache, "Neutrosophy, a new Branch of Philosophy," *Mult. valued Log.*, 2002.

[5] A. Sleem, M. Abdel-Baset, and I. El-henawy, "PyIVNS: A python based tool for Interval-valued neutrosophic operations and normalization," *SoftwareX*, 2020, doi: 10.1016/j.softx.2020.100632.

[6] F. Smarandache, M. A. Quiroz-Martínez, J. E. Ricardo, N. B. Hernández, and M. Y. Leyva Vázquez, "Application of neutrosophic offsets for digital image processing," *Investig. Operacional*, 2020.

[7] F. Smarandache, M. Abdel-Basset, and S. Broumi, "Neutrosophic Sets and Systems, vol. 40, 2021," *Neutrosophic Sets Syst.*, vol. 40, no. 1, p. 32, 2021.

[8] M. Saqlain, N. Jafar, S. Moin, M. Saeed, and S. Broumi, "Single and Multi-valued Neutrosophic Hypersoft set and Tangent Similarity Measure of Single valued Neutrosophic Hypersoft Sets," *Neutrosophic Sets Syst.*, vol. 32, pp. 317–329, 2020.

[9] M. T. Cadena, M. F. J. Burgos, and M. J. J. Montenegro, "Neutrosophic Case-Based Reasoning Method to Determine the Profitability of the Tourism Sector in the City of Riobamba," *Neutrosophic Sets Syst.*, vol. 37, no. 1, p. 13, 2020.

[10] J. L. R. Villafuerte, L. D. T. Torres, and L. T. Jimenez, "Neutrosophic Hypothesis to validate a modification for Article 630 of the Integral Organic Criminal Code of Ecuador," *Neutrosophic Sets Syst.*, vol. 37, no. 1, p. 31, 2020.

[11] M. Leyva-Vázquez, M. A. Quiroz-Martínez, Y. Portilla-Castell, J. R. Hechavarría-Hernández, and E. González-Caballero, "A New Model for the Selection of Information Technology Project in a Neutrosophic Environment," *Neutrosophic Sets Syst.*, p. 344.

[12] F. Sasangohar, A. Dhala, F. Zheng, N. Ahmadi, B. Kash, and F. Masud, "Use of telecritical care for family visitation to ICU during the COVID-19 pandemic: an interview study and sentiment analysis," *BMJ Qual. Saf.*, 2020.

[13] I. Kandasamy, W. B. Vasantha, J. M. Obbineni, and F. Smarandache, "Sentiment analysis of tweets using refined neutrosophic sets," *Comput. Ind.*, 2020, doi: 10.1016/j.compind.2019.103180.

[14] F. Smarandache, B. Teodorescu, and M. Teodorescu, "Uncertainty Communication Solution in Neutrosophic Key," *SSRN Electron. J.*, 2018, doi: 10.2139/ssrn.2731609.

[15] F. Smarandache, "The Neutrosophic Research Method in Scientific and Humanistic Fields," *Multisp. MULTISTRUCTURE. NEUTROSOPHIC Transdiscipl.*, p. 732.

[16] I. M. Hezam, M. Abdel-Baset, and F. Smarandache, "NEUTROSOPHIC OPERATIONAL RESEARCH. Volume I," 2018.

[17] J. Demšar *et al.*, "Orange: Data mining toolbox in python," *J. Mach. Learn. Res.*, 2013.

[18] A. Borg and M. Boldt, "Using VADER sentiment and SVM for predicting customer response sentiment," *Expert Syst. Appl.*, 2020, doi: 10.1016/j.eswa.2020.113746.

[19] D. J. Baughman, A. Waheed, M. N. Khan, and J. M. Nicholson, "Enhancing Value-Based Care With a Walk-in Clinic: A Primary Care Provider Intervention to Decrease Low Acuity Emergency Department Overutilization," *Cureus*, vol. 13, no. 2, 2021.

[20] R. L. Schalock *et al.*, *Intellectual disability: Definition, classification, and systems of supports*. ERIC, 2010.

[21] S. McLeod, "Social Learning Theory Bandura Social Learning Theory," *Learn. Theor.*, 2016.

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Statistical Convergence of Double Sequences in Neutrosophic Normed Spaces

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Abstract. In this paper, we define and study the notion of statistically convergent and statistically Cauchy double sequences in neutrosophic normed spaces. Moreover, we give the double statistically Cauchy sequence in neutrosophic normed space and present the double statistically completeness in connection with a neutrosophic normed space.

Keywords: Nuetrosophic normed spaces, statistical double convergence.

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1. **Introduction**

The concept of fuzzy set was originally introduced by Zadeh [20] in 1965. The fuzzy theory has become an area of active research for the last fifty years. It has a wide range of applications in the field of science and engineering, population dynamics [1], chaos control [5], computer programming [8], nonlinear dynamical systems [9], fuzzy physics [13] and more. Taking into account the concept of fuzzy set, Smarandache [16] introduced the notion of Neutrosophic set (NS) which is a new version of the idea of the classical set. The first world publication related to the concept of neutrosophy was published in 1998 and included in the literature [17].

On the other hand, Kaleva and Seikkala [10] defined the fuzzy metric spaces (FMS) as a distance between two points to be a non-negative fuzzy number. after that, In [6] some basic properties of FMS were studied and the Baire Category Theorem for FMS was proved. Furthermore, some properties such as separability, countability are given and Uniform Limit Theorem is proved in [7]. Consequently, FMS has used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decisionmaking and more. After defined of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park [15] introduced IF metric space (IFMS), that is a generalization of FMS. Then, Park used George and Veeramani's [6] work for applying t-norm and t-conorm to FMS meanwhile defining IFMS and studying its basic properties. Moreover, Bera and Mahapatra introduced the neutrosophic soft linear spaces (NSLS) [3]. Later, neutrosophic soft normed linear spaces(NSNLS) was defined by Bera and Mahapatra [2]. Besides, In [2], neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were defined and studied. Recently, Kirisci and Simsek [11] in 2020, introduced and studied the notion of statistical convergence in a neutrosophic normed spaces. Besides, they showed some interesting results.

In this paper, we extend the notion of statistical convergence on neutrosophic normed spaces by using double sequences. Moreover, we prove some of its properties and characterizations. This paper is organized as follows: In the second section, we procure some well-known notions and definitions which are useful for the developing of this paper. In the third part, we define and study the notion of statistical convergence of double sequences on neutrosophic normed spaces (NNS). And the fourth section, we put a a conclusion in which we discuss about the results showed in section 3 and some future studies.

2. **Preliminaries**

The notion of statistical convergence was defined by Fast [7] and Steinhaus [18] independently and later this notion was studied by various authors.

Let K be a subset of $\mathbb{N}$, then the asymptotic density of K, denoted by $d(K)$ is defined as follows:

$$d(K) = \lim_{n} \frac{1}{n} |k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set. A number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for each $\epsilon > 0$, the set $d(\epsilon) = \{k \leq n : |x_k - L| > \epsilon\}$ has asymptotic density zero. Then, taking into account that notion, Mursaleen and Edely [14] defined the notion of statistical convergence of double sequences. Let $K \subset \mathbb{N} \times \mathbb{N}$ be two-dimensional set of positive integers and let $K(m, n)$ be the numbers of (j, k) in K such that $j \leq n$ and $k \leq n$. Then, the two-dimensional analogue of natural density can be defined as follows:

The lower asymptotic density of the set $K \subset \mathbb{N} \times \mathbb{N}$ is defined as:

$$d_2(K) = \lim_{m,n} \inf \frac{K(m, n)}{mn}$$

In case that the sequence $(K(m, n)/mn)$ has a limit in Pringsheim's sense then we say that K has a double density and is defined as:

$$\lim_{m,n} \frac{K(m,n)}{mn} = \delta_2(K).$$

Statistical convergence for double sequence $x = (x_{kj})$ of real which was defined by [14] as: A real double sequence $x = (x_{kj})$ is said to be statistically convergent to the number L if for each $\epsilon > 0$, the set $\{(j,k), j \leq m, k \leq n : |x_{kj} - L| \geq \epsilon\}$, has a double natural density zero. In this case, we write $S_2\text{-}\lim x_{jk} = L$.

On the other hand, Triangular norms (t-norms) (TN) were initiated by Menger [13]. In the problem of computing the distance between two elements in space, Menger offered using probability distributions instead of using numbers for distance. TNs are used to generalize with the probability distribution of triangle inequality in metric space conditions. Triangular conorms (t-conorms) (TC) know as dual operations of TNs. TNs and TCs are very significant for fuzzy operations(intersections and unions).

Definition 2.1. ([13]) Give an operation $\circ : [0,1] \times [0,1] \to [0,1]$. If the operation $\circ$ is satisfying the following conditions:

(1) $s \circ 1 = s$,

(2) If $s \leq u$ and $t \leq v$, then $s \circ t \leq u \circ v$,

(3) $\circ$ is continuous,

(4) $\circ$ is continuous and associative.

Then, it is called that the operation $\circ$ is continuous TN, for $s, t, u, v \in [0,1]$.

Definition 2.2. ([13]) Give an operation $\bullet : [0,1] \times [0,1] \to [0,1]$. If the operation $\bullet$ is satisfying the following conditions:

(1) $s \bullet 0 = s$,

(2) If $s \leq u$ and $t \leq v$, then $s \bullet t \leq u \bullet v$,

(3) $\bullet$ is continuous,

(4) $\bullet$ is continuous and associative.

Then, it is called that the operation $\bullet$ is continuous TC, for $s, t, u, v \in [0,1]$.

Remark 2.3. [11]) From the above definitions, we can see that if we take $0 < \epsilon_1$, $\epsilon_2 < 1$ for $\epsilon_1 > \epsilon_2$, then there exist $0 < \epsilon_3$, $\epsilon_4 < 0,1$ such that $\epsilon_1 \circ \epsilon_3 \geq \epsilon_2$, $\epsilon_1 \geq \epsilon_4 \bullet \epsilon_2$. Moreover, if we take $\epsilon_5 \in (0,1)$, then there exist $\epsilon_6, \epsilon_7 \in (0,1)$ such that $\epsilon_6 \circ \epsilon_6 \geq \epsilon_5$ and $\epsilon_7 \bullet \epsilon_7 \leq \epsilon_5$.

Definition 2.4. ([12]) Take F be an arbitrary set, $\mathbb{N} = \{< u, Q(u), W(u), E(u) >: u \in F\}$ be a NS (neutrosophic set) such that $\mathbb{N} : F \times F \times R^+ \to [0,1]$. Let $\circ$ and $\bullet$ show the continuous TN and continuous TC, respectively. If the following conditions are satisfied, then the four-tuple $(F, \mathbb{N}, \circ, \bullet)$ is called neutrosophic metric space (NMS):

(1) $0 \leq Q(u,v,\lambda) \leq 1$, $0 \leq W(u,v,\lambda) \leq 1$, $0 \leq E(u,v,\lambda) \leq 1$ for all $\lambda \in R^+$,

(2) $Q(u,v,\lambda) + W(u,v,\lambda) + E(u,v,\lambda) \leq 3$, for$\lambda \in R^+$,

(3) $Q(u,v,\lambda)=1$, for $\lambda>0$ if and only if $u=v$,
(4) $Q(u,v,\lambda)=Q(v,u,\lambda)$, for $\lambda>0$,
(5) $Q(u,v,\lambda)\circ Q(v,y,\mu)\leq Q(u,y,\lambda+\mu)$, for all $\mu,\lambda>0$,
(6) $Q(u,v,.):[0,\infty)\to[0,1]$ is continuous,
(7) $\lim_{\lambda\to\infty}Q(u,v,\lambda)=1$, for all $\lambda>0$,
(8) $W(u,v,\lambda)=0$, for $\lambda>0$ if and only if $u=v$,
(9) $W(u,v,\lambda)=W(v,u,\lambda)$, for $\lambda>0$,
(10) $W(u,v,\lambda)\bullet W(v,y,\mu)\geq W(u,y,\lambda+\mu)$, for all $\mu,\lambda>0$,
(11) $W(u,v,.):[0,\infty)\to[0,1]$ is continuous,
(12) $\lim_{\lambda\to\infty}W(u,v,\lambda)=1$, for all $\lambda>0$,
(13) $E(u,v,\lambda)=0$, for $\lambda>0$ if and only if $u=v$,
(14) $E(u,v,\lambda)=E(v,u,\lambda)$, for $\lambda>0$,
(15) $E(u,v,\lambda)\bullet E(v,y,\mu)\geq E(u,y,\lambda+\mu)$, for all $\mu,\lambda>0$,
(16) $W(u,v,.):[0,\infty)\to[0,1]$ is continuous,
(17) $\lim_{\lambda\to\infty}E(u,v,\lambda)=1$, for all $\lambda>0$,
(18) if $\lambda\geq 0$, then $Q(u,v,\lambda)=0$, $W(u,v,\lambda)=1$ and $E(u,v,\lambda)=1$.

For all $u,v,y\in F$. Then, $N=(Q,W,E)$ is called Neutrosophic metric (NM) on F.

The notion of neutrosophic normed space (NNS) was defined by [11], as well as, the definition statistical convergence with respect to NNS was given.

Definition 2.5. ([11])Take F as a vector space $N=\{<u,G(u),B(u),Y(u)>: u\in F\}$ be a normed space (NS) such that $N:F\times R^+\to[0,1]$. Let $\circ$ and $\bullet$ show the continuous TN and continuous TC, respectively. If the following contritions are satisfied, then the four-tuple $V=(F,N,\circ,\bullet)$ is called NNS, for all $u,v\in F$, $\lambda,\mu>0$ and for each $\sigma\neq 0$:

(1) $0\leq G(u,\lambda)\leq 1$, $0\leq B(u,\lambda)\leq 1$, $0\leq Y(u,\lambda)\leq 1$ for all $\lambda\in R^+$,
(2) $G(u,\lambda)+B(u,\lambda)+Y(u,\lambda)\leq 3$, for$\lambda\in R^+$,
(3) $G(u,\lambda)=1$, for $\lambda>0$ if and only if $u=0$,
(4) $G(\sigma u,\lambda)=G(u,\dfrac{\lambda}{|\sigma|})$,
(5) $G(u,\mu)\circ G(v,\lambda)\leq G(u+v,\lambda+\mu)$,
(6) $G(u,.)$ is continuous non-decreasing function,
(7) $\lim_{\lambda\to\infty}G(u,\lambda)=1$,
(8) $B(u,\lambda)=0$, for $\lambda>0$ if and only if $u=0$,
(9) $B(\sigma u,\lambda)=B(u,\dfrac{\lambda}{|\sigma|})$,
(10) $B(u,\mu)\bullet B(v,\lambda)\geq B(u+v,\lambda+\mu)$,
(11) $B(u,.)$ is continuous non-increasing function,
(12) $\lim_{\lambda\to\infty}B(u,\lambda)=0$,

(13) $Y(u,\lambda)=0$, for $\lambda>0$ if and only if $u=0$,

(14) $Y(\sigma u,\lambda)=Y(u,\dfrac{\lambda}{|\sigma|})$,

(15) $Y(u,\mu)\bullet Y(v,\lambda)\geq Y(u+v,\lambda+\mu)$,

(16) $Y(u,.)$ is continuous non-increasing function,

(17) $\lim_{\lambda\to\infty} Y(u,\lambda)=0$,

(18) if $\lambda\leq 0$, then $G(u,\lambda)=0$, $B(u,\lambda)=1$ and $Y(u,\lambda)=1$.

Then, $N=(G,B,Y)$ is called neutrosophic norm (NN).

Example 2.6. ([11]) Let $(F,\|.\|)$ be a NS. Give the operations $\circ$ and $\bullet$ as TN $u\circ v=uv$; TC $u\bullet v=u+v-uv$. For $\lambda>\|u\|$,

$$G(u,\lambda)=\frac{\lambda}{\lambda+\|u\|},\ B(u,\lambda)=\frac{\|u\|}{\lambda+\|u\|},\ Y(u,\lambda)=\frac{\|u\|}{\lambda},$$

for all $u,v\in F$ and $\lambda>0$. If we take $\lambda\leq\|u\|$, then $G(u,\lambda)=0$, $B(u,\lambda)=1$ and $Y(u,\lambda)=1$. Then, $(F,N,\circ,\bullet)$ is NNS such that $N:F\times R^+\to[0,1]$.

Definition 2.7. ([11]) Let V be a NNS and (x_k) be a sequence in V such that $0<\epsilon<1$ and $\lambda>0$. Then, (x_k) converges to x if and only if there exists $n_0\in\mathbb{N}$ such that $G(x_k-x,\lambda)>1-\epsilon$, $B(x_k-x,\lambda)<\epsilon$ and $Y(x_k-x,\lambda)<\epsilon$. That is $\lim\limits_{k\to\infty} G(x_k-x,\lambda)=1$, $\lim\limits_{k\to\infty} B(x_k-x,\lambda)=0$ and $\lim\limits_{k\to\infty} Y(x_k-x,\lambda)=0$ as $\lambda>0$. In this case, the sequence (x_k) is said to be a convergent sequence in V. The convergent in NNS is denoted by N-$\lim x_k=L$.

Definition 2.8. ([11]) Let V be a NNS, the sequence (x_k) in V where $0<\epsilon<1$ and $\lambda>0$. Then, the sequence (x_k) is Cauchy in a NNS V if there is a $n_0\in\mathbb{N}$ such that $G(x_k-x_q,\lambda)>1-\epsilon$, $B(x_k-x_q,\lambda)<\epsilon$ and $Y(x_k-x_q,\lambda)<\epsilon$ for $k,q\geq n_0$.

Definition 2.9. ([11]) Let V be a NNS. For $\lambda>0$, $u\in F$ and $0<\epsilon<1$,

$$O(u,\epsilon,\lambda)=\{vinF: G(u-v,\lambda)>1-\epsilon,\ B(u-v,\lambda)<\epsilon,\ Y(u-v,\lambda)<\epsilon\}$$

is called open ball (OB)with center u and radius ϵ.

Definition 2.10. ([11]) The set $A\subset F$ is called neutrosophic-bounded (NB)in NNS V, if there exist $\lambda>0$, and $\epsilon\in(0,1)$ such that $G(u,\lambda)>1-\epsilon$, $B(u,\lambda)<\epsilon$ and $Y(u,\lambda)<\epsilon$ for each $u\in A$.

3. Statistical convergence of double sequences on NNS

In this section, we define and study the notion of statistical double convergence in a Neutrosophic normed space

Definition 3.1. Let V be a NNS and (x_{kj}) be a double sequence in V such that $0<\epsilon<1$ and $\lambda>0$. Then, (x_{kj}) converges to x if and only if there exists $n_0\in\mathbb{N}$ such that $G(x_{kj}-x,\lambda)>1-\epsilon$, $B(x_{kj}-x,\lambda)<\epsilon$ and $Y(x_{kj}-x,\lambda)<\epsilon$. That is $\lim\limits_{k,j\to\infty} G(x_{kj}-x,\lambda)=1$,

$\lim_{k,j\to\infty} B(x_{kj}-x,\lambda)=0$ and $\lim_{k,j\to\infty} Y(x_{kj}-x,\lambda)=0$ as $\lambda>0$. In this case, the double sequence (x_{kj}) is said to be a double convergent sequence in V. The double convergent in NNS is denoted by $N_2\text{-}\lim x_{kj}=L$.

Theorem 3.2. *Let V be a NNS and (x_{kj}) be a double sequence in V. Then, the following statements hold:*

(1) *If (x_{kj}) in V is convergent, then the limit point is unique.*
(2) *In V, if $\lim_{k,j\to\infty} x_{kj}=x$ and $\lim_{k,j\to\infty} y_{kj}=y$, then $\lim_{k,j\to\infty} x_{kj}+y_{kj}=x+y$.*
(3) *in V, if $\lim_{k,j\to\infty} x_{kj}=x$ and $\alpha\neq 0$, then $\lim_{k,j\to\infty}\alpha x_{kj}=\alpha x$.*

Proof: Since the proof of this Theorem is straightforward, we omitted it.

Definition 3.3. Let V be a NNS, the double sequence (x_{kj}) in V where $0<\epsilon<1$ and $\lambda>0$. Then, the double sequence (x_{kj}) is Cauchy in a $NNS\ V$ if there is a $n_0\in\mathbb{N}$ such that $G(x_{kj}-x_{qw},\lambda)>1-\epsilon$, $B(x_{kj}-x_{qw},\lambda)<\epsilon$ and $Y(x_{kj}-x_{qw},\lambda)<\epsilon$ for $k,j,q,w\geq n_0$. A NNS V is called complete if and only if every double Cauchy sequence (x_{kj}) is convergent to x in a NNS V.

Example 3.4. Consider G,B and Y from Example 2.6 . Then, V is a NNS. Besides,

$$\lim_{k,j,q,w\to\infty}\frac{\lambda}{\lambda+\|x_{kj}-x_{qw}\|}=1,\ \lim_{k,j,q,w\to\infty}\frac{\|x_{kj}-x_{qw}\|}{\lambda+\|x_{kj}-x_{qw}\|}=0,\ \lim_{k,j,q,w\to\infty}\frac{\|x_{kj}-x_{qw}\|}{\lambda}=0,$$

That is

$$\lim_{k,j,q,w\to\infty} G(x_{kj}-x_{qw},\lambda)=1, \lim_{k,j,q,w\to\infty} B(x_{kj}-x_{qw},\lambda)=0\ , \lim_{k,j,q,w\to\infty} Y(x_{kj}-x_{qw},\lambda)=0.$$

Therefore, we can say that the double sequence (x_{kj}) is a double Cauchy sequence in NNS V.

Remark 3.5. It is clear that every double convergent sequence in V is a double Cauchy sequence. But the inverse of this expression is not be true.

Theorem 3.6. *Let V be a NNS and (x_{kj}) be a double sequence in V. Then, the following statements hold:*

(1) *If for $u,v\in[0,1]$, we choose the continuous TN $u\circ v=\min\{u,v\}$ and the continuous TC $u\bullet v=\max\{u,v\}$, then every double Cauchy sequence is bounded in NNS V.*
(2) *Let the double sequences (x_{nmj}) and (y_{kj}) be double Cauchy and the double sequence (α_{kj}) is scalars in NNS V. Then, the double sequences $(x_{kj}+y_{kj})$ and $(\alpha_{kj}x_{kj})$ are also double Cauchy in NNS V.*
(3) *V is a complete NNS, if every double Cauchy sequence has a double convergent subsequence in NNS V.*

Proof: The proof of this Theorem is followd by the definitions of NNS, $G;B;Y$,double Cauchy sequence in V and completeness.

Definition 3.7. Let V a NNS. A double sequence (x_{kj}) is said to be statistical convergence with respect to neutrosophic normed (DSC-NN), if there exist $L \in F$ such that the set

$$K_{\epsilon_2} = \{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \epsilon \text{ or } B(x_{kj} - L, \lambda) \geq \epsilon, Y(x_{kj} - L, \lambda) \geq \epsilon\}$$

or equivalently

$$K_{\epsilon_2} = \{k \leq n, j \leq m : G(x_{kj} - L, \lambda) > 1 - \epsilon \text{ or } B(x_{kj} - L, \lambda) < \epsilon, Y(x_{kj} - L, \lambda) < \epsilon\}.$$

has double neutrosophic density (DND) zero, for every $\epsilon > 0$ and $\lambda > 0$. That is $d(K_{\epsilon_2}) = 0$ or equivalently,

$$\lim_{n,m} \frac{1}{nm} |\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \epsilon \text{ or } B(x_{kj} - L, \lambda) \geq \epsilon, Y(x_{kj} - L, \lambda) \geq \epsilon\}| = 0$$

Therefore, we write S_{N_2}-$\lim x_{kj} = L$ or $x_{kj} \to L(S_{N_2})$. The set of DSC-NN will be denoted by S_{N_2}. If $L = 0$, then we will write $S^0_{N_2}$.

Lemma 3.8. *Let V be a NNS. Then, the following statements are equivalent, for every $\epsilon > 0$ and $\lambda > 0$:*

(1) S_{N_2}-$\lim x_{kj} = L$.
(2) $\lim_{n,m} \frac{1}{nm} |\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \epsilon\}| = \lim_{n,m} \frac{1}{nm} |\{B(x_{kj} - L, \lambda) \geq \epsilon\}| = \lim_{n,m} \frac{1}{nm} |\{Y(x_{kj} - L, \lambda) \geq \epsilon\}| = 0.$
(3) $\lim_{n,m} \frac{1}{nm} |\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) < 1 - \epsilon \text{ and } B(x_{kj} - L, \lambda) < \epsilon, Y(x_{kj} - L, \lambda) < \epsilon\}| = 1.$
(4) $\lim_{n,m} \frac{1}{nm} |\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) > 1 - \epsilon\}| = \lim_{n,m} \frac{1}{nm} |\{k \leq n, j \leq m : B(x_{kj} - L, \lambda) < \epsilon\}| = \lim_{n,m} \frac{1}{nm} |\{k \leq n, j \leq m : Y(x_{kj} - L, \lambda) < \epsilon\}| = 1.$
(5) S_2-$\lim G(x_{kj} - L, \lambda) = 1,$ *and* S_2-$\lim B(x_{kj} - L, \lambda) = 0,$ S_2-$\lim Y(x_{kj} - L, \lambda) = 0.$

Proof: The poof of this Lemma is followed by the Definitions 3.7 and the notions showed in Section 2 .

Theorem 3.9. *Let V a NNS. If (x_{kj}) is DSC-NN, then S_{N_2}-$\lim x_{kj} = L$ is unique.*

Proof: Consider that S_{N_2}-$\lim x_{kj} = L_1$ and S_{N_2}-$\lim x_{kj} = L_2$ for $L_1 \neq L_2$. Now, take $\epsilon > 0$. Then, for a given $\mu > 0$, $(1 - \epsilon) \circ (1 - \epsilon) > 1 - \mu$ and $\epsilon \bullet \epsilon < \mu$. For any $\lambda > 0$. Let's write the following sets:

$$K_{G_1}(\epsilon, \lambda) := \{k \leq n, j \leq m : G(x_{kj} - L_1, \frac{\lambda}{2}) \leq 1 - \epsilon\},$$
$$K_{G_2}(\epsilon, \lambda) := \{k \leq n, j \leq m : G(x_{kj} - L_2, \frac{\lambda}{2}) \leq 1 - \epsilon\}$$
$$K_{B_1}(\epsilon, \lambda) := \{k \leq n, j \leq m : B(x_{kj} - L_1, \frac{\lambda}{2}) \leq 1 - \epsilon\},$$
$$K_{B_2}(\epsilon, \lambda) := \{k \leq n, j \leq m : B(x_{kj} - L_2, \frac{\lambda}{2}) \leq 1 - \epsilon\}$$
$$K_{Y_1}(\epsilon, \lambda) := \{k \leq n, j \leq m : Y(x_{kj} - L_1, \frac{\lambda}{2}) \leq 1 - \epsilon\},$$

$$K_{Y_2}(\epsilon,\lambda) := \{k \leq n, j \leq m : Y(x_{kj} - L_2, \frac{\lambda}{2}) \leq 1 - \epsilon\}$$

Since that S_{N_2}-lim $x_{kj} = L_1$. Then, by the Lemma 3.8 , for all $\lambda > 0$,

$$d(K_{G_1}(\mu,\lambda)) = d(K_{B_1}(\mu,\lambda)) = d(K_{Y_1}(\mu,\lambda)) = 0$$

Moreover, since we have S_{N_2}-lim $x_{kj} = L_2$, by the Lemma 3.8 , for $\lambda > 0$,

$$d(K_{G_2}(\mu,\lambda)) = d(K_{B_2}(\mu,\lambda)) = d(K_{Y_2}(\mu,\lambda)) = 0$$

Now, let

$$K_{N_2}(\mu,\lambda) := \{K_{G_1}(\mu,\lambda) \cup K_{G_2}(\mu,\lambda)\} \cap \{K_{B_1}(\mu,\lambda) \cup K_{B_2}(\mu,\lambda)\} \cap \{K_{Y_1}(\mu,\lambda) \cup K_{Y_2}(\mu,\lambda)\}.$$

Then, we can see that $d(K_{N_2}(\mu,\lambda)) = 0$ which implies $d(\mathbb{N} - K_{N_2}(\mu,\lambda)) = 1$. Then, we have the following possible situations, when we take $(k,j) \in \mathbb{N} - K_{N_2}(\mu,\lambda)$:

(1) $(k,j) \in \mathbb{N} - (K_{G_1}(\mu,\lambda) \cup K_{G_2}(\mu,\lambda))$,
(2) $(k,j) \in \mathbb{N} - (K_{B_1}(\mu,\lambda) \cup K_{B_2}(\mu,\lambda))$,
(3) $(k,j) \in \mathbb{N} - (K_{Y_1}(\mu,\lambda) \cup K_{Y_2}(\mu,\lambda))$.

First at all, consider (1). Then, we have

$$G(L_1 - L_2, \lambda) \geq G(x_{kj} - L_1, \frac{\lambda}{2}) \circ G(x_{kj} - L_2, \frac{\lambda}{2}) > 1 - \epsilon \circ (1 - \epsilon)$$

And then, since $(1-\epsilon) \circ (1-\epsilon) > 1 - \mu$,

$$G(L_1 - L_2, \lambda) > 1 - \mu \tag{1}$$

By 1 , for all $\lambda > 0$, we have that $G(L_1 - L_2, \lambda) = 1$, where $\mu > 0$ is arbitrary. This is, $L_1 = L_2$.

Secondly, for (2), if we choose $(k,j) \in \mathbb{N} - (K_{B_1}(\mu,\lambda) \cup K_{B_2}(\mu,\lambda))$, then we can write

$$B(L_1 - L_2, \lambda) \leq B(x_{kj} - L_1, \frac{\lambda}{2}) \bullet B(x_{kj} - L_2, \frac{\lambda}{2}) < \epsilon \bullet \epsilon$$

Using $\epsilon \bullet \epsilon < \mu$, we can see that $B(L_1 - L_2, \lambda) < \mu$. For all $\lambda > 0$, we get $B(L_1 - L_2, \lambda) = 0$, where $\mu > 0$ is arbitrary. Therefore, $L_1 = L_2$.

Finally, in the same way, for the situation (3), if we choose $(k,j) \in \mathbb{N} - (K_{Y_1}(\mu,\lambda) \cup K_{Y_2}(\mu,\lambda))$, then we can write

$$Y(L_1 - L_2, \lambda) \leq Y(x_{kj} - L_1, \frac{\lambda}{2}) \bullet Y(x_{kj} - L_2, \frac{\lambda}{2}) < \epsilon \bullet \epsilon$$

Using $\epsilon \bullet \epsilon < \mu$, we can see that $Y(L_1 - L_2, \lambda) < \mu$. For all $\lambda > 0$, we get $Y(L_1 - L_2, \lambda) = 0$, where $\mu > 0$ is arbitrary. Therefore, $L_1 = L_2$. And this step ends the proof.

Theorem 3.10. *If N_2-lim $x_{kj} = L$ for a NNS V. Then, S_{N_2}-lim $x_{kj} = L$.*

Proof: Let N_2-$\lim x_{kj} = L$. Then, for every $\epsilon > 0$ and $\lambda > 0$, there exist a number $n_0 \in \mathbb{N}$ such that $G(x_{kj} - L, \lambda) > 1 - \epsilon$ and $B(x_{kj} - L, \lambda) < \epsilon$, $Y(x_{kj} - L, \lambda) < \epsilon$, for all $k, j \geq n_0$. Hence, the set

$$\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \epsilon \text{ or } B(x_{kj} - L, \lambda) \geq \epsilon,\, Y(x_{kj} - L, \lambda) \geq \epsilon\}$$

has at most finitely many terms. Therefore, since every finite subset of $\mathbb{N}$ has double density zero,

$$\lim_{n,m} \frac{1}{nm} |\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \epsilon \text{ or } B(x_{kj} - L, \lambda \geq \epsilon,\, Y(x_{kj} - L, \lambda) \geq \epsilon\}| = 0$$

And this ends the proof.

Theorem 3.11. *Let V be a NNS. S_{N_2}-$\lim x_{kj} = L$ if and only if there exists an increasing index double sequence $L_2 = \{l_1, ...l_n, ...; l_1, ..., l_m, ...\} \subset \mathbb{N} \times \mathbb{N}$, while $d(L_2) = 1$, N_2-$\lim_{n,m\to\infty} x_{l_{nm}} = L$.*

Proof: Suppose that $S_{G_{N_2}}$-$\lim x_{kj} = L$. For any $\lambda > 0$ and $\mu = 1, 2, ...,$

$$P_{N_2}(\mu, \lambda) = \{k \leq n, j \leq m : G(x_{kj} - L, \lambda) > 1 - \frac{1}{\mu} \text{ and } B(x_{kj} - L, \lambda) < \frac{1}{\mu},$$
$$Y(x_{kj} - L, \lambda) < \frac{1}{\mu}\}$$

and

$$R_{N_2}(\mu, \lambda) = \{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \frac{1}{\mu} \text{ or } B(x_{kj} - L, \lambda) \geq \frac{1}{\mu},\, Y(x_{kj} - L, \lambda) \geq \frac{1}{\mu}\}.$$

Then, $d(R_{N_2}(\mu, \lambda)) = 0$, since S_{N_2}-$\lim x_{kj} = L$. Besides, for $\lambda > 0$ and $\mu = 1, 2, ...,$

$$d(P_{G_{N_2}}(\mu, \lambda)) = 1 \tag{2}$$

Now, we will prove that for $(k, j) \in PN_2(\mu, \lambda)$, N_2-$\lim x_{kj} = L$. Consider that N_2-$\lim x_{kj} \neq L$, for some $(k, j) \in P_{N_2}(\mu, \lambda)$. Then, there is $\rho > 0$ and a positive integer n_0 such that $G(x_{kj} - L, \lambda) \leq 1 - \rho$ or $B(x_{kj} - L, \lambda) \geq \rho$, $Y(x_{kj} - L, \lambda) \geq \rho$, for all $k, j \geq n_0$. Now, let $G(x_{kj} - L, \lambda) > 1 - \rho$ and $B(x_{kj} - L, \lambda) < \rho$, $Y(x_{kj} - L, \lambda) < \rho$ for all $k < n; j < m$. Hence,

$$\lim_{n,m} \frac{1}{nm} |\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) > 1 - \rho \text{ and } B(x_{kj} - L, \lambda) < \rho,\, Y(x_{kj} - L, \lambda) < \rho\}| = 0$$

Since $\rho > \frac{1}{\mu}$, we have that $d(P_{N_2}(\mu, \lambda)) = 0$, which contradicts 2 . Therefore, N_2-$\lim x_{kj} = L$.

Now, let's assume that there exists a subset $L_2 = \{l_1, ..., l_n, ...; l_1, ..., l_m, ...\} \subset \mathbb{N} \times \mathbb{N}$ such that $d(J_2) = 1$ and N_2-*displaystyle* $\lim_{n,m\to\infty} x_{l_{nm}} = L$, this means that there exits $n_0 \in \mathbb{N}$ such that $G(x_{kj} - L, \lambda) > 1 - \mu$ and $B(x_{kj} - L, \lambda) < \mu$, $Y(x_{kj} - L, \lambda) < \mu$, for every $\mu > 0$ and $\lambda > 0$. In that case,

$$R_{N_2}(\mu, \lambda) := \{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \mu \text{ or } B(x_{kj} - L, \lambda) \geq \mu, Y(x_{kj} - L, \lambda) \geq \mu\}$$
$$\subseteq \mathbb{N} \times \mathbb{N} - \{l_{n+1}, l_{n+2}, ...; l_{m+1}, l_{m+2}, ...\}$$

Therefore, $d(R_{N_2}(\mu,\lambda)) \leq 1-1=0$. Hence, $S_{N_2}\text{-}\lim x_{kj} = L$.

Now, we show some results that we obtained on double statistical Cauchy sequences in NNS.

Definition 3.12. The double sequence (x_{kj}) is said to be statistically Cauchy with respect to NN (DSCa-NN) in a NNS V, if there exit $N = N(\epsilon)$ and $M = M(\epsilon)$, for every $\epsilon > 0$ and $\lambda > 0$ such that

$$KC_\epsilon := \{k \leq n, j \leq m : G(x_{kj} - x_{NM}, \lambda) \leq 1-\epsilon \text{ or } B(x_{kj} - x_{NM}, \lambda) \geq \epsilon,\\ Y(x_{kj} - x_{NM}, \lambda) \geq \epsilon\}$$

has DND zero. That is $d(KC_\epsilon) = 0$.

Theorem 3.13. *If a double sequence (x_{kj}) is DSC-NN in a NNS V. Then, it is DSCa-NN.*

Proof: Let (x_{kj}) be DSC-NN. We have that $(1-\epsilon)\circ(1-\epsilon) > 1-\mu$ and $\epsilon \bullet \epsilon < \mu$, for a given $\epsilon > 0$, take $\mu > 0$. Then, we have

$$d(A(\epsilon,\mu)) = d(\{k \leq n, j \leq m : G(x_{kj} - L, \frac{\lambda}{2}) \leq 1-\epsilon \ or \ B(x_{kj} - L, \frac{\lambda}{2}) \geq \epsilon, Y(x_{kj} - L, \frac{\lambda}{2}) \geq \epsilon\}) \tag{3}$$

thus

$$d(A^c(\epsilon,\lambda)) = d(\{k \leq n, j \leq m : G(x_{kj} - L, \frac{\lambda}{2}) > 1-\epsilon \text{ and } B(x_{kj} - L, \frac{\lambda}{2}) < \epsilon,\\ Y(x_{kj} - L, \frac{\lambda}{2}) < \epsilon\}) = 1$$

for $\lambda > 0$. Let $q, w \in A^c(\epsilon,\lambda)$. Then,

$$G(x_{qw} - L, \lambda) > 1-\epsilon \text{ and } B(x_{qw} - L, \lambda) < \epsilon,\ Y(x_{qw} - L, \lambda) < \epsilon.$$

Let

$$B(\epsilon,\lambda) = \{k \leq n, j \leq m : G(x_{kj} - x_{qw}, \lambda) \leq 1-\mu \text{ or } B(x_{kj} - x_{qw}, \lambda) \geq \mu,\\ Y(x_{kj} - x_{qw}, \lambda) \geq \mu\}.$$

We claim that $B(\epsilon,\lambda) \subset A(\epsilon,\lambda)$. Let $a, s \in B(\epsilon,\lambda) - A(\epsilon,\lambda)$. Then,

$$G(x_{as} - x_{qw}, \lambda) \leq 1-\mu \text{ and } G(x_{as} - L, \frac{\lambda}{2}) > 1-\mu,$$

in particular $G(X_{qw} - L, \lambda) > 1-\epsilon$. Then,

$$1-\mu \geq G(x_{as} - x_{qw}, \lambda) \geq G(x_{as} - L, \frac{\lambda}{2}) \circ G(x_{qw} - L, \frac{\lambda}{2}) > (1-\epsilon)\circ(1-\epsilon) > 1-\mu,$$

this is not possible. Furthermore,

$$B(x_{as} - x_{qw}, \lambda) \geq \mu \text{ and } B(x_{as} - L, \frac{\lambda}{2}) < \mu,$$

in particular $B(x_{qw} - L, \frac{\lambda}{2}) < \epsilon$. Then,

$$\mu \leq B(x_{as} - x_{qw}, \lambda) \leq B(x_{as} - L, \frac{\lambda}{2}) \circ B(x_{qw} - L, \frac{\lambda}{2}) < \epsilon \bullet \epsilon < \mu,$$

and, this is not possible. Similarly,

$$Y(x_{as} - x_{qw}, \lambda) \geq \mu \text{ and } Y(x_{as} - L, \frac{\lambda}{2}) < \mu,$$

in particular $Y(x_{qw} - L, \frac{\lambda}{2}) < \epsilon$. Then,

$$\mu \leq Y(x_{as} - x_{qw}, \lambda) \leq Y(x_{as} - L, \frac{\lambda}{2}) \circ Y(x_{qw} - L, \frac{\lambda}{2}) < \epsilon \bullet \epsilon < \mu,$$

and, this is not possible. In this case, $B(\epsilon, \lambda) \subset A(\epsilon, \lambda)$. Then, by 3, $d(\epsilon, \lambda) = 0$ and (x_{kj}) is DSCa-NN.

Definition 3.14. Let V be a NNS. Then, V is called double statistically complete (DSC-NN), if for every DSCa-NN is DSC-NN.

Theorem 3.15. *Every NNS V is (DSC-NN)-Complete.*

Proof: Let (x_{kj}) be DSCa-NN but bot DSC-NN. Take $\mu > 0$. We have $(1 - \epsilon) \circ (1 - \epsilon) > (1 - \mu)$ and $\epsilon \bullet \epsilon < \mu$, for a given $\epsilon > 0$ and $\lambda > 0$, since (x_{kj}) is not DSC-NN,

$$G(x_{kj} - x_{NM}, \lambda) \geq G(x_{kj} - L, \frac{\lambda}{2}) \circ G(x_{NM} - L, \frac{\lambda}{2}) > (1 - \epsilon) \circ (1 - \epsilon) > 1 - \mu,$$

$$B(x_{kj} - x_{NM}, \lambda) \leq B(x_{kj} - L, \frac{\lambda}{2}) \bullet B(x_{NM} - L, \frac{\lambda}{2}) < \epsilon \bullet \epsilon < \mu,$$

$$Y(x_{kj} - x_{NM}, \lambda) \leq Y(x_{kj} - L, \frac{\lambda}{2}) \bullet Y(x_{NM} - L, \frac{\lambda}{2}) < \epsilon \bullet \epsilon < \mu$$

For

$$T(\epsilon, \lambda) = \{k \leq N, j \leq M : B_{x_{kj} - x_{NM}}(\epsilon) \leq 1 - \mu\},$$

$d(T^c(\epsilon, \lambda)) = 0$ and hence $d(T(\epsilon, \lambda)) = 1$, and this is a contradiction, since (x_{kj}) is DSCa-NN. Therefore, (x_{kj}) must be DSC-NN. In consequence, every NNS is (DSC-NN)-complete.

Lemma 3.16. *Let V be a NNS. Then, for any double sequence $(x_{kj}) \in F$, the following conditions are equivalent:*

(1) *(x_{kj}) is DSC-NN.*
(2) *(x_{kj}) is DSCa-NN.*
(3) *NNS V is (DSC-NN)-complete.*
(4) *There exits an increasing double index sequence $L_2 = (j_{nm})$ of natural numbers such that $d(L_2) = 1$ and the double subsequence $(x_{j_{nm}})$ is a DSCa-NN.*

Proof: The proof is followed directly by the Theorems 3.11 , 3.13 and 3.15 .

4. Conclusion

The purpose of this paper was to define and study the notion of double statistical convergence in neutrosophic normed space. We established some of their properties and we gave some examples associated to this notion. Furthermore, statistical Cauchy double sequence and statistically double completeness for neutrosophic norm were defined. For future work, we suggest studying these notions on spaces of sequences of functions in neutrosophic normed spaces. Besides, it would be interesting to see whether these properties are satisfied on triple sequences.

References

[1] L.C. Barros, R.C. Bassanezi and P.A. Tonelli, Fuzzy modelling in population dynamics, *Ecol Model*, 128(2000), 27–33.

[2] T. Bera and N.K. Mahapatra, Neutrosophic soft normed linear spaces, *Neutrosophic Sets and Systems*, 23(2018), 52–71.

[3] T. Bera and N.K. Mahapatra, Neutrosophic soft linear spaces, *Fuzzy Information and Engineering*, 9(2017), 299–324.

[4] H. Fast, Sur la convergence statistique, *Colloq. Math.*, 2(1951), 241–244.

[5] A.L. Fradkov and R.J. Evans, Control of chaos: methods and applications in engineering, *Chaos, Solitons and Fractals*, 29(2005), 33–56.

[6] A. George and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, 64(1994), 395–399.

[7] A. George and P. Veeramani, On some results of analysis for fuzzy metric spaces, *Fuzzy Sets and Systems*, 90(1997), 365–368.

[8] R.A. Giles, Computer program for fuzzy reasoning, *Fuzzy Sets Syst.*, 4 (1980), 221–234.

[9] L. Hong L and J.Q. Sun, Bifurcations of fuzzy nonlinear dynamical systems, *Commun Nonlinear Sci Numer Simul* 1 (2006), 1–12.

[10] O. Kaleva and S. Seikkala, On fuzzy metric spaces, *Fuzzy Sets and Systems*, 12(1984), 215–229.

[11] M. Kirisci and N. Simsek, Neutrosophic normed spaces and statistical convergence, *The Journal of Analysis*, 28(2020), 1059–1073.

[12] M. Kirisci and N. Simsek, Neutrosophic metric spaces, arXiv:1907.00798

[13] J. Madore, Fuzzy physics, *Ann Phys* 219 (1992), 187–98.

[14] M. Mursaleen and O. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, 288(2003), 223–231.

[15] J. H. Park, Intuitionistic fuzzy metric spaces, *Chaos, Solitons and Fractals*, 22(2004), 1039–1046.

[16] F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, *International Journal of Pure and Applied Mathematics*, 24(2005), 287–297.

[17] F. Smarandache, *Neutrosophy, Neutrosophic Probability, Set, and Logic*, ProQuest Information and Learning. Ann Arbor, Michigan, USA, 1998.

[18] H. Steinhaus, Sur la convergence ordinaire et la convergence asymtotique, *Collq. Math.*, 2(1951), 73–74.

[19] K. Yano, Integral formulas in Riemannian geometry. *Marcel Dekker, New York*, (1970).

[20] L. A. Zadeh, Fuzzy sets, *Inform Control*, 8(1965), 338–353.

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