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# Neutrosophic Sets and Systems

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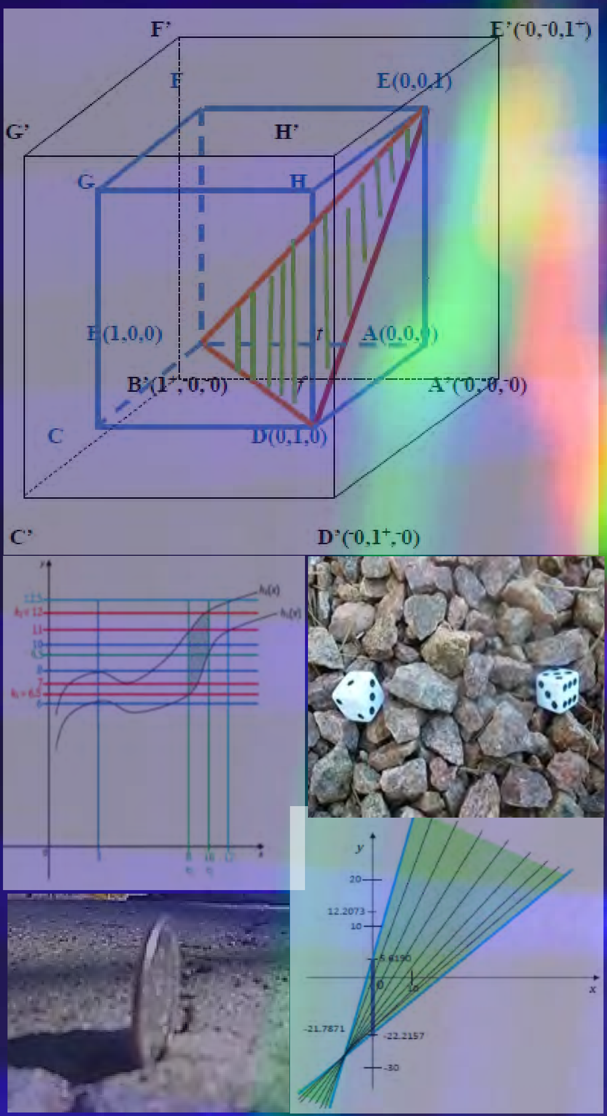


<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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# Neutrosophic Sets and Systems

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*Neutrosophy* is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

*Neutrosophic Set* and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth ($T$), a degree of indeterminacy ($I$), and a degree of falsity ($F$), where $T, I, F$ are standard or non-standard subsets of $]^-0, 1^+[$.

*Neutrosophic Probability* is a generalization of the classical probability and imprecise probability.

*Neutrosophic Statistics* is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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March 20, 2019

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# Contents

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# Decision Making Application Based on Neutrosophic Parameterized Hypersoft Set Theory

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**Abstract**. Hypersoft set is the generalization of soft set as it converts single attribute function to multi-attribute function. The core purpose of this study is to make the existing literature regarding neutrosophic parameterized soft set in line with the need of multi-attribute function. We first conceptualize the neutrosophic parameterized hypersoft set along with some of its elementary properties and operations. Then we propose decision making based algorithm with the help of this theory. Moreover, an illustrative example is presented which depicts its validity for successful application to the problems involving vagueness and uncertainties.

**Keywords:** Neutrosophic Set; Hypersoft Set; Neutrosophic Parameterized Soft Set; Neutrosophic Parameterized Hypersoft set.

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## 1. Introduction

Fuzzy sets theory (FST) [1] and intuitionistic fuzzy set theory (IFST) [2] are considered apt mathematical modes to tackle many intricate problems involving various uncertainties, in different mathematical disciplines. The former one emphasizes on the degree of true belongingness of a certain object from the initial sample space whereas the later one accentuates on degree of true membership and degree of non-membership with condition of their dependency on each other. These theories depict some kind of inadequacy regarding the provision of due status to degree of indeterminacy. Such impediment is addressed with the introduction of neutrosophic set theory (NST) [3, 4] which not only considers the due status of degree of indeterminacy but also waives off the condition of dependency. This theory is more flexible and appropriate to deal with uncertainty and vagueness. NST has attracted the keen concentration of many researchers [5–19] to further utilization in statistics, topological spaces as well as in the development of certain neutrosophic-like blended structures with other existing models for useful applications in decision making.

FST, IFST and NST have some kind of complexities which restrain them to solve problem involving uncertainty professionally. The reason for these hurdles is, possibly, the inadequacy of the parametrization tool. It demands a mathematical tool free of all such impediments to tackle such issues. This scantiness is resolved with the development of soft set theory [20] which is a new parameterized family of subsets of the universe of discourse. The researchers [21–30] studied and investigated some elementary properties, operations, laws and hybrids of SST with applications in decision making. The gluing concept of NST and SST, is studied in [31] to make the NST adequate with parameterized tool. In many real life situations, distinct attributes are further partitioned in disjoint attribute-valued sets but existing SST is insufficient for dealing with such kind of attribute-valued sets. Hypersoft set theory (HST) [32] is developed to make the SST in line with attribute-valued sets to tackle real life scenarios. Certain elementary properties, aggregation operations, laws, relations and functions of HST, are investigated by [33–35] for proper understanding and further utilization in different fields. The applications of HST in decision making is studied by [36–39] and the intermingling study of HST with complex sets, convex and concave sets is studied by [40, 41]. The core aim of this study is to develop a novel theory of embedding structure of parameterized neutrosophic set and hypersoft set with the extension of concept investigated in [42,43]. A decision-making based algorithm is proposed to solve a real life problem relating to the purchase of most suitable and appropriate product with the help of some essential operations of this presented theory. The rest of the paper is systemized as:

| | |
|---|---|
| Section 2 | Some essential definitions and terminologies are recalled. |
| Section 3 | Theory of neutrosophic parameterized hypersoft set is developed with suitable examples. |
| Section 4 | Neutrosophic decision system is constructed with proposed decision making algorithm and application. |
| Section 5 | Paper is summarized with future directions. |

## 2. **Preliminaries**

Here some basic terms are recalled from existing literature to support the proposed work. Throughout the paper, $\mathbb{X}$ , $\mathbb{P}(\mathbb{X})$ and $\mathbb{I}$ will denote the universe of discourse, power set of $\mathbb{X}$ and closed unit interval respectively.

**Definition 2.1.** [1]

A *fuzzy set* $\mathcal{X}$ defined as $\mathcal{X} = \{(\epsilon, \zeta_{\mathcal{X}}(\epsilon))|\epsilon \in \mathbb{X}\}$ such that $\zeta_{\mathcal{X}} : \mathbb{X} \to \mathbb{I}$ where $\zeta_{\mathcal{X}}(\epsilon)$ denotes the belonging value of $\epsilon \in \mathcal{X}$.

**Definition 2.2.** [2]
An *intuitionistic fuzzy set* $\mathcal{Y}$ defined as $\mathcal{Y} = \{(\beta, <\zeta_{\mathcal{Y}}(\beta), \xi_{\mathcal{Y}}(\beta)>)|\beta \in \mathbb{X}\}$ such that $\zeta_{\mathcal{Y}} : \mathbb{X} \to \mathbb{I}$ and $\xi_{\mathcal{Y}} : \mathbb{X} \to \mathbb{I}$, where $\zeta_{\mathcal{Y}}(\beta)$ and $\xi_{\mathcal{Y}}(\beta)$ denote the belonging value and not-belonging value of $\beta \in \mathcal{Y}$ with condition of $0 \leq \zeta_{\mathcal{Y}}(\beta) + \xi_{\mathcal{Y}}(\beta) \leq 1$.

**Definition 2.3.** [3]
A *neutrosophic set* $\mathcal{Z}$ defined as $\mathcal{Z} = \{(\gamma, <\mathcal{A}_{\mathcal{Z}}(\gamma), \mathcal{B}_{\mathcal{Z}}(\gamma), \mathcal{C}_{\mathcal{Z}}(\gamma)>)|\gamma \in \mathbb{X}\}$ such that $\mathcal{A}_{\mathcal{Z}}(\gamma), \mathcal{B}_{\mathcal{Z}}(\gamma), \mathcal{C}_{\mathcal{Z}}(\gamma) : \mathbb{X} \to (^{-}0, 1^{+})$, where $\mathcal{A}_{\mathcal{Z}}(\gamma), \mathcal{B}_{\mathcal{Z}}(\gamma)$ and $\mathcal{C}_{\mathcal{Z}}(\gamma)$ denote the degrees of membership, indeterminacy and non-membership of $\gamma \in \mathcal{Z}$ with condition of $^{-}0 \leq \mathcal{A}_{\mathcal{Z}}(\gamma) + \mathcal{B}_{\mathcal{Z}}(\gamma) + \mathcal{C}_{\mathcal{Z}}(\gamma) \leq 3^{+}$.

**Definition 2.4.** [20]
A pair $(\zeta_S, \Lambda)$ is called a *soft set* over $\mathbb{X}$, where $\zeta_S : \Lambda \to \mathbb{P}(\mathbb{X})$ and $\Lambda$ be a subset of a set of attributes E.

For more detail on soft set, see [21–30].

**Definition 2.5.** [32]
The pair $(\Psi, G)$ is called a hypersoft set over $\mathbb{X}$, where $G$ is the cartesian product of n disjoint sets $G_1, G_2, G_3, ...., G_n$ having attribute values of n distinct attributes $g_1, g_2, g_3, ...., g_n$ respectively and $\Psi : G \to \mathbb{P}(\mathbb{X})$.

For more definitions and operations of hypersoft set, see [33–35].

## 3. Neutrosophic Parameterized Hypersoft Set (*nphs*-set)with Application

In this section, neutrosophic parameterized hypersoft set is conceptualized and some of its fundamentals are discussed.

**Definition 3.1.** Let $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, ...., \mathcal{A}_n\}$ be a collection of disjoint attribute-valued sets corresponding to n distinct attributes $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ respectively. A NP-hypersoft set (*nphs*-set) $\Psi_{\mathcal{N}}$ over $\mathbb{X}$ is defined as

$$\Psi_{\mathcal{N}} = \{(< P_{\mathcal{N}}(g), Q_{\mathcal{N}}(g), R_{\mathcal{N}}(g) > /g, \psi_{\mathcal{N}}(g)) : g \in \mathbb{G}\}$$

where

(i) $\mathbb{G} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times .... \times \mathcal{A}_n$
(ii) $\mathcal{N}$ is a neutrosophic set over $\mathbb{G}$ with $P_{\mathcal{N}}, Q_{\mathcal{N}}, R_{\mathcal{N}} : \mathbb{G} \to \mathbb{I}$ as membership function, indeterminacy function and nonmembership function of *nphs*-set.
(iii) $\psi_{\mathcal{N}} : \mathbb{G} \to \mathbb{P}(\mathbb{X})$ is called approximate function of *nphs*-set.

Note that collection of all *nphs*-sets is represented by $\Omega_{NPHS}(\mathbb{X})$.

**Definition 3.2.** Let $\Psi_{\mathcal{N}} \in \Omega_{NPHS}(\mathbb{X})$. If $\psi_{\mathcal{N}}(g) = \phi, P_{\mathcal{N}}(g) = 0, Q_{\mathcal{N}}(g) = 1, R_{\mathcal{N}}(g) = 1$ for all $g \in \mathbb{G}$, then $\Psi_{\mathcal{N}}$ is called $\mathcal{N}$-empty *nphs*-set, denoted by $\Psi_{\Phi_{\mathcal{N}}}$. If $\mathcal{N} = \phi$, then $\mathcal{N}$-empty *nphs*-set is called an empty *nphs*-set, denoted by $\Psi_{\Phi}$.

**Definition 3.3.** Let $\Psi_{\mathcal{N}} \in \Omega_{NPHS}(\mathbb{X})$. If $\psi_{\mathcal{N}}(g) = \mathbb{X}, P_{\mathcal{N}}(g) = 1, Q_{\mathcal{N}}(g) = 0, R_{\mathcal{N}}(g) = 0$ for all $g \in \mathbb{G}$, then $\Psi_{\mathcal{N}}$ is called $\mathcal{N}$-universal *nphs*-set, denoted by $\Psi_{\tilde{\mathcal{N}}}$. If $\mathcal{N} = \mathbb{G}$ , then the $\mathcal{N}$-universal *nphs*-set is called universal *nphs*-set, denoted by $\Psi_{\tilde{\mathbb{G}}}$.

**Example 3.4.** Consider $\mathbb{X} = \{u_1, u_2, u_3, u_4, u_5\}$ and $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ with $\mathcal{A}_1 = \{a_{11}, a_{12}\}$, $\mathcal{A}_2 = \{a_{21}, a_{22}\}, \mathcal{A}_3 = \{a_{31}\}$, then
$G = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$
$G = \{(a_{11}, a_{21}, a_{31}), (a_{11}, a_{22}, a_{31}), (a_{12}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{31})\} = \{g_1, g_2, g_3, g_4\}$.
Case 1.
If $\mathcal{N}_1 = \{< 0.2, 0.3, 0.4 > /g_2, < 0, 1, 1 > /g_3, < 1, 0, 0 > /g_4\}$ and
$\psi_{\mathcal{N}_1}(g_2) = \{u_2, u_4\}$, $\psi_{\mathcal{N}_1}(g_3) = \phi$, and $\psi_{\mathcal{N}_1}(g_4) = \mathbb{X}$, then
$\Psi_{\mathcal{N}_1} = \{(< 0.2, 0.3, 0.4 > /g_2, \{u_2, u_4\}), (< 0, 1, 1 > /g_3, \phi), (< 1, 0, 0 > /g_4, \mathbb{X})\}$.
Case 2.
If $\mathcal{N}_2 = \{< 0, 1, 1 > /g_2, < 0, 1, 1 > /g_3\}, \psi_{\mathcal{N}_2}(g_2) = \phi$ and $\psi_{\mathcal{N}_2}(g_3) = \phi$, then $\Psi_{\mathcal{N}_2} = \Psi_{\Phi_{\mathcal{N}_2}}$.
Case 3.
If $\mathcal{N}_3 = \phi$ corresponding to all elements of $\mathbb{G}$, then $\Psi_{\mathcal{N}_3} = \Psi_{\Phi}$.
Case 4.
If $\mathcal{N}_4 = \{< 1, 0, 0 > /g_1, < 1, 0, 0 > /g_2\}, \psi_{\mathcal{N}_4}(g_1) = \mathbb{X}$, and $\psi_{\mathcal{N}_4}(g_2) = \mathbb{X}$, then $\Psi_{\mathcal{N}_4} = \Psi_{\tilde{\mathcal{N}_4}}$.
Case 5.
If $\mathcal{N}_5 = \mathbb{X}$ with respect to all elements of $\mathbb{G}$, then $\Psi_{\mathcal{N}_5} = \Psi_{\tilde{\mathbb{G}}}$.

**Definition 3.5.** Let $\Psi_{\mathcal{N}_1}$, $\Psi_{\mathcal{N}_2} \in \Omega_{NPHS}(\mathbb{X})$ then $\Psi_{\mathcal{N}_1}$ is an *nphs*-subset of $\Psi_{\mathcal{N}_2}$, denoted by $\Psi_{\mathcal{N}_1} \tilde{\subseteq} \Psi_{\mathcal{N}_2}$ if
$P_{\mathcal{N}_1}(g) \leq P_{\mathcal{N}_2}(g), Q_{\mathcal{N}_1}(g) \geq Q_{\mathcal{N}_2}(g), R_{\mathcal{N}_1}(g) \geq R_{\mathcal{N}_2}(g)$ and $\psi_{\mathcal{N}_1}(g) \subseteq \psi_{\mathcal{N}_2}(g)$ for all $g \in \mathbb{G}$.

**Proposition 3.6.** *Let* $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2}, \Psi_{\mathcal{N}_3} \in \Omega_{NPHS}(\mathbb{X})$ *then*

(1) $\Psi_{\mathcal{N}_1} \tilde{\subseteq} \Psi_{\tilde{\mathbb{G}}}$.
(2) $\Psi_{\Phi} \tilde{\subseteq} \Psi_{\mathcal{N}_1}$.
(3) $\Psi_{\mathcal{N}_1} \tilde{\subseteq} \Psi_{\mathcal{N}_1}$.
(4) *if* $\Psi_{\mathcal{N}_1} \tilde{\subseteq} \Psi_{\mathcal{N}_2}$ *and* $\Psi_{\mathcal{N}_2} \tilde{\subseteq} \Psi_{\mathcal{N}_3}$ *then* $\Psi_{\mathcal{N}_1} \tilde{\subseteq} \Psi_{\mathcal{N}_3}$.

**Definition 3.7.** Let $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2} \in \Omega_{NPHS}(\mathbb{X})$ then, $\Psi_{\mathcal{N}_1}$ and $\Psi_{\mathcal{N}_2}$ are *nphs*-equal, represented as $\Psi_{\mathcal{N}_1} = \Psi_{\mathcal{N}_2}$, if and only if $P_{\mathcal{N}_1}(g) = P_{\mathcal{N}_2}(g), Q_{\mathcal{N}_1}(g) = Q_{\mathcal{N}_2}(g), R_{\mathcal{N}_1}(g) = R_{\mathcal{N}_2}(g)$ and $\psi_{\mathcal{N}_1}(g) = \psi_{\mathcal{N}_2}(g)$ for all $g \in \mathbb{G}$.

**Proposition 3.8.** *Let* $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2}, \Psi_{\mathcal{N}_3} \in \Omega_{NPHS}(\mathbb{X})$ *then,*

(1) *if* $\Psi_{\mathcal{N}_1} = \Psi_{\mathcal{N}_2}$ *and* $\Psi_{\mathcal{N}_2} = \Psi_{\mathcal{N}_3}$ *then* $\Psi_{\mathcal{N}_1} = \Psi_{\mathcal{N}_3}$.

(2) *if* $\Psi_{\mathcal{N}_1} \tilde{\subseteq} \Psi_{\mathcal{N}_2}$ *and* $\Psi_{\mathcal{N}_2} \tilde{\subseteq} \Psi_{\mathcal{N}_1} \Leftrightarrow \Psi_{\mathcal{N}_1} = \Psi_{\mathcal{N}_2}$.

**Definition 3.9.** Let $\Psi_{\mathcal{N}} \in \Omega_{NPHS}(\mathbb{X})$ then, complement of $\Psi_{\mathcal{N}}$(i.e. $\Psi_{\mathcal{N}}^{\tilde{c}}$) is an *nphs*-set given as $P_{\mathcal{N}}^{\tilde{c}}(g) = 1 - P_{\mathcal{N}}(g), Q_{\mathcal{N}}^{\tilde{c}}(g) = 1 - Q_{\mathcal{N}}(g), R_{\mathcal{N}}^{\tilde{c}}(g) = 1 - R_{\mathcal{N}}(g)$ and $\psi_{\mathcal{N}}^{\tilde{c}}(g) = \mathbb{X} \setminus \psi_{\mathcal{N}}(g)$

**Proposition 3.10.** *Let* $\Psi_{\mathcal{N}} \in \Omega_{NPHS}(\mathbb{X})$ *then,*

(1) $(\Psi_{\mathcal{N}}^{\tilde{c}})^{\tilde{c}} = \Psi_{\mathcal{N}}$.

(2) $\Psi_{\phi}^{\tilde{c}} = \Psi_{\tilde{\mathbb{G}}}$.

**Definition 3.11.** Let $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2} \in \Omega_{NPHS}(\mathbb{X})$ then, union of $\Psi_{\mathcal{N}_1}$ and $\Psi_{\mathcal{N}_2}$, denoted by $\Psi_{\mathcal{N}_1} \tilde{\cup} \Psi_{\mathcal{N}_2}$, is defined by

(i) $P_{\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2}(g) = max\{P_{\mathcal{N}_1}(x), P_{\mathcal{N}_2}(g)\}$,

(ii) $Q_{\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2}(g) = min\{Q_{\mathcal{N}_1}(x), Q_{\mathcal{N}_2}(g)\}$,

(iii) $R_{\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2}(g) = min\{R_{\mathcal{N}_1}(x), R_{\mathcal{N}_2}(g)\}$,

(iv) $\psi_{\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2}(g) = \psi_{\mathcal{N}_1}(g) \cup \psi_{\mathcal{N}_2}(g)$, for all $g \in \mathbb{G}$.

**Proposition 3.12.** *Let* $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2}, \Psi_{\mathcal{N}_3} \in \Omega_{NPHS}(\mathbb{X})$ *then,*

(1) $\Psi_{\mathcal{N}_1} \tilde{\cup} \Psi_{\mathcal{N}_1} = \Psi_{\mathcal{N}_1}$,

(2) $\Psi_{\mathcal{N}_1} \tilde{\cup} \Psi_{\Phi} = \Psi_{\mathcal{N}_1}$,

(3) $\Psi_{\mathcal{N}_1} \tilde{\cup} \Psi_{\tilde{\mathbb{G}}} = \Psi_{\tilde{\mathbb{G}}}$,

(4) $\Psi_{\mathcal{N}_1} \tilde{\cup} \Psi_{\mathcal{N}_2} = \Psi_{\mathcal{N}_2} \tilde{\cup} \Psi_{\mathcal{N}_1}$,

(5) $(\Psi_{\mathcal{N}_1} \tilde{\cup} \Psi_{\mathcal{N}_2}) \tilde{\cup} \Psi_{\mathcal{N}_3} = \Psi_{\mathcal{N}_1} \tilde{\cup} (\Psi_{\mathcal{N}_2} \tilde{\cup} \Psi_{\mathcal{N}_3})$.

**Definition 3.13.** Let $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2} \in \Omega_{NPHS}(\mathbb{X})$ then intersection of $\Psi_{\mathcal{N}_1}$ and $\Psi_{\mathcal{N}_2}$, denoted by $\Psi_{\mathcal{N}_1} \tilde{\cap} \Psi_{\mathcal{N}_2}$, is an *nphs*-set defined by

(i) $P_{\mathcal{N}_1 \tilde{\cap} \mathcal{N}_2}(g) = min\{P_{\mathcal{N}_1}(x), P_{\mathcal{N}_2}(g)\}$,

(ii) $Q_{\mathcal{N}_1 \tilde{\cap} \mathcal{N}_2}(g) = max\{Q_{\mathcal{N}_1}(x), Q_{\mathcal{N}_2}(g)\}$,

(iii) $R_{\mathcal{N}_1 \tilde{\cap} \mathcal{N}_2}(g) = max\{R_{\mathcal{N}_1}(x), R_{\mathcal{N}_2}(g)\}$,

(iv) $\psi_{\mathcal{N}_1 \tilde{\cap} \mathcal{N}_2}(g) = \psi_{\mathcal{N}_1}(g) \cap \psi_{\mathcal{N}_2}(g)$, for all $g \in \mathbb{G}$.

**Proposition 3.14.** *Let* $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2}, \Psi_{\mathcal{N}_3} \in \Omega_{NPHS}(\mathbb{X})$ *then*

(1) $\Psi_{\mathcal{N}_1} \tilde{\cap} \Psi_{\mathcal{N}_1} = \Psi_{\mathcal{N}_1}$.

(2) $\Psi_{\mathcal{N}_1} \tilde{\cap} \Psi_{\Phi} = \Psi_{\Phi}$.

(3) $\Psi_{\mathcal{N}_1} \tilde{\cap} \Psi_{\tilde{\mathbb{G}}} = \Psi_{\tilde{\mathcal{N}_1}}$.

(4) $\Psi_{\mathcal{N}_1} \tilde{\cap} \Psi_{\mathcal{N}_2} = \Psi_{\mathcal{N}_2} \tilde{\cap} \Psi_{\mathcal{N}_1}$.

(5) $(\Psi_{\mathcal{N}_1} \tilde{\cap} \Psi_{\mathcal{N}_2}) \tilde{\cap} \Psi_{\Psi_{\mathcal{N}_3}} = \Psi_{\mathcal{N}_1} \tilde{\cap} (\Psi_{\mathcal{N}_2} \tilde{\cap} \Psi_{\Psi_{\mathcal{N}_3}})$.

**Remark 3.15.** Let $\Psi_{\mathcal{N}} \in \Omega_{NPHS}(\mathbb{X})$. If $\Psi_{\mathcal{N}} \neq \Psi_{\tilde{\mathbb{G}}}$, then $\Psi_{\mathcal{N}} \tilde{\cup} \Psi_{\mathcal{N}}^{\tilde{c}} \neq \Psi_{\tilde{\mathbb{G}}}$ and $\Psi_{\mathcal{N}} \tilde{\cap} \Psi_{\mathcal{N}}^{\tilde{c}} \neq \Psi_{\Phi}$

**Proposition 3.16.** *Let* $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2} \in \Omega_{NPHS}(\mathbb{X})$ *D'Morgans laws are valid*

(1) $(\Psi_{\mathcal{N}_1} \tilde{\cup} \Psi_{\mathcal{N}_2})^{\tilde{c}} = \Psi_{\mathcal{N}_1}^{\tilde{c}} \tilde{\cap} \Psi_{\mathcal{N}_2}^{\tilde{c}}$.
(2) $(\Psi_{\mathcal{N}_1} \tilde{\cap} \Psi_{\mathcal{N}_2})^{\tilde{c}} = \Psi_{\mathcal{N}_1}^{\tilde{c}} \tilde{\cup} \Psi_{\mathcal{N}_2}^{\tilde{c}}$.

*Proof.* For all $g \in \mathbb{G}$,
(1). Since $(P_{\mathcal{N}_1\tilde{\cup}\mathcal{N}_2})^{\tilde{c}}(g) = 1 - P_{\mathcal{N}_1\tilde{\cup}\mathcal{N}_2}(g)$
$= 1 - max\{P_{\mathcal{N}_1}(g), P_{\mathcal{N}_2}(g)\}$
$= min\{1 - P_{\mathcal{N}_1}(g), 1 - P_{\mathcal{N}_2}(g)\}$
$= min\{P_{\mathcal{N}_1}^{\tilde{c}}(g), P_{\mathcal{N}_2}^{\tilde{c}}(g)\}$
$= P_{\mathcal{N}_1\tilde{\cap}\mathcal{N}_2}^{\tilde{c}}(g)$
also
$(Q_{\mathcal{N}_1\tilde{\cup}\mathcal{N}_2})^{\tilde{c}}(g) = 1 - Q_{\mathcal{N}_1\tilde{\cup}\mathcal{N}_2}(g)$
$= 1 - min\{Q_{\mathcal{N}_1}(g), Q_{\mathcal{N}_2}(g)\}$
$= max\{1 - Q_{\mathcal{N}_1}(g), 1 - Q_{\mathcal{N}_2}(g)\}$
$= max\{Q_{\mathcal{N}_1}^{\tilde{c}}(g), Q_{\mathcal{N}_2}^{\tilde{c}}(g)\}$
$= Q_{\mathcal{N}_1\tilde{\cap}\mathcal{N}_2}^{\tilde{c}}(g)$
and
$(R_{\mathcal{N}_1\tilde{\cup}\mathcal{N}_2})^{\tilde{c}}(g) = 1 - R_{\mathcal{N}_1\tilde{\cup}\mathcal{N}_2}(g)$
$= 1 - min\{R_{\mathcal{N}_1}(g), R_{\mathcal{N}_2}(g)\}$
$= max\{1 - R_{\mathcal{N}_1}(g), 1 - R_{\mathcal{N}_2}(g)\}$
$= max\{R_{\mathcal{N}_1}^{\tilde{c}}(g), R_{\mathcal{N}_2}^{\tilde{c}}(g)\}$
$= R_{\mathcal{N}_1\tilde{\cap}\mathcal{N}_2}^{\tilde{c}}(g)$
and
$(\psi_{\mathcal{N}_1\tilde{\cup}\mathcal{N}_2})^{\tilde{c}}(g) = \mathbb{X} \setminus \psi_{\mathcal{N}_1\tilde{\cup}\mathcal{N}_2}(g)$
$= \mathbb{X} \setminus (\psi_{\mathcal{N}_1}(g) \cup \psi_{\mathcal{N}_2}(g))$
$= (\mathbb{X} \setminus \psi_{\mathcal{N}_1}(g)) \cap (\mathbb{X} \setminus \psi_{\mathcal{N}_2}(g))$
$= \psi_{\mathcal{N}_1}^{\tilde{c}}(g) \tilde{\cap} \psi_{\mathcal{N}_2}^{\tilde{c}}(g)$
$= \psi_{\mathcal{N}_1\,\tilde{\cap}\,\mathcal{N}_2}^{\tilde{c}}(g)$.
similarly (2) can be proved easily. □

**Proposition 3.17.** *Let* $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2}, \Psi_{\mathcal{N}_3} \in \Omega_{NPHS}(\mathbb{X})$ *then*

(1) $\Psi_{\mathcal{N}_1} \tilde{\cup} (\Psi_{\mathcal{N}_2} \tilde{\cap} \Psi_{\mathcal{N}_3}) = (\Psi_{\mathcal{N}_1} \tilde{\cup} \Psi_{\mathcal{N}_2}) \tilde{\cap} (\Psi_{\mathcal{N}_1} \tilde{\cup} \Psi_{\mathcal{N}_3})$.
(2) $\Psi_{\mathcal{N}_1} \tilde{\cap} (\Psi_{\mathcal{N}_2} \tilde{\cup} \Psi_{\mathcal{N}_3}) = (\Psi_{\mathcal{N}_1} \tilde{\cap} \Psi_{\mathcal{N}_2}) \tilde{\cup} (\Psi_{\mathcal{N}_1} \tilde{\cap} \Psi_{\mathcal{N}_3})$.

*Proof.* For all $g \in \mathbb{G}$,
(1). Since $P_{\mathcal{N}_1\tilde{\cup}(\mathcal{N}_2\tilde{\cap}\mathcal{N}_3)}(g) = max\{P_{\mathcal{N}_1}(g), P_{\mathcal{N}_2\tilde{\cap}\mathcal{N}_3}(g)\}$
$= max\{P_{\mathcal{N}_1}(g), min\{P_{\mathcal{N}_2}(g), P_{\mathcal{N}_3}(g)\}\}$

$= min\{max\{P_{\mathcal{N}_1}(g), P_{\mathcal{N}_2}(g)\}, max\{P_{\mathcal{N}_1}(g), P_{\mathcal{N}_3}(g)\}\}$
$= min\{P_{\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2}(g), P_{\mathcal{N}_1 \tilde{\cup} \mathcal{N}_3}(g)\}$
$= P_{(\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2) \tilde{\cap} (\mathcal{N}_1 \tilde{\cup} \mathcal{N}_3)}(g)$
and
$Q_{\mathcal{N}_1 \tilde{\cup} (\mathcal{N}_2 \tilde{\cap} \mathcal{N}_3)}(g) = min\{Q_{\mathcal{N}_1}(g), Q_{\mathcal{N}_2 \tilde{\cap} \mathcal{N}_3}(g)\}$
$= min\{Q_{\mathcal{N}_1}(g), max\{Q_{\mathcal{N}_2}(g), Q_{\mathcal{N}_3}(g)\}\}$
$= max\{min\{Q_{\mathcal{N}_1}(g), Q_{\mathcal{N}_2}(g)\}, min\{Q_{\mathcal{N}_1}(g), Q_{\mathcal{N}_3}(g)\}\}$
$= max\{Q_{\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2}(g), Q_{\mathcal{N}_1 \tilde{\cup} \mathcal{N}_3}(g)\}$
$= Q_{(\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2) \tilde{\cap} (\mathcal{N}_1 \tilde{\cup} \mathcal{N}_3)}(g)$
and
$R_{\mathcal{N}_1 \tilde{\cup} (\mathcal{N}_2 \tilde{\cap} \mathcal{N}_3)}(g) = min\{R_{\mathcal{N}_1}(g), R_{\mathcal{N}_2 \tilde{\cap} \mathcal{N}_3}(g)\}$
$= min\{R_{\mathcal{N}_1}(g), max\{R_{\mathcal{N}_2}(g), R_{\mathcal{N}_3}(g)\}\}$
$= max\{min\{R_{\mathcal{N}_1}(g), R_{\mathcal{N}_2}(g)\}, min\{R_{\mathcal{N}_1}(g), R_{\mathcal{N}_3}(g)\}\}$
$= max\{R_{\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2}(g), R_{\mathcal{N}_1 \tilde{\cup} \mathcal{N}_3}(g)\}$
$= R_{(\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2) \tilde{\cap} (\mathcal{N}_1 \tilde{\cup} \mathcal{N}_3)}(g)$
and
$\psi_{\mathcal{N}_1 \tilde{\cup} (\mathcal{N}_2 \tilde{\cap} \mathcal{N}_3)}(g) = \psi_{\mathcal{N}_1}(g) \cup \psi_{\mathcal{N}_2 \tilde{\cap} \mathcal{N}_3}(g)$
$= \psi_{\mathcal{N}_1}(g) \cup (\psi_{\mathcal{N}_2}(g) \cap \psi_{\mathcal{N}_3}(g))$
$= (\psi_{\mathcal{N}_1}(g) \cup \psi_{\mathcal{N}_2}(g)) \cap (\psi_{\mathcal{N}_1}(g) \cup \psi_{\mathcal{N}_3}(g))$
$= \psi_{\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2}(g) \cap \psi_{\mathcal{N}_1 \tilde{U} \mathcal{N}_3}(g)$
$= \psi_{(\mathcal{N}_1 \tilde{\cup} \mathcal{N}_2) \tilde{\cap} (\mathcal{N}_1 \tilde{\cup} \mathcal{N}_3)}(g)$
In the same way, (2) can be proved. □

**Definition 3.18.** Let $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2} \in \Omega_{NPHS}(\mathbb{X})$ then OR-operation of $\Psi_{\mathcal{N}_1}$ and $\Psi_{\mathcal{N}_2}$, denoted by $\Psi_{\mathcal{N}_1} \tilde{\oplus} \Psi_{\mathcal{N}_2}$, is an *nphs*-set defined by

(i) $P_{\mathcal{N}_1 \tilde{\oplus} \mathcal{N}_2}(g_1, g_2) = max\{P_{\mathcal{N}_1}(g_1), P_{\mathcal{N}_2}(g_2)\}$,
(ii) $Q_{\mathcal{N}_1 \tilde{\oplus} \mathcal{N}_2}(g_1, g_2) = min\{Q_{\mathcal{N}_1}(g_1), Q_{\mathcal{N}_2}(g_2)\}$,
(iii) $R_{\mathcal{N}_1 \tilde{\oplus} \mathcal{N}_2}(g_1, g_2) = min\{R_{\mathcal{N}_1}(g_1), R_{\mathcal{N}_2}(g_2)\}$,
(iv) $\psi_{\mathcal{N}_1 \tilde{\oplus} \mathcal{N}_2}(g_1, g_2) = \psi_{\mathcal{N}_1}(g_1) \cup \psi_{\mathcal{N}_2}(g_2)$, for all $(g_1, g_2) \in \mathcal{N}_1 \times \mathcal{N}_2$.

**Definition 3.19.** Let $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2} \in \Omega_{NPHS}(\mathbb{X})$ then AND-operation of $\Psi_{\mathcal{N}_1}$ and $\Psi_{\mathcal{N}_2}$, denoted by $\Psi_{\mathcal{N}_1} \tilde{\otimes} \Psi_{\mathcal{N}_2}$, is an *nphs*-set defined by

(i) $P_{\mathcal{N}_1 \tilde{\otimes} \mathcal{N}_2}(g_1, g_2) = min\{P_{\mathcal{N}_1}(g_1), P_{\mathcal{N}_2}(g_2)\}$,
(ii) $Q_{\mathcal{N}_1 \tilde{\otimes} \mathcal{N}_2}(g_1, g_2) = max\{Q_{\mathcal{N}_1}(g_1), Q_{\mathcal{N}_2}(g_2)\}$,
(iii) $R_{\mathcal{N}_1 \tilde{\otimes} \mathcal{N}_2}(g_1, g_2) = max\{R_{\mathcal{N}_1}(g_1), R_{\mathcal{N}_2}(g_2)\}$,
(iv) $\psi_{\mathcal{N}_1 \tilde{\otimes} \mathcal{N}_2}(g_1, g_2) = \psi_{\mathcal{N}_1}(g_1) \cap \psi_{\mathcal{N}_2}(g_2)$, for all $(g_1, g_2) \in \mathcal{N}_1 \times \mathcal{N}_2$.

**Proposition 3.20.** *Let* $\Psi_{\mathcal{N}_1}, \Psi_{\mathcal{N}_2}, \Psi_{\mathcal{N}_3} \in \Omega_{NPHS}(\mathbb{X})$ *then*

(1) $\Psi_{\mathcal{N}_1} \tilde{\otimes} \Psi_{\Phi} = \Psi_{\Phi}$.
(2) $(\Psi_{\mathcal{N}_1} \tilde{\otimes} \Psi_{\mathcal{N}_2}) \tilde{\otimes} \Psi_{\mathcal{N}_3} = \Psi_{\mathcal{N}_1} \tilde{\otimes} (\Psi_{\mathcal{N}_2} \tilde{\otimes} \Psi_{\mathcal{N}_3})$.
(3) $(\Psi_{\mathcal{N}_1} \tilde{\oplus} \Psi_{\mathcal{N}_2}) \tilde{\oplus} \Psi_{\mathcal{N}_3} = \Psi_{\mathcal{N}_1} \tilde{\oplus} (\Psi_{\mathcal{N}_2} \tilde{\oplus} \Psi_{\mathcal{N}_3})$.

## 4. Neutrosophic Decision Set of *nphs*-set

Having motivation from decision making methods stated in [42–50], here an algorithm is presented with the help of characterization of neutrosophic decision set on *nphs*-set which based on decision making technique and is explained with example.

**Definition 4.1.** Let $\Psi_{\mathcal{N}} \in \Omega_{NPHS}(\mathbb{X})$ then a neutrosophic decision set of $\Psi_{\mathcal{N}}$ (i.e. $\Psi^D_{\mathcal{N}}$) is represented as

$$\Psi^D_{\mathcal{N}} = \{< \mathcal{T}^D_{\mathcal{N}}(u), \mathcal{I}^D_{\mathcal{N}}(u), \mathcal{F}^D_{\mathcal{N}}(u) > /u : u \in \mathbb{X}\}$$

where $\mathcal{T}^D_{\mathcal{N}}, \mathcal{I}^D_{\mathcal{N}}, \mathcal{F}^D_{\mathcal{N}} : \mathbb{X} \to \mathbb{I}$ and

$$\mathcal{T}^D_{\mathcal{N}}(u) = \frac{1}{|\mathbb{X}|} \sum_{v \in S(\mathcal{N})} \mathcal{T}_{\mathcal{N}}(v) \Gamma_{\psi_{\mathcal{N}(v)}}(u)$$

$$\mathcal{I}^D_{\mathcal{N}}(u) = \frac{1}{|\mathbb{X}|} \sum_{v \in S(\mathcal{N})} \mathcal{I}_{\mathcal{N}}(v) \Gamma_{\psi_{\mathcal{N}(v)}}(u)$$

$$\mathcal{F}^D_{\mathcal{N}}(u) = \frac{1}{|\mathbb{X}|} \sum_{v \in S(\mathcal{N})} \mathcal{F}_{\mathcal{N}}(v) \Gamma_{\psi_{\mathcal{N}(v)}}(u)$$

where $| \bullet |$ denotes set cardinality with

$$\Gamma_{\psi_{\mathcal{N}(v)}}(u) = \begin{cases} 1 & ; \quad u \in \Gamma_{\psi_{\mathcal{N}}}(v) \\ 0 & ; \quad u \notin \Gamma_{\psi_{\mathcal{N}}}(v) \end{cases}$$

**Definition 4.2.** If $\Psi_{\mathcal{N}} \in \Omega_{NPHS}(\mathbb{X})$ with neutrosophic decision set $\Psi^D_{\mathcal{N}}$ then reduced fuzzy set of $\Psi^D_{\mathcal{N}}$ is a fuzzy set represented as

$$\mathbb{R}(\Psi^D_{\mathcal{N}}) = \left\{ \zeta_{\Psi^D_{\mathcal{N}}}(u)/u : u \in \mathbb{X} \right\}$$

where $\zeta_{\Psi^D_{\mathcal{N}}} : \mathbb{X} \to \mathbb{I}$ with $\zeta_{\Psi^D_{\mathcal{N}}}(u) = \mathcal{T}^D_{\mathcal{N}}(u) + \mathcal{I}^D_{\mathcal{N}}(u) - \mathcal{F}^D_{\mathcal{N}}(u)$

### 4.1. *Proposed Algorithm*

Once $\Psi^D_{\mathcal{N}}$ has been established, it may be indispensable to select the best single substitute from the options. Therefore, decision can be set up with the help of following algorithm.

Step 1 Determine $\mathcal{N} = \{< \mathcal{T}_{\mathcal{N}}(g), \mathcal{I}_{\mathcal{N}}(g), \mathcal{F}_{\mathcal{N}}(g) > /g : \mathcal{T}_{\mathcal{N}}(g), \mathcal{I}_{\mathcal{N}}(g), \mathcal{F}_{\mathcal{N}}(g) \in \mathbb{I}, g \in \mathbb{G}\}$,

Step 2 Find $\psi_{\mathcal{N}}(g)$

Step 3 Construct $\Psi_{\mathcal{N}}$ over $\mathbb{X}$,

Step 4 Compute $\Psi^D_{\mathcal{N}}$ ,

TABLE 1. Degrees of Membership $\mathcal{T}_{\mathcal{N}}(g_i)$

| $\mathcal{T}_{\mathcal{N}}(g_i)$ | Degree | $\mathcal{T}_{\mathcal{N}}(g_i)$ | Degree |
|---|---|---|---|
| $\mathcal{T}_{\mathcal{N}}(g_1)$ | 0.1 | $\mathcal{T}_{\mathcal{N}}(g_9)$ | 0.9 |
| $\mathcal{T}_{\mathcal{N}}(g_2)$ | 0.2 | $\mathcal{T}_{\mathcal{N}}(g_{10})$ | 0.16 |
| $\mathcal{T}_{\mathcal{N}}(g_3)$ | 0.3 | $\mathcal{T}_{\mathcal{N}}(g_{11})$ | 0.25 |
| $\mathcal{T}_{\mathcal{N}}(g_4)$ | 0.4 | $\mathcal{T}_{\mathcal{N}}(g_{12})$ | 0.45 |
| $\mathcal{T}_{\mathcal{N}}(g_5)$ | 0.5 | $\mathcal{T}_{\mathcal{N}}(g_{13})$ | 0.35 |
| $\mathcal{T}_{\mathcal{N}}(g_6)$ | 0.6 | $\mathcal{T}_{\mathcal{N}}(g_{14})$ | 0.75 |
| $\mathcal{T}_{\mathcal{N}}(g_7)$ | 0.7 | $\mathcal{T}_{\mathcal{N}}(g_{15})$ | 0.65 |
| $\mathcal{T}_{\mathcal{N}}(g_8)$ | 0.8 | $\mathcal{T}_{\mathcal{N}}(g_{16})$ | 0.85 |

Step 5 Choose the maximum of $\zeta_{\Psi^D_{\mathcal{N}}}(u)$.

**Example 4.3.** Suppose that Mr. James Peter wants to buy a mobile from a mobile market. There are eight kinds of mobiles (options) which form the set of discourse $\mathbb{X} = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8\}$. The best selection may be evaluated by observing the attributes i.e. $a_1 =$ Company, $a_2 =$ Camera Resolution, $a_3 =$ Size, $a_4 =$ RAM, and $a_5 =$ Battery power. The attribute-valued sets corresponding to these attributes are:

$B_1 = \{b_{11}, b_{12}\}$

$B_2 = \{b_{21}, b_{22}\}$

$B_3 = \{b_{31}, b_{32}\}$

$B_4 = \{b_{41}, b_{42}\}$

$B_5 = \{b_{51}\}$

then $\mathbb{G} = B_1 \times B_2 \times B_3 \times B_4 \times B_5$

$\mathbb{G} = \{g_1, g_2, g_3, g_4, ....., g_{16}\}$ where each $g_i, i = 1, 2, ..., 16$, is a 5-tuples element.

Step 1 :

From tables 1, 2, 3 we can construct $\mathcal{N}$ as

$$\mathcal{N} = \left\{ \begin{array}{l} <0.1, 0.2, 0.3> /g_1, <0.2, 0.3, 0.4> /g_2, <0.3, 0.4, 0.5> /g_3, <0.4, 0.5, 0.6> /g_4, \\ <0.5, 0.6, 0.7> /g_5, <0.6, 0.7, 0.8> /g_6, <0.7, 0.8, 0.9> /g_7, <0.8, 0.9, 0.1> /g_8, \\ <0.9, 0.1, 0.2> /g_9, <0.16, 0.27, 0.37> /g_{10}, <0.25, 0.35, 0.45> /g_{11}, <0.45, 0.55, 0.65> /g_{12}, \\ <0.35, 0.45, 0.55> /g_{13}, <0.75, 0.85, 0.95> /g_{14}, <0.65, 0.75, 0.85> /g_{15}, <0.85, 0.95, 0.96> /g_{16} \end{array} \right\}$$

Step 2 :

Table 4 presents $\psi_{\mathcal{N}}(g_i)$ corresponding to each element of $\mathbb{G}$.

Step 3 :

TABLE 2. Degrees of Indeterminacy $\mathcal{I}_{\mathcal{N}}(g_i)$

| $\mathcal{I}_{\mathcal{N}}(g_i)$ | Degree | $\mathcal{I}_{\mathcal{N}}(g_i)$ | Degree |
|---|---|---|---|
| $\mathcal{I}_{\mathcal{N}}(g_1)$ | 0.2 | $\mathcal{I}_{\mathcal{N}}(g_9)$ | 0.1 |
| $\mathcal{I}_{\mathcal{N}}(g_2)$ | 0.3 | $\mathcal{I}_{\mathcal{N}}(g_{10})$ | 0.27 |
| $\mathcal{I}_{\mathcal{N}}(g_3)$ | 0.4 | $\mathcal{I}_{\mathcal{N}}(g_{11})$ | 0.35 |
| $\mathcal{I}_{\mathcal{N}}(g_4)$ | 0.5 | $\mathcal{I}_{\mathcal{N}}(g_{12})$ | 0.55 |
| $\mathcal{I}_{\mathcal{N}}(g_5)$ | 0.6 | $\mathcal{I}_{\mathcal{N}}(g_{13})$ | 0.45 |
| $\mathcal{I}_{\mathcal{N}}(g_6)$ | 0.7 | $\mathcal{I}_{\mathcal{N}}(g_{14})$ | 0.85 |
| $\mathcal{I}_{\mathcal{N}}(g_7)$ | 0.8 | $\mathcal{I}_{\mathcal{N}}(g_{15})$ | 0.75 |
| $\mathcal{I}_{\mathcal{N}}(g_8)$ | 0.9 | $\mathcal{I}_{\mathcal{N}}(g_{16})$ | 0.95 |

TABLE 3. Degrees of Non-Membership $\mathcal{F}_{\mathcal{N}}(g_i)$

| $\mathcal{F}_{\mathcal{N}}(g_i)$ | Degree | $\mathcal{F}_{\mathcal{N}}(g_i)$ | Degree |
|---|---|---|---|
| $\mathcal{F}_{\mathcal{N}}(g_1)$ | 0.3 | $\mathcal{F}_{\mathcal{N}}(g_9)$ | 0.2 |
| $\mathcal{F}_{\mathcal{N}}(g_2)$ | 0.4 | $\mathcal{F}_{\mathcal{N}}(g_{10})$ | 0.37 |
| $\mathcal{F}_{\mathcal{N}}(g_3)$ | 0.5 | $\mathcal{F}_{\mathcal{N}}(g_{11})$ | 0.45 |
| $\mathcal{F}_{\mathcal{N}}(g_4)$ | 0.6 | $\mathcal{F}_{\mathcal{N}}(g_{12})$ | 0.65 |
| $\mathcal{F}_{\mathcal{N}}(g_5)$ | 0.7 | $\mathcal{F}_{\mathcal{N}}(g_{13})$ | 0.55 |
| $\mathcal{F}_{\mathcal{N}}(g_6)$ | 0.8 | $\mathcal{F}_{\mathcal{N}}(g_{14})$ | 0.95 |
| $\mathcal{F}_{\mathcal{N}}(g_7)$ | 0.9 | $\mathcal{F}_{\mathcal{N}}(g_{15})$ | 0.85 |
| $\mathcal{F}_{\mathcal{N}}(g_8)$ | 0.1 | $\mathcal{F}_{\mathcal{N}}(g_{16})$ | 0.96 |

With the help of step 1 and step 2, we can construct $\Psi_{\mathcal{N}}$ as

$$\Psi_{\mathcal{N}} = \left\{ \begin{array}{l} (<0.1,0.2,0.3>/g_1,\{m_1,m_2\}),(<0.2,0.3,0.4>/g_2,\{m_1,m_2,m_3\}), \\ (<0.3,0.4,0.5>/g_3,\{m_2,m_3,m_4\}),(<0.4,0.5,0.6>/g_4,\{m_4,m_5,m_6\}), \\ (<0.5,0.6,0.7>/g_5,\{m_6,m_7,m_8\}),(<0.6,0.7,0.8>/g_6,\{m_2,m_3,m_4\}), \\ (<0.7,0.8,0.9>/g_7,\{m_1,m_3,m_5\}),(<0.8,0.9,0.1>/g_8,\{m_2,m_3,m_7\}), \\ (<0.9,0.1,0.2>/g_9,\{m_2,m_7,m_8\}),(<0.16,0.27,0.37>/g_{10},\{m_6,m_7,m_8\}), \\ (<0.25,0.35,0.45>/g_{11},\{m_2,m_4,m_6\}),(<0.45,0.55,0.65>/g_{12},\{m_2,m_3,m_6\}), \\ (<0.35,0.45,0.55>/g_{13},\{m_3,m_5,m_7\}),(<0.75,0.85,0.95>/g_{14},\{m_1,m_3,m_5\}), \\ (<0.65,0.75,0.85>/g_{15},\{m_5,m_7,m_8\}),(<0.85,0.95,0.96>/g_{16},\{m_4,m_5,m_6\}) \end{array} \right\}$$

Step 4 :

From tables 5 to 8 , we can construct $\mathbb{R}(\Psi^D_{\mathcal{N}})$ as

$$\mathbb{R}(\Psi^D_{\mathcal{N}}) = \left\{ \begin{array}{l} 0.1688/m_1, 0.4625/m_2, 0.5313/m_3, 0.2488/m_4, \\ 0.3988/m_5, 0.2625/m_6, 0.4575/m_7, 0.2263/m_8 \end{array} \right\}$$

Step 5 :

Since maximum of $\zeta_{\Psi^D_{\mathcal{N}}}(m_i)$ is 0.5313 so the mobile $m_3$ is selected.

Table 4. Approximate functions $\psi_{\mathcal{N}}(g_i)$

| $g_i$ | $\psi_{\mathcal{N}}(g_i)$ | $g_i$ | $\psi_{\mathcal{N}}(g_i)$ |
|---|---|---|---|
| $g_1$ | $\{m_1, m_2\}$ | $g_9$ | $\{m_2, m_7, m_8\}$ |
| $g_2$ | $\{m_1, m_2, m_3\}$ | $g_{10}$ | $\{m_6, m_7, m_8\}$ |
| $g_3$ | $\{m_2, m_3, m_4\}$ | $g_{11}$ | $\{m_2, m_4, m_6\}$ |
| $g_4$ | $\{m_4, m_5, m_6\}$ | $g_{12}$ | $\{m_2, m_3, m_6\}$ |
| $g_5$ | $\{m_6, m_7, m_8\}$ | $g_{13}$ | $\{m_3, m_5, m_7\}$ |
| $g_6$ | $\{m_2, m_3, m_4\}$ | $g_{14}$ | $\{m_1, m_3, m_5\}$ |
| $g_7$ | $\{m_1, m_3, m_5\}$ | $g_{15}$ | $\{m_5, m_7, m_8\}$ |
| $g_8$ | $\{m_2, m_3, m_7\}$ | $g_{16}$ | $\{m_4, m_5, m_6\}$ |

Table 5. Membership values $\mathcal{T}^D_{\mathcal{N}}(m_i)$

| $m_i$ | $\mathcal{T}^D_{\mathcal{N}}(m_i)$ | $m_i$ | $\mathcal{T}^D_{\mathcal{N}}(m_i)$ |
|---|---|---|---|
| $m_1$ | 0.2188 | $m_5$ | 0.4625 |
| $m_2$ | 0.4500 | $m_6$ | 0.3263 |
| $m_3$ | 0.5188 | $m_7$ | 0.4200 |
| $m_4$ | 0.3000 | $m_8$ | 0.2763 |

Table 6. Indeterminacy values $\mathcal{I}^D_{\mathcal{N}}(m_i)$

| $m_i$ | $\mathcal{I}^D_{\mathcal{N}}(m_i)$ | $m_i$ | $\mathcal{I}^D_{\mathcal{N}}(m_i)$ |
|---|---|---|---|
| $m_1$ | 0.2688 | $m_5$ | 0.5375 |
| $m_2$ | 0.4375 | $m_6$ | 0.4025 |
| $m_3$ | 0.6188 | $m_7$ | 0.3838 |
| $m_4$ | 0.3625 | $m_8$ | 0.2150 |

Table 7. Non-Membership values $\mathcal{F}^D_{\mathcal{N}}(m_i)$

| $m_i$ | $\mathcal{F}^D_{\mathcal{N}}(m_i)$ | $m_i$ | $\mathcal{F}^D_{\mathcal{N}}(m_i)$ |
|---|---|---|---|
| $m_1$ | 0.3188 | $m_5$ | 0.6013 |
| $m_2$ | 0.4250 | $m_6$ | 0.4663 |
| $m_3$ | 0.6063 | $m_7$ | 0.3463 |
| $m_4$ | 0.4138 | $m_8$ | 0.2650 |

## 5. Conclusion

In this study, neutrosophic parameterized hypersoft set is conceptualized along with some of elementary properties and theoretic operations. A novel algorithm is proposed for decision making and is validated with the help of an illustrative example for appropriate purchasing

TABLE 8. Reduced Fuzzy membership $\zeta_{\Psi^D_{\mathcal{N}}}(m_i)$

| $m_i$ | $\zeta_{\Psi^D_{\mathcal{N}}}(m_i)$ | $m_i$ | $\zeta_{\Psi^D_{\mathcal{N}}}(m_i)$ |
|---|---|---|---|
| $m_1$ | 0.1688 | $m_5$ | 0.3988 |
| $m_2$ | 0.4625 | $m_6$ | 0.2625 |
| $m_3$ | 0.5313 | $m_7$ | 0.4575 |
| $m_4$ | 0.2488 | $m_8$ | 0.2263 |

of mobile from mobile market. Future work may include the extension of this work for other neutrosophic-like environments and the implementation for solving more real life problems in decision making.

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# The Importance of Indeterminate and Unknown Factors in Nourishing Crime: A Case Study of South Africa Using Neutrosophy

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**Abstract:** There is no doubt regarding the notion that crime is deteriorating the socio-economic structure of society. Crime poses a serious threat to human values and existence. Therefore this menace should be stopped as early as possible otherwise it would lead to unavoidable circumstances. Whenever policies are formed there are some certain factors that are always taken into consideration to stop the crime. These measures were effective but with the passage of time there seems to be a constant situation and crime seems to be at its peak. This situation has forced us to think that there may be other factors that are leading to criminal behaviour in humans. These factors may be uncertain, unknown or indeterminate. Though previous researches in this regard have taken into consideration all the known factors, the present work takes into account both known and unknown factors together with the relationship among them. Taking into account all the factors which nourish crime either directly or indirectly, here we try to model the situation mathematically using Neutrosophic Cognitive Map since it provides us with a methodology of representing known and unknown factors together. The work is carried out using graphical methods and concepts together with linear algebra. The present work takes into account the crimes which are occurring in South Africa and models this situation taking into considerations all the certain and uncertain factors. The study reveals that relative poverty & inadequate housing, limited social and cognitive abilities, exclusion from school, family violence, culture conflict, colonialism, unemployment, income inequality, violent expressions of masculinity and use of violence to 'resolve' are directly related to crime in the country. The other factors such as Adherence to social norms, the multi-racial character of the society, Racial discrimination, apartheid policy, political transition, restructuring of the criminal justice system, gathering of people, intimate partner violence & femicide and use of 'tik' (crystal meth/ methamphetamine) which were not supposed to have a direct influence on crime in the country by previous researches are also having a significant effect on crime. The present work contributes effectively in identifying the factors leading to criminal behaviour among people. This would in turn help policymakers to take necessary steps at ground level to curb the crime in the country. The work also shows the modelling of the situation using Fuzzy Cognitive Maps just to represent the effectiveness of Neutrosophic Cognitive Maps over them.

**Keywords:** Crime Analysis, Unsupervised Data, Fuzzy Logic, Fuzzy Cognitive Maps, Neutrosophy, Neutrosophic Cognitive Maps

## 1. Introduction

Crime has remained a serious challenge in the history of South Africa. The recent statistics by the police department have shown an increase in the number of crimes [18]. There are several instances where it has been noticed that criminal behaviour in humans is motivated by certain factors. The need to identify these factors more accurately, the present work is carried out using recent data from South Africa. Various crime instances have been noticed in the recent past in the country. This has motivated researchers to study criminal behaviour among people. Though most of the studies are concerned with only known factors none has focused on indeterminate and unknown factors. This study takes into account both the factors and shows how indeterminate factors play important roles in determining criminal behaviour among people. The crime in South Africa has started increasing from the mid-1980s to the early 1990s [19]. The studies at that time foretold that the crime was expected to reduce in between 1995-1996 which happened as expected but later in 1996 it again started at a large scale. Recently released report by Mid-Year Population Estimates (MYPE) 2019 shows that the population of South Africa is 56.78 million [26]. The population is not only comprised of native citizens but there exists a lot of multi-racial population. The Union contains four principal groups: Europeans, almost equally divided between British and Afrikaaner (2,643,187, according to the 1951 census); Africans or Bantus or "natives" (8,535,341); Colored, like those of mixed racial descent are known, (1,102,323); Asians, most of whom are Indians, (365,524). The multi-racial nature of society has led to various problems in the country. The crime in the country is at its peak at each and every corner. Below we show the crime statistics from the South African Police department which show how many numbers of crimes are committed annually with respect to the nature of the crime.

| Type of Crimes | 2013/14 | 2014/15 | 2015/16 | 2016/17 | 2017/18 |
|---|---|---|---|---|---|
| **Motor Vehicle Theft** | 57 415 | 67 104 | 57 783 | 47 586 | 56 526 |
| **Housebreaking/Burglary** | 940 954 | 874 606 | 844 982 | 776 933 | 832 122 |
| **Home Robbery** | 268 639 | 208 401 | 187 830 | 151 279 | 156 089 |
| **Theft of livestock, poultry and other animals** | 253 373 | 164 710 | 148 785 | 161 063 | 159 421 |
| **Theft of crops planted by the household** | 47 977 | 16 843 | 39 155 | 15 003 | 11 493 |
| **Trends in murder** | 26 529 | 18 012 | 14 930 | 16 201 | 16 809 |
| **Theft out of motor vehicle** | 208 978 | 196 236 | 192 736 | 139 432 | 130 350 |
| **Deliberate damage, burning or destruction of dwellings** | 58 452 | 60 624 | 40 892 | 46 915 | 50 426 |
| **Motor vehicle vandalism** | 54 633 | 74 824 | 67 715 | 31 907 | 40 155 |
| **Theft of bicycle** | 54 119 | 60 375 | 37 227 | 21 051 | 29 264 |
| **Theft of personal property** | 1 012 537 | 921 773 | 842 478 | 708 357 | 693 219 |
| **Robbery** | 373 148 | 348 349 | 283 544 | 294 874 | 280 526 |
| **Sexual Offences** | 62 074 | 44 464 | 29 473 | 73 842 | 28 596 |
| **Assault** | 431 043 | 431 914 | 331 913 | 318 077 | 355 739 |
| **Consumer Fraud** | 86 012 | 90 249 | 160 076 | 85 848 | 137 274 |

**Table 1 Crime statistics of South Africa**

**Source: South African Police Services http://www.statssa.gov.za**

The above table shows how crime is increasing in the country annually. The crime includes not only heinous crimes like murder, sexual assault but also includes the crime at a small level. These criminal behaviours among the people of South Africa are motivated due to several factors. However, while going through the previous researches in this regard the diversity in a population is regarded as one of the key reason for crime as explained by the experts [2]. Not only diversity but there are certainly other factors that are put forward by various researchers. These factors are regarded as certain factors throughout this study. These factors which are leading to most of the crimes in South Africa are Relative Poverty & inadequate housing [1], Limited social and cognitive abilities[1], Exclusion from school[1], Family violence [1], culture conflict [2], colonialism [9], unemployment [1], income inequality [10-12], violent expressions of masculinity [13-16], use of violence to 'resolve' conflict [10] and access to firearms [10] [17]. There exists a lot of literature that almost deals with all these factors. These certain or determinate factors have always been taken into consideration for making policies to tackle the situation of crime in South Africa. But despite considering all these factors and formulating strategies to curb crime in this country; crime appears to be the major problem at present. This situation has motivated us to inquire about the situation of this country to know what the other causes are leading to crime in this country. Through the reports by various agencies together with the opinion of the experts we came to know that there are uncertain and indeterminate factors that are increasing crime in this country more than certain factors. These factors are lack of adherence to social norms [27-28], the multi-racial character of the society [2], Racial discrimination [2], apartheid policy [2], political transition [3], restructuring of the criminal justice system [3], gathering of people at various occasions [3], perpetrating intimate partner violence (IPV) & femicide [4-6] and most importantly the use of 'tik' (crystal meth/ methamphetamine) [7-8] by people in South Africa. The data which is collected for analyzing any situation is always unsupervised [28-29] and this unsupervised data is in no way free from uncertainty and indeterminacy. The present work attempts to prove mathematically how these indeterminate and uncertain factors are related to crime in South Africa. Since the mathematical field of neutrosophy [21] [23] deals with the uncertainty among concepts; we try to model the situation of crime in South Africa using neutrosophy. Though various factors are taken into consideration in earlier researches to identify the criminal behaviour among the masses, as per knowledge none has taken into consideration the unknown and indeterminate factors. The present work in this regard seems to be more effective in knowing the behaviours by considering all known and unknown factors. There is recent research work by researchers in the field of crime in South Africa. The authors in [38] have explored whether the crime rate has been affected due to the weather conditions in the country or not. Authors in [39] have performed a multi-level model analysis to check whether criminal behaviour among the masses of South Africa is the result of internal migration or immigration. Authors in [40] have come up totally with different viewpoints. Their study seeks to argue that the extent of corruption in South African public service as being equal to a crime against human rights and dignity. Authors in [41] have studied the impact of social media on crimes in the country. The increasing xenophobic hate crime in South Africa is on its verge. The authors in [42] have studied the reasons for such crime in South Africa. The study related to property crime in South Africa is conducted in [43].

The rest of the paper is divided as follows; section 2 gives the concepts and preliminaries required to carry out this work, section 3 presents the methodology, section 4 models the situation of crime in South Africa using Neutrosophy, section 5 shows calculation and interprets the results obtained and section 6 concludes the paper.

## 2. Concepts and Preliminaries

The situation of crime could also be modelled using fuzzy logic and fuzzy cognitive maps [24] but it has several limitations [22] [34]. The fuzzy logic is based on membership functions and crisp sets. It addresses the causal relationship between the concepts. The existence of membership and non-existence of membership among various concepts is measured by Fuzzy theory but it says nothing about the indeterminate concepts. As it is a well-known fact that when we deal with unsupervised data indeterminacy and uncertainty is always present; hence it needs to be addressed while dealing with unsupervised data. Since fuzzy logic is limited to the certainty of concepts here in this study we have employed neutrosophic sets and theories for dealing with unsupervised data.

Neutrosophy [21] [23] is a field of study that is not limited to certainties but it's an emerging field that incorporates all the indeterminacy and uncertainties. A number of problems are solved using this theory all around the globe with surprising results. The recent developments in this novel field could be seen in [35] where authors have proposed a multi-criteria decision-making model for evaluating sustainable hydrogen production. In [36] authors again proposed a multi-criteria decision-making model for evaluation of the medical care system by taking various case studies to prove the feasibility of the proposed model. To describe the real cognitive information authors in [37] have proposed type-2 Neutrosophic Number TOPSIS. They have demonstrated the effectiveness of the proposed technique by taking into account several case studies. This has led us to apply this theory in analyzing the crime situation in South Africa. To apply this theory we need to understand some of the concepts and preliminaries as follows:

**Definition 1.** Let $N = \{(T, I, F): T, I, F \in (0,1)\}$ be a neutrosophic set. Let $m: P \rightarrow N$ is a mapping of a group of propositional formulas into N, i.e., each sentence $p \in P$is associated to a value in N, as it is exposed in the Equation 1, meaning that p is T% true, I% indeterminate and F% false.

$$m(p) = (T, I, F) \qquad (1)$$

Hence, it can be concluded that fuzzy logic when generalized based on some concepts of neutrosophy; it becomes neutrosophic logic according to [21]

**Definition 2.** A Neutrosophic matrix is a matrix $M = [a_{ij}]_{ij}$ where $i = 1,2,3, \dots\dots.., m$ and $j = 1,2,3, \dots\dots.., n$such that each $a_{ij} \in K(I)$ where $K(I)$ is a neutrosophic ring. Now let us understand this neutrosophic matrix by an example. Suppose each element of matrix is represented by $a + bI$ where $a$ and $b$ are real numbers and $I$ is a factor of indeterminacy.
For Example:

$$\begin{pmatrix} -1 & I & 5I \\ I & 4 & 7 \end{pmatrix} \begin{pmatrix} I & 9I & 6 \\ 0 & I & 0 \\ -4 & 7 & 5 \end{pmatrix} = \begin{pmatrix} -21I & 27I & -6 + 25I \\ -28 + I & 49 + 13I & 35 + 6I \end{pmatrix}$$

**Definition 3.** A neutrosophic graph is a graph in which there exists an indeterminate node or an indeterminate edge. Now taking reference from the Definition 2 above we can conclude that when $a_{ij} = 0$ it means there is no connection between nodes $i$ and $j$, $a_{ij} = 1$ means there is a connection between nodes $i$ and $j$ and $a_{ij} = I$ means that connection is indeterminate (unknown).
**Definition 4.** Cognitive maps are cause-effect networks, with nodes representing concepts articulated by individuals, and directional linkages capturing causal dependencies [25].
**Definition 5.** A directed graph whose nodes are represented as concepts and edges among concepts represents relationship which can be determinate &indeterminate edges; this graph is referred to as Neutrosophic Cognitive Map [20]

## 3. Methodology

The proposed methodology tries to introduce indeterminacy in Fuzzy Cognitive Maps (FCMs) [24]. This mapping would be referred as Neutrosophic Cognitive Maps (NCMs). This concept is well illustrated by W. B. Vasantha Kandasamy [20]. This concept of NCMs would be applied in modeling the situation in South Africa to study the influence of different determinate and indeterminate factors that have worsen the situation crime. To do this now let us understands NCMs. NCM is a neutrosophic graph. This is a directed graph in which dotted edge represents indeterminacy. The node of the graph is referred to various concepts. When $K_1, K_2, \dots\dots, K_n$ are n nodes of neutrosophic graph. These nodes of graph are connected using edges having weight '0' or '1' or 'I' where 'I' shows indeterminacy, '1' indicates that the node is at ON state and when it has value '0' it indicates the OFF state of the node. These NCMs are most of the time referred to as simple NCMs. The matrix corresponding to neutrosophic graph is called Neutrosophic adjacency matrix. Later this matrix is evaluated using laws of mathematics and the results obtained by this will be interpreted which would show the importance of the present work. To show the effectiveness of Neutrosophic Cognitive Maps over Fuzzy Cognitive Maps in analyzing the situation of crime in South Africa let us model the situation using FCM.

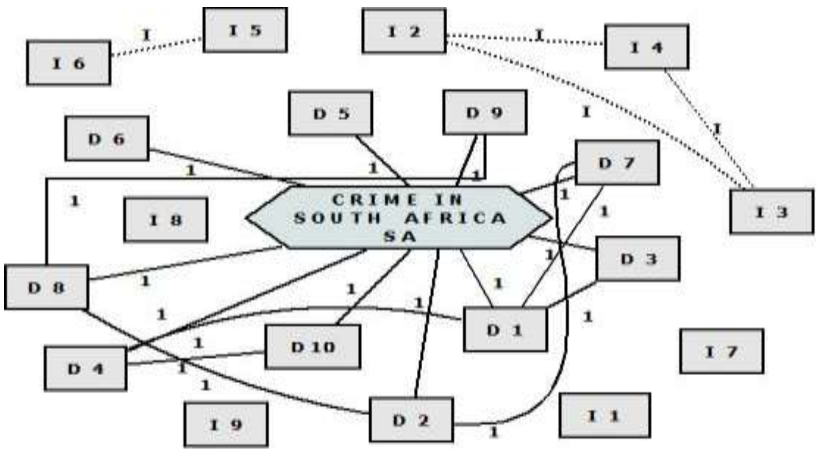


**Figure 1 Fuzzy Cognitive Map based on determinate factors affecting crime in South Africa**

The above graph is called is called Fuzzy cognitive map for studying the situation of crime in South Africa. The edges having weight '1' denotes determinate edges which show how determinate factors are nourishing crime in South Africa. Since Fuzzy does not take into consideration the indeterminate relationship therefore the indeterminate factors are not connected to the node representing crimes in South Africa. We also show how these indeterminate concepts are related to each other which are represented using dotted line with symbol 'I' denotes indeterminate edges. Now we formulate the adjacency matrix based on above graph.

| | **SA** | **D1** | **I1** | **D2** | **I2** | **D3** | **I3** | **D4** | **I4** | **D5** | **I5** | **D6** | **I6** | **D7** | **I7** | **D8** | **I8** | **D9** | **I9** | **D10** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **SA** | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| **D1** | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| **I1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D2** | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| **I2** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D3** | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **I3** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D4** | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| **I4** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D5** | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **I5** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D6** | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **I6** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D7** | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **I7** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **D8** | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| **I8** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D9** | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| **I9** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D10** | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 2 Fuzzy Adjacency Matrix based on neutrosophic cognitive map in figure 1**

The fuzzy adjacency matrix is now evaluated to know the effect of factors on the crime is South Africa. Now for this we take vector SA as on state i.e.

The state vector $\mathrm{SA}_1 = (1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ is given as input effect of $\mathrm{SA}_1$ on the combined system is $\mathrm{SA}_1\mathrm{F(E)}$. The symbol $\rightarrow$ denotes that the resultant vector is updated and threshold. The following calculation is carried out till we obtain a constant state vector or it is also referred as limit cycle.

$SA_1F(E) = (0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1) \rightarrow$

$(1\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1) = SA_2$

$SA_2F(E) = (9\ \ 4\ \ 0\ \ 3\ \ 0\ \ 2\ \ 0\ \ 2\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 3\ \ 0\ \ 3\ \ 0\ \ 2\ \ 0\ \ 2) \rightarrow$

$(1\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1) = SA_3$

We notice that $\mathrm{SA}_2 = \mathrm{SA}_3$ so further iterations are not required. $\mathrm{SA}_8$ is a fixed point or limit cycle. The significance of this limit cycle is the most since it shows a hidden pattern which is used in drawing inferences. The current results obtained above shows that when crime in South Africa is in on state all the factors such as relative poverty & inadequate housing, limited social and cognitive abilities, exclusion from school, family violence, culture conflict, colonialism, unemployment, income inequality, violent expressions of masculinity and use of violence to 'resolve' conflict are in on state. This signifies that all these factors have direct influence on crime in the country. But the factors which are put forward by the experts are other studies like Adherence to social norms, multi-racial character of the society, Racial discrimination, apartheid policy, political transition, restructuring of the criminal justice system, gathering of people, intimate partner violence (IPV) & femicide and use of 'tik' (crystal meth/ methamphetamine) are absent in this regard. So it could be clearly inferred that the FCMs take no importance of uncertain factors which could have direct influence on the concepts. Now further we try to model the situation using Neutrosophy [31-33].

## 4. Application of Neutrosophy in modeling situation of Crime in South Africa

To model the current situation of crime in South Africa we have considered some certain factor from previous researches and some of the factors are considered using expert's opinion. We have also utilized reports from various official departments to ascertain the current scenario in this country. These factors not only include certain factors but also capture some of the uncertain and indeterminate factors. Lists of the factors which are considered whether known or indeterminate are as follows:

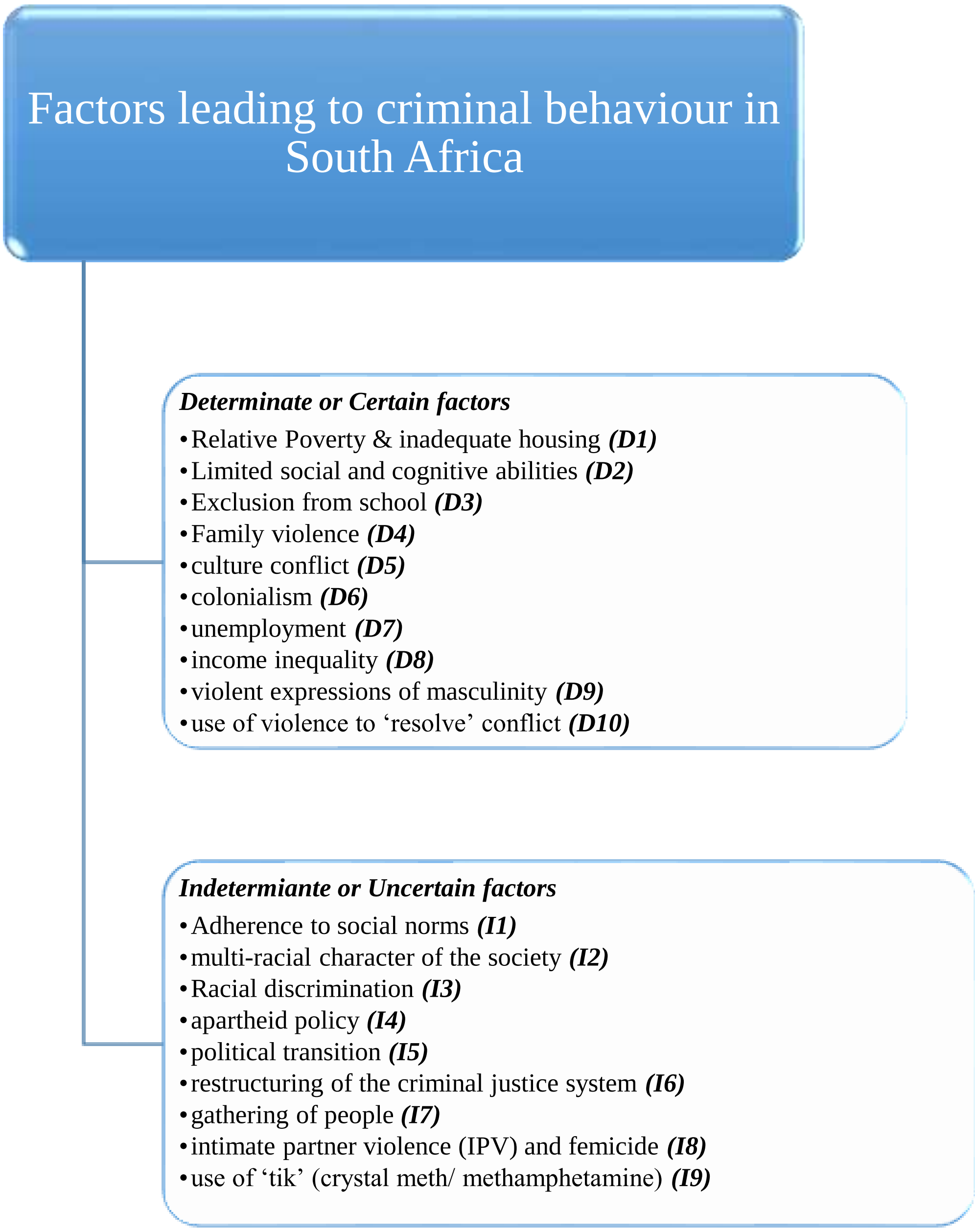


**Figure 2 Summary of factors leading to crime in South Africa based on previous researches and experts opinion**

Now we model the situation with the help of neutrosophic cognitive maps since it represents better models while analyzing the situation [30]. We try to show how these indeterminate and uncertain factors do influence the determinate and certain factors. The previous researches in this regard show that determinate factor such as Relative Poverty & inadequate housing and Exclusion from school led to the difficulty in Adherence to social norms and gathering of people at several places which are indeterminate factors [1] [3] [27-28]. Culture conflict among the people of society directly influences Adherence to social norms and racial discrimination [2] [27-28]. Unemployment is thought to be the main factors that led to crime in South Africa [1] and this unemployment results in some indeterminate factors which are also increasing crimes in the country. These factors are Adherence to social norms and gathering of people [27-28] [3]. The expression of masculinity is referred to as one of the key cause of crime in the country in many studies [13-16]. This violent expression of masculinity results in intimate partner violence (IPV) and femicide which itself is a crime [4-6]. Use of violence to 'resolve' conflict is also related to intimate partner violence (IPV) and femicide and use of 'tik' (crystal meth/ methamphetamine) [7-8] [10]. Many historical studies suggest that factors such as colonialism and apartheid have left a legacy of violence [2] [9]. This directly relates to political transition and restructuring of criminal justice system [3]. This shows how factors which always taken in considerations in various studies are linked which indeterminate and uncertain factors which most of the time are neglected. There some indeterminate factors which are interlinked like multi-racial character of the society and racial discrimination have association with apartheid policy of the country [2]. Taking all these factors and relationship among them we now model the situation of crime in South Africa using neutrosophic cognitive maps which is prominent concept of neutrosophy.

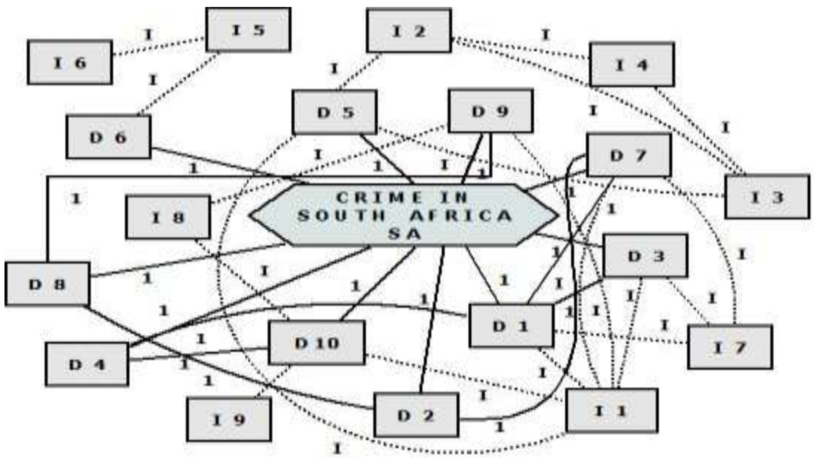


**Figure 3 Neutrosophic Cognitive Map based on determinate and indeterminate factors affecting crime in South Africa**

The above graph is called is called neutrosophic cognitive map for studying the situation of crime in South Africa. The edges having weight '1' denotes determinate edges and those edges which are shown with symbol 'I' denotes indeterminate edges.

## 5. Results

Now with the help of above cognitive map we form the neutrosophic adjacency matrix. This matrix is formulated taking in account the factors which are represented as nodes in cognitive maps and the relationship among the factors.

| | **SA** | **D1** | **I1** | **D2** | **I2** | **D3** | **I3** | **D4** | **I4** | **D5** | **I5** | **D6** | **I6** | **D7** | **I7** | **D8** | **I8** | **D9** | **I9** | **D10** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **SA** | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| **D1** | 1 | 0 | I | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | I | 0 | 0 | 0 | 0 | 0 |
| **I1** | 0 | I | 0 | 0 | 0 | I | 0 | 0 | 0 | I | 0 | 0 | 0 | I | 0 | 0 | 0 | I | 0 | I |
| **D2** | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| **I2** | 0 | 0 | 0 | 0 | 0 | 0 | I | 0 | I | I | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D3** | 1 | 1 | I | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | I | 0 | 0 | 0 | 0 | 0 |
| **I3** | 0 | 0 | 0 | 0 | I | 0 | 0 | 0 | I | I | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D4** | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| **I4** | 0 | 0 | 0 | 0 | I | 0 | I | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D5** | 1 | 0 | I | 0 | I | 0 | I | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **I5** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | I | I | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D6** | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | I | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **I6** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | I | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **D7** | 1 | 1 | I | 1 | 0 | I | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | I | 0 | 0 | 0 | 0 | 0 |
| **I7** | 0 | I | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | I | 0 | 0 | 0 | 0 | 0 | 0 |
| **D8** | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| **I8** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | I | 0 | I |
| **D9** | 1 | 0 | I | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | I | 0 | 0 | 0 |
| **I9** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | I |
| **D10** | 1 | 0 | I | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | I | 0 | I | 0 |

**Figure 4 Neutrosophic Adjacency Matrix based on neutrosophic cognitive map in figure 3**

The neutrosophic adjacency matrix is now evaluated to know the effect of factors on the crime is South Africa. Now for this we take vector SA as on state i.e.

The state vector $SA_1 = (1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ is given as input effect of $SA_1$ on the combined system is $SA_1N(E)$. The symbol $\rightarrow$ denotes that the resultant vector is updated and threshold. The following calculation is carried out till we obtain a constant state vector or it is also referred as limit cycle.

$SA_1N(E) = (0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1) \to (1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1)$
$= SA_2$

$SA_2N(E) = (10\ \ 3\ \ 6I\ \ 2\ \ I\ \ I+1\ \ I\ \ 2\ \ 0\ \ 0\ \ I\ \ 0\ \ 0\ \ 2\ \ 3I\ \ 2\ \ 2I\ \ 1\ \ I\ \ 1) \to (1\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ 0\ \ 0\ \ I\ \ 0\ \ 0\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1)$
$= SA_3$

$SA_3N(E) = (8\ \ 2I^2+4\ \ 5I\ \ 3\ \ I^2\ \ I^2+I+2\ \ I^2\ \ 3\ \ 2I^2\ \ 3I^2+1\ \ 0\ \ I^2+1\ \ I^2\ \ 2I^2+3\ \ 3I\ \ 3\ \ 2I\ \ 2I^2+2\ \ I\ \ 2I\ \ 2I^2$
$+2) \to$
$(1\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ 0\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1) = SA_4$

$SA_4N(E) = (10\ \ 2I^2+4\ \ 6I\ \ 3\ \ 2I^2+I\ \ I^2+I+2\ \ 2I^2+I\ \ 3\ \ 2I^2\ \ 3I^2+1\ \ I^2+I\ \ 1\ \ 0\ \ 2I^2+3\ \ 3I\ \ 3\ \ 2I\ \ 2I^2$
$+2\ \ I\ \ 3I^2+2) \to (1\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ 0\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1) = SA_5$

$SA_5N(E) = (10\ \ 2I^2+4\ \ 6I\ \ 3\ \ 2I^2+I\ \ I^2+I+2\ \ 2I^2+I\ \ 3\ \ 2I^2\ \ 3I^2+1\ \ I\ \ I^2+1\ \ I^2\ \ 2I^2+3\ \ 3I\ \ 3\ \ 2I\ \ 2I^2$
$+2\ \ I\ \ 3I^2+2) \to (1\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ 1\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1) = SA_6$

$SA_6N(E) = (10\ \ 2I^2+4\ \ 6I\ \ 3\ \ I^2+2I\ \ I^2+I+2\ \ 2I^2+I\ \ 3\ \ I^2+I\ \ 2I^2+I+1\ \ I^2+I\ \ I^2+1\ \ I^2\ \ 2I^2$
$+3\ \ 3I\ \ 3\ \ 2I\ \ 2I^2+2\ \ I\ \ 3I^2+2) \to (1\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1) = SA_7$

$SA_7N(E) = (10\ \ 2I^2+4\ \ 6I\ \ 3\ \ 2I^2+I\ \ I^2+I+2\ \ 2I^2+I\ \ 3\ \ 2I^2\ \ 3I^2+1\ \ I^2+I\ \ I^2+1\ \ I^2\ \ 2I^2+3\ \ 3I\ \ 3\ \ 2I\ \ 2I^2$
$+2\ \ I\ \ 3I^2+2) \to (1\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1\ \ I\ \ 1) = SA_8$

We notice that $SA_7 = SA_8$ so further iterations are not required. $SA_8$ is a fixed point or limit cycle. The significance of this limit cycle is the most since it shows a hidden pattern which is used in drawing inferences. These inferences show the joint effect of interacting knowledge. The current results obtained using NCMs is $(1\ 1\ I\ 1\ I\ 1\ I\ 1\ I\ 1\ I\ 1\ I\ 1\ I\ 1\ I\ 1\ I\ 1)$ which shows that when crime in South Africa is in on state all the factors such as relative poverty & inadequate housing, limited social and cognitive abilities, exclusion from school, family violence, culture conflict, colonialism, unemployment, income inequality, violent expressions of masculinity and use of violence to 'resolve' conflict are in on state. This signifies that all these factors have direct influence on crime in the country. The factors such as Adherence to social norms, multi-racial character of the society, Racial discrimination, apartheid policy, political transition, restructuring of the criminal justice system, gathering of people, intimate partner violence (IPV) & femicide and use of 'tik' (crystal meth/ methamphetamine) which were not supposed to have direct influence on crime in the country by previous researches are also having significant effect on crime as we have not obtained *'0'* in the limit cycle at their position but we have obtained *'I'* which shows these are having relationship with crime in the country. The previous result obtained using FCM is $(1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1)$ that clearly shows that all indeterminate and uncertain factors are absent which signifies that the study conducted using FCM is unable to represent any real life situation. This proves that NCMs are better to model real life situation than FCMs also representing the importance of indeterminate and uncertain events in analyzing any real life situation.

## 6. Conclusion

The present work is aimed at mathematically analyzing the situation of crime in South Africa. The paper contributes in a sense that it takes into account all causes (factors) whether certain (known) or uncertain (indeterminate and unknown), responsible for nourishing crime in the country. Though the previous researches have focused only on known factors, the present work emphasizes both the factors which may not be considered in previous studies. Considering and representing all the factors mathematically we tried to develop a mathematical model using neutrosophic cognitive maps so that the situation could be analyzed at ground level. The model further evaluated using some mathematical laws of calculation like graphs and linear algebra. Later the results are interpreted which shows how indeterminate and uncertain factors are giving rise to criminal behaviour in the population of South Africa. Below is the finding of our work that shows what are the certain factors nourishing crime and what are the indeterminate/uncertain factors nourishing crime:

**Known and certain factors nourishing crime:**

- relative poverty & inadequate housing,
- limited social and cognitive abilities,
- exclusion from school,
- family violence,
- culture conflict,
- colonialism,
- unemployment,
- income inequality,
- violent expressions of masculinity and
- use of violence to 'resolve' conflict

**Unknown and uncertain factors nourishing crime:**

- Adherence to social norms,
- multi-racial character of the society,
- Racial discrimination,
- apartheid policy,
- political transition,
- restructuring of the criminal justice system,
- gathering of people,
- intimate partner violence (IPV) & femicide and
- use of 'tik' (crystal meth/ methamphetamine)

This study is expected to help policymakers in taking corrective measures to curb crime in the country. The current work takes a very limited number of factors in consideration and all the work is performed manually. Future work in this regard would be modelling the situation mathematically considering a large number of factors and employing machine learning algorithms so that it may become easy to model the situation and interpret the results.

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# Neutrosophic Logic Theory and Applications

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**Abstract:** *Neutrosophic logic is a very powerful and effective concept. It has different application areas due to its ability to capture the stochasticity in many complex real-life use cases. This paper presents the main types of neutrosophic sets. It also surveys and analyzes its most common applications.*

**Keywords:** Neutrosophic Set; Single-Valued Neutrosophic Set; Multi-Valued Neutrosophic Set; Fuzzy Set

## 1. Introduction

The term Neutrosophic means neutral thought knowledge. It is a combination of two terms (Neuter) and (Sophia), wherein Latin Neuter means "Neutral" and Sophia means "Wisdom". In general, Neutrosophic set and logic are generalizations of classical fuzzy and intuitionistic fuzzy [40], while neutrosophic Probability and Statistics are generalizations of classical and imprecise probability and statistics [3]. Neutrosophic Logic (NL) is a framework for unifying many existing logics, such as fuzzy logic, paraconsistent logic, intuitionistic logic, etc. [34, 37]. The main idea of NL is to characterize each logical statement in a 3D-Neutrosophic space, where each dimension of that space represents the truth (T), the indeterminacy (I), and the falsehood (F) of the statement respectively under consideration; where T, I, and F are standard or non-standard real subsets from $]0, 1[$ with not necessarily any connection between them [2]. Many examples can be represented only by neutrosophic logic and neither by fuzzy nor intuitionistic fuzzy. One of those examples is "Voting" [36]. In general, the neutrosophic set depends on three membership functions (T, I, and F). These functions are independent, and their sum does not add up to 1. Meanwhile, it should add up to 3 [39]. Neutrosophic logic is considered a bigger umbrella of Fuzzy logic. Also, it has many applications; however, it has not been used so far alongside Q-learning. Although, by combining it with Q-learning, more realistically and flexible long-term values for Q are expected to be obtained.

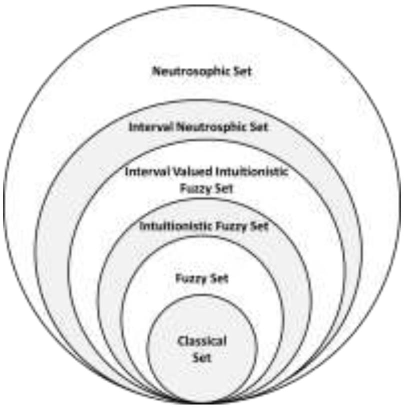


Figure-1: Relationship between neutrosophic set and other sets

Many methods were invented to deal with uncertainty. Starting from Fuzzy logic [109], which represents the "partial Truth" concept as the true value ranges between 0 and 1 according to whether it is entirely false or completely true. Meanwhile, Fuzzy logic had many drawbacks that encouraged the researchers to propose interval-valued sets to allow interval membership values within the same set. Then, an intuitionistic fuzzy set was a generalization for the traditional fuzzy sets. In an intuitionistic fuzzy set, each element has a degree of membership and even non-membershsip [110]. Meanwhile, it had drawbacks that encouraged some researchers to propose a neutrosophic set [111]. Figure-1 demonstrates the relationship between neutrosophic set and other sets, while, Table-1 lists sample advantages and disadvantages of each of these concepts [112].

Table-1: Advantages and Disadvantages of uncertainty algorithms

| Algorithm | Advantages | Disadvantages |
|---|---|---|
| **Fuzzy Sets** | - The first algorithm to deal with uncertainty<br>- Ability to solve complex problems<br>- Generates output even if only a few input data are at hand<br>- Flexible algorithm and its rules can be modified<br>- Easy to implement | - It depends on human knowledge and expertise<br>- Its rules have to be regularly visited and updated<br>- Sometimes its accuracy is not entirely reliable when it works on inaccurate inputs<br>- No single and systematic approach for solving a problem might lead to confusion<br>- Assume only crisp values for representing True/False |
| **Intuitionistic Fuzzy Sets** | - Assume a value for not only the belongingness of a number to a set (i.e., membership) but also a non-membership value | - It ignores the indeterminacy component<br>- Sometimes might generate confusing results [113]<br>- It contradicts the intuitionistic logic in some cases [113] |

| | | |
|---|---|---|
| **Interval-Valued Intuitionistic Fuzzy Sets** | - Introduce interval for representing True and False values | - It ignores the indeterminacy component |
| **Neutrosophic Sets** | - It is a generalization of other fuzzy concepts<br>- Flexible<br>- Takes into consideration the indeterminacy component that captures any vagueness and uncertainty<br>- Solves many complex problems that have incomplete and imprecise information<br>- Generates reliable results in many multi-criteria decision-making problems<br>- Ability to deal with information that comes from different data sources | - Assume single value for each component, and his might not lead to completely certain results |
| **Interval-Valued Neutrosophic Sets** | - Perfectly capture the uncertain and inconsistent information that exist in real-world<br>- Assume an interval for each neutrosophic component (T, I, F) | - There are only a few research in this area. Hence it still needs proof of its robustness |

Due to the significance of CLV and the effectiveness of Q-learning, fuzzy logic, and neutrosophic logic algorithms, many researchers compete in developing models to utilize these algorithms separately in the marketing context. Meanwhile, each of their implementations has a specific drawback. For instance, neutrosophic logic is not applied yet in a real-life marketing context to maximize CLV [11]. Also, fuzzy logic is not utilized to maximize CLV, but for many other purposes, including clustering the customer base according to their profitability level or measuring it with RFM values instead of CLV [28, 6]. Finally, Q-learning has been combined with different machine learning and deep learning algorithms for that purpose. For instance, some researchers utilized deep learning to predict Q's optimal value that maximized the long-term profitability of the customers within the firm [31, 19]. Meanwhile, these algorithms overestimated Q's action values, hence generating unrealistic actions [14].

**Single-Valued Neutrosophic Set**

Two types of membership functions for the NQL model are illustrated (Trapezoidal and Triangular). The goal is to utilize the neutrosophic model to learn the optimal Q value that maximizes long-term rewards. The stochastic nature of the problem is captured by assuming three values for Q (i.e., T, I, and F) instead of a single value, each of which follows the Trapezoidal or Triangular membership function illustrated in the upcoming sub-sections.

### 2.2.1. Trapezoidal Neutrosophic Q-Learning

In light of neutrosophic logic's definition mentioned in Section-1, which depends upon 3 core values (T, I, and F), this section illustrates how to calculate these values and how to calculate the model performance measurements [17].

Let H be a universal set. Hence, a single-valued neutrosophic set B in H is calculated in Eq. (1)

$$B = \{h, < T_B(h), I_B(h), F_B(h) > | h \in H\}, \qquad (1)$$

Where truth membership function (T_B (h)), indeterminacy membership function (I_B (h)), and falsity membership function (F_B (h)) satisfy the following conditions:

$$T_S(z) = \begin{cases} t_S\left(\frac{(z-k)}{(l-k)}\right), & k \le z \le l \\ t_S, & l \le z \le m \\ t_S\left(\frac{(n-z)}{n-m}\right), & m \le z \le n \\ 0, & otherwise \end{cases} \qquad (2)$$

$$I_S(z) = \begin{cases} \frac{l-z+(z-k')i_S}{(l-k')}, & k' \le z \le l \\ i_S, & l \le z \le m \\ \frac{z-m+(n'-z)i_S}{(n'-m)}, & m < z \le n' \\ 1, & otherwise \end{cases} \qquad (3)$$

$$F_S(z) = \begin{cases} \frac{l-z+(z-k'')f_S}{(l-k'')}, & k'' \le z \le l \\ f_S, & l \le z \le m \\ \frac{z-m+(n''-z)f_S}{(n''-m)}, & m < z \le n'' \\ 1, & otherwise \end{cases} \qquad (4)$$

Where S is a trapezoidal neutrosophic number, $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n} \in \boldsymbol{R}$. Then $\boldsymbol{S} = ([\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}];\ \boldsymbol{t_s}, \boldsymbol{i_s}, \boldsymbol{f_s})$ is called trapezoidal neutrosophic number (TrNN); and it has one of three possibilities (Positive TrNN, negative TrNN, or normalized TrNN). $\boldsymbol{m}$ is called positive TrNN, if $\boldsymbol{0} \le \boldsymbol{k} \le \boldsymbol{m} \le \boldsymbol{n}$. While, if $\boldsymbol{k} \le \boldsymbol{l} \le \boldsymbol{m} \le \boldsymbol{n} \le \boldsymbol{0}$, then $\boldsymbol{S}$ is called negative TrNN. If $\boldsymbol{0} \le \boldsymbol{k} \le \boldsymbol{l} \le \boldsymbol{m} \le \boldsymbol{n} \le \boldsymbol{1}$ and $\boldsymbol{T_s}, \boldsymbol{I_s}, \boldsymbol{F_s} \in [\boldsymbol{0}, \boldsymbol{1}]$, then $\boldsymbol{X}$ is called normalized TrNN. The membership function is demonstrated in Fig.4.

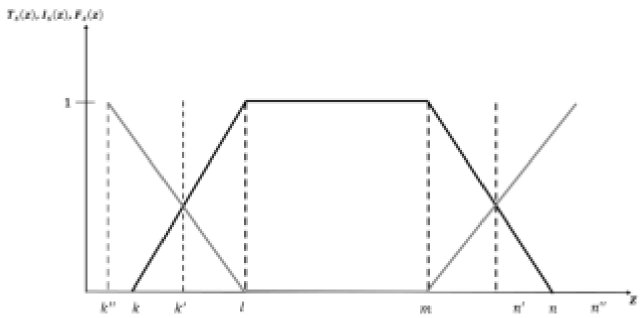


**Figure 4** TrNN membership function for truth, indeterminacy, and falsity functions

**Multi-Valued Neutrosophic Set**

Assume X is a space of points (each of which is x), then multi-valued neutrosophic set $A \, in \, X$ has membership functions ($\widetilde{T_A}(x), \widetilde{I_A}(x), \widetilde{F_A}(x)$) defined on multiple-valued as follows

$$A = \left\{\left(x, \widetilde{T_A}(x), \widetilde{I_A}(x), \widetilde{F_A}(x)\right) \middle| x \epsilon X\right\}$$

Where each of the membership functions $T, I, and \, F \; \epsilon$ [0,1], and is defined as a set of finite discrete values that satisfies the following conditions ( $0 \leq \gamma, \eta, \xi \leq 1, 0 \leq \gamma^{+} + \eta^{+} + \xi^{+} \leq 3$ ), where $\gamma \, \epsilon \, \widetilde{T_A}(x), \eta \, \epsilon \, \widetilde{I_A}(x), \xi \, \epsilon \, \widetilde{F_A}(x), \gamma^{+} = sup\widetilde{T_A}(x), \eta^{+} = sup\widetilde{I_A}(x), \xi^{+} = sup\widetilde{F_A}(x)$ [74]. Multi-valued neutrosophic can be converted to single-valued neutrosophic set $iff$ each of $\widetilde{T_A}(x), \widetilde{I_A}(x), \widetilde{F_A}(x)$ has only one value.

**Classification of Neutrosophic applications**

Neutrosophic logic plays a significant role and has many application areas [37]. This section presents a few of these applications. It starts with listing a set of theoretical contributions, then mentioning the role of neutrosophy in some practical application areas, including medicine, marketing, image processing, strategic planning, supply chain, and many other areas.

**Theoretical Contributions of Neutrosophic Logic**

Neutrosophic set is a powerful research area that proved its effectiveness and robustness in many application domains. Meanwhile, most of the contributions were theoretical and confirmed only by mathematical examples or few data-sets and were not generalized using other applications. In [50] they conducted a survey and listed the theoretical contributions of neutrosophic sets. They also proposed a method for designing the single-valued neutrosophic set. Their proposed method depended on creating neutrosophic membership functions through experimental data. Yet, their contribution needed to be applied to a real-life dataset. While in [52] they listed its applicability in medical applications. The researchers in [1] investigated different concepts, including (weighted average operator and weighted geometric operator) on neutrosophic cubic sets. This is for the sake of aggregating the neutrosophic cubic information. Their developed algorithm helped in multiple criteria decision-making. Their proposed algorithm was applied in a mathematical example to prove its usefulness and applicability. Yet, it was not implemented in a real-life business situation, which was on top of their limitations. Another theoretical contribution was done in [8]. They criticized the non-standard Neurtrsophic logic for the sake of its better understanding, although it was never used in practical applications. Their analysis we structured and well-formulated, yet it was not applied in cf situations or even in case studies to prove its robustness. An instance of a neutrosophic set called "interval neutrosophic set" was introduced in [10]. Meanwhile, their contribution was not applied to a real-life dataset. In [11] two special models of traditional neutrosophic logic, were introduced: Single-valued linguistic complex neutrosophic set, and Interval linguistic complex neutrosophic set. These models proved their applicability when they were applied in the University of Economics and Business for lecturer selection. As proposed by the researchers, this work might be extended by using different membership functions, including trapezoidal and triangular. In [18] the researcher tried to investigate some properties of neutrosophic sets and subsets.

Blending neutrosophic logic with Q learning attracted many researchers. In [22], they studied the relationship between those two algorithms. They applied their proposed algorithm in online education during Covid-19 pandemic. Their proposed model's goal was to select the best Information Communication Technology (ICT) tool as an online learning platform. Their model was illustrated through a numerical example, but it still needed to be applied in reality to prove its robustness and generalization. Interval neutrosophic sub-algebra and its properties were introduced in [26]. In [33] they focused on labor issues and tried to solve the untimely dismissal in Ecuador problem. To solve this, they integrated the IADOV method with neutrosophic logic. Their model obtained trusted results on their case study but still needed to be generalized to other datasets. In [36] they tried to test the effectiveness of neutrosophic logic in image processing. They expected to have good results when applying neutrosophic logic in imperfectly defined images. During their work, they studied different types of distance measures between neutrosophic sets. Although their work was theoretical, they expected to have outstanding results when applying their proposed model in image processing on real-life use cases. In [39] they tried to list the main concepts in single-valued neutrosophic sets. Meanwhile, they did not mention practical examples or real-life use cases to prove its practicality. While in [40], they studied the hybridization between single-values neutrosophic sets and machine learning.

In [42] they introduced a new concept in neutrosophic sub-algebra (i.e. MBJ neutrosophic sub-algebra) and listed its applications. While in [45] they introduced neutrosophic generalized topological spaces. They discussed closed and open mappings, as well as their related attributes. Meanwhile, they did not mention the practical application of these concepts. In [49] they introduced neutrosophic social structures based on the three neutrosophic components (T, I, and F). In [53] they proposed a python based open-source implementation for basic concepts of neutrosophic logic. It was an awesome contribution that might be an atom for many neutrosophic logic implementations. In [59] they focused on neutrosophic sets and neutrosophic soft-sets. They mainly studied new algebraic operations and fundamental properties of these neutrosophic sets. Their analysis was well presented and well documented, but it still needed to be applied in real-life application areas to make it more concrete and reliable. In [60] they made an extended theoretical overview over the neutrosophic set and its instances. Meanwhile, it would be great if they mentioned the applicability of these models in real-life.

On top of the applicable and powerful proposed model was that one in [62]. They combined neutrosophic logic with neural networks. Their built neural network model consisted of single input and one output. Their hidden layer contained two activation functions (i.e. Chebyshev neutrosophic orthogonal polynomial function and neutrosophic sigmoid activation function). Their model's main drawback was that it was not applied to a real-life dataset but only on illustrative examples. It could also be extended by implementing it using multiple inputs and/or multiple outputs. A contribution in empowering the multi-criteria decision-making by neutrosophic was mentioned in [65]. They utilized a bipolar neutrosophic set with both positive and negative membership functions. They illustrated their model using an illustrative example but not a real-life application. Hence, it would be recommended to apply it in the real-life use case to prove its reliability. While in [67] they proposed fuzzy equivalence concept on the standard concepts of neutrosophic sets and rough neutrosophic set for cluster analysis. Their model was illustrated through numerical examples. Applying it to real-life clustering problems was highly recommended. In [71] they had many contributions in single-valued neutrosophic sets. On top of them was that they combined single-valued neutrosophic sets with rough sets. They presented their proposed models using illustrative examples but did not mention any practical applications. This is on top of the way forward steps of their work. Also, multi-valued neutrosophic sets might be applied instead of single-valued ones. While [72] focused on the inclusion relations of neutrosophic sets. They also built a ranking method using a neutrosophic set that proved its effectiveness on set of practical examples.sMeanwhile, their model was recommended to be applied in real-life application areas to confirm this effectiveness and prove its reliability.

In [75] they proposed a model that combined data mining concepts to single-valued neutrosophic logic. They focused on machine learning and similarity measures concepts from the data mining umbrella. Other researchers could think of applying this proposed model in real-life datasets. While in [76] they combined single-valued neutrosophic (SVN) logic to a weighted correlation coefficient (CC) measure. They mentioned practical examples for their decision-making proposed method. Meanwhile, it was recommended to apply it in engineering and scientific applications. They also suggested generalizing it to other application areas. The work in [81] was also applied to combine neutrosophic set and correlation concept. But their main focus was interval neutrosophic set instead of single neutrosophic set. They illustrated their proposed model using an example. Yet, would be better to apply it in real-life use cases due to its effectiveness in empowering the decision-making process. Also in [77] they didn't apply their proposed model in real-life applications. Their model that combined SVN to minimum spanning tree was illustrated using a set of examples. They implemented it in two phases (i.e., defined the distances between SVNs, and then constructed the minimum spanning tree as a clustering algorithm for SVN. Their proposed model could be applied in many application areas. Hence it still needed generalization. The same researchers contributed in [82] by combing the correlation concept to a single neutrosophic set. Their proposed multiple attribute decision-making process ranked the alternatives in an imprecise environment. It helped to select the best option out of all options at hand. This was proved through an illustrative example. Meanwhile, it still needed to be applied in reality to prove its reliability.

Also in [78] the researchers were interested in combing SVN and even interval neutrosophic set to score function and accuracy function. Meanwhile, they did not apply their proposed models in real-life cases. A bit different than this was the work in [79]. They focused mainly on interval neutrosophic numbers. They improved the entropy formula of interval neutrosophic number. They tried to mimic the attitude of the decision-maker towards risks and under indeterminacy. They did not apply their proposed model in reality but illustrated it throughput through mathematical examples. Meanwhile, it was recommended to apply their proposed model in real-life cases to prove its robustness. In [89], they listed a set of applications related to multi-attribute decision-making and applying it under the neutrosophic environment. Highlighting that the top applications in this area were related to medical diagnosis and pattern recognition. In [90] their contribution was a bit different. They proposed a method that utilized interval –valued neutrosophic set properties to generate formal concepts of interval-valued neutrosophic concepts and even refined some parameters (i.e., $\alpha,\beta,\gamma$) cuts. They illustrated their proposals through mathematical examples not real-life applications.

**Neutrosophic Logic in Data Mining**

In [93] they utilized a multi-refined neutrosophic set in sentiment analysis. Their model consisted of two positive (T), two negative (F), and three indeterminate (I) membership functions. Their model showed outperforming results when analyzing tweets on ten different topics related to the Indian scenario and other international scenarios. While in [94] they proposed a neutrosophic association rules algorithm. sTheir algorithm-generated association rules through an item's neutrosophic attributes (i.e., T, I, F). Their model was compared to fuzzy to prove its effectiveness. Yet, it was recommended to be applied to different real-life applications to prove its effectiveness. In [95] they proposed a single-valued neutrosophic set algorithm that measured the factors that impacted students' engagement and their overall attitude in mathematics achievements. They relied upon trends of international mathematics and science study. Although their model showed outperforming results, it still needed to be generalized. In [96] they proposed a neutrosophic based Dixon's test due to its significance and applicability. Their model was illustrated through a mathematical example. In [105] they proposed a sentiment analysis model for large documents. Their model consisted of binary and ternary classifiers and combined neutrosophic logic to particle swarm optimization (PSO) algorithm. Their proposed model was tested on a real-life dataset, and the ternary classifier gave outperforming results. Meanwhile, other researchers might think of generalize it to other data sets and applications for both short and large text. While in [106] they developed a clustering model using k-means but reduced the number of attributes using rough neutrosophic sets. Their model was applied to a real-life dataset but still needed to be generalized. A bit different than this was the contribution in [108]. They proposed a domain generation algorithm using a neutrosophic set. They classified their data to benign, malicious, and indeterminacy domain names. Their model showed outperforming results when being applied to a real-life dataset. Yet, it might be generalized to other datasets.

**Neutrosophic in Blockchain**

In [97] they proposed utilized single-valued and interval-valued neutrosophic graphs in Blockchain and bitcoin application. They also listed the advantages and limitations of Blockchain graphs. While in [98] their main focus was to select the most appropriate Blockchain model for providing a secure and trustworthy healthcare Blockchain solution. Their proposed model was well-presented yet had a set of limitations, including its generalization.

**Neutrosophic Logic in marketing**

Neutrosophic PROMETHEE method in one-to-one marketing. They proved the necessity of analyzing different aspects of potential buyers, including their emotional and physiological states. One of the main obstacles of their model was the collection, and governance of the customers' emotions related data, especially under the existence of the data privacy rules exist in many companies. Consequently, collecting the proper data needed time and cost [3]. A single-valued neutrosophic set was combined with multi-criteria group decision-making in [69]. They used their hybrid proposed model in market segment selection and evaluation. Their model was effectively applied in a real-life dataset related to smart bike-sharing firm. As future work related to their contribution might be to employ weighting methods to provide the market segments' ranking. It could seven be applied in other applications. In [87] the researchers proposed a method that combined neutrosophic logic to Q learning. They compared its results to another developed model that combined fuzzy logic to Q learning. Both models were applied in two benchmark datasets. Yet, it was encouraged to apply them in real-life business cases to prove their reliability. Other researchers might also think of combining their model with neural networks to optimize the Q values. The researchers in [91] proposed a clustering algorithm based on single-valued neutrosophic and similarity measures. Their proposed model proved its effectiveness when it was applied in a car market. Meanwhile, it was recommended to be generalized to other applications.

**Neutrosophic in Medicine**

The researchers in [57] survived the contributions of neutrosophic logic in image segmentation. They focused on medical images. They recommended blinding the contributions they mentioned with deep learning for more effective, robust, and reliable outcomes. A neutrosophic login-based model was applied in the medical domain to act as a medical decision eid for the physicians. Their model outperformed the other fuzzy-based models. Their work's main limitation is its dependency on a huge amount of data for obtaining accurate and reliable outputs [2]. While in [4] they proposed a recommender system based on neutrosophic logic that contributed to medicine through predicting the disease. They could design the formulation of algebraic formulas using their proposed algebraic similarity measure. Their proposed algorithm proved its robustness when being applied to different medical datasets, including heart, Breast Cancer, Diabetes, and more from the University of California Irvin (UCI) benchmark datasets repository [6]. The limitation of their work was that it was generalized and even not applied to other real-life datasets. The researchers in [7] developed an automatic choroidal segmentation method from Enhanced Depth Imaging Optical Coherence Tomography (EDI-OCT) images. This was done in neutrosophic space. Their model started by transforming the images to the neutrosophic space, after that, calculating the weights between the nodes. Then applied Dijkstra algorithm to detect the Retinal Pigment Epithelium layer. Finally, defining the false set using a gamma homomorphic filter. Their model was applied and tested on real-life datasets and proved its robustness by obtaining relatively small and acceptable error values. Their model was well defined, and well-presented especially its experimental results.

The researchers in [9] utilized offsets and off uniforms Neutrosophic sets in image processing. Mainly for a segmentation and edge detection of an image. They did not apply their model in real-life cases, but in demonstrated examples. This is on top of the limitations of their work. While, researchers in [16] developed a multi-criteria decision-making technique using neutrosophic algorithm to help physicians in diagnosing those who suffer from heart failure. Their model was already applied to real-life case studies and proved its effectiveness when was compared to other techniques. Meanwhile, their technique might be applied to larger dataset to confirm its robustness.

In [17], they conducted a study that combined Neutrosophic logic to Convolutional Neural Networks. Their main goal was to classify a brain tumor as either benign or malignant. Their model could prove its robustness and reliability with was compared to other classifiers. Although their model was well written and presented, it still needed to be applied to a real-life dataset. While the researchers in

[20] proposed cosine similarity measures that help diagnose bipolar disorder using neutrosophic logic. They verified their proposal on numerical examples. Their technique could prove its robustness; meanwhile, it might be applied to other case studies to prove its robustness and generalizability. The researchers in [32] conducted another contribution in cosine similarity. They build improved weighted single-valued neutrosophic cosine similarity measures. Their main goal was to help in the medical diagnosis. They applied their proposed model in two real case studies. Meanwhile, it still needed to be applied in other fields and other case studies to prove its reliability. The usage of single-valued neutrosophic sets has been presented in [21]. They projected their illustration in medicine. They utilized their model to help in the disease diagnosing based on the preliminary symptoms of each patient. This is done by formulating new distance measures for single-valued neutrosophic sets. Their model has been applied in medical examples, yet it still needed to be applied to many real-life cases to prove its reliability. In comparison, the researchers in [29] illustrated the divergence measure in neutrosophic sets. They formulated this relationship and mentioned its properties. Finally, they applied their proposed model in a medical problem. Yet, their model still needed to be applied in other case studies. In [34] they introduced proposed types of distance measures for single-valued neutrosophic sets. Their proposed models were illustrated with pattern recognition, and medical case studies. Yet, these models still needed to be applied in other use cases to prove their generalization.

The contribution of neutrosophic login in the early diagnosing of COVID-19 based on the patients' medical images was discussed in [38]. Their model combined neutrosophic logic with deep learning. They applied their model to a real-life dataset. In conclusion, they encouraged using deep learning with neutrosophic logic to obtain reliable results to diagnose and overcome COVID-19. Hence, their model still needs to be applied in many datasets to increase its reliability. In [46] they introduced a refined technique of single-valued neutrosophic cosine similarity measure. They applied their proposed technique in medical diagnosis. Their model still needed to be generalized on other applications and might be extended by applying interval neutrosophic instead of single neutrosophic set. One of the most interesting application for neutrosophic logic was that one mentioned in [47]. They utilized neutrosophic statistics in analyzing dental fluorosis. Their analysis was well presented and expected to have a huge impact on improving human's tooth health (especially children). In [68] they constructed a comprehensive framework using a single-valued neutrosophic set to capture both incomplete and/or inconsistent information. They applied their proposed framework in a real-life healthcare case study. Meanwhile, it was encouraged to generalize it in other application areas, especially smart cities. In [104] they proposed a framework that combined interval neutrosophic to a neural network. They compared their interval neutrosophic rough neural network to other algorithms, and it outperformed them. Meanwhile, it still needed to be generalized using other real-life applications.

**Neutrosophic in COVID-19**

From 2020 until publishing this survey, the whole world is struggling with different generations of Coronavirus (i.e., COVID-19). Many researchers tried to contribute to developing models for dealing with COVID-19 using a different algorithm, and neutrosophic logic was on top of these utilized algorithms. In [101] they developed a framework that combined COVID-19's disruptive technologies for analyzing this pandemic virus. Their framework had many advantages, including restricting COVID-19's outbreaks and ensuring the healthcare team's safety. The power of their model was applying it in an empirical case study. Meanwhile, it would be great to be generalized on other use cases. In [99] they identified priority tanking of insurance companies related to healthcare services. They mainly built a multi-criteria performance evaluation methodology with the help of experts' opinions. Their main focus was on Turkey. Meanwhile, their model could be generalized to other countries. They mainly focused on intuitionistic fuzzy logic, but it was encouraged to utilize neutrosophic logic instead for more effective results. While in [100] contributed to COVID-19's vaccine. They identified a set of criteria and sub-criteria that helped identify priority groups for COVID-19's vaccine doses distribution. Their proposed priority groups model was well-presented and well-illustrated; meanwhile, it still needed generalization. On the other hand, the researchers in [102] focused on the diagnosis part of COVID-19. They developed a framework for this purpose. Their framework not only focused on COVID-19's early diagnosis but also on its treatment. Their proposed framework combined deep learning with a neutrosophic classifier. It was a well-presented and effective contribution, yet suffered from data availability limitations and a set of weaknesses of big data architectures. While the researchers in [103] differentiated between COVID-19 and other four chest diseases that had some common symptoms. They utilized neutrosophic logic for this purpose to diagnose COVID-19 using only the CT scan and the primary symptoms. They also studied the effect of the internet of things (IOT) in helping the medical staff monitor the spread of COVID-19. Their proposed model achieved 98% detection accuracy. Yet, it was recommended to update their study by including other COVID-19's symptoms added by the World Health Organization (WHO) related to the virus's evolution.

**Neutrosophic in Image Processing**

In [25] they proposed a neutrosophic similarity clustering algorithm. They applied their model in image processing field for segmenting gray-level images. Their model was applied in many images, both artificial and real images. This proved the effectiveness of their proposed model in image processing and computer vision. In [44] they applied neutrosophic logic in grayscale image processing. Meanwhile, they did not mention real-life use cases to prove the effectiveness of their model. Another contribution of neutrosophic logic in image processing was mentioned in [48]. It was mainly used with Dice coefficients to deal with the missing data uncertainty. Their proposed model was experimentally validated. Their model had high applicability not only in image processing but only in natural language processing. Also, in [51] they utilized neutrosophic sets in image processing. They added two operations to the traditional neutrosophic membership functions (i.e., $\alpha$-mean and $\beta$-enhancement). These added operations could reduce the indeterminacy of the set. Their model was effective as it was able to segment different types of images, even noisy ones. Hence, it was recommended to be applied in real application areas to confirm its applicability. A bit different than this, is the contribution in [55]. They utilized a neutrosophic set in grayscale images. They analyzed the effect of applying bipolar neutrosophic set in grayscale images and came up with their proposed model. Their model could extract useful information from even the noisy images, and was tested on different images, including human brain images. This could prove its effectiveness in the medical field. In [114] they proposed a particle swarm neutrosophic algorithm to cluster liver tumors in CT images. They evaluated the indeterminacy of the neutrosophic set using entropy. Their proposed showed outperforming results on both CT and non-CT images.

**Neutrosophic in Supply Chain**

In [12] the researchers combined neutrosophic techniques with AHP method. Their main goal was to support enterprise decision-making in the internet of things (IoT) era. Their proposed algorithm was applied in different enterprises, including Smart Village in Egypt and Smart City in U.K. and China. Meanwhile, involving more companies in their model validation would enrich its results. The researchers in [20] designed an interval complex neutrosophic set and listed its characteristics. To prove their model's practicality, they applied it in supplier selection related to a transportation company. Meanwhile, it still needed to be applied to more real-life datasets to prove its robustness. Another contribution in IoT field was proposed by the researchers in [24] who developed a model that helped detect cancer early. The data was extracted from a set of smart devices (i.e., sensor networks). Their model would help in the early prediction, detection, and treatment of cancer. Their proposed model was well presented, yet it still needs to be applied to many real-life case studies to prove its reliability.

The effect of neutrosophic login in Blockchain has been proved in [28]. They developed single and interval-values neutrosophic Blockchain graphs. They applied their proposed model to various graphs of Blockchain. In [30] they utilized neutrosophic sets to build a ranking technique in a supply chain environment. Their main goal is to handle the economic and environmental vague performances. Their proposed model was applied in two real-life use cases: the petroleum industry and a manufacturing firm in China. Their model was an atom for a combination of analytics and neutrosophic sets. Consequently, it could be extended to predict future trends. In [35] they proposed a fuzzy neutrosophic approach based on trapezoidal neutrosophic variables. It was a decision-making aid for supplier evaluation and selection. Their model was applied in a resilient supply chain management context, in a real-life business case study. Meanwhile, it could be applied in many other areas to test its reliability. It can also be integrated with other fuzzy tools (i.e. rough sets) to enrich its effectiveness. In [41] they proposed a model for the analysis of failure mode and effect. They applied their proposed algorithm in an empirical, real-life case study, and it showed its effectiveness and reliability yet still needed to be generalized. In [61] they had a different perspective in illustrating neutrosophic numbers (i.e., both linear and non-linear). They also tackled both neutrosophication and de-neutrosophication. They applied their proposed concepts in two application areas related to project evaluation review technique (PERT) and route selection. Their proposals could be applied in different applications, and also other neutrosophic numbers' types could be used.

The researchers in [81] developed a supply chain-related multi-criteria group decision-making method. Their proposed model combined analytical network process method to ViseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method. Their model that was developed for a neutrosophic environment that had incomplete information, utilized triangular membership function was applied in a real case study. It showed outperforming results. Its main limitation was the dependency on experts' opinions and it was hard to find those experts fulfilled the researchers' predefined requirements. Also, its dependency on a forecasting phase that needed to have large input data to have robust results. Meanwhile, it was recommended to generalize it to other applications. The researchers in [83] also utilized VIKOR method to guide the decision-making process in an uncertain environment. For this purpose, they combined VIKOR method to the cubic neutrosophic number. This combination was illustrated through an example but was recommended to be applied in real applications. The contribution in [84] was mentioned to be the first contribution that combined Delphi method to neutrosophy. It was illustrated in a hypothetical case study. Meanwhile, it was recommended to apply it in practical use cases to prove its reliability and robustness. In [86] they proposed a dynamic interval valued neutrosophic set that was applied to a university of languages and international studies. Their proposed model proved its robustness yet still needed to be applied in other application areas.

They also proposed a modified combination between correlation coefficient and single-valued neutrosophic set [86]. Their main objective was to build a decision-making method that helped in selecting the best alternative out of all these given alternatives. They applied their proposed model in an illustrative example for choosing between investment alternatives. Meanwhile, it was recommended to be applied in real-life business cases.

**Neutrosophic in Strategic Planning**

Neutrosophic analytic hierarchy process model would help select the best strategy out of many different possible strategies under the existence of vague and incomplete information situations. They integrated their model with SWOT analysis. They applied their model in "Starbucks" company. Their model proved its reliability and robustness based on the "Starbucks" dataset; meanwhile, it was flexible enough to be generalized to many other fields and industries [4].

The researchers in [13] were interested in multi-attribute decision-making. Hence, they developed an outranking approach in a bipolar neutrosophic environment. They applied their model to a real example of an investment company. Yet, they did not compare their model's results with another traditional model to prove its effectiveness. A bit similar to this is the study conducted by the researchers in [14] to empower the decision-making using neutrosophic. They mainly designed a proactive approach to analyze and then determine the set of factors that would influence suppliers' selection in supply chain management. In conclusion, they found that "Quality" was the most influential criterion in suppliers' selection. Generalizing their work might face some obstacles related to its need for a huge amount of data, large processing, and complex calculations.

On the other hand, the researchers in [15] contributed to IoT companies' decision-making by proposing a hybrid analytical hierarchal process and neutrosophic theory combination. Their proposal detected and handled the challenges of uncertainty and inconsistency.

In [23] the researchers presented the applicability of intuitionistic neutrosophic on the graph structure. They listed a set of applications using their proposed model that could boost the decision-making process. Meanwhile, they did not report applying it in a real-life case study step by step.

In [27] the researchers combined soft-sets and bipolar sets in their proposal of bipolar neutrosophic soft-sets. They illustrated their proposed model using only numerical examples but did not apply it in real-life. This is on top of their limitations. In [32] they presented some real number-based operational laws for single-valued neutrosophic numbers. They also developed a set of weighted averaging models and geometric aggregation operators. These operators have been utilized to build a multi-attribute decision-making model. Their model is illustrated on a real numerical example. Meanwhile, it still needed to be applied in many real-life use cases to prove its effectiveness and reliability. In [34] they developed a Hierarchical Neutrosophics Analytical model to solve bus routes expected to meet tourists' demands. They achieved satisfactory results. The researchers in [54] studied the applications on entropy and similarity measures in decision-making (DS), especially multi-attribute DS. They mainly focused on hamming distance, contingent, and cosine functions. They confirmed their proposed model's applicability and efficiency, but they did not mention a real-life application that proved its applicability.

In [56] they made a different contribution compared to the above contributions. They utilized rough neutrosophic multisets to support the decision-making in marketing strategy. Their proposed model could be extended to other application areas to prove its effectiveness. Other neutrosophic multisets relations could also be applied than Max, Min, and composition of two rough multisets. In [58] they proposed three neutrosophic models, i.e., single-values hypersoft set, tangent similarity measure for single-valued neutrosophic hypersoft sets, and multi-values hypersoft set. They applied their proposed models in players' match selection. Their models helped in selecting the best option for each player. Meanwhile, it could be applied in many other application areas to validate its effectiveness. In [63] they utilized logarithmic operations for two purposes. First, developing aggregation operators, second, a multi-criteria decision-making approach for a single-valued neutrosophic environment. Their model was tested on a practical case study and proved its effectiveness and selecting the best alternative. Hence, it was encouraged to be applied in other application areas to confirm its effectiveness. While in [64] their main focus was the interval neutrosophic sets. They studied its similarity and entropy. They also mentioned its applicability in multi-attribute decision-making. Their model showed its effectiveness and robustness. Yet, it was still needed to be applied in other applications to prove this effectiveness. An interesting contribution was in [66]. They studied the pathogenic hypersoft set. This concept covered most of the cases in fuzzy and neutrosophic sets. They presented this concept using an illustrative example. Then they built a multi-criteria decision-making model based on this concept and was applied in a real-life application. Meanwhile, it could be extended to other applications, especially graph theory and pattern recognition, to prove its reliability. In [70] validated the pedagogical strategy implementation through two case studies. Their main goal was to increase scientific knowledge through extending the implementation of Iadov method and neutrosophic analysis. Their model could be extended through developing a software tool that could facilitate its applicability in other areas. In [88] they utilized single valued neutrosophic set for multi-attribute decision-making process mainly for school choice. They illustrate their idea through a numerical example, yet would be great to apply it in a real-life use case to prove its effectiveness. In [92] they utilized neutrosophic normal cloud concept, cloud aggregator, and many other concepts to build a multi-criteria group decision-making model. Their model was effectively applied in a real-life use case related to an online retailer. But it would be useful to generalize it to other applications. The researchers in [107] introduced the concept of neutrosophic soft rough topology. They aimed to develop a multi-criteria decision-making method for ambiguous real-life problems. Meanwhile, their model might be integrated with other algorithms (i.e., TOPOSIS and AHP) to enhance its applicability to other application areas.

**Conclusion and Future Work**

Neutrosophic set is a very powerful and reliable algorithm. It proved its superiority in many application areas, including image processing, natural language processing, multi-criteria decision-making, strategic planning, Blockchain, and many more. In this paper, sample contributions in these research areas have been presented. Meanwhile, there are a lot of other research points that other researchers might tackle. For instance, combing deep learning, Q learning, and deep Q learning with neutrosophic is an open research area. The researchers in [87] introduced the combination of Q learning with a neutrosophic set, but it still needs a generalization to prove its robustness. Also, combing machine learning to neutrosophic sets still a rich research area. Finally, most of the paper's contributions were not generalized to different application areas to confirm their reliability, which might be another future research direction.

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# Pentapartitioned Neutrosophic *Q*-Ideals of *Q*-Algebra

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**Abstract:** In this paper, we procure the idea of pentapartitioned neutrosophic *Q*-ideal of *Q*-algebra. Then, we formulate some definitions and results on it. Further, we furnish some suitable examples.

**Keywords:** *Pentapartitioned Neutrosophic Set*; *PN-Q-Algebra*; *PN-Q-Ideal*; *PN-Q-Sub-Algebra*.

## 1. Introduction

Iseki and Tanaka [15] presented the concept of BCK-algebra in the year 1978. Later on, Negger and Kim [26] established the notion of *d*-algebra by extending the idea of BCK-algebra. In the year 1999, Negger et al. [25] defined the *d*-ideal in *d*-algebra. The notion of fuzzy set (FS) theory was established by Zadeh [28] in the year 1965. Thereafter, Atanassov [4] introduced the idea of intuitionistic fuzzy set (IFS) theory by generalizing the concept of FS. In the year 2013 F. Smarandache [27] extended the neutrosophic set to refined [n-valued] neutrosophic set, and to refined neutrosophic logic, and to refined neutrosophic probability. The notion of fuzzy *d*-ideals of *d*-algebras was studied by Jun et. al. [17] in the year 2000. The idea of intuitionistic fuzzy *d*-algebra was presented by Jun et al. [16]. In the year 2017, the concept of intuitionistic fuzzy *d*-ideal of *d*-algebra was introduced by Hasan [12]. Hasan [13] also studied the intuitionistic fuzzy *d*-filter of *d*-algebra. The notion of *Q*-algebra was grounded by Neggers et. al. [24] in the year 2001. Thereafter, Abdullah and Jawad [1] studied some new types of ideals in *Q*-algebra. Mostafa et. al. [22] introduced the notion of fuzzy *Q*-ideals in *Q*-algebras. Mostafa et. al. [23] also studied the intuitionistic fuzzy *Q*-ideals of *Q*-algebra. In the year 2005, Smarandache [27] grounded the idea of neutrosophic set by extending the IFS. Later on, the notion of neutrosophic *BCI*/*BCK*-algebras was presented by Agboola and Davvaz [2]. In the year 2016, Martina Jency and Arockiarani [21] established the notion of single valued neutrosophic ideals of *BCK*-algebras. In the year 2019, Mallick and Pramanik [20] presented the concept of pentapartitioned neutrosophic set and studied different operations on them. In this article, we procure the idea of pentapartitioned neutrosophic *Q*-ideals of *Q*-algebra.

The rest of the paper is designed as follows:
In section 2, we recall some preliminary definitions and results on *Q*-algebra, *Q*-ideal, fuzzy *Q*-algebra, fuzzy *Q*-ideal, intuitionistic fuzzy *Q*-algebra, intuitionistic fuzzy *Q*-ideal. In section-3, we introduce the notion of pentapartitioned neutrosophic *Q*-ideal of *Q*-algebra by generalizing the theory of intuitionistic fuzzy *Q*-ideal and neutrosophic *Q*-ideal. Further, we formulate some results on pentapartitioned neutrosophic *Q*-ideals of *Q*-algebra. In section 4, we conclude the work done in this article.

## 2. Relevant Definitions and Results:

Here we procure some basic definition and example which is needed for our work.

F. Smarandache[27] introduced the n-symbolic or numerical-Valued Refined Neutrosophic Logic

In general: $T$ can be split into many types of truths: $T_1$, $T_2$, ..., $T_p$, and $I$ into many types of indeterminacies: $I_1$, $I_2$, ..., $I_r$, and F into many types of falsities: $F_1$, $F_2$, ..., $F_s$,, where all p, r, s ≥ 1 are integers, and $p + r + s = n$.

All subcomponents $T_j$, $I_k$, $F_l$ are symbolic or numerical for $j\in\{1,2,\ldots,p\}$, $k\in\{1,2,\ldots,r\}$, and $l\in\{1,2,\ldots,s\}$.

If at least one $I_k = T_j \wedge F_l$ = contradiction, we get again the Extenics.

We use five valued neutrosophic logic which is the particular case of the n-valued neutrosophic logic. The details for the *n*-valued neutrosophic logic one may refer to [27].

**Definition 2.1.**[20] Assume that *W* be a fixed set. A pentapartitioned neutrosophic set *P* over *W* is defined as follows:
$Y = \{(c, \hat{A}_Y(c), \hat{C}_Y(c), \hat{E}_Y(c), \check{D}_Y(c), \hat{U}_Y(c)) : c \in W\}$, where $\hat{A}_Y(c), \hat{C}_Y(c), \hat{E}_Y(c), \check{D}_Y(c), \hat{U}_Y(c)$ ($\in ]0,1[$ ) are the truth, contradiction, ignorance, unknown and falsity membership values of each $c\in W$. So,

$$0 \leq \hat{A}_Y(c)+\hat{C}_Y(c)+\hat{E}_Y(c)+\check{D}_Y(c)+\hat{U}_Y(c) \leq 5.$$

**Definition 2.3.**[20] Suppose that $X = \{(c, \hat{A}_X(c), \hat{C}_X(c), \hat{E}_X(c), \check{D}_X(c), \hat{U}_X(c)) : c \in W\}$ and $Y = \{(c, \hat{A}_Y(c), \hat{C}_Y(c), \hat{E}_Y(c), \check{D}_Y(c), \hat{U}_Y(c)) : c \in W\}$ be two pentapartitioned neutrosophic sets over *W*. Then, $X\subseteq Y$ if and only if $\hat{A}_X(c)\leq \hat{A}_Y(c)$, $\hat{C}_X(c)\leq \hat{C}_Y(c)$, $\hat{E}_X(c)\geq \hat{E}_Y(c)$, $\check{D}_X(c)\geq \check{D}_Y(c)$, $\hat{U}_X(c)\geq \hat{U}_Y(c)$, for all $c\in W$.

**Definition 2.4.**[20] Suppose that $X = \{(c, \hat{A}_X(c), \hat{C}_X(c), \hat{E}_X(c), \check{D}_X(c), \hat{U}_X(c)) : c \in W\}$ and $Y = \{(c, \hat{A}_Y(c), \hat{C}_Y(c), \hat{E}_Y(c), \check{D}_Y(c), \hat{U}_Y(c)) : c \in W\}$ be two pentapartitioned neutrosophic sets over *W*. Then, $X \cap Y =$

{(*c*, min {$\hat{A}_X(c)$, $\hat{A}_Y(c)$}, min {$\hat{C}_X(c)$, $\hat{C}_Y(c)$}, max {$\hat{E}_X(c)$, $\hat{E}_Y(c)$}, max {$\check{D}_X(c)$, $\check{D}_Y(c)$}, max {$\hat{U}_X(c)$, $\hat{U}_Y(c)$}): $c\in W$}.

**Definition 2.5.**[20] Suppose that X = {(c, $\hat{A}_X(c)$, $\hat{C}_X(c)$, $\hat{E}_X(c)$, $\check{D}_X(c)$, $\hat{U}_X(c)$) : c ∈W} and Y = {(c, $\hat{A}_Y(c)$, $\hat{C}_Y(c)$, $\hat{E}_Y(c)$, $\check{D}_Y(c)$, $\hat{U}_Y(c)$) : c ∈W} be two pentapartitioned neutrosophic sets over *W*. Then, $X \cup Y$ = {(*c*, max {$\hat{A}_X(c)$, $\hat{A}_Y(c)$}, max {$\hat{C}_X(c)$, $\hat{C}_Y(c)$}, min {$\hat{E}_X(c)$, $\hat{E}_Y(c)$}, min {$\check{D}_X(c)$, $\check{D}_Y(c)$}, min {$\hat{U}_X(c)$, $\hat{U}_Y(c)$}): $c\in W$}.

**Definition 2.6.**[20] Suppose that X = {(c, $\hat{A}_X(c)$, $\hat{C}_X(c)$, $\hat{E}_X(c)$, $\check{D}_X(c)$, $\hat{U}_X(c)$) : c ∈W} be a pentapartitioned neutrosophic set over *W*. Then, $X^c$ = {(*c*, $\hat{U}_X(c)$, $\check{D}_X(c)$, 1-$\hat{E}_X(c)$, $\hat{C}_X(c)$, $\hat{A}_X(c)$): $q\in W$}.

**Definition 2.1.**[15] Suppose that W be a fixed set. Let 0 be a constant in W and ∗ be a binary operation defined on W. Then (W, ∗, 0) is called a BCK-algebra if the following holds:

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $x * y = y * x = 0 \Rightarrow x = y$,

(v) $0 * x = 0$, for all x, y, z ∈ *X* .

**Definition 2.2.**[15] Let W be a BCK-algebra with binary operator ∗ and a constant 0. Then I ⊆ W is called a BCK-ideal of W if the following holds:

(i) 0 ∈ I;

(ii) h ∗ d∈ I and d∈ I ⇒ h∈ I, ∀ h, d∈ W.

**Definition 2.3.**[22] Suppose that W be a fixed set. Let 0 be a constant in W and ∗ be a binary operation defined on W. Then (W, ∗, 0) is called a *Q*-algebra if the followings hold:

(i) $h * h = 0$, ∀ h ∈ W

(ii) $0 * h = h = h * 0$, ∀ h ∈ W

(iii) $(h * d) * e = (h * e) * d$, ∀ h, d, e ∈ W.

Sometime, one can refer to h ≤ d if and only if h ∗ d = 0.

**Definition 2.4.**[24] Let (W, ∗, 0) be a *Q*-algebra. Then, (W, ∗, 0) is said to be commutative *Q*-algebra if $c * (c * d) = d * (d * c)$, ∀c, d ∈W, and $d * (d * c)$ is denoted by ($c \wedge d$).

**Definition 2.5.**[24] A *Q*-algebra W is called bounded if there exist g ∈ W such that h ≤ g for all h ∈ W, i.e. h ∗ g = 0, ∀ h ∈ W.

**Definition 2.6.**[24] Let W be a *Q*-algebra with binary operator ∗ and H ($\neq 0_N$) ⊆ W. Then, H is called a *Q*-sub-algebra of W, if h, d ∈A implies h ∗ d ∈ H.

**Definition 2.7.**[1] Let W be a *Q*-algebra with binary operator ∗ and a constant 0. Then, I ⊆ W is called a *Q*-ideal of W if the following holds:

(i) 0∈ I;

(ii) (h ∗ d) ∗ e∈ I and d∈ I ⇒ h ∗ e∈ I, for all h, d, e∈ W.

**Proposition 2.1.**[1] Let (W, ∗, 0) be called a *Q*-algebra. Let I be a *Q*-ideal of W. Then, I be a BCK-ideal of W.

**Definition 2.8.**[3] A fuzzy set Y={(c,$T_Y(c)$): c∈W} over a BCK-algebra W is called the fuzzy BCK-ideal if the following two conditions holds:

(i) $T_Y(0) \geq T_Y(c)$, for all c∈ W;

(ii) $T_Y(c) \geq \min\{T_Y((c * d), T_Y(d)\}$ for all c, d∈W.

**Definition 2.9.**[22] A fuzzy set Y={(c,$T_Y$(c)): c∈W} over a *Q*-algebra W is called the fuzzy *Q*-ideal if the following holds:

(i) $T_Y(0) \geq T_Y(c)$, for all c∈ W;

(ii) $T_Y(c * d) \geq \min\{T_Y((c * h) * d), T_Y(h)\}$ for all c, h, d∈W.

**Lemma 2.1.**[22] Let (W, ∗, 0) be a *Q*-algebra. If Y = {(c,$T_Y$(c)): c∈W} is a fuzzy *Q*-ideal of W, then it is also a fuzzy BCK-ideal of W.

**Definition 2.10.**[23] An intuitionistic fuzzy set Y={(c,$T_Y$(c),$F_Y$(c)): c∈W} over a *Q*-algebra W is called the intuitionistic fuzzy *Q*-ideal if it satisfies the following inequalities:

(i) $T_Y(0) \geq T_Y(c)$, $F_Y(0) \leq F_Y(c)$, for all c∈ W;

(ii) $T_Y(c * d) \geq \min\{T_Y((c * h) * d), T_Y(h)\}$;

(iii) $F_Y(c * d) \leq \max\{F_Y((c * h) * d), F_Y(h)\}$.

**Lemma 2.2.**[23] Let Y={(c,$T_Y$(c),$F_Y$(c)): c∈W} be an intuitionistic fuzzy *Q*-ideal over a *Q*-algebra W. If $c * d \leq h$, for all c, d, h ∈W, then $T_Y(c) \geq \min\{T_Y(d), T_Y(h)\}$ and $F_Y(c) \geq \max\{F_Y(d), F_Y(h)\}$, for all c∈W.

**Lemma 2.3.**[23] Let Y={(c,$T_Y$(c),$F_Y$(c)): c∈W} be an intuitionistic fuzzy *Q*-ideal over a *Q*-algebra W. If c ≤ d, , for all c, d∈ W, then $T_Y(c) \geq T_Y(d)$ and $F_Y(c) \leq F_Y(d)$.

**Lemma 2.4.**[23] If Y={(c,$T_Y$(c),$F_Y$(c)): c∈W} be an intuitionistic fuzzy *Q*-ideal over a *Q*-algebra W, then the sets $\alpha\text{-}T_Y = \{c: c\in W, T_Y(c) \geq \alpha\}$ and $\alpha\text{-}F_Y = \{c: c\in W, F_Y(c) \leq \alpha\}$ are *Q*-ideal of *Q*-algebra W.

## 3. Pentapartitioned Neutrosophic *Q*-Ideals of *Q*-Algebra:

In this section, we procure the notion of pentapartitioned neutrosophic *Q*-ideal (PN-*Q*-Ideal) of pentapartitioned neutrosophic *Q*-algebra (PN-*Q*-Algebra). Then, we formulate some definitions and results on PN-*Q*-Ideal and PN-*Q*-Algebra.

**Definition 3.1.** Suppose that *W* be a *Q*-algebra and Y = {(c, $\hat{A}_Y$(c), $\hat{C}_Y$(c), $\hat{E}_Y$(c), $\check{D}_Y$(c), $\hat{U}_Y$(c)) : c ∈W} be a pentapartitioned neutrosophic set over W. Then, Y is said to be a pentapartitioned neutrosophic *Q*-algebra (PN-*Q*-algebra) if and only if the following holds:

(i) $\hat{A}_Y(c * d) \geq \min\{\hat{A}_Y((c * h) * d), \hat{A}_Y(h)\}$;

(ii) $\hat{C}_Y(c * d) \geq \min\{\hat{C}_Y((c * h) * d), \hat{C}_Y(h)\}$;

(iii) $\hat{E}_Y(c * d) \leq \max\{\hat{E}_Y((c * h) * d), \hat{E}_Y(h)$;

(iv) $\check{D}_Y(c * d) \leq \max\{\check{D}_Y((c * h) * d), \check{D}_Y(h)$;

(v) $\hat{U}_Y(c * d) \leq \max\{\hat{U}_Y((c * h) * d), \hat{U}_Y(h)\}$, where c, d∈ W.

By the structure [(W, Y), ∗, 0], we denotes the PN-*Q*-algebra Y over W.

**Theorem 3.1.** If {$Y_i$ : i∈Δ} be the collection of PN-*Q*-algebra's of W, then, $\bigcap_{i\in\Delta} Y_i$ is also a PN-*Q*-algebra of W.

**Proof.** Assume that {$Y_i$: i∈Δ} be a family of PN-*Q*-algebras of W. It is clear that, $\bigcap_{i\in\Delta} Y_i$ = {(c, $\wedge\hat{A}_{Y_i}$(c), $\wedge\hat{C}_{Y_i}$(c), $\vee\hat{E}_{Y_i}$(c), $\vee\check{D}_{Y_i}$(c), $\vee\hat{U}_{Y_i}$(c)) : c∈W}.

Now,

$$\wedge\hat{A}_{Y_i}(c * d) = \wedge\{\hat{A}_{Y_i}(c * d): i\in\Delta\}$$
$$\geq \wedge\{\min\{\hat{A}_{Y_i}((c * h) * d), \hat{A}_{Y_i}(h)\}\}$$
$$= \min\{\wedge\hat{A}_{Y_i}((c * h) * d), \wedge\hat{A}_{Y_i}(h)\}\}$$
$$\Rightarrow \wedge\hat{A}_{Y_i}(c * d) \geq \min\{\wedge\hat{A}_{Y_i}((c * h) * d), \wedge\hat{A}_{Y_i}(h)\}\}.$$

Now,

$\wedge\hat{C}_{Y_i}(c*d) = \wedge\{\hat{C}_{Y_i}(c*d): i\in\Delta\}$

$\geq \wedge\{\min\{\hat{C}_{Y_i}((c*h)*d),\ \hat{C}_{Y_i}(h)\}\}$

$= \min\{\wedge\hat{C}_{Y_i}((c*h)*d), \wedge\hat{C}_{Y_i}(h)\}\}$

$\Rightarrow \wedge\hat{C}_{Y_i}(c*d) \geq \min\{\wedge\hat{C}_{Y_i}((c*h)*d), \wedge\hat{C}_{Y_i}(h)\}\}$.

Now,

$\vee\hat{E}_{Y_i}(c*d) = \vee\{\hat{E}_{Y_i}(c*d): i\in\Delta\}$

$\geq \vee\{\min\{\hat{E}_{Y_i}((c*h)*d),\ \hat{E}_{Y_i}(h)\}\}$

$= \min\{\vee\hat{E}_{Y_i}((c*h)*d), \vee\hat{E}_{Y_i}(h)\}\}$

$\Rightarrow \vee\hat{E}_{Y_i}(c*d) \geq \min\{\vee\hat{E}_{Y_i}((c*h)*d), \vee\hat{E}_{Y_i}(h)\}\}$.

Now,

$\vee\check{D}_{Y_i}(c*d) = \vee\{\check{D}_{Y_i}(c*d): i\in\Delta\}$

$\geq \vee\{\min\{\check{D}_{Y_i}((c*h)*d),\ \check{D}_{Y_i}(h)\}\}$

$= \min\{\vee\check{D}_{Y_i}((c*h)*d), \vee\check{D}_{Y_i}(h)\}\}$

$\Rightarrow \vee\check{D}_{Y_i}(c*d) \geq \min\{\vee\check{D}_{Y_i}((c*h)*d), \vee\check{D}_{Y_i}(h)\}\}$.

Now,

$\vee\hat{U}_{Y_i}(c*d) = \vee\{\hat{U}_{Y_i}(c*d): i\in\Delta\}$

$\geq \vee\{\min\{\hat{U}_{Y_i}((c*h)*d),\ \hat{U}_{Y_i}(h)\}\}$

$= \min\{\vee\hat{U}_{Y_i}((c*h)*d), \vee\hat{U}_{Y_i}(h)\}\}$

$\Rightarrow \vee\hat{U}_{Y_i}(c*d) \geq \min\{\vee\hat{U}_{Y_i}((c*h)*d), \vee\hat{U}_{Y_i}(h)\}\}$.

Therefore, $\bigcap_{i\in\Delta} Y_i$ is also a PN-*Q*-algebra of W.

**Definition 3.2.** Suppose that Y = {(c, $\hat{A}_Y(c)$, $\hat{C}_Y(c)$, $\hat{E}_Y(c)$, $\check{D}_Y(c)$, $\hat{U}_Y(c)$) : c ∈W} be a pentapartitioned neutrosophic set over W. Then, Y is said to be a pentapartitioned neutrosophic *Q*-sub-algebra (PN-*Q*-sub-algebra) if and only if the following holds:

(i) $\hat{A}_Y(c*d) \geq \min\{\hat{A}_Y(c), \hat{A}_Y(d)\}$;

(ii) $\hat{C}_Y(c*d) \geq \min\{\hat{C}_Y(c), \hat{C}_Y(d)\}$;

(iii) $\hat{E}_Y(c*d) \leq \max\{\hat{E}_Y(c), \hat{E}_Y(d)\}$;

(iv) $\check{D}_Y(c*d) \leq \max\{\check{D}_Y(c), \check{D}_Y(d)\}$;

(v) $\hat{U}_Y(c*d) \leq \max\{\hat{U}_Y(c), \hat{U}_Y(d)\}$; where c, d∈ W.

**Theorem 3.2.** Let Y = {(c, $\hat{A}_Y(c)$, $\hat{C}_Y(c)$, $\hat{E}_Y(c)$, $\check{D}_Y(c)$, $\hat{U}_Y(c)$) : c ∈W} be a PN-*Q*-sub-algebra of a *Q*-algebra W. Then, the following holds:

(i) $\hat{A}_Y(0)\geq \hat{A}_Y(c)$, for all c ∈W;

(ii) $\hat{C}_Y(0)\geq \hat{C}_Y(c)$, for all c ∈W;

(iii) $\hat{E}_Y(0)\leq \hat{E}_Y(c)$, for all c ∈W;

(iv) $\check{D}_Y(0)\leq \check{D}_Y(c)$, foĈr all c ∈W;

(v) $\hat{U}_Y(0)\leq \hat{U}_Y(c)$, for all c ∈W.

**Proof.** Assume that Y = {(c, $\hat{A}_Y(c)$, $\hat{C}_Y(c)$, $\hat{E}_Y(c)$, $\check{D}_Y(c)$, $\hat{U}_Y(c)$) : c ∈W} be a PN-*Q*-sub-algebra of a *Q*-algebra W. Hence, $\hat{A}_Y(c*d) \geq \min\{\hat{A}_Y(c), \hat{A}_Y(d)\}$, $\hat{C}_Y(c*d) \geq \min\{\hat{C}_Y(c), \hat{C}_Y(d)\}$, $\hat{E}_Y(c*d) \leq \max\{\hat{E}_Y(c), \hat{E}_Y(d)\}$, $\check{D}_Y(c*d) \leq \max\{\check{D}_Y(c), \check{D}_Y(d)\}$, $\hat{U}_Y(c*d) \leq \max\{\hat{U}_Y(c), \hat{U}_Y(d)\}$, for all c, d∈ W.

Now we have,

$\hat{A}_Y(0) = \hat{A}_Y(c*c)$

$\geq \min\{\hat{A}_Y(c), \hat{A}_Y(c)\}$

$= \hat{A}_Y(c)$

$\Rightarrow \hat{A}_Y(0) \geq \hat{A}_Y(c)$, for all $c \in W$.

$\hat{C}_Y(0) = \hat{C}_Y(c * c)$

$\geq \min\{\hat{C}_Y(c), \hat{C}_Y(c)\}$

$= \hat{C}_Y(c)$

$\Rightarrow \hat{C}_Y(0) \geq \hat{C}_Y(c)$, for all $c \in W$.

$\hat{E}_Y(0) = \hat{E}_Y(c * c)$

$\leq \max\{\hat{E}_Y(c), \hat{E}_Y(d)\}$

$= \hat{E}_Y(c)$

$\Rightarrow \hat{E}_Y(0) \leq \hat{E}_Y(c)$, for all $c \in W$.

$\check{D}_Y(0) = \check{D}_Y(c * c)$

$\leq \max\{\check{D}_Y(c), \check{D}_Y(d)\}$

$= \check{D}_Y(c)$

$\Rightarrow \check{D}_Y(0) \leq \check{D}_Y(c)$, for all $c \in W$.

$\hat{U}_Y(0) = \hat{U}_Y(c * c)$

$\leq \max\{\hat{U}_Y(c), \hat{U}_Y(d)\}$

$= \hat{U}_Y(c)$

$\Rightarrow \hat{U}_Y(0) \leq \hat{U}_Y(c)$, for all $c \in W$.

**Definition 3.3.** A pentapartitioned neutrosophic set $Y = \{(c, \hat{A}_Y(c), \hat{C}_Y(c), \hat{E}_Y(c), \check{D}_Y(c), \hat{U}_Y(c)) : c \in W\}$ over a *Q*-algebra W is said to be a pentapartitioned neutrosophic *Q*-ideal (PN-*Q*-ideal) if and only if the following inequalities holds:

(i) $\hat{A}_Y(0) \geq \hat{A}_Y(c)$ & $\hat{A}_Y(c * d) \geq \min\{\hat{A}_Y((c * h) * d), \hat{A}_Y(h)\}$, for all $c, d, h \in W$;

(ii) $\hat{C}_Y(0) \geq \hat{C}_Y(c)$ & $\hat{C}_Y(c * d) \geq \min\{\hat{C}_Y((c * h) * d), \hat{C}_Y(h)\}$, for all $c, d, h \in W$;

(iii) $\hat{E}_Y(0) \leq \hat{E}_Y(c)$ & $\hat{E}_Y(c * d) \leq \max\{\hat{E}_Y((c * h) * d), \hat{E}_Y(h)\}$, for all $c, d, h \in W$;

(iv) $\check{D}_Y(0) \leq \check{D}_Y(c)$ & $\check{D}_Y(c * d) \leq \max\{\check{D}_Y((c * h) * d), \check{D}_Y(h)\}$, for all $c, d, h \in W$;

(v) $\hat{U}_Y(0) \leq \hat{U}_Y(c)$ & $\hat{U}_Y(c * d) \leq \max\{\hat{U}_Y((c * h) * d), \hat{U}_Y(h)\}$, for all $c, d, h \in W$.

**Remark 3.1.** Every PN-*Q*-ideal of a *Q*-algebra W is also a PN-*Q*-sub-algebra.

**Theorem 3.3.** Suppose that $\{D_i : i \in \Delta\}$ be a family of PN-*Q*-ideals of *Q*-algebra W. Then, $\bigcap_{i\in\Delta} D_i$ is also a PN-*Q*-ideal of *Q*-algebra W.

**Proof.** Let $\{D_i : i \in \Delta\}$ be a family of PN-*Q*-ideals of *Q*-algebra W. Therefore,

(i) $\hat{A}_{D_i}(0) \geq \hat{A}_{D_i}(c)$ & $\hat{A}_{D_i}(c * d) \geq \min\{\hat{A}_{D_i}((c * h) * d), \hat{A}_{D_i}(h)\}$, for all $c, d, h \in W$ and $i \in \Delta$;

(ii) $\hat{C}_{D_i}(0) \geq \hat{C}_{D_i}(c)$ & $\hat{C}_{D_i}(c * d) \geq \min\{\hat{C}_{D_i}((c * h) * d), \hat{C}_{D_i}(h)\}$, for all $c, d, h \in W$ and $i \in \Delta$;

(iii) $\hat{E}_{D_i}(0) \leq \hat{E}_{D_i}(c)$ & $\hat{E}_{D_i}(c * d) \leq \max\{\hat{E}_{D_i}((c * h) * d), \hat{E}_{D_i}(h)\}$, for all $c, d, h \in W$ and $i \in \Delta$;

(iv) $\check{D}_{D_i}(0) \leq \check{D}_{D_i}(c)$ & $\check{D}_{D_i}(c * d) \leq \max\{\check{D}_{D_i}((c * h) * d), \check{D}_{D_i}(h)\}$, for all $c, d, h \in W$ and $i \in \Delta$;

(v) $\hat{U}_{D_i}(0) \leq \hat{U}_{D_i}(c)$ & $\hat{U}_{D_i}(c * d) \leq \max\{\hat{U}_{D_i}((c * h) * d), \hat{U}_{D_i}(h)\}$, for all $c, d, h \in W$ and $i \in \Delta$.

Clearly, $\bigcap_{i\in\Delta} D_i = \{< c, \wedge\hat{A}_{D_i}(c), \wedge\hat{C}_{D_i}(c), \vee\hat{E}_{D_i}(c), \vee\check{D}_{D_i}(c), \vee\hat{U}_{D_i}(c) > : c \in W\}$.

Now, we have

$\hat{A}_{D_i}(0) \geq \hat{A}_{D_i}(c)$, for all $c \in W$ and $i \in \Delta$

$\Rightarrow \wedge\hat{A}_{D_i}(0) \geq \wedge\hat{A}_{D_i}(c)$.

$\hat{C}_{D_i}(0) \geq \hat{C}_{D_i}(c)$, for all c ∈W and i∈Δ

$\Rightarrow \wedge\hat{C}_{D_i}(0) \geq \wedge\hat{C}_{D_i}(c)$.

$\hat{E}_{D_i}(0) \leq \hat{E}_{D_i}(c)$, for all c ∈W and i∈Δ

$\Rightarrow \vee\hat{E}_{D_i}(0) \leq \vee\hat{E}_{D_i}(c)$.

$\check{D}_{D_i}(0) \leq \check{D}_{D_i}(c)$, for all c ∈W and i∈Δ

$\Rightarrow \vee\check{D}_{D_i}(0) \leq \vee\check{D}_{D_i}(c)$.

$\hat{U}_{D_i}(0) \leq \hat{U}_{D_i}(c)$, for all c ∈W and i∈Δ

$\Rightarrow \vee\hat{U}_{D_i}(0) \leq \vee\hat{U}_{D_i}(c)$.

Further, we have

$\hat{A}_{D_i}(c * d) \geq \min\{\hat{A}_{D_i}((c * h) * d), \hat{A}_{D_i}(h)\}$, for all c, d, h∈W and i∈Δ.

$\Rightarrow \wedge\hat{A}_{D_i}(c * d) \geq \wedge\min\{\hat{A}_{D_i}((c * h) * d), \hat{A}_{D_i}(h)\}$

$= \min\{\wedge\hat{A}_{D_i}((c * h) * d), \wedge\hat{A}_{D_i}(h)\}$

$\Rightarrow \wedge\hat{A}_{D_i}(c * d) \geq \min\{\wedge\hat{A}_{D_i}((c * h) * d), \wedge\hat{A}_{D_i}(h)\}$.

$\hat{C}_{D_i}(c * d) \geq \min\{\hat{C}_{D_i}((c * h) * d), \hat{C}_{D_i}(h)\}$, for all c, d, h∈W and i∈Δ.

$\Rightarrow \wedge\hat{C}_{D_i}(c * d) \geq \wedge\min\{\hat{C}_{D_i}((c * h) * d), \hat{C}_{D_i}(h)\}$

$= \min\{\wedge\hat{C}_{D_i}((c * h) * d), \wedge\hat{C}_{D_i}(h)\}$

$\Rightarrow \wedge\hat{C}_{D_i}(c * d) \geq \min\{\wedge\hat{C}_{D_i}((c * h) * d), \wedge\hat{C}_{D_i}(h)\}$.

$\hat{E}_{D_i}(c * d) \leq \max\{\hat{E}_{D_i}((c * h) * d), \hat{E}_{D_i}(h)\}$, for all c, d, h∈W and i∈Δ.

$\Rightarrow \vee\hat{E}_{D_i}(c * d) \leq \vee\max\{\hat{E}_{D_i}((c * h) * d), \hat{E}_{D_i}(h)\}$

$= \max\{\vee\hat{E}_{D_i}((c * h) * d), \vee\hat{E}_{D_i}(h)\}$

$\Rightarrow \vee\hat{E}_{D_i}(c * d) \leq \max\{\vee\hat{E}_{D_i}((c * h) * d), \vee\hat{E}_{D_i}(h)\}$.

$\check{D}_{D_i}(c * d) \leq \max\{\check{D}_{D_i}((c * h) * d), \check{D}_{D_i}(h)\}$, for all c, d, h∈W and i∈Δ.

$\Rightarrow \vee\check{D}_{D_i}(c * d) \leq \vee\max\{\check{D}_{D_i}((c * h) * d), \check{D}_{D_i}(h)\}$

$= \max\{\vee\check{D}_{D_i}((c * h) * d), \vee\check{D}_{D_i}(h)\}$

$\Rightarrow \vee\check{D}_{D_i}(c * d) \leq \max\{\vee\check{D}_{D_i}((c * h) * d), \vee\check{D}_{D_i}(h)\}$.

$\hat{U}_{D_i}(c * d) \leq \max\{\hat{U}_{D_i}((c * h) * d), \hat{U}_{D_i}(h)\}$, for all c, d, h∈W and i∈Δ.

$\Rightarrow \vee\hat{U}_{D_i}(c * d) \leq \vee\max\{\hat{U}_{D_i}((c * h) * d), \hat{U}_{D_i}(h)\}$

$= \max\{\vee\hat{U}_{D_i}((c * h) * d), \vee\hat{U}_{D_i}(h)\}$

$\Rightarrow \vee\hat{U}_{D_i}(c * d) \leq \max\{\vee\hat{U}_{D_i}((c * h) * d), \vee\hat{U}_{D_i}(h)\}$.

Therefore, $\bigcap_{i\in\Delta} D_i$ is a PN-*Q*-ideal of *Q*-algebra W.

**Corollary 3.1.** Assume that Y = {(c, $\hat{A}_Y$(c), $\hat{C}_Y$(c), $\hat{E}_Y$(c), $\check{D}_Y$(c), $\hat{U}_Y$(c)) : c ∈W} be a PN-*Q*-ideal of a *Q*-algebra W. Then, Y is a neutrosophic BCK-ideal of the BCK-algebra W.

**Theorem 3.4.** Assume that Y = {(c, $\hat{A}_Y$(c), $\hat{C}_Y$(c), $\hat{E}_Y$(c), $\check{D}_Y$(c), $\hat{U}_Y$(c)) : c ∈W} be a PN-*Q*-ideal over a *Q*-algebra W. If c, h, d ∈ W such that c ∗ h ≤ d, then $\hat{A}_Y(c) \geq \min\{\hat{A}_Y(h), \hat{A}_Y(d)\}$, $\hat{C}_Y(c) \geq \min\{\hat{C}_Y(h), \hat{C}_Y(d)\}$, $\hat{E}_Y(c) \leq \max\{\hat{E}_Y(h), \hat{E}_Y(d)\}$, $\check{D}_Y(c) \leq \max\{\check{D}_Y(h), \check{D}_Y(d)\}$ and $\hat{U}_Y(c) \leq \max\{\hat{U}_Y(h), \hat{U}_Y(d)\}$.

**Proof.** Let Y = {(c, $\hat{A}_Y$(c), $\hat{C}_Y$(c), $\hat{E}_Y$(c), $\check{D}_Y$(c), $\hat{U}_Y$(c)) : c ∈W} be a PN-*Q*-ideal over a *Q*-algebra W. Suppose that c, h, d ∈W such that c * h ≤ d. Therefore, (c ∗ h) ∗ d = 0.

Now, we have

$\hat{A}_Y(c) = \hat{A}_Y(c * 0) \geq \min\{\hat{A}_Y((c * h) * 0), \hat{A}_Y(h)\}$

$= \min\{\hat{A}_Y((c * h)), \hat{A}_Y(h)\}$

$\geq \min\{\min\{\hat{A}_Y((c * d) * h), \hat{A}_Y(d)\}, \hat{A}_Y(h)\}$

$= \min \{\hat{A}_Y((c * h) * d), \hat{A}_Y(d), \hat{A}_Y(h)\}$
$= \min \{\hat{A}_Y(0), \hat{A}_Y(d), \hat{A}_Y(h)\}$
$= \min \{\hat{A}_Y(h), \hat{A}_Y(d)\}$
$\Rightarrow \hat{A}_Y(c) \geq \min \{ \hat{A}_Y(h), \hat{A}_Y(d)\}$.

$\hat{C}_Y(c) = \hat{C}_Y(c * 0) \geq \min \{\hat{C}_Y((c * h) * 0), \hat{C}_Y(h)\}$
$= \min \{\hat{C}_Y((c * h)), \hat{C}_Y(h)\}$
$\geq \min \{\min \{\hat{C}_Y((c * d) * h), \hat{C}_Y(d)\}, \hat{C}_Y(h)\}$
$= \min \{\hat{C}_Y((c * h) * d), \hat{C}_Y(d), \hat{C}_Y(h)\}$
$= \min \{\hat{C}_Y(0), \hat{C}_Y(d), \hat{C}_Y(h)\}$
$= \min \{\hat{C}_Y(h), \hat{C}_Y(d)\}$
$\Rightarrow \hat{C}_Y(c) \geq \min \{ \hat{C}_Y(h), \hat{C}_Y(d)\}$.

$\hat{E}_Y(c) = \hat{E}_Y(c * 0) \leq \max \{ \hat{E}_Y((c * h) * 0), \hat{E}_Y(h)\}$
$= \max \{ \hat{E}_Y((c * h)), \hat{E}_Y(h)\}$
$\leq \max \{ \max\{ \hat{E}_Y((c * d) * h), \hat{E}_Y(d)\}, \hat{E}_Y(h)\}$
$= \max\{ \hat{E}_Y((c * h) * d), \hat{E}_Y(d), \hat{E}_Y(h)\}$
$= \max\{ \hat{E}_Y(0), \hat{E}_Y(d), \hat{E}_Y(h)\}$
$= \max\{ \hat{E}_Y(h), \hat{E}_Y(d)\}$
$\Rightarrow \hat{E}_Y(c) \leq \max\{ \hat{E}_Y(h), \hat{E}_Y(d)\}$.

$\check{D}_Y(c) = \check{D}_Y(c * 0) \leq \max \{ \check{D}_Y((c * h) * 0), \check{D}_Y(h)\}$
$= \max \{ \check{D}_Y((c * h)), \check{D}_Y(h)\}$
$\leq \max \{ \max\{ \check{D}_Y((c * d) * h), \check{D}_Y(d)\}, \check{D}_Y(h)\}$
$= \max\{ \check{D}_Y((c * h) * d), \check{D}_Y(d), \check{D}_Y(h)\}$
$= \max\{ \check{D}_Y(0), \check{D}_Y(d), \check{D}_Y(h)\}$
$= \max\{ \check{D}_Y(h), \check{D}_Y(d)\}$
$\Rightarrow \check{D}_Y(c) \leq \max\{ \check{D}_Y(h), \check{D}_Y(d)\}$.

Further, we have

$\hat{U}_Y(c) = \hat{U}_Y(c * 0) \leq \max \{ \hat{U}_Y((c * h) * 0), \hat{U}_Y(h)\}$
$= \max \{ \hat{U}_Y((c * h)), \hat{U}_Y(h)\}$
$\leq \max \{ \max\{ \hat{U}_Y((c * d) * h), \hat{U}_Y(d)\}, \hat{U}_Y(h)\}$
$= \max \{ \hat{U}_Y((c * h) * d), \hat{U}_Y(d), \hat{U}_Y(h)\}$
$= \max \{ \hat{U}_Y(0), \hat{U}_Y(d), \hat{U}_Y(h)\}$
$= \max \{ \hat{U}_Y(h), \hat{U}_Y(d)\}$
$\Rightarrow \hat{U}_Y(c) \leq \max \{\hat{U}_Y(h), \hat{U}_Y(d)\}$.

**Theorem 3.5.** Let $Y = \{(c, \hat{A}_Y(c), \hat{C}_Y(c), \hat{E}_Y(c), \check{D}_Y(c), \hat{U}_Y(c)) : c \in W\}$ be a PN-*Q*-ideal over a *Q*-algebra W. If c, h $\in$W such that $c \leq h$, then $\hat{A}_Y(c) \geq \hat{A}_Y(h)$, $\hat{C}_Y(c) \geq \hat{C}_Y(h)$, $\hat{E}_Y(c) \leq \hat{E}_Y(h)$, $\check{D}_Y(c) \leq \check{D}_Y(h)$ and $\hat{U}_Y(c) \leq \hat{U}_Y(h)$.

**Proof.** Assume that $Y = \{(c, \hat{A}_Y(c), \hat{C}_Y(c), \hat{E}_Y(c), \check{D}_Y(c), \hat{U}_Y(c)) : c \in W\}$ be a PN-*Q*-ideal over a *Q*-algebra W. Suppose c, h be two elements of W such that $c \leq h$. Therefore, $c * h=0$.

Now, we have

$\hat{A}_Y(c) = \hat{A}_Y(c * 0) \geq \min \{\hat{A}_Y((c * h) * 0), \hat{A}_Y(h)\}$
$= \min \{\hat{A}_Y((c * h)), \hat{A}_Y(h)\}$

$= \min\{\hat{A}_Y(0), \hat{A}_Y(h)\}$
$= \hat{A}_Y(h)$

$\Rightarrow \hat{A}_Y(c) \geq \hat{A}_Y(h)$.

$\hat{C}_Y(c) = \hat{C}_Y(c * 0) \geq \min\{\hat{C}_Y((c * h) * 0), \hat{C}_Y(h)\}$
$= \min\{\hat{C}_Y((c * h)), \hat{C}_Y(h)\}$
$= \min\{\hat{C}_Y(0), \hat{C}_Y(h)\}$
$= \hat{C}_Y(h)$

$\Rightarrow \hat{C}_Y(c) \geq \hat{C}_Y(h)$.

$\hat{E}_Y(c) = \hat{E}_Y(c * 0) \leq \max\{\hat{E}_Y((c * h) * 0), \hat{E}_Y(h)\}$
$= \max\{\hat{E}_Y((c * h)), \hat{E}_Y(h)\}$
$= \max\{\hat{E}_Y(0), \hat{E}_Y(h)\}$
$= \hat{E}_Y(h)$

$\Rightarrow \hat{E}_Y(c) \leq \hat{E}_Y(h)$.

$\check{D}_Y(c) = \check{D}_Y(c * 0) \leq \max\{\check{D}_Y((c * h) * 0), \check{D}_Y(h)\}$
$= \max\{\check{D}_Y((c * h)), \check{D}_Y(h)\}$
$= \max\{\check{D}_Y(0), \check{D}_Y(h)\}$
$= \check{D}_Y(h)$

$\Rightarrow \check{D}_Y(c) \leq \check{D}_Y(h)$.

$\hat{U}_Y(c) = \hat{U}_Y(c * 0) \leq \max\{\hat{U}_Y((c * h) * 0), \hat{U}_Y(h)\}$
$= \max\{\hat{U}_Y((c * h)), \hat{U}_Y(h)\}$
$= \max\{\hat{U}_Y(0), \hat{U}_Y(h)\}$
$= \hat{U}_Y(h)$

$\Rightarrow \hat{U}_Y(c) \leq \hat{U}_Y(h)$.

**Theorem 3.6.** If $Y = \{(c, \hat{A}_Y(c), \hat{C}_Y(c), \hat{E}_Y(c), \check{D}_Y(c), \hat{U}_Y(c)) : c \in W\}$ is a PN-*Q*-sub-algebra over a *Q*-algebra W, then the sets $\alpha\text{-}\hat{A}_Y = \{c: c \in W, \hat{A}_Y(c) \geq \alpha\}$, $\alpha\text{-}\hat{C}_Y = \{c: c \in W, \hat{C}_Y(c) \geq \alpha\}$, $\alpha\text{-}\hat{E}_Y = \{c: c \in W, \hat{E}_Y(c) \leq \alpha\}$, $\alpha\text{-}\check{D}_Y = \{c: c \in W, \check{D}_Y(c) \leq \alpha\}$ and $\alpha\text{-}\hat{U}_Y = \{c: c \in W, \hat{U}_Y(c) \leq \alpha\}$ are the *Q*-sub-algebra of W.

**Proof.** Suppose that $Y = \{(c, \hat{A}_Y(c), \hat{C}_Y(c), \hat{E}_Y(c), \check{D}_Y(c), \hat{U}_Y(c)) : c \in W\}$ be a PN-*Q*-sub-algebra over a *Q*-algebra W. Therefore,

(i) $\hat{A}_Y(c * d) \geq \min\{\hat{A}_Y(c), \hat{A}_Y(d)\}$;
(ii) $\hat{C}_Y(c * d) \geq \min\{\hat{C}_Y(c), \hat{C}_Y(d)\}$;
(iii) $\hat{E}_Y(c * d) \leq \max\{\hat{E}_Y(c), \hat{E}_Y(d)\}$;
(iv) $\check{D}_Y(c * d) \leq \max\{\check{D}_Y(c), \check{D}_Y(d)\}$;
(v) $\hat{U}_Y(c * d) \leq \max\{\hat{U}_Y(c), \hat{U}_Y(d)\}$; where $c, d \in W$.

Let $c, d \in \alpha\text{-}\hat{A}_Y$. This implies, $\hat{A}_Y(c) \geq \alpha$, $\hat{A}_Y(d) \geq \alpha$.
Therefore, $\hat{A}_Y(c * d) \geq \min\{\hat{A}_Y(c), \hat{A}_Y(d)\} \geq \min\{\alpha, \alpha\} \geq \alpha$.
Hence, $\alpha\text{-}\hat{A}_Y = \{c: c \in W, \hat{A}_Y(c) \geq \alpha\}$ is a *Q*-sub-algebra of W.
Let $c, d \in \alpha\text{-}\hat{C}_Y$. This implies, $\hat{C}_Y(c) \geq \alpha$, $\hat{C}_Y(d) \geq \alpha$.
Therefore, $\hat{C}_Y(c * d) \geq \min\{\hat{C}_Y(c), \hat{C}_Y(d)\} \geq \min\{\alpha, \alpha\} \geq \alpha$.
Hence, $\alpha\text{-}\hat{C}_Y = \{c: c \in W, \hat{C}_Y(c) \geq \alpha\}$ is a *Q*-sub-algebra of W.
Let $c, d \in \alpha\text{-}\hat{E}_Y$. This implies, $\hat{E}_Y(c) \leq \alpha$, $\hat{E}_Y(d) \leq \alpha$.
Therefore, $\hat{E}_Y(c * d) \leq \max\{\hat{E}_Y(c), \hat{E}_Y(d)\} \leq \max\{\alpha, \alpha\} \leq \alpha$.

Hence, $\alpha$-$\hat{E}_Y$ = {c: c$\in$ W, $\hat{E}_Y(c) \leq \alpha$} is a *Q*-sub-algebra of W.

Let c, d $\in \alpha$-$\check{D}_Y$. This implies, $\check{D}_Y(c) \leq \alpha$, $\check{D}_Y(d) \leq \alpha$.

Therefore, $\check{D}_Y(c * d) \leq \max\{\check{D}_Y(c), \check{D}_Y(d)\} \leq \max\{\alpha, \alpha\} \leq \alpha$.

Hence, $\alpha$-$\check{D}_Y$ = {c: c$\in$ W, $\check{D}_Y(c) \leq \alpha$} is a *Q*-sub-algebra of W.

Let c, d $\in \alpha$-$\hat{U}_Y$. This implies, $\hat{U}_Y(c) \leq \alpha$, $\hat{U}_Y(d) \leq \alpha$.

Therefore, $\hat{U}_Y(c * d) \leq \max\{\hat{U}_Y(c), \hat{U}_Y(d)\} \leq \max\{\alpha, \alpha\} \leq \alpha$.

Hence, $\alpha$-$\hat{U}_Y$ = {c: c$\in$ W, $\hat{U}_Y(c) \leq \alpha$} is a *Q*-sub-algebra of W.

**Theorem 3.7.** Suppose that Y = {(c, $\hat{A}_Y(c)$, $\hat{C}_Y(c)$, $\hat{E}_Y(c)$, $\check{D}_Y(c)$, $\hat{U}_Y(c)$) : c $\in$W} be a PN-*Q*-ideal of W, then the sets W($\hat{A}$)={c$\in$W: $\hat{A}_Y(c)=\hat{A}_Y(0)$}, W($\hat{C}$)={c$\in$W: $\hat{C}_Y(c)=\hat{C}_Y(0)$}, W($\hat{E}$)={c$\in$W: $\hat{E}_Y(c)= \hat{E}_Y(0)$}, W($\check{D}$)={c$\in$W: $\check{D}_Y(c)= \check{D}_Y(0)$}, and W($\hat{U}$)={c$\in$W: $\hat{U}_Y(c)= \hat{U}_Y(0)$} are *Q*-ideals of W.

**Proof.** Suppose that Y = {(c, $\hat{A}_Y(c)$, $\hat{C}_Y(c)$, $\hat{E}_Y(c)$, $\check{D}_Y(c)$, $\hat{U}_Y(c)$) : c $\in$W} be a PN-*Q*-ideal of W. Therefore,

(i) $\hat{A}_Y(0) \geq \hat{A}_Y(c)$ & $\hat{A}_Y(c * d) \geq \min\{\hat{A}_Y((c * h) * d), \hat{A}_Y(h)\}$, for all c, d, h$\in$W;

(ii) $\hat{C}_Y(0) \geq \hat{C}_Y(c)$ & $\hat{C}_Y(c * d) \geq \min\{\hat{C}_Y((c * h) * d), \hat{C}_Y(h)\}$, for all c, d, h$\in$W;

(iii) $\hat{E}_Y(0) \leq \hat{E}_Y(c)$ & $\hat{E}_Y(c * d) \leq \max\{\hat{E}_Y((c * h) * d), \hat{E}_Y(h)\}$, for all c, d, h$\in$W;

(iv) $\check{D}_Y(0) \leq \check{D}_Y(c)$ & $\check{D}_Y(c * d) \leq \max\{\check{D}_Y((c * h) * d), \check{D}_Y(h)\}$, for all c, d, h$\in$W;

(v) $\hat{U}_Y(0) \leq \hat{U}_Y(c)$ & $\hat{U}_Y(c * d) \leq \max\{\hat{U}_Y((c * h) * d), \hat{U}_Y(h)\}$, for all c, d, h$\in$W.

Since, $\hat{A}_Y(0)=\hat{A}_Y(0)$, so 0 $\in$ W($\hat{A}$).

Since, $\hat{C}_Y(0)= \hat{C}_Y(0)$, so 0 $\in$ W($\hat{C}$).

Since, $\hat{E}_Y(0)= \hat{E}_Y(0)$, so 0 $\in$ W($\hat{E}$).

Since, $\check{D}_Y(0)= \check{D}_Y(0)$, so 0 $\in$ W($\check{D}$).

Since, $\hat{U}_Y(0)= \hat{U}_Y(0)$, so 0 $\in$ W($\hat{U}$).

Let (h $*$ d) $*$ e$\in$W($\hat{A}$) and d$\in$W($\hat{A}$). Therefore, $\hat{A}_Y((h * d) * e)=\hat{A}_Y(0)$ and $\hat{A}_Y(d)=\hat{A}_Y(0)$.

It is clear that $\hat{A}_Y(0) \geq \hat{A}_Y(h * e)$ (1)

Now, we have

$\hat{A}_Y(h * e) \geq \min\{\hat{A}_Y((h * d) * e), \hat{A}_Y(d)\}= \min\{\hat{A}_Y(0), \hat{A}_Y(0)\}= \hat{A}_Y(0)$

$\Rightarrow \hat{A}_Y(h * e) \geq \hat{A}_Y(0)$ (2)

From (1) and (2), we get

$\hat{A}_Y(h * e) = \hat{A}_Y(0)$.

This implies, h $*$ e$\in$ W($\hat{A}$). Therefore, the set W($\hat{A}$)={c$\in$W: $\hat{A}_Y(c)=\hat{A}_Y(0)$} is a *Q*-ideal of W.

Similarly, it can be shown that, the sets W($\hat{C}$)={c$\in$W: $\hat{C}_Y(c)= \hat{C}_Y(0)$}, W($\hat{E}$)={c$\in$W: $\hat{E}_Y(c)= \hat{E}_Y(0)$}, W($\check{D}$)={c$\in$W: $\check{D}_Y(c)= \check{D}_Y(0)$} and W($\hat{U}$)={c$\in$W: $\hat{U}_Y(c)= \hat{U}_Y(0)$} are *Q*-ideals of W.

**Theorem 3.8.** Assume that Y = {(c, $\hat{A}_Y(c)$, $\hat{C}_Y(c)$, $\hat{E}_Y(c)$, $\check{D}_Y(c)$, $\hat{U}_Y(c)$) : c $\in$W} be a PN-*Q*-ideal of *Q*-algebra W. Then, the fuzzy sets {(c, $\hat{A}_Y(c)$): c$\in$W}, {(c, $\hat{C}_Y(c)$): c$\in$W}, {(c,1- $\hat{E}_Y(c)$): c$\in$W}, {(c,1- $\check{D}_Y(c)$): c$\in$W}, {(c,1- $\hat{U}_Y(c)$): c$\in$W} are fuzzy *Q*-ideals of W.

**Proof.** Let Y = {(c, $\hat{A}_Y(c)$, $\hat{C}_Y(c)$, $\hat{E}_Y(c)$, $\check{D}_Y(c)$, $\hat{U}_Y(c)$) : c $\in$W} be a PN-*Q*-ideal of a *Q*-algebra W. Therefore,

(i) $\hat{A}_Y(0) \geq \hat{A}_Y(c)$ & $\hat{A}_Y(c * d) \geq \min\{\hat{A}_Y((c * h) * d), \hat{A}_Y(h)\}$, for all c, d, h$\in$W;

(ii) $\hat{C}_Y(0) \geq \hat{C}_Y(c)$ & $\hat{C}_Y(c * d) \geq \min\{\hat{C}_Y((c * h) * d), \hat{C}_Y(h)\}$, for all c, d, h$\in$W;

(iii) $\hat{E}_Y(0) \leq \hat{E}_Y(c)$ & $\hat{E}_Y(c * d) \leq \max\{\hat{E}_Y((c * h) * d), \hat{E}_Y(h)\}$, for all c, d, h$\in$W;

(iv) $\check{D}_Y(0) \leq \check{D}_Y(c)$ & $\check{D}_Y(c * d) \leq \max\{\check{D}_Y((c * h) * d), \check{D}_Y(h)\}$, for all c, d, h$\in$W;

(v) $\hat{U}_Y(0) \leq \hat{U}_Y(c)$ & $\hat{U}_Y(c * d) \leq \max\{\hat{U}_Y((c * h) * d), \hat{U}_Y(h)\}$, for all c, d, h$\in$W.

It is clear that, $\hat{A}_Y(0) \geq \hat{A}_Y(c)$ & $\hat{A}_Y(c * d) \geq \min\{\hat{A}_Y((c * h) * d), \hat{A}_Y(h)\}$, for all c, d, h∈W. Therefore, the fuzzy set $\{(c, \hat{A}_Y(c)): c \in W\}$ is a fuzzy *Q*-ideal of W.
It is clear that, $\hat{C}_Y(0) \geq \hat{C}_Y(c)$ & $\hat{C}_Y(c * d) \geq \min\{\hat{C}_Y((c * h) * d), \hat{C}_Y(h)\}$, for all c, d, h∈W. Therefore, the fuzzy set $\{(c, \hat{C}_Y(c)): c \in W\}$ is a fuzzy *Q*-ideal of W.
Now, for all c, d, h∈W,
$\hat{E}_Y(c * d) \leq \max\{\hat{E}_Y((c * h) * d), \hat{E}_Y(h)\} \Rightarrow 1- \hat{E}_Y(c) \geq \min\{1- \hat{E}_Y((c * h) * d), 1-\hat{E}_Y(h)\}$
and $\hat{E}_Y(0) \leq \hat{E}_Y(c) \Rightarrow 1- \hat{E}_Y(0) \geq 1- \hat{E}_Y(c)$.
Therefore, the fuzzy set $\{(c,1-\hat{E}_Y(c)): c \in W\}$ is a fuzzy *Q*-ideal of W.
Now, for all c, d, h∈W,
$\check{D}_Y(c * d) \leq \max\{\check{D}_Y((c * h) * d), \check{D}_Y(h)\} \Rightarrow 1- \check{D}_Y(c) \geq \min\{1- \check{D}_Y((c * h) * d), 1-\check{D}_Y(h)\}$
and $\check{D}_Y(0) \leq \check{D}_Y(c) \Rightarrow 1- \check{D}_Y(0) \geq 1- \check{D}_Y(c)$.
Therefore, the fuzzy set $\{(c,1-\check{D}_Y(c)): c \in W\}$ is a fuzzy *Q*-ideal of W.
Further, for all c, d, h∈W,
$\hat{U}_Y(c * d) \leq \max\{ \hat{U}_Y((c * h) * d), \hat{U}_Y(h)\} \Rightarrow 1-\hat{U}_Y(c) \geq \min\{1- \hat{U}_Y((c * h) * d), 1-\hat{U}_Y(h)\}$
and $\hat{U}_Y(0) \leq \hat{U}_Y(c) \Rightarrow 1- \hat{U}_Y(0) \geq 1- \hat{U}_Y(c)$.
Therefore, the fuzzy set $\{(c, 1- \hat{U}_Y(c)): c \in W\}$ is a fuzzy *Q*-ideal of W.

## 4. Conclusions:

In this paper, we have established the notion of PN-*Q*-ideals of PN-*Q*-algebra. By defining PN-*Q*-ideals, we have formulated some results on PN-*Q*-algebra from the point of view of neutrosophic set. It is just the beginning of the concept of PN-*Q*-algebra. In the future, we hope that based on the notions of PN-*Q*-ideals many new investigations can be done.

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# Comparative analysis of AHP, FAHP and Neutrosophic-AHP based on multi-criteria for adopting ERPS

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**Abstract:** Management business has successfully forced enterprises to rebuild its process and adopt technology that help in integrating all process across different departments, analysis information in real-time, improve decision-making. ERP is a key information system for these purposes. There are many criteria in choice ERPS based on enterprise and application. Hence, there are many consulting firms with huge number of experts and technicians in carrying out analysis, evaluation ERPs and supporting IT-department in enterprises in selecting suitable ERPS. As many systems are semi-similar in features or semi-suitable for specific organization which leads to confusing decision making. Hence, using Multi-criteria decision method (MCDM) is essential. Using decision-making tools doesn't mean missing data or information about what decision is made for. But sometimes more information creates a confusing decision as in this case-study. The case-study covers two main folds; it provides proposed criteria of ERPS adoption and studies their weights, then decision making process that is established by AHP, FAHP and Neutrosophic-AHP. It compares between the results of these approaches and measures the priority/weight effect of adding sub-criteria. This study provides a comparative analysis of AHP, FAHP and Neutrosophic-AHP. This paper contributes in emphasize the accuracy of Neutrosophic set in decision making. It also emphasizes on importance of using multi-criteria (criteria and factors) in designing decision model special in information system that have many factors for one aspect. The paper also contribute in ERPS field by providing criteria that help decision maker board in adopting ERPS cares on enterprise's culture, vision and business processes.

**Keywords**: ERPS, AHP, Fuzzy-AHP, Neutrosophic-AHP and MCMD.

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## 1. Introduction

The basic idea of an Enterprise Recourse Planning (ERP) platform is based on one of software engineering's trends. It is *"produce applications that help developers reduce the number of lines of code which are written by the developer until they reach the zero line of code point"* [1]. This evolution in software engineering leads the Enterprise Recourse Planning system (ERPS) to appear and grow. ERP architecture varies with the evolution of technology. As ERP is one of information system type, and information management is a critical element in any system whatever its activities [2]. Further, ERP is *"business process management software that allows an organization to use a system of integrated applications to manage the business and automate many back office functions"* [1]. In the last two decades, technical development has pushed enterprises, whatever its size to rethink their process management with respect to the new dynamics and changing in business environment, customer demands rising and market competition. The implementing an ERPs become a critical and essential step must be adopted by many businesses to help in organizing and optimizing the way they do business [3–6][7].

Enterprise architecture must have business elements, their relationship to each other and environments and principles that governing its design and evolution. Where the requirements of enterprises almost change based on customers, competitors and strategic targets. So its architecture reflects that. Because ERPS is a software solution for enterprise architecture and needs, so ERPS's architecture also developed to serve that. Many enterprises migrate their ERPS's architecture form monolith to service-oriented architecture (SOA) or to Microservices (MSA), or changed from SOA to MSA. Each architecture has characteristics that do not only reflect on ERPS's performance, but also in the enterprise repetition between competitors, business and enterprise targets. Thereby, the selection of architecture is not only based on its excellent.

The choice of ERP's vendor is not an easy mission. Thereby, decision support system (DSS) and decision making system (DMS) highlight their importance. DSS uses the analytical model and database to support semi-structured business decision that is made by the decision maker, while DMS analyzes alternatives based on factors to make a recommendation/decision instead of human. Multi-criteria decision making (MCDM) studies quantitative and qualitative characteristics of alternatives, and then assigning values to intangible and tangible aspects of decisions, and estimating decision based on better or worst calculated options. Models of decision making that are supported by the decision-making community are TOPSIS, MAUT, MAVT, ELECTRE, BWM, VIKOR, PROMETHEE, AHP and ANP [8], [9]. Analytic hierarchy process (AHP) is a broadly utilized tool for MCDM. It has been generally used in complex decision because of its high flexibility [10–14].

Criteria for adopting an ERP system and studying the consistency of these criteria are related to study a qualification of adopting a decision. This paper focuses on study factors that effect of adopting ERPS and related to make decision about architecture of system software. The paper proves the accuracy of using Neutrosophic-set in decision rather than Saaty and Fuzzy sets although Fuzzy and Neutrosophic are semi-close. Further, the paper proves that consistency of decision when supported with decision model uses factors and criteria rather than model uses only criteria. Thereby, the paper addressed these proves by a case study. The case study is handled in three main parts; analysis available alternatives by SWOT analysis then make a decision by using applying two models are illustrated in figures 1 and 2, finally testing consistency of decision and criteria by three different scale sets for AHP. This study is addressed in an empirical case study. Analysis part provides a comparison between the most professional platform solutions in ERP market; Odoo and Oracle e-Business Suite (EBS). They have same system architecture; SOA. The reason of choice these ERP systems are regarded to ERP industry, where Odoo is justified as the best open source ERP, and EBS is the main licensing ERP. These studies are visualized in SWOT analysis. Eventually, Odoo is excelling Oracle e-Business Suite in some features and Oracle does. The final choice of adopting one of them refers now to enterprise criteria and culture. So, case study proposes critical criteria for purchasing an ERP system based on non-profit, governmental enterprise with multi purposes, stakeholders and beneficiaries. This paper chooses AHP because it is one of methods that used in the selection decision. The paper applies AHP and its improved versions like FAHP and Neutrosophic approaches to grantee accuracy and consistency decision after declaring the technical features and measuring their relative values. As Neutrosophic is a development of Intuitionistic Fuzzy Sets (IFS) that outline precise and improving understanding of uncertainty [15]. The study recommends using it for the decision's consistency and accuracy.

This paper helps decision maker in enterprises and researches in decision making because of comparative analysis that is provided and proposed criteria of adopting ERPS.

The proposed criteria of adopting ERPS are produced in section 3, while the comparative analysis is addressed by a case study in section 4. Further, studding the consistency of criteria that used in this decision by three scale sets; Saaty, Fuzzy and Neutrosophic sets with AHP, also weights of alternatives (decision) are provided in decision section 5.

## 2. Literature review

Critical success factors (CSF) are defined as '*An area where an organization must perform well if it is to succeed'*. That means these factors enable enterprises to achieve its goals. CSF targets things that affect quality, customer satisfaction, increase revenues, decrease cost and market share. Effective performance measures helps in monitoring performance to detect whether it is meeting enterprise's goals, how well system is doing, degree of customer's satisfaction, and finally orient enterprise to take action that improve performance and efficiency [16]. The measurement is observation and quantification, while evaluation is a paired measurement with an observation of what would be desired, and comparison is putting two evaluations against each other [17]. Although performance measurement and evaluation are ensuring the successful implementation of information systems, also ERP model consists of data models, Critical Success Factor (CSF) models and phase models [18]. Evaluation ERP solutions in post-implementation phase is under-research [19].

In [8], [20] previewed some researches that discussed the relation between criteria of ERP selection and enterprise's size, and concluded that the size does not significantly affect criteria selection, but only on the judgment importance assigned in comparisons. For example, flexibility and supplier support are two first selection criteria in large-sized enterprise, however cost and adoptability are the most important criteria for small-medium sized enterprises. [18] Mapped the critical success factors of ERP successful implementation articles since 2002 until 2016 and classified all these factors into four main classes: Organization-related, Customization of ERP, Project-related, and Individual-related. [21] Studied different roles and participations of ERP's users with factors that effect on their missions via a comparison between four companies with different industrial fields used ERP to solve problems but unfortunately, they gained new problems. [22] Mentioned what CSF means, and all different CSF's factors from 2003 to 2010. In [23] handles the classification of ERP implementation strategies (organization, technology and people), the context and conceptual model of ERP system implementation and separate between them.

All these models are not handled criteria and factors for selecting ERPS that fit enterprise's culture and strategic targets. Section 3 addressed this gap by proposing these criteria and studying their consistencies in section 4 by real case study in industry field.

the selection an ERP system is a nightmare for software consultant, system architect and enterprise managers (chief executive officer (CEO), chief financial officer (CFO), chief human resources officer (CHRO), general manager (GM), and marketing manager) due to its importance. Decision making is selecting the most suitable among multiple and convergent alternatives keeping in sight the heterogeneous decision criterion, objectives and priorities of decision maker [24]. Decision making is very important at strategic-level management. Therefore, Difficulty of decision making is a motivation for developing many approaches and tools not only to support a decision but also making it. Multi-criteria decision making (MCDM) aims to provide a model for decision problems by capturing and addressing both qualitative and quantitative characteristics of alternatives, then assigning numerical values to intangible aspects inherent to decisions, and estimating better or worst options that have difficult cost and benefits relationships.

In [8] use AHP to measure nine criteria for small-size enterprise are concluded from seven selection criteria models. In [25] used AHP with four criteria and 12subcriteria for assessing the suitability of the existing waste landfill in Zanjan, Iran. It combines AHP and Geographic information system to build suitability assessment model. This model is recommended to use in reevaluating the suitability of any old operating reservoir such as heavy industrial tanks, oil reservoirs, landfills. [26], [9] Used the criteria of updated DeLone & McLean of success IS model, apply hybrid MCDM process (AHP and TOPSIS) on it to detect that service quality is a best criterion (with its sub-criteria: on time delivery, knowledge and competency, error network, availability, access, rate delay and reliability) for two different IS in banking and construction industry sector. [19] after listed evaluation models from 1999 to 2011 it modified to updated D&M model in 2004, it proposed 23 criteria of ERP in post-implementation and 111 experts ranked them with important, essential, important but not essential. [27] Studies the correlation between the results of fuzzy-ANP and classical-ANP for software security assessment and proves that they are highly correlated. That was a motivation to apply hybrid fuzzy-ANP-TOPSIS method to get better results in decision problems in case of the uncertain and imprecise information. In spite of fuzzy-ANP-TOPSIS results, but this study recommended that "for software security assessment issue, as it complex and dynamic task faced by both developers and users, there may be better MCDM symmetrical techniques rather than Hybrid fuzzy-ANP-TOSIS".

Fuzzy sets were used with MCDM methods like in AHP to reduce uncertainty. However, it does not solve this kind of problems in decision making. Saaty and et al. dose not support fuzzy-AHP because AHP is fuzzy by itself. Neutrosophy is the origin of Neutrosophic which is care neutral (indeterminate/unknown) part as in philosophy. Its components are T, I, F. they are representing the membership (truth), indeterminacy (intermediate) and non-membership (false) values respectively. Each element in Neutrosophic set has three components which are considers a subset, contrary all other types of sets as in fuzzy set, its three component are numbers [28]. Neutrosophic set is more general than other set as fuzzy and thereby Saaty set. Neutrosophic set is more reliable in judgment and pairwise comparison for criteria and alternative especial in Multi-Criteria Group

Decision Making (MCGDM). Neutrosophic is more suitable for dealing with high degree of imprecision and incomplete information [14]. Neutrosophic set provides accurate values in decision rather than Saaty and fuzzy sets [8], [9], [29], [30].

## 3. A Proposed decision model of adopting ERPS

'Which ERP system is enterprise purchase?' This question is synonym to adopting an ERP system decision. Where there are many ERP's vendors with semi-different features. The proposed criteria of ERPS selection form is illustrated in figure 1. the proposed criteria form for purchasing ERP system that combines all desired features and nature of purchasing system process are: 1- trust vendor, 2- Support different Technical platform (on-premises, on-cloud, mobility, OS (Windows & Linux)), 3- Vendor package (deployment, recovery, training staff, maintenance and customization), 4- Low Total costs (ownership licenses, service/support, implementation, training staff cost, deployment, maintenance, consultancy and customization), 5- Upper management support, 6- Accuracy, 7- Availability, 8- Risk management and security, 9- Support different language ( Arabic and English is essential), 10- Database independency.

These criteria are ranked by experts. Experts are IT-staff, academic researchers, project management manager, external technicians and key-users in different enterprises. It designed based on the results of previous questionnaire, where the average of criterion's importance is calculated, and then criteria with average value less than 80% is eliminated. Essential vector is numbered with 8/10, more important but not essential is numbered with 5/10 and important is numbered with 3/10. The high ratio 80% is detected because selecting ERPS that supports its culture, vision and strategic goal is not easy mission. Aforementioned, Neutrosophic excels on fuzzy and saaty, thereby the Neutrosophic-AHP is suggested to use in making decision of adopting ERPS to grantee an accurate decision. Steps of Neutrosophic-AHP are illustrated in figure 1.

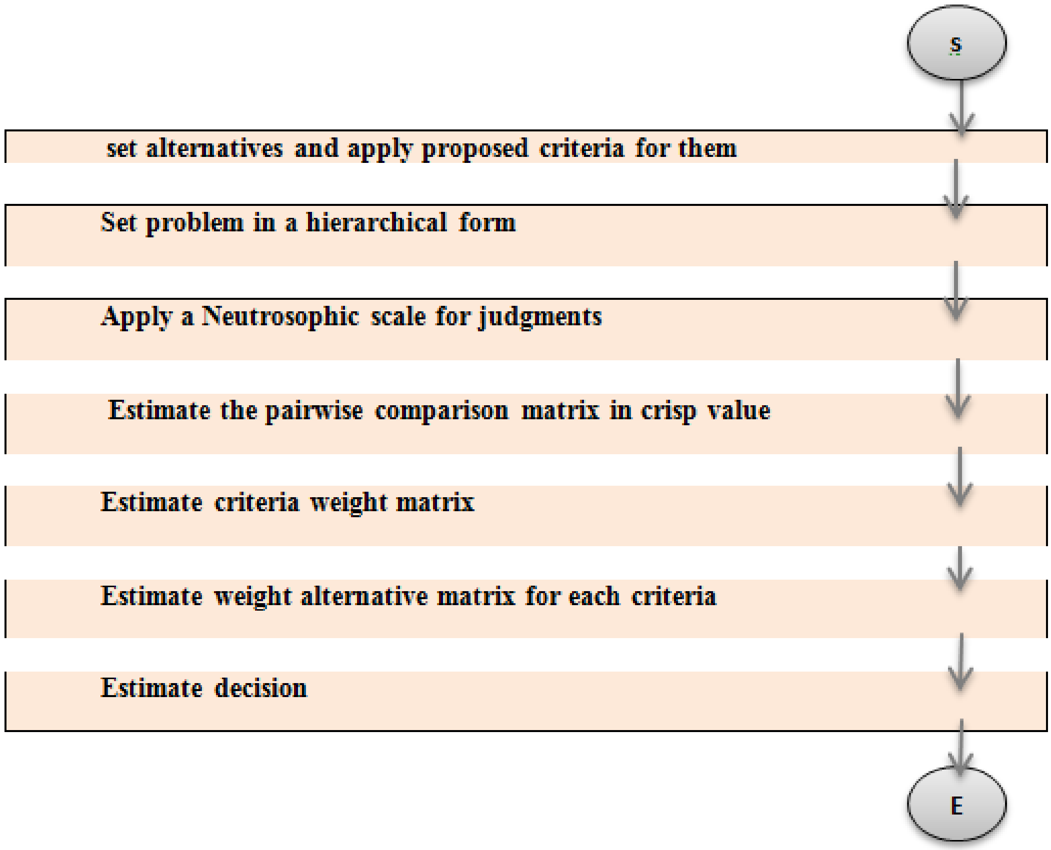


*Figure 1 flowchart of recommended set (Neutrosophic Set) for proposed model*

To prove the accuracy of proposed decision model of adopting ERPS with Neutrosophic-AHP and consistency of these criteria, the next section provides case study for applying a proposed decision model with ten criteria and 15 factors by AHP, FAHP and Neutrosophic-AHP. Briefly, the case study provides a comparative analysis and discusses an accuracy level of decision for using Neutrosophic-AHP and factors for criterion.

Steps of applying AHP [12–14] are briefly previewed in figure 2. They are

1- **Set problem in a hierarchical form**
2- **Estimate the pairwise comparison matrix**
3- **Estimate normalize pairwise comparison criteria matrix:** By Get summation of each column$\sum_{j=1}^{n} a_{ij}$. Then, divide each value in a pairwise comparison matrix to previous summation, final equation is: $C_{ij=\frac{a_{ij}}{\sum_{j=1}^{n} a_{ij}}}$ (1)
4- **Estimate weight criteria matrix:** By: calculate average value for each row
$$W_i = \frac{\sum_{j=1}^{n} C_{ij}}{n} \quad (2)$$
5- **Confirm values of weight criteria is standard by estimate consistency index (CI), consistency ratio (CR)** By: Estimate consistency from following equation

$$Consistency_{ij} = \frac{\sum_{j=1}^{n}(W_{i1} \times a_{ij})}{W_{i1}}, \text{ Then,} \quad (3)$$

calculate $\lambda_{max} = \frac{\sum Consistency_{ij}}{n},$ (4)

$$CI = \frac{\lambda_{max} - n}{(n-1)}, \qquad (5)$$

CR = CI / RI where RI is random consistency index value that detected based on a random index's table [29], [32]

6- **Repeat same steps 2, 3, 4 5 for each alternative based on each criteria to get priority weight for alternative and confirm from its consistency by estimating CR.** By: Estimate pairwise comparison matrix with same Saaty scale table, and normalized pairwise matrix, then criteria/priority weight, Estimate $\lambda_{max}$, CI and CR.
7- **Make a decision** By: Calculate decision weight by the summation of Product criteria weight matrix with alternative priority weight matrix according to the following equation:

$$D_{iv} = \sum_{i=1}^{j} CriteriaWeight_i \times PeriorityWeight_{ij} \qquad (6)$$

The biggest value of decision weight is the most suitable alternative for these criteria.

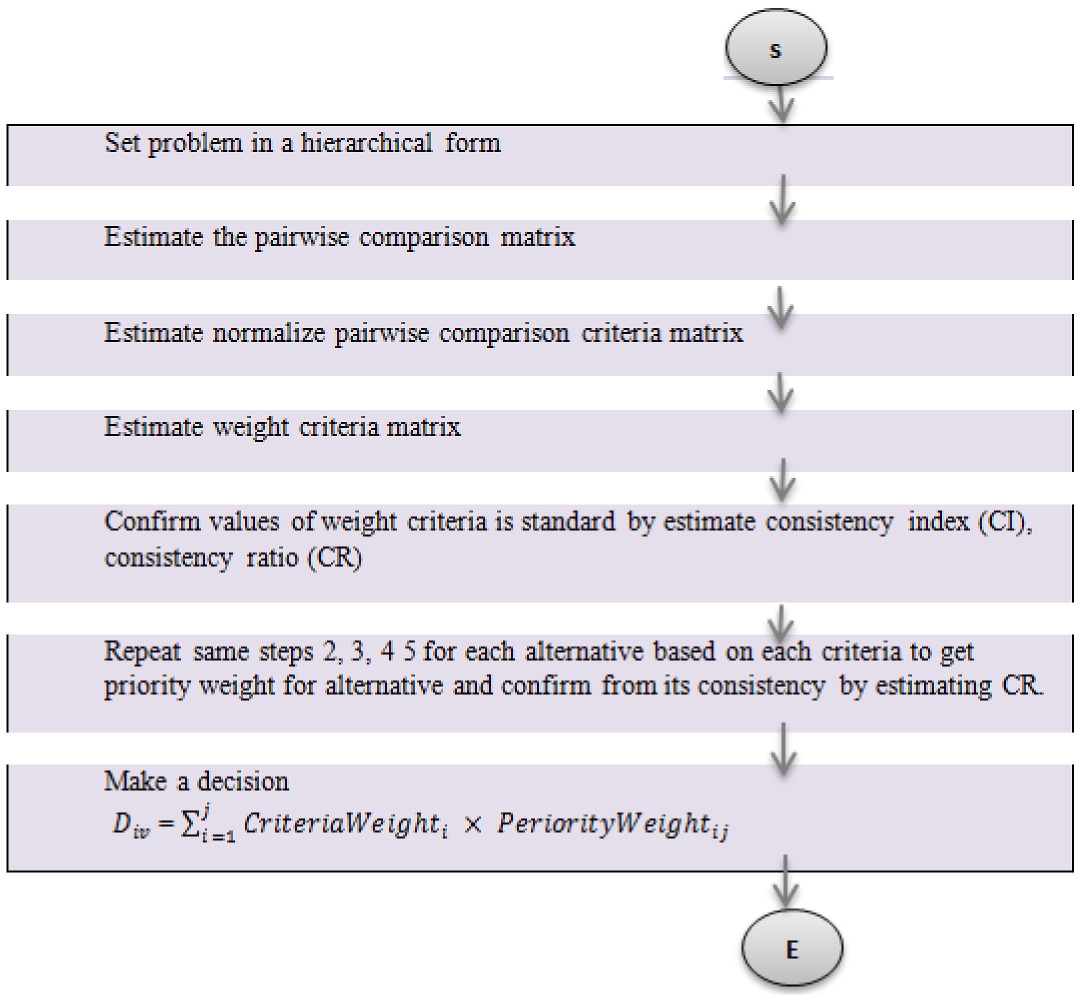


*Figure 2 flowchart of AHP steps*

Further, Steps of using FAHP [10], [11], [33] are

1- The first step is the same step in AHP except using a Fuzzy triangular scale table as in table 1.

2- **Estimate the pairwise comparison matrix ,** By: (<u>Note</u>: based on our criteria i and j = 10, matrix size= 10×10), use the same rule in step 2 in AHP except replace crisp values with fuzzy set values [10]

$$a`_{ij} = (1,1,1) \text{ when i=j} \quad (7)$$

$$a_{ij} = fuzzy\ set\ relevant\ value\ in\ fuzzy\ triangular\ table\ (L, m, u) \text{ when i≠j}$$

$$a`_{ji} = \frac{1}{a_{ij}} \quad (8)$$

After calculating the average of evaluation values for three judgments and apply rules of $a`_{ij}$, Hence, the pairwise comparison matrix in fuzzy form is created.

3- **Estimate criteria weight matrix,** By: Calculate geometric means for value as following equation:

$$ŕ = \prod_{j=1}^{n} á_{ij} \quad (9)$$

then, calculate the fuzzy weight by equation:

$$W`_i = r`i \otimes (r`1 \oplus r`_2 \oplus \ldots \oplus r`_n) \quad (10)$$

And, calculate a crisp weight by equation

$$W_i = \sum Lw_i, m_{w_i}, u\, w_i / n \quad (11)$$

Also, check the weight is normalized or not by summation all weights, if equal one it is true, else it false.

4- **Estimate weight alternative matrix for each criteria,** By: the repeat same steps in 2& 3 for alternatives after converting crisp values of table in step 6 in AHP.

5- **Estimate decision,** By: Calculate decision weight by the summation of Product criteria weight matrix with alternative priority weight matrix according to the following equation:

$$D_{iv} = \sum_{i=1}^{j} CriteriaWeight_i \times PeriorityWeight_{ij} \quad (12)$$

The biggest value of decision weight is the most suitable alternative for these criteria. These steps are summarized in figure 3.

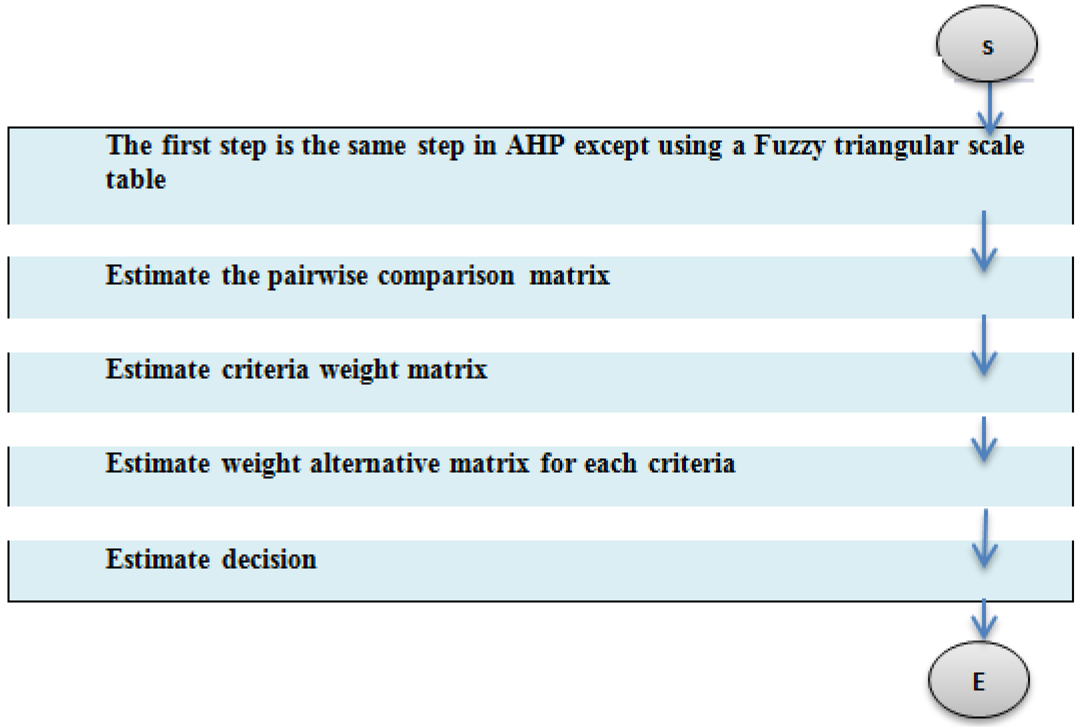


*Figure 3 flowchart of FAHP steps*

While steps of Neutrosophic-AHP [14], [15], [34] are illustrated briefly in figure 1. They are

1- The first step in AHP and FAHP is also shared with this approach, except using a triangular Neutrosophic scale in table 1.

2- **Estimate the pairwise comparison matrix in crisp value,** By: [34] (Note: based on our criteria i and j = 10, matrix size= 10×10), use the same rule in step 2 in AHP except replace values with Neutrosophic set values in table 1.

$$=< (1,1,1), 0.5, 0.5, 0.5 > \text{ when i=j} \qquad (13)$$

*these values are fuzzy set relevant value in fuzzy triangular table* $(L, m, u)$ , and T is the truth-membership, I is indeterminacy, and F is falsity membership functions of Neutrosophic set. So, pairwise-comparison matrix with Neutrosophic values is created.

To convert values of Neutrosophic form to crisp value , use the following equation:

$$s(r_{ji)} = \left|\left(l_{ij} \times m_{ij} \times u_{ij}\right)^{\frac{T_{ij}+I_{ij}+F_{ij}}{9}}\right| \quad \text{when } i \neq j \qquad (14)$$

After calculating the average of evaluation values for three judgments and applying rules of$a`_{ij}$, hence, pairwise comparison matrix in crisp values is created.

3- **Estimate criteria weight matrix,** By: Calculate weight matrix as the following equations:

(Calculate each column, then divide the previous crisp value by each summation column)

$$W_i^m = \frac{w_i}{\sum_{i=1}^{m} w_i} \quad where\ i = 1,2, \ldots m$$

Then, (calculate row average to get final criteria weight)

$$W_i = \frac{\sum_{j=1}^{m}(X_{ij})}{n} \text{ i= 1, 2, … m; j= 1,2 3 …n} \quad (15)$$

Then calculate the total summation of weight, when it equals to 1 that means they are normalization of weights. After that, check consistency of weights by calculating consistence index (CI) and consistence Ratio (CR).

4- **Estimate weight alternative matrix for each criteria,** By: repeat same steps in 2& 3 for alternatives after converting crisp values of table in step 6 in AHP, the pairwise-comparison matrix with Neutrosophic values is created.
5- **Estimate decision,** By: Calculate decision weight by the summation of Product criteria weight matrix with alternative priority weight matrix according to the following equation:

$$D_{iv} = \sum_{i=1}^{j} CriteriaWeight_i \times PeriorityWeight_{ij} \quad (16)$$

The biggest value of decision weight is the most suitable alternative for these criteria.

All equations, that are used in previous steps of AHP, FAHP and Neutrosophic-AHP, are listed in mentioned references.

## 4. An empirical application for a proposed model - Case study

'ISLAH Charitable Foundation' is a non-profit distributed enterprise in EGYPT, it starts building its management information systems. The ERP market is studied to select one fit its culture (non-profit and social organization), its vision and multiply-purposes.

Based on these criteria, Odoo13 and oracle e-business suite (EBS) are candidates. Because Odoo is an open source suite of integrated business applications with most popular open source ERP rank in 2016. While Oracle E-Business Suite is an integrated business applications enable organizations to improve decision making, and increase corporate performance. To detect which one of them is suitable. The trade-off is considered as a decision analysis and a pre-step of making a decision. The decision analysis is represented in SWOT analysis for both as declared in appendix A [36-43]. Unfortunately, this analysis caused confusion. More information and more data do not mean making a decision, but support decision-making and sometimes decision maker's confusion as in this case. That was a motivation for using decision-making tools and put structured steps for making a consistent and accuracy decision.

Analytic Hierarchy Process (AHP) is proposed to select ERP system, where it is used in many applications in project management, risk estimation, evaluation of knowledge management tools and ERPs selection [31]. To get accurate and consistent decision, the trusted decision is measured by AHP, Fuzzy AHP (FAHP) and Neutrosophic-AHP with three different scale sets that are declared in table 1 and table 2 provides random consistency index that used in consistency calculation. These approaches structure the decision problem into objective, alternatives and criteria. Regarding to the case study, the objective is purchasing a suitable ERPS, alternatives are Odoo 13 system and Oracle E-business Suite and ten criteria that are declared in previous form in section 4.1. Section 4.2 provides a comparative analysis for AHP, FAHP and Neutrosophic-AHP with using multi-criteria; ten criteria and 15 factors (sub-criteria). These sections study accuracy decisions with three different sets, and with using multi-criteria instead of only criteria. Also these sections proves consistence of adopting criteria in proposed decision model. Further, these sections discuss the proposed recommendation of using Neutrosophic-AHP in proposed decision model.

*Table 1: three Scale Set for AHP, FAHP and Neutrosophic – combined from* [10], [15], [32]

| Saaty Scale | Explanation | Fuzzy triangular scale | Neutrosophic triangular scale |
|---|---|---|---|
| 1 | Equally significant | (1, 1, 1) | <<1,1,1>; 0.50, 0.50, 0.50> |
| 3 | slightly significant | (2, 3, 4) | <<2, 3, 4>; 0.30, 0.75, 0.70> |
| 5 | String significant | (4, 5,6) | <<4, 5, 6>; 0.80, 0.15, 0.20> |
| 7 | Very strong significant | (6, 7, 8) | <<6, 7, 8>; 0.90, 0.10, 0.10> |
| 9 | absolutely significant | (9, 9, 0) | <<9, 9, 0>; 1.00, 0.00, 0.00> |
| 2 | | (1, 2, 3) | <<1, 2, 3,>; 0.40, 0.60, 0.65> |
| 4 | | (3, 4, 5) | <<3, 4, 5>; 0.35, 0.60, 0.40> |
| 6 | Sporadic values between two | (5, 6, 7) | <<5, 6, 7>; 0.70, 0.25, 0.30> |
| 8 | Close scale | (7, 8, 9) | <<7, 8, 9>; 0.85, 0.10, 0.15> |

*Table 2: part of Random consistency index that listed in* [29]

| N | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| RI (random index) | 0.0 | 0.0 | 0.58 | 0.90 | 1.12 | 1.24 | 1.32 | 1.41 | 1.45 | 1.49 | 1.52 | 1.54 | 1.56 | 1.58 | 1.59 |

**4.1 Making decision by AHP, fuzzy-AHP (FAHP) and Neutrosophic-AHP:**

The decision problem is visualized in hieratical form, as in figure 4 that represents goals, alternatives and criteria at levels. The decision with Saaty set and AHP approach recommended Odoo 13 rather than EBS.

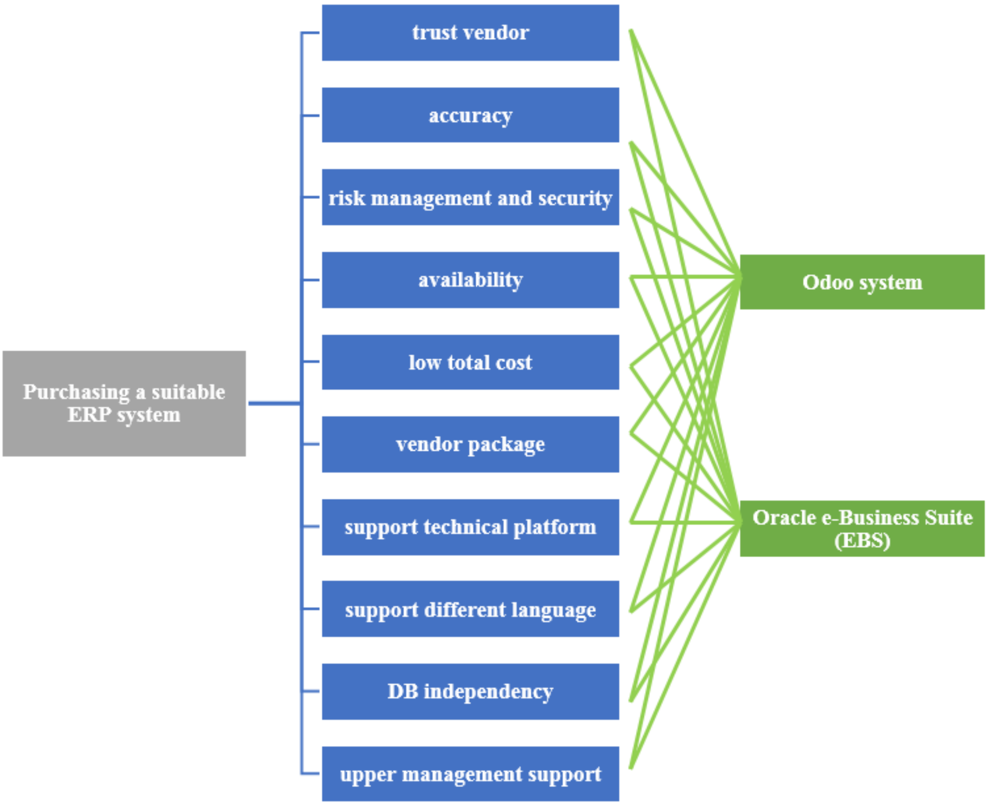


*Figure 4 Hierarchical model of purchasing decision for AHP with 10 criteria only*

Because of the complexity and uncertainty of real decision-making problems, decision makers often find that it's more realistic to assign linguistic variables to judgments rather than fixed values. Hence, presenting data using fuzzy numbers is more appropriate instead of crisp numbers [31]. Hence fuzzy-AHP is an improved version of AHP.

There are many methods to conclude priority vector such as the extent analysis method (EAM), tolerance deviation, entropy concepts, Lambda-Max method, eigenvector method, fuzzy preference programming and Fuzzy LinPreRa. The most widely applied and popular is EAM but unfortunately weights from a fuzzy comparison matrix cannot be estimated correctly. This paper uses geometric means to estimate priority vector (fuzzy weight) because it is more accurate and consistency ratio in EAM are produced after the evaluation process, this led decision makers to find it difficult to ensure continuous comparison of decisions. In addition to it requires n(n-1)/2 of pairwise comparisons [11], [31].

After applying steps of decision making by using Neutrosophic-set, the decision of using Neutrosophic-AHP is semi-agrees with FAHP, but there is high gap between AHP and both FAHP and Neutrosophic-AHP. Weights of using Odoo by AHP, FAHP and Neutrosophic-AHP are 27%, 40%, 46% respectively. While for EBS with same order of different set of AHP are 63%, 60% and 54%. Thereby, the decision stills confuses.

**4.2 Decision by using multi-criteria and AHP, fuzzy-AHP (FAHP) and Neutrosophic-AHP**

In the previous section, decision is estimated by AHP, FAHP and Neutrosophic-AHP for ten criteria, this section estimates decisions by same three scale set and AHP approach, but with sub-criteria for some of the criteria as a method to measure the effect of using sub-criteria in the decision. Here, the hierarchy of decision is at four levels; where both levels three and four for criteria and its sub respectively. Infrastructure platform and operating system are sub-criteria for 'support different technical platform' criteria. Continuous deployment, recovery, training staff, maintenance and continuous integration are sub-criteria for 'vendor package' criteria. Ownership licenses, services, implementation, consultancy, deployment and customization are sub-criteria for 'low total cost' criteria. Support different language has Arabic and English sub-criteria. Hence, the hierarchy of purchasing decision with criteria and sub-criteria are visualized in below figure 5.

Same steps of Steps of AHP, FAHP and Neutrosophic-AHP approaches in section 4.1 are applied in respectively to calculate decision. The final equations 6, 12 and 16 are applied to get values of recommendation for both alternatives. The steps are same for three approaches as declared in figures 1, 2 and 3 but the equations are different because of used scale set. Thereby, equations 13, 14 and 15 for in Neutrosophic-AHP are different on equations 7, 8 and 9 for FAHP and equations 1, 2 and 3 for AHP.

Decision's Weights of using Odoo system by AHP, FAHP and Neutrosophic-AHP are 46%, 44% and 45% in respective. While for EBS are 54%, 56% and 55% in respective. Values of these decisions are more realistic than are listed in section 4.1. This proves that, using multi-criteria (criteria and its factors) make decision more accurate and realistic.

The weights of decision with Neutrosophic-AHP with criteria model and multi-criteria model is very approximate rather than in AHP and FAHP for two cases. A decision with Neutrosophic-AHP in 10 criteria case and 10 criteria and 15 sub-criteria are 46% and 45% for Odoo, while for EBS are 54% and 55%. However, a decision with FAHP in 10 criteria case and 10 criteria and 15 sub-criteria are 60% and 44% for Odoo, while for EBS are 40% and 56%. Furthermore, a decision with AHP in 10 criteria case and 10 criteria and 15 sub-criteria are 27% and 46%, while for EBS are 63% and 54%. That proves that using Neutrosophic-AHP provide accuracy and consistency decision rather that AHP and FAHP.

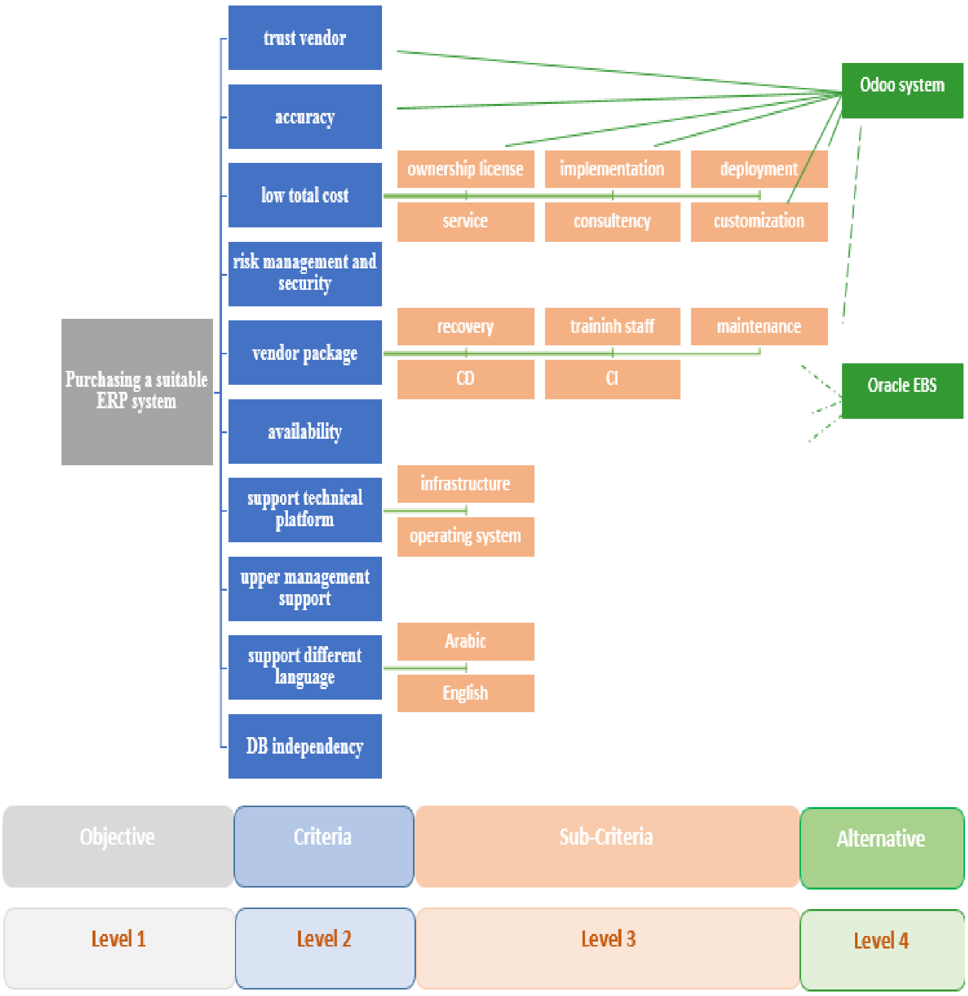


*Figure 5 a hierarchy model of purchasing ERP system decision with 10 criteria and 15 factors (sub-criteria)*

## 5. Results and discussion:

### 5.1 Choosing optimal method of MCDM

Firstly, from these three approaches' estimations, **AHP** ranked **Odoo** decision with 0.63 while **EBS** ranked with 0.27. Also, **FAHP** recommend **Odoo** system with 0.54 value while **EBS** ranked with 0.46 value. **Neutrosophic-AHP** get a decision on purchasing **Odoo** system 0.56 while the decision of purchasing **EBS** system gets 0.44 It is noted that the values of ranking Odoo system by three approaches is higher than EBS rank. Hence, the decision is purchasing Odoo system.

The weights of decision with Neutrosophic-AHP with criteria model and multi-criteria model is very approximate rather than in AHP and FAHP for two cases. A decision with Neutrosophic-AHP in 10 criteria case and 10 criteria and 15 sub-criteria are 46% and 45% for Odoo, while for EBS are 54% and 55%. However, a decision with FAHP in 10 criteria case and 10 criteria and 15 sub-criteria are 60% and 44% for Odoo, while for EBS are 40% and 56%. Furthermore, a decision with AHP in 10 criteria case and 10 criteria and 15 sub-criteria are

27% and 46%, while for EBS are 63% and 54%. That proves that using Neutrosophic-AHP provide accuracy and consistency decision rather that AHP and FAHP.

Secondly, three approaches that are used in selection decision provide same decision with different recommendation values. These results have different preference distributions despite having the same initial input from same decision makers and all used approaches agreed on the same goal. The different Scale set value of AHP method is the reason to different values for each alternative. To answer question "Which one of AHP, FAHP or Neutrosophic-AHP is accurate approach?" there are three opinions. (1) One of them is 'CI and CR measures are used to prove the consistency of decision maker preferences [12]', but CI and CR already estimated in each approach for sure that criteria's weights and alternatives' weights are consistent, so the decision for all approach is consistent. (2) Another answer is "different judgment scales are influencing the results and decisions [12]". This case study, using different scale set values, i.e. Saaty scale, triangular scale and Neutrosophic scale and they effects on stability of decision's weights in case of comparing between three values of AHP, FAHP and Neutrosophic-AHP for two alternatives. Decisions with recommendations round 63%, 54% and 56% for Odoo and 27%, 46%, and 44% for EBS with small disparity. (3) Another answer is 'using the Spearman's correlation coefficient index [13], [35]'. A Spearman's coefficient for the case study is estimated by using weights for criteria and alternatives, then ascending them order, set ranks and apply coefficient equation: $\rho = \frac{6\sum d_i^2}{n(n^2-1)}$ (where n in case study =10). For AHP, Spearman's coefficient for Odoo and EBS is same value, it equals to 0.984. For Fuzzy-AHP, Spearman's coefficient for Odoo equals to 0.975 while for EBS equals to 0.972. They are very close, where 0.003 is the disparity between two decisions in the same method. For Neutrosophic-AHP, Spearman's coefficient for Odoo equals to 0.95 while for EBS equals to 0.18. Based on values of Spearman's coefficient that are estimated for three methods; Neutrosophic set is more accurate than AHP and FAHP, but same coefficient not prove that AHP has same accuracy that FAHP has, and that conflict with many literatures that documented other that. All these correlation coefficient values are limited in the closed period [0.7, 1], that means that a strong direct correlation for all. Also, it provides values are very close for a different approach. For example, 0.012 is the difference value between an Odoo decision by AHP and FAHP. The final answer of which scale set is accurate rather other, this case study proved is '*Neutrosophic-set is the most accurate, therefore, Neutrosophic-AHP is more accurate and consistence rather than AHP and fuzzy-AHP*'.

**5.2 Effect of using sub-criteria on decision's accuracy:**

**Priority Criteria and decision consistency between criteria's levels:**

Basically, Criteria weights for criterion based its sub-criteria are calculated by average weights of sub-criteria, the next table previews difference value that main criteria get before and after estimating weights of its sub. (The importance of criteria is calculated by the average of its sub-criteria. The importance of criterion that has only two sub-criteria does not give a real value as it is seen in table 3).

*Table 3 weights of criteria that have sub-criteria (factors)*

| Criteria | Weight | Weight | Weight | Weight | Weight score | Weight score |
|---|---|---|---|---|---|---|

| that have sub criteria | score by AHP (without sub-criteria) | score by AHP (with sub-criteria) | score by FAHP (without sub-criteria) | score by FAHP (with sub) | by Neutrosophic-AHP (without sub) | by Neutrosophic-AHP (with sub) |
|---|---|---|---|---|---|---|
| Support different technical platform | 18% | 50% | 17% | 50% | 15% | 50% |
| Vendor package | 21% | 20% | 20% | 20% | 14% | 20% |
| Low total cost | 9% | 17% | 9% | 16% | 9% | 17% |
| Support different language | 3% | 50% | 3% | 50% | 6% | 50% |

In comparison criteria's rank and its importance, decision score between using one level of criteria and two levels of them (sub-criteria), to see the number of criteria's level effect on decision's quality, below tables 4 and 5 also figures 6:9 show that how factors of criterion adjust weight criteria and its consistency. Tables 4 and 5 preview how the importance of criteria is changed when sub-criteria (factors) are used in decision model. That shows the effect of sub-criteria on criterion's weight and therefore decision. Table 4 lists the criteria with its weight and rank between whole proposed criteria. The weight's criterion regards its weight. While table 5 shows how same criterion's importance is different when used factors for it. This difference reflects of alternatives' weights and final decisions

*Table 4 importance and rank of 10 criteria*

| Criteria | AHP | | FAHP | | Neutrosophic-AHP | |
|---|---|---|---|---|---|---|
| | importance | Rank | importance | Rank | importance | Rank |
| **Trust vendor** | 17% | 3 | 18% | 2 | 14% | 2 |
| **Support different Technical platform** | 18% | 2 | 17% | 3 | 15% | 1 |
| **Vendor package** | 21% | 1 | 20% | 1 | 14% | 2 |
| **Low total costs** | 9% | 5 | 9% | 5 | 9% | 5 |
| **Upper management support** | 8% | 6 | 8% | 6 | 9% | 5 |
| **accuracy** | 8% | 6 | 9% | 5 | 10% | 4 |
| **Availability** | 10% | 4 | 10% | 4 | 11% | 3 |
| **Risk management and security** | 3% | 7 | 4% | 7 | 6% | 6 |
| **Support different language** | 3% | 7 | 3% | 8 | 6% | 6 |
| **Database independency** | 2% | 8 | 3% | 8 | 5% | 7 |

*Table 5 importance of criteria that have sub-criteria for AHP, FAHP and Neutrosophic-AHP*

| Criteria and its sub-criteria | AHP | | FAHP | | Neutrosophic-AHP | |
|---|---|---|---|---|---|---|
| | Applying model without factors | Applying model with factors | Applying model without factors | Applying model with factors | Applying model without factors | Applying model with factors |
| **Support different Technical platform** | 18% | **50%** | 17% | **50%** | 15% | **50%** |

| | | | | | | |
|---|---|---|---|---|---|---|
| • Infrastructure platform | | 88% | | 87% | | 67% |
| • Operating system | | 13% | | 22% | | 33% |
| **Vendor package** | 21% | **20%** | 20% | **20%** | 14% | **20%** |
| • Continuous deployment | | 26% | | 22% | | 24% |
| • Recovery | | 31% | | 40% | | 25% |
| • Training staff | | 19% | | 11% | | 17% |
| • Maintenance | | 11% | | 11% | | 15% |
| • Continuous integration | | 14% | | 16% | | 19% |
| **Low total cost** | 9% | **17%** | 9% | **16%** | 9% | **17%** |
| • Ownership licenses | | 25% | | 31% | | 23% |
| • Services | | 4% | | 2% | | 11% |
| • Implementation | | 9% | | 7% | | 13% |
| • Consultancy | | 8% | | 8% | | 12% |
| • deployment | | 20% | | 20% | | 22% |
| • Customization | | 34% | | 30% | | 19% |
| **Support different language** | 3% | **50%** | 3% | **50%** | 6% | **50%** |
| • Arabic | | 90% | | 97% | | 50% |
| • English | | 10% | | 3% | | 50% |

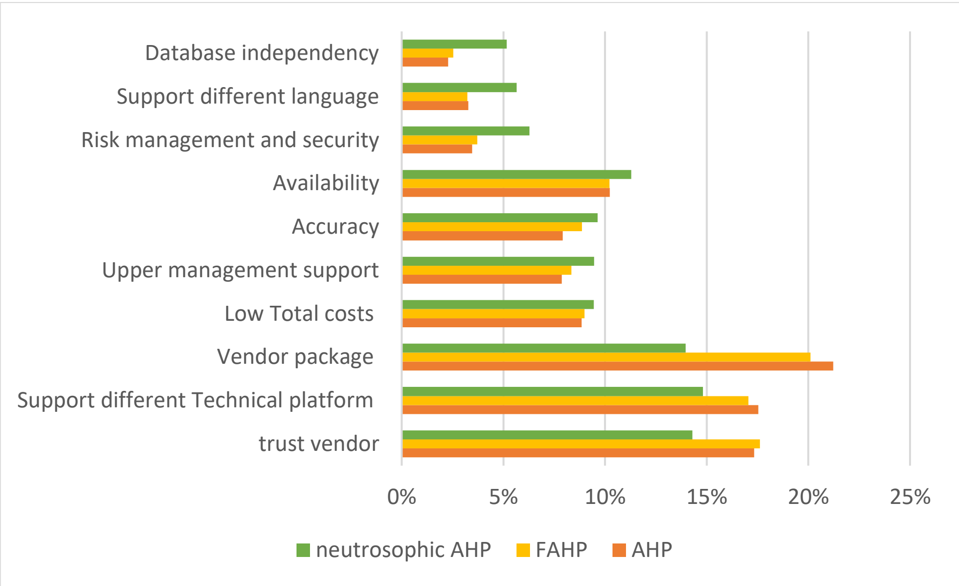


*Figure 6 Criteria's Importance*

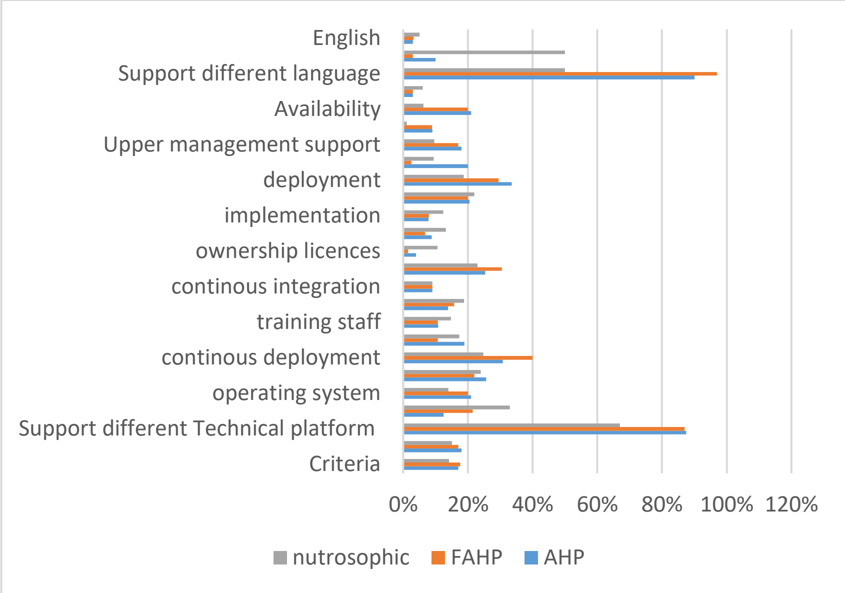


*Figure 7 Criteria and sub-criteria importance*

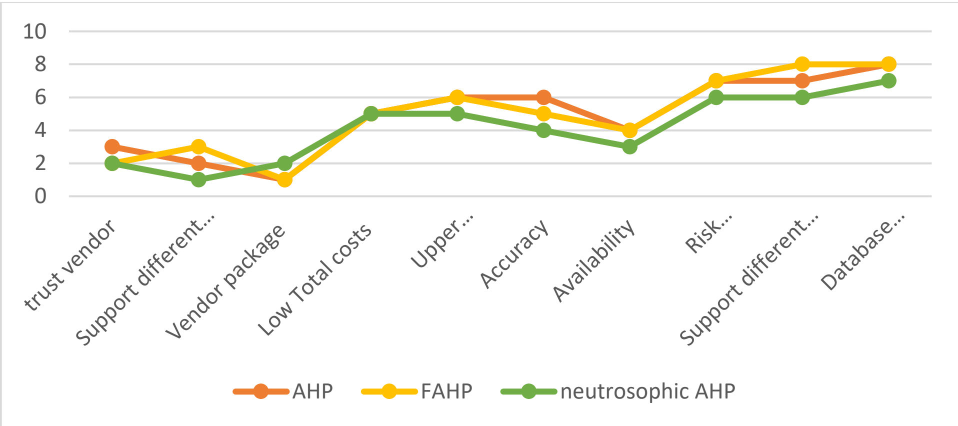


*Figure 8 Criteria's Rank for model in section 4.1(criteria only)*

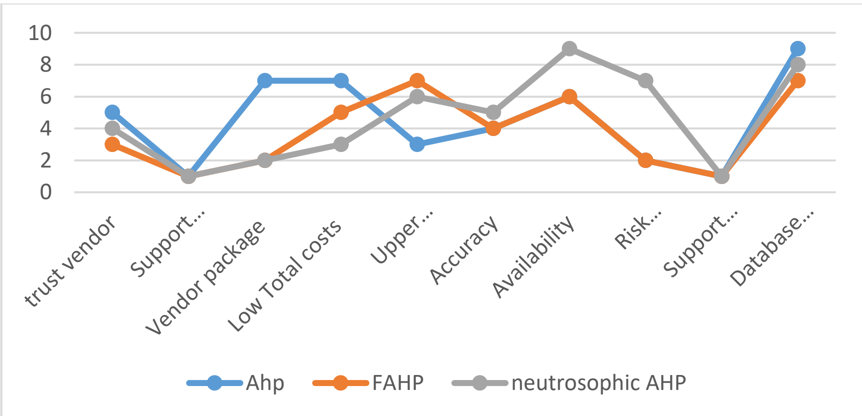


*Figure 9 Criteria's Rank for model in section 4.2 (criteria and factors)*

Consistency index confirms on the consistency of criteria and further on the decision, where it is the index of the consistency of judgments across all pairwise comparisons. The consistency of main criterion that has sub-criteria is less than consistency of criteria without its sub as listed in table 3.

*Table 3 consistency of main criteria with and without its sub-criteria (factors)*

| Criteria that have sub criteria | consistency by AHP (without factors) | consistency by AHP (with factors) |
|---|---|---|
| Support different technical platform | 13.19 | 2 |
| Vendor package | 12.88 | 4.96 |
| Low total cost | 11.12 | 6.02 |
| Support different language | 10.68 | 2 |

The other consistency of criteria that have not sub criteria are the same and are listed in table 4

*Table 4 consistency of criteria that have not sub criteria*

| Sub-criteria | Consistency by AHP | Sub-criteria | Consistency by AHP | criteria | Consistency by AHP |
|---|---|---|---|---|---|
| Infrastructure platform | 1 | Service/support | 1.005 | Trusted vendor | 11.9 |
| Operating system | 1 | Implementation | 0.691 | upper management support | 11.26 |
| Continuous deployment | 1.08 | Consultancy | 1.24 | Accuracy | 11.39 |
| Recovery | 0.88 | Deployment | 1.06 | Availability | 11.91 |
| Training staff | 1.24 | Customization | 0.88 | Risk management and security | 11.48 |
| Maintenance | 0.95 | Arabic | 1 | Database independency | 11.36 |
| Continuous integration | 0.79 | English | 1 | | |
| Ownership licenses | 1.12 | | | | |

The selecting ERP system decision based on 10 criteria regards to approximate rank of decision based on 25 criteria (10 criteria and 15 sub-criteria). Decision score based on these criteria for each method is listed in below table 5 and in following figures 10and 11.

*Table 5 decision score with three scale sets*

| systems | AHP | AHP (with factors) | FAHP | FAHP (with factors) | Neutrosophic-AHP | Neutrosophic-AHP (with factors) |
|---|---|---|---|---|---|---|
| **Odoo** | 63% | 60% | 54% | 54% | 56% | 55% |
| **EBS** | 27% | 40% | 46% | 46% | 44% | 45% |

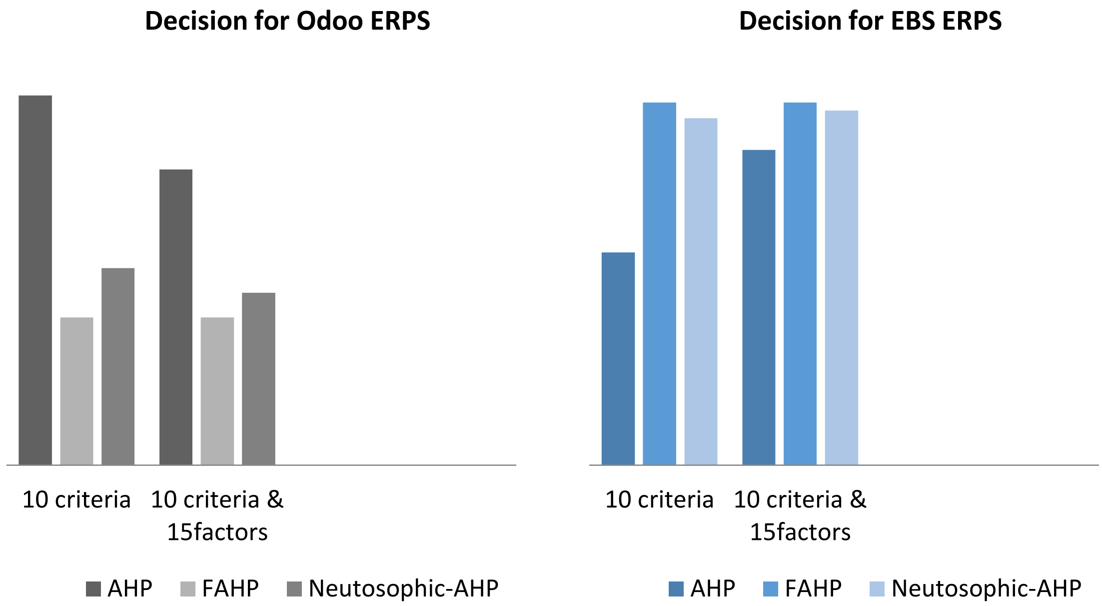


*Figure 10 decisions for two alternatives systems based on two models*

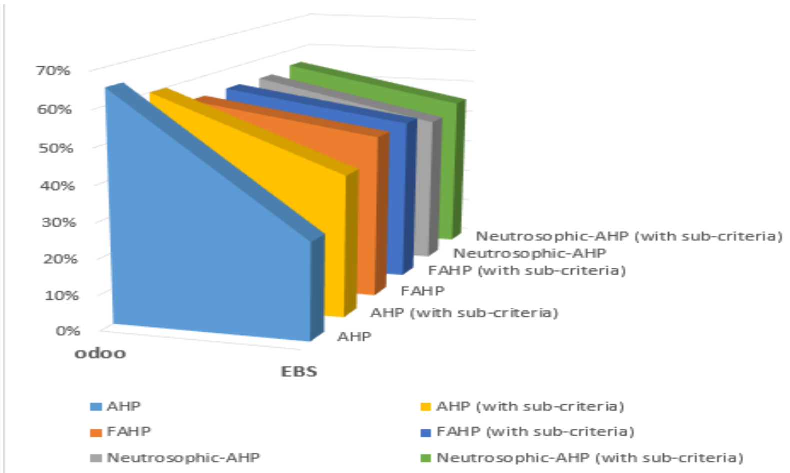


*Figure 11 rank decision of Odoo and EBS selection with and without sub-criteria*

## 6. Conclusion

The comparison chart allows enterprises to take an in-depth look at whether different software packages can meet their technical and functional requirements. Comparison Report allows buyers of business software to assess functions, features, capabilities, downside of the software solutions, but it does not help in decision making. On analysis stage, the SWOT analysis and comparisons may be not enough for detect which system is suitable as in case study, but it creates flog and confusion environment. In this inconsistency decision the Multi criteria decision making (MCDM) is solved. This paper applies three methods of it; AHP, FAHP and Neutrosophic-AHP, firstly, with 10 criteria and secondly, with 25 criteria (adding 15 sub criteria). Three approaches ranked two alternatives ERPS. The paper provides a comparative analysis for AHP, FAHP and Neutrosophic-AHP. Although many researches handle criteria of evaluating ERP but purchasing ERP almost is not found. The paper proposes criteria of adopting ERPS. Furthermore, the paper studies consistency of these criteria.

The paper studies accuracy of decision with AHP, FAHP, and Neutrosophic-AHP. This study compares making decision of adopting ERPS by these three based on 10 criteria, an based on 10 criteria and 15 sub-criteria. This study also analyzes criteria and factors by calculating their weights based on two alternatives' properties and characteristics. The paper also studies the accuracy of decision by comparing the consistency of using multi-criteria and criteria for decision model.

The paper proves that Neutrosophic-AHP is the most accuracy rather than AHP and FAHP. Also it shows effect of using criteria and its factors in decision's accuracy. The third contribution, the comparative analysis that is addressed in paper tries to fill gap between industrial and academic fields by real empirical application.

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**Appendix A**

*Table A SWOT analysis for Odoo and EBS*

| Odoo | | Oracle e-business suite | |
|---|---|---|---|
| **Strength** | **Weakness** | **Strength** | **Weakness** |
| • Flexibility to tailor the system for enterprises needs<br>• The free version of it, consider an announcement and increase availability in the ERP market, marketing for the commercial version.<br>• High modular: easy to add more module<br>• Customize created modules.<br>• Lower cost<br>• Open source<br>• Free educational version<br>• Easy to integrate with external systems<br>• Commercial edition in SaaS version<br>• Has 900+ partners over 1176 countries with 4000000+users. | • Documentation needs to improve.<br>• Odoo does not has business analytics, product design, SCM, and asset management Commercial version is not for small enterprises | • Its company has more than 130,000 employees and developers working with Oracle<br>• Oracle Company (owner) has market dominance in many technical products such as Oracle Database, Enterprise Manager, Fusion Middleware, servers, workstations, storage etc.<br>• Has the ability to integrate with different modules.<br>• Is an extremely powerful, robust, that meet the needs of virtually any business<br>• Support their products with update, continues release<br>• It offers services like SAAS, PAAS, consulting, financing etc.<br>• Oracle has its presence in 100+ countries that share in EBS using over them. | • because its effected role in technical market, Oracle has had to face many lawsuits and controversies which affected its brand image<br>• competition means limited growth in market share<br>• its user interface is not friendly<br>• not user-friendly enough than some other platforms particularly for small businesses<br>• The default tax module and sales modules found on EBS is often not adequate, leading companies to have their own custom modules built.<br>• There are also many modules for the platform, that work, but do not work as well as they do on other systems |

| Opportunities | Threats | Opportunities | Threats |
|---|---|---|---|
| • Continuous developing thanks to open source nature and partners.<br>• Cooperation with governmental organizations helps it to grow its business.<br>• Large enterprises such as Toyota and Hyundai turned to using Odoo is a motivation to cooperate with more.<br>• Odoo can work towards tapping the huge internet, different infrastructures (PC, Mobile, VM, Cloud) and grow-up of data analysis science<br>• Many add on, modules, features add easily without additional cost<br>• Its popularity increase | • Strong Competition from commercial ERP vendors such as Oracle, SAP etc.<br>• Competition from open source ERP vendors | • Because Oracle is a trusted vendor in many technology as database, that will be reflected on EBS's reputation.<br>• More brand visibility and announcement can highlight EBS<br>• Cooperation with governmental organizations<br>• (PC, Mobile, VM, Cloud) and grow-up of data analysis science | • There are strong top competitors such as: SAP, Microsoft, HP Hewlett-Packard and IBM.<br>• Competition from Open source vendors such as Odoo.<br>• Because EBS is spread over the world, market instability may reduce its profits.<br>• Increasing the competition may be decrease EBS's market dominance |

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# Neutrosophic Data Envelopment Analysis: An Application to Regional Hospitals in Tunisia

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**Abstract:** In many real-life situations, decision-making units (DMUs)—such as production processes or manufacturing or service systems—involve data related to inputs and outputs that are volatile, imprecise, or even missing. This makes it difficult to measure these DMUs' efficiency. In this context, a data envelopment analysis (DEA) is a powerful methodology to facilitate this measurement, but this is also sensitive to data: any noise or error in the data measurement can easily cause non-applicable or insignificant results. The neutrosophic theory has demonstrated its superiority over other approaches and theories in handling this type of data, and especially in its capability to consider indeterminate data. However, in the DEA context, the use of this theory remains limited to a few theoretical works. In order to filling this gap, the present paper aims to highlight the neutrosophic DEA in a real-life application. Two different neutrosophic approaches, or namely, the ranking and parametric approaches, are adjusted then applied to measure and evaluate the efficiency of 32 regional hospitals in Tunisia. These results allow a comparison of these two approaches, but more importantly, they reveal the desired efficiency measurement that permits inefficient hospitals' necessary actions. Consequently, indeterminate inputs and outputs are no longer a handicap in using the DEA.

**Keywords:** data envelopment analysis; indeterminate data; neutrosophic sets; hospital efficiency

## 1. Introduction

All organizations, whether governmental or private, need an accurate performance assessment for development, growth, and sustainability. In fact, in today's competitive environment, these organizations face pressure to convert inputs into outputs as cheaply as possible (at a given level of quality and quantity). This pressure encourages them to be efficient. Precisely, in the public sector, where the usual disciplines of a competitive market are absent, one of the key roles of government is to provide public goods and services. So that, identifying efficient providers can enhance efficiency by allowing the recognition and spread of good practice.

In seeking to evaluate the technical efficiency of a set of decision-making units (DMUs), Charnes et al. [1] proposed the data envelopment analysis (DEA) methodology. Subsequently, this technique has been used in a variety of models and applications, or in more than 4,000 publications as noted by Emrouznejad et al. [2]. In presence of several inputs and outputs, the DEA essentially uses linear programming to find a best-practice frontier for efficient DMUs that envelops all other inefficient DMUs. This methodology is especially popular because it does not require any specified production function, and can simultaneously consider many inputs and outputs.

The original DEA methodology fundamentally assumes that inputs and outputs are measured with crisp, positive values on a ratio scale, and all the required data are available. As its name indicates, this methodology is highly sensitive to data: any noise or error in data measurement can easily cause non-applicable or insignificant results. Therefore, a key to the DEA's success involves accurately measuring all factors, including inputs and outputs. However, the data related to inputs and outputs in many real-life situations—such as in production processes or manufacturing or service systems—are volatile, imprecise, or even missing. Therefore, it is desirable to use theories and methods that can handle this kind of data.

Among many approaches, such as: (1) stochastic methods; Cooper et al. [3] treated the topic of stochastic characterizations of efficiency and inefficiency in DEA using chance constrained programming formulations and constructs centered on congestion as one form of inefficiency. Khodabakhshi et al. [4] developed an input-oriented super-efficiency measure in stochastic data envelopment analysis. (2) interval DEA models; Entani and Tanaka [5] presented a method in order to improve the efficiency interval of a DMU by adjusting its given inputs and outputs. Smirlis et al. [6] introduced an approach based on interval DEA that allows the evaluation of the units with data. Jahanshahloo et al. [7]], developed an interval DEA model to obtain an efficiency interval consisting of evaluations from both the optimistic and the pessimistic viewpoints. (3) fuzzy theory; introduced by Zadeh [8], it has been mostly applied to handle imprecise, uncertain, or incomplete data in DEAs. Sengupta [9] is the first who explored the use of fuzzy set-theory in the context of data envelopment analysis. Kao and Liu [10] presented a procedure to measure the efficiencies of DMUs with fuzzy observations. authors transformed a fuzzy DEA model to a family of conventional crisp DEA models by applying the $\alpha$-cut approach. Wang et al. [11] proposed two new fuzzy DEA models constructed from the perspective of fuzzy arithmetic to deal with fuzziness in input and output data in DEA. Zerafat et al. [12] introduced the concept of ''local $\alpha$ -level'' to develop a multi-objective linear programming to measure the DEA efficiency of DMUs under uncertainty. Agarwal [13] proposed a fuzzy DEA model based on $\alpha$-cut approach to deal with the efficiency measuring and ranking problem. Kumar [14] applied fuzzy data envelopment analysis in assessing the productivity of banks. According to Hatami-Marbini et al. [15], DEA approaches using fuzzy theory can be classified into four primary categories, while Emrouznejad et al. [16] presented a taxonomy of the fuzzy DEA methods, with a classification scheme that includes six categories.

Although the fuzzy set theory has been introduced as a powerful tool to quantify vague data, a key inadequacy exists in these past methodologies. A critical problem is that fuzziness is insufficient to consider the degree of information certainty when handling real data. Smarandache [18] recently introduced the neutrosophic theory as a generalization of fuzzy theory. As this can handle vague, imprecise, incomplete, as well as indeterminate data, the neutrosophic theory is considered closer to human thinking due to its better simulation of human decision-making processes by considering indeterminate data. In fact, each element of a neutrosophic set has truth, indeterminacy, and falsity membership functions. Since Smarandache's introduction of the neutrosophic set concept, many different sets have been proposed, with the single value neutrosophic set introduced by Wang et al. [19] as the most popular. Single-valued neutrosophic numbers present a special case involving single-valued neutrosophic sets, and are important in neutrosophic, multi-attribute decision-making problems because they effectively describe an ill-known quantity (Deli and Şubaş, [20]).

The neutrosophic set theory has since been applied in many mathematical programming and multi-criteria decision-making methods, such as the following: linear programming [Abdel-Nasser et al. [21], Abdel-Basset [22]], non-linear programming (Ye et al. [23]), the analytic hierarchy process (Abdel-Basset et al. [24]), goal programming [Pramanik [25], Pramanik and Banerjee [26]], analytic hierarchy process combined with preference ranking organization method for enrichment evaluations type II method (Abdel-Basset et al. [27]), and the technique for order preference by similarity to an ideal solution (Biswas et al. [27]), among others. Abdel-Nasser and Hagar [28] also present some earlier works using multi-criteria decision-making methods in a neutrosophic environment. Further, the concept of neutrosophic sets and its extensions have been applied in a variety of fields, including computer science (Ali and Smarandache, [29]), mathematics (Salama and Alblowi, [30]), and medicine (Abdel-Basset et al. [31]).

In the DEA context, few studies to the best of our knowledge have addressed neutrosophic data. Edalatpanah [32] presented a brief DEA model with neutrosophic inputs and outputs, and suggested that the score function developed by Despotis and Smirlis [33] be used to transform the model into a crisp DEA model and solve it using any conventional method. Abdelfattah [34] also presented a DEA model with all neutrosophic inputs and outputs; the author solved this model by developing a parametric approach based on what he called the "degrees of variation" in a neutrosophic number. However, these two studies are only theoretical, and have not applied their neutrosophic DEA models to real examples to further demonstrate the importance of this research axis. Therefore, this paper aims to highlight the neutrosophic DEA approach in a real-life application through an efficiency evaluation of Tunisian regional hospitals with indeterminate data.

The remainder of the paper is organized as follows: Section 2 introduces the neutrosophic DEA model and the two approaches that will follow for its resolution. Section 3 presents the main body of the paper and its data, data adjustment, results, and analysis related to the application case. Section 4 provides a summary and the research's final conclusions.

## 2. Methodology

### *2.1. Neutrosophic DEA model*

Charnes et al. [1] developed the first DEA model to measure the relative efficiency of a set of homogenous DMUs under the assumption of constant returns to scale. First, let $x_{ij}$ and $y_{rj}$ denote the inputs and outputs of a DMU $j$, respectively, with $m$ inputs, $s$ outputs, and $n$ DMUs. The output-oriented DEA model measuring the efficiency of a given DMU $k$ is:

$$
\begin{aligned}
& Min\ E_k = \sum_{i=1}^{m} v_i x_{ik} \,/ \sum_{r=1}^{s} u_r y_{rk} \\
& Subject\ to: \\
& \sum_{i=1}^{m} v_i x_{ij} \,/ \sum_{r=1}^{s} u_r y_{rj} \geq 1, j = 1, 2, \dots, n; \\
& u_r, v_i \geq 0, \forall\, r,
\end{aligned}
\tag{1}
$$

where $u_r$ indicates the weight assigned to the output $r$, and $v_i$ is the weight assigned to the input $i$.

Model (1) is a fractional programming model converted into linear programming, as follows:

$$
\begin{aligned}
& Min\ E_k = \sum_{i=1}^{m} v_i x_{ik} \\
& Subject\ to: \\
& \sum_{r=1}^{s} u_r y_{rk} = 1 \\
& \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \geq 0, j = 1, 2, \dots, n; \\
& u_r, v_i \geq 0, \forall\, r, i
\end{aligned}
\tag{2}
$$

If any of this model's observation data related to inputs and/or outputs is imprecise, uncertain, or indeterminate, then the efficiency of the DMU $k$ will be misleading. Additionally, if this DMU lies on the efficient production function, it will reflect a doubtful reference unit for the other inefficient DMUs. A powerful approach to address this kind of problem involves relying on the neutrosophic set theory.

Assuming inputs and outputs are neutrosophic, they can be represented by triangular neutrosophic numbers, while the variables $u_r$ and $v_i$ are real numbers; thus, Model (2) will be written as follows:

$$Min\ \tilde{E}_k = \sum_{i=1}^{m} v_i \tilde{x}_{ik}$$

$$Subject\ to:$$

$$\sum_{r=1}^{s} u_r \tilde{y}_{rk} = 1 \tag{3}$$

$$\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0, j = 1, 2, \dots, n;$$

$$u_r, v_i \geq 0, \forall\ r, i$$

where $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ are triangular neutrosophic numbers, such that:

$$\tilde{x}_{ij} = \langle (x_{ij1}, x_{ij2}, x_{ij3}), t_{\tilde{x}_{ij}}, d_{\tilde{x}_{ij}}, f_{\tilde{x}_{ij}} \rangle$$

$$\tilde{y}_{rj} = \langle (y_{rj1}, y_{rj2}, y_{rj3}), t_{\tilde{y}_{rj}}, d_{\tilde{y}_{rj}}, f_{\tilde{y}_{rj}} \rangle$$

where $x_{ij1}$, $x_{ij2}$, and $x_{ij3}$ denote the lower bound, median value, and upper bound of $\tilde{x}_{ij}$, respectively; $t_{\tilde{x}_{ij}}, d_{\tilde{x}_{ij}}$, and $f_{\tilde{x}_{ij}}$ indicate the degrees of truth, indeterminacy, and falsity for $\tilde{x}_{ij}$, respectively. Subsequently, $\tilde{y}_{rj}$ is defined in a similar manner.

As Model (3) is a neutrosophic DEA model that cannot be solved using typical techniques, the author suggests using the following approaches while introducing some modifications that make them applicable in the proposed model.

*2.2. Ranking approach*

As a first alternative, we consider the approach from work by Abdel-Basset et al. [22], which was specifically developed to address neutrosophic linear programming models. This method suggests that each trapezoidal neutrosophic number $\tilde{a}$ be converted into its equivalent crisp value using the following ranking function:

$$R = \left( \frac{a^l + a^u + 2(a^{m1} + a^{m2})}{2} \right) + (t_{\tilde{a}} - d_{\tilde{a}} - f_{\tilde{a}}) \tag{4}$$

Note that $\tilde{a} = \langle (a^l, a^{m1}, a^{m2}, a^u), t_{\tilde{a}}, d_{\tilde{a}}, f_{\tilde{a}} \rangle$ is a trapezoidal neutrosophic number, where $a^l, a^{m1}, a^{m2}$, and $a^u$ are the lower bound, first and second median values, and the upper bound of $\tilde{a}$, respectively; $t_{\tilde{a}}, d_{\tilde{a}}$, and $f_{\tilde{a}}$ are the degrees of truth, indeterminacy, and falsity for the trapezoidal number; and $(t_{\tilde{a}} - d_{\tilde{a}} - f_{\tilde{a}})$ indicates the degree of confirmation.

As the DEA model can be transformed as shown into a linear programming model, we can apply this ranking function to solve Model (3). Accordingly, the input and output values in this work should be triangular neutrosophic numbers, and thus, we propose the following ranking function:

$$= \left( \frac{a^l + 2a + a^u}{4} \right) + (t_{\tilde{a}} - d_{\tilde{a}} - f_{\tilde{a}}) \tag{5}$$

Applying this ranking function to Model (3) obtains the following crisp explicit model; standard methods are then used to calculate the optimal solution:

$$Min\ \tilde{E}_k = \sum_{i=1}^{m} v_i \left[ {}^{1}\!/_{4} \left( x_{ik1} + 2x_{ik2} + x_{ik3} \right) + \left( t_{\tilde{x}_{ik}} - d_{\tilde{x}_{ik}} - f_{\tilde{x}_{ik}} \right) \right] \tag{6}$$

$Subject\ to$:

$$\sum_{r=1}^{s} u_r[{}^{1}/_{4}(y_{rk1} + 2y_{rk2} + y_{rk3}) + (t_{\tilde{y}_{rk}} - d_{\tilde{y}_{rk}} - f_{\tilde{y}_{rk}})] = 1$$

$$\sum_{r=1}^{s} u_r\left[{}^{1}/_{4}(y_{rj1} + 2y_{rj2} + y_{rj3}) + \left(t_{\tilde{y}_{rj}} - d_{\tilde{y}_{rj}} - f_{\tilde{y}_{rj}}\right)\right] - \sum_{i=1}^{m} v_i\left[{}^{1}/_{4}(x_{ij1} + 2x_{ij2} + x_{ij3}) + \left(t_{\tilde{x}_{ij}} - d_{\tilde{x}_{ij}} - f_{\tilde{x}_{ij}}\right)\right] \leq 0, j = 1, 2, \dots, n;$$

$u_r, v_i \geq 0, \forall\, r, i$

*2.3. Parametric approach*

All input and output data in Model (3) should be triangular neutrosophic numbers. Unlike the ranking approach, Abdelfattah's [34] proposed parametric approach consists of transforming these data into intervals rather than crisp values by considering the decision-makers' levels of acceptance, indeterminacy, and rejection toward the data. This approach essentially determines the degrees of variation for every single neutrosophic input or output, given by the following equation:

$$\theta_{\tilde{a}} = \frac{1}{4}\left[\frac{\alpha}{t_{\tilde{a}}} + 2\frac{(1-\beta)}{1-d_{\tilde{a}}} + \frac{(1-\gamma)}{1-f_{\tilde{a}}}\right]; \quad \theta_{\tilde{a}} \in [0, 1]; \tag{7}$$

where $t_{\tilde{a}}, d_{\tilde{a}}$, and $f_{\tilde{a}}$ indicate the degrees of truth, indeterminacy, and falsity for the triangular neutrosophic number $\tilde{a}$, respectively; $\alpha$ denotes the minimal degree of acceptance, or $\alpha \in [0, t_{\tilde{a}}]$; $\beta$ denotes the maximal degree of indeterminacy, or $\beta \in [d_{\tilde{a}}, 1]$; and $\gamma$ denotes the maximal degree of rejection, or $\gamma \in [f_{\tilde{a}}, 1]$.

Input and output values are then converted into their equivalent intervals with the following equation:

$$\tilde{a} = [a^l, a^u] = [a_1 + (a_2 - a_1)\theta_{\tilde{a}}, a_3 - (a_3 - a_2)\theta_{\tilde{a}}] \tag{8}$$

where $a_1, a_2$, and $a_3$ are the lower bound, median value, and upper bound of $\tilde{a}$, respectively.

According to this approach, Model (3) can then be transformed into two sub-models. Note that Abdelfattah [34] adopted an input-oriented DEA model, while this paper adopts an output-oriented DEA model, as its application will require. Thus, Model (3) is transformed into the following two sub-models (9a) and (9b), representing the most favorable (maximal) efficiency and the least favorable (minimal) efficiency, respectively:

$$Min\ (E_k)_\theta^u = \sum_{i=1}^{m} v_i[x_{ik1} + (x_{ik2} - x_{ik1})\theta_{\tilde{x}_{ik}}]$$

$Subject\ to$:

$$\begin{aligned}
&\sum_{r=1}^{s} u_r[y_{rk3} - (y_{rk3} - y_{rk2})\theta_{\tilde{y}_{rk}}] = 1;\\
&\sum_{i=1}^{m} v_i[x_{ik1} + (x_{ik2} - x_{ik1})\theta_{\tilde{x}_{ik}}] - \sum_{r=1}^{s} u_r[y_{rk3} - (y_{rk3} - y_{rk2})\theta_{\tilde{y}_{rk}}] \geq 0;\\
&\sum_{i=1}^{m} v_i\left[x_{ij3} - (x_{ij3} - x_{ij2})\theta_{\tilde{x}_{ij}}\right] - \sum_{r=1}^{s} u_r[y_{rj1} + (y_{rj2} - y_{rj1})\theta_{\tilde{y}_{rj}}] \geq 0, j = 1, 2, \dots, n, \quad j \neq k;\\
&u_r, v_i \geq 0, \forall\, r, i
\end{aligned} \tag{9 a}$$

$$Min\ (E_k)_\theta^l = \sum_{i=1}^{m} v_i[x_{ik3} - (x_{ik3} - x_{ik2})\theta_{\tilde{x}_{ik}}]$$

$$Subject\ to:$$

$$\begin{aligned}
&\sum_{r=1}^{s} u_r[y_{rk1} + (y_{rk2} - y_{rk1})\theta_{\tilde{y}_{rk}}] = 1 \\
&\sum_{i=1}^{m} v_i[x_{ik3} - (x_{ik3} - x_{ik2})\theta_{\tilde{x}_{ik}}] - \sum_{r=1}^{s} u_r[y_{rk1} + (y_{rk2} - y_{rk1})\theta_{\tilde{y}_{rk}}] \geq 0 \\
&\sum_{i=1}^{m} v_i\left[x_{ij1} + (x_{ij2} - x_{ij1})\theta_{\tilde{x}_{ij}}\right] - \sum_{r=1}^{s} u_r[y_{rj3} - (y_{rj3} - y_{rj2})\theta_{\tilde{y}_{rj}}] \geq 0, j = 1, 2, \dots, n, \qquad j \neq k \\
&u_r, v_i \geq 0, \forall\ r, i
\end{aligned} \qquad (9\ b)$$

$$\theta_{\tilde{x}_{ij}} = \frac{1}{4}\left[\frac{\alpha}{t_{\tilde{x}_{ij}}} + 2\frac{(1-\beta)}{1-d_{\tilde{x}_{ij}}} + \frac{(1-\gamma)}{1-f_{\tilde{x}_{ij}}}\right];\ \theta_{\tilde{y}_{rj}} = \frac{1}{4}\left[\frac{\alpha}{t_{\tilde{y}_{rj}}} + 2\frac{(1-\beta)}{1-d_{\tilde{y}_{rj}}} + \frac{(1-\gamma)}{1-f_{\tilde{y}_{rj}}}\right]$$
$$\alpha \in \left[0, min\left\{t_{\tilde{x}_{ij}}, t_{\tilde{y}_{rj}}\right\}\right];\ \beta \in \left[max\left\{d_{\tilde{x}_{ij}}, d_{\tilde{y}_{rj}}\right\}, 1\right];\ \gamma \in \left[max\left\{f_{\tilde{x}_{ij}}, f_{\tilde{y}_{rj}}\right\}, 1\right]$$

After a decision-maker sets specific values of $\alpha$, $\beta$, and $\gamma$—representing his or her minimal degree of acceptance, maximal degree of indeterminacy, and maximal degree of rejection, respectively—Models (9a) and (9b) will yield bounded intervals of efficiency scores $\left[(E_k)_{\theta_i}^l, (E_k)_{\theta_i}^u\right]$ for all evaluated DMUs.

## 3. An Application to Evaluate the Efficiency of Regional Hospitals in Tunisia

Providing suitable healthcare services is key for every society's well-being. Tunisia considers the health sector as a national priority, and invested 7% of its 2014 gross domestic product in its healthcare industry.1 This percentage is higher than the minimum 5% threshold recommended by the World Health Organization, and is equivalent to that of upper-middle-income countries. Tunisia's public health facilities are classified according to their mission, equipment, technical level, and territorial competence, categorized as: basic health centers, district hospitals, regional hospitals, and university hospital centers.

As this paper is only concerned with regional hospitals, we attempt to measure their ability to efficiently use minimum resources (inputs) to produce suitable healthcare services (outputs) using the DEA. Literature has similarly applied the DEA in this type of efficiency measurement; specifically, Kohl et al. [35] provide a noteworthy review of this issue. As some observations are not available and others are not "precise," this paper applies the concept of indeterminacy, and therefore, the approaches described in the previous section.

### *3.1. Data*

This study evaluates all 32 regional hospitals in Tunisia, with data collected from the Ministry of Public Health's 2015 health map.2

The selection of inputs and outputs to be considered is typically a subject of debate. For example, Ozcan [36] suggested that inputs include beds, a weighted service-mix, full-time equivalents, and operations expenses, and that outputs include case-mix-adjusted admissions and outpatient visits. Azreena et al. [37] systematically reviewed hospitals' inputs and outputs in measuring efficiency

[1] https://www.who.int/countries/tun/en visited on 15-July-2019

[2] http://www.santetunisie.rns.tn/images/docs/anis/stat/cartesanitaire2015.pdf visited and downloaded on 18-July-2019

using a DEA. Regarding this issue, Dyson et al. [38] stated that using significant numbers of inputs and outputs does not necessarily garner better results. These authors posit that the most important factor is the number of DMUs, as there should always be more than $2 \times (number\ of\ inputs + number\ of\ outputs)$. This study respects this rule, as three outputs and only one input are considered for the 32 regional hospitals, as follows:

| Type | Name | Explanation |
|---|---|---|
| Input | Operating Budget (OB) | The hospital's annual expenses, coming from:<br>- The state's budget in terms of salaries<br>- Contributions from the public health insurance fund (CNAM)<br>- Net revenues |
| Output 1 | Admissions | The admissions of hospitalized patients at a hospital for a given period. As hospital statistics do not distinguish between the number of admissions and the number of entries, the same patient can be re-hospitalized for the considered period and generate several entries. |
| Output 2 | Outpatient Visits | The number of times that a patient is not hospitalized overnight, but visits the hospital for diagnosis or treatment. |
| Output 3 | Emergency Visits | The number of cases calling for immediate action as registered by the hospital's emergency room/department. |

Tables 1 and 2 present the data regarding the considered inputs and outputs, respectively.

**Table 1.** Input data: 2015 operating budget of Tunisian regional hospitals (in TND)

| DMU | DMU Name: Hospital | Salaries | CNAM | Net Revenue | Total OB |
|---|---|---|---|---|---|
| 1 | Mahmoud El Matri de l'Ariana | 200,000 | 1,796,390 | 707,249 | 2,703,639 |
| 2 | Khair-Eddine | 500,000 | 1,333,887 | 233,547 | 2,067,434 |
| 3 | Hôpital Ben Arous | 100,000 | 3,611,455 | 1,529,633 | 5,241,088 |
| 4 | Menzel Bourguiba | * | 7,236,055 | 1,567,819 | 8,803,874 |
| 5 | Nabeul | 200,000 | 2,331,000 | 1,138,213 | 3,669,213 |
| 6 | Menzel Témime | **0** | 3,907,115 | 1,191,333 | 5,098,448 |
| 7 | Zaghouan | 200,000 | 2,656,103 | 864,525 | 3,720,628 |
| 8 | Jendouba | 500,000 | 4,393,450 | 1,334,386 | 6,227,836 |
| 9 | Tabarka | * | * | 1,400,000 | 1,400,000 |
| 10 | Béja | 400,000 | 5,873,920 | 1,078,012 | 7,351,932 |
| 11 | Medjez El Bab | 400,000 | 1,568,142 | 426,514 | 2,394,656 |
| 12 | M'hamed Bourguiba du Kef | 401,000 | 5,654,441 | 1,008,723 | 7,064,164 |
| 13 | Siliana | **0** | 4,869,819 | 925,074 | 5,794,893 |
| 14 | Kasserine | 1,000,000 | 5,270,338 | 1,890,662 | 8,161,000 |
| 15 | M'Saken | 200,000 | 1,586,683 | 1,020,266 | 2,806,949 |
| 16 | Moknine | 200,000 | 2,717,575 | 688,340 | 3,605,915 |
| 17 | Haj Ali Soua de Ksar Hellal | **0** | 2,025,418 | 1,053,743 | 3,079,161 |
| 18 | Kerkennah | **0** | 2,551,472 | 269,535 | 2,821,007 |

| | | | | | |
|---|---|---|---|---|---|
| 19 | Jebeniana | 300,000 | 1,576,958 | 608,216 | 2,485,174 |
| 20 | Mahres | 200,000 | 1,377,638 | 529,700 | 2,107,338 |
| 21 | Houcine Bouzaiene de Gafsa | 400,000 | 3,200,000 | 922,708 | 4,522,708 |
| 22 | Metlaoui | 100,000 | 2,216,796 | 337,818 | 2,654,614 |
| 23 | Tozeur | 100,000 | 4,032,146 | 564,305 | 4,696,451 |
| 24 | Sidi Bouzid | **0** | 6,208,290 | 1,387,163 | 7,595,453 |
| 25 | Mohamed Ben Sassi de Gabès | 700,000 | 8,249,323 | 2,380,640 | 11,329,963 |
| 26 | Kébili | 300,000 | 4,365,460 | 901,949 | 5,567,409 |
| 27 | Habib Bourguiba de Médenine | **0** | 2,884,710 | 1,754,319 | 4,639,029 |
| 28 | Sadok Mokadem de Jerba | **0** | 4,967,925 | 1,569,452 | 6,537,377 |
| 29 | Zarzis | **0** | 2,858,210 | 1,028,585 | 3,886,795 |
| 30 | Ben Guerdenne | 300,000 | 2,016,560 | 822,980 | 3,139,540 |
| 31 | Tataouine | 500,000 | 2,473,696 | 870,845 | 3,844,541 |
| 32 | Nefta | **0** | 548,000 | 560,978 | 1,108,978 |
| Minimum (missing values and zeros are not included) | | 100,000 | 548,000 | | |
| Maximum (missing values are not included) | | 1,000,000 | 8,249,323 | | |
| Median (missing values and zeros are not included) | | 300,000 | 2,858,210 | | |
| Median-minimum | | 200,000 | 2,310,210 | | |
| Maximum-medium | | 700,000 | 5,391,113 | | |

**Table 2.** Output data

| DMU | DMU Name: Hospital | Admissions | Outpatient Visits | Emergency Visits |
|---|---|---|---|---|
| 1 | Mahmoud El Matri de l'Ariana | 4,544 | 58,233 | 26,500 |
| 2 | Khair-Eddine | 607 | 59,349 | 31,623 |
| 3 | Hôpital Ben Arous | 10,162 | 96,391 | 72,402 |
| 4 | Menzel Bourguiba | 11,720 | 64,402 | 71,357 |
| 5 | Nabeul | 10,845 | 22,203 | 62,534 |
| 6 | Menzel Témime | 10,105 | 36,757 | 73,075 |
| 7 | Zaghouan | 6,933 | 44,853 | 54,254 |
| 8 | Jendouba | 15,238 | 80,248 | 98,661 |
| 9 | Tabarka | 2,229 | 10,500 | 40,073 |
| 10 | Béja | 11,115 | 54,742 | 66,014 |
| 11 | Medjez El Bab | 2,479 | 29,175 | 38,695 |
| 12 | M'hamed Bourguiba du Kef | 11,812 | 64,752 | 84,222 |
| 13 | Siliana | 13,460 | 73,979 | 56,981 |
| 14 | Kasserine | 27,006 | 61,565 | 115,607 |
| 15 | M'Saken | 2,459 | 60,712 | 76,092 |
| 16 | Moknine | 4,432 | 38,672 | 52,532 |
| 17 | Haj Ali Soua de Ksar Hellal | 3,353 | 34,494 | 73,422 |
| 18 | Kerkennah | 2,205 | 1,557 | 16,190 |

| | | | | |
|---|---|---|---|---|
| 19 | Jebeniana | 3,483 | 33,232 | 42,993 |
| 20 | Mahres | 2,722 | 34,609 | 26,950 |
| 21 | Houcine Bouzaiene de Gafsa | 15,400 | 72,357 | 143,696 |
| 22 | Metlaoui | 3,121 | 31,753 | 29,854 |
| 23 | Tozeur | 7,598 | 25,308 | 45,724 |
| 24 | Sidi Bouzid | 15,040 | 76,156 | 70,796 |
| 25 | Mohamed Ben Sassi de Gabès | 26,509 | 101,642 | 123,141 |
| 26 | Kébili | 11,608 | 33,938 | 48,268 |
| 27 | Habib Bourguiba de Médenine | 14,005 | 60,539 | 67,476 |
| 28 | Sadok Mokadem de Jerba | 18,010 | 33,846 | 58,814 |
| 29 | Zarzis | 11,070 | 26,095 | 39,655 |
| 30 | Ben Guerdenne | 6,344 | 30,529 | 44,110 |
| 31 | Tataouine | 8,498 | 24,537 | 42,080 |
| 32 | Nefta | 935 | 13,256 | 21,890 |

*3.2. De-neutrosophizing the input data*

Table 1 reveals that salary values are missing as related to the Menzel Bourguiba (DMU 4) and Tabarka (DMU 9) hospitals. Further, the latter exhibits another missing value related to the annual amount received from the CNAM public insurance fund. Additionally, the same table indicates that various hospitals—represented by DMUs 6, 13, 17, 18, 24, 27, 28, 29, and 32—recorded zero amounts for annual salaries. This data cannot be correct, as a government can delay remunerations in certain difficult circumstances, but cannot refuse to give salaries for an entire year. Hence, the OB information is incomplete, imprecise, and subsequently indeterminate, and contrary to the outputs noted in Table 2 as crisp values, the input OB for each of the previously mentioned DMUs will be treated as neutrosophic data.

By choosing to represent the neutrosophic data as triangular neutrosophic numbers, the lower bounds, median values, and upper bounds should be set. As they are not available, we calculate them as follows:

The lower bounds are the same as the obtained total values in Table 1:

$$OB^l = Salaries + CNAM + Net\ revenue$$

The median value is:

$$OB^m = OB^l + (median - minimum)$$

The upper bound is:

$$OB^u = OB^l + (maximum - median)$$

Table 3 presents all obtained values of these bounds for the considered DMUs. Further, the same table presents the degrees of truth, indeterminacy, and falsity—or $t_{\widetilde{OB}}$, $d_{\widetilde{OB}}$, and $f_{\widetilde{OB}}$, respectively—that decision-maker(s) should give subjectively.

**Table 3.** Bounds; degrees of truth, indeterminacy, and falsity; and input data degrees of variation

| DMU | $\widetilde{OB} =< (OB^l, OB^m, OB^u), t_{\widetilde{OB}}, d_{\widetilde{OB}}, f_{\widetilde{OB}} >$ | | | | | | Degrees of Variation | | |
|---|---|---|---|---|---|---|---|---|---|
| | $OB^l$ | $OB^m$ | $OB^u$ | $t_{\widetilde{OB}}$ | $d_{\widetilde{OB}}$ | $f_{\widetilde{OB}}$ | (0; 1; 1) | (0,7; 0,3; 0,4) | (0,4; 0,6; 0,7) |
| 4 | 8,803,874 | 9,003,874 | 9,503,874 | 0.9 | 0.1 | 0.3 | 0 | 0.798 | 0.440 |
| 6 | 5,098,448 | 5,298,448 | 5,798,448 | 0.8 | 0.2 | 0.2 | 0 | 0.844 | 0.469 |
| 9 | 1,400,000 | 3,910,210 | 7,491,113 | 0.7 | 0.3 | 0.4 | 0 | 1 | 0.554 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 13 | 5,794,893 | 5,994,893 | 6,494,893 | 0.8 | 0.2 | 0.2 | 0 | 0.844 | 0.469 |
| 17 | 3,079,161 | 3,279,161 | 3,779,161 | 0.8 | 0.2 | 0.2 | 0 | 0.844 | 0.469 |
| 18 | 2,821,007 | 3,021,007 | 3,521,007 | 0.8 | 0.2 | 0.2 | 0 | 0.844 | 0.469 |
| 24 | 7,595,453 | 7,795,453 | 8,295,453 | 0.8 | 0.2 | 0.2 | 0 | 0.844 | 0.469 |
| 27 | 4,639,029 | 4,839,029 | 5,339,029 | 0.8 | 0.2 | 0.2 | 0 | 0.844 | 0.469 |
| 28 | 6,537,377 | 6,737,377 | 7,237,377 | 0.8 | 0.2 | 0.2 | 0 | 0.844 | 0.469 |
| 29 | 3,886,795 | 4,086,795 | 4,586,795 | 0.8 | 0.2 | 0.2 | 0 | 0.844 | 0.469 |
| 32 | 1,108,978 | 1,308,978 | 1,808,978 | 0.8 | 0.2 | 0.2 | 0 | 0.844 | 0.469 |

The ranking approach can be now applied. However, the degrees of variation for the obtained neutrosophic numbers should be calculated for the parametric approach, and it is only sufficient to set a single value each for $\alpha \in [0, 0.7]$, $\beta \in [0.3, 1]$, and $\gamma \in [0.4, 1]$. We respect these ranges by choosing to consider three different values in the triplet $(\alpha, \beta, \gamma)$. This will yield superior in-depth analyses and interpretations of the obtained efficiencies.

Table 3 displays the obtained degrees of variation, and easily reveals that all degrees of variation for $(\alpha, \beta, \gamma) = (0; 1; 1)$ equal zero. This parallels the definition of degrees of variation given by Abdelfattah [34], in which this degree is null when decision-maker chooses to set the degree of acceptance at its minimum ($\alpha = 0$) and the degrees of indeterminacy and rejection at their maximum ($\beta = 1$ and $\gamma = 1$). The opposite case is also verified; in fact, DMU 9 has a recorded degree of variation that equals 1 when the decision-maker sets the acceptance degree at its maximum ($\alpha = 0.7$) and the degrees of indeterminacy and rejection at their minimum ($\beta = 0.3$ and $\gamma = 0.4$). Only this DMU has a degree of variation that equals 1 because this is the only one with the same time $t_{\widetilde{OB}} = 0.7$, $d_{\widetilde{OB}} = 0.3$, and $f_{\widetilde{OB}} = 0.4$.

Once the degrees of variation are set, the parametric approach can be applied to convert triangular neutrosophic values related to the input OB into their corresponding interval ranges. The ranking approach does not need these degrees of variation, as it relies only on the availability of the bounds and degrees of truth, indeterminacy, and falsity. Table 4 illustrates the intervals and crisp values of inputs yielded through the parametric and ranking approaches, respectively.

**Table 4.** De-neutrosophized input data

| **DMU** | **Parametric Approach** | | | **Ranking Approach** |
|---|---|---|---|---|
| | (0; 1; 1) | (0,7; 0,3; 0,4) | (0,4; 0,6; 0,7) | |
| 4 | [8,803,874; 9,503,874] | [8,963,398; 9,105,064] | [8,891,969; 9,283,636] | 9,078,875 |
| 6 | [5,098,448; 5,798,448] | [5,267,198; 5,376,573] | [5,192,198; 5,564,073] | 5,373,448 |
| 9 | [1,400,000; 7,491,113] | 3,910,210 | [2,789,581; 5,508,827] | 4,177,883 |
| 13 | [5,794,893; 6,494,893] | [5,963,643; 6,073,018] | [5,888,643; 6,260,518] | 6,069,893 |
| 17 | [3,079,161; 3,779,161] | [3,247,911; 3,357,286] | [3,172,911; 3,544,786] | 3,354,161 |
| 18 | [2,821,007; 3,521,007] | [2,989,757; 3,099,132] | [2,914,757; 3,286,632] | 3,096,007 |
| 24 | [7,595,453; 8,295,453] | [7,764,203; 7,873,578] | [7,689,203; 8,061,078] | 7,870,453 |
| 27 | [4,639,029; 5,339,029] | [4,807,779; 4,917,154] | [4,732,779; 5,104,654] | 4,914,029 |
| 28 | [6,537,377; 7,237,377] | [6,706,127; 6,815,502] | [6,631,127; 7,003,002] | 6,812,377 |
| 29 | [3,886,795; 4,586,795] | [4,055,545; 4,164,920] | [3,980,545; 4,352,420] | 4,161,795 |
| 32 | [1,108,978; 1,808,978] | [1,277,728; 1,387,103] | [1,202,728; 1,574,603] | 1,383,978 |

Table 4 demonstrates that the largest-interval input values are obtained when $(\alpha, \beta, \gamma) = (0, 1, 1)$, and the smallest intervals are obtained when $(\alpha, \beta, \gamma) = (0,7; 0,3; 0,4)$. Moreover, all input values obtained using the ranking approach are included in their corresponding intervals obtained by the parametric approach, except for DMU 9, when $(\alpha, \beta, \gamma) = (0{,}7; 0{,}3; 0{,}4)$. This is a favorable sign, indicating that the efficiency intervals and scores yielded using the two approaches will likely be very close.

*3.3. Results*

Table 5 provides the results from applying Models (6), (9a), and (9b) to obtain efficiency scores for Tunisia's 32 regional hospitals.

**Table 5.** Efficiency scores of Tunisian regional hospitals using the two approaches

| DMU | Parametric Approach | | | | | Ranking Approach | |
|---|---|---|---|---|---|---|---|
| | Scores | | | Ranking Index | Rank | Scores | Rank |
| | (0; 1; 1) | (0,7; 0,3; 0,4) | (0,4; 0,6; 0,7) | | | | |
| 1 | 0.950 | 0.950 | 0.950 | 0.952 | 5 | 0.950 | 5 |
| 2 | 1.000 | 1.000 | 1.000 | 1.000 | 1 | 1.000 | 1 |
| 3 | 0.880 | 0.880 | 0.880 | 0.884 | 6 | 0.880 | 6 |
| 4 | [0.395, 0.426] | [0.412, 0.419] | [0.404, 0.422] | 0.336 | 31 | 0.414 | 30 |
| 5 | 0.868 | 0.868 | 0.868 | 0.872 | 7 | 0.868 | 7 |
| 6 | [0.512, 0.582] | [0.552, 0.563] | [0.533, 0.572] | 0.516 | 24 | 0.552 | 24 |
| 7 | 0.658 | 0.658 | 0.658 | 0.643 | 16 | 0.658 | 16 |
| 8 | 0.765 | 0.765 | 0.765 | 0.764 | 11 | 0.765 | 11 |
| 9 | [0.168, 0.901] | 0.323 | [0.229, 0.452] | 0.348 | 30 | 0.302 | 31 |
| 10 | 0.455 | 0.455 | 0.455 | 0.389 | 29 | 0.455 | 29 |
| 11 | 0.594 | 0.594 | 0.594 | 0.567 | 21 | 0.594 | 21 |
| 12 | 0.535 | 0.535 | 0.535 | 0.493 | 27 | 0.535 | 26 |
| 13 | [0.664, 0.744] | [0.710, 0.723] | [0.688, 0.732] | 0.695 | 13 | 0.710 | 13 |
| 14 | 0.972 | 0.972 | 0.972 | 0.974 | 4 | 0.972 | 4 |
| 15 | 1.000 | 1.000 | 1.000 | 1.000 | 1 | 1.000 | 1 |
| 16 | 0.538 | 0.538 | 0.538 | 0.497 | 26 | 0.538 | 25 |
| 17 | [0.611, 0.750] | [0.688, 0.712] | [0.652, 0.728] | 0.669 | 15 | 0.689 | 14 |
| 18 | [0.184, 0.230] | [0.209, 0.217] | [0.197, 0.222] | 0.040 | 32 | 0.209 | 32 |
| 19 | 0.654 | 0.654 | 0.654 | 0.639 | 17 | 0.654 | 17 |
| 20 | 0.726 | 0.726 | 0.726 | 0.720 | 12 | 0.726 | 12 |
| 21 | 1.000 | 1.000 | 1.000 | 1.000 | 1 | 1.000 | 1 |
| 22 | 0.561 | 0.561 | 0.561 | 0.525 | 23 | 0.561 | 23 |
| 23 | 0.475 | 0.475 | 0.475 | 0.415 | 28 | 0.475 | 28 |
| 24 | [0.555, 0.606] | [0.584, 0.593] | [0.571, 0.598] | 0.554 | 22 | 0.585 | 22 |
| 25 | 0.687 | 0.687 | 0.687 | 0.677 | 14 | 0.687 | 15 |
| 26 | 0.612 | 0.612 | 0.612 | 0.589 | 19 | 0.612 | 19 |
| 27 | [0.770, 0.887] | [0.836, 0.855] | [0.806, 0.869] | 0.820 | 8 | 0.837 | 8 |
| 28 | [0.731, 0.809] | [0.776, 0.789] | [0.755, 0.798] | 0.765 | 9 | 0.776 | 10 |
| 29 | [0.709, 0.836] | [0.781, 0.802] | [0.747, 0.817] | 0.764 | 10 | 0.781 | 9 |
| 30 | 0.601 | 0.601 | 0.601 | 0.575 | 20 | 0.601 | 20 |
| 31 | 0.649 | 0.649 | 0.649 | 0.633 | 18 | 0.649 | 18 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 32 | [0.404, 0.658] | [0.526, 0.571] | [0.464, 0.607] | 0.503 | 25 | 0.528 | 27 |

Table 5 reveals that the two applied approaches act only on DMUs with neutrosophic data, as all other DMUs have the same unchanged efficiency scores, with no loss of information for these latter DMUs. Further, these DMUs have exactly the same crisp efficiency score whether yielded using the ranking approach or using the parametric approach for the three considered values of $(\alpha, \beta, \gamma)$.

Another inference from Table 5 is that the efficiency scores for all DMUs with neutrosophic input values obtained using the ranking approach include elements of their corresponding interval efficiency scores obtained using the parametric approach. Additionally, the largest efficiency intervals bound by the highest, best efficiencies and lowest, worst efficiencies are obtained when the acceptance degree $\alpha$ is at its minimum and the degrees of indeterminacy and falsity $\beta$ and $\gamma$ are at their maximum $(0, 1, 1)$. In contrast, the smallest efficiency intervals bound by the lowest, best and highest, worst efficiencies are obtained when the acceptance degree $\alpha$ is at its maximum and the degrees of indeterminacy and falsity $\beta$ and $\gamma$ are at their minimum $(0.7, 0.3, 0.4)$. Another noteworthy observation is that the efficiency score yielded by the ranking approach is equal or nearly equal to the lower value of the interval efficiency scores yielded by the parametric approach when $(\alpha, \beta, \gamma) = (0{,}7; 0{,}3; 0{,}4)$.

On the one hand, the hospitals' best efficiency scores—equal to 1—were achieved by hospitals represented by DMUs 2, 15, and 21. Although these hospitals exhibited relatively small OBs (Table 1), they successfully recorded important numbers, and especially in outpatient and emergency visits (Table 2). On the other hand, the worst efficiency scores were associated with the Kerkennah hospital, represented by DMU 18, with lowest and highest efficiency scores of 0.184 and 0.230 using the parametric approach, respectively. This hospital also had an efficiency score of 0.209 using the ranking approach, and ranked last according to both approaches. Although it has a relatively small OB (Table 4), in terms of this hospital's outputs, it also has relatively few admissions, outpatient visits, and emergency visits (Table 2). Logically, the region is characterized as a small island, which may be among the causes of these results. This research considers Chen and Klein's [39] ranking index in ranking DMUs using the parametric approach.

Although the Mohamed Ben Sassi de Gabès hospital (DMU 25) has generated important records in terms of output, we found it ranked only in the middle, or specifically, 14th and 15th according to the parametric and ranking approaches, respectively, with the same efficiency score of 0.687. This can be explained by the hospital's important OB values, and this can also be partially applied to the Menzel Bourguiba (DMU 4) and Kasserine hospitals (DMU 14).

### *3.4. Efficiency improvement*

Measuring efficiency is a mean rather than a goal, as the ultimate objective involves finding a way to improve efficiency among inefficient DMUs. Among the DEA's strengths is that it conveys how much an inefficient DMU should reduce the quantity of its inputs and/or increase the quantity of its outputs to be relatively more efficient than other DMUs. One way of achieving this involves using a dual model. This study determines the possible improvements that inefficient hospitals can make by using the dual of model (6) obtained by using the ranking approach. This is chosen because only one dual model should be solved rather than two when using the parametric approach; further, the two approaches have yielded nearly the same efficiency scores and DMU rankings. The explicit dual model is as follows:

$$Max\ \phi$$
$$Subject\ to: \tag{10}$$

$$\phi[{}^{1}\!/_{4}\,(y_{rk1} + 2y_{rk2} + y_{rk3}) + (t_{\tilde{y}_{rk}} - d_{\tilde{y}_{rk}} - f_{\tilde{y}_{rk}})]$$

$$- \sum_{j=1}^{n} \lambda_j \left[{}^{1}\!/_{4}\left(y_{rj1} + 2y_{rj2} + y_{rj3}\right) + \left(t_{\tilde{y}_{rj}} - d_{\tilde{y}_{rj}} - f_{\tilde{y}_{rj}}\right)\right] \leq 0, r = 1, 2, \dots, s;$$

$$\sum_{j=1}^{n} \lambda_j \left[{}^{1}\!/_{4}\left(x_{ij1} + 2x_{ij2} + x_{ij3}\right) + \left(t_{\tilde{x}_{ij}} - d_{\tilde{x}_{ij}} - f_{\tilde{x}_{ij}}\right)\right] - {}^{1}\!/_{4}\left(x_{ik1} + 2x_{ik2} + x_{ik3}\right)$$

$$- \left(t_{\tilde{x}_{ik}} - d_{\tilde{x}_{ik}} - f_{\tilde{x}_{ik}}\right) \leq 0, i = 1, 2, \dots, m;$$

$$\lambda_j \geq 0, j = 1, 2, \dots, n;$$

In this model, $\phi$ is scalar, such that $\phi^{-1}$ represents the proportional increase that will be simultaneously applied to all outputs of the $k^{th}$ DMU to make it efficient. Thus, the value of $\phi^{-1}$ obtained from resolving this model defines the efficiency score of the $k^{th}$ DMU. If ($\phi = 1$), this DMU is considered efficient, and inefficient otherwise ($\phi > 1$); $\phi^{-1} \in [0, 1]$.

The previous Section 3.3 measured the 32 regional hospitals' efficiency scores. Only three hospitals—represented by DMUs 2, 15, and 21—were found to be efficient, such that while maintaining their current input and output values, these hospitals can be considered as references for the other inefficient hospital facilities. Table 6 lists the target values of outputs for the 29 inefficient hospitals; in other words, this table provides the possible output adjustments that these latter facilities can apply to achieve perfect efficiency.

**Table 6.** Target values of outputs for inefficient DMUs to achieve perfect efficiency

| **DMU** | **DMU Name: Hospital** | **DMU of Reference** | **Benchmark** | **Target Value** | | |
|---|---|---|---|---|---|---|
| | | | | Admissions | Outpatient Visits | Emergency Visits |
| 1 | Mahmoud El Matri de l'Ariana | 2; 21 | (0.69; 0.28) | 4,784 | 61,314 | 62,486 |
| 3 | Hôpital Ben Arous | 2; 21 | (0.98; 0.71) | 11,551 | 109,563 | 133,183 |
| 4 | Menzel Bourguiba | 2; 21 | (0.40; 1.82) | 28,343 | 155,748 | 274,843 |
| 5 | Nabeul | 21 | (0.81) | 12,494 | 58,702 | 116,579 |
| 6 | Menzel Témime | 21 | (1.19) | 18,297 | 85,968 | 170,726 |
| 7 | Zaghouan | 2; 21 | (0.33; 0.67) | 10,543 | 68,208 | 106,955 |
| 8 | Jendouba | 2; 21 | (0.20; 1.29) | 19,918 | 104,896 | 191,053 |
| 9 | Tabarka | 21 | (0.92) | 14,226 | 66,840 | 132,740 |
| 10 | Béja | 2; 21 | (0.10; 1.58) | 24,404 | 120,192 | 230,253 |
| 11 | Medjez El Bab | 2; 15; 21 | (0.16; 0.41; 0.20) | 4,173 | 49,113 | 65,139 |
| 12 | M'hamed Bourguiba du Kef | 2; 21 | (0.31; 1.42) | 22,084 | 121,062 | 214,013 |
| 13 | Siliana | 2; 21 | (0.27; 1.22) | 18,947 | 104,138 | 183,740 |
| 14 | Kasserine | 21 | (1.80) | 27,789 | 130,565 | 259,292 |
| 16 | Moknine | 2; 15; 21 | (0.43; 0.18; 0.49) | 8,232 | 71,830 | 97,573 |
| 17 | Haj Ali Soua de Ksar Hellal | 21 | (0.74) | 11,421 | 53,662 | 106,569 |
| 18 | Kerkennah | 21 | (0.68) | 10,542 | 49,532 | 98,367 |
| 19 | Jebeniana | 2; 15; 21 | (0.34; 0.14; 0.31) | 5,322 | 50,776 | 65,690 |
| 20 | Mahres | 2; 21 | (0.53; 0.22) | 3,752 | 47,699 | 48,823 |
| 22 | Metlaoui | 2; 21 | (0.54; 0.34) | 5,568 | 56,647 | 65,961 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 23 | Tozeur | 21 | (1.04) | 15,992 | 75,137 | 149,216 |
| 24 | Sidi Bouzid | 2; 21 | (0.17; 1.66) | 25,730 | 130,284 | 244,398 |
| 25 | Mohamed Ben Sassi de Gabès | 21 | (2.51) | 38,579 | 181,264 | 359,977 |
| 26 | Kébili | 21 | (1.23) | 18,957 | 89,071 | 176,888 |
| 27 | Habib Bourguiba de Médenine | 21 | (1.09) | 16,732 | 78,618 | 156,129 |
| 28 | Sadok Mokadem de Jerba | 21 | (1.51) | 23,196 | 108,989 | 216,444 |
| 29 | Zarzis | 21 | (0.92) | 14,171 | 66,583 | 132,229 |
| 30 | Ben Guerdenne | 2; 21 | (0.02; 0.68) | 10,554 | 50,787 | 99,026 |
| 31 | Tataouine | 21 | (0.85) | 13,091 | 61,507 | 122,149 |
| 32 | Nefta | 15; 21 | (0.19; 0.19) | 3,370 | 25,130 | 41,497 |

For example, let us consider the Jendouba hospital, represented by DMU 8. Its efficiency score obtained by using the ranking approach is 0.765 (Table 5). This hospital can become efficient by achieving the following: 19,918 admissions, rather than 15,238; 104,896 outpatient visits, rather than 80,248; and 191,053 emergency visits, rather than 98,661 (for current and target values, refer to Tables 2 and 6, respectively). At this point, it should be noted that it is not logical to force people to visit a given hospital to make it efficient. However, an inefficient hospital can be asked to do its best to accommodate more patients based on its actual capacity to do so, given its amount of resources (inputs).

Table 6 also provides the reference hospitals that each inefficient hospital is compared with in calculating their efficiency scores, in addition to their respective possible benchmarks. Let us again consider the Jendouba hospital (DMU 8): its reference hospitals are the Khair-Eddine (DMU 2) and Houcine Bouzaiene de Gafsa hospitals (DMU 21), with a respective benchmark of (0.69; 0.28). Thus, we have:

$$\sum_{r=1}^{3} y_{r8}^{*} = 0.69 \times \sum_{r=1}^{3} y_{r2} + 0.28 \times \sum_{r=1}^{3} y_{r21} \quad (11)$$

where $\sum_{r=1}^{3} y_{r8}^{*}$ denotes the total target output of DMU 8, $\sum_{r=1}^{3} y_{r2}$ is the total current output of DMU 2, and $\sum_{r=1}^{3} y_{r21}$ is the total current output of DMU 21.

**4. Conclusions**

One requirement in using the DEA methodology to measure efficiency is that all input and output data from each DMU should be available in advance with their crisp values; otherwise, classic DEA models are inapplicable. Many approaches have been developed to handle these types of problems, such as stochastic methods, interval DEA models, and fuzzy theory. However, these approaches do not consider the information's degree of sureness, and the neutrosophic theory demonstrates its power at this moment. In fact, in addition to addressing vague, imprecise, and incomplete data, this theory can also treat indeterminate data.

In the DEA context, only two theoretical works by Edalatpanah [32] and Abdelfattah [34] have handled neutrosophic inputs and outputs. In this paper, however, a real application that consists in measuring and evaluating the efficiency of 32 regional hospitals in Tunisia in a neutrosophic environment. It was demonstrated that neutrosophic DEA can also handle real-world applications.

Two approaches were used: First, the ranking approach as inspired by Abdel-Basset et al. [22] was suggested specifically to solve linear programming models with trapezoidal neutrosophic coefficients. Consequently, a DEA model can be transformed into a linear program; this paper used this approach as a primary alternative given that a trapezoidal neutrosophic number can be reduced to a triangular neutrosophic number. Second, Abdelfattah's [34] parametric approach was used, although this paper used an output-oriented DEA model rather than one that is input-oriented. The two approaches are then compared.

- In terms of their use reveals that the ranking approach is certainly easier, as it does not need the calculation of variation degrees and relies only on the availability of bounds and truth, indeterminacy, and falsity degrees.
- In term of results, the parametric approach is favored, as this approach better interprets the obtained results by providing efficiency scores in ranges bound by the best and worst efficiency scores that a DMU cannot exceed.
- The two approaches act only on DMUs with neutrosophic data; both offer close efficiency scores for these DMUs, or specifically, efficiency scores obtained through the ranking approach are included in the corresponding efficiency intervals obtained through the parametric approach.
- The two approaches give exactly the same efficiency scores for DMUs with crisp data.

From a theoretical perspective, the two approaches applied in this paper measure DMUs' efficiency regardless of the proportion of neutrosophic data. However, such data should be minimized to allow managers to more confidently make their decisions. Moreover, as degrees of truth, indeterminacy, and falsity are subjectively provided, they should be carefully selected.

One noteworthy topic for further research could involve improving one of the existing neutrosophic approaches to solve other DEA models, such as the network DEA. Another adaptation could incorporate another statistical method to better estimate missing and doubtful data bounds' values. Further, a post-analysis of the estimate data could be performed based on obtained efficiency scores and the applied DEA model's adopted orientation.

The DEA method has already demonstrated its power in practice. With the generalization of fuzziness to include neutrosophic logic, this methodology gains the additional capability to evaluate DMUs' performance in terms of their efficiency in real-life applications.

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# Some Elementary Properties of Neutrosophic Integers

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**Abstract:** In this paper, we firstly defined a relation in the set of neutrosophic integers $Z[I]$ and proved that this relation is an equivalence relation. Thus we obtained a partition of $Z[I]$. Secondly we investigated the ordering relation in $Z[I]$ and we have seen that $Z[I]$ is not a totally ordered set. We also gave some relations of positive and negative neutrosophic integers and ordering in $Z[I]$. In the last part of the paper, we introduced the factorial of a positive neutrosophic integer.

**Keywords:** Neutrosophic integers; ordering in neutrosophic integers; factorial of a neutrosophic integer.

## 1. Introduction

Neutrosophy concept is presented by Smarandache to deal with indeterminacy in nature and science [1]. Neutrosophy has a lot of important applications in many fields and hundreds of studies have been done in these fields. One of these fields is neutrosophic number theory. Neutrosophic number theory is a mathematical way to deal with the properties of neutrosophic integers. Neutrosophic number theory was introduced in [2]. In [2], some properties of neutrosophic integers were introduced as division theorem, the form of primes in $Z[I]$.

In this study, it is obtained a partition of the set $Z[I]$ by an equivalence relation. Then it is investigated the ordering relation in $Z[I]$ and have seen that $Z[I]$ is not a totally ordered set, also given some relations of positive and negative neutrosophic integers and ordering in $Z[I]$. In the last part of the paper, we introduced the factorial of a positive neutrosophic integer.

## 2. Preliminaries

In the following, we give some elemantary definitions and results for emphasis.

**Definition 2.1** [3] Let $(R;+,.)$ be a ring and I be an indeterminate element which satisfies $I^2 = I$. The set $R[I] = \{a + bI : a, b \in R\}$ is called a neutrosophic ring generated by I and R under the binary operations of R.

For example; $Z[I] = \{a + bI : a, b \in Z\}$ is a neutrosophic ring generated by I and Z where Z is integers ring. $Z[I]$ is called neutrosophic integers ring.

**Definition 2.2** [4] Let $R[I] = \{a + bI : a, b \in R\}$ be the field of neutrosophic real numbers where R is the field of real numbers. For $a + bI, c + dI \in R[I]$,

$$a+bI\le c+dI \Leftrightarrow a\le c, a+b\le c+d\ .$$

**Theorem 2.1** [4] The relation defined in Definition 2.2 is a partial order relation.

According to Definition 2.2, we are able to define positive neutrosophic real numbers as follows:

$$a+bI\ge 0 \Leftrightarrow a\ge 0, a+b\ge 0.$$

## 3. Ordering in Neutrosophic Integers

**Definition 3.1** Let $a+bI, c+dI\in Z[I]$. If $a+b=c+d$, then the neutrosophic integers $a+bI$ and $c+dI$ are said to be equivalent and denoted by $a+bI\sim c+dI$. Then we write this with symbolically:

$$a+bI\sim c+dI \Leftrightarrow a+b=c+d\ .$$

**Example 3.1** Since $-1+1=2-2$, we have $-1+I\sim 2-2I$ and since $2+3\neq 1+2$, we have $2+3I$ is not equivalent to $1+2I$.

**Theorem 3.1** The relation $"\sim"$ is an equivalence relation.

**Proof.** It can be proved easily.

The relation $"\sim"$ the separates the set $Z[I]$ into equivalence classes. The equivalence class of any $a+bI\in Z[I]$ denoted by $\overline{a+bI}$ and

$$\overline{a+bI}=\{x+yI : x+yI\in Z[I], x+yI\sim a+bI\}\ .$$

If we match $a+bI\in Z[I]$ to the point $(a,b)$ on the cartesian plane, then the equivalence class $\overline{a+bI}$ is the set of the points $(x,y)$ where $x,y\in Z$ on the line $x+y=a+b$.

**Example 3.2**

$$\begin{aligned}\overline{0+0I}&=\{x+yI : x+yI\in Z[I], x+yI\sim 0+0I\}\\&=\{x+yI : x,y\in Z, x+y=0\}\\&=\{...,-2+2I,-1+I,0+0I,1-I,2-2I,...\}.\end{aligned}$$

$\overline{0+0I}=\bar{0}$ is the set of the points $(x,y)$ where $x,y\in Z$ on the line $x+y=0$.

$$\begin{aligned}\overline{1+0I}&=\{x+yI : x+yI\in Z[I], x+yI\sim 1+0I\}\\&=\{x+yI : x,y\in Z, x+y=1\}\\&=\{...,-2+3I,-1+2I,0+I,1-0I,2-I,...\}.\end{aligned}$$

$\overline{1+0I}=\bar{1}$ is the set of the points $(x,y)$ where $x,y\in Z$ on the line $x+y=1$.

If we define the set $D=\{\overline{a+bI} : a+bI\in Z[I]\}$, then $D=\{,,,\overline{-2},\overline{-1},\bar{0},\bar{1},\bar{2},...\}=\{\bar{m} : m\in Z\}$. For $m,n\in Z$ and $m\neq n$, we see that $\bar{m}\cap\bar{n}=\varnothing$ and $\bigcup_{m\in Z}\bar{m}=Z[I]$. Then it is also obvious that the set D is a partition of $Z[I]$.

Definition 2.2 is valid for $Z[I]$. Let's rewrite it for topic integrity:

**Definition 3.2** Let $a+bI, c+dI\in Z[I]$. If $a\le c$ and $a+b\le c+d$, we say that the neutrosophic integer $a+bI$ is less than or equal to $c+dI$ and denoted by $a+bI\le c+dI$. Shortly, we write:

$$a+bI\le c+dI \Leftrightarrow a\le c, a+b\le c+d\ .$$

Note that the relation $"\leq"$ is an partially ordering relation. Hence the set $Z[I]$ is a partially ordered set according to the relation $"\leq"$ but it is not an totally ordered set. Because, every element of $Z[I]$ can not be compared. For example; $1-2I$ and $-1+3I$ are incomparable. That is, $1-2I \nleq -1+3I$ and $-1+3I \nleq 1-2I$.

**Example 3.3** The set of $x+yI \in Z[I]$ which satisfy $1+I \leq x+yI$ on the cartesian plane is drawn below:

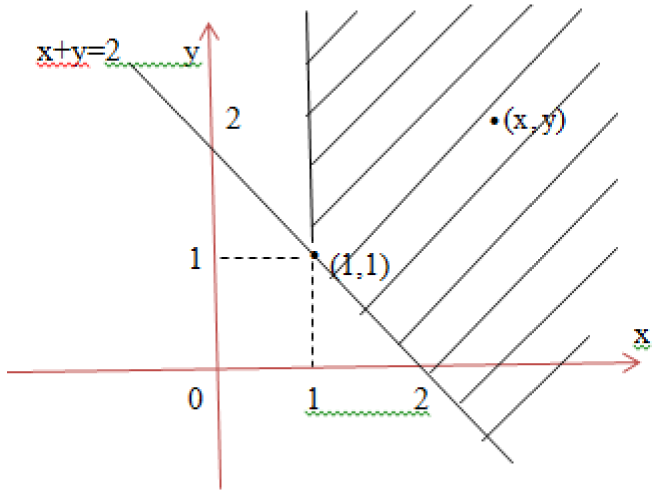


**Figure 1.** The set of $x+yI \in Z[I]$ which satisfy $1+I \leq x+yI$ on the cartesian plane.

**Corollary 2.1** Let $a+bI \in Z[I]$.

i) $a+bI \geq 0 \Leftrightarrow a \geq 0$ and $a+b \geq 0$,

ii) $a+bI \leq 0 \Leftrightarrow a \leq 0$ and $a+b \leq 0$.

**Proof.** The first relation was given in [4]. (i) and (ii) can be proven using the Definition 3.2.

If we match $a+bI \in Z[I]$ to the point $(a,b)$ on the cartesian plane, we can show the regions of positive and negative neutrosophic integers:

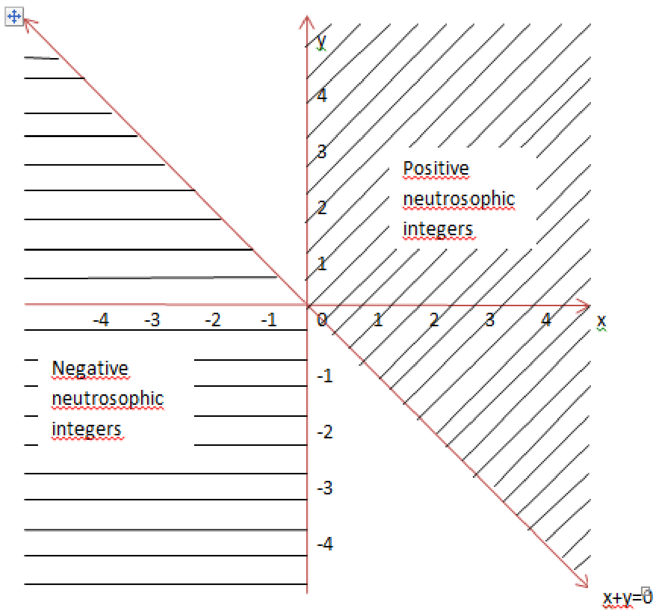


**Figure 2.** Positive and negative neutrosophic integers on cartesian plane.

We denote the set of positive neutrosophic integers by $Z[I]^{+}$. We know that the set $Z[I]^{+}$ is not totally ordered set. We can see that $1\le 1+I\le 2$ and $1\le 2-I\le 2$ but $1+I$ and $2-I$ are incomparable. $0+0I$ is the smallest element of the set $Z[I]^{+}\cup\{0+0I\}$. But the set $Z[I]^{+}$ has not smallest element.

The subsemilattice of the set $Z[I]^{+}\cup\{0+0I\}$ is given the following figure:

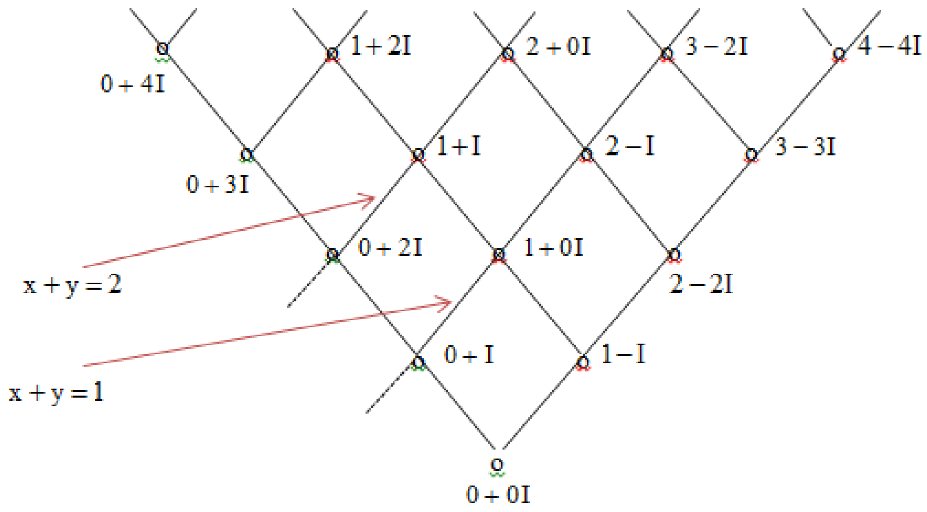


**Figure 3.** The subsemilattice of the set $Z[I]^{+}\cup\{0+0I\}$.

**Theorem 3.2** Let $x=a+bI, y=c+dI\in Z[I]$. Then $x\le y$ if and only if there exists an $u\in Z[I]$ such that $u\ge 0$ and $x+u=y$.

**Proof.** Suppose that there exists an $u\in Z[I]$ such that $u\ge 0$ and $x+u=y$. Then, if $u=u_1+u_2I$, we get $u_1\ge 0$ and $u_1+u_2\ge 0$. Also since $x+u=y$, we have $a+bI+u_1+u_2I=c+dI$. So $a+u_1=c$ and $b+u_2=d$ or $u_1=c-a$ and $u_2=d-b$. Since $u_1\ge 0$, we get $c-a\ge 0$ or $a\le c$. Also since $u_1+u_2\ge 0$, we have $c-a+d-b\ge 0$ or $a+b\le c+d$. Hence since $a\le c$ and $a+b\le c+d$, we see that $x\le y$. Conversely, let $x\le y$. Then $a+b\le c+d$. Hence we have $a\le c$ and $a+b\le c+d$ in Z. Then if we say $c-a=u_1$ and $d-b=u_2$, we see that $u_1\ge 0$ and $u_1+u_2\ge 0$. Then we have $u=u_1+u_2I\in Z[I]$ and $u\ge 0$.

$$
\begin{aligned}
x+u &= a+bI+u_1+u_2I\\
&= a+bI+c-a+(d-b)I\\
&= c+dI\\
&= y.
\end{aligned}
$$

**Example 3.4** We know that $-3+2I\le 2+I$. Then $-3+2I+5-I=2+I$ and $5-I\ge 0$.

**Theorem 3.3** Let $x=x_1+x_2I, y=y_1+y_2I, z=z_1+z_2I$ and $u=u_1+u_2I\in Z[I]$. Then

(i) $x\le y\Leftrightarrow x+z\le y+z$,

(ii) $x\le y$ and $z\le u\Rightarrow x+z\le y+u$,

(iii) $x\le y$ and $z\ge 0\Rightarrow xz\le yz$,

(iv) $x \le y$ and $z \le 0 \Rightarrow xz \ge yz$,

**Proof.** (i) Since $x+z = x_1 + z_1 + (x_2 + z_2)I$ and $y+z = y_1 + z_1 + (y_2 + z_2)I$, we have

$$\begin{aligned} x \le y &\Leftrightarrow x_1 + x_2 I \le y_1 + y_2 I \\ &\Leftrightarrow x_1 \le y_1 \text{ and } x_1 + x_2 \le y_1 + y_2 \text{ in } Z \\ &\Leftrightarrow x_1 + z_1 \le y_1 + z_1 \text{ and } x_1 + x_2 + z_1 + z_2 \le y_1 + y_2 + z_1 + z_2 \text{ for } z_1, z_2 \in Z \\ &\Leftrightarrow x_1 + z_1 + (x_2 + z_2)I \le y_1 + z_1 + (y_2 + z_2)I \\ &\Leftrightarrow x+z \le y+z. \end{aligned}$$

(ii) Since $x+z = x_1 + z_1 + (x_2 + z_2)I$ and $y+z = y_1 + u_1 + (y_2 + u_2)I$, we have

$$\begin{aligned} x \le y \text{ and } z \le u &\Rightarrow x_1 + x_2 I \le y_1 + y_2 I \text{ and } z_1 + z_2 I \le u_1 + u_2 I \\ &\Rightarrow x_1 \le y_1, x_1 + x_2 \le y_1 + y_2, z_1 \le u_1 \text{ and } z_1 + z_2 \le u_1 + u_2 \\ &\Rightarrow x_1 + z_1 \le y_1 + u_1, x_1 + x_2 + z_1 + z_2 \le y_1 + y_2 + u_1 + u_2 \\ &\Rightarrow x_1 + z_1 + (x_2 + z_2)I \le y_1 + u_1 + (y_2 + u_2)I \\ &\Rightarrow x+z \le y+u. \end{aligned}$$

(iii) Let $z = z_1 + z_2 I \ge 0$. Then $z_1 \ge 0$ and $z_1 + z_2 \ge 0$. Since $xz = x_1 z_1 + (x_1 z_2 + x_2 z_1 + x_2 z_2)I$ and $yz = y_1 z_1 + (y_1 z_2 + y_2 z_1 + y_2 z_2)I$, we have

$$\begin{aligned} x \le y &\Leftrightarrow x_1 + x_2 I \le y_1 + y_2 I \\ &\Leftrightarrow x_1 \le y_1, x_1 + x_2 \le y_1 + y_2 \\ &\Leftrightarrow x_1 z_1 \le y_1 z_1 \text{ and } (x_1 + x_2)(z_1 + z_2) \le (y_1 + y_2)(z_1 + z_2) \\ &\Leftrightarrow x_1 z_1 \le y_1 z_1 \text{ and } x_1 z_1 + x_1 z_2 + x_2 z_1 + x_2 z_2 \le y_1 z_1 + y_1 z_2 + y_2 z_1 + y_2 z_2 \\ &\Leftrightarrow x_1 z_1 + (x_1 z_2 + x_2 z_1 + x_2 z_2)I \le y_1 z_1 + (y_1 z_2 + y_2 z_1 + y_2 z_2)I \\ &\Leftrightarrow xz \le yz. \end{aligned}$$

iv) Let $z = z_1 + z_2 I \le 0$. Then $z_1 \le 0$ and $z_1 + z_2 \le 0$. Since $xz = x_1 z_1 + (x_1 z_2 + x_2 z_1 + x_2 z_2)I$ and $yz = y_1 z_1 + (y_1 z_2 + y_2 z_1 + y_2 z_2)I$, we have,

$$\begin{aligned} x \le y &\Leftrightarrow x_1 + x_2 I \le y_1 + y_2 I \\ &\Leftrightarrow x_1 \le y_1, x_1 + x_2 \le y_1 + y_2 \\ &\Leftrightarrow x_1 z_1 \ge y_1 z_1 \text{ and } (x_1 + x_2)(z_1 + z_2) \ge (y_1 + y_2)(z_1 + z_2) \\ &\Leftrightarrow x_1 z_1 \ge y_1 z_1 \text{ and } x_1 z_1 + x_1 z_2 + x_2 z_1 + x_2 z_2 \ge y_1 z_1 + y_1 z_2 + y_2 z_1 + y_2 z_2 \\ &\Leftrightarrow x_1 z_1 + (x_1 z_2 + x_2 z_1 + x_2 z_2)I \ge y_1 z_1 + (y_1 z_2 + y_2 z_1 + y_2 z_2)I \\ &\Leftrightarrow xz \ge yz. \end{aligned}$$

## 4. Factorial of a Positive Neutrosophic Number

It is known that $n! = n.(n-1)...2.1$ for a $n \in Z^+$ and $0! = 1$. This is the product of all integers less than or equal to n on the positive real axis of the coordinate system.

Now we want to extend the factorial concept in $Z$ to $Z[I]$. For $n \in Z^+$, we have $n = n + 0I \in Z[I]$. The we can write $(n+0I)! = (n+0I).(n-1+0I)...(2+0I).(1+0I)$. The numbers $n+0I, n-1+0I, ...2+0I, 1+0I$ are some positive neutrosophic integers less than or equal to $n+0I$. If we match these numbers to the points $(n,0),(n-1,0),...,(2,0),(1,0)$, we see that they are on the half line $y = 0.x = 0$.

Now we take $5+5I \in Z[I]$. Then the numbers $5+5I, 4+4I, 3+3I, 2+2I, 1+I$ are some positive neutrosophic integers less than or equal to $5+5I$. If we match these numbers to the points

$(5,5),(4,4),(3,3),(2,2),(1,1)$, we see that they are on the half line $y = x$. We can write

$$\begin{aligned}(5+5I)! &= (5+5I)(4+4I)(3+3I)(2+2I)(1+I)\\ &= 5.4.3.2.1.(1+I)^5\\ &= 5!(1+I)^5\end{aligned}$$

Now we construct $(12+16I)!$ similarly. The points $(12,16),(9,12),(6,8),(3,4)$ are on the half line $y = \frac{16}{12}x = \frac{4}{3}x$. The corresponding neutrosophic integers $12+16I, 9+12I, 6+8I, 3+4I$ are less than or equal to $12+16I$. So we can write

$$\begin{aligned}(12+16I)! &= (12+16I)(9+12I)(6+8I)(3+4I)\\ &= 4.3.2.1.(3+4I)^4\\ &= 4!(3+4I)^4\end{aligned}$$

Now we are ready to define the factorial of a positive neutrosophic integer:

**Definition 4.1** Let $a+bI \in Z[I]$. Then

$$(a+bI)! = d!\left(\frac{a}{d}+\frac{b}{d}I\right)^d$$

where $d = \gcd\{a,b\}$ (gcd:greatest common divisor).

**Example 4.1**

i) $5! = (5+0I)! = 5!\left(\frac{5}{5}+\frac{0}{5}I\right)^5 = 5!+0I$ since $\gcd\{5,0\} = 5$.

ii) $(0+5I)! = 5!\left(\frac{0}{5}+\frac{5}{5}I\right)^5 = 0+5!I$ since $\gcd\{0,5\} = 5$.

iii) $(9-3I)! = 3!\left(\frac{9}{3}-\frac{3}{3}I\right)^3 = 3!(3-I)^3$ since $\gcd\{9,-3\} = 3$.

The following Theorem and its proof were given for the neutrosophic n square matrices in [5, Theorem 3.6].

**Theorem 4.1** Let $a+bI \in Z[I]$. Then,

$$(a+bI)^n = a^n + \left((a+b)^n - a^n\right)I$$

for $n \in Z^+$.

**Proof.** We use induction on n. For n=1, the above equality is true. Suppose that the claim is true for n-1. That is, $(a+bI)^{n-1} = a^{n-1} + \left((a+b)^{n-1} - a^{n-1}\right)I$. Then we have

$$\begin{aligned}(a+bI)^n &= (a+bI)^{n-1}(a+bI)\\ &= \left(a^{n-1} + \left((a+b)^{n-1} - a^{n-1}\right)I\right)(a+bI)\\ &= a^n + \left(a^{n-1}b + (a+b)^{n-1}a - a^n + (a+b)^{n-1}b - a^{n-1}b\right)I\end{aligned}$$

$$= a^n + \left((a+b)^{n-1}(a+b) - a^n\right)I$$
$$= a^n + \left((a+b)^n - a^n\right)I$$

Therefore Theorem is true.

**Corollary 4.1** Let $a + bI \in Z[I]^+$. Then

$$(a+bI)! = d!\left\{\left(\frac{a}{d}\right)^d + \left[\left(\frac{a}{d}+\frac{b}{d}\right)^d - \left(\frac{a}{d}\right)^d\right]I\right\}$$

where $d = \gcd\{a, b\}$.

**Proof.** It is clear by Definition 4.1 and Theorem 4.1.

**5. Conclusions**

In this paper, it is obtained a partition of the set $Z[I]$ by an equivalence relation. Then, it is investigated the ordering relation in $Z[I]$ and have seen that $Z[I]$ is not a totally ordered set, also given some relations of positive and negative neutrosophic integers and ordering in $Z[I]$. In the last part of the paper, we introduced the factorial of a positive neutrosophic integer. In our future studies, we intend to continue to examine the properties of $Z[I]$.

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# An Integrated Neutrosophic SWARA and VIKOR Method for Ranking Risks of Green Supply Chain

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**Abstract**: The green supply chain (GSC) plays a vital role for companies and organizations. Though there are several risks thread GSC Hence, these risks need to be ranked for companies. So, the goal of this study ranking these risks under a neutrosophic environment due to this problem contains uncertain information. So, this study proposed a multi-criteria decision making (MCDM) for dealing with conflicting criteria and used MCDM methods. This study introduces an integrated model with Stepwise Weight Assessment Ratio Analysis (SWARA) and visekriterijumsko kompromisno rangiranje (VIKOR). The SWARA method is used to calculate the weights of criteria and the VIKOR method is used to rank the risks of GSC based on six main criteria and twenty sub-criteria with ten risks (alternatives). Then the proposed model was evaluated by a numerical example. Finally, the sensitivity analysis is conducted.

**Keywords**: Green Supply Chain (GSC), Neutrosophic, SWARA, VIKOR, Risks, SVNSs

## 1. **Introduction**

Green Supply Chain (GSC) introduces several benefits and advantages to companies like increasing the financial power and enable companies to share their market strongly by improving the capacity of the environment and reduce the negative impact of environmental[1]. The gaining advantage competitive and keep it is a vital role for the company in performing the creativities green in GSC[2, 3]. The success of establishments in the supply chain becomes more difficult[4].

There are several risks when performing the initiatives green in GSC[5]. Reduce cost and increase customer satisfaction are the goals for improving performance in the supply chain[6]. The risks of GSC make many problems in operations and reduce GSC performance[7]. There are many problems that may result from risks of GSC like negative impact of environmental, issues of quality, failure in operations, reduce performance, and disarray of supply materials[8]. So, these risks are necessary to analysis and ranking for companies for adoption the initiatives green in GSC.

The analysis and rank risks in GSC contain vague, inconsistent, and uncertain information[7]. To overcome this uncertainty and vague information some studies proposed fuzzy sets. They used linguistic terms for their assessment. But the fuzzy set cannot deal with indeterminacy [9]. So, this study proposed the neutrosophic set to overcome this uncertainty information. The neutrosophic set is generalized to fuzzy sets. It is contained with truth, indeterminacy, and false (T.I,F). The neutrosophic sets are proposed in several fields like manufacturing, healthcare and others [10-12]. In this paper proposed he single valued neutrosophic sets (SVNSs). The SVNSs is a subset of neutrosophic sets. It work with three value (T,I,F).

The GSC contains many conflict criteria. So, use multi-criteria decision-making (MCDM) to deal with this problem. This study proposed two MCDM methods. First, the SWARA method to calculate the weights of criteria. The SWARA has two advantages first, the criteria are compensatory. Second, the criteria are independent of each other. Then the VIKOR method is applied to rank the risks of GSC [13]. The VIKOR method is used to solve problems with conflicting criteria [14]. This paper used the VIKOR method for ranking the risks of GSC.

The rest of this paper is organized as follows: the literature review is presented in section 2. Section 3 introduces the methodology of this paper. Section 4 introduces the numerical example to validate the methodology. The sensitivity analysis is presented in section 5. Finally, section 6 introduces the conclusion of this paper.

## 2. **Literature Review**

There are several works to evaluate and analyze the risks of GSC[15, 16]. For instance Allen H.Hu et al.[17] used the analysis of effects and failures mode to rank and analysis the risks of the green component to with the European Union in compliance. They used the fuzzy AHP to calculate the weights of four criteria. Then the risks are ranked for each green component. Zhen-kun Yang and Jian Li [18] are ranked the risks of GSC and describe the operations of GSC. They used the fuzzy AHP to calcite the weights of the criteria and then rank the risks of GSC. The aim of their study to introduce the risk control of organization and reliability for selection of supply chain.

Dan-li Du et al. [19] used the gray theory for assessing the risks manufacturing of GSC. The aim of their study that provides stability of running the GSC and evade risks appearing. Li Qianlei [20] used the systematic analysis to recognize the risks of products of agriculture GSC and introduce measures risks management for agriculture products GSC. Xiaojun Wang et al. [7] proposed two phase fuzzy AHP for evaluation risks of GSC.

This problem contains uncertain information[21]. Wei Wang et al. [22] discuss the demand uncertainty in the GSC. Kuo-Jui Wu et al. [23] discuss uncertainty for exploring decisive factors in GSC practice. They used the fuzzy DEMATEL to overcome this uncertainty. To overcome this uncertainty information proposed single-valued neutrosophic sets. M.Abdel-Baset et al. [24] proposed single-valued neutrosophic sets to assess the GSC management practices. The risks of GSC contain different conflicting criteria so, the MCDM is proposed to deal with these criteria. [25] Morteza Yazdani proposed an integrated MCDM for GSC. Hsiu Mei Wong Chen et al. [26] proposed the fuzzy MCDM methods for GS selection.

This study proposed the SWARA method for calculating the weights of criteria. Serap Akcan and Mehmet Ali Taş [27] proposed the SWARA method for green supplier assessment to decrease environmental risk factors. Selçuk Perçin [28] proposed a fuzzy SWARA method for outsourcing provider selection. After calculating the weights of criteria then needs to rank the risks of GSC. The VIKOR method is used to rank the risks of GSC. Reza Rostamzadeh et al. [29] proposed the fuzzy VIKOR method for assessment GSC management practices. Xiaolu Zhang and Xiaoming Xing [30] introduce the VIKOR method for assessing the GSC initiatives.

From the literature review, no research takes into consideration the indeterminacy value. So, in this study introduce the SVNSs to overcome this uncertain information. Then the SWARA and VIKOR methods are not used in previous research with this problem. So, the SWARA is used to calcite the weights of criteria and the VIKOR method to rank the risks of GSC.

## 3. Methodology

The methodology of this study is proposed for ranking the risks of GSC, which it contains from two main stages. The first stage collects criteria and risks dimension and proposed the SWARA method to calculate the weights of criteria. The second stage proposed the VIKOR method to rank the risks of GSC.

### 3.1. SWARA Method

This method is used to calculate the weights of criteria. It is a relatively simple use. Fig 1. Show the SWARA steps. The steps of SWARA is organized as follow [13]:

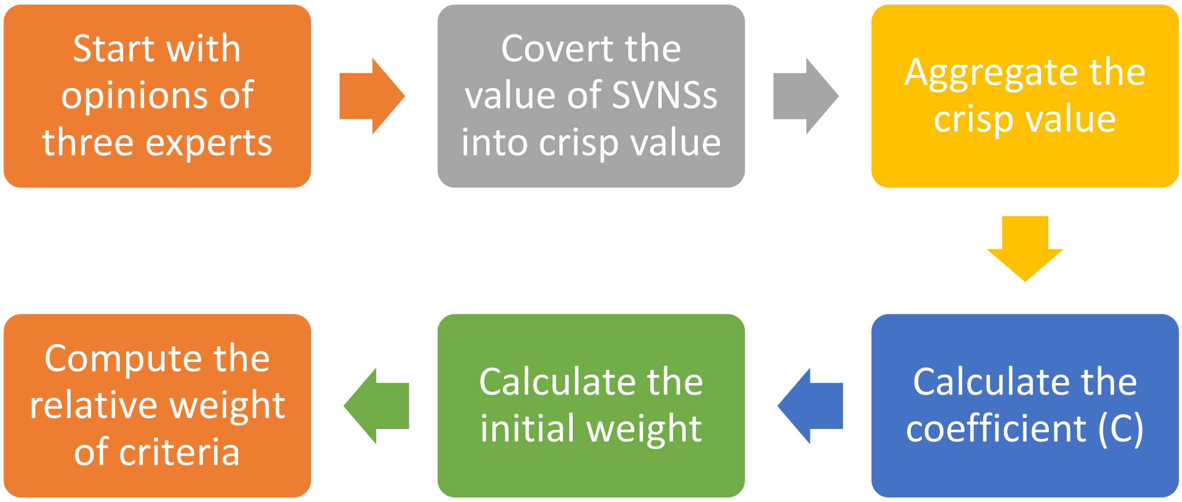


**Fig 1.** The steps of the SWARA method

**Step 1.** Start with the opinions of three experts and decision-makers with the linguistic terms in Table 1.

**Step 2.** Convert the value of single-valued neutrosophic numbers (SVNNs) into crisp value by using the following score function

$$s(P_m^D) = \frac{2 + T_m^D - I_m^D - F_m^D}{3} \quad (1)$$

$T_m^D, I_m^D, F_m^D$ Presents truth, indeterminacy, and falsity of the SVNNs and D refers to decision-makers

**Step 3**. Aggregate the crisp value to obtain one value by using the following equation

$$P_m = \frac{\sum_{D=1}^{D} P_m}{D} \quad (2)$$

**Step 4.** Calculate the coefficient (C) by using the following equation

$$C_m = \begin{cases} 1, & m = 1 \\ P_{m+1}, & m > 1 \end{cases} \quad m = 1,2,3, \ldots .n \; number\ of\ criteria \quad (3)$$

**Step 5.** Calculate the initial weight by using the following equation

$$A_m = \begin{cases} 1, & m = 1 \\ \frac{A_{m-1}}{C_m}, & m > 1 \end{cases} \quad m = 1,2,3, \ldots .n \; number\ of\ criteria \quad (4)$$

**Step 6.** Compute the relative weight of criteria by using the following equation

$$W_m = \frac{A_m}{\sum_{m=1}^{n} A_m} \quad (5)$$

### 3.2 VIKOR Method

The VIKOR method is used to rank the risks of GSC Fig 2. Show the steps of the VIKOR method. The steps of the VIKOR method is organized as follow [31]:


**Fig 2.** The steps of the VIKOR method

**Step 7.** Start with building the decision matrix between the criteria and alternatives (risks) by opinions of experts with the linguistic term in Table 1 by using the following equation. Then convert the SVNNs to the crisp value by Eq. (1). Then combine the decision matrix into one matrix by using Eq. (2).

$$\mathrm{P}^{\mathrm{D}} = \begin{bmatrix} \mathrm{P}_{11}^{\mathrm{D}} & \cdots & \mathrm{P}_{1y}^{\mathrm{D}} \\ \vdots & \ddots & \vdots \\ \mathrm{P}_{m1}^{\mathrm{D}} & \cdots & \mathrm{P}_{my}^{\mathrm{D}} \end{bmatrix} \quad m = 1,2,3, \dots n \,; y = 1,2,3, \dots x$$

(6)

**Step 8.** Calculate the best and worst solution for positive and negative criteria

Best solution $P_m{}^{+} = (P_{my})_{max}$ for positive criteria $P_m{}^{+} = (P_{my})_{min}$ for negative criteria (7)

Worst solution $P_m{}^{-} = (P_{my})_{min}$ for positive criteria $P_m{}^{-} = (P_{my})_{max}$ for negative criteria (8)

**Step 9.** Calculate the value of $g_m$ , $h_m$ by using the following equation

$$g_m = \sum_{y=1}^{x} (W_y * \frac{P_m^{+} - P_{my}}{P_m^{+} - P_m^{-}}) \tag{9}$$

$$h_m = \max_y (W_y * \frac{P_m^{+} - P_{my}}{P_m^{+} - P_m^{-}}) \tag{10}$$

**Step 10.** Calculate the value of $Z_m$ by using the following equation

$$Z_m = f * \frac{g_m - g^*}{g^- - g^*} + (1 - f) * \frac{h_m - h^*}{h^- - h^*} \tag{11}$$

Where $g^* = \min g_m$, $g^- = \max g_m$ , $h^* = \min h_m$, $h^- = \max h_m$ and $f$ is recognized as a weight for the strategy of maximum group utility, whereas $(1 - f)$ is the weight of the separate remorse. Usually, the value of f is set as 0.5. Though, $f$ can set any value from 0 to 1.

**Step 11.** Rank the risks according to ascending value of $Z_m$

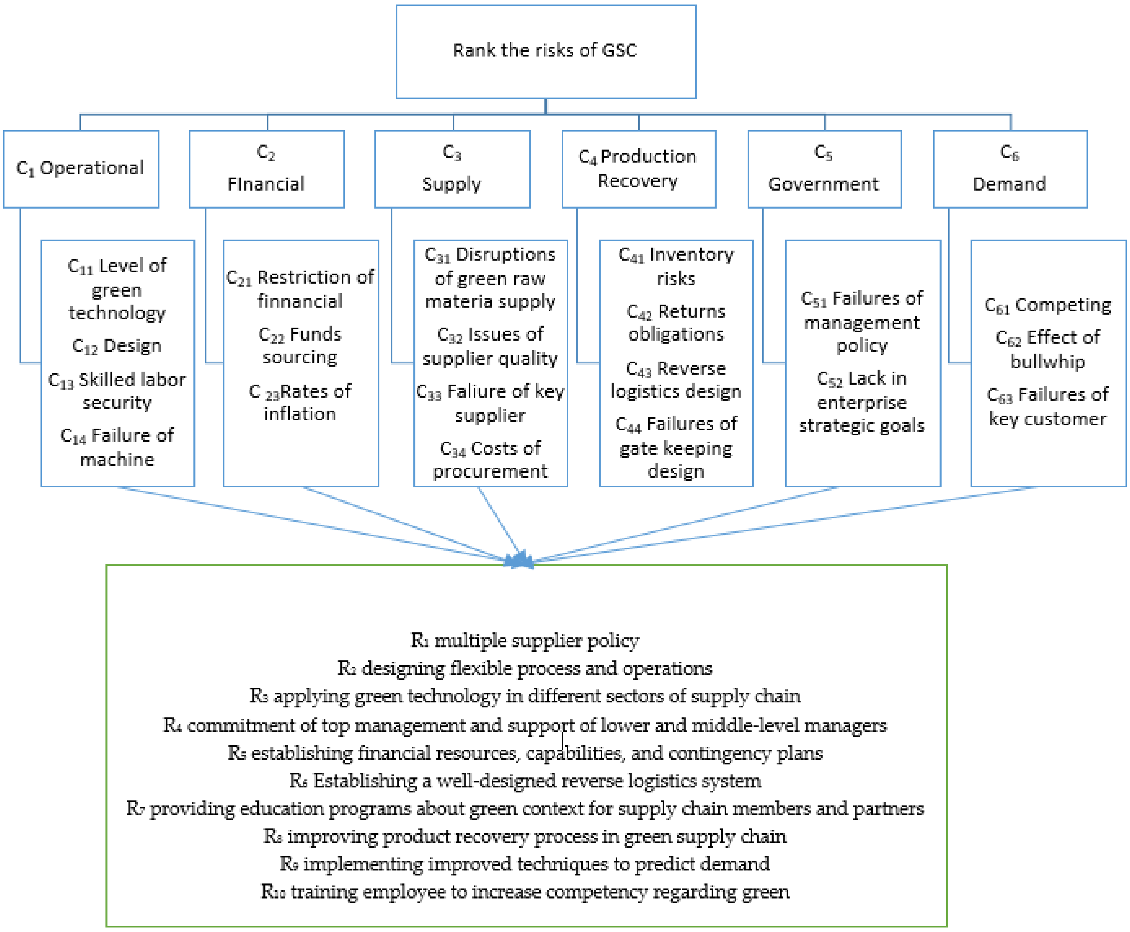


**Fig 3.** The criteria and risks (alternatives) of GSC

## 4. Numerical Example and discussion

The criteria and risks of GSC are extracted from the literature review. Fig 3. shows the criteria and alternatives of this problem. Firstly, the weights of criteria are obtained from section 3.1 by the SWARA method. This problem introduces the three decision-makers and the value of SVNNs is presented in Table 1. The SVNNs contain from (T,I,F). After taking the opinions of experts the three value (T,I,F) is converted to one value by score function by Eq. (1). Then aggregate the three values of three decision-makers into one value by using Eq. (2). Then the coefficient value is obtained by using Eq. (3). Then the initial weight is obtained by using Eq. (4). Then the weights of main and sub-criteria are obtained by using Eq. (5) in Table 2.

The weights of main criteria found that the operational risks are the highest value with 0.369 and demand risks is the lowest weight with value 0.04. The weights of sub-criteria found that the green technology level is the height weight with value 0.1617 and failures of getting keeping design risks is the lowest weight value 0.00655. Fig 4. Show the weights of the main criteria. Fig 5. Show the weights of sub-criteria.

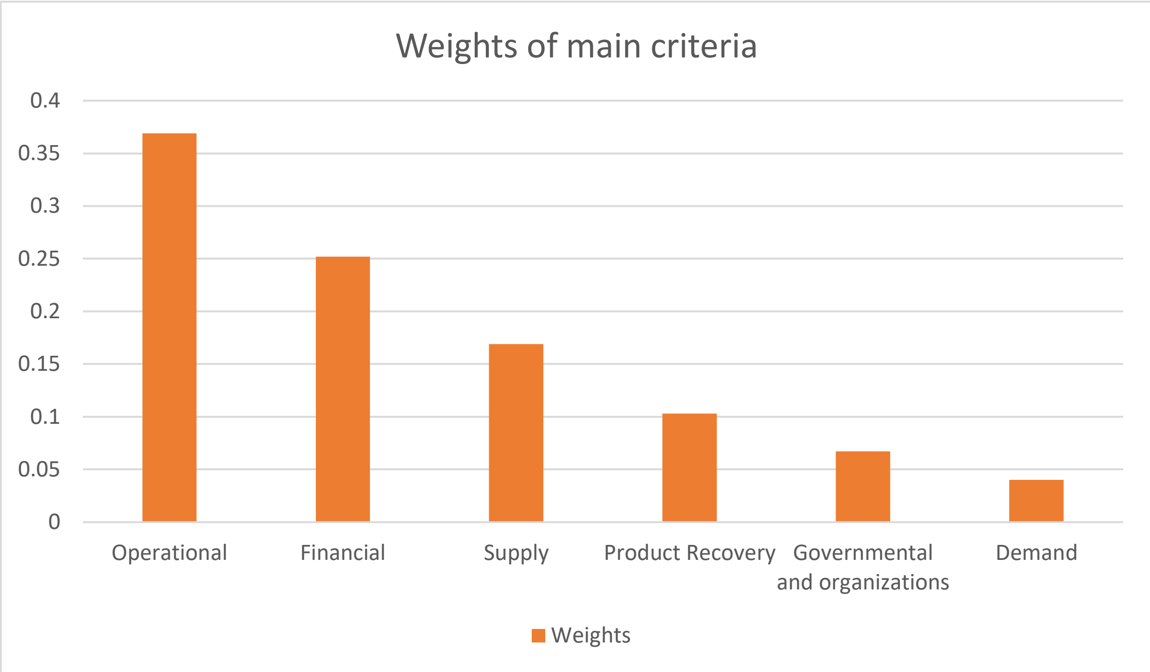


**Fig 4**. The weights of main criteria.

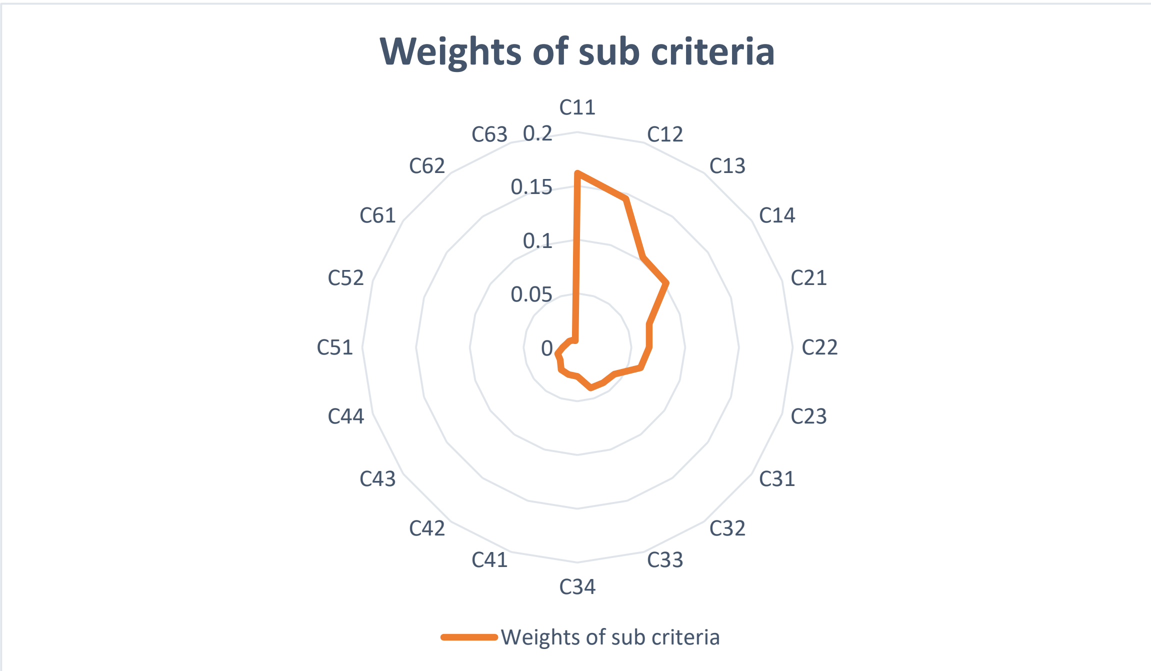


**Fig 5.** The weights of sub criteria.

**Table 1.** SVNSs scale.

| Linguistic Term | Single valued neutrosophic numbers (SVNNs) |
|---|---|

| | |
|---|---|
| Very Wicked | <0.15,0.8,0.8> |
| Wicked | <0.25,0.7,0.7> |
| Medium Wicked | <0.35,0.6,0.6> |
| Medium | <0.45,0.5,0.45> |
| Medium Moral | <0.6,0.4,0.35> |
| Moral | <0.75,0.35,0.25> |
| Very Moral | <0.85,0.2,0.2> |

**Table 2.** The weights of main and sub criteria.

| Main Criteria | Weights | Sub Criteria | Weights |
|---|---|---|---|
| | | $C_{11}$ | 0.161754 |
| $C_1$ | 0.369 | $C_{12}$ | 0.103247 |
| | | $C_{13}$ | 0.061742 |
| | | $C_{14}$ | 0.042257 |
| | | $C_{21}$ | 0.066624 |
| $C_2$ | | $C_{22}$ | 0.025625 |
| | 0.252 | $C_{23}$ | 0.010752 |
| | | $C_{31}$ | 0.145019 |
| $C_3$ | | $C_{32}$ | 0.070173 |
| | 0.169 | $C_{33}$ | 0.027164 |
| | | $C_{34}$ | 0.009644 |
| | | $C_{41}$ | 0.10184 |
| $C_4$ | | $C_{42}$ | 0.040736 |
| | | $C_{43}$ | 0.019872 |
| | 0.103 | $C_{44}$ | 0.006551 |
| $C_5$ | | $C_{51}$ | 0.02644 |
| | 0.067 | $C_{52}$ | 0.01356 |
| | | $C_{61}$ | 0.039717 |
| $C_6$ | | $C_{62}$ | 0.019219 |
| | 0.04 | $C_{63}$ | 0.008064 |

Applying the VIKOR method for ranking the risks of GSC. Start with building the decision matrix between criteria and risks with the SVNNs in Table 1 by opinions of three experts by Eq. (6). Then covert the SVNNs to the crisp value by using Eq. (1). Then combine the three decision matrix into one matrix by using Eq. (2) in Table 3. Then the best and worst solution is obtaining by using Eqs. (7,8) , the procurement criteria are the negative criteria and the rest of the criteria is positive criteria. The value of $g_m$ , $h_m$ is obtained by Eqs. (9,10) in Table 4. Then the value of $Z_m$ is obtained by using Eq. (11) in Table 4. Finally, the risks of GSC is ranking according to ascending order of $Z_m$ in Table 4.

As result of VIKOR, the $R_7$ is the highest rank and the $R_3$ is the lowest rank. Fig 6. Show the rank of risks by VIKOR method.

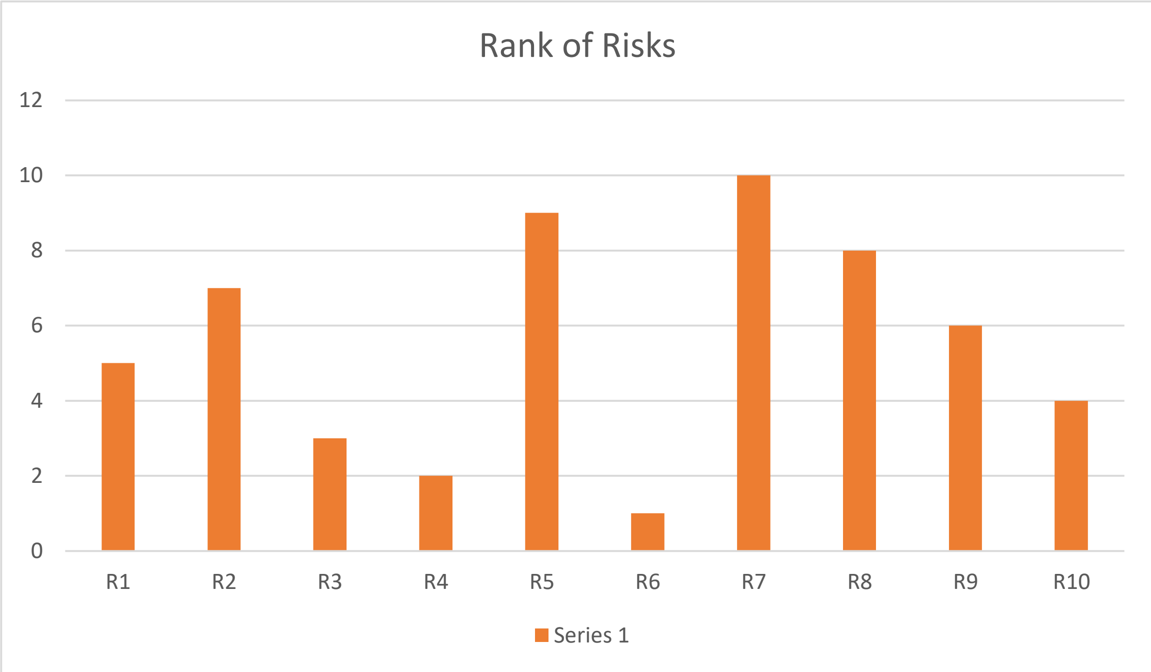


**Fig 6.** The rank of Risks GSC

**Table 3.** The decision matrix between criteria and alternatives.

| Criteria/Risks | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{14}$ | $C_{21}$ | $C_{22}$ | $C_{23}$ | $C_{31}$ | $C_{32}$ | $C_{33}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $R_1$ | 0.6055 | 0.4277 | 0.2833 | 0.6778 | 0.6778 | 0.6055 | 0.6778 | 0.6444 | 0.427 | 0.283 |
| $R_2$ | 0.6444 | 0.7833 | 0.5 | 0.6778 | 0.7500 | 0.6055 | 0.6055 | 0.7500 | 0.427 | 0.5 |
| $R_3$ | 0.3555 | 0.4277 | 0.5333 | 0.5722 | 0.6444 | 0.5 | 0.6055 | 0.5333 | 0.427 | 0.783 |
| $R_4$ | 0.2833 | 0.7833 | 0.7833 | 0.4277 | 0.7500 | 0.6778 | 0.5333 | 0.4611 | 0.677 | 0.783 |
| $R_5$ | 0.7833 | 0.6055 | 0.6055 | 0.5 | 0.4277 | 0.4277 | 0.7500 | 0.5722 | 0.750 | 0.711 |
| $R_6$ | 0.711 | 0.677 | 0.2833 | 0.283 | 0.5333 | 0.427 | 0.355 | 0.283 | 0.572 | 0.750 |
| $R_7$ | 0.816 | 0.750 | 0.6055 | 0.750 | 0.7500 | 0.750 | 0.355 | 0.638 | 0.427 | 0.783 |
| $R_8$ | 0.677 | 0.427 | 0.7833 | 0.2833 | 0.7500 | 0.533 | 0.716 | 0.783 | 0.750 | 0.283 |
| $R_9$ | 0.677 | 0.572 | 0.5 | 0.5722 | 0.3555 | 0.750 | 0.711 | 0.572 | 0.5 | 0.355 |
| $R_{10}$ | 0.716 | 0.283 | 0.4277 | 0.7830 | 0.7167 | 0.816 | 0.283 | 0.427 | 0.355 | 0.750 |

| | $C_{34}$ | $C_{41}$ | $C_{42}$ | $C_{43}$ | $C_{44}$ | $C_{51}$ | $C_{52}$ | $C_{61}$ | $C_{62}$ | $C_{63}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $R_1$ | 0.644 | 0.605 | 0.461 | 0.750 | 0.355 | 0.783 | 0.750 | 0.427 | 0.605 | 0.750 |
| $R_2$ | 0.783 | 0.427 | 0.355 | 0.677 | 0.533 | 0.644 | 0.355 | 0.283 | 0.605 | 0.572 |
| $R_3$ | 0.533 | 0.750 | 0.605 | 0.716 | 0.5 | 0.5 | 0.711 | 0.605 | 0.5 | 0.638 |
| $R_4$ | 0.605 | 0.283 | 0.783 | 0.5 | 0.750 | 0.283 | 0.750 | 0.572 | 0.750 | 0.638 |
| $R_5$ | 0.572 | 0.605 | 0.644 | 0.283 | 0.783 | 0.427 | 0.427 | 0.750 | 0.783 | 0.572 |
| $R_6$ | 0.355 | 0.283 | 0.283 | 0.283 | 0.533 | 0.716 | 0.283 | 0.750 | 0.427 | 0.533 |
| $R_7$ | 0.283 | 0.716 | 0.750 | 0.355 | 0.783 | 0.750 | 0.427 | 0.572 | 0.644 | 0.783 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $R_8$ | 0.355 | 0.750 | 0.355 | 0.427 | 0.750 | 0.5 | 0.427 | 0.711 | 0.355 | 0.2833 |
| $R_9$ | 0.750 | 0.716 | 0.572 | 0.750 | 0.750 | 0.605 | 0.716 | 0.605 | 0.605 | 0.2833 |
| $R_{10}$ | 0.427 | 0.461 | 0.605 | 0.750 | 0.572 | 0.783 | 0.644 | 0.5 | 0.750 | 0.7167 |

**Table 4.** The value of $g_m, h_m, z_m$ and rank of risks

| Risks | $g_m$ | $h_m$ | $z_m$ | Rank |
|---|---|---|---|---|
| $R_1$ | 0.46789 | 0.07342 | 0.400352 | 5 |
| $R_2$ | 0.38739 | 0.070318 | 0.287022 | 7 |
| $R_3$ | 0.50237 | 0.139849 | 0.760587 | 3 |
| $R_4$ | 0.47686 | 0.161754 | 0.834264 | 2 |
| $R_5$ | 0.33777 | 0.061229 | 0.182796 | 9 |
| $R_6$ | 0.61233 | 0.145019 | 0.919877 | 1 |
| $R_7$ | 0.20363 | 0.057324 | 0.000001 | 10 |
| $R_8$ | 0.30981 | 0.07342 | 0.206966 | 8 |
| $R_9$ | 0.4121 | 0.066624 | 0.299557 | 6 |
| $R_{10}$ | 0.47943 | 0.103247 | 0.557282 | 4 |

## 5. Sensitivity Analysis

The weights of criteria affect the rank of risks. So, this paper introduces seven scenarios for changing the weights of criteria in Table 5. Then the weights of sub-criteria are changed in Table 6. Fig 7. shows the rank of risks under different scenarios.

The next step combines the rank with different scenarios into one rank. First, the highest rank takes 10 points and the next take 9 points, and so on [31]. Then calculate the total points. Table 7. Show the aggregation rank under different scenarios. Fig 8. shows the Final rank under different scenarios.

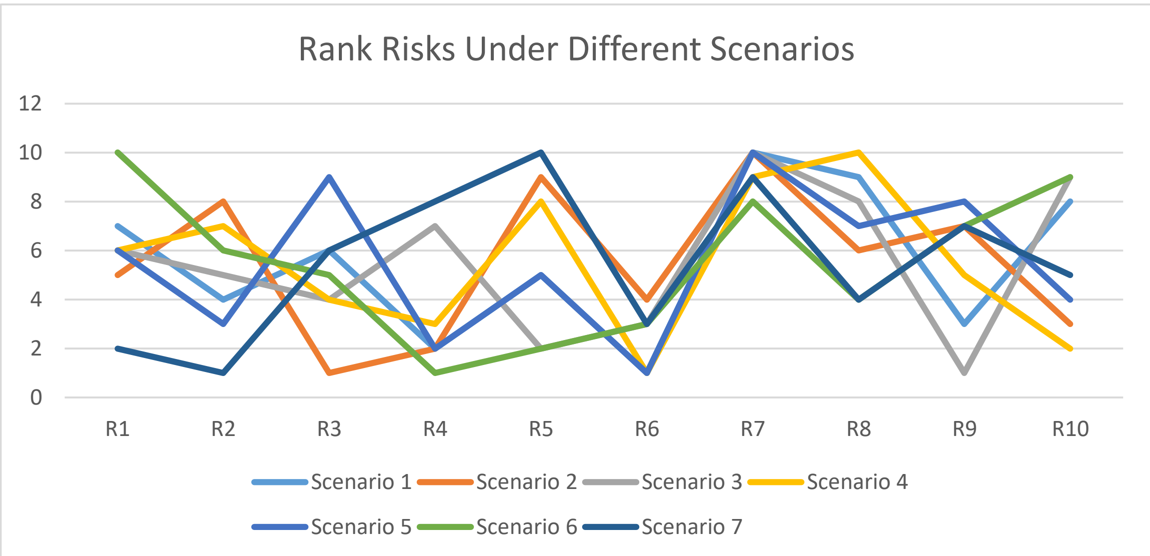


**Fig 7.** The rank of risks under seven scenarios

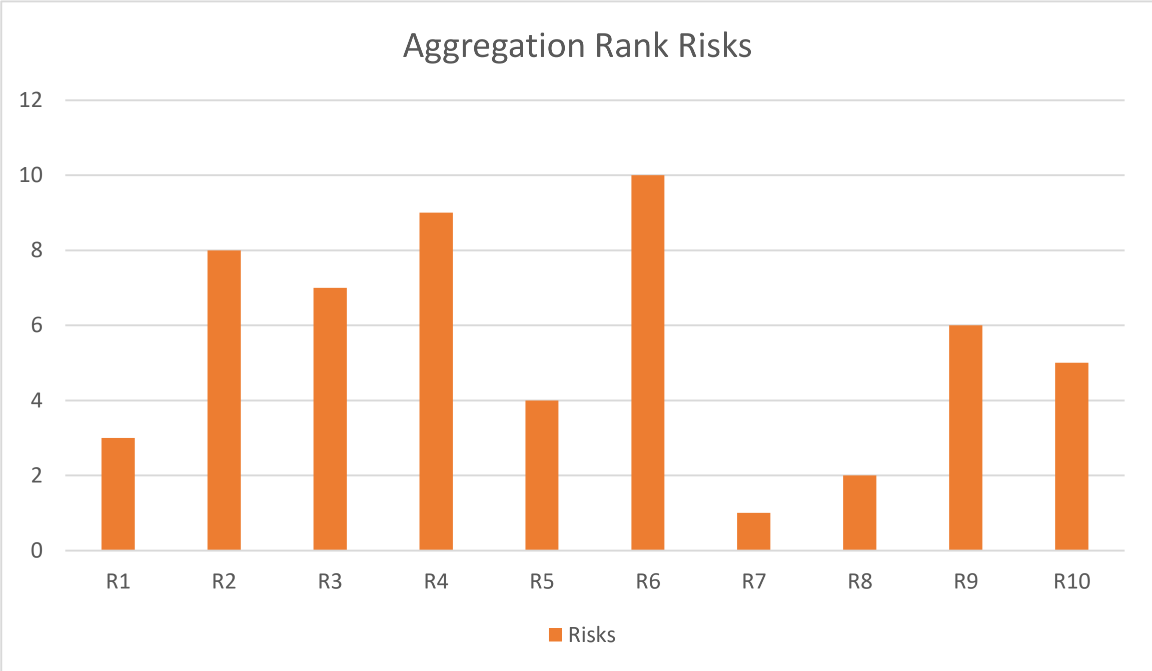


**Fig 8.** The Aggregation rank risks under different scenarios

**Table 5.** Seven scenarios of weights changes

| Scenarios | Operational | Financial | Supply | Production Recovery | Government | Demand |
|---|---|---|---|---|---|---|
| Scenario 1 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |
| Scenario 2 | 0.5 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |
| Scenario 3 | 0.1 | 0.5 | 0.1 | 0.1 | 0.1 | 0.1 |
| Scenario 4 | 0.1 | 0.1 | 0.5 | 0.1 | 0.1 | 0.1 |
| Scenario 5 | 0.1 | 0.1 | 0.1 | 0.5 | 0.1 | 0.1 |
| Scenario 6 | 0.1 | 0.1 | 0.1 | 0.1 | 0.5 | 0.1 |
| Scenario 7 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.5 |

**Table 6.** Seven scenarios of sub criteria weights

| Criteria | Scenario 1 | Scenario 2 | Scenario 3 | Scenario 4 | Scenario 5 | Scenario 6 | Scenario 7 |
|---|---|---|---|---|---|---|---|
| $C_{11}$ | 0.219178 | 0.043836 | 0.043836 | 0.043836 | 0.043836 | 0.043836 | 0.219178 |
| $C_{12}$ | 0.139901 | 0.02798 | 0.02798 | 0.02798 | 0.02798 | 0.02798 | 0.139901 |
| $C_{13}$ | 0.083661 | 0.016732 | 0.016732 | 0.016732 | 0.016732 | 0.016732 | 0.083661 |
| $C_{14}$ | 0.057259 | 0.011452 | 0.011452 | 0.011452 | 0.011452 | 0.011452 | 0.057259 |
| $C_{21}$ | 0.064683 | 0.323416 | 0.064683 | 0.064683 | 0.064683 | 0.064683 | 0.064683 |
| $C_{22}$ | 0.024878 | 0.124391 | 0.024878 | 0.024878 | 0.024878 | 0.024878 | 0.024878 |
| $C_{23}$ | 0.010439 | 0.052193 | 0.010439 | 0.010439 | 0.010439 | 0.010439 | 0.010439 |
| $C_{31}$ | 0.057547 | 0.057547 | 0.287737 | 0.057547 | 0.057547 | 0.057547 | 0.057547 |
| $C_{32}$ | 0.027846 | 0.027846 | 0.139232 | 0.027846 | 0.027846 | 0.027846 | 0.027846 |
| $C_{33}$ | 0.010779 | 0.010779 | 0.053897 | 0.010779 | 0.010779 | 0.010779 | 0.010779 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $C_{34}$ | 0.003827 | 0.003827 | 0.019135 | 0.003827 | 0.003827 | 0.003827 | 0.003827 |
| $C_{41}$ | 0.060261 | 0.060261 | 0.060261 | 0.301303 | 0.060261 | 0.060261 | 0.060261 |
| $C_{42}$ | 0.024104 | 0.024104 | 0.024104 | 0.120521 | 0.024104 | 0.024104 | 0.024104 |
| $C_{43}$ | 0.011759 | 0.011759 | 0.011759 | 0.058794 | 0.011759 | 0.011759 | 0.011759 |
| $C_{44}$ | 0.003876 | 0.003876 | 0.003876 | 0.019382 | 0.003876 | 0.003876 | 0.003876 |
| $C_{51}$ | 0.066101 | 0.066101 | 0.066101 | 0.066101 | 0.330503 | 0.066101 | 0.066101 |
| $C_{52}$ | 0.033899 | 0.033899 | 0.033899 | 0.033899 | 0.169497 | 0.033899 | 0.033899 |
| $C_{61}$ | 0.05928 | 0.05928 | 0.05928 | 0.05928 | 0.05928 | 0.296398 | 0.05928 |
| $C_{62}$ | 0.028685 | 0.028685 | 0.028685 | 0.028685 | 0.028685 | 0.143423 | 0.028685 |
| $C_{63}$ | 0.012036 | 0.012036 | 0.012036 | 0.012036 | 0.012036 | 0.060178 | 0.012036 |

**Table 7.** The aggregation rank under different scenarios

| Risks | Total Points | Rank |
|---|---|---|
| $R_1$ | 42 | 3 |
| $R_2$ | 34 | 8 |
| $R_3$ | 35 | 7 |
| $R_4$ | 25 | 9 |
| $R_5$ | 41 | 4 |
| $R_6$ | 16 | 10 |
| $R_7$ | 66 | 1 |
| $R_8$ | 48 | 2 |
| $R_9$ | 38 | 6 |
| $R_{10}$ | 40 | 5 |

## 6. Conclusions

GSC plays a vital part in enhancement the ecological performance of companies. But the GSC has many risks. So, these risks need to rank for companies and organizations. This work proposed a hybrid neutrosophic MCDM for ranking the risks of GSC using SWARA and VIKOR methods under a neutrosophic environment. The SVNSs are proposed to overcome the uncertainty of information. The SWARA is used to calculate the weights of criteria and the VIKOR method is used to rank the risks of GSC. The proposed methodology is tested by a numerical example with twenty criteria and ten risks (alternatives).

The main contributions in this study proposed a neutrosophic environment to deal with indeterminacy value due to no previous study deal with the indeterminacy value. The SWARA and VIKOR method not used in previous research. In the future study, used other MCDM methods like PRPMETHEE II and ELECTRE.

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# Compactness on Single-Valued Neutrosophic Ideal Topological Spaces

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**Abstract:** In the current paper, particular acheivments of single-valued neutrosophic continuity on a single-valued neutrosophic topological space $(\widetilde{\mathfrak{T}}, \tilde{\tau}^{\tilde{\gamma}}, \tilde{\tau}^{\tilde{\eta}}, \tilde{\tau}^{\tilde{\mu}})$ are introduced. Some necessary implications between them are illustrated. The theories of *r*-single-valued neutrosophic compact, *r*-single-valued neutrosophic ideal compact, *r*-single-valued neutrosophic quasi H-closed and *r*-single-valued neutrosophic compact modulo an single-valued neutrosophic ideal $\tilde{\mathcal{J}}$ are presented and investigated.

**Keywords:** single-valued neutrosophic (almost; weakly) continuous mapping; single-valued neutrosophic ideal (compact; quasi H-closed) and *r*-single-valued neutrosophic compact modulo.

## 1. Introduction

Using a fuzzy ideal $\tilde{\mathcal{J}}$ defined on a fuzzy topological space (FTS) $(\widetilde{\mathfrak{T}}, \tilde{\tau})$, a fuzzy ideal topological space (FITS) $(\widetilde{\mathfrak{T}}, \tilde{\tau}, \tilde{\mathcal{J}})$ is generated. It is a way of generalizing so many notions and results in $(\widetilde{\mathfrak{T}}, \tilde{\tau})$. The main definition of fuzzy topology that is related to the results in this article was established by Šostak in [1]. The notion of fuzzy ideal was created in [2]. Tripathy et al. in [3 - 6] introduced different valuble research studies on (FITS) and gave several forms of fuzzy continuities. Saber and others [7 - 11] have considered several *r*-fuzzy compactnesses in (FITS) $(\widetilde{\mathfrak{T}}, \tilde{\tau}, \tilde{\mathcal{J}})$ and several types of fuzzy continuity.

Smarandache established the idea of the neutrosophic sets [12] in 1998. In terms of neutrosophic sets, there are a membership score $(\tilde{\gamma})$, an indeterminacy score $(\tilde{\eta})$ and a non-membership score $(\tilde{\mu})$ and a neutrosophic value is in the form $(\tilde{\gamma}, \tilde{\eta}, \tilde{\mu})$. In other meaning, in explaining an event or finding of a solution to a problem, a condition is handled according to its truth, not truth and resolution. Hence, the study of neutrosophic sets and neutrosophic logic are useful for decision-making applications in neutrosophic theories and led to too many researches and studies in the field as in [12-25]. It also gives the opportunity to others to establish some approaches in decision-making for neutrosophic theory as in [26-31]. Wang et al, [32] and Kim et al, [33] presented the theory of the neutrosophoic equivalence relation single-valued. Single-valued neutrosophic ideal ($\mathcal{SVNI}$) aspects in single-valued neutrosophic topological spaces ($\mathcal{SVNTS}$), have been introduced and considered by several authors from diverse viewpoints such as in [34-37].

In this research, we foreground the idea of $r$-single-valued neutrosophic (compact, ideal compact and quasi H-closed) in ($\mathcal{SVNTS}$) in the sense of Šostak. We are working on getting some of its important characteristics and results. Moreover, we investigate some properties of single-valued neutrosophic continuous mappings. Finally, some fascinating application of neutrosophic topology in reverse logistics arises could be found as in Abdel-Baset paper articles and others [38-41].

## 2. Preliminaries

**Definition 2.1 [22]** Suppose that $\tilde{\mathfrak{T}}$ is a non-empty set. We mean by a neutrosophic set (briefly, $\mathcal{NS}$) $A$ the objects having the form

$$\mathcal{S} = \{\langle \omega, \tilde{\gamma}_{\mathcal{S}}, \tilde{\eta}_{\mathcal{S}}, \tilde{\mu}_{\mathcal{S}} \rangle : \omega \in \tilde{\mathfrak{T}}\}.$$

Anywhere $\tilde{\mu}_{\mathcal{S}}$ ,$\tilde{\eta}_{\mathcal{S}}$ *and* $\tilde{\gamma}_{\mathcal{S}}$ indicate the degree of non-membership, the degree of indeterminacy, and the degree of membership, respectively of any element $\omega \in \tilde{\mathfrak{T}}$ to the set $\mathcal{S}$.

**Definition 2.2 [32]** Suppose that $\tilde{\mathfrak{T}}$ is a universal set. For $\forall \omega \in \tilde{\mathfrak{T}}$, $0 \leq \tilde{\gamma}_{\mathcal{S}}(\omega) + \tilde{\eta}_{\mathcal{S}}(\omega) + \tilde{\mu}_{\mathcal{S}}(\omega) \leq 3$, by the meanings $\tilde{\gamma}_{\mathcal{S}}$: $\mathcal{S} \to [0.1]$, $\tilde{\eta}_{\mathcal{S}}$: $\mathcal{S} \to [0.1]$ and $\tilde{\mu}_{\mathcal{S}}$: $\mathcal{S} \to [0.1]$, a single-valued neutrosophic set (briefly, $\mathcal{SVNS}$) on $\tilde{\mathfrak{T}}$ is defined by

$$\mathcal{S} = \{\langle \omega, \tilde{\gamma}_{\mathcal{S}}, \tilde{\eta}_{\mathcal{S}}, \tilde{\mu}_{\mathcal{S}} \rangle : \omega \in \tilde{\mathfrak{T}}\}.$$

Now, $\tilde{\mu}_{\mathcal{S}}$ ,$\tilde{\eta}_{\mathcal{S}}$ and $\tilde{\gamma}_{\mathcal{S}}$ are the degrees of falsity, indeterminacy and trueness of $\omega \in \tilde{\mathfrak{T}}$, respectively. We will convey the set of all $\mathcal{SVNS}$s in $\mathcal{S}$ as $I^{\tilde{\mathfrak{T}}}$.

**Definition 2.3 [32]** The accompaniment of a $\mathcal{SVNS}$ $\mathcal{S}$ is indicated by $\mathcal{S}^c$ and is cleared by

$$\tilde{\gamma}_{\mathcal{S}^c}(\omega) = \tilde{\mu}_{\mathcal{S}}(\omega), \quad \tilde{\eta}_{\mathcal{S}^c}(\omega) = 1 - \tilde{\eta}_{\mathcal{S}}(\omega) \; and \; \tilde{\mu}_{\mathcal{S}^c}(\omega) = \tilde{\gamma}_{\mathcal{S}}(\omega).$$

for any $\omega \in \tilde{\mathfrak{T}}$,

**Definition 2.4 [41]** Let $\mathcal{S}, \mathcal{E} \in I^{\tilde{\mathfrak{T}}}$. Then,

1. $\mathcal{S} \subseteq \mathcal{E}$, if, for every $\omega \in \tilde{\mathfrak{T}}$,
$$\tilde{\gamma}_{\mathcal{S}}(\omega) \leq \tilde{\gamma}_{\mathcal{E}}(\omega), \qquad \tilde{\eta}_{\mathcal{S}}(\omega) \geq \tilde{\eta}_{\mathcal{E}}(\omega), \qquad \tilde{\mu}_{\mathcal{S}}(\omega) \geq \tilde{\mu}_{\mathcal{E}}(\omega)$$
2. $\mathcal{S} = \mathcal{E}$ if $\mathcal{S} \subseteq \mathcal{E}$ and $\mathcal{S} \supseteq \mathcal{E}$.
3. $\tilde{0} = \langle 0,1,1 \rangle$ and $\tilde{1} = \langle 1,0,0 \rangle$

**Definition 2.5 [42]** Let $\mathcal{S}, \mathcal{E} \in I^{\tilde{\mathfrak{T}}}$. Then,

1. $\mathcal{S} \cap \mathcal{E}$ is a $\mathcal{SVNS}$ in $\tilde{\mathfrak{T}}$ defined as:
$$\mathcal{S} \cap \mathcal{E} = (\tilde{\gamma}_{\mathcal{S}} \cap \tilde{\gamma}_{\mathcal{E}}, \; \tilde{\eta}_{\mathcal{S}} \cup \tilde{\eta}_{\mathcal{E}}, \; \tilde{\mu}_{\mathcal{S}} \cup \tilde{\mu}_{\mathcal{E}}).$$

Where, $(\tilde{\mu}_{\mathcal{S}} \cup \tilde{\mu}_{\mathcal{E}})(\omega) = \tilde{\mu}_{\mathcal{S}}(\omega) \cup \tilde{\mu}_{\mathcal{E}}(\omega)$ and $(\tilde{\gamma}_{\mathcal{S}} \cap \tilde{\gamma}_{\mathcal{E}})(\omega) = \tilde{\gamma}_{\mathcal{S}}(\omega) \cap \tilde{\gamma}_{\mathcal{E}}(\omega)$, for all $\omega \in \tilde{\mathfrak{T}}$,

1. $\mathcal{S} \cup \mathcal{E}$ is an $\mathcal{SVNS}$ on $\tilde{\mathfrak{T}}$ defined as:

$$\mathcal{S} \cup \mathcal{E} = (\tilde{\gamma}_{\mathcal{S}} \cup \tilde{\gamma}_{\mathcal{E}},\ \tilde{\eta}_{\mathcal{S}} \cap \tilde{\eta}_{\mathcal{E}},\ \tilde{\mu}_{\mathcal{S}} \cap \tilde{\mu}_{\mathcal{E}}).$$

**Definition 2.6 [21]** Suppose that $\tilde{\mathfrak{T}}$ is a nonempty set and $\mathcal{S} \in I^{\tilde{\mathfrak{T}}}$ is having the form $\mathcal{S} = \{\langle \omega, \tilde{\gamma}_{\mathcal{S}}, \tilde{\eta}_{\mathcal{S}}, \tilde{\mu}_{\mathcal{S}} \rangle : \omega \in \tilde{\mathfrak{T}}\}$ on $\tilde{\mathfrak{T}}$. Then,

1. $\left(\bigcap_{j\in\triangle} \mathcal{S}_j\right)(\omega) = \left(\bigcap_{j\in\triangle} \tilde{\gamma}_{\mathcal{S}_j}(\omega),\ \bigcup_{j\in\triangle} \tilde{\eta}_{\mathcal{S}_j}(\omega),\ \bigcup_{j\in\triangle} \tilde{\mu}_{\mathcal{S}_j}(\omega)\right),$

2. $\left(\bigcup_{j\in\triangle} \mathcal{S}_j\right)(\omega) = \left(\bigcup_{j\in\triangle} \tilde{\gamma}_{\mathcal{S}_j}(\omega),\ \bigcap_{j\in\triangle} \tilde{\eta}_{\mathcal{S}_j}(\omega),\ \bigcap_{j\in\triangle} \tilde{\mu}_{\mathcal{S}_j}(\omega)\right).$

**Definition 2.7 [34]** Let $s, t, k \in I_0$ and $s + t + k \le 3$. A single-valued neutrosophic point ($\mathcal{SVNP}$) $x_{s,t,k}$ of $\tilde{\mathfrak{T}}$ is the $\mathcal{SVNS}$ in $I^{\tilde{\mathfrak{T}}}$ for every $\omega \in \mathcal{S}$, defined by

$$x_{s,t,k}(\omega) = \begin{cases} (s,t,k), & if\ x = \omega, \\ (0,1,1), & if\ x \neq \omega. \end{cases}$$

A $\mathcal{SVNP}$ $x_{s,t,k}$ is supposed to belong to a $\mathcal{SVNS}$ $\mathcal{S} = \{\langle \omega, \tilde{\gamma}_{\mathcal{S}}, \tilde{\eta}_{\mathcal{S}}, \tilde{\mu}_{\mathcal{S}} \rangle : \omega \in \tilde{\mathfrak{T}}\} \in I^{\tilde{\mathfrak{T}}}$, (notion: $x_{s.t.p} \in \mathcal{S}$ iff $s < \tilde{\gamma}_{\mathcal{S}}$, $t \ge \tilde{\eta}_{\mathcal{S}}$ and $k \ge \tilde{\mu}_{\mathcal{S}}$), and the set off all $\mathcal{SVNP}$ in $\tilde{\mathfrak{T}}$ indicated by $\mathcal{SVNP}(\tilde{\mathfrak{T}})$. $x_{s,t,k} \in \mathcal{SVNP}(\tilde{\mathfrak{T}})$ quasi-coincident with a $\mathcal{SVNS}$ $\mathcal{S} \in I^{\tilde{\mathfrak{T}}}$ denoted by $x_{s,t,k} q \mathcal{S}$, if

$$s + \tilde{\gamma}_{\mathcal{S}} > 1,\ t + \tilde{\eta}_{\mathcal{S}} \le 1\ ,\ k + \tilde{\mu}_{\mathcal{S}} \le 1.$$

For every $\mathcal{S}, \mathcal{E} \in I^{\tilde{\mathfrak{T}}}$ $\mathcal{S}$ is quasi-coincident with $\mathcal{E}$ indicated by $\mathcal{S}\mathrm{q}\mathcal{E}$, if there exists $x_{s,t,k} \in I^{\tilde{\mathfrak{T}}}$ s.t

$$\tilde{\gamma}_{\mathcal{E}} + \tilde{\gamma}_{\mathcal{S}} > 1,\ \tilde{\eta}_{\mathcal{E}} + \tilde{\eta}_{\mathcal{S}} \le 1 \text{ and } \tilde{\mu}_{\mathcal{E}} + \tilde{\mu}_{\mathcal{S}} \le 1.$$

**Definition 2.8 [25]** Let $\tilde{\tau}^{\tilde{\gamma}}, \tilde{\tau}^{\tilde{\eta}}, \tilde{\tau}^{\tilde{\mu}} : I^{\tilde{\mathfrak{T}}} \to I$ be mappings satisfying the following conditions:

1. $\tilde{\tau}^{\tilde{\gamma}}(\underline{0}) = \tilde{\tau}^{\tilde{\gamma}}(\underline{1}) = 1$ and $\tilde{\tau}^{\tilde{\eta}}(\underline{0}) = \tilde{\tau}^{\tilde{\eta}}(\underline{1}) = \tilde{\tau}^{\tilde{\mu}}(\underline{0}) = \tilde{\tau}^{\tilde{\mu}}(\underline{1}) = 0,$
2. $\tau^{\tilde{\gamma}}(\mathcal{S} \cap \mathcal{E}) \ge \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}) \cap \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E})$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S} \cap \mathcal{E}) \le \tau^{\tilde{\eta}}(\mathcal{S}) \cup \tilde{\tau}^{\tilde{\eta}}(\mathcal{E})$ and $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S} \cap \mathcal{E}) \le \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}) \cup \tilde{\tau}^{\tilde{\mu}}(\mathcal{E})$, for every $\mathcal{S}, \mathcal{E} \in I^{\tilde{\mathfrak{T}}}$,
3. $\tilde{\tau}^{\tilde{\gamma}}(\cup_{j\in\Gamma} \mathcal{S}_j) \ge \cap_{j\in\Gamma} \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j)$, $\tilde{\tau}^{\tilde{\eta}}(\cup_{i\in\Gamma} \mathcal{S}_j) \le \cup_{j\in\Gamma} \tau^{\tilde{\eta}}(\mathcal{S}_j)$ and $\tilde{\tau}^{\tilde{\mu}}(\cup_{j\in\Gamma} \mathcal{S}_j) \le \cup_{j\in\Gamma} \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j)$, for every $\{\mathcal{S}_j,\ j \in \Gamma\} \in I^{\tilde{\mathfrak{T}}}$.

Then $(\tilde{\tau}^{\tilde{\gamma}}, \tilde{\tau}^{\tilde{\eta}}, \tilde{\tau}^{\tilde{\mu}})$ is called single valued neutrosophic topology $\mathcal{SVNT}$. Usually, we will write $\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$ for $(\tilde{\tau}^{\tilde{\gamma}}, \tilde{\tau}^{\tilde{\eta}}, \tilde{\tau}^{\tilde{\mu}})$ and it will cause no indistinctness.

**Definition 2.9 [34]** Let $(\tilde{\mathfrak{T}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNTS}$. Then, for all $\mathcal{S} \in I^{\tilde{\mathfrak{T}}}$ and $r \in I_0$, the single valued neutrosophic (closure and interior) of $\mathcal{S}$ are define by:

$$C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}.r) = \bigcap \{\mathcal{E} \in I^{\tilde{\mathfrak{T}}} : \mathcal{S} \le \mathcal{E},\quad \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}^c) \ge r,\quad \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}^c) \le 1 - r,\quad \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}^c) \le 1 - r\}$$

$$int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}.r) = \bigcup \{\mathcal{E} \in I^{\tilde{\mathfrak{T}}} : \mathcal{S} \ge \mathcal{E},\quad \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}) \ge r,\quad \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}) \le 1 - r,\quad \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}) \le 1 - r\}.$$

**Definition 2.10 [34]** A mapping $\tilde{\mathcal{J}}^{\tilde{\gamma}}, \tilde{\mathcal{J}}^{\tilde{\eta}}, \tilde{\mathcal{J}}^{\tilde{\mu}} : I^{\tilde{\mathfrak{T}}} \to I$ is said to be $\mathcal{SVNJ}$ on $\tilde{\mathfrak{T}}$ if it satisfies the next three conditions for $\mathcal{S}, \mathcal{E} \in I^{\tilde{\mathfrak{T}}}$:

1. $\tilde{\mathcal{J}}^{\tilde{\eta}}(\tilde{0}) = \tilde{\mathcal{J}}^{\tilde{\mu}}(\tilde{0}) = 0$, $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\tilde{0}) = 1$,
2. If $\mathcal{S} \le \mathcal{E}$ then $\tilde{\mathcal{J}}^{\tilde{\eta}}(\mathcal{E}) \ge \tilde{\mathcal{J}}^{\tilde{\eta}}(\mathcal{S})$, $\tilde{\mathcal{J}}^{\tilde{\mu}}(\mathcal{E}) \ge \tilde{\mathcal{J}}^{\tilde{\mu}}(\mathcal{S})$ and $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\mathcal{E}) \le \tilde{\mathcal{J}}^{\tilde{\gamma}}(\mathcal{S})$.
3. $\tilde{\mathcal{J}}^{\tilde{\eta}}(\mathcal{S} \cup \mathcal{E}) \le \tilde{\mathcal{J}}^{\tilde{\eta}}(\mathcal{E}) \cup \tilde{\mathcal{J}}^{\tilde{\eta}}(\mathcal{E})$, $\tilde{\mathcal{J}}^{\tilde{\mu}}(\mathcal{S} \cup \mathcal{E}) \le \tilde{\mathcal{J}}^{\tilde{\mu}}(\mathcal{S}) \cup \tilde{\mathcal{J}}^{\tilde{\mu}}(\mathcal{E})$ and $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\mathcal{S} \cup \mathcal{E}) \ge \tilde{\mathcal{J}}^{\tilde{\gamma}}(\mathcal{S}) \cap \tilde{\mathcal{J}}^{\tilde{\gamma}}(\mathcal{E})$.

Then, $(\tilde{\mathfrak{T}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is said to be a single-valued neutrosophic ideal topological space ($\mathcal{SVNJTS}$).

**Definition 2.12 [36]** A mapping $f:(\widetilde{\mathfrak{X}}_1,\ \tilde{\tau}_1^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}) \to (\widetilde{\mathfrak{X}}_2,\ \tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}})$ from an $\mathcal{SVNTS}$ $(\widetilde{\mathfrak{X}}_1,\ \tilde{\tau}_1^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}})$ into another $\mathcal{SVNTS}$ $(\widetilde{\mathfrak{X}}_2,\ \tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}})$ is said to be single-valued neutrosophic continuous (briefly, $\mathcal{SVN}$-continuous) if and only if $\tilde{\tau}_2^{\widetilde{\gamma}}(\mathcal{S}) \le \tilde{\tau}_1^{\widetilde{\gamma}}(f^{-1}(\mathcal{S}))$, $\tilde{\tau}_2^{\widetilde{\eta}}(\mathcal{S}) \ge \tilde{\tau}_1^{\widetilde{\eta}}(f^{-1}(\mathcal{S}))$ and $\tilde{\tau}_2^{\widetilde{\mu}}(\mathcal{S}) \ge \tilde{\tau}_1^{\widetilde{\mu}}(f^{-1}(\mathcal{S}))$, for every $\mathcal{S} \in I^{\widetilde{\mathfrak{X}}_2}$.

## 3. Single-Valued Neutrosophic (almost , weakly) Continuous Mappings

This section is dedicated to present the concepts of the single-valued neutrosophic (almost and weakly) mappings (briefly $\mathcal{SVN}-$ almost continuous, $\mathcal{SVN} - weakly\ continuous$) mappings, respectively. It is also devoted to mark out the concepts of single-valued neutrosophic ( preopen , regular-open ) sets (briefly, $r - SVNPO$, $r - SVNRO$) sets, respectively.

**Definition 3.1.** Let $(\widetilde{\mathfrak{X}}, \tilde{\tau}^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}})$ be an $\mathcal{SVNTS}$ and $r \in I_0$. Then, $\mathcal{S} \in I^{\widetilde{\mathfrak{X}}}$ is said to be:

1. $r - SVNPO\ set$ iff $\mathcal{S} \le int_{\tilde{\tau}^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}\left(C_{\tilde{\tau}^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(\mathcal{S}, r), r\right)$,
2. $r - SVNRO$ set if $\mathcal{S} = int_{\tilde{\tau}^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}\left(C_{\tilde{\tau}^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(\mathcal{S}, r), r\right)$.

The complement of $r - SVNPO$ (resp, $r - SVNRO$) are said to be $r - SVNPC$ (resp, $r - SVNRC$), respectively.

**Remark 3.2.** Let $(\widetilde{\mathfrak{X}}, \tilde{\tau}^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}})$ be an $\mathcal{SVNTS}$ and $r \in I_0$, if $\mathcal{S}$ is an $r - SVNRO$ set, then $\mathcal{S}$ is $r - SVNPO$.

**Example 3.3.** Let $\widetilde{\mathfrak{X}} = \{a, b\}$. Define $\mathcal{E}_1, \mathcal{E}_2 \in I^{\widetilde{\mathfrak{X}}}$ as follows:

$\mathcal{E}_1 = \langle(0\cdot 5, 0.4, 0\cdot 5), (0\cdot 5, 0.4, 0\cdot 5), (0\cdot 5, 0.5, 0\cdot 5)\rangle$, $\mathcal{E}_2 = \langle(0\cdot 4, 0\cdot 4, 0.4), (0\cdot 5, 0\cdot 4, 0.4), (0\cdot 5.0\cdot 5, .4)\rangle$.

Define $\tilde{\tau}^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}} : I^{\widetilde{\mathfrak{X}}} \to I$ as follows:

$$\tilde{\tau}^{\widetilde{\gamma}}(\mathcal{S}) = \begin{cases} 1, & if\ \mathcal{S} = \tilde{0}, \\ 1, & if\ \mathcal{S} = \tilde{1}, \\ \frac{1}{2}, & if\ \mathcal{S} = \mathcal{E}_1, \\ 0, & otherwise \end{cases} \qquad \tilde{\tau}^{\widetilde{\eta}}(\mathcal{S}) = \begin{cases} 0, & if\ \mathcal{S} = \tilde{0}, \\ 0, & if\ \mathcal{S} = \tilde{1}, \\ \frac{1}{2}, & if\ \mathcal{S} = \{\mathcal{E}_1, \mathcal{E}_2\}, \\ 1, & otherwise \end{cases}$$

$$\tilde{\tau}^{\widetilde{\eta}}(\mathcal{S}) = \begin{cases} 0, & if\ \mathcal{S} = \tilde{0}, \\ 0, & if\ \mathcal{S} = \tilde{1}, \\ \frac{1}{2}, & if\ \mathcal{S} = \{\mathcal{E}_1, \mathcal{E}_2\}, \\ 1, & otherwise \end{cases}$$

Let, $\mathcal{E}_3 = \left\{\langle\omega, (0\cdot 5, 0.5, 0\cdot 1), (0\cdot 6, 0.3, 0\cdot 1), (0\cdot 6, 0.3, 0\cdot 1)\rangle : \omega \in \widetilde{\mathfrak{X}}\right\}$. Then, $\mathcal{E}_3$ is $\frac{1}{2} - SVNPO$ set but it is not $\frac{1}{2} - SVNRO$ set because, $\mathcal{E}_3 \ne int_{\tilde{\tau}^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}\left(C_{\tilde{\tau}^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}\left(\mathcal{E}_3, \frac{1}{2}\right), \frac{1}{2}\right) = \tilde{1}$.

**Lemma 3.4.** Let $\mathcal{S}$ be an $\mathcal{SVNS}$ in an $\mathcal{SVNTS}$ $(\widetilde{\mathfrak{X}}, \tilde{\tau}^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}})$. Then, for each $r \in I_0$.

1. If $\mathcal{S}$ is $r - SVNRO$ set (resp, $r - SVNRC\ set$), then $[\tilde{\tau}^{\widetilde{\gamma}}(\mathcal{S}) \ge r,\ \tilde{\tau}^{\widetilde{\eta}}(\mathcal{S}) \le 1 - r\ ,\ \tilde{\tau}^{\widetilde{\mu}}(\mathcal{S}) \le 1 - r]$ (resp, $[\tilde{\tau}^{\widetilde{\gamma}}(\mathcal{S}^c) \ge r,\ \tilde{\tau}^{\widetilde{\eta}}(\mathcal{S}^c) \le 1 - r,\ \tilde{\tau}^{\widetilde{\mu}}(\mathcal{S}^c) \le 1 - r]$),
2. $\mathcal{S}$ is $r - SVNRO$ set if and only if $\mathcal{S}^c$ is $r - SVNRC$ set.

**Proof.** Follows directly from Definition 3.1.

**Lemma 3.5.** Let $(\widetilde{\mathfrak{X}}, \tilde{\tau}^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}})$ be an $\mathcal{SVNTS}$. Then,

1. the union of two $r - SVNRC$ sets is $r - SVNRC$,
2. the intersection of two $r - SVNRO$ sets, is $r - SVNRO$.

**Proof.** (1) Let $\mathcal{S}, \mathcal{E}$ be any two $r - SVNRC$ sets. By Lemma 3.4, $[\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1 - r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1 - r]$ and $[\tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}^c) \geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}^c) \leq 1 - r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}^c) \leq 1 - r]$. Then,

$$\tilde{\tau}^{*\tilde{\gamma}}(\mathcal{S} \cup \mathcal{E}) \geq \tilde{\tau}^{*\tilde{\gamma}}(\mathcal{S}) \cap \tilde{\tau}^{*\tilde{\gamma}}(\mathcal{E}),\ \tilde{\tau}^{*\tilde{\eta}}(\mathcal{S} \cup \mathcal{E}) \leq \tilde{\tau}^{*\tilde{\eta}}(\mathcal{S}) \cup \tilde{\tau}^{*\tilde{\eta}}(\mathcal{E}),\ \tilde{\tau}^{*\tilde{\mu}}(\mathcal{S} \cup \mathcal{E}) \leq \tilde{\tau}^{*\tilde{\mu}}(\mathcal{S}) \cup \tilde{\tau}^{*\tilde{\mu}}(\mathcal{E}),$$

but $int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S} \cup \mathcal{E}, \mathrm{r}) \leq \mathcal{S} \cup \mathcal{E}$, this suggests that

$$C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S} \cup \mathcal{E}, \mathrm{r}), \mathrm{r}\big) \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S} \cup \mathcal{E}, r) = \mathcal{S} \cup \mathcal{E}.$$

Now,

$$\mathcal{S} = C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r}), \mathrm{r}\big) \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S} \cup \mathcal{E}, \mathrm{r}), \mathrm{r}\big),$$

and

$$\mathcal{E} = C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}, \mathrm{r}), \mathrm{r}\big) \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S} \cup \mathcal{E}, \mathrm{r}), \mathrm{r}\big).$$

Thus, $\mathcal{S} \cup \mathcal{E} \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S} \cup \mathcal{E}, \mathrm{r}), \mathrm{r}\big)$. So, $\mathcal{S} \cup \mathcal{E} = C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S} \cup \mathcal{E}, \mathrm{r}), \mathrm{r}\big)$. Hence, $\mathcal{S} \cup \mathcal{E}$ $r - SVNRC$ set.

(2) It can be ascertained by the same method.

**Theorem 3.6.** Let $(\widetilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNTS}$, Then,

1. If $\mathcal{S} \in I^{\widetilde{\mathfrak{X}}}$ s.t, $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1 - r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1 - r$, then, $int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r})$ is $r - SVNRO$ set,
2. If $\mathcal{S} \in I^{\widetilde{\mathfrak{X}}}$ s.t, $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}) \geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}) \leq 1 - r$ and $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}) \leq 1 - r$ , then, $C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r})$ is $r - SVNRC$ set.

**Proof.** (1) Suppose that $\mathcal{S} \in I^{\widetilde{\mathfrak{X}}}$ such that, $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1 - r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1 - r$. Clearly,

$$int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r}) \leq int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r}), \mathrm{r}\big),$$

this denotes that, $int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r}) \leq int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r}), \mathrm{r}\big), r)$. Now, since,

$$\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1 - r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1 - r,$$

then $C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r}), \mathrm{r}\big) \leq \mathcal{S}$; therefore,

$$int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r}) \geq int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r}), \mathrm{r}\big), r).$$

Then, $int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r}) = int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}\big(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r}), \mathrm{r}\big), r)$. Hence, $int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, \mathrm{r})$ is $r - SVNRO$ set.

(2) Similar to the proof of (1).

**Definition 3.7.** A mapping $f: (\widetilde{\mathfrak{X}}_1,\ \tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \to (\widetilde{\mathfrak{X}}_2,\ \tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ from an $\mathcal{SVNTS}$ $(\widetilde{\mathfrak{X}}_1,\ \tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ into another $\mathcal{SVNTS}$ $(\widetilde{\mathfrak{X}}_2,\ \tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is called:

1. $SVN -$ *almost continuous* iff $\tilde{\tau}_1^{\tilde{\gamma}}(f^{-1}(\mathcal{S})) \geq \mathrm{r},\ \tilde{\tau}_1^{\tilde{\eta}}\big(f^{-1}(\mathcal{S})\big) \leq 1 - r,\ \tilde{\tau}_1^{\tilde{\mu}}\big(f^{-1}(\mathcal{S})\big) \leq 1 - \mathrm{r}$, for each $r - SVNRO$ set $\mathcal{S}$ of $\widetilde{\mathfrak{X}}_2$,
2. $SVN -$ *weakly continuous* iff $\tilde{\tau}_2^{\tilde{\gamma}}(\mathcal{S}) \geq \mathrm{r},\ \tilde{\tau}_2^{\tilde{\eta}}(\mathcal{S}) \leq 1 - r$ and $\tilde{\tau}_2^{\tilde{\mu}}(\mathcal{S}) \leq 1 - \mathrm{r}$, implies $\tilde{\tau}_1^{\tilde{\gamma}}(f^{-1}(\mathcal{S})) \geq \mathrm{r}$, $\tilde{\tau}_1^{\tilde{\eta}}\big(f^{-1}(\mathcal{S})\big) \leq 1 - r,\ \tilde{\tau}_1^{\tilde{\mu}}\big(f^{-1}(\mathcal{S})\big) \leq 1 - \mathrm{r}$, , for each $\mathcal{S} \in I^{\widetilde{\mathfrak{X}}_2}$.

**Remark 3.8.** From Definition 3.7, it is clear that the next implications are correct for $r \in I_0$:

$$SVN -\ almost\ continuous\ mapping$$

$$\Uparrow$$

$$SVN -\ continuous\ mapping$$

$$\Downarrow$$

$$SVN -\ weakly\ continuous\ mapping$$

However, the one-sided suggestions are not correct in general, as presented by the next example.

**Example 3.9.** Suppose that $\widetilde{\mathfrak{X}} = \{a, b, c\}$. Define $\mathcal{E}_1, \mathcal{E}_2 \in I^{\widetilde{\mathfrak{X}}}$ as follows:

$\mathcal{E}_1 = \langle(0\cdot 5, 0.4, 0\cdot 5), (0\cdot 5, 0.4, 0\cdot 5), (0\cdot 5, 0.5, 0\cdot 5)\rangle,\ \mathcal{E}_2 = \langle(0\cdot 5, 0\cdot 4, 0.4), (0\cdot 5, 0\cdot 4, 0.4), (0\cdot 5, 0\cdot 5, .4)\rangle,$

$\mathcal{E}_3 = \langle(0\cdot 3, 0.6, 0\cdot 5), (0\cdot 3, 0.6, 0\cdot 5), 0\cdot 3, 0.6, 0\cdot 5\rangle,\quad \mathcal{E}_4 = \langle(0\cdot 4, 0\cdot 4, 0.4), (0\cdot 5, 0\cdot 4, 0.4), (0\cdot 5.0\cdot 5, .4)\rangle.$

We difine an $\tilde{\tau}_1^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}, \tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}} : I^{\widetilde{\mathfrak{X}}} \to I$ as follows:

$$\tilde{\tau}_1^{\widetilde{\gamma}}(\mathcal{S}) = \begin{cases} 1, & if\ \mathcal{S} = \tilde{0}, \\ 1, & if\ \mathcal{S} = \tilde{1}, \\ \frac{1}{2}, & if\ \mathcal{S} = \mathcal{E}_2, \\ 0, & otherwise \end{cases} \qquad \tilde{\tau}_2^{\widetilde{\gamma}}(\mathcal{S}) = \begin{cases} 1, & if\ \mathcal{S} = \tilde{0}, \\ 1, & if\ \mathcal{S} = \tilde{1}, \\ \frac{1}{2}, & if\ \mathcal{S} = \{\mathcal{E}_2, \mathcal{E}_4\}, \\ 0, & otherwise \end{cases}$$

$$\tilde{\tau}_1^{\widetilde{\eta}}(\mathcal{S}) = \begin{cases} 0, & if\ \mathcal{S} = \tilde{0}, \\ 0, & if\ \mathcal{S} = \tilde{1}, \\ \frac{1}{2}, & if\ \mathcal{S} = \{\mathcal{E}_1, \mathcal{E}_2\}, \\ 1, & otherwise \end{cases} \qquad \tilde{\tau}_2^{\widetilde{\eta}}(\mathcal{S}) = \begin{cases} 0, & if\ \mathcal{S} = \tilde{0}, \\ 0, & if\ \mathcal{S} = \tilde{1}, \\ \frac{1}{2}, & if\ \mathcal{S} = \{\mathcal{E}_2, \mathcal{E}_4\}, \\ 1, & otherwise \end{cases}$$

$$\tilde{\tau}_1^{\widetilde{\mu}}(\mathcal{S}) = \begin{cases} 0, & if\ \mathcal{S} = \tilde{0}, \\ 0, & if\ \mathcal{S} = \tilde{1}, \\ \frac{1}{2}, & if\ \mathcal{S} = \{\mathcal{E}_2, \mathcal{E}_3\}, \\ 1, & otherwise \end{cases} \qquad \tilde{\tau}_2^{\widetilde{\mu}}(\mathcal{S}) = \begin{cases} 0, & if\ \mathcal{S} = \tilde{0}, \\ 0, & if\ \mathcal{S} = \tilde{1}, \\ \frac{1}{2}, & if\ \mathcal{S} = \{\mathcal{E}_2, \mathcal{E}_4\}, \\ 1, & otherwise \end{cases}$$

Then, the identity mapping, $f: (\widetilde{\mathfrak{X}}_1, \tilde{\tau}_1^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}) \to (\widetilde{\mathfrak{X}}_2, \tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}})$ is $\mathcal{SVN}-$ almost continuous, but it is not $\mathcal{SVN}-$ continuou. Since, $\tilde{\tau}_2^{\widetilde{\gamma}}(\mathcal{E}_4) = \frac{1}{2}$ and $\mathcal{E}_4$ is not $\frac{1}{2} - SVNO$ set in $\widetilde{\mathfrak{X}}_1$, because, $\tilde{\tau}_1^{\widetilde{\gamma}}(f^{-1}(\mathcal{E}_4)) = 0 \ngeq \frac{1}{2}$, $\tilde{\tau}_1^{\widetilde{\eta}}(f^{-1}(\mathcal{E}_4)) = 1 \nleq \frac{1}{2}$ and $\tilde{\tau}_1^{\widetilde{\mu}}(f^{-1}(\mathcal{E}_4)) = 1 \ngeq \frac{1}{2}$. Hence, [ $\tilde{\tau}_2^{\widetilde{\gamma}}(\mathcal{E}_4) = \frac{1}{2} \nleq 0 = \tilde{\tau}_1^{\widetilde{\gamma}}(f^{-1}(\mathcal{E}_4))$ , $\tilde{\tau}_2^{\widetilde{\eta}}(\mathcal{E}_4) = \frac{1}{2} \ngeq 1 = \tilde{\tau}_1^{\widetilde{\eta}}(f^{-1}(\mathcal{E}_4))$ , $\tilde{\tau}_2^{\widetilde{\mu}}(\mathcal{E}_4)\frac{1}{2} \ngeq 1 = \tilde{\tau}_1^{\widetilde{\mu}}(f^{-1}(\mathcal{E}_4))$].

**Theorem 3.10**. Let $f: (\widetilde{\mathfrak{X}}_1, \tilde{\tau}_1^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}) \to (\widetilde{\mathfrak{X}}_2, \tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}})$ be a mapping from an $\mathcal{SVNTS}$ $(\widetilde{\mathfrak{X}}_1, \tilde{\tau}_1^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}})$ into another $\mathcal{SVNTS}$ $(\widetilde{\mathfrak{X}}_2, \tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}})$. Then the next statements are equivalent:

1. $f$ is $\mathcal{SVN}-$ almost continuous,
2. $\tilde{\tau}_1^{\widetilde{\gamma}}\left(\left(f^{-1}(\mathcal{S})\right)^c\right) \geq \mathrm{r}$, $\tilde{\tau}_1^{\widetilde{\eta}}\left(\left(f^{-1}(\mathcal{S})\right)^c\right) \leq 1-r$, $\tilde{\tau}_1^{\widetilde{\mu}}\left(\left(f^{-1}(\mathcal{S})\right)^c\right) \leq 1-\mathrm{r}$, for any $r - SVNRC$ set $\mathcal{S}$ of $\widetilde{\mathfrak{X}}_2$,
3. $f^{-1}(\mathcal{S}) \leq int_{\tilde{\tau}_1^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(f^{-1}(int_{\tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(C_{\tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(\mathcal{S}, \mathrm{r}), \mathrm{r})), r)$, for any $\mathcal{S}$ of $\widetilde{\mathfrak{X}}_2$ such that $\tilde{\tau}_2^{\widetilde{\gamma}}(\mathcal{S}) \geq \mathrm{r}$, $\tilde{\tau}_2^{\widetilde{\eta}}(\mathcal{S}) \leq 1-r$ and $\tilde{\tau}_2^{\widetilde{\mu}}(\mathcal{S}) \leq 1-\mathrm{r}$,
4. $C_{\tilde{\tau}_1^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(f^{-1}(C_{\tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(int_{\tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(\mathcal{S}, \mathrm{r}), \mathrm{r})), r) \leq f^{-1}(\mathcal{S})$, for any $\mathcal{S}$ of $\widetilde{\mathfrak{X}}_2$ such that $\tilde{\tau}_2^{\widetilde{\gamma}}(\mathcal{S}) \geq \mathrm{r}$, $\tilde{\tau}_2^{\widetilde{\eta}}(\mathcal{S}) \leq 1-r$ and $\tilde{\tau}_2^{\widetilde{\mu}}(\mathcal{S}) \leq 1-\mathrm{r}$.

**Proof.** (1)⇒(2). Let $\mathcal{S}$ be an $r - SVNRC$ set of $\widetilde{\mathfrak{X}}_2$ Then by Lemma 3.4, $\mathcal{S}^{c}$ is $r - SVNRO$ set in $\widetilde{\mathfrak{X}}_2$. By (1), we obtain

$$\tilde{\tau}_1^{\widetilde{\gamma}}(f^{-1}(\mathcal{S}^c)) = \tilde{\tau}_1^{\widetilde{\gamma}}((f^{-1}(\mathcal{S}))^c) \geq r, \qquad \tilde{\tau}_1^{\widetilde{\eta}}(f^{-1}(\mathcal{S}^c)) = \tilde{\tau}_1^{\widetilde{\eta}}((f^{-1}(\mathcal{S}))^c) \leq 1-r,$$
$$\tilde{\tau}_1^{\widetilde{\mu}}(f^{-1}(\mathcal{S}^c)) = \tilde{\tau}_1^{\widetilde{\mu}}((f^{-1}(\mathcal{S}))^c) \leq 1-r.$$

(2)⇒(1). It is analogous to the proof of (1)⇒(2).

(1)⇒(3). Since, $[\tilde{\tau}_2^{\widetilde{\gamma}}(\mathcal{S}) \geq r,\ \tilde{\tau}_2^{\widetilde{\eta}}(\mathcal{S}) \leq 1-r,\ \tilde{\tau}_2^{\widetilde{\mu}}(\mathcal{S}) \leq 1-r]$, then, $\mathcal{S} = int_{\tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(\mathcal{S}, r) \leq int_{\tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(C_{\tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(\mathcal{S}, r), r)$, and hence, $\mathrm{f}^{-1}(\mathcal{S}) = \mathrm{f}^{-1}(\mathrm{int}_{\tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(\mathrm{C}_{\tilde{\tau}_2^{\widetilde{\gamma}\widetilde{\eta}\widetilde{\mu}}}(\mathcal{S}, \mathrm{r}), \mathrm{r}))$, since

$$\tilde{\tau}_2^{\tilde{\gamma}}\left([C_{\tilde{\tau}_2^{\tilde{\gamma}}}(\mathcal{S},r)]^c\right) \geq r\,, \quad \tilde{\tau}_2^{\tilde{\eta}}\left([C_{\tilde{\tau}_2^{\tilde{\eta}}}(\mathcal{S},r)]^c\right) \leq 1-r\,, \quad \tilde{\tau}_2^{\tilde{\mu}}\left([C_{\tilde{\tau}_2^{\tilde{\mu}}}(\mathcal{S},r)]^c\right) \leq 1-r\,,$$

then by Theorem 3.6 $int_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S},r),r)$ is $r-SVNRO$ set. So,

$$\tilde{\tau}_1^{\tilde{\gamma}}(f^{-1}(int_{\tilde{\tau}_2^{\tilde{\gamma}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}}}(\mathcal{S},r),r))) \geq r,\ \tilde{\tau}_1^{\tilde{\eta}}(f^{-1}(int_{\tilde{\tau}_2^{\tilde{\eta}}}(C_{\tilde{\tau}_2^{\tilde{\eta}}}(\mathcal{S},r),r))) \leq 1-r,\ \tilde{\tau}_1^{\tilde{\mu}}(f^{-1}(int_{\tilde{\tau}_2^{\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\mu}}}(\mathcal{S},r),r))) \leq 1-r.$$

Therefore, $f^{-1}(\mathcal{S}) \leq f^{-1}(int_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}}}(\mathcal{S},r),r)) = int_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(int_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S},r),r)).$

(3)⇒(1). Let $\mathcal{S}$ be an $r-SVNRO$ set of $\widetilde{\mathfrak{X}}_2$. Then, we get

$$f^{-1}(\mathcal{S}) \leq int_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(int_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S},\mathrm{r}),\mathrm{r})),r) = int_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}),\mathrm{r});$$

this suggests that, $f^{-1}(\mathcal{S}) = int_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}),\mathrm{r})$, then

$$\tilde{\tau}_1^{\tilde{\gamma}}(f^{-1}(\mathcal{S})) = \tilde{\tau}_1^{\tilde{\gamma}}(int_{\tilde{\tau}_1^{\tilde{\gamma}}}(f^{-1}(\mathcal{S}),\mathrm{r})) \geq r, \quad \tilde{\tau}_1^{\tilde{\eta}}(f^{-1}(\mathcal{S})) = \tilde{\tau}_1^{\tilde{\eta}}(int_{\tilde{\tau}_1^{\tilde{\eta}}}(f^{-1}(\mathcal{S}),\mathrm{r})) \leq 1-r,$$

$$\tilde{\tau}_1^{\tilde{\mu}}(f^{-1}(\mathcal{S})) = \tilde{\tau}_1^{\tilde{\mu}}(int_{\tilde{\tau}_1^{\tilde{\mu}}}(f^{-1}(\mathcal{S}),\mathrm{r})) \leq 1-r.$$

Therefore, $f$ is $\mathcal{SVN}-$ *almost continuous*.

(2)⇔(4). Can be proved similarly.

**Theorem 3.11**. Let $f\colon(\widetilde{\mathfrak{X}}_1,\ \tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \to (\widetilde{\mathfrak{X}}_2,\ \tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be a map from an $\mathcal{SVNTS}$ $(\widetilde{\mathfrak{X}}_1,\ \tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ into another $\mathcal{SVNTS}$ $(\widetilde{\mathfrak{X}}_2,\ \tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$. Then the following are equivalent:

1. $f$ is $\mathcal{SVN}-$ *weakly continuous*,
2. $f(C_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S},\mathrm{r})) \leq C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f(\mathcal{S}),\mathrm{r})$ for each $\mathcal{S} \in I^{\widetilde{\mathfrak{X}}_1}$

Proof. (1) ⇒ (2). : Let $\mathcal{S} \in I^{\widetilde{\mathfrak{X}}_1}$. Then,

$$f^{-1}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f(\mathcal{S}),\mathrm{r}))) = f^{-1}\left[\bigcap\left\{\mathcal{E} \in I^{\widetilde{\mathfrak{X}}_2}\colon \tilde{\tau}_2^{\tilde{\gamma}}(\mathcal{E}^c) \geq r, \tilde{\tau}_2^{\tilde{\eta}}(\mathcal{E}^c) \leq 1-r, \tilde{\tau}_2^{\tilde{\mu}}(\mathcal{E}^c) \leq 1-r, \quad \mathcal{E} \geq f(\mathcal{S})\right\}\right]$$

$$\geq f^{-1}\left[\bigcap\left\{\mathcal{E} \in I^{\widetilde{\mathfrak{X}}_2}\colon \tilde{\tau}_1^{\tilde{\gamma}}(f^{-1}(\mathcal{E}^c)) \geq r, \tilde{\tau}_1^{\tilde{\eta}}(f^{-1}(\mathcal{E}^c)) \leq 1-r, \tilde{\tau}_1^{\tilde{\mu}}(f^{-1}(\mathcal{E}^c)) \leq 1-r, \quad \mathcal{E} \geq f(\mathcal{S})\right\}\right]$$

$$\geq f^{-1}\left[\bigcap\left\{\mathcal{E} \in I^{\widetilde{\mathfrak{X}}_2}\colon \tilde{\tau}_1^{\tilde{\gamma}}\left(\left(f^{-1}(\mathcal{E})\right)^c\right) \geq r, \tilde{\tau}_1^{\tilde{\eta}}\left(\left(f^{-1}(\mathcal{E})\right)^c\right) \leq 1-r, \tilde{\tau}_1^{\tilde{\mu}}\left(\left(f^{-1}(\mathcal{E})\right)^c\right) \leq 1-r, \quad \mathcal{E} \geq f(\mathcal{S})\right\}\right]$$

$$\geq \bigcap\left\{f^{-1}(\mathcal{E}) \in I^{\widetilde{\mathfrak{X}}_1}\colon \tilde{\tau}_1^{\tilde{\gamma}}\left(\left(f^{-1}(\mathcal{E})\right)^c\right) \geq r, \tilde{\tau}_1^{\tilde{\eta}}\left(\left(f^{-1}(\mathcal{E})\right)^c\right) \leq 1-r, \tilde{\tau}_1^{\tilde{\mu}}\left(\left(f^{-1}(\mathcal{E})\right)^c\right) \leq 1-r, \quad f^{-1}(\mathcal{E}) \geq \mathcal{S}\right\}$$

$$\geq \bigcap\left\{\mathcal{D} \in I^{\widetilde{\mathfrak{X}}_1}\colon \tilde{\tau}_1^{\tilde{\gamma}}(\mathcal{D}^c) \geq r, \tilde{\tau}_1^{\tilde{\eta}}(\mathcal{D}^c) \leq 1-r, \tilde{\tau}_1^{\tilde{\mu}}(\mathcal{D}^c) \leq 1-r, \ \ \mathcal{D} \geq \mathcal{S}\right\} = C_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S},\mathrm{r}).$$

Hence, $f(C_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S},\mathrm{r})) \leq f(f^{-1}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f(\mathcal{S}),\mathrm{r}))) \leq C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f(\mathcal{S}),\mathrm{r}).$

(2)⇒(1). It is similar to that of (1)⇒(2).

**Corollary 3.12.** Let $f\colon\widetilde{\mathfrak{X}}_1 \to \widetilde{\mathfrak{X}}_2$ be an $\mathcal{SVN}-$ *continuous* mapping with respect to the $\mathcal{SVNT}$s $\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$ and $\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$ respectively. Then , for each $\mathcal{S} \in I^{\widetilde{\mathfrak{X}}_1}$, $f(C_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S},\mathrm{r})) \leq C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f(\mathcal{S}),\mathrm{r}).$

**Theorem 3.13.** Let $f\colon\widetilde{\mathfrak{X}}_1 \to \widetilde{\mathfrak{X}}_2$ be an $\mathcal{SVN}-$ *continuous* mapping with respect to the $\mathcal{SVNT}$ $\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$ and $\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$, respectively. Then , for any $\mathcal{S} \in I^{\widetilde{\mathfrak{X}}_2}$, $C_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}),\mathrm{r})) \leq f^{-1}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}),\mathrm{r})).$

**Proof.** Let $\mathcal{S} \in I^{\widetilde{\mathfrak{X}}_2}$. We get from Theorem 3.12, $C_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}),\mathrm{r})) \leq f^{-1}(f(C_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}),\mathrm{r})) \leq f^{-1}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S},\mathrm{r})).$

Hence, $C_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}),r)) \leq f^{-1}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S},r))$, for every $\mathcal{S} \in I^{\tilde{\mathfrak{X}}_2}$.

## 4. Compactness on Single-Valued Neutrosophic Ideal Topological Spaces

This section aims to establish new notions of $r$-single-valued neutrosophic aspects called (compact, ideal compact, ideal quasi H-closed, compact modulo an single-valued neutrosophic ideal) (briefly, $r-\mathcal{SVN}-compact$, $r-\mathcal{SVNJ}-compact$, $r-\mathcal{SVNJ}-quasi\ H-closed$, $r-\mathcal{SVNC}(\mathcal{J})-compact$) in $\mathcal{SVNJTS}$.

**Definition 4.1.** Let $(\tilde{\mathfrak{X}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},)$ be an $\mathcal{SVNTS}$ and $r \in I_0$. Then $\tilde{\mathfrak{X}}$ is called $r-\mathcal{SVN}-compact$ iff for every family $\{\mathcal{S}_j \in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j) \geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j) \leq 1-r,\ j \in \Gamma\}$ such that $\cup_{j\in\Gamma}\mathcal{S}_j = \tilde{1}$, there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\cup_{j\in\Gamma_0}\mathcal{S}_j = \tilde{1}$.

**Definition 4.2.** Let $(\tilde{\mathfrak{X}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNJTS}$ and $r \in I_0$. Then,

(1) $\tilde{\mathfrak{X}}$ is called $r-\mathcal{SVNJ}-compact$ (resp., $r-\mathcal{SVNJ}-quasi\ H-closed$ ) iff every family, $\{\mathcal{S}_j \in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j) \geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j) \leq 1-r,\ j \in \Gamma\}$ such that $\cup_{j\in\Gamma}\mathcal{S}_j = \tilde{1}$, there exists a finite subse $\Gamma_0 \subseteq \Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\cup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \geq r$, $\tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\cup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \leq 1-r$, $\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\cup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \leq 1-r$ (resp., $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\cup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r)\right]^c\right) \geq r$, $\tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\cup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r)\right]^c\right) \leq 1-r$, $\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\cup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)\right]^c\right) \leq 1-r$).

(2) $\tilde{\mathfrak{X}}$ is called $r-\mathcal{SVNC}(\mathcal{J})-$compact if for any $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1-r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1-r$ and every family $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}_j) \geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}_j) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}_j) \leq 1-r,\ j \in \Gamma\}$ such that $\mathcal{S} \leq \cup_{j\in\Gamma}\mathcal{E}_j$, there exists a finite subse $\Gamma_0 \subseteq \Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\mathcal{S} \cap \left[\cup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j,r)\right]^c\right) \geq r$, $\tilde{\mathcal{J}}^{\tilde{\eta}}\left(\mathcal{S} \cap \left[\cup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j,r)\right]^c\right) \leq 1-r$, $\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S} \cap \left[\cup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j,r)\right]^c\right) \leq 1-r$.

**Definition 4.3.** Let $(\tilde{\mathfrak{X}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNJTS}$ and $\mathcal{S} \in I^{\tilde{\mathfrak{X}}}$. Then $\mathcal{S}$ is called $r-\mathcal{SVNJ}-compact$ iff every family $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}_j) \geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}_j) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}_j) \leq 1-r,\ j \in \Gamma\}$ such that $\mathcal{S} \leq \cup_{j\in\Gamma}\mathcal{E}_j$, there exists a finite subse $\Gamma_0 \subseteq \Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\mathcal{S} \cap \left[\cup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right) \geq r$, $\tilde{\mathcal{J}}^{\tilde{\eta}}\left(\mathcal{S} \cap \left[\cup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right) \leq 1-r$, $\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S} \cap \left[\cup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right) \leq 1-r$.

**Theorem 4.4.** Let $(\tilde{\mathfrak{X}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNJTS}$ and $r \in I_0$. Then,

*(1)* $r-\mathcal{SVN}-compact \Rightarrow r-\mathcal{SVNJ}-compact$,

*(2)* $r-\mathcal{SVNJ}-compact \Rightarrow r-\mathcal{SVNC}(\mathcal{J})-compact$,

*(3)* $r-\mathcal{SVNJ}-compact \Rightarrow r-SVNI-\ quasi\ H-closed$.

**Proof.** (1) For every family $\{\mathcal{S}_j \in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j) \geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j) \leq 1-r,\ j \in \Gamma\}$ such that $\cup_{j\in\Gamma}\mathcal{S}_j = \tilde{1}$. By $r-\mathcal{SVN}-$compactness of $\tilde{\mathfrak{X}}$, there exists a finite subse $\Gamma_0 \subseteq \Gamma$ such that $\cup_{j\in\Gamma_0}\mathcal{S}_j = \tilde{1}$. Now, since $\left[\cup_{j\in\Gamma_0}\mathcal{S}_j\right]^c = \tilde{0}$, we have $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\cup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \geq r$, $\tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\cup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \leq 1-r$, $\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\cup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \leq 1-r$.

(2) For every $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1-r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1-r$ and evrey family $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}_j) \geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}_j) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}_j) \leq 1-r,\ j \in \Gamma\}$ such that $\mathcal{S} \leq \cup_{j\in\Gamma}\mathcal{E}_j$. By $r-\mathcal{SVNJ}-$compactness of $\mathcal{S}$, there exists a finite subse $\Gamma_0 \subseteq \Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\mathcal{S} \cap \left[\cup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right) \geq r$, $\tilde{\mathcal{J}}^{\tilde{\eta}}\left(\mathcal{S} \cap \left[\cup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right) \leq 1-r$, $\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S} \cap \left[\cup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right) \leq 1-r$. Since, $\mathcal{S} \cap \left[\cup_{j\in\Gamma_0}\mathcal{E}_j\right]^c \geq \mathcal{S} \cap \left[\cup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,r)\right]^c$, we have

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j,r)\right]^c\right) \geq r, \quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j,r)\right]^c\right) \leq 1-r, \quad \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j,r)\right]^c\right) \leq 1-r$$

Hence, $\widetilde{\mathfrak{T}}$ is $r-\mathcal{SVNC}(\mathcal{I})-compact$.

(3) Let $\{\mathcal{S}_j \in I^{\widetilde{\mathfrak{T}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j) \geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j) \leq 1-r: j \in \Gamma\}$ be a family such that $\bigcup_{j\in\Gamma}\mathcal{S}_j = \tilde{1}$. By $r-\mathcal{SVNI}-$compactness of $(\widetilde{\mathfrak{T}}, \widetilde{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \mathfrak{I}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}})$, there exists a finite subfamily $\Gamma_0 \subseteq \Gamma$ such that $\mathfrak{I}^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \geq r$, $\mathfrak{I}^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \leq 1-r$, $\mathfrak{I}^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \leq 1-r$. Since, $\left[\bigcup_{j\in\Gamma_0}\mathcal{S}_j\right]^c \geq \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j, r)\right]^c$, we have

$$\mathfrak{I}^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j, r)\right]^c\right) \geq r, \quad \mathfrak{I}^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j, r)\right]^c\right) \leq 1-r, \quad \mathfrak{I}^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j, r)\right]^c\right) \leq 1-r$$

Hence, $\widetilde{\mathfrak{T}}$ is $r-SVNI-\ quasi\ H-closed$.

**Theorem 4.5.** The next statements are equivalent in an $\mathcal{SVNITS}$ $(\widetilde{\mathfrak{T}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \mathfrak{I}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$:

(1) $\widetilde{\mathfrak{T}}$ is $\mathrm{r}-\mathcal{SVNI}-$compact,

(2) For any family $\{\mathcal{S}_j \in I^{\widetilde{\mathfrak{T}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j^c) \geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j^c) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j^c) \leq 1-r,\ j \in \Gamma\}$ with $\bigcap_{j\in\Gamma}\mathcal{S}_j = \tilde{0}$, there exists a finite subset $\Gamma_0 \subseteq \Gamma$ with $\mathfrak{I}^{\tilde{\gamma}}\left(\bigcap_{j\in\Gamma_0}\mathcal{S}_j\right) \geq r$, $\mathfrak{I}^{\tilde{\eta}}\left(\bigcap_{j\in\Gamma_0}\mathcal{S}_j\right) \leq 1-r$, $\mathfrak{I}^{\tilde{\mu}}\left(\bigcap_{j\in\Gamma_0}\mathcal{S}_j\right) \leq 1-r$.

**Proof.** (1)⇒(2). For each family $\{\mathcal{S}_j \in I^{\widetilde{\mathfrak{T}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j^c) \geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j^c) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j^c) \leq 1-r,\ j \in \Gamma\}$ with $\bigcap_{j\in\Gamma}\mathcal{S}_j = \tilde{0}$. Then, $\bigcup_{j\in\Gamma}\mathcal{S}_j^c = \tilde{1}$. By $r-\mathcal{SVNI}-$compactness of $(\widetilde{\mathfrak{T}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \mathfrak{I}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$, there exists a finite subse $\Gamma_0 \subseteq \Gamma$ such that $\mathfrak{I}^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{S}_j^c\right]^c\right) \geq r$, $\mathfrak{I}^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{S}_j^c\right]^c\right) \leq 1-r$, $\mathfrak{I}^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{S}_j^c\right]^c\right) \leq 1-r$, this implies that,

$$\mathfrak{I}^{\tilde{\gamma}}\left(\bigcap_{j\in\Gamma_0}\mathcal{S}_j\right) \geq r, \quad \mathfrak{I}^{\tilde{\eta}}\left(\bigcap_{j\in\Gamma_0}\mathcal{S}_j\right) \leq 1-r, \quad \mathfrak{I}^{\tilde{\mu}}\left(\bigcap_{j\in\Gamma_0}\mathcal{S}_j\right) \leq 1-r.$$

(2)⇒(1). Let $\{\mathcal{S}_j \in I^{\widetilde{\mathfrak{T}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j) \geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j) \leq 1-r,\ j \in \Gamma\}$ be a family such that $\bigcup_{j\in\Gamma}\mathcal{S}_j = \tilde{1}$. Then, $\bigcap_{j\in\Gamma}\mathcal{S}_j^c = \tilde{0}$, by (2), there exists a finite subse $\Gamma_0 \subseteq \Gamma$ such that $\mathfrak{I}^{\tilde{\gamma}}\left(\bigcap_{j\in\Gamma_0}\mathcal{S}_j^c\right) \geq r$, $\mathfrak{I}^{\tilde{\eta}}\left(\bigcap_{j\in\Gamma_0}\mathcal{S}_j^c\right) \leq 1-r$, $\mathfrak{I}^{\tilde{\mu}}\left(\bigcap_{j\in\Gamma_0}\mathcal{S}_j^c\right) \leq 1-r$ this implies that $\mathfrak{I}^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \geq r$, $\mathfrak{I}^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \leq 1-r$, $\mathfrak{I}^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{S}_j\right]^c\right) \leq 1-r$. Therefore $(\widetilde{\mathfrak{T}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \mathfrak{I}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $r-\mathcal{SVNI}-compact$.

**Remark 4.6.** Let $(\widetilde{\mathfrak{T}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \mathfrak{I}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNITS}$. The simplest $\mathcal{SVNI}$ on $\widetilde{\mathfrak{T}}$ is $\mathfrak{I}_0^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}: I^{\widetilde{\mathfrak{T}}} \longrightarrow I$, where

$$\mathfrak{I}_0^{\tilde{\gamma}}(\mathcal{S}) = \begin{cases} 1, & if\ \mathcal{S} = \tilde{0} \\ 0, & otherwise, \end{cases} \qquad \mathfrak{I}_0^{\tilde{\eta}}(\mathcal{S}) = \begin{cases} 0, & if\ \mathcal{S} = \tilde{0} \\ 1, & otherwise, \end{cases} \qquad \mathfrak{I}_0^{\tilde{\mu}}(\mathcal{S}) = \begin{cases} 0, & if\ \mathcal{S} = \tilde{0} \\ 1, & otherwise, \end{cases}$$

If $\mathfrak{I}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}} = \mathfrak{I}_0^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$ then $r-\mathcal{SVN}-$compact and $r-\mathcal{SVNI}-$compact are equivalent

**Definition 4.7**. An $\mathcal{SVNTS}$ $(\widetilde{\mathfrak{T}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is said to be *r*-single-valued neutrosophic regular ($r-\mathcal{SVN}-$regular) iff for every $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}) \leq 1-r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}) \leq 1-r$ and $r \in I_0$,

$$\mathcal{S} = \bigcup\{\mathcal{E} \in I^{\widetilde{\mathfrak{T}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}) \geq r, \quad \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}) \leq 1-r, \quad \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}) \leq 1-r, \quad C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}, \mathrm{r}) = \mathcal{S}\}.$$

**Theorem 4.8.** Let $(\widetilde{\mathfrak{T}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \mathfrak{I}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $r-\mathcal{SVNI}-$ quasi $\mathrm{H}-$closed and $r-\mathcal{SVN}-$regular. Then $(\widetilde{\mathfrak{T}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \mathfrak{I}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $r-\mathcal{SVNI}-$compact.

**Proof.** For every family $\{\mathcal{S} \in I^{\widetilde{\mathfrak{T}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j) \geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j) \leq 1-r,\ j \in \Gamma\}$ such that $\bigcup_{j\in\Gamma}\mathcal{S}_j = \tilde{1}$. By $r-\mathcal{SVN}-$regularity of $(\widetilde{\mathfrak{T}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \mathfrak{I}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$, for any $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j) \leq 1-r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j) \leq 1-r$, we have

$$\mathcal{S}_j = \bigcup_{j_\Delta \in \Delta_j} \{\mathcal{S}_{j_\Delta}:\ \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_{j_\Delta}) \geq r,\quad \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_{j_\Delta}) \leq 1-r,\quad \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_{j_\Delta}) \leq 1-r,\quad C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_{j_\Delta}, \mathrm{r}) \leq \mathcal{S}_j\}.$$

Thus, $\cup_{j\in\Gamma}(\cup_{j_\Delta\in\Delta_j} \mathcal{S}_{j_\Delta}) = \tilde{1}$. Since $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $r-\mathcal{SVNJ}$- quasi H-closed, there exists a finite subset $K \times \Delta_K$ such that

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\bigcup_{k\in K}(\bigcup_{k_\Delta\in\Delta_k} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_{k_\Delta}, r))\right]^c\right) \geq r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\bigcup_{k\in K}(\bigcup_{k_\Delta\in\Delta_k} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_{k_\Delta}, r))\right]^c\right) \leq 1-r, \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\bigcup_{k\in K}(\bigcup_{k_\Delta\in\Delta_k} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_{k_\Delta}, r))\right]^c\right) \leq 1-r.$$

For each $k \in K$, since $\cup_{k_\Delta\in\Delta_k} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_{k_\Delta}, r) \leq \mathcal{S}_k$. It implies that $\left[\cup_{k\in K}(\cup_{k_\Delta\in\Delta_k} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_{k_\Delta}, r))\right]^c \geq [\cup_{k\in K} \mathcal{S}_k]^c$. Thus,

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\bigcup_{k\in K} \mathcal{S}_k\right]^c\right) \geq \tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\bigcup_{k\in K}(\bigcup_{k_\Delta\in\Delta_k} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_{k_\Delta}, r))\right]^c\right) \geq r,\quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\bigcup_{k\in K} \mathcal{S}_k\right]^c\right) \leq \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\bigcup_{k\in K}(\bigcup_{k_\Delta\in\Delta_k} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_{k_\Delta}, r))\right]^c\right) \leq 1-r$$

$$\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\bigcup_{k\in K} \mathcal{S}_k\right]^c\right) \leq \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\bigcup_{k\in K}(\bigcup_{k_\Delta\in\Delta_k} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_{k_\Delta}, r))\right]^c\right) \leq 1-r.$$

Hence, $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $r-\mathcal{SVNJ}-$ compact.

**Definition 4.9.** A family $\{\mathcal{S}_j\}_{j\in\Gamma}$ in $\tilde{\mathfrak{X}}$ has the finite intersection property $(\boldsymbol{I-FIP})$ iff the intersection of no finite sub-family $\Gamma_0 \subseteq \Gamma$ s.t $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\cap_{j\in\Gamma_0} \mathcal{S}_j) \geq r$, $\tilde{\mathcal{J}}^{\tilde{\eta}}(\cap_{j\in\Gamma_0} \mathcal{S}_j) \leq 1-r$, $\tilde{\mathcal{J}}^{\tilde{\mu}}(\cap_{j\in\Gamma_0} \mathcal{S}_j) \leq 1-r$.

**Theorem 4.10.** An $\mathcal{SVNJTS}$ $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $\mathrm{r}-\mathcal{SVNJ}-compact$, iff every family $\{\mathcal{S}_j \in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j^c) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j^c) \leq 1-r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j^c) \leq 1-r,\ j \in \Gamma\}$ having the finite intersection property $(\boldsymbol{I-FIP})$ has a non-empty intersection.

**Proof.** Obvious.

**Theorem 4.11**. Suppose that $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is an $\mathcal{SVNJTS}$, $\mathcal{S}$ is $r-\mathcal{SVNJ}-compact$. Then for every collection $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \mathcal{E}_j \leq int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j, r), r),\ j \in \Gamma\}$ with $\mathcal{S} \leq \cup_{j\in\Gamma} \mathcal{E}_j$, there exists a finite subset $\Gamma_0 \subseteq \Gamma$ s.t,

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j, r), r)\right]^c\right) \geq r,\quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j, r), r)\right]^c\right) \leq 1-r$$

$$\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j, r), r)\right]^c\right) \leq 1-r.$$

**Proof.** Let $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \mathcal{E}_j \leq int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j, r), r),\ j \in \Gamma\}$ with $\mathcal{S} \leq \cup_{j\in\Gamma} \mathcal{E}_j$.Then, $\mathcal{S} \leq \cup_{j\in\Gamma} int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j, r), r)$, $[\tilde{\tau}^{\tilde{\gamma}}\left(int_{\tilde{\tau}^{\tilde{\gamma}}}(C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j, r), r)\right) \geq r$, $\tilde{\tau}^{\tilde{\eta}}\left(int_{\tilde{\tau}^{\tilde{\eta}}}(C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j, r), r)\right) \leq 1-r$, $\tilde{\tau}^{\tilde{\mu}}\left(int_{\tilde{\tau}^{\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j, r), r)\right) \leq 1-r]$. By $\mathrm{r}-\mathcal{SVNJ}$ $-compactness$ of $\mathcal{S}$, there exists a finite subset $\Gamma_0 \subseteq \Gamma$ s.t,

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j, r), r)\right]^c\right) \geq r,\quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j, r), r)\right]^c\right) \leq 1-r$$

$$\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S}\cap\left[\bigcup_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,r),r)\right]^c\right)\le 1-r.$$

**Definition 4.12.** Let $(\tilde{\mathfrak{T}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNTS}$ and $\mathcal{S}\in I^{\tilde{\mathfrak{T}}}$. Then $\mathcal{S}$ is called $r$-single-valued neutrosophic locally closed iff $\mathcal{S}=\mathcal{E}\cap\mathcal{D}$ where $[\tilde{\tau}^{\tilde{\gamma}}(\mathcal{E})\ge r,\tilde{\tau}^{\tilde{\eta}}(\mathcal{E})\le 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{E})\le 1-r]$, $[\tilde{\tau}^{\tilde{\gamma}}(\mathcal{D}^c)\ge r,\tilde{\tau}^{\tilde{\eta}}(\mathcal{D}^c)\le 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{D}^c)\le 1-r]$.

**Lemma 4.13.** Let $(\tilde{\mathfrak{T}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNTS}$ and $\mathcal{S}\in I^{\tilde{\mathfrak{T}}}$. Then $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S})\ge r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S})\le 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S})\le 1-r$ iff $\mathcal{S}$ both $r$-single-valued neutrosophic locally closed and $r-SVNPO$ set.

**Proof.** It is trivial.

**Lemma 4.14.** If $\mathcal{S}$ is $r-\mathcal{SVNJ}-$ compact, then for every collection $\{\mathcal{E}_j\in I^{\tilde{\mathfrak{T}}}\colon \mathcal{E}_j\ is\ both\ r-SVNPO\ and\ r-single-valued\ neutrosophic\ locally\ closed\ sets,\ j\in\Gamma\}$ with $\mathcal{S}\le\bigcup_{j\in\Gamma}(\mathcal{E}_j)$, there exists a finite subfamily $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\mathcal{S}\cap[\bigcup_{j\in\Gamma_0}\mathcal{E}_j]^c\right)\ge r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\mathcal{S}\cap[\bigcup_{j\in\Gamma_0}\mathcal{E}_j]^c\right)\le 1-r,\ \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S}\cap[\bigcup_{j\in\Gamma_0}\mathcal{E}_j]^c\right)\le 1-r.$

**Proof.** Follows from Lemma 4.13.

**Theorem 4.15**. Let $(\tilde{\mathfrak{T}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNJTS}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ are $\mathrm{r}-\mathcal{SVNJ}-compact$. Then, $\mathcal{S}\cup\mathcal{E}$ is $\mathrm{r}-\mathcal{SVNJ}-compact$ subset relative to $\tilde{\mathfrak{T}}$.

**Proof.** Let $\{\mathcal{E}_j\in I^{\tilde{\mathfrak{T}}}\colon \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}_j)\ge r,\tilde{\tau}^{\tilde{\eta}}(\mathcal{E}_j)\le 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}_j)\le 1-r,\ j\in\Gamma\}$ be a family such that $\mathcal{S}_1\cup\mathcal{S}_2\le\bigcup_{j\in\Gamma}\mathcal{E}_j$. Then $\mathcal{S}_1\le\bigcup_{j\in\Gamma}\mathcal{E}_j$ and $\mathcal{S}_2\le\bigcup_{j\in\Gamma}\mathcal{E}_j$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are $r-\mathcal{SVNJ}-compact$, there exists a finite subset $\Gamma_0\subseteq\Gamma$ such that

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\mathcal{S}_k\cap\left[\bigcup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right)\ge r,\qquad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\mathcal{S}_k\cap\left[\bigcup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right)\le 1-r,\qquad \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S}_k\cap\left[\bigcup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right)\le 1-r,$$

for $k=1,2$, since $\left(\mathcal{S}_1\cap[\bigcup_{j\in\Gamma_0}\mathcal{E}_j]^c\right)\cup\left(\mathcal{S}_2\cap[\bigcup_{j\in\Gamma_0}\mathcal{E}_j]^c\right)=(\mathcal{S}_1\cup\mathcal{S}_2)\cap[\bigcup_{j\in\Gamma_0}\mathcal{E}_j]^c$. Then,

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left((\mathcal{S}_1\cup\mathcal{S}_2)\cap\left[\bigcup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right)\ge r,\qquad \tilde{\mathcal{J}}^{\tilde{\eta}}\left((\mathcal{S}_1\cup\mathcal{S}_2)\cap\left[\bigcup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right)\le 1-r,\qquad \tilde{\mathcal{J}}^{\tilde{\mu}}\left((\mathcal{S}_1\cup\mathcal{S}_2)\cap\left[\bigcup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right)\le 1-r.$$

This shown that $(\mathcal{S}_1\cup\mathcal{S}_2)$ is $r-\mathcal{SVNJ}-$ compact.

**Theorem 4.16.** Suppose $(\tilde{\mathfrak{T}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNJTS}$, $r\in I_0$. Then the next statements are equivalent:

(1) $(\tilde{\mathfrak{T}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $r-\mathcal{SVNJ}-\ quasi\ H-closed$,

(2) For every collection $\{\mathcal{S}_j\in I^{\tilde{\mathfrak{T}}}\colon \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j^c)\ge r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j^c)\le 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j^c)\le 1-r,\ j\in\Gamma\}$ with $\bigcap_{j\in\Gamma}\mathcal{S}_j=\tilde{0}$, there exists $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\bigcap_{j\in\Gamma_0}int_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r))\ge r$, $\tilde{\mathcal{J}}^{\tilde{\eta}}(\bigcap_{j\in\Gamma_0}int_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r))\le 1-r$, $\tilde{\mathcal{J}}^{\tilde{\mu}}(\bigcap_{j\in\Gamma_0}int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r))\le 1-r$,

(3) $\bigcap_{j\in\Gamma}\mathcal{S}_j\ne\tilde{0}$, holds for any collection $\{\mathcal{S}_j\in I^{\tilde{\mathfrak{T}}}\colon \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j^c)\ge r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j^c)\le 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j^c)\le 1-r,\ j\in\Gamma\}$ such that $\{int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r)\colon \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j^c)\ge r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j^c)\le 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j^c)\le 1-r,\ j\in\Gamma\}$ has the $\boldsymbol{I-FIP}$,

(4) For any collection $\{\mathcal{S}_j\in I^{\tilde{\mathfrak{T}}}\colon \mathcal{S}_j$ is $r-SVNRO\ sets,\ j\in\Gamma\}$ such taht $\bigcup_{j\in\Gamma}\mathcal{S}_j=\tilde{1}$, there exists $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left([\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r)]^c\right)\ge r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}\left([\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r)]^c\right)\le 1-r,\ \tilde{\mathcal{J}}^{\tilde{\mu}}\left([\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)]^c\right)\le 1-r,$

(5) For every collection $\{\mathcal{S}_j \in I^{\tilde{\mathfrak{X}}}: \mathcal{S}_j \text{ is } r-SVNRC\ set,\ j\in\Gamma\}$ such taht $\bigcap_{j\in\Gamma}\mathcal{S}_j=\tilde{0}$, there exists $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r))\geq r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r))\leq 1-r,\ \tilde{\mathcal{J}}^{\tilde{\mu}}(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r))\leq 1-r,$

(6) $\bigcap_{j\in\Gamma}\mathcal{S}_j\neq\tilde{0}$ , holds for every collection $\{\mathcal{S}_j \in I^{\tilde{\mathfrak{X}}}: \mathcal{S}_j \text{ is } r-SVNRC\ set,\ j\in\Gamma\}$ such taht $\{int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r): \mathcal{S}_j \text{ is } r-SVNRC\ set,\ j\in\Gamma\}$ has the $\boldsymbol{I-FIP}$.

Proof. (1)⇒(2). Let $\{\mathcal{S}_j\in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j^c)\geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j^c)\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j^c)\leq 1-r,\ j\in\Gamma\}$ be a family with $\bigcap_{j\in\Gamma}\mathcal{S}_j=\tilde{0}$. Then, $\bigcup_{j\in\Gamma}\mathcal{S}_j^c=\tilde{1}$. Since, $(\tilde{\mathfrak{X}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $r-\mathcal{SVNJ}-\ quasi\ H-closed$, there exists $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}([\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j^c,r)]^c)\geq r$ , $\tilde{\mathcal{J}}^{\tilde{\eta}}([\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j^c,r)]^c)\leq 1-r$ , $\tilde{\mathcal{J}}^{\tilde{\mu}}([\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j^c,r)]^c)\leq 1-r$ . Since, $[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j^c,r)]^c=\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r)$, we have

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)\right)\geq r,\quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)\right)\leq 1-r,\quad \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)\right)\leq 1-r.$$

(2)⇒(1). Let $\{\mathcal{S}_j\in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j)\geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j)\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j)\leq 1-r, j\in\Gamma\}$ be a family s.t $\bigcup_{j\in\Gamma}\mathcal{S}_j=\tilde{1}$. Then, $\bigcap_{j\in\Gamma}\mathcal{S}_j^c=\tilde{0}$ and by hypothesis, there exists $\Gamma_0\subseteq\Gamma$ s.t, $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j^c,r))\geq r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j^c,r))\leq 1-r,\ \tilde{\mathcal{J}}^{\tilde{\mu}}(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j^c,r))\leq 1-r$. Since, $\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j^c,r)=[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r)]^c$,

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r)\right]^c\right)\geq r,\quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r)\right]^c\right)\leq 1-r,\quad \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)\right]^c\right)\leq 1-r.$$

Thus, $(\tilde{\mathfrak{X}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $r-\mathcal{SVNJ}$- *quasi H-closed,*

(1) ⇒ (3). For any family $\{\mathcal{S}_j\in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j^c)\geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j^c)\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j^c)\leq 1-r,\ j\in\Gamma\}$ such that $\{int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r): \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j^c)\geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j^c)\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j^c)\leq 1-r,\ j\in\Gamma\}$ has the $\boldsymbol{I-FIP}$. If $\bigcap_{j\in\Gamma}\mathcal{S}_j=\tilde{0}$, then $\bigcup_{j\in\Gamma}\mathcal{S}_j^c=\tilde{1}$. Since $(\tilde{\mathfrak{X}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $r-\mathcal{SVNJ}$- *quasi H-closed*, there exists a finite subset $\Gamma_0\subseteq\Gamma$ such that

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j^c,r)\right]^c\right)\geq r,\quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j^c,r)\right]^c\right)\leq 1-r,\quad \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j^c,r)\right]^c\right)\leq 1-r.$$

Since, $[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j^c,r)]^c=\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)$, we have

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r)\right)\geq r,\quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r)\right)\leq 1-r,\quad \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)\right)\leq 1-r.$$

Which is a contradiction.

(3)⇒(1). For any family $\{\mathcal{S}_j\in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j)\geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j)\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j)\leq 1-r,\ j\in\Gamma\}$ such that $\bigcup_{j\in\Gamma}\mathcal{S}_j=\tilde{1}$, with the property that for no finite $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}([\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r)]^c)\geq r, \tilde{\mathcal{J}}^{\tilde{\eta}}([\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r)]^c)\leq 1-r$, $\tilde{\mathcal{J}}^{\tilde{\mu}}([\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)]^c)\leq 1-r$. Since,

$$\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r)\right]^c=\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j^c,r).$$

The family $\{int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j^c,r): \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j)\geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j)\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j)\leq 1-r,\ j\in\Gamma\}$ has the $\boldsymbol{I-FIP}$ .By (3). $\bigcap_{j\in\Gamma}\mathcal{S}_j^c\neq\tilde{0}$, Then, $\bigcup_{j\in\Gamma}\mathcal{S}_j\neq\tilde{1}$. It is a contradiction.

(1)⇒(4). Let $\{\mathcal{S}_j\}_{j\in\Gamma}$ be a family of $r-SVNRO$ set such that $\bigcup_{j\in\Gamma}\mathcal{S}_j=\tilde{1}$. Then, $\bigcup_{j\in\Gamma} int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r),\mathrm{r})=\tilde{1}$, since, $\tilde{\tau}^{\tilde{\gamma}}(int_{\tilde{\tau}^{\tilde{\gamma}}}(C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r),\mathrm{r}))\geq r,\ \tilde{\tau}^{\tilde{\eta}}(int_{\tilde{\tau}^{\tilde{\eta}}}(C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r),\mathrm{r}))\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(int_{\tilde{\tau}^{\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r),\mathrm{r}))\leq 1-r$ and $\tilde{\mathfrak{X}}$ is $r-\mathcal{SVNJ}$- *quasi H-closed*, there exists a finite subset $\Gamma_0\subseteq\Gamma$ such that

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(int_{\tilde{\tau}^{\tilde{\gamma}}}(C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r),\mathrm{r}),r)\right]^c\right)\geq r,\quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}\left(int_{\tilde{\tau}^{\tilde{\eta}}}\left(C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r),\mathrm{r}\right)\right)\right]^c\right)\leq 1-r,$$

$$\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}\left(int_{\tilde{\tau}^{\tilde{\mu}}}\left(C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r),\mathrm{r}\right),r\right)\right]^c\right)\leq 1-r.$$

Since, for $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j)\geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j)\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j)\leq 1-r$ we have $C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r),\mathrm{r}),r)=C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r)$. Hence, $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r)\right]^c\right)\geq r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r)\right]^c\right)\leq 1-r,\ \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)\right]^c\right)\leq 1-r.$

(4)⇒(5). Let $\{\mathcal{S}_j\in I^{\tilde{\mathfrak{T}}}:\ j\in\Gamma\}$ be a family of $r-SVNRC$ sets such that $\bigcap_{j\in\Gamma}\mathcal{S}_j=\tilde{0}$. Then, $\bigcup_{j\in\Gamma}\mathcal{S}_j^c=\tilde{1}$, and $\{\mathcal{S}_j^c\in I^{\tilde{\mathfrak{T}}}:\ j\in\Gamma\}$ is a family of $r-SVNRO$ sets. By (4), there will be a finite subset $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j^c,r)\right]^c\right)\geq r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j^c,r)\right]^c\right)\leq 1-r,\ \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j^c,r)\right]^c\right)\leq 1-r$, Thus,

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r)\right)\geq r,\quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r)\right)\leq 1-r,\quad \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)\right)\leq 1-r.$$

(5)⇒(1). Let $\{\mathcal{S}_j\in I^{\tilde{\mathfrak{T}}}:\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j)\geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j)\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j)\leq 1-r,\ j\in\Gamma\}$ be a family such that $\bigcup_{j\in\Gamma}\mathcal{S}_j=\tilde{1}$. Then, $\bigcup_{j\in\Gamma} int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r),\mathrm{r})=\tilde{1}$. Thus, $\bigcap_{j\in\Gamma} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j^c,r),\mathrm{r})=\tilde{0}$ and $C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j^c,r),\mathrm{r})$ is $r-SVNRC$. For the hypothesis, there exists $\Gamma_0\subseteq\Gamma$ such that

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}}}(C_{\tilde{\tau}^{\tilde{\gamma}}}(int_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j^c,r),\mathrm{r}),r)\right)\geq r,\quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\eta}}}(C_{\tilde{\tau}^{\tilde{\eta}}}(int_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j^c,r),\mathrm{r}),r)\right)\leq 1-r,$$

$$\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\mu}}}(int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j^c,r),\mathrm{r}),r)\right)\leq 1-r$$

Since, for $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j)\geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j)\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j)\leq 1-r$ we have $C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r),\mathrm{r}),r)=C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r)$, and hence, $\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j^c,r),\mathrm{r}),r)=[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,r)]^c$. Therefore, $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r)\right]^c\right)\geq r$, $\tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r)\right]^c\right)\leq 1-r$ , $\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)\right]^c\right)\leq 1-r$ ). Hence, $(\tilde{\mathfrak{T}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $r-\mathcal{SVNJ}-$ $quasi\ H-closed,$

(6)⇔(4) is proved similarly like (3)⇔(1).

**Theorem 4.17.** Let $(\tilde{\mathfrak{T}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNJTS}$ and $r\in I_0$, Then the next statements are equivalent:

(1) $(\tilde{\mathfrak{T}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $r-\mathcal{SVNJ}-\ quasi\ H-closed,$

(2) For any family $\{\mathcal{S}_j\in I^{\tilde{\mathfrak{T}}}:\mathcal{S}_j\leq int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,\mathrm{r}),\mathrm{r})\}$ with $\bigcup_{j\in\Gamma}\mathcal{S}_j=\tilde{1}$, there exists a finite subset $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r)\right]^c\right)\geq r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r)\right]^c\right)\leq 1-r,\ \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r)\right]^c\right)\leq 1-r),$

(3) For any family $\{\mathcal{S}_j\in I^{\tilde{\mathfrak{T}}}:\ \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j^c)\geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}_j^c)\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}_j^c)\leq 1-r,\ j\in\Gamma\}$ such that $\bigcap_{j\in\Gamma}\mathcal{S}_j=\tilde{0}$, there exists a finite subset $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{S}_j,r))\geq r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{S}_j,r))\leq 1-r$, $\tilde{\mathcal{J}}^{\tilde{\mu}}(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{S}_j,r))\leq 1-r).$

**Proof.** Obvious.

**Theorem 4.18**. Let $(\tilde{\mathfrak{T}},\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an $\mathcal{SVNJTS}$ and $r\in I_0$, Then the next statements are equivalent:

(1) $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathfrak{I}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is $r - \mathcal{SVNC}(\mathcal{I}) - compact$,

(2) For each family $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}_j^c) \geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}_j^c) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}_j^c) \leq 1-r,\ j \in \Gamma\}$ and every $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1-r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1-r$ with $\bigcap_{j\in\Gamma} \mathcal{E}_j\, \bar{q} \mathcal{S}$, there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\tilde{\mathfrak{I}}^{\tilde{\gamma}}(\mathcal{S} \cap \bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j, r)) \geq r$, $\tilde{\mathfrak{I}}^{\tilde{\eta}}(\mathcal{S} \cap \bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j, r)) \leq 1-r$, $\tilde{\mathfrak{I}}^{\tilde{\mu}}(\mathcal{S} \cap \bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j, r)) \leq 1-r$.

(3) $\bigcap_{j\in\Gamma} \mathcal{E}_j\, q \mathcal{S}$ holds for each family $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}_j^c) \geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}_j^c) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}_j^c) \leq 1-r,\ j \in \Gamma\}$ and any $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1-r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1-r$ with $\{int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j, r)\ \mathrm{q}\mathcal{S},\ j \in \Gamma\}$ has the $\boldsymbol{I-FIP}$,

(4) For each family $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \mathcal{E}_j\ is\ r - SVNRO,\ j \in \Gamma\}$ and any $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1-r$. $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1-r$ with $\mathcal{S} \leq \bigcup_{j\in\Gamma} \mathcal{E}_j$, there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that,

$$\tilde{\mathfrak{I}}^{\tilde{\gamma}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j, r)\right]^c\right) \geq r, \tilde{\mathfrak{I}}^{\tilde{\eta}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j, r)\right]^c\right) \leq 1-r, \tilde{\mathfrak{I}}^{\tilde{\mu}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j, r)\right]^c\right) \leq 1-r.$$

(5) For each family $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \mathcal{E}_j \text{ is r} - SVNRC,\ j \in \Gamma\}$ and any $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1-r$, with $\bigcap_{j\in\Gamma} \mathcal{E}_j\, \bar{q} \mathcal{S}$, there exists $\Gamma_0 \subseteq \Gamma$ such that,

$$\tilde{\mathfrak{I}}^{\tilde{\gamma}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j, r) \cap \mathcal{S}\right) \geq r, \quad \tilde{\mathfrak{I}}^{\tilde{\eta}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j, r) \cap \mathcal{S}\right) \leq 1-r, \tilde{\mathfrak{I}}^{\tilde{\mu}}\left(\bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j, r) \cap \mathcal{S}\right) \leq 1-r,$$

(6) $\bigcap_{j\in\Gamma} \mathcal{E}_j\, q \mathcal{S}$ holds for each family $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \mathcal{E}_j\ is\ r - SVNRC,\ j \in \Gamma\}$ and any $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r, \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1-r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1-r$ such taht $\{int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j, r) \cap \mathcal{S}:\ j \in \Gamma\}$ has the $\boldsymbol{I-FIP}$.

**Proof.** (1)⇒(2). Let $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}_j^c) \geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}_j^c) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}_j^c) \leq 1-r,\ j \in \Gamma\}$and $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1-r$ with $\bigcap_{j\in\Gamma} \mathcal{E}_j\, \bar{q} \mathcal{S}$. Then, $\tilde{\gamma}_{\bigcap_{j\in\Gamma}\mathcal{E}_j} + \tilde{\gamma}_{\mathcal{S}} \leq 1$, $\tilde{\eta}_{\bigcap_{j\in\Gamma}\mathcal{E}_j} + \tilde{\eta}_{\mathcal{S}} \geq 1$, $\tilde{\mu}_{\bigcap_{j\in\Gamma}\mathcal{E}_j} + \tilde{\mu}_{\mathcal{S}} \geq 1$. It implies that $\mathcal{S} \leq \bigcup_{j\in\Gamma} \mathcal{E}_j^c$. By $r - \mathcal{SVNC}(\mathcal{I}) - compactness$ of $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathfrak{I}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$, there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that,

$$\tilde{\mathfrak{I}}^{\tilde{\gamma}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j^c, r)\right]^c\right) \geq r, \quad \tilde{\mathfrak{I}}^{\tilde{\eta}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j^c, r)\right]^c\right) \leq 1-r, \quad \tilde{\mathfrak{I}}^{\tilde{\mu}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j^c, r)\right]^c\right) \leq 1-r.$$

Since, $\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j^c, r)\right]^c = \mathcal{S} \cap \bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j, r)$. Then

$$\tilde{\mathfrak{I}}^{\tilde{\gamma}}\left(\mathcal{S} \cap \bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j, r)\right) \geq r, \quad \tilde{\mathfrak{I}}^{\tilde{\eta}}\left(\mathcal{S} \cap \bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j, r)\right) \leq 1-r, \quad \tilde{\mathfrak{I}}^{\tilde{\mu}}\left(\mathcal{S} \cap \bigcap_{j\in\Gamma_0} int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j, r)\right) \leq 1-r.$$

(2)⇒(3). It is trivial.

(3) ⇒ (1). Let $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}: \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}_j) \geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}_j) \leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}_j) \leq 1-r,\ j \in \Gamma\}$ be a family and $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1-r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1-r$ such that $\mathcal{S} \leq \bigcup_{j\in\Gamma} \mathcal{E}_j$ with property that for no finite subfamily $\Gamma_0$ of $\Gamma$ one has, $\tilde{\mathfrak{I}}^{\tilde{\gamma}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j, r)\right]^c\right) \geq r$, $\tilde{\mathfrak{I}}^{\tilde{\eta}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j, r)\right]^c\right) \leq 1-r$, $\tilde{\mathfrak{I}}^{\tilde{\mu}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j, r)\right]^c\right) \leq 1-r$. Since, $\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j, r)\right]^c = \bigcap_{j\in\Gamma_0}\{int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j^c, r) \cap \mathcal{S}$, the family $\{\bigcap_{j\in\Gamma}\{int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j^c, r) \cap \mathcal{S},\ \ j \in \Gamma\}$ has the $\boldsymbol{I-FIP}$, By (3), $\bigcap_{j\in\Gamma} \mathcal{E}_j^c\, q\mathcal{S}$ implies that $\bigcup_{j\in\Gamma} \mathcal{E}_j \leq \mathcal{S}$. It is a contradiction.

(1)⇒(4). Let $\{\mathcal{E}_j \in I^{\tilde{\mathfrak{X}}}:\ j \in \Gamma\}$ be a family of $r - SVNRO\ sets$ and $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1-r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1-r$ with $\mathcal{S} \leq \bigcup_{j\in\Gamma} \mathcal{E}_j$. Then, $\mathcal{S} \leq \bigcup_{j\in\Gamma} int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j, r), r)$. By $r - \mathcal{SVNC}(\mathcal{I}) - compactness$ of $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathfrak{I}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$, there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that,

$$\tilde{\mathfrak{I}}^{\tilde{\gamma}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\gamma}}}(int_{\tilde{\tau}^{\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j, r), r), r)\right]^c\right) \geq r, \quad \tilde{\mathfrak{I}}^{\tilde{\eta}}\left(\mathcal{S} \cap \left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\eta}}}(int_{\tilde{\tau}^{\tilde{\eta}}}(C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j, r), r), r)\right]^c\right) \leq 1-r,$$

$$\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S}\cap\left[\bigcup_{j\in\Gamma_0} C_{\tilde{\tau}^{\tilde{\mu}}}(int_{\tilde{\tau}^{\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j,r),r),r)\right]^c\right)\leq 1-r$$

Since, for $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}_j)\geq r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{E}_j)\leq 1-r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{E}_j)\leq 1-r,\ C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,r),\mathrm{r}),r)=C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,r)$. Therefore, $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\mathcal{S}\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j,r)\right]^c\right)\geq r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\mathcal{S}\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j,r)\right]^c\right)\leq 1-r,\ \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S}\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j,r)\right]^c\right)\leq 1-r$.

(4)⇒(1). It is trivial.

(4)⇒(5). Let $\{\mathcal{E}_j\}_{j\in\Gamma}$ be a family of $r-SVNRC$ sets and every $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c)\geq r,\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c)\leq 1-r\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c)\leq 1-r$ such that $\bigcap_{j\in\Gamma}\mathcal{E}_j\,\bar{q}\mathcal{S}$. Then, $\mathcal{S}\leq\bigcup_{j\in\Gamma}\mathcal{E}_j^c$ and $\{\mathcal{E}_j^c\in I^{\tilde{\mathfrak{X}}}:\ j\in\Gamma\}$ be a family of $r-SVNRO$ sets. By (4), there exists a finite subset $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\mathcal{S}\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j^c,r)\right]^c\right)\geq r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\mathcal{S}\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j^c,r)\right]^c\right)\leq 1-r$ , $\tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S}\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j^c,r)\right]^c\right)\leq 1-r$ implies that

$$\tilde{\mathcal{J}}^{\tilde{\gamma}}\left(\mathcal{S}\cap\bigcap_{j\in\Gamma_0}int_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j,r)\right)\geq r,\quad \tilde{\mathcal{J}}^{\tilde{\eta}}\left(\mathcal{S}\cap\bigcap_{j\in\Gamma_0}int_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j,r)\right)\leq 1-r,\quad \tilde{\mathcal{J}}^{\tilde{\mu}}\left(\mathcal{S}\cap\bigcap_{j\in\Gamma_0}int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j,r)\right)\leq 1-r.$$

(5)⇒(6). Let $\{\mathcal{E}_j\}_{j\in\Gamma}$ be a family of $r-SVNRC\ sets$ and every $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c)\geq r,\ \tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c)\leq 1-r,\ \tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c)\leq 1-r$ such taht $\{int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,r)\cap\mathcal{S}:\ j\in\Gamma\}$ has the $\boldsymbol{I-FIP}$. If $\bigcap_{j\in\Gamma}\mathcal{E}_j\,\bar{q}\mathcal{S}$. By (5), there exists a finite subset $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\bigcap_{j\in\Gamma_0}int_{\tilde{\tau}^{\tilde{\gamma}}}(\mathcal{E}_j,r)\cap\mathcal{S})\geq r,\ \tilde{\mathcal{J}}^{\tilde{\eta}}(\bigcap_{j\in\Gamma_0}int_{\tilde{\tau}^{\tilde{\eta}}}(\mathcal{E}_j,r)\cap\mathcal{S})\leq 1-r,\ \tilde{\mathcal{J}}^{\tilde{\mu}}(\bigcap_{j\in\Gamma_0}int_{\tilde{\tau}^{\tilde{\mu}}}(\mathcal{E}_j,r)\cap\mathcal{S})\leq 1-r$. It is a contradiction.

(6)⇒(4). It is trivial.

**Theorem 4.19.** Let $\left(\tilde{\mathfrak{X}}_1,\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$, $\left(\tilde{\mathfrak{X}}_2,\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$ be two $\mathcal{SVNJTS}'s$ and $f:\tilde{\mathfrak{X}}_1\longrightarrow\tilde{\mathfrak{X}}_2$ a surjective $\mathcal{SVN}$-continuous. If $\left(\tilde{\mathfrak{X}}_1,\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$ is $r-\mathcal{SVNJ}_1-compact$ and $\tilde{\mathcal{J}}_1^{\tilde{\gamma}}(\mathcal{S})\leq\tilde{\mathcal{J}}_2^{\tilde{\gamma}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\eta}}(\mathcal{S})\geq\tilde{\mathcal{J}}_2^{\tilde{\eta}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\mu}}(\mathcal{S})\geq\tilde{\mathcal{J}}_2^{\tilde{\mu}}(f(\mathcal{S}))$. Then, $\left(\tilde{\mathfrak{X}}_2,\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$ is $r-\mathcal{SVNJ}_2-compact$.

**Proof.** Let $\left\{\mathcal{E}_j\in I^{\tilde{\mathfrak{X}}}:\tilde{\tau}_2^{\tilde{\gamma}}(\mathcal{E}_j)\geq r,\tilde{\tau}_2^{\tilde{\eta}}(\mathcal{E}_j)\leq 1-r,\ \tilde{\tau}_2^{\tilde{\mu}}(\mathcal{E}_j)\leq 1-r,\ j\in\Gamma\right\}$ be a family such that $\bigcup_{j\in\Gamma}\mathcal{E}_j=\tilde{1}$. Then, $\bigcup_{j\in\Gamma}f^{-1}(\mathcal{E}_j)=\tilde{1}$. Since, $f$ is $\mathcal{SVN}-continuous$, for each $j\in\Gamma$, $\tilde{\tau}_1^{\tilde{\gamma}}(f^{-1}(\mathcal{E}_j))\geq r$, $\tilde{\tau}_1^{\tilde{\eta}}(f^{-1}(\mathcal{E}_j))\leq 1-r$, $\tilde{\tau}_1^{\tilde{\mu}}(f^{-1}(\mathcal{E}_j))\leq 1-r$. By $r-\mathcal{SVNJ}_1-compactness$ of $\left(\tilde{\mathfrak{X}}_1,\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$, there exists a finite $\Gamma_0\subseteq\Gamma$ such that $\tilde{\mathcal{J}}_1^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0}f^{-1}(\mathcal{E}_j)\right]^c\right)\geq r,\ \tilde{\mathcal{J}}_1^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0}f^{-1}(\mathcal{E}_j)\right]^c\right)\leq 1-r,\ \tilde{\mathcal{J}}_1^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0}f^{-1}(\mathcal{E}_j)\right]^c\right)\leq 1-r$. Since $\tilde{\mathcal{J}}_1^{\tilde{\gamma}}(\mathcal{S})\leq\tilde{\mathcal{J}}_2^{\tilde{\gamma}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\eta}}(\mathcal{S})\geq\tilde{\mathcal{J}}_2^{\tilde{\eta}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\mu}}(\mathcal{S})\geq\tilde{\mathcal{J}}_2^{\tilde{\mu}}(f(\mathcal{S}))$, for $j\in\Gamma_0$, $\tilde{\mathcal{J}}_2^{\tilde{\gamma}}\left(f(\left[\bigcup_{j\in\Gamma_0}f^{-1}(\mathcal{E}_j)\right]^c)\right)\geq r,\ \tilde{\mathcal{J}}_2^{\tilde{\eta}}\left(f(\left[\bigcup_{j\in\Gamma_0}f^{-1}(\mathcal{E}_j)\right]^c)\right)\leq 1-r$, $\tilde{\mathcal{J}}_2^{\tilde{\mu}}\left(f(\left[\bigcup_{j\in\Gamma_0}f^{-1}(\mathcal{E}_j)\right]^c)\right)\leq 1-r$. From the surjectively of $f$ we obtain $f\left(\left[\bigcup_{j\in\Gamma_0}f^{-1}(\mathcal{E}_j)\right]^c\right)=\left[\bigcup_{j\in\Gamma_0}\mathcal{E}_j\right]^c$. Hence, $\tilde{\mathcal{J}}_2^{\tilde{\gamma}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right)\geq r,\ \tilde{\mathcal{J}}_2^{\tilde{\eta}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right)\leq 1-r,\ \tilde{\mathcal{J}}_2^{\tilde{\mu}}\left(\left[\bigcup_{j\in\Gamma_0}\mathcal{E}_j\right]^c\right)\leq 1-r$. Thus, $\left(\tilde{\mathfrak{X}}_2,\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$ is $r-\mathcal{SVNJ}_2-compact$.

**Theorem 4.20**. Let $\left(\tilde{\mathfrak{X}}_1,\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$, $\left(\tilde{\mathfrak{X}}_2,\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$ be two $\mathcal{SVNJTS}'s$ and $f:\tilde{\mathfrak{X}}_1\longrightarrow\tilde{\mathfrak{X}}_2$ a surjective $\mathcal{SVN}$-continuous. If $\left(\tilde{\mathfrak{X}}_1,\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$ is $r-\mathcal{SVNC}(\mathcal{J})_1-compact$ and $\tilde{\mathcal{J}}_1^{\tilde{\gamma}}(\mathcal{S})\leq\tilde{\mathcal{J}}_2^{\tilde{\gamma}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\eta}}(\mathcal{S})\geq\tilde{\mathcal{J}}_2^{\tilde{\eta}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\mu}}(\mathcal{S})\geq\tilde{\mathcal{J}}_2^{\tilde{\mu}}(f(\mathcal{S}))$. Then, $\left(\tilde{\mathfrak{X}}_2,\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$ is $r-\mathcal{SVNC}(\mathcal{J})_2-compact$.

**Proof.** Let $\tilde{\tau}_2^{\tilde{\gamma}}(\mathcal{S})\geq r,\ \tilde{\tau}_2^{\tilde{\eta}}(\mathcal{S})\leq 1-r,\ \tilde{\tau}_2^{\tilde{\mu}}(\mathcal{S})\leq 1-r$ and every family $\{\mathcal{E}_j\in I^{\tilde{\mathfrak{X}}}:\tilde{\tau}_2^{\tilde{\gamma}}(\mathcal{E}_j)\geq r,\ \ \tilde{\tau}_2^{\tilde{\eta}}(\mathcal{E}_j)\leq 1-r\}$ with $\mathcal{S}\leq\bigcup_{j\in\Gamma}\mathcal{E}_j$. Then, $f^{-1}(\mathcal{S})\leq\bigcup_{j\in\Gamma}f^{-1}(\mathcal{E}_j)$. Since, $f$ is $\mathcal{SVN}$- continuous for each $j\in\Gamma$, $\tilde{\tau}_1^{\tilde{\gamma}}(f^{-1}(\mathcal{E}_j))\geq r$, $\tilde{\tau}_1^{\tilde{\eta}}(f^{-1}(\mathcal{E}_j))\leq 1-r,\ \tilde{\tau}_1^{\tilde{\mu}}(f^{-1}(\mathcal{E}_j))\leq 1-r$. By $r-\mathcal{SVNC}(\mathcal{J})_1-compactness$ of $\left(\tilde{\mathfrak{X}}_1,\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$, there exists a finite $\Gamma_0\subseteq\Gamma$ such that

$$\tilde{\mathcal{J}}_1^{\tilde{\gamma}}\left(f^{-1}(\mathcal{S})\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}_1^{\tilde{\gamma}}}(f^{-1}(\mathcal{E}_j),\mathrm{r})\right]^c\right)\geq r,\qquad \tilde{\mathcal{J}}_1^{\tilde{\eta}}\left(f^{-1}(\mathcal{S})\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}_1^{\tilde{\eta}}}(f^{-1}(\mathcal{E}_j),\mathrm{r})\right]^c\right)\leq 1-r,$$

$$\tilde{\mathcal{J}}_1^{\tilde{\mu}}\left(f^{-1}(\mathcal{S})\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}_1^{\tilde{\mu}}}(f^{-1}(\mathcal{E}_j),\mathrm{r})\right]^c\right)\leq 1-r.$$

Since, $f$ is $\mathcal{SVN}$ - continuous mapping, $C_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}_j,\mathrm{r})\leq f^{-1}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_j,\mathrm{r}))$ for every $\mathcal{S}\in I^{\tilde{\mathfrak{X}}_2}$. Therefore, $f^{-1}(\mathcal{S})\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{E}_j,\mathrm{r})\right]^c=f^{-1}(\mathcal{S})\cap\left[\bigcup_{j\in\Gamma_0}f^{-1}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,\mathrm{r}))\right]^c$. Hence,

$$\tilde{\mathcal{J}}_1^{\tilde{\gamma}}\left(f^{-1}(\mathcal{S}_j)\cap\left[\bigcup_{j\in\Gamma_0}f^{-1}(C_{\tilde{\tau}_2^{\tilde{\gamma}}}(\mathcal{S},\mathrm{r}))\right]^c\right)\geq r,\qquad \tilde{\mathcal{J}}_1^{\tilde{\eta}}\left(f^{-1}(\mathcal{S}_j)\cap\left[\bigcup_{j\in\Gamma_0}f^{-1}(C_{\tilde{\tau}_2^{\tilde{\eta}}}(\mathcal{S},\mathrm{r}))\right]^c\right)\leq 1-r,$$

$$\tilde{\mathcal{J}}_1^{\tilde{\mu}}\left(f^{-1}(\mathcal{S}_j)\cap\left[\bigcup_{j\in\Gamma_0}f^{-1}(C_{\tilde{\tau}_2^{\tilde{\mu}}}(\mathcal{S},\mathrm{r}))\right]^c\right)\leq 1-r.$$

Since, $\tilde{\mathcal{J}}_1^{\tilde{\gamma}}(\mathcal{S})\leq\tilde{\mathcal{J}}_2^{\tilde{\gamma}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\eta}}(\mathcal{S})\geq\tilde{\mathcal{J}}_2^{\tilde{\eta}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\mu}}(\mathcal{S})\geq\tilde{\mathcal{J}}_2^{\tilde{\mu}}(f(\mathcal{S}))$, for each $j\in\Gamma_0$ we have,

$$\tilde{\mathcal{J}}_2^{\tilde{\gamma}}\left(f[f^{-1}(\mathcal{S}_j)\cap\left[\bigcup_{j\in\Gamma_0}f^{-1}\left(C_{\tilde{\tau}_2^{\tilde{\gamma}}}(\mathcal{S},\mathrm{r})\right)\right]^c]\right)\geq r,\qquad \tilde{\mathcal{J}}_2^{\tilde{\eta}}\left(f[f^{-1}(\mathcal{S}_j)\cap\left[\bigcup_{j\in\Gamma_0}f^{-1}\left(C_{\tilde{\tau}_2^{\tilde{\eta}}}(\mathcal{S},\mathrm{r})\right)\right]^c]\right)\leq 1-r,$$

$$\tilde{\mathcal{J}}_2^{\tilde{\mu}}\left(f[f^{-1}(\mathcal{S}_j)\cap\left[\bigcup_{j\in\Gamma_0}f^{-1}\left(C_{\tilde{\tau}_2^{\tilde{\mu}}}(\mathcal{S},\mathrm{r})\right)\right]^c]\right)\leq 1-r.$$

Since, $f$ is surjective,

$$\tilde{\mathcal{J}}_2^{\tilde{\gamma}}\left(\mathcal{S}_j\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}_2^{\tilde{\gamma}}}(\mathcal{S},\mathrm{r})\right]^c\right)\geq r,\qquad \tilde{\mathcal{J}}_2^{\tilde{\eta}}\left(\mathcal{S}_j\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}_2^{\tilde{\eta}}}(\mathcal{S},\mathrm{r})\right]^c\right)\leq 1-r,\qquad \tilde{\mathcal{J}}_2^{\tilde{\mu}}\left(\mathcal{S}_j\cap\left[\bigcup_{j\in\Gamma_0}C_{\tilde{\tau}_2^{\tilde{\mu}}}(\mathcal{S},\mathrm{r})\right]^c\right)\leq 1-r.$$

Thus, $\left(\tilde{\mathfrak{X}}_2,\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$ is $r-\mathcal{SVN}(\mathcal{J})_2-compact$.

**Theorem 4.21.** The image of an $r-\mathcal{SVNJ}_1-compact$ under a surjective $\mathcal{SVN}-almost\ continuous$ mapping and $\tilde{\mathcal{J}}_1^{\tilde{\gamma}}(\mathcal{S})\leq\tilde{\mathcal{J}}_2^{\tilde{\gamma}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\eta}}(\mathcal{S})\geq\tilde{\mathcal{J}}_2^{\tilde{\eta}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\mu}}(\mathcal{S})\geq\tilde{\mathcal{J}}_2^{\tilde{\mu}}(f(\mathcal{S}))$ is $r-\mathcal{SVNC}(\mathcal{J})_2-compact$.

**Proof.** Let $\mathcal{S}\in \mathrm{I}^{\tilde{\mathfrak{X}}_1}$ be an $r-\mathcal{SVNJ}_1-compact$ in $\left(\tilde{\mathfrak{X}}_1,\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$ and $f:\left(\tilde{\mathfrak{X}}_1,\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)\to\left(\tilde{\mathfrak{X}}_2,\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}},\tilde{\mathcal{J}}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$ a surjective $\mathcal{SVN}-almost\ continuous$. If $\tilde{\tau}_2^{\tilde{\gamma}}(\mathcal{S}^c)\geq r,\tilde{\tau}_2^{\tilde{\eta}}(\mathcal{S}^c)\leq 1-r$, $\tilde{\tau}_2^{\tilde{\mu}}(\mathcal{S}^c)\leq 1-r$ and each family $\{\mathcal{E}_j\in I^{\tilde{\mathfrak{X}}}$: $\tilde{\tau}_2^{\tilde{\gamma}}(\mathcal{E}_j)\geq r$, $\tilde{\tau}_2^{\tilde{\eta}}(\mathcal{E}_j)\leq 1-r$, $\tilde{\tau}_2^{\tilde{\mu}}(\mathcal{E}_j)\leq 1-r\}$ with $f(\mathcal{S})\leq\bigcup_{j\in\Gamma}\mathcal{E}_j$, then $f(\mathcal{S})\leq\bigcup_{j\in\Gamma}int_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,r),r)$ and

since for $j\in\Gamma$,

$$int_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(int_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,r),r),r),\mathrm{r})=int_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,r),r).$$

By $\mathcal{SVN}-almost\ continuous$ of $f$ we have $\mathcal{S}\leq\bigcup_{j\in\Gamma}f^{-1}(int_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,r),r))$ and

$$\tilde{\tau}_1^{\tilde{\gamma}}\left(f^{-1}(int_{\tilde{\tau}_2^{\tilde{\gamma}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}}}(\mathcal{E}_j,r),r))\right)\geq r,\qquad \tilde{\tau}_2^{\tilde{\eta}}\left(f^{-1}(int_{\tilde{\tau}_2^{\tilde{\eta}}}(C_{\tilde{\tau}_2^{\tilde{\eta}}}(\mathcal{E}_j,r),r))\right)\leq 1-r,$$

$$\tilde{\tau}_1^{\tilde{\mu}}\left(f^{-1}(int_{\tilde{\tau}_2^{\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\mu}}}(\mathcal{E}_j,r),r))\right) \leq 1-r.$$

By $r-\mathcal{SVNJ}_1-compactness$ of $\mathcal{S}$ in $\left(\tilde{\mathfrak{X}}_1, \tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathcal{J}}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}\right)$, there exists a finite $\Gamma_0 \subseteq \Gamma$ such that

$$\tilde{\mathcal{J}}_1^{\tilde{\gamma}}\left(\mathcal{S}_j \cap \left[\bigcup_{j\in\Gamma_0} f^{-1}(int_{\tilde{\tau}_2^{\tilde{\gamma}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}}}(\mathcal{E}_j,r),r))\right]^c\right) \geq r, \quad \tilde{\mathcal{J}}_1^{\tilde{\eta}}\left(\mathcal{S}_j \cap \left[\bigcup_{j\in\Gamma_0} f^{-1}(int_{\tilde{\tau}_2^{\tilde{\eta}}}(C_{\tilde{\tau}_2^{\tilde{\eta}}}(\mathcal{E}_j,r),r))\right]^c\right) \leq 1-r,$$

$$\tilde{\mathcal{J}}_1^{\tilde{\mu}}\left(\mathcal{S}_j \cap \left[\bigcup_{j\in\Gamma_0} f^{-1}(int_{\tilde{\tau}_2^{\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\mu}}}(\mathcal{E}_j,r),r))\right]^c\right) \leq 1-r.$$

Since $\tilde{\mathcal{J}}_1^{\tilde{\gamma}}(\mathcal{S}) \leq \tilde{\mathcal{J}}_2^{\tilde{\gamma}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\eta}}(\mathcal{S}) \geq \tilde{\mathcal{J}}_2^{\tilde{\eta}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\mu}}(\mathcal{S}) \geq \tilde{\mathcal{J}}_2^{\tilde{\mu}}(f(\mathcal{S}))$, we have

$$\tilde{\mathcal{J}}_2^{\tilde{\gamma}}\left(f(\mathcal{S}_j \cap \left[\bigcup_{j\in\Gamma_0} f^{-1}(int_{\tilde{\tau}_2^{\tilde{\gamma}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}}}(\mathcal{E}_j,r),r))\right]^c)\right) \geq r, \quad \tilde{\mathcal{J}}_2^{\tilde{\eta}}\left(f(\mathcal{S}_j \cap \left[\bigcup_{j\in\Gamma_0} f^{-1}(int_{\tilde{\tau}_2^{\tilde{\eta}}}(C_{\tilde{\tau}_2^{\tilde{\eta}}}(\mathcal{E}_j,r),r))\right]^c)\right) \leq 1-r,$$

$$\tilde{\mathcal{J}}_2^{\tilde{\mu}}\left(f(\mathcal{S}_j \cap \left[\bigcup_{j\in\Gamma_0} f^{-1}(int_{\tilde{\tau}_2^{\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\mu}}}(\mathcal{E}_j,r),r))\right]^c)\right) \leq 1-r.$$

By surjectively of $f$, $f(\mathcal{S}_j \cap \left[\bigcup_{j\in\Gamma_0} f^{-1}(int_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,r),r))\right]^c) = f(\mathcal{S}_j) \cap \left[\bigcup_{j\in\Gamma_0}(C_{\tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}_j,r)\right]^c$. Thus,

$$\tilde{\mathcal{J}}_2^{\tilde{\gamma}}\left(f(\mathcal{S}_j) \cap \left[\bigcup_{j\in\Gamma_0}(C_{\tilde{\tau}_2^{\tilde{\gamma}}}(\mathcal{E}_j,r)\right]^c\right) \geq r, \quad \tilde{\mathcal{J}}_2^{\tilde{\eta}}\left(f(\mathcal{S}_j) \cap \left[\bigcup_{j\in\Gamma_0}(C_{\tilde{\tau}_2^{\tilde{\eta}}}(\mathcal{E}_j,r)\right]^c\right) \leq 1-r,$$

$$\tilde{\mathcal{J}}_2^{\tilde{\mu}}\left(f(\mathcal{S}_j) \cap \left[\bigcup_{j\in\Gamma_0}(C_{\tilde{\tau}_2^{\tilde{\mu}}}(\mathcal{E}_j,r)\right]^c\right) \leq 1-r.$$

and hence, $f(\mathcal{S})$ is $\mathrm{r}-\mathcal{SVNC}(\mathcal{J})_2-compact$.

**Theorem 4.22.** The image of an $r-\mathcal{SVNJ}_1-compact$ under a surjective $\mathcal{SVN}-weakly\ continuous$ mapping and $\tilde{\mathcal{J}}_1^{\tilde{\gamma}}(\mathcal{S}) \leq \tilde{\mathcal{J}}_2^{\tilde{\gamma}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\eta}}(\mathcal{S}) \geq \tilde{\mathcal{J}}_2^{\tilde{\eta}}(f(\mathcal{S}))$, $\tilde{\mathcal{J}}_1^{\tilde{\mu}}(\mathcal{S}) \geq \tilde{\mathcal{J}}_2^{\tilde{\mu}}(f(\mathcal{S}))$, is $r-\mathcal{SVNJ}_2$ −quasi H-closed.

**Proof**. Similar to proof of Theorem 4.21.

## 5. Conclusions

In the current research paper, we found some results of single-valued neutrosophic continuous mappings called almost continuous and weakly continuous. These instances are kinds of some generalizations of fuzzy continuity in view of the definition of $\tilde{S}$ostak. We brought counterexamples whenever such properties fail to be preserved. We also introduced and studied several kinds of *r*-single-valued neutrosophic compactness defined on the single-valued neutrosophic ideal topological spaces.

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# An Introduction to Neutro-Prime Topology and Decision Making Problem

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**Abstract:** In this industrial era the innovation of industrial machines had a significant impact on industrial evaluation which minimize manpower, time consumption for product making, material wastage. Heavy usage of machines leads to the occurrence of some faults in it. Such damaged parts of each machine have been identified to overhaul which is defined as a set called neutro-prime set under its topological structure. Some related properties of such space have been proved and some are disproved with counterexamples. Also, the idea of interior and closure dealt with this space with few basic properties. This article provides a decision-making process to identify the best fit of those damages under a neutrosophic environment and the priority is given to the heavily damaged machine. We also use step by step algorithm and formulae to compute machine values. Our objective is to demonstrate that our proposed algorithm can calculate key measurements for fault diagnostic in machines as well as to provide fair and reliable forecasted outcomes.

**Keywords:** Neutro-prime sets; neutro-prime topological spaces; neutro-prime interior; neutro-prime closure; neutro-prime absolute complement; decision making.

## 1. Introduction

Throughout history, the relation between humans and machines became most important in moral, ethical, social, economic, and the environment. Machines have confirmed to grasp the key to further developments we humans so extremely need. In the process of doing so, a machine whether or not in continuous use will get damaged and worn-out. In our daily life, we need to reduce the risk of its expensive cost, bad maintenance, and repair parts.

The principles of three autonomous membership degrees such as truth, falsity, and indeterminacy, committed to each element of a set which categorized to neutrosophic set (NS) as instigated by (1998) Smarandache [20, 21], which is an explanation of a fuzzy set (FS) defined by (1965) Zadeh [33], and intuitionistic fuzzy set (IFS) generated by (1986) Atanassov [32]. It is an active organization that hypothesizes the notion of all other sets introduced before. It goes out to be a treasured mathematical implement to observe unformed, damaged, indistinct facts. In recent years many researchers have further expanded and developed the theory and application of NSs [1, 2, 3, 5, 6, 14, 16-19]. Also, (2017) Smarandache [22] originated a new trend set called plithogenic set and others developed [4, 9, 12, 15].

Topology plays a vital role among many sets such as FS, NS, soft sets (SS), neutrosophic soft set (NSS), etc., These types of sets are extended by different researchers [7, 10, 11, 13, 23-27, 29, 30, 31] and its application in decision making (DM) problems [8]. Chinnadurai and Sindhu [28] introduced the notion of prime sets (PSs) and prime-topological spaces (PTSs) (2020), as one of the mathematical utensils for dealing with the subsets of the universe set.

The major achievements of this research are:

- Initiating a neutrosophic environment on prime sets under a topological space.
- Demonstrating the decision-making problem for analyzing the amount of damage in machines.
- An outcome of the proposed algorithm fits in a better way with the number of faults in machines by diagnosis the set values.

To overcome the disadvantages of machines, solving algorithms are presented in this study. A decision-making process delivers to identify the best fit of those damages under a neutrosophic environment and the priority is given to the heavily damaged machine with the use of step by step algorithm and formulae to compute machine values. The main tool used to find the faults in machines are complement and absolute complements of the specified set.

The structure of this study is as follows: Some significant definitions interrelated to the study are presented in part 2. Part 3 introduces the definition of neutro-prime sets, neutro-prime topological spaces, neutro-prime interior and neutro-prime closure with fundamental properties, and related examples. Part 4 explains the DM problem to repair the sample machines with some damages. The algorithm and formulae are presented to find the final result. Finally, the contributions of this study are concluded with future works in part 5.

## 2. Preliminaries

In this part, some essential definitions connected to this work are pointed.

**Definition 2.1** Let $W$ be a non-empty set and $w \in W$. A NS $D$ in $W$ is characterized by a truth-membership function $T_D$, an indeterminacy-membership function $I_D$, and a false-membership function $F_D$ which are subsets of [0, 1] and is defined as

$$D = \{\langle w, T_D(w), I_D(w), F_D(w)\rangle : w \in W\},$$

where

$$0 \le \sup T_D(w) + \sup I_D(w) + \sup F_D(w) \le 3.$$

**Definition 2.2** Let $NS(W)$ denote the family of all NSs over $W$ and $\tau_n \subset NS(W)$. Then $\tau_\eta$ is called a neutrosophic topology (NT) on $W$ if it satisfies the following conditions

(i) $0_n, 1_n \in \tau_n$, where null NS $0_n = \{\langle w, 0, 0, 1\rangle : w \in W\}$ and an absolute NS $1_n = \{\langle w, 1, 1, 0\rangle : w \in W\}$.

(ii) the intersection of any finite number of members of $\tau_n$ belongs to $\tau_n$.

(iii) the union of any collection of members of $\tau_n$ belongs to $\tau_n$.

Then the pair $(W, \tau_n)$ is called a NTS.

Every member of $\tau_n$ is called $\tau_n$-open neutrosophic set ($\tau_n$- ONS). An NS is called $\tau_n$-closed ($\tau_n$- CNS) if and only if its complement is $\tau_n$- ONS.

**Definition 2.3** Let $D$ be a NS over $W$. Then the complement of is denoted by $D'$ and defined by

$$D' = \{\langle w, F_D(w), 1 - I_D(w), T_D(w)\rangle : w \in W\}.$$

Clearly, $(D')' = D$.

**Definition 2.4** Let $(W, \tau)$ be a topological space (TS), where $W$ is the universe and $\tau$ is a topology. Let $K$ be a proper nonempty subset of $W$. Let $D$ be a $\tau$ -open set, where $D \neq \phi, W$. Then the prime set (PS) over $W$ is denoted by $\xi$ and defined by $\xi = \{\phi, W, K : K \cap D \neq \phi\}$.

**Definition 2.5** Let $(W, \tau)$ be a TS. Then $\tau_p$ is called a prime topology (PT) if it satisfies the following conditions

(i) $\phi, W \in \tau_p$ .

(ii) the intersection of any finite number of members of $\tau_p$ belongs to $\tau_p$.

(iii) the union of any collection of members of $\tau_p$ belongs to $\tau_p$.

Then the pair $(W, \tau_p)$ is called a prime topological space (PTS).

Every member of $\tau_p$ is called $\tau_p$-prime open set ($\tau_p$- POS). The complement of every $\tau_p$- POS of $W$ is called the $\tau_p$-prime closed set ($\tau_p$- PCS) of $W$ and this collection is denoted by $\tau_p{}^*$.

**Example 2.6** Let $W = \{w_1, w_2, w_3\}$ with topology $\tau = \{\phi, W, \{w_1\}\}$.
Clearly, $(W, \tau)$ is a TS over $W$.
Then

$$\tau_p = \{\phi, W, \{w_1\}, \{w_1, w_2\}, \{w_1, w_3\}\} = PS(W)$$

and its members are $\tau_p$- POSs.
Thus $(W, \tau_p)$ is a PTS over $W$.
Then

$$\tau_p{}^* = \{\phi, W, \{w_2, w_3\}, \{w_3\}, \{w_2\}\}$$

and its members are $\tau_p$- PCSs, whose complements are $\tau_p$- POSs.

**Definition 2.7** Let $W$ be a set of universe and $w_i \in W$ where $i \in I$. Let $D$ be a NS over $W$. Then the subset of NS (sub-NS) $D$ is denoted as $\xi_D(W^*)$ and defined as

$$\xi_D(W^*) = \{\langle w_i, T_D(w_i), I_D(w_i), F_D(w_i)\rangle, \langle (w_i, w_j), \max(T_D(w_i), T_D(w_j)), \max(I_D(w_i), I_D(w_j)), \min(F_D(w_i), F_D(w_j))\rangle\}.$$

where $i, j \in I$ and $i \neq j$.
Clearly, $(w_i, w_j) = (w_j, w_i)$.

**Example 2.8** Let $W = \{w_1, w_2, w_3\}$ be a set of features of the washing machine, where $w_1$ = energy efficiency, $w_2$ = capacity, $w_3$ =price. Let $D$ be a NS over $W$, defined as

$$D = \{\langle w_1, .7, .5, .4\rangle, \langle w_2, .2, .7, .9\rangle, \langle w_3, .4, .1, .3\rangle\}.$$

Then the sub-NS $D$ is

$$\xi_D(W^*) = \{\langle w_1, .7, .5, .4\rangle, \langle w_2, .2, .7, .9\rangle, \langle w_3, .4, .1, .3\rangle, \langle (w_1, w_2), .7, .7, .4\rangle, \langle (w_1, w_3), .7, .5, .3\rangle, \langle (w_2, w_3), .4, .7, .3\rangle\}.$$

**Definition 2.9** Let $W$ be a set of universe and $w_i \in W$ where $i \in I$. Let $V$ be any proper nonempty subset of $W$, say $\{w_i\}$ and $\{w_i, w_j\}$. Let $D$ be a NS over $W$. Then the subset of NS $D$ with respect to $w_i$ (sub-NS $D_{w_i}$ ) and $w_i, w_j$ (sub-NS $D_{w_i, w_j}$ ) are denoted as $\xi_D(w_i)$ and $\xi_D(w_i, w_j)$, and defined as

$$\xi_D(w_i) = \{\langle w_i, T_D(w_i), I_D(w_i), F_D(w_i)\rangle, \langle (w_i, w_j), \max(T_D(w_i), T_D(w_j)), \max(I_D(w_i), I_D(w_j)), \min(F_D(w_i), F_D(w_j))\rangle,$$
$$\langle w_k, T_D(0_n), I_R(0_n), F_R(0_n)\rangle, \langle (w_k, w_l), T_D(0_n), I_R(0_n), F_D(0_n)\rangle\}$$

where $i \in I$, $j \in I - \{i\}$, $k, l \in I - \{i, j\}$ and $k \neq l$

and

$$\xi_D(w_i, w_j) = \{\langle w_i, T_D(w_i), I_D(w_i), F_D(w_i)\rangle, \langle w_j, T_D(w_j), I_D(w_j), F_D(w_j)\rangle, \langle w_k, T_D(0_n), I_D(0_n), F_D(0_n)\rangle,$$
$$\langle (w_i, w_j), \max(T_D(w_i), T_D(w_j)), \max(I_D(w_i), I_D(w_j)), \min(F_D(w_i), F_D(w_j))\rangle,$$
$$\langle (w_i, w_k), \max(T_D(w_i), T_D(w_k)), \max(I_D(w_i), I_D(w_k)), \min(F_D(w_i), F_D(w_k))\rangle,$$
$$\langle (w_j, w_k), \max(T_D(w_j), T_D(w_k)), \max(I_D(w_j), I_D(w_k)), \min(F_D(w_j), F_D(w_k))\rangle\}$$

where $i, j, k \in I$ and $i \neq j \neq k$, respectively.

**Example 2.10** Let $W = \{w_1, w_2, w_3\}$. Let *D* and *F* be two NSs over *W* and are defined as follows

$$D = \{\langle w_1, .1, .2, .8\rangle, \langle w_2, .4, .7, .3\rangle, \langle w_3, .6, .5, .2\rangle\}$$

and

$$F = \{\langle w_1, .6, .5, .3\rangle, \langle w_2, .9, .8, .1\rangle, \langle w_3, .7, .6, .1\rangle\}.$$

Then sub-NS $D_{w_2, w_3}$ and sub-NS $F_{w_2}$ are defined as

$$\xi_D(w_2, w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .4, .7, .3\rangle, \langle w_3, .6, .5, .2\rangle, \langle w_{1,2}, .4, .7, .3\rangle, \langle w_{1,3}, .6, .5, .2\rangle, \langle w_{2,3}, .6, .7, .2\rangle\}$$ and

$$\xi_F(w_2) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .9, .8, .1\rangle, \langle w_3, 0, 0, 1\rangle, \langle w_{1,2}, .9, .8, .1\rangle, \langle w_{1,3}, 0, 0, 1\rangle, \langle w_{2,3}, .9, .8, .1\rangle\}$$, respectively.

## 3. Neutro-Prime Topology

In this part, the new type of set is initiated as neutro-prime sets and defined its topological space as neutro-prime topological spaces. Some of its basic properties are examined with illustrative examples.

**Definition 3.1** Let $(W, \tau_n)$ be a neutrosophic topological space (NTS), where *W* is the universe and $\tau_n$ is a neutrosophic topology (NT). Let *D* be a $\tau_n$-open neutrosophic set, where $D \neq \phi, W$. Let *V* be any proper nonempty subset of *W*. Then

$$\eta_p D(V) = \{\xi_D(V^*) : V \cap V^* \neq \phi\},$$

for all proper nonempty subset $V^*$ of *W*.

Thus the elements belongs to $\eta_p D(V)$ are said to be neutro-prime sets (NPSs) over *W* and denoted by $\xi_D(V^*)$.

**Example 3.2** Let $W = \{w_1, w_2, w_3\}$ be a set of features of the washing machine, where $w_1$ = energy efficiency, $w_2$ = capacity, $w_3$ =price. Let *D* be a NS over *W*, defined as

$$D = \{\langle w_1, .7, .5, .4\rangle, \langle w_2, .2, .7, .9\rangle, \langle w_3, .4, .1, .3\rangle\}.$$

Then NPS

$$\eta_p D(w_3) = \{\xi_D(w_3), \xi_D(w_1, w_3), \xi_D(w_2, w_3)\},$$

where

$$\xi_D(w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .4, .1, .3\rangle, \langle (w_1, w_2), 0, 0, 1\rangle, \langle (w_1, w_3), .7, .5, .3\rangle, \langle (w_2, w_3), .4, .7, .3\rangle\},$$

$$\xi_D(w_1, w_3) = \{\langle w_1, .7, .5, .4\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .4, .1, .3\rangle, \langle (w_1, w_2), .7, .7, .4\rangle, \langle (w_1, w_3), .7, .5, .3\rangle, \langle (w_2, w_3), .4, .7, .3\rangle\}$$

and

$$\xi_D(w_2, w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .2, .7, .9\rangle, \langle w_3, .4, .1, .3\rangle, \langle (w_1, w_2), .7, .7, .4\rangle, \langle (w_1, w_3), .7, .5, .3\rangle, \langle (w_2, w_3), .4, .7, .3\rangle\}.$$

**Definition 3.3** Let *W* be a set of the universe and *V* be any proper nonempty subset of the *W*. Then the null NPS is denoted as $0_{np}$ and defined as

$$0_{np} = \{\langle V, T_R(V) = 0, I_R(V) = 0, F_R(V) = 1\rangle : \forall V\}.$$

**Definition 3.4** Let *W* be a set of the universe and *V* be any proper nonempty subset of the *W*. Then the absolute NPS is denoted as $1_{np}$ and defined as

$$1_{np} = \{\langle V, T_R(V) = 1, I_R(V) = 1, F_R(V) = 0\rangle : \forall V\}.$$

**Definition 3.5** Let $\xi_D(V_1^*)$ and $\xi_D(V_2^*)$ be two NPSs over *W*. Then their union is denoted as $\xi_D(V_1^*) \cup \xi_D(V_2^*) = \xi_D(V_{1\vee 2}^*)$ and is defined as

$$\xi_D(V_{1\vee 2}^*) = \left\{\left\langle V_{1\vee 2}^*, \max\left(T_R(V_1^*), T_R(V_2^*)\right), \max\left(I_R(V_1^*), I_R(V_2^*)\right), \min\left(F_R(V_1^*), F_R(V_2^*)\right)\right\rangle\right\}.$$

**Definition 3.6** Let $\xi_D(V_1^*)$ and $\xi_D(V_2^*)$ be two NPSs over *W*. Then their intersection is denoted as $\xi_D(V_1^*) \cap \xi_D(V_2^*) = \xi_D(V_{1\wedge 2}^*)$ and is defined as

$$\xi_D(V_{1\wedge 2}^*) = \left\{\left\langle V_{1\wedge 2}^*, \min\left(T_R(V_1^*), T_R(V_2^*)\right), \min\left(I_R(V_1^*), I_R(V_2^*)\right), \max\left(F_R(V_1^*), F_R(V_2^*)\right)\right\rangle\right\}.$$

**Definition 3.7** Let $\xi_D(V^*)$ be a NPS over *W*. Then its complement is denoted as $\xi_D(V^*)'$ and is defined as

$$\xi_D(V^*)' = \left\{\left\langle V^*, T_D(V^*), 1 - I_D(V^*), F_D(V^*)\right\rangle\right\}.$$

Clearly, the complement of $\xi_D(V^*)'$ equals $\xi_D(V^*)$. i.e., $\left(\xi_D(V^*)'\right)' = \xi_D(V^*)$.

**Definition 3.8** Let $\xi_D(V_1^*)$ and $\xi_D(V_2^*)$ be two NPSs over *W*. Then $\xi_D(V_1^*)$ is said to be a neutro-prime subset of $\xi_D(V_2^*)$ if

$$T_R(V_1^*) \le T_R(V_2^*),\ T_R(I_1^*) \le T_R(I_2^*),\ F_R(V_1^*) \ge F_R(V_2^*).$$

It is denoted by $\xi_D(V_1^*) \subseteq \xi_D(V_2^*)$.

Also $\xi_D(V_1^*)$ is said to be neutro-prime equal to $\xi_D(V_2^*)$ if $\xi_D(V_1^*)$ is a neutro-prime subset of $\xi_D(V_2^*)$ and $\xi_D(V_2^*)$ is a neutro-prime subset of $\xi_D(V_1^*)$. It is denoted by $\xi_D(V_1^*) = \xi_D(V_2^*)$.

**Proposition 3.9** Let $\xi_D(V_1^*)$, $\xi_D(V_2^*)$ and $\xi_D(V_3^*)$ be NPSs over *W*. Then,

(i) $\xi_D(V_1^*) \cup 0_{np} = \xi_D(V_1^*)$.

(ii) $\xi_D(V_1^*) \cup 1_{np} = 1_{np}$.

(iii) $\xi_D(V_1^*) \cup \left[\xi_D(V_2^*) \cup \xi_D(V_3^*)\right] = \left[\xi_D(V_1^*) \cup \xi_D(V_2^*)\right] \cup \xi_D(V_3^*)$.

(iv) $\xi_D(V_1^*) \cup \left[\xi_D(V_2^*) \cap \xi_D(V_3^*)\right] = \left[\xi_D(V_1^*) \cup \xi_D(V_2^*)\right] \cap \left[\xi_D(V_1^*) \cup \xi_D(V_3^*)\right]$.

Proof. Straightforward.

**Proposition 3.10** Let $\xi_D(V_1^*)$, $\xi_D(V_2^*)$ and $\xi_D(V_3^*)$ be NPSs over *W*. Then,

(i) $\xi_D(V_1^*) \cap 0_{np} = 0_{np}$.

(ii) $\xi_D(V_1^*) \cap 1_{np} = \xi_D(V_1^*)$.

(iii) $\xi_D(V_1^*) \cap \left[\xi_D(V_2^*) \cap \xi_D(V_3^*)\right] = \left[\xi_D(V_1^*) \cap \xi_D(V_2^*)\right] \cap \xi_D(V_3^*)$.

(iv) $\xi_D(V_1^*) \cap \left[\xi_D(V_2^*) \cup \xi_D(V_3^*)\right] = \left[\xi_D(V_1^*) \cap \xi_D(V_2^*)\right] \cup \left[\xi_D(V_1^*) \cap \xi_D(V_3^*)\right]$.

Proof. Straightforward.

**Proposition 3.11** Let $\xi_D(V_1^*)$ and $\xi_D(V_2^*)$ be two NPSs over *W*. Then,

(i) $\left[\xi_D(V_1^*)\cup\xi_D(V_2^*)\right]' = \xi_D(V_1^*)'\cap\xi_D(V_2^*)'$.

(ii) $\left[\xi_D(V_1^*)\cap\xi_D(V_2^*)\right]' = \xi_D(V_1^*)'\cup\xi_D(V_2^*)'$.

Proof. Straightforward.

**Proposition 3.12** Let $\xi_D(V_1^*)$, $\xi_D(V_2^*)$ and $\xi_F(V_1^*)$ be NPSs over *W* for NSs *D* and *F*. Then,

(i) $D\subseteq F \Rightarrow \xi_D(V_1^*)\subseteq\xi_F(V_1^*)$.

(ii) $\xi_D(V_1^*)\cup\xi_D(V_2^*)=\xi_D(V_1^*\cup V_2^*)$.

(iii) $\xi_D(V_1^*)\cap\xi_D(V_2^*)\subseteq\xi_D(V_1^*)$ and $\xi_D(V_1^*)\cap\xi_D(V_2^*)\subseteq\xi_D(V_2^*)$.

(iv) $\xi_D(V_1^*)\cup\xi_D(V_2^*)\supseteq\xi_D(V_1^*)$ and $\xi_D(V_1^*)\cup\xi_D(V_2^*)\supseteq\xi_D(V_2^*)$.

(v) $\xi_D(V_1^*)\subseteq\xi_D(V_2^*)\Rightarrow\xi_D(V_1^*)'\subseteq\xi_D(V_2^*)'$.

Proof. Straightforward.

**Definition 3.13** Let $(W,\tau_n)$ be a NTS. Let *NPS*(*W*) be the collection of NPSs $\xi_D(V^*)$ over *W* and *D* be a $\tau_n$-open neutrosophic set (ONS), where $D\neq\phi, W$. Then $\tau_{np}\subset NPS(W)$ is called a neutro-prime topology (NPT) if it satisfies the following conditions

(i) $0_{np}, 1_{np}\in\tau_{np}$.

(ii) the union of any collection of members of $\tau_{np}$ belongs to $\tau_{np}$.

(iii) the intersection of any finite number of members of $\tau_{np}$ belongs to $\tau_{np}$.

Then the pair $(W,\tau_{np})$ is said to be a neutro-prime topological space (NPTS).

Every member of $\tau_{np}$ is said to be a $\tau_{np}$-neutro-prime open set ($\tau_{np}$- NPOS). The complement of every $\tau_{np}$- NPOS of *W* is said to be a $\tau_{np}$-neutro-prime closed set ($\tau_{np}$- NPCS) of *W* and this collection is denoted by $\tau_{np}^*$.

**Example 3.14** Let $W=\{w_1,w_2,w_3\}$ and $\tau_n=\{0_n,1_n,D,F\}$ where *D* and *F* are NSs over *W* and are defined as follows

$$D=\{\langle w_1,.1,.2,.8\rangle,\langle w_2,.4,.7,.3\rangle,\langle w_3,.6,.5,.2\rangle\}$$

and

$$F=\{\langle w_1,.6,.5,.3\rangle,\langle w_2,.9,.8,.1\rangle,\langle w_3,.7,.6,.1\rangle\}.$$

Thus $(W,\tau_n)$ is a NTS over *W*.

Here NPSs are

$$\eta_p D(w_1,w_3)=\{\xi_D(w_1),\xi_D(w_3),\xi_D(w_1,w_2),\xi_D(w_1,w_3),\xi_D(w_2,w_3)\},$$

where

$$\xi_D(w_1)=\{\langle w_1,.1,.2,.8\rangle,\langle w_2,0,0,1\rangle,\langle w_3,0,0,1\rangle,\langle (w_1,w_2),.4,.7,.3\rangle,\langle (w_1,w_3),.6,.5,.2\rangle,\langle (w_2,w_3),0,0,1\rangle\},$$

$$\xi_D(w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.6,.5,.2\rangle,\langle (w_1,w_2),0,0,1\rangle,\langle (w_1,w_3),.6,.5,.2\rangle,\langle (w_2,w_3),.6,.7,.2\rangle\},$$

$$\xi_D(w_1,w_2)=\{\langle w_1,.1,.2,.8\rangle,\langle w_2,.4,.7,.3\rangle,\langle w_3,0,0,1\rangle,\langle (w_1,w_2),.4,.7,.3\rangle,\langle (w_1,w_3),.6,.5,.2\rangle,\langle (w_2,w_3),.6,.7,.2\rangle\},$$

$$\xi_D(w_1,w_3)=\{\langle w_1,.1,.2,.8\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.6,.5,.2\rangle,\langle (w_1,w_2),.4,.7,.3\rangle,\langle (w_1,w_3),.6,.5,.2\rangle,\langle (w_2,w_3),.6,.7,.2\rangle\},$$

$$\xi_D(w_2,w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,.4,.7,.3\rangle,\langle w_3,.6,.5,.2\rangle,\langle (w_1,w_2),.4,.7,.3\rangle,\langle (w_1,w_3),.6,.5,.2\rangle,\langle (w_2,w_3),.6,.7,.2\rangle\},$$

and

$$\eta_p F(w_2)=\{\xi_F(w_2),\xi_F(w_1,w_2),\xi_F(w_2,w_3)\},$$

where

$$\xi_F(w_2)=\{\langle w_1,0,0,1\rangle,\langle w_2,.9,.8,.1\rangle,\langle w_3,0,0,1\rangle,\langle (w_1,w_2),.9,.8,.1\rangle,\langle (w_1,w_3),0,0,1\rangle,\langle (w_2,w_3),.9,.8,.1\rangle\},$$

$\xi_F(w_1,w_2)=\{\langle w_1,.6,.5,.3\rangle,\langle w_2,.9,.8,.1\rangle,\langle w_3,0,0,1\rangle,\langle (w_1,w_2),.9,.8,.1\rangle,\langle (w_1,w_3),.7,.6,.1\rangle,\langle (w_2,w_3),.9,.8,.1\rangle\}$,
$\xi_F(w_2,w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,.9,.8,.1\rangle,\langle w_3,.7,.6,.1\rangle,\langle (w_1,w_2),.9,.8,.1\rangle,\langle (w_1,w_3),.7,.6,.1\rangle,\langle (w_2,w_3),.9,.8,.1\rangle\}$.
Then

$\tau_{np}=\{0_{np},1_{np},\xi_D(w_1,w_2),\xi_F(w_1,w_2)\}$ is a NPT.

Thus ($W$, $\tau_{np}$ ) is a NPTS over $W$.

Also, the complement of the NPT $\tau_{np}$ is

$\tau_{np}{}^{*}=\{0_{np},1_{np},\xi_D(w_1,w_2)',\xi_F(w_1,w_2)'\}$,

where
$\xi_D(w_1,w_2)'=\{\langle w_1,.8,.8,.1\rangle,\langle w_2,.3,.3,.4\rangle,\langle w_3,1,1,0\rangle,\langle (w_1,w_2),.3,.3,.4\rangle,\langle (w_1,w_3),.2,.5,.6\rangle,\langle (w_2,w_3),.2,.3,.6\rangle\}$
and
$\xi_F(w_1,w_2)'=\{\langle w_1,.3,.5,.6\rangle,\langle w_2,.1,.2,.9\rangle,\langle w_3,1,1,0\rangle,\langle (w_1,w_2),.1,.2,.9\rangle,\langle (w_1,w_3),.1,.4,.7\rangle,\langle (w_2,w_3),.1,.2,.9\rangle\}$.

**Remark 3.15** The collection of NPS $\eta_p D(V)$ can generate one or more NPT, which is illustrated in the following example.

**Example 3.16** Consider Example 3.14.
Here

$1\tau_{np}=\{0_{np},1_{np},\xi_D(w_1,w_2),\xi_F(w_1,w_2)\}$ and

$2\tau_{np}=\{0_{np},1_{np},\xi_D(w_2,w_3),\xi_F(w_2,w_3)\}$ are NPTs

Thus ($W$, $1\tau_{np}$ ) and ($W$, $2\tau_{np}$ ) are NPTSs over $W$.

Also, the complement of the NPTs $1\tau_{np}$ and $2\tau_{np}$ are

$1\tau_{np}{}^{*}=\{0_{np},1_{np},\xi_D(w_1,w_2)',\xi_F(w_1,w_2)'\}$ and

$2\tau_{np}{}^{*}=\{0_{np},1_{np},\xi_D(w_2,w_3)',\xi_F(w_2,w_3)'\}$ , respectively,

where
$\xi_D(w_1,w_2)'=\{\langle w_1,.8,.8,.1\rangle,\langle w_2,.3,.3,.4\rangle,\langle w_3,1,1,0\rangle,\langle (w_1,w_2),.3,.3,.4\rangle,\langle (w_1,w_3),.2,.5,.6\rangle,\langle (w_2,w_3),.2,.3,.6\rangle\}$
$\xi_F(w_1,w_2)'=\{\langle w_1,.3,.5,.6\rangle,\langle w_2,.1,.2,.9\rangle,\langle w_3,1,1,0\rangle,\langle (w_1,w_2),.1,.2,.9\rangle,\langle (w_1,w_3),.1,.4,.7\rangle,\langle (w_2,w_3),.1,.2,.9\rangle\}$
and
$\xi_D(w_2,w_3)'=\{\langle w_1,1,1,0\rangle,\langle w_2,.3,.3,.4\rangle,\langle w_3,.2,.5,.6\rangle,\langle (w_1,w_2),.3,.3,.4\rangle,\langle (w_1,w_3),.2,.5,.6\rangle,\langle (w_2,w_3),.2,.3,.6\rangle\}$
$\xi_F(w_2,w_3)'=\{\langle w_1,1,1,0\rangle,\langle w_2,.1,.2,.9\rangle,\langle w_3,.1,.4,.7\rangle,\langle (w_1,w_2),.1,.2,.9\rangle,\langle (w_1,w_3),.1,.4,.7\rangle,\langle (w_2,w_3),.1,.2,.9\rangle\}$.

**Definition 3.17** A NPT $\tau_{np}$ is said to be a neutro-prime discrete topology if $\tau_{np}=NPS(W)$ for all the subsets of $W$.

**Definition 3.18** A NPT $\tau_{np}$ is said to be a neutro-prime indiscrete topology if $\tau_{np}$ contains only $0_{np}$ and $1_{np}$.

**Proposition 3.19** Let ($W$, $1\tau_{np}$ ) and ($W$, $2\tau_{np}$ ) be two NPTSs over $W$ and let $1\tau_{np}\bigcap 2\tau_{np}=\{D\in NPS(W):D\in 1\tau_{np}\bigcap 2\tau_{np}\}$. Then $1\tau_{np}\bigcap 2\tau_{np}$ is also a NPT over $W$.

Proof. Let ($W$, $1\tau_{np}$ ) and ($W$, $2\tau_{np}$ ) be two NPTSs over $W$.

(i) $0_{np},1_{np}\in 1\tau_{np}\bigcap 2\tau_{np}$.

(ii) Let $D_1,D_2\in 1\tau_{np}\bigcap 2\tau_{np}$.

Then $D_1,D_2\in 1\tau_{np}$ and $D_1,D_2\in 2\tau_{np}$.

$\Rightarrow D_1 \cap D_2 \in 1\tau_{np}$ and $D_1 \cap D_2 \in 2\tau_{np}$.

$\Rightarrow D_1 \cap D_2 \in 1\tau_{np} \cap 2\tau_{np}$.

(iii) Let $D_i \in 1\tau_{np} \cap 2\tau_{np}$, $i \in \mathrm{I}$.

Then $D_i \in 1\tau_{np}$ and $D_i \in 2\tau_{np}$, $i \in \mathrm{I}$.

$\Rightarrow \bigcup_{i \in \mathrm{I}} D_i \in 1\tau_{np}$ and $\Rightarrow \bigcup_{i \in \mathrm{I}} D_i \in 2\tau_{np}$, $i \in \mathrm{I}$.

$\Rightarrow \bigcup_{i \in \mathrm{I}} D_i \in 1\tau_{np} \cup 2\tau_{np}$.

Thus $1\tau_{np} \cap 2\tau_{np}$ is also a NPT over *W*.

**Remark 3.20** The union of two NPTs need not be a NPT. The following example illustrates this remark.

**Example 3.21** Consider Example 3.14.
Here NPTs are

$$1\tau_{np} = \{0_{np}, 1_{np}, \xi_D(w_1, w_2), \xi_F(w_1, w_2)\},$$

where

$$\xi_D(w_1, w_2) = \{\langle w_1, .1, .2, .8\rangle, \langle w_2, .4, .7, .3\rangle, \langle w_3, 0, 0, 1\rangle, \langle (w_1, w_2), .4, .7, .3\rangle, \langle (w_1, w_3), .6, .5, .2\rangle, \langle (w_2, w_3), .6, .7, .2\rangle\},$$

$$\xi_F(w_1, w_2) = \{\langle w_1, .6, .5, .3\rangle, \langle w_2, .9, .8, .1\rangle, \langle w_3, 0, 0, 1\rangle, \langle (w_1, w_2), .9, .8, .1\rangle, \langle (w_1, w_3), .7, .6, .1\rangle, \langle (w_2, w_3), .9, .8, .1\rangle\}$$

and

$$2\tau_{np} = \{0_{np}, 1_{np}, \xi_D(w_2, w_3), \xi_F(w_2, w_3)\},$$

where

$$\xi_D(w_2, w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .4, .7, .3\rangle, \langle w_3, .6, .5, .2\rangle, \langle (w_1, w_2), .4, .7, .3\rangle, \langle (w_1, w_3), .6, .5, .2\rangle, \langle (w_2, w_3), .6, .7, .2\rangle\},$$

$$\xi_F(w_2, w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .9, .8, .1\rangle, \langle w_3, .7, .6, .1\rangle, \langle (w_1, w_2), .9, .8, .1\rangle, \langle (w_1, w_3), .7, .6, .1\rangle, \langle (w_2, w_3), .9, .8, .1\rangle\}.$$

Thus $(W, 1\tau_{np})$ and $(W, 2\tau_{np})$ are NPTSs over *W*.

Clearly,

$$1\tau_{np} \cup 2\tau_{np} = \{0_{np}, 1_{np}, \xi_D(w_1, w_2), \xi_F(w_1, w_2), \xi_D(w_2, w_3), \xi_F(w_2, w_3)\}.$$

Then

$$\xi_D(w_1, w_2) \cap \xi_D(w_2, w_3)$$
$$= \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .4, .7, .3\rangle, \langle w_3, 0, 0, 1\rangle, \langle (w_1, w_2), .4, .7, .3\rangle, \langle (w_1, w_3), .6, .5, .2\rangle, \langle (w_2, w_3), .6, .7, .2\rangle\}$$

Thus $1\tau_{np} \cup 2\tau_{np}$ is not a NPTS, since $\xi_D(w_1, w_2) \cap \xi_D(w_2, w_3) \notin 1\tau_{np} \cup 2\tau_{np}$.

Hence the union of two NPTs need not be a NPT.

**Proposition 3.22** Let $\xi_D(V_1^*)$ and $\xi_D(V_2^*)$ be two $\tau_{np}$-NPOSs over *W*. Then

(i) $\left(\xi_D(V_1^*) \cup \xi_D(V_2^*)\right)' = \xi_D(V_1^*)' \cap \xi_D(V_2^*)'$.

(ii) $\left(\xi_D(V_1^*) \cap \xi_D(V_2^*)\right)' = \xi_D(V_1^*)' \cup \xi_D(V_2^*)'$.

Proof. Straightforward.

**Definition 3.23** Let $(W, \tau_{np})$ be a NPTS over *W*. Let $\xi_D(V^*)$ be any NPSs over *W*. Then the neutro-prime interior of $\xi_D(V^*)$ is denoted by $\mathrm{int}_{np}\left(\xi_D(V^*)\right)$ and defined by

$$\mathrm{int}_{np}\left(\xi_D(V^*)\right) = \cup\{\xi_D(U^*) : \xi_D(U^*) \in \tau_{np} \quad \textit{and} \quad \xi_D(U^*) \subseteq \xi_D(V^*)\}$$

Clearly, it is the union of all $\tau_{np}$-NPOSs contained in $\xi_D(V^*)$.

**Definition 3.24** Let $(W, \tau_{np})$ be a NPTS over *W*. Let $\xi_D(V^*)$ be any NPSs over *W*. Then the neutro-prime closure of $\xi_D(V^*)$ is denoted by $cl_{np}(\xi_D(V^*))$ and defined by

$$cl_{np}(\xi_D(V^*)) = \bigcap\{\xi_D(U^*) : \xi_D(U^*) \in \tau_{np}^* \quad and \quad \xi_D(U^*) \supseteq \xi_D(V^*)\}$$

Clearly, it is the intersection of all $\tau_{np}^*$-NPCSs containing $\xi_D(V^*)$.

**Example 3.25** Let $W = \{w_1, w_2, w_3\}$ and $\tau_n = \{0_n, 1_n, A, B, C, D\}$ where *A, B, C,* and *D* are NSs over *W* and are defined as follows

$$A = \{\langle w_1, .1, .2, .3\rangle, \langle w_2, .4, .5, .6\rangle, \langle w_3, .2, .4, .6\rangle\},$$
$$B = \{\langle w_1, .4, .5, .6\rangle, \langle w_2, .7, .8, .2\rangle, \langle w_3, .1, .2, .3\rangle\},$$
$$C = \{\langle w_1, .1, .2, .6\rangle, \langle w_2, .4, .5, .6\rangle, \langle w_3, .1, .2, .6\rangle\} \text{ and}$$
$$D = \{\langle w_1, .4, .5, .3\rangle, \langle w_2, .7, .8, .2\rangle, \langle w_3, .2, .4, .3\rangle\}.$$

Here $A \bigcup B = D$, $A \bigcup D = D$, $A \bigcup C = A$, $B \bigcup D = D$, $B \bigcup C = B$, $D \bigcup C = D$ and $A \bigcap B = A$, $A \bigcap D = A$, $A \bigcap C = C$, $B \bigcap D = B$, $B \bigcap C = C$, $D \bigcap C = C$.

Then *A, B, C,* and *D* are $\tau_n$- ONSs over *W*.

Thus $(W, \tau_n)$ is a NTS over *W*.

Here NPSs are

$$\eta_p A(w_2) = \{\xi_A(w_2), \xi_A(w_1, w_2), \xi_A(w_2, w_3)\},$$

where

$$\xi_A(w_2) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .4, .5, .6\rangle, \langle w_3, 0, 0, 1\rangle, \langle (w_1, w_2), .4, .5, .3\rangle, \langle (w_1, w_3), 0, 0, 1\rangle, \langle (w_2, w_3), .4, .5, .6\rangle\},$$
$$\xi_A(w_1, w_2) = \{\langle w_1, .1, .2, .3\rangle, \langle w_2, .4, .5, .6\rangle, \langle w_3, 0, 0, 1\rangle, \langle (w_1, w_2), .4, .5, .3\rangle, \langle (w_1, w_3), .2, .4, .3\rangle, \langle (w_2, w_3), .4, .5, .6\rangle\},$$
$$\xi_A(w_2, w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .4, .5, .6\rangle, \langle w_3, .2, .4, .6\rangle, \langle (w_1, w_2), .4, .5, .3\rangle, \langle (w_1, w_3), .2, .4, .3\rangle, \langle (w_2, w_3), .4, .5, .6\rangle\};$$

$$\eta_p B(w_1) = \{\xi_B(w_1), \xi_B(w_1, w_2), \xi_B(w_1, w_3)\},$$

where

$$\xi_B(w_1) = \{\langle w_1, .4, .5, .6\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, 0, 0, 1\rangle, \langle (w_1, w_2), .7, .8, .2\rangle, \langle (w_1, w_3), .4, .5, .3\rangle, \langle (w_2, w_3), 0, 0, 1\rangle\},$$
$$\xi_B(w_1, w_2) = \{\langle w_1, .4, .5, .6\rangle, \langle w_2, .7, .8, .2\rangle, \langle w_3, 0, 0, 1\rangle, \langle (w_1, w_2), .7, .8, .2\rangle, \langle (w_1, w_3), .4, .5, .3\rangle, \langle (w_2, w_3), .7, .8, .2\rangle\}$$
$$\xi_B(w_1, w_3) = \{\langle w_1, .4, .5, .6\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .1, .2, .3\rangle, \langle (w_1, w_2), .7, .8, .2\rangle, \langle (w_1, w_3), .4, .5, .3\rangle, \langle (w_2, w_3), .7, .8, .2\rangle\};$$

$$\eta_p C(w_1, w_3) = \{\xi_C(w_1), \xi_C(w_3), \xi_C(w_1, w_2), \xi_C(w_1, w_3), \xi_C(w_2, w_3)\},$$

where

$$\xi_C(w_1) = \{\langle w_1, .1, .2, .6\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, 0, 0, 1\rangle, \langle (w_1, w_2), .4, .5, .6\rangle, \langle (w_1, w_3), .1, .2, .6\rangle, \langle (w_2, w_3), 0, 0, 1\rangle\},$$
$$\xi_C(w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .1, .2, .6\rangle, \langle (w_1, w_2), 0, 0, 1\rangle, \langle (w_1, w_3), .1, .2, .6\rangle, \langle (w_2, w_3), .4, .5, .6\rangle\},$$
$$\xi_C(w_1, w_2) = \{\langle w_1, .1, .2, .6\rangle, \langle w_2, .4, .5, .6\rangle, \langle w_3, 0, 0, 1\rangle, \langle (w_1, w_2), .4, .5, .6\rangle, \langle (w_1, w_3), .1, .2, .3\rangle, \langle (w_2, w_3), .4, .5, .6\rangle\},$$
$$\xi_C(w_1, w_3) = \{\langle w_1, .1, .2, .6\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .1, .2, .6\rangle, \langle (w_1, w_2), .4, .5, .6\rangle, \langle (w_1, w_3), .1, .2, .3\rangle, \langle (w_2, w_3), .4, .5, .6\rangle\},$$
$$\xi_C(w_2, w_3) = \{\langle w_1, .0, 0, 1\rangle, \langle w_2, .4, .5, .6\rangle, \langle w_3, 1, .2, .6\rangle, \langle (w_1, w_2), .4, .5, .6\rangle, \langle (w_1, w_3), .1, .2, .3\rangle, \langle (w_2, w_3), .4, .5, .6\rangle\};$$

$$\eta_p D(w_3) = \{\xi_D(w_3), \xi_D(w_1, w_3), \xi_D(w_2, w_3)\},$$

where

$$\xi_D(w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .2, .4, .3\rangle, \langle (w_1, w_2), 0, 0, 1\rangle, \langle (w_1, w_3), .4, .5, .3\rangle, \langle (w_2, w_3), .7, .8, .2\rangle\},$$
$$\xi_D(w_1, w_3) = \{\langle w_1, .4, .5, .3\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .2, .4, .3\rangle, \langle (w_1, w_2), .7, .8, .2\rangle, \langle (w_1, w_3), .4, .5, .3\rangle, \langle (w_2, w_3), .7, .8, .2\rangle\},$$
$$\xi_D(w_2, w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .7, .8, .2\rangle, \langle w_3, .2, .4, .3\rangle, \langle (w_1, w_2), .7, .8, .2\rangle, \langle (w_1, w_3), .4, .5, .3\rangle, \langle (w_2, w_3), .7, .8, .2\rangle\}.$$

Then

$$\tau_{np} = \{0_{np}, 1_{np}, \xi_B(w_1), \xi_C(w_1)\} \text{ is a NPT.}$$

Thus $(W, \tau_{np})$ is a NPTS over *W*.

Also, the complement of the NPT $\tau_{np}$ is

$$\tau_{np}{}^{*} = \{0_{np}, 1_{np}, \xi_B(w_1)', \xi_C(w_1)'\},$$

where

$$\xi_B(w_1)' = \{\langle w_1,.6,.5,.4\rangle, \langle w_2,1,1,0\rangle, \langle w_3,1,1,0\rangle, \langle (w_1,w_2),.2,.2,.7\rangle, \langle (w_1,w_3),.3,.5,.4\rangle, \langle (w_2,w_3),1,1,0\rangle\}$$

and

$$\xi_C(w_1)' = \{\langle w_1,.6,.8,.1\rangle, \langle w_2,1,1,0\rangle, \langle w_3,1,1,0\rangle, \langle (w_1,w_2),.6,.5,.4\rangle, \langle (w_1,w_3),.6,.8,.1\rangle, \langle (w_2,w_3),1,1,0\rangle\}.$$

Consider a NPS for NS *B*,

$$\xi_B(w_1,w_3) = \{\langle w_1,.4,.5,.6\rangle, \langle w_2,0,0,1\rangle, \langle w_3,.1,.2,.3\rangle, \langle (w_1,w_2),.7,.8,.2\rangle, \langle (w_1,w_3),.4,.5,.3\rangle, \langle (w_2,w_3),.7,.8,.2\rangle\}.$$

Clearly,

$$\xi_B(w_1,w_3) \supseteq 0_{np}, \xi_B(w_1).$$

Thus

$$\text{int}_{np}(\xi_B(w_1,w_3)) = 0_{np} \cup \xi_B(w_1) = \xi_B(w_1).$$

Also,

$$\xi_B(w_1,w_3) \subseteq 1_{np}.$$

Thus

$$cl_{np}(\xi_B(w_1,w_3)) = 1_{np}.$$

Consider a NPS for NS *C*,

$$\xi_C(w_3) = \{\langle w_1,0,0,1\rangle, \langle w_2,0,0,1\rangle, \langle w_3,.1,.2,.6\rangle, \langle (w_1,w_2),0,0,1\rangle, \langle (w_1,w_3),.1,.2,.6\rangle, \langle (w_2,w_3),.4,.5,.6\rangle\}.$$

Clearly,

$$\xi_C(w_3) \supseteq 0_{np}.$$

Thus

$$\text{int}_{np}(\xi_C(w_3)) = 0_{np}.$$

Also,

$$\xi_C(w_3) \subseteq 1_{np}, \xi_C(w_1)'.$$

Thus

$$cl_{np}(\xi_C(w_3)) = 1_{np} \cap \xi_C(w_1)' = \xi_C(w_1)'.$$

**Theorem 3.26** Let $(W, \tau_{np})$ be a NPTS over *W*. Let $\xi_D(V_1^*)$ and $\xi_F(V_1^*)$ be NPSs over *W* for NSs *D* and *F*. Then

(i) $\text{int}_{np}(\xi_D(V_1^*)) \subseteq \xi_D(V_1^*)$ and $\text{int}_{np}(\xi_D(V_1^*))$ is the largest $\tau_{np}$-NPOS.

(ii) $D \subseteq F \Rightarrow \text{int}_{np}(\xi_D(V_1^*)) \subseteq \text{int}_{np}(\xi_F(V_1^*))$.

(iii) $\text{int}_{np}(\xi_D(V_1^*))$ is an $\tau_{np}$-NPOS.

(iv) $\xi_D(V_1^*)$ is a $\tau_{np}$-NPOS iff $\text{int}_{np}(\xi_D(V_1^*)) = \xi_D(V_1^*)$.

(v) $\text{int}_{np}(\text{int}_{np}(\xi_D(V_1^*))) = \text{int}_{np}(\xi_D(V_1^*))$.

(vi) $\text{int}_{np}(0_{np}) = 0_{np}$ and $\text{int}_{np}(1_{np}) = 1_{np}$.

(vii) $\text{int}_{np}(\xi_D(V_1^*) \cap \xi_F(V_1^*)) = \text{int}_{np}(\xi_D(V_1^*)) \cap \text{int}_{np}(\xi_F(V_1^*))$.

(viii) $\text{int}_{np}(\xi_D(V_1^*) \cup \xi_F(V_1^*)) \subseteq \text{int}_{np}(\xi_D(V_1^*)) \cup \text{int}_{np}(\xi_F(V_1^*))$.

Proof. Follows from Definition 3.23.

**Theorem 3.27** Let $(W, \tau_{np})$ be a NPTS over $W$. Let $\xi_D(V_1^*)$ and $\xi_F(V_1^*)$ be NPSs over $W$ for NSs $D$ and $F$. Then

(i) $\xi_D(V_1^*) \subseteq cl_{np}\left(\xi_D(V_1^*)\right)$ and $cl_{np}\left(\xi_D(V_1^*)\right)$ is the smallest $\tau_{np}$-NPCS.

(ii) $D \subseteq F \Rightarrow cl_{np}\left(\xi_D(V_1^*)\right) \subseteq cl_{np}\left(\xi_F(V_1^*)\right)$.

(iii) $cl_{np}\left(\xi_D(V_1^*)\right)$ is an $\tau_{np}$-NPCS.

(iv) $\xi_D(V_1^*)$ is a $\tau_{np}$-NPCS iff $cl_{np}\left(\xi_D(V_1^*)\right) = \xi_D(V_1^*)$.

(v) $cl_{np}\left(cl_{np}\left(\xi_D(V_1^*)\right)\right) = cl_{np}\left(\xi_D(V_1^*)\right)$.

(vi) $cl_{np}(0_{np}) = 0_{np}$ and $cl_{np}(1_{np}) = 1_{np}$.

(vii) $cl_{np}\left(\xi_D(V_1^*) \cup \xi_F(V_1^*)\right) = cl_{np}\left(\xi_D(V_1^*)\right) \cup cl_{np}\left(\xi_F(V_1^*)\right)$.

(viii) $cl_{np}\left(\xi_D(V_1^*) \cap \xi_F(V_1^*)\right) \subseteq cl_{np}\left(\xi_D(V_1^*)\right) \cap cl_{np}\left(\xi_F(V_1^*)\right)$.

Proof. Follows from Definition 3.24.

**Theorem 3.28** Let $(W, \tau_{np})$ be a NPTS over $W$. Let $\xi_D(V^*)$ be a NPS over $W$ for a NS $D$. Then

(i) $\left(\text{int}_{np}\left(\xi_D(V_1^*)\right)\right)' = cl_{np}\left(\xi_D(V^*)'\right)$.

(ii) $\left(cl_{np}\left(\xi_D(V_1^*)\right)\right)' = \text{int}_{np}\left(\xi_D(V^*)'\right)$.

Proof. Follows from Definitions 3.23 and 3.24.

## 4. Decision Making in NPTS

In this section, the real-life application dealt to repair the sample machines with some damages. To repair it, priority is given to the high damaged machine. The solving techniques are given in the algorithm and formulae for evaluation are given. Some examples are considered to decide on these DM problems.

**Definition 4.1** Let $\xi_D(V^*)$ be a $\tau_{np}$-NPOS over $W$ of a NPTS $(W, \tau_{np})$. Then the neutro-prime absolute complement of $\xi_D(V^*)$ is denoted as $\tilde{\xi}_D\left((V^*)'\right)$ and defined as $\tilde{\xi}_D\left((V^*)'\right) = \tilde{\xi}_D\left((W - V^*)\right)$.

Thus the collection of $\tilde{\xi}_D\left((V^*)'\right)$ is denoted as $\tilde{\tau}_{np}$ and defined as $\tilde{\tau}_{np} = \left\{0_{np}, 1_{np}, \tilde{\xi}_D\left((V^*)'\right)\right\}$. The elements belong to $\tilde{\xi}_D\left((V^*)'\right)$ are said to be neutro-prime absolute open sets (NPAOSs) over $(W, \tau_{np})$ and the complement of NPOSs are said to be neutro-prime absolute closed sets (NPACSs) over $(W, \tau_{np})$ and denote the collection by $\tilde{\tau}_{np}{}^*$.

**Example 4.2** Let $W = \{w_1, w_2, w_3\}$ and $\tau_n = \{0_n, 1_n, D\}$ where $D$ is a NS over $W$ and are defined as follows

$$D = \left\{\langle w_1, .9, .4, .6\rangle, \langle w_2, .6, .5, .1\rangle, \langle w_3, .7, .8, .1\rangle\right\}.$$

Thus $(W, \tau_n)$ is a NTS over $W$.

Then NPS

$$\eta_p D(w_3) = \{\xi_D(w_3), \xi_D(w_1, w_3), \xi_D(w_2, w_3)\},$$

where

$$\xi_D(w_3) = \left\{\langle w_1, 0, 0, 1\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .7, .8, .1\rangle, \langle (w_1, w_2), 0, 0, 1\rangle, \langle (w_1, w_3), .9, .8, .1\rangle, \langle (w_2, w_3), .7, .8, .1\rangle\right\},$$

$$\xi_D(w_1, w_3) = \left\{\langle w_1, .9, .4, .6\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .7, .8, .1\rangle, \langle (w_1, w_2), .9, .5, .1\rangle, \langle (w_1, w_3), .9, .8, .1\rangle, \langle (w_2, w_3), .7, .8, .1\rangle\right\}$$

and

$\xi_D(w_2, w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .6, .5, .1\rangle, \langle w_3, .7, .8, .1\rangle, \langle (w_1, w_2), .9, .5, .1\rangle, \langle (w_1, w_3), .9, .8, .1\rangle, \langle (w_2, w_3), .7, .8, .1\rangle\}$.

Then

$\tau_{np} = \{0_{np}, 1_{np}, \xi_D(w_1, w_3)\}$ is a NPT.

Thus $(W, \tau_{np})$ is a NPTS over $W$.

Also, the complement of the NPT $\tau_{np}$ is

$\tau_{np}{}^* = \{0_{np}, 1_{np}, \xi_D(w_1, w_3)'\}$,

where

$\xi_D(w_1, w_3)' = \{\langle w_1, .6, .6, .9\rangle, \langle w_2, 1, 1, 0\rangle, \langle w_3, .1, .2, .7\rangle, \langle (w_1, w_2), .1, .5, .9\rangle, \langle (w_1, w_3), .1, .2, .9\rangle, \langle (w_2, w_3), .1, .2, .7\rangle\}$.

Then NPAOSs over $(W, \tau_{np})$ is

$\tilde{\tau}_{np} = \{0_{np}, 1_{np}, \tilde{\xi}_D((w_1, w_3)')\} = \{0_{np}, 1_{np}, \tilde{\xi}_D(w_2)\}$,

where

$\tilde{\xi}_D((w_1, w_3)') = \tilde{\xi}_D(w_2) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .6, .5, .1\rangle, \langle w_3, 0, 0, 1\rangle, \langle (w_1, w_2), .9, .5, .1\rangle, \langle (w_1, w_3), 0, 0, 1\rangle, \langle (w_2, w_3), .7, .8, .1\rangle\}$.

Also, NPACSs over $(W, \tau_{np})$ is

$\tilde{\tau}_{np}{}^* = \{0_{np}, 1_{np}, \tilde{\xi}_D((w_1, w_3)')'\}$

where

$\tilde{\xi}_D((w_1, w_3)')' = \tilde{\xi}_D(w_2)' = \{\langle w_1, 1, 1, 0\rangle, \langle w_2, .1, .5, .6\rangle, \langle w_3, 1, 1, 0\rangle, \langle (w_1, w_2), .1, .5, .9\rangle, \langle (w_1, w_3), 1, 1, 0\rangle, \langle (w_2, w_3), .1, .2, .7\rangle\}$.

**Definition 4.3** Let $W$ be a set of universe and $w \in W$. Let $D$ be a NS over $W$ and $U$ be any proper non-empty subset of $W$. Let $\xi_D(U)$ be a $\tau_{np}$-NPOS over $W$ of a NFTS $(W, \tau_{np})$.

Then the value of $D$ with respect to $U$ is denoted by $Val[D(U)]$ and is calculated by the formula

$$Val[D(U)] = \left| \frac{\left[\sum_i \left(\tilde{T}_D(U')\right)_i - \sum_i \left(F_D(U)'\right)_i\right] + \left[\sum_i \left(T_D(U)'\right)_i - \sum_i \left(\tilde{F}_D(U')\right)_i\right]}{2} \times \left[1 - \frac{\sum_i \left(\tilde{I}_D(U')\right)_i - \sum_i \left(I_D(U)'\right)_i}{2}\right] \right|, \quad (4.3.1)$$

where

$\sum_i (T_D(U)')_i$, $\sum_i (I_D(U)')_i$ and $\sum_i (F_D(U)')_i$ are the sum of all truth, indeterminacy and falsity values of $\xi_D(U)'$ respectively, and

$\sum_i (\tilde{T}_R(U'))_i$, $\sum_i (\tilde{I}_R(U'))_i$ and $\sum_i (\tilde{F}_D(U'))_i$ are the sum of all truth, indeterminacy, and falsity values of $\tilde{\xi}_D((U'))$ respectively.

Then the grand value of $D$ is denoted by $GV[D]$ and is calculated by the formula

$$GV[D] = \sum_i Val[D(U_i)], \text{ for all } i. \quad (4.3.2)$$

**Algorithm**

**Step 1:** List the set of machines for the sample.

**Step 2:** List some of its damaged parts as the universe $W$, where $w \in W$.

**Step 3:** Go through the damages of the machines.
**Step 4:** Define each machines as NSs, say *M*.
**Step 5:** Collect these NSs which defines a NT $\tau_n$ and so $(W, \tau_n)$ is a NTS.
**Step 6:** Define NPSs for each NS with respect to their damaged parts, say $\xi_M(U)$, where *U* is a proper non-empty subset of *W*.

**Step 7:** Define all possible NPTs $\tau_{np}$ and $\xi_M(U) \in \tau_{np}$, where *U* is a proper non-empty singleton subset of *W*.

**Step 8:** Define NPTSs $(W, \tau_{np})$ for all possible NPTs $\tau_{np}$.

**Step 9:** Find the complement and neutro-prime absolute complement of each NPTs.
**Step 10:** Calculate *Val*[*M*(*U*)] for all *M* with respect to some *U*, by using the formula 4.3.1.
**Step 11:** Tabulate all the estimated values of *Val*[*M*(*U*)].
**Step 12:** Calculate *GV*[*M*] for all *M*, by using the formula 4.3.2.
**Step 13:** Tabulate all the estimated values of *GV*[*M*].
**Step 14:** Select the highest value among all the *GV*[*M*].
**Step 15:** If two or more *GV*[*M*] are similar for a particular *U*, replace that *U* with some other damaged parts and repeat the process.
**Step 16:** End the process, till getting a unique *GV*[*M*].

**Example 4.4** Consider the problem that a technician came to repair damaged machines. Let *M*I, *M*II, *M*III, and *M*IV be sample machines whose damages to be repaired. Let $W = \{p_1, p_2, p_3\}$ be some parts of each damaged machine, where $p_1$–part 1, $p_2$–part 1 and $p_3$–part 3. Here the technician gives priority to the high damaged machine and to repair it initially.
1. Let *M*I, *M*II, *M*III, and *M*IV be sample machines whose damages to be repaired.
2. Let $W = \{p_1, p_2, p_3\}$ be the universe, where $p_1$–part 1, $p_2$–part 2, and $p_3$–part 3.
3. The technician goes through the damages on each machine.
4. Define *M*I, *M*II, *M*III, and *M*IV as NSs.

$$MI = \{\langle p_1,.7,.6,.3\rangle, \langle p_2,.4,.5,.4\rangle, \langle p_3,.5,.3,.4\rangle\},$$

$$MII = \{\langle p_1,.6,.3,.1\rangle, \langle p_2,.6,.8,.4\rangle, \langle p_3,.2,.2,.7\rangle\},$$

$$MIII = \{\langle p_1,.7,.6,.1\rangle, \langle p_2,.6,.8,.4\rangle, \langle p_3,.5,.3,.4\rangle\} \quad \text{and}$$

$$MIV = \{\langle p_1,.6,.3,.3\rangle, \langle p_2,.4,.5,.4\rangle, \langle p_3,.2,.2,.7\rangle\}.$$

5. Thus $\tau_n = \{0_n, 1_n, MI, MII, MIII, MIV\}$ is a NT and so $(W, \tau_n)$ is a NTS.
6. Define NPSs for each NS with respect to their damaged parts as follows:

$$\eta_p MI(p_1, p_3) = \{\xi_{MI}(p_1), \xi_{MI}(p_3), \xi_{MI}(p_1, p_2), \xi_{MI}(p_1, p_3), \xi_{MI}(p_2, p_3)\},$$

where

$$\xi_{MI}(p_1) = \{\langle p_1,.7,.6,.3\rangle, \langle p_2,0,0,1\rangle, \langle p_3,0,0,1\rangle, \langle (p_1, p_2),.7,.6,.3\rangle, \langle (p_1, p_3),.7,.6,.3\rangle, \langle (p_2, p_3),0,0,1\rangle\},$$

$\xi_{MI}(p_3)=\{\langle p_1,0,0,1\rangle,\langle p_2,0,0,1\rangle,\langle p_3,.5,.3,.4\rangle,\langle (p_1,p_2),0,0,1\rangle,\langle (p_1,p_3),.7,.6,.3\rangle,\langle (p_2,p_3),.5,.5,.4\rangle\}$,
$\xi_{MI}(p_1,p_2)=\{\langle p_1,.7,.6,.3\rangle,\langle p_2,.4,.5,.4\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.7,.6,.3\rangle,\langle (p_1,p_3),.7,.6,.3\rangle,\langle (p_2,p_3),.5,.5,.4\rangle\}$,
$\xi_{MI}(p_1,p_3)=\{\langle p_1,.7,.6,.3\rangle,\langle p_2,0,0,1\rangle,\langle p_3,.5,.3,.4\rangle,\langle (p_1,p_2),.7,.6,.3\rangle,\langle (p_1,p_3),.7,.6,.3\rangle,\langle (p_2,p_3),.5,.5,.4\rangle\}$,
$\xi_{MI}(p_2,p_3)=\{\langle p_1,0,0,1\rangle,\langle p_2,.4,.5,.4\rangle,\langle p_3,.5,.3,.4\rangle,\langle (p_1,p_2),.7,.6,.3\rangle,\langle (p_1,p_3),.7,.6,.3\rangle,\langle (p_2,p_3),.5,.5,.4\rangle\}$;

$$\eta_p MII(p_2)=\{\xi_{MII}(p_2),\xi_{MII}(p_1,p_2),\xi_{MII}(p_2,p_3)\},$$

where
$\xi_{MII}(p_2)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.6,.8,.1\rangle,\langle (p_1,p_3),0,0,1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\}$,
$\xi_{MII}(p_1,p_2)=\{\langle p_1,.6,.3,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.6,.8,.1\rangle,\langle (p_1,p_3),.6,.3,.1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\}$,
$\xi_{MII}(p_2,p_3)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,.2,.2,.7\rangle,\langle (p_1,p_2),.6,.8,.1\rangle,\langle (p_1,p_3),.6,.3,.1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\}$;

$$\eta_p MIII(p_1,p_2)=\{\xi_{MIII}(p_1),\xi_{MIII}(p_2),\xi_{MIII}(p_1,p_2),\xi_{MIII}(p_1,p_3),\xi_{MIII}(p_2,p_3)\},$$

where
$\xi_{MIII}(p_1)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,0,0,1\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),0,0,1\rangle\}$,
$\xi_{MIII}(p_2)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),0,0,1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\}$,
$\xi_{MIII}(p_1,p_2)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\}$,
$\xi_{MIII}(p_1,p_3)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,0,0,1\rangle,\langle p_3,.5,.3,.4\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\}$,
$\xi_{MIII}(p_2,p_3)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,.5,.3,.4\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\}$
and

$$\eta_p MIV(p_1)=\{\xi_{MIV}(p_1),\xi_{MIV}(p_1,p_2),\xi_{MIV}(p_1,p_3)\},$$

where
$\xi_{MIV}(p_1)=\{\langle p_1,.6,.3,.3\rangle,\langle p_2,0,0,1\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.6,.5,.3\rangle,\langle (p_1,p_3),.6,.3,.3\rangle,\langle (p_2,p_3),0,0,1\rangle\}$,
$\xi_{MIV}(p_1,p_2)=\{\langle p_1,.6,.3,.3\rangle,\langle p_2,.4,.5,.4\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.6,.5,.3\rangle,\langle (p_1,p_3),.6,.3,.3\rangle,\langle (p_2,p_3),.4,.5,.4\rangle\}$,
$\xi_{MIV}(p_1,p_3)=\{\langle p_1,.6,.3,.3\rangle,\langle p_2,0,0,1\rangle,\langle p_3,.2,.2,.7\rangle,\langle (p_1,p_2),.6,.5,.3\rangle,\langle (p_1,p_3),.6,.3,.3\rangle,\langle (p_2,p_3),.4,.5,.4\rangle\}$.

7. The possible NPTs for all proper non-empty singleton subset of $W$ are defined as follows:
$1\tau_{np}=\{0_{np},1_{np},\xi_{MI}(p_1),\xi_{MIII}(p_1),\xi_{MIV}(p_1)\}$ ,
$2\tau_{np}=\{0_{np},1_{np},\xi_{MII}(p_2),\xi_{MIII}(p_2)\}$ and
$3\tau_{np}=\{0_{np},1_{np},\xi_{MI}(p_3)\}$.
8. Thus $(W,1\tau_{np})$, $(W,2\tau_{np})$ and $(W,3\tau_{np})$ are NPTSs over $W$.
9. The complement and neutro-prime absolute complement of NPT $1\tau_{np}$ are as follows,

$$1\tau_{np}^{*}=\{0_{np},1_{np},\xi_{MI}(p_1)',\xi_{MIII}(p_1)',\xi_{MIV}(p_1)'\},$$

where
$\xi_{MI}(p_1)'=\{\langle p_1,.3,.4,.7\rangle,\langle p_2,1,1,0\rangle,\langle p_3,1,1,0\rangle,\langle (p_1,p_2),.3,.4,.7\rangle,\langle (p_1,p_3),.3,.4,.7\rangle,\langle (p_2,p_3),1,1,0\rangle\}$
$\xi_{MIII}(p_1)'=\{\langle p_1,.1,.4,.7\rangle,\langle p_2,1,1,0\rangle,\langle p_3,1,1,0\rangle,\langle (p_1,p_2),.1,.2,.7\rangle,\langle (p_1,p_3),.1,.4,.7\rangle,\langle (p_2,p_3),1,1,0\rangle\}$
$\xi_{MIV}(p_1)'=\{\langle p_1,.3,.7,.6\rangle,\langle p_2,1,1,0\rangle,\langle p_3,1,1,0\rangle,\langle (p_1,p_2),.3,.5,.6\rangle,\langle (p_1,p_3),.3,.7,.6\rangle,\langle (p_2,p_3),1,1,0\rangle\}$
and

$$1\tilde{\tau}_{np}=\{0_{np},1_{np},\tilde{\xi}_{MI}(p_2,p_3),\tilde{\xi}_{MIII}(p_2,p_3),\tilde{\xi}_{MIV}(p_2,p_3)\},$$

where
$\tilde{\xi}_{MI}(p_2,p_3)=\{\langle p_1,0,0,1\rangle,\langle p_2,.4,.5,.4\rangle,\langle p_3,.5,.3,.4\rangle,\langle (p_1,p_2),.7,.6,.3\rangle,\langle (p_1,p_3),.7,.6,.3\rangle,\langle (p_2,p_3),.5,.5,.4\rangle\}$
$\tilde{\xi}_{MIII}(p_2,p_3)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,.5,.3,.4\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\}$

$\tilde{\xi}_{MIV}(p_2, p_3) = \{\langle p_1, 0, 0, 1\rangle, \langle p_2, .4, .5, .4\rangle, \langle p_3, .2, .2, .7\rangle, \langle (p_1, p_2), .6, .5, .3\rangle, \langle (p_1, p_3), .6, .3, .3\rangle, \langle (p_2, p_3), .4, .5, .4\rangle\}.$

The complement and neutro-prime absolute complement of NPT $2\tau_{np}$ are as follows,

$2\tau_{np}{}^* = \{0_{np}, 1_{np}, \xi_{MII}(p_2)', \xi_{MIII}(p_2)'\},$

where

$\xi_{MII}(p_2)' = \{\langle p_1, 1, 1, 0\rangle, \langle p_2, .4, .2, .6\rangle, \langle p_3, 1, 1, 0\rangle, \langle (p_1, p_2), .1, .2, .6\rangle, \langle (p_1, p_3), 1, 1, 0\rangle, \langle (p_2, p_3), .4, .2, .6\rangle\}$

$\xi_{MIII}(p_2)' = \{\langle p_1, 1, 1, 0\rangle, \langle p_2, .4, .2, .6\rangle, \langle p_3, 1, 1, 0\rangle, \langle (p_1, p_2), .1, .2, .7\rangle, \langle (p_1, p_3), 1, 1, 0\rangle, \langle (p_2, p_3), .4, .2, .6\rangle\}$

and

$2\tilde{\tau}_{np} = \{0_{np}, 1_{np}, \tilde{\xi}_{MII}(p_1, p_3), \tilde{\xi}_{MIII}(p_1, p_3)\},$

where

$\tilde{\xi}_{MII}(p_1, p_3) = \{\langle p_1, .6, .3, .1\rangle, \langle p_2, 0, 0, 1\rangle, \langle p_3, .2, .2, .7\rangle, \langle (p_1, p_2), .6, .8, .1\rangle, \langle (p_1, p_3), .6, .3, .1\rangle, \langle (p_2, p_3), .6, .8, .4\rangle\}$

$\tilde{\xi}_{MIII}(p_1, p_3) = \{\langle p_1, .7, .6, .1\rangle, \langle p_2, 0, 0, 1\rangle, \langle p_3, .5, .3, .4\rangle, \langle (p_1, p_2), .7, .8, .1\rangle, \langle (p_1, p_3), .7, .6, .1\rangle, \langle (p_2, p_3), .6, .8, .4\rangle\}.$

The complement and neutro-prime absolute complement of NPT $3\tau_{np}$ are as follows,

$3\tau_{np}{}^* = \{0_{np}, 1_{np}, \xi_{MI}(p_3)'\},$

where

$\xi_{MI}(p_3)' = \{\langle p_1, 1, 1, 0\rangle, \langle p_2, 1, 1, 0\rangle, \langle p_3, .4, .7, .5\rangle, \langle (p_1, p_2), 1, 1, 0\rangle, \langle (p_1, p_3), .3, .4, .7\rangle, \langle (p_2, p_3), .4, .5, .5\rangle\}$

and

$3\tilde{\tau}_{np} = \{0_{np}, 1_{np}, \tilde{\xi}_{MI}(p_1, p_2)\},$

where

$\tilde{\xi}_{MI}(p_1, p_2) = \{\langle p_1, .7, .6, .3\rangle, \langle p_2, .4, .5, .4\rangle, \langle p_3, 0, 0, 1\rangle, \langle (p_1, p_2), .7, .6, .3\rangle, \langle (p_1, p_3), .7, .6, .3\rangle, \langle (p_2, p_3), .5, .5, .4\rangle\}.$

10. By using the formula 4.3.1, evaluated the values of all machines with respect to each proper non-empty singleton subset of *W*.

i.e. *Val*[*M*I($p_i$)], *Val*[*M*II($p_i$)], *Val*[*M*III($p_i$)] and *Val*[*M*IV($p_i$)], for $i = 1, 2, 3$.

11. These estimated values are tabulated in the following table.

**Table 4.4.1.** Value Table

| | $p_1$ | $p_2$ | $p_3$ |
|---|---|---|---|
| ***M*I** | 2.025 | 0 | 3.645 |
| ***M*II** | 0 | 2.3 | 0 |
| ***M*III** | 1.59 | 3.6425 | 0 |
| ***M*IV** | 1.35 | 0 | 1.59 |

12. By using the formula 4.3.2, evaluated the grand values of all machines.

i.e. *GV*[*M*I], *GV*[*M*II], *GV*[*M*III], and *GV*[*M*IV].

13. These estimated values are tabulated in the following table.

**Table 4.4.2.** Grand Value Table

| | $p_1$ | $p_2$ | $p_3$ | **GV** |
|---|---|---|---|---|
| ***M*I** | 2.025 | 0 | 3.645 | **5.67** |
| ***M*II** | 0 | 2.3 | 0 | 2.3 |
| ***M*III** | 1.59 | 3.6425 | 0 | 5.2325 |
| ***M*IV** | 1.35 | 0 | 1.59 | 1.35 |

14. Thus $GV[MI]$ is the highest value.

Hence the technician gives priority to repairing the damaged machine $M$I.

**Example 4.5** Consider the problem explained in Example 4.4.

1. Let $M$I, $M$II, and $M$III be sample machines whose damages to be repaired.
2. Let $W=\{p_1,p_2,p_3\}$ be the universe, where $p_1$ –part 1, $p_2$ –part 2, and $p_3$ –part 3.
3. The technician goes through the damages on each machine.
4. Define $M$I, $M$II, and $M$III as NSs.

$$MI=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,.7,.6,.1\rangle\},$$

$$MII=\{\langle p_1,.6,.3,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,.2,.2,.7\rangle\} \text{ and}$$

$$MIII=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,.7,.6,.1\rangle\}\ .$$

5. Thus $\tau_n=\{0_n,1_n,MI,MII,MIII\}$ is a NT and so $(W,\tau_n)$ is a NTS.
6. Define NPSs for each NS with respect to their damaged parts as follows:

$$\eta_p MI(p_3)=\{\xi_{MI}(p_3),\xi_{MI}(p_1,p_3),\xi_{MI}(p_2,p_3)\},$$

where

$$\xi_{MI}(p_3)=\{\langle p_1,0,0,1\rangle,\langle p_2,0,0,1\rangle,\langle p_3,.7,.6,.1\rangle,\langle (p_1,p_2),0,0,1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),.7,.8,.1\rangle\},$$

$$\xi_{MI}(p_1,p_3)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,0,0,1\rangle,\langle p_3,.7,.6,.1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),.7,.8,.1\rangle\},$$

$$\xi_{MI}(p_2,p_3)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),.7,.8,.1\rangle\};$$

$$\eta_p MII(p_2)=\{\xi_{MII}(p_2),\xi_{MII}(p_1,p_2),\xi_{MII}(p_2,p_3)\},$$

where

$$\xi_{MII}(p_2)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.6,.8,.1\rangle,\langle (p_1,p_3),0,0,1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\},$$

$$\xi_{MII}(p_1,p_2)=\{\langle p_1,.6,.3,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.6,.8,.1\rangle,\langle (p_1,p_3),.6,.3,.1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\},$$

$$\xi_{MII}(p_2,p_3)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,.2,.2,.7\rangle,\langle (p_1,p_2),.6,.8,.1\rangle,\langle (p_1,p_3),.6,.3,.1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\}$$

and

$$\eta_p MIII(p_1)=\{\xi_{MIII}(p_1),\xi_{MIII}(p_1,p_2),\xi_{MIII}(p_1,p_3)\},$$

where

$$\xi_{MIII}(p_1)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,0,0,1\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),0,0,1\rangle\},$$

$$\xi_{MIII}(p_1,p_2)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),.7,.8,.1\rangle\},$$

$$\xi_{MIII}(p_1,p_3)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,0,0,1\rangle,\langle p_3,.7,.6,.1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),.7,.8,.1\rangle\},$$

7. The possible NPTs for all proper non-empty singleton subset of $W$ are defined as follows:

$$1\tau_{np}=\{0_{np},1_{np},\xi_{MIII}(p_1)\}\ ,$$

$$2\tau_{np}=\{0_{np},1_{np},\xi_{MII}(p_2)\} \text{ and}$$

$$3\tau_{np}=\{0_{np},1_{np},\xi_{MI}(p_3)\}.$$

8. Thus $(W,1\tau_{np})$, $(W,2\tau_{np})$ and $(W,3\tau_{np})$ are NPTSs over $W$.
9. The complement and neutro-prime absolute complement of NPT $1\tau_{np}$ are as follows,

$$1\tau_{np}{}^{*}=\{0_{np},1_{np},\ \xi_{MIII}(p_1)'\},$$

where

$\xi_{MIII}(p_1)' = \{\langle p_1,.1,.4,.7\rangle,\langle p_2,1,1,0\rangle,\langle p_3,1,1,0\rangle,\langle (p_1,p_2),.1,.2,.7\rangle,\langle (p_1,p_3),.1,.4,.7\rangle,\langle (p_2,p_3),1,1,0\rangle\}$

and

$$1\tilde{\tau}_{np} = \{0_{np}, 1_{np}, \tilde{\xi}_{MIII}(p_2,p_3)'\},$$

where

$\tilde{\xi}_{MIII}(p_2,p_3) = \{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,.5,.3,.4\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\}.$

The complement and neutro-prime absolute complement of NPT $2\tau_{np}$ are as follows,

$$2\tau_{np}^{*} = \{0_{np}, 1_{np}, \xi_{MII}(p_2)'\},$$

where

$\xi_{MII}(p_2)' = \{\langle p_1,1,1,0\rangle,\langle p_2,.4,.2,.6\rangle,\langle p_3,1,1,0\rangle,\langle (p_1,p_2),.1,.2,.6\rangle,\langle (p_1,p_3),1,1,0\rangle,\langle (p_2,p_3),.4,.2,.6\rangle\}$

and

$$2\tilde{\tau}_{np} = \{0_{np}, 1_{np}, \tilde{\xi}_{MII}(p_1,p_3)\},$$

where

$\tilde{\xi}_{MII}(p_1,p_3) = \{\langle p_1,.6,.3,.1\rangle,\langle p_2,0,0,1\rangle,\langle p_3,.2,.2,.7\rangle,\langle (p_1,p_2),.6,.8,.1\rangle,\langle (p_1,p_3),.6,.3,.1\rangle,\langle (p_2,p_3),.6,.8,.4\rangle\}.$

The complement and neutro-prime absolute complement of NPT $3\tau_{np}$ are as follows,

$$3\tau_{np}^{*} = \{0_{np}, 1_{np}, \xi_{MI}(p_3)'\},$$

where

$\xi_{MI}(p_3)' = \{\langle p_1,1,1,0\rangle,\langle p_2,1,1,0\rangle,\langle p_3,.1,.4,.7\rangle,\langle (p_1,p_2),1,1,0\rangle,\langle (p_1,p_3),.1,.4,.7\rangle,\langle (p_2,p_3),.1,.2,.7\rangle\}$

and

$$3\tilde{\tau}_{np} = \{0_{np}, 1_{np}, \tilde{\xi}_{MI}(p_1,p_2)\},$$

where

$\tilde{\xi}_{MI}(p_1,p_2) = \{\langle p_1,.7,.6,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_3,0,0,1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_3),.7,.6,.1\rangle,\langle (p_2,p_3),.7,.8,.1\rangle\}.$

10. By using the formula 4.3.1, evaluated the values of all machines with respect to each proper non-empty singleton subset of *W*.
    i.e. *Val*[*M*I($p_i$)], *Val*[*M*II($p_i$)] and *Val*[*M*III($p_i$)], for $i$ = 1, 2, 3.
11. These estimated values are tabulated in the following table.

**Table 4.5.1.** Value Table

| | **p1** | **p2** | **p3** |
|---|---|---|---|
| ***M*I** | 0 | 0 | 3.92 |
| ***M*II** | 0 | 2.3 | 0 |
| ***M*III** | 3.92 | 0 | 0 |

12. By using the formula 4.3.2, evaluated the grand values of all machines.
    i.e. *GV*[*M*I], *GV*[*M*II] and *GV*[*M*III].
13. These estimated values are tabulated in the following table.

**Table 4.5.2.** Grand Value Table

| | **p1** | **p2** | **p3** | **GV** |
|---|---|---|---|---|
| ***M*I** | 0 | 0 | 3.92 | **3.92** |
| ***M*II** | 0 | 2.3 | 0 | 2.3 |
| ***M*III** | 3.92 | 0 | 0 | **3.92** |

14. Thus both *GV*[*M*I] and *GV*[*M*III] are the highest value.

15. In this situation, replace part 3 ( $p_3$ ) with some other damaged part, say $p_4$, and repeat the process.

1. Let *M*I, *M*II, and *M*III be sample machines whose damages to be repaired.
2. Let $W=\{p_1,p_2,p_4\}$ be the universe, where $p_1$ –part 1, $p_2$ –part 2, and $p_4$ –part 4.
3. The technician goes through the damages on each machine.
4. Define *M*I, *M*II, and *M*III as NSs.

$$MI=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_4,.5,.3,.4\rangle\},$$

$$MII=\{\langle p_1,.6,.3,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_4,.4,.3,.6\rangle\} \text{ and}$$

$$MIII=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_4,.5,.3,.4\rangle\} .$$

5. Thus $\tau_n=\{0_n,1_n,MI,MII,MIII\}$ is a NT and so $(W,\tau_n)$ is a NTS.
6. Define NPSs for each NS with respect to their damaged parts as follows:

$$\eta_p MI(p_4)=\{\xi_{MI}(p_4),\xi_{MI}(p_1,p_4),\xi_{MI}(p_2,p_4)\},$$

where

$$\xi_{MI}(p_4)=\{\langle p_1,0,0,1\rangle,\langle p_2,0,0,1\rangle,\langle p_4,.5,.3,.4\rangle,\langle (p_1,p_2),0,0,1\rangle,\langle (p_1,p_4),.7,.6,.1\rangle,\langle (p_2,p_4),.6,.8,.4\rangle\},$$
$$\xi_{MI}(p_1,p_4)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,0,0,1\rangle,\langle p_4,.5,.3,.4\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_4),.7,.6,.1\rangle,\langle (p_2,p_4),.6,.8,.4\rangle\},$$
$$\xi_{MI}(p_2,p_4)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_4,.5,.3,.4\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_4),.7,.6,.1\rangle,\langle (p_2,p_4),.6,.8,.4\rangle\};$$

$$\eta_p MII(p_2)=\{\xi_{MII}(p_2),\xi_{MII}(p_1,p_2),\xi_{MII}(p_2,p_4)\},$$

where

$$\xi_{MII}(p_2)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_4,0,0,1\rangle,\langle (p_1,p_2),.6,.8,.1\rangle,\langle (p_1,p_4),0,0,1\rangle,\langle (p_2,p_4),.6,.8,.4\rangle\},$$
$$\xi_{MII}(p_1,p_2)=\{\langle p_1,.6,.3,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_4,0,0,1\rangle,\langle (p_1,p_2),.6,.8,.1\rangle,\langle (p_1,p_4),.6,.3,.1\rangle,\langle (p_2,p_4),.6,.8,.4\rangle\},$$
$$\xi_{MII}(p_2,p_4)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_4,.4,.3,.6\rangle,\langle (p_1,p_2),.6,.8,.1\rangle,\langle (p_1,p_4),.6,.3,.1\rangle,\langle (p_2,p_4),.6,.8,.4\rangle\}$$

and

$$\eta_p MIII(p_1,p_2)=\{\xi_{MIII}(p_1),\xi_{MIII}(p_2),\xi_{MIII}(p_1,p_2),\xi_{MIII}(p_1,p_4),\xi_{MIII}(p_2,p_4)\},$$

where

$$\xi_{MIII}(p_1)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,0,0,1\rangle,\langle p_4,0,0,1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_4),.7,.6,.1\rangle,\langle (p_2,p_4),0,0,1\rangle\},$$
$$\xi_{MIII}(p_2)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_4,0,0,1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_4),0,0,1\rangle,\langle (p_2,p_4),.6,.8,.4\rangle\},$$
$$\xi_{MIII}(p_1,p_2)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_4,0,0,1\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_4),.7,.6,.1\rangle,\langle (p_2,p_4),.6,.8,.4\rangle\},$$
$$\xi_{MIII}(p_1,p_4)=\{\langle p_1,.7,.6,.1\rangle,\langle p_2,0,0,1\rangle,\langle p_4,.5,.3,.4\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_4),.7,.6,.1\rangle,\langle (p_2,p_4),.6,.8,.4\rangle\},$$
$$\xi_{MIII}(p_2,p_4)=\{\langle p_1,0,0,1\rangle,\langle p_2,.6,.8,.4\rangle,\langle p_4,.5,.3,.4\rangle,\langle (p_1,p_2),.7,.8,.1\rangle,\langle (p_1,p_4),.7,.6,.1\rangle,\langle (p_2,p_4),.6,.8,.4\rangle\}$$

7. The possible NPTs for all proper non-empty singleton subset of *W* are defined as follows:

$1\tau_{np}=\{0_{np},1_{np},\xi_{MIII}(p_1)\}$ ,

$2\tau_{np}=\{0_{np},1_{np},\xi_{MII}(p_2),\xi_{MIII}(p_2)\}$ and

$3\tau_{np}=\{0_{np},1_{np},\xi_{MI}(p_4)\}$.

8. Thus $(W,1\tau_{np})$, $(W,2\tau_{np})$ and $(W,3\tau_{np})$ are NPTSs over *W*.

9. The complement and neutro-prime absolute complement of NPT $1\tau_{np}$ are as follows,

$$1\tau_{np}{}^{*} = \{0_{np}, 1_{np}, \xi_{MIII}(p_1)'\},$$

where

$$\xi_{MIII}(p_1)' = \{\langle p_1, .1, .4, .7\rangle, \langle p_2, 1, 1, 0\rangle, \langle p_4, 1, 1, 0\rangle, \langle (p_1, p_2), .1, .2, .7\rangle, \langle (p_1, p_4), .1, .4, .7\rangle, \langle (p_2, p_4), 1, 1, 0\rangle\}$$

and

$$1\tilde{\tau}_{np} = \{0_{np}, 1_{np}, \tilde{\xi}_{MIII}(p_2, p_4)'\},$$

where

$$\tilde{\xi}_{MIII}(p_2, p_4) = \{\langle p_1, 0, 0, 1\rangle, \langle p_2, .6, .8, .4\rangle, \langle p_4, .5, .3, .4\rangle, \langle (p_1, p_2), .7, .8, .1\rangle, \langle (p_1, p_4), .7, .6, .1\rangle, \langle (p_2, p_4), .6, .8, .4\rangle\}.$$

The complement and neutro-prime absolute complement of NPT $2\tau_{np}$ are as follows,

$$2\tau_{np}{}^{*} = \{0_{np}, 1_{np}, \xi_{MII}(p_2)', \xi_{MIII}(p_2)'\},$$

where

$$\xi_{MII}(p_2)' = \{\langle p_1, 1, 1, 0\rangle, \langle p_2, .4, .2, .6\rangle, \langle p_4, 1, 1, 0\rangle, \langle (p_1, p_2), .1, .2, .6\rangle, \langle (p_1, p_4), 1, 1, 0\rangle, \langle (p_2, p_4), .4, .2, .6\rangle\}$$

$$\xi_{MIII}(p_2)' = \{\langle p_1, 1, 1, 0\rangle, \langle p_2, .4, .2, .6\rangle, \langle p_4, 1, 1, 0\rangle, \langle (p_1, p_2), .1, .2, .7\rangle, \langle (p_1, p_4), 1, 1, 0\rangle, \langle (p_2, p_4), .4, .2, .6\rangle\}$$

and

$$2\tilde{\tau}_{np} = \{0_{np}, 1_{np}, \tilde{\xi}_{MII}(p_1, p_4)\tilde{\xi}_{MIII}(p_1, p_4)\},$$

where

$$\tilde{\xi}_{MII}(p_1, p_4) = \{\langle p_1, .6, .3, .1\rangle, \langle p_2, 0, 0, 1\rangle, \langle p_4, .2, .2, .7\rangle, \langle (p_1, p_2), .6, .8, .1\rangle, \langle (p_1, p_4), .6, .3, .1\rangle, \langle (p_2, p_4), .6, .8, .4\rangle\}$$

$$\tilde{\xi}_{MIII}(p_1, p_4) = \{\langle p_1, .7, .6, .1\rangle, \langle p_2, 0, 0, 1\rangle, \langle p_4, .5, .3, .4\rangle, \langle (p_1, p_2), .7, .8, .1\rangle, \langle (p_1, p_4), .7, .6, .1\rangle, \langle (p_2, p_4), .6, .8, .4\rangle\}.$$

The complement and neutro-prime absolute complement of NPT $3\tau_{np}$ are as follows,

$$3\tau_{np}{}^{*} = \{0_{np}, 1_{np}, \xi_{MI}(p_4)'\},$$

where

$$\xi_{MI}(p_4)' = \{\langle p_1, 1, 1, 0\rangle, \langle p_2, 1, 1, 0\rangle, \langle p_4, .4, .7, .5\rangle, \langle (p_1, p_2), 1, 1, 0\rangle, \langle (p_1, p_4), .1, .4, .7\rangle, \langle (p_2, p_4), .4, .2, .6\rangle\}$$

and

$$3\tilde{\tau}_{np} = \{0_{np}, 1_{np}, \tilde{\xi}_{MI}(p_1, p_2)\},$$

where

$$\tilde{\xi}_{MI}(p_1, p_2) = \{\langle p_1, .7, .6, .1\rangle, \langle p_2, .6, .8, .4\rangle, \langle p_4, 0, 0, 1\rangle, \langle (p_1, p_2), .7, .8, .1\rangle, \langle (p_1, p_4), .7, .6, .1\rangle, \langle (p_2, p_4), .6, .8, .4\rangle\}.$$

10. By using the formula 4.3.1, evaluated the values of all machines with respect to each proper non-empty singleton subset of *W*.

i.e. *Val*[MI($p_i$)], *Val*[MII($p_i$)] and *Val*[MIII($p_i$)], for $i$ = 1, 2, 3.

11. These estimated values are tabulated in the following table.

**Table 4.5.3.** Value Table

| | $p_1$ | $p_2$ | $p_4$ |
|---|---|---|---|
| *M*I | 0 | 0 | 4.8675 |
| *M*II | 0 | 2.665 | 0 |
| *M*III | 1.59 | 3.6425 | 0 |

12. By using the formula 4.3.2, evaluated the grand values of all machines.

i.e. *GV*[*M*I], *GV*[*M*II] and *GV*[*M*III].

13. These estimated values are tabulated in the following table.

**Table 4.5.4.** Grand Value Table

| | $p_1$ | $p_2$ | $p_4$ | GV |
|---|---|---|---|---|
| ***M*I** | 0 | 0 | 4.8675 | 4.8675 |
| ***M*II** | 0 | 2.665 | 0 | 2.665 |
| ***M*III** | 1.59 | 3.6425 | 0 | **5.2325** |

14. Thus *GV*[*M*III] is the highest value.

Hence the technician gives priority to repairing the damaged machine *M*III.

## 5. Conclusions

The major contribution of this work is initiating a neutrosophic environment on prime sets under a topological space. Some related properties of NPTSs have been proved and some are disproved with counterexamples. Also, the idea of interior and closure dealt with such space with few basic properties. The novelty of this study is to merge two different poles. The decision-making problem is demonstrated with an example to analyze the number of faults in industrial machines. Sample machines are represented as NSs and their damages are represented as NPSs under its topological space. The values of fault machines are detected by finding the complement and absolute complement of each NPS. The various values of faults are taken as different subsets, for analysis. The proposed algorithm analyzes through the NPSs and finds the best suitable set values which indicate the heavy damage in machines. The lower fault machines are neglected by decision-making problems.

The primary results of this study are:

- Prime set is studied under the environment of neutrosophic.
- Related properties are stated with proof and also disproved in counter examples.
- The intersection of two NPTs is a NPT but not for its union.
- Demonstrating the decision-making problem of analyzing the number of damages in machines.
- The complement and absolute complement of NPSs are evaluated to find the best fit of fault by diagnosis the machines.

In the future, this set may develop more genetic algorithms to predict multi-criteria DM problems. Many more sets like soft sets, rough sets, crisp sets, etc., can be developed on NPTSs. More ideas may be claimed and investigated to achieve a deeper understanding of the economic and social consequences of robotization.

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# A novel score and accuracy function for neutrosophic sets and their real-world applications to multi-criteria decision-making process

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**Abstract:** The purpose of this work is to understand the ranking order of the neutrosophic sets, where the uncertain or ambiguous information/data is stored in the terms of three independent variables i.e., degree of truthfulness, degree of indeterminacy, and degree of falseness. There exist many ranking tools in decision-making (DM) like score function (SF) and accuracy function (AF) that help to rank the single-valued neutrosophic set (SVNS) and the interval-valued neutrosophic set (IVNS) to make a better choice among all the available alternatives. An intensive study about all the existing score functions and an accuracy function reveals that the existing ranking method for SVNS and IVNS in DM problems holds for certain kinds of neutrosophic information and has its limitations. To validate these observations some well-defined examples are chosen, illustrating that the existing score functions and accuracy functions are like special cases for certain kinds of neutrosophic data. Since nothing in this world is an ultimate truth, hence this existing gap is a real motivation to come up with a more efficient SF and AF that would rank SVNS and IVNS in the real-life problems to make a better selection among all the available alternatives in DM problems in an efficient way. Hence, a new SF and AF have been proposed and multi-criteria decision-making (MCDM) method is developed based on these proposed SF and AF. Furthermore, a real-life problem from our immediate surroundings is taken and solved successfully, and also a comparative analysis of the solutions for the existing problems is made in detail.

**Keywords:** Accuracy function (AF), average operators, IVNS, multi-criteria decision-making (MCDM), score function (SF), SVNS.

## 1. Introduction

Humans among all the living beings have evolved most intelligently since their existence and the reason behind it is a proper and timely DM in their environment. In this computer age, the scientific world is in continuous motion and whatever is new today is old in another hour, and the information or statistical data available is not always crisp, definite, constant, determinate, and consistent. Thus to deal with such kind of firsthand information, a new theory was evolved in 1965 by a great philosopher Zadeh, who had a farsighted vision and a penetrating understanding of the known and unknown data. He has come up with his intriguing theory of sets claiming that in real-life uncertainty is the only thing that is certain in life and named it as a fuzzy set (FS) [1] which deals with the concept of belongingness. This theory was reluctantly accepted in that period (i.e. in and around 1965) which tells that the available data is not always a real-value but it beholds the hand of uncertainty together and the study of this uncertainty or vagueness would be able to bring a huge revolution in the coming time with the real-life MCDM and MADM problems [2, 3]. After the acceptance of this theory of fuzzy sets, later with time, the scientific and intellectual world developed a keen interest in this concept of fuzziness, and then onwards various and wide extensions of fuzzy sets are propounded like- Atanassov proposed an intuitionistic fuzzy set (IFS) [4] who considered together both the concept of the degree of belongingness and the degree of non-belongingness. Since it is not always possible to evaluate any information in an exact value so, to define such data sometimes it is expressed in the interval, thus IFS was later expanded by Atanassov and Gargov to an interval-valued intuitionistic fuzzy set (IVIFS) [5]. Yager developed a Pythagorean fuzzy set (PFS) [6-8] which was extended by Zhang to an interval-valued Pythagorean fuzzy set (IVPFS) [9]. Smarandache introduced another extension of the FS as neutrosophic sets [10-12] which was more like a philosophical approach stating that together with membership and non-membership there is also an existence of one more component and he named it as indeterminacy such that, all these three values are independent of each other. IFS did not tell or explain indeterminate or inconsistent sets of information and hence neutrosophic set (NS) was able to handle such indeterminate data in a more efficient way. To apply this philosophy of NS into the real-world application Wang et al. [13, 14] proposed the concept of SVNSs and IVNSs along with their operators and properties respectively.

Since FS theory and its extensions lagged to deal with indeterminacy and inconsistent set of data therefore neutrosophic sets have successfully overcome these fuzzy drawbacks. A lot of exploration has been made till now in the area of SVNSs and IVNSs like neutrosophic sets are successfully applied in fuzzy linear optimization by using an important DM technique as linear programming by various researchers say Hezam et al. [15], Abdel-Basset et al. [16-17], Pramanik [18], Ye [19], Nafei et al. [20], Khatter [21], Bera et al. [22], Basumatary et al. [23], etc. Cubic fuzzy sets (CFSs) are introduced by YB Jun et al. [24] and, then YB Jun et al. [25] and M. Alia et al. [26] have extended CFSs to the neutrosophic environment and proposed neutrosophic cubic fuzzy sets (NCFSs) along with some of their basic operations. Recently, Ajay et al. [27] proposed aggregation operators on NCFSs. JC Kely [28] in 1963 introduced bitopological spaces which were extended in other fuzzy environments by many other researchers like Kandil et al. [29], Lee et al. [30] and Mwchahary et al. [31] recently proposed the concept and the propositions of neutrosophic bitopological spaces. Abdel-Basset et al. [32] proposed a method using quality function deployment (QFD) and plithogenic aggregation operations, and also, Abdel-Basset and Rehab [33] proposed a methodology based on plithogenic MCDM approach, utilizing both, techniques for order preference by similarity to ideal solution (TOPSIS) and criteria importance through inter-criteria correlation (CRITIC) techniques and applied in the study of telecommunications equipment categories. Lately, Nabeeh et al. [34] have contributed a lot in decision-making problems undertaken in the neutrosophic field like they have developed a neutrosophic MCDM framework to deal with inconsistent data related to environmental problems. Nabeeh et al. [35] have used integrated neutrosophic and TOSIS to deal with the personnel selection process. Nabeeh et al. [36] have applied the neutrosophic analytical hierarchy process (AHP) of the internet of things (IoT) in enterprises to estimate influential factors. Abdel-Basset et al. [37] proposed a hybrid combination of AHP and neutrosophic theory to deal with the uncertainty of IoT-based enterprises.

DM is a procedure that helps in selecting the best possible alternative among the set of feasible solutions. Since the world is in continuous motion, the societal structure is growing every second, and we need to make decisions under all these factors i.e., peer pressure, the vagueness of the imprecise data, limited funds, high-risk factor, environmental factors, biases, etc. which influences the DM of a decision-maker. There influencing factors are directly or indirectly associated with the unpredictability of the set of data which could be indeterminate, inconsistent, or uncertain, etc., occurring in different fields of life like economics, engineering, medical sciences, computer sciences, management sciences, psychology, meteorology, sociology, decision making. Since neutrosophic sets are quite efficient in dealing with indeterminate and inconsistent sets of data hence, many researchers in literature [15-42] have applied neutrosophic sets in real-life applications and can provide a more satisfactory solution to real-world applications like telecommunication, supply chain management, environment, personnel selection, enterprises, signal processing, pattern recognition, medical diagnosis, social sciences, engineering, management sciences, artificial intelligence, robotics, computer networks, DM, etc. MCDM helps the decision-maker to make his preferences by taking care of each criterion of the available alternatives, rank them by using some MCDM tools, and choose the best among the available alternatives.

This paper is the outcome of a deep study that has been made to understand the various existing ranking orders like SF and AF of the extensions of fuzzy sets like IFS [43-45], PFS [6-8], IVPFS [46-54], NS [17, 55-65], trapezoidal interval-valued neutrosophic numbers [66], etc., by the various researchers. After a rigorous analysis, it has been observed that the existing SF and AF [67, 68] for comparing single-valued neutrosophic sets (SVNSs) and the interval-valued neutrosophic sets (IVNSs) are more efficient for some special cases of SVNSs and IVNSs. Some well-defined counter-examples are chosen where the rating value of uncertainty is taken as SVNSs and IVNSs to claim that, the existing SF and AF rank these SVNSs and IVNSs correctly only to a certain limit. Hence to fulfill all the restrictions of the existing SF and AF, there is a need to propose a new SF and AF which would act as a helpful tool in the real-world DM problems. Taking this notion as an inspiration, an effort is made to suggest a new SF and AF for efficiently comparing SVNSs and IVNSs. Furthermore, based on these proposed SF and AF, an MCDM method is developed to solve the real-life applications and to validate these proposed SF and AF, the exact result of the real-life problem taken from our immediate surroundings is solved successfully in which the preference rating values are expressed by SVNSs and IVNSs and also, a detailed comparative analysis of the solutions with the existing approaches is presented respectively.

This paper is presented in the following manner: Section 2 - preliminaries; Section 3 - proposed SF and AF for SVNSs and IVNSs; Section 4 - MCDM method is proposed; Section 5 - real-life problem considered and solved; Section 6 - discussion and a comparative analysis of the obtained solutions; Section 7 – managerial insights; Section 8-conclusions.

## 2. Preliminaries - SVNSs, IVNSs and its SF and AF

This section states some requisites while dealing with SVNSs and IVNSs in the real-life application for the DM process.

**Definition 2.1 [1]** A set $\widetilde{A} = \{\langle x, \mu_{\widetilde{A}}(x)\rangle \mid x \in X,\ 0 \le \mu_{\widetilde{A}}(x) \le 1\}$, defined on the universal set $X$, is said to be an FS, where $\mu_{\widetilde{A}}(x)$ represents the degree of membership of the element $x$ in $\widetilde{A}$.

**Definition 2.2 [10]** A set $\widetilde{A}^N = \{\langle x, T_{\widetilde{A}^N}(x),\ I_{\widetilde{A}^N}(x), F_{\widetilde{A}^N}(x)\rangle | x \in X,\ 0^- \le T_{\widetilde{A}^N}(x) \le 1^+,\ 0^- \le I_{\widetilde{A}^N}(x) \le 1^+,\ 0^- \le F_{\widetilde{A}^N} \le 1^+,\ T_{\widetilde{A}^N}(x) + I_{\widetilde{A}^N}(x) + F_{\widetilde{A}^N}(x) \le 3^+\}$, defined on the universal set $X$, is said to be an NS, where, $T_{\widetilde{A}^N}(x), I_{\widetilde{A}^N}(x)$, and $F_{\widetilde{A}^N}(x)$ represents the degree of truth-membership, the degree of indeterminacy-membership and degree of falsity-membership respectively of the element $x$ in $\widetilde{A}^N$ as a real standard or real non-standard subsets of $]0^-, 1^+[$.

**Definition 2.3 [13]** A set $\widetilde{A}^N = \{\langle x, T_{\widetilde{A}^N}(x), I_{\widetilde{A}^N}(x), F_{\widetilde{A}^N}(x)\rangle | x \in X,\ 0 \le T_{\widetilde{A}^N}(x) \le 1,\ 0 \le I_{\widetilde{A}^N}(x) \le 1,\ 0 \le F_{\widetilde{A}^N} \le 1,\ T_{\widetilde{A}^N}(x) + I_{\widetilde{A}^N}(x) + F_{\widetilde{A}^N}(x) \le 3\}$, defined on the universal set $X$, is said to be an SVNS, where, $T_{\widetilde{A}^N}(x), I_{\widetilde{A}^N}(x)$, and $F_{\widetilde{A}^N}(x)$ represents the degree of truth-membership, the degree of indeterminacy-membership and degree of falsity-membership respectively of the element $x$ in $\widetilde{A}^N$. For convenience, we may write the single-valued neutrosophic number (SVNN) as $\widetilde{A}^N = \langle \alpha, \beta, \gamma\rangle$.

**Definition 2.4 [14]** A set $\widetilde{A}^N = \{\langle x, [T^L_{\widetilde{A}^N}(x), T^U_{\widetilde{A}^N}(x)], [I^L_{\widetilde{A}^N}(x), I^U_{\widetilde{A}^N}(x)], [F^L_{\widetilde{A}^N}(x), F^U_{\widetilde{A}^N}(x)]\rangle | x \in X,\ 0 \le T^L_{\widetilde{A}^N}(x) \le T^U_{\widetilde{A}^N}(x) \le 1,\ 0 \le I^L_{\widetilde{A}^N}(x) \le I^U_{\widetilde{A}^N}(x) \le 1,\ 0 \le F^L_{\widetilde{A}^N}(x) \le F^U_{\widetilde{A}^N}(x) \le 1,\ T^U_{\widetilde{A}^N}(x) + I^U_{\widetilde{A}^N}(x) + F^U_{\widetilde{A}^N}(x) \le 3\}$, defined on the universal set $X$, is said to be an IVNS, where, $[T^L_{\widetilde{A}^N}(x), T^U_{\widetilde{A}^N}(x)]$, $[I^L_{\widetilde{A}^N}(x), I^U_{\widetilde{A}^N}(x)]$ and $[F^L_{\widetilde{A}^N}(x), F^U_{\widetilde{A}^N}(x)]$ represents the intervals of the degree of truth-membership, the degree of indeterminacy-membership and the degree of falsity-membership respectively of the element $x$ in $\widetilde{A}^N$. For convenience, we may write interval-valued neutrosophic number (IVNN) as $\widetilde{A}^N = \langle [\alpha_1, \alpha_2)], [\beta_1, \beta_2], [\gamma_1, \gamma_2]\rangle$.

**Definition 2.5 [67]** Average operator for SVNSs:

Since $\widetilde{A}^N = \{\langle x, T_{\widetilde{A}^N}(x),\ I_{\widetilde{A}^N}(x), F_{\widetilde{A}^N}(x)\rangle | x \in X\}$ is an SVNS, then let $\widetilde{A}^N_k (k = 1,2, \dots, n)$ be n numbers of SVNSs.

**(i)** Weighted arithmetic average operator (WAM) for SVNSs is defined as

$$AO_{WA} = (\widetilde{A}^N_1, \widetilde{A}^N_2, \dots, \widetilde{A}^N_n) = \sum_{k=1}^n w_k \widetilde{A}^N_k$$

$$= \left(1 - \prod_{k=1}^n \left(1 - T_{\widetilde{A}^N_k}(x)\right)^{w_k}, \prod_{k=1}^n \left(I_{\widetilde{A}^N_k}(x)\right)^{w_k}, \prod_{k=1}^n \left(F_{\widetilde{A}^N_k}(x)\right)^{w_k}\right) \quad (1)$$

where $w_k$ denotes the weight vector of SVNSs $\widetilde{A}^N_k (k = 1,2, \dots, n)$ and satisfies the conditions such that $w_k \in [0,1]$ and $\sum_{k=1}^n w_k = 1$.

**(ii)** Weighted geometric average operator (WGM) for SVNSs is defined as

$$AO_{WG} = (\widetilde{A}^N_1, \widetilde{A}^N_2, \dots, \widetilde{A}^N_n) = \prod_{k=1}^n w_k \widetilde{A}^N_k$$

$$= \left(\prod_{k=1}^n \left(T_{\widetilde{A}^N_k}(x)\right)^{w_k}, 1 - \prod_{k=1}^n \left(1 - I_{\widetilde{A}^N_k}(x)\right)^{w_k}, 1 - \prod_{k=1}^n \left(1 - F_{\widetilde{A}^N_k}(x)\right)^{w_k}\right) \quad (2)$$

where $w_k$ denotes the weight vector of SVNSs $\widetilde{A}_k^N (k = 1,2, \dots, n)$ and satisfies the conditions such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$.

**Definition 2.6 [67]** Average operator for IVNSs:

Since $\widetilde{A}^N = \{\langle x, [T_{\widetilde{A}^N}^L(x), T_{\widetilde{A}^N}^U(x)], [I_{\widetilde{A}^N}^L(x), I_{\widetilde{A}^N}^U(x)], [F_{\widetilde{A}^N}^L(x), F_{\widetilde{A}^N}^U(x)]\rangle | x \in X\}$ is an IVNS, then let $\widetilde{A}_k^N (k = 1,2, \dots, n)$ be $n$ numbers of IVNSs.

**(i)** Weighted arithmetic average operator (WAM) for IVNSs is defined as

$$AO_{WA} = (\widetilde{A}_1^N, \widetilde{A}_2^N, \dots, \widetilde{A}_n^N) = \sum_{k=1}^{n} w_k \widetilde{A}_k^N = \left(\left[1 - \prod_{k=1}^{n}\left(1 - T_{\widetilde{A}^N}^L(x)\right)^{w_k},\ 1 - \prod_{k=1}^{n}\left(1 - T_{\widetilde{A}^N}^U(x)\right)^{w_k}\right],\ \left[\prod_{k=1}^{n}\left(I_{\widetilde{A}^N}^L(x)\right)^{w_k},\ \prod_{k=1}^{n}\left(I_{\widetilde{A}^N}^U(x)\right)^{w_k}\right],\ \left[\prod_{k=1}^{n}\left(F_{\widetilde{A}^N}^L(x)\right)^{w_k}, \prod_{k=1}^{n}\left(F_{\widetilde{A}^N}^U(x)\right)^{w_k}\right]\right) \tag{3}$$

where $w_k$ denotes the weight vector of IVNSs $\widetilde{A}_k^N (k = 1,2, \dots, n)$ and satisfies the conditions such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$.

**(ii)** Weighted geometric average operator (WGM) for IVNSs is defined as

$$AO_{WG} = (\widetilde{A}_1^N, \widetilde{A}_2^N, \dots, \widetilde{A}_n^N) = \prod_{k=1}^{n} w_k \widetilde{A}_k^N$$

$$= \left(\left[\prod_{k=1}^{n}\left(T_{\widetilde{A}^N}^L(x)\right)^{w_k}, \prod_{k=1}^{n}\left(T_{\widetilde{A}^N}^U(x)\right)^{w_k}\right],\ \left[1 - \prod_{k=1}^{n}\left(1 - I_{\widetilde{A}^N}^L(x)\right)^{w_k}, 1 - \prod_{k=1}^{n}\left(1 - I_{\widetilde{A}^N}^U(x)\right)^{w_k}\right], \left[1 - \prod_{k=1}^{n}\left(1 - F_{\widetilde{A}^N}^L(x)\right)^{w_k}, 1 - \prod_{k=1}^{n}\left(1 - F_{\widetilde{A}^N}^U(x)\right)^{w_k}\right]\right) \tag{4}$$

where $w_k$ denotes the weight vector of IVNSs $\widetilde{A}_k^N (k = 1,2, \dots, n)$ and satisfies the conditions such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$.

SF and AF are defined as a metric method for ranking SVNSs and IVNSs which clearly and precisely order the available alternatives and helps in choosing the best alternative among all the present alternatives.

**Definition 2.7** SF and AF for ranking SVNS

Let $\widetilde{A}^N = \{\langle x, T_{\widetilde{A}^N}(x), I_{\widetilde{A}^N}(x), F_{\widetilde{A}^N}(x)\rangle | x \in X\}$ be an SVNS, then SF and AF for SVNS is defined as,

**(i)** Existing SVNS SF [67] is defined as

$$\sigma_S(\widetilde{A}^N) = \frac{1 + T_{\widetilde{A}^N} - 2I_{\widetilde{A}^N} - F_{\widetilde{A}^N}}{2}, \quad \text{where } \sigma_S(\widetilde{A}^N) \in [0,1]. \tag{5}$$

**(ii)** Existing SVNS AF [67] is defined as

$$\sigma_A(\widetilde{A}^N) = T_{\widetilde{A}^N} - I_{\widetilde{A}^N}(1 - T_{\widetilde{A}^N}) - F_{\widetilde{A}^N}(1 - I_{\widetilde{A}^N}), \quad \text{where } \sigma_A(\widetilde{A}^N) \in [-1,1]. \tag{6}$$

**(iii)** Existing SVNS SF [68] is defined as

$$\tau_S(\widetilde{A}^N) = \frac{1 + (T_{\widetilde{A}^N} - 2I_{\widetilde{A}^N} - F_{\widetilde{A}^N})(2 - T_{\widetilde{A}^N} - F_{\widetilde{A}^N})}{2}, \quad \text{where } \tau_S(\widetilde{A}^N) \in [0,1]. \tag{7}$$

**(iv)** Existing SVNS AF [68] is defined as

$$\tau_A(\widetilde{A}^N) = T_{\widetilde{A}^N} - 2I_{\widetilde{A}^N} - F_{\widetilde{A}^N}, \quad \text{where } \tau_A(\widetilde{A}^N) \in [-1,1]. \tag{8}$$

**Definition 2.8** SF and AF for ranking IVNS

Let $\widetilde{A}^N = \{\langle x, [T_{\widetilde{A}^N}^L(x), T_{\widetilde{A}^N}^U(x)], [I_{\widetilde{A}^N}^L(x), I_{\widetilde{A}^N}^U(x)], [F_{\widetilde{A}^N}^L(x), F_{\widetilde{A}^N}^U(x)]\rangle | x \in X\}$ be an IVNS, then SF and AF for IVNS is defined as,

**(i)** Existing IVNS SF [67] is defined as

$$\chi_S(\tilde{A}^N) = \frac{2+T^L_{\tilde{A}^N}+T^U_{\tilde{A}^N}-2I^L_{\tilde{A}^N}-2I^U_{\tilde{A}^N}-F^L_{\tilde{A}^N}-F^U_{\tilde{A}^N}}{4}, \quad \text{where } \chi_S(\tilde{A}^N) \in [0,1]. \quad (9)$$

**(ii)** Existing IVNS AF [67] is defined as

$$\chi_A(\tilde{A}^N) = \frac{1}{2}\left(T^L_{\tilde{A}^N} + T^U_{\tilde{A}^N} - I^U_{\tilde{A}^N}(1 - T^U_{\tilde{A}^N}) - I^L_{\tilde{A}^N}(1 - T^L_{\tilde{A}^N}) - F^U_{\tilde{A}^N}(1 - I^L_{\tilde{A}^N}) - F^L_{\tilde{A}^N}(1 - I^U_{\tilde{A}^N})\right),$$

where $\chi_A(\tilde{A}^N) \in [-1,1]$. (10)

**(iii)** Existing IVNS SF [68] is defined as

$$\psi_S(\tilde{A}^N) = \frac{4+\left(T^L_{\tilde{A}^N}+T^U_{\tilde{A}^N}-2I^L_{\tilde{A}^N}-2I^U_{\tilde{A}^N}-F^L_{\tilde{A}^N}-F^U_{\tilde{A}^N}\right)\left(4-T^L_{\tilde{A}^N}-T^U_{\tilde{A}^N}-F^L_{\tilde{A}^N}-F^U_{\tilde{A}^N}\right)}{8}, \text{ where } \psi_S(\tilde{A}^N) \in [0,1]. \quad (11)$$

## 3. Proposed SF and AF for SVNSs and IVNSs

This section of the paper suggests a new SF and AF to obtain the correct ranking order of all the available alternatives of SVNSs and IVNSs and helps to choose the best alternative among all.

### 3.1 Proposed SF and AF for SVNSs

#### 3.1.1 Proposed SF for SVNSs

Let $\tilde{A}^N = \{\langle x, T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x), F_{\tilde{A}^N}(x)\rangle | x \in X\}$ be an SVNS, then an SF in terms of the degree of truth-membership, the degree of indeterminacy-membership and the degree of falsity-membership respectively for SVNS, is defined by:

$$\varphi_S(\tilde{A}^N) = \frac{1+\left(T_{\tilde{A}^N}-2I_{\tilde{A}^N}-F_{\tilde{A}^N}\right)}{2\left(2-T_{\tilde{A}^N}-F_{\tilde{A}^N}\right)}, \quad where\ \varphi_S(\tilde{A}^N) \in [0,1]\ and\ T_{\tilde{A}^N} + F_{\tilde{A}^N} \neq 2. \quad (12)$$

Clearly, it is observed that if $T_{\tilde{A}^N} + F_{\tilde{A}^N} = 1$, then $\varphi_S(\tilde{A}^N) = \sigma_S(\tilde{A}^N)$, therefore $T_{\tilde{A}^N} + F_{\tilde{A}^N} \neq 1$.

To validate the claim of the proposed SF $(\text{Eq.}\,(12))$, some well-defined SVNNs are chosen and evaluated. Let us consider the following examples.

**Example 1.** Let $\tilde{A}^N_1 = \langle T_{\tilde{A}^N_1}(x), I_{\tilde{A}^N_1}(x), F_{\tilde{A}^N_1}(x)\rangle$, and $\tilde{A}^N_2 = \langle T_{\tilde{A}^N_2}(x), I_{\tilde{A}^N_2}(x), F_{\tilde{A}^N_2}(x)\rangle$ be any two SVNNs, then the desirable alternative is selected according to the obtained value of SF using Eq. (12) among $\tilde{A}^N_1$ and $\tilde{A}^N_2$.

- **(i)** Let $\tilde{A}^N_1 = \langle 0.6, 0.3, 0.0\rangle$ and $\tilde{A}^N_2 = \langle 0.2, 0.1, 0.0\rangle$, then $\tilde{A}^N_1 > \tilde{A}^N_2$ in accordance with proposed SF $(\text{Eq.}\,(12))$.
- **(ii)** Let $\tilde{A}^N_1 = \langle 0.6,0.2,0.2\rangle$ and $\tilde{A}^N_2 = \langle 0.3,0.1,0.1\rangle$ , then $\tilde{A}^N_1 > \tilde{A}^N_2$ in accordance with proposed SF $(\text{Eq.}\,(12))$.
- **(iii)** Let $\tilde{A}^N_1 = \langle 0.1,0.0,0.1\rangle$ and $\tilde{A}^N_2 = \langle 0.3,0.0,0.3\rangle$ , then $\tilde{A}^N_2 > \tilde{A}^N_1$ in accordance with proposed SF $(\text{Eq.}\,(12))$.

For a deliberate comparison among various existing metric methods, for finding the correct ranking order of Example 1, a systematic tabular representation of the function values of various metric methods is presented in Table 1.

**Table 1. SF ($\varphi_S(\tilde{A}^N)$) values in comparison with various existing metric methods**

| SVNNs | $\sigma_S(\tilde{A}^N)$ | $\sigma_A(\tilde{A}^N)$ | $\tau_S(\tilde{A}^N)$ | $\tau_A(\tilde{A}^N)$ | $\varphi_S(\tilde{A}^N)$ |
|---|---|---|---|---|---|
| $\tilde{A}_1^N = \langle 0.6, 0.3, 0.0\rangle$ | 0.5 | 0.48 | 0.5 | 0.0 | 0.4545 |
| $\tilde{A}_2^N = \langle 0.2, 0.1, 0.0\rangle$ | 0.5 | 0.12 | 0.5 | 0.0 | 0.2777 |
| | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{\boldsymbol{A}}_1^N > \tilde{\boldsymbol{A}}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{\boldsymbol{A}}_1^N > \tilde{\boldsymbol{A}}_2^N$ |
| $\tilde{A}_1^N = \langle 0.6, 0.2, 0.2\rangle$ | 0.5 | 0.36 | 0.5 | 0.0 | 0.4166 |
| $\tilde{A}_2^N = \langle 0.3, 0.1, 0.1\rangle$ | 0.5 | 0.14 | 0.5 | 0.0 | 0.3125 |
| | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{\boldsymbol{A}}_1^N > \tilde{\boldsymbol{A}}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{\boldsymbol{A}}_1^N > \tilde{\boldsymbol{A}}_2^N$ |
| $\tilde{A}_1^N = \langle 0.1, 0.0, 0.1\rangle$ | 0.5 | 0.0 | 0.5 | 0.0 | 0.2777 |
| $\tilde{A}_2^N = \langle 0.3, 0.0, 0.1\rangle$ | 0.5 | 0.0 | 0.5 | 0.0 | 0.3571 |
| (Adopted from [61]) | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{\boldsymbol{A}}_1^N = \tilde{\boldsymbol{A}}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{\boldsymbol{A}}_2^N > \tilde{\boldsymbol{A}}_1^N$ |

From Table 1, it is expressed that there may exist several ranking methods for SVNNs which can rank the alternatives, $\tilde{A}_1^N$, $\tilde{A}_2^N$, and suggest which of the alternative is better among both. It has been observed that sometimes, the existing metric methods [67, 68] may or may not fail to rank, but the proposed SF $(\text{Eq.}(12))$ is providing desirable results respectively. Hence, it claims the validity of the proposed SF $(\text{Eq.}(12))$, stating that it is reasonable.

**3.1.2 Proposed AF for SVNSs**

It is observed that there may exist several SVNNs where, $T_{\tilde{A}^N} + F_{\tilde{A}^N} = 1$, then sometimes the proposed SF, Eq. (12) may or may not be able to rank the SVNNs desirably. Some of SVNNs $\tilde{A}_1^N$ and $\tilde{A}_2^N$, exhibiting such nature are considered as follows:

**Example 2.** Let $\tilde{A}_1^N = \langle T_{\tilde{A}_1^N}(x), I_{\tilde{A}_1^N}(x), F_{\tilde{A}_1^N}(x)\rangle$, and $\tilde{A}_2^N = \langle T_{\tilde{A}_2^N}(x), I_{\tilde{A}_2^N}(x), F_{\tilde{A}_2^N}(x)\rangle$ be any two SVNNs, then the desirable alternative is selected according to the obtained value of SF using Eq. (12) among $\tilde{A}_1^N$ and $\tilde{A}_2^N$.

**(i)** Let $\tilde{A}_1^N = \langle 0.6, 0.1, 0.4\rangle$ and $\tilde{A}_2^N = \langle 0.8, 0.3, 0.2\rangle$, then $\tilde{A}_1^N = \tilde{A}_2^N$ in accordance with proposed SF $(\text{Eq.}(12))$. While it is obvious that $\mathrm{A}_1 \neq \mathrm{A}_2$.

**(ii)** Let $\tilde{A}_1^N = \langle 0.9,0.4,0.1\rangle$ and $\tilde{A}_2^N = \langle 0.7,0.2,0.3\rangle$, then $\tilde{A}_1^N = \tilde{A}_2^N$ in accordance with proposed SF $(\text{Eq.}(12))$. While it is obvious that $A_1 \neq A_2$.

For a deliberate comparison among various existing metric methods, for finding the correct ranking order of Example 2, a systematic tabular representation of the function values of various metric methods is presented below in Table 2.

**Table 2. SF ($\varphi_S(\tilde{A}^N)$) value of some special SVNNs using various existing methods**

| SVNNs | $\sigma_S(\tilde{A}^N)$ | $\tau_S(\tilde{A}^N)$ | $\tau_A(\tilde{A}^N)$ | $\varphi_S(\tilde{A}^N)$ |
|---|---|---|---|---|
| $\tilde{A}_1^N = \langle 0.6, 0.1, 0.4\rangle$ | 0.5 | 0.5 | 0.0 | 0.5 |
| $\tilde{A}_2^N = \langle 0.8, 0.3, 0.2\rangle$ | 0.5 | 0.5 | 0.0 | 0.5 |
| | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{\boldsymbol{A}}_1^N = \tilde{\boldsymbol{A}}_2^N$ |
| $\tilde{A}_1^N = \langle 0.9,0.4,0.1\rangle$ | 0.5 | 0.5 | 0.0 | 0.5 |
| $\tilde{A}_2^N = \langle 0.7, 0.2, 0.3\rangle$ | 0.5 | 0.5 | 0.0 | 0.5 |
| | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{\boldsymbol{A}}_1^N = \tilde{\boldsymbol{A}}_2^N$ |

From Table 2, it is observed that if there exist SVNNs where, $T_{\tilde{A}^N} + F_{\tilde{A}^N} = 1$, then sometimes may or may not the proposed SF, Eq. (12) lacks in providing a desirable solution.

Thus, to overcome this restriction, there is a need to find a new function that would be helpful to rank such alternatives $\tilde{A}_1^N$ and $\tilde{A}_2^N$ appropriately. Hence, a novel AF is proposed as follows:

$$\varphi_A(\tilde{A}^N) = 1 - I_{\tilde{A}^N} - 2F_{\tilde{A}^N}, \quad (13)$$

*where* $\varphi_A(\tilde{A}^N) \in [-1, 1]$, $T_{\tilde{A}^N} + F_{\tilde{A}^N} = 1$, and $I_{\tilde{A}^N} \neq 0$.

To validate the claim of the proposed AF $(\text{Eq.}(13))$, Example 2 is considered again and evaluated using the proposed AF $(\text{Eq.}(13))$ as follows:

**(i)** For Example 2 (i), where $\tilde{A}_1^N = \langle 0.6, 0.1, 0.4\rangle$ and $\tilde{A}_2^N = \langle 0.8, 0.3, 0.2\rangle$, then $\tilde{A}_2^N > \tilde{A}_1^N$ in accordance with the proposed AF $(\text{Eq.}(13))$.

**(ii)** For Example 2 (ii), where $\tilde{A}_1^N = \langle 0.9, 0.4, 0.1\rangle$ and $\tilde{A}_2^N = \langle 0.7, 0.2, 0.3\rangle$, then $\tilde{A}_1^N > \tilde{A}_2^N$ in accordance with proposed AF $(\text{Eq.}(13))$.

For a deliberate comparison among various existing metric methods for finding the correct ranking order of Example 2, a systematic tabular representation is presented in Table 3.

**Table 3. AF ($\varphi_A(\tilde{A}^N)$) values in comparison with various metric methods**

| SVNNs | $\sigma_S(\tilde{A}^N)$ | $\tau_S(\tilde{A}^N)$ | $\tau_A(\tilde{A}^N)$ | $\varphi_S(\tilde{A}^N)$ | $\varphi_A(\tilde{A}^N)$ | $\sigma_A(\tilde{A}^N)$ |
|---|---|---|---|---|---|---|
| $\tilde{A}_1^N = \langle 0.6,0.1,0.4\rangle$ | 0.5 | 0.5 | 0.0 | 0.5 | 0.1 | 0.2 |
| $\tilde{A}_2^N = \langle 0.8,0.3,0.2\rangle$ | 0.5 | 0.5 | 0.0 | 0.5 | 0.3 | 0.6 |
| | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\boldsymbol{\tilde{A}_2^N > \tilde{A}_1^N}$ | $\boldsymbol{\tilde{A}_2^N > \tilde{A}_1^N}$ |
| $\tilde{A}_1^N = \langle 0.9,0.4,0.1\rangle$ | 0.5 | 0.5 | 0.0 | 0.5 | 0.4 | 0.8 |
| $\tilde{A}_2^N = \langle 0.7,0.2,0.3\rangle$ | 0.5 | 0.5 | 0.0 | 0.5 | 0.2 | 0.4 |
| | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ | $\boldsymbol{\tilde{A}_1^N > \tilde{A}_2^N}$ | $\boldsymbol{\tilde{A}_1^N > \tilde{A}_2^N}$ |

From Table 3, it is expressed that if there exist some SVNNs, exhibiting some peculiar behavior, then it can be ranked accordingly by using Eq. (13), and hence it produces desirable results and helps in selecting a better alternative among available SVNNs. Hence, it claims the validity of the proposed AF $(\text{Eq.}(13))$, stating that it is reasonable.

Furthermore, it is observed that, if any of the necessary condition of AF $(\text{Eq.}(13))$ is violated i.e.,

**(i)** $T_{\tilde{A}^N} + F_{\tilde{A}^N} = 1$, or

**(ii)** $I_{\tilde{A}^N} \neq 0$,

then the proposed AF $(\text{Eq.}(13))$, may or may not give a desirable result. Let us consider the following examples:

**Example 3.** Let $\tilde{A}_1^N = \langle T_{\tilde{A}_1^N}(x), I_{\tilde{A}_1^N}(x), F_{\tilde{A}_1^N}(x)\rangle$, and $\tilde{A}_2^N = \langle T_{\tilde{A}_2^N}(x), I_{\tilde{A}_2^N}(x), F_{\tilde{A}_2^N}(x)\rangle$ be any two SVNNs, then the desirable alternative is selected according to the obtained value of AF using Eq. (13) among these two SVNNs $\tilde{A}_1^N$ and $\tilde{A}_2^N$.

**(i)** Let $\tilde{A}_1^N = \langle 0.1,0.0,0.9\rangle$ and $\tilde{A}_2^N = \langle 0.5,0.0,0.9\rangle$, then $\tilde{A}_1^N = \tilde{A}_2^N = -0.8$ in accordance with proposed AF $(\text{Eq.}(13))$. While, it is obvious that $A_1 \neq A_2$, but we can see that the necessary conditions of AF, Eq. (13) are violated.

**(ii)** Let $\tilde{A}_1^N = \langle 0.1,0.0,0.4\rangle$ and $\tilde{A}_2^N = \langle 0.7,0.2,0.3\rangle$, then $\tilde{A}_1^N = \tilde{A}_2^N = 0.2$ in accordance with proposed AF $(\text{Eq.}\,(13))$. While, it is obvious that $A_1 \neq A_2$, but we can see that the necessary conditions of AF, Eq. (13) are violated.

Therefore, if we are using AF, Eq. (13), to obtain a reasonable solution then the must condition of AF, Eq. (13) should be necessarily followed. To validate this claim and to understand more precisely, let us consider some other example as follows:

**Example 4.** Let $\tilde{A}_1^N = \langle T_{\tilde{A}_1^N}(x), I_{\tilde{A}_1^N}(x), F_{\tilde{A}_1^N}(x)\rangle$, and $\tilde{A}_2^N = \langle T_{\tilde{A}_2^N}(x), I_{\tilde{A}_2^N}(x), F_{\tilde{A}_2^N}(x)\rangle$ be any two SVNNs, then the desirable alternative is selected according to the obtained value of AF using Eq. (13) among these two SVNNs $\tilde{A}_1^N$ and $\tilde{A}_2^N$.

**(i)** Let $\tilde{A}_1^N = \langle 0.7, 0.2, 0.3\rangle$ and $\tilde{A}_2^N = \langle 0.6, 0.0, 0.4\rangle$ be any two SVNNs, then on applying AF Eq. (13), the obtained values of $\tilde{A}_1^N$ and $\tilde{A}_2^N$ are $\tilde{A}_1^N = \tilde{A}_2^N = 0.2$ respectively.

**(ii)** Let $\tilde{A}_1^N = \langle 0.7, 0.2, 0.3\rangle$ and $\tilde{A}_2^N = \langle 0.6, 0.000001, 0.4\rangle$ be any two SVNNs, then on applying AF Eq. (13), the obtained values of $\tilde{A}_1^N = 0.2$ and $\tilde{A}_2^N = 0.1999999$ respectively.

Thus, we observe that in SVNN $\tilde{A}_2^N = \langle 0.6, 0.0, 0.4\rangle$, of Example 4 (i), $I_{\tilde{A}_2^N}(x) = 0.0$, the must condition of AF Eq. (13) i.e., $I_{\tilde{A}_2^N} \neq 0$, is violated, whereas in, SVNN $\tilde{A}_2^N = \langle 0.6, 0.000001, 0.4\rangle$ of Example 4 (ii), $I_{\tilde{A}_2^N}$ is nearly zero but is strictly not zero, i.e., $I_{\tilde{A}_2^N} = 0.0000001 \neq 0$, therefore, when AF Eq. (13) is applied on $\tilde{A}_2^N = \langle 0.6, 0.0, 0.4\rangle$ of Example 4 (i), the obtained value of $\tilde{A}_2^N$ is $\tilde{A}_2^N = 0.2$, and when AF Eq. (13) is applied on $\tilde{A}_2^N = \langle 0.6, 0.000001, 0.4\rangle$ of Example 4 (ii), the obtained value of $\tilde{A}_2^N$ is $\tilde{A}_2^N = 0.1999999$. Hence, we conclude that in Example 4 (i) alternatives $\tilde{A}_1^N$ and $\tilde{A}_2^N$ are equal i.e., $\tilde{A}_1^N = \tilde{A}_2^N = 0.2$ while in Example 4 (ii) alternatives $\tilde{A}_1^N$ is greater than $\tilde{A}_2^N$ i.e., $\tilde{A}_1^N > \tilde{A}_2^N$.

Therefore, we conclude from the above Example 3 and Example 4 that there may or may not exist several such SVNNs violating the must condition of the proposed AF, then Eq. (13) may or may not give an appropriate result. To handle such cases where the must condition for AF, Eq. (13) are not satisfied, then we can find the solution of such SVNNs from the AF, Eq. (6) of the literature [67] which can handle such special SVNNs in a better way.

Hence, it is claimed that the proposed AF Eq. (13) is simple but has restrictions in handling some special SVNNs, then such SVNNs can be ranked more appropriately by using the various other existing metric methods [67, 68], and the proposed SF Eq. (12) respectively. To validate the claim, Example 3 has been evaluated by using various other existing metric methods [67, 68], the proposed SF Eq. (12), and we conclude that in both the SVNSs, i.e., Example 3(i) and Example 3(ii), the desirable solution is $\tilde{A}_2^N$.

Thus, the detailed comparative analysis of Example 3 has been made using various other existing metric methods [67, 68] and the proposed SF Eq. (12), as shown below in Table 4.

**Table 4. SF and AF values of various existing metric methods**

| SVNNs | $\sigma_S(\tilde{A}^N)$ | $\sigma_A(\tilde{A}^N)$ | $\tau_S(\tilde{A}^N)$ | $\tau_A(\tilde{A}^N)$ | $\varphi_S(\tilde{A}^N)$ | $\varphi_A(\tilde{A}^N)$ |
|---|---|---|---|---|---|---|
| $\tilde{A}_1^N = \langle 0.1,0.0,0.9\rangle$ | 0.1 | −0.8 | 0.1 | −0.8 | 0.1 | −0.8 |
| $\tilde{A}_2^N = \langle 0.5,0.0,0.9\rangle$ | 0.3 | −0.4 | 0.38 | −0.4 | 0.5 | −0.8 |
| | $\tilde{A}_2^N > \tilde{A}_1^N$ | $\tilde{A}_2^N > \tilde{A}_1^N$ | $\tilde{A}_2^N > \tilde{A}_1^N$ | $\tilde{A}_2^N > \tilde{A}_1^N$ | $\tilde{A}_2^N > \tilde{A}_1^N$ | $\tilde{A}_1^N = \tilde{A}_2^N$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| $\tilde{A}_1^N = \langle 0.1,0.0,0.4\rangle$<br>$\tilde{A}_2^N = \langle 0.7,0.2,0.3\rangle$ | 0.35<br>0.5<br>$\tilde{A}_2^N > \tilde{A}_1^N$ | −0.3<br>0.4<br>$\tilde{A}_2^N > \tilde{A}_1^N$ | 0.2750<br>0.5<br>$\tilde{A}_2^N > \tilde{A}_1^N$ | −0.3<br>0.0<br>$\tilde{A}_2^N > \tilde{A}_1^N$ | 0.2333<br>0.5<br>$\tilde{A}_2^N > \tilde{A}_1^N$ | 0.2<br>0.2<br>$\tilde{A}_2^N = \tilde{A}_1^N$ |

Hence, based on the existing metric methods [67, 68] for comparing any two SVNSs $\tilde{A}_1^N = \langle T_{\tilde{A}_1^N}(x), I_{\tilde{A}_1^N}(x), F_{\tilde{A}_1^N}(x)\rangle$ and $\tilde{A}_2^N = \langle T_{\tilde{A}_2^N}(x), I_{\tilde{A}_2^N}(x), F_{\tilde{A}_2^N}(x)\rangle$ using SF $\varphi_S(\tilde{A}^N)$ and AF $\varphi_A(\tilde{A}^N)$, a comparison method can be defined as follows:

- If $\varphi_S(\tilde{A}_1^N) > \varphi_S(\tilde{A}_2^N)$ then $\tilde{A}_1^N > \tilde{A}_2^N$.
- If $\varphi_S(\tilde{A}_1^N) < \varphi_S(\tilde{A}_2^N)$ then $\tilde{A}_1^N < \tilde{A}_2^N$.
- If $\varphi_S(\tilde{A}_1^N) = \varphi_S(\tilde{A}_2^N)$ then check $\varphi_A(\tilde{A}^N)$ in the next step.
  - ✓If $\varphi_A(\tilde{A}_1^N) > \varphi_A(\tilde{A}_2^N)$ then $\tilde{A}_1^N > \tilde{A}_2^N$.
  - ✓If $\varphi_A(\tilde{A}_1^N) < \varphi_A(\tilde{A}_2^N)$ then $\tilde{A}_1^N < \tilde{A}_2^N$.
  - ✓If $\varphi_A(\tilde{A}_1^N) = \varphi_A(\tilde{A}_2^N)$ implies $\tilde{A}_1^N = \tilde{A}_2^N$ for special SVNNs, then check $\sigma_A(\tilde{A}^N)$ in the next step.
    - If $\sigma_A(\tilde{A}_1^N) > \sigma_A(\tilde{A}_2^N)$ then $\tilde{A}_1^N > \tilde{A}_2^N$.
    - If $\sigma_A(\tilde{A}_1^N) < \sigma_A(\tilde{A}_2^N)$ then $\tilde{A}_1^N < \tilde{A}_2^N$.
    - If $\sigma_A(\tilde{A}_1^N) = \sigma_A(\tilde{A}_2^N)$ implies $\tilde{A}_1^N = \tilde{A}_2^N$.

Thus, the proposed SF, Eq. (12) and the proposed AF, Eq. (13) can handle most of the SVNNs concerning its conditions and hence, are helpful in the DM process in a far better manner, and can give answers where the existing methods were having trouble in deriving the conclusions.

To validate the claim of the proposed SF, Eq. (12) and the proposed AF, Eq. (13), a detailed analysis of its properties are presented as follows:

**Property 3.1.** For SVNS $\tilde{A}^N = \langle \tilde{T}^N(x), \tilde{I}^N(x), \tilde{F}^N(x)\rangle$ the value of the proposed SF $\varphi_S(\tilde{A}^N)$, Eq. (12) lies between [0,1] i.e., $\varphi_S(\tilde{A}^N) \in [0,1]$.

**Property 3.2.** For SVNS $\tilde{A}^N = \langle \tilde{T}^N(x), \tilde{I}^N(x), \tilde{F}^N(x)\rangle$ or SVNN $\tilde{A}^N = \langle \alpha, \beta, \gamma\rangle$ (for convenience), if $\alpha + \gamma = 2$, then on using proposed SF, Eq. (12) no conclusion can be drawn.

*Proof:* Let us consider an example given below:

Let $\tilde{A}^N = \langle 1, 0.7, 1\rangle$ be any SVNN, where $\alpha + \gamma = 2$, then from the proposed SF, Eq. (12), we have

$$\varphi_S(\tilde{A}^N) = \frac{1+(\alpha-2(\beta)-\gamma)}{2\,(2-(\alpha+\gamma))} = \frac{1+(1-2(0.7)-1)}{2\,(2-1-1)} = \frac{1-1.4}{0} = \infty.$$

Since $\varphi_S(\tilde{A}^N) \in [0,1]$, hence, no conclusion can be drawn. Thus, for any SVNN $\tilde{A}^N = \langle \alpha, \beta, \gamma\rangle$, SF $\varphi_S(\tilde{A}^N)$ holds if $\alpha + \gamma \neq 2$

**Property 3.3.** For SVNS $\tilde{A}^N = \langle \tilde{T}^N(x), \tilde{I}^N(x), \tilde{F}^N(x)\rangle$ or SVNN $\tilde{A}^N = \langle \alpha, \beta, \gamma\rangle$ (for convenience), if $\alpha + \gamma = 1$, then proposed SF, Eq. (12) reduces to SF, Eq. (5) i.e., $\varphi_S(\tilde{A}^N) = \sigma_S(\tilde{A}^N)$.

*Proof:* Let $\alpha + \gamma = 1$, then from the proposed SF, Eq. (12), we have

$$\varphi_S(\tilde{A}^N) = \frac{1+(\alpha-2(\beta)-\gamma)}{2\,(2-(\alpha+\gamma))} = \frac{1+(\alpha-2(\beta)-\gamma)}{2\,(2-1)} = \frac{1+(\alpha-2(\beta)-\gamma)}{2} = \sigma_S(\tilde{A}^N).$$

**Property 3.4.** For SVNS $\tilde{A}^N = \langle \tilde{T}^N(x), \tilde{I}^N(x), \tilde{F}^N(x)\rangle$ or SVNN $\tilde{A}^N = \langle \alpha, \beta, \gamma\rangle$ (for convenience), the proposed SF, Eq. (12) is having a relationship with existing SF, $\sigma_S(\tilde{A}^N)$, and existing AF, $\tau_A(\tilde{A}^N)$ as follows:

**(i)** $\varphi_S(\tilde{A}^N) = \frac{1+(\alpha-2(\beta)-\gamma)}{2\,(2-\alpha-\gamma)} = \frac{\sigma_S(\tilde{A}^N)}{(2-\alpha-\gamma)}$

**(ii)** $\varphi_S(\tilde{A}^N) = \frac{1+(\alpha-2(\beta)-\gamma)}{2\,(2-\alpha-\gamma)} = \frac{1+\tau_A(\tilde{A}^N)}{2\,(2-\alpha-\gamma)}$.

**Property 3.5.** *One property*: If SVNN $\tilde{A}^N = \langle 1,0,0\rangle$, then $\varphi_S(\tilde{A}^N) = 1$, i.e., the maximum value of SVNN $\tilde{A}^N$ is 1.

*Proof*: Let $\tilde{A}^N = \langle 1,0,0\rangle$ be any SVNN, then from Eq. (12), we have

$$\varphi_S(\tilde{A}^N) = \frac{1+(1-2(0)-0)}{2\,(2-1-0)} = 1.$$

**Property 3.6.** *Zero property*: If SVNN $\tilde{A}^N = \langle 0,0,1\rangle$, then $\varphi_S(\tilde{A}^N) = 0$, i.e., the minimum value of SVNN $\tilde{A}^N$ is 0.

*Proof*: Let $\tilde{A}^N = \langle 0,0,1\rangle$ be any SVNN, then from Eq. (12), we have

$$\varphi_S(\tilde{A}^N) = \frac{1+(0-2(0)-1)}{2\,(2-0-0)} = 0.$$

**Property 3.7.** For any subset of SVNS $\tilde{A}^N = \langle \tilde{T}^N(x), \tilde{I}^N(x), \tilde{F}^N(x)\rangle$ or $\tilde{A}^N = \langle \alpha, \beta, \gamma\rangle$ (for convenience), the value of $\varphi_S(\tilde{A}^N) = \alpha - \beta$, if $\alpha + \gamma = 1$.

*Proof*: Let $\tilde{A}^N = \langle \alpha, \beta, \gamma\rangle$ be any subset of SVNS and $\alpha + \gamma = 1$.

**(i)** Let $\tilde{A}^N = \langle \alpha, \beta, 1-\alpha\rangle$, then from Eq. (12), we have

$$\varphi_S(\tilde{A}^N) = \frac{1+\big(\alpha-2(\beta)-(1-\alpha)\big)}{2\,\big(2-\alpha-(1-\alpha)\big)} = \frac{1+(2\alpha-2\beta-1)}{2} = \alpha-\beta.$$

**(ii)** Let $\tilde{A}^N = \langle 1-\gamma, \beta, \gamma\rangle$, then from Eq. (12), we have

$$\varphi_S(\tilde{A}^N) = \frac{1+\big((1-\gamma)-2\beta-\gamma\big)}{2\,(2-(1-\gamma)-\gamma)} = \frac{1+(1-2\beta-2\gamma)}{2} = \frac{2(1-\beta-\gamma)}{2} = \alpha-\beta.$$

**Property 3.8.** For SVNS $\tilde{A}^N = \langle \tilde{T}^N(x), \tilde{I}^N(x), \tilde{F}^N(x)\rangle$, the value of the proposed AF $\varphi_A(\tilde{A}^N)$, Eq. (13) lies between $[-1,1]$ i.e., $\varphi_A(\tilde{A}^N) \in [-1,1]$, provided $\alpha + \gamma = 1$, and $\beta \neq 0$.

**Property 3.9.** For SVNS $\tilde{A}^N = \langle \tilde{T}^N(x), \tilde{I}^N(x), \tilde{F}^N(x)\rangle$ or $\tilde{A}^N = \langle \alpha, \beta, \gamma\rangle$ (for convenience), the proposed AF, $\varphi_A(\tilde{A}^N) = 1 - \beta - 2\gamma$, is having a relation with SF Eq. (5) and AF, Eq. (6) as follows: i.e., $\varphi_A(\tilde{A}^N) = 1 - (\sigma_A - \sigma_S) - 3\gamma$ provided $\alpha + \gamma = 1$, and $\beta \neq 0$.

*Proof*: Let $\tilde{A}^N = \langle \alpha, \beta, \gamma\rangle$ be any SVNS, and $\alpha + \gamma = 1$, $\beta \neq 0$, then we have

$$\varphi_A(\tilde{A}^N) = 1 - (\sigma_A - \sigma_S) - 3\gamma$$

$$= 1 - \left\{\big(\alpha - \beta(1-\alpha) - \gamma(1-\beta)\big) - \left(\frac{1+\alpha-2\beta-\gamma}{2}\right)\right\} - 3\gamma$$

$$= 1 - \left\{(\alpha - \beta + \alpha\beta - \gamma + \beta\gamma) - \left(\frac{1+1-\gamma-2\beta-\gamma}{2}\right)\right\} - 3\gamma$$

$$= 1 - \left\{(1-\gamma-\beta+(1-\gamma)\beta - \gamma + \beta\gamma) - \left(\frac{2-2\gamma-2\beta}{2}\right)\right\} - 3\gamma$$

$$= 1 - \{(1-\gamma-\beta+\beta-\beta\gamma-\gamma+\beta\gamma) - (1-\gamma-\beta)\} - 3\gamma$$

$$= 1 - (-\gamma+\beta) - 3\gamma$$

$= 1 - \beta - 2\gamma$.

### 3.2 Proposed SF for IVNSs

Let $\tilde{A}^N = \{\langle x, [T^L_{\tilde{A}^N}(x), T^U_{\tilde{A}^N}(x)], [I^L_{\tilde{A}^N}(x), I^U_{\tilde{A}^N}(x)], [F^L_{\tilde{A}^N}(x), F^U_{\tilde{A}^N}(x)]\rangle | x \in X\}$ be an IVNS, then a new SF in terms of the degree of truth-membership, the degree of indeterminacy-membership, and the degree of falsity-membership respectively for IVNS are defined by:

$$\omega_S(\tilde{A}^N) = \frac{2 + \left(T^L_{\tilde{A}^N} + T^U_{\tilde{A}^N} - 2I^L_{\tilde{A}^N} - 2I^U_{\tilde{A}^N} - F^L_{\tilde{A}^N} - F^U_{\tilde{A}^N}\right)}{2\left(4 - T^L_{\tilde{A}^N} - T^U_{\tilde{A}^N} - F^L_{\tilde{A}^N} - F^U_{\tilde{A}^N}\right)} \quad (14)$$

*where* $\omega_S(\tilde{A}^N) \in [0,1]$ *and* $T^L_{\tilde{A}^N} + T^U_{\tilde{A}^N} + F^L_{\tilde{A}^N} + F^U_{\tilde{A}^N} \neq 4$, as $0 \le \mathrm{T}^{\mathrm{L}}_{\tilde{A}^N}(\mathrm{x}) \le \mathrm{T}^{\mathrm{U}}_{\tilde{A}^N}(\mathrm{x}) \le 1$, $0 \le \mathrm{F}^{\mathrm{L}}_{\tilde{A}^N}(\mathrm{x}) \le \mathrm{F}^{\mathrm{U}}_{\tilde{A}^N}(\mathrm{x}) \le 1$.

Clearly, it is observed that if $T^L_{\tilde{A}^N} + T^U_{\tilde{A}^N} + F^L_{\tilde{A}^N} + F^U_{\tilde{A}^N} = 2$, then $\omega_S(\tilde{A}^N) = \chi_S(\tilde{A}^N)$.

To validate the claim of the proposed SF $(\text{Eq.}(14))$, some well-defined IVNNs are chosen and evaluated. Let us consider the following examples.

**Example 5.** Let $\tilde{A}^N_1 = \langle [T^L_{\tilde{A}^N_1}(x), T^U_{\tilde{A}^N_1}(x)], [I^L_{\tilde{A}^N_1}(x), I^U_{\tilde{A}^N_1}(x)], [F^L_{\tilde{A}^N_1}(x), F^U_{\tilde{A}^N_1}(x)]\rangle$ and $\tilde{A}^N_2 = \langle [T^L_{\tilde{A}^N_2}(x), T^U_{\tilde{A}^N_2}(x)], [I^L_{\tilde{A}^N_2}(x), I^U_{\tilde{A}^N_2}(x)], [F^L_{\tilde{A}^N_2}(x), F^U_{\tilde{A}^N_2}(x)]\rangle$ be any two IVNNs, then the desirable alternative is selected according to the obtained value of SF using Eq. (14) among $\tilde{A}^N_1$ and $\tilde{A}^N_2$.

**(i)** Let $\tilde{A}^N_1 = \langle[0.4,0.5],[0.1,0.2],[0.1,0.2]\rangle$ and $\tilde{A}^N_2 = \langle[0.48,0.52],[0.0,0.2],[0.2,0.4]\rangle$ then $\tilde{A}^N_2 > \tilde{A}^N_1$ in accordance with proposed SF $(\text{Eq.}(14))$.

**(ii)** Let $\tilde{A}^N_1 = \langle[0.4,0.6],[0.125,0.125],[0.1,0.4]\rangle$ and $\tilde{A}^N_2 = \langle[0.23,0.67],[0.1125,0.1125],[0.05,0.4]\rangle$ then $\tilde{A}^N_1 > \tilde{A}^N_2$ in accordance with proposed SF $(\text{Eq.}(14))$.

For a deliberate comparison among various existing metric methods, for finding the correct ranking order of Example 5, a systematic tabular representation of the function values of various metric methods is presented in Table 5.

**Table 5. SF $\left(\omega_S(\tilde{A}^N)\right)$ values in comparison with various existing metric methods**

| IVNNs | $\chi_S(\tilde{A}^N)$ | $\psi_S(\tilde{A}^N)$ | $\omega_S(\tilde{A}^N)$ | $\chi_A(\tilde{A}^N)$ |
|---|---|---|---|---|
| $\tilde{A}^N_1 = \langle[0.4,0.5],[0.1,0.2],[0.1,0.2]\rangle$ | 0.5 | 0.5 | 0.3571 | 0.24 |
| $\tilde{A}^N_2 = \langle[0.48,0.52],[0.0,0.2],[0.2,0.4]\rangle$ | 0.5 | 0.5 | 0.4167 | 0.27 |
| | $\tilde{A}^N_1 = \tilde{A}^N_2$ | $\tilde{A}^N_1 = \tilde{A}^N_2$ | $\boldsymbol{\tilde{A}^N_2 > \tilde{A}^N_1}$ | $\tilde{A}^N_2 > \tilde{A}^N_1$ |
| $\tilde{A}^N_1 = \langle[0.4,0.6],[0.125,0.125],[0.1,0.4]\rangle$ | 0.5 | 0.5 | 0.4 | 0.2188 |
| $\tilde{A}^N_2 = \langle[0.23,0.67],[0.1125,0.1125],[0.05,0.4]\rangle$ | 0.5 | 0.5 | 0.3774 | 0.1885 |
| | $\tilde{A}^N_1 = \tilde{A}^N_2$ | $\tilde{A}^N_1 = \tilde{A}^N_2$ | $\boldsymbol{\tilde{A}^N_1 > \tilde{A}^N_2}$ | $\tilde{A}^N_1 > \tilde{A}^N_2$ |

From Table 5, it is expressed that there may exist several ranking methods for IVNNs which can rank the alternatives, $\tilde{A}^N_1$, $\tilde{A}^N_2$, and suggest which of the alternative is better among both. It has been observed that sometimes, the existing metric methods [67, 68] may or may not fail to rank, but the proposed SF $(\text{Eq.}(14))$ is providing desirable results. Hence, it claims the validity of the proposed SF $(\text{Eq.}(14))$, stating that it is reasonable.

Also, it is observed that there may exist several IVNNs where, $T^L_{\tilde{A}^N} + T^U_{\tilde{A}^N} + F^L_{\tilde{A}^N} + F^U_{\tilde{A}^N} = 2$, then the proposed SF, Eq. (14) reduces to the existing SF (Eq. (9)) [67]. Some of IVNNs exhibiting such nature are considered as follows:

**Example 6.** Let $\tilde{A}^N = \langle[0.22, 0.78], [0.1, 0.3], [0.3, 0.7]\rangle$, then $\omega_S(\tilde{A}^N) = \chi_S(\tilde{A}^N) = 0.3000$ in accordance with obtained value of SF on using $(\text{Eq.}(14))$ and $(\text{Eq.}(9))$.

**Example 7.** Let $\tilde{A}^N = \langle[0.45, 0.55], [0.1, 0.2], [0.4, 0.6]\rangle$, then $\omega_S(\tilde{A}^N) = \chi_S(\tilde{A}^N) = 0.3500$ in accordance with obtained value of SF on using $(\text{Eq.}(14))$ and $(\text{Eq.}(9))$.

For a deliberate comparison among various existing metric methods, for finding the correct score value of Example 6 and Example 7, a systematic tabular representation of the function values of various metric methods is presented in Table 6.

**Table 6. Function values of some special IVNNs using various existing methods**

| **IVNNs** | $\chi_S(\tilde{A}^N)$ | $\psi_S(\tilde{A}^N)$ | $\omega_S(\tilde{A}^N)$ | $\chi_A(\tilde{A}^N)$ |
|---|---|---|---|---|
| $\tilde{A}^N = \langle[0.22, 0.78], [0.1, 0.3], [0.3, 0.7]\rangle$ | 0.3000 | 0.3000 | 0.3000 | 0.0080 |
| $\tilde{A}^N = \langle[0.45, 0.55], [0.1, 0.2], [0.4, 0.6]\rangle$ | 0.3500 | 0.3500 | 0.3500 | −0.0025 |

Also, it is observed that there may exist several IVNNs where, $T^L_{\tilde{A}^N} + T^U_{\tilde{A}^N} + F^L_{\tilde{A}^N} + F^U_{\tilde{A}^N} = 4$, then the proposed score functions Eq. (14), have its limitation. Let us consider an example of IVNNs exhibiting such nature as follows:

**Example 8.** Let $\tilde{A}^N_1 = \langle[1, 1], [0.2, 0.7], [1, 1]\rangle$ and $\tilde{A}^N_2 = \langle[1, 1], [0.5, 0.9], [1, 1]\rangle$ be any two IVNNs, then $\tilde{A}^N_1 = \tilde{A}^N_2 = \infty$ on using the proposed SF, Eq. (14), since it is violating the must condition for the proposed SF Eq. (14) hence, no conclusion can be drawn.

Furthermore, on analysis, it is observed that the existing AF $\chi_A(\tilde{A}^N)$, (Eq. (10) ) [67] is successful in giving a desirable solution for such IVNNs, where sometimes all the existing [67, 68] and the proposed SF i.e., Eq. (9), Eq. (11) and Eq. (14) may or may not be able to give an appropriate solution. To validate the claim, above stated Example 8 is evaluated using AF $\chi_A(\tilde{A}^N)$, (Eq. (10) ) [67] which states that, $\tilde{A}^N_2 > \tilde{A}^N_1$ i.e., $\tilde{A}^N_2$ is the best alternative among $\tilde{A}^N_2$ and $\tilde{A}^N_1$ as shown in Table 7.

For a deliberate comparison among various existing metric methods, for finding the correct ranking order of some special SVNN concerning an existing AF $\chi_A(\tilde{A}^N)$, (Eq. (10) ) [67], is presented in Table 7 as follows:

**Table 7. Function values of some special IVNNs using various existing metric methods**

| **IVNNs** | $\chi_S(\tilde{A}^N)$ | $\psi_S(\tilde{A}^N)$ | $\omega_S(\tilde{A}^N)$ | $\chi_A(\tilde{A}^N)$ |
|---|---|---|---|---|
| $\tilde{A}^N_1 = \langle[1, 1], [0.2, 0.7], [1, 1]\rangle$ | 0.05 | 0.5 | ∞ | −0.05 |
| $\tilde{A}^N_2 = \langle[1, 1], [0.5, 0.9], [1, 1]\rangle$ | −0.2 $(\alpha_S(\tilde{A}^N) \notin [0, 1])$ | 0.5 | ∞ | 0.7 |

Thus, from Table 7, it is concluded that there may or may not exists several such SVNNs for which sometimes all the existing [67, 68] and the proposed SF i.e., Eq. (9), Eq. (11) and Eq. (14) may not suggest an appropriate solution among $\tilde{A}_1^N$ and $\tilde{A}_2^N$ but the existing AF $\chi_A(\tilde{A}^N)$, (Eq. (10) ) [67] is successful in providing a satisfactory solution so far.

Hence, based on the existing metric methods [67, 68] for comparing any two IVNSs $\tilde{A}_1^N = \langle [T^L_{\tilde{A}_1^N}(x), T^U_{\tilde{A}_1^N}(x)], [I^L_{\tilde{A}_1^N}(x), I^U_{\tilde{A}_1^N}(x)], [F^L_{\tilde{A}_1^N}(x), F^U_{\tilde{A}_1^N}(x)] \rangle$ and $\tilde{A}_2^N = \langle [T^L_{\tilde{A}_2^N}(x), T^U_{\tilde{A}_2^N}(x)], [I^L_{\tilde{A}_2^N}(x), I^U_{\tilde{A}_2^N}(x)], [F^L_{\tilde{A}_2^N}(x), F^U_{\tilde{A}_2^N}(x)] \rangle$ using SF and AF, a comparison method can be defined as follows:

- If $\omega_S(\tilde{A}_1^N) > \omega_S(\tilde{A}_2^N)$ then $\tilde{A}_1^N > \tilde{A}_2^N$.
- If $\omega_S(\tilde{A}_1^N) < \omega_S(\tilde{A}_2^N)$ then $\tilde{A}_1^N < \tilde{A}_2^N$.
- If $\omega_S(\tilde{A}_1^N) = \omega_S(\tilde{A}_2^N)$ or no conclusion can be drawn, then check $\chi_A(\tilde{A}^N)$ in the next step.
    - If $\chi_A(\tilde{A}_1^N) > \chi_A(\tilde{A}_2^N)$ then $\tilde{A}_1^N > \tilde{A}_2^N$.
    - If $\chi_A(\tilde{A}_1^N) < \chi_A(\tilde{A}_2^N)$ then $\tilde{A}_1^N < \tilde{A}_2^N$.
    - If $\chi_A(\tilde{A}_1^N) = \chi_A(\tilde{A}_2^N)$ then $\tilde{A}_1^N = \tilde{A}_2^N$.

Thus, the proposed SF, Eq. (14) can handle most of the IVNNs along with its conditions and hence, is helpful in the DM process in a much better way, also can give answers where the existing methods were not leading the solution to anywhere.

To validate the claim of the proposed SF, Eq. (14), a detailed analysis of its properties are presented as follows:

**Property 3.10.** For IVNS $\tilde{A}^N = \langle [T^L_{\tilde{A}^N}(x), T^U_{\tilde{A}^N}(x)], [I^L_{\tilde{A}^N}(x), I^U_{\tilde{A}^N}(x)], [F^L_{\tilde{A}^N}(x), F^U_{\tilde{A}^N}(x)] \rangle$ the value of the proposed SF $\omega_S(\tilde{A}^N)$, Eq. (14) lies between [0,1] i.e., $\omega_S(\tilde{A}^N) \in [0,1]$.

**Property 3.11.** For IVNS $\tilde{A}^N = \langle [T^L_{\tilde{A}^N}(x), T^U_{\tilde{A}^N}(x)], [I^L_{\tilde{A}^N}(x), I^U_{\tilde{A}^N}(x)], [F^L_{\tilde{A}^N}(x), F^U_{\tilde{A}^N}(x)] \rangle$ or $\tilde{A}^N = \langle [\alpha_1, \alpha_2)], [\beta_1, \beta_2], [\gamma_1, \gamma_2] \rangle$ (for convenience), if $\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 = 4$, then on using proposed SF, Eq. (14) no conclusion can be drawn.

*Proof:* Let us consider an example given below:

Let $\tilde{A}^N = \langle\langle [1,1], [0.25, 0.571], [1,1] \rangle\rangle$ be any IVNN, where $\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 = 4$, then from the proposed SF, Eq. (14), we have

$$\omega_S(\tilde{A}^N) = \frac{2+(1+1-2(0.25)-2(0.571)-1-1)}{2\,(4-1-1-1-1)} = \frac{2+(-1.6420)}{0} = \frac{0.3580}{0} = \infty.$$

Since $\omega_S(\tilde{A}^N) \in [0,1]$, hence, no conclusion can be drawn. Thus, for any IVNN $\tilde{A}^N = \langle [\alpha_1, \alpha_2)], [\beta_1, \beta_2], [\gamma_1, \gamma_2] \rangle$, SF $\omega_S(\tilde{A}^N)$ holds if $\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 \neq 4$.

**Property 3.12.** For IVNS $\tilde{A}^N = \langle [T^L_{\tilde{A}^N}(x), T^U_{\tilde{A}^N}(x)], [I^L_{\tilde{A}^N}(x), I^U_{\tilde{A}^N}(x)], [F^L_{\tilde{A}^N}(x), F^U_{\tilde{A}^N}(x)] \rangle$ or $\tilde{A}^N = \langle [\alpha_1, \alpha_2)], [\beta_1, \beta_2], [\gamma_1, \gamma_2] \rangle$ (for convenience), if $\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 = 2$, , then proposed SF, Eq. (14) reduces to SF, Eq. (9) i.e., $\omega_S(\tilde{A}^N) = \chi_S(\tilde{A}^N)$.

*Proof:* Let $\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 = 2$, then from the proposed SF, Eq. (14), we have

$$\omega_S(\tilde{A}^N) = \frac{2+(\alpha_1+\alpha_2-2\beta_1-\beta_2-\gamma_1-\gamma_2)}{2\,(4-\alpha_1-\alpha_2-\gamma_1-\gamma_2)} = \frac{2+(\alpha_1+\alpha_2-2\beta_1-\beta_2-\gamma_1-\gamma_2)}{2\,(4-(\alpha_1+\alpha_2+\gamma_1+\gamma_2))} = \frac{2+(\alpha_1+\alpha_2-2\beta_1-\beta_2-\gamma_1-\gamma_2)}{2\,(4-2)}$$

$$= \frac{2+(\alpha_1+\alpha_2-2\beta_1-\beta_2-\gamma_1-\gamma_2)}{4}$$

$= \chi_S(\tilde{A}^N)$.

**Property 3.13.** For IVNS $\tilde{A}^N = \langle [T^L_{\tilde{A}^N}(x), T^U_{\tilde{A}^N}(x)], [I^L_{\tilde{A}^N}(x), I^U_{\tilde{A}^N}(x)], [F^L_{\tilde{A}^N}(x), F^U_{\tilde{A}^N}(x)] \rangle$ or $\tilde{A}^N = \langle [\alpha_1, \alpha_2)], [\beta_1, \beta_2], [\gamma_1, \gamma_2] \rangle$ (for convenience), the proposed SF $\omega_S(\tilde{A}^N)$, Eq. (14) is having a relation with SF $\chi_S(\tilde{A}^N)$, Eq. (9) as follows: i.e., $\omega_S(\tilde{A}^N) = \frac{\chi_S}{(4-\alpha_1-\alpha_2-\gamma_1-\gamma_2)}$.

**Property 3.14.** *One property*: If IVNN $\tilde{A}^N = \langle [1, 1], [0, 0], [0,0] \rangle$, then $\omega_S(\tilde{A}^N) = 1$, i.e., the maximum value of IVNN $\tilde{A}^N$ is 1.

*Proof:* Let $\tilde{A}^N = \langle [1, 1], [0, 0], [0,0] \rangle$ be any IVNN, then from Eq. (14), we have

$$\omega_S(\tilde{A}^N) = \frac{2+(1+1-0-0-0-0)}{2\,(4-1-1-0-0)} = \frac{4}{4} = 1.$$

**Property 3.15.** *Zero property*: If IVNN $\tilde{A}^N = \langle [0, 0], [0, 0], [1,1] \rangle$, then $\omega_S(\tilde{A}^N) = 0$, i.e., the minimum value of IVNN $\tilde{A}^N$ is 0.

*Proof:* Let $\tilde{A}^N = \langle [0, 0], [0, 0], [1,1] \rangle$ be any IVNN, then from Eq. (14), we have

$$\omega_S(\tilde{A}^N) = \frac{2+(0+0-0-0-1-1)}{2\,(4-0-0-1-1)} = \frac{0}{4} = 0.$$

## 4. MCDM method based on proposed SF and AF under neutrosophic environment

In this section MCDM method is proposed for both SVNSs and IVNSs using proposed SF and proposed AF, which is pictorially presented in Figure 1.

### 4.1. MCDM method based on proposed SF and AF under SVNSs

Let us consider an MCDM problem having $m$ number of alternatives i.e., $\tilde{A}^N = \{\tilde{A}^N_1, \tilde{A}^N_2, \ldots, \tilde{A}^N_m\}$ which are evaluated on $n$ number of criteria i.e., $\tilde{G}^N = \{\tilde{G}^N_1, \tilde{G}^N_2, \ldots, \tilde{G}^N_n\}$. Suppose that the weight allotted to each criterion by the decision-maker is $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ . Also, the characteristics of an alternatives $\tilde{A}^N_i (i = 1,2, \ldots, m)$ per criterion $\tilde{G}^N_j (j = 1,2, \ldots, n)$ can be represented by an SVNS i.e., $\tilde{A}^N_i = \left\{ \langle \tilde{G}^N_j, T_{\tilde{A}^N_i}(\tilde{G}^N_j), I_{\tilde{A}^N_i}(\tilde{G}^N_j), F_{\tilde{A}^N_i}(\tilde{G}^N_j) \rangle \,|\, \tilde{G}^N_j \in \tilde{G}^N \right\}$, where $T_{\tilde{A}^N_i}(\tilde{G}^N_j) + I_{\tilde{A}^N_i}(\tilde{G}^N_j) + F_{\tilde{A}^N_i}(\tilde{G}^N_j) \leq 3$ and $T_{\tilde{A}^N_i}(\tilde{G}^N_j) \geq 0$, $I_{\tilde{A}^N_i}(\tilde{G}^N_j) \geq 0$, $F_{\tilde{A}^N_i}(\tilde{G}^N_j) \geq 0$, for all $i = 1 \; to \; m$ and $j = 1 \; to \; n$, for convenience it is denoted as $\Psi_{ij} = \langle \alpha_{ij}, \beta_{ij}, \gamma_{ij} \rangle$. The single-valued neutrosophic decision matrix (SVNDM) derived from the collected single-valued neutrosophic data available for $m$ number of alternatives with respect to $n$ number of the criterion is represented as

$D = (\Psi_{ij})_{m \times n} = (\langle \alpha_{ij}, \beta_{ij}, \gamma_{ij} \rangle)_{m \times n}$ i.e.,

$$(\Psi_{ij})_{m \times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} G_1 & G_2 & & G_n \end{array} \\ \begin{bmatrix} \langle \alpha_{11}, \beta_{11}, \gamma_{11} \rangle & \langle \alpha_{12}, \beta_{12}, \gamma_{12} \rangle & \cdots & \langle \alpha_{1n}, \beta_{1n}, \gamma_{1n} \rangle \\ \langle \alpha_{21}, \beta_{21}, \gamma_{21} \rangle & \langle \alpha_{22}, \beta_{22}, \gamma_{22} \rangle & \cdots & \langle \alpha_{2n}, \beta_{2n}, \gamma_{2n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \alpha_{m1}, \beta_{m1}, \gamma_{m1} \rangle & \langle \alpha_{m2}, \beta_{m2}, \gamma_{m2} \rangle & \cdots & \langle \alpha_{mn}, \beta_{mn}, \gamma_{mn} \rangle \end{bmatrix} \end{array}.$$

For evaluating the MCDM problem for SVNSs we need a step-wise procedure which is summarized as follows:

**Step 1:** Check that all the criteria of the SVNDM, $D$ are of the same type or not.

**Case (i)** If all the criteria are of the same type then go to Step 2.

**Case (ii)** If some criteria are of benefit type and others are of cost types then normalize the SVNDM by transforming the cost criterion into benefit criterion, in the following manner: If the $p^{th}$ criterion is cost criterion then replace all the elements $\langle \alpha_{ip}, \beta_{ip}, \gamma_{ip} \rangle$ of the $p^{th}$ column of the decision matrix, $D$ with $\langle \gamma_{ip}, 1-\beta_{ip}, \alpha_{ip} \rangle$.

**Step 2:** Evaluate the SVNSs $\Psi_{ij}$ for each $\tilde{A}_i^N$ into an SVNN $\Psi_i$ using WAM, Eq. (1), or the WGM, Eq. (2).

**Step 3:** After aggregating (by applying either of the approaches i.e., WAM or WGM) according to Step 2, now obtain the crisp value of $\Psi_i$ $(i = 1,2,\dots,m)$ by using SF $\varphi_S(\tilde{A}^N)$, Eq. (12) or AF $\varphi_A(\tilde{A}^N)$, Eq. (13).

**Step 4:** After Step 3, rank all the alternatives as per the obtained value of $\varphi_S(\tilde{A}^N)$ or $\varphi_A(\tilde{A}^N)$ and choose the best alternative.

**4.2. MCDM method based on proposed SF and AF under IVNSs**

Let us consider an MCDM problem having $m$ number of alternatives i.e., $\tilde{A}^N = \{\tilde{A}_1^N, \tilde{A}_2^N, \dots, \tilde{A}_m^N\}$ which are evaluated on $n$ number of criteria i.e., $\tilde{G}^N = \{\tilde{G}_1^N, \tilde{G}_2^N, \dots, \tilde{G}_n^N\}$. Suppose that the weight allotted to each criterion by the decision-maker is $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. Also, the characteristics of an alternatives $\tilde{A}_i^N (i = 1,2,\dots,m)$ as per criterion $\tilde{G}_j^N (j = 1,2,\dots,n)$ can be represented by an IVNS i.e., $\tilde{A}_i^N = \left\{\langle \tilde{G}_j^N, \left[T_{\tilde{A}_i^N}^L(\tilde{G}_j^N), T_{\tilde{A}_i^N}^U(\tilde{G}_j^N)\right], \left[I_{\tilde{A}_i^N}^L(\tilde{G}_j^N), I_{\tilde{A}_i^N}^U(\tilde{G}_j^N)\right], \left[F_{\tilde{A}_i^N}^L(\tilde{G}_j^N), F_{\tilde{A}_i^N}^U(\tilde{G}_j^N)\right]\rangle \,|\, \tilde{G}_j^N \in \tilde{G}^N\right\}$, where $T_{\tilde{A}_i^N}^U(\tilde{G}_j^N) + I_{\tilde{A}_i^N}^U(\tilde{G}_j^N) + F_{\tilde{A}_i^N}^U(\tilde{G}_j^N) \leq 3$ and $0 \leq T_{\tilde{A}_i^N}^L(\tilde{G}_j^N) \leq T_{\tilde{A}_i^N}^U(\tilde{G}_j^N) \leq 1$, $0 \leq I_{\tilde{A}_i^N}^L(\tilde{G}_j^N) \leq I_{\tilde{A}_i^N}^U(\tilde{G}_j^N) \leq 1$, $0 \leq F_{\tilde{A}_i^N}^L(\tilde{G}_j^N) \leq F_{\tilde{A}_i^N}^U(\tilde{G}_j^N) \leq 1$, for all $i = 1 \text{ to } m$ and $j = 1 \text{ to } n$, for convenience it is denoted as $\Psi_{ij} = \langle \alpha_{ij}, \beta_{ij}, \gamma_{ij} \rangle = \langle [\alpha_{ij}^L, \alpha_{ij}^U], [\beta_{ij}^L, \beta_{ij}^U], [\gamma_{ij}^L, \gamma_{ij}^U] \rangle$. The interval-valued neutrosophic decision-matrix (IVNDM) derived from the collected interval-valued neutrosophic data available for $m$ number of alternatives for $n$ number of the criterion is represented as

$$D = (\Psi_{ij})_{m\times n} = (\langle \alpha_{ij}, \beta_{ij}, \gamma_{ij} \rangle)_{m\times n} = (\langle [\alpha_{ij}^L, \alpha_{ij}^U], [\beta_{ij}^L, \beta_{ij}^U], [\gamma_{ij}^L, \gamma_{ij}^U] \rangle)_{m\times n} \text{ i.e.,}$$

$$(\Psi_{ij})_{m\times n} = \begin{matrix} & G_1 & G_2 & & G_n \\ A_1 & \langle \alpha_{11}, \beta_{11}, \gamma_{11} \rangle & \langle \alpha_{12}, \beta_{12}, \gamma_{12} \rangle & \cdots & \langle \alpha_{1n}, \beta_{1n}, \gamma_{1n} \rangle \\ A_2 & \langle \alpha_{21}, \beta_{21}, \gamma_{21} \rangle & \langle \alpha_{22}, \beta_{22}, \gamma_{22} \rangle & \cdots & \langle \alpha_{2n}, \beta_{2n}, \gamma_{2n} \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & \langle \alpha_{m1}, \beta_{m1}, \gamma_{m1} \rangle & \langle \alpha_{m2}, \beta_{m2}, \gamma_{m2} \rangle & \cdots & \langle \alpha_{mn}, \beta_{mn}, \gamma_{mn} \rangle \end{matrix}.$$

For evaluating the MCDM problem for IVNSs we need a step-wise procedure which is summarized as follows:

**Step 1:** Check that all the criteria of the IVNDM, $D$ are of the same type or not.

**Case (i)** If all the criteria are of the same type then go to Step 2.

**Case (ii)** If some criteria are of benefit types and others are of cost types then normalize the IVNDM by transforming the cost criterion into benefit criterion, in the following manner: If the

$p^{th}$ criterion is of cost type then replace all the elements $\langle \alpha_{ip}, \beta_{ip}, \gamma_{ip} \rangle$ of the $p^{th}$ column of the decision matrix, $D$ with $\langle \gamma_{ip}, 1-\beta_{ip}, \alpha_{ip} \rangle$.

**Step 2:** Evaluate the IVNSs $\Psi_{ij}$ for each $\tilde{A}_i^N$ into an IVNN $\Psi_i$ using WAM, Eq. (3) or the WGM, Eq. (4).

**Step 3:** After aggregating (by applying either of the approaches i.e., WAM or WGM) according to Step 2, now obtain the crisp value of $\Psi_i$ $(i = 1,2, \ldots, m)$ by using SF $\omega_S(\tilde{A}^N)$, Eq. (14) or AF $\chi_A(\tilde{A}^N)$, Eq. (10).

**Step 4:** After Step 3, rank all the alternatives as per the obtained value of $\omega_S(\tilde{A}^N)$ or $\chi_A(\tilde{A}^N)$ and choose the best alternative.

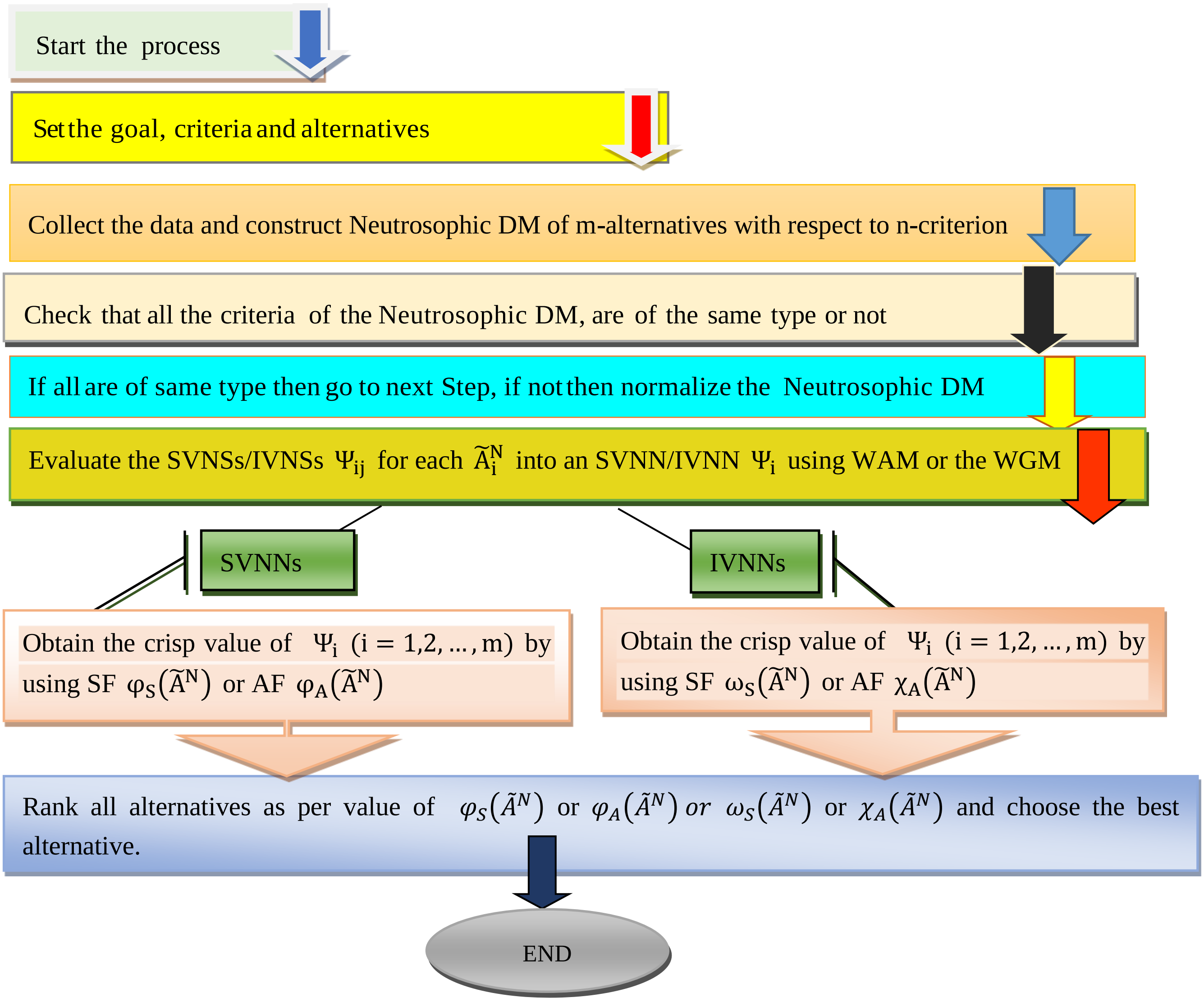


**Figure 1. The flowchart of the proposed MCDM method**

## 5. Real-world problem on SVNSs and IVNSs

In this section, a very common example is taken from a real-life which helps in validating the proposed approach.

**Example 5.1.** Consider an MCDM problem of selecting a pre-school for the first time, by the parents of a kindergarten child. To make the best selection, parents have collected the data in terms of the neutrosophic set (SVNSs or IVNSs) of 05 possible pre-schools, as per their liking, which are their prospective alternatives $\tilde{A}^N = \{\tilde{A}_1^N, \tilde{A}_2^N, \tilde{A}_3^N, \tilde{A}_4^N, \tilde{A}_5^N\}$ respectively. The data of these 05 possible alternatives are based on 03 different criteria $\tilde{G}^N = \{\tilde{G}_1^N, \tilde{G}_2^N, \tilde{G}_3^N\}$ where $\tilde{G}_1^N$ represents "near to the house, and safety of the child", $\tilde{G}_2^N$ represents "fee, infrastructure, and rapport" and $\tilde{G}_3^N$ represents "teaching methods in terms of effective learning concerning, cognitive, conative, affective, and physical activity" and the weight vectors are chosen for each criterion is $w_j = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$. Thus, when these five schools w.r.t the above-stated criteria are assessed by the parents (decision-maker), using the above-mentioned procedure stated in Section 4.1 and Section 4.2 as represented pictorially in Figure 2, the best alternative is obtained.

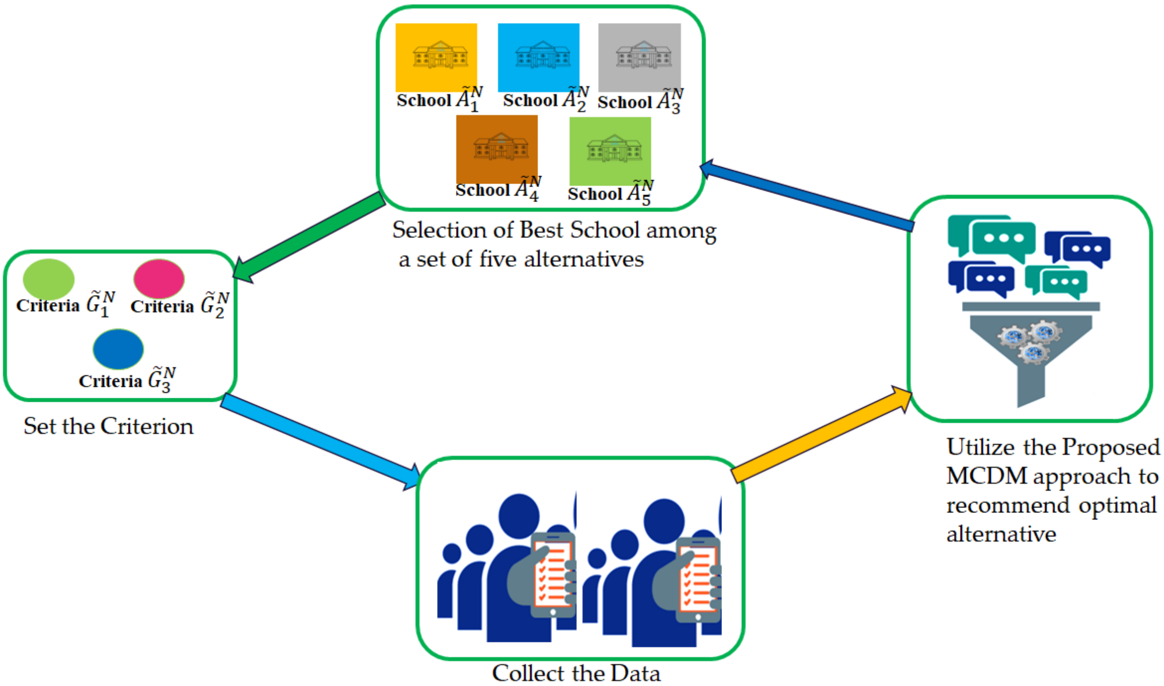


**Figure 2. A Framework of Proposed MCDM approach for a real-life problem**

### 5.1 Real-world problem on SVNSs

On applying the procedure mentioned in Section 4.1 on Example 5.1, where the collected data by the decision-maker is in terms of SVNSs, the best solution is derived as follows:

**Step 1:** Using the Step 1 of Section 4.1, the obtained SVNDM, $D$ as per the collected SVNS information is presented in Table 8.

**Table 8. SVNDM**

| | $\tilde{G}_1^N$ | $\tilde{G}_2^N$ | $\tilde{G}_3^N$ |
|---|---|---|---|
| $\tilde{A}_1^N$ | $\langle 0.6,0.3,0.0\rangle$ | $\langle 0.6,0.1,0.4\rangle$ | $\langle 0.4,0.3,0.8\rangle$ |
| $\tilde{A}_2^N$ | $\langle 0.2,0.1,0.0\rangle$ | $\langle 0.8,0.3,0.2\rangle$ | $\langle 0.9,0.2,0.6\rangle$ |
| $\tilde{A}_3^N$ | $\langle 0.4,0.2,0.3\rangle$ | $\langle 0.4,0.2,0.3\rangle$ | $\langle 0.2,0.2,0.5\rangle$ |
| $\tilde{A}_4^N$ | $\langle 0.3,0.2,0.3\rangle$ | $\langle 0.5,0.2,0.3\rangle$ | $\langle 0.5,0.3,0.2\rangle$ |
| $\tilde{A}_5^N$ | $\langle 0.7,0.0,0.1\rangle$ | $\langle 0.6,0.1,0.2\rangle$ | $\langle 0.4,0.3,0.2\rangle$ |

**Step 2:** Using the Step 2 of Section 4.1, the obtained aggregated SVNN $\Psi_i$, for SVNSs $\Psi_{ij}$, for each $\tilde{A}_i^N$ using WAM, Eq. (1) and the WGM, Eq. (2) are shown in Table 9.

**Table 9. SVNN using $AO_{WA}$ and $AO_{WG}$**

| | $AO_{WA}$ | $AO_{WG}$ |
|---|---|---|
| $\Psi_1$ | $\langle 0.5421, 0.2080, 0.0\rangle$ | $\langle 0.5241, 0.2388, 0.5068\rangle$ |
| $\Psi_2$ | $\langle 0.7480, 0.1817, 0.0\rangle$ | $\langle 0.5241, 0.2042, 0.3160\rangle$ |
| $\Psi_3$ | $\langle 0.3396, 0.2, 0.3557\rangle$ | $\langle 0.3175, 0.2, 0.3743\rangle$ |
| $\Psi_4$ | $\langle 0.4407, 0.2289, 0.2621\rangle$ | $\langle 0.4217, 0.2348, 0.2681\rangle$ |
| $\Psi_5$ | $\langle 0.5840, 0.0, 0.1587\rangle$ | $\langle 0.5518, 0.1427, 0.1680\rangle$ |

**Step 3:** After Step 2, using the Step 3 of Section 4.1, the score value, $\varphi_S$ for each $\Psi_i$ $(i = 1,2, \dots, m)$ are obtained by using Eq. (12) as follows:

***Approach 1 (Using WAM).*** Aggregated SVNN $\Psi_i$ $(i = 1,2, \dots, m)$ on using, WAM Eq. (1), the obtained score values $\varphi_S(\tilde{A}_i^N)$ are as follows:

$\varphi_S(\Psi_1) = 0.3862$, $\varphi_S(\Psi_2) = 0.5530$, $\varphi_S(\Psi_3) = 0.2238$, $\varphi_S(\Psi_4) = 0.2778$, $\varphi_S(\Psi_5) = 0.5668$.

***Approach 2 (Using WGM).*** Aggregated SVNN $\Psi_i$ $(i = 1,2, \dots, m)$ on using, WGM Eq. (2), the obtained score values $\varphi_S(\tilde{A}_i^N)$ are as follows:

$\varphi_S(\Psi_1) = 0.2785$, $\varphi_S(\Psi_2) = 0.3448$, $\varphi_S(\Psi_3) = 0.2076$, $\varphi_S(\Psi_4) = 0.2610$, $\varphi_S(\Psi_5) = 0.4290$.

**Step 4:** According to the obtained values of SF in Step 3, the following results are deduced, i.e.,

**(i)** For approach 1, the obtained ranking order of the alternatives is $\tilde{A}_5^N > \tilde{A}_2^N > \tilde{A}_1^N > \tilde{A}_4^N > \tilde{A}_3^N$, hence, $\tilde{A}_5^N$ is the best alternative according to the obtained score value $\varphi_S(\tilde{A}^N)$ for each $\Psi_i$ $(i = 1,2, \dots, m)$.

**(ii)** For approach 2, the obtained ranking order of the alternatives is $\tilde{A}_5^N > \tilde{A}_2^N > \tilde{A}_1^N > \tilde{A}_4^N > \tilde{A}_3^N$, hence, $\tilde{A}_5^N$ is the best alternative according to the obtained score value $\varphi_S(\tilde{A}^N)$ for each $\Psi_i$ $(i = 1,2, \dots, m)$.

Furthermore, to validate the above results obtained from the proposed method $\varphi_S(\tilde{A}^N)$, a detailed comparative analysis of given data in Table 9 is done with the existing methods $\sigma_S(\tilde{A}^N)$ [67], and $\tau_S(\tilde{A}^N)$ [68], and the obtained values of their respective score functions are represented in Table 10 and Table 11, on applying both the approaches of aggregation i.e., WAM and WGM respectively.

According to the obtained values of the SF on using the existing methods $\sigma_S(\tilde{A}^N)$ [67], $\tau_S(\tilde{A}^N)$ [68], and the proposed method $\varphi_S(\tilde{A}^N)$ for WAM, the obtained ranking order of all the alternatives is the same, i.e., $\tilde{A}_5^N > \tilde{A}_2^N > \tilde{A}_1^N > \tilde{A}_4^N > \tilde{A}_3^N$, as shown in Table 10, hence we conclude that $\tilde{A}_5^N$ is the best alternative.

**Table 10. Comparative analysis of SF of various methods for WAM**

| | **SVNNs** | $\boldsymbol{\sigma_S(\tilde{A}^N)}$ | $\boldsymbol{\tau_S(\tilde{A}^N)}$ | $\boldsymbol{\varphi_S(\tilde{A}^N)}$ |
|---|---|---|---|---|
| $\varphi_S(\Psi_1)$ | $\langle 0.5421, 0.2080, 0.0\rangle$ | 0.5631 | 0.5919 | 0.3862 |
| $\varphi_S(\Psi_2)$ | $\langle 0.7480, 0.1817, 0.0\rangle$ | 0.6923 | 0.7408 | 0.5530 |
| $\varphi_S(\Psi_3)$ | $\langle 0.3396, 0.2, 0.3557\rangle$ | 0.2919 | 0.2286 | 0.2238 |
| $\varphi_S(\Psi_4)$ | $\langle 0.4407, 0.2289, 0.2621\rangle$ | 0.3604 | 0.3189 | 0.2778 |
| $\varphi_S(\Psi_5)$ | $\langle 0.5840, 0.0, 0.1587\rangle$ | $\mathbf{0.7127}$ | $\mathbf{0.7674}$ | $\mathbf{0.5668}$ |
| *Ranking order* | | $\tilde{A}_5^N > \tilde{A}_2^N > \tilde{A}_1^N > \tilde{A}_4^N > \tilde{A}_3^N$ | $\tilde{A}_5^N > \tilde{A}_2^N > \tilde{A}_1^N > \tilde{A}_4^N > \tilde{A}_3^N$ | $\tilde{A}_5^N > \tilde{A}_2^N > \tilde{A}_1^N > \tilde{A}_4^N > \tilde{A}_3^N$ |

Similarly, according to the obtained values of the SF on using the existing methods $\sigma_S(\tilde{A}^N)$ [67], $\tau_S(\tilde{A}^N)$ [68] and the proposed method $\varphi_S(\tilde{A}^N)$ for WGM, the obtained ranking order of the best and the second-best alternatives is the same as shown in Table 11, thus we conclude that $\tilde{A}_5^N$ is the best alternative.

**Table 11. Comparative analysis of SF of various methods for WGM**

| | **SVNNs** | $\boldsymbol{\sigma_S(\tilde{A}^N)}$ | $\boldsymbol{\tau_S(\tilde{A}^N)}$ | $\boldsymbol{\varphi_S(\tilde{A}^N)}$ |
|---|---|---|---|---|
| $\varphi_S(\Psi_1)$ | $\langle 0.5241, 0.2388, 0.5068\rangle$ | 0.2699 | 0.2770 | 0.2785 |
| $\varphi_S(\Psi_2)$ | $\langle 0.5241, 0.2042, 0.3160\rangle$ | 0.3999 | 0.3838 | 0.3448 |
| $\varphi_S(\Psi_3)$ | $\langle 0.3175, 0.2, 0.3743\rangle$ | 0.2716 | 0.2012 | 0.2076 |
| $\varphi_S(\Psi_4)$ | $\langle 0.4217, 0.2348, 0.2681\rangle$ | 0.3420 | 0.2930 | 0.2610 |
| $\varphi_S(\Psi_5)$ | $\langle 0.5518, 0.1427, 0.1680\rangle$ | $\mathbf{0.5492}$ | $\mathbf{0.5630}$ | $\mathbf{0.4290}$ |
| *Ranking order* | | $\tilde{A}_5^N > \tilde{A}_2^N > \tilde{A}_4^N > \tilde{A}_3^N > \tilde{A}_1^N$ | $\tilde{A}_5^N > \tilde{A}_2^N > \tilde{A}_4^N > \tilde{A}_1^N > \tilde{A}_3^N$ | $\tilde{A}_5^N > \tilde{A}_2^N > \tilde{A}_1^N > \tilde{A}_4^N > \tilde{A}_3^N$ |

Hence, we can conclude that the proposed score function $\varphi_S(\tilde{A}^N)$ is justified and is giving reasonable results on applying in real-world applications.

**5.2 Real-world problem on IVNSs**

On applying the procedure mentioned in Section 4.2 on Example 5.1, where the rating value of the collected data by the decision-maker is in terms of IVNSs, the best solution is derived as follows:

**Step 1:** Using the Step 1 of Section 4.2, the obtained IVNDM, $D$ as per the collected IVNS information is presented in Table 12.

**Table 12. IVNDM**

| | $\tilde{G}_1^N$ | $\tilde{G}_2^N$ | $\tilde{G}_3^N$ |
|---|---|---|---|
| $\tilde{A}_1^N$ | $\langle[0.1, 0.5], [0.1, 0.2], [0, 0]\rangle$ | $\langle[0.1, 0.5], [0, 0.1], [0.2, 0.2]\rangle$ | $\langle[0.1, 0.3], [0.1, 0.2], [0.3, 0.5]\rangle$ |
| $\tilde{A}_2^N$ | $\langle[0.1, 0.1], [0.05, 0.95], [0,0]\rangle$ | $\langle[0.1, 0.7], [0.1, 0.2], [0.5, 0.15]\rangle$ | $\langle[0.1, 0.8], [0.1, 0.1], [0.3, 0.3]\rangle$ |
| $\tilde{A}_3^N$ | $\langle[0.1, 0.3], [0.1, 0.1], [0.15,0.15]\rangle$ | $\langle[0.1, 0.3], [0.1, 0.1], [0, 0.3]\rangle$ | $\langle[0.1, 0.1], [0.1, 0.1], [0.2, 0.3]\rangle$ |
| $\tilde{A}_4^N$ | $\langle[0.11, 0.19], [0.05, 0.15], [0, 0.27]\rangle$ | $\langle[0.1, 0.4], [0.1, 0.1], [0.15, 0.15]\rangle$ | $\langle[0.1, 0.4], [0.1, 0.2], [0.05, 0.15]\rangle$ |
| $\tilde{A}_5^N$ | $\langle[0.1, 0.6], [0, 0], [0.02, .08]\rangle$ | $\langle[0.1, 0.5], [0, 0.1], [0.1, 0.1]\rangle$ | $\langle[0.1, 0.3], [0.1, 0.2], [0.1, 0.1]\rangle$ |

**Step 2:** Using the Step 2 of Section 4.2, the obtained aggregated IVNN $\Psi_i$, for IVNSs $\Psi_{ij}$, for each $\tilde{A}_i^N$ using WAM, Eq. (3) and the WGM, Eq. (4) are shown in Table 13.

**Table 13. IVNN using $AO_{WA}$ and $AO_{WG}$**

| | $AO_{WA}$ | $AO_{WG}$ |
|---|---|---|
| $\Psi_1$ | $\langle[0.1, 0.4407], [0, 0.1587], [0,0]\rangle$ | $\langle[0.1, 0.4217], [0.0678, 0.1680], [0.1757, 0.2632]\rangle$ |
| $\Psi_2$ | $\langle[0.1, 0.6220], [0.0794, 0.2668], [0,0]\rangle$ | $\langle[0.1, 0.3826], [0.0836, 0.6698], [0.1271, 0.1589]\rangle$ |
| $\Psi_3$ | $\langle[0.1, 0.2388], [0.1, 0.1], [0, 0.2381]\rangle$ | $\langle[0.1, 0.2080], [0.1, 0.1], [0.1206, 0.2532]\rangle$ |
| $\Psi_4$ | $\langle[0.1033, 0.3369], [0.0794, 0.1442], [0, 0.1825]\rangle$ | $\langle[0.1032, 0.3121], [0.0836, 0.1510], [0.0688, 0.1920]\rangle$ |
| $\Psi_5$ | $\langle[0.1, 0.4808], [0, 0], [0.0585, 0.0928]\rangle$ | $\langle[0.1, 0.4481], [0.0345, 0.1037], [0.0741, 0.0934]\rangle$ |

**Step 3:** After Step 2, using the Step 3 of Section 4.2, the score value, $\omega_S(\tilde{A}^N)$ for each $\Psi_i$ $(i = 1,2, \dots, m)$ are obtained by using Eq. (14) as follows:

***Approach 1 (Using WAM).*** Aggregated IVNN $\Psi_i$ $(i = 1,2, \dots, m)$ on using, WAM Eq. (3), the obtained score values $\omega_S(\tilde{A}_i^N)$ are as follows:

$\omega_S(\Psi_1) = 0.3214,\ \omega_S(\Psi_2) = 0.3096,\ \omega_S(\Psi_3) = 0.2484,\ \omega_S(\Psi_4) = 0.2680,\ \omega_S(\Psi_5) = 0.3717.$

***Approach 2 (Using WGM).*** Aggregated IVNN $\Psi_i$ $(i = 1,2, \dots, m)$ on using, WGM Eq. (4), the obtained score values $\omega_S(\tilde{A}_i^N)$ are as follows:

$\omega_S(\Psi_1) = 0.2651,\ \omega_S(\Psi_2) = 0.1067,\ \omega_S(\Psi_3) = 0.2312,\ \omega_S(\Psi_4) = 0.2535,\ \omega_S(\Psi_5) = 0.3203.$

**Step 4:** According to the obtained values of SF in Step 3, the following results are deduced, i.e.,

**(i)** For approach 1, the obtained ranking order of the alternatives is $\tilde{A}_5^N > \tilde{A}_1^N > \tilde{A}_2^N > \tilde{A}_4^N > \tilde{A}_3^N$, hence, $\tilde{A}_5^N$ is the best alternative according to the obtained score value $\omega_S(\tilde{A}^N)$ for each $\Psi_i$ $(i = 1,2, \dots, m)$.

**(ii)** For approach 2, the obtained ranking order of the alternatives is $\tilde{A}_5^N > \tilde{A}_1^N > \tilde{A}_4^N > \tilde{A}_3^N > \tilde{A}_2^N$ hence $\tilde{A}_5^N$ is the best alternative according to the obtained score value $\omega_S(\tilde{A}^N)$ for each $\Psi_i$ $(i = 1,2, \dots, m)$.

Furthermore, to validate the above-obtained results from the proposed method $\omega_S(\tilde{A}^N)$, a detailed comparative analysis of given data in Table 13 is done with the existing methods $\chi_S(\tilde{A}^N)$ [67], $\psi_S(\tilde{A}^N)$ [68], and the obtained values of their respective score functions are represented in Table 14 and Table 15, on applying both the approaches of aggregation i.e., WAM and WGM respectively.

According to the obtained values of the SF on using the existing methods $\chi_S(\tilde{A}^N)$ [67], $\psi_S(\tilde{A}^N)$ [68] and the proposed method $\omega_S(\tilde{A}^N)$ for WAM, the obtained ranking order of the first three alternatives is the same, i.e., $\tilde{A}_5^N > \tilde{A}_1^N > \tilde{A}_2^N$, as shown in Table 14, we conclude that $\tilde{A}_5^N$ is the best alternative.

**Table 14. Comparative analysis of SF of various methods for WAM**

| | **SVNNs** | $\chi_S(\tilde{A}^N)$ | $\psi_S(\tilde{A}^N)$ | $\omega_S(\tilde{A}^N)$ |
|---|---|---|---|---|
| $\omega_S(\Psi_1)$ | $\langle[0.1, 0.4407], [0, 0.1587], [0,0]\rangle$ | 0.5558 | 0.5966 | 0.3214 |
| $\omega_S(\Psi_2)$ | $\langle[0.1, 0.6220], [0.0794, 0.2668], [0,0]\rangle$ | 0.5074 | 0.5121 | 0.3096 |
| $\omega_S(\Psi_3)$ | $\langle[0.1, 0.2388], [0.1, 0.1], [0, 0.2381]\rangle$ | 0.4252 | 0.3719 | 0.2484 |
| $\omega_S(\Psi_4)$ | $\langle[0.1033, 0.3369], [0.0794, 0.1442], [0, 0.1825]\rangle$ | 0.4526 | 0.4200 | 0.2680 |
| $\omega_S(\Psi_5)$ | $\langle[0.1, 0.4808], [0, 0], [0.0585, 0.0928]\rangle$ | **0.6074** | **0.6754** | **0.3717** |
| | *Ranking order* | $\tilde{A}_5^N > \tilde{A}_1^N > \tilde{A}_2^N > \tilde{A}_4^N > \tilde{A}_3^N$ | $\tilde{A}_5^N > \tilde{A}_1^N > \tilde{A}_2^N > \tilde{A}_4^N > \tilde{A}_3^N$ | $\tilde{A}_5^N > \tilde{A}_1^N > \tilde{A}_2^N > \tilde{A}_4^N > \tilde{A}_3^N$ |

Similarly, according to the obtained values of the SF on using the existing methods $\chi_S(\tilde{A}^N)$ [67], $\psi_S(\tilde{A}^N)$ [68] and the proposed method $\omega_S(\tilde{A}^N)$ for WGM, the obtained ranking order suggests that $\tilde{A}_5^N$ is the best alternative by all the existing and the proposed method, as shown below in Table 15, hence, we conclude that $\tilde{A}_5^N$ is the best alternative.

**Table 15. Comparative analysis of SF of various methods for WGM**

| | **SVNNs** | $\chi_S(\tilde{A}^N)$ | $\psi_S(\tilde{A}^N)$ | $\omega_S(\tilde{A}^N)$ |
|---|---|---|---|---|
| $\omega_S(\Psi_1)$ | $\langle[0.1, 0.4217], [0.0678, 0.1680], [0.1757, 0.2632]\rangle$ | 0.4028 | 0.3523 | 0.2651 |
| $\omega_S(\Psi_2)$ | $\langle[0.1, 0.3826], [0.0836, 0.6698], [0.1271, 0.1589]\rangle$ | 0.1724 | −0.0292 | 0.1067 |
| $\omega_S(\Psi_3)$ | $\langle[0.1, 0.2080], [0.1, 0.1], [0.1206, 0.2532]\rangle$ | 0.3836 | 0.3068 | 0.2312 |

| | | | | |
|---|---|---|---|---|
| $\omega_S(\Psi_4)$ | $\langle [0.1032, 0.3121], [0.0836, 0.1510], [0.0688, 0.1920] \rangle$ | 0.4213 | 0.3692 | 0.2535 |
| $\omega_S(\Psi_5)$ | $\langle [0.1, 0.4481], [0.0345, 0.1037], [0.0741, 0.0934] \rangle$ | **0.5261** | **0.5428** | **0.3203** |
| | *Ranking order* | $\tilde{A}_5^N > \tilde{A}_4^N > \tilde{A}_1^N > \tilde{A}_3^N > \tilde{A}_2^N$ | $\tilde{A}_5^N > \tilde{A}_4^N > \tilde{A}_1^N > \tilde{A}_3^N > \tilde{A}_2^N$ | $\tilde{A}_5^N > \tilde{A}_1^N > \tilde{A}_4^N > \tilde{A}_3^N > \tilde{A}_2^N$ |

Hence, we can conclude that the proposed score function $\omega_S(\tilde{A}^N)$ is justified and is giving reasonable results on applying in real-world applications.

## 6. Discussion and Comparative Analysis

In this section, the SVNSs and IVNSs from the existing literature [67-69] are considered and solved by the existing and the proposed method, and the obtained solutions are presented in Table 16 given below. The obtained Table 16 argues well that the proposed methods are giving the same or the better results for all the considered problems reasonably and also it highlights, that the existing methods [67, 68] are behaving well for some particular SVNSs or IVNSs but fails under certain restrictions, then to deal with such SVNSs or IVNSs the proposed approaches works well and a desirable conclusion can be drawn respectively. Hence, it is claimed that the proposed SF and AF are better to evaluate MCDM problems and can be easily applied in solving real-life problems.

**Table 16. A comparative analysis of SVNSs and IVNSs with various existing metric methods**

| **SVNNs** | | | | | | |
|---|---|---|---|---|---|---|
| | $\boldsymbol{\sigma_S(\tilde{A}^N)}$ | $\boldsymbol{\sigma_A(\tilde{A}^N)}$ | $\boldsymbol{\tau_S(\tilde{A}^N)}$ | $\boldsymbol{\tau_A(\tilde{A}^N)}$ | $\boldsymbol{\varphi_S(\tilde{A}^N)}$ | $\boldsymbol{\varphi_A(\tilde{A}^N)}$ |
| $\tilde{A}_1^N = \langle 0.5, 0.2, 0.6 \rangle$<br>$\tilde{A}_2^N = \langle 0.6, 0.4, 0.2 \rangle$<br>(Adopted from [67]) | 0.25<br>0.3<br>$\tilde{A}_2^N > \tilde{A}_1^N$ | $-0.08$<br>0.32<br>$\tilde{A}_2^N > \tilde{A}_1^N$ | 0.2750<br>0.2600<br>$\tilde{A}_1^N > \tilde{A}_2^N$ | $-0.5$<br>$-0.4$<br>$\tilde{A}_2^N > \tilde{A}_1^N$ | 0.2778<br>0.25<br>$\tilde{A}_1^N > \tilde{A}_2^N$ | $-0.4$<br>0.2<br>$\tilde{A}_2^N > \tilde{A}_1^N$ |
| $\tilde{A}_1^N = \langle 0.5, 0.2, 0.6 \rangle$<br>$\tilde{A}_2^N = \langle 0.2, 0.2, 0.3 \rangle$<br>(Adopted from [68]) | 0.25<br>0.25<br>$\tilde{A}_1^N = \tilde{A}_2^N$ | $-0.08$<br>$-0.2$<br>$\tilde{A}_1^N > \tilde{A}_2^N$ | 0.2750<br>0.1250<br>$\tilde{A}_1^N > \tilde{A}_2^N$ | $-0.5$<br>$-0.5$<br>$\tilde{A}_1^N = \tilde{A}_2^N$ | 0.2778<br>0.1667<br>$\tilde{A}_1^N > \tilde{A}_2^N$ | $-0.4$<br>0.2<br>$\tilde{A}_2^N > \tilde{A}_1^N$ |
| $\tilde{A}_1^N = \langle 0.5, 0.0, 0.2 \rangle$<br>$\tilde{A}_2^N = \langle 0.4, 0.0, 0.1 \rangle$<br>(Adopted from [68]) | 0.65<br>0.65<br>$\tilde{A}_1^N = \tilde{A}_2^N$ | 0.3<br>0.3<br>$\tilde{A}_1^N = \tilde{A}_2^N$ | 0.6950<br>0.7250<br>$\tilde{A}_2^N > \tilde{A}_1^N$ | 0.3<br>0.3<br>$\tilde{A}_1^N = \tilde{A}_2^N$ | 0.5<br>0.433<br>$\tilde{A}_1^N > \tilde{A}_2^N$ | 0.6<br>0.8<br>$\tilde{A}_2^N > \tilde{A}_1^N$ |
| $\tilde{A}_1^N = \langle 0.8, 0.1, 0.6 \rangle$<br>$\tilde{A}_2^N = \langle 0.8, 0.2, 0.4 \rangle$ | 0.5 | 0.24 | 0.5 | 0.0 | 0.8333 | $-0.3$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| (Adopted from [69]) | 0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.44<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ | 0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.0<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.625<br>$\widetilde{A}_1^N > \widetilde{A}_2^N$ | 0.0<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ |
| $\widetilde{A}_1^N = \langle 0.1, 0.0, 0.1 \rangle$<br>$\widetilde{A}_2^N = \langle 0.3, 0.0, 0.3 \rangle$<br>(Adopted from [69]) | 0.5<br>0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.0<br>0.0<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.5<br>0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.0<br>0.0<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.2778<br>0.3571<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ | 0.8<br>0.4<br>$\widetilde{A}_1^N > \widetilde{A}_2^N$ |
| **IVNNs** | | | | | | |
| | | | $\chi_S(\widetilde{A}^N)$ | $\psi_S(\widetilde{A}^N)$ | $\omega_S(\widetilde{A}^N)$ | $\chi_A(\widetilde{A}^N)$ |
| $\widetilde{A}_1^N = \langle [0.6, 0.4], [0.3, 0.1], [0.1, 0.3] \rangle$<br>$\widetilde{A}_2^N = \langle [0.1, 0.6], [0.2, 0.3], [0.1, 0.4] \rangle$<br>(Adopted from [67]) | | | 0.45<br>0.3<br>$\widetilde{A}_1^N > \widetilde{A}_2^N$ | 0.4375<br>0.22<br>$\widetilde{A}_1^N > \widetilde{A}_2^N$ | 0.3462<br>0.2143<br>$\widetilde{A}_1^N > \widetilde{A}_2^N$ | 0.26<br>0.005<br>$\widetilde{A}_1^N > \widetilde{A}_2^N$ |
| $\widetilde{A}_1^N = \langle [0.4, 0.6], [0.2, 0.3], [0.5, 0.7] \rangle$<br>$\widetilde{A}_2^N = \langle [0.2, 0.7], [0.1, 0.2], [0.1, 0.3] \rangle$<br>(Adopted from [68]) | | | 0.2<br>0.4750<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ | 0.23<br>0.4663<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ | 0.2222<br>0.3519<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ | $-0.0750$<br>0.2050<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ |
| $\widetilde{A}_1^N = \langle [0.1, 0.7], [0.05, 0.15], [0.1, 0.3] \rangle$<br>$\widetilde{A}_2^N = \langle [0.2, 0.8], [0.05, 0.15], [0.2, 0.4] \rangle$<br>(Adopted from [69]) | | | 0.5<br>0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.5<br>0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.2857<br>0.4167<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ | 0.17<br>0.19<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ |
| $\widetilde{A}_1^N = \langle [0.1, 0.7], [0.1, 0.1], [0.1, 0.3] \rangle$<br>$\widetilde{A}_2^N = \langle [0.2, 0.8], [0.1, 0.1], [0.2, 0.4] \rangle$<br>(Adopted from [69]) | | | 0.5<br>0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.5<br>0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.3571<br>0.4167<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ | 0.16<br>0.18<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ |
| $\widetilde{A}_1^N = \langle [0.1, 0.7], [0.0, 0.2], [0.1, 0.3] \rangle$<br>$\widetilde{A}_2^N = \langle [0.2, 0.8], [0.0, 0.2], [0.2, 0.4] \rangle$<br>(Adopted from [69]) | | | 0.5<br>0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.5<br>0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.3571<br>0.4167<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ | 0.18<br>0.2<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ |
| $\widetilde{A}_1^N = \langle [0.1, 0.2], [0.0, 0.0], [0.1, 0.2] \rangle$<br>$\widetilde{A}_2^N = \langle [0.4, 0.5], [0.0, 0.0], [0.4, 0.5] \rangle$<br>(Adopted from [69]) | | | 0.5<br>0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.5<br>0.5<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ | 0.2941<br>0.4545<br>$\widetilde{A}_2^N > \widetilde{A}_1^N$ | 0.0<br>0.0<br>$\widetilde{A}_1^N = \widetilde{A}_2^N$ |

## 7. Managerial insights

The study adopted DM when multiple criteria are involved and the decision-maker is supposed to find the best alternative among all the present alternatives on the basis of their corresponding criteria. The data involved indeterminacy and inconsistent sets of information, since neutrosophic sets deal with such data in the best possible manner, so the neutrosophic environment has been chosen to deal with the real-life problem. On using the proposed MCDM methodology all the available alternatives are evaluated under the neutrosophic environment, by the proposed SF and AF which lead to the best option available in the alternative based on their criteria. The application of the proposed MCDM methodology based on SF and AF provides a judicious solution to the decision-maker by considering all the available information in real-world applications in comparison to all the existing metric methods. Hence, the proposed MCDM methodology is more reliable in terms of its derived solutions.

## 8. Conclusions

In this paper, a new SF and AF for SVNSs and IVNSs are proposed, also, an MCDM method is developed based on the new ranking tools for SVNSs and IVNSs. In the proposed MCDM method, the score value of the aggregated SVNSs or IVNSs is obtained by applying the proposed SF or AF. According to the obtained results on applying the proposed SF or AF, the alternatives are ordered and the most desirable alternative i.e., the alternative with the highest value of SF or AF is chosen in the DM problem. To illustrate the efficiency and the validity of the proposed SF and AF for SVNSs and IVNSs a real-life application is solved successfully and the obtained results sync completely with the existing methods [67, 68]. Since neutrosophic sets are efficient enough to consider indeterminate and inconsistent information/data, hence, our proposed method would play an effective role in dealing with MCDM problems in several real-life applications like personnel selection, enterprises, signal processing, pattern recognition, medical diagnosis, engineering, management, DM, etc. having indeterminacy and inconsistent set of data. The only limitations of the proposed method would be that the data must be analyzed properly and all the restrictions should be followed in order to derive an accurate solution to the problem on applying the proposed SF and AF. Moreover, neutrosophic sets are still being in their prime have a lot to reveal concerning real-life applications. In the future, these proposed SF and AF for SVNSs and IVNSs will be expanded and will be generalized to the other domains of the neutrosophic sets like-refined neutrosophic sets, neutrosophic soft sets, neutrosophic cubic fuzzy sets, and their applications.

## Compliance with Ethical Standards

**Conflict of Interest**

The authors declare that they do not have any financial or associative interest indicating a conflict of interest in about submitted work.

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# A Neutrosophic Cognitive Map Based Approach to Explore the Health Deterioration Factors

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**Abstract:** Neutrosophy has gained an exponential fame in recent years among researchers and academician especially when there is a need to deal with different and difficult situations. Out of several neutrosophy-sets related concepts, the researchers have used neutrosophic cognitive maps for identifying the hidden and indeterminate factors that can influence a particular situation or can significantly affect any problems which involve making decisions. In our study, we have used neutrosophic cognitive maps to explore the factors which can lead to health deterioration. The present method not only illustrates the way neutrosophic cognitive maps can be used but also suggest ways which can help the masses in finding out the health affecting factors and to control it. It is believed that the proposed method can be helpful for analyzing many such a situation and can set a benchmark in using soft computing for healthcare.

**Keywords:** neutrosophy sets, neutrosophic logic, neutrosophic cognitive maps, healthcare

## 1. Introduction

Neutrosophic logic was devised by Florentine Smarandache [1]. Neutrosophic set is more general and complex concept, can be considered as an extended fuzzy logic wherein indeterminacy factor is also included. This particular feature makes neutrosophy more robust and applicable in various domain of real life. The idea of neutrosophic logic presents a necessary part in solving everyday problems. It is a logic in which each proposition is considered to have the percentage of truth, indeterminacy and falsity in subsets T, I, F respectively where T, I, F are neutrosophic components. In this logic instead of only numbers, we use subsets of T, I and F which are estimated by non-standard subsets. A neutrosophic directed graph representing the causal relationship between concepts like policies, events etc. as nodes and causalities indeterminate as edges are called neutrosophic cognitive maps [2].

In this paper, we used these neutrosophic cognitive maps on the factors which affect health overall. The current generation has seen a rapid increase in health deterioration cases [3]. Due to this, health in general, has been affected drastically. The health of people is influenced by various factors. The climate also plays a major role in determining whether a person is healthy or not. The factors that have a vital effect on health include our locality, heredity, our resources and literacy, and our relations with friends and family. There are few other factors like accessibility and use of healthcare co-operations which have a relatively lesser effect on health.

Some other factors that are supposed to be indeterminate are physical exercises, inadequate drainage systems [4]. Neutrosophic cognitive maps are used to depict this condition mathematically as to how these indeterminate factors have an effect on health. A health improvement approach is represented using neutrosophic cognitive maps in this article.

A neutrosophic set has the capacity of being a general framework for interpreting uncertainty in data sets. It contributes to overcoming the limitations of uncertainty and inconsistency that circles environment and affect the judgment of the decision maker [5]. Hence, neutrosophic logic not only handles the misinterpretations of decision-makers but also the environmental factors of uncertainty circumstances [6].

We can also consolidate multiple decision makers' aspects to accomplish the ideal prospects by managing the confliction and biasness between them [7]. Mohamed Abdel-Basset in his research used neutrosophic theory effectively to solve transition complexities of enterprises based on IOT [8].

Section 2 and Section 3 illustrates the background concepts of neutrosophic cognitive maps (NCM) and factors that can influence health of individuals, whereas in Section 4 NCM based approach to identify the various indeterminate factors which can influence human health is discussed which is followed by discussion and conclusion sections respectively.

## 2. Background

Healthcare is defined as efforts made to maintain or cure physical, mental, or emotional well-being especially by trained and authorized specialists. According to WHO, health is not merely the absence of disease, but it is a state of complete fitness. Lately, we have witnessed a massive increase in health deterioration overall [9]. This has led to the worsening of mental and physical health progressively over time. There are many factors which merge to affect the health of people and societies. The conditions and environment determine whether people are healthy or not. In general, factors like our locality, the climate' nature, genetics, revenue, literacy level, and connections with family and friends have significant impacts on one's health, on the other hand the more generally viewed factors like access and health care services usage oftentimes have less of an impact. Factors such as physical exercises, inadequate drainage systems are neglected by researchers as they are assumed to be indeterminate. To explain how this indeterminate affect health, we represent this situation mathematically using Neutrosophic Cognitive Maps (NCM). It illustrates the extent of dependencies of factors affecting health. In our study, we propose a health improvement approach using neutrosophic cognitive maps. In [4] Florentin Smarandache explains a logic in which each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F is called Neutrosophic Logic. This logic is also considered as the generalized form of Intuitionistic fuzzy logic [10]. According to Charles Ashbacher [11] incomplete information on a variable, proposition or event one has $T + I + F < 1$ in intuitionistic logic. We use NCM's as in [12] describes the efficiency of NCM technique vs Fuzzy Cognitive Maps (FCM) in situation analysis to deal with the unpredictability and indeterminacy.

## 3. Factors affecting health

### *3.1 Air Pollution*

Air Pollution is regarded as being instrumental in bringing complications in the health of the individual. It has been clearly observed that there has been a massive increase in mortality rate and hospital admissions that are associated with respiratory diseases which is further related to exposure of human body to harmful air borne participate matter and ozone [1]. This leads to deteriorating effects on human health which can be both short-term and long-term. It has wide range of effects which may include minor respiratory irritations, heart disease, lung cancer and other lung diseases etc. [2]. And bad health also causes air pollution in some instances like usage of the ambulance for patient convenience so, both health and air pollution are bi-directional [13].

*3.2 Family Genetics*

The knowledge of our genetic structure and other associated characteristics can help us in many ways. There are numerous diseases which are passed on from one generation to another, so members of the family have a potential risk to develop some diseases which have been in their ancestors as well which is mainly due to the fact that highly penetrant genetic mutations are transmitted through generations. These diseases include cardiovascular diseases, diabetes and several cancers as well [14].

*3.3 Unhygienic Livelihood*

Poor hygiene is the reason for transmission and spread of disease as it becomes home of several bacteria and viruses. It reduces human wellbeing, social and economic development. On the contrary, maintaining good hygiene and sanitation results in good impact on health and help in reducing diseases like diarrhea [15].

In slum areas, generally large population lives with less facilities available like no proper sanitation facilities and limited supply of water and the hygiene is not maintained in those areas which results in severe health problems [16]. The hygienic and sanitary condition of many fish retail markets are very poor that may have an adverse effect on fish retailer’s health [17].

*3.4 Impure Water*

Contamination of water can occur anywhere in lakes, rivers, wells to modern water tanks that supplies water to the citizens homes. More than 500,000 deaths recorded every year due to contaminated drinking water.

According to World Health Organization more than one third wealth of Sub Saharan African poor people is spent on waterborne diseases like Diarrhea, Malaria and worm infections. In countries which are not yet developed fully are having problem with contaminated supply of drinking water with bacteria, which results in several diseases [10]. Water quality should be checked on a regular basis so that it will not impact the health of people.

*3.5 Junk Food*

Food without nutrition can be called as junk, most of them are good at taste but bad for the health of an individual as some of them contains beverages like salt, oils and large amount of sugar. It can be fried, burgers soft drinks and some packaged food. They contain high calories due to that the body cannot intake nutritious food with vitamins and minerals.

According to food institutes analytical data, millennials solely spend 15 percent of their budgets on dining out. In contrast to some 40 years back people now spend half of their food budget on eating in restaurants. A couple of years back only 38 percent of food budget were spending on eating outside home.

The increased prevalence of cardiovascular risk and other diseases like obesity and diabetes is a result of changing food habits i.e., consuming junk food [18]. Similarly, due to bad health i.e., contagious disease food will also get affected so the process is vice versa [19].

*3.6 Smoking and Alcohol Consumption*

Dr Stanley Chia said cigarettes contain four thousand plus syntheses and 400 toxic substances that involve carbon monoxide, tar, DDT, arsenic and formaldehyde. On the other hand, heavy alcohol consumption also leads to numerous critical health diseases. Over drinking can begin instant difficulties like nausea and vomiting, alcohol poisoning, blurred eyesight, impaired judgment and acute intoxication.

Furthermore, the combined consumption of cigarettes and alcohol is hazardous for the brain. Scientists showed neural harm in particular brain areas due to mutual use of tobacco and alcohol.
For a man who inhaled more than 25 cigarettes every day had a higher danger of diabetes of 1.94 collated to non-smokers. And the man who absorbed 30.0-49.9 g of alcohol diurnal had a relevant risk for diabetes of 0.61 [20].

*3.7 Inadequate Drainage System*

According to WHO, significant environmental health step to reduce disease is decreased monsoon water and household wastewater. The unmanaged drainage system of rainwater gives putrid pools that provide replication places for viruses. Environmental hygiene standards are not perfect in developing countries and it is also a global concern which helps in the extent of disease. Outcomes propose that drainage and sewerage might be a vital effect on diarrhea, nutritional status, and for intestinal nematode, health impact was most effective [21].

Risky hygiene methods have big impacts on people's health. 297000 children are dying annually from diarrheal because of bad sanitation, unsafe drinking water, and poor hygiene [22].

*3.8 Physical Exercise*

Exercise gives strength to our heart and enhances our blood transmission. The extended bloodstream and boosts the levels of oxygen in the human body. It also benefits heart hazards like coronary artery disease, heart attack, and high cholesterol.

Average physical exercise improves the cardiovascular system and also helps in an overcome from a physical disorder such as osteoporosis, renal disease, and diabetes. Now research showed us physical exercise has benefits for psychological health as well. And the most significant effect of exercise is on cognition. It also deals with neurodegenerative diseases like AD etc. and also reduces the risk of depression [23].

Better quality of life is joined with physical activity and exercise [24]. With the help of physical activity, exercise capacity and body fitness are improved which may give many health profits [25-27].

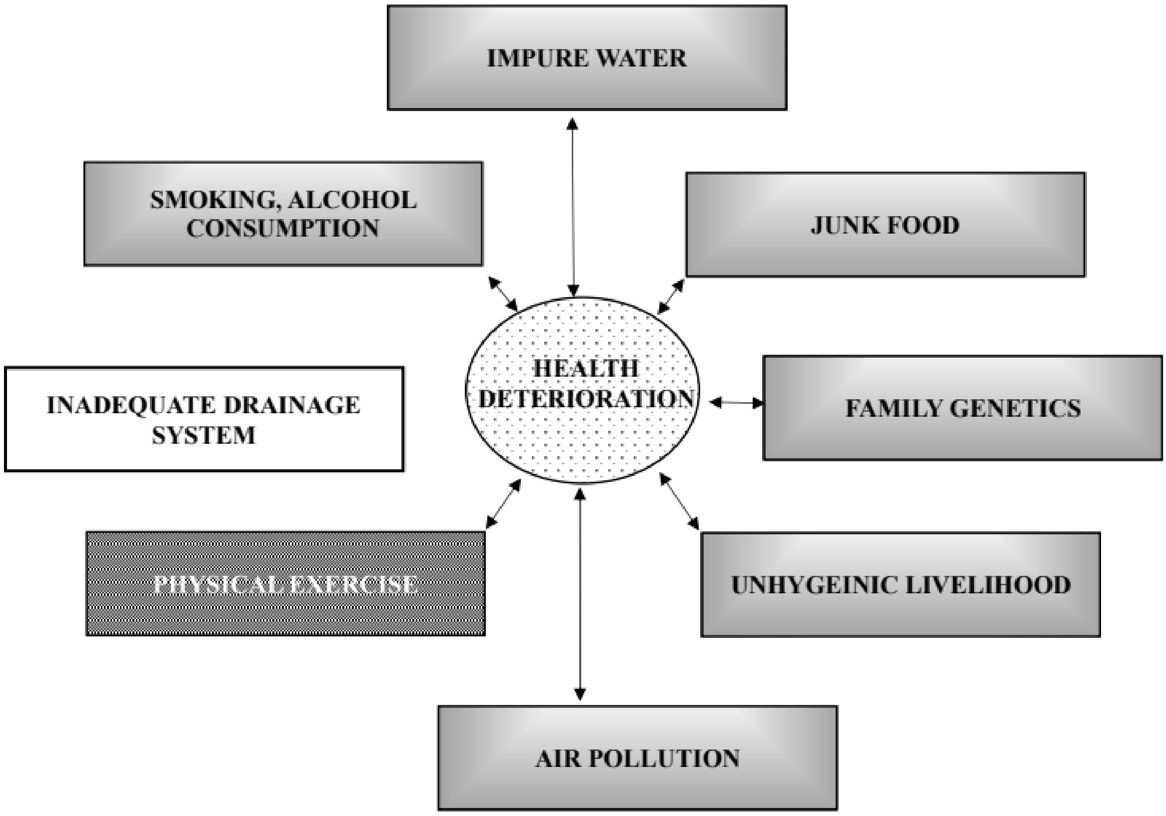


**Figure 1**: Factors affecting health

## 4. Neutrosophic Cognitive Map Based Approach to Identify Causes for Heath Deterioration

A Neutrosophic Cognitive Map is a graph in which direct edges represents relationship or indeterminates, dotted edges represent indeterminacy and concepts represented by nodes [28, 29]. Assume, (Ci) & (Cj) indicate nodes of the neutrosophic graph. The edge directed from (Ci) to (Cj) is the connection of both the nodes that indicate the causality of (Ci) on (Cj).

All the edges in the map are assigned with a weight in the set {-1, 0, 1, I}. Maps with the weight of the edge {-1, 0,1, I} are described as simple Neutrosophic Cognitive Maps [3, 30].

Assume C1, C2, C3, ........., Cn are nodes of the graph, so the matrix N(E) is defined as N(E) = (eij), where (eij) is weight of (CiCj) directed edge, where eij H {- 1, 0, 1, I} in this neutrosophic cognitive map, N(E) is the neutrosophic adjacency matrix.

Now here we present a situation through a graphical model as shown in figure (2). We take various factors in India which have a vital role in health deterioration. Indeterminacy plays a critical role in practical living as affirmed by W.B. Vasantha Kandasamy [4, 31]. Hence, in this condition, when data under analysis include indeterminate concepts, we are unable to form mathematical expression by any other method except NCMs because NCM shows the importance of indeterminacy in the situation. An inadequate drainage system leads to unhygienic livelihood as well as affects the purity of water. Here inadequate drainage system is an indeterminate factor. To show the dependency of this indeterminate factor on health deterioration we use Neutrosophic Cognitive Maps. Indeterminacy is represented in figure (1) [32].

Dotted lines depict the indeterminate connection among the nodes.

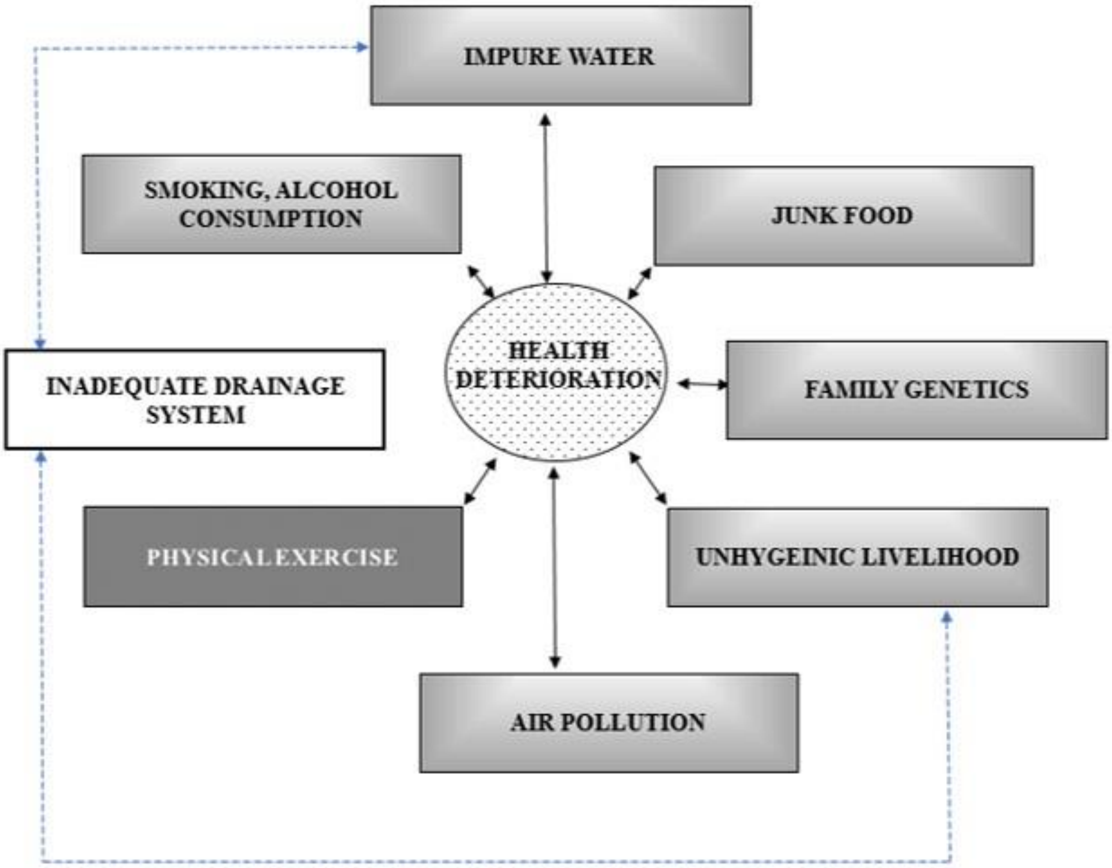


**Figure 2**: Indeterminate factor affecting health

Let us consider the following nodes:

F1 → Air pollution

F2 → Family Genetics

F3→ Unhygienic Livelihood

F4→ Impure Water

F5 → Junk Food

F6 → Smoking &Alcohol Consumption

F7 → Inadequate Drainage System

F8 → Physical Exercise

F9→ Health Deterioration

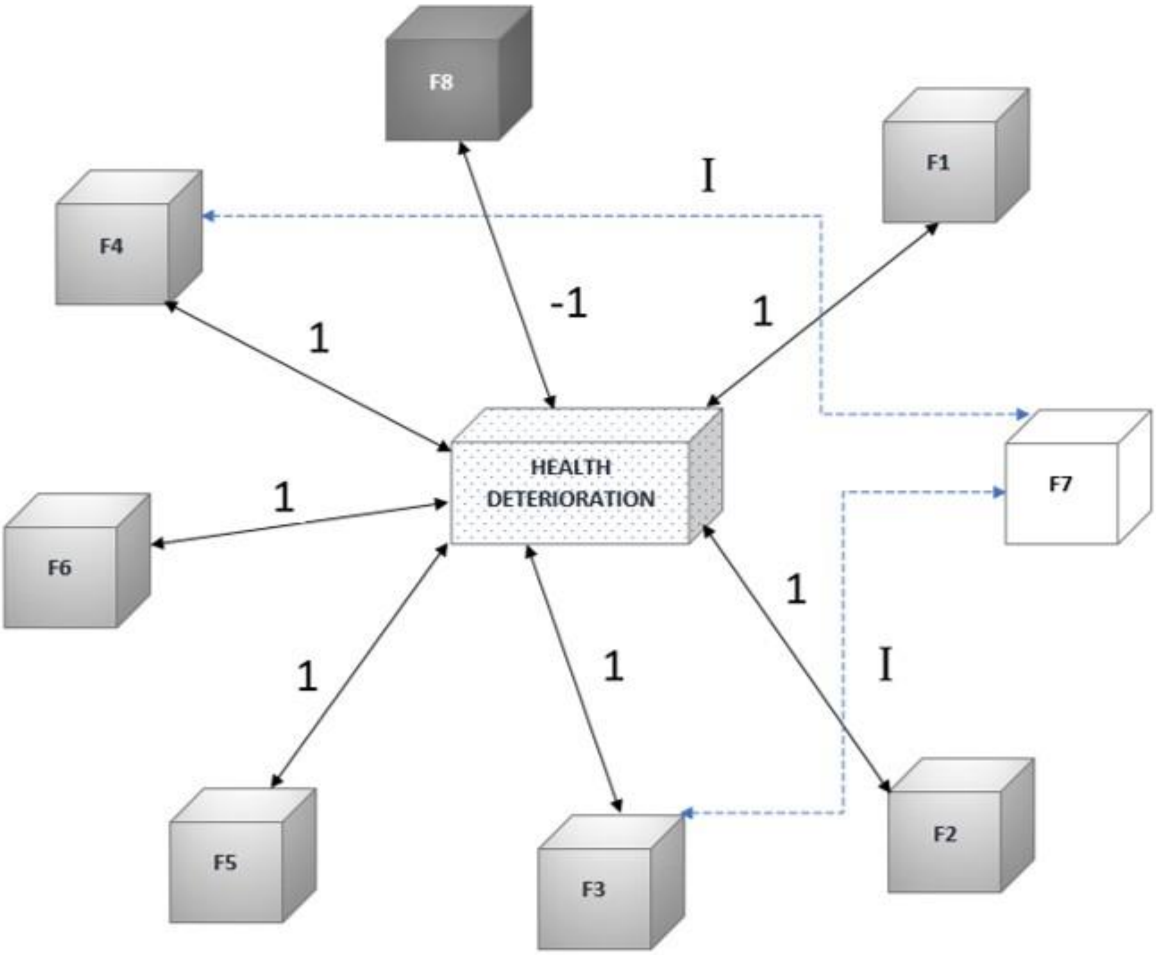


**Figure 3**: Symbolic representation of NCM model

Neutrosophic Cognitive Maps (NCM) express the presence or absence of relationships among concepts and show indeterminate relations among the concepts as shown above. Further, we describe the Neutrosophic Augmented Matrix F(M) in Figure 4.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & I & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 0 \end{pmatrix}$$

**Figure4**: Related connection matrix to the graph in Figure3.

Assume we consider the state vector (The detail of state vector assumption is given in [4]) to be Y1 i.e.:
Y1 = (0 0 0 0 0 0 0 0 1)

Now, we see how it affects F(M). After thresholding and updating the following resultant vector is incurred.

Y1F(M) = (1 1 1 1 1 1 0 -1 1) → (1 1 1 1 1 1 0 0 1) = Y2
Y2F(M) = (1 1 1 1 1 1 I -1 6) → (1 1 1 1 1 1 I 0 1) = Y3
Y3F(M) = (1 1 1+I 1+I 1 1 I -1 6) →( 1 1 1 1 1 1 I 0 1) = Y3

The symbol `→` signifies the thresholder and updated resultant vector. This depicts that health gets affected by air pollution, family genetics, unhygienic livelihood, impure water, junk food, smoking & alcohol consumption and the factor inadequate drainage system is indeterminate to health deterioration.

## 5. Discussion

In this paper, we have stated factors which affect health like air pollution, impure water, etc. To analyze the dependency of each factor including indeterminate factors like inadequate drainage system we have formulated a mathematical expression.
(1 1 1 1 1 1 I 0 1) = Y3

If we examined these dependencies without mathematical expression, then we would have to set up huge labs and perform extensive researches. To overcome this, we use a soft computing approach which makes this interpretation very simple.

In our problem, we can employ two approaches of soft computing: Fuzzy Cognitive Maps (FCM) and Neutrosophic Cognitive Maps (NCM). We haven't used FCM here as in [12] because it has not been successful to associate the indeterminate relations among concepts. NCM is used as it not only represents presence or absence of relationships within concepts but also represents indeterminate relations among the concepts [33, 34]. The research in the literature has stated mostly 5-7 factors, whereas we have identified one more factor, in other words approximately 14% coverage has been increased.

## 6. Conclusion

In this paper, we have tried to come up with a soft computing-based technique to better investigate the factors which could influence the health of a person significantly but has not been addressed adequately in the literature. Some of the factors which affect health are indeterminate but they are essential for measuring health deterioration. We have used a powerful tool, neutrosophic logic that applies over those indeterminate factors which are important but are not affecting health deterioration directly. We have got an increased 14% coverage of the indeterminate factors which can help in making people aware of these so that the masses can get benefits.

In the future, neutrosophy can be applied in various fields namely, expert systems, soft computing techniques in e-commerce and e-learning, reliability theory, image segmentation, robotics etc. which will enhance them eventually. It can also be used widely in situational analysis. Neutrosophy paved its way into research because the universe is filled with indeterminacy. This logic can ease researchers and developers in building algorithms involved in decision making wherein indeterminate factors can be taken into consideration [35, 36].

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# Neutrosophic Beta Distribution with Properties and Applications

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**Abstract:** This research is an extension of classical statistics distribution theory as the theory did not deal with the problems having ambiguity, impreciseness, or indeterminacy. An important life-time distribution called Beta distribution from classical statistics is proposed by considering the indeterminate environment and named the new proposed distribution as neutrosophic beta distribution. Various distributional properties like mean, variance, moment generating function, r-th moment order statistics that includes smallest order statistics, largest order statistics, joint order statistics, and median order statistics are derived. The parameters of the proposed distribution are estimated via maximum likelihood method. Proposed distribution is applied on two real data sets and goodness of fit is assessed through AIC and BIC criteria's. The estimates of the proposed distribution suggested a better fit than the classical form of Beta distribution and recommended to use when the data in the interval form follows a Beta distribution and have some sort of indeterminacy.

## 1. Introduction

The real world is full of ambiguity, unclear, uncertain situations, and problems and a particular value cannot be assigned to the characteristics of the statistics in such an imprecise situation [1]. In such situations, classical probability fails to provide accurate results [2]. In recent times, several developments have been made to model such imprecise situations by considering fuzzy logic and neutrosophy [3-6]. Smarandache [7] proposed neutrosophic statistics that deal with indeterminacy or some part unclear aspect present in the data. He introduced the concept of neutrosophic logic in 1995 by representing the components as T, I, F that represents a true part, undetermined part, and falsehood. Several researchers have contributed to the theory of neutrosophic statistics both methodologically and applied form e.g., Alhabib [8] design the time-series theory under indeterminacy, Aslam et al. [9-14] extended the theory of control charts and sampling plans under indeterminacy environment and presented several neutrosophic control charts and sampling plans, [1, 15, 16] applied the neutrosophic theory in engineering problems. Various researchers have been done in terms of neutrosophic probability distributions to calculate indeterminacy in real-life problems and produced better results in comparison to classical statistics. Alhasan and Smarandache [17] proposed Neutrosophic Weibull distribution in terms of neutrosophic statistics. This distribution gives more space in the applied area due to its wide applicability in classical statistics that helps in solving more problems that have been ignored in classical statistics under indeterminacy.

Neutrosophic Uniform, Neutrosophic exponential, and Neutrosophic Poisson have been developed and solved numerically [5, 18]. Normal distribution and binomial distribution in terms of neutrosophy are explored in detail through many examples by [19]. Aslam and Ahtisham [2] proposed the neutrosophic form of Raleigh distribution. In this research, we proposed an important life-time Beta distribution in the form of neutrosophy and extended the applications of the classical beta distribution when the data is in interval form and has some form of indeterminacy. Several properties are explored under the newly proposed distribution and explained the applications with the help of simulated and real-life data examples.

## 2. Neutrosophic Form

In classical data, there is the crisp value or specific values to deal with but in neutrosophic statistics, data can be in any form because indeterminacy can occur in any form and it depends upon the type of problem we are solving. The form of Neutrosophic number in terms of extension of classical statistics has a standard form and is shown as follow:

$$X = a + i$$

where, a = determined/known part of the data and, i = uncertain/ indeterminacy part of the data. a and i can be any real number. $\mu_N \in [\mu_L, \mu_U]$

## 3. Some existing neutrosophic continuous probability distributions

The followings are the extended classical distributions with a neutrosophic logic in literature:

### 3.1 Neutrosophic Weibull Distribution

The probability density function of neutrosophic Weibull distribution is:

$$f_N(X) = \frac{\beta_N}{\alpha_N} X^{\beta_N - 1} e^{-\left(\frac{X}{\alpha_N}\right)^{\beta_N}}, \qquad X > 0, \alpha_N > 0, \beta_N > 0 \tag{1}$$

### 3.2 Neutrosophic Gamma Distribution

The probability density function of the neutrosophic Gamma distribution is:

$$f(t_N) = \frac{{b_N}^{a_N}}{\Gamma a_N} {t_N}^{a_N - 1} e^{-b_N t_N}; \;\; t_N, b_N, a_N > 0 \tag{2}$$

### 3.3 Neutrosophic Exponential Distribution

The probability density function of the neutrosophic exponential distribution is:

$$f_N(x) = \lambda_N e^{-x\lambda_N} \; ; x>0 \, , \lambda_N > 0 \tag{3}$$

### 3.4 Neutrosophic Normal Distribution

Probability density function of neutrosophic normal distribution is:

$$X_N \sim N_N(\mu_N, {\sigma^2}_N) = \frac{1}{\sigma_N\sqrt{2\pi}} \exp\left(-\frac{(x-\mu_N)^2}{2{\sigma^2}_N}\right); X, \mu_N, \sigma_N > 0 \tag{4}$$

## 4. Neutrosophic Beta Distribution

A neutrosophic Beta Distribution (N-Beta) of a continuous variable X can or cannot be a classical beta distribution of X, having or not having its mean or parameters imprecise or unclear. Consider X as the classical random variable which has a neutrosophic beta distribution having neutrosophic parameters $\alpha_N , \beta_N$ i.e. X→N-beta $(x; \alpha_N , \beta_N)$, then pdf is as follow:

$$f_N(X) = \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)} \qquad where\ X > 0 \tag{5}$$

$\alpha_N , \beta_N$ are the neutrosophic shape parameters.

with cdf

$$F_N(X) = I_x(\alpha_N, \beta_N) = \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)} \tag{6}$$

5. **Mathematical Properties**

Various properties of neutrosophic beta distribution for the r.v X→N-beta $(x; \alpha_N , \beta_N)$ have been derived and the results are shown as follow:

**Mean:** $E_N(X) = \frac{\alpha_N}{\alpha_N + \beta_N}$ (7)

**Variance:** $V_N(X) = \frac{\alpha_N \beta_N}{(\alpha_N + \beta_N + 1)(\alpha_N + \beta_N)^2}$ (8)

**R-th Moment:** $E_N(X^r) = \prod_{i=0}^{r-1} \frac{(\alpha_N + i)}{(\alpha_N + \beta_N + i)}$ (9)

**Moment Generating Function:** $E_N(e^{tx}) = \sum_{k=0}^{\infty} \frac{t^k \alpha_N{}^{(k)}}{k!\,(\alpha_N + \beta_N)^{(k)}}$ (10)

**Hazard Rate Function:** $h_N(X) = \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N) - \beta(X, \alpha_N, \beta_N)}$ (11)

**Survival Function:** $S_N(X) = 1 - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}$ (12)

**R-th Order Statistics:** Let $X_1, X_2, \dots, X_r$ be the random sample from N-beta and let $X_{(1)}, X_{(2)}, \dots, X_{(r)}$ be the corresponding order statistics. $R^{th}$ order statistics of neutrosophic beta distribution can be given as:

$$f_{N_{r,n}}(x) = \frac{1}{B(r, n-r+1)} \left[\frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{r-1} \left[1 - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{n-r} * \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)} \tag{13}$$

**Smallest Order Statistics:** $f_{N_{1:n}}(x) = n_N \left[1 - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{n-1} * \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)}$ (14)

**Largest Order Statistics:** $f_{N_{n:n}}(x) = n_N \left[\frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{n-1} * \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)}$ (15)

**Joint Order Statistics:**

$$f_{N_{m+1:n}}(x) = \frac{n_N!}{(i-1)!\,(n_N - j)!(i-j-1)!} * \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)} * \frac{Y^{\alpha_N - 1}(1-Y)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)} * \left[\frac{\beta(Y, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)} - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{j-i-1} \left[1 - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{n_N - j} * \left[\frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{i-1} \tag{16}$$

**Median Order Statistics:**

$$f_{N_{m+1:n}}(x) = \frac{(2m+1)!}{m! n_N!} \left[\frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{m} \left[1 - \frac{\beta(X, \alpha_N, \beta_N)}{\beta(\alpha_N, \beta_N)}\right]^{m} * \frac{X^{\alpha_N - 1}(1-X)^{\beta_N - 1}}{\beta(\alpha_N, \beta_N)} \tag{17}$$

## 6. Parameter Estimation

In this section, the parameters of neutrosophic beta distribution has been calculated through the maximum likelihood method. Estimated parameters are given below:

$$\frac{\partial lnL}{\partial \alpha_N} = \sum_{i=1}^{n} lnx_i - \frac{n.\partial \ln[\beta(\alpha_N,\beta_N)]}{\partial \alpha_N} = 0 \qquad (18)$$

$$\frac{\partial lnL}{\partial \beta_N} = \sum_{i=1}^{n} \ln(1-x_i) - n.\frac{\partial \ln[\beta(\alpha_N,\beta_N)]}{\partial \beta_N} = 0 \qquad (19)$$

Theoretical estimation of parameters is not possible but they can be estimated mathematical simulation.

## 7. APPLICATIONS

In this section parameters of the proposed distribution will be estimated with the help of real-life data examples and the goodness of fit of the proposed distribution will be assessed by using the AIC and BIC criterias.

**Case Study 1:** Data has been taken from [19] and related to the exceedances of "flood peaks (in m 3/s) of the Wheaton river near Carcross in Yukon Territory, Canada". Parameters of the neutrosophic beta distribution are calculated by the MLE method and to see the performance of this model goodness of fit is calculated. We assumed beta as a neutrosophic parameter and alpha as a classical parameter. For comparison purposes classical beta distribution will also be used. The results are as follow:

Table 1. Parameter estimation and goodness of fit of N-Beta $(\alpha_N, \beta_N)$.

| Distribution | MLE Estimates | | AIC | BIC |
|---|---|---|---|---|
| | $\hat{\alpha}$ | $\hat{\beta}$ | | |
| N-beta | 0.8597 | [1.9450,1.276] | [278.608, 83.8750] | [362.965, 53.9857] |
| Beta | 0.9096 | 1.316 | 398.437 | 446.587 |

It can be seen that in terms of neutrosophy, neutrosophic statistics is more accurate to give results in terms of impreciseness instead of ignoring impreciseness and uncertainty. Hence we can say that the proposed neutrosophic Beta distribution is more accurate than the classical beta distribution when the data is in interval form and contains some sort of indeterminacy.

**Case Study 2:** Another real-life application has been done on data of automobiles that have been taken from an automobile manufacturing company in Korea. Twenty eight uncertain data observations have been taken [20]. Parameters of the neutrosophic Beta distribution are calculated by the MLE method and to see the performance of this model goodness of fit AIC, BIC method is used. We assumed beta as a neutrosophic parameter and alpha are taken as a classical parameter. For comparison purpose, classical beta distribution is used. The results are as follow:

From the results presented in TABLE 2, it can be seen that the proposed neutrosophic Beta distribution is more accurate as compared to classical distribution.

Table 2. Parameter estimation and goodness of fit of automobile data for N-Beta ($\alpha_N, \beta_N$).

| **Distribution** | **MLE Estimates** | | **AIC** | **BIC** |
|---|---|---|---|---|
| | $\hat{\alpha}$ | $\hat{\beta}$ | | |
| N-beta | 1.6386 | [2.8437,0.8561] | [728.068, 96.1830] | [658.7123, 49.1298] |
| Beta | 3.6587 | 9.5462 | 898.3621 | 846.8352 |

## 8. CONCLUSION

A generalization of the classical beta distribution has been proposed in the form of neutrosophic beta distribution by considering the interval form of the data occurring in many real-life situations. we derived several properties of the proposed distribution that include the measure of location, measure of spread R-th moment, survival function, hazard rate function, and common forms of order statistics. Neutrosophic beta distribution is applied to two real-life data sets to see if they behave well as compared to classical beta distribution and found better. We conclude that in real life situations having some sort of uncertainty, and indeterminacy in it and follow the beta distribution than in such situation our proposed form of the beta distribution will perform better and provide more realistic results by coping the indeterminacy of the data.

**Conflicts of Interest:** The authors declare no conflict of interest.

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# Interval Valued Intuitionistic Neutrosophic Soft Set and its Application on Diagnosing Psychiatric Disorder by Using Similarity Measure

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**Abstract**. The primary focus of this manuscript comprises three sections. Initially, we discuss the notion of an interval-valued intuitionistic neutrosophic soft set. We impose an intuitionistic condition between the membership grades of truth and falsity such that their supremum sum does not exceed unity. Similarly, for indeterminacy, the membership grade is in interval from the closed interval $[0, 1]$. Hence in this case, the supremum sum of membership grades of truth, indeterminacy, and falsity does not exceed two. We present the notion of necessity, possibility, concentration, and dilation operators and establish some of its properties. Second, we define the similarity measure between two interval-valued intuitionistic neutrosophic soft sets. Also, we discuss its superiority by comparing it with existing methods. Finally, we develop an algorithm and illustrate with an example of diagnosing psychiatric disorders. Even though the similarity measure plays a vital role in diagnosing psychiatric disorders, existing methods deal hardly in diagnosing psychiatric disorders. By nature, most of the psychiatric disorder behaviors are ambivalence. Hence, it is vital to capture the membership grades by using interval-valued intuitionistic neutrosophic soft set. In this manuscript, we provide a solution in diagnosing psychiatric disorders, and the proposed similarity measure is valuable and compatible in diagnosing psychiatric disorders in any neutrosophic environment.

**Keywords:** neutrosophic set, intuitionistic neutrosophic set; similarity measures; decision making.

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## 1. Introduction

Zadeh [35] coined the notion of a fuzzy set (FS) to the world. In FS theory, the membership grade of each element in a set is specified by a real number from the closed interval of $[0, 1]$. Later, Atanassov [3] defined the notion of an intuitionistic FS (IFS) as an extension of FS. In IFS theory, the elements are assumed to posses both membership and non-membership grades with the condition that their sum does not exceed unity. Also, Atanassov [5] established some properties of IFS. Both FS and IFS theories have significant roles in handling decision-making problems. But in today's decision-making scenarios, the primary focus of the decision-makers (DMs) is to select the best option under different precise or imprecise criteria. DMs may fall short of adequate level of knowledge of the problem in cognitive terms and therefore have difficulty in selecting the right object. This difficulty is overcome by the use of the notion of neutrosophic set (NS), which is characterized by the grades of truth, indeterminacy and falsity membership for each element of the set. Smarandache [27] presented the concept of NS. DMs applied this concept widely to show the importance of truth, indeterminacy, and falsity information on which humans handle the decisions. Wang et al. [30] defined the concept of single-valued NS (SVNS) with the restricted conditions for the membership grades to facilitate the real-life applications and to overcome the constraints faced in neutrosophic theory. We cite some recent developments of IFS, NS theories, and also on similarity measures (SMs) below. Beg and Tabasam [7] introduced the concept of comparative linguistic expression based on hesitant IFSs. Anita et al. [2] developed an application to solve multi-criteria decision-making (MCDM) problems using interval-valued IFS of root type. Jianming et al. [22] defined the concepts of weighted aggregation operators in neutrosophic cubic sets (NCSs) and provided applications in MCDM. Majid Khan et al. [24] presented the notions of algebraic and Einstein operators on NCSs and developed an MCDM application based on these operators. Hashim et al. [20] introduced SMs in neutrosophic bipolar FS with a purpose to build a children's hospital with the help of the HOPE foundation. Chinnadurai et al. [15] presented the concept of unique ranking by using the parameters in a neutrosophic environment. Chinnadurai and Bobin [16] used prospect theory in real-life applications to solve MCDM problems. Saranya et al. [25] introduced an application for finding the similarity value of any two NSs in the neutrosophic environment by using programming language. Broumi and Smarandache [9] developed SMs using Hausdorff distance. Liu et al. [23] introduced the concept of SMs using Euclidean distance. Chahhtterjee et al. [13] presented various concepts of SMs in neutrosophic environment. Shahzadi et al. [26] diagnosed the medical symptoms using SVNSs. Hamidi and Borumand [19] developed the concept of neutrosophic graphs to analyze the sensor networks.

Smarandache [28] studied the concept of soul in psychology by using neutrosophic theory. Christianto and Smarandache [17] reviewed the concept of cultural psychology as one of the seven applications using neutrosophic logic. Chicaiza et al. [14] studied the concept of emotional intelligence of the students using neutrosophic psychology. Abdel-Basset et al. [1] used cosine SMs in bipolar and interval-valued bipolar SVNS to diagnose bipolar disorder behaviors. The domination of NS and SVNS in psychology is clear from the above-cited works. Hence, in this research, we enlighten the concept of neutrosophic theory in the field of psychology. In general, the psychotherapist considers that each person holds a mental structure called the self, which acts as an origin of personality. Winnicott [33] introduced the concept of true self and false self. He applied the thought of true-self to sense out the individual's aliveness or conceiving real. True-self is always be in part or hidden completely. The notion of false-self acts as a defense mechanism to protect the true-self by hibernating it. He explained that the behavior of a person in society is a gentle and self-conscious attitude because of the defense mechanism of false-self. In brief, we can establish the separation of true and false self on a continuum between the normal and the pathological behaviors. There always exists a doubtful amount of unconscious substance that pertains to the whole of the self. The above statements explain the need for an intuitionistic neutrosophic set (INS) to deal with this new concept. We cite the literature review of INS and its role in decision making below. Bhowmik and Pal [8] presented the concept of INS and studied its properties. Broumi and Smarandache [10] defined the concept of intuitionistic neutrosophic soft set (INSS) and established some of its properties. Both INS and INSS have a significant role in handling decision-making problems. They defined the restricted conditions as i) the minimum of membership grades between truth and indeterminacy to be less than or equal to 0.5, ii) the minimum of membership grades between truth and falsity to be less than or equal to 0.5, and iii) the minimum of membership grades between falsity and indeterminacy to be less than or equal to 0.5, such that the sum of membership grades of truth, indeterminacy, and falsity cannot exceed two. Let us consider an example $\mathcal{N} = \langle 0.4, 0.7, 0.6\rangle$. Now according to INS definition, we have $\min\{0.4, 0.6\} < 0.5,\ \min\{0.4, 0.7\} < 0.5$ and $\min\{0.6, 0.7\} \nless 0.5$ but satisfies the condition $0 < 0.4 + 0.7 + 0.6 < 2$. It is evident that the given example is not an INS. However, the DM may have a situation where the membership grades of falsity and indeterminacy are greater than 0.5. Similarly, let us consider another example $\mathcal{N} = \langle 0.7, 0.8, 0.3\rangle$. According to INS definition, we have $\min\{0.7, 0.3\} < 0.5,\ \min\{0.8, 0.3\} < 0.5$ and $\min\{0.7, 0.8\} \nless 0.5$ but satisfies the condition $0 < 0.7+0.8+0.3 < 2$. It is clear that the given example is not an INS. However, the DM may have a situation where the membership grades of truth and falsity are greater than 0.5. Therefore DM may have some constraints while handling these information in INS environment. Now, when the membership grades are in interval as well the membership grades of true and false are a continuum and the membership grade of indeterminacy is independent, it becomes a challenge to input the grades during decision-making with the help of INS and INSS. Hence, it is very clear that there is a need for a new INS with simplified conditions. So, we introduce an interval-valued INS (IVINS) and interval-valued INSS (IVINSS) to effectively handle the decision-making problems.

One of the purpose of this study is to bring out the importance of IVINSS when experts provide membership grades in truth, indeterminacy, and falsity in a restricted environment. In recent years, human beings face many decisions-making problems in multiple fields and analyzing the psychiatric disorder of the subject is one of them. Similarly, SM plays a significant factor in diagnosing psychiatric disorders, but hardly no existing methods deal with it. Therefore, it is necessary to provide a working model for determining the same. DMs look for many novel extensions of the NS to compete with other working models. So, we propose a new extension of NS and a model for diagnosing the psychiatric disorders using SM, which provides an advantage for the DMs to make well-defined decisions.

The manuscript enfolds the following sections. Section 2 provides a glimpse of existing definitions. Section 3 introduces the concept of IVINSS and some basic definitions related to IVINSS. Section 4 and 5 deal with the necessity, possibility, and two new operators ($\pm$ and $\mp$) with their properties. Section 6 explains the concept of $\mathcal{N}_\epsilon$, $\mathcal{N}_{\epsilon,\rho}$ and $\mathcal{I}_{\epsilon,\rho}$ operators on IVINSS. Section 7 provides insight on concentration and dilation operators on SINSS. Section 8 highlights the concept of SM with a new definition and also with a comparison study to show the importance of the proposed method. Section 9 wraps up with a conclusion.

## 2. **Preliminaries**

We discuss some essential definitions required for this study. Let us consider the following notations throughout this manuscript unless otherwise specified . Let $\mathcal{V}$ represent universe and $v \in \mathcal{V}$, $\mathcal{Q}$ be a set of parameters, $\mathcal{S} \subseteq \mathcal{Q}$, C[0,1] denotes the set of all closed sub intervals of [0,1] and $\mathcal{I}^N$ be the set of all IVINS over $\mathcal{V}$.

**Definition 2.1.** [6] An interval-valued IFS (IVIFS) is a set of the form $\mathcal{F} = \{\langle v, \alpha_{\mathcal{F}}(v), \gamma_{\mathcal{F}}(v)\rangle\}$, where $\alpha_{\mathcal{F}}(v) : \mathcal{V} \to C[0,1]$ and $\gamma_{\mathcal{F}}(v) : \mathcal{V} \to C[0,1]$ are the interval-valued membership and non-membership grades respectively. The lower and upper ends of $\alpha_{\mathcal{F}}(v)$ and $\gamma_{\mathcal{F}}(v)$ are denoted by $\underline{\alpha}_{\mathcal{F}}(v)$, $\overline{\alpha}_{\mathcal{F}}(v)$ and $\underline{\gamma}_{\mathcal{F}}(v)$, $\overline{\gamma}_{\mathcal{F}}(v)$, where $0 \leq \overline{\alpha}_{\mathcal{F}}(v)+\overline{\gamma}_{\mathcal{F}}(v) \leq 1$ and $\underline{\alpha}_{\mathcal{F}}(v), \underline{\gamma}_{\mathcal{F}}(v) \geq 0$.

**Definition 2.2.** [31] An interval-valued NS (IVNS) is a set of the form $\mathcal{N} = \{v, \langle \alpha_{\mathcal{N}}(v), \beta_{\mathcal{N}}(v), \gamma_{\mathcal{N}}(v)\rangle\}$, where $\alpha_{\mathcal{N}}(v) : \mathcal{V} \to C[0,1]$, $\beta_{\mathcal{N}}(v) : \mathcal{V} \to C[0,1]$ and $\gamma_{\mathcal{N}}(v) : \mathcal{V} \to C[0,1]$ are the interval-valued membership of truth, indeterminacy and falsity respectively. The lower and upper limits of $\alpha_{\mathcal{N}}(v)$, $\beta_{\mathcal{N}}(v)$ and $\gamma_{\mathcal{N}}(v)$ are denoted by $\underline{\alpha}_{\mathcal{N}}(v)$, $\overline{\alpha}_{\mathcal{N}}(v)$, $\underline{\beta}_{\mathcal{N}}(v)$, $\overline{\beta}_{\mathcal{N}}(v)$, and $\underline{\gamma}_{\mathcal{N}}(v)$, $\overline{\gamma}_{\mathcal{N}}(v)$, where $0 \leq \overline{\alpha}_{\mathcal{N}}(v) + \overline{\beta}_{\mathcal{N}}(v) + \overline{\gamma}_{\mathcal{N}}(v) \leq 3$.

## 3. **Interval-valued intuitionistic neutrosophic soft set**

We present the notion of IVINSS and investigate some of its properties. We generalize the operations and properties on IVINSS by the concepts discussed in [3] and [21].

**Definition 3.1.** An IVINS in $\mathcal{V}$ is a set of the form $\mathcal{I} = \{\langle v, \alpha_{\mathcal{I}}(v), \beta_{\mathcal{I}}(v), \gamma_{\mathcal{I}}(v)\rangle\}$, where $\alpha_{\mathcal{I}}(v) : \mathcal{V} \to C[0,1]$, $\beta_{\mathcal{I}}(v) : \mathcal{V} \to C[0,1]$ and $\gamma_{\mathcal{I}}(v) : \mathcal{V} \to C[0,1]$. $\alpha_{\mathcal{I}}(v)$, $\beta_{\mathcal{I}}(v)$ and $\gamma_{\mathcal{I}}(v)$ are closed sub intervals of [0,1], representing the membership grades of truth, indeterminacy and falsity of the element $v \in \mathcal{V}$. The lower and upper ends of $\alpha_{\mathcal{I}}(v)$, $\beta_{\mathcal{I}}(v)$ and $\gamma_{\mathcal{I}}(v)$ are denoted, respectively by $\underline{\alpha}_{\mathcal{I}}(v)$, $\overline{\alpha}_{\mathcal{I}}(v)$, $\underline{\beta}_{\mathcal{I}}(v)$, $\overline{\beta}_{\mathcal{I}}(v)$, and $\underline{\gamma}_{\mathcal{I}}(v)$, $\overline{\gamma}_{\mathcal{I}}(v)$, where $0 \leq \overline{\alpha}_{\mathcal{I}}(v) + \overline{\gamma}_{\mathcal{I}}(v) \leq 1$ and $\underline{\alpha}_{\mathcal{I}}(v), \underline{\beta}_{\mathcal{I}}(v), \underline{\gamma}_{\mathcal{I}}(v) \geq 0$, $0 \leq \overline{\alpha}_{\mathcal{I}}(v) + \overline{\beta}_{\mathcal{I}}(v) + \overline{\gamma}_{\mathcal{I}}(v) \leq 2$, $\forall\, v \in \mathcal{V}$.

**Example 3.2.** Let $\mathcal{V} = \{v_1, v_2, v_3\}$ be a non-empty set. Then an IVINS on $\mathcal{V}$ can be represented as,

$$\mathcal{I} = \Big\{ \langle v_1, [0.3, 0.4], [0.7, 0.8], [0.1, 0.2] \rangle, \langle v_2, [0.4, 0.5], [0.8, 0.9], [0.2, 0.3] \rangle, \langle v_3, [0.6, 0.7], [0.2, 0.3], [0.2, 0.3] \rangle \Big\}.$$

**Definition 3.3.** A pair $(\Omega, \mathcal{S})$ is called IVINSS over $\mathcal{V}$, where $\Omega : \mathcal{S} \to \mathcal{I}^N$. Thus for any parameter $q \in \mathcal{S}$, $\Omega(q)$ is an IVINSS.

**Example 3.4.** Let $\mathcal{V} = \{v_1, v_2, v_3\}$ represent clients with cognitive disorders and $\mathcal{S} = \{q_1, q_2, q_3\}$ represent symptoms which stand for inability of motor coordination (IMC), loss of memory (LM) and identity confusion (IC) respectively. An IVINSS $(\Omega, \mathcal{S})$ is a collection of subsets of $\mathcal{V}$, given by a psychiatrist based on the description in Table 1.

Table 1. Shows client with cognitive disorders in IVINSS $(\Omega, \mathcal{S})$ form.

| $\mathcal{V}$ | IMC($q_1$) | LM($q_2$) | IC($q_3$) |
|---|---|---|---|
| $v_1$ | $\langle [0.2, 0.4], [0.4, 0.5], [0.4, 0.5] \rangle$ | $\langle [0.3, 0.4], [0.5, 0.6], [0.3, 0.5] \rangle$ | $\langle [0.2, 0.3], [0.5, 0.8], [0.6, 0.7] \rangle$ |
| $v_2$ | $\langle [0.4, 0.6], [0.3, 0.5], [0.1, 0.2] \rangle$ | $\langle [0.7, 0.8], [0.2, 0.5], [0.1, 0.2] \rangle$ | $\langle [0.6, 0.7], [0.7, 0.8], [0.1, 0.2] \rangle$ |
| $v_3$ | $\langle [0.6, 0.7], [0.2, 0.7], [0.1, 0.2] \rangle$ | $\langle [0.1, 0.3], [0.6, 0.7], [0.5, 0.6] \rangle$ | $\langle [0.2, 0.3], [0.7, 0.8], [0.4, 0.5] \rangle$ |

**Definition 3.5.** Let $(\Omega_1, \mathcal{S}_1)$ and $(\Omega_2, \mathcal{S}_2)$ be two IVINSS over $\mathcal{V}$. Then,
(i) $(\Omega_1, \mathcal{S}_1)$ OR $(\Omega_2, \mathcal{S}_2)$ is an IVINSS represented as $(\Omega_1, \mathcal{S}_1) \vee (\Omega_2, \mathcal{S}_2) = (\Omega_\vee, \mathcal{S}_1 \times \mathcal{S}_2)$, where $\Omega_\vee(q_1, q_2) = \Omega_1(q_1) \cup \Omega_2(q_2)$, $\forall\, (q_1, q_2) \in \mathcal{S}_1 \times \mathcal{S}_2$.

$$\begin{aligned}\Omega_\vee(q_1, q_2) = \Big\langle & [\vee(\underline{\alpha}_{\Omega_1(q_1)}(v), \underline{\alpha}_{\Omega_2(q_2)}(v)), \vee(\overline{\alpha}_{\Omega_1(q_1)}(v), \overline{\alpha}_{\Omega_2(q_2)}(v))], \\ & [\vee(\underline{\beta}_{\Omega_1(q_1)}(v), \underline{\beta}_{\Omega_2(q_2)}(v)), \vee(\overline{\beta}_{\Omega_1(q_1)}(v), \overline{\beta}_{\Omega_2(q_2)}(v))], \\ & [\wedge(\underline{\gamma}_{\Omega_1(q_1)}(v), \underline{\gamma}_{\Omega_2(q_2)}(v)), \wedge(\overline{\alpha}_{\Omega_1(q_1)}(v), \overline{\alpha}_{\Omega_2(q_2)}(v))], \Big\rangle, \forall\, (q_1, q_2) \in \mathcal{S}_1 \times \mathcal{S}_2.\end{aligned}$$

(ii) $(\Omega_1, \mathcal{S}_1)$ AND $(\Omega_2, \mathcal{S}_2)$ is an IVINSS represented as $(\Omega_1, \mathcal{S}_1) \wedge (\Omega_2, \mathcal{S}_2) = (\Omega_\wedge, \mathcal{S}_1 \times \mathcal{S}_2)$, where $\Omega_\wedge(q_1, q_2) = \Omega_1(q_1) \cap \Omega_2(q_2)$, $\forall\, (q_1, q_2) \in \mathcal{S}_1 \times \mathcal{S}_2$.

$$\begin{aligned}\Omega_\wedge(q_1, q_2) = \Big\langle & [\wedge(\underline{\alpha}_{\Omega_1(q_1)}(v), \underline{\alpha}_{\Omega_2(q_2)}(v)), \wedge(\overline{\alpha}_{\Omega_1(q_1)}(v), \overline{\alpha}_{\Omega_2(q_2)}(v))], \\ & [\wedge(\underline{\beta}_{\Omega_1(q_1)}(v), \underline{\beta}_{\Omega_2(q_2)}(v)), \wedge(\overline{\beta}_{\Omega_1(q_1)}(v), \overline{\beta}_{\Omega_2(q_2)}(v))], \\ & [\vee(\underline{\gamma}_{\Omega_1(q_1)}(v), \underline{\gamma}_{\Omega_2(q_2)}(v)), \vee(\overline{\gamma}_{\Omega_1(q_1)}(v), \overline{\gamma}_{\Omega_2(q_2)}(v))] \Big\rangle, \forall\, (q_1, q_2) \in \mathcal{S}_1 \times \mathcal{S}_2.\end{aligned}$$

**Definition 3.6.** Let $(\Omega_1, \mathcal{S}_1)$ and $(\Omega_2, \mathcal{S}_2)$ be two IVINSS over $\mathcal{V}$. Then,
(i) $(\Omega_1, \mathcal{S}_1)$ union $(\Omega_2, \mathcal{S}_2)$ is an IVINSS represented as $(\Omega_1, \mathcal{S}_1) \Cup (\Omega_2, \mathcal{S}_2) = (\Omega_{\Cup}, \mathcal{S}_{\Cup})$, where $\mathcal{S}_{\Cup} = \mathcal{S}_1 \cup \mathcal{S}_2$ and $\forall\ q \in \mathcal{S}_{\Cup}$,

$$\Omega_{\Cup}(q) = \begin{cases} \left\{\left\langle v, (\alpha_{\Omega_1(q)}(v), \beta_{\Omega_1(q)}(v), \gamma_{\Omega_1(q)}(v))\right\rangle; & \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\right\}, \\ \left\{\left\langle v, (\alpha_{\Omega_2(q)}(v), \beta_{\Omega_2(q)}(v), \gamma_{\Omega_2(q)}(v))\right\rangle; & \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\right\}, \\ \left\{\left\langle v, [\vee(\underline{\alpha}_{\Omega_1(q)}(v), \underline{\alpha}_{\Omega_2(q)}(v)), \vee(\overline{\alpha}_{\Omega_1(q)}(v), \overline{\alpha}_{\Omega_2(q)}(v))],\right.\right. & \\ \quad [\vee(\underline{\beta}_{\Omega_1(q)}(v), \underline{\beta}_{\Omega_2(q)}(v)), \vee(\overline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_2(q)}(v))], & \\ \quad \left.\left.[\wedge(\underline{\gamma}_{\Omega_1(q)}(v), \underline{\gamma}_{\Omega_2(q)}(v)), \wedge(\overline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v))]\right\rangle; \right. & \text{if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\big\}. \end{cases}$$

(ii) $(\Omega_1, \mathcal{S}_1)$ intersection $(\Omega_2, \mathcal{S}_2)$ is an IVINSS represented as $(\Omega_1, \mathcal{S}_1) \Cap (\Omega_2, \mathcal{S}_2) = (\Omega_{\Cap}, \mathcal{S}_{\Cap})$, where $\mathcal{S}_{\Cap} = \mathcal{S}_1 \cup \mathcal{S}_2$ and $\forall\ q \in \mathcal{S}_{\Cap}$,

$$\Omega_{\Cap}(q) = \begin{cases} \left\{\left\langle v, (\alpha_{\Omega_1(q)}(v), \beta_{\Omega_1(q)}(v), \gamma_{\Omega_1(q)}(v))\right\rangle; & \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\right\}, \\ \left\{\left\langle v, (\alpha_{\Omega_2(q)}(v), \beta_{\Omega_2(q)}(v), \gamma_{\Omega_2(q)}(v))\right\rangle; & \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\right\}, \\ \left\{\left\langle v, [\wedge(\underline{\alpha}_{\Omega_1(q)}(v), \underline{\alpha}_{\Omega_2(q)}(v)), \wedge(\overline{\alpha}_{\Omega_1(q)}(v), \overline{\alpha}_{\Omega_2(q)}(v))],\right.\right. & \\ \quad [\wedge(\underline{\beta}_{\Omega_1(q)}(v), \underline{\beta}_{\Omega_2(q)}(v)), \wedge(\overline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_2(q)}(v))], & \\ \quad \left.\left.[\vee(\underline{\gamma}_{\Omega_1(q)}(v), \underline{\gamma}_{\Omega_2(q)}(v)), \vee(\overline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v))]\right\rangle; \right. & \text{if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\big\}. \end{cases}$$

**Definition 3.7.** The complement of an IVINSS $(\Omega, \mathcal{S})$ is represented as,

$$(\Omega, \mathcal{S})^c = \left\{\left\langle v, \gamma_{\Omega(q)}(v), [(1 - \overline{\beta}_{\Omega(q)}(v)), (1 - \underline{\beta}_{\Omega(q)}(v))], \alpha_{\Omega(q)}(v)\right\rangle; \text{and } q \in \mathcal{S}\right\}.$$

**Theorem 3.8.** *Let $(\Omega_1, \mathcal{S}_1)$ and $(\Omega_2, \mathcal{S}_2)$ be two IVINSS over $\mathcal{V}$. Then,*
*(i) $\big((\Omega_1, \mathcal{S}_1) \vee (\Omega_2, \mathcal{S}_2)\big)^c = (\Omega_1, \mathcal{S}_1)^c \wedge (\Omega_2, \mathcal{S}_2)^c$;*
*(ii) $\big((\Omega_1, \mathcal{S}_1) \wedge (\Omega_2, \mathcal{S}_2)\big)^c = (\Omega_1, \mathcal{S}_1)^c \vee (\Omega_2, \mathcal{S}_2)^c$.*

*Proof.* We give the proof of (i), and proof of (ii) is analogous.
(i) $(\Omega_1, \mathcal{S}_1) \vee (\Omega_2, \mathcal{S}_2) = (\Omega_\vee, \mathcal{S}_1 \times \mathcal{S}_2)$.
$\big((\Omega_1, \mathcal{S}_1) \vee (\Omega_2, \mathcal{S}_2)\big)^c = (\Omega_\vee, \mathcal{S}_1 \times \mathcal{S}_2)^c$.

$$\begin{aligned} \Omega_\vee^c(q_1, q_2) = \big\langle & [\wedge(\underline{\gamma}_{\Omega_1(q_1)}(v), \underline{\gamma}_{\Omega_2(q_2)}(v)), \wedge(\overline{\gamma}_{\Omega_1(q_1)}(v), \overline{\gamma}_{\Omega_2(q_2)}(v))], \\ & [\wedge((1 - \overline{\beta}_{\Omega_1(q_1)}(v)), (1 - \overline{\beta}_{\Omega_2(q_2)}(v))), \wedge((1 - \underline{\beta}_{\Omega_1(q_1)}(v)), (1 - \underline{\beta}_{\Omega_2(q_2)}(v)))], \\ & [\vee(\underline{\alpha}_{\Omega_1(q_1)}(v), \underline{\alpha}_{\Omega_2(q_2)}(v)), \vee(\overline{\alpha}_{\Omega_1(q_1)}(v), \overline{\alpha}_{\Omega_2(q_2)}(v))]\big\rangle, \forall\ (q_1, q_2) \in \mathcal{S}_1 \times \mathcal{S}_2. \end{aligned}$$

and $(\Omega_1, \mathcal{S}_1)^c \wedge (\Omega_2, \mathcal{S}_2)^c = (\Omega_\wedge, \mathcal{S}_1 \times \mathcal{S}_2)$.

$$\begin{aligned} \Omega_\wedge(q_1, q_2) = \big\langle & [\wedge(\underline{\gamma}_{\Omega_1(q_1)}(v), \underline{\gamma}_{\Omega_2(q_2)}(v)), \wedge(\overline{\gamma}_{\Omega_1(q_1)}(v), \overline{\gamma}_{\Omega_2(q_2)}(v))], \\ & [\wedge((1 - \overline{\beta}_{\Omega_1(q_1)}(v)), (1 - \overline{\beta}_{\Omega_2(q_2)}(v))), \wedge((1 - \underline{\beta}_{\Omega_1(q_1)}(v)), (1 - \underline{\beta}_{\Omega_2(q_2)}(v)))], \\ & [\vee(\underline{\alpha}_{\Omega_1(q_1)}(v), \underline{\alpha}_{\Omega_2(q_2)}(v)), \vee(\overline{\alpha}_{\Omega_1(q_1)}(v), \overline{\alpha}_{\Omega_2(q_2)}(v))]\big\rangle, \forall\ (q_1, q_2) \in \mathcal{S}_1 \times \mathcal{S}_2. \end{aligned}$$

Thus $\big((\Omega_1,\mathcal{S}_1)\vee(\Omega_2,\mathcal{S}_2)\big)^c=(\Omega_1,\mathcal{S}_1)^c\wedge(\Omega_2,\mathcal{S}_2)^c$. □

**Definition 3.9.** Let $\mathcal{S}_1,\mathcal{S}_2\subseteq\mathcal{Q}$. $(\Omega_1,\mathcal{S}_1)$ is an interval-valued intuitionistic neutrosophic soft subset (IVINSSS) of $(\Omega_2,\mathcal{S}_2)$ represented as $(\Omega_1,\mathcal{S}_1)\Subset(\Omega_2,\mathcal{S}_2)$ if and only if (iff)
(i) $\mathcal{S}_1\subseteq\mathcal{S}_2$;
(ii)$\Omega_1(q)$ is an IVINSSS of $\Omega_2(q)$ that is for all $q\in\mathcal{S}_1$,
$\underline{\alpha}_{\Omega_1(q)}(v)\le\underline{\alpha}_{\Omega_2(q)}(v)$, $\overline{\alpha}_{\Omega_1(q)}(v)\le\overline{\alpha}_{\Omega_2(q)}(v)$; $\underline{\beta}_{\Omega_1(q)}(v)\le\underline{\beta}_{\Omega_2(q)}(v)$ ,
$\overline{\beta}_{\Omega_1(q)}(v)\le\overline{\beta}_{\Omega_2(q)}(v)$ , $\underline{\gamma}_{\Omega_1(q)}(v)\ge\underline{\gamma}_{\Omega_2(q)}(v)$ and $\overline{\gamma}_{\Omega_1(q)}(v)\ge\overline{\gamma}_{\Omega_2(q)}(v)$.
Also, $(\Omega_2,\mathcal{S}_2)$ is called an interval-valued intuitionistic neutrosophic soft superset of $(\Omega_1,\mathcal{S}_1)$ and represented as $(\Omega_2,\mathcal{S}_2)\Supset(\Omega_1,\mathcal{S}_1)$.

**Definition 3.10.** If $(\Omega_1,\mathcal{S}_1)$ and $(\Omega_2,\mathcal{S}_2)$ are two IVINSS, then $(\Omega_1,\mathcal{S}_1)=(\Omega_2,\mathcal{S}_2)$ iff $(\Omega_1,\mathcal{S}_1)\Subset(\Omega_2,\mathcal{S}_2)$ and $(\Omega_2,\mathcal{S}_2)\Subset(\Omega_1,\mathcal{S}_1)$.

## 4. Necessity ($\oplus$) and possibility ($\ominus$) operators on IVINSS

We provide the definition of $\oplus$ and $\ominus$ operators on IVINSS and its properties. We generalize these operations and some properties on IVINSS using the concepts discussed in [3] and [21].

**Definition 4.1.** If $(\Omega,\mathcal{S})$ is an IVINSS over $\mathcal{V}$ and $\Omega:\mathcal{S}\to\mathcal{I}^N$, then,
(i) the necessity operator ($\oplus$) is represented as,

$$\oplus(\Omega,\mathcal{S})=\left\{\left\langle v,\alpha_{\oplus\Omega(q)}(v),\beta_{\oplus\Omega(q)}(v),\gamma_{\oplus\Omega(q)}(v)\right\rangle;\ q\in\mathcal{S}\right\}.$$

Here, $\alpha_{\oplus\Omega(q)}(v)=[\underline{\alpha}_{\Omega(q)}(v),\overline{\alpha}_{\Omega(q)}(v)]$, $\beta_{\oplus\Omega(q)}(v)=[\underline{\beta}_{\Omega(q)}(v),\overline{\beta}_{\Omega(q)}(v)]$ and
$\gamma_{\oplus\Omega(q)}(v)=[(1-\overline{\alpha}_{\Omega(q)}(v)),(1-\underline{\alpha}_{\Omega(q)}(v))]$, are the membership grades of truth, indeterminacy and falsity for the object $v$ on the parameter $q$.
(ii) the possibility operator ($\ominus$) is represented as,

$$\ominus(\Omega,\mathcal{S})=\left\{\left\langle v,\alpha_{\ominus\Omega(q)}(v),\beta_{\ominus\Omega(q)}(v),\gamma_{\ominus\Psi(q)}(v)\right\rangle;\ q\in\mathcal{S}\right\}.$$

Here, $\alpha_{\ominus\Omega(q)}(v)=[(1-\overline{\gamma}_{\Omega(q)}(v)),(1-\underline{\gamma}_{\Omega(q)}(v))]$, $\beta_{\ominus\Omega(q)}(v)=[\underline{\beta}_{\Omega(q)}(v),\overline{\beta}_{\Omega(q)}(v)]$ and
$\gamma_{\ominus\Omega(q)}(v)=[\underline{\gamma}_{\Omega(q)}(v),\overline{\gamma}_{\Omega(q)}(v)]$, are the membership grades of truth, indeterminacy and falsity for the object $v$ on the parameter $q$.

**Example 4.2.** (i) The IVINSS $\oplus(\Omega,\mathcal{S})$ for Example 3.4 is shown in Table 2.

TABLE 2. Shows client with cognitive disorders using $\oplus$ operator.

| $\mathcal{V}$ | IMC($q_1$) | LM($q_2$) | IC($q_3$) |
|---|---|---|---|
| $v_1$ | $\langle[0.2,0.4],[0.4,0.5],[0.6,0.8]\rangle$ | $\langle[0.3,0.4],[0.5,0.6],[0.6,0.7]\rangle$ | $\langle[0.2,0.3],[0.5,0.8],[0.7,0.8]\rangle$ |
| $v_2$ | $\langle[0.4,0.6],[0.3,0.5],[0.4,0.6]\rangle$ | $\langle[0.7,0.8],[0.2,0.5],[0.2,0.3]\rangle$ | $\langle[0.6,0.7],[0.7,0.8],[0.3,0.4]\rangle$ |
| $v_3$ | $\langle[0.6,0.7],[0.2,0.7],[0.3,0.4]\rangle$ | $\langle[0.1,0.3],[0.6,0.7],[0.7,0.9]\rangle$ | $\langle[0.2,0.3],[0.7,0.8],[0.7,0.8]\rangle$ |

TABLE 3. Shows client with cognitive disorders using $\ominus$ operator.

| $\mathcal{V}$ | IMC($q_1$) | LM($q_2$) | IC($q_3$) |
|---|---|---|---|
| $v_1$ | $\langle[0.5,0.6],[0.4,0.5],[0.4,0.5]\rangle$ | $\langle[0.5,0.7],[0.5,0.6],[0.3,0.5]\rangle$ | $\langle[0.3,0.4],[0.5,0.8],[0.6,0.7]\rangle$ |
| $v_2$ | $\langle[0.8,0.9],[0.3,0.5],[0.1,0.2]\rangle$ | $\langle[0.8,0.9],[0.2,0.5],[0.1,0.2]\rangle$ | $\langle[0.8,0.9],[0.7,0.8],[0.1,0.2]\rangle$ |
| $v_3$ | $\langle[0.8,0.9],[0.2,0.7],[0.1,0.2]\rangle$ | $\langle[0.4,0.5],[0.6,0.7],[0.5,0.6]\rangle$ | $\langle[0.5,0.6],[0.7,0.8],[0.4,0.5]\rangle$ |

(ii) The IVINSS $\ominus(\Omega,\mathcal{S})$ for Example 3.4 is shown in Table 3.

**Theorem 4.3.** *Let* $(\Omega_1,\mathcal{S}_1)$ *and* $(\Omega_2,\mathcal{S}_2)$ *be two IVINSS over* $\mathcal{V}$. *Then,*
*(i)* $\oplus\left((\Omega_1,\mathcal{S}_1)\Cup(\Omega_2,\mathcal{S}_2)\right) = \oplus(\Omega_1,\mathcal{S}_1)\Cup\oplus(\Omega_2,\mathcal{S}_2)$;
*(ii)* $\oplus\left((\Omega_1,\mathcal{S}_1)\Cap(\Omega_2,\mathcal{S}_2)\right) = \oplus(\Omega_1,\mathcal{S}_1)\Cap\oplus(\Omega_2,\mathcal{S}_2)$;
*(iii)* $\oplus\oplus(\Omega_1,\mathcal{S}_1)=\oplus(\Omega_1,\mathcal{S}_1)$.

*Proof.* We present the proof of (i), and proof of (ii) is analogous.
(i) Let $(\Omega_1,\mathcal{S}_1)\Cup(\Omega_2,\mathcal{S}_2) = (\Omega_{\Cup},\mathcal{S}_{\Cup})$, where $\mathcal{S}_{\Cup}=\mathcal{S}_1\cup\mathcal{S}_2$, $\forall\ q\in\mathcal{S}_{\Cup}$.
Consider,

$$\oplus\Psi_{\Cup}(q)$$

$$= \begin{cases} \big\{\big\langle v,[\underline{\alpha}_{\Omega_1(q)}(v),\overline{\alpha}_{\Omega_1(q)}(v)],[\underline{\beta}_{\Omega_1(q)}(v),\overline{\beta}_{\Omega_1(q)}(v)], \\ \qquad [(1-\overline{\alpha}_{\Omega_1(q)}(v)),(1-\underline{\alpha}_{\Omega_1(q)}(v))]\big\rangle; \qquad \text{if } q\in\mathcal{S}_1-\mathcal{S}_2\big\}, \\ \big\{\big\langle v,[\underline{\alpha}_{\Omega_2(q)}(v),\overline{\alpha}_{\Omega_2(q)}(v)],[\underline{\beta}_{\Omega_2(q)}(v),\overline{\beta}_{\Omega_2(q)}(v)], \\ \qquad [(1-\overline{\alpha}_{\Omega_2(q)}(v)),(1-\underline{\alpha}_{\Omega_2(q)}(v))]\big\rangle; \qquad \text{if } q\in\mathcal{S}_2-\mathcal{S}_1\big\}, \\ \big\{\big\langle v,\vee(\alpha_{\Omega_1(q)}(v),\alpha_{\Omega_2(q)}(v)),\vee(\beta_{\Omega_1(q)}(v),\beta_{\Omega_2(q)}(v)), \\ \quad [(1-\vee(\overline{\alpha}_{\Omega_1(q)}(v),\overline{\alpha}_{\Omega_2(q)}(v))),(1-\vee(\underline{\alpha}_{\Omega_1(q)}(v),\overline{\alpha}_{\Omega_2(q)}(v)))]\big\rangle; \qquad \text{if } q\in\mathcal{S}_1\cap\mathcal{S}_2\big\}. \end{cases}$$

$$= \begin{cases} \big\{\big\langle v,[\underline{\alpha}_{\Omega_1(q)}(v),\overline{\alpha}_{\Omega_1(q)}(v)],[\underline{\beta}_{\Omega_1(q)}(v),\overline{\beta}_{\Omega_1(q)}(v)], \\ \qquad [(1-\overline{\alpha}_{\Omega_1(q)}(v)),(1-\underline{\alpha}_{\Omega_1(q)}(v))]\big\rangle; \qquad \text{if } q\in\mathcal{S}_1-\mathcal{S}_2\big\}, \\ \big\{\big\langle v,[\underline{\alpha}_{\Omega_2(q)}(v),\overline{\alpha}_{\Omega_2(q)}(v)],[\underline{\beta}_{\Omega_2(q)}(v),\overline{\beta}_{\Omega_2(q)}(v)], \\ \qquad [(1-\overline{\alpha}_{\Omega_2(q)}(v)),(1-\underline{\alpha}_{\Omega_2(q)}(v))]\big\rangle; \qquad \text{if } q\in\mathcal{S}_2-\mathcal{S}_1\big\}, \\ \big\{\big\langle v,\vee(\alpha_{\Omega_1(q)}(v),\alpha_{\Omega_2(q)}(v)),\vee(\beta_{\Omega_1(q)}(v),\beta_{\Omega_2(q)}(v)), \\ \quad [\wedge((1-\overline{\alpha}_{\Omega_1(q)}(v)),(1-\overline{\alpha}_{\Omega_2(q)}(v))),\wedge((1-\underline{\alpha}_{\Omega_1(q)}(v)),(1-\overline{\alpha}_{\Omega_2(q)}(v)))]\big\rangle; \\ \qquad \text{if } q\in\mathcal{S}_1\cap\mathcal{S}_2\big\}. \end{cases}$$

We know that,

$$\oplus(\Omega_1,\mathcal{S}_1) = \big\{\big\langle v,[\underline{\alpha}_{\Omega_1(q)}(v),\overline{\alpha}_{\Omega_1(q)}(v)],[\underline{\beta}_{\Omega_1(q)}(v),\overline{\beta}_{\Omega_1(q)}(v)], [(1-\overline{\alpha}_{\Omega_1(q)}(v)),(1-\underline{\alpha}_{\Omega_1(q)}(v))]\big\rangle; q\in\mathcal{S}_1\big\},$$

$$\oplus(\Omega_2,\mathcal{S}_2) = \big\{\big\langle v,[\underline{\alpha}_{\Omega_2(q)}(v),\overline{\alpha}_{\Omega_2(q)}(v)],[\underline{\beta}_{\Omega_2(q)}(v),\overline{\beta}_{\Omega_2(q)}(v)], [(1-\overline{\alpha}_{\Omega_2(q)}(v)),(1-\underline{\alpha}_{\Omega_2(q)}(v))]\big\rangle; q\in\mathcal{S}_2\big\},$$

Let $\oplus(\Omega_1, \mathcal{S}_1) \Cup \oplus(\Omega_2, \mathcal{S}_2) = (\Omega_{\oplus\Cup}, \mathcal{S}_{\oplus\Cup})$, where $\mathcal{S}_{\oplus\Cup} = \mathcal{S}_1 \cup \mathcal{S}_2$.

For $q \in \mathcal{S}_{\oplus\Cup}$,

$$\Omega_{\oplus\Cup}(q)$$

$$= \begin{cases} \{\langle v, [\underline{\alpha}_{\Omega_1(q)}(v), \overline{\alpha}_{\Omega_1(q)}(v)], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], \\ \qquad [(1-\overline{\alpha}_{\Omega_1(q)}(v)), (1-\underline{\alpha}_{\Omega_1(q)}(v))]\rangle; \qquad \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\}, \\ \{\langle v, [\underline{\alpha}_{\Omega_2(q)}(v), \overline{\alpha}_{\Omega_2(q)}(v)], [\underline{\beta}_{\Omega_2(q)}(v), \overline{\beta}_{\Omega_2(q)}(v)], \\ \qquad [(1-\overline{\alpha}_{\Omega_2(q)}(v)), (1-\underline{\alpha}_{\Omega_2(q)}(v))]\rangle; \qquad \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\}, \\ \{\langle v, \vee(\alpha_{\Omega_1(q)}(v), \alpha_{\Omega_2(q)}(v)), \vee(\beta_{\Omega_1(q)}(v), \beta_{\Omega_2(q)}(v)), \\ \quad [\wedge((1-\overline{\alpha}_{\Omega_1(q)}(v)), (1-\overline{\alpha}_{\Omega_2(q)}(v))), \wedge((1-\underline{\alpha}_{\Omega_1(q)}(v)), (1-\overline{\alpha}_{\Omega_2(q)}(v)))]\rangle; \\ \qquad \text{if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\}. \end{cases}$$

Thus $\oplus((\Omega_1, \mathcal{S}_1) \Cup (\Omega_2, \mathcal{S}_2)) = \oplus(\Omega_1, \mathcal{S}_1) \Cup \oplus(\Omega_2, \mathcal{S}_2)$.

(iii) $\oplus \oplus (\Omega_1, \mathcal{S}_1)$

$$= \oplus \{\langle v, [\underline{\alpha}_{\Omega_1(q)}(v), \overline{\alpha}_{\Omega_1(q)}(v)], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], [(1-\overline{\alpha}_{\Omega_1(q)}(v)), (1-\underline{\alpha}_{\Omega_1(q)}(v))]\rangle; q \in \mathcal{S}_1\}$$

$$= \{\langle v, [\underline{\alpha}_{\Omega_1(q)}(v), \overline{\alpha}_{\Omega_1(q)}(v)], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], [(1-\overline{\alpha}_{\Omega_1(q)}(v)), (1-\underline{\alpha}_{\Omega_1(q)}(v))]\rangle; q \in \mathcal{S}_1\}$$

$$= \oplus(\Omega_1, \mathcal{S}_1).$$

□

**Theorem 4.4.** *Let $(\Omega_1, \mathcal{S}_1)$ and $(\Omega_2, \mathcal{S}_2)$ be two IVINSS over $\mathcal{V}$. Then,*

*(i) $\ominus((\Omega_1, \mathcal{S}_1) \Cup (\Omega_2, \mathcal{S}_2)) = \ominus(\Omega_1, \mathcal{S}_1) \Cup \ominus(\Omega_2, \mathcal{S}_2)$;*

*(ii) $\ominus((\Omega_1, \mathcal{S}_1) \Cap (\Omega_2, \mathcal{S}_2)) = \ominus(\Omega_1, \mathcal{S}_1) \Cap \ominus(\Omega_2, \mathcal{S}_2)$;*

*(iii) $\ominus \ominus (\Omega_1, \mathcal{S}_1) = \ominus(\Omega_1, \mathcal{S}_1)$.*

*Proof.* We give the proof of (i), and proof of (ii) is analogous.

(i) Let $(\Omega_1, \mathcal{S}_1) \Cup (\Omega_2, \mathcal{S}_2) = (\Omega_{\Cup}, \mathcal{S}_{\Cup})$, where $\mathcal{S}_{\Cup} = \mathcal{S}_1 \cup \mathcal{S}_2$, $\forall\ q \in \mathcal{S}_{\Cup}$.

Consider, $\ominus\Psi_{\Cup}(q)$

$$= \begin{cases} \{\langle v, [(1-\overline{\gamma}_{\Omega_1(q)}(v)), (1-\underline{\gamma}_{\Omega_1(q)}(v))], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], \\ \qquad [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)]\rangle; \qquad \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\}, \\ \{\langle v, [(1-\overline{\gamma}_{\Omega_2(q)}(v)), (1-\underline{\gamma}_{\Omega_2(q)}(v))], [\overline{\beta}_{\Omega_2(q)}(v), \overline{\beta}_{\Omega_2(q)}(v)] \\ \qquad [\overline{\gamma}_{\Omega_2(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v)]\rangle; \qquad \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\}, \\ \{\langle v, [(1-\wedge(\overline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v))), (1-\wedge(\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v)))], \\ \qquad \wedge(\beta_{\Omega_1(q)}(v), \beta_{\Omega_2(q)}(v)), \wedge(\gamma_{\Omega_1(q)}(v), \gamma_{\Omega_2(q)}(v))\rangle; \qquad \text{if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\}. \end{cases}$$

$$= \begin{cases} \{\langle v, [(1-\overline{\gamma}_{\Omega_1(q)}(v)), (1-\underline{\gamma}_{\Omega_1(q)}(v))], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], \\ \qquad [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\}, \\ \{\langle v, [(1-\overline{\gamma}_{\Omega_2(q)}(v)), (1-\underline{\gamma}_{\Omega_2(q)}(v))], [\overline{\beta}_{\Omega_2(q)}(v), \overline{\beta}_{\Omega_2(q)}(v)] \\ \qquad [\overline{\gamma}_{\Omega_2(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\}, \\ \{\langle v, [\vee((1-\overline{\gamma}_{\Omega_1(q)}(v)), (1-\overline{\gamma}_{\Omega_2(q)}(v))), \vee((1-\underline{\gamma}_{\Omega_1(q)}(v)), (1-\overline{\gamma}_{\Omega_2(q)}(v)))], \\ \qquad \vee(\beta_{\Omega_1(q)}(v), \beta_{\Omega_2(q)}(v)), \wedge(\gamma_{\Omega_1(q)}(v), \gamma_{\Omega_2(q)}(v))\rangle; \quad \text{if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\}. \end{cases}$$

We know that,

$$\ominus(\Omega_1, \mathcal{S}_1) = \{\langle v, [(1-\overline{\gamma}_{\Omega_1(q)}(v)), (1-\underline{\gamma}_{\Omega_1(q)}(v))], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], \\ [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)]\rangle; q \in \mathcal{S}_1\},$$

$$\ominus(\Omega_2, \mathcal{S}_2) = \{\langle v, [(1-\overline{\gamma}_{\Omega_2(q)}(v)), (1-\underline{\gamma}_{\Omega_2(q)}(v))], [\underline{\beta}_{\Omega_2(q)}(v), \overline{\beta}_{\Omega_2(q)}(v)], \\ [\underline{\gamma}_{\Omega_2(q)}(v), \overline{\gamma}_{\Omega(q)}(v)]\rangle; q \in \mathcal{S}_2\},$$

Let $\ominus(\Omega_1, \mathcal{S}_1) \Cup \ominus(\Omega_2, \mathcal{S}_2) = (\Omega_{\ominus\Cup}, \mathcal{S}_{\ominus\Cup})$, where $\mathcal{S}_{\ominus\Cup} = \mathcal{S}_1 \cup \mathcal{S}_2$.

For $q \in \mathcal{S}_{\ominus\Cup}$,

$\Omega_{\ominus\Cup}(q)$

$$= \begin{cases} \{\langle v, [(1-\overline{\gamma}_{\Omega_1(q)}(v)), (1-\underline{\gamma}_{\Omega_1(q)}(v))], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], \\ \qquad [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\}, \\ \{\langle v, [(1-\overline{\gamma}_{\Omega_2(q)}(v)), (1-\underline{\gamma}_{\Omega_2(q)}(v))], [\overline{\beta}_{\Omega_2(q)}(v), \overline{\beta}_{\Omega_2(q)}(v)] \\ \qquad [\overline{\gamma}_{\Omega_2(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\}, \\ \{\langle v, [\vee((1-\overline{\gamma}_{\Omega_1(q)}(v)), (1-\overline{\gamma}_{\Omega_2(q)}(v))), \vee((1-\underline{\gamma}_{\Omega_1(q)}(v)), (1-\overline{\gamma}_{\Omega_2(q)}(v)))], \\ \qquad \vee(\beta_{\Omega_1(q)}(v), \beta_{\Omega_2(q)}(v)), \wedge(\gamma_{\Omega_1(q)}(v), \gamma_{\Omega_2(q)}(v))\rangle; \quad \text{if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\}. \end{cases}$$

Thus $\ominus((\Omega_1, \mathcal{S}_1) \Cup (\Omega_2, \mathcal{S}_2)) = \ominus(\Omega_1, \mathcal{S}_1) \Cup \ominus(\Omega_2, \mathcal{S}_2)$.

(iii) $\ominus\ominus(\Omega_1, \mathcal{S}_1)$

$$= \ominus\{\langle v, [(1-\overline{\gamma}_{\Omega_1(q)}(v)), (1-\underline{\gamma}_{\Omega_1(q)}(v))], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)]\rangle; q \in \mathcal{S}_1\}$$

$$= \{\langle v, [(1-\overline{\gamma}_{\Omega_1(q)}(v)), (1-\underline{\gamma}_{\Omega_1(q)}(v))], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)]\rangle; q \in \mathcal{S}_1\}$$

$$= \ominus(\Omega_1, \mathcal{S}_1).$$

□

**Theorem 4.5.** *Let $(\Omega, \mathcal{S})$ be an IVINSS over $\mathcal{V}$. Then,*

*(i)* $\ominus\oplus(\Omega, \mathcal{S}) = \oplus(\Omega, \mathcal{S})$;

*(ii)* $\oplus\ominus(\Omega, \mathcal{S}) = \ominus(\Omega, \mathcal{S})$.

*Proof.* (i)$\ominus \oplus (\Omega, \mathcal{S})$

$$= \big\{\big\langle v, [(1-(1-\underline{\alpha}_{\Omega(q)}(v)), (1-(1-\overline{\alpha}_{\Omega(q)}(v))], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [(1-\overline{\alpha}_{\Omega(q)}(v)), (1-\underline{\alpha}_{\Omega(q)}(v))]\big\rangle; q \in \mathcal{S}\big\}$$

$$= \big\{\big\langle v, [\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [(1-\overline{\alpha}_{\Omega(q)}(v)), (1-\underline{\alpha}_{\Omega(q)}(v))]\big\rangle; q \in \mathcal{S}\big\}$$

$$= \oplus (\Omega, \mathcal{S}).$$

(ii) $\oplus \ominus (\Omega, \mathcal{S})$

$$= \big\{\big\langle v, [(1-\overline{\gamma}_{\Omega(q)}(v)), (1-\underline{\gamma}_{\Omega(q)}(v))], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [(1-(1-\underline{\gamma}_{\Omega(q)}(v)), (1-(1-\overline{\gamma}_{\Omega(q)}(v))]\big\rangle; q \in \mathcal{S}\big\}$$

$$= \big\{\big\langle v, [(1-\overline{\gamma}_{\Omega(q)}(v)), (1-\underline{\gamma}_{\Omega(q)}(v))], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\underline{\gamma}_{\Omega(q)}(v), \overline{\gamma}_{\Omega(q)}(v)]\big\rangle; q \in \mathcal{S}\big\}$$

$$= \ominus (\Omega, \mathcal{S}).$$

□

**Theorem 4.6.** *Let $(\Omega_1, \mathcal{S}_1)$ and $(\Omega_2, \mathcal{S}_2)$ be two IVINSS over $\mathcal{V}$. Then,*
*(i) $\oplus ((\Omega_1, \mathcal{S}_1) \wedge (\Omega_2, \mathcal{S}_2)) = \oplus(\Omega_1, \mathcal{S}_1) \wedge \oplus(\Omega_2, \mathcal{S}_2)$;*
*(ii) $\oplus ((\Omega_1, \mathcal{S}_1) \vee (\Omega_2, \mathcal{S}_2)) = \oplus(\Omega_1, \mathcal{S}_1) \vee \oplus(\Omega_2, \mathcal{S}_2)$;*
*(iii) $\ominus ((\Omega_1, \mathcal{S}_1) \wedge (\Omega_2, \mathcal{S}_2)) = \ominus(\Omega_1, \mathcal{S}_1) \wedge \oplus(\Omega_2, \mathcal{S}_2)$;*
*(iv) $\ominus ((\Omega_1, \mathcal{S}_1) \vee (\Omega_2, \mathcal{S}_2)) = \ominus(\Omega_1, \mathcal{S}_1) \vee \oplus(\Omega_2, \mathcal{S}_2)$.*

*Proof.* We present the proofs of (i) and (iii), and proofs of (ii) and (iv) are analogous.
(i) $\oplus ((\Omega_1, \mathcal{S}_1) \wedge (\Omega_2, \mathcal{S}_2))$

$$= \big\{\big\langle v, [\wedge(\underline{\alpha}_{\Omega_1(q_1)}(v), \underline{\alpha}_{\Omega_2(q_2)}(v)), \wedge(\overline{\alpha}_{\Omega_1(q_1)}(v), \overline{\alpha}_{\Omega_2(q_2)}(v))], [\wedge(\underline{\beta}_{\Omega_1(q_1)}(v), \underline{\beta}_{\Omega_2(q_2)}(v)), \wedge(\overline{\beta}_{\Omega_1(q_1)}(v), \overline{\beta}_{\Omega_2(q_2)}(v))], [(1-\wedge(\underline{\alpha}_{\Omega_1(q_1)}(v), \underline{\alpha}_{\Omega_2(q_2)}(v))), (1-\wedge(\overline{\alpha}_{\Omega_1(q_1)}(v), \overline{\alpha}_{\Omega_2(q_2)}(v)))], \forall (q_1, q_2) \in \mathcal{S}_1 \times \mathcal{S}_2\big\}.$$

$$= \big\{\big\langle v, [\wedge(\underline{\alpha}_{\Omega_1(q_1)}(v), \underline{\alpha}_{\Omega_2(q_2)}(v)), \wedge(\overline{\alpha}_{\Omega_1(q_1)}(v), \overline{\alpha}_{\Omega_2(q_2)}(v))], [\wedge(\underline{\beta}_{\Omega_1(q_1)}(v), \underline{\beta}_{\Omega_2(q_2)}(v)), \wedge(\overline{\beta}_{\Omega_1(q_1)}(v), \overline{\beta}_{\Omega_2(q_2)}(v))], [\vee((1-\overline{\alpha}_{\Omega_1(q_1)}(v)), (1-\overline{\alpha}_{\Omega_2(q_2)}(v))), \vee((1-\underline{\alpha}_{\Omega_1(q_1)}(v)), (1-\underline{\alpha}_{\Omega_2(q_2)}(v)))], \forall (q_1, q_2) \in \mathcal{S}_1 \times \mathcal{S}_2\big\}.$$

Also,

$$\oplus(\Omega_1, \mathcal{S}_1) = \big\{\big\langle v, [\underline{\alpha}_{\Omega_1(q_1)}(v), \overline{\alpha}_{\Omega_1(q_1)}(v)], [\underline{\beta}_{\Omega_1(q_1)}(v), \overline{\beta}_{\Omega_1(q_1)}(v)], [(1-\overline{\alpha}_{\Omega_1(q_1)}(v)), (1-\underline{\alpha}_{\Omega_1(q_1)}(v))]\big\rangle;\ q_1 \in \mathcal{S}_1\big\},$$

$$\oplus(\Omega_2,\mathcal{S}_2)=\big\{\big\langle v,[\underline{\alpha}_{\Omega_2(q_2)}(v),\overline{\alpha}_{\Omega_2(q_2)}(v)],[\underline{\beta}_{\Omega_2(q_2)}(v),\overline{\beta}_{\Omega_2(q_2)}(v)],$$
$$[(1-\overline{\alpha}_{\Omega_2(q_2)}(v)),(1-\underline{\alpha}_{\Omega_2(q_2)}(v))]\big\rangle;\ q_2\in\mathcal{S}_2\big\}.$$

Therefore, we have

$\oplus(\Omega_1,\mathcal{S}_1)\wedge\oplus(\Omega_2,\mathcal{S}_2)$

$$=\big\{\big\langle v,[\wedge(\underline{\alpha}_{\Omega_1(q_1)}(v),\underline{\alpha}_{\Omega_2(q_2)}(v)),\wedge(\overline{\alpha}_{\Omega_1(q_1)}(v),\overline{\alpha}_{\Omega_2(q_2)}(v))],$$
$$[\wedge(\underline{\beta}_{\Omega_1(q_1)}(v),\underline{\beta}_{\Omega_2(q_2)}(v)),\wedge(\overline{\beta}_{\Omega_1(q_1)}(v),\overline{\beta}_{\Omega_2(q_2)}(v))],[\vee((1-\overline{\alpha}_{\Omega_1(q_1)}(v)),(1-\overline{\alpha}_{\Omega_2(q_2)}(v))),$$
$$\vee((1-\underline{\alpha}_{\Omega_1(q_1)}(v)),(1-\underline{\alpha}_{\Omega_2(q_2)}(v)))],\forall(q_1,q_2)\in\mathcal{S}_1\times\mathcal{S}_2\big\}.$$
$$=\ \oplus\left((\Omega_1,\mathcal{S}_1)\wedge(\Omega_2,\mathcal{S}_2)\right).$$

(iii) $\ominus\left((\Omega_1,\mathcal{S}_1)\wedge(\Omega_2,\mathcal{S}_2)\right)$

$$=\big\{\big\langle v,[(1-\vee(\overline{\gamma}_{\Omega_1(q_1)}(v),\overline{\gamma}_{\Omega_2(q_2)}(v)),(1-\vee(\underline{\gamma}_{\Omega_1(q_1)}(v),\underline{\alpha}_{\Omega_2(q_2)}(v))],$$
$$[\vee(\underline{\beta}_{\Omega_1(q_1)}(v),\underline{\beta}_{\Omega_2(q_2)}(v)),\vee(\overline{\beta}_{\Omega_1(q_1)}(v),\overline{\beta}_{\Omega_2(q_2)}(v))],$$
$$[\vee(\underline{\gamma}_{\Omega_1(q_1)}(v)),\underline{\gamma}_{\Omega_2(q_2)}(v)),\vee(\overline{\gamma}_{\Omega_1(q_1)}(v),\overline{\gamma}_{\Omega_2(q_2)}(v))],\forall(q_1,q_2)\in\mathcal{S}_1\times\mathcal{S}_2\big\}.$$
$$=\big\{\big\langle v,[\wedge((1-\overline{\gamma}_{\Omega_1(q_1)}(v),(1-\overline{\gamma}_{\Omega_2(q_2)}(v))),\wedge((1-\underline{\gamma}_{\Omega_1(q_1)}(v)),(1-\underline{\gamma}_{\Omega_2(q_2)}(v)))],$$
$$[\vee(\underline{\beta}_{\Omega_1(q_1)}(v),\underline{\beta}_{\Omega_2(q_2)}(v)),\vee(\overline{\beta}_{\Omega_1(q_1)}(v),\overline{\beta}_{\Omega_2(q_2)}(v))],$$
$$[\vee(\underline{\gamma}_{\Omega_1(q_1)}(v),\underline{\gamma}_{\Omega_2(q_2)}(v)),\vee(\overline{\gamma}_{\Omega_1(q_1)}(v),\overline{\gamma}_{\Omega_2(q_2)}(v))],\forall(q_1,q_2)\in\mathcal{S}_1\times\mathcal{S}_2\big\}.$$

Also,

$$\ominus(\Omega_1,\mathcal{S}_1)=\big\{\big\langle v,[(1-\overline{\gamma}_{\Omega_1(q_1)}(v)),(1-\underline{\gamma}_{\Omega_1(q_1)}(v))],$$
$$[\underline{\beta}_{\Omega_1(q_1)}(v),\overline{\beta}_{\Omega_1(q_1)}(v)],[\underline{\gamma}_{\Omega_1(q_1)}(v),\overline{\gamma}_{\Omega_1(q_1)}(v)]\big\rangle;\ q_1\in\mathcal{S}_1\big\},$$

$$\ominus(\Omega_2,\mathcal{S}_2)=\big\{\big\langle v,[(1-\overline{\gamma}_{\Omega_2(q_2)}(v)),(1-\underline{\gamma}_{\Omega_2(q_2)}(v))],$$
$$[\underline{\beta}_{\Omega_2(q_2)}(v),\overline{\beta}_{\Omega_2(q_2)}(v)],[\underline{\gamma}_{\Omega_2(q_2)}(v),\overline{\gamma}_{\Omega_2(q_2)}(v)]\big\rangle;\ q_2\in\mathcal{S}_2\big\}.$$

Therefore, we have

$\ominus(\Omega_1,\mathcal{S}_1)\wedge\ominus(\Omega_2,\mathcal{S}_2)$

$$=\big\{\big\langle v,[\wedge((1-\overline{\gamma}_{\Omega_1(q_1)}(v)),(1-\overline{\gamma}_{\Omega_2(q_2)}(v)),\wedge((1-\underline{\gamma}_{\Omega_1(q_1)}(v)),(1-\underline{\gamma}_{\Omega_2(q_2)}(v))],$$
$$[\vee(\underline{\beta}_{\Omega_1(q_1)}(v),\underline{\beta}_{\Omega_2(q_2)}(v)),\vee(\overline{\beta}_{\Omega_1(q_1)}(v),\overline{\beta}_{\Omega_2(q_2)}(v))],$$
$$[\vee(\underline{\gamma}_{\Omega_1(q_1)}(v)),\underline{\gamma}_{\Omega_2(q_2)}(v)),\vee(\overline{\gamma}_{\Omega_1(q_1)}(v),\overline{\gamma}_{\Omega_2(q_2)}(v)))],\forall(q_1,q_2)\in\mathcal{S}_1\times\mathcal{S}_2\big\}.$$
$$=\ \ominus\left((\Omega_1,\mathcal{S}_1)\wedge(\Omega_2,\mathcal{S}_2)\right).$$

□

## 5. $\pm$ and $\mp$ operators on IVINSS

We provide the definition of two new operators ($\pm$ and $\mp$) on IVINSS and discuss some of their properties. We generalize these operations and properties on IVINSS using the concepts given in [5].

**Definition 5.1.** Let $(\Omega_1, \mathcal{S}_1)$ and $(\Omega_2, \mathcal{S}_2)$ be two IVINSS over $\mathcal{V}$. Then,
(i) the operator $\pm$ is represented as $(\Omega_1, \mathcal{S}_1) \pm (\Omega_2, \mathcal{S}_2) = (\Omega_{\pm}, \mathcal{S}_{\pm})$, where $\mathcal{S}_{\pm} = \mathcal{S}_1 \cup \mathcal{S}_2$. $\forall q \in \mathcal{S}_{\pm}$,

$$\Omega_{\pm}(q) = \begin{cases} \left\{\left\langle v, [\underline{\alpha}_{\Omega_1(q)}(v) + \overline{\alpha}_{\Omega_1(q)}(v)], [\underline{\beta}_{\Omega_1(q)}(v) + \overline{\beta}_{\Omega_1(q)}(v)], [\underline{\gamma}_{\Omega_1(q)}(v) + \overline{\gamma}_{\Omega_1(q)}(v)]\right\rangle; \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\right\}, \\ \left\{\left\langle v, [\underline{\alpha}_{\Omega_2(q)}(v) + \overline{\alpha}_{\Omega_2(q)}(v)], [\underline{\beta}_{\Omega_2(q)}(v) + \overline{\beta}_{\Omega_2(q)}(v)], [\underline{\gamma}_{\Omega_2(q)}(v) + \overline{\gamma}_{\Omega_2(q)}(v)]\right\rangle; \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\right\}, \\ \left\{\left\langle v, \left[\frac{\underline{\alpha}_{\Omega_1(q)}(v) + \underline{\alpha}_{\Omega_2(q)}(v)}{2}, \frac{\overline{\alpha}_{\Omega_1(q)}(v) + \overline{\alpha}_{\Omega_2(q)}(v)}{2}\right], \left[\frac{\underline{\beta}_{\Omega_1(q)}(v) + \underline{\beta}_{\Omega_2(q)}(v)}{2}, \frac{\overline{\beta}_{\Omega_1(q)}(v) + \overline{\beta}_{\Omega_2(q)}(v)}{2}\right], \right.\right. \\ \qquad \left.\left. \left[\frac{\underline{\gamma}_{\Omega_1(q)}(v) + \underline{\gamma}_{\Omega_2(q)}(v)}{2}, \frac{\overline{\gamma}_{\Omega_1(q)}(v) + \overline{\gamma}_{\Omega_2(q)}(v)}{2}\right]\right\rangle; \text{ if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\right\}. \end{cases}$$

(ii) the operator $\mp$ is represented as $(\Omega_1, \mathcal{S}_1) \mp (\Omega_2, \mathcal{S}_2) = (\Omega_{\mp}, \mathcal{S}_{\mp})$, where $\mathcal{S}_{\mp} = \mathcal{S}_1 \cup \mathcal{S}_2$. $\forall q \in \mathcal{S}_{\mp}$,

$$\Omega_{\mp}(q) = \begin{cases} \left\{\left\langle v, [\underline{\alpha}_{\Omega_1(q)}(v) + \overline{\alpha}_{\Omega_1(q)}(v)], [\underline{\beta}_{\Omega_1(q)}(v) + \overline{\beta}_{\Omega_1(q)}(v)], [\underline{\gamma}_{\Omega_1(q)}(v) + \overline{\gamma}_{\Omega_1(q)}(v)]\right\rangle; \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\right\}, \\ \left\{\left\langle v, [\underline{\alpha}_{\Omega_2(q)}(v) + \overline{\alpha}_{\Omega_2(q)}(v)], [\underline{\beta}_{\Omega_2(q)}(v) + \overline{\beta}_{\Omega_2(q)}(v)], [\underline{\gamma}_{\Omega_2(q)}(v) + \overline{\gamma}_{\Omega_2(q)}(v)]\right\rangle; \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\right\}, \\ \left\{\left\langle v, \left[\frac{2\underline{\alpha}_{\Omega_1(q)}(v).\underline{\alpha}_{\Omega_2(q)}(v)}{\underline{\alpha}_{\Omega_1(q)}(v) + \underline{\alpha}_{\Omega_2(q)}(v)}, \frac{2\overline{\alpha}_{\Omega_1(q)}(v).\overline{\alpha}_{\Omega_2(q)}(v)}{\overline{\alpha}_{\Omega_1(q)}(v) + \overline{\alpha}_{\Omega_2(q)}(v)}\right], \left[\frac{\underline{\beta}_{\Omega_1(q)}(v) + \underline{\beta}_{\Omega_2(q)}(v)}{2}, \frac{\overline{\beta}_{\Omega_1(q)}(v) + \overline{\beta}_{\Omega_2(q)}(v)}{2}\right], \right.\right. \\ \qquad \left.\left. \left[\frac{2\underline{\gamma}_{\Omega_1(q)}(v).\underline{\gamma}_{\Omega_2(q)}(v)}{\underline{\gamma}_{\Omega_1(q)}(v) + \underline{\gamma}_{\Omega_2(q)}(v)}, \frac{2\overline{\gamma}_{\Omega_1(q)}(v).\overline{\gamma}_{\Omega_2(q)}(v)}{\overline{\gamma}_{\Omega_1(q)}(v) + \overline{\gamma}_{\Omega_2(q)}(v)}\right]\right\rangle; \text{ if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\right\}. \end{cases}$$

**Example 5.2.** Consider that a psychiatrist has conducted two counseling sessions for the clients. Assume the psychiatrist has given the values in the IVINSS form for the first session $(\Omega_1, \mathcal{S}_1)$, as in Table 1 and for the second session $(\Omega_2, \mathcal{S}_2)$ in Table 4. Now we calculate the combined results of the two sessions using $(\Omega_1, \mathcal{S}_1) \pm (\Omega_2, \mathcal{S}_2)$, $(\Omega_1, \mathcal{S}_1) \mp (\Omega_2, \mathcal{S}_2)$ and present the results in Table 5 and 6 respectively.

TABLE 4. Shows client with cognitive disorders in IVINSS $(\Omega_2, \mathcal{S}_2)$ form.

| $\mathcal{V}$ | IMC$(q_1)$ | LM$(q_2)$ | IC$(q_3)$ |
|---|---|---|---|
| $v_1$ | $\langle[0.1, 0.3], [0.6, 0.7], [0.2, 0.3]\rangle$ | $\langle[0.2, 0.3], [0.7, 0.8], [0.4, 0.6]\rangle$ | $\langle[0.3, 0.4], [0.70.9], [0.4, 0.5]\rangle$ |
| $v_2$ | $\langle[0.3, 0.5], [0.5, 0.8], [0.2, 0.4]\rangle$ | $\langle[0.6, 0.7], [0.5, 0.6], [0.2, 0.3]\rangle$ | $\langle[0.5, 0.6], [0.30.5], [0.2, 0.3]\rangle$ |
| $v_3$ | $\langle[0.5, 0.6], [0.6, 0.9], [0.3, 0.4]\rangle$ | $\langle[0.2, 0.4], [0.9, 1.0], [0.3, 0.4]\rangle$ | $\langle[0.1, 0.2], [0.20.4], [0.3, 0.4]\rangle$ |

(i) The IVINSS $(\Omega_1, \mathcal{S}_1) \pm (\Omega_2, \mathcal{S}_2)$ is shown in Table 5.

(ii) The IVINSS $(\Omega_1, \mathcal{S}_1) \mp (\Omega_2, \mathcal{S}_2)$ is given in Table 6.

Table 5. Representation of clients with cognitive disorders in IVINSS $(\Omega_1, \mathcal{S}_1) \pm (\Omega_2, \mathcal{S}_2)$ form.

| $\mathcal{V}$ | IMC($q_1$) | LM($q_2$) | IC($q_3$) |
|---|---|---|---|
| $v_1$ | $\langle[0.15, 0.35], [0.50, 0.60], [0.30, 0.40]\rangle$ | $\langle[0.25, 0.35], [0.60, 0.70], [0.35, 0.55]\rangle$ | $\langle[0.25, 0.35], [0.60, 0.85], [0.50, 0.60]\rangle$ |
| $v_2$ | $\langle[0.35, 0.55], [0.40, 0.65], [0.15, 0.30]\rangle$ | $\langle[0.65, 0.75], [0.35, 0.55], [0.15, 0.25]\rangle$ | $\langle[0.55, 0.65], [0.50, 0.65], [0.15, 0.25]\rangle$ |
| $v_3$ | $\langle[0.55, 0.65], [0.40, 0.80], [0.20, 0.30]\rangle$ | $\langle[0.15, 0.35], [0.75, 0.85], [0.40, 0.50]\rangle$ | $\langle[0.15, 0.25], [0.45, 0.60], [0.35, 0.45]\rangle$ |

Table 6. Shows clients with cognitive disorders in IVINSS $(\Omega_1, \mathcal{S}_1) \mp (\Omega_2, \mathcal{S}_2)$ form.

| $\mathcal{U}$ | IMC($q_1$) | LM($q_2$) | IC($q_3$) |
|---|---|---|---|
| $v_1$ | $\langle[0.13, 0.34], [0.50, 0.60], [0.26, 0.37]\rangle$ | $\langle[0.24, 0.34], [0.60, 0.70], [0.34, 0.54]\rangle$ | $\langle[0.24, 0.34], [0.60, 0.85], [0.48, 0.58]\rangle$ |
| $v_2$ | $\langle[0.34, 0.54], [0.40, 0.65], [0.13, 0.26]\rangle$ | $\langle[0.64, 0.74], [0.35, 0.55], [0.13, 0.24]\rangle$ | $\langle[0.54, 0.64], [0.50, 0.65], [0.13, 0.24]\rangle$ |
| $v_3$ | $\langle[0.54, 0.64], [0.40, 0.80], [0.15, 0.26]\rangle$ | $\langle[0.13, 0.34], [0.75, 0.85], [0.37, 0.48]\rangle$ | $\langle[0.13, 0.24], [0.45, 0.60], [0.34, 0.44]\rangle$ |

**Proposition 5.3.** *Let $(\Omega_1, \mathcal{S}_1)$ and $(\Omega_2, \mathcal{S}_2)$ be non-empty over $\mathcal{V}$. Then,*
*(i)* $(\Omega_1, \mathcal{S}_1) \pm (\Omega_2, \mathcal{S}_2) = (\Omega_2, \mathcal{S}_2) \pm (\Omega_1, \mathcal{S}_1)$;
*(ii)* $[(\Omega_1, \mathcal{S}_1)^c \pm (\Omega_2, \mathcal{S}_2)^c]^c = (\Omega_1, \mathcal{S}_1) \pm (\Omega_1, \mathcal{S}_1)$.

*Proof.* (i) Proof straightforward.
(ii) Let

$$(\Omega_1, \mathcal{S}_1) = \left\{\left\langle v, [\underline{\alpha}_{\Omega_1(q)}(v) + \overline{\alpha}_{\Omega_1(q)}(v)], [\underline{\beta}_{\Omega_1(q)}(v) + \overline{\beta}_{\Omega_1(q)}(v)], [\underline{\gamma}_{\Omega_1(q)}(v) + \overline{\gamma}_{\Omega_1(q)}(v)]\right\rangle; q \in \mathcal{S}_1\right\},$$

and

$$(\Omega_2, \mathcal{S}_2) = \left\{\left\langle v, [\underline{\alpha}_{\Omega_2(q)}(v) + \overline{\alpha}_{\Omega_2(q)}(v)], [\underline{\beta}_{\Omega_2(q)}(v) + \overline{\beta}_{\Omega_2(q)}(v)], [\underline{\gamma}_{\Omega_2(q)}(v) + \overline{\gamma}_{\Omega_2(q)}(v)]\right\rangle; q \in \mathcal{S}_2\right\}$$

be two IVINSS.
Then, $[(\Omega_1, \mathcal{S}_1)^c \pm (\Omega_2, \mathcal{S}_2)^c]$

$$= \begin{cases} \left\{\left\langle v, [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)], [(1 - \overline{\beta}_{\Omega_1(q)}(v)), (1 - \underline{\beta}_{\Omega_1(q)}(v))], \right.\right. \\ \qquad \left.\left. [\underline{\alpha}_{\Omega_1(q)}(v), \overline{\alpha}_{\Omega_1(q)}(v)]\right\rangle; \quad \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\right\}, \\ \left\{\left\langle v, [\underline{\gamma}_{\Omega_2(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v)], [(1 - \overline{\beta}_{\Omega_2(q)}(v)), (1 - \underline{\beta}_{\Omega_2(q)}(v))], \right.\right. \\ \qquad \left.\left. [\underline{\alpha}_{\Omega_2(q)}(v), \overline{\alpha}_{\Omega_2(q)}(v)]\right\rangle; \quad \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\right\}, \\ \left\{\left\langle v, \left[\frac{\underline{\gamma}_{\Omega_1(q)}(v)+\underline{\gamma}_{\Omega_2(q)}(v)}{2}, \frac{\overline{\gamma}_{\Omega_1(q)}(v)+\overline{\gamma}_{\Omega_2(q)}(v)}{2}\right], \left[\frac{(1-\underline{\beta}_{\Omega_1(q)}(v))+(1-\underline{\beta}_{\Omega_2(q)}(v))}{2}, \right.\right.\right. \\ \qquad \left.\left.\left. \frac{(1-\overline{\beta}_{\Omega_1(q)}(v))+(1-\overline{\beta}_{\Omega_2(q)}(v)}{2}\right], \left[\frac{\underline{\alpha}_{\Omega_1(q)}(v)+\underline{\alpha}_{\Omega_2(q)}(v)}{2}, \frac{\overline{\alpha}_{\Omega_1(q)}(v)+\overline{\alpha}_{\Omega_2(q)}(v)}{2}\right]\right\rangle; \text{if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\right\}. \end{cases}$$

Now consider, $[(\Omega_1,\mathcal{S}_1)^c \pm (\Omega_2,\mathcal{S}_2)^c]^c$

$$= \begin{cases} \{\langle v, [\underline{\alpha}_{\Omega_1(q)}(v), \overline{\alpha}_{\Omega_1(q)}(v)], [(1-\overline{\beta}_{\Omega_1(q)}(v)), (1-\underline{\beta}_{\Omega_1(q)}(v))], \\ \qquad [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\}, \\ \{\langle v, [\underline{\alpha}_{\Omega_2(q)}(v), \overline{\alpha}_{\Omega_2(q)}(v)], [(1-\overline{\beta}_{\Omega_2(q)}(v)), (1-\underline{\beta}_{\Omega_2(q)}(v))], \\ \qquad [\underline{\gamma}_{\Omega_2(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\}, \\ \Big\{\Big\langle v, \Big[\frac{\underline{\alpha}_{\Omega_1(q)}(v)+\underline{\alpha}_{\Omega_2(q)}(v)}{2}, \frac{\overline{\alpha}_{\Omega_1(q)}(v)+\overline{\alpha}_{\Omega_2(q)}(v)}{2}\Big]\Big[\frac{(1-\underline{\beta}_{\Omega_1(q)}(v))+(1-\underline{\beta}_{\Omega_2(q)}(v))}{2}, \\ \quad \frac{(1-\overline{\beta}_{\Omega_1(q)}(v))+(1-\overline{\beta}_{\Omega_2(q)}(v)}{2}\Big], \Big[\frac{\underline{\gamma}_{\Omega_1(q)}(v)+\underline{\gamma}_{\Omega_2(q)}(v)}{2}, \frac{\overline{\gamma}_{\Omega_1(q)}(v)+\overline{\gamma}_{\Omega_2(q)}(v)}{2}\Big]\Big\rangle; \quad \text{if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\Big\}. \end{cases}$$

Hence $[(\Omega_1,\mathcal{S}_1)^c \pm (\Omega_2,\mathcal{S}_2)^c]^c = (\Omega_1,\mathcal{S}_1) \pm (\Omega_2,\mathcal{S}_2)$. □

**Proposition 5.4.** *Let $(\Omega_1,\mathcal{S}_1)$ and $(\Omega_2,\mathcal{S}_2)$ be non-empty over $\mathcal{V}$. Then,*
*(i) $(\Omega_1,\mathcal{S}_1) \mp (\Omega_2,\mathcal{S}_2) = (\Omega_2,\mathcal{S}_2) \mp (\Omega_1,\mathcal{S}_1)$;*
*(ii) $[(\Omega_1,\mathcal{S}_1)^c \mp (\Omega_2,\mathcal{S}_2)^c]^c = (\Omega_1,\mathcal{S}_1) \mp (\Omega_1,\mathcal{S}_1)$.*

*Proof.* (i) Consider, $(\Omega_1,\mathcal{S}_1) \mp (\Omega_2,\mathcal{S}_2)$

$$= \begin{cases} \{\langle v, [\underline{\alpha}_{\Omega_1(q)}(v), \overline{\alpha}_{\Omega_1(q)}(v)], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\}, \\ \{\langle v, [\underline{\alpha}_{\Omega_2(q)}(v), \overline{\alpha}_{\Omega_2(q)}(v)], [\underline{\beta}_{\Omega_2(q)}(v), \overline{\beta}_{\Omega_2(q)}(v)], [\underline{\gamma}_{\Omega_2(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\}, \\ \{\Big\langle v, \Big[\frac{2\underline{\alpha}_{\Omega_1(q)}(v).\underline{\alpha}_{\Omega_2(q)}(v)}{\underline{\alpha}_{\Omega_1(q)}(v)+\underline{\alpha}_{\Omega_2(q)}(v)}, \frac{2\overline{\alpha}_{\Omega_1(q)}(v).\overline{\alpha}_{\Omega_2(q)}(v)}{\overline{\alpha}_{\Omega_1(q)}(v)+\overline{\alpha}_{\Omega_2(q)}(v)}\Big], \Big[\frac{\underline{\beta}_{\Omega_1(q)}(v)+\underline{\beta}_{\Omega_2(q)}(v)}{2}, \frac{\overline{\beta}_{\Omega_1(q)}(v)+\overline{\beta}_{\Omega_2(q)}(v)}{2}\Big], \\ \qquad \Big[\frac{2\underline{\gamma}_{\Omega_1(q)}(v).\underline{\gamma}_{\Omega_2(q)}(v)}{\underline{\gamma}_{\Omega_1(q)}(v)+\underline{\gamma}_{\Omega_2(q)}(v)}, \frac{2\overline{\gamma}_{\Omega_1(q)}(v).\overline{\gamma}_{\Omega_2(q)}(v)}{\overline{\gamma}_{\Omega_1(q)}(v)+\overline{\gamma}_{\Omega_2(q)}(v)}\Big]\Big\rangle; \text{ if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\}. \end{cases}$$

$$= \begin{cases} \{\langle v, [\underline{\alpha}_{\Omega_1(q)}(v), \overline{\alpha}_{\Omega_1(q)}(v)], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\}, \\ \{\langle v, [\underline{\alpha}_{\Omega_2(q)}(v), \overline{\alpha}_{\Omega_2(q)}(v)], [\underline{\beta}_{\Omega_2(q)}(v), \overline{\beta}_{\Omega_2(q)}(v)], [\underline{\gamma}_{\Omega_2(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\}, \\ \{\Big\langle v, \Big[\frac{2\underline{\alpha}_{\Omega_2(q)}(v).\underline{\alpha}_{\Omega_1(q)}(v)}{\underline{\alpha}_{\Omega_2(q)}(v)+\underline{\alpha}_{\Omega_1(q)}(v)}, \frac{2\overline{\alpha}_{\Omega_2(q)}(v).\overline{\alpha}_{\Omega_1(q)}(v)}{\overline{\alpha}_{\Omega_2(q)}(v)+\overline{\alpha}_{\Omega_1(q)}(v)}\Big], \Big[\frac{\underline{\beta}_{\Omega_2(q)}(v)+\underline{\beta}_{\Omega_1(q)}(v)}{2}, \frac{\overline{\beta}_{\Omega_2(q)}(v)+\overline{\beta}_{\Omega_1(q)}(v)}{2}\Big], \\ \qquad \Big[\frac{2\underline{\gamma}_{\Omega_2(q)}(v).\underline{\gamma}_{\Omega_1(q)}(v)}{\underline{\gamma}_{\Omega_2(q)}(v)+\underline{\gamma}_{\Omega_1(q)}(v)}, \frac{2\overline{\gamma}_{\Omega_2(q)}(v).\overline{\gamma}_{\Omega_1(q)}(v)}{\overline{\gamma}_{\Omega_2(q)}(v)+\overline{\gamma}_{\Omega_1(q)}(v)}\Big]\Big\rangle; \quad \text{if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\}. \end{cases}$$

Hence $(\Omega_1,\mathcal{S}_1) \mp (\Omega_2,\mathcal{S}_2) = (\Omega_2,\mathcal{S}_2) \mp (\Omega_1,\mathcal{S}_1)$.

(ii) Consider, $(\Omega_1,\mathcal{S}_1)^c \mp (\Omega_2,\mathcal{S}_2)^c$

$$= \begin{cases} \{\langle v, [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)], [(1-\overline{\beta}_{\Omega_1(q)}(v)), (1-\underline{\beta}_{\Omega_1(q)}(v))], \\ \qquad [\underline{\alpha}_{\Omega_1(q)}(v), \overline{\alpha}_{\Omega_1(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\}, \\ \{\langle v, [\underline{\gamma}_{\Omega_2(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v)], [(1-\overline{\beta}_{\Omega_2(q)}(v)), (1-\underline{\beta}_{\Omega_2(q)}(v))], \\ \qquad [\underline{\alpha}_{\Omega_2(q)}(v), \overline{\alpha}_{\Omega_2(q)}(v)]\rangle; \quad \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\}, \\ \Big\{\Big\langle v, \Big[\frac{2\underline{\gamma}_{\Omega_1(q)}(v).\underline{\gamma}_{\Omega_2(q)}(v)}{\underline{\gamma}_{\Omega_1(q)}(v)+\underline{\gamma}_{\Omega_2(q)}(v)}, \frac{2\overline{\gamma}_{\Omega_1(q)}(v).\overline{\gamma}_{\Omega_2(q)}(v)}{\overline{\gamma}_{\Omega_1(q)}(v)+\overline{\gamma}_{\Omega_2(q)}(v)}\Big], \Big[\frac{(1-\underline{\beta}_{\Omega_1(q)}(v))+(1-\underline{\beta}_{\Omega_2(q)}(v))}{2}, \\ \quad \frac{(1-\overline{\beta}_{\Omega_1(q)}(v))+(1-\overline{\beta}_{\Omega_2(q)}(v)}{2}\Big], \Big[\frac{2\underline{\alpha}_{\Omega_1(q)}(v).\underline{\alpha}_{\Omega_2(q)}(v)}{\underline{\alpha}_{\Omega_1(q)}(v)+\underline{\alpha}_{\Omega_2(q)}(v)}, \frac{2\overline{\alpha}_{\Omega_1(q)}(v).\overline{\alpha}_{\Omega_2(q)}(v)}{\overline{\alpha}_{\Omega_1(q)}(v)+\overline{\alpha}_{\Omega_2(q)}(v)}\Big]\Big\rangle; \text{if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\Big\}. \end{cases}$$

Then, $[(\Omega_1,\mathcal{S}_1)^c \mp (\Omega_2,\mathcal{S}_2)^c]^c$

$$= \begin{cases} \big\{\big\langle v, [\underline{\alpha}_{\Omega_1(q)}(v), \overline{\alpha}_{\Omega_1(q)}(v)], [\underline{\beta}_{\Omega_1(q)}(v), \overline{\beta}_{\Omega_1(q)}(v)], \\ \qquad [\underline{\gamma}_{\Omega_1(q)}(v), \overline{\gamma}_{\Omega_1(q)}(v)]\big\rangle; \quad \text{if } q \in \mathcal{S}_1 - \mathcal{S}_2\big\}, \\ \big\{\big\langle v, [\underline{\alpha}_{\Omega_2(q)}(v), \overline{\alpha}_{\Omega_2(q)}(v)], [\underline{\beta}_{\Omega_2(q)}(v), \overline{\beta}_{\Omega_2(q)}(v)], \\ \qquad [\underline{\gamma}_{\Omega_2(q)}(v), \overline{\gamma}_{\Omega_2(q)}(v)]\big\rangle; \quad \text{if } q \in \mathcal{S}_2 - \mathcal{S}_1\big\}, \\ \Big\{\Big\langle v, \Big[\frac{2\underline{\alpha}_{\Omega_1(q)}(v).\underline{\alpha}_{\Omega_2(q)}(v)}{\underline{\alpha}_{\Omega_1(q)}(v)+\underline{\alpha}_{\Omega_2(q)}(v)}, \frac{2\overline{\alpha}_{\Omega_1(q)}(v).\overline{\alpha}_{\Omega_2(q)}(v)}{\overline{\alpha}_{\Omega_1(q)}(v)+\overline{\alpha}_{\Omega_2(q)}(v)}\Big], \Big[\frac{(1-\underline{\beta}_{\Omega_1(q)}(v))+(1-\underline{\beta}_{\Omega_2(q)}(v))}{2}, \\ \frac{(1-\overline{\beta}_{\Omega_1(q)}(v))+(1-\overline{\beta}_{\Omega_2(q)}(v)}{2}\Big], \Big[\frac{2\underline{\gamma}_{\Omega_1(q)}(v).\underline{\gamma}_{\Omega_2(q)}(v)}{\underline{\gamma}_{\Omega_1(q)}(v)+\underline{\gamma}_{\Omega_2(q)}(v)}, \frac{2\overline{\gamma}_{\Omega_1(q)}(v).\overline{\gamma}_{\Omega_2(q)}(v)}{\overline{\gamma}_{\Omega_1(q)}(v)+\overline{\gamma}_{\Omega_2(q)}(v)}\Big]\Big\rangle; \quad \text{if } q \in \mathcal{S}_1 \cap \mathcal{S}_2\Big\}. \end{cases}$$

Hence $[(\Omega_1,\mathcal{S}_1)^c \mp (\Omega_2,\mathcal{S}_2)^c]^c = (\Omega_1,\mathcal{S}_1) \mp (\Omega_2,\mathcal{S}_2)$. □

## 6. $\mathcal{N}_\epsilon$, $\mathcal{N}_{\epsilon,\rho}$ and $\mathcal{I}_{\epsilon,\rho}$ operators on IVINSS

In this section, we define the operators $\mathcal{N}_\epsilon$, $\mathcal{N}_{\epsilon,\rho}$ and $\mathcal{I}_{\epsilon,\rho}$ on IVINSS and discuss some of their properties in detail. We generalize these operations and properties on IVINSS by the concepts discussed in [4].

**Definition 6.1.** Let $\epsilon \in [0,1]$. Then the operator $\mathcal{N}_\epsilon(\Omega,\mathcal{S})$ is represented as,
$\mathcal{N}_\epsilon(\Omega,\mathcal{S})$

$$\begin{aligned} &= \big\{\big\langle v, [\underline{\alpha}_{\Omega(q)}(v) + \epsilon(1-\underline{\alpha}_{\Omega(q)}(v) - \underline{\gamma}_{\Omega(q)}(v)), \overline{\alpha}_{\Omega(q)}(v) + \epsilon(1-\overline{\alpha}_{\Omega(q)}(v) - \overline{\gamma}_{\Omega(q)}(v))], \\ &\quad [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\overline{\gamma}_{\Omega(q)}(v) + (1-\epsilon)(1-\overline{\alpha}_{\Omega(q)}(v) - \overline{\gamma}_{\Omega(q)}(v)), \\ &\quad \underline{\gamma}_{\Omega(q)}(v) + (1-\epsilon)(1-\underline{\alpha}_{\Omega(q)}(v) - \underline{\gamma}_{\Omega(q)}(v))]\big\rangle; q \in \mathcal{S}\big\}. \\ &= \big\{\big\langle v, [\underline{\alpha}_{\Omega(q)}(v) + \epsilon(\underline{\pi}_{\Omega(q)}(v)), \overline{\alpha}_{\Omega(q)}(v) + \epsilon(\overline{\pi}_{\Omega(q)}(v))], \\ &\quad [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\overline{\gamma}_{\Omega(q)}(v) + (1-\epsilon)(\overline{\pi}_{\Omega(q)}(v)), \\ &\quad \underline{\gamma}_{\Omega(q)}(v) + (1-\epsilon)(\underline{\pi}_{\Omega(q)}(v))]\big\rangle; q \in \mathcal{S}\big\}, \end{aligned}$$

where $\underline{\pi}_{\Omega(q)}(v) = (1-\underline{\alpha}_{\Omega(q)}(v) - \underline{\gamma}_{\Omega(q)}(v))$and $\overline{\pi}_{\Omega(q)}(v) = (1-\overline{\alpha}_{\Omega(q)}(v) - \overline{\gamma}_{\Omega(q)}(v))$.

**Proposition 6.2.** *Let $\epsilon, \rho \in [0,1]$ and $\epsilon \le \rho$. Then for every IVINSS $(\Omega,\mathcal{S})$ the following hold:*
*(i) $\mathcal{N}_\epsilon(\Omega,\mathcal{S}) \Subset \mathcal{N}_\rho(\Omega,\mathcal{S})$;*
*(ii) $\mathcal{N}_0(\Omega,\mathcal{S}) = \oplus(\Omega,\mathcal{S})$;*
*(iii) $\mathcal{N}_1(\Omega,\mathcal{S}) = \ominus(\Omega,\mathcal{S})$.*

*Proof.* (i) $\mathcal{N}_{\epsilon}(\Omega,\mathcal{S})$

$$= \{\langle v, [\underline{\alpha}_{\Omega(q)}(v)+\epsilon(\underline{\pi}_{\Omega(q)}(v)), \overline{\alpha}_{\Omega(q)}(v)+\epsilon(\overline{\pi}_{\Omega(q)}(v))], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\overline{\gamma}_{\Omega(q)}(v)+(1-\epsilon)(\overline{\pi}_{\Omega(q)}(v)), \underline{\gamma}_{\Omega(q)}(v)+(1-\epsilon)(\underline{\pi}_{\Omega(q)}(v))]\rangle; q\in\mathcal{S}\}, \text{and}$$

$\mathcal{N}_{\rho}(\Omega,\mathcal{S})$

$$= \{\langle v, [\underline{\alpha}_{\Omega(q)}(v)+\rho(\underline{\pi}_{\Omega(q)}(v)), \overline{\alpha}_{\Omega(q)}(v)+\rho(\overline{\pi}_{\Omega(q)}(v))], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\overline{\gamma}_{\Omega(q)}(v)+(1-\rho)(\overline{\pi}_{\Omega(q)}(v)), \underline{\gamma}_{\Omega(q)}(v)+(1-\rho)(\underline{\pi}_{\Omega(q)}(v))]\rangle; q\in\mathcal{S}\}.$$

Since $\epsilon \leq \rho$, we have

$$(\overline{\alpha}_{\Omega(q)}(v)+\epsilon(\overline{\pi}_{\Omega(q)}(v)) \leq (\overline{\alpha}_{\Omega(q)}(v)+\rho(\overline{\pi}_{\Omega(q)}(v)).$$

Also, $(1-\epsilon) \geq (1-\rho)$, we have

$$(\overline{\gamma}_{\Omega(q)}(v)+(1-\epsilon)(\overline{\pi}_{\Omega(q)}(v)) \geq (\overline{\gamma}_{\Omega(q)}(v)+(1-\rho)(\overline{\pi}_{\Omega(q)}(v)).$$

Hence $\mathcal{N}_{\epsilon}(\Omega,\mathcal{S}) \Subset \mathcal{N}_{\rho}(\Omega,\mathcal{S})$.
(ii) Consider, $\epsilon = 0$
$\mathcal{N}_{0}(\Omega,\mathcal{S})$

$$\begin{aligned}
&= \{\langle v, [\underline{\alpha}_{\Omega(q)}(v)+0, \overline{\alpha}_{\Omega(q)}(v)+0], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [(\underline{\gamma}_{\Omega(q)}(v)+1.(\underline{\pi}_{\Omega(q)}(v))),\\
&\quad (\overline{\gamma}_{\Omega(q)}(v)+1.(\overline{\pi}_{\Omega(q)}(v)))]\rangle; q\in\mathcal{S}\}\\
&= \{\langle v, [\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [(1-\overline{\alpha}_{\Omega(q)}(v)), (1-\underline{\alpha}_{\Omega(q)}(v))]\rangle; q\in\mathcal{S}\}\\
&= \oplus(\Omega,\mathcal{S}).
\end{aligned}$$

Hence $\mathcal{N}_{0}(\Omega,\mathcal{S}) = \oplus(\Omega,\mathcal{S})$.
(iii) Consider, $\epsilon = 1$
$\mathcal{N}_{1}(\Omega,\mathcal{S})$

$$\begin{aligned}
&= \{\langle v, [(\underline{\alpha}_{\Omega(q)}(v)+\underline{\pi}_{\Omega(q)}(v)), (\overline{\alpha}_{\Omega(q)}(v)+\overline{\pi}_{\Omega(q)}(v))], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\
&\quad [\underline{\gamma}_{\Omega(q)}(v)+0, \overline{\gamma}_{\Omega(q)}(v)+0]\rangle; q\in\mathcal{S}\}\\
&= \{\langle v, [(1-\overline{\gamma}_{\Omega(q)}(v)), (1-\underline{\gamma}_{\Omega(q)}(v))], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\underline{\gamma}_{\Omega(q)}(v), \overline{\gamma}_{\Omega(q)}(v)]\rangle; q\in\mathcal{S}\}\\
&= \oplus(\Omega,\mathcal{S}).
\end{aligned}$$

Hence $\mathcal{N}_{1}(\Omega,\mathcal{S}) = \ominus(\Omega,\mathcal{S})$. □

**Remark 6.3.** The operator $\mathcal{N}_{\epsilon}$ is an extension of $\oplus$ and $\ominus$ operators.

**Definition 6.4.** Let $\epsilon, \rho \in [0,1]$ and $\epsilon + \rho \leq 1$. Then the operator $\mathcal{N}_{\epsilon,\rho}(\Omega, \mathcal{S})$ is represented as,

$$\mathcal{N}_{\epsilon,\rho}(\Omega, \mathcal{S})$$

$$= \{\langle v, [\underline{\alpha}_{\Omega(q)}(v) + \epsilon(\underline{\pi}_{\Omega(q)}(v)), \overline{\alpha}_{\Omega(q)}(v) + \epsilon(\overline{\pi}_{\Omega(q)}(v))],$$
$$[\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\underline{\gamma}_{\Omega(q)}(v) + \rho(\underline{\pi}_{\Omega(q)}(v)), \overline{\gamma}_{\Omega(q)}(v) + \rho(\overline{\pi}_{\Omega(q)}(v)]\rangle; q \in \mathcal{S}\},$$

where $\underline{\pi}_{\Omega(q)}(v) = (1 - \underline{\alpha}_{\Omega(q)}(v) - \underline{\gamma}_{\Omega(q)}(v))$and $\overline{\pi}_{\Omega(q)}(v) = (1 - \overline{\alpha}_{\Omega(q)}(v) - \overline{\gamma}_{\Omega(q)}(v))$.

**Theorem 6.5.** *Let $\epsilon, \rho, \sigma \in [0,1]$ and $\epsilon + \rho \leq 1$. Then for every IVINSS $(\Omega, \mathcal{S})$ the following hold:*

*(i) $\mathcal{N}_{\epsilon,\rho}(\Omega, \mathcal{S})$ is an IVINSS;*
*(ii) If $0 \leq \sigma \leq \epsilon$ then $\mathcal{N}_{\sigma,\rho}(\Omega, \mathcal{S}) \Subset \mathcal{N}_{\epsilon,\rho}(\Omega, \mathcal{S})$;*
*(iii) If $0 \leq \sigma \leq \rho$ then $\mathcal{N}_{\epsilon,\rho}(\Omega, \mathcal{S}) \Subset \mathcal{N}_{\epsilon,\sigma}(\Omega, \mathcal{S})$;*
*(iv) $\mathcal{N}_{\epsilon}(\Omega, \mathcal{S}) = \mathcal{N}_{\epsilon,(1-\epsilon)}(\Omega, \mathcal{S})$;*
*(v) $\oplus(\Omega, \mathcal{S}) = \mathcal{N}_{0,1}(\Omega, \mathcal{S})$;*
*(vi) $\ominus(\Omega, \mathcal{S}) = \mathcal{N}_{1,0}(\Omega, \mathcal{S})$;*
*(vii) $(\mathcal{N}_{\epsilon,\rho}(\Omega, \mathcal{S})^c)^c = (\mathcal{N}_{\rho,\epsilon}(\Omega, \mathcal{S}))$.*

*Proof.* (i) Consider,

$$\frac{\overline{\alpha}_{\Omega(q)}(v) + \epsilon(\overline{\pi}_{\Omega(q)}(v))}{2} + \overline{\beta}_{\Omega(q)}(v) + \frac{\overline{\gamma}_{\Omega(q)}(v) + \rho(\overline{\pi}_{\Omega(q)}(v))}{2}$$
$$= \frac{\overline{\alpha}_{\Omega(q)}(v) + \overline{\gamma}_{\Omega(q)}(v)}{2} + \overline{\beta}_{\Omega(q)}(v) + (\epsilon + \rho)\frac{(\overline{\pi}_{\Omega(q)}(v))}{2}$$
$$\leq \frac{\overline{\alpha}_{\Omega(q)}(v) + \overline{\gamma}_{\Omega(q)}(v)}{2} + \overline{\beta}_{\Omega(q)}(v) + \frac{(1 - \overline{\alpha}_{\Omega(q)}(v) - \overline{\gamma}_{\Omega(q)}(v)}{2}$$
$$\leq \frac{1}{2} + 1 < 2, \qquad \text{since } \epsilon + \rho \leq 1 \text{ and } \overline{\beta}_{\Omega(q)}(v) \leq 1$$

Hence $\mathcal{N}_{\epsilon,\rho}(\Omega, \mathcal{S})$ is an IVINSS.

(ii) Consider,

$$\mathcal{N}_{\sigma,\rho}(\Omega, \mathcal{S}) = \{\langle v, [\underline{\alpha}_{\Omega(q)}(v) + \sigma(\underline{\pi}_{\Omega(q)}(v)), \overline{\alpha}_{\Omega(q)}(v) + \sigma(\overline{\pi}_{\Omega(q)}(v))], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],$$
$$[\underline{\gamma}_{\Omega(q)}(v) + \rho(\underline{\pi}_{\Omega(q)}(v)), \overline{\gamma}_{\Omega(q)}(v) + \rho(\overline{\pi}_{\Omega(q)}(v)]\rangle; q \in \mathcal{S}\}$$

$$\mathcal{N}_{\epsilon,\rho}(\Omega, \mathcal{S}) = \{\langle v, [\underline{\alpha}_{\Omega(q)}(v) + \epsilon(\underline{\pi}_{\Omega(q)}(v)), \overline{\alpha}_{\Omega(q)}(v) + \epsilon(\overline{\pi}_{\Omega(q)}(v))], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],$$
$$[\underline{\gamma}_{\Omega(q)}(v) + \rho(\underline{\pi}_{\Omega(q)}(v)), \overline{\gamma}_{\Omega(q)}(v) + \rho(\overline{\pi}_{\Omega(q)}(v)]\rangle; q \in \mathcal{S}\}$$

Now, $\underline{\alpha}_{\Omega(q)}(v) + \sigma(\underline{\pi}_{\Omega(q)}(v)) \leq \underline{\alpha}_{\Omega(q)}(v) + \epsilon(\underline{\pi}_{\Omega(q)}(v))$, since $\sigma \leq \epsilon$
Similarly, $\overline{\alpha}_{\Omega(q)}(v) + \sigma(\underline{\pi}_{\Omega(q)}(v)) \leq \overline{\alpha}_{\Omega(q)}(v) + \epsilon(\underline{\pi}_{\Omega(q)}(v))$.
Hence $\mathcal{N}_{\sigma,\rho}(\Omega, \mathcal{S}) \Subset \mathcal{N}_{\epsilon,\rho}(\Omega, \mathcal{S})$.
(iii) Similar to proof (ii).

(iv) Consider,

$$\begin{aligned}\mathcal{N}_{\epsilon,1-\epsilon}(\Omega,\mathcal{S}) = \big\{\big\langle v, [&\underline{\alpha}_{\Omega(q)}(v) + \epsilon(\underline{\pi}_{\Omega(q)}(v)), \overline{\alpha}_{\Omega(q)}(v) + \epsilon(\overline{\pi}_{\Omega(q)}(v))], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\ &[\underline{\gamma}_{\Omega(q)}(v) + (1-\epsilon)(\underline{\pi}_{\Omega(q)}(v)), \overline{\gamma}_{\Omega(q)}(v) + (1-\epsilon)(\overline{\pi}_{\Omega(q)}(v)]\big\rangle; q \in \mathcal{S}\big\}\\ =&\mathcal{N}_{\epsilon}(\Omega,\mathcal{S}).\end{aligned}$$

Hence $\mathcal{N}_{\epsilon}(\Omega,\mathcal{S}) = \mathcal{N}_{\epsilon,(1-\epsilon)}(\Omega,\mathcal{S})$.

(v) Let $\epsilon = 0$ and $\rho = 1$,

$$\begin{aligned}\mathcal{N}_{0,1}(\Omega,\mathcal{S}) =& \big\{\big\langle v, [\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\underline{\gamma}_{\Omega(q)}(v) + \underline{\pi}_{\Omega(q)}(v),\\ &\overline{\gamma}_{\Omega(q)}(v) + \overline{\pi}_{\Omega(q)}(v)]\big\rangle; q \in \mathcal{S}\big\}\\ =&\big\{\big\langle v, [\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [(1-\overline{\alpha}_{\Omega(q)}(v)),\\ &(1-\underline{\alpha}_{\Omega(q)}(v)]\big\rangle; q \in \mathcal{S}\big\}\\ =& \oplus(\Omega,\mathcal{S}).\end{aligned}$$

Hence $\oplus(\Omega,\mathcal{S}) = \mathcal{N}_{0,1}(\Omega,\mathcal{S})$.

(vi)Let $\alpha = 1$ and $\beta = 0$,

$$\begin{aligned}\mathcal{N}_{1,0}(\Omega,\mathcal{S}) =& \big\{\big\langle v, [\underline{\alpha}_{\Omega(q)}(v) + \underline{\pi}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v) + \overline{\pi}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\ &[\underline{\gamma}_{\Omega(q)}(v), \overline{\gamma}_{\Omega(q)}(v)]\big\rangle; q \in \mathcal{S}\big\}\\ =&\big\{\big\langle v, [(1-\overline{\gamma}_{\Omega(q)}(v)), (1-\underline{\gamma}_{\Omega(q)}(v))], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\underline{\gamma}_{\Omega(q)}(v),\\ &\overline{\gamma}_{\Omega(q)}(v)]\big\rangle; q \in \mathcal{S}\big\}\\ =& \ominus(\Omega,\mathcal{S}).\end{aligned}$$

Hence $\ominus(\Omega,\mathcal{S}) = \mathcal{N}_{1,0}(\Omega,\mathcal{S})$.

(vii) Consider,

$$\begin{aligned}\mathcal{N}_{\epsilon,\rho}(\Omega,\mathcal{S})^c = \big\{\big\langle v, [&\underline{\gamma}_{\Omega(q)}(v) + \epsilon(\underline{\pi}_{\Omega(q)}(v)), \overline{\gamma}_{\Omega(q)}(v) + \epsilon(\overline{\pi}_{\Omega(q)}(v))],\\ &[\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\underline{\alpha}_{\Omega(q)}(v) + \rho(\underline{\pi}_{\Omega(q)}(v)), \overline{\alpha}_{\Omega(q)}(v) + \rho(\overline{\pi}_{\Omega(q)}(v)]\big\rangle; q \in \mathcal{S}\big\}\\ (\mathcal{N}_{\epsilon,\rho}(\Omega,\mathcal{S})^c)^c = \big\{\big\langle v, [&\underline{\alpha}_{\Omega(q)}(v) + \rho(\underline{\pi}_{\Omega(q)}(v)), \overline{\alpha}_{\Omega(q)}(v) + \rho(\overline{\pi}_{\Omega(q)}(v)],\\ &[\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\underline{\gamma}_{\Omega(q)}(v) + \epsilon(\underline{\pi}_{\Omega(q)}(v)), \overline{\gamma}_{\Omega(q)}(v) + \epsilon(\overline{\pi}_{\Omega(q)}(v))]\big\rangle; q \in \mathcal{S}\big\}\\ =&(\mathcal{N}_{\rho,\epsilon}(\Omega,\mathcal{S})).\end{aligned}$$

Hence $(\mathcal{N}_{\epsilon,\rho}(\Omega,\mathcal{S})^c)^c = (\mathcal{N}_{\rho,\epsilon}(\Omega,\mathcal{S}))$. □

**Remark 6.6.** If $\epsilon + \rho = 1$, then $\mathcal{N}_{\epsilon,\rho}(\Omega,\mathcal{S}) = \mathcal{N}_{\epsilon}(\Omega,\mathcal{S})$.

**Definition 6.7.** Let $\epsilon, \rho \in [0,1]$ and $\epsilon + \rho \leq 1$. Then the operator $\mathcal{I}_{\epsilon,\rho}(\Omega,\mathcal{S})$ is represented as,

$$\mathcal{I}_{\epsilon,\rho}(\Omega,\mathcal{S}) = \big\{\big\langle v, [\epsilon.\underline{\alpha}_{\Omega(q)}(v), \epsilon.\overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\rho.\underline{\gamma}_{\Omega(q)}(v), \rho.\overline{\gamma}_{\Omega(q)}(v)]\big\rangle; q \in \mathcal{S}\big\}$$

**Theorem 6.8.** *Let $\epsilon, \rho, \gamma \in [0,1]$. Then for every IVINSS $(\Omega, \mathcal{S})$ the following hold:*
*(i) $\mathcal{I}_{\epsilon,\rho}(\Omega, \mathcal{S})$ is an IVINSS;*
*(ii) If $\epsilon \leq \sigma$ then $\mathcal{I}_{\epsilon,\rho}(\Omega, \mathcal{S}) \Subset \mathcal{I}_{\sigma,\rho}(\Omega, \mathcal{S})$;*
*(iii) If $\rho \leq \sigma$ then $\mathcal{I}_{\epsilon,\rho}(\Omega, \mathcal{S}) \supset \mathcal{I}_{\epsilon,\sigma}(\Omega, \mathcal{S})$;*
*(iv) If $\delta \in [0,1]$ then $\mathcal{I}_{\epsilon,\rho}(\mathcal{I}_{\sigma,\delta}(\Omega, \mathcal{S})) = \mathcal{I}_{\epsilon\sigma,\rho\delta}(\Omega, \mathcal{S}) = \mathcal{I}_{\sigma,\delta}(\mathcal{I}_{\epsilon,\rho}(\Omega, \mathcal{S}))$;*
*(v)$(\mathcal{I}_{\epsilon,\rho}(\Omega, \mathcal{S})^c)^c = (\mathcal{I}_{\rho,\epsilon}(\Omega, \mathcal{S}))$.*

*Proof.* (i) Proof straightforward.
(ii) Consider,

$$\mathcal{I}_{\epsilon,\rho}(\Omega, \mathcal{S}) = \Big\{\Big\langle v, [\epsilon.\underline{\alpha}_{\Omega(q)}(v), \epsilon.\overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\rho.\underline{\gamma}_{\Omega(q)}(v), \rho.\overline{\gamma}_{\Omega(q)}(v)]\Big\rangle; q \in \mathcal{S}\Big\},$$
$$\mathcal{I}_{\sigma,\rho}(\Omega, \mathcal{S}) = \Big\{\Big\langle v, [\sigma.\underline{\alpha}_{\Omega(q)}(v), \sigma.\overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\rho.\underline{\gamma}_{\Omega(q)}(v), \rho.\overline{\gamma}_{\Omega(q)}(v)]\Big\rangle; q \in \mathcal{S}\Big\}.$$

Since $\epsilon \leq \sigma$, $\epsilon.\underline{\alpha}_{\Omega(q)}(v) \leq \sigma.\underline{\alpha}_{\Omega(q)}(v)$ and $\epsilon.\overline{\alpha}_{\Omega(q)}(v) \leq \sigma.\overline{\alpha}_{\Omega(q)}(v)$.
Hence $\mathcal{I}_{\epsilon,\rho}(\Omega, \mathcal{S}) \Subset \mathcal{I}_{\sigma,\rho}(\Omega, \mathcal{S})$.
(iii) Consider,

$$\mathcal{I}_{\epsilon,\rho}(\Omega, \mathcal{S}) = \Big\{\Big\langle v, [\epsilon.\underline{\alpha}_{\Omega(q)}(v), \epsilon.\overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\rho.\underline{\gamma}_{\Omega(q)}(v), \rho.\overline{\gamma}_{\Omega(q)}(v)]\Big\rangle; q \in \mathcal{S}\Big\},$$
$$\mathcal{I}_{\epsilon,\sigma}(\Omega, \mathcal{S}) = \Big\{\Big\langle v, [\epsilon.\underline{\alpha}_{\Omega(q)}(v), \epsilon.\overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\sigma.\underline{\gamma}_{\Omega(q)}(v), \sigma.\overline{\gamma}_{\Omega(q)}(v)]\Big\rangle; q \in \mathcal{S}\Big\}.$$

Since, $\rho \leq \sigma$, $\rho.\underline{\gamma}_{\Omega(q)}(v) \leq \sigma.\underline{\gamma}_{\Omega(q)}(v)$ and $\rho.\overline{\gamma}_{\Omega(q)}(v) \leq \sigma.\overline{\gamma}_{\Omega(q)}(v)$.
Hence $\mathcal{I}_{\epsilon,\rho}(\Omega, \mathcal{S}) \supset \mathcal{I}_{\epsilon,\rho}(\Omega, \mathcal{S})$.
(iv) Consider,

$$\begin{aligned}
\mathcal{I}_{\epsilon,\rho}(\mathcal{I}_{\sigma,\delta}(\Omega, \mathcal{S})) &= \Big\{\Big\langle v, [\epsilon\sigma.\underline{\alpha}_{\Omega(q)}(v), \epsilon\sigma.\overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\
&\quad [\rho\delta.\underline{\gamma}_{\Omega(q)}(v), \rho\delta.\overline{\gamma}_{\Omega(q)}(v)]\Big\rangle; q \in \mathcal{S}\Big\},\\
&= \mathcal{I}_{\epsilon\sigma,\rho\delta}(\Omega, \mathcal{S}).\\
\mathcal{I}_{\sigma,\delta}(\mathcal{I}_{\epsilon,\rho}(\Omega, \mathcal{S})) &= \Big\{\Big\langle v, [\sigma\epsilon.\underline{\alpha}_{\Omega(q)}(v), \sigma\epsilon.\overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\
&\quad [\delta\rho.\underline{\gamma}_{\Omega(q)}(v), \delta\rho.\overline{\gamma}_{\Omega(q)}(v)]\Big\rangle; q \in \mathcal{S}\Big\},\\
&= \Big\{\Big\langle v, [\epsilon\sigma.\underline{\alpha}_{\Omega(q)}(v), \epsilon\sigma.\overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\
&\quad [\rho\delta.\underline{\gamma}_{\Omega(q)}(v), \rho\delta.\overline{\gamma}_{\Omega(q)}(v)]\Big\rangle; q \in \mathcal{S}\Big\},\\
&= \mathcal{I}_{\epsilon\sigma,\rho\delta}(\Omega, \mathcal{S})).
\end{aligned}$$

Hence $\mathcal{I}_{\epsilon,\rho}(\mathcal{I}_{\sigma,\delta}(\Omega,\mathcal{S})) = \mathcal{I}_{\epsilon\sigma,\rho\delta}(\Omega,\mathcal{S}) = \mathcal{I}_{\sigma,\delta}(\mathcal{I}_{\epsilon,\rho}(\Omega,\mathcal{S}))$.

(v) Consider,

$$\begin{aligned}
(\Omega,\mathcal{S})^c &= \big\{\big\langle v, [\underline{\gamma}_{\Omega(q)}(v), \overline{\gamma}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)]\big\rangle; q \in \mathcal{S}\big\},\\
\mathcal{I}_{\epsilon,\rho}(\Omega,\mathcal{S})^c &= \big\{\big\langle v, [\epsilon.\underline{\gamma}_{\Omega(q)}(v), \epsilon.\overline{\gamma}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\rho.\underline{\alpha}_{\Omega(q)}(v), \rho.\overline{\alpha}_{\Omega(q)}(v)]\big\rangle; q \in \mathcal{S}\big\},\\
(\mathcal{I}_{\epsilon,\rho}(\Omega,\mathcal{S})^c)^c &= \big\{\big\langle v, [\rho.\underline{\alpha}_{\Omega(q)}(v), \rho.\overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\epsilon.\underline{\gamma}_{\Omega(q)}(v), \epsilon.\overline{\gamma}_{\Omega(q)}(v)]\big\rangle; q \in \mathcal{S}\big\},\\
&= \mathcal{I}_{\rho,\epsilon}(\Omega,\mathcal{S}).
\end{aligned}$$

Hence $(\mathcal{I}_{\epsilon,\rho}(\Omega,\mathcal{S})^c)^c = \mathcal{I}_{\rho,\epsilon}(\Omega,\mathcal{S})$. □

## 7. Concentration ($\mathcal{CO}$) and dilation ($\mathcal{DO}$) operators on IVINSS

We provide the definition of ($\mathcal{CO}$) and ($\mathcal{DO}$) on IVINSS and discuss their properties in detail. We generalize these operations and properties on IVINSS by the concepts discussed in [32], [18] and [2].

**Definition 7.1.** Let $(\Omega,\mathcal{S})$ be an IVINSS over $\mathcal{V}$. Then,

(i) the $\mathcal{CO}$ of $(\Omega,\mathcal{S})$ is represented as,

$$\begin{aligned}
\mathcal{C}(\Omega,\mathcal{S}) = \big\{\big\langle v, &[\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\
&[1-(1-\underline{\gamma}_{\Omega(q)}(v))^2, 1-(1-\overline{\gamma}_{\Omega(q)}(v))^2]\big\rangle; q \in \mathcal{S}\big\};
\end{aligned}$$

(ii) the $\mathcal{DO}$ of $(\Omega,\mathcal{S})$ is represented as,

$$\begin{aligned}
\mathcal{D}(\Omega,\mathcal{S}) = \big\{\big\langle v, &[(\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}, (\overline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\
&[1-(1-\underline{\gamma}_{\Omega(q)}(v))^{\frac{1}{4}}, 1-(1-\overline{\gamma}_{\Omega(q)}(v))^{\frac{1}{4}}]\big\rangle; q \in \mathcal{S}\big\};
\end{aligned}$$

**Proposition 7.2.** *Let $\mathcal{V}$ denote a non-empty set and $(\Omega,\mathcal{S})$ be an IVINSS over $\mathcal{V}$.*

*(i) If $[\underline{\pi}_{\Omega(q)}(v), \overline{\pi}_{\Omega(q)}(v)]$ = [0,0], then $[\underline{\pi}_{\mathcal{C}\Omega(q)}(v), \overline{\pi}_{\mathcal{C}\Omega(q)}(v)]$ = [0,0] iff $[\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)]$ = [0,0] or $[\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)]$ = [1,1] ;*

*(ii) $\oplus[\mathcal{C}(\Omega,\mathcal{S})] = \mathcal{C}[\oplus(\Omega,\mathcal{S})]$ iff $[\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)]$ = [0,0] or $[\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)]$ = [1,1] ;*

*(iii) $\ominus[\mathcal{C}(\Omega,\mathcal{S})] = \mathcal{C}[\ominus(\Omega,\mathcal{S})]$ iff $[\underline{\gamma}_{\Omega(q)}(v), \overline{\gamma}_{\Omega(q)}(v)]$ = [0,0] or $[\underline{\gamma}_{\Omega(q)}(v), \overline{\gamma}_{\Omega(q)}(v)]$ = [1,1] .*

*Proof.* (i) If $[\underline{\pi}_{\Omega(q)}(v), \overline{\pi}_{\Omega(q)}(v)] = [0,0]$

$\Rightarrow 1-\underline{\alpha}_{\Omega(q)}(v)-\underline{\gamma}_{\Omega(q)}(v) = 0$ and $1-\overline{\alpha}_{\Omega(q)}(v)-\overline{\gamma}_{\Omega(q)}(v) = 0$

$\Rightarrow \underline{\alpha}_{\Omega(q)}(v) + \underline{\gamma}_{\Omega(q)}(v) = 1$ and $\overline{\alpha}_{\Omega(q)}(v) + \overline{\gamma}_{\Omega(q)}(v) = 1$

$$\begin{aligned}\mathcal{C}(\Omega, \mathcal{S}) = \big\{\big\langle v, [\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\ [1-(1-\underline{\gamma}_{\Omega(q)}(v))^2, 1-(1-\overline{\gamma}_{\Omega(q)}(v))^2]\big\rangle; q \in \mathcal{S}\big\}.\\ = \big\{\big\langle v, [\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\ [1-(\underline{\alpha}_{\Omega(q)}(v))^2, 1-(\overline{\alpha}_{\Omega(q)}(v))^2]\big\rangle; q \in \mathcal{S}\big\}.\end{aligned}$$

If $\underline{\pi}_{\mathcal{C}\Omega(q)}(v) = 0 \Leftrightarrow 1 - \underline{\alpha}_{\Omega(q)}(v) - (1 - (\underline{\alpha}_{\Omega(q)}(v))^2) = 0.$
Then $\underline{\alpha}_{\Omega(q)}(v)(\underline{\alpha}_{\Omega(q)}(v) - 1) = 0 \Leftrightarrow \underline{\alpha}_{\Omega(q)}(v) = 0$ or $\underline{\alpha}_{\Omega(q)}(v) = 1.$
Similarly, $\overline{\alpha}_{\Omega(q)}(v) = 0$ or $\overline{\alpha}_{\Omega(q)}(v) = 1.$
(ii) We know that, $\oplus[\mathcal{C}(\Omega, \mathcal{S})]$

$$= \big\{\big\langle v, [\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [(1-\overline{\alpha}_{\Omega(q)}(v)), (1-\underline{\alpha}_{\Omega(q)}(v))]\big\rangle; q \in \mathcal{S}\big\} \quad (1)$$

Also,
$\mathcal{C}[\oplus(\Omega, \mathcal{S})]$

$$\begin{aligned}&= \big\{\big\langle v, [\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [1-(1-(1-\underline{\alpha}_{\Omega(q)}(v)))^2,\\ &\quad 1-(1-(1-\overline{\alpha}_{\Omega(q)}(v)))^2]\big\rangle; q \in \mathcal{S}\big\} \quad (2)\\ &= \big\{\big\langle v, [\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [1-(\underline{\alpha}_{\Omega(q)}(v))^2, 1-(\overline{\alpha}_{\Omega(q)}(v))^2]\big\rangle; q \in \mathcal{S}\big\}.\end{aligned}$$

From (1) and (2), we conclude that

$$\begin{aligned}\oplus[\mathcal{C}(\Omega, \mathcal{S})] = \mathcal{C}[\oplus(\Omega, \mathcal{S})] &\Leftrightarrow 1 - \underline{\alpha}_{\Omega(q)}(v) = 1 - (\underline{\alpha}_{\Omega(q)}(v))^2\\ &\Leftrightarrow \underline{\alpha}_{\Omega(q)}(v)(\underline{\alpha}_{\Omega(q)}(v) - 1) = 0\\ &\Leftrightarrow \underline{\alpha}_{\Omega(q)}(v) = 0 \text{ or } \underline{\alpha}_{\Omega(q)}(v) = 1.\end{aligned}$$

Similarly, $\overline{\alpha}_{\Omega(q)}(v) = 0$ or $\overline{\alpha}_{\Omega(q)}(v) = 1$ .
(iii) Proof is similar to (ii). □

**Proposition 7.3.** *Let $\mathcal{V}$ denote a non-empty set and $(\Omega, \mathcal{S})$ be an IVINSS over $\mathcal{V}$.*
*(i) If $[\underline{\pi}_{\Omega(q)}(v), \overline{\pi}_{\Omega(q)}(v)]$ = [0,0], then $[\underline{\pi}_{\mathcal{D}\Omega(q)}(v), \overline{\pi}_{\mathcal{C}\Omega(q)}(v)]$ = [0,0] iff $[\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)]$ = [0,0] or $[\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)]$ = [1,1] ;*
*(ii) $\oplus[\mathcal{D}(\Omega, \mathcal{S})] = \mathcal{D}[\oplus(\Omega, \mathcal{S})]$ iff $[\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)]$ = [0,0] or $[\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)]$ = [1,1] ;*
*(iii) $\ominus[\mathcal{D}(\Omega, \mathcal{S})] = \mathcal{D}[\ominus(\Omega, \mathcal{S})]$ iff $[\underline{\gamma}_{\Omega(q)}(v), \overline{\gamma}_{\Omega(q)}(v)]$ = [0,0] or $[\underline{\gamma}_{\Omega(q)}(v), \overline{\gamma}_{\Omega(q)}(v)]$ = [1,1] .*

*Proof.* (i) If $[\underline{\pi}_{\Omega(q)}(v), \overline{\pi}_{\Omega(q)}(v)]$ = [0,0]
$\Rightarrow 1 - \underline{\alpha}_{\Omega(q)}(v) - \underline{\gamma}_{\Omega(q)}(v) = 0$ and $1 - \overline{\alpha}_{\Omega(q)}(v) - \overline{\gamma}_{\Omega(q)}(v) = 0$

$\Rightarrow \underline{\alpha}_{\Omega(q)}(v) + \underline{\gamma}_{\Omega(q)}(v) = 1$ and $\overline{\alpha}_{\Omega(q)}(v) + \overline{\gamma}_{\Omega(q)}(v) = 1$

$$\begin{aligned}\mathcal{D}(\Omega, \mathcal{S}) = \big\{\big\langle v, &[(\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}, (\overline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\ &[1-(1-\underline{\gamma}_{\Omega(q)}(v))^{\frac{1}{4}}, 1-(1-\overline{\gamma}_{\Omega(q)}(v))^{\frac{1}{4}}]\big\rangle; q \in \mathcal{S}\big\}\\ =\big\{\big\langle v, &[(\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}, (\overline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\ &[1-(\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{4}}, 1-(\overline{\alpha}_{\Omega(q)}(v))^{\frac{1}{4}}]\big\rangle; q \in \mathcal{S}\big\}.\end{aligned}$$

If $\underline{\pi}_{\mathcal{D}\Omega(q)}(v) = 0 \Leftrightarrow 1 - (\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}} - 1 + (\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{4}} = 0 \Leftrightarrow (\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{4}} = (\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}$.
Then $\underline{\alpha}_{\Omega(q)}(v)(\underline{\alpha}_{\Omega(q)}(v) - 1) = 0 \Leftrightarrow \underline{\alpha}_{\Omega(q)}(v) = 0$ or $\underline{\alpha}_{\Omega(q)}(v) = 1$.
Similarly, $\overline{\alpha}_{\Omega(q)}(v) = 0$ or $\overline{\alpha}_{\Omega(q)}(v) = 1$.

(ii) We know that,

$$\begin{aligned}\oplus[\mathcal{D}(\Omega, \mathcal{S})] = \big\{\big\langle v, &[(\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}, (\overline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\ &[1-(\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}, 1-(\overline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}]\big\rangle; q \in \mathcal{S}\big\}.\end{aligned} \tag{3}$$

Also,

$$\begin{aligned}\mathcal{D}[\oplus(\Omega, \mathcal{S})] = \big\{\big\langle v, &[(\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}, (\overline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\ &[1-(1-(1-\underline{\alpha}_{\Omega(q)}(v)))^{\frac{1}{4}}, 1-(1-(1-\overline{\alpha}_{\Omega(q)}(v)))^{\frac{1}{4}}\big\rangle; q \in \mathcal{S}\big\}\\ = \big\{\big\langle v, &[(\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}, (\overline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)],\\ &[1-(\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{4}}, 1-(\overline{\alpha}_{\Omega(q)}(v))^{\frac{1}{4}}]\big\rangle; q \in \mathcal{S}\big\}.\end{aligned} \tag{4}$$

From (3) and (4), we conclude that

$$\begin{aligned}\oplus[\mathcal{D}(\Omega, \mathcal{S})] = \mathcal{D}[\oplus(\Omega, \mathcal{S})] &\Leftrightarrow 1 - (\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}} = 1 - (\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{4}}.\\ &\Leftrightarrow \underline{\alpha}_{\Omega(q)}(v)(1-\underline{\alpha}_{\Omega(q)}(v)) = 0.\\ &\Leftrightarrow \underline{\alpha}_{\Omega(q)}(v) = 0 \text{ or } \underline{\alpha}_{\Omega(q)}(v) = 1.\end{aligned}$$

Similarly, $\overline{\alpha}_{\Omega(q)}(v) = 0$ or $\overline{\alpha}_{\Omega(q)}(v) = 1$.
(iii) Proof is similar to (ii). □

**Proposition 7.4.** *For any IVINSS* $(\Omega, \mathcal{S})$*,* $\mathcal{C}(\Omega, \mathcal{S}) \Subset (\Omega, \mathcal{S}) \Subset \mathcal{D}(\Omega, \mathcal{S})$.

*Proof.* Consider,

$$(\Omega, \mathcal{S}) = \left\{\left\langle v, [\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [\underline{\gamma}_{\Omega(q)}(v), \overline{\gamma}_{\Omega(q)}(v)]\right\rangle;\ q \in \mathcal{S}\right\}.$$

$$\begin{aligned}\mathcal{C}(\Omega, \mathcal{S}) = \big\{\big\langle v, &[\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v)], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [1-(1-\underline{\gamma}_{\Omega(q)}(v))^2,\\ &1-(1-\overline{\gamma}_{\Omega(q)}(v))^2]\big\rangle; q \in \mathcal{S}\big\};\end{aligned}$$

Since, $\underline{\gamma}_{\Omega(q)}(v), \overline{\gamma}_{\Omega(q)}(v) \in [0,1], (1-(1-\underline{\gamma}_{\Omega(q)}(v))^2) \geq \underline{\gamma}_{\Omega(q)}(v)$and
$(1-(1-\overline{\gamma}_{\Omega(q)}(v))^2) \geq \overline{\gamma}_{\Omega(q)}(v)$.

$$\text{Hence } \mathcal{C}(\Omega,\mathcal{S}) \Subset (\Omega,\mathcal{S}). \tag{5}$$

$$\mathcal{D}(\Omega,\mathcal{S}) = \Big\{\Big\langle v, [(\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}, (\overline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}], [\underline{\beta}_{\Omega(q)}(v), \overline{\beta}_{\Omega(q)}(v)], [1-(1-\underline{\gamma}_{\Omega(q)}(v))^{\frac{1}{4}}, 1-(1-\overline{\gamma}_{\Omega(q)}(v))^{\frac{1}{4}}]\Big\rangle; q \in \mathcal{S}\Big\}.$$

Since, $\underline{\alpha}_{\Omega(q)}(v), \overline{\alpha}_{\Omega(q)}(v), \underline{\gamma}_{\Omega(q)}(v), \overline{\gamma}_{\Omega(q)}(v) \in [0,1]$,
$\underline{\alpha}_{\Omega(q)}(v) \leq (\underline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}$, $\overline{\alpha}_{\Omega(q)}(v) \leq (\overline{\alpha}_{\Omega(q)}(v))^{\frac{1}{2}}$, $\underline{\gamma}_{\Omega(q)}(v) \geq (1-(1-\underline{\gamma}_{\Omega(q)}(v))^{\frac{1}{4}})$ and $\overline{\gamma}_{\Omega(q)}(v) \geq (1-(1-\overline{\gamma}_{\Omega(q)}(v))^{\frac{1}{4}})$.

$$\text{Hence } (\Omega,\mathcal{S}) \Subset \mathcal{D}(\Omega,\mathcal{S}). \tag{6}$$

From (5) and (6), we get $\mathcal{C}(\Omega,\mathcal{S}) \Subset (\Omega,\mathcal{S}) \Subset \mathcal{D}(\Omega,\mathcal{S})$. □

## 8. **Similarity measures between IVINSS**

We provide a new similarity measure (SM) between IVINSS and explain its use with an application. We illustrate the working model with an algorithm and examples. Also, we bring out the importance of the proposed SM by comparing with existing SMs.

**Definition 8.1.** Let $\mathcal{V} = \{v_1, v_2, ..., v_n\}$ represent the universe and $\mathcal{S} = \{q_1, q_2, ....q_m\}$ represent the parameters. Then the SM between IVINSSs $(\Omega_1,\mathcal{S})$ and $(\Omega_2,\mathcal{S})$ is represented as,

$$S_M \langle(\Omega_1,\mathcal{S}),(\Omega_2,\mathcal{S})\rangle$$
$$= 1 - \frac{1}{4m}\sum_{i=1}^{m}\sum_{j=1}^{n}\Bigg[\frac{|\underline{\alpha}_{\Omega_1(q_i)}(v_j) - \underline{\alpha}_{\Omega_2(q_i)}(v_j)|}{2+\underline{\alpha}_{\Omega_1(q_i)}(v_j)+\underline{\alpha}_{\Omega_2(q_i)}(v_j)} + \frac{|\overline{\alpha}_{\Omega_1(q_i)}(v_j) - \overline{\alpha}_{\Omega_2(q_i)}(v_j)|}{2+\overline{\alpha}_{\Omega_1(q_i)}(v_j)+\overline{\alpha}_{\Omega_2(q_i)}(v_j)} + \frac{|\underline{\beta}_{\Omega_1(q_i)}(v_j) - \underline{\beta}_{\Omega_2(q_i)}(v_j)|}{2+\underline{\beta}_{\Omega_1(q_i)}(v_j)+\underline{\beta}_{\Omega_2(q_i)}(v_j)} +$$
$$\frac{|\overline{\beta}_{\Omega_1(q_i)}(v_j) - \overline{\beta}_{\Omega_2(q_i)}(v_j)|}{2+\overline{\beta}_{\Omega_1(q_i)}(v_j)+\overline{\beta}_{\Omega_2(q_i)}(v_j)} + \frac{|\underline{\gamma}_{\Omega_1(q_i)}(v_j) - \underline{\gamma}_{\Omega_2(q_i)}(v_j)|}{2+\underline{\gamma}_{\Omega_1(q_i)}(v_j)+\underline{\gamma}_{\Omega_2(q_i)}(v_j)} + \frac{|\overline{\gamma}_{\Omega_1(q_i)}(v_j) - \overline{\gamma}_{\Omega_2(q_i)}(v_j)|}{2+\overline{\gamma}_{\Omega_1(q_i)}(v_j)+\overline{\gamma}_{\Omega_2(q_i)}(v_j)} +$$
$$\left|\frac{(\underline{\alpha}_{\Omega_1(q_i)}(v_j) - \underline{\gamma}_{\Omega_1(q_i)}(v_j))}{2} - \frac{(\underline{\alpha}_{\Omega_2(q_i)}(v_j) - \underline{\gamma}_{\Omega_2(q_i)}(v_j))}{2}\right| +$$
$$\left|\frac{(\overline{\alpha}_{\Omega_1(q_i)}(v_j) - \overline{\gamma}_{\Omega_1(q_i)}(v_j))}{2} - \frac{(\overline{\alpha}_{\Omega_2(q_i)}(v_j) - \overline{\gamma}_{\Omega_2(q_i)}(v_j))}{2}\right|\Bigg].$$

### 8.1. *Comparison analysis with existing SMs*

In this section, we analyze some existing SMs on IVINSS. DMs apply SM to identify the most similar pattern between the precise and imprecise values. The framework of existing measures are given below.

(i) $\mathcal{S}_Y(\Omega_1,\Omega_2)$ [11]

$$= 1-\frac{1}{n}\sum_{i=1}^{n} w_j\Big[|\underline{\alpha}_{\Omega_1(q_i)}(v_j)-\underline{\alpha}_{\Omega_2(q_i)}(v_j)|+|\overline{\alpha}_{\Omega_1(q_i)}(v_j)-\overline{\alpha}_{\Omega_2(q_i)}(v_j)|+|\underline{\beta}_{\Omega_1(q_i)}(v_j)-\underline{\beta}_{\Omega_2(q_i)}(v_j)|$$
$$+|\overline{\beta}_{\Omega_1(q_i)}(v_j)-\overline{\beta}_{\Omega_2(q_i)}(v_j)|+|\underline{\gamma}_{\Omega_1(q_i)}(v_j)-\underline{\gamma}_{\Omega_2(q_i)}(v_j)|+|\overline{\gamma}_{\Omega_1(q_i)}(v_j)-\overline{\gamma}_{\Omega_2(q_i)}(v_j)|\Big].$$

(ii) $\mathcal{S}_C(\Omega_1,\Omega_2)$ [11]

$$= 1-\frac{1}{n}\sum_{i=1}^{n}\frac{(\tilde{\alpha}_{\Omega_1(q_i)}(v_j))(\tilde{\alpha}_{\Omega_2(q_i)}(v_j))+(\tilde{\beta}_{\Omega_1(q_i)}(v_j))(\tilde{\beta}_{\Omega_2(q_i)}(v_j))+(\tilde{\gamma}_{\Omega_1(q_i)}(v_j))(\tilde{\gamma}_{\Omega_1(q_i)}(v_j))}{(\sqrt{(\tilde{\alpha}_{\Omega_1(q_i)}(v_j))^2+(\tilde{\beta}_{\Omega_1(q_i)}(v_j))^2+(\tilde{\gamma}_{\Omega_1(q_i)}(v_j))^2})(\sqrt{(\tilde{\alpha}_{\Omega_2(q_i)}(v_j))^2+(\tilde{\beta}_{\Omega_2(q_i)}(v_j))^2+(\tilde{\gamma}_{\Omega_2(q_i)}(v_j))^2})},$$

where $\tilde{\alpha}_{\Omega_1(q_i)}(v_j)=\underline{\alpha}_{\Omega_1(q_i)}(v_j)+\overline{\alpha}_{\Omega_1(q_i)}(v_j), \tilde{\alpha}_{\Omega_2(q_i)}(v_j)=\underline{\alpha}_{\Omega_2(q_i)}(v_j)+\overline{\alpha}_{\Omega_2(q_i)}(v_j)$,
$\tilde{\beta}_{\Omega_1(q_i)}(v_j)=\underline{\beta}_{\Omega_1(q_i)}(v_j)+\overline{\beta}_{\Omega_1(q_i)}(v_j), \tilde{\beta}_{\Omega_2(q_i)}(v_j)=\underline{\beta}_{\Omega_2(q_i)}(v_j)+\overline{\beta}_{\Omega_2(q_i)}(v_j)$,
$\tilde{\gamma}_{\Omega_1(q_i)}(v_j)=\underline{\gamma}_{\Omega_1(q_i)}(v_j)+\overline{\gamma}_{\Omega_1(q_i)}(v_j)$ and $\tilde{\gamma}_{\Omega_2(q_i)}(v_j)=\underline{\gamma}_{\Omega_2(q_i)}(v_j)+\overline{\gamma}_{\Omega_2(q_i)}(v_j)$.

(iii) $\mathcal{S}_T(\Omega_1,\Omega_2)$ [12]

$$=\frac{\sum_{i=1}^{n}\Big(\begin{array}{l} min(\underline{\alpha}_{\Omega_1(q_i)}(v_j),\underline{\alpha}_{\Omega_2(q_i)}(v_j))+min(\overline{\alpha}_{\Omega_1(q_i)}(v_j),\overline{\alpha}_{\Omega_2(q_i)}(v_j))+min(\underline{\beta}_{\Omega_1(q_i)}(v_j),\underline{\beta}_{\Omega_2(q_i)}(v_j))\\ \quad +min(\overline{\beta}_{\Omega_1(q_i)}(v_j),\overline{\beta}_{\Omega_2(q_i)}(v_j))+min(\underline{\gamma}_{\Omega_1(q_i)}(v_j),\underline{\gamma}_{\Omega_2(q_i)}(v_j))+min(\overline{\gamma}_{\Omega_1(q_i)}(v_j),\overline{\gamma}_{\Omega_2(q_i)}(v_j))\end{array}\Big)}{\sum_{i=1}^{n}\Big(\begin{array}{l} max(\underline{\alpha}_{\Omega_1(q_i)}(v_j),\underline{\alpha}_{\Omega_2(q_i)}(v_j))+max(\overline{\alpha}_{\Omega_1(q_i)}(v_j),\overline{\alpha}_{\Omega_2(q_i)}(v_j))+max(\underline{\beta}_{\Omega_1(q_i)}(v_j),\underline{\beta}_{\Omega_2(q_i)}(v_j))\\ \quad +max(\overline{\beta}_{\Omega_1(q_i)}(v_j),\overline{\beta}_{\Omega_2(q_i)}(v_j))+max(\underline{\gamma}_{\Omega_1(q_i)}(v_j),\underline{\gamma}_{\Omega_2(q_i)}(v_j))+max(\overline{\gamma}_{\Omega_1(q_i)}(v_j),\overline{\gamma}_{\Omega_2(q_i)}(v_j))\end{array}\Big)},$$

(iv) $\mathcal{S}_H(\Omega_1,\Omega_2)$ [12]

$$=\frac{1}{6}\sum_{i=1}^{n} w_j\Big[|\underline{\alpha}_{\Omega_1(q_i)}(v_j)-\underline{\alpha}_{\Omega_2(q_i)}(v_j)|+|\overline{\alpha}_{\Omega_1(q_i)}(v_j)-\overline{\alpha}_{\Omega_2(q_i)}(v_j)|+|\underline{\beta}_{\Omega_1(q_i)}(v_j)-\underline{\beta}_{\Omega_2(q_i)}(v_j)|$$
$$+|\overline{\beta}_{\Omega_1(q_i)}(v_j)-\overline{\beta}_{\Omega_2(q_i)}(v_j)|+|\underline{\gamma}_{\Omega_1(q_i)}(v_j)-\underline{\gamma}_{\Omega_2(q_i)}(v_j)|+|\overline{\gamma}_{\Omega_1(q_i)}(v_j)-\overline{\gamma}_{\Omega_2(q_i)}(v_j)|\Big].$$

(v) $\mathcal{S}_E(\Omega_1,\Omega_2)$ [12]

$$=\Big(\sum_{i=1}^{n} w_j\Big[|\underline{\alpha}_{\Omega_1(q_i)}(v_j)-\underline{\alpha}_{\Omega_2(q_i)}(v_j)|^2+|\overline{\alpha}_{\Omega_1(q_i)}(v_j)-\overline{\alpha}_{\Omega_2(q_i)}(v_j)|^2+|\underline{\beta}_{\Omega_1(q_i)}(v_j)-\underline{\beta}_{\Omega_2(q_i)}(v_j)|^2$$
$$+|\overline{\beta}_{\Omega_1(q_i)}(v_j)-\overline{\beta}_{\Omega_2(q_i)}(v_j)|^2+|\underline{\gamma}_{\Omega_1(q_i)}(v_j)-\underline{\gamma}_{\Omega_2(q_i)}(v_j)|^2+|\overline{\gamma}_{\Omega_1(q_i)}(v_j)-\overline{\gamma}_{\Omega_2(q_i)}(v_j)|^2\Big]\Big)^{\frac{1}{2}}.$$

(vi) $\mathcal{S}_{C_1}(\Omega_1,\Omega_2)$ [34]

$$=\frac{1}{n}\sum_{i=1}^{n}\mathrm{Cos}\Big[\frac{\pi}{12}\Big(|\underline{\alpha}_{\Omega_1(q_i)}(v_j)-\underline{\alpha}_{\Omega_2(q_i)}(v_j)|+|\overline{\alpha}_{\Omega_1(q_i)}(v_j)-\overline{\alpha}_{\Omega_2(q_i)}(v_j)|+|\underline{\beta}_{\Omega_1(q_i)}(v_j)-\underline{\beta}_{\Omega_2(q_i)}(v_j)|$$
$$+|\overline{\beta}_{\Omega_1(q_i)}(v_j)-\overline{\beta}_{\Omega_2(q_i)}(v_j)|+|\underline{\gamma}_{\Omega_1(q_i)}(v_j)-\underline{\gamma}_{\Omega_2(q_i)}(v_j)|+|\overline{\gamma}_{\Omega_1(q_i)}(v_j)-\overline{\gamma}_{\Omega_2(q_i)}(v_j)|\Big)\Big].$$

(vii) $\mathcal{S}_{C_2}(\Omega_1,\Omega_2)$ [34]

$$=\frac{1}{n}\sum_{i=1}^{n}\mathrm{Cos}\Big[\frac{\pi}{4}\Big(|\underline{\alpha}_{\Omega_1(q_i)}(v_j)-\underline{\alpha}_{\Omega_2(q_i)}(v_j)|\vee|\underline{\beta}_{\Omega_1(q_i)}(v_j)-\underline{\beta}_{\Omega_2(q_i)}(v_j)|\vee|\underline{\gamma}_{\Omega_1(q_i)}(v_j)-\underline{\gamma}_{\Omega_2(q_i)}(v_j)|$$
$$+|\overline{\alpha}_{\Omega_1(q_i)}(v_j)-\overline{\alpha}_{\Omega_2(q_i)}(v_j)|\vee|\overline{\beta}_{\Omega_1(q_i)}(v_j)-\overline{\beta}_{\Omega_2(q_i)}(v_j)|\vee|\overline{\gamma}_{\Omega_1(q_i)}(v_j)-\overline{\gamma}_{\Omega_2(q_i)}(v_j)|\Big)\Big].$$

**Example 8.2.** Consider the following values, as in Table 7. Table 8 shows the superiority of the proposed SM than the existing SMs. It illustrates that the proposed SM can identify similar patterns (refer third column) even when the existing SMs have some limitations (refer first column). For computation purpose, let us consider $w_j$=1 for $\mathcal{S}_Y(\Omega_1,\Omega_2)$ and $\mathcal{S}_H(\Omega_1,\Omega_2)$.

Table 7. Shows precise and imprecise values.

| Precise value | Imprecise values |
|---|---|
| $\Omega = \langle [0.20, 0.30], [0.50, 0.60], [0.30, 0.50] \rangle$ | $\Omega_1 = \langle [0.60, 0.70], [0.50, 0.60], [0.10, 0.20] \rangle$, $\Omega_2 = \langle [0.50, 0.60], [0.40, 0.50], [0.10, 0.20] \rangle$. |
| $\Omega = \langle [0.60, 0.70], [0.70, 0.80], [0.20, 0.30] \rangle$ | $\Omega_1 = \langle [0.50, 0.60], [0.30, 0.60], [0.10, 0.20] \rangle$, $\Omega_2 = \langle [0.60, 0.70], [0.40, 0.50], [0.20, 0.30] \rangle$. |
| $\Omega = \langle [0.40, 0.50], [0.80, 0.90], [0.30, 0.40] \rangle$ | $\Omega_1 = \langle [0.50, 0.60], [0.50, 0.60], [0.20, 0.30] \rangle$, $\Omega_2 = \langle [0.30, 0.40], [0.40, 0.50], [0.30, [0.40] \rangle$. |
| $\Omega = \langle [0.69, 0.75], [0.75, 0.85], [0.15, 0.25] \rangle$ | $\Omega_1 = \langle [0.55, 0.65], [0.55, 0.66], [0.21, 0.29] \rangle$, $\Omega_2 = \langle [0.58, 0.69], [0.57, 0.68], [0.22, 0.29] \rangle$. |
| $\Omega = \langle [0.69, 0.75], [0.75, 0.85], [0.15, 0.25] \rangle$ | $\Omega_1 = \langle [0.55, 0.65], [0.54, 0.66], [0.21, 0.29] \rangle$, $\Omega_2 = \langle [0.58, 0.69], [0.46, 0.68], [0.22, 0.29] \rangle$. |
| $\Omega = \langle [0.20, 0.30], [0.50, 0.60], [0.30, 0.50] \rangle$ | $\Omega_1 = \langle [0.40, 0.50], [0.40, 0.50], [0.10, 0.20] \rangle$, $\Omega_2 = \langle [0.50, 0.60], [0.50, 0.60], [0.20, 0.30] \rangle$. |

Table 8. Analysis of existing SMs.

| Existing SMs | Proposed SMs | Similar pattern |
|---|---|---|
| $\mathcal{S}_Y(\Omega, \Omega_1) = \mathcal{S}_Y(\Omega, \Omega_2) = 0.7833$, $\mathcal{S}_H(\Omega, \Omega_1) = \mathcal{S}_H(\Omega, \Omega_2) = 0.2166$, $\mathcal{S}_{C_1}(\Omega, \Omega_1) = \mathcal{S}_{C_1}(\Omega, \Omega_2) = 0.9426$. | $\mathcal{S}_M(\Omega, \Omega_1) = 0.7198$, $\mathcal{S}_M(\Omega, \Omega_2) = 0.7435$ $\mathcal{S}_M(\Omega, \Omega_2) > \mathcal{S}_M(\Omega, \Omega_1) \Rightarrow \Omega_2$ | $\Omega_2$ |
| $\mathcal{S}_{C_2}(\Omega, \Omega_1) = \mathcal{S}_{C_2}(\Omega, \Omega_2) = 0.9877$ | $\mathcal{S}_M(\Omega, \Omega_1) = 0.9154$, $\mathcal{S}_M(\Omega, \Omega_2) = 0.9530$ $\mathcal{S}_M(\Omega, \Omega_2) > \mathcal{S}_M(\Omega, \Omega_1) \Rightarrow \Omega_2$ | $\Omega_2$ |
| $\mathcal{S}_Y(\Omega, \Omega_1) = \mathcal{S}_Y(\Omega, \Omega_2) = 0.8333$, $\mathcal{S}_H(\Omega, \Omega_1) = \mathcal{S}_H(\Omega, \Omega_2) = 0.1666$, $\mathcal{S}_{C_1}(\Omega, \Omega_1) = \mathcal{S}_{C_1}(\Omega, \Omega_2) = 0.9659$. | $\mathcal{S}_M(\Omega, \Omega_1) = 0.8698$, $\mathcal{S}_M(\Omega, \Omega_2) = 0.8964$ $\mathcal{S}_M(\Omega, \Omega_2) > \mathcal{S}_M(\Omega, \Omega_1) \Rightarrow \Omega_2$ | $\Omega_2$ |
| $\mathcal{S}_C(\Omega, \Omega_1) = \mathcal{S}_C(\Omega, \Omega_2) = 0.9937$ | $\mathcal{S}_M(\Omega, \Omega_1) = 0.9003$, $\mathcal{S}_M(\Omega, \Omega_2) = 0.8702$ $\mathcal{S}_M(\Omega, \Omega_1) > \mathcal{S}_M(\Omega, \Omega_2) \Rightarrow \Omega_1$ | $\Omega_1$ |
| $\mathcal{S}_T(\Omega, \Omega_1) = \mathcal{S}_T(\Omega, \Omega_2) = 0.800$ | $\mathcal{S}_M(\Omega, \Omega_1) = 0.8995$, $\mathcal{S}_M(\Omega, \Omega_2) = 0.8611$ $\mathcal{S}_M(\Omega, \Omega_1) > \mathcal{S}_M(\Omega, \Omega_2) \Rightarrow \Omega_1$ | $\Omega_1$ |
| $\mathcal{S}_E(\Omega, \Omega_1) = \mathcal{S}_E(\Omega, \Omega_2) = 0.1957$ | $\mathcal{S}_M(\Omega, \Omega_1) = 0.7851$, $\mathcal{S}_M(\Omega, \Omega_2) = 0.8060$ $\mathcal{S}_M(\Omega, \Omega_2) > \mathcal{S}_M(\Omega, \Omega_1) \Rightarrow \Omega_2$ | $\Omega_2$ |

**Theorem 8.3.** *Let* $(\Omega_1, \mathcal{S})$ *and* $(\Omega_2, \mathcal{S})$ *be two IVINSS over* $\mathcal{V}$. *Then,*
*(i)* $0 \leq S_M((\Omega_1, \mathcal{S}), (\Omega_2, \mathcal{S})) \leq 1$;
*(ii)* $S_M((\Omega_1, \mathcal{S}), (\Omega_2, \mathcal{S})) = S_M((\Omega_2, \mathcal{S}), (\Omega_1, \mathcal{S}))$;
*(iii)* $S_M((\Omega_1, \mathcal{S}), (\Omega_2, \mathcal{S})) = 1$ *iff* $(\Omega_1, \mathcal{S}) = (\Omega_2, \mathcal{S})$.

*Proof.* Proof straightforward □

8.2. *Diagnosing psychiatric disorder for people with COVID-19*

In this section, we present an application on diagnosing psychiatric disorder for people with COVID-19 using IVINSS. Let us consider the SM between two IVINSS over different universes with the same set of parameters. We use this to analyze the psychiatric disorder problem. We have proposed an algorithm and illustrated the technique with a suitable example.

8.3. *Description of the problem*

Let $\mathcal{V} = \{v_1, v_2, ..., v_n\}$ represent universe and $\mathcal{S} = \{q_1, q_2, ..., q_m\}$ represent the parameters. Also, let the precise values $(\Omega, \mathcal{S})$ describe the elements of the universe in IVINSS form given by the psychiatrist for each stage. Let the psychiatrist define the norms to identify the levels (low or moderate or high) associated with psychiatric disorder as in Table 10. Let $(\Omega_i, \mathcal{S})$, $(i = 1, 2, ..., t)$ denote the imprecise values. Each $(\Omega_i, \mathcal{S})$ is in IVINSS form representing the alternatives based on the observations on the subject by the psychiatrist made in relation to each element of the universe and for each element of the parameter set. Now the problem is to identify the level associated with $(\Omega_i, \mathcal{S})$ to the precise information $(\Omega, \mathcal{S})$.

8.4. *A new method to diagnose psychiatric disorder*

Let's assume that $(\Omega, \mathcal{S})$ and $(\Omega_i, \mathcal{S})$ represent the precise and imprecise values, respectively in IVINSS form. By using Definition 7.1, the psychiatrist identifies the SM value associated with $(\Omega_i, \mathcal{S})$ $(i = 1, 2, ..., t)$ to the precise information $(\Omega, \mathcal{S})$. Now, the psychiatrist compares the obtained SM value with the norms (Table 10) and interprets on the level of psychiatric disorder for each subject.

8.5. *Algorithm for diagnosing psychiatric disorder*

An algorithm is given below for diagnosing psychiatric disorder based on SM between IVINSS.

**Step 1**: Construct the precise values $(\Omega, \mathcal{S})$ and the norms based on the evaluation of psychiatrist for diagnosing psychiatric disorder.

**Step 2**: Construct the imprecise values $(\Omega_i, \mathcal{S})$, $(i = 1, 2, ..., t)$ by observing the behavior of the subjects.

**Step 3**: Compute the SM between $(\Omega, \mathcal{S})$ and $(\Omega_i, \mathcal{S})$.

**Step 4**: Compare the calculated SM value between $(\Omega, \mathcal{S})$ and $(\Omega_i, \mathcal{S})$ with the norms.

**Step 5**: Identify the level associated with each subject to diagnose the psychiatric disorder.

**Example 8.4.** Let $\mathcal{V} = \{o_1, o_2, o_3\}$ represent the sessions conducted by a psychiatrist. Let $C_1$, $C_2$ and $C_3$ represent the subjects and $\mathcal{S} = \{q_1, q_2, q_3, q_4, q_5\}$ represent the parameters, $q_1 =$

feeling sad or low, $q_2$ = confused thinking, $q_3$ = extreme mood changes, $q_4$ = excessive fears or worries and $q_5$ = sleeping problems. The psychiatrist has to diagnose the psychiatric disorder based on the norms associated with each subject.

**Step 1.** Construct the precise values $(\Omega, \mathcal{S})$ as in Table 9 and the norms as in Table 10 based on the evaluation of psychiatrist for diagnosing psychiatric disorder.

TABLE 9. Representation of precise values $(\Omega, \mathcal{S})$ in IVINSS form for each session.

| $\mathcal{V}$ | $o_1$ | $o_2$ | $o_3$ |
|---|---|---|---|
| $q_1$ | $\langle[0.6, 0.7], [0.7, 0.8], [0.1, 0.2]\rangle$ | $\langle[0.7, 0.8], [0.7, 0.8], [0.1, 0.2]\rangle$ | $\langle[0.6, 0.7], [0.8, 0.9], [0.1, 0.2]\rangle$ |
| $q_2$ | $\langle[0.5, 0.6], [0.8, 0.9], [0.2, 0.3]\rangle$ | $\langle[0.7, 0.8], [0.8, 0.9], [0.1, 0.2]\rangle$ | $\langle[0.5, 0.6], [0.9, 1.0], [0.2, 0.3]\rangle$ |
| $q_3$ | $\langle[0.4, 0.5], [0.7, 0.8], [0.3, 0.4]\rangle$ | $\langle[0.2, 0.3], [0.9, 1.0], [0.4, 0.5]\rangle$ | $\langle[0.4, 0.5], [0.8, 0.9], [0.4, 0.5]\rangle$ |
| $q_4$ | $\langle[0.3, 0.4], [0.6, 0.7], [0.4, 0.5]\rangle$ | $\langle[0.4, 0.5], [0.7, 0.8], [0.3, 0.4]\rangle$ | $\langle[0.6, 0.7], [0.8, 0.9], [0.1, 0.2]\rangle$ |
| $q_5$ | $\langle[0.2, 0.3], [0.5, 0.6], [0.4, 0.5]\rangle$ | $\langle[0.5, 0.6], [0.7, 0.8], [0.1, 0.2]\rangle$ | $\langle[0.4, 0.5], [0.9, 1.0], [0.1, 0.3]\rangle$ |

TABLE 10. Norms for NPD.

| Range of SM values | Levels of psychiatric disorder |
|---|---|
| $0.00 \leq S_M \langle(\Omega, \mathcal{S}), (\Omega_i, \mathcal{S})\rangle < 0.40$ | Low |
| $0.40 \leq S_M \langle(\Omega, \mathcal{S}), (\Omega_i, \mathcal{S})\rangle < 0.75$ | Moderate |
| $0.75 \leq S_M \langle(\Omega, \mathcal{S}), (\Omega_i, \mathcal{S})\rangle \leq 1.00$ | High |

**Step 2.** Now construct the imprecise values $(\Omega_i, \mathcal{S})$, $(i = 1, 2, ..., t)$ by observing the behavior of the subjects $C_1$, $C_2$ and $C_3$ respectively, as in Table 11, 12 and 13.

TABLE 11. Representation of imprecise values $(\Omega_1, \mathcal{S})$ for the first subject in SINSS form for each session.

| $\mathcal{V}$ | $o_1$ | $o_2$ | $o_3$ |
|---|---|---|---|
| $q_1$ | $\langle[0.8, 0.9], [0.7, 0.8], [0.0, 0.1]\rangle$ | $\langle[0.2, 0.3], [0.4, 0.5], [0.1, 0.2]\rangle$ | $\langle[0.1, 0.2], [0.7, 0.8], [0.1, 0.2]\rangle$ |
| $q_2$ | $\langle[0.7, 0.8], [0.6, 0.7], [0.1, 0.2]\rangle$ | $\langle[0.1, 0.2], [0.3, 0.4], [0.2, 0.3]\rangle$ | $\langle[0.2, 0.3], [0.8, 0.9], [0.1, 0.2]\rangle$ |
| $q_3$ | $\langle[0.6, 0.7], [0.8, 0.9], [0.2, 0.3]\rangle$ | $\langle[0.3, 0.4], [0.5, 0.6], [0.3, 0.4]\rangle$ | $\langle[0.3, 0.4], [0.9, 1.0], [0.4, 0.5]\rangle$ |
| $q_4$ | $\langle[0.6, 0.7], [0.7, 0.8], [0.1, 0.2]\rangle$ | $\langle[0.2, 0.4], [0.2, 0.3], [0.4, 0.5]\rangle$ | $\langle[0.2, 0.3], [0.9, 1.0], [0.2, 0.3]\rangle$ |
| $q_5$ | $\langle[0.6, 0.7], [0.9, 1.0], [0.2, 0.3]\rangle$ | $\langle[0.1, 0.3], [0.4, 0.5], [0.1, 0.2]\rangle$ | $\langle[0.1, 0.3], [0.8, 0.9], [0.3, 0.4]\rangle$ |

**Step 3.** By using Definition 7.1, calculate the $S_M \langle(\Omega, \mathcal{S}), (\Omega_i, \mathcal{S})\rangle$.

The values are as below:

$S_M \langle(\Omega, \mathcal{S}), (\Omega_1, \mathcal{S})\rangle = 0.396$, $S_M \langle(\Omega, \mathcal{S}), (\Omega_2, \mathcal{S})\rangle = 0.663$, $S_M \langle(\Omega, \mathcal{S}), (\Omega_3, \mathcal{S})\rangle = 0.772$.

**Step 4.** Now compare the calculated values of $S_M \langle(\Omega, \mathcal{S}), (\Omega_i, \mathcal{S})\rangle$ with Table 10.

The level of psychiatric disorder for the first subject shows low, for the second average and the third high.

**Step 5.** We can conclude from the above observation that the psychiatrist to start the next set of treatment sessions for the subjects $C_2$ and $C_3$ to lower the level of psychiatric disorder.

Table 12. Representation of imprecise values $(\Omega_2, \mathcal{S})$ for the second subject in SINSS form for each session.

| $\mathcal{V}$ | $o_1$ | $o_2$ | $o_3$ |
|---|---|---|---|
| $q_1$ | $\langle[0.2, 0.3], [0.7, 0.8], [0.2, 0.3]\rangle$ | $\langle[0.7, 0.8], [0.8, 0.9], [0.1, 0.2]\rangle$ | $\langle[0.7, 0.8], [0.7, 0.8], [0.1, 0.2]\rangle$ |
| $q_2$ | $\langle[0.4, 0.5], [0.8, 0.9], [0.1, 0.2]\rangle$ | $\langle[0.5, 0.6], [0.9, 1.0], [0.2, 0.3]\rangle$ | $\langle[0.4, 0.5], [0.8, 0.9], [0.1, 0.2]\rangle$ |
| $q_3$ | $\langle[0.4, 0.5], [0.9, 1.0], [0.4, 0.5]\rangle$ | $\langle[0.4, 0.5], [0.8, 0.9], [0.3, 0.4]\rangle$ | $\langle[0.3, 0.4], [0.9, 1.0], [0.4, 0.5]\rangle$ |
| $q_4$ | $\langle[0.6, 0.7], [0.7, 0.8], [0.1, 0.2]\rangle$ | $\langle[0.3, 0.4], [0.9, 1.0], [0.4, 0.5]\rangle$ | $\langle[0.6, 0.7], [0.9, 1.0], [0.2, 0.3]\rangle$ |
| $q_5$ | $\langle[0.5, 0.6], [0.7, 0.8], [0.2, 0.3]\rangle$ | $\langle[0.2, 0.3], [0.8, 0.9], [0.1, 0.2]\rangle$ | $\langle[0.5, 0.6], [0.8, 0.9], [0.3, 0.4]\rangle$ |

Table 13. Representation of imprecise values $(\Omega_3, \mathcal{S})$ for the third subject in SINSS form for each session.

| $\mathcal{V}$ | $o_1$ | $o_2$ | $o_3$ |
|---|---|---|---|
| $q_1$ | $\langle[0.7, 0.8], [0.7, 0.8], [0.1, 0.2]\rangle$ | $\langle[0.7, 0.8], [0.8, 0.9], [0.1, 0.2]\rangle$ | $\langle[0.7, 0.8], [0.8, 0.9], [0.1, 0.2]\rangle$ |
| $q_2$ | $\langle[0.4, 0.5], [0.8, 0.9], [0.1, 0.2]\rangle$ | $\langle[0.6, 0.7], [0.9, 1.0], [0.1, 0.2]\rangle$ | $\langle[0.5, 0.6], [0.7, 0.8], [0.2, 0.3]\rangle$ |
| $q_3$ | $\langle[0.2, 0.3], [0.6, 0.7], [0.2, 0.3]\rangle$ | $\langle[0.1, 0.3], [0.8, 0.9], [0.2, 0.3]\rangle$ | $\langle[0.1, 0.2], [0.7, 0.8], [0.1, 0.3]\rangle$ |
| $q_4$ | $\langle[0.4, 0.5], [0.7, 0.8], [0.1, 0.2]\rangle$ | $\langle[0.4, 0.5], [0.9, 1.0], [0.1, 0.2]\rangle$ | $\langle[0.6, 0.7], [0.9, 1.0], [0.1, 0.2]\rangle$ |
| $q_5$ | $\langle[0.1, 0.2], [0.8, 0.9], [0.2, 0.3]\rangle$ | $\langle[0.3, 0.4], [0.8, 0.9], [0.1, 0.2]\rangle$ | $\langle[0.5, 0.6], [0.8, 0.9], [0.2, 0.3]\rangle$ |

## 9. **Conclusion**

In this manuscript, we outline the notions of IVINS, IVINSS, and establish some of their properties. Also, we show the effectiveness of the proposed SM by comparing it with existing SMs. In today's complicated psychiatric disorder behaviors, SM plays a significant role in diagnosing the same. So, we propose a diagnosing method based on the SM for diagnosing psychiatric disorder with IVINSSs. In this method, we predict the psychiatric behavior of the subjects represented in the IVINSS form. We can apply this concept to other hybrid sets for diagnosing psychiatric disorders. Our future study would be the applications of neutrosophics in sociology [29].

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# Pairwise Pythagorean Neutrosophic P-spaces (with dependent neutrosophic components between T and F)

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**Abstract**. In this paper, we study the pairwise Pythagorean Neutrosophic (for shortly,Pairwise PN) bitopological spaces (with T and F are dependent neutrosophic components). We also study the pairwise PN P-spaces and the conditions under which PN bitopological spaces become pairwise PN P-spaces are investigated.

**Keywords:** PN bitopology, Pairwise PN P-spaces,PN Baire space.

## 1. **Introduction**

Fuzzy sets were introduced by Zadeh [17] and he discussed only membership function. The fuzzy topology concept was first introduced by C.L.Chang [3] in 1968.After the extensions of fuzzy set theory Atanassov [1] generalized this concept and introduced a new concept called intuitionistic fuzzy set (IFS). Yager [13] familiarized the model of Pythagorean fuzzy set. However, in the some practical problems, the sum of membership degree and non-membership degree to which an alternative satisfying attribute provided by decision maker(DM) may be bigger than 1, but their square sum is less than or equal to 1.

IFS was failed to deal with indeterminate and inconsistent information which exist in beliefs system, therefore, Smarandache [9] in 1995 introduced new concept known as neutrosophic set(NS) which generalizes fuzzy sets and intuitionistic fuzzy sets and so on. A neutrosophic set includes truth membership, falsity membership and indeterminacy membership. In 2006, F.Smarandache[9] introduced, for the first time, the degree of dependence (and consequently the degree of independence) between the components of the fuzzy set, and also between the components of the neutrosophic set. In 2016, the refined neutrosophic set was generalized to the degree of dependence or independence of subcomponents [10]. In neutrosophic set [10], if truth membership and falsity membership are dependent and indeterminacy is independent.Sometimes in real life, we face many problems which cannot be handled by using neutrosophic for example when $T_A(x) + I_A(x) + F_A(x) > 2$. Pythagorean neutrosophic sets [PN-sets]with T and F are dependent neutrosophic components [PNS] of condition is as their square sum does not exceeds 2.Jansi and Mohana[6]was studied about PN-sets.In 2003, A.K.Mishra [8] introduced the concept of P-spaces.The concept of P-spaces in fuzzy setting was defined by G.Balasubramanian[11].Almost P-spaces in classical topology was introduced by R.Levy[7] .

In this paper we study the pairwise PN P-spaces.Also we studied the conditions under which PN bitopological spaces become pairwise PN P-spaces are investigated.

## 2. preliminaries

**Definition 2.1. (Pythagorean Fuzzy Set)**[14] Let X be a non-empty set. A PF set A is an object of the form $A = \{(x, P_A(x), Q_A(x)) : x \in X\}$ where the function $P_A : X \to [0, 1]$ and $Q_A : X \to [0, 1]$ denote respectively the degree of membership and degree of non-membership of each element $x \in X$ to the set P, and $0 \leq (P_A(x))^2 + (Q_A(x))^2 \leq 1$ for each $x \in X$.

**Definition 2.2.** [10] Let X be a non-empty set. A neutrosophic set A on X is an object of the form: $A = \{(x, P_A(x), Q_A(x), R_A(x)) : x \in X\}$, Where $P_A(x), Q_A(x), R_A(x) \in [0, 1]$,$0 \leq P_A(x) + Q_A(x) + R_A(x) \leq 2$,for all x in X. $P_A(x)$ is the degree of membership, $Q_A(x)$ is the degree of inderminancy and $R_A(x)$ is the degree of non-membership. Here $P_A(x)$ and $R_A(x)$ are dependent components and $Q_A(x)$ is an independent components.

**Definition 2.3.** [6] (Pythagorean neutrosophic [PN]-sets (with T and F are dependent neutrosophic components))[13] Let X be a non-empty set. PN-set $A = \{(X, P_A(x), Q_A(x), R_A(x) : x \in X\}$ where $P_A : X \to [0, 1], Q_A : X \to [0, 1]$ and $R_A : X \to [0, 1]$ are the mappings such that $0 \leq P_A^2(x)+Q_A^2(x)+R_A^2(x) \leq 2$ and $P_A(x)$ denote the membership degree,$Q_A(x)$ denote the Indeterminacy and $R_A(x)$ denote the non-membership degree.Here $P_A$ and $R_A$ are dependent neutrosophic components and $Q_A$ is an independent components.

**Definition 2.4.** [6] Let $A = (P_A, Q_A, R_A)$ and $B = (P_B, Q_B, R_B)$ be two PNSs,then their operations are defined as follows:
(1)$A \subseteq B$ if and if $P_A(x) \leq P_B(x), Q_A(x) \geq Q_B(x), R_A(x) \geq R_B(x)$
(2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
(3) $A \cup B = \{(x, max(P_A, P_B), min(Q_A, Q_B), min(R_A, R_B) : x \in X\}$
(4)$A \cap B = \{(x, min(P_A, P_B), max(Q_A, Q_B), max(R_A, R_B) : x \in X\}$
(5)$A^C = \{(x, R_A, 1 - Q_A, P_A) : x \in X\}$.

## 3. Pairwise Pythagorean Neutrosophic [Pairwise PN] P-Spaces(with T and F are dependent neutrosophic components)

**Definition 3.1.** A Pythagorean neutrosophic (with T and F are dependent neutrosophic components) topology (PNT in Short) on X is a family $PN\tau$ of PNs in X satisfying the following axioms:

(1) $0_X, 1_X \in PN\tau$

(2)$G_1 \cap G_2 \in PN\tau$, for any $G_1, G_2 \in PN\tau$

(3) $\cup G_i \in PN\tau$ for any family $\{G_i/i \in J\} \subseteq PN\tau$.

In this case the pair $(X, PN\tau)$ is called a Pythagorean neutrosophic sets with T and F are dependent neutrosophic components topological space (PNTS in Short) and any BPFTS in $PN\tau$ is known as a Pythagorean neutrosophic sets with T and F are dependent neutrosophic components open set (PNOS in Short) in X.

The Complement $A^c$ of a PNOS A in a PNTS $(X, PN\tau)$ is called a Pythagorean neutrosophic sets with T and F are dependent neutrosophic components closed set (PNCS in Short) in X.

**Definition 3.2.** Let $(X, PN\tau)$ be a PNTS and be a PN in X. Then the PN interior and closure of a PN closure are defined by

$PNint(A) = \cup\{G/GisaPNOSinXandG \subseteq A\}$

$PNcl(A) = \cap\{K/KisaPNCSinXandA \subseteq K\}$.

Note that for any PN $A$ in $(X, PN\tau)$, we have $(PNcl(A))^c = PNint(A^c)$ and $(PNint(A))^c = PNcl(A^c)$.

**Definition 3.3.** A set X on which are defined two (arbitrary)PN topologies $PN\tau_1$ and $PN\tau_2$ is called PN bitopological spaces and denoted by $(X, PN\tau_1, PN\tau_2)$.
We shall write $PNint_{PN\tau_i}(A)$ and $PNcl_{PN\tau_i}(A)$ to mean respectively the PN interior and PN closure of PN set A with respect to the PN topology $PN\tau_i$ in $(X, PN\tau_1, PN\tau_2)$.

**Definition 3.4.** A PN $A$ in a PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is called a pairwise PN open set if $A \in PN\tau_i(i = 1, 2)$.The complement of pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$ is called a pairwise PN closed set.

**Definition 3.5.** A PN $A$ in a PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is called a pairwise PN dense set if $PNcl_{PN\tau_1}PNcl_{PN\tau_2}(A) = PNcl_{PN\tau_2}PNcl_{PN\tau_1}(A) = 1_X$ in $PNcl_{PN\tau_1}PNcl_{PN\tau_2}(A)$.

**Definition 3.6.** A PN $A$ in a PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is called pairwise PN nowhere dense if $PNint_{PN\tau_1}PNcl_{PN\tau_2}(A) = PNint_{PN\tau_2}PNcl_{PN\tau_1}(A) = 0_X$ in $PNint_{PN\tau_1}PNcl_{PN\tau_2}(A)$.

**Definition 3.7.** Let $(X, PN\tau_1, PN\tau_2)$ be a PN bitopological space.A PN $A$ in $(X, PN\tau_1, PN\tau_2)$ is called a pairwise PN first category set if $A = \bigcup_{i=1}^{\infty}(A_i)$, where $(\lambda_i)$'s are pairwise PN nowhere dense sets in $(X, PN\tau_1, PN\tau_2)$.Any other PN set in $(X, PN\tau_1, PN\tau_2)$ is said to be a pairwise PN second category set in $(X, PN\tau_1, PN\tau_2)$.

**Definition 3.8.** If $A$ is a pairwise PN first category set in a PN bitopological space $(X, PN\tau_1, PN\tau_2)$, then the PN $A^c$ is called a pairwise PN residual set in $(X, PN\tau_1, PN\tau_2)$.

**Definition 3.9.** A PN $A$ in a PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is called pairwise PN $F_\sigma$-set in $(X, PN\tau_1, PN\tau_2)$ if $A = \bigcup_{i=1}^{\infty}(A_i)$ where $(A_i)^c \in PN\tau_i$.

**Definition 3.10.** A PN $A$ in a PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is called pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$ if $A = \bigcap_{i=1}^{\infty}(A_i)$ where $A_i \in PN\tau_i$ .

**Definition 3.11.** A PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is called pairwise PN first category space if the PN set $1_X$ is a pairwise PN first category set in $(X, PN\tau_1, PN\tau_2)$.That is, $1_X = \bigcup_{i=1}^{\infty}(A_i)$, where $A_i$'s are pairwise PN nowhere dense sets in $(X, PN\tau_1, PN\tau_2)$.
Otherwise $(X, PN\tau_1, PN\tau_2)$ will be called a pairwise PN second category space.

**Definition 3.12.** A PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is called a pairwise PN Baire if $PNint_{PN\tau_i}(\bigcup_{k=1}^{\infty}(A_k))$
$= 0_X, (i = 1, 2)$, where $A_k$'s are pairwise PN nowhere dense sets in $(X, PN\tau_1, PN\tau_2)$.

**Definition 3.13.** A PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is called a pairwise PN P-space if every non-zero pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$ is a pairwise open set in $(X, PN\tau_1, PN\tau_2)$.That is, if $A = \bigcap_{k=1}^{\infty}(A_k)$,where $(A_k)'$s are pairwise PN open sets in $(X, PN\tau_1, PN\tau_2)$, then $A$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$.

**Proposition 3.14.** If $A$ is a pairwise PN $F_\sigma$-set in a pairwise PN P-space $(X, PN\tau_1, PN\tau_2)$,then $A$ is a pairwise PN closed set in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* Let $A$ be a pairwise PN $F_\sigma$-set in $(X, PN\tau_1, PN\tau_2)$.Then $A^c$ is a pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$.Since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space, the pairwise PN $G_\delta$-set $(A^c)$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$.
Hence $A$ is a pairwise PN closed sets in $(X, PN\tau_1, PN\tau_2)$. □

**Proposition 3.15.** If the PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space, then $PNcl_{PN\tau_i}(\bigcap_{k=1}^{\infty}(A_k)) = \bigcup_{k=1}^{\infty} PNcl_{PN\tau_i}(A_k), (i = 1, 2)$ where $A_k$'s are pairwise PN closed sets in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* Let the PN sets $(A_k)$'s ($k = 1$ to $\infty$) be pairwise PN closed sets in $(X, PN\tau_1, PN\tau_2)$. Then, $PNcl_{PN\tau_i}(A_k), (i = 1, 2)$ in $(X, PN\tau_1, PN\tau_2)$.Let $A = \bigcup_{k=1}^{\infty}(A_k)$.Then, $A$ is a pairwise PN $F_\sigma$-set in $(X, PN\tau_1, PN\tau_2)$.Since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space, by proposition 3.12, $A$ is a pairwise PN closed sets in $(X, PN\tau_1, PN\tau_2)$.
Then, $PNcl_{PN\tau_i}(A) = A(i = 1, 2)$.
Now $PNcl_{PN\tau_i}(\bigcup_{k=1}^{\infty}(A_k)) = \bigcup_{k=1}^{\infty}(A_k) = \bigcup_{k=1}^{\infty} PNcl_{PN_tau_i}(A_k)$
and hence $PNcl_{A_i}(\bigcup_{k=1}^{\infty}(A_k)) = \bigcup_{k=1}^{\infty} PNcl_{PN\tau_i}(A_k)$, where $(A_k)$'s are pairwise PN closed sets in $(X, PN\tau_1, PN\tau_2)$. □

**Proposition 3.16.** If the PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space, then $PNint_{PN\tau_i}(\bigcap_{k=1}^{\infty}(A_k)) = \bigcap_{k=1}^{\infty} PNint_{PN\tau_i}(A_k)$ where $(A_k)$'s are pairwise PN open sets in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* Let the PN sets $(A_k)$'s $(k = 1to\infty)$ be pairwise PN open sets in $(X, PN\tau_1, PN\tau_2)$.
Then,$(A_k)^c$'s $(k = 1to\infty)$ be pairwise PN closed sets in $(X, PN\tau_1, PN\tau_2)$.
Since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space, by proposition 3.13,
$PNcl_{PN\tau_i}(\bigcup_{k=1}^{\infty}(A_k)^c) = \bigcup_{k=1}^{\infty} PNcl_{PN\tau_i}(A_k)^c$ in $(X, PN\tau_1, PN\tau_2)$.
Then, by definition 3.2, $PNcl_{PN\tau_i}(\bigcap_{k=1}^{\infty}(A_k)^c) = (\bigcup_{k=1}^{\infty}(PNint_{PN\tau_i}(A_k)))^c$and
hence $(PNint_{PN\tau_i}(\bigcap_{k=1}^{\infty}(A_k))^c = (\bigcap_{k=1}^{\infty} PNint_{PN\tau_i}(A_k))^c$.
Therefore $PNint_{PN\tau_i}(\bigcap_{k=1}^{\infty}(A_k) = \bigcap_{k=1}^{\infty}(A_k))$ where $(A_k)$'s are pairwise PN open sets in $(X, PN\tau_1, PN\tau_2)$. □

**Example 3.17.** Let $X = \{a, b\}$.The PN sets $A_1$, $A_2$ and $A_3$ are defined on X as follows:

$A_1$={(x,(0.2,0.3),(0.4,0.4),(0.7,0.6))}
$A_2$={(x,(0.8,0.6),(0.3,0.5),(0.4,0.1))}
$A_3$={(x,(0.2,0.1),(0.6,0.5),(0.9,0.7))}
$A_4$={(x,(0.6,0.2),(0.3,0.4),(0.5,0.5))}

Then, $PN\tau_1$=$\{0_X, A_1, A_2, A_3, A_1 \cup A_2, A_1 \cup A_3, A_2 \cup A_3, A_1 \cap A_2, A_1 \cap A_3, A_2 \cap A_3, A_1 \cup (A_2 \cap A_3), A_3 \cap (A_1 \cup A_2), A_2 \cup (A_1 \cap A_3), A_1 \cup A_2 \cup A_3, 1_X\}$ and

$PN\tau_2$=$\{0_X, A_1, A_3, A_4, A_1 \cup A_3, A_1 \cup A_4, A_3 \cup A_4, A_1 \cap A_3, A_3 \cap A_4, A_1 \cup (A_3 \cap A_4), A_4 \cap (A_1 \cup A_3), A_3 \cup (A_1 \cap A_4), A_1 \cup A_3 \cup A_4, 1_X\}$are PN topology on X.Now the PN sets $A_1, A_3, A_1 \cup A_3, A_1 \cap A_2, A_1 \cap A_3, A_1 \cup (A_2 \cap A_3), A_1 \cap A_4, A_3 \cap A_4, A_1 \cup (A_3 \cap A_4), A_3 \cap (A_1 \cup A_4)$ are pairwise PN open sets in $(X, PN\tau_1, PN\tau_2)$.Now the PN sets

$\alpha = A_1 \cap (A_1 \cap A_2) \cap (A_3 \cap (A_1 \cup A_2)) \cap (A_1 \cup (A_3 \cap A_4))$ and $\gamma = (A_1 \cup A_3) \cap (A_1 \cup (A_3 \cap A_4)) \cap (A_3 \cap (A_1 \cup A_4))$ are pairwise PN $G_\delta$-sets in $(X, PN\tau_1, PN\tau_2)$ and $\alpha \in PN\tau_i (i = 1, 2)$ and $\gamma \in PN\tau_i (i = 1, 2)$ shows that the PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space.

**Definition 3.18.** A PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is called a pairwise PN submaximal space if each pairwise PN dense set in $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$.That is, if $A$ is a pairwise PN dense in a PN bitopological space $(X, PN\tau_1, PN\tau_2)$, then $A \in PN\tau_i (i = 1, 2)$.

**Proposition 3.19.** If $(A_i)'s(i = 1to\infty)$ be pairwise PN dense sets in $(X, PN\tau_1, PN\tau_2)$.
Since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN submaximal space and pairwise PN P-space $(X, PN\tau_1, PN\tau_2)$, then $\bigcap_{i=1}^{\infty}(A_i)$ is a pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* Let $(A_i)$'s $(i = 1to\infty)$ be pairwise PN dense sets in $(X, PN\tau_1, PN\tau_2)$.Since $(X, PN\tau_1, PN\tau_2)$ is a submaximal space, $(A_i)$'s are pairwise PN open sets in $(X, PN\tau_1, PN\tau_2)$.Then, $\bigcap_{i=1}^{\infty}(A_i)$ is a pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$.
Since the PN bitopologcal space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space, $\bigcap_{i=1}^{\infty}(A_i)$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$. □

**Proposition 3.20.** If each pairwise PN $G_\delta$-set is a pairwise PN dense set in a pairwise PN submaximal space $(X, PN\tau_1, PN\tau_2)$, then $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space.

*Proof.* Let $A$ be a pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$.By hypothesis, $A$ is a pairwise PN dense set in $(X, PN\tau_1, PN\tau_2)$.Since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN submaximal space,the pairwise PN dense set $A$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$.Hence the pairwise $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$.Thus the PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space. □

**Proposition 3.21.** If $PNint_{PN\tau_i} PNint_{PN\tau_j}(A) = 0_X, (i, j = 1, 2 and i \neq j)$ where $A$ is a pairwise PN $F_\sigma$-set in a pairwise PN submaximal space $(X, PN\tau_1, PN\tau_2)$,
then $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space.

*Proof.* Let $A$ be a pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$.Then, $A^c$ is a pairwise PN $F_\sigma$-set in $(X, PN\tau_1, PN\tau_2)$.
By hypothesis,
$PNint_{PN\tau_i} PNint_{PN\tau_j}(A^c) = 0_X, (i, j = 1, 2 and i \neq j)$ in $(X, PN\tau_1, PN\tau_2)$.
This implies that $(PNcl_{PN\tau_i} PNcl_{PN\tau_j}(A))^c = 0_X$ and thus
$PNcl_{PN\tau_i} PNcl_{PN\tau_j}(A) = 1_X$.
Hence $A$ is a pairwise PN dense set in $(X, PN\tau_1, PN\tau_2)$.Since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN submaximal space, the pairwise PN dense set $A$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$.Hence the pairwise PN $G_\delta$-set $A$ in $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$.Therefore $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space. □

**Proposition 3.22.** If each pairwise PN $F_\sigma$-set is a pairwise PN nowhere dense set in a pairwise PN submaximal space $(X, PN\tau_1, PN\tau_2)$, then $(X, PN\tau_1, PN\tau_2)$ is a PN P-space.

*Proof.* Let $A$ be a pairwise PN $F_\sigma$-set in a pairwise PN submaximal space $(X, PN\tau_1, PN\tau_2)$ such that $PNint_{PN\tau_i} PNcl_{PN\tau_j}(A) = 0_X, (i, j = 1, 2 and i \neq j)$.
But $PNint_{PN\tau_i}(A) \subseteq PNint_{PN\tau_i} PNcl_{PN\tau_j}(A)$, implies that
$PNint_{PN\tau_i}(A) \subseteq 0_X$.
That is, $PNint_{PN\tau_i}(A) = 0_X$.
Then, $PNint_{PN\tau_i} PNint_{PN\tau_j}(A) = PNint_{PN\tau_i}(A) = 0_X, (i, j = 1, 2 and i \neq j)$.
Thus, $PNint_{PN\tau_i} PNint_{PN\tau_j}(A) = 0_X$, for a pairwise PN $F_\sigma$-set $A$ in a pairwise PN submaximal space $(X, PN\tau_1, PN\tau_2)$.
Then, by proposition 3.18, the PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space. □

**Proposition 3.23.** If $PNcl_{PN\tau_i} PNint_{PN\tau_j}(A) = 1_X, (i, j = 1, 2 and i \neq j)$ for each pairwise $G_\sigma$-set $A$ in a pairwise PN submaximal space $(X, PN\tau_1, PN\tau_2)$,then $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space.

*Proof.* Let $A$ be a pairwise PN $F_\sigma$-set in the PN bitopological space $(X, PN\tau_1, PN\tau_2)$.
Then $A^c$ is a pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$.
By hypothesis,
$PNcl_{PN\tau_i} PNint_{PN\tau_j}(A^c) = 1_X, (i, j = 1, 2 and i \neq j)$ in $(X, PN\tau_1, PN\tau_2)$.
This implies that $(PNint_{PN\tau_i} PNcl_{PN\tau_j}(A))^c = 1_X$ in $(X, PN\tau_1, PN\tau_2)$ and
hence $PNint_{PN\tau_i} PNcl_{PN\tau_j}(A) = 0_X$ in $(X, PN\tau_1, PN\tau_2)$.
Then $A$ is a pairwise PN nowhere dense set in $(X, PN\tau_1, PN\tau_2)$.Thus, the pairwise PN $F_\sigma$-set $A$ is a pairwise PN nowhere dense set in a pairwise PN submaximal space

$(X, PN\tau_1, PN\tau_2)$.Hence, by proposition 3.19, the PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space. □

**Proposition 3.24.** If $A$ is a pairwise PN residual set in a pairwise PN submaximal space $(X, PN\tau_1, PN\tau_2)$, then $A$ is a pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* Let $A$ be a pairwise PN residual set in a pairwise PN submaximal space $(X, PN\tau_1, PN\tau_2)$.Then, $A^c$ is a pairwise PN first category set in $(X, PN\tau_1, PN\tau_2)$ and hence $A^c = \bigcup_{k=1}^{\infty}(A_k)$, where the PN $(A_k)$'s are pairwise PN nowhere dense sets in $(X, PN\tau_1, PN\tau_2)$.Since $(A_k)$'s are pairwise PN nowhere dense sets in $(X, PN\tau_1, PN\tau_2)$, $PNint_{PN\tau_i}PNcl_{PN\tau_j}(A_k) = 0_X, (i, j = 1, 2 and i \neq j)$.
But $PNint_{PN\tau_i}(A_k) \subseteq PNint_{PN\tau_i}PNcl_{PN\tau_j}(A_k)$,
implies that $PNint_{PN\tau_i}(A_K) \subseteq 0_X$.That is, $PNint_{PN\tau_i}(A_k) = 0_X$.

Thus, $PNint_{PN\tau_i}PNint_{PN\tau_j}(A_k) = 0_X$ in $(X, PN\tau_1, PN\tau_2)$.
Then, $PNcl_{PN\tau_i}PNcl_{PN\tau_j}((A_k)^c) = (PNint_{PN\tau_i}PNint_{PN\tau_j}(A_k))^c = 1_X$.
Hence $(A_k)^c$'s are pairwise PN dense sets in $(X, PN\tau_1, PN\tau_2)$.Since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN submaximal space, the pairwise PN dense sets $(A_k)^c$'s are pairwise PN open sets in $(X, PN\tau_1, PN\tau_2)$.
Then, $A_k)$'s are pairwise PN closed sets in $(X, PN\tau_1, PN\tau_2)$.Hence $A^c = \bigcup_{k=1}^{\infty}(A_k)$, where the PN $(A_k)$'s are pairwise PN closed sets in $(X, PN\tau_1, PN\tau_2)$,implies that $A^c$ is a pairwise PN $F_\sigma$-set in $(X, PN\tau_1, PN\tau_2)$.Therefore $A$ is a pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$. □

**Proposition 3.25.** If $A$ is a pairwise PN residual set in a pairwise PN submaximal and pairwise PN P-space $(X, PN\tau_1, PN\tau_2)$, then $A$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* Let $A$ be a pairwise PN residual set in a pairwise PN submaximal space $(X, PN\tau_1, PN\tau_2)$.Since the PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is a PN submaximal space, by proposition 3.21,
$A$ is a pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$.
Since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space, the pairwise PN $G_\delta$-set $A$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$.Therefore, the pairwise PN residual set $A$ in a pairwise PN submaximal and pairwise PN P-space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$. □

**Proposition 3.26.** If $A$ is a pairwise PN nowhere dense set in a pairwise submaximal space $(X, PN\tau_1, PN\tau_2)$, then $A$ is a pairwise PN closed set in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* Let $A$ be a pairwise PN nowhere dense set in the bitopological space $(X, PN\tau_1, PN\tau_2)$.
Then, $PNint_{PN\tau_i} PNcl_{PN\tau_j}(A) = 0_X, (i, j = 1, 2 and i \neq j)$.
But, $PNint_{PN\tau_i}(A) \subseteq PNint_{PN\tau_i} PNcl_{PN\tau_j}(A)$, implies that $PNint_{PN\tau_i}(A) \subseteq 0_X$.
That is, $PNint_{PN\tau_i}(A) = 0_X$.
Thus, $PNint_{PN\tau_i} PNint_{PN\tau_j}(A) = 0_X$,in $(X, PN\tau_1, PN\tau_2)$.
Then, $PNcl_{PN\tau_i} PNcl_{PN\tau_j}(A^c) = (PNint_{PN\tau_i} PNint_{PN\tau_j}(A))^c = 1_X$.
Hence $A^c$ is a pairwise PN dense set in $(X, PN\tau_1, PN\tau_2)$.
Since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN submaximal space, the pairwise PN dense set $A^c$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$.Thus, $A$ is a pairwise PN closed set in $(X, PN\tau_1, PN\tau_2)$. □

**Proposition 3.27.** If $(A_k)$'s $(k = 1 to \infty)$ are pairwise PN nowhere dense sets in a pairwise PN submaximal and pairwise PN P-space $(X, PN\tau_1, PN\tau_2)$ such that $\bigcup_{k=1}^{\infty}(A_k) \neq 1_X$, then $PNcl_{PN\tau_i}(\bigcup_{k=1}^{\infty}(A_k)) = \bigcup_{k=1}^{\infty}(A_k)$ in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* Let $(A_k)$'s $(k = 1 to \infty)$ be pairwise PN nowhere dense sets in $(X, PN\tau_1, PN\tau_2)$.
Since the PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN submaximal space, by proposition 3.23, $(A_k)$'s are pairwise PN closed sets in $(X, PN\tau_1, PN\tau_2)$ and hence $(A_k)^c$'s are pairwise PN open sets in $(X, PN\tau_1, PN\tau_2)$.
Let $A = \bigcap_{k=1}^{\infty}(A_k)^c$.Then, $A$ is a non-zero pairwise PN $G_\delta$-set in $(X, PN\tau_1, PN\tau_2)$.(For, if $A = 0_X$, then $\bigcap_{k=1}^{\infty}(A_k)^c = 0_X$, will imply that $\bigcup_{k=1}^{\infty}(A_k) = 1_X$, which is a contradiction to the hypothesis).Since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN P-space, the pairwise PN $G_\delta$-set $A$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$ and hence, $PNint_{PN\tau_i}(A) = A$.
This implies that $PNint_{PN\tau_i}(\bigcap_{k=1}^{\infty}(A_k)^c) = \bigcap_{k=1}^{\infty}(A_k)^c$.
Then, $(PNcl_{PN\tau_i}(\bigcup_{k=1}^{\infty}(A_k))^c = (\bigcup_{k=1}^{\infty}(A_k))^c$ in $(X, PN\tau_1, PN\tau_2)$.
Hence $PNcl_{PN\tau_i}(\bigcup_{k=1}^{\infty}(A_k) = \bigcup_{k=1}^{\infty}(A_k)$ in $(X, PN\tau_1, PN\tau_2)$. □

**Proposition 3.28.** If $A$ is a pairwise PN first category set in a pairwise PN submaximal and pairwise PN P-space $(X, PN\tau_1, PN\tau_2)$, then $A$ is a pairwise PN closed set in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* Let $A(\neq 1_X)$ be a pairwise PN first category set in $(X, PN\tau_1, PN\tau_2)$.Then, $A = \bigcup_{k=1}^{\infty}(A_k)$, where $(A_k)$'s are pairwise PN nowhere dense sets in $(X, PN\tau_1, PN\tau_2)$.
Since the PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN submaximal and pairwise PN P-space, by proposition 3.24, we have
$PNcl_{PN\tau_i}(\bigcup_{k=1}^{\infty}(A_k)) = \bigcup_{k=1}^{\infty}(A_k)$ in $(X, PN\tau_1, PN\tau_2)$ and hence
$PNcl_{PN\tau_i}(A) = A$.Thus the pairwise PN first category set $A$ is a pairwise PN closed set in $(X, PN\tau_1, PN\tau_2)$. □

**Theorem 3.29.** Let $(X, PN\tau_1, PN\tau_2)$ be a PN bitopological space .Then the following are equivalent:

(i) $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN Baire space
(ii) $PNint_{PN\tau_i}(A) = 0_X, (i = 1, 2)$,for each pairwise PN first category set $A$ in $(X, PN\tau_1, PN\tau_2)$.
(iii) $PNcl_{PN\tau_i}(A) = 1_X, (i = 1, 2)$,for each pairwise PN residual set $A$ in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* (i) ⇒ (ii) Let $A$ be a pairwise PN first category set in $(X, PN\tau_1, PN\tau_2)$.
Then $A = \bigcup_{k=1}^{\infty}(A_k)$, where $A_i$'s are pairwise PN nowhere dense sets in $(X, PN\tau_1, PN\tau_2)$.
Now $PNint_{PN\tau_i}(A) = PNint_{PN\tau_i}(\bigcup_{k=1}^{\infty}(A_k)) = 0_X, (i = 1, 2)$, (since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN Baire space).
Therefore $PNint_{PN\tau_i}(A) = 0_X$, where $A_k$'s are pairwise PN nowhere dense sets in $(X, PN\tau_1, PN\tau_2)$.

(ii) ⇒ (iii).Let $B$ be a pairwise PN residual set in $(X, PN\tau_1, PN\tau_2)$.
Then $B^c$ is a pairwise PN first category set in $(X, PN\tau_1, PN\tau_2)$.
By hypothesis, $PNint_{PN\tau_i}(B^c) = 0_X, (i = 1, 2)$,
which implies that $(PNcl_{PN\tau_i}(B))^c = 0_X$.
Hence $PNcl_{PN\tau_i}(B) = 1_X, (i = 1, 2)$.

(iii) ⇒ (i).Let $A$ be a pairwise PN first category set in $(X, PN\tau_1, PN\tau_2)$.Then $A = \bigcup_{k=1}^{\infty}(A_k)$, where $A_k$'s are pairwise PN nowhere dense sets in $(X, PN\tau_1, PN\tau_2)$.
Now n$A$ is a pairwise PN first category set in $(X, PN\tau_1, PN\tau_2)$ implies that $A^c$ is a pairwise PN residual set in $(X, PN\tau_1, PN\tau_2)$.
By hypothesis, we have $PNcl_{PN\tau_i}(A^c) = 1_X, (i = 1, 2)$,
which implies that $(PNint_{PN\tau_i}(A))^c = 0_X, (i = 1, 2)$.
Then $PNint_{PN\tau_i}(A) = 1_X$.That is, $PNint_{PN\tau_i}(\bigcup_{k=1}^{\infty}(A_k)) = 0_X$, where $A_k$'s are pairwise PN nowhere dense sets in $(X, PN\tau_1, PN\tau_2)$.
Hence the PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN Baire space. □

**Proposition 3.30.** If the PN bitopological space $PNint_{PN\tau_i}(A) = 0_X, (i = 1, 2)$,for each pairwise PN first category set $A$ in $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN Baire space and pairwise PN submaximal space, then each pairwise PN first category set is a pairwise PN nowhere dense set in $PNint_{PN\tau_i}(A) = 0_X, (i = 1, 2)$,for each pairwise PN first category set $A$ in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* Let $A$ be a pairwise PN first category set in $(X, PN\tau_1, PN\tau_2)$.Since the PN bitopological space $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN submaximal and pairwise PN P-space, by proposition 3.25 , $A$ is a pairwise PN closed set in $(X, PN\tau_1, PN\tau_2)$.Then $PNcl_{PN\tau_i}(A) = A, (i = 1, 2)$ in $(X, PN\tau_1, PN\tau_2)$.
Also since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN Baire space, by Theorem 3.26,
$PNint_{PN\tau_i}(A) = 0_X, (i = 1, 2)$, for the pairwise PN first category set $A$ in $(X, PN\tau_1, PN\tau_2)$.
Now $PNint_{PN\tau_i} PNcl_{PN\tau_j}(A) = PNint_{PN\tau_i}(A) = 0_X, )i = 1, 2)$.
Therefore, the pairwise PN first category set $A$ is a pairwise PN nowhere dense set in $(X, PN\tau_1, PN\tau_2)$. □

**Proposition 3.31.** If $A$ is a pairwise PN residual set in a pairwise PN submaximal, pairwise PN Baire and pairwise PN P-space $(X, PN\tau_1, PN\tau_2)$,
then $A$ is a pairwise PN open and pairwise PN dense set in $(X, PN\tau_1, PN\tau_2)$.

*Proof.* Let $A$ be a pairwise PN residual set in $(X, PN\tau_1, PN\tau_2)$.
Since $(X, PN\tau_1, PN\tau_2)$ is a pairewise PN submaximal and pairwise PN P-space, by proposition 3.25, $A$ is a pairwise PN open set in $(X, PN\tau_1, PN\tau_2)$.Also since $(X, PN\tau_1, PN\tau_2)$ is a pairwise PN Baire space, by Theorem 3.26,
$PNcl_{PN\tau_i}(A) = 1_X, (i = 1, 2)$, for the pairwise PN residual set $A$ in $(X, PN\tau_1, PN\tau_2)$ and hence $A$ is a pairwise PN open and a pairewise PN dense set in $(X, PN\tau_1, PN\tau_2)$. □

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# Pythagorean Neutrosophic Ideals in Semigroups

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**Abstract**. In this paper, we introduce the notion of Pythagorean neutrosophic ideals, Pythagorean neutrosophic bi-ideal, Pythagorean neutrosophic interior ideal, Pythagorean neutrosophic (1,2) ideal of semigroups and some of them interesting properties.

**Keywords:** Pythagorean fuzzy set; Neutrosophic set; fuzzy ideals; semigroup.

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## 1. **Introduction**

After the introduction of the fuzzy set by Zadeh [11], several researchers conducted experiments on the generalizations of the notion of a fuzzy set. The concept of the intuitionistic fuzzy set was introduced by Atanassov [1,2] as a generalization of the fuzzy set. Jun et al. [4,5] considered the fuzzification of interior ideals in semigroups and the notion of an intuitionistic fuzzy interior ideal of a semigroup S, and its properties were investigated. Kuroki [8] discussed some properties of fuzzy ideals and fuzzy bi-ideals in the semigroup. Jun et al. [6] considered the fuzzification of (1,2)-ideals in semigroups and investigated its properties. Yager [9, 10] introduced the Pythagorean fuzzy set as a generalization of the fuzzy set. After its existence, several researchers also studied the properties of fuzzy ideals of the semigroup. Yager and Abbasov [37] initiated the notion of Pythagorean fuzzy set and this concept could be considered as a successful generalization of intuitionistic fuzzy sets. The main difference between intuitionistic fuzzy sets and Pythagorean fuzzy sets is that, in the latter case, the sum of membership and non-membership grades is greater than 1, however, the sum of their squares belongs to the unit interval [0,1]. Analogously, in this novel pattern, the associated uncertainty of membership grade and non-membership grade can be explained in a valuable method that than of intuitionistic fuzzy set. Gun et al. [7] introduced the new concept of spherical fuzzy set and discuss the new operations. Smarandache [13] introduced the new concept of neutrosophic set. Khan et.al [12] introduced the Neutrosophic N-Structures and their application in semigroups. The neutrosophic theories have received greater attention in recent years [14]-[32]. Abdel-Basset et al. [33] proposed a new hybrid multi-criteria decision-making (MCDM) using Analytical Hierarchy Process(AHP) and Preference Ranking Organization Method for Enrichment Evaluations (PROMETHEE)-II approach for optimal offshore wind power station location selection. Abdel-Basset et al. [34] Provided a neutrosophic PROMETHEE technique for MCDM problems to describe fuzzy information efficiently. Abdel-Basset et al. [35] discussed how smart internet of things technology can assist medical staff in monitoring the spread of COVID-19. Abdel-Basset et al. [36] studied a comprehensive evaluation of the sustainability of hydrogen production options through the use of a MCDM model.

In this paper, we discuss the properties of Pythagorean neutrosophic ideals in semigroups.

## 2. Preliminaries

**Definition 2.1.** [3] Let $S$ be a semigroup. $M$ and $N$ be subsets of $S$, the product of $M$ and $N$ is defined as $MN = \{mn \in S \mid m \in M \ and \ n \in N\}$ A non- empty subset $M$ of $S$ is called a sub-semigroup of $S$ if $MM \subseteq M$. A non-empty subset $M$ of $S$ is called a left (resp. right) ideal of $S$ if $SM \subseteq M$ (resp.$MS \subseteq M$). A is called a two sided ideal of $S$ if it is both a left ideal and right ideal of $S$. A sub- semigroup $M$ of $S$ is called a bi-ideal of $S$ if $MSM \subseteq M$. A sub-semigroup $M$ of $S$ is called a (1,2) ideal of $S$ if $MSM^2 \subseteq M$. A semigroup $S$ is said to be (2,2)- regular if $m \in m^2Sm^2$ for any $m \in S$. A semigroup $S$ is called regular if for each element $m \in S$ there exists $x \in S$ such that $m = mxm$. A semigroup $S$ is said to be completely regular if, for any $m \in S$, there exists $x \in S$ such that $m = mxm$ and $mx = xm$. For a semigroup $S$, is completely regular if and only if(iff) $S$ is a union of groups iff $S$ is (2,2)-regular. By a fuzzy set $\mu$ in a non-empty set $S$ we mean a function $\mu : S \to [0, 1]$, and the complement of $\mu$, denoted by $\overline{\mu}$, is the fuzzy set in $S$ given by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in S$.

**Definition 2.2.** [9] Let $X$ be a universe of discourse, A **Pythagorean fuzzy set** (PFS) $P = \{z, \vartheta_p(x), \omega_p(x)/z \in X\}$ where $\vartheta : X \to [0, 1]$ and $\omega : X \to [0, 1]$ represent the degree of membership and non-membership of the object $z \in X$ to the set $P$ subset to the condition $0 \leq (\vartheta_p(z))^2 + (\omega_p(z))^2 \leq 1$ for all $z \in X$. For the sake of simplicity a PFS is denoted as $P = (\vartheta_p(z), \omega_p(z))$.

**Definition 2.3.** [13] Let $X$ be a universe of discourse, A **Neutrosophic set** (NS) $N = \{z, \vartheta_N(z), \omega_N(z), \psi_N(z)/z \in X\}$ where $\vartheta : X \to [0, 1]$, $\omega : X \to [0, 1]$ and $\psi : X \to [0, 1]$ represent the degree of truth membership, indeterminacy-membership and false-membership of the object $z \in X$ to the set $N$ subset to the condition $0 \leq (\vartheta_N(z)) + (\omega_N(z)) + (\psi_N(z)) \leq 3$ for all $z \in X$. For the sake of simplicity a NS is denoted as $N = (\vartheta_N(z), \omega_N(z), \psi_N(z))$.

## 3. **Pythagorean neutrosophic set**

**Definition 3.1.** Let $X$ be a universe of discourse, A **Pythagorean neutrosophic set** (PNS) $P_N = \{z, \mu_p(z), \zeta_p(z), \psi_p(z)/z \in X\}$ where $\mu : X \to [0,1]$, $\zeta : X \to [0,1]$ and $\psi : X \to [0,1]$ represent the degree of membership, non-membership and inderminancy of the object $z \in X$ to the set $P_N$ subset to the condition $0 \leq (\mu_p(z))^2 + (\zeta_p(z))^2 + (\psi_p(z))^2 \leq 2$ for all $z \in X$. For the sake of simplicity a PNS is denoted as $P_N = (\mu_p(z), \zeta_p(z), \psi_p(z))$.

**Definition 3.2.** Let X be a nonempty set and I the unit interval [0, 1]. A Pythagorean neutrosophic set with neutrosophic components [PNS] $P_{N_1}$ and $P_{N_2}$ of the form $P_{N_1} = (z, \mu_{p_1}(z), \zeta_{p_1}(z), \psi_{p_1}(z)/z \in X)$ *and* $P_{N_2} = (z, \mu_{p_2}(z), \zeta_{p_2}(z), \psi_{p_2}(z)/z \in X)$. Then
1)$P_N^c = (z, \psi_{p_1}(z), \zeta_{p_1}(z), \mu_{p_1}(z)/z \in X)$
2)$P_{N_1} \cup P_{N_2} = \{z, max(\mu_{P_1}(z), \mu_{P_2}(z)), max(\zeta_{P_1}(z), \zeta_{P_2}(z)), min(\psi_{P_1}(z), \psi_{P_2}(z))/z \in X\}$
3)$P_{N_1} \cap P_{N_2} = \{z, min(\mu_{P_1}(z), \mu_{P_2}(z)), min(\zeta_{P_1}(z), \zeta_{P_2}(z)), max(\psi_{P_1}(z), \psi_{P_2}(z))/z \in X\}$

## 4. **Pythagorean neutrosophic ideals in semigroups**

In this section, let S denote a semigroup unless otherwise specified. We discuss the details of Pythagorean neutrosophic ideals in semigroups.

**Definition 4.1.** A Pythagorean neutrosophic (PNS) $P_N = (\mu_p, \zeta_p, \psi_p)$ in $S$ is called an Pythagorean neutrosophic sub-semigroup of $S$, if
(i) $\mu_p(x_1x_2) \leq max\{\mu_p(x_1), \mu_p(x_2)\}$
(ii) $\zeta_p(x_1x_2) \geq max\{\zeta_p(x_1), \zeta_p(x_2)\}$
(iii) $\psi_p(x_1x_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$ for all $x_1, x_2 \in S$.

**Definition 4.2.** A PNS $P = (\mu_p, \zeta_p, \psi_p)$ in $S$ is called an Pythagorean neutrosophic left ideal of $S$, if
(i) $\mu_p(x_1x_2) \leq \mu_p(x_2)$
(ii) $\zeta_p(x_1x_2) \geq \zeta_p(x_2)$
(iii) $\psi_p(x_1x_2) \leq \psi_p(x_2)$ for all $x_1, x_2 \in S$.
A Pythagorean neutrosophic right ideal of $S$ is defined in an analogous way.An PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ in $S$ is called an Pythagorean neutrosophic ideal of $S$, if it is both a Pythagorean neutrosophic left and Pythagorean neutrosophic right ideal of $S$.It is clear that any Pythagorean neutrosophic left(resp. right) ideal of $S$ is a Pythagorean neutrosophic sub-semigroup of $S$.

**Definition 4.3.** A Pythagorean neutrosophic sub-semigroup $P_N = (\mu_p, \zeta_p, \psi_p)$ of $S$ is called an Pythagorean neutrosophic bi-ideal(PNBI) of $S$.
(i) $\mu_p(x_1ux_2) \leq max\{\mu_p(x_1), \mu_p(x_2)\}$

(ii) $\zeta_p(x_1ux_2) \geq max\{\zeta_p(x_1), \zeta_p(x_2)\}$

(ii) $\psi_p(x_1ux_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$ for all $u, x_1, x_2 \in S$.

**Theorem 4.4.** *If $\{P_i\}_{i\in I}$ is a family of PNBI of $S$, then $\cap P_i$ is an PNBI of $S$. Where $\cap P_i = (\vee\mu_{p_i}, \vee\zeta_{p_i}, \vee\psi_{p_i})$ and $\vee\mu_{p_i} = sup\{\mu_{p_i}(x_1)|i \in I, x_1 \in S\}$, $\vee\zeta_{p_i} = sup\{\zeta_{p_i}(x_1)|i \in I, x_1 \in S\}$, $\vee\psi_{p_i} = sup\{\psi_{p_i}(x_1)|i \in I, x_1 \in S\}$.*

*Proof.* Let $x_1, x_2 \in S$. Then we have

$$\begin{aligned}\vee\mu_{p_i}(x_1x_2) &\leq \vee\{max\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\ &= max\{max\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\ &= max\{max\{\mu_{p_i}(x_1)\}, max\{\mu_{p_i}(x_2)\}\}\\ &= max\{\vee\mu_{p_i}(x_1), \vee\mu_{p_i}(x_2)\}\end{aligned}$$

$$\begin{aligned}\vee\zeta_{p_i}(x_1x_2) &\geq \vee\{max\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\ &= max\{max\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\ &= max\{max\{\zeta_{p_i}(x_1)\}, max\{\zeta_{p_i}(x_2)\}\}\\ &= max\{\wedge\zeta_{p_i}(x_1), \wedge\zeta_{p_i}(x_2)\}\end{aligned}$$

$$\begin{aligned}\vee\psi_{p_i}(x_1x_2) &\leq \vee\{max\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\ &= max\{max\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\ &= max\{max\{\psi_{p_i}(x_1)\}, max\{\psi_{p_i}(x_2)\}\}\\ &= max\{\vee\psi_{p_i}(x_1), \vee\psi_{p_i}(x_2)\}.\end{aligned}$$

Hence $\cap P_i$ is an Pythagorean neutrosophic sub-semigroup of $S$.

Next for $u, x_1, x_2 \in S$, we obtain

$$\begin{aligned}\vee\mu_{p_i}(x_1ux_2) &\leq \vee\{min\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\ &= max\{max\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\ &= max\{max\{\mu_{p_i}(x_1)\}, max\{\mu_{p_i}(x_2)\}\}\\ &= max\{\vee\mu_{p_i}(x_1), \vee\mu_{p_i}(x_2)\}\end{aligned}$$

$$\begin{aligned}\vee\zeta_{p_i}(x_1ux_2) &\geq \vee\{min\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\ &= max\{max\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\ &= max\{max\{\zeta_{p_i}(x_1)\}, max\{\zeta_{p_i}(x_2)\}\}\\ &= max\{\vee\zeta_{p_i}(x_1), \vee\zeta_{p_i}(x_2)\}\end{aligned}$$

$$\begin{aligned}\vee\psi_{p_i}(x_1ux_2) &\leq \vee\{max\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\ &= max\{max\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\ &= max\{max\{\psi_{p_i}(x_1)\}, max\{\psi_{p_i}(x_2)\}\}\\ &= max\{\vee\psi_{p_i}(x_1), \vee\psi_{p_i}(x_2)\}.\end{aligned}$$

Hence $\cap P_i$ is an PNBI of $S$.

This completes the proof. □

**Theorem 4.5.** *Every Pythagorean neutrosophic left(right) ideal of $S$ is an Pythagorean neutrosophic bi-ideal of $S$.*

*Proof.* Let $P_N = (\mu_p, \zeta_p, \psi_p)$ is a Pythagorean neutrosophic left ideal of $S$ and $u, x_1, x_2 \in S$. Then

$$\mu_p(x_1ux_2) = \mu_p(x_1ux_2)$$
$$\leq \mu_p(x_2)$$
$$\mu_p(x_1ux_2) \leq max\{\mu_p(x_1, \mu_p(x_2))\}$$
$$\zeta_p(x_1ux_2) = \zeta_p(x_1ux_2)$$
$$\geq \zeta_p(x_2)$$
$$\zeta_p(x_1ux_2) \geq max\{\zeta_p(x_1, \zeta_p(x_2))\}$$
$$\psi_p(x_1ux_2) = \psi_p(x_1ux_2)$$
$$\leq \psi_p(x_2)$$
$$\psi_p(x_1ux_2) \leq max\{\psi_p(x_1, \psi_p(x_2))\}$$

Thus $P_N = (\mu_p, \zeta_p, \psi_p)$ is PNBI of $S$.

The right case is provided in an analogous way. □

**Theorem 4.6.** *Every Pythagorean neutrosophic bi-ideal of a group $S$ is constant.*

*Proof.* Let $P_N = (\mu_p, \zeta_p, \psi_p)$ be an PNBI of a group $S$ and let $x_1$ be any element of $S$. Then

$$\begin{aligned}\mu_p(x_1) &= \mu_p(ex_1e)\\ &\leq max\{\mu_p(e), \mu_p(e)\}\\ &= \mu_p(e)\\ &= \mu_p(ee)\\ &= \mu_p(x_1x_1^{-1})(x_1^{-1}x_1)\\ &= \mu_p(x_1(x_1^{-1}x_1^{-1})x_1)\\ &\leq max\{\mu_p(x_1, \mu_p(x_1))\}\\ &= \mu_p(x_1)\end{aligned}$$

$$\begin{aligned}\zeta_p(x_1) &= \zeta_p(ex_1e)\\ &\geq max\{\zeta_p(e), \zeta_p(e)\}\\ &= \zeta_p(e)\\ &= \zeta_p(ee)\\ &= \zeta_p(x_1x_1^{-1})(x_1^{-1}x_1)\\ &= \zeta_p(x_1(x_1^{-1}x_1^{-1})x_1)\\ &\geq max\{\zeta_p(x_1, \zeta_p(x_1))\}\\ &= \zeta_p(x_1)\end{aligned}$$

and

$$\begin{aligned}\psi_p(x_1) &= \psi_p(ex_1e)\\ &\leq max\{\psi_p(e), \psi_p(e)\}\\ &= \psi_p(e)\\ &= \psi_p(ee)\\ &= \psi_p(x_1x_1^{-1})(x_1^{-1}x_1)\\ &= \psi_p(x_1(x_1^{-1}x_1^{-1})x_1)\\ &\leq max\{\psi_p(x_1, \psi_p(x_1))\}\\ &= \psi_p(x_1).\end{aligned}$$

Where $e$ is the identity of $S$. It follows that $\mu_p(x_1) = \mu_p(e)$, $\zeta_p(x_1) = \zeta_p(e)$ and $\psi_p(x_1) = \psi_p(e)$ which means that $P_N = (\mu_p, \zeta_p, \psi_p)$ is constant. □

**Theorem 4.7.** *If an PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ in $S$ is an PNBI of $S$, then so is $\Box P_N = (\mu_p, \zeta_p, \overline{\psi}_p)$.*

*Proof.* It is sufficient to show that $\overline{\psi}_p$ satisfies the conditions in Definition 3.1 and Definition 3.4. For any $u, x_1, x_2 \in S$, we have

$$\begin{aligned}\overline{\psi}_p(x_1x_2) &= 1 - \psi_p(x_1x_2)\\ &\leq 1 - min\{\psi_p(x_1), \psi_p(x_2)\}\\ &= max\{1 - \psi_p(x_1), 1 - \psi_p(x_2)\}\\ &= max\{\overline{\psi}_p(x_1), \overline{\psi}_p(x_2)\}\end{aligned}$$

and

$$\begin{aligned}\overline{\psi}_p(x_1ux_2) &= 1 - \psi_p(x_1ux_2)\\ &\leq 1 - min\{\psi_p(x_1), \psi_p(x_2)\}\\ &= max\{1 - \psi_p(x_1), 1 - \psi_p(x_2)\}\\ &= max\{\overline{\psi}_p(x_1), \overline{\psi}_p(x_2)\}.\end{aligned}$$

Therefore $\Box P_N$ is an PNBI of $S$. □

**Definition 4.8.** A Pythagorean neutrosophic sub-semigroup $P_N = (\mu_p, \zeta_p, \psi_p)$ of $S$ is called a Pythagorean neutrosophic (1,2) ideal of $S$. If

(i) $\mu_p(x_1u(x_2x_3)) \leq max\{\mu_p(x_1), \mu_p(x_2), \mu_p(x_3)\}$

(ii) $\zeta_p(x_1u(x_2x_3)) \geq max\{\zeta_p(x_1), \zeta_p(x_2), \zeta_p(x_3)\}$

(iii) $\psi_p(x_1u(x_2x_3)) \leq max\{\psi_p(x_1), \psi_p(x_2), \psi_p(x_3)\}$ $u, x_1, x_2, x_3 \in S$.

**Theorem 4.9.** *Every PNBI is a Pythagorean neutrosophic (1,2) ideal of $S$.*

*Proof.* Let PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ be an PNBI of $S$ and let $u, x_1, x_2, x_3 \in S$.
Then

$$\mu_p(x_1u(x_2x_3)) = \mu_p((x_1ux_2)x_3)$$

$$
\begin{aligned}
&\leq max\left\{\mu_p(x_1ux_2),\mu_p(x_3)\right\}\\
&\leq max\left\{max\left\{\mu_p(x_1),\mu_p(x_2)\right\},\mu_p(x_3)\right\}\\
&= max\left\{\mu_p(x_1),\mu_p(x_2),\mu_p(x_3)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\zeta_p(x_1u(x_2x_3)) &= \zeta_p((x_1ux_2)x_3)\\
&\geq max\left\{\zeta_p(x_1ux_2),\zeta_p(x_3)\right\}\\
&\geq max\left\{max\left\{\zeta_p(x_1),\zeta_p(x_2)\right\},\zeta_p(x_3)\right\}\\
&= max\left\{\zeta_p(x_1),\zeta_p(x_2),\zeta_p(x_3)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_p(x_1u(x_2x_3)) &= \psi_p((x_1ux_2)x_3)\\
&\leq max\left\{\psi_p(x_1ux_2),\psi_p(x_3)\right\}\\
&\leq max\left\{max\left\{\psi_p(x_1),\psi_p(x_2)\right\},\psi_p(x_3)\right\}\\
&= max\left\{\psi_p(x_1),\psi_p(x_2),\psi_p(x_3)\right\}.
\end{aligned}
$$

Hence $P_N = (\mu_p, \zeta_p, \psi_p)$ is a Pythagorean neutrosophic (1,2) ideal of $S$. □

To consider the converse of theorem next theorem, we need to strengthen the condition of a semigroup $S$.

**Theorem 4.10.** *If $S$ is a regular semigroup, then every Pythagorean neutrosophic (1,2) ideal of $S$ is an PNBI of $S$.*

*Proof.* Assume that a semigroup $S$ is regular and let $P_N = (\mu_p, \zeta_p, \psi_p)$ be an Pythagorean neutrosophic (1,2) ideal of $S$. Let $u, x_1, x_2, x_3 \in S$. Since $S$ is regular, we have $x_1u \in (x_1Sx_1)S \subseteq x_1Sx_1$, which implies that $x_1u = x_1Sx_1$ for some $s \in S$.
Thus

$$
\begin{aligned}
\mu_p(x_1ux_2) &= \mu_p((x_1sx_1)x_2)\\
&= \mu_p(x_1s(x_1x_2))\\
&\leq max\left\{\mu_p(x_1),\mu_p(x_1),\mu_p(x_2)\right\}\\
&= max\left\{\mu_p(x_1),\mu_p(x_2)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\zeta_p(x_1ux_2) &= \zeta_p((x_1sx_1)x_2)\\
&= \zeta_p(x_1s(x_1x_2))\\
&\geq max\left\{\zeta_p(x_1),\zeta_p(x_1),\zeta_p(x_2)\right\}\\
&= max\left\{\zeta_p(x_1),\zeta_p(x_2)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_p(x_1ux_2) &= \psi_p((x_1sx_1)x_2)\\
&= \psi_p(x_1s(x_1x_2))\\
&\leq max\left\{\psi_p(x_1),\psi_p(x_1),\psi_p(x_2)\right\}
\end{aligned}
$$

$= max\{\psi_p(x_1), \psi_p(x_2)\}.$

Therefore $P_N = (\zeta_p, \psi_p)$ is PNBI of $S$. □

**Theorem 4.11.** *A PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ is an PNBI of $S$ if and only if $\mu_p$, $\zeta_p$ and $\overline{\psi_p}$ are FBI of $S$.*

*Proof.* Let $P_N = (\mu_p, \zeta_p, \psi_p)$ be an PNBI of $S$. Then clearly $\mu_p$ is a FBI of $S$. Let $u, x_1, x_2 \in S$. Then

$$\begin{aligned}\overline{\psi_p}(x_1x_2) &= 1 - \psi_p(x_1x_2)\\ &\geq 1 - max\{\psi_p(x_1), \psi_p(x_2)\}\\ &= min\{(1-\psi_p(x_1)), (1-\psi_p(x_2))\}\\ &= min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}\end{aligned}$$

$$\begin{aligned}\overline{\psi_p}(x_1ux_2) &= 1 - \psi_p(x_1ux_2)\\ &\geq 1 - max\{\psi_p(x_1), \psi_p(x_2)\}\\ &= min\{(1-\psi_p(x_1)), (1-\psi_p(x_2))\}\\ &= min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}.\end{aligned}$$

Hence $\overline{\psi_p}$ is a fuzzy bi-ideal of $S$.

Conversely, suppose that $\zeta_p$ and $\overline{\psi_p}$ are FBI of $S$. Let $u, x_1, x_2 \in S$.

Then

$$\begin{aligned}1 - \psi_p(x_1x_2) &= \overline{\psi_p}(x_1x_2)\\ &\leq min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}\\ &= min\{(1-\psi_p(x_1)), (1-\psi_p(x_2))\}\\ &= max\{\psi_p(x_1), \psi_p(x_2)\}\end{aligned}$$

$$\begin{aligned}1 - \psi_p(x_1ux_2) &= \overline{\psi_p}(x_1ux_2)\\ &\geq min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}\\ &= 1 - max\{\psi_p(x_1), \psi_p(x_2)\}.\end{aligned}$$

Which implies that $\psi_p(x_1x_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$ and $\psi_p(x_1ux_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$

This completes the proof. □

**Definition 4.12.** A PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ in $S$ is called an Pythagorean neutrosophic interior ideal(PNII) of $S$ if it satisfies

(i) $\mu_p(x_1ux_2) \leq \mu_p(u)$

(ii) $\zeta_p(x_1ux_2) \geq \zeta_p(u)$

(iii) $\psi_p(x_1ux_2) \leq \psi_p(u)$ $u, x_1, x_2 \in S$.

**Theorem 4.13.** *If $\{P_i\}_{i\in I}$ is a family of PNII of $S$, then $\cap P_i$ is a PNII of $S$. Where $\cap P_i = (\vee\mu_{p_i}, \vee\zeta_{p_i}, \vee\psi_{p_i})$ and $\vee\mu_{p_i}(x_1) = sup\{\mu_{p_i}(x_1) | i \in I, x_1 \in S\}$, $\vee\zeta_{p_i}(x_1) = sup\{\zeta_{p_i}(x_1) | i \in I, x_1 \in S\}$, $\vee\psi_{p_i}(x_1) = sup\{\psi_{p_i}(x_1) | i \in I, x_1 \in S\}$.*

*Proof.* Let $u, x_1, x_2 \in S$.

Then

$\vee\mu_{p_i}(x_1x_2) \leq max\{max\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}$
$= (\vee\mu_{p_i}(x_1)) \vee (\vee\mu_{p_i}(x_2))$

$\vee\zeta_{p_i}(x_1x_2) \geq max\{max\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}$
$= (\vee\zeta_{p_i}(x_1)) \vee (\vee\zeta_{p_i}(x_2))$

and

$\vee\psi_{p_i}(x_1x_2) \leq max\{max\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}$
$= (\vee\psi_{p_i}(x_1)) \vee (\vee\psi_{p_i}(x_2))$

$\vee\mu_{p_i}(x_1ux_2) \leq \vee\mu_{p_i}(u)$

$\vee\zeta_{p_i}(x_1ux_2) \geq \vee\zeta_{p_i}(u)$

and

$\vee\psi_{p_i}(x_1ux_2) \leq \vee\psi_{p_i}(u)$.

Hence $\cap P_i$ is an PNII of $S$. □

**Definition 4.14.** Let $P_N = (\mu_p, \zeta_p, \psi_p)$ is a PNS of $S$ and let $\alpha \in [0, 1]$ then the sets. $\mu_{p,\alpha} = \{x_1 \in S : \mu_p(x_1)\alpha\}$, $\zeta_{p,\alpha} = \{x_1 \in S : \zeta_p(x_1)\alpha\}$ and $\psi_{p,\alpha} = \{x_1 \in S : \psi_p(x_1)\alpha\}$ are called a $\mu_p$-level $\alpha$-cut, $\zeta_p$-level $\alpha$-cut and $\psi_p$-level $\alpha$-cut of $K$ respectively.

**Theorem 4.15.** *If an PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ in $S$ is an PNII of $S$, then the $\mu$-level $\alpha$-cut $\mu_{p,\alpha}$, $\zeta$-level $\alpha$-cut $\zeta_{p,\alpha}$and $\psi$-level $\alpha$-cut $\psi_{p,\alpha}$ of $P_N$ are interior ideal of $S$, for every $\alpha \in Im(\mu_p) \cap Im(\zeta_p) \cap Im(\psi_p) \subseteq [0, 1]$.*

*Proof.* Let $\alpha \in Im(\mu_p) \cap Im(\zeta_p) \cap Im(\psi_p) \subseteq [0, 1]$.

let $x_1, x_2 \in \mu_{p,\alpha}$ then $\mu_p(x_1) \leq \alpha$ and $\mu_p(x_2) \leq \alpha$. It follows from that

$\mu_p(x_1x_2) \leq \mu_p(x_1) \vee \mu_p(x_2) \leq \alpha$. So that $x_1, x_2 \in \mu_{p,\alpha}$.

If $x_1, x_2 \in \zeta_{p,\alpha}$ then $\zeta_p(x_1) \geq \alpha$ and $\zeta_p(x_2) \geq \alpha$. It follows from that.

$\zeta_p(x_1x_2) \geq \zeta_p(x_1) \vee \zeta_p(x_2) \geq \alpha$. So that $x_1, x_2 \in \zeta_{p,\alpha}$.

If $x_1, x_2 \in \psi_{p,\alpha}$, then $\psi_p(x_1) \leq \alpha$ and $\psi_p(x_2) \leq \alpha$ and so $\psi_p(x_1x_2) \leq \psi_p(x_1) \vee \psi_p(x_2) \leq \alpha$, that is $x_1, x_2 \in \psi_{p,\alpha}$.

Hence $\mu_{p,\alpha}$, $\zeta_{p,\alpha}$ and $\psi_{p,\alpha}$ are sub-semigroup of $S$. Now let $x_1x_2 \in S$ and $u \in \mu_{p,\alpha}$. Then $\mu_p(x_1ux_2) \leq \mu_p(u) \leq \alpha$ and so $x_1ux_2 \in \mu_{p,\alpha}$.

If $u \in \zeta_{p,\alpha}$. Then $\zeta_p(x_1ux_2) \geq \zeta_p(u) \geq \alpha$ and so $x_1ux_2 \in \zeta_{p,\alpha}$.

If $u \in \psi_{p,\alpha}$. Then $\psi_p(x_1ux_2) \leq \psi_p(u) \leq \alpha$ thus $x_1ux_2 \in \psi_{p,\alpha}$.

Therefore $\mu_{p,\alpha}$,$\zeta_{p,\alpha}$ and $\psi_{p,\alpha}$ are interior ideal of $S$. □

**Theorem 4.16.** *A PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ is and PNII of $S$ if and only if $\mu_p, \zeta_p, \overline{\psi}_p$ are fuzzy interior ideal (FII) of $S$.*

*Proof.* Let $P_N = (\mu_p, \zeta_p, \psi_p)$ be an PNII of $S$. Then clearly $\mu_p$ is FII of $S$. Let $u, x_1, x_2 \in S$. Then

$$\begin{aligned}\overline{\psi_p}(x_1x_2) &= 1 - \psi_p(x_1x_2)\\ &\geq 1 - (\psi_p(x_1)) \vee \psi_p(x_2)\\ &= (1 - \psi_p(x_1)) \wedge (1 - \psi_p(x_2))\\ &= \overline{\psi_k}(x_1) \wedge \overline{\psi_p}(x_2)\end{aligned}$$

$$\begin{aligned}\overline{\psi_p}(x_1ux_2) &= 1 - \psi_p(x_1ux_2)\\ &\geq 1 - (\psi_p(u))\\ &= \overline{\psi_p}(u)\end{aligned}$$

$\overline{\psi_k}$ is a FII of $S$.

Conversely.

Suppose that $\zeta_p$ and $\overline{\psi_p}$ are FII of $S$. Let $u, x_1, x_2 \in S$.

$$\begin{aligned}1 - \psi_p(x_1x_2) &= \overline{\psi_p}(x_1x_2)\\ &\geq \overline{\psi_p}(x_1) \wedge \overline{\psi_p}(x_2)\\ &= (1 - \psi_p(x_1)) \wedge (1 - \psi_p(x_2))\\ &= 1 - \psi_p(x_1) \vee \psi_p(x_2)\\ &= 1 - \psi_p(x_1ux_2) = \overline{\psi_p}(x_1ux_2)\\ &\geq \overline{\psi_p}(u) = 1 - \psi_p(u)\end{aligned}$$

which implies $\psi_p(x_1x_2) \leq \psi_p(x_1) \vee \psi_p(x_2)$

and

$\psi_p(x_1ux_2) \leq \psi_p(u)$

This completes the proof. □

## 5. Conclusions

In this paper Pythagorean neutrosophic sub-semigroup, Pythagorean neutrosophic left(resp.right) ideal, Pythagorean neutrosophic ideal, Pythagorean neutrosophic bi-ideal, Pythagorean neutrosophic interior ideal and investigated some properties.

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# Simplified Neutrosophic Multiplicative Refined Sets and Their Correlation Coefficients with Application in Medical Pattern Recognition

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**Abstract**. In this paper, the notion of simplified neutrosophic multiplicative refined set (aka, simplified neutrosophic multiplicative multi-set) is introduced and some basic operational relations are investigated. The correlation coefficient is one of the most frequently used tools to provide the strength of relationship between two fuzzy/neutrosophic (refined) sets. Two different methods are proposed to calculate the correlation coefficients between two simplified neutrosophic multiplicative refined sets. Further, the effectiveness of these methods is demonstrated by dealing with the medical pattern recognition problem under the simplified neutrosophic multiplicative refined set environment.

**Keywords:** Neutrosophic sets; Simplified neutrosophic multiplicative sets; Simplified neutrosophic multiplicative refined sets; Simplified neutrosophic multiplicative multi-sets; Correlation coefficient; Pattern recognition

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## 1. Introduction

Most of the problems of real-life involve uncertainty or unknown data, and traditional mathematical tools cannot deal with such problems. The fuzzy sets (FSs), originated by Zadeh [48], is a useful tool to cope with vagueness and ambiguity. In 1986, Atanassov [6] initiated the theory of intuitionistic fuzzy sets (IFSs) extending the FSs. In the following years, many authors studied the the fuzzy set extensions [9, 19, 20, 30, 40] and their matrix representations [23, 25, 27–29]. However, the FSs and their extensions failed to cope with indeterminate and inconsistent information which exist in beliefs system, therefore, Smarandache [36] proposed new concept named neutrosophic set (NS) which generalizes the FSs and IFSs. Later, Wang et al. [42] and Ye [46] introduced specific descriptions of NSs known as single-valued neutrosophic set (SVNS) and simplified neutrosophic set (SNS), motivated from a practical point of view and can be used in real scientific and engineering applications. These theories of NSs have proven useful in the different fields such as medical diagnosis [3,16], decision making [1–5,15,21,26] and so on. In 1995, Smarandache put forward that in some cases the degrees of truth-membership, indeterminacy-membership and falsity-membership in the structure of (single-valued/simplified) NS can be not only in the interval [0,1] but also less than 0 or greater than 1, and presented some real world arguments supporting this assertion. Based on this idea, he introduced the concepts of neutrosophic oversets (when some neutrosophic components are $> 1$), neutrosophic undersets (when some neutrosophic component is $< 0$), and neutrosophic offsets (when some neutrosophic components are off the interval [0,1], i.e. some neutrosophic components $> 1$ and some neutrosophic components $< 0$) and studied their fundamentals [35, 37, 38].

The multi-set theory was introduced by Yager [45] as generalization of the set theory and then the multi-set was improved by Calude et al. [8]. Occasionally, several authors made a number of generalizations of the multi-set theory. Sebastian and Ramakrishnan [33] described a multi fuzzy set (mFS) which is a generalization of the multi-set. In [11, 34], the authors presented an extension of the notion of mFS to an intuitionstic fuzzy set which was termed to be an intuitionstic fuzzy multi-sets (IFmS). As the concepts of mFS and IFmS failed to deal with indeterminacy, Smarandache [39] extended the classical neutrosophic logic to n-valued refined neutrosophic logic, by refining each neutrosophic component. Meanwhile, Ye and Ye [47] proposed the concept of neutrosophic multi-set (NmS) (aka, neutrosophic refined set (NRS)) and investigated their characteristic properties. Deli et al. [10] studied some aspects of NRSs such as intersection, union, convex and strongly convex in a new way. In recent years, many seminal articles on the NmSs/NRSs have been published [7, 22, 41].

In spite of the fact that the FSs, IFSs and NSs are effective mathematical tools for dealing with uncertainties, these sets use the 0-1 scale, which is distributed symmetrically and uniformly. But, there are real-life issues that need to be scaled as unsymmetrically and non-uniformly. The grading system of universities is the most obvious example of such situations [17]. In dealing with such problems that need to be scaled unsymmetrically and non-uniformly while assigning the variable grades, Saaty [31] proposed the 1-9 scale (or $\frac{1}{9} - 9$ scale) as a useful tool. These different scales lead to the modelling of multiplicative preference relation [32]. In 2013, Xia et al. [44] proposed the idea of intuitionistic multiplicative sets (IMSs) and the intuitionistic multiplicative preference relations (IMPRs). Further, they gave a comparison between 0.1-0.9 and $\frac{1}{9} - 9$ scales as in Table 1.

Table 1. The comparison between 0.1-0.9 and $\frac{1}{9} - 9$ scales [44]

| $\frac{1}{9} - 9$ scale | 0.1-0.9 scale | Meaning |
|---|---|---|
| $\frac{1}{9}$ | 0.1 | Extremely not preferrred |
| $\frac{1}{7}$ | 0.2 | Very strongly not preferrred |
| $\frac{1}{5}$ | 0.3 | Strongly not preferrred |
| $\frac{1}{3}$ | 0.4 | Moderately not preferrred |
| 1 | 0.5 | Equally preferrred |
| 3 | 0.6 | Moderately preferrred |
| 5 | 0.7 | Strongly preferrred |
| 7 | 0.8 | Very strongly preferrred |
| 9 | 0.9 | Extremely preferrred |
| Other values between $\frac{1}{9}$ and 9 | Other values between 0 and 1 | Intermediate values used to present compromise |

Recently, the theoretical aspects of IMSs and IMPRs have been studied in detail [12–14, 18, 43]. In 2019, Köseoğlu et al. [24] put forward that the IMSs cannot handle real-life problems, which include the indeterminate information in addition to the truth-membership information and falsity-membership information of IMS. To eradicate this restriction, they introduced the concepts of simplified neutrosophic multiplicative set (SNMS) and simplified neutrosophic multiplicative preference relations (SNMPRs).

Moreover, they gave several formulas for measuring the distance between two SNMSs.
There are two main objectives underlying this study. The first is to initiate the theory of simplified neutrosophic multiplicative refined set (SNMRS) (aka, simplified neutrosophic multiplicative multi-set). Obviously, the concept of SNMRS is a generalization of IMSs and SNMSs. The second is to propose novel correlation coefficients to numerically determine the relationship between two SNMRSs. By using the proposed correlation coefficients, the ranking of all alternatives (objects) can be achieved. The layout of rest of this paper is presented as follows: In Section 2, the concepts of NSs, SNSs and SNMSs are given. In Section 3, the SNMRSs are conceptualized and their fundamentals such as subset, complement, intersection, union and aggregation operators are studied. In Section 4, the conceptual approaches of correlation coefficients between two SNMRSs are proposed and their characteristic properties are discussed. In Section 5, an example are given to validate the proposed correlation measures and the comparative analysis is presented to demonstrate their effectiveness. In Section 6, the conclusion of this study is summarized.

## 2. Preliminaries

In this section, some basic concepts of neutrosophic sets, simplified neutrosophic sets and simplified neutrosophic multiplicative sets are recalled.

Let $\mathcal{E}$ be a space of points (object) with a generic element denoted by $\varepsilon$.

**Definition 2.1.** ( [36]) A neutrosophic set (NS) $\mathcal{N}$ in $\mathcal{E}$ is characterized by a truth-membership function $t_{\mathcal{N}} : \mathcal{E} \to ]0^-, 1^+[$, an indeterminacy-membership function $\imath_{\mathcal{N}} : \mathcal{E} \to ]0^-, 1^+[$, and a falsity-membership function $f_{\mathcal{N}} : \mathcal{E} \to ]0^-, 1^+[$. $t_{\mathcal{N}}(\varepsilon)$, $\imath_{\mathcal{N}}(\varepsilon)$ and $f_{\mathcal{N}}(\varepsilon)$ are real standard or non-standard subsets of $]0^-, 1^+[$. There is no restriction on the sum of $t_{\mathcal{N}}(\varepsilon)$, $\imath_{\mathcal{N}}(\varepsilon)$ and $f_{\mathcal{N}}(\varepsilon)$, so $0^- \leq \sup t_{\mathcal{N}}(\varepsilon) + \sup \imath_{\mathcal{N}}(\varepsilon) + \sup f_{\mathcal{N}}(\varepsilon) \leq 3^+$ for $\varepsilon \in \mathcal{E}$.

However, Wang et al. [42] and Ye [46] stated the difficulty of using the NSs of non-standard intervals in practice, and introduced the simplified neutrosophic sets as follows.

**Definition 2.2.** ( [46]) An NS $\mathcal{N}$ is characterized by a truth-membership function $t_{\mathcal{N}} : \mathcal{E} \to [0, 1]$, an indeterminacy-membership function $\imath_{\mathcal{N}} : \mathcal{E} \to [0, 1]$, and a falsity-membership function $f_{\mathcal{N}} : \mathcal{E} \to [0, 1]$. $t_{\mathcal{N}}(\varepsilon)$, $\imath_{\mathcal{N}}(\varepsilon)$ and $f_{\mathcal{N}}(\varepsilon)$ are singleton subintervals/subsets in the standard interval $[0, 1]$, then it is termed to be a simplified neutrosophic set (SNS) and described as

$$\mathcal{N} = \{(\varepsilon, \langle t_{\mathcal{N}}(\varepsilon), \imath_{\mathcal{N}}(\varepsilon), f_{\mathcal{N}}(\varepsilon)\rangle) : \ \varepsilon \in \mathcal{E}\} \tag{1}$$

This kind of NS is is named a single-valued neutrosophic set (SVNS) by Wang et al. [42]. Throughout this paper, we will use the term "simplified neutrosophic set (SNS)".

**Definition 2.3.** ( [39, 47]) A simplified neutrosophic refined set (SNRS) $\widetilde{\mathcal{N}}$ can be defined as follows:

$$\widetilde{\mathcal{N}} = \{(\varepsilon, \langle (t^1_{\widetilde{\mathcal{N}}}(\varepsilon), t^2_{\widetilde{\mathcal{N}}}(\varepsilon), ..., t^q_{\widetilde{\mathcal{N}}}(\varepsilon)), (\imath^1_{\widetilde{\mathcal{N}}}(\varepsilon), \imath^2_{\widetilde{\mathcal{N}}}(\varepsilon), ..., \imath^q_{\widetilde{\mathcal{N}}}(\varepsilon)), (f^1_{\widetilde{\mathcal{N}}}(\varepsilon), f^2_{\widetilde{\mathcal{N}}}(\varepsilon), ..., f^q_{\widetilde{\mathcal{N}}}(\varepsilon))\rangle) : \ \varepsilon \in \mathcal{E}\} \tag{2}$$

where

$$t^1_{\widetilde{\mathcal{N}}}, t^2_{\widetilde{\mathcal{N}}}, ..., t^q_{\widetilde{\mathcal{N}}} : \mathcal{E} \to [0,1],\ \ \imath^1_{\widetilde{\mathcal{N}}}, \imath^2_{\widetilde{\mathcal{N}}}, ..., \imath^q_{\widetilde{\mathcal{N}}} : \mathcal{E} \to [0,1], \text{ and } f^1_{\widetilde{\mathcal{N}}}, f^2_{\widetilde{\mathcal{N}}}, ..., f^q_{\widetilde{\mathcal{N}}} : \mathcal{E} \to [0,1]$$

such that

$$0^- \leq \sup t^i_{\widetilde{\mathcal{N}}}(\varepsilon) + \sup \imath^i_{\widetilde{\mathcal{N}}}(\varepsilon) + \sup f^i_{\widetilde{\mathcal{N}}}(\varepsilon) \leq 3^+ \ \forall i \in I_q = \{1, 2, ..., q\}.$$

for each $\varepsilon \in \mathcal{E}$. Further, the truth-membership sequence $(t^1_{\widetilde{\mathcal{N}}}(\varepsilon), t^2_{\widetilde{\mathcal{N}}}(\varepsilon), ..., t^q_{\widetilde{\mathcal{N}}}(\varepsilon))$ may be in decreasing/increasing order, and the corresponding indeterminacy-membership sequence $(\imath^1_{\widetilde{\mathcal{N}}}(\varepsilon), \imath^2_{\widetilde{\mathcal{N}}}(\varepsilon), ..., \imath^q_{\widetilde{\mathcal{N}}}(\varepsilon))$ and falsity-membership sequence $(f^1_{\widetilde{\mathcal{N}}}(\varepsilon), f^2_{\widetilde{\mathcal{N}}}(\varepsilon), ..., f^q_{\widetilde{\mathcal{N}}}(\varepsilon))$. Also, $q$ is termed to be the dimension of SNMS $\widetilde{\mathcal{N}}$.

*Note 1.* In the literature, SNRSs are also referred to as simplified neutrosophic multisets (SNmSs).

**Definition 2.4.** ( [24]) A simplified neutrosophic multiplicative set (SNMS) $\mathcal{M}$ in $\mathcal{E}$ is defined as

$$\mathcal{M} = \{(\varepsilon, \langle \zeta_{\mathcal{M}}(\varepsilon), \eta_{\mathcal{M}}(\varepsilon), \vartheta_{\mathcal{M}}(\varepsilon)\rangle) :\ \varepsilon \in \mathcal{E}\}, \tag{3}$$

which assigns to each element $\varepsilon$ a truth-membership information $\zeta_{\mathcal{M}}(\varepsilon)$, an indeterminacy-membership information $\eta_{\mathcal{M}}(\varepsilon)$, and a falsity-membership information $\vartheta_{\mathcal{M}}(\varepsilon)$ with conditions

$$\frac{1}{9} \leq \zeta_{\mathcal{M}}(\varepsilon), \eta_{\mathcal{M}}(\varepsilon), \vartheta_{\mathcal{M}}(\varepsilon) \leq 9 \text{ and } 0 < \zeta_{\mathcal{M}}(\varepsilon)\vartheta_{\mathcal{M}}(\varepsilon) \leq 1. \tag{4}$$

for each $\varepsilon \in \mathcal{E}$.

*Note 1.* In 1995, Smarandache put forward that in some cases the degrees of truth-membership, indeterminacy-membership and falsity-membership in the structure of (single-valued/simplified) NS can be not only in the interval [0,1] but also greater than 1. Thus, he described the truth-membership function, indeterminacy-membership function and falsity-membership function as $t_{\mathcal{N}}, \imath_{\mathcal{N}}, f_{\mathcal{N}} : \mathcal{E} \to [0, \Omega]$ where $0 < 1 < \Omega$ and $\Omega$ is named overlimit. He called this extended type of (single-valued/simplified) NSs as neutrosophic overset [37, 38]. It is noted that the SNMSs are particular case of the neutrosophic oversets.

## 3. Simplified Neutrosophic Multiplicative Refined Sets

In this section, we initiate the theory of simplified neutrosophic multiplicative refined sets. Also, we derive some basic operations on simplified neutrosophic multiplicative refined sets and study the related properties.

**Definition 3.1.** Let $\mathcal{E}$ be a space of points (object) with a generic element denoted by $\varepsilon$. A simplified neutrosophic multiplicative refined set (SNMRS) $\widetilde{\mathcal{M}}$ in $\mathcal{E}$ is defined as

$$\begin{aligned} \widetilde{\mathcal{M}} &= \{(\varepsilon, \langle(\zeta^1_{\widetilde{\mathcal{M}}}(\varepsilon), \zeta^2_{\widetilde{\mathcal{M}}}(\varepsilon), ..., \zeta^q_{\widetilde{\mathcal{M}}}(\varepsilon)), (\eta^1_{\widetilde{\mathcal{M}}}(\varepsilon), \eta^2_{\widetilde{\mathcal{M}}}(\varepsilon), ..., \eta^q_{\widetilde{\mathcal{M}}}(\varepsilon)), (\vartheta^1_{\widetilde{\mathcal{M}}}(\varepsilon), \vartheta^2_{\widetilde{\mathcal{M}}}(\varepsilon), ..., \vartheta^q_{\widetilde{\mathcal{M}}}(\varepsilon))\rangle) :\ \varepsilon \in \mathcal{E}\} \\ &= \{\langle \varepsilon, (\zeta^i_{\widetilde{\mathcal{M}}}(\varepsilon))_{i \in I_q}, (\eta^i_{\widetilde{\mathcal{M}}}(\varepsilon))_{i \in I_q}, (\vartheta^i_{\widetilde{\mathcal{M}}}(\varepsilon))_{i \in I_q}\rangle :\ \varepsilon \in \mathcal{E}\} \end{aligned} \tag{5}$$

which assigns to each element $\varepsilon$ a sequence of truth-membership information $\zeta^i_{\widetilde{\mathcal{M}}}(\varepsilon)$ $(i = 1, 2, ..., q)$, a sequence of indeterminacy-membership information $\eta^i_{\widetilde{\mathcal{M}}}(\varepsilon)$ $(i = 1, 2, ..., q)$, and a sequence of falsity-membership information $\vartheta^i_{\widetilde{\mathcal{M}}}(\varepsilon)$ $(i = 1, 2, ..., q)$ with conditions

$$\frac{1}{9} \leq \zeta^i_{\widetilde{\mathcal{M}}}(\varepsilon), \eta^i_{\widetilde{\mathcal{M}}}(\varepsilon), \vartheta^i_{\widetilde{\mathcal{M}}}(\varepsilon) \leq 9 \text{ and } 0 < \zeta^i_{\widetilde{\mathcal{M}}}(\varepsilon)\vartheta^i_{\widetilde{\mathcal{M}}}(\varepsilon) \leq 1 \ \ \forall i \in I_q \tag{6}$$

for each $\varepsilon \in \mathcal{E}$. Further, the truth-membership sequence $(\zeta^i_{\widetilde{\mathcal{M}}}(\varepsilon))_{i\in I_q} = (\zeta^1_{\widetilde{\mathcal{M}}}(\varepsilon), \zeta^2_{\widetilde{\mathcal{M}}}(\varepsilon), ..., \zeta^q_{\widetilde{\mathcal{M}}}(\varepsilon))$ may be in decreasing/increasing order, and the corresponding indeterminacy-membership sequence $(\eta^i_{\widetilde{\mathcal{M}}}(\varepsilon))_{i\in I_q} = (\eta^1_{\widetilde{\mathcal{M}}}(\varepsilon), \zeta^2_{\widetilde{\mathcal{M}}}(\varepsilon), ..., \zeta^q_{\widetilde{\mathcal{M}}}(\varepsilon))$ and falsity-membership sequence $(\vartheta^i_{\widetilde{\mathcal{M}}}(\varepsilon))_{i\in I_q} = (\vartheta^1_{\widetilde{\mathcal{M}}}(\varepsilon), \vartheta^2_{\widetilde{\mathcal{M}}}(\varepsilon), ..., \vartheta^q_{\widetilde{\mathcal{M}}}(\varepsilon))$. Also, $q$ is termed to be the dimension of SNMRS $\widetilde{\mathcal{M}}$. For convenience, any element of $\widetilde{\mathcal{M}}$ can be represented as $\psi = \langle (\zeta^i_{\widetilde{\mathcal{M}}})_{i\in I_q}, (\eta^i_{\widetilde{\mathcal{M}}})_{i\in I_q}, (\vartheta^i_{\widetilde{\mathcal{M}}})_{i\in I_q} \rangle$ and it is said to be a simplified neutrosophic multiplicative refined number (SNMRN).

From now on, $SNMRS(\mathcal{E}, q)$ denotes the collection of all $q$-dimension SNMRSs in $\mathcal{E}$.

**Example 3.2.** Assume that $\mathcal{E} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ is the universal set where all the elements represent some drugs suitable for different infections such as bronchitis, sinusitis, skin infections and ear infections. We can easily categorize these drugs according to their side effects. Thus, the (3-dimension) SNMRS is given as follows:

$$\widetilde{\mathcal{M}} = \left\{ \begin{array}{l} (\varepsilon_1, \langle (\frac{1}{4}, 1, 4), (1, \frac{1}{5}, \frac{1}{2}), (3, \frac{2}{5}, \frac{1}{4}) \rangle), (\varepsilon_2, \langle (1, 3, 5), (\frac{1}{2}, \frac{1}{4}, 2), (1, \frac{1}{4}, \frac{1}{5}) \rangle), \\ (\varepsilon_3, \langle (\frac{1}{9}, \frac{1}{6}, \frac{1}{2}), (9, 1, \frac{1}{5}), (2, 1, \frac{1}{2}) \rangle), (\varepsilon_4, \langle (\frac{5}{4}, 4, 5), (\frac{1}{4}, 4, 5), (\frac{1}{5}, \frac{1}{5}, \frac{1}{9}) \rangle) \end{array} \right\}.$$

Consider $(\varepsilon_1, \langle (\frac{1}{4}, 1, 4), (1, \frac{1}{5}, \frac{1}{2}), (3, \frac{2}{5}, \frac{1}{4}) \rangle) \in \widetilde{\mathcal{M}}$. Then, $(\frac{1}{4}, 1, 4)$ means the sequence of truth-membership information (scaled between $\frac{1}{9}$ and 9) of side effects of drug $\varepsilon_1$. The sequences of indeterminacy-membership information and falsity-membership information of $\varepsilon_1$ can be interpreted similarly.

**Definition 3.3.** Let $\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2 \in SNMRS(\mathcal{E}, q)$.

**(a):** If for each $\varepsilon \in \mathcal{E}$,

$$\zeta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon) \le \zeta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon),\ \eta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon) \ge \eta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon),\ \vartheta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon) \ge \vartheta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\ \ \forall i \in I_q$$

then $\widetilde{\mathcal{M}}_1$ is an SNMR subset of $\widetilde{\mathcal{M}}_2$, denoted by $\widetilde{\mathcal{M}}_1 \subseteq \widetilde{\mathcal{M}}_2$.

**(b):** If for each $\varepsilon \in \mathcal{E}$,

$$\zeta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon) = \zeta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon),\ \eta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon) = \eta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon),\ \vartheta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon) = \vartheta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\ \ \forall i \in I_q$$

then the SNMRSs $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ are equal, denoted by $\widetilde{\mathcal{M}}_1 = \widetilde{\mathcal{M}}_2$. That is, $\widetilde{\mathcal{M}}_1 = \widetilde{\mathcal{M}}_2$ iff $\widetilde{\mathcal{M}}_1 \subseteq \widetilde{\mathcal{M}}_2$ and $\widetilde{\mathcal{M}}_2 \subseteq \widetilde{\mathcal{M}}_1$.

**(c):** The complement of $\widetilde{\mathcal{M}}$, is denoted and defined as

$$\widetilde{\mathcal{M}}^c = \{(\varepsilon, \langle (\vartheta^i_{\widetilde{\mathcal{M}}}(\varepsilon))_{i\in I_q}, (\frac{1}{\eta^i_{\widetilde{\mathcal{M}}}(\varepsilon)})_{i\in I_q}, (\zeta^i_{\widetilde{\mathcal{M}}}(\varepsilon))_{i\in I_q} \rangle) :\ \varepsilon \in \mathcal{E}\}.$$

where $(\frac{1}{\eta^i_{\widetilde{\mathcal{M}}}(\varepsilon)})_{i\in I_q}$ represents the sequence $(\frac{1}{\eta^1_{\widetilde{\mathcal{M}}}(\varepsilon)}, \frac{1}{\eta^2_{\widetilde{\mathcal{M}}}(\varepsilon)}, ..., \frac{1}{\eta^q_{\widetilde{\mathcal{M}}}(\varepsilon)})$

**(d):** The intersection of $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$, denoted by $\widetilde{\mathcal{M}}_1 \cap \widetilde{\mathcal{M}}_2$, is described as

$$\widetilde{\mathcal{M}}_1 \cap \widetilde{\mathcal{M}}_2 = \left\{ \left( \varepsilon, \left\langle \begin{array}{l} \left(\min\{\zeta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q}, \\ \left(\max\{\eta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \eta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q}, \\ \left(\max\{\vartheta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \vartheta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q} \end{array} \right\rangle \right) :\ \varepsilon \in \mathcal{E} \right\}.$$

**(e):** The union of $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$, denoted by $\widetilde{\mathcal{M}}_1 \cup \widetilde{\mathcal{M}}_2$, is described as

$$\widetilde{\mathcal{M}}_1 \cup \widetilde{\mathcal{M}}_2 = \left\{ \left( \varepsilon, \left\langle \begin{array}{c} \left(\max\{\zeta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q}, \\ \left(\min\{\eta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \eta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q}, \\ \left(\min\{\vartheta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \vartheta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q} \end{array} \right\rangle \right) : \varepsilon \in \mathcal{E} \right\}.$$

For the sequences of truth-membership information in definitions of intersection and union of SNMRSs, $\left(\min\{\zeta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q} = (\min\{\zeta^1_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^1_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}, \min\{\zeta^2_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^2_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}, ..., \min\{\zeta^q_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^q_{\widetilde{\mathcal{M}}_2}(\varepsilon)\})$ and $\left(\max\{\zeta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q} = (\max\{\zeta^1_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^1_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}, \max\{\zeta^2_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^2_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}, ..., \max\{\zeta^q_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^q_{\widetilde{\mathcal{M}}_2}(\varepsilon)\})$. It can be considered similar matches for the indeterminacy-membership information and falsity-membership information in Definition 3.3 (d) and (e).

**Theorem 3.4.** *Let $\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2, \widetilde{\mathcal{M}}_3 \in SNMRS(\mathcal{E}, q)$.*

**(i):** *If $\widetilde{\mathcal{M}}_1 \star \widetilde{\mathcal{M}}_2$ and $\widetilde{\mathcal{M}}_2 \star \widetilde{\mathcal{M}}_3$ then $\widetilde{\mathcal{M}}_1 \star \widetilde{\mathcal{M}}_3$ for each $\star \in \{\subseteq, =\}$.*
**(ii):** *If $\widetilde{\mathcal{M}}_1 \star \widetilde{\mathcal{M}}_2$ then $(\widetilde{\mathcal{M}}_1 \bullet \widetilde{\mathcal{M}}_3) \star (\widetilde{\mathcal{M}}_2 \bullet \widetilde{\mathcal{M}}_3)$ for each $\star \in \{\subseteq, =\}$ and $\bullet \in \{\cap, \cup\}$.*
**(iii):** *$\widetilde{\mathcal{M}}_1 \bullet \widetilde{\mathcal{M}}_2 = \widetilde{\mathcal{M}}_2 \bullet \widetilde{\mathcal{M}}_1$ for each $\bullet \in \{\cap, \cup\}$.*
**(iv):** *$\widetilde{\mathcal{M}}_1 \bullet (\widetilde{\mathcal{M}}_2 \bullet \widetilde{\mathcal{M}}_3) = (\widetilde{\mathcal{M}}_1 \bullet \widetilde{\mathcal{M}}_2) \bullet \widetilde{\mathcal{M}}_3$ for each $\bullet \in \{\cap, \cup\}$.*
**(v):** *$\widetilde{\mathcal{M}}_1 \bullet (\widetilde{\mathcal{M}}_2 \blacklozenge \widetilde{\mathcal{M}}_3) = (\widetilde{\mathcal{M}}_1 \bullet \widetilde{\mathcal{M}}_2) \blacklozenge (\widetilde{\mathcal{M}}_1 \bullet \widetilde{\mathcal{M}}_3)$ for each $\bullet, \blacklozenge \in \{\cap, \cup\}$.*
**(vi):** *$(\widetilde{\mathcal{M}}_1 \bullet \widetilde{\mathcal{M}}_2)^c = \widetilde{\mathcal{M}}_1^c \blacklozenge \widetilde{\mathcal{M}}_2^c$ for each $\bullet, \blacklozenge \in \{\cap, \cup\}$ and $\bullet \neq \blacklozenge$.*

*Proof.* Let us prove the properties (vi) for $\bullet = \cap$ and $\blacklozenge = \cup$.
**(iv):** From Definition 3.3 (c) and (d), we can write

$$(\widetilde{\mathcal{M}}_1 \cap \widetilde{\mathcal{M}}_2)^c = \left\{ \left( \varepsilon, \left\langle \begin{array}{c} \left(\max\{\vartheta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \vartheta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q}, \\ \left(\frac{1}{\max\{\eta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \eta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}}\right)_{i\in I_q}, \\ \left(\min\{\zeta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q} \end{array} \right\rangle \right) : \varepsilon \in \mathcal{E} \right\} \tag{7}$$

For the right side of the equality, we can write

$$\widetilde{\mathcal{M}}_k^c = \{\langle \varepsilon, (\vartheta^i_{\widetilde{\mathcal{M}}_k}(\varepsilon))_{i\in I_q}, (\frac{1}{\eta^i_{\widetilde{\mathcal{M}}_k}(\varepsilon)})_{i\in I_q}, (\zeta^i_{\widetilde{\mathcal{M}}_k}(\varepsilon))_{i\in I_q} \rangle : \varepsilon \in \mathcal{E}\}.$$

for $k = 1, 2$ and so

$$\widetilde{\mathcal{M}}_1^c \cup \widetilde{\mathcal{M}}_2^c = \left\{ \left( \varepsilon, \left\langle \begin{array}{c} \left(\max\{\vartheta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \vartheta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q}, \\ \left(\min\{\frac{1}{\eta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon)}, \frac{1}{\eta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)}\}\right)_{i\in I_q}, \\ \left(\max\{\zeta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \zeta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}\right)_{i\in I_q} \end{array} \right\rangle \right) : \varepsilon \in \mathcal{E} \right\} \tag{8}$$

Since $\eta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \eta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon) \in [\frac{1}{9}, 9]$ for all $i \in I_q$, the equality $\frac{1}{\max\{\eta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon), \eta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)\}} = \min\{\frac{1}{\eta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon)}, \frac{1}{\eta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon)}\}$ is valid. So, from the Eqs. (7) and (8), we deduce that $(\widetilde{\mathcal{M}}_1 \cap \widetilde{\mathcal{M}}_2)^c = \widetilde{\mathcal{M}}_1^c \cup \widetilde{\mathcal{M}}_2^c$. Proceeding with similar calculations, it can be demonstrated that $(\widetilde{\mathcal{M}}_1 \cup \widetilde{\mathcal{M}}_2)^c = \widetilde{\mathcal{M}}_1^c \cap \widetilde{\mathcal{M}}_2^c$.
By using similar techniques, the properties (i)-(v) can be proved, therefore they are omitted. □

**Definition 3.5.** For $\psi = \langle (\zeta^i)_{i\in I_q}, (\eta^i)_{i\in I_q}, (\vartheta^i)_{i\in I_q} \rangle$, the score, accuracy and certainty functions of $\psi$ are described respectively as follows:

$$f_S(\psi) = \frac{1}{q} \sum_{i\in I_q} \frac{\zeta^i}{\eta^i \vartheta^i} \tag{9}$$

$$f_A(\psi) = \frac{1}{q} \sum_{i\in I_q} \zeta^i \vartheta^i \tag{10}$$

and

$$f_C(\psi) = \frac{1}{q} \sum_{i\in I_q} \zeta^i \tag{11}$$

To compare two SNMRNs $\psi_1$ and $\psi_2$, the steps detailed below can be followed:

**(1):** if $f_S(\psi_1) > f_S(\psi_2)$ then $\psi_1 \succ \psi_2$,
**(2):** if $f_S(\psi_1) < f_S(\psi_2)$ then $\psi_1 \prec \psi_2$,
**(3):** if $f_S(\psi_1) = f_S(\psi_2)$ then
- **(i):** if $f_A(\psi_1) > f_A(\psi_2)$ then $\psi_1 \succ \psi_2$,
- **(ii):** if $f_A(\psi_1) < f_A(\psi_2)$ then $\psi_1 \prec \psi_2$,
- **(iii):** if $f_A(\psi_1) = f_A(\psi_2)$ then
  - **(a):** if $f_C(\psi_1) > f_C(\psi_2)$ then $\psi_1 \succ \psi_2$,
  - **(b):** if $f_C(\psi_1) < f_C(\psi_2)$ then $\psi_1 \prec \psi_2$,
  - **(c):** if $f_C(\psi_1) = f_C(\psi_2)$ then $\psi_1 = \psi_2$.

**Example 3.6.** If we take $\psi_1 = \langle (\frac{1}{4}, \frac{1}{2}, 1), (3, \frac{3}{4}, 3), (1, 1, \frac{1}{2}) \rangle$ and $\psi_2 = \langle (\frac{1}{2}, 1, 1), (\frac{3}{4}, 3, 3), (1, \frac{1}{2}, \frac{1}{4}) \rangle$ then we get $f_S(\psi_1) = f_S(\psi_2) = \frac{17}{36}$. Since the score values of $\psi_1$ and $\psi_2$ are equal, by using the accuracy function, we obtain $f_A(\psi_1) = f_A(\psi_2) = \frac{5}{12}$. By considering the certainty function, we calculate as $f_C(\psi_1) = \frac{17}{12}$ and $f_C(\psi_2) = \frac{3}{4}$. Thus, we have $\psi_1 \succ \psi_2$ since $f_C(\psi_1) > f_C(\psi_2)$.

**Definition 3.7.** Let $\psi = \langle (\zeta^i)_{i\in I_q}, (\eta^i)_{i\in I_q}, (\vartheta^i)_{i\in I_q} \rangle$, $\psi_1 = \langle (\zeta_1^i)_{i\in I_q}, (\eta_1^i)_{i\in I_q}, (\vartheta_1^i)_{i\in I_q} \rangle$ and $\psi_2 = \langle (\zeta_2^i)_{i\in I_q}, (\eta_2^i)_{i\in I_q}, (\vartheta_2^i)_{i\in I_q} \rangle$ be three SNMRNs and $\omega > 0$ be a real number. Then, some operational laws of SNMRNs are described as follows.

**(a):**

$$\psi_1 \oplus \psi_2 = \Big\langle \ \big(\tfrac{(1+2\zeta_1^i)(1+2\zeta_2^i)-1}{2}\big)_{i\in I_q}, \big(\tfrac{2\eta_1^i\eta_2^i}{(2+\eta_1^i)(2+\eta_2^i)-\eta_1^i\eta_2^i}\big)_{i\in I_q}, \big(\tfrac{2\vartheta_1^i\vartheta_2^i}{(2+\vartheta_1^i)(2+\vartheta_2^i)-\vartheta_1^i\vartheta_2^i}\big)_{i\in I_q} \ \Big\rangle.$$

**(b):**

$$\psi_1 \otimes \psi_2 = \Big\langle \ \big(\tfrac{2\zeta_1^i\zeta_2^i}{(2+\zeta_1^i)(2+\zeta_2^i)-\zeta_1^i\zeta_2^i}\big)_{i\in I_q}, \big(\tfrac{(1+2\eta_1^i)(1+2\eta_2^i)-1}{2}\big)_{i\in I_q}, \big(\tfrac{(1+2\vartheta_1^i)(1+2\vartheta_2^i)-1}{2}\big)_{i\in I_q} \ \Big\rangle.$$

**(c):**

$$\omega\psi = \Big\langle \ \big(\tfrac{(1+2\zeta^i)^\omega-1}{2}\big)_{i\in I_q}, \big(\tfrac{2(\eta^i)^\omega}{(2+\eta^i)^\omega-(\eta^i)^\omega}\big)_{i\in I_q}, \big(\tfrac{2(\vartheta^i)^\omega}{(2+\vartheta^i)^\omega-(\vartheta^i)^\omega}\big)_{i\in I_q} \ \Big\rangle.$$

**(d):**

$$\psi^\omega = \Big\langle \ \big(\tfrac{2(\zeta^i)^\omega}{(2+\zeta^i)^\omega-(\zeta^i)^\omega}\big)_{i\in I_q}, \big(\tfrac{(1+2\eta^i)^\omega-1}{2}\big)_{i\in I_q}, \big(\tfrac{(1+2\vartheta^i)^\omega-1}{2}\big)_{i\in I_q} \ \Big\rangle.$$

**(e):**

$$\psi^c = \Big\langle \ (\vartheta^i)_{i\in I_q}, (\tfrac{1}{\eta^i})_{i\in I_q}, (\zeta^i)_{i\in I_q} \ \Big\rangle.$$

**Example 3.8.** We consider $\psi_1$ and $\psi_2$ given in Example 3.6. Then, we obtain

$$\begin{aligned}\psi_1 \oplus \psi_2 &= \left\langle \begin{array}{l} \big(\frac{(1+2\times\frac{1}{4})(1+2\times\frac{1}{2})-1}{2}, \frac{(1+2\times\frac{1}{2})(1+2\times 1)-1}{2}, \frac{(1+2\times 1)(1+2\times 1)-1}{2}\big), \\ \big(\frac{2\times 3\times\frac{3}{4}}{(2+3)(2+\frac{3}{4})-3\times\frac{3}{4}}, \frac{2\times\frac{3}{4}\times 3}{(2+\frac{3}{4})(2+3)-\frac{3}{4}\times 3}, \frac{2\times 3\times 3}{(2+3)(2+3)-3\times 3}\big), \\ \big(\frac{2\times 1\times 1}{(2+1)(2+1)-1\times 1}, \frac{2\times 1\times\frac{1}{2}}{(2+1)(2+\frac{1}{1})-1\times\frac{1}{2}}, \frac{2\times\frac{1}{2}\times\frac{1}{4}}{(2+\frac{1}{2})(2+\frac{1}{4})-\frac{1}{2}\times\frac{1}{4}}\big) \end{array} \right\rangle \\ &= \langle (1, \frac{5}{2}, 4), (\frac{9}{23}, \frac{9}{23}, \frac{9}{8}), (\frac{1}{4}, \frac{1}{7}, \frac{1}{22}) \rangle\end{aligned}$$

and

$$\psi_1^c = \langle (1, 1, \frac{1}{2}), (\frac{1}{3}, \frac{4}{3}, \frac{1}{3}), (\frac{1}{4}, \frac{1}{2}, 1) \rangle.$$

**Theorem 3.9.** *Let* $\psi = \langle (\zeta^i)_{i\in I_q}, (\eta^i)_{i\in I_q}, (\vartheta^i)_{i\in I_q} \rangle$, $\psi_1 = \langle (\zeta_1^i)_{i\in I_q}, (\eta_1^i)_{i\in I_q}, (\vartheta_1^i)_{i\in I_q} \rangle$ *and* $\psi_2 = \langle (\zeta_2^i)_{i\in I_q}, (\eta_2^i)_{i\in I_q}, (\vartheta_2^i)_{i\in I_q} \rangle$ *be three SNMRNs and* $\omega, \omega_1, \omega_2 > 0$ *be real numbers, then the following properties are valid.*

**(i):** $\psi_1 \bullet \psi_2 = \psi_2 \bullet \psi_1$ *for each* $\bullet \in \{\oplus, \otimes\}$.

**(ii):** $\omega(\psi_1 \oplus \psi_2) = \omega\psi_1 \oplus \omega\psi_2$.

**(iii):** $(\psi_1 \otimes \psi_2)^\omega = \psi_1^\omega \otimes \psi_2^\omega$.

**(iv):** $\omega_1\psi \oplus \omega_2\psi = (\omega_1 + \omega_2)\psi$.

**(v):** $\psi^{\omega_1} \otimes \psi^{\omega_2} = \psi^{\omega_1+\omega_2}$.

**(vi):** $(\psi_1 \bullet \psi_2)^c = \psi_1^c \blacklozenge \psi_2^c$ *for each* $\bullet, \blacklozenge \in \{\oplus, \otimes\}$ *and* $\bullet \neq \blacklozenge$.

*Proof.* Considering Definition 3.7, they can be achieved with simple calculations and so omitted. □

## 4. Correlation Coefficients for SNMRSs

In this section, we propose some types of correlation coefficients for the SNMRSs, which can be applied to real-life problems.

Suppose that $\widetilde{\mathcal{M}}_1 = \{\langle \varepsilon_j, (\zeta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon_j))_{i\in I_q}, (\eta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon_j))_{i\in I_q}, (\vartheta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon_j))_{i\in I_q} \rangle : \varepsilon_j \in \mathcal{E}\}$ and $\widetilde{\mathcal{M}}_2 = \{\langle \varepsilon_j, (\zeta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon_j))_{i\in I_q}, (\eta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon_j))_{i\in I_q}, (\vartheta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon_j))_{i\in I_q} \rangle : \varepsilon_j \in \mathcal{E}\}$ be any two $q$-dimension SNMRSs in the universal set $\mathcal{E} = \{\varepsilon_j : j = 1, 2, ..., m\}$ where $\zeta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon_j), \eta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon_j), \vartheta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon_j), \zeta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon_j), \eta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon_j), \vartheta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon_j) \in [\frac{1}{9}, 9]$, $0 < \zeta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon_j)\vartheta^i_{\widetilde{\mathcal{M}}_1}(\varepsilon_j) \leq 1$ and $0 < \zeta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon_j)\vartheta^i_{\widetilde{\mathcal{M}}_2}(\varepsilon_j) \leq 1$ $(i = 1, 2, ..., q)$ for each $\varepsilon_j \in \mathcal{E}$. The informational energies of SNMRSs $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ are defined as

$$\mathfrak{T}(\widetilde{\mathcal{M}}_1) = \frac{1}{q} \sum_{i\in I_q} \sum_{j=1}^{m} \Big( \frac{\zeta^i_{\mathcal{M}_1}(\varepsilon_j)}{2(1+\zeta^i_{\mathcal{M}_1}(\varepsilon_j))} + \frac{\eta^i_{\mathcal{M}_1}(\varepsilon_j)}{2(1+\eta^i_{\mathcal{M}_1}(\varepsilon_j))} + \frac{\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)}{2(1+\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))} \Big) \tag{12}$$

and

$$\mathfrak{T}(\widetilde{\mathcal{M}}_2) = \frac{1}{q} \sum_{i\in I_q} \sum_{j=1}^{m} \Big( \frac{\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}{2(1+\zeta^i_{\mathcal{M}_2}(\varepsilon_j))} + \frac{\eta^i_{\mathcal{M}_2}(\varepsilon_j)}{2(1+\eta^i_{\mathcal{M}_2}(\varepsilon_j))} + \frac{\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)}{2(1+\vartheta^i_{\mathcal{M}_2}(\varepsilon_j))} \Big) \tag{13}$$

The correlation of the SNMRSs $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ is described as

$$\mathfrak{C}(\widetilde{\mathcal{M}}_1,\widetilde{\mathcal{M}}_2)=\frac{1}{q}\sum_{i\in I_q}\sum_{j=1}^{m}\left(\begin{array}{c}\frac{2\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\zeta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\zeta^i_{\mathcal{M}_2}(\varepsilon_j))-\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}\\+\frac{2\eta^i_{\mathcal{M}_1}(\varepsilon_j)\eta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\eta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\eta^i_{\mathcal{M}_2}(\varepsilon_j))-\eta^i_{\mathcal{M}_1}(\varepsilon_j)\eta^i_{\mathcal{M}_2}(\varepsilon_j)}\\+\frac{2\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\vartheta^i_{\mathcal{M}_2}(\varepsilon_j))-\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)}\end{array}\right) \tag{14}$$

It is clear that the Eq. (14) has the following axioms.

(1) $\mathfrak{C}(\widetilde{\mathcal{M}}_1,\widetilde{\mathcal{M}}_1)=\mathfrak{T}(\widetilde{\mathcal{M}}_1)$.
(2) $\mathfrak{C}(\widetilde{\mathcal{M}}_1,\widetilde{\mathcal{M}}_2)=\mathfrak{C}(\widetilde{\mathcal{M}}_2,\widetilde{\mathcal{M}}_1)$.

The correlation coefficients between two SNMRs $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ are defined as follows.

**Definition 4.1.** Let $\widetilde{\mathcal{M}}_1,\widetilde{\mathcal{M}}_2\in SNMRS(\mathcal{E},q)$. Then, the (type-1) correlation coefficient between $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ is denoted and defined as

$$\begin{aligned}\kappa_1(\mathcal{M}_1,\mathcal{M}_2) &= \frac{\mathfrak{C}(\widetilde{\mathcal{M}}_1,\widetilde{\mathcal{M}}_2)}{\sqrt{\mathfrak{T}(\widetilde{\mathcal{M}}_1)\cdot\mathfrak{T}(\widetilde{\mathcal{M}}_2)}}\\ &= \frac{\frac{1}{q}\sum_{i\in I_q}\sum_{j=1}^{m}\left(\begin{array}{c}\frac{2\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\zeta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\zeta^i_{\mathcal{M}_2}(\varepsilon_j))-\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}\\+\frac{2\eta^i_{\mathcal{M}_1}(\varepsilon_j)\eta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\eta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\eta^i_{\mathcal{M}_2}(\varepsilon_j))-\eta^i_{\mathcal{M}_1}(\varepsilon_j)\eta^i_{\mathcal{M}_2}(\varepsilon_j)}\\+\frac{2\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\vartheta^i_{\mathcal{M}_2}(\varepsilon_j))-\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)}\end{array}\right)}{\sqrt{\frac{1}{q}\sum_{i\in I_q}\sum_{j=1}^{m}\left(\begin{array}{c}\frac{\zeta^i_{\mathcal{M}_1}(\varepsilon_j)}{2(1+\zeta^i_{\mathcal{M}_1}(\varepsilon_j))}\\+\frac{\eta^i_{\mathcal{M}_1}(\varepsilon_j)}{2(1+\eta^i_{\mathcal{M}_1}(\varepsilon_j))}\\+\frac{\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)}{2(1+\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))}\end{array}\right)}\times\sqrt{\frac{1}{q}\sum_{i\in I_q}\sum_{j=1}^{m}\left(\begin{array}{c}\frac{\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}{2(1+\zeta^i_{\mathcal{M}_2}(\varepsilon_j))}\\+\frac{\eta^i_{\mathcal{M}_2}(\varepsilon_j)}{2(1+\eta^i_{\mathcal{M}_2}(\varepsilon_j))}\\+\frac{\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)}{2(1+\vartheta^i_{\mathcal{M}_2}(\varepsilon_j))}\end{array}\right)}}\end{aligned} \tag{15}$$

**Theorem 4.2.** *Let $\widetilde{\mathcal{M}}_1,\widetilde{\mathcal{M}}_2\in SNMRS(\mathcal{E},q)$. For the (type-1) correlation coefficient between $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$, the following properties are satisfied.*

**(i):** $\widetilde{\mathcal{M}}_1=\widetilde{\mathcal{M}}_2\Rightarrow\kappa_1(\mathcal{M}_1,\mathcal{M}_2)=1$.
**(ii):** $\kappa_1(\mathcal{M}_1,\mathcal{M}_2)=\kappa_1(\mathcal{M}_2,\mathcal{M}_1)$.
**(iii):** $\frac{1}{9}\leq\kappa_1(\mathcal{M}_1,\mathcal{M}_2)\leq 9$.

*Proof.* The proofs of (i) and (ii) are obvious from the Eq. (15). Let us prove the assertion (iii).
**(iii):** Let $\widetilde{\mathcal{M}}_1,\widetilde{\mathcal{M}}_2\in SNMRS(\mathcal{E},q)$. Since for each $\varepsilon_j\in\mathcal{E}$,

$$\frac{1}{9}\leq\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\leq 9\quad\forall i\in I_q \tag{16}$$

it implies that

$$\frac{1}{20}\leq\frac{1}{2(1+\zeta^i_{\mathcal{M}_1}(\varepsilon_j))}\leq\frac{9}{20}\quad\forall i\in I_q \tag{17}$$

Hence, we obtain

$$\frac{(\zeta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{20} \leq \frac{(\zeta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{2(1+\zeta^i_{\mathcal{M}_1}(\varepsilon_j))} \leq \frac{9(\zeta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{20} \quad \forall i \in I_q \tag{18}$$

for each $\varepsilon_j \in \mathcal{E}$. Likewise, for the indeterminacy-membership and falsity-membership, we obtain the following inequalities:

$$\frac{(\eta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{20} \leq \frac{(\eta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{2(1+\eta^i_{\mathcal{M}_1}(\varepsilon_j))} \leq \frac{9(\eta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{20} \quad \forall i \in I_q \tag{19}$$

and

$$\frac{(\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{20} \leq \frac{(\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{2(1+\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))} \leq \frac{9(\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{20} \quad \forall i \in I_q \tag{20}$$

for each $\varepsilon_j \in \mathcal{E}$. By adding Eqs. (18), (19) and (20), we have

$$\begin{aligned}\frac{(\zeta^i_{\mathcal{M}_1}(\varepsilon_j))^2+(\eta^i_{\mathcal{M}_1}(\varepsilon_j))^2+(\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{20} &\leq \frac{(\zeta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{2(1+\zeta^i_{\mathcal{M}_1}(\varepsilon_j))}+\frac{(\eta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{2(1+\eta^i_{\mathcal{M}_1}(\varepsilon_j))}+\frac{(\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))^2}{2(1+\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))}\\ &\leq \frac{9((\zeta^i_{\mathcal{M}_1}(\varepsilon_j))^2+(\eta^i_{\mathcal{M}_1}(\varepsilon_j))^2+(\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))^2)}{20} \quad \forall i \in I_q \end{aligned} \tag{21}$$

for each $\varepsilon_j \in \mathcal{E}$. By using Eq. (12), we have the following inequality for informational energy of SNMRS $\mathcal{M}_1$.

$$\frac{1}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\begin{pmatrix}(\zeta^i_{\mathcal{M}_1}(\varepsilon_j))^2\\+(\eta^i_{\mathcal{M}_1}(\varepsilon_j))^2\\+(\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))^2\end{pmatrix} \leq \mathfrak{T}(\widetilde{\mathcal{M}}_1) \leq \frac{9}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\begin{pmatrix}(\zeta^i_{\mathcal{M}_1}(\varepsilon_j))^2\\+(\eta^i_{\mathcal{M}_1}(\varepsilon_j))^2\\+(\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))^2\end{pmatrix} \tag{22}$$

Similarly, we can obtain the following inequality for informational energy of SNMRS $\mathcal{M}_2$.

$$\frac{1}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\begin{pmatrix}(\zeta^i_{\mathcal{M}_2}(\varepsilon_j))^2\\+(\eta^i_{\mathcal{M}_2}(\varepsilon_j))^2\\+(\vartheta^i_{\mathcal{M}_2}(\varepsilon_j))^2\end{pmatrix} \leq \mathfrak{T}(\widetilde{\mathcal{M}}_2) \leq \frac{9}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\begin{pmatrix}(\zeta^i_{\mathcal{M}_2}(\varepsilon_j))^2\\+(\eta^i_{\mathcal{M}_2}(\varepsilon_j))^2\\+(\vartheta^i_{\mathcal{M}_2}(\varepsilon_j))^2\end{pmatrix} \tag{23}$$

On the other hand, we can easily deduce that

$$\frac{\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}{20} \leq \frac{2\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\zeta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\zeta^i_{\mathcal{M}_2}(\varepsilon_j))-\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)} \leq \frac{9\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}{20} \quad \forall i \in I_q \tag{24}$$

for each $\varepsilon_j \in \mathcal{E}$, and so

$$\begin{aligned}\frac{1}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j) &\leq \frac{1}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\frac{2\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\zeta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\zeta^i_{\mathcal{M}_2}(\varepsilon_j))-\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}\\ &\leq \frac{9}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)\end{aligned} \tag{25}$$

Likewise, for the indeterminacy-membership and falsity-membership, the following results can be obtained:

$$\frac{1}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\eta^i_{\mathcal{M}_1}(\varepsilon_j)\eta^i_{\mathcal{M}_2}(\varepsilon_j) \leq \frac{1}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\frac{2\eta^i_{\mathcal{M}_1}(\varepsilon_j)\eta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\eta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\eta^i_{\mathcal{M}_2}(\varepsilon_j))-\eta^i_{\mathcal{M}_1}(\varepsilon_j)\eta^i_{\mathcal{M}_2}(\varepsilon_j)} \leq \frac{9}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\eta^i_{\mathcal{M}_1}(\varepsilon_j)\eta^i_{\mathcal{M}_2}(\varepsilon_j) \quad (26)$$

and

$$\frac{1}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)\vartheta^i_{\mathcal{M}_2}(\varepsilon_j) \leq \frac{1}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\frac{2\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\vartheta^i_{\mathcal{M}_2}(\varepsilon_j))-\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)} \leq \frac{9}{20q}\sum_{i\in I_q}\sum_{j=1}^{m}\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)\vartheta^i_{\mathcal{M}_2}(\varepsilon_j) \quad (27)$$

So, by using Eq. (15), we obtain

$$\frac{\frac{1}{20q}\xi}{\frac{9}{20q}(\sqrt{\mu}\times\sqrt{\nu})} \leq \kappa_1(\mathcal{M}_1,\mathcal{M}_2) \leq \frac{\frac{9}{20q}\xi}{\frac{1}{20q}(\sqrt{\mu}\times\sqrt{\nu})} \quad (28)$$

and so

$$\frac{1}{9}\frac{\xi}{\sqrt{\mu}\times\sqrt{\nu}} \leq \kappa_1(\mathcal{M}_1,\mathcal{M}_2) \leq 9\frac{\xi}{\sqrt{\mu}\times\sqrt{\nu}} \quad (29)$$

where

$$\xi=\sum_{i\in I_q}\sum_{j=1}^{m}\begin{pmatrix}\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)\\ +\eta^i_{\mathcal{M}_1}(\varepsilon_j)\eta^i_{\mathcal{M}_2}(\varepsilon_j)\\ +\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)\end{pmatrix},\ \mu=\sum_{i\in I_q}\sum_{j=1}^{m}\begin{pmatrix}(\zeta^i_{\mathcal{M}_1}(\varepsilon_j))^2\\ +(\eta^i_{\mathcal{M}_1}(\varepsilon_j))^2\\ +(\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))^2\end{pmatrix},\ \nu=\sum_{i\in I_q}\sum_{j=1}^{m}\begin{pmatrix}(\zeta^i_{\mathcal{M}_2}(\varepsilon_j))^2\\ +(\eta^i_{\mathcal{M}_2}(\varepsilon_j))^2\\ +(\vartheta^i_{\mathcal{M}_2}(\varepsilon_j))^2\end{pmatrix}$$

Thus, we conclude that $\frac{1}{9} \leq \kappa_1(\mathcal{M}_1,\mathcal{M}_2) \leq 9$. □

**Definition 4.3.** Let $\widetilde{\mathcal{M}}_1,\widetilde{\mathcal{M}}_2 \in SNMRS(\mathcal{E},q)$. Then, the (type-2) correlation coefficient between $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ is denoted and defined as

$$\kappa_2(\mathcal{M}_1,\mathcal{M}_2) = \frac{\mathfrak{C}(\widetilde{\mathcal{M}}_1,\widetilde{\mathcal{M}}_2)}{\max\{\mathfrak{T}(\widetilde{\mathcal{M}}_1),\mathfrak{T}(\widetilde{\mathcal{M}}_2)\}} = \frac{\frac{1}{q}\sum_{i\in I_q}\sum_{j=1}^{m}\begin{pmatrix}\frac{2\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\zeta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\zeta^i_{\mathcal{M}_2}(\varepsilon_j))-\zeta^i_{\mathcal{M}_1}(\varepsilon_j)\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}\\ +\frac{2\eta^i_{\mathcal{M}_1}(\varepsilon_j)\eta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\eta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\eta^i_{\mathcal{M}_2}(\varepsilon_j))-\eta^i_{\mathcal{M}_1}(\varepsilon_j)\eta^i_{\mathcal{M}_2}(\varepsilon_j)}\\ +\frac{2\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)}{(2+\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))(2+\vartheta^i_{\mathcal{M}_2}(\varepsilon_j))-\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)}\end{pmatrix}}{\max\left\{\frac{1}{q}\sum_{i\in I_q}\sum_{j=1}^{m}\begin{pmatrix}\frac{\zeta^i_{\mathcal{M}_1}(\varepsilon_j)}{2(1+\zeta^i_{\mathcal{M}_1}(\varepsilon_j))}\\ +\frac{\eta^i_{\mathcal{M}_1}(\varepsilon_j)}{2(1+\eta^i_{\mathcal{M}_1}(\varepsilon_j))}\\ +\frac{\vartheta^i_{\mathcal{M}_1}(\varepsilon_j)}{2(1+\vartheta^i_{\mathcal{M}_1}(\varepsilon_j))}\end{pmatrix},\frac{1}{q}\sum_{i\in I_q}\sum_{j=1}^{m}\begin{pmatrix}\frac{\zeta^i_{\mathcal{M}_2}(\varepsilon_j)}{2(1+\zeta^i_{\mathcal{M}_2}(\varepsilon_j))}\\ +\frac{\eta^i_{\mathcal{M}_2}(\varepsilon_j)}{2(1+\eta^i_{\mathcal{M}_2}(\varepsilon_j))}\\ +\frac{\vartheta^i_{\mathcal{M}_2}(\varepsilon_j)}{2(1+\vartheta^i_{\mathcal{M}_2}(\varepsilon_j))}\end{pmatrix}\right\}} \quad (30)$$

**Theorem 4.4.** *Let $\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2 \in SNMRS(\mathcal{E}, q)$. For the (type-2) correlation coefficient between $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$, the following properties are valid.*

**(i):** $\widetilde{\mathcal{M}}_1 = \widetilde{\mathcal{M}}_2 \Rightarrow \kappa_2(\mathcal{M}_1, \mathcal{M}_2) = 1$.

**(ii):** $\kappa_2(\mathcal{M}_1, \mathcal{M}_2) = \kappa_2(\mathcal{M}_2, \mathcal{M}_1)$.

**(iii):** $\frac{1}{9} \leq \kappa_2(\mathcal{M}_1, \mathcal{M}_2) \leq 9$.

*Proof.* They can be demonstrated similar to the proof of Theorem 4.2. □

## 5. An Application of Correlation Coefficients of SNMRSs in Medical Pattern Recognition

In order to demonstrate the application of the proposed correlation coefficients, we consider the following medical pattern recognition problem under the SNMRS environment.

**Example 5.1.** Scientists divided coronaviruses into four sub-groupings, called alpha, beta, gamma and delta. Five of beta viruses can infect people: OC43, HKU1, MERS-CoV, SARS-CoV and SARS-CoV-2 (COVID-19). Specially, we focus on three dangerous types of beta viruses: (1) MERS-CoV, (2) SARS-CoV and (3) SARS-CoV-2. We consider the patterns of MERS-CoV, SARS-CoV and SARS-CoV-2 based on the symptoms which are specified by experts as sequences of truth-membership information, indeterminacy-membership information and falsity-membership information (they are scaled between $\frac{1}{9}$ and 9) as a result of investigation and experiments. Suppose that the patterns of MERS-CoV, SARS-CoV and SARS-CoV-2 for the symptoms $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3$ (i.e., $\mathcal{E} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$) are given as follows respectively.

$$\mathcal{P}_1 = \left\{ \begin{array}{l} (\varepsilon_1, \langle (2,4,5,7,8), (\frac{3}{5}, 1, 2, \frac{5}{2}, 6), (\frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \frac{1}{8}, \frac{1}{9}) \rangle), \\ (\varepsilon_2, \langle (\frac{2}{9}, \frac{1}{2}, 1, 3, 4), (\frac{1}{4}, 2, \frac{3}{4}, 1, 5), (\frac{1}{8}, 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}) \rangle), \\ (\varepsilon_3, \langle (\frac{1}{9}, \frac{1}{3}, \frac{1}{2}, 1, 4), (\frac{1}{5}, 7, \frac{2}{3}, \frac{1}{5}, \frac{1}{2}), (9, 3, 2, 1, \frac{1}{6}) \rangle) \end{array} \right\},$$

$$\mathcal{P}_2 = \left\{ \begin{array}{l} (\varepsilon_1, \langle (\frac{1}{8}, \frac{1}{2}, 2, 3, 7), (3, \frac{1}{2}, 1, \frac{1}{5}, \frac{1}{7}), (8, 1, \frac{1}{6}, \frac{1}{7}, \frac{1}{6}) \rangle), \\ (\varepsilon_2, \langle (\frac{1}{3}, \frac{1}{2}, 1, 4, 6), (\frac{1}{3}, \frac{1}{4}, \frac{1}{8}, 2, \frac{1}{3}), (6, 2, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}) \rangle), \\ (\varepsilon_3, \langle (\frac{1}{3}, \frac{1}{2}, 2, 3, 9), (6, \frac{1}{2}, \frac{1}{3}, 4, \frac{1}{7}), (1, 2, \frac{1}{4}, \frac{1}{5}, \frac{1}{9}) \rangle) \end{array} \right\}$$

and

$$\mathcal{P}_3 = \left\{ \begin{array}{l} (\varepsilon_1, \langle (1,4,5,6,9), (\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, 2, 1), (\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}) \rangle), \\ (\varepsilon_2, \langle (\frac{1}{2}, 2, 3, 5, 8), (\frac{1}{4}, \frac{1}{9}, 1, 3, 2), (\frac{1}{6}, \frac{1}{2}, \frac{1}{4}, \frac{1}{7}, \frac{1}{8}) \rangle), \\ (\varepsilon_3, \langle (\frac{1}{3}, \frac{1}{2}, 1, 2, 3), (\frac{1}{4}, \frac{1}{7}, 1, \frac{1}{2}, 1), (2, 2, \frac{1}{6}, \frac{1}{6}, \frac{1}{9}) \rangle) \end{array} \right\}.$$

Experts (or doctors) often come across slightly different versions (i.e., unknown patterns) of viruses: MERS-CoV, SARS-CoV and SARS-CoV-2. Suppose that an expert come across an unknown pattern $\mathcal{P}$ which will be reorganized as an SNMRS in $\mathcal{E}$, where

$$\mathcal{P} = \left\{ \begin{array}{l} (\varepsilon_1, \langle (2,4,5,7,9), (\frac{1}{3}, \frac{1}{5}, \frac{1}{8}, 2, 2), (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{7}, \frac{1}{9}) \rangle), \\ (\varepsilon_2, \langle (\frac{1}{4}, \frac{1}{2}, 3, 4, 9), (\frac{1}{2}, \frac{1}{9}, 1, 3, 4), (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{9}) \rangle), \\ (\varepsilon_3, \langle (\frac{1}{5}, \frac{1}{3}, 1, 2, 4), (\frac{1}{4}, \frac{1}{8}, 1, \frac{1}{2}, 1), (1, 2, \frac{1}{6}, \frac{1}{6}, \frac{1}{9}) \rangle) \end{array} \right\}.$$

The motivation of this problem is to classify the pattern $\mathcal{P}$ in one of the classes $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$. For this purpose, the correlation coefficients $\kappa_1$ and $\kappa_2$ described in Eqs. (4.1) and (4.3) can be used.
By calculating the the (type-1) correlation coefficients between $\mathcal{P}$ and $\mathcal{P}_k$ $(k = 1, 2, 3)$, we can get

$$\kappa_1(\mathcal{P}_1, \mathcal{P}) = 2.507027, \kappa_1(\mathcal{P}_2, \mathcal{P}) = 2.078367 \text{ and } \kappa_1(\mathcal{P}_3, \mathcal{P}) = 2.727818$$

As a result of (type-1) correlation coefficients, the ranking of $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ is obtained as $\mathcal{P}_2 \prec \mathcal{P}_1 \prec \mathcal{P}_3$, and thus it is most convenient to classify the pattern $\mathcal{P}$ with the pattern $\mathcal{P}_3$ (SARS-CoV-2).
Similarly, by using Eq. (4.3), we have the following (type-2) correlation coefficients between $\mathcal{P}$ and $\mathcal{P}_k$ $(k = 1, 2, 3)$

$$\kappa_2(\mathcal{P}_1, \mathcal{P}) = 2.365829, \kappa_2(\mathcal{P}_2, \mathcal{P}) = 2.032161 \text{ and } \kappa_2(\mathcal{P}_3, \mathcal{P}) = 2.718598$$

Consequently, the ranking of these three patterns is $\mathcal{P}_2 \prec \mathcal{P}_1 \prec \mathcal{P}_3$, and therefore it is most convenient to classify the pattern $\mathcal{P}$ with the pattern $\mathcal{P}_3$.

*Comparison and Discussion:* In 2018, Garg [14] proposed new correlation coefficients for IMSs and presented their applications in handling decision making. For Examples 1, 2 and 3 in Section 4 of [14], if we assume the 1-dimension simplified neutrosophic multiplicative refined value (i.e., simplified neutrosophic multiplicative value) $\langle \rho, 1, \sigma \rangle$ instead of the priority value $\langle \rho, \sigma \rangle$ of alternative under the IMS environment then the proposed (type-1 and type-2) correlation coefficients (in this paper) can be applied to these problems and the comparison results in Table 2 are obtained.

Table 2. Results of comparing the proposed ones with the correlation coefficients of IMSs

| Problems | Ranking for correlation coefficients of IMSs | Ranking for correlation coefficients of SNMRSs |
|---|---|---|
| Example 1 in [14] | $K_1(X_4, X^*) > K_1(X_1, X^*) > K_1(X_3, X^*) > K_1(X_2, X^*)$ | $\kappa_1(X_4, X^*) > \kappa_1(X_1, X^*) > \kappa_1(X_3, X^*) > \kappa_1(X_2, X^*)$ |
| | $K_2(X_4, X^*) > K_2(X_1, X^*) > K_2(X_3, X^*) > K_2(X_2, X^*)$ | $\kappa_2(X_4, X^*) > \kappa_2(X_1, X^*) > \kappa_2(X_3, X^*) > \kappa_2(X_2, X^*)$ |
| Example 2 in [14] | $K_1(C_2, P) > K_1(C_1, P) > K_1(C_3, P)$ | $\kappa_1(C_2, P) > \kappa_1(C_1, P) > \kappa_1(C_3, P)$ |
| | $K_2(C_2, P) > K_2(C_1, P) > K_2(C_3, P)$ | $\kappa_2(C_2, P) > \kappa_2(C_1, P) > \kappa_2(C_3, P)$ |
| Example 3 in [14] | $K_1(P, Q_2) > K_1(P, Q_1) > K_1(P, Q_5) > K_1(P, Q_3) > K_1(P, Q_4)$ | $\kappa_1(P, Q_2) > \kappa_1(P, Q_1) > \kappa_1(P, Q_5) > \kappa_1(P, Q_3) > \kappa_1(P, Q_4)$ |
| | $K_2(P, Q_2) > K_2(P, Q_5) > K_2(P, Q_1) > K_2(P, Q_3) > K_2(P, Q_4)$ | $\kappa_2(P, Q_2) > \kappa_2(P, Q_5) > \kappa_2(P, Q_1) > \kappa_2(P, Q_3) > \kappa_2(P, Q_4)$ |

In 2016, Broumi and Deli [7] studied the correlation measure of (simplified) neutrosophic refined sets and applied them to the problems of medical diagnosis and pattern recognition. For Examples 4.1 and 4.2 in Section 4 of [7], considering the matches between $0 - 1$ and $\frac{1}{9} - 9$ scales given in Table 1 in the Introduction for the priority value $\langle (T^1, T^2, ..., T^p), (I^1, I^2, ..., I^p), (F^1, F^2, ..., F^p) \rangle$ of alternative under the (simplified) NRS environment, we can apply the proposed (type-1 and type-2) correlation coefficients to these problems and the comparison results are presented in Table 3.

Table 3. Results of comparing the proposed ones with the correlation coefficients of NRSs

| Problems | Ranking for correlation coefficient of NRSs | Ranking for correlation coefficients of SNMRSs |
|---|---|---|
| Example 4.1 in [7] | $\rho_{NRS}(P_1, D_2) > \rho_{NRS}(P_1, D_3) > \rho_{NRS}(P_1, D_4) > \rho_{NRS}(P_1, D_1)$ | $\kappa_1(P_1, D_2) > \kappa_1(P_1, D_3) > \kappa_1(P_1, D_4) > \kappa_1(P_1, D_1)$ |
| | $\rho_{NRS}(P_2, D_3) > \rho_{NRS}(P_2, D_2) > \rho_{NRS}(P_2, D_1) > \rho_{NRS}(P_2, D_4)$ | $\kappa_1(P_2, D_3) > \kappa_1(P_2, D_2) > \kappa_1(P_2, D_1) > \kappa_1(P_2, D_4)$ |
| | $\rho_{NRS}(P_3, D_3) > \rho_{NRS}(P_3, D_2) > \rho_{NRS}(P_3, D_4) > \rho_{NRS}(P_3, D_1)$ | $\kappa_1(P_3, D_3) > \kappa_1(P_3, D_2) > \kappa_1(P_3, D_4) > \kappa_1(P_3, D_1)$ |
| Example 4.2 in [7] | $\rho_{NRS}(Pat.I, Pat.III) > \rho_{NRS}(Pat.II, Pat.III)$ | $\kappa_1(Pat.I, Pat.III) > \kappa_1(Pat.II, Pat.III)$ <br> $\kappa_2(Pat.I, Pat.III) > \kappa_2(Pat.II, Pat.III)$ |

Consequently, we can say that the correlation coefficients of SNMRSs are generalized forms of correlation coefficients of both IMSs and NRSs (by considering Table 1 in the Introduction). These support that the range of application areas of the proposed correlation coefficients is quite wide and therefore advantageous in many situations.

## 6. Conclusions

In this paper, we have established a new extension of SNMS named as SNMRS which is more efficient and flexible structure to deal with ambiguity. The space for SNMRSs is larger than those of IMSs and SNMSs. We have founded some significant results in the framework of SNMRS. We have presented new correlation coefficients under the SNMRS environment and their application in medical pattern recognition. We hope that the findings in this study will be helpful for researchers handling with various real-life problems that involve uncertainties. Further, the proposed approaches may be extended in new directions including information fusion, aggregation and measures. The next research will aim to explore the real-life applications related to the concepts based on the extensions of SNMRSs.

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# Extension of TOPSIS Method under Single-Valued Neutrosophic $N$-Soft Environment

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**Abstract:**
In this paper, we discuss aggregation operators for single-valued neutrosophic $N$-soft numbers. Further, we develop single-valued neutrosophic $N$-soft TOPSIS method based on single-valued neutrosophic $N$-soft aggregate operators in order to cumulate the decisions of all experts according to the worth of experts' opinion and parameters related to each alternative. For the final decision, we use revised closeness index depending upon the distance measures of alternatives from single-valued neutrosophic $N$-soft positive ideal solution and single-valued neutrosophic $N$-soft negative ideal solution. A numerical example is described to illustrate the importance of the proposed method. A comparison of single-valued neutrosophic $N$-soft TOPSIS method with single-valued neutrosophic TOPSIS method ensures the significance and trustworthiness of the proposed model.

**Keywords:** $N$-soft set, single-valued neutrosophic $N$-soft sets, TOPSIS method, MAGDM.

## 1 Introduction

In many field of life, the evaluation process is certainly switch from binary evaluation ($\{0, 1\}$) to non-binary evaluation ($\{0, 1, \ldots, N-1\}$), that is, we are using the system of 5-stars, 4-stars or 3-stars instead of yes or no, in many disciplines of mathematical social sciences. Keeping in view the importance of ranking system, Fatima et al. [7] introduced $N$-soft sets and decision making methods to handle problems basis on non-binary evaluations. Apparently, $N$-soft set is an extension of soft set presented by Molodtsov [4], described all type of parametrization, while in $N$-soft sets grades are assigned to the parameters that actually representing the level of alternatives with respect to the attributes. Further, Akram et al. [28, 30] extended the concept of $N$-soft sets to fuzzy $N$-soft sets and intuitionistic fuzzy $N$-soft sets $(IFNS_fS)$. The intuitionistic fuzzy $N$-soft set is describing the level of alternatives as well as the degree of membership and non-membership with their sum less than equal to zero. The Pythagorean fuzzy set (PFS) was firstly presented by Yager [39] in which squares sum of degree of membership and non-membership should not exceed one. Zhang [21] introduced the notion of Pythagorean fuzzy $N$-soft sets $(PFNS_fS)$.
Human decision nature has indeterminacy within the judgments of yes or no that is actually prescribed the indecision for the related object. Since the PFSs and IFSs are not able to handle such part of decision nature independently, with limited range. Therefore, PFS and IFS will not be applicable. This is the origin of neutrosophic sets (NSs) presented by Smarandache [13] in 1999. Later on, Wang et al. [20] developed the concept of single-valued neutrosophic sets (SVNSs) to deal real life scientific problems having indeterminate information.Moreover, Singh [35, 36] presented theory of three-way and multi-granular based $n$-valued neutrosophic logics introduced by Smarandache [15] in 2014. On the other hand, Maji [34] and Jana et al. [2] combined the concept of soft sets with NSs and SVNSs, respectively. Many researchers work on TOPSIS method, like Chen [3], Chu and Kysely [41] and Alguliyev [38] extended the TOPSIS method in fuzzy environment for solving multi-attribute group decision making problems. Moreover, Gupta et al. [33] and Shen et al. [12] introduced the extended version of intuitionistic fuzzy TOPSIS method. Akram et al. [31, 29] developed a theoretical description for the Pythagorean fuzzy TOPSIS method. Similarly, and also motivated by SVNSs , Sahin and Yigider [40] used a single-valued neutrosophic-TOPSIS method to find the best supplier for production industry. Riaz et al. [32] being inspired by $N$-soft sets, presented a model of neutrosophic $N$-soft sets $(NNS_fSs)$ with TOPSIS method that used relations and composition for evaluating the $NNS_f$ positive ideal solution and negative ideal solution. They used similarity measures and choice function for solving MADM problem in medical diagnosis. In this paper, we discuss aggregation operators for single-valued neutrosophic $N$-soft numbers. Further, we develop single-valued neutrosophic $N$-soft TOPSIS method based on single-valued neutrosophic $N$-soft aggregate operators in order to cumulate the decisions of all experts according to the worth of experts' opinion and parameters related to each alternative. For the final decision, we use revised closeness index depending upon the distance measures of alternatives from single-valued neutrosophic $N$-soft positive ideal solution and single-valued neutrosophic $N$-soft negative ideal solution. A numerical example is described to illustrate the importance of the proposed method.
The rest of the paper is organized as follows: In Section 2, we represent the concept of $SVNNS_fS$ with related example. In Section 3, we define $SVNNS_fN$ with some properties and operations, like score function, accuracy function, comparison between two $SVNNS_fNs$, sum and

product of $SVNNS_fNs$, inclusively. Section 4, describes intellectual basics for the $SVNNS_fS$-TOPSIS method for solving real life problems within an algorithm. Section 5, presenting a MAGDM problem, which is sorted out using $SVNNS_fS$-TOPSIS. In Section 6, we compare the proposed model with the SVN-TOPSIS method. In Section 7 we give conclusions about the paper and future directions for research.

**Definition 1.** *[13] Let $Y$ be non-empty set. A neutrosophic set $(NS)$ $\rho$ over the universe of discourse $Y$ is defined as:*

$$\rho = \langle y, \beta_\rho(y), \gamma_\rho(y), \delta_\rho(y) : y \in Y\rangle,$$

*where, $\beta_\rho(y)$, $\gamma_\rho(y)$ and $\delta_\rho(y)$ are degree of satisfaction, degree of indeterminacy and degree of dissatisfaction, respectively, belongs to non-standard interval $]^-0, 1^+[$, for every $y \in Y$.*

**Definition 2.** *[20] Let $Y$ be non-empty set. A single-valued neutrosophic set $(SVNS)$ $\rho$ over the universe of discourse $Y$ is defined as:*

$$\rho = \langle y, \beta_\rho(y), \gamma_\rho(y), \delta_\rho(y) : y \in Y\rangle,$$

*where, $\beta_\rho(y)$, $\gamma_\rho(y)$ and $\delta_\rho(y) \in [0, 1]$. For every $y \in Y$, $\beta_\rho(y)$, $\gamma_\rho(y)$ and $\delta_\rho(y)$, the degree of the satisfaction, degree of indeterminacy and degree of dissatisfaction, respectively, without any restriction on $\beta_\rho(y)$, $\gamma_\rho(y)$ and $\delta_\rho(y)$ or we can say that for all $y \in Y$,*

$$0 \leq \beta_\rho(y) + \gamma_\rho(y) + \delta_\rho(y) \leq 3.$$

*The triplet $(\beta_\rho(y), \gamma_\rho(y), \delta_\rho(y))$ is called single-valued neutrosophic number $(SVNN)$.*

**Definition 3.** *[4] Let $X$ be a non-empty set and $E \subseteq A$, $A$ be a set of parameters. A pair $(\urcorner, E)$ is called soft set $S_fS$ over $X$ denoted as:*

$$(\urcorner, E) = \{\langle e_i, \urcorner(e)\rangle : \forall e_i \in E\},$$

*if $\urcorner : E \to P(X)$, where $P(X)$ represents the family of all subsets of $X$.*

**Definition 4.** *Let $X$ be a non-empty set and $E \subseteq A$, $A$ be a set of parameters. A pair $(\Upsilon, E)$ is called single-valued neutrosophic soft set $(SVNS_fS)$ over $X$, if $\Upsilon : E \to \mathcal{P}(X)$ is a mapping, which is denoted as:*

$$\Upsilon(e_i) = \{\langle x_j, (\beta_{ij}, \gamma_{ij}, \delta_{ij})\rangle : x_j \in X\},$$

*where, $\mathcal{P}(X)$ represents the family of all SVNSs over $X$ and $\beta_{ij}, \gamma_{ij}, \delta_{ij}$, which belongs to unit closed interval, are satisfying the condition*

$$0 \leq \beta_{ij} + \gamma_{ij} + \delta_{ij} \leq 3, \forall x_j \in X.$$

**Definition 5.** *[7] Let $X$ be a non-empty set and $E \subseteq A$, $A$ be a set of parameters. Let $O = \{0, 1, 2, \ldots, N-1\}$ be a set of ordered grades with $N \in \{2, 3, \ldots\}$. A triple $(H, E, N)$ is called $N$-soft set $(NS_fS)$ over $X$ if $H : E \to 2^{U\times G}$ is a mapping , with the property that for each $e_i \in E$ and $x_j \in X$ there exist a unique $(x_j, o_i^j) \in X \times O$ such that $(x_j, o_i^j) \in H(e_j), x_j \in X, o_i^j \in O$.*

## 2 Single-valued neutrosophic $N$-soft numbers

**Definition 6.** *Let $X$ be a non-empty set and $E \subseteq A$, $A$ be a set of parameters. Let $O = \{0, 1, 2, \ldots, N-1\}$ be a set of ordered grades with $N \in \{2, 3, \ldots\}$. Let $H : E \to 2^{X\times O}$ be an $NS_fS$ on $X$, and $T : E \to \mathcal{P}(SVNN)$, be a mapping, that $\mathcal{P}(SVNN)$ denotes the collection of single-valued neutrosophic numbers of $X$, then a triple $(H_T, E, N)$ is called a single-valued neutrosophic $N$-soft set $(SVNNS_fS)$ on $X$, if $H_T : E \to (2^{X\times O} \times \mathcal{P}(SVNN))$ is a mapping, which is defined as:*

$$\begin{aligned} H_T(e_i) = & \{\langle (H(e_i), T(e_i))\rangle : e_i \in E, H(e_i) \in 2^{W\times G}, T(e_i) \times \mathcal{P}(SVNN)\}, \\ = & \{\langle ((x_j, o_i^j), (\beta_{e_i}(x_j), \gamma_{e_i}(x_j), \delta_{e_i}(x_j)))\rangle\}, \\ = & \{\langle ((x_j, o_i^j), (\beta_{ij}, \gamma_{ij}, \delta_{ij}))\rangle\}, \end{aligned}$$

*where, $o_i^j$ denotes the level of attribute for the element $x_j$ and $\beta_{ij}, \gamma_{ij}, \delta_{ij} \in [0, 1]$, satisfying the condition*

$$0 \leq \beta_{ij} + \gamma_{ij} + \beta_{ij} \leq 3, \text{for all } x_j \text{ belongs to } X.$$

**Example 1.** *Mr. and Mrs. Bean decided to gift their child a bicycle on his 17th birthday because he needed a conveyance to go to college. For this purpose, they visited plenty of websites online, among these websites they found a website named as "Cycling weekly". This website provided ratings of bicycles according to the parameters filtered by Mr. and Mrs. Bean. For the selection of a best bicycle based on ratings, we will use* $SVNNS_fS$.
*Let* $X = \{x_1 =$ *Merida Mission Road 7000-E,* $x_2 =$ *Bianchi Infinity XE Ultegra Disc,* $x_3 =$ *Strider 12,* $x_4 =$ *Scott Iddict RC Pro,* $x_5 =$ *Willier Cento 10 SL* $\}$ *be the set of five bicycles and the set of parameters be* $E = \{e_1 =$ *Framework (stiffness and comfort frame),* $e_2 =$ *weight,* $e_3 =$ *Shape and quality,* $e_4 =$ *Cost price* $\}$. *Following the ratings of bicycles according to the parameters, a 6-soft set is organized in Table 1, where*

*Five checkmarks means 'Infinitely Good',*

*Four checkmarks means 'Extremely Good',*

*Three checkmarks means 'Good',*

*Two checkmarks means 'Bad',*

*One checkmarks means 'Extremely Bad',*

*Big dot means 'Infinitely Bad'*

*This level assessment by checkmarks can be represented by numbers as* $O = \{0, 1, 2, 3, 4, 5\}$, *where*

0 *means* "$\bullet$",

1 *means* "✓",

2 *means* "✓✓",

3 *means* "✓✓✓",

4 *means* "✓✓✓✓".

5 *means* "✓✓✓✓✓".

Table 1: Evaluation data provided by the Website

| $X/E$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $x_1$ | ✓ | ✓ | ✓✓✓ | ✓✓✓ |
| $x_2$ | ✓✓ | $\bullet$ | ✓✓✓ | ✓✓✓✓ |
| $x_3$ | ✓✓✓✓ | ✓✓✓ | ✓✓✓✓ | ✓✓✓✓✓ |
| $x_4$ | ✓✓✓✓✓ | ✓✓✓✓ | ✓✓✓✓✓ | ✓✓✓ |
| $x_5$ | ✓✓✓ | ✓✓ | ✓✓✓✓✓ | ✓✓✓✓ |

*Table 2 can be adopted as natural convention of 5-soft set model.*

Table 2: A 6-soft set

| $X/E$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $x_1$ | 1 | 1 | 3 | 3 |
| $x_2$ | 2 | 0 | 3 | 4 |
| $x_3$ | 4 | 3 | 4 | 5 |
| $x_4$ | 5 | 4 | 5 | 3 |
| $x_5$ | 3 | 2 | 5 | 4 |

*In coalition with the Definition 6, we describe for example* $(x_3, o_2^3 = 3) \in H(e_2)$ *and* $(x_5, o_4^5 = 4) \in H(e_4)$. *This form of data is enough when it is extracted from real data, however, when there is ambiguity in the data and experts wants to describe the viewpoint of customers based on their satisfaction, hesitancy and dissatisfaction then we* $SVNNS_fSs$ *are appropriate which provide us information, how these grades are given* *to bicycles. The evaluation of bicycles follow this grading criteria;*

$$\begin{aligned}
&when\ o_i^j = 0, && -1.000 \leq S_T < -0.787,\\
&when\ o_i^j = 1, && -0.787 \leq S_T < -0.400,\\
&when\ o_i^j = 2, && -0.400 \leq S_T < 0.000,\\
&when\ o_i^j = 3, && 0.000 \leq S_T < 0.400,\\
&when\ o_i^j = 4, && 0.400 \leq S_T < 0.787,\\
&when\ o_i^j = 5, && 0.787 \leq S_T < 1.000.
\end{aligned}$$

*According to above grading criteria, we can obtain Table 3.*

Table 3: Grading criteria

| $o_i^j/T$ | Satisfaction degree | Indeterminacy degree | Dissatisfaction degree |
|---|---|---|---|
| grades | $\beta_{ij}$ | $\gamma_{ij}$ | $\delta_{ij}$ |
| $o_i^j = 0$ | $[0.00, 0.15]$ | $[0, 0.450)$ | $[0.90, 1.00)$ |
| $o_i^j = 1$ | $[0.15, 0.30)$ | $(0, 0.020)$ | $(0.70, 0.90)$ |
| $o_i^j = 2$ | $[0.30, 0.50)$ | $[0, 0.140)$ | $(0.50, 0.70]$ |
| $o_i^j = 3$ | $[0.50, 0.70)$ | $(0, 0.070]$ | $[0.30, 0.50]$ |
| $o_i^j = 4$ | $(0.70, 0.90]$ | $[0, 0.070)$ | $[0.15, 0.30)$ |
| $o_i^j = 5$ | $(0.90, 1.00]$ | $[0, 0.017)$ | $[0.00, 0.15)$ |

*Using Table 3 and Definition 6, a $SVN6S_fS$ that is also arranged in Table 4, is defined as:*

$$\begin{aligned}
(\beta_{e_1}, \gamma_{e_1}, \delta_{e_1}) &= \{((x_1, 1), (0.160, 0.300, 0.870)), ((x_2, 2), (0.320, 0.015, 0.600)),\\
&\quad ((x_3, 4), (0.750, 0.012, 0.170)), ((x_4, 5), (0.950, 0.011, 0.120)),\\
&\quad ((x_5, 3), (0.550, 0.030, 0.420))\} \in SVNNS_fS,\\
(\beta_{e_2}, \gamma_{e_2}, \delta_{e_2}) &= \{((x_1, 1), (0.270, 0.017, 0.710)), ((x_2, 0), (0.120, 0.300, 0.950)),\\
&\quad ((x_3, 3), (0.560, 0.012, 0.380)), ((x_4, 4), (0.870, 0.025, 0.230)),\\
&\quad ((x_5, 2), (0.400, 0.120, 0.620))\} \in SVNNS_fS,\\
(\beta_{e_3}, \gamma_{e_3}, \delta_{e_3}) &= \{((x_1, 3), (0.520, 0.020, 0.350)), ((x_2, 3), (0.650, 0.010, 0.370)),\\
&\quad ((x_3, 4), (0.760, 0.033, 0.210)), ((x_4, 5), (0.970, 0.013, 0.040)),\\
&\quad ((x_5, 5), (0.920, 0.014, 0.14))\} \in SVNNS_fS,
\end{aligned}$$

$$\begin{aligned}
(\beta_{e_4}, \gamma_{e_4}, \delta_{e_4}) &= \{((x_1, 3), (0.550, 0.030, 0.360)), ((x_2, 4), (0.750, 0.032, 0.200)), && (1)\\
&\quad ((x_3, 5), (0.910, 0.016, 0.140)), ((x_4, 3), (0.660, 0.017, 0.360)), && (2)\\
&\quad ((x_5, 4), (0.780, 0.040, 0.290))\} \in SVNNS_fS. && (3)\\
& && (4)
\end{aligned}$$

**Definition 7.** *Let $H_T(e_i) = \{\langle((x_j, o_i^j), (\beta_{ij}, \gamma_{ij}, \delta_{ij}))\rangle\}$ be a $SVNNS_fS$. Then the single-valued neutrosophic $N$-soft number $(SVNNS_fN)$ is defined as:*

$$\rho_{ij} = (o_i^j, (\beta_{ij}, \gamma_{ij}, \delta_{ij})),$$

*where $\beta_{ij}, \gamma_{ij}$ and $\delta_{ij}$, belong to unit interval, are the degree of membership, indeterminacy and non-membership, respectively.*

**Remark 8.** *We see that:*

1. *For $N = 2$, $SVNNS_fS$ becomes single-valued neutrosophic soft set.*
2. *When $|E| = 1$, $SVNNS_fS$ becomes single-valued neutrosophic set.*

Table 4: A $SVN6S_fS$ $(H_T, E, 6)$

| $(H_T, E, 6)$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $x_1$ | $(1, (0.160, 0.300, 0.870))$ | $(1, (0.270, 0.017, 0.710))$ | $(3, (0.520, 0.020, 0.350))$ | $(3, (0.550, 0.030, 0.360))$ |
| $x_2$ | $(2, (0.320, 0.015, 0.600))$ | $(0, (0.120, 0.300, 0.950))$ | $(3, (0.650, 0.010, 0.370))$ | $(4, (0.750, 0.032, 0.200))$ |
| $x_3$ | $(4, (0.750, 0.012, 0.170))$ | $(3, (0.560, 0.012, 0.380))$ | $(4, (0.760, 0.033, 0.210))$ | $(5, (0.910, 0.016, 0.140))$ |
| $x_4$ | $(5, (0.950, 0.011, 0.120))$ | $(4, (0.870, 0.025, 0.230))$ | $(5, (0.970, 0.013, 0.040))$ | $(3, (0.660, 0.017, 0.360))$ |
| $x_5$ | $(3, (0.550, 0.030, 0.420))$ | $(2, (0.400, 0.120, 0.620))$ | $(5, (0.920, 0.014, 0.140))$ | $(4, (0.780, 0.040, 0.290))$ |

**Definition 9.** *Consider a $SVNNS_fN$ $\rho_{ij} = (o_i^j, (\beta_{ij}, \gamma_{ij}, \delta_{ij}))$. The score function $Sc(\rho_{ij})$ is defined as:*

$$Sc(\rho_{ij}) = (\frac{o_i^j}{N-1}) + \beta_{ij} - \gamma_{ij} - \delta_{ij},$$

*where $Sc(\rho) \in [-2, 2]$. The accuracy function $Ac(\rho_{ij})$ is defined as:*

$$Ac(\rho_{ij}) = (\frac{o_i^j}{N-1}) + \beta_{ij} + \gamma_{ij} + \delta_{ij},$$

*where $Ac(\rho) \in [0, 4]$, respectively.*

**Definition 10.** *Let $\rho_{ij} = (o_i^j, (\beta_{ij}, \gamma_{ij}, \delta_{ij}))$ and $\rho_{kj} = (o_k^j, (\beta_{kj}, \gamma_{kj}, \delta_{kj}))$, be two $SVNNS_fNs$.*

1. *If $Sc(\rho_{ij}) < Sc(\rho_{kj})$, then $\rho_{ij} < \rho_{kj}$,*
2. *If $Sc(\rho_{ij}) > Sc(\rho_{kj})$, then $\rho_{ij} > \rho_{kj}$,*
3. *If $Sc(\rho_{ij}) = Sc(\rho_{kj})$, then*

   **(i)** *$Ac(\rho_{ij}) < Ac(\rho_{kj})$, then $\rho_{ij} < \rho_{kj}$,*

   **(ii)** *$Ac(\rho_{ij}) > Ac(\rho_{kj})$, then $\rho_{ij} > \rho_{kj}$,*

   **(iii)** *$Ac(\rho_{ij}) = Ac(\rho_{kj})$, then $\rho_{ij} \sim \rho_{kj}$.*

**Definition 11.** *Let $\rho_{ij} = (o_i^j, (\beta_{ij}, \gamma_{ij}, \delta_{ij}))$ and $\rho_{kj} = (o_k^j, (\beta_{kj}, \gamma_{kj}, \delta_{kj}))$ be two $SVNNS_fNs$ and $\zeta > 0$. The operations for $SVNNS_fNs$ can be defined as:*

$$\begin{aligned}
\rho_{ij} \cup \rho_{kj} &= \left(\max(o_i^j, o_k^j), (\max(\beta_{ij}, \beta_{kj}), \min(\gamma_{ij}, \gamma_{kj}), \min(\delta_{ij}, \delta_{kj}))\right), \\
\rho_{ij} \cap \rho_{kj} &= \left(\min(o_i^j, o_k^j), (\min(\beta_{ij}, \beta_{kj}), \max(\gamma_{ij}, \gamma_{kj}), \max(\delta_{ij}, \delta_{kj}))\right), \\
\zeta\rho_{ij} &= \left(o_i^j, 1 - (1 - \beta_{ij})^\zeta, \gamma_{ij}^\zeta, \delta_{ij}^\zeta\right), \\
\rho_{ij}^\zeta &= \left(o_i^j, \beta_{ij}^\zeta, 1 - (1 - \gamma_{ij})^\zeta, 1 - (1 - \delta_{ij})^\zeta\right), \\
\rho_{ij} \bigoplus \rho_{kj} &= \left(\max(o_i^j, o_k^j), \beta_{ij} + \beta_{kj} - \beta_{ij}\beta_{kj}, \gamma_{ij}\gamma_{kj}, \delta_{ij}\delta_{kj}\right), \\
\rho_{ij} \bigotimes \rho_{kj} &= \left(\min(o_i^j, o_k^j), \beta_{ij}\beta_{kj}, \gamma_{ij} + \gamma_{kj} - \gamma_{ij}\gamma_{kj}, \delta_{ij} + \delta_{kj} - \delta_{ij}\delta_{kj}\right).
\end{aligned}$$

**Definition 12.** *Let $\rho_{ij} = (o_i^j, (\beta_{ij}, \gamma_{ij}, \delta_{ij}))$ and $\rho_{kj} = (o_k^j, (\beta_{kj}, \gamma_{kj}, \delta_{kj}))$ be any two $SVNNS_fNs$, then the following properties hold:*

1. $\rho_{ij} \bigoplus \rho_{kj} = \rho_{kj} \bigoplus \rho_{ij}$,
2. $\rho_{ij} \bigotimes \rho_{kj} = \rho_{kj} \bigotimes \rho_{ij}$,
3. $\zeta\rho_{ij} \bigoplus \zeta\rho_{kj} = \zeta(\rho_{kj} \bigoplus \rho_{ij}), \zeta > 0$,
4. $\zeta_1\rho_{ij} \bigoplus \zeta_1\rho_{ij} = (\zeta_1 + \zeta_2)\rho_{ij}, \zeta_1, \zeta_2 > 0$,
5. $\rho_{ij}^\zeta \bigotimes \rho_{kj}^\zeta = (\rho_{kj} \bigotimes \rho_{ij})^\zeta, \zeta > 0$,
6. $\rho_{ij}^{\zeta_1} \bigotimes \rho_{ij}^{\zeta_1} = \rho_{ij}^{(\zeta_1+\zeta_2)}. \zeta_1, \zeta_2 > 0.$

*Proof.* 1. $\rho_{ij} \bigoplus \rho_{kj}$

$$\begin{aligned}
&= \left(\max(o_i^j, o_k^j), \beta_{ij} + \beta_{kj} - \beta_{ij}\beta_{kj}, \gamma_{ij}\gamma_{kj}, \delta_{ij}\delta_{kj}\right), \\
&= \left(\max(o_k^j, o_i^j), \beta_{kj} + \beta_{ij} - \beta_{kj}\beta_{ij}, \gamma_{kj}\gamma_{ij}, \delta_{kj}\delta_{ij}\right), \\
&= \rho_{kj} \bigoplus \rho_{ij}.
\end{aligned}$$

2. $\rho_{ij} \bigotimes \rho_{kj}$

$$\begin{aligned}
&= \left(\min(o_i^j, o_k^j), \beta_{ij}\beta_{ij}, \gamma_{ij} + \gamma_{kj} - \gamma_{ij}\gamma_{kj}, \delta_{ij} + \delta_{kj} - \delta_{ij}\delta_{kj}\right) \\
&= \left(\min(o_k^j, o_i^j), \beta_{kj}\beta_{ij}, \gamma_{kj} + \gamma_{ij} - \gamma_{kj}\gamma_{ij}, \delta_{kj} + \delta_{ij} - \delta_{kj}\delta_{ij}\right) \\
&= \rho_{kj} \bigotimes \rho_{ij}.
\end{aligned}$$

3. $\zeta\rho_{ij} \bigoplus \zeta\rho_{kj}$

$$\begin{aligned}
&= \left(o_i^j, [1-(1-\beta_{ij})^\zeta], \gamma_{ij}^\zeta, \delta_{ij}^\zeta) \bigoplus (o_k^j, [1-(1-\beta_{kj})^\zeta], \gamma_{kj}^\zeta, \delta_{kj}^\zeta\right) \\
&= \left(\max(o_i^j, o_k^j), [1-(1-\beta_{ij})^\zeta] + [1-(1-\beta_{kj})^\zeta] - [1-(1-\beta_{ij})^\zeta][1-(1-\beta_{kj})^\zeta], \gamma_{ij}^\zeta\gamma_{kj}^\zeta, \delta_{ij}^\zeta\delta_{kj}^\zeta\right) \\
&= \left(\max(o_i^j, o_k^j), [1-(1-\beta_{ij} + \beta_{kj} - \beta_{ij}\beta_{kj})^\zeta], (\gamma_{ij}\gamma_{kj})^\zeta, (\delta_{ij}\delta_{kj})^\zeta\right) \\
&= \zeta(\max(o_i^j, o_k^j), \beta_{ij} + \beta_{kj} - \beta_{ij}\beta_{kj}, \gamma_{ij}\gamma_{kj}, \delta_{ij}\delta_{kj}) \\
&= \zeta\left(\rho_{ij} \bigoplus \rho_{kj}\right).
\end{aligned}$$

4. $\zeta_1\rho_{ij} \bigoplus \zeta_1\rho_{ij}$

$$\begin{aligned}
&= \left(o_i^j, 1-(1-\beta_{ij})^{\zeta_1}, \gamma_{ij}^{\zeta_1}, \delta_{ij}^{\zeta_1}) \bigoplus (o_i^j, 1-(1-\beta_{ij})^{\zeta_2}, \gamma_{ij}^{\zeta_2}, \delta_{ij}^{\zeta_2}\right) \\
&= \left(\max(o_i^j, o_i^j), [1-(1-\beta_{ij})^{\zeta_1}] + [1-(1-\beta_{ij})^{\zeta_2}] - [1-(1-\beta_{ij})^{\zeta_1}][1-(1-\beta_{ij})^{\zeta_2}], \gamma_{ij}^{\zeta_1}\gamma_{ij}^{\zeta_2}, \delta_{ij}^{\zeta_1}\delta_{ij}^{\zeta_2}\right) \\
&= \left(o_i^j, 1-(1-\beta_{ij})^{\zeta_1+\zeta_2}, \gamma_{ij}^{\zeta_1+\zeta_2}, \delta_{ij}^{\zeta_1+\zeta_2}\right) \\
&= (\zeta_1 + \zeta_2)\rho_{ij}.
\end{aligned}$$

5. $\rho_{ij}^\zeta \bigotimes \rho_{kj}^\zeta$

$$\begin{aligned}
&= \left(o_i^j, \beta_{ij}^\zeta, [1-(1-\gamma_{ij})^\zeta], [1-(1-\delta_{ij})^\zeta]) \bigotimes (o_k^j, \beta_{kj}^\zeta, [1-(1-\gamma_{kj})^\zeta], [1-(1-\delta_{kj})^\zeta]\right) \\
&= \Big( \min(o_i^j, o_k^j), \beta_{ij}^\zeta\beta_{kj}^\zeta, [1-(1-\gamma_{ij})^\zeta] + [1-(1-\gamma_{kj})^\zeta] - [1-(1-\gamma_{ij})^\zeta][1-(1-\gamma_{kj})^\zeta] \\
&\quad , [1-(1-\delta_{ij})^\zeta] + [1-(1-\delta_{kj})^\zeta] - [1-(1-\delta_{ij})^\zeta][1-(1-\delta_{kj})^\zeta]\Big). \\
&= \left(\min(o_i^j, o_k^j), (\beta_{ij}\beta_{kj})^\zeta, [1-(1-\gamma_{ij} + \gamma_{kj} - \gamma_{ij}\gamma_{kj})^\zeta], [1-(1-\gamma_{ij} + \gamma_{kj} - \gamma_{ij}\gamma_{kj})^\zeta]\right) \\
&= (\rho_{kj} \bigotimes \rho_{ij})^\zeta.
\end{aligned}$$

6. $\rho_{ij}^{\zeta_1} \bigotimes \rho_{ij}^{\zeta_1}$

$$
\begin{aligned}
&= \Big(o_i^j, \beta_{ij}^{\zeta_1}, [1-(1-\gamma_{ij})^{\zeta_1}], [1-(1-\delta_{ij})^{\zeta_1}]\Big) \bigotimes \Big(o_k^j, \beta_{ij}^{\zeta_2}, [1-(1-\gamma_{ij})^{\zeta_2}], [1-(1-\delta_{ij})^{\zeta_2}]\Big) \\
&= \Big(\min(o_i^j, o_i^j), \beta_{ij}^{\zeta_1}\beta_{ij}^{\zeta_2}, [1-(1-\gamma_{ij})^{\zeta_1}] + [1-(1-\gamma_{ij})^{\zeta_2}] - [1-(1-\gamma_{ij})^{\zeta_1}][1-(1-\gamma_{ij})^{\zeta_2}] \\
&\quad , [1-(1-\delta_{ij})^{\zeta_1}] + [1-(1-\delta_{ij})^{\zeta_2}] - [1-(1-\delta_{ij})^{\zeta_1}][1-(1-\delta_{ij})^{\zeta_2}]\Big) \\
&= \Big(o_i^j, \beta_{ij}^{(\zeta_1+\zeta_2)}, [1-(1-\gamma_{ij})^{(\zeta_1+\zeta_2)}], [1-(1-\delta_{ij})^{(\zeta_1+\zeta_2)}]\Big) \\
&= \rho_{ij}^{(\zeta_1+\zeta_2}.
\end{aligned}
$$

□

**Definition 13.** *Let $\rho_{ij} = \rho_{ij} = (o_i^j, (\beta_{ij}, \gamma_{ij}, \delta_{ij}))$ $(i = 1, 2, \ldots, l)$ be a collection of $SVNNS_fNs$ and $\theta_i$ be the weight vectors $(WV)$ of $\rho_{ij}$ with $\theta_i > 0$ and $\sum_{i=1}^{l} \theta_i = 1$. The single-valued neutrosophic $N$-soft weighted average operator $(SVNNS_fWA)$ is a mapping $SVNNS_fWA : \mathcal{B}^l \to \mathcal{B}$, where $\mathcal{B}$ is the set of $SVNNS_fNs$, defined as follows:*

$$
\begin{aligned}
SVNNS_fWA(\rho_{1j}, \rho_{2j}, \ldots, \rho_{lj}) &= \bigoplus_{i=1}^{l} (\theta_i \rho_{ij}) && (5) \\
&= \left(\max_{i=1}^{l}(o_i^j), 1 - \Pi_{i=1}^{l}(1-\beta_{ij})^{\theta_i}, \Pi_{i=1}^{l}(\gamma_{ij})^{\theta_i}, \Pi_{i=1}^{l}(\delta_{ij})^{\theta_i}\right). && (6)
\end{aligned}
$$

**Definition 14.** *Let $\rho_{ij} = \rho_{ij} = (o_i^j, (\beta_{ij}, \gamma_{ij}, \delta_{ij}))$ $(i = 1, 2, \ldots, l)$ be a collection of $SVNNS_fNs$ and $\theta_i$ be the weight vectors $(WV)$ of $\rho_{ij}$ with $\theta_i > 0$ and $\sum_{i=1}^{l} \theta_i = 1$. The single-valued neutrosophic $N$-soft ordered weighted average operator $(SVNNS_fOWA)$ is a mapping $SVNNS_fOWA :$ $\mathcal{B}^l \to \mathcal{B}$, where $\mathcal{B}$ is the set of $SVNNS_fNs$, defined as follows:*

$$
\begin{aligned}
SVNNS_fOWA(\rho_{1j}, \rho_{2j}, \ldots, \rho_{lj}) &= \left(\theta_1\rho_{\phi(1j)} \bigoplus \theta_2\rho_{\phi(2j)} \bigoplus \ldots \bigoplus \theta_l\rho_{\phi(lj)}\right) \\
&= \left(\max_{i=1}^{l}(o_i^j), 1 - \Pi_{i=1}^{l}(1-\beta_{\phi(1j)})^{\theta_i}, \Pi_{i=1}^{l}(\gamma_{\phi(1j)})^{\theta_i}, \Pi_{i=1}^{l}(\delta_{\phi(1j)})^{\theta_i}\right),
\end{aligned}
$$

*where, $(\phi(1j), \phi(2j), \ldots, \phi(lj))$ is a permutation of $(1j, 2j, \ldots, lj)$ such that $\rho_{\phi(ij)} \geq \rho_{\phi(kj)}$, for all $i < k$, $(i, k = 1, 2, \ldots, l)$ and $(j = 1, 2, \ldots, m)$.*

**Definition 15.** *Let $\rho_{ij} = \rho_{ij} = (o_i^j, (\beta_{ij}, \gamma_{ij}, \delta_{ij}))$ $(i = 1, 2, \ldots, l)$ be a collection of $SVNNS_fNs$ and $\theta_i$ be the weight vectors $(WV)$ of $\rho_{ij}$ with $\theta_i > 0$ and $\sum_{i=1}^{l} \theta_i = 1$. The single-valued neutrosophic $N$-soft weighted geometric operator $(SVNNS_fWG)$ is a mapping $SVNNS_fWG : \mathcal{B}^l \to \mathcal{B}$, where $\mathcal{B}$ is the set of $SVNNS_fNs$, defined as follows:*

$$
\begin{aligned}
SVNNS_fWG(\rho_{1j}, \rho_{2j}, \ldots, \rho_{lj}) &= \bigotimes_{i=1}^{l} (\rho_{ij})^{\theta_i} && (7) \\
&= \left(\min_{i=1}^{l}(o_i^j), \Pi_{i=1}^{l}(\beta_{ij})^{\theta_i}, 1 - \Pi_{i=1}^{l}(1-\gamma_{ij})^{\theta_i}, 1 - \Pi_{i=1}^{l}(1-\delta_{ij})^{\theta_i}\right). && (8)
\end{aligned}
$$

**Definition 16.** *Let $\rho_{ij} = \rho_{ij} = (o_i^j, (\beta_{ij}, \gamma_{ij}, \delta_{ij}))$ $(i = 1, 2, \ldots, l)$ be a collection of $SVNNS_fNs$ and $\theta_i$ be the weight vectors $(WV)$ of $\rho_{ij}$ with $\theta_i > 0$ and $\sum_{i=1}^{l} \theta_i = 1$. The single-valued neutrosophic $N$-soft ordered weighted geometric operator $(SVNNS_fOWG)$ is a mapping $SVNNS_fOWG : \mathcal{B}^l \to \mathcal{B}$, where $\mathcal{B}$ is the set of $SVNNS_fNs$, defined as follows:*

$$SVNNS_fOWG(\rho_{1j},\rho_{2j},\ldots,\rho_{lj}) = \left(\rho_{\phi(1j)^{\theta_1}} \bigotimes \rho_{\phi(2j)^{\theta_2}} \bigotimes \cdots \bigotimes \rho_{\phi(lj)^{\theta_l}}\right)$$
$$= \left(\min_{i=1}^{l}(o_i^j), \Pi_{i=1}^{l}(\beta_{\phi(1j)})^{\theta_i}, 1-\Pi_{i=1}^{l}(1-\gamma_{\phi(1j)})^{\theta_i}, 1-\Pi_{i=1}^{l}(1-\delta_{\phi(1j)})^{\theta_i}\right),$$

*where,* $(\phi(1j),\phi(2j),\ldots,\phi(lj))$ *is a permutation of* $(1j,2j,\ldots,lj)$ *such that* $\rho_{\phi(ij)} \geq \rho_{\phi(kj)}$, *for all* $i<k$, $(i,k=1,2,\ldots,l)$ *and* $(j=1,2,\ldots,m)$.

# 3 Single-valued neutrosophic $N$-soft TOPSIS method

In this section, we extend TOPSIS method to the environment of $SVNNS_fSs$ that will be used to find out an alternative that is nearest to the positive ideal solution (PIS) and farthest from the negative ideal solution (NIS) as the feasible solution of MAGDM problem. Let $E=\{E_1,E_2,E_3,\ldots,E_m\}$ denote the set of attributes decided by the experts $\widetilde{D}_1,\widetilde{D}_2,\widetilde{D}_3,\ldots,\widetilde{D}_p$, for the alternatives $X=\{X_1,X_2,X_3,\ldots,X_q\}$, according to the MAGDM problems. The experts decisions weighted through the weight vector $\theta=(\theta_1,\theta_2,\theta_3,\ldots,\theta_p)^T$ with $\sum_{r=1}^{p}\theta=1$, where $\theta_r\in[0,1]$. The step by step procedure for $SVNNS_f$-TOPSIS method is as follows:

## 3.1 Formulation of decision matrices of each experts

Each expert assigns ranking, corresponding to each linguistic term, to the alternatives after thoroughly observing the attributes and MAGDM problem. The ranking provided by the experts is actually denoting $NS_fS$ related to each expert. According to the proficiencies of the MAGDM problem, grading criteria defined by the experts according to which $SVNNS_fN$ is assigned to $NS_fS$, that is associated with each expert. Further, a single-valued neutrosophic $N$-soft decision matrix $(SVNNS_fDM)$ $G^{(r)}=(G_{ij}^{(r)})_{j\times i}$, is assembled by rth expert $\widetilde{D}_r$. So $p$ $SVNNS_fDMs$, $G^{(1)},G^{(2)},\ldots,G^{(p)}$, are formed as follows:

$$G^{(r)}=\begin{pmatrix} (o_1^{1(r)},\beta_{11}^{(r)},\gamma_{11}^{(r)},\delta_{11}^{(r)}) & (o_2^{1(r)},\beta_{12}^{(r)},\gamma_{12}^{(r)},\delta_{12}^{(r)}) & \ldots & (o_m^{1\ (r)},\beta_{1m}^{(r)},\gamma_{1m}^{(r)},\delta_{1m}^{(r)}) \\ (o_1^{2(r)},\beta_{21}^{(r)},\gamma_{21}^{(r)},\delta_{21}^{(r)}) & (o_2^{2(r)},\beta_{22}^{(r)},\gamma_{22}^{(r)},\delta_{22}^{(r)}) & \ldots & (o_m^{2\ (r)},\beta_{2m}^{(r)},\gamma_{2m}^{(r)},\delta_{2m}^{(r)}) \\ \vdots & \vdots & \ddots & \vdots \\ (o_1^{q(r)},\beta_{q1}^{(r)},\gamma_{q1}^{(r)},\delta_{q1}^{(r)}) & (o_2^{q}(r),\beta_{q2}^{(r)},\gamma_{q2}^{(r)},\delta_{q2}^{(r)}) & \ldots & (o_m^{q\ (r)},\beta_{qm}^{(r)},\gamma_{qm}^{(r)},\delta_{qm}^{(r)}) \end{pmatrix},$$

where, $G_{ij}^{(r)}=((o_i^j)^{(r)},\beta_{ij}^{(r)},\gamma_{ij}^{(r)},\delta_{ij}^{(r)})$, $j=\{1,2,3,\ldots,q\}$, $i=\{1,2,3,\ldots,m\}$ and $r=\{1,2,3,\ldots,p\}$.

## 3.2 Formulation of aggregated single-valued neutrosophic $N$-soft decision matrix

The $SVNNS_fWA$ operator or $SVNNS_fWG$ operator, given in Equations 5 and 7, are used to summarize the $SVNNS_fDMs$ related to each expert, known as aggregated single-valued neutrosophic $N$-soft decision matrix $(ASVNNS_fDM)$, is calculated as follows:

$$\mathbb{G}=SVNNS_fWA(G_{ij}^{(1)},G_{ij}^{(2)},\ldots,G_{ij}^{(r)});$$

or

$$\mathbb{G}=SVNNS_fWG(G_{ij}^{(1)},G_{ij}^{(2)},\ldots,G_{ij}^{(r)});$$

The $ASVNN_fSDM$ denoted as:

$$\mathbb{G}=\begin{pmatrix} (o_1^1,\beta_{11},\gamma_{11},\delta_{11}) & (o_2^1,\beta_{12},\gamma_{12},\delta_{12}) & \ldots & (o_m^1,\beta_{1m},\gamma_{1m},\delta_{1m}) \\ (o_1^2,\beta_{21},\gamma_{21},\delta_{21}) & (o_2^2,\beta_{22},\gamma_{22},\delta_{22}) & \ldots & (o_m^2,\beta_{2m},\gamma_{2m},\delta_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ (o_1^q,\beta_{q1},\gamma_{q1},\delta_{q1}) & (o_2^q,\beta_{q2},\gamma_{q2},\delta_{q2}) & \ldots & (o_m^q,\beta_{qm},\gamma_{qm},\delta_{qm}) \end{pmatrix}.$$

## 3.3 Calculation for weight vector of attributes

The value and importance of the attributes variate according to the MAGDM problem. The experts assigned rank to each attribute as weightage, keeping in view the expertise of the alternatives in the MAGDM problem. Using grading criteria, $SVNNS_fN$ assigned to each rank, i.e., $\mu_i^{(r)}=(o_i^{(r)},\beta_i^{(r)},\gamma_i^{(r)},\delta_i^{(r)})$ be the weight of ith attribute given by the rth expert in the decision maker panel. The weight vector $\mu=(\mu_1,\mu_2,\ldots,\mu_m)^T=(o_i,\beta_i,\gamma_i,\delta_i)$ is accumulated, by using the $SVNNS_fWA$ operator or $SVNNS_fWG$ operator given in Equations 5 and 7, as follows:

$$\mu_i=SVNNS_fWA(\mu_1^{(r)},\mu_2^{(r)},\ldots,\mu_m^{(r)});$$

or

$$\mu_i = SVNNS_f WG(\mu_1^{(r)}, \mu_2^{(r)}, \ldots, \mu_m^{(r)}).$$

## 3.4 Formulation of aggregated weighted single-valued neutrosophic $N$-soft decision matrix

The $ASVNN_f SDM$ and the weightage $\mu_i$ corresponding to each attribute $E_i$ are used to calculate the aggregated weighted single-valued neutrosophic $N$-soft decision matrix
$(AWSVNNS_f DM)$ as follows:

$$\begin{aligned}\widetilde{\mathbb{G}} &= \mathbb{G} \bigotimes \mu_i \\ &= (\min((o_i^j), o_i), \beta_{ij}\beta_i, \gamma_{ij} + \gamma_i - \gamma_{ij}\gamma_i, \delta_{ij} + \gamma_i - \gamma_{ij}\gamma_i) \\ &= (\widetilde{o}_i^j, \widetilde{\beta}_{ij}, \widetilde{\gamma}_{ij}, \widetilde{\delta}_{ij}).\end{aligned}$$

So that the $AWSVNNS_f DM$ is:

$$\widetilde{\mathbb{G}} = \begin{pmatrix} (\widetilde{o}_1^1, \widetilde{\beta}_{11}, \widetilde{\gamma}_{11}, \widetilde{\delta}_{11}) & (\widetilde{o}_2^1, \widetilde{\beta}_{12}, \widetilde{\gamma}_{12}, \widetilde{\delta}_{12}) & \ldots & (\widetilde{o}_m^1, \widetilde{\beta}_{1m}, \widetilde{\gamma}_{1m}, \widetilde{\delta}_{1m}) \\ (\widetilde{o}_1^2, \widetilde{\beta}_{21}, \widetilde{\gamma}_{21}, \widetilde{\delta}_{21}) & (\widetilde{o}_2^2, \widetilde{\beta}_{22}, \widetilde{\gamma}_{22}, \widetilde{\delta}_{22}) & \ldots & (\widetilde{o}_m^2, \widetilde{\beta}_{2m}, \widetilde{\gamma}_{2m}, \widetilde{\delta}_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ (\widetilde{o}_1^q, \widetilde{\beta}_{q1}, \widetilde{\gamma}_{q1}, \widetilde{\delta}_{q1}) & (\widetilde{o}_2^q, \widetilde{\beta}_{q2}, \widetilde{\gamma}_{q2}, \widetilde{\delta}_{q2}) & \ldots & (\widetilde{o}_m^q, \widetilde{\beta}_{qm}, \widetilde{\gamma}_{qm}, \widetilde{\delta}_{qm}) \end{pmatrix}.$$

## 3.5 Formulation of single-valued neutrosophic $N$-soft ideal solution

The score value and the accuracy value are used to evaluate the single-valued neutrosophic positive ideal solution $SVNN_f S$-PIS and single-valued neutrosophic $N$-soft negative ideal solution $SVNNS_f$-NIS on the basis of cost-type attributes and benefit-type attributes. Let $\mathfrak{A}_\mathfrak{c}$ and $\mathfrak{A}_\mathfrak{b}$ be the collection of cost-type attributes and benefit-type attributes, respectively, that are chosen according to the nature of the MAGDM problem. Now, relative to the attribute $E_i$ the $SVNNS_f$-PIS can be calculated as follows:

$$\overline{\mathbb{G}}_i = \begin{cases} \max\limits_{j=1}^{q} \widetilde{\mathcal{G}}_{ij}, & \text{if } E_i \in \mathfrak{A}_\mathfrak{b}, \\ \min\limits_{j=1}^{q} \widetilde{\mathcal{G}}_{ij}, & \text{if } E_i \in \mathfrak{A}_\mathfrak{c}, \end{cases} \tag{9}$$

and the $SVNNS_f$-NIS is computed as:

$$\underline{\mathbb{G}}_i = \begin{cases} \max\limits_{j=1}^{q} \widetilde{\mathcal{G}}_{ij}, & \text{if } E_i \in \mathfrak{A}_\mathfrak{c}, \\ \min\limits_{j=1}^{q} \widetilde{\mathcal{G}}_{ij}, & \text{if } E_i \in \mathfrak{A}_\mathfrak{b}. \end{cases} \tag{10}$$

The $SVNNS_f$-$PIS$ and $SVNNS_f$-$NIS$ are denoted as: $\overline{\mathbb{G}}_i = (\overline{o}_i, \overline{\beta}_i, \overline{\gamma}_i, \overline{\delta}_i)$, and $\underline{\mathbb{G}}_i = (\underline{o}_i, \underline{\beta}_i, \underline{\gamma}_i, \underline{\delta}_i)$, respectively.

## 3.6 Evaluation of normalized Euclidean distance

To find out best solution, we have to identify the nearest and farthest alternative from the $SVNNS_f$-$PIS$ and $SVNNS_f$-$NIS$, respectively. For this purpose, we computed normalized Euclidean distance of $SVNNS_f$-$PIS$ and $SVNNS_f$-$NIS$ from each alternative, simultaneously, as follows:

$$d(\overline{\mathbb{G}}_i, X_j) = \Big(\frac{1}{4i}\sum_{i=1}^{m}\Big[((\frac{\overline{o}_i}{N-1}) - (\frac{\widetilde{o}_i^j}{N-1}))^2 + (\overline{\beta}_i - \widetilde{\beta}_{ij})^2 + (\overline{\gamma}_i - \widetilde{\gamma}_{ij})^2 + (\overline{\delta}_i - \widetilde{\delta}_{ij})^2\Big]\Big).$$

The normalized Euclidean distance between the $SVNNS_f$-NIS and any of the alternative $X_j$, can be evaluated as follows:

$$d(\underline{\mathbb{G}}_i, X_j) = \Big(\frac{1}{4i}\sum_{i=1}^{m}\Big[((\frac{\underline{o}_i}{N-1}) - (\frac{\widetilde{o}_i^j}{N-1}))^2 + (\underline{\beta}_i - \widetilde{\beta}_{ij})^2 + (\underline{\gamma}_i - \widetilde{\gamma}_{ij})^2 + (\underline{\delta}_i - \widetilde{\delta}_{ij})^2\Big]\Big).$$

## 3.7 Computation of revised closeness index

We have to use some ranking index to compare the alternatives as we have alternatives having maximum distance from $SVNNS_f$-PIS as well as the minimum distance from $SVNNS_f$-NIS. Therefore, the revised closeness index modified by Gundogdu and Kahraman [11] for the selection of optimal solution is as follows:

$$\psi(X_j) = \frac{d(\overline{\mathbb{G}}_i, X_j)}{\min_j d(\overline{\mathbb{G}}_i, X_j)} - \frac{d(\underline{\mathbb{G}}_i, X_j)}{\max_j d(\underline{\mathbb{G}}_i, X_j)}, \tag{11}$$

where, $i = 1, 2, \ldots, m$.
Clearly, the closed index in Equation 14, generates zero or negative outputs, therefore we prefer this modified relation given in Equation 11 for $SVNNS_f$-TOPSIS method as it gives zero or positive results.

## 3.8 Order of alternatives

The alternatives are arranged in ascending order with respect to the revised closeness index and the alternative with lowest value is considered as the most suitable solution of the MAGDM problem.
The algorithm and the flowchart of the proposed $SVNNS_f$-TOPSIS method is given in Algorithm 1. For solving a MAGDM problem, the Algorithm 1 is given as:

Algorithm 1: Steps to deal MAGDM problem by $SVNN_f$-TOPSIS method

1. **Input**:
   $X$ : Set of alternatives,
   $E$ : Set of attributes,
   $\theta$ : WV for experts $\widetilde{D}_r$,
   $NS_fS : (H, E, N)$ with $O = \{0, 1, 2, 3, \ldots, N-1\}$, $N \in \{1, 2, 3, \ldots\}$,
2. Construct the $SVNNS_fDM\ G^{(r)}$, corresponding to each ordered grade for the element $X_j$.
3. Evaluate the $ASVNNS_fDM$ using equation

$$\mathbb{G}_{ij} = \left( \max_{r=1}^{p} (o_i^j)^{(r)}, 1 - \prod_{r=1}^{p} (1 - (\beta_{ij}^{(r)}))^{\theta_r}, \prod_{r=1}^{p} (\gamma_{ij}^{(r)})^{\theta_r}, \prod_{r=1}^{p} (\delta_{ij}^{(r)})^{\theta_r} \right).$$

4. Calculating the weight vector $\mu = (\mu_1, \mu_2, \ldots, \mu_m)^T$ for attributes as follows:

$$\mu_i = \left( \max_{r=1}^{p} (o_i^{(r)}), 1 - \prod_{r=1}^{p} (1 - (\beta_i^{(r)}))^{\theta_r}, \prod_{r=1}^{p} (\gamma_i^{(r)})^{\theta_r}, \prod_{r=1}^{p} (\delta_i^{(r)})^{\theta_r} \right).$$

5. Compute the $AWSVNNS_fDM$ using $ASVNNS_fDM$ and the weight vector of attributes $\mu_i$, as follows:

$$\widetilde{\mathbb{G}} = (\min((o_i^j), o_i), \beta_{ij}\beta_i, \gamma_{ij} + \gamma_i - \gamma_{ij}\gamma_i, \delta_{ij} + \delta_i - \delta_{ij}\delta_i).$$

6. Identify the $SVNNS_f$ PIS and $SVNNS_f$ NIS, using Equations (9) and (10).
7. Compute the normalized Euclidean distance of $CSVNNS_f$ PIS and $CSVNNS_f$ NIS from each alternative, respectively.
8. Calculate the revised closeness index.
9. Rank the alternatives in ascending order with respect to the revised closeness index.

**Output**: The alternative with least revised closeness index will be the decision.

# 4 Application

In this section, we solve a multi-attribute group decision making (MADM) problem using $SVNNS_f - TOPSIS$ method for the selection of branch manager in Quiqup company(courier company), UAE.

## 4.1 Selection for the post of branch manager in Quiqup company, UAE

The courier companies are serving as a bridge between the sellers and the customers that enhance the M-Commerce which is a shopping online through smartphone. M-Commerce has enabled us to have a lot of free time that we can sell or buy anything, anytime within a seconds and through courier companies. In UAE, the online shopping arena has been making tremendous growth in the past 10-years. For this purpose there are so many companies in UAE, one of them is Quiqup company in which courier drivers are specifically appointed for placing orders at the right place where the branch manager has to look after the overall records of the couriers. For the post of branch manager, three decision makers shortlisted five courier drivers for further evaluations. The experts $\widetilde{D}_1$, $\widetilde{D}_2 and \widetilde{D}_3$ analyzed courier drivers, named as $\{X_1 = Bahzad, X_2 = Naqash, X_3 = Zakwan, X_4 = Soreach, X_5 = Waqas\}$, on the basis of the following parameters $\{E_1 =$ Experience, $X_2 =$ Education, $X_3 =$ courier services , $X_4 =$ Fines and Expenditures, $X_5 =$ Behaviour $\}$. The weight vector for the experts is $\theta = (0.4, 0.3, 0.3)^T$ according to this MAGDM problem.

**Step 1:** According to these attributes each expert model 6-soft set in Table 5, where,

Five stars means 'Infinitely Good',

Four stars means 'Extremely Good',

Three stars means 'Good',

Two stars means 'Bad',

One stars means 'Extremely Bad',

Big dot means 'Infinitely Bad'

Table 3 represents the grading criteria, used for assigning the $SVNNS_fN$ corresponding to each rank by the expert $\widetilde{D}_1$, $\widetilde{D}_2$ and $\widetilde{D}_3$ arranged in Tables 6, 7 and 8, respectively.

Table 5: Experts' opinion related to parameters

| $Parameters$ | $Alternatives$ | $\widetilde{D}_1$ | $\widetilde{D}_2$, | $\widetilde{D}_3$ |
|---|---|---|---|---|
| $E_1$ | $X_1$ | $\ast\ast = 2$ | $\ast\ast\ast = 3$ | $\ast = 1$ |
| | $X_2$ | $\ast = 1$ | $\ast\ast = 2$ | $\ast\ast = 2$ |
| | $X_3$ | $\ast\ast\ast\ast\ast = 5$ | $\ast\ast\ast\ast\ast = 5$ | $\ast\ast\ast\ast = 4$ |
| | $X_4$ | $\ast\ast = 2$ | $\ast\ast\ast = 3$ | $\ast\ast\ast = 3$ |
| | $X_5$ | $\bullet = 0$ | $\ast = 1$ | $\ast\ast = 2$ |
| $E_2$ | $X_1$ | $\ast\ast\ast\ast = 4$ | $\ast\ast\ast\ast\ast = 5$ | $\ast\ast\ast = 3$ |
| | $X_2$ | $\ast = 1$ | $\bullet = 0$ | $\ast = 1$ |
| | $X_3$ | $\ast\ast\ast\ast = 4$ | $\ast\ast\ast\ast\ast = 5$ | $\ast\ast\ast\ast\ast = 5$ |
| | $X_4$ | $\ast\ast\ast = 3$ | $\ast = 1$ | $\bullet = 0$ |
| | $X_5$ | $\ast\ast = 2$ | $\ast = 1$ | $\ast\ast = 2$ |
| $E_3$ | $X_1$ | $\ast\ast\ast = 3$ | $\ast\ast\ast\ast = 4$ | $\ast\ast\ast\ast\ast = 5$ |
| | $X_2$ | $\ast\ast\ast\ast = 4$ | $\ast\ast\ast = 3$ | $\ast\ast\ast\ast = 4$ |
| | $X_3$ | $\ast\ast\ast\ast\ast = 5$ | $\ast\ast\ast\ast\ast = 5$ | $\ast\ast\ast\ast = 4$ |
| | $X_4$ | $\ast = 1$ | $\ast\ast\ast = 3$ | $\ast\ast\ast = 3$ |
| | $X_5$ | $\ast\ast\ast = 3$ | $\ast\ast\ast\ast = 4$ | $\ast\ast\ast = 3$ |
| $E_4$ | $X_1$ | $\ast\ast\ast\ast = 4$ | $\ast\ast\ast = 3$ | $\ast\ast\ast\ast\ast = 5$ |
| | $X_2$ | $\ast\ast\ast\ast = 4$ | $\ast\ast\ast\ast\ast = 5$ | $\ast\ast\ast\ast\ast = 5$ |
| | $X_3$ | $\ast\ast\ast = 3$ | $\ast\ast = 2$ | $\ast\ast\ast\ast = 4$ |
| | $X_4$ | $\ast\ast\ast\ast = 4$ | $\ast\ast\ast = 3$ | $\ast\ast\ast\ast = 4$ |
| | $X_5$ | $\ast\ast\ast\ast\ast = 5$ | $\ast\ast\ast\ast = 4$ | $\ast\ast\ast\ast = 4$ |
| $E_5$ | $X_1$ | $\ast\ast\ast\ast = 4$ | $\ast\ast\ast\ast = 4$ | $\ast\ast\ast\ast = 4$ |
| | $X_2$ | $\ast\ast = 2$ | $\ast\ast = 2$ | $\ast\ast = 2$ |
| | $X_3$ | $\ast\ast\ast\ast\ast = 5$ | $\ast\ast\ast\ast\ast = 5$ | $\ast\ast\ast\ast\ast = 5$ |
| | $X_4$ | $\ast\ast\ast = 3$ | $\ast\ast\ast = 3$ | $\ast\ast\ast = 3$ |
| | $X_5$ | $\ast = 1$ | $\ast = 1$ | $\ast = 1$ |

Table 6: $SVNNS_fDM$ of expert $\widetilde{D}_1$

| $(H_J^{(1)}, E, 6)$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $X_1$ | $(2, (0.410, 0.125, 0.610))$ | $(4, (0.710, 0.030, 0.250))$ | $(3, (0.690, 0.068, 0.480))$ | $(4, (0.720, 0.040, 0.260))$ | $(4, (0.730, 0.050, 0.270))$ |
| $X_2$ | $(1, (0.290, 0.018, 0.810))$ | $(1, (0.280, 0.017, 0.790))$ | $(4, (0.740, 0.060, 0.220))$ | $(4, (0.750, 0.550, 0.170))$ | $(2, (0.460, 0.132, 0.160))$ |
| $X_3$ | $(5, (0.980, 0.010, 0.020))$ | $(4, (0.870, 0.012, 0.160))$ | $(5, (0.970, 0.015, 0.016))$ | $(3, (0.680, 0.0350, 0.410))$ | $(5, (0.990, 0.010, 0.014))$ |
| $X_4$ | $(2, (0.430, 0.129, 0.630))$ | $(3, (0.660, 0.036, 0.430))$ | $(1, (0.270, 0.016, 0.780))$ | $(4, (0.760, 0.057, 0.180))$ | $(3, (0.670, 0.034, 0.420))$ |
| $X_5$ | $(0, (0.500, 0.300, 0.800))$ | $(2, (0.420, 0.127, 0.620))$ | $(3, (0.650, 0.037, 0.440))$ | $(5, (0.910, 0.016, 0.140))$ | $(1, (0.260, 0.015, 0.770))$ |

**Step 2:** The $ASVNNS_fDM$ formulated by aggregation formula defined in Algorithm 1(3). The accumulated opinions of all experts is shown in Table 9.

**Step 3:** According to the MAGDM problem, experts assigned ratings to parameters to explain their significance related to each alternatives. Further, the ratings are replaced by $SVNNS_fNs$, shown in Table 10, and the weight vector $\mu$ cumulated using Algorithm 1(step 4) is

Table 7: $SVNNS_fDM$ of expert $\widetilde{D}_2$

| $(H_J^{(2)}, E, 6)$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $X_1$ | $(3,(0.640,0.038,0.450))$ | $(5,(0.950,0.015,0.130))$ | $(4,(0.780,0.058,0.190))$ | $(3,(0.630,0.039,0.460))$ | $(4,(0.790,0.059,0.210))$ |
| $X_2$ | $(2,(0.440,0.130,0.640))$ | $(0,(0.510,0.310,0.810))$ | $(3,(0.620,0.040,0.470))$ | $(5,(0.920,0.016,0.140))$ | $(2,(0.470,0.133,0.670))$ |
| $X_3$ | $(5,(0.980,0.011,0.009))$ | $(5,(0.995,0.008,0.007))$ | $(5,(0.975,0.007,0.006))$ | $(2,(0.450,0.131,0.650))$ | $(5,(0.960,0.004,0.040))$ |
| $X_4$ | $(3,(0.610,0.041,0.480))$ | $(1,(0.250,0.014,0.760))$ | $(3,(0.620,0.042,0.490))$ | $(3,(0.630,0.043,0.350))$ | $(3,(0.640,0.044,0.360))$ |
| $X_5$ | $(1,(0.240,0.013,0.750))$ | $(1,(0.230,0.012,0.740))$ | $(4,(0.810,0.061,0.220))$ | $(4,(0.820,0.062,0.230))$ | $(1,(0.220,0.011,0.730))$ |

Table 8: $SVNNS_fDM$ of expert $\widetilde{D}_3$

| $(H_J^{(3)}, E, 6)$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $X_1$ | $(1,(0.210,0.010,0.720))$ | $(3,(0.510,0.045,0.370))$ | $(5,(0.915,0.013,0.120))$ | $(5,(0.925,0.014,0.100))$ | $(4,(0.830,0.064,0.250))$ |
| $X_2$ | $(2,(0.490,0.135,0.550))$ | $(1,(0.200,0.009,0.710))$ | $(4,(0.820,0.063,0.240))$ | $(5,(0.930,0.010,0.110))$ | $(2,(0.480,0.134,0.680))$ |
| $X_3$ | $(4,(0.710,0.015,0.165))$ | $(5,(0.970,0.005,0.006))$ | $(4,(0.840,0.065,0.260))$ | $(4,(0.850,0.066,0.270))$ | $(5,(0.983,0.005,0.050))$ |
| $X_4$ | $(3,(0.520,0.046,0.380))$ | $(0,(0.520,0.320,0.820))$ | $(3,(0.530,0.047,0.390))$ | $(4,(0.860,0.067,0.280))$ | $(3,(0.540,0.048,0.290))$ |
| $X_5$ | $(2,(0.350,0.136,0.560))$ | $(2,(0.360,0.137,0.570))$ | $(3,(0.550,0.049,0.330))$ | $(4,(0.870,0.068,0.290))$ | $(1,(0.190,0.008,0.700))$ |

Table 9: Aggregated single-valued neutrosophic $N$-soft decision matrix

| $\mathbb{G}$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $X_1$ | $(3,(0.466,0.043,0.572))$ | $(5,(0.821,0.026,0.219))$ | $(5,(0.801,0.042,0.245))$ | $(5,(0.778,0.030,0.250))$ | $(4,(0.780,0.056,0.242))$ |
| $X_2$ | $(2,(0.398,0.060,0.677))$ | $(1,(0.354,0.040,0.776))$ | $(4,(0.729,0.052,0.293))$ | $(5,(0.878,0.058,0.142))$ | $(2,(0.468,0.132,0.668))$ |
| $X_3$ | $(5,(0.964,0.011,0.026))$ | $(5,(0.971,0.008,0.024))$ | $(5,(0.957,0.016,0.022))$ | $(4,(0.680,0.065,0.434))$ | $(5,(0.981,0.006,0.028))$ |
| $X_4$ | $(3,(0.522,0.066,0.504))$ | $(3,(0.511,0.044,0.616))$ | $(3,(0.480,0.029,0.557))$ | $(4,(0.756,0.054,0.254))$ | $(3,(0.630,0.040,0.362))$ |
| $X_5$ | $(2,(0.379,0.082,0.715))$ | $(2,(0.344,0.056,0.646))$ | $(4,(0.699,0.047,0.321))$ | $(5,(0.874,0.036,0.200))$ | $(1,(0.229,0.011,0.738))$ |

given as follows:

$$\mu = \begin{pmatrix} (5,(0.932,0.027,0.204)) \\ (3,(0.815,0.037,0.541)) \\ (4,(0.914,0.026,0.266)) \\ (4,(0.525,0.047,0.499)) \\ (5,(0.657,0.035,0.278)) \end{pmatrix}.$$

Table 10: Ratings of experts about parameters

| | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $\widetilde{D}_1$ | $(4,(0.820,0.040,0.250))$ | $(3,(0.600,0.020,0.40))$ | $(4,(0.800,0.025,0.200))$ | $(1,(0.200,0.040,0.850))$ | $(2,(0.350,0.100,0.600))$ |
| $\widetilde{D}_2$ | $(5,(0.920,0.010,0.550))$ | $(2,(0.370,0.090,0.550))$ | $(3,(0.660,0.030,0.410))$ | $(4,(0.760,0.030,0.220))$ | $(4,(0.750,0.020,0.210))$ |
| $\widetilde{D}_3$ | $(3,(0.680,0.061,0.041))$ | $(1,(0.270,0.030,0.554))$ | $(4,(0.770,0.025,0.230))$ | $(2,(0.360,0.120,0.670))$ | $(5,(0.950,0.015,0.127))$ |

**Step 4:** We used $\mathbb{G}$ and weight vector $\mu$ of parameters for availing the $AWSVNNS_fDM$ summarized in Table 11.

Table 11: Aggregated weighted single-valued neutrosophic $N$-soft decision matrix

| $\mathbb{G}$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $X_1$ | $(3,(0.430,0.068,0.659))$ | $(3,(0.699,0.062,0.641))$ | $(4,(0.732,0.066,0.446))$ | $(4,(0.408,0.076,0.624))$ | $(4,(0.512,0.089,0.452))$ |
| $X_2$ | $(2,(0.367,0.085,0.742))$ | $(1,(0.288,0.075,0.897))$ | $(4,(0.666,0.076,0.481))$ | $(4,(0.460,0.102,0.570))$ | $(2,(0.307,0.162,0.760))$ |
| $X_3$ | $(5,(0.890,0.038,0.224))$ | $(3,(0.791,0.044,0.552))$ | $(4,(0.874,0.042,0.282))$ | $(4,(0.460,0.108,0.716))$ | $(5,(0.644,0.040,0.298))$ |
| $X_4$ | $(3,(0.481,0.091,0.605))$ | $(3,(0.416,0.079,0.824))$ | $(3,(0.438,0.054,0.674))$ | $(4,(0.396,0.098,0.626))$ | $(3,(0.414,0.074,0.539))$ |
| $X_5$ | $(2,(0.350,0.106,0.773))$ | $(2,(0.280,0.090,0.838))$ | $(4,(0.638,0.072,0.502))$ | $(4,(0.458,0.081,0.599))$ | $(1,(0.150,0.046,0.810))$ |

**Step 5:** The parameters experiences, customer services, education and behaviour are benefit-type parameters while the fines and expenditures is cost-type parameter .Keeping in view the nature of parameters and applying Equation 9 and 10 $SVNNS_f$-PIS and $SVNNS_f$-NIS are evaluated, arranged in Table 12.

Table 12: $SVNNS_f$-PIS and $CSVNNS_f$-NIS

| Attribute | $CSVNNS_f$-PIS | $SVNNS_f$-NIS |
|---|---|---|
| $z_1$ | $(5,(0.890,0.038,0.224))$ | $(2,(0.350,0.106,0.773))$ |
| $z_2$ | $(3,(0.791,0.044,0.552))$ | $(1,(0.288,0.075,0.897))$ |
| $z_3$ | $(4,(0.874,0.042,0.282))$ | $(3,(0.438,0.054,0.674))$ |
| $z_4$ | $(4,(0.460,0.108,0.716))$ | $(4,(0.460,0.102,0.570))$ |
| $z_5$ | $(5,(0.644,0.040,0.298))$ | $(1,(0.150,0.046,0.810))$ |

**Step 6:** The normalized Euclidean distance, from each alternative to $SVNNS_f$-PIS and $SVNNS_f$-NIS, is given in Table 13.

Table 13: Normalized Euclidean distance from ideal solution

| Alternative | $d(\overline{\mathbb{G}}_k, X_j)$ | $d(\underline{\mathbb{G}}_k, X_j)$ |
|---|---|---|
| $X_1$ | 0.0361 | 0.0640 |
| $X_2$ | 0.1122 | 0.0390 |
| $X_3$ | 0.0005 | 0.1540 |
| $X_4$ | 0.0679 | 0.0302 |
| $X_5$ | 0.1327 | 0.0070 |

**Step 7:** The revised closeness index of each alternative is calculated by utilizing Equation 11, given in Table 14.

Table 14: Revised closeness index of each alternative

| Alternative | $\psi(X_j)$ |
|---|---|
| $X_1$ | 6.8044 |
| $X_2$ | 22.186 |
| $X_3$ | 0 |
| $X_4$ | 13.3838 |
| $X_5$ | 26.4945 |

**Step 8:** Since $X_3$ has minimum revised closeness index, therefore Zakwan is the most suitable courier driver for branch manager post. The ranking of alternatives is shown in Table 15.

Table 15: Ranking according to the revised closeness index

| Alternative | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
|---|---|---|---|---|---|
| Ranking | 2 | 4 | 1 | 3 | 5 |

## 5 Comparison

In this section, we solve the MAGDM problem "selection for the post of branch manager in Quiqup company, UAE" using single-valued neutrosophic TOPSIS method, proposed by Sahi and Yigider. [40], to demonstrate the significance of the proposed model. The solution by single-valued neutrosophic TOPSIS method is as follows:

**Step 1** The linguistic term corresponding to each rank asses by the experts, are same as given in Table 5. To apply the SVN-TOPSIS method the grading part is excluded from the $SVNNS_fN$ and SVNNs are assigning by each expert $\widetilde{D}_1$, $\widetilde{D}_2$ and $\widetilde{D}_3$, are arranged in Tables 16,17 and 18, respectively, according to the grading criteria define in Table 3.

**Step 2** Using the weight vector of experts $\theta = (0.4, 0.3, 0.3)^T$ and single-valued neutrosophic weighted average $(SVNWA)$ operator [40], we can calculate the aggregated single-valued neutrosophic decision matrix $(ASVNDM)$, whose entries are evaluated by the formula defined as follows:

$$\mathbb{G}_{ij} = \left(1 - \prod_{r=1}^{p}(1-(\beta_{ij}^{(r)}))^{\theta_r}, \prod_{r=1}^{p}(\gamma_{ij}^{(r)})^{\theta_r}, \prod_{r=1}^{p}(\delta_{ij}^{(r)})^{\theta_r}\right).$$

Table 16: $SVNDM$ of expert $\widetilde{D}_1$

| | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $X_1$ | $(0.410, 0.125, 0.610)$ | $(0.710, 0.030, 0.250)$ | $(0.690, 0.068, 0.480)$ | $(0.720, 0.040, 0.260)$ | $(0.730, 0.050, 0.270)$ |
| $X_2$ | $(0.290, 0.018, 0.810)$ | $(0.280, 0.017, 0.790)$ | $(0.740, 0.060, 0.220)$ | $(0.750, 0.550, 0.170)$ | $(0.460, 0.132, 0.160)$ |
| $X_3$ | $(0.980, 0.010, 0.020)$ | $(0.870, 0.012, 0.160)$ | $(0.970, 0.015, 0.016)$ | $(0.680, 0.035, 0.410)$ | $(0.990, 0.010, 0.014)$ |
| $X_4$ | $(0.430, 0.129, 0.630)$ | $(0.660, 0.036, 0.430)$ | $(0.270, 0.016, 0.780)$ | $(0.760, 0.057, 0.180)$ | $(0.670, 0.034, 0.420)$ |
| $X_5$ | $(0.500, 0.300, 0.800)$ | $(0.420, 0.127, 0.620)$ | $(0.650, 0.037, 0.440)$ | $(0.910, 0.016, 0.140)$ | $(0.260, 0.015, 0.770)$ |

Table 17: $SVNDM$ of expert $\widetilde{D}_2$

| | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $X_1$ | $(0.640, 0.038, 0.450)$ | $(0.950, 0.015, 0.130)$ | $(0.780, 0.058, 0.190)$ | $(0.630, 0.039, 0.460)$ | $(0.790, 0.059, 0.210)$ |
| $X_2$ | $(0.440, 0.130, 0.640)$ | $(0.510, 0.310, 0.810)$ | $(0.620, 0.040, 0.470)$ | $(0.920, 0.016, 0.140)$ | $(0.470, 0.133, 0.670)$ |
| $X_3$ | $(0.980, 0.011, 0.009)$ | $(0.995, 0.008, 0.007)$ | $(0.975, 0.007, 0.006)$ | $(0.450, 0.131, 0.650)$ | $(0.960, 0.004, 0.040)$ |
| $X_4$ | $(0.610, 0.041, 0.480)$ | $(0.250, 0.014, 0.760)$ | $(0.620, 0.042, 0.490)$ | $(0.630, 0.043, 0.350)$ | $(0.640, 0.044, 0.360)$ |
| $X_5$ | $(0.240, 0.013, 0.750)$ | $(0.230, 0.012, 0.740)$ | $(0.810, 0.061, 0.220)$ | $(0.820, 0.062, 0.230)$ | $(0.220, 0.011, 0.730)$ |

Table 18: $SVNDM$ of expert $\widetilde{D}_3$

| | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $X_1$ | $(0.210, 0.010, 0.720)$ | $(0.510, 0.045, 0.370)$ | $(0.915, 0.013, 0.120)$ | $(0.925, 0.014, 0.100)$ | $(0.830, 0.064, 0.250)$ |
| $X_2$ | $(0.490, 0.135, 0.550)$ | $(0.200, 0.009, 0.710)$ | $(0.820, 0.063, 0.240)$ | $(0.930, 0.010, 0.110)$ | $(0.480, 0.134, 0.680)$ |
| $X_3$ | $(0.710, 0.015, 0.165)$ | $(0.970, 0.005, 0.006)$ | $(0.840, 0.065, 0.260)$ | $(0.850, 0.066, 0.270)$ | $(0.983, 0.005, 0.050)$ |
| $X_4$ | $(0.520, 0.046, 0.380)$ | $(0.520, 0.320, 0.820)$ | $(0.530, 0.047, 0.390)$ | $(0.860, 0.067, 0.280)$ | $(0.540, 0.048, 0.290)$ |
| $X_5$ | $(0.350, 0.136, 0.560)$ | $(0.360, 0.137, 0.570)$ | $(0.550, 0.049, 0.330)$ | $(0.870, 0.068, 0.290)$ | $(0.190, 0.008, 0.700)$ |

The $ASVNDM$ is arranged in Table 19.

Table 19: Aggregated single-valued neutrosophic decision matrix

| $\mathbb{G}$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $X_1$ | $(0.466, 0.043, 0.572)$ | $(0.821, 0.026, 0.219)$ | $(0.801, 0.042, 0.245)$ | $(0.778, 0.030, 0.250)$ | $(0.780, 0.056, 0.242)$ |
| $X_2$ | $(0.398, 0.060, 0.677)$ | $(0.354, 0.040, 0.776)$ | $(0.729, 0.052, 0.293)$ | $(0.878, 0.058, 0.142)$ | $(0.468, 0.132, 0.668)$ |
| $X_3$ | $(0.964, 0.011, 0.026)$ | $(0.971, 0.008, 0.024)$ | $(0.957, 0.016, 0.022)$ | $(0.680, 0.065, 0.434))$ | $(0.981, 0.006, 0.028)$ |
| $X_4$ | $(0.522, 0.066, 0.504)$ | $(0.511, 0.044, 0.616)$ | $(0.480, 0.029, 0.557)$ | $(0.756, 0.054, 0.254)$ | $(0.630, 0.040, 0.362)$ |
| $X_5$ | $(0.379, 0.082, 0.715)$ | $(0.344, 0.056, 0.646)$ | $(0.699, 0.047, 0.321)$ | $(0.874, 0.036, 0.200)$ | $(0.229, 0.011, 0.738)$ |

**Step 3** The experts opinion about the importance of attributes are given in Table 20. The experts opinion are combined using $(SVNWA)$ operator [40], to formulate the weight vector $\mu$ for the attributes, defined as follows:

$$\mathbb{G}_{ij} = \left(1 - \prod_{r=1}^{p}(1 - (\beta_{ij}^{(r)}))^{\theta_r}, \prod_{r=1}^{p}(\gamma_{ij}^{(r)})^{\theta_r}, \prod_{r=1}^{p}(\delta_{ij}^{(r)})^{\theta_r}\right).$$

Thus we have,

$$\mu = \begin{pmatrix} (0.932, 0.027, 0.204) \\ (0.815, 0.037, 0.541) \\ (0.914, 0.026, 0.266) \\ (0.525, 0.047, 0.499) \\ (0.657, 0.035, 0.278) \end{pmatrix}.$$

**Step 4** The aggregated weighted single-valued neutrosophic decision matrix (AWSVNDM) arranged in Table 21, where the entries of AWSVNDM

Table 20: Ratings of experts about parameters in single-valued neutrosophic environment

| | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $\widetilde{D}_1$ | $(0.820, 0.040, 0.250)$ | $(0.600, 0.020, 0.400)$ | $(0.800, 0.025, 0.200)$ | $(0.200, 0.040, 0.850)$ | $(0.350, 0.100, 0.600)$ |
| $\widetilde{D}_2$ | $(0.920, 0.010, 0.550)$ | $(0.370, 0.090, 0.550)$ | $(0.660, 0.030, 0.410)$ | $(0.760, 0.030, 0.220)$ | $(0.750, 0.020, 0.210)$ |
| $\widetilde{D}_3$ | $(0.680, 0.061, 0.041)$ | $(0.270, 0.030, 0.554)$ | $(0.770, 0.025, 0.230)$ | $(0.360, 0.120, 0.670)$ | $(0.950, 0.015, 0.127)$ |

are calculated using the formula:

$$\widetilde{\mathbb{G}} = (\beta_{ij}\beta_i, \gamma_{ij} + \gamma_i - \gamma_{ij}\gamma_i, \delta_{ij} + \delta_i - \delta_{ij}\delta_i).$$

Table 21: Aggregated weighted single-valued neutrosophic decision matrix

| $\mathbb{G}$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| $X_1$ | $(0.430, 0.068, 0.659)$ | $(0.699, 0.062, 0.641)$ | $(0.732, 0.066, 0.446)$ | $(0.408, 0.076, 0.624)$ | $(0.512, 0.089, 0.452)$ |
| $X_2$ | $(0.367, 0.085, 0.742)$ | $(0.288, 0.075, 0.897)$ | $(0.666, 0.076, 0.481)$ | $(0.460, 0.102, 0.570)$ | $(0.307, 0.162, 0.760)$ |
| $X_3$ | $(0.890, 0.038, 0.224)$ | $(0.791, 0.044, 0.552)$ | $(0.874, 0.042, 0.282)$ | $(0.460, 0.108, 0.716)$ | $(0.644, 0.040, 0.298)$ |
| $X_4$ | $(0.481, 0.091, 0.605)$ | $(0.416, 0.079, 0.824)$ | $(0.438, 0.054, 0.674)$ | $(0.396, 0.098, 0.626)$ | $(0.414, 0.074, 0.539)$ |
| $X_5$ | $(0.350, 0.106, 0.773)$ | $(0.280, 0.090, 0.838)$ | $(0.638, 0.072, 0.502)$ | $(0.458, 0.081, 0.599)$ | $(0.150, 0.046, 0.810)$ |

**Step 5** To evaluate the single-valued neutrosophic positive ideal solution (SVN-PIS) and negative ideal solution (SVN-NIS) are to be calculated by the formula:

$$\overline{\mathbb{G}}_i = \begin{cases} (\max_j \widetilde{\beta}_{ij}, \min_j \widetilde{\gamma}_{ij}, \min_j \widetilde{\delta}_{ij}), & \text{if } E_i \in \mathfrak{A}_{\mathfrak{b}}, \\ (\min_j \widetilde{\beta}_{ij}, \max_j \widetilde{\gamma}_{ij}, \max_j \widetilde{\delta}_{ij}), & \text{if } E_i \in \mathfrak{A}_{\mathfrak{c}}, \end{cases}$$

and

$$\underline{\mathbb{G}}_i = \begin{cases} (\max_j \widetilde{\beta}_{ij}, \min_j \widetilde{\gamma}_{ij}, \min_j \widetilde{\delta}_{ij}), & \text{if } E_i \in \mathfrak{A}_{\mathfrak{c}}, \\ (\min_j \widetilde{\beta}_{ij}, \max_j \widetilde{\gamma}_{ij}, \max_j \widetilde{\delta}_{ij}), & \text{if } E_i \in \mathfrak{A}_{\mathfrak{b}}, \end{cases}$$

So that, the SVN-PIS and SVN-NIS found given in Table 22.

Table 22: SVN-PIS and SVN-NIS

| Parameters | SVN-PIS | SVN-NIS |
|---|---|---|
| $E_1$ | $(0.802, 0.038, 0.028)$ | $(0.340, 0.106, 0.703)$ |
| $E_2$ | $(0.452, 0.052, 0.554)$ | $(0.166, 0.184, 0.848)$ |
| $E_3$ | $(0.720, 0.034, 0.260)$ | $(0.370, 0.042, 0.591)$ |
| $E_4$ | $(0.458, 0.089, 0.634)$ | $(0.458, 0.250, 0.541)$ |
| $E_5$ | $(0.426, 0.032, 0.758)$ | $(0.786, 0.046, 0.271)$ |

**Step 6** The Euclidean distance of each alternative from SVN-PIS and SVN-NIS, evaluated by Equations 12 and 13, respectively, is given in Table 23.

$$d_E(\overline{\mathbb{G}}_i, X_j) = \sqrt{\left(\frac{1}{3}\sum_{i=1}^{m}\left[(\overline{\beta}_i - \widetilde{\beta}_{ij})^2 + (\overline{\gamma}_i - \widetilde{\gamma}_{ij})^2 + (\overline{\delta}_i - \widetilde{\delta}_{ij})^2\right]\right)}. \tag{12}$$

and

$$d_E(\underline{\mathbb{G}}_i, X_j) = \sqrt{\left(\frac{1}{3}\sum_{i=1}^{m}\left[(\underline{\beta}_i - \widetilde{\beta}_{ij})^2 + (\underline{\gamma}_i - \widetilde{\gamma}_{ij})^2 + (\underline{\delta}_i - \widetilde{\delta}_{ij})^2\right]\right)}. \tag{13}$$

Table 23: single-valued neutrosophic Euclidean distance

| Alternative | $d(\overline{\mathbb{G}}_k, X_j)$ | $d(\underline{\mathbb{G}}_k, X_j)$ |
|---|---|---|
| $X_1$ | 0.4159 | 0.4830 |
| $X_2$ | 0.6753 | 0.2120 |
| $X_3$ | 0.0369 | 0.7092 |
| $X_4$ | 0.5759 | 0.2898 |
| $X_5$ | 0.7239 | 0.1084 |

**Step 7:** The revised closeness index of each alternative, evaluated by Equations 14, is tabulated in Table 24 and the ratings are tabulated in Table 25 in descending order, according to which $X_3$ is the best choice for the post of branch manager in Quiqup company, UAE.

$$\psi(X_j) = \frac{d(\underline{\mathbb{G}}_i, X_j)}{d(\overline{\mathbb{G}}_i, X_j) + d(\underline{\mathbb{G}}_i, X_j)} \tag{14}$$

where, $i = 1, 2, \ldots, m$.

Table 24: Revised closeness index of each alternative

| Alternative | $\psi(X_j)$ |
|---|---|
| $X_1$ | 0.5373 |
| $X_2$ | 0.23900 |
| $X_3$ | 0.9505 |
| $X_4$ | 0.3347 |
| $X_5$ | 0.1302 |

Table 25: Ranking in single-valued neutrosophic environment

| Alternative | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
|---|---|---|---|---|---|
| Ranking | 2 | 4 | 1 | 3 | 5 |

## 5.1 Discussion

1. We conclude that the comparison of the proposed $SVNNS_f$-TOPSIS method with the existing technique SVN-TOPSIS method results the same courier driver for the post of branch manager as well as the order of ranking of the remaining alternatives remain same, given in Table 26.
   The accuracy and reliability of the outcomes in comparison proves the superiority of the the proposed method from the $SVN$ methods.

Table 26: Comparison

| Method | Ranking | Best candidate |
|---|---|---|
| SVN-TOPSIS [40] | $X_3 > X_1 > X_4 > X_2 > X_5$ | $X_3$ |
| $SVNNS_f$-TOPSIS (Proposed) | $X_3 > X_1 > X_4 > X_2 > X_5$ | $X_3$ |

2. The $SVNNS_f$-TOPSIS method has ability to handle MAGDM problems under the framework of $IFNS_fS$ and $PFNS_fS$ but these models have no capacity to deal the hesitancy opinion of human nature independently.
3. The existing models, specifically the generalized model SVNSs are impotent to handle modern problems described by parameterized rating systems but our model has potential to grip such type of modern problems.
4. By substituting $N = 2$, we switch from $SVNNS_f$ environment to $SVNS_f$ environment so that the $SVNNS_f$-TOPSIS method could be applied to the $SVNS_f$ environment in a satisfactory manners.

# 6 Conclusion

In this paper, we have mainly contributed in TOPSIS method precisely for group decision making under the most generalized environment of $SVNNS_fSs$. The $SVNNS_f$-TOPSIS method is an advanced technique to evaluate the optimal alternative nearest to $SVNNS_f$-PIS and farthest from $SVNNS_f$-NIS. For the extension of TOPSIS method, we have presented the aggregate operators to assess the $SVNNS_f$ aggregated and weighted aggregated decision matrix that are further used to spot the $SVNNS_f$-PIS and $SVNNS_f$-NIS heeding the benefit and cost type parameters. We have defined normalized Euclidian distance for $SVNNS_fNs$ so that we can evaluate the revised closeness index regarding to each alternative. We have illustrated practical examples of the MAGDM problem that is the selection of the branch manager post in Quiqup company, UAE, to intimate the application of the proposed method and have performed the comparison with SVN-TOPSIS technique that signify the legitimacy of the proposed method. For future direction, we can apply the presented method to solve many other MAGDM problems like for designer selection or management system. We can develop theory for the following techniques under the $SVNNS_f$-framework: (1) AHP method (2) VIKOR method.

**Ethical approval:** This article does not contain any studies with human participants or animals performed by any of the authors.

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