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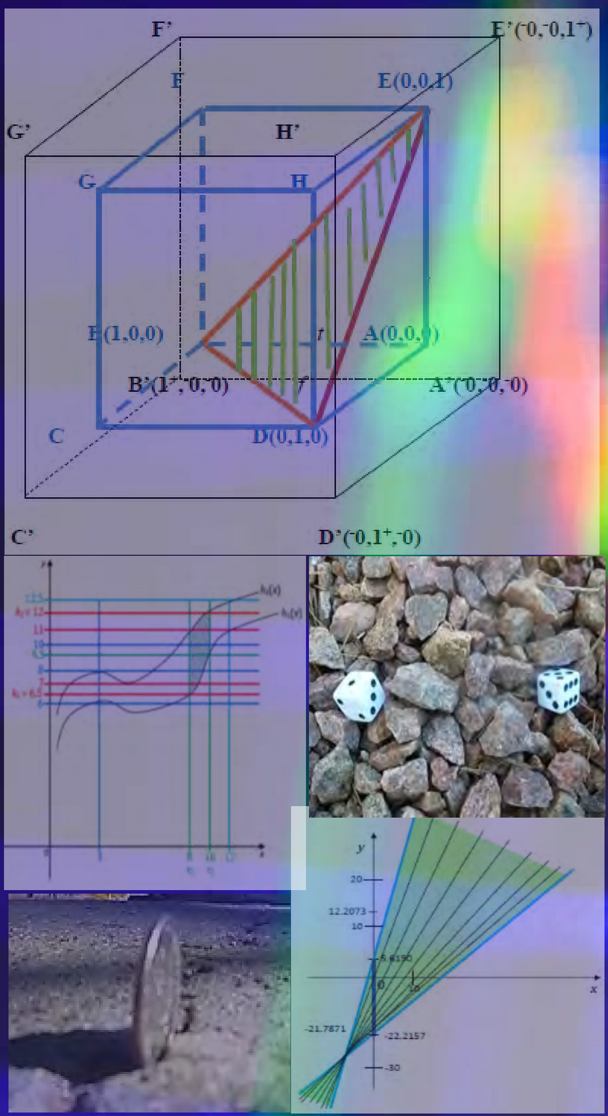

<A> <neutA> <antiA>

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"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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March 20, 2019

Prof. Florentin Smarandache
Univ New Mexico, Gallup Campus

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Thank you very much.

Sincerely,

Marian Hollingsworth

Marian Hollingsworth
Director, Publisher Relations

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The Algebraic Creativity in The Neutrosophic Square Matrices

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Abstract: The objective of this paper is to study algebraic properties of neutrosophic matrices, where a necessary and sufficient condition for the invertibility of a square neutrosophic matrix is presented by defining the neutrosophic determinant. On the other hand, this work introduces the concept of neutrosophic Eigen values and vectors with an easy algorithm to compute them. Also, this article finds a necessary and sufficient condition for the diagonalization of a neutrosophic matrix.

Keywords: Neutrosophic matrix, neutrosophic Eigen value, neutrosophic determinant, neutrosophic inverse, diagonalization of neutrosophic matrices

1. Introduction

Neutrosophy is a general form of logic founded by Smarandache to deal with indeterminacy in all fields of knowledge science. We find many applications in, decision making [2,3,23], optimization theory [1], topology [7], medical studies [26,27], energy studies [25], and number theory [16],

Recently, there is an increasing interesting in algebraic applications of neutrosophy such as neutrosophic modules [11,17], spaces [4,18], rings [14,16], and their generalizations [5,6,19].

After the emergence of the neutrosophic logic at 1995 there were a lot of applications to handle the indeterminacy notion. It is common for anyone to say that an unknown data is indeterminate than saying it is not exist as well in mathematics. Because when that the unknown data is not exist to a common mind it means that this data is absent does not exist. However, indeterminacy is suitable, for we can say to any layman, "We cannot determine what you ask for", but we cannot say, "your inquiry is not exist". Therefore, when we are in a moderate position as we cannot perceive $\emptyset$ for un known data, so we felt it is appropriate under these circumstances to introduce the notion of indeterminacy I where $I^2 = I$. Using this indeterminacy, we construct some notion regarding neutrosophic matrices, which can be used in neutrosophic models. Researchers have already defined the concept of neutrosophic matrices and have used them in Neutrosophic Cognitive Maps model and in the Neutrosophic Relational Equations models, which are analogous to Fuzzy Cognitive Map and Fuzzy Relational Equations models respectively.

In [21], Kandasamy et al, proposed for the first time the notion of bi-matrices. Also, a minimal study of their properties can be found in [8,12,13].

In this essay and for the first time sheds the light on the notion of determinant of a neutrosophic matrix, and we find the form of its inverse and illustrate them with examples. Also, we introduce easy algorithms to find Eigen values and vectors for neutrosophic matrices, with a direct application into the problem of diagonalization.

Neutrosophic matrices are useful in the study of indeterminacy and they have many important properties in algebra, from this point of view we introduce this work.

All matrices through this paper are defined over a neutrosophic field $F(I)$.

2. Preliminaries

Definition 2.1 [24]**:** Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $\{x, (\mu_A(x), \delta_A(x), \gamma_A(x)) : x \in X\}$, where $\mu_A(x)$, $\delta_A(x)$ *and* $\gamma_A(x)$ represent the degree of membership, the degree of indeterminacy, and the degree of non-membership respectively of each element $x \in X$ to the set A .

Definition 2.2 [10]: Let K be a field, the neutrosophic file generated by $\langle K \cup I \rangle$ which is denoted by $K(I) = \langle K \cup I \rangle$.

Definition 2.3 [9]: Classical neutrosophic number has the form $a + bI$ where a, b are real or complex numbers and I is the indeterminacy such that $0 \cdot I = 0$ and $I^2 = I$ which results that $I^n = I$ for all positive integers n.

Definition 2.4 (Neutrosophic matrix) [16]. Let $M_{m\times n} = \{(a_{ij}) : a_{ij} \in K(I)\}$, where $K(I)$ is a neutrosophic field. We call to be the neutrosophic matrix.

3. Main discussion

Definition 3.1:

Let $M = A + BI$ a neutrosophic $n$square matrix, where A and B are two n squares matrices, then M is called an invertible neutrosophic n square matrix, if and only if there exists an n square matrix $S = S_1 + S_2 I$, where S_1 and S_2 are two n square matrices such that

$S \cdot M = M \cdot S = U_{n\times n}$, where $U_{n\times n}$ denotes the $n \times n$ identity matrix.

Definition 3.2:

Let $M = A + BI$ be a neutrosophic n square matrix. The determinant of M is defined as

$detM = detA + I[det(A + B) - detA]$.

Theorem 3.3:

Let $M = A + BI$ a neutrosophic square $n \times n$ matrix, where A, B are two squares $n \times n$ matrices, then M is invertible if and only if A and $A + B$ are invertible matrices and

$M^{-1} = A^{-1} + I[(A + B)^{-1} - A^{-1}]$.

Proof:

If A and $A + B$ are invertible matrices, then $(A + B)^{-1}$, A^{-1} are existed, and

$M^{-1} = A^{-1} + I[(A + B)^{-1} - A^{-1}]$ exists too. Now to prove $M^{-1}$is the inverse of M,

$MM^{-1} = (A + BI) \cdot (A^{-1} + I[(A + B)^{-1} - A^{-1}])$

$$= AA^{-1} + I[A(A + B)^{-1} - AA^{-1} + B \cdot A^{-1} + B(A + B)^{-1} - BA^{-1}]$$

$$= U_{n\times n} + I[(A + B)(A + B)^{-1} - U_{n\times n}]$$

$$= U_{n\times n} + I[U_{n\times n} - U_{n\times n}] = U_{n\times n} = M^{-1}M.$$

conversely, we suppose that M is invertible, thus there is a matrix $S = S_1 + S_2 I$, with the property $M \cdot S = S \cdot M = U_{n\times n}$.

$MS = (A + BI)(S_1 + S_2 I) = AS_1 + I[(A + B)(S_1 + S_2) - AS_1] = U_{n\times n} + 0_{n\times n}$=$SM$. Hence, we get:

(a)$S_1 A = AS_1 = U_{n\times n}, thus\ A\ is\ invertible\ and\ A^{-1} = S_1$.

(b)$(A + B)(S_1 + S_2) - AS_1 = (S_1 + S_2)(A + B) - S_1 A = O_{n\times n}$, thus,

$(S_1 + S_2)(A + B) = (A + B)(S_1 + S_2) = AS_1 = U_{n\times n}$. This implies that $(A + B)$ is invertible.

Theorem 3.4:

M is invertible matrix if and only if $detM \neq 0$.

Proof:

From **Theorem 3.3** we find that M is invertible matrix if and only if $A+B, A$ are two invertible matrices, hence $det[A+B] \neq 0, detA \neq 0$ which means

$detM = detA + I[\det(A+B) - detA] \neq 0$.

Example 3.5:

Consider the following neutrosophic matrix

$M = A + BI = \begin{pmatrix} 1 & -1+I \\ I & 2+I \end{pmatrix}$. Where $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(a)$detA = 2, A + B = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}, \det(A+B) = 3, detM = 2 + I[3-2] = 2 + I \neq 0, hence\ M\ is\ invertible.$

(b) We have $A^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, (A+B)^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}, thus\ M^{-1} = (A^{-1}) + I[(A+B)^{-1} - A^{-1}]$

$= \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} + I \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} - \frac{1}{2}I \\ -\frac{1}{3}I & \frac{1}{2} - \frac{1}{6}I \end{pmatrix}.$

(c) We can compute $MM^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = U_{2\times2}$.

Theorem 3.6:

Let $M = A + BI$ be a neutrosophic n square matrix, were A and B are two n square matrices, then

3.6.1) $M^r = A^r + I[(A+B)^r - A^r]$.

3.6.2)$M\ is\ nilpotent\ if\ and\ only\ if\ A, A+B\ are\ nilpotent.$

3.6.3)$M\ isi\ dempotent\ if\ and\ only\ if\ A, A+B\ are\ idempotent.$

Proof:

(3.6.1) By using mathematical induction, it easy to see $P(r=1)$ is true.

Suppose$P(k)$, then we must prove $P(k+1)$ is true like the following

$M^{k+1} = M^k \cdot M = (A^k + I[(A+B)^k - A^k]) \cdot (A + IB)$

$= A^{k+1} + I[(A^k \cdot B + (A+B)^k \cdot A + (A+B)^k \cdot B - A^k \cdot A - A^k \cdot B)]$

$= A^{k+1} + I[(A+B)^k \cdot (A+B) - A^{k+1}]$

$= A^{k+1} + I[(A+B)^{k+1} - A^{k+1}]$.

(2) $M$is nilpotent $if\ and\ only\ if\ \exists\, r \in N^{+};\ M^{r} = 0$, this is equivalent to

$A^{r} + I[(A+B)^{r} - A^{r}] = 0$, thus

$A^{r} = (A+B)^{r} = 0$. Which is equivalent to

$A, A+B$ are nilpotent.

(3) The proof is similar to (2).

Theorem 3.7:

Let $M = A + BI$ and $N = C + DI$ be two neutrosophic n square matrices, then

(3.7.1) $det(M \cdot N) = detM \cdot detN$.

(3.7.2) $\det(M^{-1}) = (detM)^{-1}$.

(3.7.3) $detM$ =1 if and only if $detA = det(A+B) = 1$.

Proof:

(a) $M \cdot N = A \cdot C + I[B \cdot C + B \cdot D + A \cdot D]$

$= A \cdot C + I[(A+B)(C+D) - A \cdot C]$.

$\det(M \cdot N) = \det(A \cdot C) + I\big[\det\big((A+B)(C+D)\big) - \det(A \cdot C)\big]$,

$= \det A \cdot \det C + I[\det(A+B) \cdot \det(C+D) - \det(A \cdot C)]$,

$= \det A \cdot \det C + I[\det(A+B) \cdot \det(C+D) - \det A \cdot \det C]$,

$= (\det A + I[\det(A+B) - \det A]) \cdot (\det C + I[\det(C+D) - \det C])$,

$= \det M \cdot \det N$.

(b) We have

$\det(MM^{-1}) = \det(U_{n\times n}) = 1, thus\ detM.\det(M^{-1}) = 1, so\ that\ \det(M^{-1}) = (detM)^{-1}$.

(c) $detM = 1$ is equivalent to $\det A + I[\det(A+B) - \det A] = 1,$ thus it is equivalent to

$detA = det(A+B) = 1$.

Remark: The result in the section (c) can be generalized easily to the following fact:

$detM = detA$ if and only if $detA = det(A+B)$.

Definition 3.8:

Let $M = A + BI$ be a neutrosophic n square matrix, where A and B are two n squarematrices. M is satisfying the orthogonality property if and only if $M \cdot M^{T} = U_{n\times n}$.

Theorem 3.9:

Let $M = A + BI$ a neutrosophic n square matrix, then

(a) $M\ is$ orthogonal $if\ and\ only\ if\ A, B$ are two orthogonal matrices .

(b) $If\ M\ is$ orthogonal, then $detM \in \{1, -1, -1 + 2I, 1 - 2I\}$.

Proof:

(a) $M\ is$ orthogonal neutrosophic matrix $if\ and\ only\ if\ M^T = M^{-1}$, this is equivalent to

$A^T + B^T I = A^{-1} + I[(A + B)^{-1} - A^{-1}]$, thus

$A^{-1} = A^T, (A + B)^{-1} - A^{-1} = B^T$. This is equivalent to

$A^{-1} = A^T and\ (A + B)^{-1} = B^T + A^{-1} = B^T + A^T = (A + B)^T$. Thus the proof is complete.

(b) If M is orthogonal, we get that $det(M \cdot M^T) = det(U_{n\times n}) = 1$. This implies

$detM \cdot detM^T = 1,$

$(\det M)^2 = 1$, hence

$detM \in \{1, -1, -1 + 2I, 1 - 2I\}$.

Definition 3.10:

Let $M = A + BI$ be a square neutrosophic matrix, we say that M is diagonalizable if and only if there is an invertible neutrosophic matrix $S = C + DI$ such that $S^{-1}MS = D$. Where D is a diagonal neutrosophic matrix$\left(i.e.\ d_{ij} = 0\ \ \forall i \neq j, and\ d_{ij} \neq 0\ \ \forall i = j\right)$.

Theorem 3.11:

Let $M = A + BI$ be any square neutrosophic matrix. Then M is diagonalizable if and only if $A, A + B$ are diagonalizable.

Proof:

Consider a diagonalizable neutrosophic matrix M, then there exists an invertible matrix S such that $S^{-1}MS = K\left(k_{ij}\right)$(3.11,1).

Now, to compute the entries elements k_{ij} , solve (3.1.11) as follows:

$\left[C^{-1} + I[(C + D)^{-1} - C^{-1}]\right](A + BI)(C + DI) = \left[C^{-1} + I[(C + D)^{-1} - C^{-1}]\right]\left[AC + I[(A + B)(C + D) - AC]\right] = C^{-1}AC + I[(C + D)^{-1}(A + B)(C + D) - C^{-1}AC] = D_1 + (D_2 - D_1)I = K$. Where K is a diagonal matrix, thus D_1, D_2 are diagonal, and $A, A + B$ are diagonalizable. Conversely, assume that $A, A + B$ are diagonalizable, then there are C, D, where $C^{-1}AC = D_1, D^{-1}(A + B)D = D_2$. Put $S = C + (D - C)I$.

Now we compute $S^{-1}MS = [C^{-1} + I[D^{-1} - C^{-1}]](A + BI)(C + (D - C)I)$

$= [C^{-1} + I[D^{-1} - C^{-1}]][AC + I[(A + B)(D) - AC]] = C^{-1}AC + I[D^{-1}(A + B)D - C^{-1}AC]$

$= D_1 + (D_2 - D_1)I = K$. Thus, M is diagonalizable, that is because D_1, D_2 are diagonal matrices.

Remark 3.12:

If C is the diagonalization matrix of A, and D is the diagonalization matrix of $A + B$, then

$S = C + (D - C)I$ is the diagonalization matrix of $M = A + BI$.

Example 3.13:

Consider the neutrosophic matrix defined in Example 3.5, we have:

(a) A is a diagonalizable matrix. Its diagonalization matrix is $C = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, the corresponding diagonal matrix is $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, we can see that $C^{-1}AC = D_1$. Also, the diagonalization matrix of $A + B$ is $D = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$, the corresponding diagonal matrix is $D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. It is easy to check that $D^{-1}(A + B)D = D_2$.

(b) Since $A, A + B$ are diagonalizable, then M is diagonalizable. The neutrosophic diagonalization matrix of M is $S = C + (D - C)I = \begin{pmatrix} 1 & 1 - I \\ -\frac{1}{2}I & -1 + 2I \end{pmatrix}$. The corresponding diagonal matrix is

$L = D_1 + I[D_2 - D_1] = \begin{pmatrix} 1 & 0 \\ 0 & 2 + I \end{pmatrix}$.

(c) It is easy to see that $S^{-1} = C^{-1} + I[D^{-1} - C^{-1}] = \begin{pmatrix} 1 & 1 - I \\ \frac{1}{2}I & -1 + 2I \end{pmatrix}$.

(d) We can compute $S^{-1}MS = \begin{pmatrix} 1 & 1 - I \\ \frac{1}{2}I & -1 + 2I \end{pmatrix}\begin{pmatrix} 1 & -1 + I \\ I & 2 + I \end{pmatrix}\begin{pmatrix} 1 & 1 - I \\ -\frac{1}{2}I & -1 + 2I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 + I \end{pmatrix} = L$.

Definition 3.14:

Let $M = A + BI$ be a $n$square neutrosophic matrix over the neutrosophic field $F(I)$, we say that $Z = X + YI$ is a neutrosophic Eigen vector if and only if $MZ = (a + bI)Z$. The neutrosophic number $a + bI$ is called the Eigen value of the eigen vector Z.

Theorem 3.15:

Let $M = A + BI$ be a n square neutrosophic matrix, then $a + bI$ is an eigen value of M if and only if a is an eigen value of A, and $a + b$ is an eigen value of $A + B$. As well as, the eigen vector of $M$is $Z = X + YI$ if and only if X is the corresponding eigen vector of A, and $X + Y$ is the corresponding eigen vector of $A + B$.

Proof:

We suppose that $Z = X + YI$ is an eigen vector of M with the corresponding eigen value $a + bI$, hence $MZ = (a + bI)Z$, this implies

$(A + BI)(X + YI) = (a + bI)(X + YI), thus\ AX + I[(A + B)(X + Y) - AX] = aX + I[(a + b)(X + Y) - aX]$. We get:

$AX = aX, (A + B)(X + Y) = (a + b)(X + Y)$, so that X is an eigen vector of A, $X + Y$ is an eigen vector of $A + B$. The corresponding eigen value of X is a, and the corresponding eigen value of $X + Y$ is $a + b$.

For the converse, we assume that X is an eigen vector of A with a as the corresponding eigen value, and $X + Y$ is an eigen vector of $A + B$ with $a + b$ as the corresponding eigen value, so that we get $AX = aX, (A + B)(X + Y) = (a + b)(X + Y)$.

Let us compute

$MZ = (A + BI)(X + YI) = AX + I[(A + B)(X + Y) - AX]$

$= aX + I[(a + b)(X + Y) - aX] = (a + bI)(X + YI) = (a + bI)Z$. Thus $Z = X + YI$ is an eigen vector of M with $a + bI$ as a neutrosophic eigen value.

Theorem 3.16:

The eigen values of a neutrosophic matrix $M = A + BI$ can be computed by solving the neutrosophic equation $det(M - (a + bI)\ U_{n\times n}) = 0$.

Proof:

We have $det(M - (a + bI)U_{n\times n}) = \det([A - aU_{n\times n}] + I[B - bU_{n\times n}])$

$= \det([A - aU_{n\times n}] + I[\det((A + B) - (a + b)U_{n\times n}) - \det[A - aU_{n\times n}]]$. Thus, the equation $det(M - (a + bI)\ U_{n\times n}) = 0$ is equivalent to

$\det([A - aU_{n\times n}] = 0$ (3,16,1)$, and$ $[\det((A + B) - (a + b)U_{n\times n}) - \det[A - aU_{n\times n}] = 0$ (3.16.2).

From equation (3,16,1), we get a as eigen value of A, and from (3.16.2) we get

$[\det((A + B) - (a + b)U_{n\times n}) = \det[A - aU_{n\times n}] = 0$, thus$a + b$ is an eigen value of $A + B$.

Example 3.17:

Consider M the neutrosophic matrix defined in Example 3.5, we have

(a) The eigen values of the matrix A are {1,2}, and {1,3} for the matrix $A+B$. This implies that the eigen values of the neutrosophic matrix M are

$\{1+(3-1)I, 1+(1-1)I, 2+(3-2)I, 2+(1-2)I\} = \{1+2I, 1, 2+I, 2-I\}$.

(b) If we solved the equation $\det(M-(a+bI)U_{n\times n})$=0 has been solved, the same values will be gotten.

(c) The eigen vectors of A are $\{(1,0),(1,-1)\}$,the eigen vectors of $A+B$ are $\{(1,-1/2),(0,1)\}$. Thus, the neutrosophic eigen vectors of M are

$\left\{(1,0)+I[(0,1)-(1,0)], (1,0)+I\left[\left(1,-\frac{1}{2}\right)-(1,0)\right], (1,-1)+I[(0,1)-(1,-1)], (1,-1)+\right.$

$\left.I\left[\left(1,-\frac{1}{2}\right)-(1,-1)\right]\right\} = \{(1,0)+I(-1,1), (1,0)+I(0,-1/2), (1,-1)+I(-1,2), (1,-1)+I(0,1/2)\} =$

$\{(1-I, I), (1,-1/2\ I), (1-I,-1+2I), (1,-1+1/2\ I)\}$.

To determine the neutrosophic eigen vectors using Theorem 3.15. let X be an eigen vector of $A, and\ Y$ be an eigen vector of $A+B$, hence $X+[(Y)-X]I = X+(Y-X)I$ is an Eigen vector of $M = A+BI$.

Conclusion

In this article, we have determined necessary and sufficient conditions for the invertibility and diagonalization of neutrosophic matrices. Also, we have found an easy algorithm to compute the inverse of a neutrosophic matrix and its Eigen values and vectors.

As a future research direction, we aim to find the representation of neutrosophic matrices by linear transformations in neutrosophic vector spaces.

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Decision-Making Approach Based on Correlation Coefficient with its Properties Under Interval-Valued Neutrosophic hypersoft set environment

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Abstract: The correlation coefficient between two variables plays an essential part in statistics. In addition, the preciseness in the assessment of correlation relies on information from the set of discourse. The data collected for various statistical studies is full of ambiguities. In this article, we investigated some fundamental concepts that strengthen the current research structure, such as soft sets, hypersoft sets, neutrosophic hypersoft set (NHSS), and interval-valued neutrosophic hypersoft set (IVNHSS). The IVNHSS is an extension of the interval-valued neutrosophic soft set. The main objective of this paper is to develop the concept of correlation and weighted correlation coefficients for IVNHSS. We also, discuss the desirable properties of correlation and weighted correlation coefficients under the IVNHSS environment in the following research. Also, develop a decision-making technique based on the proposed correlation coefficient. Through the developed methodology, a technique for solving decision-making concerns is planned. Moreover, an application of the projected methods is presented for the selection of a medical superintendent in a public hospital.

Keywords: Hypersoft set, NHSS, IVNHSS, correlation coefficient, weighted correlation coefficient

1. Introduction

Correlation plays a vital role in statistics and engineering; through correlation analysis, the joint relationship of two variables can be used to evaluate the interdependence of two variables. Although probabilistic methods have been applied to various practical engineering problems, there are still some obstacles to probabilistic strategies. For example, the probability of the process depends on the large amount of data collected, which is random. However, large complex systems have many fuzzy uncertainties, so it is difficult to obtain accurate probability events. Therefore, due to limited quantitative information, results based on probability theory do not always provide useful information for experts. In addition, in practical applications, sometimes there is not enough data to correctly process standard statistical data. Due to the aforementioned obstacles, results based on probability theory are not always available to experts. Therefore, probabilistic methods are usually insufficient to resolve such inherent uncertainties in the data. Many researchers in the world have proposed and suggested different methods to solve problems that contain uncertainty. First, Zadeh developed the concept of a fuzzy set (FS) [1] to solve those problems that contain uncertainty and ambiguity. It can be seen that in some cases, FS cannot solve this situation. To overcome such situations, Turksen [2] proposed the idea of interval-valued fuzzy sets (IVFS). In some cases, we must carefully consider membership as a non-member value in the proper representation of objects that cannot be processed by FS or IVFS under conditions of uncertainty. To overcome these difficulties, Atanasov proposed the idea of intuitionistic fuzzy sets (IFSs) [3]. The theory proposed by Atanassov only deals with insufficient data due to membership and non-membership values, but IFS cannot deal with incompatible and imprecise information.

Molodtsov [4] proposed a general mathematical tool to deal with uncertain, ambiguous, and undefined substances, called soft sets (SS). Maji et al. [5] Expanded the work of SS and developed some operations with properties. In [6], they also use SS theory to make decisions. Ali etc. [7] Modified the Maji method of SS and developed some new operations with its properties. By using different operators, they proved De Morgan's laws [8] under the SS environment. Cagman and Enginoglu [9] proposed the concept of soft matrices with operations and discussed their properties. They also introduced a decision-making method to solve problems that contain uncertainty. In [10], they modified the operation proposed by Molodtsov's SS. Maji et al. [11] proposed the concept of fuzzy soft set (FSS) by combining FS and SS. They also proposed an Intuitionistic Fuzzy Soft Set (IFSS) with basic operations and attributes [12]. Atanassov and Gargov [13] extended the theory of IFS and established a new concept called Interval Valued Intuitionistic Fuzzy Set (IVIFS). Zulqarnain et al. [14] utilized the intuitionistic fuzzy soft matrices for disease diagnosis. Yang et al. [15] proposed the concept of interval-valued fuzzy soft sets with operations (IVFSS) and proved some important results by combining IVFS and SS, and they also used the developed concepts for decision-making. Jiang et al. [16] proposed the concept of interval-valued intuitionistic fuzzy soft sets (IVIFSS) by extending IVIFS. They also proposed the necessity and possibility operations for IVIFSS with their properties. Zulqarnain and Saeed [17] developed some operations for interval-valued fuzzy soft matrix (IVFSM) and proposed a decision-making technique to solve the decision making problem. They also applied the IVFSM for decision making [18], a comparison among fuzzy soft matrices and IVFSM in [19]. Ma and Rani [20] constructed an algorithm based on IVIFSS and used the developed algorithm for decision-making. Zulqarnain et al. [21] developed the aggregation operators for IVIFSS. They also extended the TOPSIS technique under IVIFSS and utilized the presented approach to solving multi-attribute decision making problem. Zulqarnain et al. [22] utilized fuzzy TOPSIS to solve the multi-criteria decision-making (MCDM) problem.

Maji [23] offered the idea of a neutrosophic soft set (NSS) with necessary operations and properties. The idea of the possibility NSS was developed by Karaaslan [24] and introduced a possibility of neutrosophic soft decision-making method to solve those problems which contain uncertainty based on And-product. Broumi [25] developed the generalized NSS with some operations and properties and used the proposed concept for decision making. To solve MCDM problems with single-valued Neutrosophic numbers (SVNNs) presented by Deli and Subas in [26], they constructed the concept of cut sets of SVNNs. Based on the correlation of IFS, the term CC of SVNSs [27] was introduced. In [28] the idea of simplified NSs introduced with some operational laws and aggregation operators such as weighted arithmetic and weighted geometric average operators. They constructed an MCDM method on the base of proposed aggregation operators. Zulqarnain et al. [29] presented the generalized version of neutrosophic TOPSIS and utilized the considered technique to solve the MCDM problem. Hung and Wu [30] proposed the centroid method to calculate the CC of IFSs and extended the proposed method to IVIFS. Bustince and Burillo [31] introduced the correlation and CC of IVIFS and proved the decomposition theorems on the correlation of IVIFS. Hong [32] and Mitchell [33] also established the CC for IFSs and IVIFSs respectively. Garg and Arora introduced the correlation measures on IFSS and constructed the TOPSIS technique on developed correlation measures [34]. Huang and Guo [35] gave an improved CC on IFS with their properties, they also established the coefficient of IVIFS. Singh et al. [36] developed the one- and two- parametric generalization of CC on IFS and used the proposed technique in multi-attribute group decision-making problems. Zulqarnain et al. [37] proposed the aggregation operators for Pythagorean fuzzy soft sets and developed a decision-making approach to solving multi-criteria decision making problems. Sometimes experts considered the sub-attributes of the given attributes in the decision-making process. In such situations, all the above-discussed theories cannot provide any information to experts about sub-attributes of the given attributes.

To overcome the above-mentioned limitations Smarandche [38] extended the concept of soft sets to hypersoft sets (HSS) by replacing function F of one parameter to multi-parameter (sub-attributes) function defined on the cartesian product of n different attributes. The established HSS is more flexible than soft sets and more suitable for decision-making environments. He also presented the further extension of HSS, such as crisp HSS, fuzzy HSS, intuitionistic fuzzy HSS, neutrosophic HSS, and plithogenic HSS. Nowadays, the HSS theory and its extensions rapidly progress, many researchers developed different operators and properties based on HSS and its extensions [39-42]. Abdel-Basset et al. [43] plithogenic set theory was used to eliminate uncertainty and to evaluate the financial performance of the manufacturing industry. They then used the VIKOR and TOPSIS methods to determine the weight of the financial ratio using the AHP method to achieve this goal. Abdel-Basset et al. [44] presented an effective combination of plithogenic aggregate operations and quality feature deployment procedures. The advantage of this combination is to improve accuracy, as a result, summarizes the decision-makers. Zulqarnain et al. [45] extended the TOPSIS technique to an intuitionistic fuzzy hypersoft set and developed some aggregation operators under-considered environment. They also established a decision-making approach based on developed TOPSIS to solve the MADM problem.

Basset et al. [46] proposed the type 2 neutrosophic numbers with some operational laws. They also developed the aggregation operators for type 2 neutrosophic numbers and developed the decision-making technique based on developed operators to solve the MADM problem. Basset et al. [47] established the AHP and VIKOR methods for neutrosophic numbers and utilized them for supplier selection. Basset et al. [48] presented the robust ranking technique under a neutrosophic environment for the green supplier chain management. Basset et al. [49] presented a neutrosophic multi-criteria decision-making technique to aid the patient and physician to know if a patient is suffering from heart failure Smarandache's NHSS is unable to solve those problems where the truthness, indeterminacy, and falsity object of any sub-attribute is given in interval form. We know that generally, the values vary, for example, medical experts generate the report of any patient we can observe that the HP level of blood varies from 0-17.5, these values can not be handled by NHSS. To handle the above-discussed environment we need to develop IVNHSS. The developed IVNHSS competently deals with uncertain problems comparative to NHSS and other existing studies. The main objective of this research is to introduce CC and WCC for IVNHSS.

The following research is organized as follows: In Section 2, we review some basic definitions used in the following sequels, such as SS, NSS, NHSS, and IVNHSS, etc. Section 3, established the notions of CC and WCC under IVNHSS and discussed their desirable properties. An algorithm and decision-making method developed in section 4 is based on the proposed CC. We also used the established approach to solve decision making problems in an uncertain environment. Finally, the conclusion is made in section 5.

2. Preliminaries

In this section, we recollect some basic definitions which are helpful to build the structure of the following manuscript such as soft set, hypersoft set, and neutrosophic hypersoft set.

Definition 2.1 [4]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathsf{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathsf{A})$ is called a soft set over $\mathcal{U}$ and its mapping is given as

$$\mathcal{F}: \mathsf{A} \rightarrow \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F}, \mathsf{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \; if \; e \notin \mathsf{A}\}$$

Definition 2.2 [38]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3,..., k_n\}$,(n ≥ 1) be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of multi-attributes, where $1 \leq h \leq \alpha, 1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}})$ is said to be HSS over $\mathcal{U}$ and its mapping is defined as

$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} \rightarrow \mathcal{P}(\mathcal{U})$.

It is also defined as

$(\mathcal{F}, \ddot{\mathsf{A}}) = \{\breve{a}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}): \breve{a} \in \ddot{\mathsf{A}}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) \in \mathcal{P}(\mathcal{U})\}$

Definition 2.3 [38]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3,..., k_n\}$,(n ≥ 1) be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$ and $NS^{\mathcal{U}}$ be a collection of all neutrosophic subsets over $\mathcal{U}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}})$ is said to be NHSS over $\mathcal{U}$ and its mapping is defined as

$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} \rightarrow NS^{\mathcal{U}}$.

It is also defined as

$(\mathcal{F}, \ddot{\mathsf{A}}) = \{(\breve{a}, \mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a})): \breve{a} \in \ddot{\mathsf{A}}, \mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a}) \in NS^{\mathcal{U}}\}$, where $\mathcal{F}_{\ddot{\mathsf{A}}}(\breve{a}) = \{\langle \delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta), \gamma_{\mathcal{F}(\breve{a})}(\delta) \rangle : \delta \in \mathcal{U}\}$, where $\sigma_{\mathcal{F}(\breve{a})}(\delta)$, $\tau_{\mathcal{F}(\breve{a})}(\delta)$, and $\gamma_{\mathcal{F}(\breve{a})}(\delta)$ represent the truth, indeterminacy, and falsity grades of the attributes such as $\sigma_{\mathcal{F}(\breve{a})}(\delta)$, $\tau_{\mathcal{F}(\breve{a})}(\delta)$, $\gamma_{\mathcal{F}(\breve{a})}(\delta) \in [0, 1]$, and $0 \leq \sigma_{\mathcal{F}(\breve{a})}(\delta) + \tau_{\mathcal{F}(\breve{a})}(\delta) + \gamma_{\mathcal{F}(\breve{a})}(\delta) \leq 3$.

Example 2.4

Consider the universe of discourse $\mathcal{U} = \{\delta_1, \delta_2\}$ and $\mathfrak{L} = \{\ell_1 = Teaching\ methdology, \ell_2 = Subjects, \ell_3 = Classes\}$ be a collection of attributes with following their corresponding attribute values are given as teaching methodology = $L_1 = \{a_{11} = project\ base, a_{12} = class\ discussion\}$, Subjects = $L_2 = \{a_{21} = Mathematics, a_{22} = Computer\ Science, a_{23} = Statistics\}$, and Classes = $L_3 = \{a_{31} = Masters, a_{32} = Doctorol\}$. Let $\ddot{\mathsf{A}} = L_1 \times L_2 \times L_3$ be a set of attributes

$\ddot{\mathsf{A}} = L_1 \times L_2 \times L_3 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}, a_{23}\} \times \{a_{31}, a_{32}\}$

$$= \begin{Bmatrix} (a_{11}, a_{21}, a_{31}), (a_{11}, a_{21}, a_{32}), (a_{11}, a_{22}, a_{31}), (a_{11}, a_{22}, a_{32}), (a_{11}, a_{23}, a_{31}), (a_{11}, a_{23}, a_{32}), \\ (a_{12}, a_{21}, a_{31}), (a_{12}, a_{21}, a_{32}), (a_{12}, a_{22}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{12}, a_{23}, a_{31}), (a_{12}, a_{23}, a_{32}), \end{Bmatrix}$$

$\ddot{\mathsf{A}} = \{\breve{a}_1, \breve{a}_2, \breve{a}_3, \breve{a}_4, \breve{a}_5, \breve{a}_6, \breve{a}_7, \breve{a}_8, \breve{a}_9, \breve{a}_{10}, \breve{a}_{11}, \breve{a}_{12}\}$

Then the NHSS over $\mathcal{U}$ is given as follows

$(\mathcal{F}, \ddot{\mathsf{A}}) =$

$$\begin{Bmatrix} (\breve{a}_1, (\delta_1, (.6, .3, .8)), (\delta_2, (.9, .3, .5))), (\breve{a}_2, (\delta_1, (.5, .2, .7)), (\delta_2, (.7, .1, .5))), (\breve{a}_3, (\delta_1, (.5, .2, .8)), (\delta_2, (.4, .3, .4))), \\ (\breve{a}_4, (\delta_1, (.2, .5, .6)), (\delta_2, (.5, .1, .6))), (\breve{a}_5, (\delta_1, (.8, .4, .3)), (\delta_2, (.2, .3, .5))), (\breve{a}_6, (\delta_1, (.9, .6, .4)), (\delta_2, (.7, .6, .8))), \\ (\breve{a}_7, (\delta_1, (.6..5, .3)), (\delta_2, (.4, .2, .8))), (\breve{a}_8, (\delta_1, (.8, .2, .5)), (\delta_2, (.6, .8, .4))), (\breve{a}_9, (\delta_1, (.7, .4, .9)), (\delta_2, (.7..3, .5))), \\ (\breve{a}_{10}, (\delta_1, (.8, .4, .6)), (\delta_2, (.7, .2, .9))), (\breve{a}_{11}, (\delta_1, (.8, .4, .5)), (\delta_2, (.4, .2, .5))), (\breve{a}_5, (\delta_1, (.7, .5, .8)), (\delta_2, (.7, .5, .9))) \end{Bmatrix}$$

Definition 2.5 [42]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}$, $(n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{A} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$ and $IVNS^{\mathcal{U}}$ be a collection of all interval-valued neutrosophic subsets over $\mathcal{U}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{A})$ is said to be IVNHSS over $\mathcal{U}$ and its mapping is defined as

$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{A} \rightarrow IVNS^{\mathcal{U}}$.

It is also defined as

$(\mathcal{F}, \ddot{A}) = \{(\breve{a}_k, \mathcal{F}_{\ddot{A}}(\breve{a}_k)) : \breve{a}_k \in \ddot{A}, \mathcal{F}_{\ddot{A}}(\breve{a}_k) \in NS^{\mathcal{U}}\}$, where $\mathcal{F}_{\ddot{A}}(\breve{a}) = \{\langle \delta, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta), \tau_{\mathcal{F}(\breve{a}_k)}(\delta), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta) \rangle : \delta \in \mathcal{U}\}$, where $\sigma_{\mathcal{F}(\breve{a}_k)}(\delta)$, $\tau_{\mathcal{F}(\breve{a}_k)}(\delta)$, and $\gamma_{\mathcal{F}(\breve{a}_k)}(\delta)$ represent the interval truth, indeterminacy, and falsity grades of the attributes such as $\sigma_{\mathcal{F}(\breve{a}_k)}(\delta) = \left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)\right]$, $\tau_{\mathcal{F}(\breve{a}_k)}(\delta) = \left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)\right]$, $\gamma_{\mathcal{F}(\breve{a}_k)}(\delta) = \left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)\right]$, where $\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)$, $\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta)$, $\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta) \subseteq [0, 1]$, and $0 \leq \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta) \leq 3$.

Simply an interval-valued neutrosophic hypersoft number (IVNHSN) can be expressed as $\boldsymbol{\mathcal{F}} = \left\{\left[\boldsymbol{\sigma}^{\ell}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}), \boldsymbol{\sigma}^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta})\right], \left[\boldsymbol{\tau}^{\ell}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}), \boldsymbol{\tau}^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta})\right], \left[\boldsymbol{\gamma}^{\ell}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}), \boldsymbol{\gamma}^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta})\right]\right\}$, where $0 \leq \boldsymbol{\sigma}^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}) + \boldsymbol{\tau}^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}) + \boldsymbol{\gamma}^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}) \leq 3$.

3. Correlation Coefficient for Interval-Valued Neutrosophic Hypersoft Set

In this section, the concept of correlation coefficient and weighted correlation coefficient on NHSS has been proposed with some basic properties.

Definition 3.1

Let $(\mathcal{F}, \ddot{A}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G}, \ddot{\mathbb{A}}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs defined over a universe of discourse $\mathcal{U}$. Then, the informational interval neutrosophic energies of $(\mathcal{F}, \ddot{A})$ and $(\mathcal{G}, \ddot{\mathbb{A}})$ can be described as follows:

$$\varsigma_{IVNHSS}(\mathcal{F}, \ddot{A}) = \sum_{k=1}^{m} \sum_{i=1}^{n} \left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 \right) \quad (1)$$

$$\varsigma_{IVNHSS}(\mathcal{G}, \ddot{\mathbb{A}}) = \sum_{k=1}^{m} \sum_{i=1}^{n} \left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 \right). \quad (2)$$

Definition 3.2

Let $(\mathcal{F}, \ddot{A}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G}, \ddot{\mathbb{A}}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs defined over a universe of discourse $\mathcal{U}$. Then, the correlation measure between $(\mathcal{F}, \ddot{A})$ and $(\mathcal{G}, \ddot{\mathbb{A}})$ can be described as follows:

$\boldsymbol{\mathcal{C}}_{\boldsymbol{IVNHSS}}((\boldsymbol{\mathcal{F}}, \ddot{A}), (\boldsymbol{\mathcal{G}}, \ddot{\mathbb{A}})) =$

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}\boldsymbol{\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i) +} \\ \boldsymbol{\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)}\end{pmatrix} \tag{3}$$

Proposition 3.3

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{Ѫ}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs and $\mathcal{C}_{IVNHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ}))$ be a correlation between them, then the following properties hold.

1. $\mathcal{C}_{IVNHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ})) = \varsigma_{IVNHSS}(\mathcal{F},\ddot{A})$
2. $\mathcal{C}_{IVNHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ})) = \varsigma_{IVNHSS}(\mathcal{G},\ddot{Ѫ})$

Proof: The proof is trivial.

Definition 3.4

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{Ѫ}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs, then correlation coefficient between them given as $\delta_{IVNHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ}))$ and expressed as follows:

$$\delta_{IVNHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ})) = \frac{\mathcal{C}_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{Ѫ}))}{\sqrt{\varsigma_{IVNHSS}(\mathcal{F},\ddot{A})} * \sqrt{\varsigma_{IVNHSS}(\mathcal{G},\ddot{Ѫ})}} \tag{4}$$

$$\boldsymbol{\delta_{IVNHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ}))} = \frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}\boldsymbol{\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i) +} \\ \boldsymbol{\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)}\end{pmatrix}}{\sqrt{\boldsymbol{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}} \sqrt{\boldsymbol{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)}}} \tag{5}$$

Proposition 3.5

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{Ѫ}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs, then CC satisfies the following properties

1. $0 \leq \delta_{IVNHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ})) \leq 1$
2. $\delta_{IVNHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ})) = \delta_{IVNHSS}((\mathcal{G},\ddot{Ѫ}), (\mathcal{F},\ddot{A}))$
3. If $(\mathcal{F},\ddot{A}) = (\mathcal{G},\ddot{Ѫ})$, that is $\forall\ i,\ k,\ \sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\ \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\ \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, and $\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\ \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, then $\delta_{NHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ})) = 1$.

Proof 1. $\delta_{IVNHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ})) \geq 0$ is trivial, here we only need to prove that $\delta_{IVNHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ})) \leq 1$.

From equation 3, we have

$$\delta_{IVNHSS}((\mathcal{F},\ddot{A}), (\mathcal{G},\ddot{Ѫ})) = \sum_{k=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\end{pmatrix}$$

$$=$$

$$\sum_{k=1}^{m}\begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1) + \tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1) + \\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\end{pmatrix}$$

$$+$$

$$\sum_{k=1}^{m}\begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\end{pmatrix}$$

$$+\\ \vdots\\ +$$

$$\sum_{k=1}^{m}\begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\end{pmatrix}$$

$$\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))$$

$$=$$

$$\left\{\begin{matrix}\begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_1)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_1)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_1)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_1)+\tau^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_1)*\tau^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_1)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_1)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_1)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_1)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_1)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_1)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_1)\end{pmatrix}+\\ \begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_1)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_1)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_1)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_1)+\tau^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_1)*\tau^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_1)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_1)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_1)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_1)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_1)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_1)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_1)\end{pmatrix}\\ +\\ \vdots\\ +\\ \begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_1)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_1)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_1)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_1)+\tau^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_1)*\tau^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_1)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_1)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_1)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_1)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_1)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_1)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_1)\end{pmatrix}\end{matrix}\right\}$$

$$+$$

$$\left\{\begin{matrix}\begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_2)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_2)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_2)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_2)+\tau^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_2)*\tau^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_2)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_2)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_2)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_2)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_2)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_2)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_2)\end{pmatrix}+\\ \begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_2)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_2)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_2)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_2)+\tau^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_2)*\tau^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_2)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_2)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_2)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_2)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_2)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_2)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_2)\end{pmatrix}\\ +\\ \vdots\\ +\\ \begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_2)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_2)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_2)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_2)+\tau^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_2)*\tau^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_2)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_2)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_2)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_2)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_2)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_2)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_2)\end{pmatrix}\end{matrix}\right\}$$

$$+\\ \vdots\\ +$$

$$\left\{\begin{matrix}\begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_n)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_n)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_n)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_n)+\tau^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_n)*\tau^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_n)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_n)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_n)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_n)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_n)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_n)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_n)\end{pmatrix}+\\ \begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_n)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_n)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_n)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_n)+\tau^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_n)*\tau^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_n)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_n)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_n)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_n)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_n)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_n)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_n)\end{pmatrix}\\ +\\ \vdots\\ +\\ \begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_n)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_n)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_n)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_n)+\tau^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_n)*\tau^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_n)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_n)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_n)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_n)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_n)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_n)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_n)\end{pmatrix}\end{matrix}\right\}$$

$$= \sum_{k=1}^{m}\left(\begin{array}{c}\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)+\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)\\ +\cdots+\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)\end{array}\right)+$$
$$\sum_{k=1}^{m}\left(\begin{array}{c}\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)+\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)\\ +\cdots+\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)\end{array}\right)$$
$$\sum_{k=1}^{m}\left(\begin{array}{c}\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)+\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)\\ +\cdots+\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)\end{array}\right)$$

By using Cauchy-Schwarz inequality

$\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))^2 \leq$

$$\sum_{k=1}^{m}\left\{\begin{array}{c}\left(\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\\ +\left(\left(\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\\ +\left(\left(\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\end{array}\right\}$$
$$\times\sum_{k=1}^{m}\left\{\begin{array}{c}\left(\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\\ +\left(\left(\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\\ +\left(\left(\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\end{array}\right\}$$

$\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))^2 \leq$

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)+\left(\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)+\left(\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)\right)$$
$$\times\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)+\left(\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)+\left(\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)\right)$$

$\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))^2 \leq \varsigma_{IVNHSS}(\mathcal{F},\ddot{A}) \times \varsigma_{IVNHSS}(\mathcal{G},\ddot{\mathbb{A}})$.

Therefore, $\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))^2 \leq \varsigma_{IVNHSS}(\mathcal{F},\ddot{A}) \times \varsigma_{IVNHSS}(\mathcal{G},\ddot{\mathbb{A}})$. Hence, by using definition 3.4, we have

$\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) \leq 1$. So, $0 \leq \delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) \leq 1$.

Proof 2. The proof is obvious.

Proof 3. From equation 5, we have

$$\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = \frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\end{array}\right)}{\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)}}$$

As we know that

$\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, and $\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$. We get

$\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) =$

$$\frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}{\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}}$$

$\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = 1$

Thus, prove the required result.

Definition 3.6

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs. Then, their correlation coefficient is given as $\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))$ and defined as follows:

$$\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = \frac{\mathcal{C}_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))}{max\{\varsigma_{IVNHSS}(\mathcal{F},\ddot{A}),\varsigma_{IVNHSS}(\mathcal{G},\ddot{\mathbb{A}})\}} \quad (6)$$

$\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) =$

$$\frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\end{array}\right)}{max\left\{\begin{array}{c}\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right),\\ \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)\end{array}\right\}} \quad (7)$$

Proposition 3.7

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs. Then, CC satisfies the following properties

1. $0 \leq \delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) \leq 1$
2. $\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = \delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))$
3. If $(\mathcal{F},\ddot{A}) = (\mathcal{G},\ddot{\mathbb{A}})$, that is $\forall\ i,\ k,\ \sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\ \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\ \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, and $\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\ \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$. Then, $\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = 1$.

Proof 1. $\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) \geq 0$ is trivial, here we only need to prove that $\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) \leq 1$.

From equation 3, we have

$\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) =$

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\end{array}\right)$$

$$=$$

$$\sum_{k=1}^{m}\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\end{array}\right)$$

$$+$$

$$\sum_{k=1}^{m}\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\end{array}\right)$$

$$+$$
$$\vdots$$
$$+$$

$$\sum_{k=1}^{m}\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\end{array}\right)$$

$$\delta^{1}_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))$$

$$=$$

$$\left\{\begin{array}{c}
\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_1) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_1) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_1) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_1) + \tau^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_1) * \tau^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_1) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_1) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_1) + \\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_1) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_1) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_1) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_1)\end{array}\right) + \\
\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_1) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_1) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_1) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_1) + \tau^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_1) * \tau^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_1) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_1) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_1) + \\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_1) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_1) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_1) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_1)\end{array}\right) \\
+ \\ \vdots \\ + \\
\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_1) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_1) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_1) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_1) + \tau^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_1) * \tau^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_1) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_1) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_1) + \\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_1) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_1) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_1) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_1)\end{array}\right)
\end{array}\right\}$$

$$+$$

$$\left\{\begin{array}{c}
\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_2) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_2) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_2) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_2) + \tau^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_2) * \tau^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_2) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_2) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_2) + \\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_2) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_2) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_2) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_2)\end{array}\right) + \\
\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_2) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_2) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_2) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_2) + \tau^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_2) * \tau^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_2) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_2) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_2) + \\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_2) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_2) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_2) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_2)\end{array}\right) \\
+ \\ \vdots \\ + \\
\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_2) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_2) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_2) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_2) + \tau^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_2) * \tau^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_2) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_2) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_2) + \\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_2) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_2) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_2) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_2)\end{array}\right)
\end{array}\right\}$$

$$+ \\ \vdots \\ +$$

$$\left\{\begin{array}{c}
\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_n) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_n) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_n) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_n) + \tau^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_n) * \tau^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_n) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_n) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_n) + \\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_1)}(\delta_n) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_1)}(\delta_n) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_1)}(\delta_n) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_1)}(\delta_n)\end{array}\right) + \\
\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_n) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_n) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_n) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_n) + \tau^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_n) * \tau^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_n) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_n) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_n) + \\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_2)}(\delta_n) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_2)}(\delta_n) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_2)}(\delta_n) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_2)}(\delta_n)\end{array}\right) \\
+ \\ \vdots \\ + \\
\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_n) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_n) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_n) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_n) + \tau^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_n) * \tau^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_n) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_n) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_n) + \\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_m)}(\delta_n) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_m)}(\delta_n) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_m)}(\delta_n) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_m)}(\delta_n)\end{array}\right)
\end{array}\right\}$$

$$=$$

$$\sum_{k=1}^{m}\left(\begin{array}{c}\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right) + \left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right) \\ + \cdots + \left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n) * \sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n) + \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n) * \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)\end{array}\right) +$$

$$\sum_{k=1}^{m}\left(\begin{array}{c}\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right) + \left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2) * \tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right) \\ + \cdots + \left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n) * \tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n) + \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n) * \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)\end{array}\right)$$

$$\sum_{k=1}^{m}\left(\begin{array}{c}\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right) + \left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right) \\ + \cdots + \left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n) * \gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n) + \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n) * \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)\end{array}\right)$$

By using Cauchy-Schwarz inequality

$$\delta^{1}_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))^2 \leq$$

$$\sum_{k=1}^{m}\left\{\begin{array}{l}\left(\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\\ +\left(\left(\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\\ +\left(\left(\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\end{array}\right\}$$

$$\times\sum_{k=1}^{m}\left\{\begin{array}{l}\left(\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\\ +\left(\left(\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\\ +\left(\left(\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_1)\right)^2\right)+\left(\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_2)\right)^2\right)+\cdots+\left(\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_n)\right)^2\right)\right)\end{array}\right\}$$

$\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathcal{A}}))^2 \leq$

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)+\left(\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)+\left(\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)\right)$$

$$\times\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)+\left(\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)+\left(\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)\right)$$

$\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathcal{A}}))^2 \leq \varsigma_{IVNHSS}(\mathcal{F},\ddot{A}) \times \varsigma_{IVNHSS}(\mathcal{G},\ddot{\mathcal{A}})$.

Therefore, $\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathcal{A}}))^2 \leq \varsigma_{IVNHSS}(\mathcal{F},\ddot{A}) \times \varsigma_{IVNHSS}(\mathcal{G},\ddot{\mathcal{A}})$. Hence, by using definition 3.4, we have

$\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathcal{A}})) \leq 1$. So, $0 \leq \delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathcal{A}})) \leq 1$.

Proof 2. The proof is obvious.

Proof 3. From equation 5, we have

$$\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathcal{A}})) = \frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\end{array}\right)}{\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)}}$$

As we know that

$\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)=\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)=\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)=\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)=\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, and $\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)=\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)=\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$. We get

$\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathcal{A}})) =$

$$\frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}{\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}}$$

$\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathcal{A}})) = 1$

Thus, prove the required result.

Definition 3.8

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i,\left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right],\left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right],\left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathcal{A}}) = \left\{\left(\delta_i,\left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right],\left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right],\left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs. Then, their weighted correlation coefficient is given as $\delta_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathcal{A}}))$ and defined as follows:

$$\delta_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathcal{A}})) = \frac{\mathcal{C}_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathcal{A}}))}{\sqrt{\varsigma_{WIVNHSS}(\mathcal{G},\ddot{\mathcal{A}})}*\sqrt{\varsigma_{WIVNHSS}(\mathcal{G},\ddot{\mathcal{A}})}} \quad (8)$$

$$\delta_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = \frac{\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\end{array}\right)\right)}{\sqrt{\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)\right)}\sqrt{\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)\right)}} \quad (9)$$

Definition 3.9

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i,\left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right],\left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right],\left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i,\left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right],\left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right],\left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs. Then, their weighted correlation coefficient is given as $\delta^1_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))$ and defined as follows:

$$\delta^1_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = \frac{\mathcal{C}_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))}{max\{\varsigma_{WIVNHSS}(\mathcal{F},\ddot{A}),\varsigma_{WIVNHSS}(\mathcal{G},\ddot{\mathbb{A}})\}} \quad (10)$$

$$\delta^1_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = \frac{\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\begin{array}{c}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\end{array}\right)\right)}{max\left\{\begin{array}{c}\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)\right),\\ \sum_{k=1}^{m}\dot{\omega}_k\left(\sum_{k=1}^{m}\Omega_k\left(\sum_{i=1}^{n}\gamma_i\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)\right)\right)\end{array}\right\}} \quad (11)$$

If we consider $\Omega = \{\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\}$ and $\gamma = \{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\}$, then $\delta_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))$ and $\delta^1_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))$ are reduced to $\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))$ and $\delta^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))$ respectively.

Proposition 3.10

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i,\left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right],\left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right],\left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i,\left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right],\left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right],\left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs, then WCC between satisfies the following properties

1. $0 \leq \delta_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) \leq 1$
2. $\delta_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = \delta_{WIVNHSS}((\mathcal{G},\ddot{\mathbb{A}}),(\mathcal{F},\ddot{A}))$
3. If $(\mathcal{F},\ddot{A}) = (\mathcal{G},\ddot{\mathbb{A}})$, that is $\forall\ i,\ k,\ \sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\ \sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, $\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\ \tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, and $\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\ \gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i) = \gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)$, then $\delta_{WIVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = 1$.

Proof 1. Similar to proposition 3.5.

4. Application of Correlation Coefficient for Decision Making Under IVNHSS Environment

In this section, we proposed the algorithm based on CC under IVNHSS and utilize the proposed approach for decision making in real-life problems.

4.1 Algorithm for Correlation Coefficient under IVNHSS

Step 1. Pick out the set containing sub-attributes of parameters.

Step 2. Construct the IVNHSS according to experts in form of IVNHSNs.

Step 3. Find the informational interval neutrosophic energies for IVNHSS.

Step 4. Calculate the correlation between IVNHSSs by using the following formula

$\boldsymbol{\mathcal{C}_{IVNHSS}}((\boldsymbol{\mathcal{F}},\ddot{A}),(\boldsymbol{\mathcal{G}},\ddot{\mathbb{A}})) =$

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\sigma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\sigma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\tau^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\\ \gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma^{\mathcal{U}}_{\mathcal{F}(\breve{a}_k)}(\delta_i)*\gamma^{\mathcal{U}}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\end{pmatrix}$$

Step 5. Calculate the CC between IVNHSSs by using the following formula

$$\delta_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}})) = \frac{\mathcal{C}_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{G},\ddot{\mathbb{A}}))}{\sqrt{\varsigma_{IVNHSS}(\mathcal{F},\ddot{A})}*\sqrt{\varsigma_{IVNHSS}(\mathcal{G},\ddot{\mathbb{A}})}}$$

Step 6. Choose the alternative with a maximum value of CC.

Step 7. Analyze the ranking of the alternatives.

A flowchart of the above-presented algorithm can be seen in figure 1.

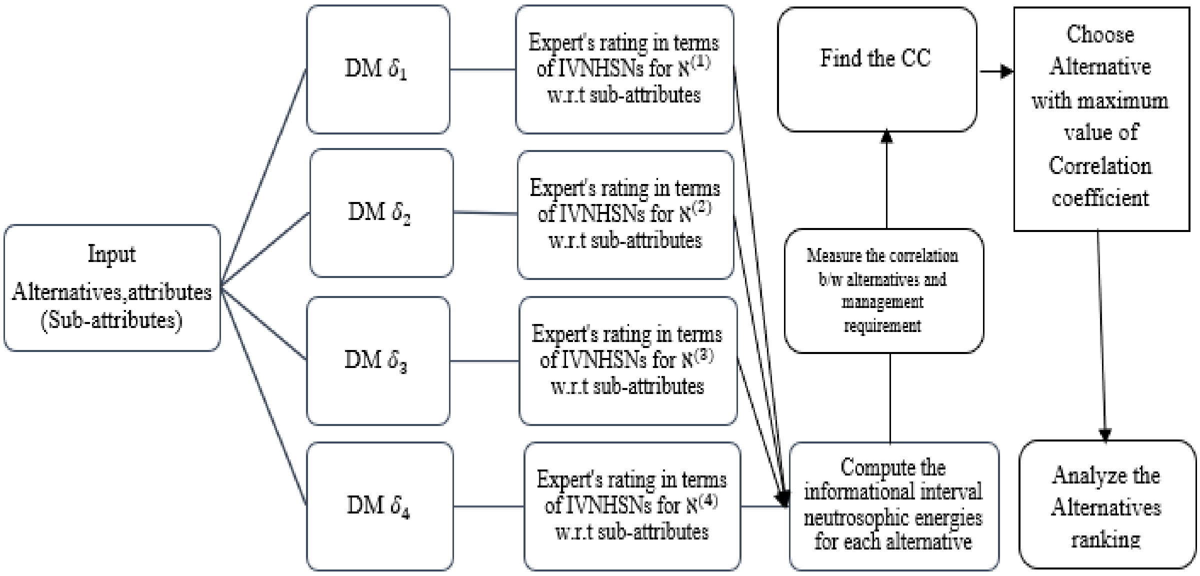

Figure 1: Flowchart for correlation coefficient under IVNHSS

4.1 Problem Formulation and Application of IVNHSS For Decision Making

Ministry of health advertises for the one vacant position of medical superintendent (MS) in hospital. Several medical experts apply for the post of MS, but referable probabilistic along with experience simply four experts are considered for further evaluation such as $\aleph = \{\aleph^1, \aleph^2, \aleph^3, \aleph^4\}$ be a set of alternatives. The secretary of the health department hires a committee of four decision-makers (DM) $\mathcal{U} = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ for the selection of MS. The team of DM decides the criteria (attributes) for the selection of MS position such as $\mathfrak{L} = \{\ell_1 = Experiance, \ell_2 = Dealing\ skills, \ell_3 = Qualification\}$ be a collection of attributes and their corresponding sub-attribute are given as Experience = ℓ_1 = $\{a_{11} = more\ than\ 20, a_{12} = less\ than\ 20\}$, Dealing skills = ℓ_2 = $\{a_{21} = public\ dealing, a_{22} = Staff\ dealing\}$, and Qualification = ℓ_3 = $\{a_{31} = Doctoral\ degree\ in\ medical\ education, a_{32} = Masters\ degree\ in\ medical\ education\}$. Let $\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3$ be a set of sub-attributes

$\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}\} \times \{a_{31}, a_{32}\}$

$= \begin{Bmatrix}(a_{11}, a_{21}, a_{31}), (a_{11}, a_{21}, a_{32}), (a_{11}, a_{22}, a_{31}), (a_{11}, a_{22}, a_{32}),\\ (a_{12}, a_{21}, a_{31}), (a_{12}, a_{21}, a_{32}), (a_{12}, a_{22}, a_{31}), (a_{12}, a_{22}, a_{32})\end{Bmatrix}$, $\mathfrak{L}' = \{\breve{a}_1, \breve{a}_2, \breve{a}_3, \breve{a}_4, \breve{a}_5, \breve{a}_6, \breve{a}_7, \breve{a}_8\}$ be a set of all multi sub-attributes. Each DM will evaluate the ratings of each alternative in the form of IVNHSNs under the considered multi sub-attributes. The developed method to find the best alternative is as follows.

4.1.1. Application of IVNHSS For Decision Making

Assume $\aleph = \{\aleph^1, \aleph^2, \aleph^3, \aleph^4\}$ be a set of alternatives who are shortlisted for interview and $\mathfrak{L} = \{\ell_1 = Experiance, \ell_2 = Dealing\ skills, \ell_3 = Qualification\}$ be a set of parameters for the selection of MS. Experience = ℓ_1 = $\{a_{11} = more\ than\ 20, a_{12} = \ less\ than\ 20\}$, Dealing skills = ℓ_2 = $\{a_{21} = public\ dealing, a_{22} = \ Staff\ dealing\}$, and Qualification = ℓ_3 = $\{a_{31} = Doctoral\ degree\ in\ medical\ education, a_{32} = \ Masters\ degree\ in\ medical\ education\}$. Let $\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3$ be a set of sub-attributes. The health ministry defines a criterion for the selection of MS for all alternatives in terms of IVNHSNs given in Table 1.

Table 1. Decision Matrix of Concerning Department

$\wp$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	([.3,.5], [.2,.4], [.2,.6])	([.2,.3], [.5,.7], [.1,.3])	([.5,.6], [.1,.3], [.4,.6])	([.2,.4], [.3,.5], [.3,.6])	([.2,.3], [.2,.4], [.4,.5])	([.4,.6], [.1,.3], [.2,.4])	([.6,.7], [.2,.3], [.3,.4])	([.4,.5], [.5,.8], [.1,.2])
δ_2	([.5,.6], [.1,.3], [.4,.6])	([.5,.7], [.1,.2], [.4,.6])	([.2,.4], [.3,.4], [.2,.5])	([.6,.8], [.1,.2], [.3,.5])	([.4,.6], [.4,.5], [.3,.5])	([.3,.5], [.4,.5], [.1,.3])	([.1,.2], [.5,.8], [.2,.4])	([.5,.7], [.1,.2], [.5,.6])
δ_3	([.2,.4], [.5,.6], [.4,.6])	([.2,.4], [.3,.4], [.2,.5])	([.4 .6], [.2,.3], [.1,.4])	([.2,.5], [.2,.3], [.1,.6])	([.3,.4], [.2,.5], [.5,.7])	([.3,.5], [.4,.5], [.1,.3])	([.2,.4], [.7,.8], [.1,.2])	([.1,.2], [.7,.8], [.2,.3])
δ_4	([.2,.3], [.5,.7], [.1,.3])	([.3,.4], [.2,.5], [.5,.7])	([.2,.4], [.3,.5], [.3,.6])	([.5,.7], [.1,.2], [.4,.6])	([.4,.6], [.1,.3], [.2,.4])	([.1,.2], [.5,.8], [.2,.4])	([.2,.4], [.3,.4], [.2,.5])	([.5,.6], [.1,.3], [.4,.6])

Table 2. Decision Matrix for Alternative $\aleph^{(1)}$

$\aleph^{(1)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	([.2,.4], [.4,.5], [.3,.4])	([.3,.4], [.4,.5], [.2,.5])	([.3,.6], [.2,.3], [.1,.2])	([.2,.4], [.4,.6], [.1,.2])	([.1,.3], [.6,.7], [.2,.3])	([.4,.5], [.2,.5], [.2,.3])	([.6,.7], [.1,.2], [.2,.3])	([.4,.6], [.2,.3], [.4,.5])
δ_2	([.3,.4], [.2,.5], [.5,.7])	([.4,.7], [.1,.2], [.1,.2])	([.4,.5], [.2,.5], [.1,.2])	([.5,.7], [.1,.2], [.2,.4])	([.6,.8], [.1,.2], [.1,.5])	([.2,.4], [.7,.8], [.1,.2])	([.2,.4], [.3,.5], [.3,.6])	([.3,.4], [.4,.5], [.2,.4])
δ_3	([.5,.6], [.2,.3], [.4,.5])	([.5,.7], [.1,.2], [.2,.4])	([.7,.8], [.1,.2], [.2,.4])	([.1,.3], [.1,.5], [.2,.5])	([.1,.4], [.2,.4], [.1,.2])	([.2,.5], [.2,.4], [.3,.5])	([.3,.5], [.2,.4], [.4,.6])	([.5,.7], [.1,.2], [.5,.6])
δ_4	([.3,.5], [.3,.4], [.6,.7])	([.2,.4], [.3,.4], [.2,.5])	([.2,.4], [.7,.8], [.1,.2])	([.4,.7], [.1,.2], [.1,.2])	([.5,.6], [.2,.3], [.4,.5])	([.2,.4], [.3,.5], [.3,.6])	([.4,.6], [.2,.3], [.4,.5])	([.1,.3], [.1,.5], [.2,.5])

Table 3. Decision Matrix for Alternative $\aleph^{(2)}$

$\aleph^{(2)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	([.2,.4], [.4,.6], [.4,.5])	([.2,.3], [.4,.6], [.3,.5])	([.1,.2], [.6,.8], [.2,.5])	([.4,.5], [.2,.5], [.1,.2])	([.2,.3], [.4,.6], [.3,.5])	([.1,.2], [.6,.8], [.2,.5])	([.7,.8], [.1,.2], [.2,.3])	([.1,.3], [.6,.7], [.2,.5])
δ_2	([.4,.5], [.2,.5], [.1,.2])	([.5,.7], [.1,.2], [.2,.4])	([.1,.3], [.6,.7], [.2,.6])	([.1,.4], [.2,.5], [.4,.6])	([.1,.4], [.2,.4], [.1,.2])	([.1,.2], [.2,.5], [.4,.6])	([.1,.4], [.2,.5], [.4,.6])	([.1,.4], [.2,.5], [.4,.6])
δ_3	([.3,.4], [.2,.6], [.4,.6])	([.2,.4], [.3,.4], [.2,.5])	([.4,.5], [.2,.5], [.1,.2])	([.1,.2], [.2,.5], [.4,.6])	([.3,.5], [.3,.5], [.6,.7])	([.3,.5], [.3,.5], [.6,.7])	([.1,.2], [.2,.5], [.4,.6])	([.5,.7], [.1,.2], [.2,.4])
δ_4	([.2,.4], [.4,.5], [.6,.8])	([.3,.5], [.3,.5], [.6,.7])	([.1,.2], [.2,.5], [.4,.6])	([.1,.4], [.2,.4], [.1,.2])	([.4,.5], [.2,.5], [.1,.2])	([.1,.2], [.2,.5], [.4,.6])	([.4,.5], [.2,.5], [.1,.2])	([.1,.2], [.2,.5], [.4,.6])

Table 4. Decision Matrix for Alternative $\aleph^{(3)}$

$\aleph^{(3)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	([.6,.7], [.1,.2], [.3,.5])	([.6,.8], [.1,.2], [.2,.3])	([.6,.7], [.3,.5], [.1,.2])	([.7,.8], [.1,.2], [.2,.5])	([.6,.7], [.1,.2], [.1,.2])	([.5,.8], [.1,.2], [.2,.4])	([.1,.3], [.6,.7], [.2,.5])	([.7,.8], [.1,.2], [.2,.3])
δ_2	([.5,.7], [.3,.4], [.2,.3])	([.5,.7], [.2,.5], [.2,.3])	([.5,.6], [.3,.4], [.1,.2])	([.7,.8], [.3,.5], [.1,.3])	([.1,.2], [.2,.5], [.4,.6])	([.1,.4], [.2,.5], [.4,.6])	([.4,.6], [.2,.3], [.1,.2]])	([.4,.6], [.2,.3], [.1,.2])
δ_3	([.2,.4], [.3,.4], [.2,.5])	([.4,.7], [.2,.3], [.3,.7])	([.4,.6], [.2,.3], [.1,.2])	([.3,.5], [.3,.5], [.6,.7])	([.6,.8], [.1,.2], [.1,.2])	([.7,.8], [.1,.2], [.2,.4])	([.1,.2], [.2,.5], [.4,.6])	([.6,.8], [.1,.2], [.1,.3])
δ_4	([.6,.8], [.3,.4], [.1,.2])	([.5,.7], [.1,.2], [.4,.5])	([.1,.2], [.2,.5], [.4,.6])	([.5,.6], [.3,.4], [.1,.2])	([.2,.4], [.3,.4], [.2,.5])	([.1,.3], [.6,.7], [.2,.5])	([.7,.8], [.1,.2], [.2,.5])	([.4,.6], [.2,.3], [.1,.2]])

Table 5. Decision Matrix for Alternative $\aleph^{(4)}$

$\aleph^{(4)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$

$\boldsymbol{\delta_1}$	([.3,.5],[.2,.4],[.1,.2])	([.3,.6],[.1,.2],[.4,.7])	([.4,.7],[.3,.4],[.2,.3])	([.7,.8],[.2,.4],[.3,.5])	([.5,.7],[.3,.4],[.2,.4])	([.4,.6],[.2,.5],[.3,.4])	([.2,.3],[.5,.7],[.2,.4])	([.5,.7],[.2,.4],[.3,.5])
$\boldsymbol{\delta_2}$	([.4,.5],[.5,.7],[.2,.4])	([.4,.7],[.3,.5],[.2,.4])	([.5,.8],[.3,.4],[.2,.3])	([.2,.4],[.2,.3],[.4,.5])	([.3,.5],[.2,.3],[.3,.5])	([.2,.4],[.2,.3],[.3,.6])	([.5,.8],[.3,.6],[.2,.3])	([.4,.6],[.2,.3],[.1,.2]])
$\boldsymbol{\delta_3}$	([. 2,.4],[. 3,.4],[. 2,.5])	([.4,.6],[.2,.3],[.3,.5])	([.3,.5],[.3,.5],[.1,.2])	([.3,.5],[.4,.6],[.6,.7])	([.5,.7],[.1,.2],[.4,.5])	([.4,.6],[.3,.5],[.1,.2])	([.6,.7],[.1,.2],[.3,.5])	([.2,.5],[.2,.3],[.4,.6])
$\boldsymbol{\delta_4}$	([.1,.2],[.2,.5],[.4,.6])	([.5,.7],[.2,.4],[.1,.3])	([.3,.5],[.2,.5],[.1,.3])	([.4,.6],[.2,.5],[.3,.4])	([.5,.8],[.3,.4],[.2,.3])	([.4,.6],[.2,.3],[.1,.2]])	([.4,.7],[.3,.5],[.2,.4])	([. 2,.4],[. 3,.4],[. 2,.5])

By using Tables 1-5, compute the correlation coefficient between $\delta_{IVNHSS}(\wp, \aleph^{(1)})$, $\delta_{IVNHSS}(\wp, \aleph^{(2)})$, $\delta_{IVNHSS}(\wp, \aleph^{(3)})$, $\delta_{IVNHSS}(\wp, \aleph^{(4)})$ by using equation 5 given as follows:
$\delta_{IVNHSS}(\wp, \aleph^{(1)})$ = .99701, $\delta_{IVNHSS}(\wp, \aleph^{(2)})$ = .99822, $\delta_{IVNHSS}(\wp, \aleph^{(3)})$ = .99986, and $\delta_{IVNHSS}(\wp, \aleph^{(4)})$ = .99759. This shows that $\delta_{IVNHSS}(\wp, \aleph^{(3)}) > \delta_{IVNHSS}(\wp, \aleph^{(2)}) > \delta_{IVNHSS}(\wp, \aleph^{(4)}) > \delta_{IVNHSS}(\wp, \aleph^{(1)})$. It can be seen from this ranking alternative $\aleph^{(3)}$ is the most suitable alternative. Therefore $\aleph^{(3)}$ is the best alternative for the vacant position of associate professor, the ranking of other alternatives given as $\aleph^{(3)} > \aleph^{(2)} > \aleph^{(4)} > \aleph^{(1)}$. Graphical results of alternatives ratings can be seen in figure 2.

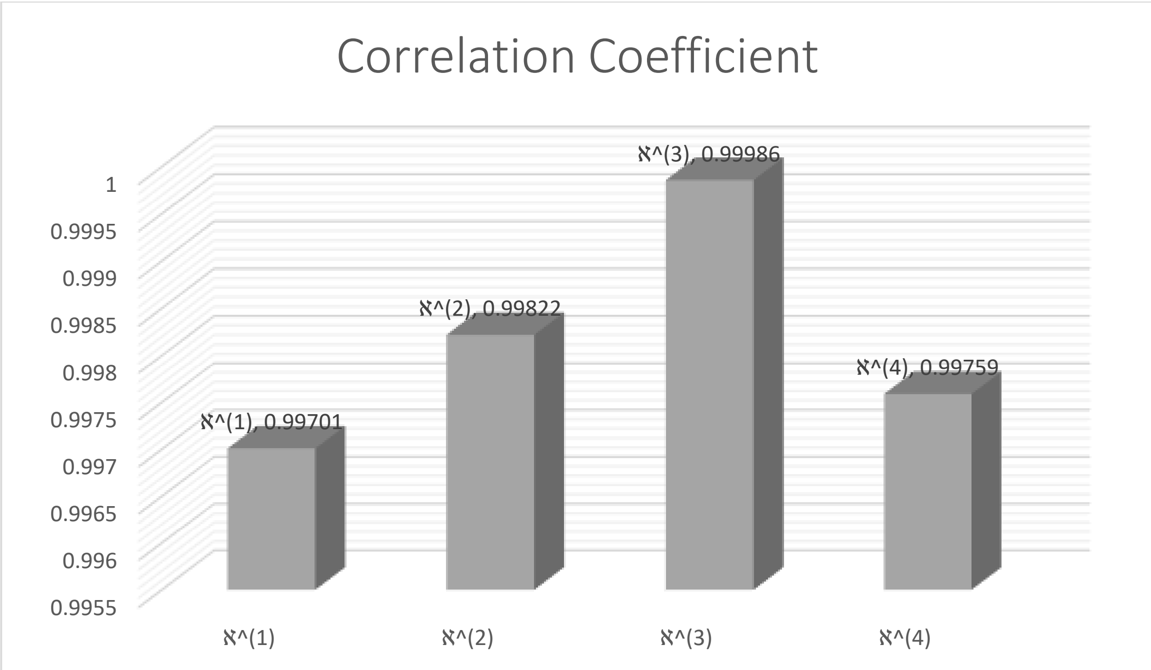

Figure 2: Alternatives rating based on correlation coefficient under IVNHSS

5. Conclusion

The interval-valued neutrosophic hypersoft set is a novel concept that is an extension of the interval-valued neutrosophic soft set. In this manuscript, we studied some basic concepts which were necessary to build the structure of the article. We introduced the correlation and weighted correlation coefficients under the IVNHSS environment. Some basic properties based on developed CC under IVNHSS were also introduced. A decision-making approach has been developed based on the established correlation coefficient and presented an algorithm under IVNHSS. Finally, a numerical illustration has been described to solve the decision-making problem by using the proposed technique. In the future, the correlation coefficient, the TOPSIS method based on correlation coefficient under IVNHSS can be presented. Future research will concentration on presenting numerous other operators under the IVNHSS environment to solve decision-making issues. Many other structures such as topological, algebraic, ordered structures, etc. can be developed and discussed under-considered environment.

6. Acknowledgment

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The Selection of LASER as Surgical Instrument in Medical using Neutrosophic Soft Set with Generalized Fuzzy TOPSIS, WSM and WPM along with MATLAB Coding

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Abstract: Lasers are medical devises and widely used in surgery to treat; diseased blood vessels, reduce blood loss, infection reduction, and many other purposes. Whereas, Lasers has many types based on the construction materials. Thus, the right selection of laser for surgery is very important to accomplish complex medical tasks. With the development of MCDM techniques and neutrosophic soft set, this problem can be solved with more accuracy and precision. The aim of this paper is to select the right type of laser for specific surgeries. To, select the right choice, six different laser types and seven criteria are taken. To find the best alternative, generalized TOPSIS, WSM, and WPM along with MATLAB coding techniques are used. Results are the same and showing the right selection of the same alternative which is already being used in the field of surgery. This shows that in the future, these techniques can be applied in the selection of medical equipment too.

Keywords: Accuracy Function, Fuzzy Soft Set, Neutrosophic Soft Set, MCDM, MATLAB, WSM, WPM, TOPSIS

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1. Introduction

All anesthetists need to have fundamental information on laser material science and how laser radiation can associate with the careful condition, including the patient, sedative mechanical assembly and careful group. Lasers are finding expanding application in both medication and medical procedure and their utilization offers ascend to a few perils. The majority of these risks emerge as an immediate consequence of the idea of laser radiation. The role of laser as a safe, non-corrosive, non-toxic surgical tool in hospitals is very important. The approach of current century is to advance the medical technology and equipment as a result, the procedures become less invasive, and low cost for treatment. Due to this importance laser is in the spotlight.

The most commonly used type of laser (CO_2 laser) was designed by C. Kumar [1, 2], it has crossed many stages to become important tool in surgical instrument [2, 3]. The instrument designer, Uzi Sharon, was the person who joined the light emission noticeable (red) helium-neon laser with the undetectable light emission CO_2 laser. The gadget from the mid-1970s was outfitted with the necessities of clinical medical procedure [4]. Isaac Kaplan is famous for "father of laser medical procedure" who created various laser-careful procedures that assisted with characterizing new fundamental conditions in plastic and reconstructive medical procedure [5].

In current century, laser treatments have a bigger number; major surgeries, skin care, ENT procedures, gall bladder removal procedure and many more [6, 7]. Additionally, lasers made a difference to build up another interventional method also to traditional medical procedure, the supposed in situ coagulation which can be performed cursorily, interstitially or intravascularly [8, 9, 10]. In this issue of Photonics and Lasers in Medicine, Philipp et al. [11] present information of 450 patients determined to have pyogenic granuloma who were dealt with utilizing the Nd: YAG laser (1064 nm) in impression strategy or on the other hand by direct coagulation. The outcomes mirror the significant skills of this division in applying the in-situ coagulation, guaranteeing not just supported helpful achievement yet in addition an incredible corrective result.

With the development of fuzzy sets [12] decision making becomes easier but later on this theory was extended by [13] named as Intuitionistic fuzzy number theory. To deal with more precision, accuracy and indeterminacy this idea was extended by [14] called as neutrosophy theory. To, discuss the applications of these theory number of developments were made but the most important one is the theory of soft set [15]. Later on, fuzzy, intuitionist and neutrosophy theories were extended to fuzzy softset [16], intuitionistic soft set [17] and neutrosophic soft set [18]. In different fields the applications of these theories are presented by many researchers [19-26], but with the development of TOPSIS, WSM and WPM techniques [27-32] it becomes more powerful tool to solve the MCDM problems [33-38]. Many other novel works under neutrosophic environment are done along with real life applications [43-46]. In object selection, neutrosophic sets are widely used for accuracy [47-49].

Now the question arises why we are using these techniques in this case study? To get the answer of this question, firstly you need to know the attribute and alternatives; since laser are of many types having different properties which makes it a perfect problem to apply the above-mentioned MCDM techniques. The neutrosophic theory is used for more accuracy thus the techniques to solve MCDM problems under neutrosophic environment can be applied.

1.1 Contribution / Motivation

LASER is widely used in all over the fields of sciences, especially in the field of medicine LASERS play revolutionary role. There are many kinds used in medical field for various surgeons, in surgery LASERS are used to cut deeply and cauterize. Producing precise and accurate surgical cut. Ablate tissues and cells from the surface. Internal surgery of patients without visible wound. To evaporate the damaged cells, there are countless uses of LASER in medical field.

In this research five construction-based types of LASERS are being discussed and we are finding which type is more efficient and accurate in the surgical field using mathematical tools along with the use of MATLAB.

1.2 The paper presentation

The layout of this research is presented in Figure:1. Section 1, consists of introduction of the topic, literature review and the motivation along with contribution. Section 2, preliminaries are presented in this section. In third section, algorithms of TOPSIS, WSM and WPM are listed along with flowcharts. In section 5, the case study of LASER selection is done using TOPSIS algorithm and in section 6, the case study is solved with the help of WSM and WPM using MATLAB code. Finally, result discussion is done and the present research is concluded with future directions.

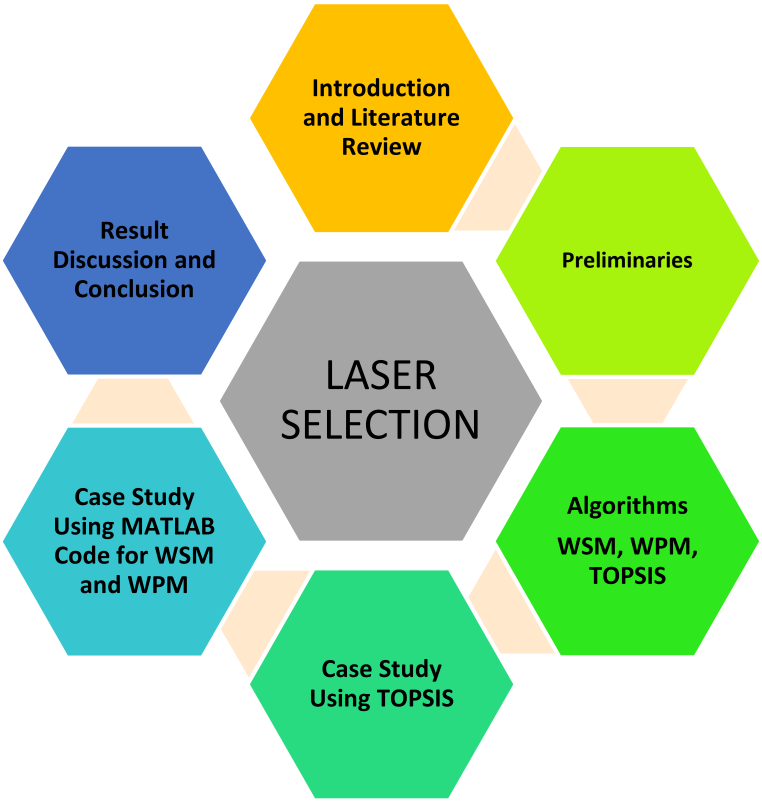

Figure 1: The layout of the paper

2.Preliminaries

Definition 2.1: Linguistic Set [39]

Let A= $\{a_0, a_1, a_2, \ldots, a_n\}$ be finite and fully ordered set of discrete terms where $n \in N$.

Example: Let us consider a set A= $\{a_1, a_2, \ldots, a_5\}$ every element representing a specific linguistic term value, which are as; *"None", "low effective", "moderate effective", "effective", "high effective"*

Definition 2.2: Fuzzy Set [12]

In fuzzy set, an element "∂" is assigned a degree of membership from [0,1]. Mathematically, represented as $\mu_\partial \in [0,1]$.

Definition 2.3: Neutrosophic Set [14]

Let τ be an initial universal set and E be a set of parameters. Let's consider, A⊂ E. Let P(τ) represents the set of all neutrosophic sets over τ, where F is a mapping given by

$$F : A \longrightarrow P(\tau)$$

Definition 2.4: MCDM [42]

Multi-criteria decision makings are very complex. To find out the best option MCDM techniques are used like, TOPSIS, VIKOR, AHP, ELECTREE, WSM, WPM, etc.

Definition 2.5: Accuracy Function [41]

The process /mathematical form of conversion of neutrosophic numbers N into crisp numbers is said to be accuracy function.

$$A(N) = \frac{[T(x) + I(x) + F(x)]}{3} : x \epsilon N$$

Definition 2.6: TOPSIS [33]

TOPSIS is an acronym that stands for 'Technique of Order Preference Similarity to the Ideal Solution' and is a pretty straight forward MCDA method. As the name implies, the method is based on finding an ideal and an anti-ideal solution and comparing the distance of each one of the alternatives to those.

Definition 2.7: LASER [1]

Laser stands for light amplification by stimulated emission of radiation, A laser is a physical device that radiate light through a process of optical amplification via stimulated emission of electromagnetic radiation.

3. Algorithms

In this section three algorithm are presented to solve MCDM problem under neutrosophic environment.

3.1 Generalized Fuzzy TOPSIS Algorithm

The TOPSIS technique [33] is mainly used for the ranking of alternatives in MCDM and MAGDM problems. In this method crisp/fuzzy/intuitionistic numbers were used to select the best alternative. Thus, technique of TOPSIS was extended for the Neutrosophic environment and said to be Generalized Fuzzy TOPSIS. The stepwise algorithm of generalized fuzzy TOPSIS is presented in Figure: 2.

Step: 1 Consideration of problem.

Step: 2 The formulation and assumptions of the problem.

Step: 3 Construction of linguistic decision matrix.

Step: 4 Assigning of neutrosophic numbers (NN's) to each linguistic value.

Step: 5 Conversion of neutrosophic numbers into crisp using accuracy function.

Step: 6 Now apply TOPSIS algorithm. (Presented below)

TOPSIS Algorithm {**Step 1:** Construct the Normalized Decision Matrix to transform the various attribute dimensions into non-dimensional attributes, which allows comparison across the attributes.

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}$$

Step 2: Construct the Weighted Normalized Decision Matrix.

Assume we have a set of weights for each criteria w_j for j = 1, 2, 3…n. Multiply each column of the normalized decision matrix r_{ij} by its associated weight. An element of the new matrix is:

$$V_{ij} = w_j \, r_{ij}$$

Step 3: Determine Ideal and Negative-Ideal Solutions, A⁺ = { $V_1, \dots, V_n$}, where

V_j^+ = {max (V_{ij}) if j ∊ J; min (V_{ij}) if j ∊ J^+}

J^+ Associated with the criteria having a positive impact.

A^- = { $V_1, \dots, V_n$}, where V_j^- = {min (V_{ij}) if j ∊ J; max (V_{ij}) if j ∊ J^-}

J^- Associated with the criteria having a negative impact.

Step 4: Calculate the Separation Measure:

- Ideal Separation

$$S_i^+ = \sqrt{\sum_{j=1}^{n}(V_{ij} - V_j^+)^2} \qquad i = 1,2,3, \dots, m$$

- Negative Ideal Separation

$$S_i^- = \sqrt{\sum_{j=1}^{n}(V_{ij} - V_j^-)^2} \qquad i = 1,2,3, \dots, m$$

Step 5: Calculate the Relative Closeness to the Ideal Solution

$$C_i^* = \frac{S_i^-}{(S_i^+ + S_i^-)}, \quad 0 < C_i^* < 1, \qquad i = 1,2,3, \dots, m\ .$$

$$C_i^* = 1, \qquad if\ A_i = A^+$$

$$C_i^* = 0, \qquad if\ A_i = A^-$$

Step 6: Rank the preference order a set of alternatives can now be preference ranked according to the descending order of C_i^* } *End of TOPSIS algorithm*.

Step: 7 Rank the alternatives.

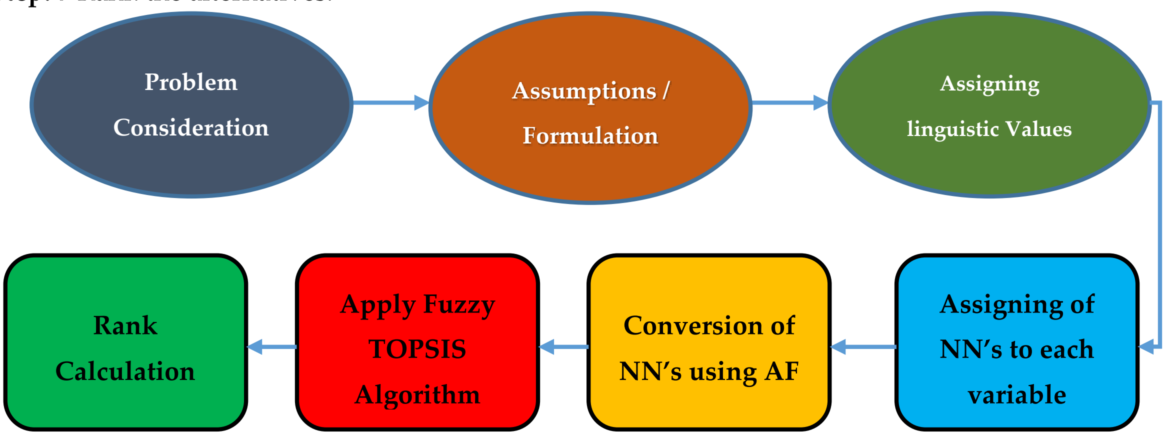

Figure 2: Flowchart for generalized fuzzy TOPSIS

3.2 Weighted Sum Model (WSM) Algorithm [27]

The WSM is commonly used for single dimensional problems. In this method the weighted sum performance rating of each alternative is calculated using the algorithm. The stepwise procedure is shown in Figure 3;

Step 1: Construction of decision matrix M from the given problem.

Step 2: Construction of normalized decision matrix $\Re = [r_{kj}]_{m\times n} k = 1,2,3, \ldots, m \text{ and } j = 1,2,3, \ldots, n$

Step 3: Construction of weighted normalized decision matrix $\wp = [w_j r_{kj}]_{m\times n} \text{ and } \sum_{j=1}^{n} w_j = 1$

Step 4: Calculation of S_k^{WSM} ; $k = 1,2,3, \ldots, m$ score of each alternative.

$$S_k^{WSM} = \sum_{j=1}^{n} w_j r_{kj} \,; k = 1,2,3, \ldots, m \text{ and } j = 1,2,3, \ldots, n$$

Step 5: Selection of best alternative i.e. $max\ (S_k^{WSM}\ ; k = 1,2,3, \ldots, m)$

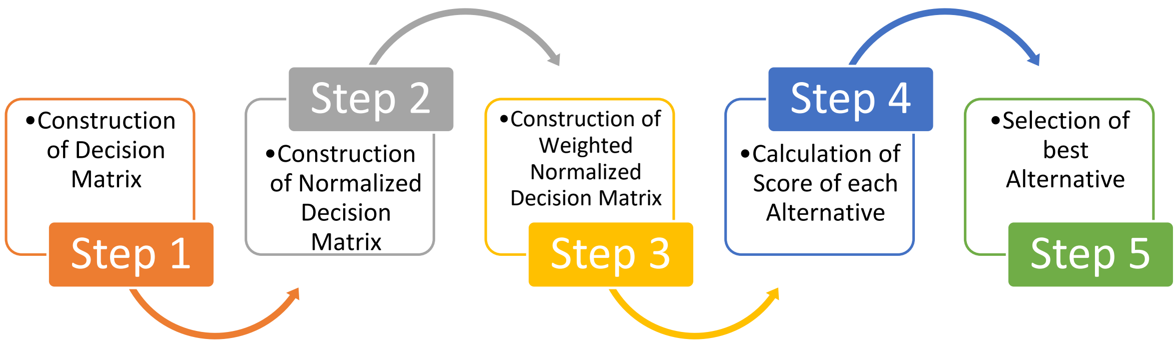

Figure 3: Flowchart for WSM algorithm

3.3 Weighted Product Model (WPM) Algorithm [29]

The WPM is mainly used to find best alternative in MCDM problems. In this method the alternatives are simplified by multiplying a number of ratios of each criterion. This method is some time known as dimensionless analysis. The stepwise procedure is shown in Figure 4;

Step 1: Construction of decision matrix $\mathcal{M}$ from the given problem.

Step 2: Construction of normalized decision matrix $\mathbb{R} = [r_{kj}]_{m\times n} k = 1,2,3, \ldots, m \text{ and } j = 1,2,3, \ldots, n$

Step 3: Construction of weighted normalized decision matrix $\mathbb{N} = [{r_{kj}}^{w_j}]_{m\times n} \text{ and } \sum_{j=1}^{n} w_j = 1$

Step 4: Calculation of S_k^{WPM} ; $k = 1,2,3, \ldots, m$ score of each alternative.

$$S_k^{WSM} = \prod_{j=1}^{n} {r_{kj}}^{w_j} \,; k = 1,2,3, \ldots, m \text{ and } j = 1,2,3, \ldots, n$$

Step 5: Selection of best alternative i.e. $max\ (S_k^{WPM}\ ; k = 1,2,3, \ldots, m)$

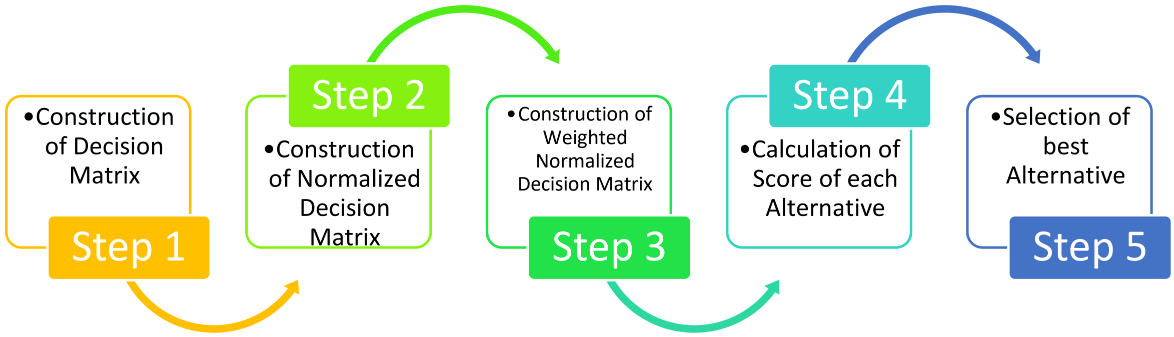

Figure 4: Flowchart for WPM algorithm

4: Case Study

In this section a case study of LASER selection for the surgery in medical is considered and the selection is made by applying all the above-mentioned algorithms.

4.1 Problem Formulation

LASERS are widely used in all over the fields of sciences, especially in the field of medicine and surgery. In surgery, LASERS are used to cut deeply and cauterize. Producing precise and accurate surgical cut. Ablate tissues and cells from the surface. Internal surgery of patients without visible wound. To evaporate the damaged cells, there are countless uses of LASER in medical field.

4.2 Parameters

Selection is a complex issue, to resolve this problem criteria and alternative plays an important role. Following criteria and alternatives are considered in this problem formulation.

Criteria's of Each Laser						
C_1 Construction Type	C_2 Wavelength	C_3 Frequency	C_4 Delivery System	C_5 Medium	C_6 Pumping Method	C_7 Interaction

Lasers as Alternatives					
L_1	L_2	L_3	L_4	L_5	L_6
Argon	KTP	Helium Neon	YAG	YSGG	Diode

4.3 Assumptions

Consider $K = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ decision makers who will assign linguistic values from Table .1 according to his own interest, knowledge and experience, to the above-mentioned criteria and alternatives and shown in Table.2.

Sr # No	Linguistic variable	Code	Neutrosophic Number
1	None	N	(0.0, 0.1,0.5)
2	Low Effective	LE	(0.2,0.4,0.8)
3	Moderate Effective	ME	(0.4,0.2,0.5)

4	Effective	E	(0.6,0.2,0.3)
5	High Effective	HE	(1.0,0.0,0.1)

Table 1: Linguistic variables, codes and neutrosophic numbers

4.4 Application of Proposed Generalized Fuzzy TOPSIS Algorithm

Step 1: Presented in section 4.1

Step 2: Presented in section 4.2 and 4.3.

Step 3: Assigning linguistic variables to each alternatives and criteria's / attributes.

	Strategies	κ_1	κ_2	κ_3	κ_4
C_1= Construction Type	L_1	N	ME	E	ME
	L_2	LE	E	HE	E
	L_3	ME	HE	N	HE
	L_4	E	ME	N	N
	L_5	HE	N	LE	LE
	L_6	N	LE	ME	ME
C_2= Wavelength	L_1	LE	ME	E	E
	L_2	ME	E	HE	HE
	L_3	E	HE	N	ME
	L_4	HE	ME	LE	E
	L_5	N	E	ME	HE
	L_6	LE	HE	E	ME
C_3= Frequency	L_1	ME	N	HE	E
	L_2	E	LE	ME	HE
	L_3	HE	ME	E	N
	L_4	ME	E	HE	LE
	L_5	E	HE	LE	ME
	L_6	HE	ME	E	N
C_4= Delivery System	L_1	N	E	HE	LE
	L_2	LE	HE	N	E
	L_3	ME	E	E	HE
	L_4	E	HE	N	LE
	L_5	HE	N	LE	E
	L_6	N	ME	LE	ME
C_5=Medium	L_1	LE	E	E	E
	L_2	ME	HE	LE	HE
	L_3	E	ME	HE	N

	L_4	HE	E	LE	HE
	L_5	ME	HE	E	N
	L_6	HE	N	LE	E
C_6= Pumping Method	L_1	LE	E	E	E
	L_2	ME	HE	LE	HE
	L_3	E	ME	HE	N
	L_4	HE	E	LE	HE
	L_5	ME	HE	E	N
	L_6	HE	N	LE	E
C_7= Interaction	L_1	LE	E	E	E
	L_2	ME	HE	LE	HE
	L_3	E	ME	HE	N
	L_4	HE	E	LE	HE
	L_5	ME	HE	E	N
	L_6	HE	N	LE	E

Table 2: Each decision maker, will assign linguistic values to each attribute, from Table .1

Step 4: Substitution of neutrosophic numbers (NNs) to each linguistic variable.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
L_1	(0.0, 0.1,0.5)	(1.0, 0.0, 0.1)	(0.6, 0.2,0.3)	(0.6, 0.2,0.3)	(0.4, 0.2,0.5)	(0.0, 0.1,0.5)	(0.6, 0.2,0.3)
L_2	(0.2, 0.4,0.8)	(0.4, 0.2,0.5)	(0.0, 0.1,0.5)	(1.0, 0.0, 0.1)	(0.6, 0.2,0.3)	(0.2, 0.4,0.8)	(0.0, 0.1,0.5)
L_3	(0.4, 0.2,0.5)	(0.0, 0.1,0.5)	(0.2, 0.4,0.8)	(1.0, 0.0, 0.1)	(0.6, 0.2,0.3)	(0.4, 0.2,0.5)	(1.0, 0.0, 0.1)
L_4	(0.6, 0.2,0.3)	(1.0, 0.0, 0.1)	(0.4, 0.2,0.5)	(0.2, 0.4,0.8)	(1.0, 0.0, 0.1)	(0.6, 0.2,0.3)	(0.0, 0.1,0.5)
L_5	(1.0, 0.0, 0.1)	(0.2, 0.4,0.8)	(0.6, 0.2,0.3)	(0.4, 0.2,0.5)	(0.0, 0.1,0.5)	(1.0, 0.0, 0.1)	(0.4, 0.2,0.5)

L_6	(0.4, 0.2,0.5)	(0.0, 0.1,0.5)	(1.0, 0.0, 0.1)	(0.6, 0.2,0.3)	(0.0, 0.1,0.5)	(0.4, 0.2,0.5)	(0.6, 0.2,0.3)

Table: 3 Assign neutrosophic number to each linguistic value from table 1.

Step 5: Conversion of fuzzy neutrosophic numbers NNs of step 4, into fuzzy numbers by using accuracy function.

$$A(F) = \{ x = \frac{[T_x+I_x+F_x]}{3} \}$$

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
L_1	0.200	0.366	0.366	0.366	0.366	0.200	0.366
L_2	0.400	0.366	0.200	0.366	0.366	0.466	0.200
L_3	0.366	0.200	0.466	0.366	0.366	0.366	0.3666
L_4	0.366	0.366	0.366	0.466	0.366	0.366	0.200
L_5	0.366	0.466	0.366	0.366	0.200	0.366	0.366
L_6	0.366	0.200	0.366	0.366	0.200	0.366	0.366

Table: 4 After applied accuracy function the obtain result converted into fuzzy value

Step 6: Now we apply algorithm of TOPSIS to obtain relative closeness.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
L_1	0.2	0.1	0.1	0.1	0	0.266	0.0006
L_2	0	0.1	0.266	0.1	0	0	0.1666
L_3	0.034	0.266	0	0.1	0	0.1	0
L_4	0.034	0.1	0.1	0	0	0.1	0.1666
L_5	0.034	0	0.1	0.1	0.166	0.1	0.0006
L_6	0.034	0.266	0.1	0.1	0.166	0.1	0.0006

Table: 5 Normalized decision matrices

Step 6.1: Calculation of weighted normalized matrix

weights	0.1	0.3	0.1	0.1	0.1	0.1	0.2
	C_1	C_2	C_3	C_4	C_5	C_6	C_7

L_1	0.2	0.1	0.1	0.1	0	0.266	0.0006
L_2	0	0.1	0.266	0.1	0	0	0.1666
L_3	0.034	0.266	0	0.1	0	0.1	0
L_4	0.034	0.1	0.1	0	0	0.1	0.1666
L_5	0.034	0	0.1	0.1	0.166	0.1	0.0006
L_6	0.034	0.266	0.1	0.1	0.166	0.1	0.0006

Table: 5 Weighted normalized decision matrices

Step 6.2: Calculation of the ideal best and ideal worst value,

v_j^+ =Indicates the ideal (best)

v_j^- = Indicates the ideal (worst)

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
L_1	0.094677	0.072439	0.030048	0.044721	0	0.079928	0.000509
L_2	0	0.072439	0.079928	0.044721	0	0	0.14142
L_3	0.016095	0.192688	0	0.044721	0	0.030048	0
L_4	0.016095	0.072439	0.030048	0	0	0.030048	0.14142
L_5	0.016095	0	0.030048	0.044721	0.070711	0.030048	0.000509
L_6	0.016095	0.192688	0.030048	0.044721	0.070711	0.030048	0.000509
v_j^+	0.211244	0.41414	0.3328	0.223607	0.234759	0.3328	0.23561
v_j^-	0.094677	0.192688	0.079928	0.044721	0.070711	0.079928	0.14142

Table: 6 Ideal worst and Ideal best values

Step 7: Calculation of rank.

$$p_i = \frac{s_{ij}^-}{s_{ij}^+ + s_{ij}^-}$$

	s_j^+	s_j^-	$p_i = \frac{s_{ij}^-}{s_{ij}^+ + s_{ij}^-}$	**Rank**
L_1	0.130184	0.130821	0.501	3
L_2	0.153048	0.116773	0.433	4
L_3	0.112088	0.198464	0.639	2

L_4	0.158502	0.080059	0.336	5
L_5	0.213991	0.056231	0.208	6
L_6	0.093076	0.200726	0.683	1

Table: 7 Calculation of rank by relative closeness

5. Case Study using WSM and WPM MATLAB Code [43]

To run the WSM and WPM MATLAB code for the case study, the variable used in coding are defined by;

X: this is defined as decision matrix and presented in Table: 4.

W: this is defined as weight of each attribute and presented in Table: 5.

Wcriteria: < (0,1,1,0,0,0,0) >

$\boldsymbol{i} = 1,2,3,4,5,6 \;\; and \; \boldsymbol{j} = 1,2,3,4,5,6,7$

MATLAB COMMAND

```
Xval=length(X(:,1));
for i=1:Xval
for j= 1:length(W)
if Wcriteria(1,j)== 0
Y(i,j)=min(X(:,j))/X(i,j);
else
Y(i,j)=X(i,j)/max(X(:,j));
end
end
end
for i=1:Xval
PWSM(i,1)=sum(Y(i,:).*W);
PWPM(i,1)=prod(Y(i,:).^W);
End
```

Results

Preference score of WSM = < (0.65641, 0.70442, 0.809, 0.72181, 0.66273, 0.83398) >

Preference score of WPM = < (0.63378, 0.66305, 0.77807, 0.69813, 0.62619, 0.80708) >

6. Result Discussion

To check the validity or applicability of algorithms in neutrosophic soft set and MCDM environment the case study of Laser selection is considered in which six lasers are considered based on the construction material. Firstly, using the generalized neutrosophic TOPSIS technique the ranking of alternatives is calculated. Secondly, WSM and WPM techniques are applied using MATLAB code to calculate the rank. In these calculations, the ranking of each laser with respect to each criterion is calculated which are shown in Table 8 and Figure 5. Result shows that the above-mentioned techniques can be used to rank medical equipment too.

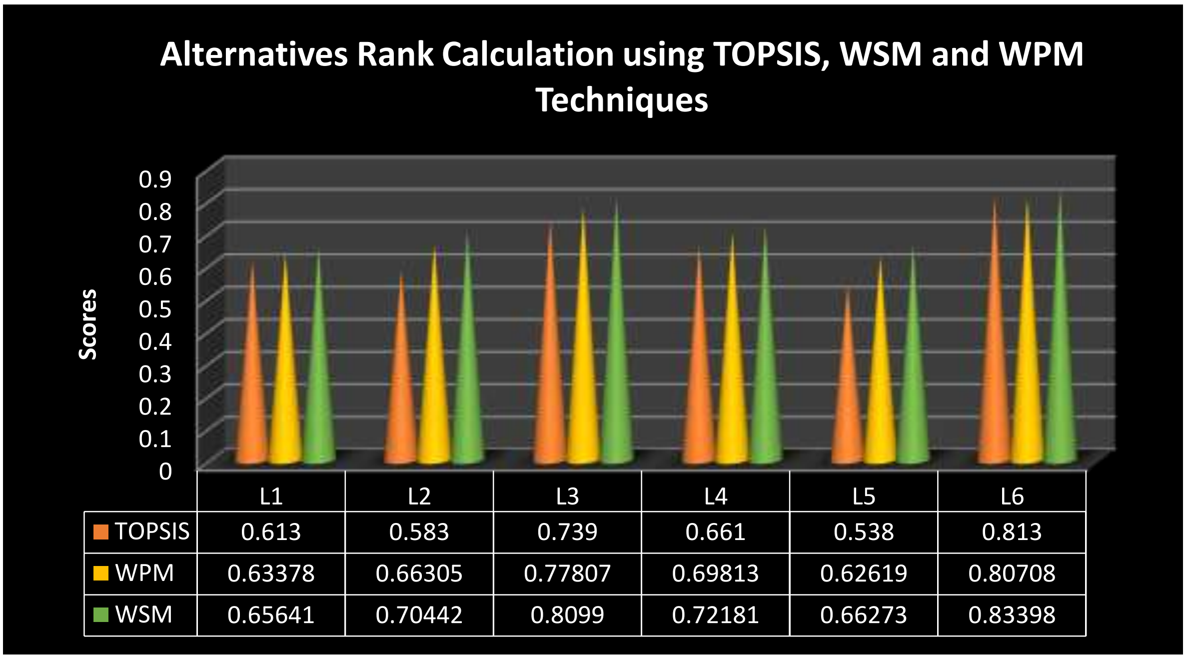

Figure 5: Ranking comparison of alternatives

Graphical and tabular comparison is presented in Table 8 and in Figure 5, which shows that under TOPSIS, WSM and WPM technique *L6* is best alternative whereas, *L5* is the worst selection respectively.

Alternative	TOPSIS	WPM	WSM
L1	0.613	0.63378	0.65641
L2	0.583	0.66305	0.70442
L3	0.739	0.77807	0.8099
L4	0.661	0.69813	0.72181
L5	0.538	0.62619	0.66273
L6	0.813	0.80708	0.83398

Table: 8 Alternatives rank comparison using WSM, WPM and TOPSIS

5. Conclusions

Lasers are medical devices that used a precisely focused beam of lights to treat or remove tissues or blood vessels etc. Based on construction material, lasers are divided into five main categories which also have different parameters and attributes. Thus, considering it as a case study, MCDM techniques are applied in the neutrosophic soft set environment. The results calculated using WSM, WPM and TOPSIS are the same. The lasers which are being used in medical filed for the surgery already have the same ranks. This shows that this technique is very helpful to rank the medical equipment in the future with more accuracy and precision.

This work can't be compared; as no one has applied these techniques to rank laser in medical surgery. In our forthcoming work, we will provide more application of these techniques in medical filed like nebulizer, infusion pumps and suction devices etc. In future, this study can be used in some more medical equipment selection.

Conflicts of Interest

The authors declare no conflict of interest.

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Generalized Hamming Similarity Measure Based on Neutrosophic Quadruple Numbers and Its Applications to Law Sciences

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Abstract: Neutrosophic quadruple numbers are the newest field studied in neutrosophy. Neutrosophic quadruple numbers, using the certain extent known data of an object or an idea, help us uncover their known part and moreover they allow us to evaluate the unknown part by the trueness, indeterminacy and falsity values. In this study, we generalized Hamming similarity measures for the generalized set-valued neutrosophic quadruple sets and numbers. We showed that generalized Hamming measure satisfies the similarity measure condition. Also, we generalized an algorithm for the generalized set-valued neutrosophic quadruple sets and numbers, we gave a multi-criteria decision making application for using the this generalized algorithm. In this application, we examined which of the laws established in different situations were more efficient. Furthermore, we obtained different result compared to previous algorithm and previous similarity measure based on single-valued neutrosophic numbers. Therefore, we have shown that generalized set-valued neutrosophic quadruplet sets and numbers, a new field of neutrosophic theory, are more useful for decision-making problems in law science and more precise results are obtained. The application in this study can be developed and used in decision-making applications for law science and other sciences.

Keywords: Neutrosophic quadruple sets, generalized set valued neutrosophic quadruple sets and numbers, Hamming similarity measure, decision-making applications, law applications

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1 Introduction

Smarandache proposed the neutrosophic logic and the neutrosophic set [3] in 1998. Neutrosophic logic and neutrosophic sets have a degree of membership T, a degree of indeterminacy I and a degree of non-membership F. These degrees are defined independently. Thus, neutrosophic theory is generalized of fuzzy theory [4] and intuitionistic fuzzy theory [5]. Also, many researchers have studied neutrosophic

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theory [6 - 19]. Recently, Smarandache extended the neutrosophic set to refined (n-valued) neutrosophic set, and to refined neutrosophic logic, and to refined neutrosophic probability, i.e. the truth value T is refined\split into types of sub-truths such as T_1, T_2, …, similarly indeterminacy I is refined\split into types of sub-indeterminacies I_1, I_2, …, and the falsehood F is refined\split into sub-falsehoods F_1, F_2, … [20]; Peng et al. obtained multi-parametric similarity measure for neutrosophic set [21]; Ye et al. introduced similarity measures of single-valued neutrosophic sets [22]; Uluçay et al. studied MCDM-problems with neutrosophic multi-sets [23]; Kandasamy et al. studied refined neutrosophic sets [24]; Hashmi et al. obtained m-Polar neutrosophic topology [25]; Aslan et al. studied Neutrosophic Modeling of Talcott Parsons's Action [2].

Decision-making applications and similarity measures are very important in neutrosophic theory. Thus, many researchers studied based on decision-making applications in neutrosophic theory. Recently, Tian et al. obtained a multi-criteria decision-making method based on neutrosophic theory [28]; Saqlain et al. studied single and multi-valued neutrosophic hypersoft set [29]; Roy et al. introduced similarity Measures of Quadripartitioned single-valued bipolar neutrosophic sets [30]; Uluçay et al. obtained decision-making method based on neutrosophic soft expert graphs [31]; Şahin et al. studied interval valued neutrosophic sets and applications [32]; Nabeeh et al. obtained an integrated neutrosophic-TOPSIS approach and its application to personnel selection [41]; Nabeeh et al. studied neutrosophic multi-criteria decision-making approach for IoT-Based enterprises [42]; Abdel-Basset et al. obtained utilizing neutrosophic theory to solve transition difficulties of IoT-Based enterprises [43].

In 2015, Smarandache discussed neutrosophic quadruple sets and neutrosophic quadruple numbers [1]. A neutrosophic quadruple set is a generalized form of a neutrosophic set. A neutrosophic quadruple set is denoted by {(x, yT, zI, tF): x, y, z, t ∈ ℝ or ℂ}. Here, x is referred to as the known part, (yT, zI, tF) as the unknown part and T, I and F are the usual tools of the neutrosophic logic. So, neutrosophic quadruple sets are generalized of neutrosophic sets. Furthermore, researchers have studied neutrosophic quadruple sets and numbers [33 - 36]. Recently, Rezaei et al. studied neutrosophic quadruple a-ideals [38]; Mohseni et al. obtained commutative neutrosophic quadruple ideals [39]; Kandasamy et al. introduced neutrosophic quadruple algebraic codes [40]. Also, Şahin et al. introduced generalized set-valued neutrosophic quadruple sets and numbers [37]. A generalized set-valued neutrosophic quadruple set denoted by

$$G_{s_i} = \{(K_{s_i}, L_{s_i}T_{s_i}, M_{s_i}I_{s_i}, N_{s_i}F_{s_i}): K_{s_i}, L_{s_i}, M_{s_i}, N_{s_i} \in \mathrm{P(X)};\ \mathrm{i} = 1, 2, 3, \dots, \mathrm{n}\}.$$

Where T_i, I_i and F_i have their usual neutrosophic logic; X is a nonempty set, P(X) is power set of X, K_{s_i} is called the known part and $(L_{s_i}T_{s_i}, M_{s_i}I_{s_i}, N_{s_i}F_{s_i})$ is called the unknown part. Thanks to this definition, neutrosophic quadruple sets have become available in the field of decision-making application. Most importantly, this definition, which has a more general structure than neutrosophic sets, will find more application areas and will give more objective results to many problems with the help of the known part, unknown part and K, L, M, N sets.

As in many branches of science, many uncertainties are encountered in terms of application and decision-making in law science. In order to cope with these uncertainties, mostly known classical methods are inadequate or cause wrong decisions to be made. In addition, many criteria should be considered in determining the laws in law science. In addition, it is clear that unknown situations will arise in the implementation of laws prepared for known situations. For all these reasons, in this study, we have prepared an application in order to determine which of the different legal applications with multiple criteria will yield most effective results. For this application, we generalized Hamming similarity measures for the generalized set-valued neutrosophic quadruple sets (GsvNQs) and numbers (GsvNQn) since GsvNQs and GsvNQn are more useful then neutrosophic sets. Also, we generalized an algorithm [2] (based on single valued neutrosophic number (SvNn) and set (SvNs)) for the GsvNQs and GsvNQn. Also, we gave a multi-criteria decision-making application using this generalized algorithm. In this application, we examined which of the laws established in different situations were more efficient. Furthermore, we obtained different result compared to previous algorithm and previous similarity measure based on SvNn thanks to structure of GsvNQs and GsvNQn.

In this paper, in Section 2, we examined neutrosophic sets [3, 8], Hamming similarity measure [22], GsvNQs and properties [33]. In section 3, we defined firstly generalized Hamming similarity measure based on GsvNQn. In Section 4, we firstly generalized an algorithm [2] for GsvNQn. In Section 5, we give a multi-criteria decision making application using the generalized algorithm in Section 4. In Section 6, we compared the results of the generalized algorithm in Section 5 with the results of algorithm (based on single valued neutrosophic set and Hamming similarity measure [22]) [2]. In Section 6, we give conclusions.

2 Preliminaries

Definition 1: [3] Let E be the universal set. For $\forall x \in E, 0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$, by the help of the functions $T_A: E \rightarrow]^-0, 1^+[$, $I_A: E \rightarrow]^-0, 1^+[$ and $F_A: E \rightarrow]^-0, 1^+[$ a neutrosophic set A on E is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\} .$$

Here, $T_A(x), I_A(x)$ and $F_A(x)$ are the degrees of trueness, indeterminacy and falsity of $x \in E$ respectively.

Definition 2: [8] Let E be the universal set. For $\forall x \in E, 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, using the functions $T_A: E \to [0,1]$, $I_A: E \to [0,1]$ and $F_A: E \to [0,1]$, a SvNs A on E is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\} .$$

Here, $T_A(x), I_A(x)$ and $F_A(x)$ are the degrees of trueness, indeterminacy and falsity of $x \in E$ respectively.

Definition 3: [22] Let

$$A_1 = <T_{A_1}(x), I_{A_1}(x), F_{A_1}(x)> \text{ and } A_2 = <T_{A_2}(x), I_{A_2}(x), F_{A_2}(x)>$$

be two SvNns, $S: A_1 \times A_2 \to [0,1]$ be a function. The Hamming similarity measure between A_1 and A_2 denoted by $S(A_1, A_2)$ such that

$$S(A_1, A_2) = \frac{1}{3}[\left|T_{A_1}(x) - T_{A_2}(y)\right| + \left|I_{A_1}(x) - I_{A_2}(y)\right| + \left|F_{A_1}(x) - F_{A_2}(y)\right|]$$

Theorem 1: [22] Let A_1 and A_2 be two SvNns, $S: A_1 \times A_2 \to [0,1]$ be a Hamming similarity measure. $S(A_1, A_2)$ satisfies below properties.

i. $0 \leq S(A_1, A_2) \leq 1$,
ii. $S(A_1, A_2) = 1$ if and only if $A_1 = A_2$,
iii. $S(A_1, A_2) = S(A_2, A_1)$,
iv. If $A_1 \subseteq A_2 \subseteq A_3 \in E$, then $S(A_1, A_3) \leq S(A_1, A_2)$ and $S(A_1, A_3) \leq S(A_2, A_3)$.

Definition 4: [1] Neutrosophic quadruple number is a number of the form

(k, lT, mI, nF)

Here, T, I and F are used as the ordinary neutrosophic logical tools and k, l, m, n ∈ ℝ or ℂ. For a neutrosophic quadruple number (k, lT, mI, nF), k is named the known part and (lT, mI, nF) is named the unknown part where k represents any asset such as a number, an idea, an object, etc. Also,

NQ = {(k, lT, mI, nF): k, l, m, n ∈ ℝ or ℂ}

is defined by neutrosophic quadruple set.

Definition 5: [33] Let X be a set and P(X) be power set of X. A GsvNQs is a set of the form

$$G_{s_i} = \{(A_{s_i}, B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i}): A_{s_i}, B_{s_i}, C_{s_i}, D_{s_i} \in P(X); i = 1, 2, 3, \dots, n\}$$

Where, T_i, I_i and F_i have their usual neutrosophic logic means and GsvNQn defined by

$$G_{N_i} = (A_{s_i}, B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i}).$$

As in neutrosophic quadruple number, for a GsvNQn $(A_{s_i}, B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i})$, representing any entity which may be a number, an idea, an object, etc.; A_{s_i} is called the known part and $(B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i})$ is called the unknown part.

Definition 6: [33] Let

$$G_{N_1} = (A_{s_1}, B_{s_1}T_{s_1}, C_{s_1}I_{s_1}, D_{s_1}F_{s_1}) \text{ and } G_{N_2} = (A_{s_2}, B_{s_2}T_{s_1}, C_{s_2}I_{s_2}, D_{s_2}F_{s_2})$$

be two GsvNQns. A_{s_1}=A_{s_2}, A_{s_1}=A_{s_2}, A_{s_1}=A_{s_2}, A_{s_1}=A_{s_2} and $T_{s_1} = T_{s_2}$, $I_{s_1} = I_{s_2}$, $F_{s_1} = F_{s_2}$ if and only if we say G_{N_1} is a equal to G_{N_2} and denote it by $G_{N_1} = G_{N_2}$.

Definition 7: [33] Let

$$G_{N_1} = (A_{s_1}, B_{s_1}T_{s_1}, C_{s_1}I_{s_1}, D_{s_1}F_{s_1}) \text{ and } G_{N_2} = (A_{s_2}, B_{s_2}T_{s_1}, C_{s_2}I_{s_2}, D_{s_2}F_{s_2})$$

be two GsvNQns. $A_{s_1} \subset A_{s_2}$, $A_{s_1} \subset A_{s_2}$, $A_{s_1} \subset A_{s_2}$, $A_{s_1} \subset A_{s_2}$ and $T_{s_1} \le T_{s_2}$, $I_{s_1} \le I_{s_2}$, $F_{s_1} \le F_{s_2}$, if and only if we say G_{N_1} is a subset of G_{N_2} and denote it by $G_{N_1} \subset G_{N_2}$.

3 Generalized Hamming Similarity Measure for Generalized Set-Valued Neutrosophic Quadruple Numbers

Now, we define generalized Hamming similarity measure for GsvNQn. Also, we assume that T, I, F ∈ [0, 1], as in SvNn, in this paper.

Definition 8: Let X be a non – empty set,

$$G_{N_1} = (A_{s_1}, B_{s_1}T_{s_1}, C_{s_1}I_{s_1}, D_{s_1}F_{s_1}) \text{ and } G_{N_2} = (A_{s_2}, B_{s_2}T_{s_1}, C_{s_2}I_{s_2}, D_{s_2}F_{s_2})$$

be two GsvNQns, $S_H : G_{N_1} \times G_{N_j} \to [0, 1]$ be a function. Then,

$$S_H(G_{N_1}, G_{N_2}) = 1 - \frac{1}{2}\left[\frac{|T_1-T_2|+|I_1-I_2|+|F_1-F_2|}{3} + \frac{4-\left[\frac{s(K_1\cap K_2)}{\max\{s(K_1\cup K_2),1\}}+\frac{s(L_1\cap L_2)}{\max\{s(L_1\cup L_2),1\}}+\frac{s(M_1\cap M_2)}{\max\{s(M_1\cup M_2),1\}}+\frac{s(N_1\cap N_2)}{\max\{s(N_1\cup N_2),1\}}\right]}{4}\right]$$

is called generalized Hamming similarity measure for GsvNQns.

Where, s(A) is the number of element of A ∈ X.

Theorem 2: Let X be a non – empty set;

$$G_{N_1} = (A_{s_1}, B_{s_1}T_{s_1}, C_{s_1}I_{s_1}, D_{s_1}F_{s_1}),\ G_{N_2} = (A_{s_2}, B_{s_2}T_{s_1}, C_{s_2}I_{s_2}, D_{s_2}F_{s_2}) \text{ and } G_{N_3} = (A_{s_3}, B_{s_3}T_{s_3}, C_{s_3}I_{s_3}, D_{s_3}F_{s_3})$$

be three GsvNQns, $S_H : G_{N_1} \times G_{N_j} \to [0, 1]$ be generalized Hamming similarity measure in Definition 8. Then, S_H satisfies the below conditions.

i) $S_H (G_{N_1}, G_{N_2}) \in [0, 1]$

ii) $S_H (G_{N_1}, G_{N_2}) = 1 \Leftrightarrow G_{N_1} = G_{N_2}$

iii) $S_H (G_{N_1}, G_{N_2}) = S_H (G_{N_1}, G_{N_2})$

iv) If $G_{N_1} \subset G_{N_2} \subset G_{N_3}$, then

$S_H (G_{N_1}, G_{N_3}) \le S_H (G_{N_1}, G_{N_2})$ and $S_H (G_{N_1}, G_{N_3}) \le S_H (G_{N_2}, G_{N_3})$.

Proof:

i) Let $G_{N_1} = G_{N_2}$. Then,

$S_H(G_{N_1}, G_{N_1}) =$

$$1-\frac{1}{2}\left[\frac{|T_1-T_1|+|I_1-I_1|+|F_1-F_1|}{3}+\frac{4-\left[\frac{s(K_1\cap K_i)}{\max\{s(K_1\cup K_1),1\}}+\frac{s(L_1\cap L_1)}{\max\{s(L_1\cup L_1),1\}}+\frac{s(M_1\cap M_1)}{\max\{s(M_1\cup M_1),1\}}+\frac{s(N_1\cap N_1)}{\max\{s(N_1\cup N_1),1\}}\right]}{4}\right]$$

$$= 1-\frac{1}{2}\left[\frac{0+0+0}{3}+\frac{4-[1+1+1+1]}{4}\right] \quad (1)$$

Thus, $\max\{S_H(G_{N_1}, G_{N_1})\} = 1$.

Now, let $K_1 \cap K_2 = \emptyset$, $L_1 \cap L_2 = \emptyset$, $M_1 \cap M_2 = \emptyset$, $N_1 \cap N_2 = \emptyset$, $|T_1 - T_2| = 1$, $|I_1 - I_2| = 1$ and $|F_1 - F_2| = 1$. Then,

$$S_H(G_{N_1}, G_{N_2}) = 1-\frac{1}{2}\left[\frac{|T_1-T_2|+|I_1-I_2|+|F_1-F_2|}{3}+\frac{4-\left[\frac{s(K_1\cap K_2)}{\max\{s(K_1\cup K_2),1\}}+\frac{s(L_1\cap L_2)}{\max\{s(L_1\cup L_2),1\}}+\frac{s(M_1\cap M_2)}{\max\{s(M_1\cup M_2),1\}}+\frac{s(N_1\cap N_2)}{\max\{s(N_1\cup N_2),1\}}\right]}{4}\right]$$

$$= 1-\frac{1}{2}\left[\frac{1+1+1}{3}+\frac{4-[0+0+0+0]}{4}\right]$$

$$= 0.$$

Thus, $\min\{S_H(G_{N_1}, G_{N_1})\} = 0$. Hence, we obtain

$$S_H\,(G_{N_1}, G_{N_2}) \in [0, 1].$$

ii) Let $G_{N_1} = G_{N_2}$. From (1), we obtain $S_H(G_{N_i}, G_{N_j}) = 1$. We assume that

$$S_H(G_{N_i}, G_{N_j}) = 1-\frac{1}{2}\left[\frac{|T_1-T_2|+|I_1-I_2|+|F_1-F_2|}{3}+\frac{4-\left[\frac{s(K_1\cap K_2)}{\max\{s(K_1\cup K_2),1\}}+\frac{s(L_1\cap L_2)}{\max\{s(L_1\cup L_2),1\}}+\frac{s(M_1\cap M_2)}{\max\{s(M_1\cup M_2),1\}}+\frac{s(N_1\cap N_2)}{\max\{s(N_1\cup N_2),1\}}\right]}{4}\right]$$

$$= 1.$$

Where, it must be

$$\frac{1}{2}\left[\frac{|T_1-T_2|+|I_1-I_2|+|F_1-F_2|}{3}+\frac{4-\left[\frac{s(K_1\cap K_2)}{\max\{s(K_1\cup K_2),1\}}+\frac{s(L_1\cap L_2)}{\max\{s(L_1\cup L_2),1\}}+\frac{s(M_1\cap M_2)}{\max\{s(M_1\cup M_2),1\}}+\frac{s(N_1\cap N_2)}{\max\{s(N_1\cup N_2),1\}}\right]}{4}\right] = 0.$$

Thus,

$$|T_1 - T_2| + |I_1 - I_2| + |F_1 - F_2| = 0$$

and

$$\left[\frac{s(K_1\cap K_2)}{\max\{s(K_1\cup K_2),1\}}+\frac{s(L_1\cap L_2)}{\max\{s(L_1\cup L_2),1\}}+\frac{s(M_1\cap M_2)}{\max\{s(M_1\cup M_2),1\}}+\frac{s(N_1\cap N_2)}{\max\{s(N_1\cup N_2),1\}}\right] = 4.$$
(2)

From (2), we obtain that

$$|T_1 - T_2| = |I_1 - I_2| = |F_1 - F_2| = 0$$

and

$$\frac{s(K_1\cap K_2)}{\max\{s(K_1\cup K_2),1\}} = \frac{s(L_1\cap L_2)}{\max\{s(L_1\cup L_2),1\}} = \frac{s(M_1\cap M_2)}{\max\{s(M_1\cup M_2),1\}} = \frac{s(N_1\cap N_2)}{\max\{s(N_1\cup N_2),1\}} = 1.$$

Thus, we have that

$$T_1 = T_2,\ I_1 = I_2,\ F_1 = F_2,\ K_1 = K_2,\ L_1 = L_2,\ M_1 = M_2,\ N_1 = N_2.$$

Therefore, from Definition 6; we obtain

$$G_{N_1} = G_{N_2}$$

iii)

$$S_H(G_{N_1}, G_{N_2}) = 1 - \frac{1}{2}\left[\frac{|T_1-T_2|+|I_1-I_2|+|F_1-F_2|}{3} + \frac{4-\left[\frac{s(K_1\cap K_2)}{\max\{s(K_1\cup K_2),1\}}+\frac{s(L_1\cap L_2)}{\max\{s(L_1\cup L_2),1\}}+\frac{s(M_1\cap M_2)}{\max\{s(M_1\cup M_2),1\}}+\frac{s(N_1\cap N_2)}{\max\{s(N_1\cup N_2),1\}}\right]}{4}\right]$$

$$= 1 - \frac{1}{2}\left[\frac{|T_2-T_1|+|I_2-I_1|+|F_2-F_1|}{3} + \frac{4-\left[\frac{s(K_2\cap K_1)}{\max\{s(K_2\cup K_1),1\}}+\frac{s(L_2\cap L_1)}{\max\{s(L_2\cup L_1),1\}}+\frac{s(M_2\cap M_1)}{\max\{s(M_2\cup M_1),1\}}+\frac{s(N_2\cap N_1)}{\max\{s(N_2\cup N_1),1\}}\right]}{4}\right]$$

$$= S_H(G_{N_2}, G_{N_1}).$$

iv) Let $G_{N_1} \subset G_{N_2} \subset G_{N_3}$. From Definition 7, we obtain that

$T_1 < T_2 < T_3$,

$I_1 < I_2 < T_3$,

$F_1 < F_2 < T_3$,

$K_1 \subset K_2 \subset K_3$,

$L_1 \subset L_2 \subset L_3$,

$M_1 \subset M_2 \subset M_3$,

$N_1 \subset N_2 \subset N_3$. (3)

From (3), we have that

$$\frac{s(K_1\cap K_2)}{\max\{s(K_1\cup K_2),1\}}+\frac{s(L_1\cap L_2)}{\max\{s(L_1\cup L_2),1\}}+\frac{s(M_1\cap M_2)}{\max\{s(M_1\cup M_2),1\}}+\frac{s(N_1\cap N_2)}{\max\{s(N_1\cup N_2),1\}} >$$
$$\frac{s(K_1\cap K_3)}{\max\{s(K_1\cup K_3),1\}}+\frac{s(L_1\cap L_3)}{\max\{s(L_1\cup L_3),1\}}+\frac{s(M_1\cap M_3)}{\max\{s(M_1\cup M_3),1\}}+\frac{s(N_1\cap N_3)}{\max\{s(N_1\cup N_3),1\}}.$$
(4)

Also, from (4), we have that

$$|T_1 - T_2| + |I_1 - I_2| + |F_1 - F_2| < |T_1 - T_3| + |I_1 - I_3| + |F_1 - F_3|.$$
(5)

Thus, from (4) and (5), we obtain that

$$\frac{1}{2}\left[\frac{|T_1-T_2|+|I_1-I_2|+|F_1-F_2|}{3} + \frac{4-\left[\frac{s(K_1\cap K_2)}{\max\{s(K_1\cup K_2),1\}}+\frac{s(L_1\cap L_2)}{\max\{s(L_1\cup L_2),1\}}+\frac{s(M_1\cap M_2)}{\max\{s(M_1\cup M_2),1\}}+\frac{s(N_1\cap N_2)}{\max\{s(N_1\cup N_2),1\}}\right]}{4}\right] <$$
$$\frac{1}{2}\left[\frac{|T_1-T_3|+|I_1-I_3|+|F_1-F_3|}{3} + \frac{4-\left[\frac{s(K_1\cap K_3)}{\max\{s(K_1\cup K_3),1\}}+\frac{s(L_1\cap L_3)}{\max\{s(L_1\cup L_3),1\}}+\frac{s(M_1\cap M_3)}{\max\{s(M_1\cup M_3),1\}}+\frac{s(N_1\cap N_3)}{\max\{s(N_1\cup N_3),1\}}\right]}{4}\right]. \quad (6)$$

Hence, from (6), we have that

$$1 - \frac{1}{2}\left[\frac{|T_1-T_3|+|I_1-I_3|+|F_1-F_3|}{3} + \frac{4-\left[\frac{s(K_1\cap K_3)}{\max\{s(K_1\cup K_3),1\}}+\frac{s(L_1\cap L_3)}{\max\{s(L_1\cup L_3),1\}}+\frac{s(M_1\cap M_3)}{\max\{s(M_1\cup M_3),1\}}+\frac{s(N_1\cap N_3)}{\max\{s(N_1\cup N_3),1\}}\right]}{4}\right] <$$
$$1 - \frac{1}{2}\left[\frac{|T_1-T_2|+|I_1-I_2|+|F_1-F_2|}{3} + \frac{4-\left[\frac{s(K_1\cap K_2)}{\max\{s(K_1\cup K_2),1\}}+\frac{s(L_1\cap L_2)}{\max\{s(L_1\cup L_2),1\}}+\frac{s(M_1\cap M_2)}{\max\{s(M_1\cup M_2),1\}}+\frac{s(N_1\cap N_2)}{\max\{s(N_1\cup N_2),1\}}\right]}{4}\right].$$

Therefore, we obtain $S_H(G_{N_1}, G_{N_3}) \leq S_H(G_{N_1}, G_{N_2})$.

Also, $S_H\ (G_{N_1}, G_{N_3}) \leq S_H\ (G_{N_2}, G_{N_3})$ can be proved similar to $S_H(G_{N_1}, G_{N_3}) \leq S_H(G_{N_1}, G_{N_2})$.

Example 1: Let X = {k, l, m, n, p, r} be a set, G_{N_1}= ({k, l, m}, {k, l}(0.7), {m, l}(0.4), {n, p, r}(0.1)), G_{N_2}= ({k, l, m, n, r}, {k, l, m, n}(0.8), {n, r}(0.2), {p}(0.2)) be two GsvNQns and $S_H(G_{N_1}, G_{N_2})$ be generalized Hamming similarity measure for GsvNQns. Then,

$$S_H(G_{N_1}, G_{N_2}) = 1 - \frac{1}{2}\left[\frac{|T_1-T_2|+|I_1-I_2|+|F_1-F_2|}{3} + \frac{4-\left[\frac{s(K_1\cap K_2)}{\max\{s(K_1\cup K_2),1\}}+\frac{s(L_1\cap L_2)}{\max\{s(L_1\cup L_2),1\}}+\frac{s(M_1\cap M_2)}{\max\{s(M_1\cup M_2),1\}}+\frac{s(N_1\cap N_2)}{\max\{s(N_1\cup N_2),1\}}\right]}{4}\right]$$

$$= 1 - \frac{1}{2}\left[\frac{|0.7-0.8|+|0.4-0.2|+|0.1-0.2|}{3} + \frac{4-\left[\frac{3}{\max\{5,1\}}+\frac{2}{\max\{4,1\}}+\frac{0}{\max\{4,1\}}+\frac{1}{\max\{3,1\}}\right]}{4}\right]$$

= 0.6125.

Where,

T_1 = 0.7, I_1 = 0.4, F_1 = 0.1; K_1 = {k, l, m}, L_1 = {k, l}, M_1 = {l, m}, N_1 = {n, p, r};

T_2 = 0.8, I_2 = 0.2, F_2 = 0.2; K_2 = {k, l, m, n, r}, L_2 = {k, l, m, n}, M_2 = {n, r}, N_2 = {p}.

4 Algorithm for Multi-Criteria Decision-Making Application

In this section, we rearranged the algorithm in Aslan et al. [2] for GsvNQns. Also, in this new algorithm, we used generalized Hamming similarity measure in section 3. So, we use the GsvNQns and generalized Hamming similarity measure instead of SvNns and similarity measure in algorithm [2]. Also, we assume that X is a nonempty set.

Step 1: The criteria are determined by considering the application. Let the set of criteria of laws be

$$K = \{k_1, k_2, \ldots, k_m\}.$$

Step 2: The weight values of the criteria for the application. Let the set of weight values be

$$W = \{w_1, w_2, \ldots, w_m\}.$$

Where,

the weight value of criterion k_1 is w_1,

the weight value of criterion k_2 is w_2,

the weight value of criterion k_3 is w_3,

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.

the weight value of criterion k_m is w_m,

Also, $w_i \epsilon [0, 1]$ and $\sum_{i=1}^{m} w_i = 1$.

Step 3: The ideal object is determined as GsvNQs according to criterias in Step 1 such that

$I = \{k_1:(A_{I_1}, B_{I_1}T_{I_1}, C_{I_1}I_{I_1}, D_{I_1}F_{I_1}), k_2:(A_{I_2}, B_{I_2}T_{I_2}, C_{I_2}I_{I_2}, D_{I_2}F_{I_2}), \ldots, k_m:(A_{I_m}, B_{I_m}T_{I_m}, C_{I_m}I_{I_m}, D_{I_m}F_{I_m}),$ $A_{I_i}, B_{I_i}, C_{I_i}, D_{I_i} \in P(X); i = 1, 2, 3, \ldots, m\}$.

Step 4: The n objects are determined as GsvNQs according to criterias in Step 1 such that

$O_1 = \{k_1:(A_{O_{1_1}}, B_{O_{1_1}}T_{O_{1_1}}, C_{O_{1_1}}I_{O_{1_1}}, D_{O_{1_1}}F_{O_{1_1}}), k_2:(A_{O_{1_2}}, B_{O_{1_2}}T_{O_{1_2}}, C_{O_{1_2}}I_{O_{1_2}}, D_{O_{1_2}}F_{O_{1_2}}), \ldots,$
$k_m:(A_{O_{1_m}}, B_{O_{1_m}}T_{O_{1_m}}, C_{O_{1_m}}I_{O_{1_m}}, D_{O_{1_m}}F_{O_{1_m}}), \quad A_{O_{1_i}}, B_{O_{1_i}}, C_{O_{1_i}}, D_{O_{1_i}} \in P(X); i = 1, 2, 3, \ldots, m\}$

$O_2 = \{k_1:(A_{O_{2_1}}, B_{O_{2_1}}T_{O_{2_1}}, C_{O_{2_1}}I_{O_{2_1}}, D_{O_{2_1}}F_{O_{2_1}}), k_2:(A_{O_{2_2}}, B_{O_{2_2}}T_{O_2}, C_{O_{2_2}}I_{O_{2_2}}, D_{O_{2_2}}F_{O_{2_2}}), \ldots,$
$k_m:(A_{O_{2_m}}, B_{O_{2_m}}T_{O_{2_m}}, C_{O_{2_m}}I_{O_{2_m}}, D_{O_{2_m}}F_{O_{2_m}}), \quad A_{O_{2_i}}, B_{O_{2_i}}, C_{O_{2_i}}, D_{O_{2_i}} \in P(X); i = 1, 2, 3, \ldots, m\}$

.

.

.

$O_n = \{k_1:(A_{O_{n_1}}, B_{O_{n_1}}T_{O_{n_1}}, C_{O_{n_1}}I_{O_{n_1}}, D_{O_{n_1}}F_{O_{n_1}}), k_2:(A_{O_{n_2}}, B_{O_{n_2}}T_{O_{n_2}}, C_{O_{n_2}}I_{O_{n_2}}, D_{O_{n_2}}F_{O_{n_2}}), \ldots,$
$k_m:(A_{O_{n_m}}, B_{O_{n_m}}T_{O_{n_m}}, C_{O_{n_m}}I_{O_{n_m}}, D_{O_{n_m}}F_{O_{n_m}}), \quad A_{O_{n_i}}, B_{O_{n_i}}, C_{O_{n_i}}, D_{O_{n_i}} \in P(X); i = 1, 2, 3, \ldots, m\}$

Step 5: The objects given in Step 4 are stated in the form of table (Table 1).

Table 1. Table of objects

	k_1	k_2	…	k_m
O_1	$(A_{O_{1_1}}, B_{O_{1_1}}T_{O_{1_1}}, C_{O_{1_1}}I_{O_{1_1}}, D_{O_{1_1}}F_{O_{1_1}})$	$(A_{O_{1_2}}, B_{O_{1_2}}T_{O_{1_2}}, C_{O_{1_2}}I_{O_{1_2}}, D_{O_{1_2}}F_{O_{1_2}})$	…	$(A_{O_{1_m}}, B_{O_{1_m}}T_{O_{1_m}}, C_{O_{1_m}}I_{O_{1_m}}, D_{O_{1_m}}F_{O_{1_m}})$
O_2	$(A_{O_{2_1}}, B_{O_{2_1}}T_{O_{2_1}}, C_{O_{2_1}}I_{O_{2_1}}, D_{O_{2_1}}F_{O_{2_1}})$	$(A_{O_{2_2}}, B_{O_{2_2}}T_{O_2}, C_{O_{2_2}}I_{O_{2_2}}, D_{O_{2_2}}F_{O_{2_2}})$	…	$(A_{O_{2_m}}, B_{O_{2_m}}T_{O_{2_m}}, C_{O_{2_m}}I_{O_{2_m}},D_{O_{2_m}}F_{O_{2_m}})$
.	.	.	…	.
.	.	.	.	.
.	.	.	.	.
O_n	$(A_{O_{n_1}}, B_{O_{n_1}}T_{O_{n_1}}, C_{O_{n_1}}I_{O_{n_1}}, D_{O_{n_1}}F_{O_{n_1}})$	$(A_{O_{n_2}}, B_{O_{n_2}}T_{O_{n_2}}, C_{O_{n_2}}I_{O_{n_2}}, D_{O_{n_2}}F_{O_{n_2}})$	…	$(A_{O_{n_m}}, B_{O_{n_m}}T_{O_{n_m}}, C_{O_{n_m}}I_{O_{n_m}},D_{O_{n_m}}F_{O_{n_m}})$

Step 6: In this step, the similarity value of the criteria of the ideal object and the criteria of other objects are calculated by using Table 1 with S_H in Section 3. So, $S_H(I_{k_j}, O_{i_{k_j}})$ is calculated for i = 1, 2, …, n; j = 1, 2, …, m. After all calculations, Table 2 is obtained.

Table 2. Similarity of the criterias of object to the criteria of ideal object

	k_1	k_2	…	k_m
O_1	$S_H(I_{k_1}, O_{1_{k_1}})$	$S_H(I_{k_2}, O_{1_{k_2}})$	…	$S_H(I_{k_m}, O_{1_{k_m}})$
O_2	$S_H(I_{k_1}, O_{2_{k_1}})$	$S_H(I_{k_2}, O_{2_{k_2}})$	…	$S_H(I_{k_m}, O_{2_{k_m}})$
.	.	.	…	.
.	.	.	.	.
.	.	.	.	.
O_n	$S_H(I_{k_1}, O_{n_{k_1}})$	$S_H(I_{k_2}, O_{n_{k_2}})$	…	$S_H(I_{k_m}, O_{n_{k_m}})$

Step 7: The weight value of each criterion given in Step 2 is multiplied by the similarity values in Table 2. Hence, the weighted similarity of the criterias of object to the criteria of ideal object in Table 3 is obtained.

Table 3. Weighted Similarity of the criterias of object to the criteria of ideal object

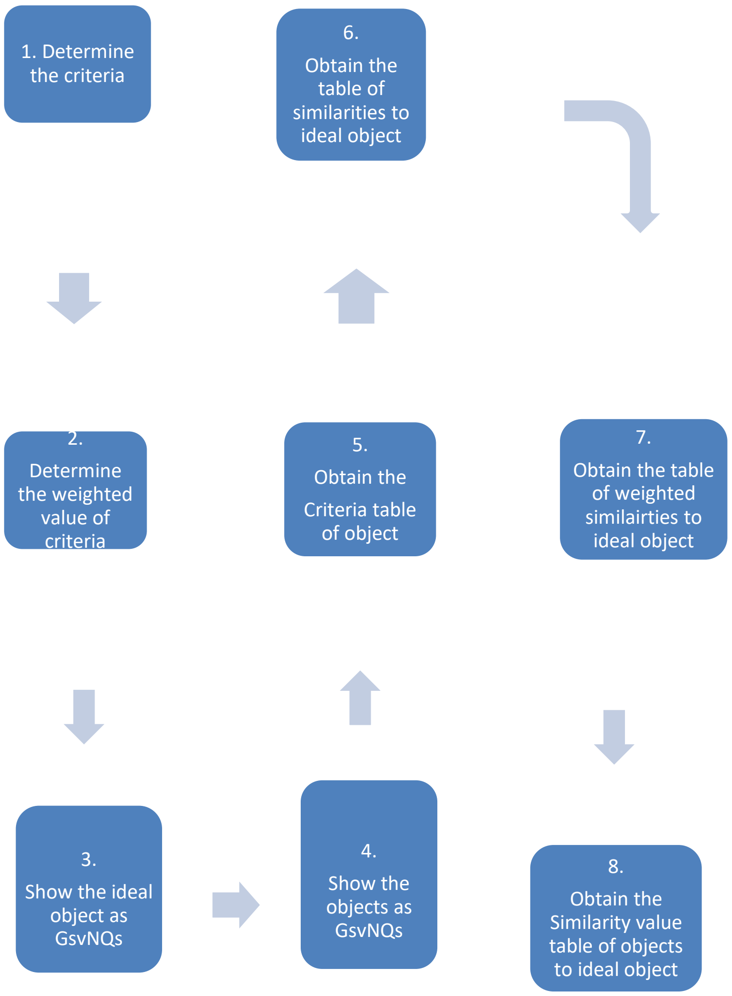
	w_1k_1	w_2k_2	…	w_mk_m
O_1	$w_1.S_H(I_{k_1}, O_{1_{k_1}})$	$w_2.S_H(I_{k_2}, O_{1_{k_2}})$	…	$w_m.S_H(I_{k_m}, O_{1_{k_m}})$
O_2	$w_1.S_H(I_{k_1}, O_{2_{k_1}})$	$w_2.S_H(I_{k_2}, O_{2_{k_2}})$	…	$w_m.S_H(I_{k_m}, O_{2_{k_m}})$
.	.	.	…	.
.	.	.	.	.
.	.	.	.	.
O_n	$w_1.S_H(I_{k_1}, O_{n_{k_1}})$	$w_2.S_H(I_{k_2}, O_{n_{k_2}})$	…	$w_m.S_H(I_{k_m}, O_{n_{k_m}})$

Step 8: In this last step, the weighted similarity values for each objects given in Table 7 are added and the similarity ratio of each law over the ideal law is obtained. So,

S_{H^t} (I, O_t) = $\sum_{t=1}^{m} w_t.\,S_H(I_{k_t}, O_{n_{k_t}})$ is calculated for k = 1, 2, …, m. After all calculations, Table 4 is obtained.

Table 4. The similarity value of the object' to the ideal object

	Similarity Value
O_1	S_{H^1} (I, O_1)
O_2	S_{H^2} (I, O_2)
.	.
.	.
.	.
O_n	S_{H^n} (I, O_n)



Graph 1: Diagram of the algorithm.

5 Multi-Criteria Decision-Making Application

We assume that four different state laws should be created to make use of night watchmen in places where police are inactive at night in four different states. We used the algorithm in Section 4 to find out which law in which state is more effective after a period of time.

Step 1: Let $K = \{k_1,\ k_2,\ k_3\}$ be set of criterias such that

k_1 = life safety

k_2 = property safety

k_3 = cost

Step 2: Let W = {0.6, 0.3, 0.1} be set of the weight values such that

0.6 for the criterion k_1

0.3 for the criterion k_2

0.1 for the criterion k_3

Step 3: Let the ideal law of state be I such that

$$I = \left\{\begin{matrix} k_1: (\{p_1, \dots, p_4, q_1, \dots, q_4, r_1, \dots, r_4, t_1, \dots, t_4\}, \{p_1, \dots, p_4, q_1, \dots, q_4, r_1, \dots, r_4, t_1, \dots, t_4\}(1),\ \emptyset(0),\ \emptyset(0)), \\ k_2: (\{p_1, \dots, p_4, q_1, \dots, q_4, r_1, \dots, r_4, t_1, \dots, t_4\}, \{p_1, \dots, p_4, q_1, \dots, q_4, r_1, \dots, r_4, t_1, \dots, t_4\}(1),\ \emptyset(0),\ \emptyset(0)), \\ k_3: (\{p_1, \dots, p_4, q_1, \dots, q_4, r_1, \dots, r_4, t_1, \dots, t_4\}, \{p_1, \dots, p_4, q_1, \dots, q_4, r_1, \dots, r_4, t_1, \dots, t_4\}(1),\ \emptyset(0),\ \emptyset(0)) \end{matrix}\right\}$$

Where, $\{p_1, \dots, p_4, q_1, \dots, q_4, r_1, \dots, r_4, t_1, \dots, t_4\}$ is known part and

$\{p_1, \dots, p_4, q_1, \dots, q_4, r_1, \dots, r_4, t_1, \dots, t_4\}(1),\ \emptyset(0),\ \emptyset(0)$ is unknown part for each criteria. Where, T = 1, I = 0 and F = 0. This means that this law gave exactly the desired result. Therefore, this law is the ideal law.

Also,

p_1: Pedestrian police with night watchmen who drive a vehicle from 7.00 p.m to 10.00 p.m

p_2: Pedestrian night watchmen with police who drive a vehicle from 1.00 a.m to 4.00 a.m

p_3: Pedestrian police with pedestrian night watchmen from 7.00 p.m to 10.00 p.m

p_4: Police who drive a vehicle with night watchmen who drive a vehicle from 7.00 p.m to 10.00 p.m

q_1: Police who drive a vehicle with night watchmen who drive a vehicle from 7.00 p.m to 10.00 p.m

q_2: Pedestrian police with pedestrian night watchmen from 1.00 a.m to 4.00 a.m

q_3: Pedestrian night watchmen with police who drive a vehicle from 7.00 p.m to 10.00 p.m

q_4: Pedestrian police with night watchmen who drive a vehicle from 7.00 p.m to 10.00 p.m

r_1: Pedestrian police with pedestrian night watchmen from 7.00 p.m to 10.00 p.m

r_2: Police who drive a vehicle with night watchmen who drive a vehicle from 1.00 a.m to 4.00 a.m

r_3: Pedestrian police with night watchmen who drive a vehicle from 7.00 p.m to 10.00 p.m

r_4: Pedestrian night watchmen with police who drive a vehicle from 7.00 p.m to 10.00 p.m

t_1: Pedestrian night watchmen with police who drive a vehicle from 7.00 p.m to 10.00 p.m

t_2: Pedestrian police with night watchmen who drive a vehicle from 1.00 a.m to 4.00 a.m

t_3: Police who drive a vehicle with night watchmen who drive a vehicle from 7.00 p.m to 10.00 p.m

t_4: Pedestrian police with pedestrian night watchmen from7.00 p.m to 10.00 p.m

Step 4: Let L = {L_1, L_2, L_3, L_4} be set of law of states such that

$$L_1=\left\{\begin{matrix} k_1: (\{p_1,p_2,p_3,p_4\}, \{p_1,p_2\}(0.8),\ \{p_4\}(0.2),\ \{p_3\}(0.1)), \\ k_2: (\{p_1,p_2,p_3,p_4\}, \{p_3\}(0.8),\ \{p_1\}(0.3),\ \{,p_2,p_4\}(0.1)), \\ k_3: (\{p_1,p_2,p_3,p_4\}, \{p_1,p_2,p_3,\}(0.9),\ \emptyset(0),\ \{p_4\}(0.3)) \end{matrix}\right\}$$

$$L_2=\left\{\begin{matrix} k_1: (\{q_1,q_2,q_3,q_4\}, \{q_1,q_2,q_3\}(0.8),\ \{q_4\}(0.4),\ \emptyset(0)), \\ k_2: (\{q_1,q_2,q_3,q_4\}, \{q_1,q_2,q_3\}(0.5),\ \emptyset(0),\ \{q_4\}(0.4)), \\ k_3: (\{q_1,q_2,q_3,q_4\}, \{q_3,q_4\}(0.4),\ \{q_1\}(0.1),\ \{q_2\}(0.7)) \end{matrix}\right\}$$

$$L_3=\left\{\begin{matrix} k_1: (\{r_1,r_2,r_3,r_4\}, \{r_1\}(0.9),\ \{r_2,r_3\}(0.2),\ \{r_4\}(0.3)), \\ k_2: (\{r_1,r_2,r_3,r_4\}, \{r_1,r_2,r_3,r_4\}(0.9),\ \emptyset(0),\ \emptyset(0)), \\ k_3: (\{r_1,r_2,r_3,r_4\}, \{r_1,r_4\}(0.6),\ \{r_2\}(0.4),\ \{r_3\}(0.3)) \end{matrix}\right\}$$

$$L_4=\left\{\begin{matrix} k_1: (\{t_1,t_2,t_3,t_4\}, \{t_4\}(0.9),\ \{t_1,t_2\}(0.1),\ \{t_3\}(0.1)), \\ k_2: (\{t_1,t_2,t_3,t_4\}, \{t_2,t_4\}(0.7),\ \{t_3\}(0.5),\ \{t_1\}(0.2)), \\ k_3: (\{t_1,t_2,t_3,t_4\}, \{t_1,t_2,t_4\}(0.4),\ \{t_3\}(0.5),\ \emptyset(0)) \end{matrix}\right\}$$

Step 5: We obtain Table 5 according to Step 4

Table 5. Table of laws

	k_1	k_2	k_3
L_1	$\left(\{p_1,p_2,p_3,p_4\},\{p_1,p_2\}(0.8), \{p_4\}(0.2),\{p_3\}(0.1)\right)$	$(\{p_1,p_2,p_3,p_4\},\{p_3\}(0.8),\{p_1\}(0.3), \{,p_2,p_4\}(0.1)$	$\left(\{p_1,p_2,p_3,p_4\},\{p_1,p_2,p_3,\}(0.9),\emptyset(0), \{p_4\}(0.3)\right)$
L_2	$\left(\{q_1,q_2,q_3,q_4\},\{q_1,q_2,q_3\}(0.8), \{q_4\}(0.4),\emptyset(0)\right)$	$(\{q_1,q_2,q_3,q_4\},\{q_1,q_2,q_3\}(0.5),\emptyset($ $\{q_4\}(0.4)$	$\left(\{q_1,q_2,q_3,q_4\},\{q_3,q_4\}(0.4),\{q_1\}(0.1), \{q_2\}(0.7)\right)$
L_3	$(\{r_1,r_2,r_3,r_4\},\{r_1\}(0.9),\{r_2,r_3\}(0.2$ $\{r_4\}(0.3)$	$(\{r_1,r_2,r_3,r_4\},\{r_1,r_2,r_3,r_4\}(0.9),\emptyset($ $\emptyset(0)$	$\left(\{r_1,r_2,r_3,r_4\},\{r_1,r_4\}(0.6),\{r_2\}(0.4), \{r_3\}(0.3)\right)$
L_4	$(\{t_1,t_2,t_3,t_4\},\{t_4\}(0.9),\{t_1,t_2\}(0.1$ $\{t_3\}(0.1)$	$(\{t_1,t_2,t_3,t_4\},\{t_2,t_4\}(0.7),\{t_3\}(0.5$ $\{t_1\}(0.2)$	$\left(\{t_1,t_2,t_3,t_4\},\{t_1,t_2,t_4\}(0.4),\{t_3\}(0.5), \emptyset(0)\right)$

Step 6: We obtain similarity of the criterias of law to the criteria of ideal law in Table 6.

Table 6. Similarity of the criterias of law to the criteria of ideal law

	k_1	k_2	k_3
L_1	0.322917	0.306250	0.339583
L_2	0.400000	0.350000	0.266667
L_3	0.400000	0.483333	0.316667
L_4	0.450000	0.333333	0.316667

Step 7: We obtain weighted similarity of the criterias of law to the criterias of ideal law in Table 7.

Table 7. Weighted similarity of the criterias of law to the criterias of ideal law

	$(0.6).k_1$	$(0.3).k_2$	$(0.1).k_3$
L_1	0.19375	0.091875	0.033958
L_2	0.24000	0.10500	0.026667
L_3	0.24000	0.144999	0.031667
L_4	0.27000	0.099990	0.031667

Step 8: We obtain similarity value of the object' to the ideal object in Table 8.

Table 8. The similarity value of the law' to the ideal law

	Similarity value
L_1	S_{H^1} (I, L_1) = 0.319583
L_2	S_{H^2} (I, L_2) = 0.371667
L_3	S_{H^3} (I, L_3) = 0.41666
L_4	S_{H^4} (I, L_4) = 0.31958

From Table 8, the laws that work best are L_3, L_2, L_1 and L_4, respectively.

6 Comparison Method

In this section, we compared the results of the generalized algorithm based on the generalized Hamming similarity measure and GsvNQn with the results of the algorithm [2] based on the Hamming similarity measure and SvNn.

If only the T, I, F components of the GsvNQns are in Section 5, we obtain in Table 9.

Table 9. Table of laws based on only (T, I , F)

	k_1	k_2	k_3
L_1	(0.8, 0.2, 01)	(0.8, 0.3, 0.1)	(0.9, 0.0, 0.3)
L_2	(0.8, 0.4, 0.0)	(0.5, 0.0, 0.4)	(0.4, 0.1, 0.7)
L_3	(0.9, 0.2, 0.3)	(0.9, 0.0, 0.0)	(0.6, 0.4, 0.3)
L_4	(0.9, 0.1, 01)	(0.7, 0.5, 0.2)	(0.4, 0.5, 0.0)

If we used the Hamming similarity measure [22] with algorithm [2] according to Table 9, we obtain Table 10 for choosing the best laws.

Table 10. The similarity value of the law' to the ideal law according to Hamming similarity measure [22] and SvNn

	Similarity value
L_1	S_{H^1} (I, L_1) = 0.826656
L_2	S_{H^2} (I, L_2) = 0.74333
L_3	S_{H^3} (I, L_3) = 0.833333
L_4	S_{H^4} (I, L_4) = 0.80333

From Table 10, the laws that work best are L_3, L_1, L_4 and L_2, respectively. Thus, we obtain different result from Section 5.

7 Discussion and Conclusions

In this study, we firstly generalized Hamming similarity measures for the GsvNQn. We showed that generalized Hamming measure satisfies the similarity measure condition. Also, we firstly generalized an algorithm (based on SvNn) for the GsvNQn and we gave a multi-criteria decision-making application using this generalized algorithm. In this application, we examined which of the laws established in different states were more efficient.

From Table 8, if we use generalized Hamming similarity measure and GsvNQn we obtain the laws that work best are

$$L_3,\ L_2,\ L_1 \text{ and } L_4$$

respectively.

From Table 10, if we use Hamming similarity measure and SvNn, we obtain the laws that work best are

$$L_3,\ L_1,\ L_4 \text{ and } L_2$$

respectively. Thus, we obtain different results according to Hamming similarity measure and SvNn in this paper. In addition, the result we obtained in Table 8 is more valid because the generalized set-valued neutrosophic quadruple numbers contain components (T, I, F) of neutrosophic sets and have more extensive components (known part, unknown part) than neutrosophic sets. As can be seen in this study, it is clear that generalized set-valued neutrosophic structures will give more objective results than both the applications using classical structures and the applications using neutrosophic structures.

Also, using this study or revising this application researchers can also work on other law applications and other science applications for decision-making problems. Furthermore, there are a lot of similarity measure for neutrosophic sets. Researchers can generalize the other similarity measures of neutrosophic set according to GsvNQn. Also, in this paper, we use single-valued neutrosophic component T, I, F ∈ [0, 1] (as in SvNn). Researchers can study generalized set-valued neutrosophic quadruple set according to bipolar neutrosophic component or interval valued neutrosophic component and researchers can use these structures for decision-making applications.

Abbreviations

SvNn: Single valued neutrosophic number

SvNs: Single valued neutrosophic set

GsvNQn: Generalized set valued neutrosophic quadruple number

GsvNQs: Generalized set valued neutrosophic quadruple set

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The safety assessment in dynamic conditions using interval neutrosophic sets

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Abstract: In this paper, the notions of three operators, Basic Belief Assignment Operator, Dynamic Basic Belief Assignment Operator, and Dynamic Weight Vector Operator in interval neutrosophic set are defined and presented. The procedure based on Dynamic Basic Belief Assignment and Dynamic Weight Vector using Dezert-Smarandache Theory is developed to solve the dynamic decision-making problems in a neutrosophic environment where criteria values take the form of interval neutrosophic numbers collected at various periods. Practical applications for validating the proposed method and assessing system safety are given taking an example from the marine industry. The results indicate that the proposed methodology provides a feasible solution for monitoring and enhancing the safety of systems working in complex and dynamically changing environment. The model can be applied to solve multicriteria decision-making problems in diversified areas that require dynamic data.

Keywords: Basic Belief Assignment Operator; Dezert-Smarandache Theory; Dynamic Basic Belief Assignment Operator; Dynamic Weight Vector; Evidential Reasoning; Interval Neutrosophic Number

1. Introduction

Multi-Criteria Decision Making (MCDM) involves either selecting the best alternative or prioritizing them after evaluating for the laid down criteria. MCDM takes the required data from records. In case the data are unreliable or scarce, experts' judgments are used for analysis. Such data contain a lot of uncertainty and hence conventional crisp techniques do not work. To overcome the limitation of crisp sets, Zadeh [1, 2] proposed the concept of a fuzzy set. The fuzzy sets were further extended to Interval Valued Fuzzy Set (IVFS) [3], Intuitionistic Fuzzy Set (IFS) [4], and Interval Valued Intuitionistic Fuzzy Set (IVIFS) [5]. The fuzzy sets are extensively used in solving MCDM problems [6-18]. But, none of the above fuzzy sets could explain the indeterminacy component associated with the membership of an element. The fuzzy sets cannot handle the possibility of the statement being true is 0.6, the statement being false is 0.4 and the statement not being sure is 0.3. Smarandache [19] developed the concept of neutrosophic sets where indeterminacy is explicitly characterized that overcome the prime limitation of fuzzy set. Neutrosophic set is defined as, a set A in a universal set X is characterized independently by a truth membership function $T_A(X)$, indeterminacy membership function $I_A(X)$, and falsity membership function $F_A(X)$, wherein

X are real or nonstandard subsets of $\left]^{-}0,1^{+}\right[$. In neutrosophic notation, the above example can be characterized as $A = \{\langle 0.6, 0.3, 0.4 \rangle\}$. To use neutrosophic sets in practical applications, Wang [20, 21] proposed the concept of a Single Valued Neutrosophic set (SVNS) and an Interval Neutrosophic set (INS). Neutrosophic sets have wide applications in decision-making problems [22-26]. Triangular neutrosophic numbers [27, 28], pentagonal fuzzy neutrosophic numbers [29-32], cylindrical neutrosophic numbers [33] are other forms of neutrosophic numbers used in solving MCDM problems. N-valued neutrosophic sets [34], bipolar neutrosophic sets [35], and neutrosophic refined sets [36] are also very popular among researchers. Neutrosophic sets are further generalized into plithogenic sets [37] which are currently used to solve real-life problems [38, 39].

Most of the MCDM problems are solved by taking static data that must be available in advance for assessment. But, most of the time we need to make decisions in dynamic conditions where scenarios change very often. Several techniques and methods have been proposed in the past to solve such dynamic decision-making problems [40-46]. Decision making in dynamic conditions requires a fusion of information gathered at different periods, different operating conditions, and even by different teams of experts [47]. Amongst the most popular theories of information fusion is the Dempster-Shafer theory of evidential reasoning [48]. But, this theory suffers from a major limitation under highly conflicting conditions and gives counter-intuitive results [49-51]. Dezert-Smarandache [52] proposed a new DSm rule of combination (DSmT). The classic DSm rule is simple and corresponds to the Free DSm model. Like D-S theory, the classic DSm rule exhibits the commutative and associative properties. It does not use the renormalization process and hence does not suffer from the problems faced by the D-S rule.

Neutrosophic PROMETHEE techniques [53], IoT based fog computing model [54], and neutrosophic analytical hierarchy process [55, 56] are effectively used to solve MCDM problems with fuzzy information. Neutrosophic sets in combination with rough sets are used to segregate and apply only the precise/complete data to enhance the quality of service in smart cities [57]. In this paper, a model is proposed to assess the safety of engineering systems in dynamic conditions. Decision-making in safety (risk) assessment is based on data collected from experts' ambiguous judgment. We have to rely on experts' judgments because the past data are either incomplete, imprecise, or not reliable. The neutrosophic sets are preferred in this study because they can very easily handle the hesitancy part of the experts' judgment. The third component of indeterminacy in the neutrosophic set eliminates the major limitation of a fuzzy set that cannot handle the hesitancy. The model used the INS because of its greater flexibility and precision over single valued neutrosophic sets. The fusion of information in dynamic conditions is done using DSmT of information fusion.

Three operators, Basic Belief Assignment Operator (BBAO), Dynamic Basic Belief Assignment Operator (DBBAO), and Dynamic weight Vector Operator (DWVO) are proposed in this study to get the basic belief assignments from Interval Neutrosophic Number (INN) and to combine the information in a dynamic environment. We have also suggested the utility of the proposed model to solve real-life problems.

1.1. The motivation for the study

Most of the multi-criteria decision-making problems are solved in static conditions where the data are available beforehand. But, in reality, there are situations when we need to use data collected in different periods. This requires the model to be robust which can be used dynamically and iteratively to ascertain the benefits of the actions taken. Moreover, we need to avoid uncertainty due to incomplete, imprecise, and missing data. Neutrosophic set has the potential to eliminate such uncertainty. In this paper, a model is proposed using neutrosophic numbers wherein the data collected in dynamic conditions can be suitably incorporated.

1.2. The novelty of the work

Neutrosophic sets are used to develop a model to assess the risk/safety of the system dynamically in a complex uncertain environment using an evidential reasoning approach. The primary purpose is to develop,

1. Basic Belief Assignment Operator (BBAO)
2. Dynamic Basic Belief Assignment Operator (DBBAO)
3. Dynamic Weight Vector Operator (DWVO)
4. A model using Dezert Smarandache's theory to solve the dynamic decision-making problems

2. Preliminaries

2.1. Neutrosophic Set

Smarandache [19] proposed and developed the concept of a neutrosophic set as an improvement of a fuzzy set. The neutrosophic sets become popular over fuzzy sets due to their indeterminacy component which handles the hesitancy efficiently and in a better way than even the highest level fuzzy set i.e. IVIFS. The neutrosophic set contains three independent components namely, the truth membership T, the Indeterminacy membership I, and the Falsity membership F. SVNS and INS help us represent the real world with uncertain, imprecise, incomplete, and inconsistent information.

2.2. Set Definition

Definition 2.1 [19]: Let U represent a universe of discourse. A neutrosophic set is:

$$A = \{\langle x : T_A(X), I_A(X), F_A(X), x \in U \rangle\}$$

Where $T_A(X), I_A(X), F_A(X), x \in [0,1]$ and

$$0^- \leq \sup(T_A(X)) + \sup(I_A(X)) + \sup(F_A(X)) \leq 3^+$$

Definition 2.2 [47]: A Dynamic Single-Valued Neutrosophic Set (DSVNS) is:

$$A = \{x \in U; x(T_x(t), I_x(t), F_x(t))\} \text{ for all } x \in A:$$

$$T_x, I_x, F_x : [0, \infty) \rightarrow [0,1]$$

where T_x, I_x, F_x are continuous functions whose arguments is time (t).
A Dynamic Interval Valued Neutrosophic Set (DIVNS) is:

$$x([T_x^L(t), T_x^U(t)], [I_x^L(t), I_x^U(t)], [F_x^L(t), F_x^U(t)]) \text{ where } t \geq 0$$

$$T_x^L(t) < T_x^U(t), I_x^L(t) < I_x^U(t), F_x^L(t) < F_x^U(t) \text{ and}$$

$$[T_x^L(t), T_x^U(t)], [I_x^L(t), I_x^U(t)], [F_x^L(t), F_x^U(t)] \subseteq [0,1]$$

In DIVNS, all intervals are changing w.r.t. time (t).

2.3. Set theoretic operations of DIVNS

Let us consider two DIVN numbers:

$$a(t)=\left\{\left\langle T_x^A(t_1),I_x^A(t_1),F_x^A(t_1)\right\rangle,\ldots,\left\langle T_x^A(t_k),I_x^A(t_k),F_x^A(t_k)\right\rangle\right\}$$

$$b(t)=\left\{\left\langle T_x^B(t_1),I_x^B(t_1),F_x^B(t_1)\right\rangle,\ldots,\left\langle T_x^B(t_k),I_x^B(t_k),F_x^B(t_k)\right\rangle\right\}$$

where $t=\{t_1,t_2,\ldots,t_k\}$ is a time sequence at each time $t_l, 1\le l\le k$

Definition 2.3 [47]: Addition of Dynamic Interval Valued Neutrosophic Numbers (DIVNN):

$$a(t)\oplus b(t)=\left\{\begin{array}{l}\left\langle T_x^A(t_1)+T_x^B(t_1)-T_x^A(t_1)\times T_x^B(t_1),I_x^A(t_1)\times I_x^B(t_1),F_x^A(t_1)\times F_x^B(t_1)\right\rangle,\ldots,\\ \left\langle T_x^A(t_k)+T_x^B(t_k)-T_x^A(t_k)\times T_x^B(t_k),I_x^A(t_k)\times I_x^B(t_k),F_x^A(t_k)\times F_x^B(t_k)\right\rangle\end{array}\right\} \quad (1)$$

Multiplication of DIVNN

$$a(t)\otimes b(t)=\left\{\begin{array}{l}\left\langle T_x^A(t_1)\times T_x^B(t_1),I_x^A(t_1)+I_x^B(t_1)-I_x^A(t_1)\times I_x^B(t_1),F_x^A(t_1)+F_x^B(t_1)-F_x^A(t_1)\times F_x^B(t_1)\right\rangle,\ldots,\\ \left\langle T_x^A(t_k)\times T_x^B(t_k),I_x^A(t_k)+I_x^B(t_k)-I_x^A(t_k)\times I_x^B(t_k),F_x^A(t_k)+F_x^B(t_k)-F_x^A(t_k)\times F_x^B(t_k)\right\rangle\end{array}\right\}$$

(2)

Scalar Multiplication of DIVNN

$$\alpha\times a(t)=\left\{\begin{array}{l}\left\langle 1-\left(1-T_x^A(t_1)\right)^\alpha,I_x^A(t_1)^\alpha,F_x^A(t_1)^\alpha\right\rangle,\ldots,\\ \left\langle 1-\left(1-T_x^A(t_k)\right)^\alpha,I_x^A(t_k)^\alpha,F_x^A(t_k)^\alpha\right\rangle\end{array}\right\} \quad (3)$$

Power of the DIVNN

$$a(t)^\alpha=\left\{\begin{array}{l}\left\langle T_x^A(t_1)^\alpha,1-\left(1-I_x^A(t_1)\right)^\alpha,1-\left(1-F_x^A(t_1)\right)^\alpha\right\rangle,\ldots,\\ \left\langle T_x^A(t_k)^\alpha,1-\left(1-I_x^A(t_k)\right)^\alpha,1-\left(1-F_x^A(t_k)\right)^\alpha\right\rangle\end{array}\right\} \quad (4)$$

*2.4. Dezert-Smarandache Theo*ry

Dezert-Smarandache [52] developed the theory of information fusion (DSmT) for dealing with imprecise, uncertain, and conflicting sources of information. It overcame three limitations of D-S theory i.e. accepting Shafer's model for the fusion problem under consideration which requires all hypotheses to be mutually exclusive and exhaustive, the third middle excluded principle, and the acceptance of Dempster's rule of combination as the framework for the combination of independent sources of information. DSmT starts with a free DSm model and is denoted as $M^f(\Theta)$, and considers Θ only as a frame of exhaustive elements, $\theta_i, i=1,\ldots,n$ which can potentially overlap. The free DSm model is commutative and associative.

Definition 2.4 [52]: Let $\Theta=\{\theta_1,\ldots,\theta_n\}$ be a finite set of n exhaustive elements. The hyper-power set D^Θ is defined as the set of all composite subsets built from elements of Θ with $\cup$ and $\cap$ operators such that

1. $\phi,\theta_1,\ldots,\theta_n\in D^\Theta$
2. If $A,B\in D^\Theta$, then $A\cap B\in D^\Theta$ and $A\cup B\in D^\Theta$
3. No other elements belong to D^Θ, except those obtained by rules 1 and 2.

When there is no constraint on the elements of the frame, the classic model is called free DSm model, $M^f(\Theta)$ of two independent sources of evidence over the same frame Θ with belief functions associated with generalized basic belief assignments $m_1(\cdot)$ and $m_2(\cdot)$ and is given by

$$\forall C \neq \phi \in D^{\Theta}, m_M f_{(\theta)}(C) \equiv m(C) = m_1(A) \oplus m_2(B) = \sum_{\substack{A,B \in D^{\Theta} \\ (A \cap B) = C}} m_1(A) m_2(B) \tag{5}$$

This rule is extended for $k \geq 2$ independent sources as,

$$\forall C \neq \phi \in D^{\Theta}, m_M f_{(\theta)}(C) \equiv m(C) = [m_1 \oplus \ldots \oplus m_k](C) = \sum_{\substack{x_1, x_2, \ldots, x_k \in D^{\Theta} \\ (x_1 \cap x_2 \cap \ldots \cap x_k) = A}} \prod_{i=1}^{k} m_i(X_i) \tag{6}$$

and $\quad m_{M^{f(\theta)}}(\phi) = 0$

3. Basic Belief Assignment (BBA), Dynamic Basic Belief Assignment (DBBA) and Dynamic Weight Vector (DWV)

3.1. Basic Belief Assignment (BBA)

Consider an interval neutrosophic set. To use the neutrosophic number in the DSmT evidential reasoning approach, we need to convert the neutrosophic number into its corresponding BBA. BBA or mass function assigns evidence to a preposition. BBAO is proposed to transform the interval neutrosophic number into their corresponding BBA's i.e. $m(T), m(F)$ and $m(I)$.

$$m(\cdot) \equiv BBA(\cdot) = \frac{mean(\cdot)}{sum\_of\_the\_mean(\cdot)} \tag{7}$$

where mean $(\cdot)$ finds the mean of the neutrosophic component interval given by

$$mean(\cdot) = \frac{(\cdot)^L + (\cdot)^U}{2} \tag{8}$$

and $sum\_of\_the\_mean(\cdot)$ gives the summation of the means of all the three components of INS.

3.2. Dynamic Basic Belief Assignment (DBBA)

Consider $A = \{A_1, A_2, \ldots, A_v\}$, $C = \{C_1, C_2, \ldots, C_n\}$, and $D = \{D_1, D_2, \ldots, D_h\}$ be the sets of alternatives, criteria and decision makers [47]. For a decision maker $D_q; q = 1, \ldots, h$, the evaluation characteristic of an alternative $A_a; a = 1,..., v$ on a criterion $C_p; p = 1,..., n$ in time sequence $t_l = \{t_1, t_2,..., t_k\}$ is represented by

$$X_{apq}(t_l) = \{[T^L_{apq}(X_{t_l}), T^U_{apq}(X_{t_l})], [I^L_{apq}(X_{t_l}), I^U_{apq}(X_{t_l})], [F^L_{apq}(X_{t_l}), F^U_{apq}(X_{t_l})]\} \tag{9}$$

DBBA for the above neutrosophic number is obtained by DBBAO and DSmT of information fusion. Since DSmT is closed on $\cup$ and $\cap$, so also truthness and falsity components are exclusive, both the belief components of $T \cup F$ and $T \cap F$ are assigned to $T \cup F$.

Dynamic basic belief mass,

$$m_{D_{ap}}(C) \equiv DBBAO(C) = \sum_{\substack{x_1, x_2, ..., x_l \in D^\Theta \\ x_1 \cap x_2 \cap .. \cap x_l = C}} \left[\prod_{l=1}^{k} \left[\sum_{\substack{x_1, x_2, ..., x_q \in D^\Theta \\ x_1 \cap x_2 \cap .. \cap x_q = C_a}} \left[\prod_{q=1}^{h} m_{lq}(X_{t_l}) \right] \right] \right] \tag{10}$$

for $a = 1,.., v$ and $p = 1,..., n$

3.3. Dynamic Weight Vector (DWV)

Decision-makers assess various alternatives w.r.t. assigned criteria. These criteria, in turn, are also evaluated to decide their importance by a group of decision-makers in different periods. These are generally expressed in linguistic terms. These are to be converted into neutrosophic numbers and aggregated to get the dynamic weight vector for information fusion. This is done by horizontal integration of neutrosophic numbers for all the decision-makers in all periods using DWVO.

Consider $C = \{C_1, C_2,..., C_n\}$ and $D = \{D_1, D_2,..., D_h\}$ be the sets of criteria and decision makers [47]. For a decision maker $D_q; q = 1,..., h$, the evaluation characteristic of a criterion $C_p; p = 1,..., n$ in time sequence $t_l = \{t_1, t_2,..., t_k\}$ is represented by

$$X_{pq}(t_l) = \{[T^L_{pq}(X_{t_l}), T^U_{pq}(X_{t_l})], [I^L_{pq}(X_{t_l}), I^U_{pq}(X_{t_l})], [F^L_{pq}(X_{t_l}), F^U_{pq}(X_{t_l})]\} \tag{11}$$

The averaged aggregation is,

$$\bar{X}_p = \{[\bar{T}^L_p(X), \bar{T}^U_p(X)], [\bar{I}^L_p(X), \bar{I}^U_p(X)], [\bar{F}^L_p(X), \bar{F}^U_p(X)]\} \tag{12}$$

where

$$\bar{T}_p(X) = \left[\left\langle 1 - \left[\prod_{i=1}^{k} \left[1 - \left[1 - \prod_{q=1}^{h} \left(1 - T^L_{iq}(X)\right)^{1/h} \right] \right] \right]^{1/k} \right\rangle, \left\langle 1 - \left[\prod_{i=1}^{k} \left[1 - \left[1 - \prod_{q=1}^{h} \left(1 - T^U_{iq}(X)\right)^{1/h} \right] \right] \right]^{1/k} \right\rangle \right]$$

(13)

$$\bar{I}_p(X)=\left[\left\langle\prod_{i=1}^{k}\left[\prod_{q=1}^{h}\left[I_{iq}^{L}(X)\right]^{1/h}\right]^{1/k}\right\rangle,\left\langle\prod_{i=1}^{k}\left[\prod_{q=1}^{h}\left[I_{iq}^{U}(X)\right]^{1/h}\right]^{1/k}\right\rangle\right] \tag{14}$$

and

$$\bar{F}_p(X)=\left[\left\langle\prod_{i=1}^{k}\left[\prod_{q=1}^{h}\left[F_{iq}^{L}(X)\right]^{1/h}\right]^{1/k}\right\rangle,\left\langle\prod_{i=1}^{k}\left[\prod_{q=1}^{h}\left[F_{iq}^{U}(X)\right]^{1/h}\right]^{1/k}\right\rangle\right] \tag{15}$$

The dynamic weight vector is a column vector $W=(w_d)_{n\times 1}$ and obtained by DWVO using the averaged aggregation,

$$\bar{w}_d \equiv DWVO(\bar{X}_p)=\frac{mean(\bar{T}_p(X))+mean(\bar{I}_p(X))+mean(\bar{F}_p(X))}{\sum_{p=1}^{n}\left[sum\_of\_the\_mean(\bar{X}_p)\right]} \tag{16}$$

4. Dynamic information fusion

Two methods are given below, one to dynamically evaluate and rank the alternatives and the second one to assess the safety of systems dynamically in a complex and uncertain environment.

4.1 Method to evaluate and rank the alternatives

Consider $A=\{A_1,A_2,...,A_v\}$, $C=\{C_1,C_2,...,C_n\}$, $D=\{D_1,D_2,...,D_h\}$ and $t=\{t_1,t_2,...,t_k\}$ be the sets of alternatives, criteria, decision-makers and periods. The proposed steps are:

Step 1: Let $'h'$ decision-makers evaluate $'v'$ alternatives w.r.t. $'n'$ criteria in $'k'$ periods as per the suitability ratings given in Table 1. Represent the evaluated characteristics in a matrix $\left(X_{apq}(t_l)\right)_{v\times k}$ given by,

$$X_{apq}(t_l)=\left\{\left[T_{apq}^{L}(X_{t_l}),T_{apq}^{U}(X_{t_l})\right],\left[I_{apq}^{L}(X_{t_l}),I_{apq}^{U}(X_{t_l})\right],\left[F_{apq}^{L}(X_{t_l}),F_{apq}^{U}(X_{t_l})\right]\right\} \tag{17}$$

$a=1,...,v$; $p=1,...,n$; $q=1,...,h$; $l=1,...,k$

Table 1. Suitability ratings as linguistic variables

Linguistic terms	INS
Very_Poor (Ve_Po)	([0.1, 0.2], [0.6, 0.7], [0.7, 0.8])
Poor (Po)	([0.2, 0.3], [0.5, 0.6], [0.6, 0.7])
Medium (Me)	([0.3, 0.5], [0.4, 0.6], [0.4, 0.5])
Good (Go)	([0.5, 0.6], [0.4, 0.5], [0.3, 0.4])
Very_Good (Ve_Go)	([0.6, 0.7], [0.2, 0.3], [0.2, 0.3])

Step 2: Applying DSmT on the evaluated characteristic matrix and using DBBAO, get the dynamic mass of an alternative $'a'$ for a criterion $'p'$ using Eq. (10).

Step 3: Let $'h'$ decision-makers evaluate $'n'$ criteria in $'k'$ periods as per their weights given in Table 2.

Table 2. Importance weights as linguistic variables

Linguistic terms	INS
Unimportant (U_IPA)	([0.1, 0.2], [0.4, 0.5], [0.6, 0.7])
Ordinary_Important (O_IPA)	([0.2, 0.4, [0.5, 0.6], [0.4, 0.5])
Important (IPA)	([0.4, 0.6], [0.4, 0.5], [0.3, 0.4])
Very_Important (V_IPA)	([0.6, 0.8], [0.3, 0.4], [0.2, 0.3])
Absolutely_Important (A_IPA)	([0.7, 0.9], [0.2, 0.3], [0.1, 0.2])

Step 4: Find the averaged aggregation of all the $'n'$ criteria as given by $'h'$ decision-makers in $'k'$ periods using Eq. (12).

Step 5: Calculate the dynamic weight vector using Eq. (16).

Step 6: Obtain the weighted dynamic basic belief assignments (m_{wD}) for all the alternatives from the dynamic basic belief assignments (m_D) and the dynamic weight vector $(\overline{w}_D)$ of the criteria.

$$m_{wD_{ap}}(X) = \overline{w}_d \times m_{D_{ap}}(X) \qquad \text{for } a = 1,\dots,v \text{ and } p = 1,\dots,n \tag{18}$$

Step 7: Synthesize the information using weighted dynamic basic belief assignments w.r.t. criteria and applying the classic DSmT of information fusion to get the dynamic belief masses for all the alternatives which are further normalized to get the final belief masses.

$$m_{D_a}(C) = \sum_{\substack{X_1, X_2, \dots, X_n \in D^\Theta \\ X_1 \cap X_2 \cap .. X_n = C}} \left[\prod_{p=1}^{n} m_{wd_a}(X) \right] \qquad \text{for } a = 1,\dots,v \tag{19}$$

Step 8: To rank the alternatives and choose the best one, compare it with the ideal alternative using the similarity measure. The similarity measure proposed by Jiang [58] using the correlation coefficient of belief functions is used.

The flowchart of all the steps to evaluate and rank the alternatives is shown in Fig. 1.

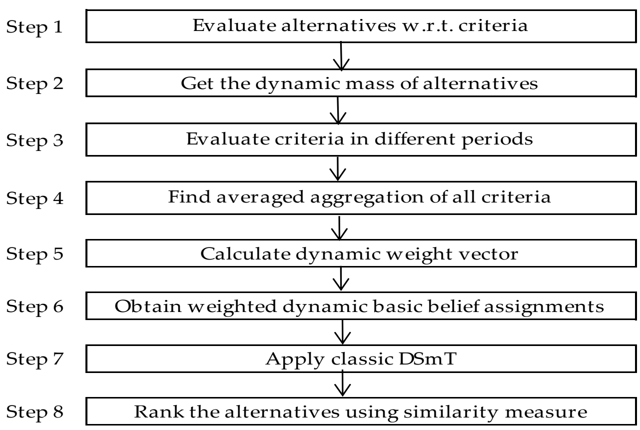

Fig.1. The flowchart to evaluate and rank the alternatives

Definition 4.1. [58]: Consider a discernment frame Θ of N elements. If we denote the mass of two pieces of evidence by m_1 and m_2, then the correlation coefficient is defined as,

$$r_{BPA}(m_1,m_2)=\frac{c(m_1,m_2)}{\sqrt{c(m_1,m_1),c(m_2,m_2)}} \quad (20)$$

where the correlation coefficient $r_{BPA}\in[0,1]$ and $c(m_1,m_2)$ is the degree of correlation denoted as:

$$c(m_1,m_2)=\sum_{i=1}^{2^n}\sum_{j=1}^{2^n} m_1(A_i)m_2(A_j)\frac{|A_i\cap A_j|}{|A_i\cup A_j|} \quad (21)$$

and $i,j=1,...,2^n; A_i, A_j$ are the focal elements of mass and $|\cdot|$ is the cardinality of a subset.

The higher value of the correlation coefficient indicates that the belief masses are close to each other. The ideal and the best interval neutrosophic number is, $\alpha^*=\{\langle(1,1),(0,0),(0,0)\rangle\}$.

The correlation coefficient r_i calculated between α^* and any other INN is an unscaled distance. Higher the value of r_i indicates the two numbers are closer to each other. $r_i=1$ indicates α^* is the same as the number. r_i can be normalized as,

$$\beta_i=\frac{r_i}{\sum_{i=1}^{4} r_i} \quad (22)$$

where, $\beta_i(i=1,2,3,4)$ represents the degree of matching between α^* and the given neutrosophic number.

4.2 Method for assessing system safety

Consider $F=\{F_1,F_2,...,F_v\}$, $D=\{D_1,D_2,...,D_h\}$ and $t=\{t_1,t_2,...,t_k\}$ be the sets of failure modes of a system, decision-makers and periods. The proposed steps for assessing system safety are,

Step 1: Let $'h'$ decision-makers identify $'v'$ failure modes of a system.

Step 2: The decision-maker's views are collected on all the $'v'$ failure modes in $'k'$ periods as per the suitability ratings in linguistic terms from Table 1. The evaluated characteristic by $'q'$ decision-maker on failure mode $'a'$ in a period $'l'$ is represented in a matrix form as,

$$(X_{aq}(t_l))_{v\times k}=\{[T^L_{aq}(X_{t_l}),T^U_{aq}(X_{t_l})],[I^L_{aq}(X_{t_l}),I^U_{aq}(X_{t_l})],[F^L_{aq}(X_{t_l}),F^U_{aq}(X_{t_l})]\} \quad (23)$$

$a = 1,\ldots,v$; $q = 1,\ldots,h$; $l = 1,\ldots,k$

Step 3: Horizontal integration is done using DBBAO and by applying DSmT on the evaluated characteristic matrix to get the dynamic mass of all the failure modes.

Step 4: Vertical integration of the dynamic masses of all the failure modes is done using DSmT to get the final dynamic mass of the system.

Step 5: The obtained dynamic mass of the system from step 4 above, is mapped back to the safety expressions of $'Poor'$, $'Average'$, $'Good'$ or using Eqs. (20) – (22). The mapping of dynamic mass with safety expressions gives a distributed assessment in combination of more than one safety expressions. Safety expressions in linguistic terms are shown in Table 3. The neutrosophic safety expressions are converted to their BBA's using BBAO to use the similarity measure.
The flowchart of all the steps for assessing system safety is shown in Fig. 2.

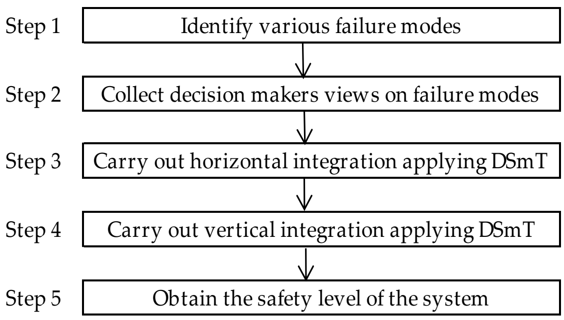

Fig.2. The flowchart for assessing system safety

Table 3. Safety expressions

Linguistic terms	INS
Poor (P)	([0.1, 0.2], [0.2, 0.3], [0.8, 0.9])
Average (A)	([0.4, 0.5, [0.4, 0.5], [0.6, 0.7])
Good (G)	([0.6, 0.7], [0.4, 0.5], [0.4, 0.5])
Excellent (E)	([0.8, 0.9], [0.2, 0.3], [0.1, 0.2])

5. Applications

Two numerical examples are discussed in this section, the first one to validate and demonstrate the proposed method. The second example shows the application of the proposed method to estimate the safety level of the systems on-board the ship.

Example 1: This example is taken from Thong et.al. [47] to evaluate lecturers' performance in the case study of ULIS-VNU. Consider five lecturers i.e. $A_1, A_2, \ldots, A_5$ and three decision-makers i.e. D_1, D_2, D_3. Five lecturers are evaluated with respect to 6 criteria: total publications (C_1), teaching student evaluations (C_2), personality characteristics (C_3), professional society (C_4), teaching experience (C_5), fluency of foreign language (C_6).

Suitability ratings as given by three decision-makers for lecturers versus defined criteria in three different periods are given in Table 4. Their dynamic basic belief assignments are shown at the right end in Table 4.

Table 4. Suitability ratings for lecturers

Criteria	Lecturers	Decision makers									Dynamic Basic Belief masses
		t_1			t_2			t_3			(T, F, TUF)
		D_1	D_2	D_3	D_1	D_2	D_3	D_1	D_2	D_3	
C_1	A_1	Me	Go	Go	Go	Go	Go	Go	Ve_Go	Go	(0.537456, 0.167772, 0.294773)
	A_2	Go	Go	Ve_Go	Ve_Go	Go	Ve_Go	Ve_Go	Go	Ve_Go	(0.677230, 0.089733, 0.233037)
	A_3	Me	Go	Go	Go	Go	Go	Go	Go	Ve_Go	(0.551952, 0.157914, 0.290134)
	A_4	Go	Me	Go	Go	Go	Go	Go	Go	Go	(0.506046, 0.189117, 0.304836)
	A_5	Me	Go	Me	Go	Go	Me	Go	Go	Go	(0.445630, 0.231638, 0.322731)
C_2	A_1	Go	Go	Go	Ve_Go	Go	Go	Go	Go	Go	(0.545188, 0.161556, 0.293256)
	A_2	Ve_Go	Go	Ve_Go	Me	Go	Go	Ve_Go	Go	Go	(0.587106, 0.137222, 0.275673)
	A_3	Ve_Go	Go	Go	Go	Me	Go	Go	Me	Go	(0.495164, 0.194219, 0.310617)
	A_4	Go	Go	Go	Go	Ve_Go	Go	Go	Go	Ve_Go	(0.592985, 0.134266, 0.272749)
	A_5	Ve_Go	Go	Go	Go	Ve_Go	Go	Go	Go	Me	(0.516687, 0.172812, 0.310501)
C_3	A_1	Ve_Go	Ve_Go	Go	Go	Ve_Go	Go	Go	Me	Go	(0.547366, 0.152625, 0.300009)
	A_2	Go	Ve_Go	Go	Ve_Go	Go	Ve_Go	Go	Go	Ve_Go	(0.639759, 0.107299, 0.252942)
	A_3	Go	Ve_Go	Ve_Go	Go	Go	Go	Go	Ve_Go	Go	(0.605431, 0.125957, 0.268611)
	A_4	Go	Go	Go	Ve_Go	Go	Go	Ve_Go	Go	Go	(0.577997, 0.142833, 0.279170)
	A_5	Ve_Go	Go	Go	Go	Ve_Go	Go	Go	Go	Go	(0.564545, 0.147920, 0.287535)
C_4	A_1	Me	Go	Me	Go	Go	Me	Me	Go	Me	(0.374181, 0.293782, 0.332038)
	A_2	Go	Me	Go	Go	Me	Go	Go	Me	Go	(0.456588, 0.224954, 0.318457)
	A_3	Go	Go	Go	Go	Go	Me	Go	Go	Ve_Go	(0.542148, 0.163564, 0.294288)
	A_4	Me	Po	Me	Go	Me	Me	Go	Go	Me	(0.335600, 0.325733, 0.338667)
	A_5	Me	Me	Po	Me	Me	Me	Me	Go	Me	(0.279417, 0.384679, 0.335904)
C_5	A_1	Me	Go	Me	Me	Go	Go	Go	Me	Go	(0.427180, 0.248386, 0.324434)
	A_2	Go	Ve_Go	Go	Ve_Go	Go	Go	Go	Ve_Go	Go	(0.597962, 0.130556, 0.271483)
	A_3	Go	Go	Me	Go	Go	Go	Go	Ve_Go	Go	(0.527769, 0.173730, 0.298501)
	A_4	Ve_Go	Go	Go	Ve_Go	Go	Go	Ve_Go	Go	Go	(0.597962, 0.130556, 0.271483)
	A_5	Go	Go	Go	Go	Go	Go	Go	Ve_Go	Go	(0.557417, 0.155493, 0.287090)
C_6	A_1	Ve_Go	Go	Go	Ve_Go	Go	Ve_Go	Ve_Go	Go	Ve_Go	(0.668533, 0.094153, 0.2237315)
	A_2	Go	Go	Go	Go	Ve_Go	Go	Go	Go	Ve_Go	(0.592985, 0.134266, 0.272749)
	A_3	Ve_Go	Go	Ve_Go	Ve_Go	Go	Ve_Go	Ve_Go	Go	Ve_Go	(0.693488, 0.081472, 0.225040)
	A_4	Go	Ve_Go	Go	Go	Ve_Go	Go	Go	Go	Go	(0.564545, 0.147920, 0.287535)
	A_5	Go	Go	Go	Ve_Go	Go	Go	Go	Ve_Go	Go	(0.577997, 0.142833, 0.279170)

The evaluation of criteria by decision-makers as per their importance is shown in Table 5. The right end column of Table 5 shows the dynamic weight vector.

Table 5. Evaluation of criteria by decision makers

Criteria	Decision makers									Dynamic Weight vector
	t_1			t_2			t_3			
	D_1	D_2	D_3	D_1	D_2	D_3	D_1	D_2	D_3	
C_1	IPA	IPA	IPA	IPA	V_IPA	IPA	V_IPA	IPA	V_IPA	0.166934
C_2	V_IPA	V_IPA	IPA	V_IPA	V_IPA	V_IPA	A_IPA	V_IPA	V_IPA	0.166570
C_3	IPA	IPA	V_IPA	IPA	IPA	V_IPA	V_IPA	IPA	V_IPA	0.167202
C_4	IPA	V_IPA	IPA	IPA	O_IPA	IPA	IPA	IPA	IPA	0.165894
C_5	IPA	IPA	IPA	V_IPA	IPA	V_IPA	IPA	IPA	IPA	0.166197
C_6	V_IPA	V_IPA	IPA	IPA	IPA	IPA	V_IPA	V_IPA	IPA	0.167202

The final normalized weighted dynamic belief masses of lecturers are given in Table 6. Table 7 gives the normalized correlation coefficients of all the alternatives w.r.t. the best and ideal neutrosophic number.

Table 6. Final normalized weighted dynamic belief masses

Lecturers	Normalized Weighted Dynamic Belief masses
A_1	(0.697808, 0.078287, 0.223905)
A_2	(0.760933, 0.050429, 0.188578)
A_3	(0.796668, 0.042129, 0.161202)
A_4	(0.701103, 0.077390, 0.221507)
A_5	(0.662146, 0.097506, 0.240348)

Table 7. Normalized correlation coefficients

Lecturers	Normalised correlation coefficients
$r_1(\alpha^*,A_1)$	0.198675
$r_2(\alpha^*,A_2)$	0.202459
$r_3(\alpha^*,A_3)$	0.204062
$r_4(\alpha^*,A_4)$	0.198903
$r_5(\alpha^*,A_5)$	0.195901

Referring to Table 7, the order of best performed lecturer to the least performed lecturer is $A_3 > A_2 > A_4 > A_1 > A_5$. The ranking order given by [47] is $A_2 > A_3 > A_4 > A_1 > A_5$. Except for the first two alternatives, the ranking order for the rest of other alternatives is in line with [47].

Example 2(a): An example from Ship is taken to illustrate how dynamically we can monitor the safety level of systems in a complex and uncertain environment using a neutrosophic set. Failure modes of Steering Gear on board ship are monitored periodically after maintenance and the safety level of the system is assessed. Steering Gear failure is common in the maritime industry and resulted in very serious accidents in the past causing major damage to the ship and its crew. This demands periodic maintenance to ensure and maintain the smooth functioning of the ship's steering gear. Two experts from the marine field (two Chief Engineers on the ship with sea sailing experience of over 20 years) were asked to analyze the steering gear system and identify the common failure modes of the system. Equal weights are assigned to the two experts. Experts identified five critical failure modes (Fig. 3) and their safety level using linguistic terms from Table 1 in two different periods. The evaluated characteristic matrix by experts in linguistic terms is given in Table 8.

Table 8. Evaluated characteristic matrix for failure modes

Failure Modes	Experts			
	t_1		t_2	
	D_1	D_2	D_1	D_2
F_1	Me	Me	Go	Go
F_2	Go	Go	Go	Go
F_3	Me	Go	Me	Go
F_4	Po	Po	Me	Me
F_5	Me	Me	Me	Go

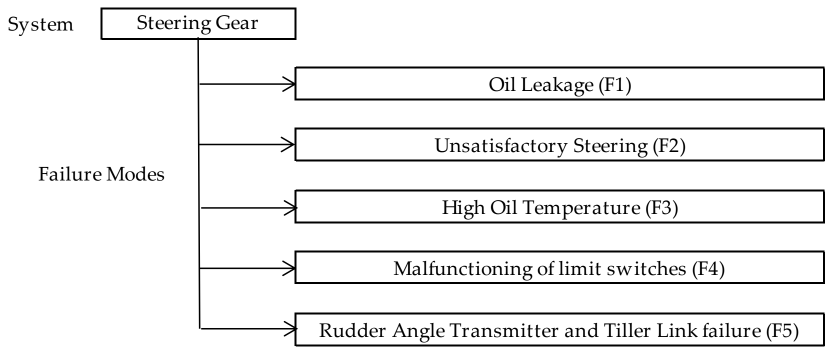

Fig.3. Steering Gear system with failure modes

Dynamic masses of all the failure modes are obtained by horizontal integration using DSmT and DBBAO. These are given in Table 9.

Table 9. Dynamic belief masses for the failure modes

Failure Modes	Dynamic Belief masses		
	m(T)	m(F)	m(T, F)
F_1	0.379971	0.281706	0.338324
F_2	0.473601	0.212394	0.314005
F_3	0.38612	0.283486	0.330394
F_4	0.194758	0.481636	0.323606
F_5	0.341035	0.323677	0.335288

Vertical integrating all the masses of failure mode using DSmT, we get the system's dynamic belief masses as,

$m(T)=0.325928$, $m(F)=0.340302$, and $m(T,F)=0.33377$

The safety score of the system is mapped back to the safety expressions using similarity measures. The safety level of the system obtained is,

$\beta_{poor}=0.23539$, $\beta_{Average}=0.266823$, $\beta_{Good}=0.265923$, $\beta_{Excellent}=0.231864$

From the above results, it is seen that the steering gear system is assessed as $'Average'$ with a belief of 26.68 %, as $'Good'$ with a belief of 26.59 %, as $'Poor'$ with a belief of 23.54 % and as $'Excellent'$ with a belief of 23.19%.

The result in graphical form is shown in Fig. 4.

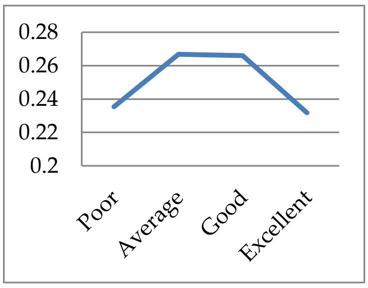

Fig. 4. System safety level

Example 2(b): The system safety level of the same example above is assessed in one more period after the regular maintenance. The two experts' views at time t_3 are given in Table 10.

Table 10. Evaluated characteristic matrix for failure modes at time t_3

Failure Modes	Experts	
	t3	
	D1	D2
F1	Ve_Go	Go
F2	Ve_Go	Ve_Go
F3	Go	Ve_Go
F4	Go	Go
F5	Go	Ve_Go

System safety level after including the third period t_3 is,

$\beta_{poor}=0.190742$, $\beta_{Average}=0.255002$, $\beta_{Good}=0.277084$, $\beta_{Excellent}=0.277172$

The results show that after inclusion of the third period, the steering gear system is assessed as $'Excellent'$ with a belief of 27.72 %, as $'Good'$ with a belief of 27.71 %, as $'Average'$ with a belief of 25.50 % and as $'Poor'$ with a belief of 19.07%. With periodic maintenance of the system, the safety level can be improved. Fig. 5. shows the result in graphical form.

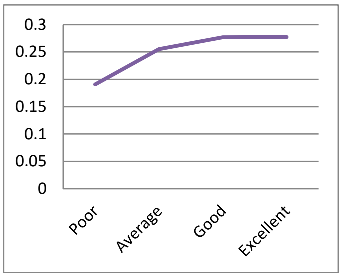

Fig. 5. System safety level (including the third period)

6. Conclusion

This paper proposed three operators Basic Belief Assignment Operator, Dynamic Basic Belief Assignment Operator (DBBAO), and Dynamic Weight Vector Operator (DWVO) to get Basic Belief Assignment (BBA), Dynamic Basic Belief Assignment (DBBA), and Dynamic Weight Vector (DWV) from the Interval Neutrosophic Number (INN). Methods are proposed with these operators in combination with Dezert-Smarandache Theory (DSmT) of information fusion to take decisions dynamically in the complex uncertain neutrosophic environments using INS. The feasibility and application of proposed methods are shown by examples from the marine industry. The method proposed can be used to monitor the systems' performance dynamically.

The main benefits of the proposed model are handling of fuzzy/vague data, converting the fuzzy data in their basic belief masses, combining the evidence using theory of information fusion and monitoring of the system periodically with different sets of data in dynamic conditions. Researchers can use this model to solve multi-criteria decision-making problems in various diversified research areas which requires data to be collected dynamically like autonomous ships, medical diagnostic support systems, weather forecasting, improving safety in transportation, etc. As future research, this model can be developed further using a plithogenic set which is an extension of a neutrosophic set.

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Hausdorff Measures on Generalized Set Valued Neutrosophic Quadruple Numbers and Decision Making Applications for Adequacy of Online Education

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Abstract: In this paper, we develop a new method of decision-making algorithm with Hausdorff distance and Hausdorff similarity measures based on generalized set-valued neutrosophic quadruple numbers. To establish the algorithm, we define Hausdorff distance measure and Hausdorff similarity measure on generalized set-valued neutrosophic quadruple. Next, we give a new method of decision-making application for impact of online learning on the learner. Also, we obtain different result from some previous applications (based on neutrosophic sets) for decision making algorithm. Thanks to our decision-making algorithm and similarity measure, researchers can obtain new applications for other decision making problems.

Keywords: Generalized set – valued neutrosophic quadruple sets, Hausdorff measures, decision making applications, adequacy of online education application

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1 Introduction

The rapid population growth experienced in the world at the end of the twentieth century and the inadequacy of classical learning-teaching (education-training) activities and methods in this respect led to new searches in the field of education. As a result of these searches, online education programs have been developed. Online education program is the name given to the study carried out with the curriculum prepared by educational institutions in a certain order to help students practice education alone. In the most general sense, we can define online education as the education practices that are structured on environments where teachers and students are separated from each other in terms of time and space. In this study, we will define a new similarity measure for generalized set-valued neutrosophic quadruple numbers to assess the competence of online education and remove uncertainties and provide a more objective assessment, and show the requirements for the similarity measure. Some of the environmental factors that affect the competence of online education are infrastructure, course material, and course hours. The difference of the similarity measure we will define from other similarity measures is that we add set operations on it. These set operations caused the result of the similarity measure to be seen more clearly. Similarities between human beings, a medicine or a new law to be exemplified can be examples of the assets we are talking about. In this report, some criteria will be selected to evaluate the adequacy of online education and the weight values of these criteria will be determined. A community of experts will then be created and an ideal (I) student template will be prepared for the assessment of online education, using generalized set-valued neutrosophic quadruples and numbers. Then, experts will be able to evaluate other students' criteria as generalized set - valued neutrosophic quadruple sets and numbers with the help of this ideal student. The evaluation result of each student will be handled separately and evaluation results of each will be obtained. Thus, an objective assessment will be made.

Smarandache defined neutrosophic logic and neutrosophic sets [1] in 1998. In terms of neutrosophic logic and neutrosophic sets, there is a membership degree (T), an indeterminacy degree (I) and a non-membership degree (F). These degrees are defined independently. A neutrosophic value is in the form (T, I, F). In other words, in explaining an event or finding a solution to a problem, a condition is handled according to its accuracy, inaccuracy and uncertainty. Therefore, neutrosophic logic and the neutrosophic sets help us find solutions to many uncertainties around us and in explaining complexity. Also, the distance measures and similarity measures are useful for decision making applications in neutrosophic theory. Therefore, many researchers studied neutrosophic theory [2-25] and decision making for neutrosophic theory [25-31]. Recently, Uluçay et al. [6] introduced neutrosophic multi-groups and applications; Uluçay [7] introduced a new similarity function of trapezoidal fuzzy multiple numbers based on multiple criteria decision making; Şahin et al. [8] obtained some weighted arithmetic operators and geometric operators with SVNSs and their application to multi-criteria decision making problems; Şahin et al. [9] studied some new operations of (α, β, γ) interval cut set of interval valued neutrosophic sets; Şahin et al. [10] obtained refined neutrosophic hierarchical clustering methods; Sahin et al. [11] studied extension principle based on neutrosophic multi-fuzzy sets and algebraic operations; Şahin et al. [12] introduced neutrosophic triplet partial g-metric space; Şahin et al. [13] introduced neutrosophic triplet normed ring space; Şahin et al. [14] studied neutrosophic quadruple theory; Broumi et al. [15] obtained Hausdroff distance and similarity measure for neutrosophic set and numbers; Şahin et al. [16] studied combined classic–neutrosophic sets and double neutrosophic sets; Şahin et al.[17] obtained decision-making applications in professional proficiencies in neutrosophic theory; Uluçay et al. [18] introduced decision-making method based on neutrosophic soft expert graphs; Ulucay et al. [19] studied an outranking approach for MCDM-problems with neutrosophic multi-sets; Hassan et al. [32] studied Q-neutrosophic soft expert set and its application in decision making; Bakbak et al. [33] obtained a theoretic approach to decision making problems in architecture with neutrosophic soft set; Şahin et al. [34] introduced neutrosophic triplet metric topology; Aslan et al. [35] introduced neutrosophic modeling of Talcott Parsons's action; Şahin et al. [36] studied an outperforming approach for multi-criteria decision-making problems with interval-valued bipolar neutrosophic sets; Abdel-Basset et al. studied a new hybrid multi-criteria decision- making approach for location selection of sustainable offshore wind energy stations [37]; Abdel-Basset et al. introduced neutrosophic theory based security approach for fog and mobile-edge computing [38]; Abdel-Basset et al. studied a model for the effective COVID-19 identification in uncertainty environment using primary symptoms and CT scans [39]; Abdel-Basset et al. introduced evaluation of sustainable hydrogen production options using an advanced hybrid MCDM approach [40].

Smarandache [20] discussed the neutrosophic quadruple set and the neutrosophic quadruple number. Neutrosophic quadruple sets are a generalized form of neutrosophic set. A neutrosophic quadruple set is represented by {(k, lT, mI, nF): k, l, m, n $\in \mathbb{R}$ or $\mathbb{C}$}. Here k is named as the known part, (lT, mI, nF) is named as the unknown part and T, I, F have the usual neutrosophic logic tools. Also, Şahin et al. [21] introduced generalized set-valued neutrosophic quadruple sets. Unlike neutrosophic quadruple set and number, in a generalized set-valued neutrosophic set and numbers; k, l, m and n are sets and T, I and F are not fixed. Thus, generalized set-valued neutrosophic set and numbers are more useful for decision making applications.

The organization of this paper is as follows: In section 2, some basis conception of the neutrosophic sets [1, 4], Hausdorff measures [15], the concept of neutrosophic quadruple sets [20, 21], Euclid measures [23] and Dice measures [22]. By adding set operations to the known Hausdorff distance measurement, we will obtain a larger set point Hausdorff distance measurement based on generalized set-valued neutrosophic quadruple numbers so that we can more clearly deal with the problems we encounter in section 3. In section 4, we will write an algorithm that we can use on sets of neutrosophic quadruple. Later, we will show the operability of Hausdorff's distance measurement, which we developed, by writing a numerical example with a neutrosophic quadruple structure. The example we gave in section 4 was calculated with other distance measurements in Seciton 5. and then, as a result of this calculation, we will comparison that the distance measurement we developed gives different results. Section 6 presents final conclusions and further research.

2 Preliminaries

Definition 2.1: [1] Let E be the universal set. For $\forall x \in E, 0^{-} \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$, by the help of the functions $T_A: E \to]^{-}0, 1^{+}[$, $I_A: E \to]^{-}0, 1^{+}[$ and $F_A: E \to]^{-}0, 1^{+}[$ a neutrosophic set A on E is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\} .$$

Here, $T_A(x), I_A(x)$ and $F_A(x)$ are the degrees of trueness, indeterminacy and falsity of $x \in E$ respectively. Where, $^{-}0 = 0 - \varepsilon$ and $1^{+} = 1 + \varepsilon$.

Definition 2.2: [4] Let E be the universal set. For $\forall x \in E, 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, using the functions $T_A: E \to [0,1]$, $I_A: E \to [0,1]$ and $F_A: E \to [0,1]$, a single-valued neutrosophic set A on E is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\} .$$

Here, $T_A(x), I_A(x)$ and $F_A(x)$ are the degrees of trueness, indeterminacy and falsity of $x \in E$, respectively.

Definition 2.3: [15] Let A_1 = <$T_{A_1}(x)$, $I_{A_1}(x)$, $F_{A_1}(x)$> and A_2 = <$T_{A_2}(x)$, $I_{A_2}(x)$, $F_{A_2}(x)$> be two single – valued neutrosophic numbers. The Hausdorff distance measure between A_1 and A_2, which is shown as $d_h(A_1, A_2)$ is defined as

$$d_h = (A_1, A_2) = max\{|T_{A_1}(x) - T_{A_2}(x)|, |I_{A_1}(x) - I_{A_2}(x)|, |F_{A_1}(x) - F_{A_2}(x)|\}.$$

Also, the Hausdorff similarity measure between A_1 and A_2, which is shown as $S_h(A_1, A_2)$ is defined as

$$S_h = (A_1, A_2) = 1 - max\{|T_{A_1}(x) - T_{A_2}(x)|, |I_{A_1}(x) - I_{A_2}(x)|, |F_{A_1}(x) - F_{A_2}(x)|\}.$$

Theorem 2.4: [15] Let X_1, X_2 and X_3 be three single – valued neutrosophic sets, d_h be Hausdorff distance measure. Then the following properties hold.

i. $0 \le d_H(X_1, X_2) \le 1$
ii. $X_1 = X_2$ if and only if $d_H(X_1, X_2) = 0$
iii. $d_H(X_1, X_2) = d_H(X_2, X_1)$
iv. If $X_1 \sqsubseteq X_2 \sqsubseteq X_3$, then $d_H(X_1, X_2) \le d_H(X_1, X_3)$ and $d_H(X_1, X_3) \le d_H(X_2, X_3)$.

Theorem 2.5: [15]: Let A_1, A_2 and A_3 be three single – valued neutrosophic sets, S_h be Hausdorff similarity measure. Then the following properties hold.

i. $0 \le S_h(A_1, A_2) \le 1$
ii. $S_h(A_1, A_2) = 1 \Leftrightarrow A_1 = A_2$
iii. $S_h(A_1, A_2) = S_h(A_2, A_1)$
iv. If $A_1 \subseteq A_2 \subseteq A_3 \in E$, then $S_h(A_1, A_3) \le S_h(A_1, A_2)$ and $S_h(A_1, A_3) \le S_h(A_2, A_3)$.

Definition 2.6: [20] NQN is a number of the form (k, lT, mI, nF). Here, T, I and F are used as the ordinary neutrosophic logical tools and k, l, m, n ∈ ℝ or ℂ. NQ = {(k, lT, mI, nF): k, l, m, n ∈ ℝ or ℂ} is defined by neutrosophic quadruple set.

For a neutrosophic quadruple number (k, lT, mI, nF), k is named the known part and (lT, mI, nF) is named the unknown part where k represents any asset such as a number, an idea, an object, etc.

Definition 2.7: [21] Let X be a set and P(X) be power set of X. A generalized set – valued neutrosophic quadruple set is a set of the form $G_{s_i} = \{(A_{s_i}, B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i}): A_{s_i}, B_{s_i}, C_{s_i}, D_{s_i} \in$ P(X); i = 1, 2, 3, … , n}.

Where T_i, I_i and F_i have their usual neutrosophic logic means and generalized set – valued neutrosophic quadruple number defined by

$$G_{N_i} = (A_{s_i}, B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i}).$$

As in neutrosophic quadruple number, for a generalized set – valued neutrosophic quadruple number $(A_{s_i}, B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i})$ representing any entity which may be a number, an idea, an object, etc.; A_{s_i} is called the known part and $(B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i})$ is called the unknown part.

Definition 2.8: [20] Let $G_{N_i} = (A_{s_i}, B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i})$ and $G_{N_j} = (A_{s_j}, B_{s_j}T_{s_j}, C_{s_j}I_{s_j}, D_{s_j}F_{s_j})$ be two generalized set – valued neutrosophic quadruple numbers. $A_{s_i} \subseteq A_{s_j}$, $A_{s_i} \subseteq A_{s_j}$, $A_{s_i} \subseteq A_{s_j}$, $A_{s_i} \subseteq A_{s_j}$ and $T_{s_i} \leq T_{s_j}$, $I_{s_i} \leq I_{s_j}$, $F_{s_i} \leq F_{s_j}$, then we say G_{N_i} is a subset of G_{N_J} and denote it by $G_{N_i} \subseteq G_{N_J}$.

Definition 2.9: [23] Let A_1 = <$T_{A_1}(x)$, $I_{A_1}(x)$, $F_{A_1}(x)$> and A_2 = <$T_{A_2}(x)$, $I_{A_2}(x)$, $F_{A_2}(x)$> be two single – valued neutrosophic numbers. The Euclid similarity measure between A_1 and A_2, which is shown as $d_E(A_1, A_2)$ is defined as

$$d_E(A_1, A_2) = 1 - \frac{1}{3}\sum_{j=1}^{n}\sqrt{\left(T_{A_1}(x) - T_{A_2}(x)\right)^2 + \left(I_{A_1}(x) - I_{A_2}(x)\right)^2 + \left(F_{A_1}(x) - F_{A_2}(x)\right)^2}.$$

Definition 2.10: [22] Let A_1 = <$T_{A_1}(x)$, $I_{A_1}(x)$, $F_{A_1}(x)$> and A_2 = <$T_{A_2}(x)$, $I_{A_2}(x)$, $F_{A_2}(x)$> be two single – valued neutrosophic numbers. The Dice similarity measure between A_1 and A_2, which is shown as $d_E(A_1, A_2)$ is defined as

$$S_{D1}(A_1, A_2) = 1 - \frac{2[\left(T_{A_1}(x).T_{A_2}(x) + I_{A_1}(x).I_{A_2}(x) + F_{A_1}(x).F_{A_2}(x)\right)]}{\left(\left(T_{A_1}(x)\right)^2 + \left(I_{A_1}(x)\right)^2 + \left(F_{A_1}(x)\right)^2\right) + \left(\left(T_{A_2}(x)\right)^2 + \left(I_{A_2}(x)\right)^2 + \left(F_{A_2}(x)\right)^2\right)}$$

3 Hausdorff Measures Based on Generalized Set-Valued Neutrosophic Quadruple Numbers and Sets

In this paper, we take T, I, F ∈ [0, 1] like single valued neutrosophic numbers in Definition 2.2.

Definition 3.1: $G_{N_1} = (A_{s_1}, B_{s_1}T_{s_1}, C_{s_1}I_{s_1}, D_{s_1}F_{s_1})$ and $G_{N_2} = (A_{s_2}, B_{s_2}T_{s_2}, C_{s_2}I_{s_2}, D_{s_2}F_{s_2})$ be two generalized set – valued neutrosophic quadruple number. We define a function $d_{QHN}: G_{N_1} \times G_{N_2} \rightarrow [0,1]$ such that

$$d_{QHN}(G_{N_1}, G_{N_2}) = d_{QHN}\left((A_{s_1}, B_{s_1}T_{s_1}, C_{s_1}I_{s_1}, D_{s_1}F_{s_1}), (A_{s_2}, B_{s_2}T_{s_2}, C_{s_2}I_{s_2}, D_{s_2}F_{s_2})\right)$$

$$= \frac{1}{2}\left[max\{|T_{s_1} - T_{s_2}|, |I_{s_1} - I_{s_2}|, |F_{s_1} - F_{s_2}|\} + \frac{1}{4}\left(\frac{max\{s(A_{s_1} \setminus A_{s_2}), s(A_{s_2} \setminus A_{s_1})\}}{max\{s(A_{s_1}), s(A_{s_2}), 1\}} + \frac{max\{s(B_{s_1} \setminus B_{s_2}), s(B_{s_2} \setminus B_{s_1})\}}{max\{s(B_{s_1}), s(B_{s_2}), 1\}} + \frac{max\{s(C_{s_1} \setminus C_{s_2}), s(C_{s_2} \setminus C_{s_1})\}}{max\{s(C_{s_1}), s(C_{s_2}), 1\}} + \frac{max\{s(D_{s_1} \setminus D_{s_2}), s(D_{s_2} \setminus D_{s_1})\}}{max\{s(D_{s_1}), s(D_{s_2}), 1\}} \right) \right]$$

Then, d_{QHN} is called a Hausdorff distance measure on generalized set-valued neutrosophic quadruple numbers.

Where, s(A) is number of element of set A.

Also, we generalized Hausdorff distance measure for generalized set-valued neutrosophic quadruple numbers in Definition 3.1.

Theorem 3.2: Let $G_{N_1} = (A_{s_1}, B_{s_1}T_{s_1}, C_{s_1}I_{s_1}, D_{s_1}F_{s_1})$, $G_{N_2} = (A_{s_2}, B_{s_2}T_{s_2}, C_{s_2}I_{s_2}, D_{s_2}F_{s_2})$ and $G_{N_3} = (A_{s_3}, B_{s_3}T_{s_3}, C_{s_3}I_{s_3}, D_{s_3}F_{s_3})$ be two generalized set – valued neutrosophic quadruple numbers. Then, d_{QHN} satisfies the below conditions.

i) $d_{QHN}(G_{N_1}, G_{N_2}) \in [0,1]$

ii) $d_{QHN}(G_{N_1}, G_{N_2}) = 0 \Leftrightarrow G_{N_1} = G_{N_2}$

iii) $d_{QHN}(G_{N_1}, G_{N_2}) = d_{QHN}(G_{N_2}, G_{N_1})$

iv) If $G_{N_1} \subset G_{N_2} \subset G_{N_3}$, then

$d_{QHN}(G_{N_1}, G_{N_2}) \leq d_{QHN}(G_{N_1}, G_{N_3})$ and $d_{QHN}(G_{N_2}, G_{N_3}) \leq d_{QHN}(G_{N_1}, G_{N_3})$.

Proof:

i)

Let $G_{N_1} = G_{N_2}$. From Definition 2.8,

$A_{s_1} = A_{s_2}$, $B_{s_1} = B_{s_2}$, $C_{s_1} = C_{s_2}$, $D_{s_1} = D_{s_2}$, $T_{s_1} = T_{s_2}$, $I_{s_1} = I_{s_2}$ and $F_{s_1} = F_{s_2}$ .Thus, we have

$$\begin{aligned} d_{QHN}(G_{N_1}, G_{N_1}) &= \frac{1}{2}\Bigg[max\{|T_{s_1} - T_{s_1}|, |I_{s_1} - I_{s_1}|, |F_{s_1} - F_{s_1}|\} \\ &+ \frac{1}{4}\left(\frac{max\{s(A_{s_1} \setminus A_{s_1}), s(A_{s_1} \setminus A_{s_1})\}}{max\{s(A_{s_1}), s(A_{s_1}), 1\}} + \frac{max\{s(B_{s_1} \setminus B_{s_1}), s(B_{s_1} \setminus B_{s_1})\}}{max\{s(B_{s_1}), s(B_{s_1}), 1\}}\right. \\ &+ \left.\frac{max\{s(C_{s_1} \setminus C_{s_1}), s(C_{s_1} \setminus C_{s_1})\}}{max\{s(C_{s_1}), s(C_{s_1}), 1\}} + \frac{max\{s(D_{s_1} \setminus D_{s_1}), s(D_{s_1} \setminus D_{s_1})\}}{max\{s(D_{s_1}), s(D_{s_1}), 1\}}\right)\Bigg] \\ &= \frac{1}{2}\left[0 + \frac{1}{4}\left(\frac{0}{s(A_{s_1})} + \frac{0}{s(B_{s_1})} + \frac{0}{s(C_{s_1})} + \frac{0}{s(D_{s_1})}\right)\right] = 0 \end{aligned}$$

Let $G_{N_1} \neq G_{N_2}$. We have $A_{s_1} \neq A_{s_2}$, $B_{s_1} \neq B_{s_2}$, $C_{s_1} \neq C_{s_2}$, $D_{s_1} \neq D_{s_2}$, $T_{s_1} \neq T_{s_2}$, $I_{s_1} \neq I_{s_2}$, $F_{s_1} \neq F_{s_2}$.

In this case, $d_{QHN}(G_{N_1}, G_{N_2}) > 0$.

Let $G_{N_1} \neq \emptyset$ and $G_{N_2} = \emptyset$. So,

$G_{N_1} = (A_{s_1}, B_{s_1}T_{s_1}, C_{s_1}I_{s_1}, D_{s_1}F_{s_1})$, $G_{N_2} = \emptyset = (\emptyset, \emptyset T_{s_2}, \emptyset I_{s_2}, \emptyset F_{s_2})$. Since we are looking for the highest value of the result, we take $T_{s_1} = I_{s_1} = F_{s_1} = 1$ and $T_{s_2} = I_{s_2} = F_{s_2} = 0$.

$$\begin{aligned} d_{QHN}(G_{N_1}, G_{N_2}) &= \frac{1}{2}\Bigg[max\{|T_{s_1} - 0|, |I_{s_1} - 0|, |F_{s_1} - 0|\} \\ &+ \frac{1}{4}\left(\frac{max\{s(A_{s_1} \setminus \emptyset), s(\emptyset \setminus A_{s_1})\}}{max\{s(A_{s_1}),\ 0, 1\}} + \frac{max\{s(B_{s_1} \setminus \emptyset), s(\emptyset \setminus B_{s_1})\}}{max\{s(B_{s_1}), 0, 1\}}\right. \\ &+ \left.\frac{max\{s(C_{s_1} \setminus \emptyset), s(\emptyset \setminus C_{s_1})\}}{max\{s(C_{s_1}), 0, 1\}} + \frac{max\{s(D_{s_1} \setminus \emptyset), s(\emptyset \setminus D_{s_1})\}}{max\{s(D_{s_1}), 0, 1\}}\right)\Bigg] \\ &= \frac{1}{2}\left[1 + \frac{1}{4}(1 + 1 + 1 + 1)\right] = 1 \end{aligned}$$

As the highest value of $d_{QHN}(G_{N_1}, G_{N_2})$ is 1 and the lowest value is 0, $d_{QHN}(G_{N_1}, G_{N_2}) \in [0, 1]$.

ii) $d_{QHN}(G_{N_1}, G_{N_2}) = 0 \Leftrightarrow G_{N_1} = G_{N_2}$

(⇒): If $d_{QHN}(G_{N_1}, G_{N_2}) = \frac{1}{2}\Big[max\{|T_{s_1} - T_{s_2}|, |I_{s_1} - I_{s_2}|, |F_{s_1} - F_{s_2}|\} + \frac{1}{4}\Big(\frac{max\{s(A_{s_1} \setminus A_{s_2}), s(A_{s_2} \setminus A_{s_1})\}}{max\{s(A_{s_1}), s(A_{s_2}), 1\}} + \frac{max\{s(B_{s_1} \setminus B_{s_2}), s(B_{s_2} \setminus B_{s_1})\}}{max\{s(B_{s_1}), s(B_{s_2}), 1\}} + \frac{max\{s(C_{s_1} \setminus C_{s_2}), s(C_{s_2} \setminus C_{s_1})\}}{max\{s(C_{s_1}), s(C_{s_2}), 1\}} + \frac{max\{s(D_{s_1} \setminus D_{s_2}), s(D_{s_2} \setminus D_{s_1})\}}{max\{s(D_{s_1}), s(D_{s_2}), 1\}}\Big)\Big] = 0$,

then

$max\{|T_{s_1} - T_{s_2}|, |I_{s_1} - I_{s_2}|, |F_{s_1} - F_{s_2}|\} = 0$ and

$$\frac{1}{4}\left(\frac{max\{s(A_{s_1} \setminus A_{s_2}), s(A_{s_2} \setminus A_{s_1})\}}{max\{s(A_{s_1}), s(A_{s_2}), 1\}} + \frac{max\{s(B_{s_1} \setminus B_{s_2}), s(B_{s_2} \setminus B_{s_1})\}}{max\{s(B_{s_1}), s(B_{s_2}), 1\}} + \frac{max\{s(C_{s_1} \setminus C_{s_2}), s(C_{s_2} \setminus C_{s_1})\}}{max\{s(C_{s_1}), s(C_{s_2}), 1\}} + \frac{max\{s(D_{s_1} \setminus D_{s_2}), s(D_{s_2} \setminus D_{s_1})\}}{max\{s(D_{s_1}), s(D_{s_2}), 1\}}\right) = 0$$

If $|T_{s_1} - T_{s_2}| = 0$, then $T_{s_1} = T_{s_2}$; if $|I_{s_1} - I_{s_2}| = 0$, then $I_{s_1} = I_{s_2}$; if $|F_{s_1} - F_{s_2}| = 0$, then $F_{s_1} = F_{s_2}$ and if

$$\frac{max\{s(A_{s_1}\backslash A_{s_2}), s(A_{s_2}\backslash A_{s_1})\}}{max\{s(A_{s_1}), s(A_{s_2}), 1\}} = 0, \frac{max\{s(B_{s_1}\backslash B_{s_2}), s(B_{s_2}\backslash B_{s_1})\}}{max\{s(B_{s_1}), s(B_{s_2}), 1\}} = 0,$$

$$\frac{max\{s(C_{s_1}\backslash C_{s_2}), s(C_{s_2}\backslash C_{s_1})\}}{max\{s(C_{s_1}), s(C_{s_2}), 1\}} = 0, \frac{max\{s(D_{s_1}\backslash D_{s_2}), s(D_{s_2}\backslash D_{s_1})\}}{max\{s(D_{s_1}), s(D_{s_2}), 1\}} = 0,$$

then

$A_{s_1} = A_{s_2}, B_{s_1} = B_{s_2}, C_{s_1} = C_{s_2}, D_{s_1} = D_{s_2}$.

Then, from Definition 2.8, we obtain that $G_{N_1} = G_{N_2}$.

($\Leftarrow$):

Let $G_{N_1} = G_{N_2}$. From i, we have $d_{QHN}(G_{N_1}, G_{N_1}) = 0$.

iii)

$$\begin{aligned} d_{QHN}(G_{N_1}, G_{N_2}) = & \frac{1}{2}\Bigg[max\{|T_{s_1} - T_{s_2}|, |I_{s_1} - I_{s_2}|, |F_{s_1} - F_{s_2}|\} \\ & + \frac{1}{4}\Bigg(\frac{max\{s(A_{s_1} \setminus A_{s_2}), s(A_{s_2} \setminus A_{s_1})\}}{max\{s(A_{s_1}), s(A_{s_2}), 1\}} + \frac{max\{s(B_{s_1} \setminus B_{s_2}), s(B_{s_2} \setminus B_{s_1})\}}{max\{s(B_{s_1}), s(B_{s_2}), 1\}} \\ & + \frac{max\{s(C_{s_1} \setminus C_{s_2}), s(C_{s_2} \setminus C_{s_1})\}}{max\{s(C_{s_1}), s(C_{s_2}), 1\}} + \frac{max\{s(D_{s_1} \setminus D_{s_2}), s(D_{s_2} \setminus D_{s_1})\}}{max\{s(D_{s_1}), s(D_{s_2}), 1\}}\Bigg)\Bigg] \\ = & \frac{1}{2}\Bigg[max\{|T_{s_2} - T_{s_1}|, |I_{s_2} - I_{s_1}|, |F_{s_2} - F_{s_1}|\} \\ & + \frac{1}{4}\Bigg(\frac{max\{s(A_{s_2} \setminus A_{s_1}), s(A_{s_1} \setminus A_{s_2})\}}{max\{s(A_{s_2}), s(A_{s_1}), 1\}} + \frac{max\{s(B_{s_2} \setminus B_{s_1}), s(B_{s_1} \setminus B_{s_2})\}}{max\{s(B_{s_2}), s(B_{s_1}), 1\}} \\ & + \frac{max\{s(C_{s_2} \setminus C_{s_1}), s(C_{s_1} \setminus C_{s_2})\}}{max\{s(C_{s_2}), s(C_{s_1}), 1\}} + \frac{max\{s(D_{s_2} \setminus D_{s_1}), s(D_{s_1} \setminus D_{s_2})\}}{max\{s(D_{s_2}), s(D_{s_1}), 1\}}\Bigg)\Bigg] = d_{QHN}(G_{N_2}, G_{N_1}). \end{aligned}$$

iv) Let $G_{N_1} \subset G_{N_2} \subset G_{N_3}$. From Definition 2.8, we obtain $A_{s_1} \subset A_{s_2} \subset A_{s_3}$, $B_{s_1} \subset B_{s_2} \subset B_{s_3}$, $C_{s_1} \subset C_{s_2} \subset C_{s_3}$, $D_{s_1} \subset D_{s_2} \subset D_{s_3}$. Also, we have

$s(A_{s_1}) \le s(A_{s_2}) \le s(A_{s_3})$, $s(B_{s_1}) \le s(B_{s_2}) \le s(B_{s_3})$, $s(C_{s_1}) \le s(C_{s_2}) \le s(C_{s_3})$, $s(D_{s_1}) \le s(D_{s_2}) \le s(D_{s_3})$, and

$$\frac{max\{s(A_{s_1}\setminus A_{s_2}),s(A_{s_2}\setminus A_{s_1})\}}{max\{s(A_{s_1}),s(A_{s_2}),1\}}+\frac{max\{s(B_{s_1}\setminus B_{s_2}),s(B_{s_2}\setminus B_{s_1})\}}{max\{s(B_{s_1}),s(B_{s_2}),1\}}+\frac{max\{s(C_{s_1}\setminus C_{s_2}),s(C_{s_2}\setminus C_{s_1})\}}{max\{s(C_{s_1}),s(C_{s_2}),1\}}+\frac{max\{s(D_{s_1}\setminus D_{s_2}),s(D_{s_2}\setminus D_{s_1})\}}{max\{s(D_{s_1}),s(D_{s_2}),1\}}\le \frac{max\{s(A_{s_1}\setminus A_{s_3}),s(A_{s_3}\setminus A_{s_1})\}}{max\{s(A_{s_1}),s(A_{s_3}),1\}}+\frac{max\{s(B_{s_1}\setminus B_{s_3}),s(B_{s_3}\setminus B_{s_1})\}}{max\{s(B_{s_1}),s(B_{s_3}),1\}}+\frac{max\{s(C_{s_1}\setminus C_{s_3}),s(C_{s_3}\setminus C_{s_1})\}}{max\{s(C_{s_1}),s(C_{s_3}),1\}}+\frac{max\{s(D_{s_1}\setminus D_{s_3}),s(D_{s_3}\setminus D_{s_1})\}}{max\{s(D_{s_1}),s(D_{s_3}),1\}}$$

since

$s(A_{s_1}\setminus A_{s_2}) = s(A_{s_2}\setminus A_{s_3}) = s(A_{s_1}\setminus A_{s_3}) = \emptyset$

$s(B_{s_1}\setminus B_{s_2}) = s(B_{s_2}\setminus B_{s_3}) = s(B_{s_1}\setminus B_{s_3}) = \emptyset$

$s(C_{s_1}\setminus C_{s_2}) = s(C_{s_2}\setminus C_{s_3}) = s(C_{s_1}\setminus C_{s_3}) = \emptyset$

$s(D_{s_1}\setminus D_{s_2}) = s(D_{s_2}\setminus D_{s_3}) = s(D_{s_1}\setminus D_{s_3}) = \emptyset$

$s(A_{s_2}\setminus A_{s_1}) \le s(A_{s_3}\setminus A_{s_1})$, $s(B_{s_2}\setminus B_{s_1}) \le s(B_{s_3}\setminus B_{s_1})$, $s(C_{s_2}\setminus C_{s_1}) \le s(C_{s_3}\setminus C_{s_1})$, $s(D_{s_2}\setminus D_{s_1}) \le s(D_{s_3}\setminus D_{s_1})$,

$s(A_{s_3}\setminus A_{s_2}) \le s(A_{s_3}\setminus A_{s_1})$, $s(B_{s_3}\setminus B_{s_2}) \le s(B_{s_3}\setminus B_{s_1})$, $s(C_{s_3}\setminus C_{s_2}) \le s(C_{s_3}\setminus C_{s_1})$, $s(D_{s_3}\setminus D_{s_2}) \le s(D_{s_3}\setminus D_{s_1})$.

Also, from Definition 2.8, we obtain

$|T_{s_1}-T_{s_2}| \le |T_{s_1}-T_{s_3}|$, $|I_{s_1}-I_{s_2}| \le |I_{s_1}-I_{s_3}|$, $|F_{s_1}-F_{s_2}| \le |F_{s_1}-F_{s_3}|$,

$|T_{s_2}-T_{s_3}| \le |T_{s_1}-T_{s_3}|$, $|I_{s_2}-I_{s_3}| \le |I_{s_1}-I_{s_3}|$, $|F_{s_2}-F_{s_3}| \le |F_{s_1}-F_{s_3}|$.

Thus, we have $d_{QHN}(G_{N_1}, G_{N_2}) \le d_{QHN}(G_{N_1}, G_{N_3})$.

Where, $d_{QHN}(G_{N_2}, G_{N_3}) \le d_{QHN}(G_{N_1}, G_{N_3})$ can be shown similar to $d_{QHN}(G_{N_1}, G_{N_2}) \le d_{QHN}(G_{N_1}, G_{N_3})$.

Definition 3.3: Let $G_{N_1} = (A_{s_1}, B_{s_1}T_{s_1}, C_{s_1}I_{s_1}, D_{s_1}F_{s_1})$ and $G_{N_2} = (A_{s_2}, B_{s_2}T_{s_2}, C_{s_2}I_{s_2}, D_{s_2}F_{s_2})$ be two generalized set – valued neutrosophic quadruple numbers. We define a function $d_{HD}: G_{N_1} \times G_{N_2} \to [0,1]$ such that

$$S_{QHN}: (G_{N_1}, G_{N_2}) = S_{QHN}\ ((A_{s_1}, B_{s_1}T_{s_1}, C_{s_1}I_{s_1}, D_{s_1}F_{s_1}), (A_{s_2}, B_{s_2}T_{s_2}, C_{s_2}I_{s_2}, D_{s_2}F_{s_2}))$$

$$= 1-\frac{1}{2}\Bigg[max\{|T_{s_1}-T_{s_2}|, |I_{s_1}-I_{s_2}|, |F_{s_1}-F_{s_2}|\} + \frac{1}{4}\Bigg(\frac{max\{s(A_{s_1}\setminus A_{s_2}), s(A_{s_2}\setminus A_{s_1})\}}{max\{s(A_{s_1}), s(A_{s_2}), 1\}} + \frac{max\{s(B_{s_1}\setminus B_{s_2}), s(B_{s_2}\setminus B_{s_1})\}}{max\{s(B_{s_1}), s(B_{s_2}), 1\}} + \frac{max\{s(C_{s_1}\setminus C_{s_2}), s(C_{s_2}\setminus C_{s_1})\}}{max\{s(C_{s_1}), s(C_{s_2}), 1\}} + \frac{max\{s(D_{s_1}\setminus D_{s_2}), s(D_{s_2}\setminus D_{s_1})\}}{max\{s(D_{s_1}), s(D_{s_2}), 1\}}\Bigg)\Bigg]$$

Then, S_{QHN} is called a Hausdorff similarity measure on generalized set - valued neutrosophic quadruple numbers.

Where, s(A) is number of element of set A.

Also, we generalized Hausdorff similarity measure for generalized set - valued neutrosophic quadruple numbers in Definition 3.3.

Theorem 3.4: Let $G_{N_1} = \left(A_{s_1}, B_{s_1}T_{s_1}, C_{s_1}I_{s_1}, D_{s_1}F_{s_1}\right)$, $G_{N_2} = (A_{s_2}, B_{s_2}T_{s_2}, C_{s_2}I_{s_2}, D_{s_2}F_{s_2})$ and $G_{N_3} = (A_{s_3}, B_{s_3}T_{s_3}, C_{s_3}I_{s_3}, D_{s_3}F_{s_3})$ be three generalized set – valued neutrosophic quadruple numbers. Then, S_{QHN} satisfies the below conditions.

i) $S_{QHN}(G_{N_1}, G_{N_2}) \in [0, 1]$

ii) $S_{QHN}(G_{N_1}, G_{N_2}) = 1 \Leftrightarrow G_{N_1} = G_{N_2}$

iii) $S_{QHN}(G_{N_1}, G_{N_2}) = S_{QHN}(G_{N_2}, G_{N_1})$

iv) If $G_{N_1} \subset G_{N_2} \subset G_{N_3}$, then

$S_{QHN}(G_{N_1}, G_{N_3}) \leq S_{QHN}(G_{N_1}, G_{N_2})$ and $S_{QHN}\left(G_{N_1}, G_{N_3}\right) \leq S_{QHN}\left(G_{N_2}, G_{N_3}\right)$.

Proof:

i) From Theorem 3.2,

when $d_{QHN}(G_{N_1}, G_{N_1}) = 0$, $S_{QHN}(G_{N_1}, G_{N_1}) = 1 - d_{QHN}(G_{N_1}, G_{N_1}) = 1 - 0 = 1$.

when $d_{QHN}(G_{N_1}, G_{N_1}) = 1$, $S_{QHN}(G_{N_1}, G_{N_1}) = 1 - d_{QHN}(G_{N_1}, G_{N_1}) = 1 - 1 = 0$.

Then, $S_{QHN}(G_{N_1}, G_{N_2}) \in [0, 1]$.

ii)

From Theorem 3.2,

(⇒): If $S_{QHN}(G_{N_1}, G_{N_2}) = 1$, then $S_{QHN}(G_{N_1}, G_{N_2}) = 1 - d_{QHN}(G_{N_1}, G_{N_2})$

$$d_{QHN}(G_{N_1}, G_{N_2}) = 1 - S_{QHN}(G_{N_1}, G_{N_2})$$

$$d_{QHN}(G_{N_1}, G_{N_2}) = 1 - 1 = 0.$$

From Theorem 3.2,

(⇐): If $G_{N_1} = G_{N_2}$, then $d_{QHN}(G_{N_1}, G_{N_2}) = 0$.

Since $S_{QHN}(G_{N_1}, G_{N_2}) = 1 - d_{QHN}(G_{N_1}, G_{N_2}) = 1 - 0 = 1$, one can write $S_{QHN}(G_{N_1}, G_{N_2}) = 1$.

iii) From Theorem 3.2,

Since $S_{QHN}(G_{N_1}, G_{N_2}) = 1 - d_{QHN}(G_{N_1}, G_{N_2})$ and $d_{QHN}(G_{N_1}, G_{N_2}) = d_{QHN}(G_{N_2}, G_{N_1})$,

$S_{QHN}(G_{N_1}, G_{N_2}) = 1 - d_{QHN}(G_{N_1}, G_{N_2}) = 1 - d_{QHN}(G_{N_2}, G_{N_1}) = S_{QHN}(G_{N_2}, G_{N_1})$.

iv) Let $G_{N_1} \subset G_{N_2} \subset G_{N_3}$.

From Theorem 3.2, if $A_{s_1} \subset A_{s_2} \subset A_{s_3}$, $B_{s_1} \subset B_{s_2} \subset B_{s_3}$, $C_{s_1} \subset C_{s_2} \subset C_{s_3}$, $D_{s_1} \subset D_{s_2} \subset D_{s_3}$, then $d_{QHN}(G_{N_1}, G_{N_2}) \leq d_{QHN}(G_{N_1}, G_{N_3})$ and $d_{QHN}(G_{N_2}, G_{N_3}) \leq d_{QHN}(G_{N_1}, G_{N_3})$.

$$d_{QHN}(G_{N_1}, G_{N_2}) \leq d_{QHN}(G_{N_1}, G_{N_3})$$

$$-d_{QHN}(G_{N_1}, G_{N_2}) \geq -d_{QHN}(G_{N_1}, G_{N_3})$$

$$1 - d_{QHN}(G_{N_1}, G_{N_2}) \geq 1 - d_{QHN}(G_{N_1}, G_{N_3})$$

$$S_{QHN}(G_{N_1}, G_{N_2}) \geq S_{QHN}(G_{N_1}, G_{N_3}).$$

Also, $S_{QHN}(G_{N_1}, G_{N_3}) \leq S_{QHN}(G_{N_2}, G_{N_3})$ can be shown similar to $S_{QHN}(G_{N_1}, G_{N_2}) \geq S_{QHN}(G_{N_1}, G_{N_3})$.

Example 3.5: Let $X = (\{x_2, x_3, x_5, x_6\}, \{x_1, x_6, x_8\}(1), \emptyset(0), \emptyset(0))$ and

$X_1 = \left(\{x_1, x_3, x_5, x_7, x_9\}, \{x_2, x_4, x_5, x_6, x_7\}(0{,}4), \{x_2, x_3, x_7\}(0{,}1), \{x_4, x_5\}(0{,}2)\right)$ be two generalized set-valued neutrosophic quadruple numbers.

We calculate $d_{QHN}(X, X_1)$, namely the distance between X and X_1.

$$d_{QHN}(X, X_1) = \frac{1}{2}\Bigg[max\{|\mathrm{T} - T_1|, |\mathrm{I} - I_1|, |\mathrm{F} - F_1|\} + \frac{1}{4}\left(\frac{max\{s(\mathrm{A} \setminus A_1), s(A_1 \setminus \mathrm{A})\}}{max\{s(\mathrm{A}), s(A_1), 1\}} + \frac{max\{s(\mathrm{B} \setminus B_1), s(B_1 \setminus \mathrm{B})\}}{max\{s(\mathrm{B}), s(B_1), 1\}} + \frac{max\{s(\mathrm{C} \setminus C_1), s(C_1 \setminus \mathrm{C})\}}{max\{s(\mathrm{C}), s(C_1), 1\}} + \frac{max\{s(\mathrm{D} \setminus D_1), s(D_1 \setminus \mathrm{D})\}}{max\{s(\mathrm{D}), s(D_1), 1\}}\right)\Bigg].$$

$$d_{QHN}(X, X_1) = \frac{1}{2}\Bigg[max\{|1 - 0{,}4|, |0 - 0{,}1|, |0 - 0{,}2|\} + \frac{1}{4}\Bigg(\frac{max\{s(\{x_2, x_3, x_5, x_6\} \setminus \{x_1, x_3, x_5, x_7, x_9\}), s(\{x_1, x_3, x_5, x_7, x_9\} \setminus \{x_2, x_3, x_5, x_6\})\}}{max\{s(\{x_2, x_3, x_5, x_6\}), s(\{x_1, x_3, x_4\}), 1\}} + \frac{max\{s(\{x_1, x_6, x_8\} \setminus \{x_2, x_4, x_5, x_6, x_7\}), s(\{x_2, x_4, x_5, x_6, x_7\} \setminus \{x_1, x_6, x_8\})\}}{max\{s(\{x_1, x_6, x_8\}), s(\{x_2, x_4, x_5, x_6, x_7\}), 1\}} + \frac{max\{s(\emptyset \setminus \{x_2, x_3, x_7\}), s(\{x_2, x_3, x_7\} \setminus \emptyset)\}}{max\{s(\{\}), s(\{x_2, x_3, x_7\}), 1\}} + \frac{max\{s(\emptyset \setminus \{x_4, x_5\}), s(\{x_4, x_5\} \setminus \emptyset)\}}{max\{s(\{\}, s(\{x_4, x_5\}), 1\}}\Bigg)\Bigg]$$

$$d_{QHN}(X, X_1) = \frac{1}{2}\left[max\{0.6, 0.1, 0.2\} + \frac{1}{4}\left(\frac{max\{2,3\}}{max\{4,3,1\}} + \frac{max\{2,4\}}{max\{3,5,1\}} + \frac{max\{0,3\}}{max\{0,3,1\}} + \frac{max\{0,2\}}{max\{0,2,1\}}\right)\right]$$

$$= \frac{1}{2}\left[0.6 + \frac{1}{4}\left(\frac{3}{4} + \frac{4}{5} + \frac{3}{3} + \frac{2}{2}\right)\right] = 0.74375.$$

As $d_{QHN}(X, X_1) = 0.74375$, $S_{QHN}(X, X_1) = 1 - d_{QHN}(X, X_1) = 1 - 0.74375 = 0.25625$.

4 Decision Making Applications for Adequacy of Online Education

Now, we give an algorithm based on the generalized set-valued neutrosophic quadruple numbers and Hausdroff measures on the generalized set-valued neutrosophic quadruple numbers for multi-criteria decision making method applications.

Algorithm 4.1:

Step 1: The criteria are determined. The criteria set get K.

$$K = \{k_1, k_2, \dots, k_n\} \ (n \in \mathbb{N})$$

The weight values of the criteria determined to $W = \{w_1, w_2, \dots, w_n\} \ (n \in \mathbb{N})$ and $\sum_{i=1}^{n} w_i = 1,\ w_i \in \mathbb{N}$. Where,

w_1 is the weight of criterion k_1,

w_2 is the weight of criterion k_2,

w_3 is the weight of criterion k_3,

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w_n is the weight of criterion k_n.

Step 2: Let I be the ideal status. For the generalized set – valued neutrosophic quadruple numbers, we define I such that

$$I = \{k_1: (P(X),\ P(X)T_1,\ \emptyset I_1,\ \emptyset F_1),\ k_2: (P(Y),\ P(Y)T_2,\ \emptyset I_2,\ \emptyset F_2), \dots ,\ k_n: (P(Z),\ P(Z)T_n,\ \emptyset I_n,\ \emptyset F_{n_i})\}.$$

Where,

$$T_1 = T_2 = \dots = T_n = 1$$

$$I_1 = I_2 = \cdots = I_n = 0$$

$$F_1 = F_2 = \cdots = F_n = 0\ .$$

Step 3: The adequacy of the efficiency of the criteria should be assessed by samples references according to each criterion and each status should be identified as a generalized set valued neutrosophic quadruple numbers.

Let the sets of the samples references be

$$A_1 = \{k_1: (X_{1_1}, X_{1_2}T_{1_1}, X_{1_3}I_{1_1}, X_{1_4}F_{1_1}), k_2: (Y_{1_1}, Y_{1_2}T_{1_2}, Y_{1_3}I_{1_2}, Y_{1_4}F_{1_2}), \ldots, \quad k_n: (Z_{1_1}, Z_{1_2}T_{1_n}, Z_{1_3}I_{1_n}, Z_{1_4}F_{1_n})\}$$

$$A_2 = \{k_1: (X_{2_1}, X_{2_2}T_{2_1}, X_{2_3}I_{2_1}, X_{2_4}F_{2_1}), k_2: (Y_{2_1}, Y_{2_2}T_{2_2}, Y_{2_3}I_{2_2}, Y_{2_4}F_{2_2}), \ldots, \quad k_n: (Z_{2_1}, Z_{2_2}T_{2_n}, Z_{2_3}I_{2_n}, Z_{2_4}F_{2_n})\}$$

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$$A_n = \{k_1: (X_{n_1}, X_{n_2}T_{n_1}, X_{n_3}I_{n_1}, X_{n_4}F_{n_1}), k_2: (Y_{n_1}, Y_{n_2}T_{n_2}, Y_{n_3}I_{n_2}, Y_{n_4}F_{n_2}), \ldots, \quad k_m: (Z_{n_1}, Z_{n_2}T_{n_n}, Z_{n_3}I_{n_n}, Z_{n_4}F_{n_n})\}$$

and each samples reference is evaluated according to each criterion. Here;

$$X_{i_j} \in P(X), Y_{i_j} \in P(Y), \ldots, Z_{i_j} \in P(Z) \quad (i = 1, 2, 3, \ldots, n)\ (j = 1, 2, \ldots, n)$$

Step 4: The sample reference criteria are given as generalized set valued neutrosophic quadruple numbers in Step 4. Now show them in Table 1.

Table 1. Example reference criterion table

	k_1	k_2	…	k_n
A_1	$(X_{1_1}, X_{1_2}T_{1_1}, X_{1_3}I_{1_1}, X_{1_4}F_{1_1})$	$(Y_{1_1}, Y_{1_2}T_{1_2}, Y_{1_3}I_{1_2}, Y_{1_4}F_{1_2})$	…	$(Z_{1_1}, Z_{1_2}T_{1_n}, Z_{1_3}I_{1_n}, Z_{1_4}F_{1_n})$
A_2	$(X_{2_1}, X_{2_2}T_{2_1}, X_{2_3}I_{2_1}, X_{2_4}F_{2_1})$	$(Y_{2_1}, Y_{2_2}T_{2_2}, Y_{2_3}I_{2_2}, Y_{2_4}F_{2_2})$	…	$(Z_{2_1}, Z_{2_2}T_{2_n}, Z_{2_3}I_{2_n}, Z_{2_4}F_{2_n})$

.	.	.	…	.
.	.	.	…	.
.	.	.	…	.
A_n	$(X_{n_1}, X_{n_2}T_{n_1}, X_{n_3}I_{n_1}, X_{n_4}F_{n_1})$	$(Y_{n_1}, Y_{n_2}T_{n_2}, Y_{n_3}I_{n_2}, Y_{n_4}F_{n_2})$	…	$(Z_{n_1}, Z_{n_2}T_{n_n}, Z_{n_3}I_{n_n}, Z_{n_4}F_{n_n})$

Step 5: Let's calculate the similarity values of the sample references with the I ideal criterion. While doing this, calculate $S_{QHN}\left(I_{k_j}, A_{i_{k_j}}\right)$ in Table 2.

Table 2. The I ideal criterion and the similarity values of the sample references

	k_1	k_2	…	k_n
A_1	$S_{QHN}(I_{k_1}, A_{1_{k_1}})$	$S_{QHN}(I_{k_2}, A_{1_{k_2}})$	…	$S_{QHN}(I_{k_n}, A_{1_{k_n}})$
A_2	$S_{QHN}(I_{k_1}, A_{2_{k_1}})$	$S_{QHN}(I_{k_2}, A_{2_{k_2}})$	…	$S_{QHN}(I_{k_n}, A_{2_{k_n}})$
.	.	.	…	.
.	.	.	.	.
.	.	.	.	.
A_n	$S_{QHN}(I_{k_1}, A_{n_{k_1}})$	$S_{QHN}(I_{k_2}, A_{n_{k_2}})$	…	$S_{QHN}(I_{k_n}, A_{n_{k_n}})$

Step 6: In this last step ın the similarity found, it is multiplied by the weight value of a criterion. For this, use the k-th weight value for each of the similarity values in the k-th column (k = 1, 2, …, n). Thus,

get the weighted similarity table in Table 3. The sum of A_i in Table 3 will be given as ${S_{QHN}}^i$ similarity value. (i = 1, 2, … n) ($n \in \mathbb{N}$)

Table 3. Weighted similarity table

	w_1k_1	w_2k_2	…	w_nk_n	$\sum_{i=1}^{n} w_ik_i = {S_{QHN}}^i$
A_1	$w_1.S_{QHN}(I_{k_1}, A_{1_{k_1}})$	$w_2.S_{QHN}(I_{k_2}, A_{1_{k_2}})$	…	$w_n.S_{QHN}(I_{k_n}, A_{1_{k_n}})$	${S_{QHN}}^1$
A_2	$w_1.S_{QHN}(I_{k_1}, A_{2_{k_1}})$	$w_2.S_{QHN}(I_{k_2}, A_{2_{k_2}})$	…	$w_n.S_{QHN}(I_{k_n}, A_{2_{k_n}})$	${S_{QHN}}^2$
. . .	. . .	. . .	… . .	. . .	. . .
A_n	$w_1.S_{QHN}(I_{k_1}, A_{n_{k_1}})$	$w_2.S_{QHN}(I_{k_2}, A_{n_{k_2}})$	…	$w_n.S_{QHN}(I_{k_n}, A_{n_{k_n}})$	${S_{QHN}}^n$

Example 4.2: The similarity measure is an important mathematical tool to deal with the problems we encounter in daily life. One of the bad consequences of the epidemic that affects the whole world is that we have to stop education. Therefore, education and training institutions have temporarily started online education practices so that students do not stay away from education. Of course, it has been seen that future online education does not have the same effect on students. Some of the factors that negatively affect students in this process are the environment, internet infrastructure, and the materials used in the course. In this section, the new similarity measure is applied to an online education problem. The generalized set-valued neutrosophic quadruple number is just a tool to deal with such cases, and for each evaluations for an alternative under the criterias can be considered as a generalized set-valued

neutrosophic quadruple number. Now, In the example below, 4 criteria and weight values of these criteria are determined in the first step. In step 2, the ideal set I to be referenced is written. In step 3, how to determine the efficiency of online courses, 10 student sets will be determined and these sets will be written as generalized set-valued neutrosophic quadruple number. The similarity values of these student sets with ideal set I are calculated and the results are multiplied by the weight values of the criteria. The similarity values of each criterion are added and the student with the best result is determined by finding the similarity values of each student separately.

Step 1: Let the set of criteria to be considered in evaluating the students' efficiency in online education be K.

$$K = \{k_1, k_2, k_3, k_4\}.$$

k_1 : Communication. The criterion weight values w_1= 0.4

k_2 : Lesson plan. The criterion weight values $w_2 = 0.2$

k_3 : Attendance. The criterion weight values $w_3 = 0.1$

k_4 : Source of Knowledge. The criterion weight values $w_4 = 0.3$

Step 2: For the I ideal student, in the generalized set valued neutrosophic quadruple set

$I = \{ k_1\text{: } \big(\{x_1,\ x_2,\ x_3,\ x_4,\ x_5\}, \{x_1,\ x_2,\ x_3,\ x_4,\ x_5\}(1),\ \emptyset(0),\ \emptyset(0)\big),$

$k_2\text{: } \big(\{y_1,\ y_2,\ y_3,\ y_4,\ y_5\}, \{y_1,\ y_2,\ y_3,\ y_4,\ y_5\}(1),\ \emptyset(0),\ \emptyset(0)\big),$

$k_3\text{: } \big(\{z_1,\ z_2,\ z_3,\ z_4,\ z_5\}, \{z_1,\ z_2,\ z_3,\ z_4,\ z_5\}(1),\ \emptyset(0),\ \emptyset(0)\big),$

$k_4\text{: } \big(\{t_1,\ t_2,\ t_3,\ t_4,\ t_5\}, \{t_1,\ t_2,\ t_3,\ t_4,\ t_5\}(1),\ \emptyset(0),\ \emptyset(0)\big)\},$

Step 3: Each student whose adequacy of the efficiency of the online lessons will be evaluated according to each criterion and each student is determined as a generalized set valued neutrosophic quadruple number.

Let the set of the students be A = $\{A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ A_6,\ A_7,\ A_8,\ A_9,\ A_{10}\}$.

$A_1 = \{k_1: (\{x_1, x_2, x_4, x_5\}, \{x_2, x_4\}(0.4), \{x_1, x_2, x_4\}(0.2), \{x_1, x_2\}(0.3)), k_2: (\{y_1, y_2, y_3, y_5\},$

$\{y_1, y_2, y_3\}(0.5), \{y_2, y_3, y_5\}(0.2), \{y_1, y_5\}(0.4)), k_3: (\{z_2, z_3, z_4, z_5\}, \{z_2, z_4, z_5\}(0.3), \{z_2\}(0.2),$

$\{z_2, z_3, z_5\}(0.4)), k_4: (\{t_1, t_2, t_3, t_4, t_5\}, \{t_1\}(0.2), \{t_4, t_5\}(0.3), \{t_4\}(0.2))\}$

$A_2 = \{k_1: (\{x_2, x_3\}, \{x_3\}(0.5), \{x_2, x_3\}(0.3), \{x_3\}(0.1)), k_2: (\{y_1, y_3, y_4, y_5\}, \{y_1, y_4, y_5\}(0.7), \{y_3\}(0.1),$
$\{y_1\}(0.1)), k_3: (\{z_5\}, \{z_5\}(0.6), \{z_5\}(0.3), \emptyset(0.1)), k_4: (\{t_1, t_2, t_3\}, \{t_1, t_2\}(0.5), \{t_2, t_3\}(0.2),$

$\{t_1, t_3\}(0.2))\}$

$A_3 = \{k_1: (\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3\}(0.6), \{x_1, x_2, x_3, x_4\}(0.2), \{x_1, x_3, x_4\}(0.3)), k_2: (\{y_1, y_2, y_3, y_4\},$

$\{y_1, y_2, y_3, y_4\}(0.09), \{y_1, y_2, y_3\}(0.05), \{y_1, y_2, y_3, y_4\}(0.01)), k_3: (\{z_5\}, \{z_5\}(0.4), \emptyset(0.1), \emptyset(0.3)),$

$k_4: (\{t_1, t_2, t_3, t_4\}, \{t_1, t_2, t_3, t_4\}0.7), \emptyset(0.7), \emptyset(0.7))\}$

$A_4 = \{k_1: (\{x_1, x_2, x_3, x_4\}, \emptyset(0.5), \emptyset(0.1), \emptyset(0.1)), k_2: (\{y_1, y_2, y_4, y_5\}, \{y_1, y_4, y_5\}(0.4), \emptyset(0.6),$

$\{y_1, y_2\}(0.8)), k_3: (\{z_1, z_2, z_3, z_4\}, \{z_1, z_2\}(0.8), \emptyset(0.7), \{z_3\}(0.3)), k_4: (\{t_1, t_2, t_3, t_4, t_5\},$

$\{t_1, t_4\}(0.6), \{t_1, t_3, t_4, t_5\}(0.1), \{t_1, t_4\}(0.2))\}$

$A_5 = \{k_1: (\{x_1, x_2, x_3\}, \{x_2, x_3\}(0.9), \{x_1, x_2, x_3\}(0.02), \{x_1, x_3\}(0.1)), k_2: (\{y_2, y_3\}, \{y_2, y_3\}(0.7),$

$\{y_3\}(0.6), \{y_2, y_3\}(0.4)), k_3: (\{z_3, z_4, z_5\}, \{z_3\}(0.1), \{z_5\}(0.1), \{z_3, z_5\}(0.2)), k_4: (\{t_1, t_2, t_5\}, \emptyset(0.8),$

$\{t_1, t_5\}(0.9), \{t_5\}(0.9))\}$

$A_6 = \{k_1: (\{x_1, x_3\}, \{x_5\}(0.04), \{x_3, x_4\}(0,06), \emptyset(0.003)), k_2: (\{y_2, y_3, y_4, y_5\},$

$\emptyset(0.07), \emptyset(0.02), \{y_2\}(0.01)), k_3: (\{z_2\}, \{z_2, z_3, z_4\}(0.4),$

$\{z_2, z_3, z_4, z_5\}(0.02), \emptyset(0.02)), k_4: (\emptyset, \{t_1, t_5\}(0.004), \{t_3, t_4\}(0.02),$

$\{t_5\}(0.5))\}$

$A_7 = \{k_1: (\{x_1, x_3, x_4, x_5\}, \{x_5\}(0.9), \{x_5\}(0.8), \emptyset(0.08)), k_2: (\{y_1, y_2, y_3, y_4, y_5\},$

$\{y_1, y_2, y_3, y_4, y_5\}(0.1), \emptyset(0.1), \emptyset(0.1)), k_3: (\emptyset, \{z_4, z_5\}(0.6), \emptyset(0.3), \{z_4\}(0.9)),$

$k_4: (\emptyset, \emptyset(0.9), \emptyset\ (0.9), \emptyset(0.1))\}$

$A_8 = \{k_1: (\{x_1, x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4, x_5\}(0.3), \{x_1, x_2, x_3, x_4, x_5\}(0.5), \{x_1, x_2, x_3, x_4, x_5\}(0.2)),$

$k_2: (\{y_4\}, \emptyset(0.8), \{y_4\}(0.3), \emptyset(0.1)), k_3: (\{z_2\}, \emptyset(0.2), \emptyset(0.02), \emptyset(0.1)),$

$k_4: (\{t_1, t_5\}, \emptyset(0.9), \emptyset(0.1), \{t_5\}(0.1))\}$

$A_9 = \{k_1: (\emptyset, \emptyset(0.2), \emptyset(0.2), \emptyset(0.1)), k_2: (\{y_1, y_3, y_4\}, \{y_3, y_4\}(0.6), \{y_4\}(0.03), \{z_1, z_3, y_4\}(0.09)),$

$k_3: (\{z_2, z_3, z_4, z_5\}, \{z_2\}(0.1), \{z_2, z_3, z_4, z_5\}(0.7), \{z_3, z_4, z_5\}(0.7)),$

$k_4: (\{t_1, t_2, t_3, t_4, t_5\}, \{t_4\}(0.6), \{t_3\}(0.9), \{t_2\}(0.9))\}$

$A_{10} = \{k_1: (\emptyset, \emptyset(0.01), \emptyset(0.02), \emptyset(0.02)), k_2: (\{y_1, y_2, y_3, y_4, y_5\}, \{y_1, y_2, y_3, y_4, y_5\}(0.4),$

$\emptyset(0.1), \{y_3\}(0.1)), k_3: (\emptyset, \emptyset(0.01), \emptyset(0.01), \emptyset(0.01)), k_4: (\emptyset, \emptyset(0.05), \emptyset(0.05), \emptyset(0.1))\}$

Step 4: We show the criteria of the students which were given as neutrosophic quadruple sets in Table 4.

Table 4. Student criteria table

	k_1	k_2	k_3	k_4
A_1	$(\{x_1, x_2, x_4, x_5\},$ $\{x_2, x_4\}(0.4),$ $\{x_1, x_2, x_4\}(0.2),$ $\{x_1, x_2\}(0.3))$	$(\{y_1, y_2, y_3, y_5\},$ $\{y_1, y_2, y_3\}(0.5),$ $\{y_2, y_3, y_5\}(0.2),$ $\{y_1, y_5\}(0.4))$	$(\{z_2, z_3, z_4, z_5\},$ $\{z_2, z_4, z_5\}(0.3),$ $\{z_2\}(0.2),$ $\{z_2, z_3, z_5\}(0.4))$	$(\{t_1, t_2, t_3, t_4, t_5\}$ $\{t_1\}(0.2),$ $\{t_4, t_5\}(0.3),$ $\{t_4\}(0.2))$

A_2	$(\{x_2, x_3\},$ $\{x_3\}(0.5),$ $\{x_2, x_3\}(0.3),$ $\{x_3\}(0.1))$	$(\{y_1, y_3, y_4, y_5\},$ $\{y_1, y_4, y_5\}(0.7),$ $\{y_3\}(0.1),$ $\{y_1\}(0.1))$	$(\{z_5\},$ $\{z_5\}(0.6),$ $\{z_5\}(0.3),$ $\emptyset(0.1))$	$(\{t_1, t_2, t_3\},$ $\{t_1, t_2\}(0.5),$ $\{t_2, t_3\}(0.2),$ $\{t_1, t_3\}(0.2))$
A_3	$(\{x_1, x_2, x_3, x_4\},$ $\{x_1, x_2, x_3\}(0.6),$ $\{x_1, x_2, x_3,$ $x_4\}(0.2),$ $\{x_1, x_3, x_4\}(0.3))$	$(\{y_1, y_2, y_3, y_4\},$ $\{y_1, y_2, y_3,$ $y_4\}(0.09),$ $\{y_1, y_2, y_3\}(0.05),$ $\{y_1, y_2, y_3,$ $y_4\}(0.01))$	$(\{z_5\},$ $\{z_5\}(0.4),$ $\emptyset(0.1),$ $\emptyset(0.3))$	$(\{t_1, t_2, t_3, t_4\},$ $\{t_1, t_2, t_3, t_4\}0.7),$ $\emptyset(0.7),$ $\emptyset(0.7))$
A_4	$(\{x_1, x_2, x_3, x_4\},$ $\emptyset(0.5),$ $\emptyset(0.1),$ $\emptyset(0.1))$	$(\{y_1, y_2, y_4, y_5\},$ $\{y_1, y_4, y_5\}(0.4),$ $\emptyset(0.6),$ $\emptyset(0.8))$	$(\{z_1, z_2, z_3, z_4\},$ $\{z_1, z_2\}(0.8),$ $\emptyset(0.7),$ $\{z_3\}(0.3))$	$(\{t_1, t_2, t_3, t_4, t_5\},$ $\{t_1, t_4\}(0.6),$ $\{t_1, t_3, t_4,$ $t_5\}(0.1),$ $\{t_1, t_4\}(0.2))$
A_5	$(\{x_1, x_2, x_3\},$ $\{x_2, x_3\}(0.9),$ $\{x_1, x_2,$ $x_3\}(0.02),$ $\{x_1, x_3\}(0.1))$	$(\{y_2, y_3\},$ $\{y_2, y_3\}(0.7),$ $\{y_3\}(0.6),$ $\{y_2, y_3\}(0.4))$	$(\{z_3, z_4, z_5\},$ $\{z_3\}(0.1),$ $\{z_5\}(0.1),$ $\{z_3, z_5\}(0.2))$	$(\{t_1, t_2, t_5\},$ $\emptyset(0.8),$ $\{t_1, t_5\}(0.9),$ $\{t_5\}(0.9))$

A_6	$(\{x_1, x_3\},$ $\{x_5\}(0.04),$ $\{x_3, x_4\}(0.06),$ $\emptyset(0.003))$	$(\{y_2, y_3, y_4, y_5\},$ $\emptyset(0.07),$ $\emptyset(0.02),$ $\{y_2\}(0.01))$	$(\{z_2\},$ $\{z_2, z_3, z_4\}(0.4),$ $\{z_2, z_3, z_4, z_5\}(0.02),$ $\emptyset(0.02))$	$\emptyset,$ $\{t_1, t_5\}(0.004),$ $\{t_3, t_4\}(0.02),$ $\{t_5\}(0.5))$
A_7	$(\{x_1, x_3, x_4, x_5\},$ $\{x_5\}(0.9),$ $\{x_5\}(0.8),$ $\emptyset(0.08))$	$(\{y_1, y_2, y_3, y_4, y_5\},$ $\{y_1, y_2, y_3, y_4, y_5\}(0.1), \emptyset(0.1),$ $\emptyset(0.1))$	$(\emptyset,$ $\{z_4, z_5\}(0.6),$ $\emptyset(0.3),$ $\{z_4\}(0.9))$	$(\emptyset,$ $\emptyset(0.9),$ $\emptyset(0.9),$ $\emptyset(0.1))$
A_8	$(\{x_1, x_2, x_3, x_4, x_5\},$ $\{x_1, x_2, x_3, x_4, x_5\}(0.3),$ $\{x_1, x_2, x_3, x_4, x_5\}(0.5),$ $\{x_1, x_2, x_3, x_4, x_5\}(0.2))$	$(\{y_4\},$ $\emptyset(0.8),$ $\{y_4\}(0.3),$ $\emptyset(0.01))$	$(\{z_3\},$ $\emptyset(0.2),$ $\emptyset(0.02),$ $\emptyset(0.1))$	$(\{t_1, t_5\},$ $\emptyset(0.9),$ $\emptyset(0.1),$ $\{t_5\}(0.1))$
A_9	$(\emptyset,$ $\emptyset(0.2),$ $\emptyset\ (0.2),$	$(\{y_1, y_3, y_4\},$ $\{y_3, y_4\}(0.6),$ $\{y_4\}(0.03),$	$(\{z_2, z_3, z_4, z_5\},$ $\{z_2\}(0.1),$ $\{z_2, z_3, z_4,$	$(\{t_1, t_2, t_3, t_4, t_5\},$ $\{t_5\}(0.6),$ $\{t_3\}(0.9),$

	$\emptyset(0.1))$	$\{y_1, y_3, y_4\}(0.09))$	$z_5\}(0.7)$, $\{z_3, z_4, z_5\}(0.7))$	$\{t_2\}(0.9))$
A_{10}	$(\emptyset, \emptyset(0.01)$, $\emptyset(0.02)$, $\emptyset(0.02))$	$(\{y_1, y_2, y_3, y_4, y_5\}$, $\{y_1, y_2, y_3, y_4, y_5\}(0.4), \emptyset(0.1)$, $\{y_3\}(0.1))$	$(\emptyset$, $\emptyset(0.01)$, $\emptyset(0.01)$, $\emptyset(0.01))$	$(\emptyset$, $\emptyset(0.05)$, $\emptyset(0.05)$, $\emptyset(0.1))$

Step 5: We calculate the individual evaluation values of the students given in Table 4 with respect to the criteria values of the I ideal student given in Step 3, one by one, using the measure of similarity. Thus, we obtain Table 5.

Table 5. Similarity table

	k_1	k_2	k_3	k_4
A_1	0.4750	0.3500	0.5250	0.4625
A_2	0.6750	0.6175	0.7000	0.5125
A_3	0.7250	0.9125	0.4750	0.5000
A_4	0,7750	0.4375	0.4875	0.7125
A_5	0.8250	0.5375	0.3875	0.3750
A_6	0.1270	0.1600	0.400	0.127
A_7	0.9500	0.8000	0.9750	0.5500

A_8	0.6000	0.6325	0.3125	0.7375
A_9	0.3000	0.6500	0.4500	0.4000
A_{10}	0.3050	0.5875	0.2550	0.2750

Step 6: We multiply the weights of all criteria in Step 2 with each of the similarity values in Table 6. The sum of the similarity values of each criterion is given as the similarity value of our student set.

Table 6. Weighted similarity table

	$0{,}4 * k_1$	$0{,}2 * k_2$	$0{,}1 * k_3$	$0{,}3 * k_4$	$\sum_{i=1}^{4} w_i k_i = S_{QHN}{}^{i}$
A_1	0.19000	0.07000	0.05250	0.13875	$S_{QHN}{}^{1}(\mathrm{I}, A_1) = 0.45125$
A_2	0.27000	0.12350	0.00700	0.15375	$S_{QHN}{}^{2}(\mathrm{I}, A_2) = 0.555425$
A_3	0.29000	0.18250	0.04750	0.15000	$S_{QHN}{}^{3}(\mathrm{I}, A_3) = 0.67000$
A_4	0.31000	0.08750	0.04875	0.21375	$S_{QHN}{}^{4}(\mathrm{I}, A_4) = 0.66000$
A_5	0.33000	0.10750	0.03875	0.11250	$S_{QHN}{}^{5}(\mathrm{I}, A_5) = 0.58875$
A_6	0.05080	0.03200	0.04000	0.03810	$S_{QHN}{}^{6}(\mathrm{I}, A_6) = 0.1609$
A_7	0.38000	0.16000	0.09750	0.16500	$S_{QHN}{}^{7}(\mathrm{I}, A_7) = 0.80250$
A_8	0.24000	0.12650	0.03125	0.22125	$S_{QHN}{}^{8}(\mathrm{I}, A_8) = 0.62025$
A_9	0.12000	0.13000	0.04500	0.12000	$S_{QHN}{}^{9}(\mathrm{I}, A_9) = 0.41500$
A_{10}	0.12200	0.11750	0.02550	0.08250	$S_{QHN}{}^{10}(\mathrm{I}, A_{10}) = 0.34750$

$$A_7 > A_3 > A_4 > A_8 > A_5 > A_2 > A_1 > A_9 > A_{10} > A_6$$

Similarity values of each student were calculated. According to the results, the most efficient student in online education is A_7 student with similarity value of 0.80250.

5 Numerical Comparison Analysis

In this section, we will compare the results of Euclid similarity measure [23], Dice similarity measure [22] and Hausdorff similarity measure [15] using the only values (T, I, F) for which we calculate the similarity value with the Hausdorff measures based on generalized set-valued neutrosophic quadruple numbers.

i) The result of calculating the similarity value of the students calculated in 4.2 with Hausdorff similarity measure [15] in Table 7.

Table 7. Result according to Hausdorff similarity measure [15]

A_1	0,650
A_2	0.450
A_3	0.628
A_4	0,550
A_5	0,520
A_6	0,228
A_7	0,860
A_8	0,450
A_9	0,760
A_{10}	0,283

$$A_7 > A_9 > A_1 > A_3 > A_4 > A_5 > A_2 = A_8 > A_{10} > A_6$$

ii) The result of calculating the similarity value of the students calculated in 4.2 with Euclid similarity measure [23] in Table 8.

Table 8. Result according to Euclid similarity measure [23]

A_1	0,7463
A_2	0.8244
A_3	0.7407
A_4	0,7827
A_5	0,7689
A_6	0,5308
A_7	0,6981
A_8	0,8129
A_9	0,6836
A_{10}	0,6980

$$A_2 > A_8 > A_4 > A_5 > A_1 > A_3 > A_7 > A_{10} > A_9 > A_6$$

iii) The result of calculating the similarity value of the students calculated in 4.2 with Dice similarity measure [22] in Table 9.

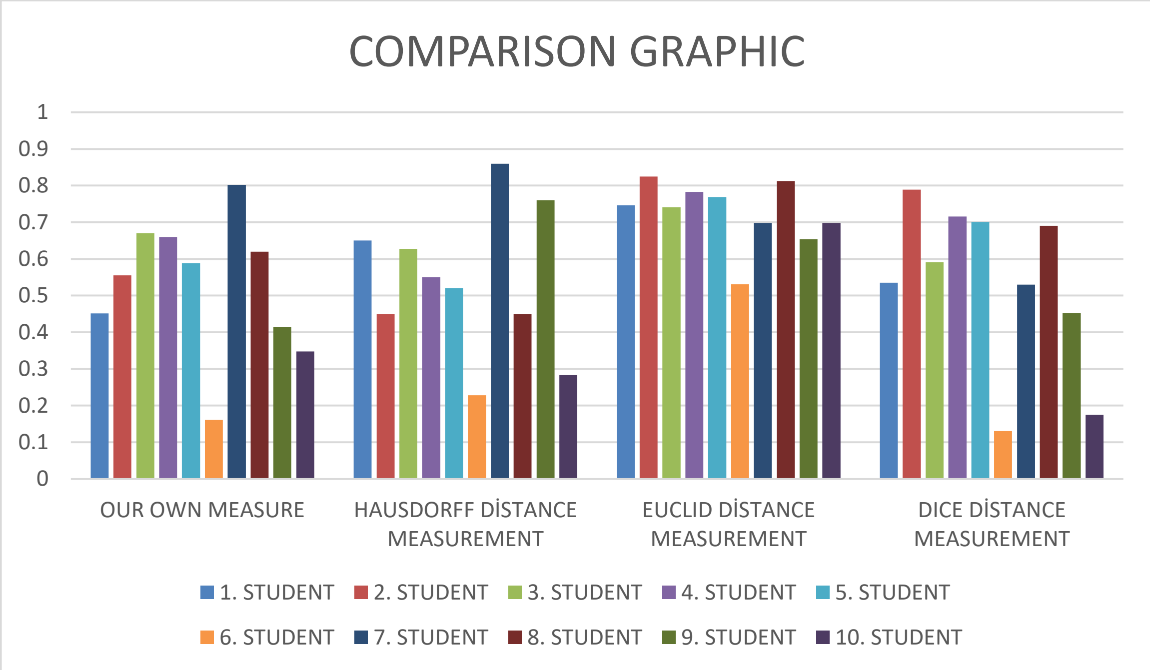
Table 9. Result according to Dice similarity measure [22]

A_1	0,5349
A_2	0.7892
A_3	0.5911
A_4	0,7162
A_5	0,7008
A_6	0,1303
A_7	0,5301
A_8	0,6908
A_9	0,4528
A_{10}	0,1748

$$A_2 > A_4 > A_5 > A_8 > A_3 > A_1 > A_7 > A_9 > A_{10} > A_6$$

From i, ii, iii; we obtain Graphic 1.

Graphic 1: Comparison of similarity measures



6 Conclusions

In this study, a new decision making application based on generalized set – valued neutrosophic quadruple numbers has been developed to calculate the efficiency of students participating in online education, which is applied to students who have to take a break from their education. We define some measures for generalized set-valued neutrosophic quadruple sets. We proved that this similarity measures satisfies the similarity conditions. Using this similarity measure, we developed an algorithm to evaluate the adequacy of online education applied to ensure that students' education is not interrupted by the epidemic, and we gave an example through this algorithm. In the developed algorithm and in the example given, we determined the highest efficiency student among the students taking courses with online education by using the generalized set-valued neutrosophic quadruple numbers. Also, we obtain different result from some previous applications (based on neutrosophic sets) for decision making algorithm. In future, we will discuss the following integration of the related topics;

1) This measure and algorithm we have obtained can be used not only for online education, but also to evaluate the competence of any newly designed application, the competence of the people who will enter the profession and its effect on a law.

2) For proposed method the effect of a drug on a particular disease.

3) For proposed method more than one expert opinion can be obtained and different weight values can be created for each expert.

4) In addition, criteria and criterion weights can be selected as desired in proposed method.

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Solvability of System of Neutrosophic Soft Linear Equations

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Abstract. This article exposes a system of Neutrosophic Soft Linear Equations (NSLE) of the form $A \otimes x = b$ and is said to be solvable if $A \otimes x(A; b) = b$ holds, otherwise unsolvable. We derive conditions under which the above system is solvable and further using Chebychev Approximation we find a prinicipal solution if the given systen is not solvable.

Keywords: Neutrosophic Soft Set (NSS), Neutrosophic Soft Matrix(NSM), Neutrosophic Soft Eigenvector(NSEv), System of Neutrosophic Soft Linear Equation(NSLE), Chebychev distance.

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1. Introduction

In human judgment, the importance of relations is almost self-evident. But the problem is mainly to pass from a vague and customary concept to a precisely formulated one. The theory of fuzzy sets is a step in such a direction and we believe that a straightforward study of fuzzy relations deserves to be developed for a better interpretation and explanation of real-world problems. The system of fuzzy relation equations is an important topic in fuzzy set theory. Sanchez [29] first introduced fuzzy relation equations with sup-inf composition in complete Brouwerian lattices. Since then, many authors investigated the methods for solving fuzzy relation equations with different composite operators over various special Brouwerian lattices. Among them, for finite fuzzy relation equations with sup-inf composition, Higashi et,al. [10] showed that the solution set can be determined by minimal solutions and the greatest solution in the linear lattice [0,1]. The solvability and unique solvability of linear systems in the max-min algebra which is one of the most important fuzzy algebra, and the related question of the strong regularity of max-min matrices was considered in [5, 6]. Cechlarova [7] studied the unique solvability of linear systems of equation over the max-min fuzzy algebra on the unit real interval. In 2010 Sriram and Murugadas discussed the relation between row space, column space and regularity of Intuitionistic Fuzzy Matrix(IFM) etc.(see [25,26,31–35]). Pradhan and Pal [27] introduced the concepts that the Intuitionistic Fuzzy Relation Equation of the form $A \otimes x = b$ is consistent when the coefficient IFM A is regular.

But all these theories have their inherent difficulties as pointed out by Molodtsove [24]. The reason for these difficulties is, possibly, the inadequacy of the parameterization tools of the theories. The fuzzy soft set representation of the intuitionistic fuzzy soft set has been studied by Maji et.al, [23]. Likewise, Rajarajeswari et.al [28], proposed new definitions for intuitionistic fuzzy soft matrices and its sort.

The notion of Neutrosophic Set (NS) was introduced by Smarandache [30]. Deli [8] defined Neutrosophic parameterized Neutrosophic soft sets (npn-soft sets) which is the combination of NS and a soft set. Deli and Broumi [9] redefined the notion of NS in a new way and put forward the concept of NSM and different types of matrices in neutrosophic soft theory. They have introduced some new operations and properties on these matrices. For recent development of NS in decision making theory see the work done by Abdel Basset et.al, [1–3] and N . Nabeeh et.al, [18–20]. The minimal solution of NSM was done by Kavitha et.al, [12] based on the notion of NSM given by Sumathi and Arokiarani [4]. As the time goes some works on NSM were done by Kavitha et.al, [13–15,17]. The Monotone interval fuzzy neutrosophic soft eigenproblem and Monotone fuzzy neutrosophic soft eigenspace structures in max-min algebra were investigated by Murugadas et.al, [21,22]. Also, two kinds of fuzzy neutrosophic soft matrices are presented by Uma et.al, [36].

In this paper, we will concentrate on the solvability of the system of NSLEs be solvable of the form $A \otimes x(A;b) = b$. We derived the maximum solution for a system of NSLEs and we define that particular solution $x(A;b)$ as principal solution. In the concluding section-5, we have tried to give an algorithm for coefficient NSM A of an unsolvable system, $A \otimes x = b$ to get a principal solution.

2. **Preliminaries**

In this section, some elementary aspects that are necessary for this paper are introduced.

Definition 2.1. [30] A neutrosophic set A on the universe of discourse X is defined as $A = \{\langle x,\ T_A(x),\ I_A(x),\ F_A(x)\rangle, x \in X\}$, where $T, I, F : X \to]^-0, 1^+[$ and
$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$. (1)]

From philosophical point of view the NS set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application especially in Scientific and Engineering problems it is difficult to use NS with value from real standard or non-standard subset of $]^{-}0, 1^{+}[$. Hence we consider the NS which takes the value from the subset of $[0, 1]$. Therefore we can rewrite equation (1) as $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. In short an element $\widetilde{a}$ in the NS A, can be written as $\widetilde{a} = \langle a^T, a^I, a^F\rangle$, where a^T denotes degree of truth, a^I denotes degree of indeterminacy, a^F denotes degree of falsity such that $0 \leq a^T + a^I + a^F \leq 3$.

Definition 2.2. [4] A NS A on the universe of discourse X is defined as $A = \{x, \langle T_A(x), I_A(x), F_A(x)\rangle, x \in X\}$, where $T, I, F : X \to [0, 1]$ and $0 \leq T_A(x)+I_A(x)+F_A(x) \leq 3$.

Definition 2.3. [24] Let U be the initial universe set and E be a set of parameter. Consider a non-empty set $A, A \subset E$. Let $P(U)$ denotes the set of all NSs of U. The collection (F, A) is termed to be the NSS over U, where F is a mapping given by $F : A \to P(U)$. Here after we simply consider A as NSS over U instead of (F, A).

Definition 2.4. [4] Let $U = \{c_1, c_2, ..., c_m\}$ be the universal set and E be the set of parameters given by $E = \{e_1, e_2, ..., e_m\}$. Let $A \subset E$. A pair (F, A) be a NSS over U. Then the subset of $U \times E$ is defined by $R_A = \{(u, e);\ e \in A,\ u \in F_A(e)\}$
which is called a relation form of $(F_A,\ E)$. The membership function, indeterminacy membership function and non membership function are written by
$T_{R_A} : U \times E \to [0, 1]$, $I_{R_A} : U \times E \to [0, 1]$ and $F_{R_A} : U \times E \to [0, 1]$ where $T_{R_A}(u, e) \in [0, 1]$, $I_{R_A}(u, e) \in [0, 1]$ and $F_{R_A}(u, e) \in [0, 1]$ are the membership value, indeterminacy value and non membership value respectively of $u \in U$ for each $e \in E$.
If $[(T_{ij},\ I_{ij},\ F_{ij})] = [T_{ij}(u_i, e_j),\ I_{ij}(u_i, e_j)\ , F_{ij}(u_i, e_j)]$ we define a matrix

$$[\langle T_{ij}\ , I_{ij},\ F_{ij}\rangle]_{m\times n} = \begin{bmatrix} \langle T_{11},\ I_{11},\ F_{11}\rangle & \cdots & \langle T_{1n},\ I_{1n},\ F_{1n}\rangle \\ \langle T_{21},\ I_{21},\ F_{21}\rangle & \cdots & \langle T_{2n},\ I_{2n},\ F_{2n}\rangle \\ \vdots & \vdots & \vdots \\ \langle T_{m1},\ I_{m1},\ F_{m1}\rangle & \cdots & \langle T_{mn},\ I_{mn},\ F_{mn}\rangle \end{bmatrix}.$$

Which is called an $m \times n$ $FNSM$ of the NSS (F_A, E) over U.

Definition 2.5. [36] Let $A = (\langle a^T_{ij},\ a^I_{ij},\ a^F_{ij}\rangle),\ B = \langle(b^T_{ij},\ b^I_{ij},\ b^F_{ij}\rangle) \in \mathcal{N}_{(m,n)}$, NSM of order $m \times n$) and $\mathcal{N}_{(n)}$-denotes a square NSM of order n. The component wise addition and component wise multiplication is defined as
$A \oplus B = (sup\{a^T_{ij},\ b^T_{ij}\},\ sup\{a^I_{ij},\ b^I_{ij}\},\ inf\{a^F_{ij},\ b^F_{ij}\})$
$A \otimes B = (inf\{a^T_{ij},\ b^T_{ij}\},\ inf\{a^I_{ij},\ b^I_{ij}\},\ sup\{a^F_{ij},\ b^F_{ij}\})$

Definition 2.6. Let $A \in \mathcal{N}_{(m,n)},\ B \in \mathcal{N}_{(n,p}$, the composition of A and B is defined as

$$A \circ B = \left(\sum_{k=1}^{n} (a_{ik}^T \wedge\ b_{kj}^T),\ \sum_{k=1}^{n} (a_{ik}^I \wedge\ b_{kj}^I),\ \prod_{k=1}^{n} (a_{ik}^F \vee\ b_{kj}^F) \right)$$

equivalently we can write the same as

$$= \left(\bigvee_{k=1}^{n} (a_{ik}^T \wedge b_{kj}^T),\ \bigvee_{k=1}^{n} (a_{ik}^I \wedge b_{kj}^I),\ \bigwedge_{k=1}^{n} (a_{ik}^F \vee b_{kj}^F) \right).$$

The product $A \circ B$ is defined if and only if the number of columns of A is same as the number of rows of B. Then A and B are said to be conformable for multiplication. We shall use AB instead of $A \circ B$.

Where $\sum(a_{ik}^T \wedge\ b_{kj}^T)$ means max-min operation and
$\prod\limits_{k=1}^{n} (a_{ik}^F \vee\ b_{kj}^F)$ means min-max operation.

Definition 2.7. [16] Let V_n will denote the set of all n-tuples $(\langle v_1^T, v_1^I, v_1^F \rangle, ..., \langle v_n^T, v_n^I, v_n^F \rangle)$ over $[0,1]^3$

An element of V_n is called a Neutrosophic Soft vector (NSV) of dimension n.

Definition 2.8. [16] If $A \in \mathcal{N}_{(m,n)}$ and $X \in \mathcal{N}_{(n,m)}$ satisfies the relation $AXA = A$ then X is called a generalized inverse(g-inverse) of A which is denoted by A^-. The g-inverse of an NSM is not necessarily unique. We denote the set of all g-inverses of A by $A\{1\}$.

Definition 2.9. [16] Let $A = \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \in \mathcal{N}_{(m,n)}$. Then the element $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle$ is called the (i,j) entry of A. Let $A_{i*}(A_{*j})$ denote the i^{th} row (column) of A. The row space $\mathcal{R}(A)$ of A is the subspace of V_n generated by rows $\{A_{i*}\}$ of A. The column space $\mathcal{C}(A)$ of A is the space of V_m generated by the columns $\{A_{*j}\}$ of A.

Definition 2.10. [16] For NSM $A, X \in \mathcal{N}_{(m \times n)}$, are said to be a Moore-Penrose of A, if $AXA = A, XAX = X, (AX)^t = AX$ and $(XA)^t = XA$.

3. Results

Definition 3.1. (Linear combination of NSVs)

Let $S = \{\langle a_1^T, a_1^I, a_1^F \rangle, \langle a_2^T, a_2^I, a_2^F \rangle, ..., \langle a_p^T, a_p^I, a_p^F \rangle\}$ be a set of NSV of dimension n. The linear combination of elements of the set S is a finite sum $\sum\limits_{i=1}^{p} \langle c_i^T, c_i^I, c_i^F \rangle \langle a_i^T, a_i^I, a_i^F \rangle$ where $\langle a_i^T, a_i^I, a_i^F \rangle \in S$ and $\langle c_i^T, c_i^I, c_i^F \rangle \in [0,1]^3$. The set of all linear combinations of the elements of S is calld the span of S, denoted by $\langle S \rangle$.

Here we illustrate the above concept.

Example 3.2. Let $S = \{\langle a_1^T, a_1^I, a_1^F \rangle, \langle a_2^T, a_2^I, a_2^F \rangle, \langle a_3^T, a_3^I, a_3^F \rangle\}$ be a subset of V_3, where
$\langle a_1^T, a_1^I, a_1^F \rangle = (\langle 0.8, 0.7, 0.2 \rangle, \langle 0.6, 0.5, 0.4 \rangle, \langle 0.4, 0.3, 0.6 \rangle)$,

$\langle a_2^T, a_2^I, a_2^F\rangle = (\langle 0.5, 0.4, 0.6\rangle, \langle 0.5, 0.4, 0.6\rangle, \langle 0.4, 0.3, 0.6\rangle)$,
and $\langle a_3^T, a_3^I, a_3^F\rangle = (\langle 0.7, 0.6, 0.3\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.9, 0.8, 0.1\rangle)$. Then

$$\begin{aligned}\langle S\rangle = \{&\langle c_1^T, c_1^I, c_1^F\rangle(\langle 0.8, 0.7, 0.2\rangle, \langle 0.6, 0.5, 0.2\rangle, \langle 0.4, 0.3, 0.6\rangle)\\ &+\langle c_2^T, c_2^I, c_2^F\rangle(\langle 0.5, 0.4, 0.6\rangle, \langle 0.5, 0.4, 0.6\rangle, \langle 0.4, 0.3, 0.6\rangle\\ &+\langle c_3^T, c_3^I, c_3^F\rangle(\langle 0.7, 0.6, 0.3\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.9, 0.8, 0.1\rangle\}.\end{aligned}$$

Definition 3.3 (Dependenece of NSVs)**.** A set S of NSVs is independent if and only if each element of S can be expressed as a linear combination of other elements of S, that is, no element $s \in S$ is a linear combination of $S \setminus \{s\}$. If a vector α can be expressed by some other vectors, then the vector α is called dependent otherwise it is called independent. These terminologies are similar to classical vectors.

An independent and dependent set of vectors are illustrated below.

Example 3.4. Let $S = \{\langle a_1^T, a_1^I, a_1^F\rangle, \langle a_2^T, a_2^I, a_2^F\rangle, \langle a_3^T, a_3^I, a_3^F\rangle\}$ be a subset of V_3, where
$\langle a_1^T, a_1^I, a_1^F\rangle = (\langle 0.8, 0.7, 0.2\rangle, \langle 0.6, 0.5, 0.4\rangle, \langle 0.4, 0.3, 0.6\rangle)$,
$\langle a_2^T, a_2^I, a_2^F\rangle = (\langle 0.5, 0.4, 0.6\rangle, \langle 0.5, 0.4, 0.6\rangle, \langle 0.4, 0.3, 0.6\rangle)$, and
$\langle a_3^T, a_3^I, a_3^F\rangle = (\langle 0.7, 0.6, 0.3\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.9, 0.8, 0.1\rangle)$.

Here the set S is an independent set.
If not then $\langle a_1^T, a_1^I, a_1^F\rangle = \langle \alpha^T, \alpha^I, \alpha^F\rangle\langle a_2^T, a_2^I, a_2^F\rangle + \langle \beta^T, \beta^I, \beta^F\rangle\langle a_3^T, a_3^I, a_3^F\rangle$
for $\langle \alpha^T, \alpha^I, \alpha^F\rangle, \langle \beta^T, \beta^I, \beta^F\rangle \in \mathcal{N}$. So

$$\begin{aligned}\langle a_1^T, a_1^I, a_1^F\rangle &= \langle \alpha^T, \alpha^I, \alpha^F\rangle(\langle 0.5, 0.4, 0.6\rangle, \langle 0.5, 0.4, 0.6\rangle, \langle 0.4, 0.3, 0.6\rangle)\\ &\quad +\langle \beta^T, \beta^I, \beta^F\rangle(\langle 0.7, 0.6, 0.3\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.9, 0.8, 0.1\rangle\\ &= (\langle \max\{\min(0.5, \alpha^T), \min(0.7, \beta^T)\}, \max\{\min(0.4, \alpha^I), \min(0.6, \beta^I)\},\\ &\quad \min\{\max(0.6, \alpha^F), \max(0.3, \beta^F)\}\rangle,\\ &\quad (\langle \max\{\min(0.5, \alpha^T), \min(0.7, \beta^T)\}, \max\{\min(0.4, \alpha^I), \min(0.6, \beta^I)\},\\ &\quad \min\{\max(0.6, \alpha^F), \max(0.3, \beta^F)\}\rangle,\\ &(\langle \max\{\min(0.4, \alpha^T), \min(0.9, \beta^T)\}, \max\{\min(0.3, \alpha^I), \min(0.8, \beta^I)\},\\ &\quad \min\{\max(0.6, \alpha^F), \max(0.1, \beta^F)\}\rangle).\end{aligned}$$

It is not possible to find any $\langle \alpha^T, \alpha^I, \alpha^F\rangle, \langle \beta^T, \beta^I, \beta^F\rangle \in \mathcal{N}$ such that the corresponding coefficients on both sides will be equal. That is,
$\langle a_1^T, a_1^I, a_1^F\rangle \neq \langle \alpha^T, \alpha^I, \alpha^F\rangle\langle a_2^T, a_2^I, a_2^F\rangle + \langle \beta^T, \beta^I, \beta^F\rangle\langle a_3^T, a_3^I, a_3^F\rangle$. Similarly,
$\langle a_2^T, a_2^I, a_2^F\rangle \neq \langle \alpha^T, \alpha^I, \alpha^F\rangle\langle a_1^T, a_1^I, a_1^F\rangle + \langle \beta^T, \beta^I, \beta^F\rangle\langle a_3^T, a_3^I, a_3^F\rangle$ and
$\langle a_3^T, a_3^I, a_3^F\rangle \neq \langle \alpha^T, \alpha^I, \alpha^F\rangle\langle a_2^T, a_2^I, a_2^F\rangle + \langle \beta^T, \beta^I, \beta^F\rangle\langle a_1^T, a_1^I, a_1^F\rangle$. So the set S is independent.

Let $S = \{\langle a_1^T, a_1^I, a_1^F\rangle, \langle a_2^T, a_2^I, a_2^F\rangle\}$ be a subset of V_3,
where $\langle a_1^T, a_1^I, a_1^F\rangle = (\langle 0.7, 0.6, 0.3\rangle, \langle 0.5, 0.4, 0.5\rangle, \langle 0.6, 0.5, 0.4\rangle)$ and
$\langle a_2^T, a_2^I, a_2^F\rangle = (\langle 0.8, 0.7, 0.2\rangle, \langle 0.5, 0.4, 0.5\rangle, \langle 0.6, 0.5, 0.4\rangle)$.
Here $\langle a_1^T, a_1^I, a_1^F\rangle = \langle c^T, c^I, c^F\rangle(\langle a_2^T, a_2^I, a_2^F\rangle)$ for $\langle c^T, c^I, c^F\rangle = \langle 0.7, 0.6, 0.3\rangle$. So S is a dependent set.

Definition 3.5 (Basis)**.** Let W be an Neutrosophic Soft Subspace of V_n and S be a subset of W such that the elements of S are independent. If every element of W can be expressed uniquely as a linear combination of the elements of S, then S is called a basis of neutrosophic soft subspace W.

Definition 3.6 (Standard basis)**.** A basis B of an Neutrosophic Soft Vector Space (NSVS) W is a standard basis if and only if whenever
$\langle b_i^T, b_i^I, b_i^F\rangle = \sum_{j=1}^{n} \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle\langle b_j^T, b_j^I, b_j^F\rangle$ for $\langle b_i^T, b_i^I, b_i^F\rangle, \langle b_j^T, b_j^I, b_j^F\rangle \in \mathcal{N}$
and $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \in [1, 0]$ then $\langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle\langle b_i^T, b_i^I, b_i^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle$.

Example 3.7. Let $S = \{\langle a_1^T, a_1^I, a_1^F\rangle, \langle a_2^T, a_2^I, a_2^F\rangle, \langle a_3^T, a_3^I, a_3^F\rangle\}$ be a subset of V_3 given by
$a_1 = (\langle 0.5, 0.4, 0.5\rangle, \langle 0.5, 0.4, 0.5\rangle, \langle 0.5, 0.4, 0.5\rangle)$ and
$a_2 = (\langle 0.5, 0.4, 0.5\rangle, \langle 0.6, 0.5, 0.4\rangle, \langle 0.8, 0.7, 0.2\rangle)$ and
$a_3 = (\langle 0.4, 0.3, 0.6\rangle, \langle 0.4, 0.3, 0.6\rangle, \langle 0.8, 0.7, 0.2\rangle)$.
Then S is independent set, since
$\langle a_1^T, a_1^I, a_1^F\rangle \neq \langle c_1^T, c_1^I, c_1^F\rangle(\langle a_2^T, a_2^I, a_2^F\rangle) + \langle c_2^T, c_2^I, c_2^F\rangle(\langle a_3^T, a_3^I, a_3^F\rangle)$,
$\langle a_2^T, a_2^I, a_2^F\rangle \neq \langle c_3^T, 3_3^I, c_3^F\rangle\langle a_1^T, a_1^I, a_1^F\rangle + \langle c_4^T, c_4^I, c_4^F\rangle\langle a_3^T, a_3^I, a_3^F\rangle$ and
$\langle a_3^T, a_3^I, a_3^F\rangle \neq \langle c_5^T, c_5^I, c_5^F\rangle(\langle a_1^T, a_1^I, a_1^F\rangle) + \langle c_6^T, c_6^I, c_6^F\rangle(\langle a_2^T, a_2^I, a_2^F\rangle)$.
So $\{\langle a_1^T, a_1^I, a_1^F\rangle, \langle a_2^T, a_2^I, a_2^F\rangle, \langle a_3^T, a_3^I, a_3^F\rangle\}$ is a basis for $\langle S\rangle$.
Now this is a standard basis . For, $\langle a_1^T, a_1^I, a_1^F\rangle = \langle c_{11}^T, c_{11}^I, c_{11}^F\rangle(\langle a_1^T, a_1^I, a_1^F\rangle) + \langle c_{12}^T, c_{12}^I, c_{12}^F\rangle(\langle a_2^T, a_2^I, a_2^F\rangle) + \langle c_{13}^T, c_{13}^I, c_{13}^F\rangle(\langle a_3^T, a_3^I, a_3^F\rangle)$ holds if $\langle c_{11}^T, c_{11}^I, c_{11}^F\rangle = \langle 0.8, 0.7, 0.2\rangle$, $\langle c_{12}^T, c_{12}^I, c_{12}^F\rangle = \langle 0.5, 0.4, 0.5\rangle$ and $\langle c_{13}^T, c_{13}^I, c_{13}^F\rangle = \langle 0.6, 0.5, 0.4\rangle$.
Also $\langle a_1^T, a_1^I, a_1^F\rangle = \langle c_{11}^T, c_{11}^I, c_{11}^F\rangle(\langle a_1^T, a_1^I, a_1^F\rangle)$ for $\langle c_{11}^T, c_{11}^I, c_{11}^F\rangle = \langle 0.8, 0.7, 0.2\rangle$.
Similarly for $\langle a_2^T, a_2^I, a_2^F\rangle$ and $\langle a_3^T, a_3^I, a_3^F\rangle$.

4. **Solvability**

In this section, we are going to study the system of NSLEs of the form,

$$A \otimes x = b \ (1)$$

that is

$$\langle \max_j \min(a_{ij}^T, x_j^T), \max_j \min(a_{ij}^I, x_j^I), \min_j \max(a_{ij}^F, x_j^F)\rangle = \langle b_i^T, b_i^I, b_i^F\rangle \ (2)$$

where the NSM $A \in \mathcal{N}_{(m\times n)}$ and the NSV $b \in \mathcal{N}_{(m)}$ are given and the NSV $x \in \mathcal{N}_{(n)}$ is unknown.

The solution set of the system defined in (1) for a given NSM A and an NSV b will be denoted by $S(A,b) = \{x \in \mathcal{N}_n | A \otimes x = b\}$.
Now our aim is to find whether the system (1) is solvable, that is, whether the solution set $S(A,b)$ is non-empty.

Lemma 4.1. Let us consider the system of NSLE $A \otimes x = b$.
If $\max\limits_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) < \langle b^T_j, b^I_j, b^F_j\rangle$ for some k, then $S(A,b) = \phi$, that is the sysem is not solvable.
Proof: If $\max\limits_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) < \langle b^T_j, b^I_j, b^F_j\rangle$ for some j, then
$\min\limits_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) \leq \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \leq \max\limits_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) < (\langle b^T_j, b^I_j, b^F_j\rangle)$
Hence, $\langle \max\limits_j \min(a^T_{ij}, x^T_i), \max\limits_j \min(a^I_{ij}, x^I_i), \min\limits_j \max(a^F_{ij}, x^F_i)\rangle < (\langle b^T_j, b^I_j, b^F_j\rangle)$ for some j, and by equation (2) no values $\langle x^T_i, x^I_i, x^F_i\rangle$ exists that satisfy the equation (1). Therefor $S(A,b) = \phi$.

Remark 4.2. Let us consider the condition of the Lemma 4.1 be
$\max\limits_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) > (\langle b^T_j, b^I_j, b^F_j\rangle)$ for some j. Then according to the proof of the Lemma 4.1, $\min\limits_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle, \langle x^T_i, x^I_i, x^F_i\rangle) \geq \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \geq \max(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) > (\langle b^T_j, b^I_j, b^F_j\rangle)$ implies the only possibility is, $\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle$ are same for all i. Then two case may arises,
Case-1: If $\langle b^T_j, b^I_j, b^F_j\rangle$ are equal for all j. Then the system reduce to one equation. So that the system is solvable.
Case-2: If $\langle b^T_j, b^I_j, b^F_j\rangle$ are different for some j. Then the equation of the system will be such that, all have the same left side with some different right side. Hence the system is not solvable.

Example 4.3. Let us consider the system of NSLEs $A \otimes x = b$ where,

$$A = \begin{bmatrix} \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.3\ 0.2\ 0.7\rangle \\ \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.6\ 0.5\ 0.4\rangle \\ \langle 0.8\ 0.7\ 0.2\rangle & \langle 0.4\ 0.3\ 0.6\rangle \end{bmatrix} \text{ and}$$

$$b = \begin{bmatrix} \langle 0.4\ 0.3\ 0.6\rangle \\ \langle 1, 1, 0\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle \end{bmatrix}.$$

Here for $j = 2$,
$\max\{\langle 0.3\ 0.2\ 0.7\rangle, \langle 0.6\ 0.5\ 0.4\rangle, \langle 0.4\ 0.3\ 0.6\rangle\} = \langle 0.6\ 0.5\ 0.4\rangle < \langle 1, 1, 0\rangle$. Hence by Lemma 4.1, the system of NSLEs $A \otimes x = b$ is not solvable.

The following theorem deduce the fact its solvability of a system of NSLEs of the form (1) depends upon the characteristics of the coefficient NSM A.

Theorem 4.4. The system of NSLEs of the form (1) has a solution if the non-zero rows of the coefficient NSM A forms a standard basis for the row space of itself.

Proof: As the non-zero rows of the NSM A forms a standard basis for the row space of A, then the NSM A be regular. That is there exists a g-inverse A^- of A such that $A \otimes A^- \otimes A = A$. Now, $A \otimes x = b$ gives $A \otimes A^- \otimes A \otimes x = b$.

That implies, $A \otimes A^- \otimes b = b$. Which shows, $(A^- \otimes b)$ is a solution of the given sytem. Therefore the system of NSLE is solvable.

Example 4.5. Let us consider the system of NSLEs $A \otimes x = b$. with

$$A = \begin{bmatrix} \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.5\ 0.4\ 0.5\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.8\ 0.7\ 0.2\rangle \end{bmatrix}$$

$$X = [\langle x_1^T, x_1^I, x_1^F\rangle, \langle x_2^T, x_2^I, x_2^F\rangle, \langle x_3^T, x_3^I, x_3^F\rangle]^T \text{ and}$$

$$b = \begin{bmatrix} \langle 0.6\ 0.5\ 0.4\rangle \\ \langle 0.5, 0.4, 0.5\rangle \end{bmatrix}.$$

Here the non-zero rows of the NSM S are linearly independent and form s standard basis . So
A is regular and one of its $g-$ inverse is

$$A^- = \begin{bmatrix} \langle 0.8\ 0.7\ 0.2\rangle & \langle 0.5\ 0.4\ 0.5\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.5\ 0.4\ 05\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.8\ 0.7\ 0.2\rangle \end{bmatrix}$$

$$x = A^- b = \begin{bmatrix} \langle 0.6\ 0.5\ 0.4\rangle \\ \langle 0.5\ 0.4\ 0.3\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle \end{bmatrix}$$

This is one of the solution of the above system of NSLEs.

The assertion of the $g-$inverse of a NSM A is not unique. So the solution of a system of NSLEs may have many solution. Among these solutions the maximum solution is defined as follows.

Definition 4.6. Any arbitrary element $\bar{x}$ of $S(A, b)$ is called a maximum solution of the system $A \otimes x = b$ if for all $x \in S(A, b), x \geq \bar{x}$ implies $x = \bar{x}$.

The following theorem demonstrate how to find the maximum solution of the system of NSLEs.

Theorem 4.7. If for a system of NSLEs $A \otimes x = b$ has a solution denoted by $\bar{x}(A, b)$ and is defined by

$$\bar{x} = \langle \bar{x}^T, \bar{x}^I, \bar{x}^F \rangle = \begin{cases} \langle 1, 1, 0 \rangle \ if \ \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \leq \langle b_j^T, b_j^I, b_j^F \rangle \ \forall i \\ \min\{\langle b_j^T, b_j^I, b_j^F \rangle\} \ if \ \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle > \langle b_j^T, b_j^I, b_j^F \rangle, \end{cases}$$

is the maximum solution.

Proof: As the system of NSLEs $A \otimes x = b$ has a solution, so it is consistent, then $\bar{x}$ is a solution of the system. If $\bar{x}$ is not a solution, then $A \otimes x \neq b$ and therefore $\max_j \min(a_{ij}^T, x_j^T), \max_j \min(a_{ij}^I, x_j^I) \min_j \max(a_{ij}^F, x_j^F) \neq (\langle b_{j_0}^T, b_{j_0}^I, b_{j_0}^F \rangle)$ for at least one j_0. The above definition of $\bar{x}$,
since $\langle \bar{x}_i^T, \bar{x}_i^I, \bar{x}_i^F \rangle \leq \langle b_j^T, b_j^I, b_j^F \rangle$ for each j, so
$\langle \bar{x}_i^T, \bar{x}_i^I, \bar{x}_i^F \rangle \leq \langle b_{j_0}^T, b_{j_0}^I, b_{j_0}^F \rangle$. By our assumption, $\max_j(\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle < \langle b_{j_0}^T, b_{j_0}^I, b_{j_0}^F \rangle)$ for some j_0 and by Lemma 4.1 it follows that $S(A, b) = \phi$, which is a contradiction. Hence $\bar{x}$ is a solution of the system $A \otimes x = b$.
Now let us prove that $\bar{x}$ is a maximum solution. If possible let us assume that $y = \langle y^T, y^I, y^F \rangle$ be a solution of the system such that $y > \bar{x}$, that is
$\langle y_{i_0}^T, y_{i_0}^I, y_{i_0}^F \rangle > \langle \bar{x}_{i_0}^T, \bar{x}_{i_0}^I, \bar{x}_{i_0}^F \rangle$ for at least one i_0.
Therefore by definition of $\bar{x}$, we have $\langle y_{i_0}^T, y_{i_0}^I, y_{i_0}^F \rangle > \min(\langle b_j^T, b_j^I, b_j^F \rangle)$ when $\langle a_{i_0 j}^T, a_{i_0 j}^I, a_{i_0 j}^F \rangle > \langle b_j^T, b_j^I, b_j^F \rangle$ for some j. Again, since $S(A, b) \neq \emptyset$, by Lemma 4.1,
$\max_i(\langle a_{ij_0}^T, a_{ij_0}^I, a_{ij_0}^F \rangle > \langle b_{j_0}^T, b_{j_0}^I, b_{j_0}^F \rangle)$ for each j_0.
Hence, $\langle b_{j_0}^T, b_{j_0}^I, b_{j_0}^F \rangle \neq \langle \max_i \min(a_{ij_0}^T, y_i^T), \max_i \min(a_{ij_0}^I, y_i^I), \min_i \max(a_{ij_0}^F, y_i^F)$, which contradicts our assumption $y \in S(A, b)$.

Therefore, $\bar{x}$ is the maximum solution of the system of NSLEs $A \otimes x = b$.

Example 4.8. Given

$$A = \begin{bmatrix} \langle 0.7 \ 0.6 \ 0.3 \rangle & \langle 0.6 \ 0.5 \ 0.4 \rangle & \langle 0.5 \ 0.4 \ 0.5 \rangle \\ \langle 0.5 \ 0.4 \ 0.5 \rangle & \langle 0.6 \ 0.5 \ 0.4 \rangle & \langle 0.8 \ 0.7 \ 0.2 \rangle \end{bmatrix} \text{ and}$$

$$b = \begin{bmatrix} \langle 0.5 \ 0.4 \ 0.5 \rangle \\ \langle 0.6, 0.5, 0.4 \rangle \end{bmatrix}.$$

From the definition of maximum solution,
$x_1 = \langle 0.5 \ 0.4 \ 0.5 \rangle, x_2 = \langle 0.6 \ 0.5 \ 0.4 \rangle$,
$x_3 = \langle 0.5 \ 0.4 \ 0.5 \rangle$. So $\bar{x} = [\langle 0.5 \ 0.4 \ 0.5 \rangle, \langle 0.6 \ 0.5 \ 0.4 \rangle, \langle 0.5 \ 0.4 \ 0.5 \rangle]^T$. Thus, $S(A, b) \neq \phi$ and $A \otimes \bar{x} = b$ hold. Hence $x = [\langle 0.5 \ 0.4 \ 0.5 \rangle, \langle 0.6 \ 0.5 \ 0.4 \rangle, \langle 0.5 \ 0.4 \ 0.5 \rangle]^t = \bar{x}$ is the maximum solution.

Now we consider the definition 2.10 of Moore-Penrose Inverse.

Theorem 4.9. Let us consider a system of NSLEs (1). The system must have a solution, that is, must be consistent if the coefficient NSM A is a symmetric and idempotent of order n.

Proof: Since A is symmetric and idempotent square NSM, that is A itself is a Moore-Penrose inverse. That is, $A = A^{+}$. So in the case the solution will be
$x = A^{+}b = Ab$.

Example 4.10. Consider the system of NSLEs $A \otimes x = b$ where,

$$A = \begin{bmatrix} \langle 0.8\ 0.7\ 0.2\rangle & \langle 0.6\ 0.5\ 0.4\rangle \\ \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.7\ 0.6\ 0.3\rangle \end{bmatrix} \text{ and}$$

$$b = \begin{bmatrix} \langle 0.8\ 0.7\ 0.2\rangle \\ \langle 0.6, 0.5, 0.4\rangle \end{bmatrix}.$$

Here, $A^{T} = A$ and $A^{2} = A$, that is, the NSM A is symmetric and idempotent. So the Moore-Penrose inverse A^{+} of A is itself A. Then the solution will be
$x = A^{+}b = Ab = [\langle 0.8\ 0.7\ 0.2\rangle, \langle 0.6, 0.5, 0.4\rangle]^{t}$.

5. Chebychev Approximation

In this section, we describe an algorithm by which we approach the right hand side of the system of NSLEs $A \otimes x = b$ by successively changing the original NSM $A \in \mathcal{N}_{m\times n}$ to a NSM $D \in \mathcal{N}_{m\times n}$ such that $D \otimes x = b$ is solvable.

Let us consider the solution or tolerable solution $x^{'}(A; b)$ of the system of NSLEs

$$A \otimes x = b \ as \ x^{'}(A; b) = \begin{cases} \langle 1, 1, 0\rangle \ if \ \langle a^{T}_{ij}, a^{I}_{ij}, a^{F}_{ij}\rangle \leq \langle b^{T}_{i}, b^{I}_{i}, b^{F}_{i}\rangle \ \forall i \\ \min\{\langle b^{T}_{i}, b^{I}_{i}, b^{F}_{i}\rangle\} \ if \ \langle a^{T}_{ij}, a^{I}_{ij}, a^{F}_{ij}\rangle > \langle b^{T}_{i}, b^{I}_{i}, b^{F}_{i}\rangle \ (3) \end{cases}$$

Now if we define that the system (1) is solvable if and only if (3) is its solution, that is $A \otimes x^{'}(A, b) = b$ holds, but in general $A \otimes x^{'}(A; b) \leq b$ holds always. So our aim is, by changing the NSM A and retain the right hand side of the system same to make the system solvable.

First we have to define some importent Definitions.

Definition 5.1. The Chebychev distance of two NSMs $A, B \in \mathcal{N}_{(m\times n)}$ is denoted by $\rho(A, B)$ and is defined by
$\rho(A, B) = \langle \max\limits_{i,j} |a^{T}_{ij} - b^{T}_{ij}|, \max\limits_{i,j} |a^{I}_{ij} - b^{I}_{ij}|, \min\limits_{ij} |a^{F}_{i,j} - b^{F}_{ij}|\rangle$.
The Chebychev distance of a NSM $A \in \mathcal{N}_{(m\times n)}$ and the set $S \in \mathcal{N}_{(m\times n)}$ is defined by $\rho(A, S) = \inf\limits_{B\in S} \rho(A, B)$.

Definition 5.2. We say that a NSM $B \in \mathcal{N}_{(m\times n)}$ is closer to a NSV $v \in \mathcal{N}_{(m)}$ than a NSM $A \in \mathcal{N}_{(m\times n)}$ if
$\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \leq \langle b^T_{ij}, b^I_{ij}, b^F_{ij}\rangle \leq \langle v^T_i, v^I_i, v^F_i\rangle$ or $\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \geq \langle b^T_{ij}, b^I_{ij}, b^F_{ij}\rangle \geq \langle v^T_i, v^I_i, v^F_i\rangle$ for all indices $i \in M$ and $j \in N$ and we denote by $A \rightarrow B \leftarrow v$.

Lemma 5.3. Let us consider two NSMs $A, C \in \mathcal{N}_{(m\times n)}$ and the NSV $b \in \mathcal{N}_{(m)}$ such that $A \rightarrow C \leftarrow b$. Then $x'(C; b) \geq x'(A; b)$.

Proof: From the definition of the solution of the system of NSLEs of the form $A \otimes x = b$ we have,

$$x'(C;b) = \begin{cases} \langle 1,1,0\rangle \; if \; \langle c^T_{ij}, c^I_{ij}, c^F_{ij}\rangle \leq \langle b^T_i, b^I_i, b^F_i\rangle \; \forall i \\ \min\{\langle b^T_i, b^I_i, b^F_i\rangle\} \; if \; \langle c^T_{ij}, c^I_{ij}, c^F_{ij}\rangle > \langle b^T_i, b^I_i, b^F_i\rangle \end{cases}$$

and

$$x'(A;b) = \begin{cases} \langle 1,1,0\rangle \; if \; \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \leq \langle b^T_i, b^I_i, b^F_i\rangle \; \forall i \\ \min\{\langle b^T_i, b^I_i, b^F_i\rangle\} \; if \; \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle > \langle b^T_i, b^I_i, b^F_i\rangle. \end{cases}$$

Now, as $A \rightarrow C \leftarrow b$, we have
$\{i; \langle c^T_{ij}, c^I_{ij}, c^F_{ij}\rangle > \langle b^T_i, b^I_i, b^F_i\rangle\} \subseteq \{i; \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle > \langle b^T_i, b^I_i, b^F_i\rangle\}$ for each $j \in N$. So $x'(C; b) \geq x'(A; b)$.

Lemma 5.4. Let A and C be two NSMs of order $(m \times n)$ and $b \in \mathcal{N}_{(m)}$ be a NSV with $A \rightarrow C \leftarrow b$. If $A \otimes x = b$ is solvable then $C \otimes x = b$ is solvable.

Proof: From our assumption, solvability of $A \otimes x = b$ means that $A \otimes x'(A, b) = b$. Then i^{th} equation of which gives,

$$\sum_{j=1}^{n} \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \otimes x'_j(A; b) = b_i. \; (4)$$

Le us suppose that in (4) the equality has been achieved in term k.
Thus, $\langle a^T_{ik}, a^I_{ik}, a^F_{ik}\rangle \otimes x'(A; b) = b_i$ which is only possible if
$\langle a^T_{ik}, a^I_{ik}, a^F_{ik}\rangle \geq \langle b^T_i, b^I_i, b^F_i\rangle$ as well as $x'_k(X; b) \geq b_i$.

Since, $A \rightarrow C \leftarrow b$, we get $\langle a^T_{ik}, a^I_{ik}, a^F_{ik}\rangle \geq \langle c^T_{ik}, c^I_{ik}, c^F_{ik}\rangle \geq \langle b^T_i, b^I_i, b^F_i\rangle$ and Lemma 5.3 gives, $x'_k(C; b) \geq x'_k(A; b) \geq b_i$. This implies, $\langle c^T_{ik}, c^I_{ik}, c^F_{ik}\rangle \otimes x'_k(C; b) \geq b_j$. Again for any NSM $C, C \otimes x'(C; b) \leq b_i$.
Hence the only possibility is, $C \otimes x'(C; b) = b$, that is, $C \otimes b = b$ is solvable.

Lemma 5.5. Let us consider the system of NSLE $A \otimes x = b$ and $x'(A; b)$ be its tolerable solution. If there exists a NSM D such that, $D \otimes x = b$ is solvable with $\rho(A, D) = \delta$, then there exists NSM C such that $A \rightarrow C \leftarrow b$ and $\rho(A, C) \leq \delta$ with $C \otimes x = b$ is solvable.

Proof: The NSM C can be chosen in three different way.

Case-1: If $\langle b_i^T, b_i^I, b_i^F\rangle \leq \langle a_i^T, a_i^I, a_i^F\rangle \leq \langle d_i^T, d_i^I, d_i^F\rangle$ or
$\langle b_i^T, b_i^I, b_i^F\rangle \geq \langle a_i^T, a_i^I, a_i^F\rangle \geq \langle d_i^T, d_i^I, d_i^F\rangle$, we set

$$\begin{aligned} c_{ij} = \langle c_{ij}^T, c_{ij}^I, c_{ij}^F\rangle &= \langle \max\{b_i^T, a_{ij}^T - (d_{ij}^T - a_{ij}^T)\}, \max\{b_i^I, a_{ij}^I - (d_{ij}^I - a_{ij}^I)\}, \\ &\quad \min\{b_i^F, a_{ij}^F + (a_{ij}^F - d_{ij}^F)\}\rangle \\ &= \langle \max\{b_i^T, (2a_{ij}^T - d_{ij}^T)\}, \max\{b_i^I, (2a_{ij}^I - d_{ij}^I)\}, \\ &\quad \min\{b_i^F, (2a_{ij}^F - d_{ij}^F)\}\rangle, \text{ or} \end{aligned}$$

$$\begin{aligned} c_{ij} = \langle c_{ij}^T, c_{ij}^I, c_{ij}^F\rangle &= \langle \min\{b_i^T, a_{ij}^T + (a_{ij}^T - d_{ij}^T)\}, \min\{b_i^I, a_{ij}^I + (a_{ij}^I - d_{ij}^I)\}, \\ &\quad \max\{b_i^F, a_{ij}^F - (d_{ij}^F - a_{ij}^F)\}\rangle \\ &= \langle \min\{b_i^T, (2a_{ij}^T - d_{ij}^T)\}, \min\{b_i^I, (2a_{ij}^I - d_{ij}^I)\}, \\ &\quad \max\{b_i^F, (2a_{ij}^F - d_{ij}^F)\}\rangle, \end{aligned}$$

respectively.

Case-2: If $\langle a_i^T, a_i^I, a_i^F\rangle \leq \langle d_i^T, d_i^I, d_i^F\rangle \leq \langle b_i^T, b_i^I, b_i^F\rangle$ or

$$\langle a_i^T, a_i^I, a_i^F\rangle \geq \langle d_i^T, d_i^I, d_i^F\rangle \geq \langle b_i^T, b_i^I, b_i^F\rangle,$$

then take $c_{ij} = d_{ij}$

Case-3: If $\langle a_i^T, a_i^I, a_i^F\rangle \leq \langle b_i^T, b_i^I, b_i^F\rangle \leq \langle d_i^T, d_i^I, d_i^F\rangle$ or

$$\langle a_i^T, a_i^I, a_i^F\rangle \geq \langle b_i^T, b_i^I, b_i^F\rangle \geq \langle d_i^T, d_i^I, d_i^F\rangle,$$

then take $c_{ij} = b_{ij}$

Now from the construction of C by the above three cases, it is obviouse that $\rho(A;C) \leq \delta$ and $A \to C \leftarrow b$. More over, $D \to C \leftarrow b$, hence by Lemma 5.4, $C \otimes x = b$ is solvable.

Definition 5.6. For a given NSM $A \in \mathcal{N}_{(m\times n)}$ and the NSV $b \in \mathcal{N}_{(n)}$ we denote the NSM $D \in \mathcal{N}_{(m\times n)}$ by $(A, \Delta \to b)$ such that for each $i \in \{1,2,3,...,m\}$ and $j \in \{1,2,3,...,n\}$,

$$\langle d_{ij}^T, d_{ij}^I, d_{ij}^F\rangle = \begin{cases} \min\{a_{ij}^T + \Delta^T, b_i^T\}, \min\{a_{ij}^I + \Delta^I, b_i^I\}, \max\{a_{ij}^F - \Delta^F, b_i^F\} \ if \ a_{ij} < b_i \\ \max\{a_{ij}^T - \Delta^T, b_i^T\}, \max\{a_{ij}^I - \Delta^I, b_i^I\}, \min\{a_{ij}^F + \Delta^F, b_i^F\} \ if \ a_{ij} \geq b_i \end{cases}$$

It is obvious that, $A \to (A, \Delta \to b) \leftarrow b$ for any non-negative $\Delta = \langle \Delta^T, \Delta^I, \Delta^F\rangle$. More over as Δ increase, we finally arrive at a NSM D such that $d_{ij} = b_i$ for all $i \in M, j \in N$, which satisfy the condition, $D \otimes x'(D;b) = b$. So computation of the NSM D is an iterative process, which can be described by the following flowchart.

Algorithm **MATRIX**

begin $k = 0; \Delta_k = \langle 0, 0, 0 \rangle;\ A(\Delta_k) = A;$

compute $x^{'}(A; b);$

If $A \otimes x^{'}(A; b) \neq b$ then

repeat $\Delta_{k+1} = \langle \Delta^T_{k+1}, \Delta^I_{k+1}, \Delta^F_{k+1} \rangle$

$= \langle \Delta^T_k + \min\{|A(\delta_k)_{ij} - b^T_i|; A(\delta^T_k)_{ij} \neq b^T_i\},$

$\Delta^I_k + \min\{|A(\delta_k)_{ij} - b^I_i|; A(\delta^I_k)_{ij} \neq b^I_i\},$

$\Delta^F_k + \min\{|A(\delta_k)_{ij} - b^F_i|; A(\delta^F_k)_{ij} \neq b^F_i\}\rangle,$

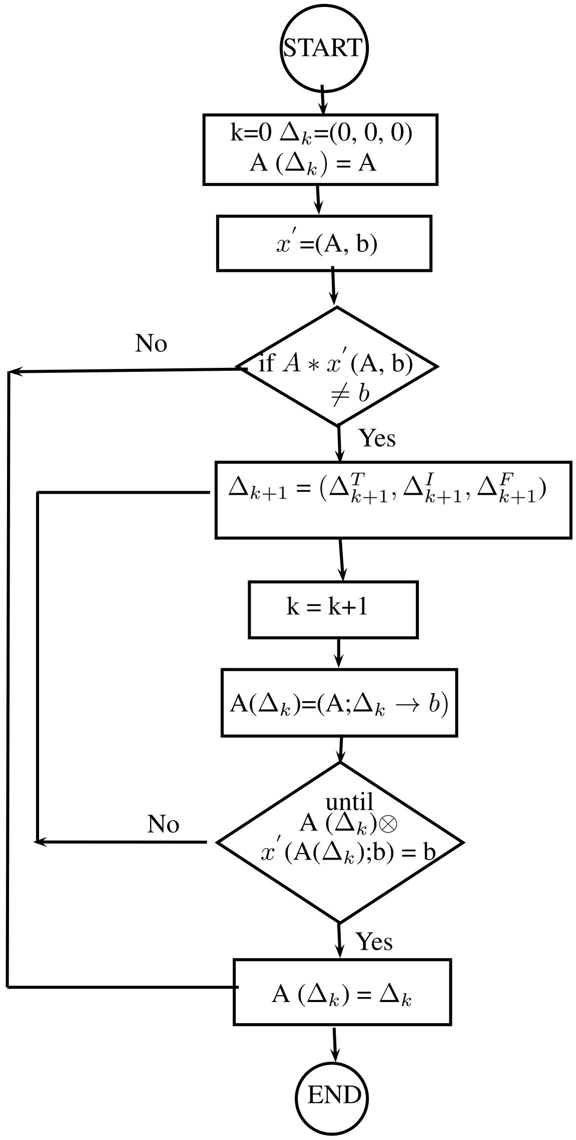
$k = k + 1;$

$A(\Delta_k) = (A; \delta_k \to b)$

until $A(\delta_k) \otimes x^{'}(A(\delta_k); b) = b;$

output: $A(\delta_k); \Delta_k$

end MATRIX.



The following example illustrate the concept of the above flowchart.

Let us consider the system of NSLEs $A \otimes x = b$ where,

Example 5.7. $A = \begin{bmatrix} \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.4\ 0.5\ 0.6\rangle & \langle 0.2\ 0.1\ 0.8\rangle \\ \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.2\ 0.1\ 08\rangle & \langle 0.9\ 0.8\ 0.1\rangle & \langle 0.1\ 0.1\ 0.9\rangle & \langle 0.6\ 0.5\ 0.4\rangle \\ \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.8\ 0.7\ 0.2\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.2\ 0.1\ 0.8\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.3\ 0.2\ 0.7\rangle \end{bmatrix}$

and

$$b = \begin{bmatrix} \langle 0.4\ 0.3\ 0.6\rangle \\ \langle 0.9\ 0.8\ 0.1\rangle \\ \langle 0.3\ 0.2\ 0.7\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle \end{bmatrix}.$$

The corresponding tolerable solution will be
$x^{'}(A;b) = [\langle 0.5\ 0.4\ 0.5\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 0.5\ 0.4\ 0.5\rangle]^t$ but

$A \otimes x^{'}(A;b) \leq b$ so the system is unsolvable.

In the first iteration,

$\Delta_1\langle 0.1, 0.1, 0.9\rangle, A(\Delta_1) =$

$$\begin{bmatrix} \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.4\ 0.5\ 0.6\rangle & \langle 0.3\ 0.2\ 0.7\rangle \\ \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.3\ 0.2\ 07\rangle & \langle 0.9\ 0.8\ 0.1\rangle & \langle 0.2\ 0.1\ 0.8\rangle & \langle 0.7\ 0.6\ 0.3\rangle \\ \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.3\ 0.2\ 0.7\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.6\ 0.5\ 0.4\rangle \end{bmatrix}$$

and
$x^{'}(A(\Delta_1);b) = [\langle 1\ 1\ 0\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 0.5\ 0.4\ 0.5\rangle, \langle 0.5\ 0.4\ 0.5\rangle]^t$.

Here , $A \otimes x^{'}(A(\Delta_1);b) \leq b$.
In the second iteration,

$\Delta_2 = \langle 0.2\ 0.2\ 0.8\rangle, A(\Delta_2) =$

$$\begin{bmatrix} \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.4\ 0.5\ 0.6\rangle & \langle 0.4\ 0.3\ 0.6\rangle \\ \langle 0.9\ 0.8\ 0.1\rangle & \langle 0.5\ 0.4\ 05\rangle & \langle 0.9\ 0.8\ 0.1\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.9\ 0.8\ 0.1\rangle \\ \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.3\ 0.2\ 0.7\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.5\ 0.4\ 0.5\rangle \end{bmatrix}$$

and
$x^{'}(A(\Delta_2);b) = [\langle 1\ 1\ 0\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 1\ 1\ 0\rangle, \langle 1\ 1\ 0\rangle, \langle 1\ 1\ 0\rangle]^t$.

In this case, $A \otimes x^{'}(A(\Delta_2);b) = b$. So $D = A(\Delta_2)$ is the Chebychev best approximation of the coefficient NSM A of the given system and $x^{'}(A(\Delta_2);b)$ is the principal solution.

6. **Conclusion**

In this piece of work, we try to find the conditions under which a system of NSLE is solvable. We have provided necessary examples to describe the theory. Further using the Chebychev approximation discussed the principal solution when the given system (1) has no solution. As a future work we are trying to apply this theory in all operation research problems.

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Some Fundamental Operations on Interval Valued Neutrosophic Hypersoft Set with Their Properties

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Abstract: Multi-criteria decision-making (MCDM) focuses on coordination, choice and planning issues, including multi-criteria. the neutrosophic soft set cannot handle environments involving multiple attributes. In order to overcome these obstacles, the neutrosophic hypersoft set (NHSS) and Interval Value neutrosophic hypersoft set (IVNHSS) are defined. In this paper, we extend the concept of IVNHSS with basic properties. We also developed some basic operations on IVNHSS such as union, intersection, addition, difference, Truth-favorite, and False-favorite, etc. with their desirable properties. Finally, the necessity and possibility operations on IVNHSS with properties are presented in the following research.

Keywords: Soft set; Neutrosophic Set; Interval-valued neutrosophic set; Hypersoft set; Interval-valued neutrosophic hypersoft set.

1. Introduction

Anxiety performs a dynamic part in lots of areas of life such as modeling, medicine, and engineering. However, people have raised a general question, that is, how can we verbalize anxiety in mathematical modeling. Several investigators all over the world have recommended and advised different methodologies to minimize uncertainty. First of all, Zadeh planned the idea of fuzzy sets [1] to resolve these complications which contain anxiety as well as ambiguity. It is seen that sometimes; fuzzy sets can't deal with scenarios. To overcome such scenarios, Turksen [2] suggested the concept of interval-valued fuzzy sets (IVFS). In some cases, we need to debate the suitable representation of the object under the circumstances of anxiety and uncertainty, and regard its unbiased membership value and non-membership value of the suitable representation of the object, that cannot be processed by these fuzzy sets or IVFS. To overcome such concerns, Atanassov projected the theory of IFS in [3]. The theory proposed by Atanassov only considers membership and non-membership values to deal with insufficient data, but the IFS theory cannot deal with incompatible and imprecise information. To deal with this incompatible and imprecise data, Smarandache proposed the idea of NS [4]. Molodtsov [5] proposed a general mathematical tool to deal with uncertain, ambiguous, and undefined substances, called soft sets (SS). Maji et al. [6] extended the work of SS and defined some operations and their attributes. In [7], they also use SS theory to make decisions. Ali et al. [8] Modified the Maji method of SS and developed some new operations with its properties. In [9], they proved De Morgan's SS theory and law by using different operators. Cagman and Enginoglu [10] proposed the concept of soft matrices with operations and discussed their properties. They also introduced a decision-making method to solve problems that contain uncertainty. In [11], they modified the actions proposed by Molodtsov's SS. In [12], the author proposed some new operations for soft matrices, such as soft difference product, soft restricted difference product, soft extended difference product, and weak extended difference product.

Maji [13] put forward the idea of NSS with necessary operations and characteristics. The idea of Possibility NSS was proposed by Karaaslan [14] and introduced a neutrosophic soft decision method to solve those uncertain problems based on And-product. Broumi [15] developed a generalized NSS with certain operations and properties and used the proposed concept for decision-making. To solve the MCDM problem with single-valued neutrosophic numbers proposed by Deli and Subas in [16], they constructed the concept of the cut set of single-valued neutrosophic numbers. Based on the correlation of IFS, the term correlation coefficient of SVNS is introduced [17]. In [18], the idea of simplifying NS introduced some algorithms and aggregation operators, such as weighted arithmetic operators and weighted geometric average operators. They constructed the MCDM method based on the proposed aggregation operator. Zulqarnain et al. [19] extended the fuzzy TOPSIS technique to the Neutrosophic TOPSIS technique and used the developed approach to solve the MCDM problem. Abdel-basset et al [20] presented the integration of TOPSIS methodology decision-making test as well as evaluation laboratory (DEMATEL) solution (TOPSIS) CIIC environment delivers a new method to pick out the proper project. Abdel-basset Mohamed [21] developed an MCDM model to discover along with display screen cancer addressing obscure, anxiety, the incompleteness of reported signs as well as handicapping apparently within cancer or replaceable ailments in the signs and symptoms. Abdel-Basset et al. [22] raised the issue of assessment of the smart emergency response techniques is interpreted as MCDM problem. they suggested a framework by combining three common MCDM strategies which are AHP, TOPSIS, and VIKOR.

All the above-mentioned studies cannot deal with the problems in which attributes of the alternates have their corresponding sub-attributes. To handle such compilations Smarandache [23] generalized the SS to HSS by converting the function to a multi-attribute function to deal with uncertainty. Saqlain et al. [24] developed the generalization of TOPSIS for the NHSS, by using accuracy function they transformed the fuzzy neutrosophic numbers to crisp form. Zulqarnain et al. [25] extended the notion of NHSSs and presented the generalized operations for NHSSs, they also developed the necessity and possibility operations and discussed their desirable features. In [26], the author's proposed the fuzzy Plithogenic hypersoft set in matrix form with some basic operations and properties. Saqlain et al. [27] proposed the aggregate operators on NHSS. In [28], the author extended the NHSS approach and introduced IVNHSS, m-polar, and m-polar IVNHSS. Zulqarnain et al. [29] presented the intuitionistic fuzzy hypersoft set, they developed the TOPSIS technique by developing a correlation coefficient to solve multi-attribute decision making problems. Many other novel researchers are done under neutrosophic environment and their applications in everyday life [30-34].

The following research is organized as follows: Some basic definitions recalled in section 2, which are used in the following research such as SS, NS, NSS, HSS, NHSS, and IVNHSS. We present different operators on IVNHSS such as union, intersection, addition, difference, extended union, extended intersection, truth-favorite, and false-favorite operations in section 3 with properties and prove the De Morgan laws by using union and intersection operators. We also proposed the necessity and possibility operators, OR, and operations with some properties in section 4.

2. Preliminaries

In this section, we recollect some basic definitions such as SS, NSS, NHSS, and IVNHSS which use in the following sequel.

Definition 2.1 [5]

The soft set is a pair (F, Λ) over $\mathbb{U}$ if and only if F: $\Lambda \rightarrow P(\mathbb{U})$ is a mapping. That is the parameterized family of subsets of $\mathbb{U}$ known as a SS.

Definition 2.2 [4]

Let $\mathbb{U}$ be a universe and Λ be an NS on $\mathbb{U}$ is defined as $\Lambda = \{< u, u_A(u), v_A(u), w_A(u) > : u \in \mathbb{U}\}$, where $u,\ v,\ w$: $\mathbb{U} \rightarrow\ ]0^-,\ 1^+[$ and $0^- \leq u_\Lambda(u) + v_\Lambda(u) + w_\Lambda(u) \leq 3^+$.

Definition 2.3 [13]

Let $\mathbb{U}$ and Ĕ̦ are universal set and set of attributes respectively. Let P($\mathbb{U}$) be the set of Neutrosophic values of $\mathbb{U}$ and $\Lambda \subseteq$ Ĕ̦. A pair (F, Λ) is called an NSS over $\mathbb{U}$ and its mapping is given as

F: $\Lambda \rightarrow (\mathbb{U})$

Definition 2.4 [35]

Let $\mathbb{U}$ be a universal set, then interval valued neutrosophic set can be expressed by the set $\boldsymbol{A} = \{< \boldsymbol{u}, \boldsymbol{u_A}(\boldsymbol{u}), \boldsymbol{v_A}(\boldsymbol{u}), \boldsymbol{w_A}(\boldsymbol{u}) > : \boldsymbol{u} \in \mathbb{U}\}$, where $\boldsymbol{u_A},\ \boldsymbol{v_A}$, and $\boldsymbol{w_A}$ are truth, indeterminacy and falsity membership functions for $\boldsymbol{A}$ respectively, $\boldsymbol{u_A},\ \boldsymbol{v_A}$, and $\boldsymbol{w_A} \subseteq [0, 1]$ for each $\boldsymbol{u} \in \mathbb{U}$. Where

$\boldsymbol{u_A}(\boldsymbol{u}) = \left[\boldsymbol{u_A^L}(\boldsymbol{u}), \boldsymbol{u_A^U}(\boldsymbol{u})\right]$

$\boldsymbol{v_A}(\boldsymbol{u}) = \left[\boldsymbol{v_A^L}(\boldsymbol{u}), \boldsymbol{v_A^U}(\boldsymbol{u})\right]$

$\boldsymbol{w_A}(\boldsymbol{u}) = \left[\boldsymbol{w_A^L}(\boldsymbol{u}), \boldsymbol{w_A^U}(\boldsymbol{u})\right]$

For each point $\boldsymbol{u} \in \mathbb{U}$, $0 \leq \boldsymbol{u_A}(\boldsymbol{u}) + \boldsymbol{v_A}(\boldsymbol{u}) + \boldsymbol{w_A}(\boldsymbol{u}) \leq 3$ and IVN($\mathbb{U}$) represents the family of all interval valued neutrosophic sets.

Definition 2.5 [23]

Let $\mathbb{U}$ be a universal set and $P(\mathbb{U})$ be a power set of $\mathbb{U}$ and for $n \geq 1$, there are n distinct attributes such as $k_1,\ k_2,\ k_3,\ \ldots,\ k_n$ and $K_1,\ K_2,\ K_3,\ \ldots,\ K_n$ are sets for corresponding values attributes respectively with following conditions such as $K_i \cap K_j = \emptyset\ (i \neq j)$ and $i, j \in \{1,2,3 \ldots n\}$. Then the pair (F, $K_1 \times K_2 \times K_3 \times \ldots \times K_n$) is said to be HSS over $\mathbb{U}$ where F is a mapping from $K_1 \times K_2 \times K_3 \times \ldots \times K_n$ to $P(\mathbb{U})$.

Definition 2.6 [23]

Let $\mathbb{U}$ be a universal set and $P(\mathbb{U})$ be a power set of $\mathbb{U}$ and for $n \geq 1$, there are n distinct attributes such as $k_1,\ k_2,\ k_3,\ \ldots,\ k_n$ and $K_1,\ K_2,\ K_3,\ \ldots,\ K_n$ are sets for corresponding values attributes respectively with following conditions such as $K_i \cap K_j = \emptyset\ (i \neq j)$ and $i, j \in \{1,2,3 \ldots n\}$. Then the pair (F, Λ) is said to be NHSS over $\mathbb{U}$ if there exists a relation $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \Lambda$. F is a mapping from $K_1 \times K_2 \times K_3 \times \ldots \times K_n$ to $P(\mathbb{U})$ and F$(K_1 \times K_2 \times K_3 \times \ldots \times K_n) = \{< u, u_A(u), v_A(u), w_A(u) > : u \in \mathbb{U}\}$ where u, v, w are membership values for truthness, indeterminacy and falsity respectively such that $u,\ v,\ w$: $\mathbb{U} \rightarrow\ ]0^-,\ 1^+[$ and $0^- \leq u_\Lambda(u) + v_\Lambda(u) + w_\Lambda(u) \leq 3^+$.

Definition 2.7 [28]

Let $\mathbb{U}$ be a universal set and $P(\mathbb{U})$ be a power set of $\mathbb{U}$ and for $n \geq 1$, there are n distinct attributes such as $k_1,\ k_2,\ k_3,\ \ldots,\ k_n$ and $K_1,\ K_2,\ K_3,\ \ldots,\ K_n$ are sets for corresponding values attributes respectively with following conditions such as $K_i \cap K_j = \emptyset\ (i \neq j)$ and $i, j \in \{1,2,3 \ldots n\}$. Then the pair $(F,\ A)$ is said to be IVNHSS over $\mathbb{U}$ if there exists a relation $K_1 \times K_2 \times K_3 \times \ldots \times K_n = A$. Where

F: $K_1 \times K_2 \times K_3 \times \ldots \times K_n \rightarrow (\mathbb{U})$ and

$F(K_1 \times K_2 \times K_3 \times \ldots \times K_n) = \{< u, [u_A^L(u), u_A^U(u)],\ [v_A^L(u), v_A^U(u)],\ [w_A^L(u), w_A^U(u)] > : u \in \mathbb{U}\}$, where $u_A^L,\ v_A^L$, and w_A^L are lower and $u_A^U,\ v_A^U$, and w_A^U are upper membership values for truthiness, indeterminacy, and falsity respectively for A and $[u_A^L(u), u_A^U(u)]$, $[v_A^L(u), v_A^U(u)]$, $[w_A^L(u), w_A^U(u)] \subseteq [0, 1]$ and $0 \leq sup u_A(u) + sup v_A(u) + sup\ w_A(u) \leq 3$ for each $u \in \mathbb{U}$.

Example 1 Assume $\mathbb{U} = \{u_1,\ u_2\}$ be a universe of discourse and $E = \{x_1,\ x_2,\ x_3,\ x_4\}$ be a set of attributes. Consider F_A be an IVNHSS over $\mathbb{U}$ can be expressed as follows

$$F_A = \{(x_1, \{\langle u_1, [.6, .8], [.5, 0.9], [.1, .4]\rangle, \langle u_2, [.4, .7], [.3, .9], [.2, .6]\rangle\}),$$
$$(x_2, \{\langle u_1, [.4, .7], [.3, .9], [.3, .5]\rangle, \langle u_2, [0, .3], [.6, .8], [.3, .7]\rangle\}),$$
$$(x_3, \{\langle u_1, [.2, .9], [.1, .5], [.7, .8]\rangle, \langle u_2, [.4, .9], [.1, .6], [.5, .7]\rangle\}),$$
$$(x_4, \{\langle u_1, [.6, .9], [.6, .9], [1, 1]\rangle, \langle u_2, [.5, .9], [.6, .8], [.1, .8]\rangle\})\}.$$

Tablur representation of IVNHSS F_A over $\mathbb{U}$ given as follows

Table 1: Tablur representation of IVNHSS F_A

$\mathbb{U}$	u_1	u_1
x_1	$\langle [.6, .8], [.5, .9], [.1, .4]\rangle$	$\langle [.4, .7], [.3, .9], [.2, .6]\rangle$
x_2	$\langle [.4, .7], [.3, .9], [.3, .5]\rangle$	$\langle [0, .3], [.6, .8], [.3, .7]\rangle$
x_3	$\langle [.2, .9], [.1, .5], [.7, .8]\rangle$	$\langle [.4, .9], [.1, .6], [.5, .7]\rangle$
x_4	$\langle [.6, .9], [.6, .9], [1, 1]\rangle$	$\langle [.5, .9], [.6, .8], [.1, .8]\rangle$

3. Operations on Interval Valued Neutrosophic Hypersoft Set with Properties

In this section, we extend the concept of IVNHSS and introduce some fundamental operations on IVNHSS with their properties.

Definition 3.1

Let F_A and $G_B \in$ IVNHSS over $\mathbb{U}$, then $F_A \subseteq G_B$ if

$inf\,u_A(u) \leq inf\,u_B(u),\ sup\,u_A(u) \leq sup\,u_B(u)$

$inf\,v_A(u) \geq inf\,v_B(u),\ sup\,v_A(u) \geq sup\,v_B(u)$

$inf\,w_A(u) \geq inf\,w_B(u),\ sup\,w_A(u) \geq sup\,w_B(u)$

Example 2 Assume $\mathbb{U} = \{u_1,\ u_2\}$ be a universe of discourse and $E = \{x_1,\ x_2,\ x_3,\ x_4\}$ be a set of attributes. Consider G_B be an IVNHSS over $\mathbb{U}$ can be expressed as follows and F_A given in example 1

$$G_B = \{(x_1, \{\langle u_1, [.6, .9], [.3, .7], [.1, .3]\rangle, \langle u_2, [.6, .9], [.3, .5], [.1, .4]\rangle\}),$$
$$(x_2, \{\langle u_1, [.6, .8], [.2, .5], [.2, .3]\rangle, \langle u_2, [.3, .5], [.4, .7], [.1, .4]\rangle\}),$$
$$(x_3, \{\langle u_1, [.4, .9], [.1, .3], [.4, .6]\rangle, \langle u_2, [.6, 1], [.1, .4], [.3, .4]\rangle\}),$$
$$(x_4, \{\langle u_1, [.7, .9], [.4, .6], [.6, 1]\rangle, \langle u_2, [.5, .7], [.4, .7], [.1, .4]\rangle\})\}.$$

Thus

$F_A \subseteq G_B$.

Definition 3.2

Let $F_A \in$ IVNHSS over $\mathbb{U}$, then

i. Empty IVNHSS can be represented as $F_{\breve{0}}$, and defined as follows $F_{\breve{0}} = \{< u, [0, 0],\ [1, 1],\ [1, 1] >: u \in \mathbb{U}\}$.
ii. Universal IVNHSS can be represented as $F_{\breve{E}}$, and defined as follows $F_{\breve{E}} = \{< u, [0, 0],\ [1, 1],\ [1, 1] >: u \in \mathbb{U}\}$.

iii. The complement of IVNHSS can be defined as follows $F_A^c = \{< u, [w_A^L(u), w_A^U(u)], [1 - v_A^U(u), 1 - v_A^L(u)], [u_A^L(u), u_A^U(u)] >: u \in \mathbb{U}\}$.

Example 3 Assume $\mathbb{U} = \{u_1, u_2\}$ be a universe of discourse and $E = \{x_1, x_2, x_3, x_4\}$ be a set of attributes. The tabular representation of $F_{\breve{0}}$ and $F_{\breve{E}}$ given as follows in table 2 and table 3 respectively.

Table 2:Tablur representation of IVNHSS $F_{\breve{0}}$

$\mathbb{U}$	u_1	u_1
x_1	$\langle[0,0],[1,1],[1,1]\rangle$	$\langle[0,0],[1,1],[1,1]\rangle$
x_2	$\langle[0,0],[1,1],[1,1]\rangle$	$\langle[0,0],[1,1],[1,1]\rangle$
x_3	$\langle[0,0],[1,1],[1,1]\rangle$	$\langle[0,0],[1,1],[1,1]\rangle$
x_4	$\langle[0,0],[1,1],[1,1]\rangle$	$\langle[0,0],[1,1],[1,1]\rangle$

Table 3:Tablur representation of IVNHSS $F_{\breve{E}}$

$\mathbb{U}$	u_1	u_1
x_1	$\langle[1,1],[0,0],[0,0]\rangle$	$\langle[1,1],[0,0],[0,0]\rangle$
x_2	$\langle[1,1],[0,0],[0,0]\rangle$	$\langle[1,1],[0,0],[0,0]\rangle$
x_3	$\langle[1,1],[0,0],[0,0]\rangle$	$\langle[1,1],[0,0],[0,0]\rangle$
x_4	$\langle[1,1],[0,0],[0,0]\rangle$	$\langle[1,1],[0,0],[0,0]\rangle$

Proposition 3.3

If $F_A \in$ IVNHSS, then

1. $(F_A^c)^c = F_A$
2. $(F_{\breve{0}})^c = F_{\breve{E}}$
3. $(F_{\breve{E}})^c = F_{\breve{0}}$

Proof 1 Let $F_A = \{< u, [u_A^L(u), u_A^U(u)], [v_A^L(u), v_A^U(u)], [w_A^L(u), w_A^U(u)] >: u \in \mathbb{U}\}$ be an IVNHSS. Then by using definition 3.3(iii), we have

$F_A^c = \{< u, [w_A^L(u), w_A^U(u)], [1 - v_A^U(u), 1 - v_A^L(u)], [u_A^L(u), u_A^U(u)] >: u \in \mathbb{U}\}$

Thus

$(F_A^c)^c = \{< u, [u_A^L(u), u_A^U(u)], [1 - (1 - v_A^L(u)), 1 - (1 - v_A^U(u))], [w_A^L(u), w_A^U(u)] >: u \in \mathbb{U}\}$

$(F_A^c)^c = \{< u, [u_A^L(u), u_A^U(u)], [v_A^L(u), v_A^U(u)], [w_A^L(u), w_A^U(u)] >: u \in \mathbb{U}\}$

$(F_A^c)^c = F_A$

Proof 2

As we know that $F_{\breve{0}} = \{< u, [0,0], [1,1], [1,1] >: u \in \mathbb{U}\}$

By using definition 3.3(iii), we get

$(F_{\breve{0}})^c = \{< u, [1,1], [0,0], [0,0] >: u \in \mathbb{U}\} = F_{\breve{E}}$.

Similarly, we can prove 3.

Definition 3.4

Let F_A and G_B ∈ IVNHSS over $\mathbb{U}$, then

$$F_A \cup G_B = \left\{\begin{matrix} (< u, [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}], \\ [min\{inf\,v_A(u),\ inf\,v_B(u)\}, min\{sup\,v_A(u),\ sup\,v_B(u)\}], \\ [min\{inf\,w_A(u),\ inf\,w_B(u)\}, min\{sup\,w_A(u),\ sup\,w_B(u)\}] >/ u \in \mathbb{U}) \end{matrix}\right\}. \quad (1)$$

Example 4 Assume $\mathbb{U} = \{u_1,\ u_2\}$ be a universe of discourse and $E = \{x_1,\ x_2,\ x_3,\ x_4\}$ be a set of attributes. Consider F_A and G_B are IVNHSS over $\mathbb{U}$ can be given as follows

$$\begin{aligned} F_A = \{&(x_1, \{\langle u_1, [.6, .8], [.5, .9], [.1, .4]\rangle, \langle u_2, [.4, .7], [.3, .9], [.2, .6]\rangle\}), \\ &(x_2, \{\langle u_1, [.4, .7], [.3, .9], [.3, .5]\rangle, \langle u_2, [.2, .8], [.6, .8], [.3, .7]\rangle\}), \\ &(x_3, \{\langle u_1, [.2, .9], [.1, .5], [.4, .7]\rangle, \langle u_2, [.4, .9], [.1, .6], [.5, .7]\rangle\}), \\ &(x_4, \{\langle u_1, [.6, .9], [.6, .9], [1, 1]\rangle, \langle u_2, [.5, .9], [.6, .8], [.1, .8]\rangle\})\} \end{aligned}$$

$$\begin{aligned} G_B = \{&(x_1, \{\langle u_1, [.5, .7], [.5, .7], [.4, .6]\rangle, \langle u_2, [.3, .9], [.3, .6], [.4, .7]\rangle\}), \\ &(x_2, \{\langle u_1, [.3, .8], [.4, .5], [.4, .9]\rangle, \langle u_2, [.4, .7], [.5, .9], [.4, .6]\rangle\}), \\ &(x_3, \{\langle u_1, [.3, .5], [.2, .6], [.3, .8]\rangle, \langle u_2, [.3, 1], [.2, .7], [.3, .8]\rangle\}), \\ &(x_4, \{\langle u_1, [.4, .6], [.7, .8], [.4, 1]\rangle, \langle u_2, [.4, .8], [.3, .6], [.2, .6]\rangle\})\} \end{aligned}$$

Then

$$\begin{aligned} F_A \cup G_B = \{&(x_1, \{\langle u_1, [.6, .8], [.5, .7], [.1, .4]\rangle, \langle u_2, [.4, .9], [.3, .6], [.2, .6]\rangle\}), \\ &(x_2, \{\langle u_1, [.4, .8], [.3, .5], [.3, .5]\rangle, \langle u_2, [.4, .8], [.5, .8], [.3, .6]\rangle\}), \\ &(x_3, \{\langle u_1, [.3, .9], [.1, .5], [.3, .7]\rangle, \langle u_2, [.4, 1], [.1, .6], [.3, .7]\rangle\}), \\ &(x_4, \{\langle u_1, [.6, .9], [.6, .8], [.4\ 1]\rangle, \langle u_2, [.5, .9], [.3, .6], [.1, .6]\rangle\})\} \end{aligned}$$

Proposition 3.5

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, $\mathcal{H}_{\breve{C}}$ ∈ IVNHSS over $\mathbb{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{0}} = \mathcal{F}_{\breve{0}}$
3. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{E}} = \mathcal{F}_{\breve{A}}$
4. $\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cup \mathcal{F}_{\breve{A}}$
5. $(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \cup \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \cup (\mathcal{G}_{\breve{B}} \cup \mathcal{H}_{\breve{C}})$

Proof By using definition 3.4 we can prove easily.

Definition 3.6

Let F_A and G_B ∈ IVNHSS over $\mathbb{U}$, then

$$F_A \cap G_B = \left\{\begin{matrix} < u, [min\{inf\,u_A(u),\ inf\,u_B(u)\}, min\{sup\,u_A(u),\ sup\,u_B(u)\}], \\ [max\{inf\,v_A(u),\ inf\,v_B(u)\}, max\{sup\,v_A(u),\ sup\,v_B(u)\}], \\ [max\{inf\,w_A(u),\ inf\,w_B(u)\}, max\{sup\,w_A(u), sup\,w_B(u)\}] >/ u \in \mathbb{U}) \end{matrix}\right\}. \quad (2)$$

Example 5 Reconsider example 4

$$\begin{aligned} F_A = \{&(x_1, \{\langle u_1, [.6, .8], [.5, .9], [.1, .4]\rangle, \langle u_2, [.4, .7], [.3, .9], [.2, .6]\rangle\}), \\ &(x_2, \{\langle u_1, [.4, .7], [.3, .9], [.3, .5]\rangle, \langle u_2, [.2, .8], [.6, .8], [.3, .7]\rangle\}), \end{aligned}$$

$(x_3, \{\langle u_1, [.2, .9], [.1, .5], [.4, .7]\rangle, \langle u_2, [.4, .9], [.1, .6], [.5, .7]\rangle\}),$

$(x_4, \{\langle u_1, [.6, .9], [.6, .9], [1, 1]\rangle, \langle u_2, [.5, .9], [.6, .8], [.1, .8]\rangle\})\}$

$G_B = \{(x_1, \{\langle u_1, [.5, .7], [.5, .7], [.4, .6]\rangle, \langle u_2, [.3, .9], [.3, .6], [.4, .7]\rangle\}),$

$(x_2, \{\langle u_1, [.3, .8], [.4, .5], [.4, .9]\rangle, \langle u_2, [.4, .7], [.5, .9], [.4, .6]\rangle\}),$

$(x_3, \{\langle u_1, [.3, .5], [.2, .6], [.3, .8]\rangle, \langle u_2, [.3, 1], [.2, .7], [.3, .8]\rangle\}),$

$(x_4, \{\langle u_1, [.4, .6], [.7, .8], [.4, 1]\rangle, \langle u_2, [.4, .8], [.3, .6], [.2, .6]\rangle\})\}$

Then

$F_A \cap G_B = \{(x_1, \{\langle u_1, [.5, .7], [.5, .9], [.4, .6]\rangle, \langle u_2, [.3, .7], [.3, .9], [.4, .7]\rangle\}),$

$(x_2, \{\langle u_1, [.3, .7], [.4, .9], [.4, .9]\rangle, \langle u_2, [.2, .7], [.6, .9], [.4, .7]\rangle\}),$

$(x_3, \{\langle u_1, [.2, .5], [.2, .6], [.4, .8]\rangle, \langle u_2, [.3, .9], [.2, .7], [.5, .8]\rangle\}),$

$(x_4, \{\langle u_1, [.4, .6], [.7, .9], [1, 1]\rangle, \langle u_2, [.4, .8], [.6, .8], [.2, .8]\rangle\})\}$

Proposition 3.7

Let $\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}, \mathcal{H}_{\breve{C}} \in$ IVNHSS over $\mathbb{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{0}} = \mathcal{F}_{\breve{A}}$
3. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{E}} = \mathcal{F}_{\breve{E}}$
4. $\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cap \mathcal{F}_{\breve{A}}$
5. $(\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) \cap \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \cap (\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}})$

Proof By using definition 3.6 we can prove easily.

Proposition 3.8

Let F_A and $G_B \in$ IVNHSS over $\mathbb{U}$, then

1. $(F_A \cup G_B)^C = {F_A}^C \cap {G_B}^C$
2. $(F_A \cap G_B)^C = {F_A}^C \cup {G_B}^C$

Proof 1 As we know that

$F_A = \{< u, u_A(u), v_A(u), w_A(u) > : u \in \mathbb{U}\}$ and $G_B = \{< u, u_B(u), v_B(u), w_B(u) > : u \in \mathbb{U}\}$. Where

$u_A(u) = [inf\, u_A(u),\ sup\, u_A(u)]$ or $[u_A^L(u), u_A^U(u)]$, $u_A^L(u) = inf\, u_A(u)$ and $u_A^U(u) = sup\, u_A(u)$

$v_A(u) = [inf\, v_A(u),\ sup\, v_A(u)]$ or $[v_A^L(u), v_A^U(u)]$, $v_A^L(u) = inf\, v_A(u)$ and $v_A^U(u) = sup\, v_A(u)$

$w_A(u) = [inf\, w_A(u),\ sup\, w_A(u)]$ or $[w_A^L(u), w_A^U(u)]$, $w_A^L(u) = inf\, w_A(u)$ and $w_A^U(u) = sup\, w_A(u)$

$u_B(u) = [inf\, u_B(u),\ sup\, u_B(u)]$ or $[u_B^L(u), u_B^U(u)]$, $u_B^L(u) = inf\, u_B(u)$ and $u_B^U(u) = sup\, u_B(u)$

$v_B(u) = [inf\, v_B(u),\ sup\, v_B(u)]$ or $[v_B^L(u), v_B^U(u)]$, $v_B^L(u) = inf\, v_B(u)$ and $v_B^U(u) = sup\, v_B(u)$

$w_B(u) = [inf\, w_B(u),\ sup\, w_B(u)]$ or $[w_B^L(u), w_B^U(u)]$, $w_B^L(u) = inf\, w_B(u)$ and $w_B^U(u) = sup\, w_B(u)$

Then by using Equation 1

$$F_A \cup G_B = \left\{ \begin{array}{c} (< u, [max\{inf\, u_A(u),\ inf\, u_B(u)\}, max\{sup\, u_A(u),\ sup\, u_B(u)\}], \\ [min\{inf\, v_A(u),\ inf\, v_B(u)\}, min\{sup\, v_A(u),\ sup\, v_B(u)\}], \\ [min\{inf\, w_A(u),\ inf\, w_B(u)\}, min\{sup\, w_A(u),\ sup\, w_B(u)\}] > / \ u \in \mathbb{U}) \end{array} \right\}$$

By using definition 3.3(iii), we get

$$(F_A \cup G_B)^C = \left\{\begin{array}{c} (< u, [min\{inf\,w_A(u),\ inf\,w_B(u)\}, min\{sup\,w_A(u),\ sup\,w_B(u)\}], \\ [1 - min\{sup\,v_A(u),\ sup\,v_B(u)\}, 1 - min\{inf\,v_A(u),\ inf\,v_B(u)\}], \\ [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}] >/ u \in \mathbb{U}) \end{array}\right\}$$

Now

$$F_A{}^C = \{< u, [inf\,w_A(u), sup\,w_A(u)], [1 - sup\,v_A(u), 1 - inf\,v_A(u)], [inf\,u_A(u), sup\,u_A(u)] > : u \in \mathbb{U}\}$$

$$G_B{}^C = \{< u, [inf\,w_B(u), sup\,w_B(u)], [1 - sup\,v_B(u), 1 - inf\,v_B(u)], [inf\,u_B(u), sup\,u_B(u)] > : u \in \mathbb{U}\}$$

$$F_A{}^C \cap G_B{}^C = \left\{\begin{array}{c} (< u, [min\{inf\,w_A(u),\ inf\,w_B(u)\}, min\{sup\,w_A(u),\ sup\,w_B(u)\}], \\ [max\{1 - sup\,v_A(u),\ 1 - sup\,v_B(u)\}, max\{1 - inf\,v_A(u),\ 1 - inf\,v_B(u)\}], \\ [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}] >/ u \in \mathbb{U}) \end{array}\right\}$$

$$F_A{}^C \cap G_B{}^C = \left\{\begin{array}{c} (< u, [min\{inf\,w_A(u),\ inf\,w_B(u)\}, min\{sup\,w_A(u),\ sup\,w_B(u)\}], \\ [1 - min\{sup\,v_A(u), sup\,v_B(u)\}, 1 - min\{inf\,v_A(u),\ inf\,v_B(u)\}], \\ [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}] >/ u \in \mathbb{U}) \end{array}\right\}$$

Hence

$(F_A \cup G_B)^C = F_A{}^C \cap G_B{}^C$

Proof 2

Similar to assertion 1.

Proposition 3.9

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, $\mathcal{H}_{\breve{C}} \in$ IVNHSS over $\mathbb{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cup (\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}}) = (\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \cap (\mathcal{F}_{\breve{A}} \cup \mathcal{H}_{\breve{C}})$
2. $\mathcal{F}_{\breve{A}} \cap (\mathcal{G}_{\breve{B}} \cup \mathcal{H}_{\breve{C}}) = (\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) \cup (\mathcal{F}_{\breve{A}} \cap \mathcal{H}_{\breve{C}})$
3. $\mathcal{F}_{\breve{A}} \cup (\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) = \mathcal{F}_{\breve{A}}$
4. $\mathcal{F}_{\breve{A}} \cap (\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) = \mathcal{F}_{\breve{A}}$

Proof 1 From Equation 2, we have

$$\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}} = \left\{\begin{array}{c} (< u, [min\{inf\,u_B(u),\ inf\,u_C(u)\}, min\{sup\,u_B(u),\ sup\,u_C(u)\}], \\ [max\{inf\,v_B(u),\ inf\,v_C(u)\}, max\{sup\,v_B(u),\ sup\,v_C(u)\}], \\ [max\{inf\,w_B(u),\ inf\,w_C(u)\}, max\{sup\,w_B(u),\ sup\,w_C(u)\}] >/ u \in \mathbb{U}) \end{array}\right\}$$

$\mathcal{F}_{\breve{A}} \cup (\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}})=$

$$\left\{\begin{array}{c} (< u, [\max\{inf\,u_A(u), min\{inf\,u_B(u),\ inf\,u_C(u)\}\}, \max\{sup\,u_A(u), min\{sup\,u_B(u), sup\,u_C(u)\}\}], \\ [\min\{inf\,v_A(u), max\{inf\,v_B(u),\ inf\,v_C(u)\}\}, \min\{sup\,v_A(u), max\{sup\,v_B(u), sup\,v_C(u)\}\}], \\ [min\{inf\,w_A(u), max\{inf\,w_B(u),\ inf\,w_C(u)\}\}, min\{sup\,w_A(u), max\{sup\,w_B(u),\ sup\,w_C(u)\}\}] >/ u \in \mathbb{U}) \end{array}\right\}$$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \left\{\begin{array}{c} (< u, [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}], \\ [min\{inf\,v_A(u),\ inf\,v_B(u)\}, min\{sup\,v_A(u),\ sup\,v_B(u)\}], \\ [min\{inf\,w_A(u),\ inf\,w_B(u)\}, min\{sup\,w_A(u),\ sup\,w_B(u)\}] >/ u \in \mathbb{U}) \end{array}\right\}.$$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{H}_{\breve{C}} = \left\{\begin{array}{c} (< u, [max\{inf\,u_A(u),\ inf\,u_C(u)\}, max\{sup\,u_A(u),\ sup\,u_C(u)\}], \\ [min\{inf\,v_A(u),\ inf\,v_C(u)\}, min\{sup\,v_A(u),\ sup\,v_C(u)\}], \\ [min\{inf\,w_A(u),\ inf\,w_C(u)\}, min\{sup\,w_A(u),\ sup\,w_C(u)\}] >/ u \in \mathbb{U}) \end{array}\right\}.$$

$(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \cap (\mathcal{F}_{\breve{A}} \cup \mathcal{H}_{\breve{C}}) =$

$$\left\{\begin{array}{c}(<u,[\min\{max\{inf\,\mathcal{u}_A(u),\ inf\,\mathcal{u}_B(u)\},max\{inf\,\mathcal{u}_A(u),\ inf\,\mathcal{u}_C(u)\},\min\{max\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\},max\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_C(u)\}],\\ [\max\{min\{inf\,\mathcal{v}_A(u),\ inf\,\mathcal{v}_B(u)\},min\{inf\,\mathcal{v}_A(u),\ inf\,\mathcal{v}_C(u)\},\max\{min\{sup\mathcal{v}_A(u),\ sup\mathcal{v}_B(u)\},min\{sup\mathcal{v}_A(u),\ sup\mathcal{v}_C(u)\}],\\ [max\{min\{inf\,\mathcal{w}_A(u),\ inf\,\mathcal{w}_B(u)\},min\{inf\,\mathcal{w}_A(u),\ inf\,\mathcal{w}_C(u)\},max\{min\,\{sup\mathcal{w}_A(u),\ sup\mathcal{w}_B(u)\},sup\mathcal{w}_A(u),\ sup\mathcal{w}_C(u)\}]>/\,u\in\mathbb{U})\end{array}\right\}$$

$(\mathcal{F}_{\breve{A}}\ \cup\ \mathcal{G}_{\breve{B}})\ \cap\ (\mathcal{F}_{\breve{A}}\ \cup\ \mathcal{H}_{\breve{C}}) =$

$$\left\{\begin{array}{c}(<u,[\max\{inf\,\mathcal{u}_A(u),min\{inf\,\mathcal{u}_B(u),\ inf\,\mathcal{u}_C(u)\}\},\max\{sup\mathcal{u}_A(u),min\{sup\mathcal{u}_B(u),sup\mathcal{u}_C(u)\}\}],\\ [\min\{inf\,\mathcal{v}_A(u),max\{inf\,\mathcal{v}_B(u),\ inf\,\mathcal{v}_C(u)\}\},\min\{sup\mathcal{v}_A(u),max\{sup\mathcal{v}_B(u),sup\mathcal{v}_C(u)\}\}],\\ [min\{inf\,\mathcal{w}_A(u),max\{inf\,\mathcal{w}_B(u),\ inf\,\mathcal{w}_C(u)\}\},min\{sup\mathcal{w}_A(u),max\{sup\mathcal{w}_B(u),\ sup\mathcal{w}_C(u)\}\}]>/\,u\in\mathbb{U})\end{array}\right\}$$

Hence

$\mathcal{F}_{\breve{A}}\ \cup\ (\mathcal{G}_{\breve{B}}\cap\ \mathcal{H}_{\breve{C}}) = (\mathcal{F}_{\breve{A}}\ \cup\ \mathcal{G}_{\breve{B}})\ \cap\ (\mathcal{F}_{\breve{A}}\ \cup\ \mathcal{H}_{\breve{C}})$.

Similarly, we can prove other results.

Definition 3.10

Let $F_A,\ G_B\ \in$ IVNHSS, then their extended union is

$$\mathcal{u}\ (F_A\ \cup\ G_B) = \begin{cases}[inf\,\mathcal{u}_A(u),sup\mathcal{u}_A(u)] & if\ u\ \in\ A-B\\ [inf\,\mathcal{u}_B(u),sup\mathcal{u}_B(u)] & if\ u\ \in\ B-A\\ [max\{inf\,\mathcal{u}_A(u),\ inf\,\mathcal{u}_B(u)\},max\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\}] & if\ u\ \in A\cap B\end{cases}$$

$$\mathcal{v}\ (F_A\ \cup\ G_B) = \begin{cases}[inf\,\mathcal{v}_A(u),sup\mathcal{v}_A(u)] & if\ u\ \in\ A-B\\ [inf\,\mathcal{v}_B(u),sup\mathcal{v}_B(u)] & if\ u\ \in\ B-A\\ [min\{inf\,\mathcal{v}_A(u),\ inf\,\mathcal{v}_B(u)\},min\{sup\mathcal{v}_A(u),\ sup\mathcal{v}_B(u)\}] & if\ u\ \in A\cap B\end{cases}$$

$$\mathcal{w}\ (F_A\ \cup\ G_B) = \begin{cases}[inf\,\mathcal{w}_A(u),sup\mathcal{w}_A(u)] & if\ u\ \in\ A-B\\ [inf\,\mathcal{w}_B(u),sup\mathcal{w}(u)] & if\ u\ \in\ B-A\\ [min\{inf\,\mathcal{w}_A(u),\ inf\,\mathcal{w}_B(u)\},min\{sup\mathcal{w}_A(u),\ sup\mathcal{w}_B(u)\}] & if\ u\ \in A\cap B\end{cases}$$

Definition 3.11

Let $F_A,\ G_B\ \in$ IVNHSS, then their extended intersection is

$$\mathcal{u}\ (F_A\ \cap\ G_B) = \begin{cases}[inf\,\mathcal{u}_A(u),sup\mathcal{u}_A(u)] & if\ u\ \in\ A-B\\ [inf\,\mathcal{u}_B(u),sup\mathcal{u}_B(u)] & if\ u\ \in\ B-A\\ [min\{inf\,\mathcal{u}_A(u),\ inf\,\mathcal{u}_B(u)\},min\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\}] & if\ u\ \in A\cap B\end{cases}$$

$$\mathcal{v}\ (F_A\ \cap\ G_B) = \begin{cases}[inf\,\mathcal{v}_A(u),sup\mathcal{v}_A(u)] & if\ u\ \in\ A-B\\ [inf\,\mathcal{v}_B(u),sup\mathcal{v}_B(u)] & if\ u\ \in\ B-A\\ [max\{inf\,\mathcal{v}_A(u),\ inf\,\mathcal{v}_B(u)\},max\{sup\mathcal{v}_A(u),\ sup\mathcal{v}_B(u)\}] & if\ u\ \in A\cap B\end{cases}$$

$$\mathcal{w}\ (F_A\ \cap\ G_B) = \begin{cases}[inf\,\mathcal{w}_A(u),sup\mathcal{w}_A(u)] & if\ u\ \in\ A-B\\ [inf\,\mathcal{w}_B(u),sup\mathcal{w}(u)] & if\ u\ \in\ B-A\\ [max\{inf\,\mathcal{w}_A(u),\ inf\,\mathcal{w}_B(u)\},max\{sup\mathcal{w}_A(u),\ sup\mathcal{w}_B(u)\}] & if\ u\ \in A\cap B\end{cases}$$

Definition 3.12

Let F_A and $G_B\ \in$ IVNHSS over $\mathbb{U}$, then their difference defined as follows

$$F_A\ \setminus\ G_B\ = \left\{\begin{array}{c}(<u,[min\{inf\,\mathcal{u}_A(u),\ inf\,\mathcal{u}_B(u)\},min\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\}],\\ [max\{inf\,\mathcal{v}_A(u),\ 1-\sup\mathcal{v}_B(u)\},max\{sup\mathcal{v}_A(u),\ 1-\inf\mathcal{v}_B(u)\}],\\ [max\{inf\,\mathcal{w}_A(u),\ inf\,\mathcal{w}_B(u)\},max\{sup\mathcal{w}_A(u),\ sup\mathcal{w}_B(u)\}]>/\,u\in\mathbb{U})\end{array}\right\}. \quad (3)$$

Example 6 Reconsider example 4

$$F_A\ \setminus\ G_B\ = \{(x_1,\{\langle u_1,[.5,.7],[.5,.9],[.4,.6]\rangle,\langle u_2,[.3,.7],[.4,.9],[.4,.7]\rangle\}),$$
$$(x_2,\{\langle u_1,[.3,.7],[.5,.9],[.4,.9]\rangle,\langle u_2,[.2,.7],[.6,.8],[.4,.7]\rangle\}),$$

$$(x_3, \{\langle u_1, [.2, .5], [.4, .8], [.4, .8]\rangle, \langle u_2, [.3, .9], [.3, .8], [.5, .8]\rangle\}),$$
$$(x_4, \{\langle u_1, [.4, .6], [.6, .9], [1, 1]\rangle, \langle u_2, [.4, .8], [.6, .8], [.2, .8]\rangle\})\}$$

Definition 3.13

Let F_A and G_B ∈ IVNHSS over $\mathbb{U}$, then their addition defined as follows

$$F_A + G_B = \left\{ \begin{matrix} (< u, [min\{inf\,u_A(u) + inf\,u_B(u), 1\}, min\{sup\,u_A(u) + sup\,u_B(u), 1\}], \\ [min\{inf\,v_A(u) + inf\,v_B(u), 1\}, min\{sup\,v_A(u) + sup\,v_B(u), 1\}], \\ [min\{inf\,w_A(u) + inf\,w_B(u), 1\}, min\{sup\,w_A(u) + sup\,w_B(u), 1\}] >/ u \in \mathbb{U}) \end{matrix} \right\}. \quad (4)$$

Example 7 Reconsider example 4

$$F_A + G_B = \{(x_1, \{\langle u_1, [1.0, 1.0], [1.0, 1.0], [0.5, 1.0]\rangle, \langle u_2, [0.7, 1.0], [0.6, 1.0], [0.6, 1.0]\rangle\}),$$
$$(x_2, \{\langle u_1, [0.7, 1.0], [0.7, 1.0], [0.7, 1.0]\rangle, \langle u_2, [0.6, 1.0], [1.0, 1.0], [0.7, 1.0]\rangle\}),$$
$$(x_3, \{\langle u_1, [0.5, 1.0], [0.3, 1.0], [0.7, 1.0]\rangle, \langle u_2, [0.7, 1.0], [0.3, 1.0], [0.8, 1.0]\rangle\}),$$
$$(x_4, \{\langle u_1, [1.0, 1.0], [1.0, 1.0], [1.0, 1.0]\rangle, \langle u_2, [0.9, 1.0], [0.9, 1.0], [0.3, 1.0]\rangle\})\}.$$

Definition 3.14

Let F_A ∈ IVNHSS over $\mathbb{U}$, then its scalar multiplication is represented as $F_A.\breve{a}$, where $\breve{a} \in [0, 1]$ and defined as follows

$$F_A.\breve{a} = \left\{ \begin{matrix} (< u, [min\{inf\,u_A(u).\breve{a}, 1\}, min\{sup\,u_A(u).\breve{a}, 1\}], \\ [min\{inf\,v_A(u).\breve{a}, 1\}, min\{sup\,v_A(u).\breve{a}, 1\}], \\ [min\{inf\,w_A(u).\breve{a}, 1\}, min\{sup\,w_A(u).\breve{a}, 1\}] >/ u \in \mathbb{U}) \end{matrix} \right\}. \quad (5)$$

Definition 3.15

Let F_A ∈ IVNHSS over $\mathbb{U}$, then its scalar division is represented as $F_A/\breve{a}$, where $\breve{a} \in [0, 1]$ and defined as follows

$$F_A/\breve{a} = \left\{ \begin{matrix} (< u, [min\{inf\,u_A(u)/\breve{a}, 1\}, min\{sup\,u_A(u)/\breve{a}, 1\}], \\ [min\{inf\,v_A(u)/\breve{a}, 1\}, min\{sup\,v_A(u)/\breve{a}, 1\}], \\ [min\{inf\,w_A(u)/\breve{a}, 1\}, min\{sup\,w_A(u)/\breve{a}, 1\}] >/ u \in \mathbb{U}) \end{matrix} \right\}. \quad (6)$$

Definition 3.16

Let F_A ∈ IVNHSS over $\mathbb{U}$, then Truth-Favorite operator on F_A is denoted by $\widetilde{\Delta}F_A$ and defined as follows

$$\widetilde{\Delta}F_A = \left\{ \begin{matrix} (< u, [min\{inf\,u_A(u) + inf\,v_A(u), 1\}, min\{sup\,u_A(u) + sup\,v_A(u), 1\}], [0, 0], \\ [inf\,w_A(u), sup\,w_A(u)] >/ u \in \mathbb{U}) \end{matrix} \right\}. \quad (7)$$

Example 8 Reconsider example 1

$$\widetilde{\Delta}F_A = \{(x_1, \{\langle u_1, [1, 1], [0, 0], [.1, .4]\rangle, \langle u_2, [.7, 1], [0, 0], [.2, .6]\rangle\}),$$
$$(x_2, \{\langle u_1, [.7, 1], [0, 0], [.3, .5]\rangle, \langle u_2, [.6, 1], [0, 0], [.3, .7]\rangle\}),$$
$$(x_3, \{\langle u_1, [.3, 1], [0, 0], [.7, .8]\rangle, \langle u_2, [.5, 1], [0, 0], [.5, .7]\rangle\}),$$
$$(x_4, \{\langle u_1, [1, 1], [0, 0], [1, 1]\rangle, \langle u_2, [1, 1], [0, 0], [.1, .8]\rangle\})\}$$

Proposition 3.17

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$ ∈ IVNHSS over $\mathbb{U}$, then

1. $\widetilde{\Delta}\widetilde{\Delta}\mathcal{F}_{\breve{A}} = \widetilde{\Delta}\mathcal{F}_{\breve{A}}$
2. $\widetilde{\Delta}(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \subseteq \widetilde{\Delta}\mathcal{F}_{\breve{A}} \cup \widetilde{\Delta}\mathcal{G}_{\breve{B}}$

3. $\widetilde{\Delta}(\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) \subseteq \widetilde{\Delta}\mathcal{F}_{\breve{A}} \cap \widetilde{\Delta}\mathcal{G}_{\breve{B}}$
4. $\widetilde{\Delta}(\mathcal{F}_{\breve{A}} + \mathcal{G}_{\breve{B}}) = \widetilde{\Delta}\mathcal{F}_{\breve{A}} + \widetilde{\Delta}\mathcal{G}_{\breve{B}}$

Proof of the above proposition is easily obtained by using definitions 3.4, 3.6, 3.13, and 3.16.

Definition 3.18

Let F_A ∈ IVNHSS over $\mathbb{U}$, then False-Favorite operator on F_A is denoted by $\tilde{\nabla}F_A$ and defined as follows

$$\tilde{\nabla}F_A = \left\{\begin{matrix}(< u, [inf\,u_A(u), sup\,u_A(u)], [0, 0], \\ [min\{inf\,w_A(u) + inf\,v_A(u), 1\}, min\{sup\,w_A(u) + sup\,v_A(u), 1\}] >/ u \in \mathbb{U})\end{matrix}\right\}. \quad (8)$$

Example 9 Reconsider example 1

$$\tilde{\nabla}F_A = \{(x_1, \{\langle u_1, [.6, .8], [0, 0], [.6, 1]\rangle, \langle u_2, [.4, .7], [0, 0], [.5, 1]\rangle\}),$$
$$(x_2, \{\langle u_1, [.4, .7], [0, 0], [.6, 1]\rangle, \langle u_2, [0, .3], [0, 0], [.9, 1]\rangle\}),$$
$$(x_3, \{\langle u_1, [.2, .9], [0, 0], [.8, 1]\rangle, \langle u_2, [.4, .9], [0, 0], [.6, 1]\rangle\}),$$
$$(x_4, \{\langle u_1, [.6, .9], [0, 0], [1, 1]\rangle, \langle u_2, [.5, .9], [0, 0], [.7, 1]\rangle\})\}$$

Proposition 3.19

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$ ∈ IVNHSS over $\mathbb{U}$, then

1. $\tilde{\nabla}\tilde{\nabla}\mathcal{F}_{\breve{A}} = \tilde{\nabla}\mathcal{F}_{\breve{A}}$
2. $\tilde{\nabla}(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \subseteq \tilde{\nabla}\mathcal{F}_{\breve{A}} \cup \tilde{\nabla}\mathcal{G}_{\breve{B}}$
3. $\tilde{\nabla}(\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) \subseteq \tilde{\nabla}\mathcal{F}_{\breve{A}} \cap \tilde{\nabla}\mathcal{G}_{\breve{B}}$
4. $\tilde{\nabla}(\mathcal{F}_{\breve{A}} + \mathcal{G}_{\breve{B}}) = \tilde{\nabla}\mathcal{F}_{\breve{A}} + \tilde{\nabla}\mathcal{G}_{\breve{B}}$

Proof of the above proposition is easily obtained by using definitions 3.4, 3.6, 3.13, and 3.18.

4. Necessity and Possibility Operations on IVNHSS

In this section, some further operations on IVNHSS are developed such as OR-Operation, And-Operation, necessity, and possibility operations with some properties.

Definition 4.1

Let F_A and G_B ∈ IVNHSS over $\mathbb{U}$, then OR-Operator is represented by $F_A \vee G_B$ and defined as follows

$$u\,(F_{A\times B}) = [max\{inf\,u_A(u), inf\,u_B(u)\}, max\{sup\,u_A(u), sup\,u_B(u)\}],$$
$$v\,(F_{A\times B}) = [min\{inf\,v_A(u), inf\,v_B(u)\}, min\{sup\,v_A(u), sup\,v_B(u)\}],$$
$$w\,(F_{A\times B}) = [min\{inf\,w_A(u), inf\,w_B(u)\}, min\{sup\,w_A(u), sup\,w_B(u)\}].$$

Definition 4.2

Let F_A and G_B ∈ IVNHSS over $\mathbb{U}$, then And-Operator is represented by $F_A \wedge G_B$ and defined as follows

$$u\,(F_{A\times B}) = [min\{inf\,u_A(u), inf\,u_B(u)\}, min\{sup\,u_A(u), sup\,u_B(u)\}],$$
$$v\,(F_{A\times B}) = [max\{inf\,v_A(u), inf\,v_B(u)\}, max\{sup\,v_A(u), sup\,v_B(u)\}],$$
$$w\,(F_{A\times B}) = [max\{inf\,w_A(u), inf\,w_B(u)\}, max\{sup\,w_A(u), sup\,w_B(u)\}].$$

Proposition 4.3

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, $\mathcal{H}_{\breve{C}}$ ∈ IVNHSSs, then

1. $\mathcal{F}_{\check{A}} \vee \mathcal{G}_{\check{B}} = \mathcal{G}_{\check{B}} \vee \mathcal{F}_{\check{A}}$
2. $\mathcal{F}_{\check{A}} \wedge \mathcal{G}_{\check{B}} = \mathcal{G}_{\check{B}} \wedge \mathcal{F}_{\check{A}}$
3. $\mathcal{F}_{\check{A}} \vee (\mathcal{G}_{\check{B}} \vee \mathcal{H}_{\check{C}}) = (\mathcal{F}_{\check{A}} \vee \mathcal{G}_{\check{B}}) \vee \mathcal{H}_{\check{C}}$
4. $\mathcal{F}_{\check{A}} \wedge (\mathcal{G}_{\check{B}} \wedge \mathcal{H}_{\check{C}}) = (\mathcal{F}_{\check{A}} \wedge \mathcal{G}_{\check{B}}) \wedge \mathcal{H}_{\check{C}}$
5. $(\mathcal{F}_{\check{A}} \vee \mathcal{G}_{\check{B}})^c = \mathcal{F}^c(\check{A}) \wedge \mathcal{G}^c(\check{B})$
6. $(\mathcal{F}_{\check{A}} \wedge \mathcal{G}_{\check{B}})^c = \mathcal{F}^c(\check{A}) \vee \mathcal{G}^c(\check{B})$

Proof We can prove easily by using definitions 4.1 and 4.2.

Definition 4.4

Let $F_A \in$ IVNHSS over $\mathbb{U}$, then necessity operator IVNHSS represented as $\oplus F_A$ and defined as follows

$\oplus F_A = \{< u, [inf\,u_A(u), sup\,u_A(u)], [inf\,v_A(u), sup\,v_A(u)], [1 - sup\,u_A(u), 1 - inf\,u_A(u)] > : u \in \mathbb{U}\}$

Example 10 Reconsider example 1

$$\begin{aligned}\oplus F_A = \{&(x_1, \{\langle u_1, [.6, .8], [.5, 0.9], [.2, .4]\rangle, \langle u_2, [.4, .7], [.3, .9], [.3, .6]\rangle\}),\\ &(x_2, \{\langle u_1, [.4, .7], [.3, .9], [.3, .6]\rangle, \langle u_2, [0, .3], [.6, .8], [.7, 1]\rangle\}),\\ &(x_3, \{\langle u_1, [.2, .9], [.1, .5], [.1, .8]\rangle, \langle u_2, [.4, .9], [.1, .6], [.1, .6]\rangle\}),\\ &(x_4, \{\langle u_1, [.6, .9], [.6, .9], [.1, .4]\rangle, \langle u_2, [.5, .9], [.6, .8], [.1, .5]\rangle\})\}\end{aligned}$$

Definition 4.5

Let $F_A \in$ IVNHSS over $\mathbb{U}$, then possibility operator on IVNHSS represented as $\otimes F_A$ and defined as follows

$\otimes F_A = \{(< u, [1 - sup\,w_A(u), 1 - inf\,w_A(u)],\ [inf\,v_A(u),\ sup\,v_A(u)],\ [inf\,w_A(u),\ sup\,w_A(u)] >/\ u \in \mathbb{U}\})\}$

Example 11 Reconsider example 1

$$\begin{aligned}\otimes F_A = \{&(x_1, \{\langle u_1, [.6, .9], [.5, 0.9], [.1, .4]\rangle, \langle u_2, [.4, .8], [.3, .9], [.2, .6]\rangle\}),\\ &(x_2, \{\langle u_1, [.5, .7], [.3, .9], [.3, .5]\rangle, \langle u_2, [.3, .7], [.6, .8], [.3, .7]\rangle\}),\\ &(x_3, \{\langle u_1, [.2, .3], [.1, .5], [.7, .8]\rangle, \langle u_2, [.3, .5], [.1, .6], [.5, .7]\rangle\}),\\ &(x_4, \{\langle u_1, [0, 0], [.6, .9], [1, 1]\rangle, \langle u_2, [.2, .9], [.6, .8], [.1, .8]\rangle\})\}\end{aligned}$$

Proposition 4.6

Let $\mathcal{F}_{\check{A}}$ and $\mathcal{G}_{\check{B}} \in$ IVNHSS over $\mathbb{U}$, then

1. $\oplus (\mathcal{F}_{\check{A}} \cup \mathcal{G}_{\check{B}}) = \oplus \mathcal{F}_{\check{A}} \cup \oplus \mathcal{G}_{\check{B}}$
2. $\oplus (\mathcal{F}_{\check{A}} \cap \mathcal{G}_{\check{B}}) = \oplus \mathcal{F}_{\check{A}} \cap \oplus \mathcal{G}_{\check{B}}$

Proof 1. As we know that

$\mathcal{F}_{\check{A}} = \{(< u, [inf\,u_A(u),\ sup\,u_A(u)],\ [inf\,v_A(u),\ sup\,v_A(u)],\ [inf\,w_A(u),\ sup\,w_A(u)] >/\ u \in \mathbb{U}\})\}$ and

$\mathcal{G}_{\check{B}} = \{(< u, [inf\,u_B(u),\ sup\,u_B(u)],\ [inf\,v_B(u),\ sup\,v_B(u)],\ [inf\,w_B(u),\ sup\,w_B(u)] >/\ u \in \mathbb{U}\})\}$

Then by using definition 3.5, we get

$$\mathcal{F}_{\check{A}} \cup \mathcal{G}_{\check{B}} = \left\{\begin{array}{c}(< u, [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}],\\ [min\{inf\,v_A(u),\ inf\,v_B(u)\}, min\{sup\,v_A(u),\ sup\,v_B(u)\}],\\ [min\{inf\,w_A(u),\ inf\,w_B(u)\}, min\{sup\,w_A(u),\ sup\,w_B(u)\}] >/\ u \in \mathbb{U})\end{array}\right\}.$$

By using the necessity operator, we get

$$\oplus\ (\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) = \left\{\begin{matrix} (< u, [max\{inf\mathcal{u}_A(u),\ inf\mathcal{u}_B(u)\}, max\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\}], \\ [min\{inf\mathcal{v}_A(u),\ inf\mathcal{v}_B(u)\}, min\{sup\mathcal{v}_A(u),\ sup\mathcal{v}_B(u)\}], \\ [1 - max\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\}, 1 - max\{inf\mathcal{u}_A(u),\ inf\mathcal{u}_B(u)\}] >/ u \in \mathbb{U}) \end{matrix}\right\}.$$

$\oplus\ (\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) =$

$$\left\{\begin{matrix} (< u, [max\{inf\mathcal{u}_A(u),\ inf\mathcal{u}_B(u)\}, max\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\}], \\ [min\{inf\mathcal{v}_A(u),\ inf\mathcal{v}_B(u)\}, min\{sup\mathcal{v}_A(u),\ sup\mathcal{v}_B(u)\}], \\ [min\{1 - sup\mathcal{u}_A(u),\ 1 - sup\mathcal{u}_B(u)\}, min\{1 - inf\mathcal{u}_A(u),\ 1 - inf\mathcal{u}_B(u)\}] >/ u \in \mathbb{U}) \end{matrix}\right\}.$$

$\oplus\ \mathcal{F}_{\breve{A}} = \{(< u, [inf\mathcal{u}_A(u),\ sup\mathcal{u}_A(u)],\ [inf\mathcal{v}_A(u),\ sup\mathcal{v}_A(u)],\ [1 - sup\mathcal{u}_A(u), 1 - inf\mathcal{u}_A(u)] >/ u \in \mathbb{U}\})\}$ and

$\oplus\ \mathcal{G}_{\breve{B}} = \{(< u, [inf\mathcal{u}_B(u),\ sup\mathcal{u}_B(u)],\ [inf\mathcal{v}_B(u),\ sup\mathcal{v}_B(u)],\ [1 - sup\mathcal{u}_B(u), 1 - inf\mathcal{u}_B(u)] >/ u \in \mathbb{U}\})\}$

Again, by using definition 3.5 we get

$\oplus\ \mathcal{F}_{\breve{A}} \cup \oplus\ \mathcal{G}_{\breve{B}} =$

$$\left\{\begin{matrix} (< u, [max\{inf\mathcal{u}_A(u),\ inf\mathcal{u}_B(u)\}, max\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\}], \\ [min\{inf\mathcal{v}_A(u),\ inf\mathcal{v}_B(u)\}, min\{sup\mathcal{v}_A(u),\ sup\mathcal{v}_B(u)\}], \\ [min\{1 - sup\mathcal{u}_A(u),\ 1 - sup\mathcal{u}_B(u)\}, min\{1 - inf\mathcal{u}_A(u),\ 1 - inf\mathcal{u}_B(u)\}] >/ u \in \mathbb{U}) \end{matrix}\right\}$$

Hence

$\oplus\ (\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) = \oplus\ \mathcal{F}_{\breve{A}} \cup \oplus\ \mathcal{G}_{\breve{B}}$

Similarly, we can prove assertion 2.

Proposition 4.7

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}} \in$ IVNHSS, then we have the following

1. $\oplus(\mathcal{F}_{\breve{A}} \wedge \mathcal{G}_{\breve{B}}) = \oplus\mathcal{F}_{\breve{A}} \wedge \oplus\mathcal{G}_{\breve{B}}$
2. $\oplus(\mathcal{F}_{\breve{A}} \vee \mathcal{G}_{\breve{B}}) = \oplus\mathcal{F}_{\breve{A}} \vee \oplus\mathcal{G}_{\breve{B}}$
3. $\otimes\ (\mathcal{F}_{\breve{A}} \wedge \mathcal{G}_{\breve{B}}) = \otimes\ \mathcal{F}_{\breve{A}} \wedge \otimes\ \mathcal{G}_{\breve{B}}$
4. $\otimes\ (\mathcal{F}_{\breve{A}} \vee \mathcal{G}_{\breve{B}}) = \otimes\ \mathcal{F}_{\breve{A}} \vee \otimes\ \mathcal{G}_{\breve{B}}$

Proof By using definitions 4.1, 4.2, 4.4, and 4.5 the proof of the above proposition can be obtained easily.

5. Conclusion

In this paper, we study NHSS and IVNHSS with some basic definitions and examples. We extend the work on IVNHSS and proposed some fundamental operations on IVNHSS such as union, intersection, extended union, extended intersection, addition, and difference, etc. are developed with their properties and proved the De Morgan laws by using union, intersection, OR-operation, and And-Operation. We also developed the addition, difference, scalar multiplication, Truth-Favorite, and False-Favorite operators on IVNHSS. Finally, the concept of necessity and possibility operations on IVNHSS with properties are presented. For future trends, we can develop the interval-valued neutrosophic hypersoft matrices by using proposed operations and use them for decision making. Furthermore, several other operators such as weighted average, weighted geometric, interaction weighted average, interaction weighted geometric, etc. can be developed with their decision-making approaches to solve MCDM problems.

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Interval-Valued Complex Neutrosophic Soft Set and its Applications in Decision-Making

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Abstract: In this paper, the notion of the interval valued complex neutrosophic soft set (IV-CNSS) is defined as a combination of interval valued complex neutrosophic set and soft set. Then, we introduce the IV-CNSS's operations such as union, intersection and complement. To explore the study further, we present some basic operational rules, and investigate their properties. A new algorithm is developed by transforming the interval complex neutrosophic soft values from complex space to the real space using a practical formula which gives a decision-making with a simple computational process without the need to carry out directed operations by complex numbers. This algorithm is then applied to evaluate two kinds of a certain product from a manufacturer and choose the most suitable one. IV-CNSS provides an interval-based membership structure to handle the uncertain data. This feature allows decision makers to record their hesitancy in assigning membership values which in turn best catch the obscurity and the complexity of such data.

Keywords: soft set; neutrosophic set; interval complex neutrosophic set; interval neutrosophic set; complex neutrosophic soft set.

1. Introduction

In 1999, the model of neu-trosophic set(NS) presented by Smarandache **[1]** as a popularization of fuzzy set **[3]**, intuitionistic fuzzy set **[6]**, interval-valued fuzzy sets[8]and interval-valued intuitionistic fuzzy sets **[13]** . it's also an important method and powerful tool to deal with incomplete, indeterminate, and inconsistent information in some real-life problem. The notion of neu-trosophic soft set(NSS) was grounded by Maji **[11]**, as a popularization of a soft set **[2]**, fuzzy soft set **[4]**, and intuitionist fuzzy soft set [6]. In some real life applications, such as decision-making processes, he also used this concept**[11].** This concept(NSS) deals with indeterminate data, while when the relationships are indefinite, the fuzzy soft set and the intuitionistic fuzzy soft set fail to work..Since the neu-trosophic set is difficult to use explicitly in real-life implementations, Maji,first of all proposed the idea of single-valued neu-trosophic soft set and supplied its theoretical practices and properties. Additionally, in numerous real-life problems, the degrees of membership, non membership, and ind-eterminacy of a proven statement may be suitably given by interval forms, instead of real numbers. Deli **[14],** deals with this case, proposed connotation of the interval neutrosophic soft set , which is described by the degrees of membership , falsehood membership and indeterminacy, that are values of which are intervals rather than true numbers. Mukherjee **[17]**, The numerous similarity measures of interval valued neutrosophic soft sets were presented and their application in problems with pattern recognition. Abdel-Basset et al **[19-26]**, suggested a solution to supply change problems, professional selection problems, time-cost tradeoffs, and leveling problems in construction using a neutrosophic environment. Recent studies have focused on designing systems using complex fuzzy sets**[30]** in (NSS) and (INSS)**[15,29,35,36,37,39,42**].To design and model real-life applications in a better way, the 'complex' feature is used to manage uncertainty and periodicity data at the same time. By adding to the concept of a complex fuzzy set, complex-valued non-membership grade **[30],** the definition of complex intuitionist fuzzy soft set was introduced by Kumar**[31].** A complex neutrosophic soft set was proposed by Broumi et all **[29],** which is an extension type of a complex fuzzy soft set and a complex intuitionistic soft .The complex neutrosophic soft set will deal with the redundant existence of insecurity, incompleteness, indeterminacy, inconsistency in periodic data. The advantage of complex neutrosophic soft sets over the neutrosophic soft sets is the fact that, in addition to the membership degree provided by the neutrosophic soft sets and represented in the complex neutrosophic soft sets by amplitude, The phase, which is an attribute degree characterizing the amplitude, is also given by the complex neutrosophic soft sets. Yet it is not easy to find a crisp (exact) neutrosophic soft membership degree in many real-life applications (as in the single-value neutrosophic soft set), because we deal with unclear and ambiguous details. So we must establish a new notion to solve this, which uses a neutrosophic soft membership degree interval. In this article, we first describe complex interval neutrosophical soft sets (IV-CNSSs) as a generalization the concept of the soft set, complex fuzzy soft set, interval valued complex fuzzy soft set [32,33,34], complex intuitionistic fuzzy soft set, interval complex valued intuitionistic fuzzy soft sets. We then add such definitions and operations of interval complex neutrosophic soft sets. Several properties of IV-CNSSs have been established that are related to activities. The goal of this paper is also to explore decision-making on the basis of interval-value complex neutrosophical soft sets. We develop an adaptable approach to decision-making based on interval-complex valued neutrosophic soft sets and include examples to demonstrate the established approach. the novelty of this work can be viewed:

- In this work, we have combined all of the following concepts Interval, Complex setting, Neutrosophic set, and Soft set .Thus we got a new model is interval valued complex neutrosophic soft set (IV-CNSS).

- We have used this hybrid model to solve one of the famous real-life problems, which is the decision-making problem.

The rest of this article is organized as follows. Section 2 recalls some basic concepts of neutrosophic set, soft set, complex fuzzy soft set, neutrosophic soft set, complex neutrosophic set, and their operations. Section 3 presents the formulation of the interval-valued complex neutrosophic set and some examples. Section 4 presenting Set-Theoretic Operations of Interval Valued Complex Neutrosophic Soft Set(IV-CNSs). Section 5 presenting Operational rules of operation Interval Valued Complex Neutrosophic Soft Sets (IV-CNSs). Section6 we introduce an application of our concept to a decision-making problem. Section 7 delineates conclusions and suggests further studies.

2. Preliminaries

Now, we present the basic meanings of the neutrosophic set in this section**[1,5],** soft set theory **[2],** and complex fuzzy soft set **[31,34],** signal valued complex neutrosophic soft set **[12]** that is helpful for subsequent discussions.

Let X be a space of points (objects) denoted as x with generic elements in X.

Definition 2.1. [5] A neutrosophical set A is an entity that has the structure $A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X\}$ where the functions T, I, F:[$^-0$,$^+1$], denote the membership functions of truth, indeterminacy, and falsehood, respectively, of the $x \in X$ element with regard to set. The condition must satisfy these membership functions $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$. The $T_A(x), I_A(x)$and$F_A(x)$ functions are actual normal or non-standard subsets of the [$^-0$,$^+1$] interval.

Definition 2.2. **[2]** Let U be a discourse universe and A be a parameter set. Let a power set of U be *P(U)*. A soft set over U is called an ordered pair *(F,A),* where F is a mapping given by$F : A \rightarrow P(U)$.

The parameterized family of subsets of the U set is a soft set. All $F(e)$, $e \in E$ set from this family can be interpreted as a set consisting of soft set *e*-elements (F, E) or as an *e*-approximate soft set member.

Definition 2.3. [34] Let U be an initial set and E be a set of parameters. Let $P(U)$ denote the power set of the complex fuzzy sets of U and let $A \subset E$. A pair (F, A) is called a *complex fuzzy soft set* over U, where F is a mapping given by $F : A \rightarrow P(U)$ such that

$$F(e_i) = \{(h_k, r_k(x).\, e^{iarg_k(x)}) | i \text{ is the number of parameter and } k\text{: is the number of sets}\}$$

Definition 2.4. [11] Let U be an initial universe,E be a set of parameters and let $NP(U)$ denotes the set of all neutrosophic sets. Then the pair(S, A) is termed to be the neutrosophic soft set over U , where S is a mapping given by $S: A \rightarrow NP(U)$.

Definition 2.5. [12] Let U be a universal set, E be a set of parameters under, $A \subseteq E$ and for all $x \in U$, Ψ_A is a complex neutrosophic set over U. Then a single-value complex neutrosophic soft set S_A over U is then defined as a mapping $S_A: E \rightarrow CN(U)$, Where the complex neutrosophic sets in U are denoted by *CN(U),* and $\Psi_A(x) = \emptyset$ If x *no belong* A Here $\Psi_A(x)$ is referred to here as a complex neutrosophical approximate function ofS_A and the values of S_A are referred to as the x-elements of the CNSS for all $x \in U$. Here S_A then be represented in the following manner by a series of ordered pairs:

$$S_A = \{\langle x, \Psi_A(x) \rangle : x \in E, \Psi_A(x) \in CN(U)\}$$

Where $\Psi_A(x) = (\langle x, p_A(x).\, e^{i\omega_A(x)}, q_A(x).\, e^{i\varphi_A(x)}, r_A(x).\, e^{i\gamma_A(x)} \rangle)$,

$p_A, q_A, r_A$are real-valued and lie in [0,1] and $\omega_A, \varphi_A, \gamma_A \in (0,2\pi]$. This is achieved in order to ensure that the CNSS model description refers to the original form of the complex fuzzy set on which the CNSS model is centered.

Definition 2.6. [12] Over the universe U, let S_A and S_B be two complex neutrosophic sets,we define the operations of complement, subset, union and intersection as follows.The complement of S_A, denoted by $S^c{}_{A,}$ *is a CNSS* defined by $S^c{}_A = \left\{\left(x, \Psi^c{}_{A,}(x)\right) : x \in U\right\}$, where $\Psi^c{}_A(x)$ is the complex neutrosophic complement of $\Psi_A(x)$.

It is said that S_A is a CNS – subset of S_B and denoted by $S_A \subseteq S_B$ for all $x \in U$, $\Psi_A(x) \subseteq \Psi_B(x)$, that is conditions are satisfied:

$p_A(e) \leq p_B(e), q_A(e) \leq q_B(e), r_A(e) \leq r_B(e)$

$And\ \ \omega_A(e) \leq \omega_B(e), \varphi_A(e) \leq \varphi_B(e), \gamma_A(e) \leq \gamma_B(e)$

iii) The union(intersection) of $S_A$and S_B, denoted as $S_A \cup (\cap) S_B$ is defined as: $S_C = S_A \cup (\cap) S_B = \{(x, \Psi_A\ (x) \cup (\cap) \Psi_B\ (x)) : x \in U\}$

$$S_C(e) = \begin{Bmatrix} (x, \Psi_A\ (x)) \text{ if } \ e \in A - B \\ (x, \Psi_B\ (x)) \text{ if } \ e \in B - A \\ (x, \Psi_A\ (x) \cup (\cap) \Psi_B\ (x)) \text{ if } \ e \in A \cup (\cap) B \end{Bmatrix}$$

where $C = A \cup (\cap) B, x \in U, and$

$$\Psi_A\ (x) \cup (\cap) \Psi_B\ (x) = \begin{cases} p_A(x) \vee\ (\wedge) p_B(x). e^{j(\omega_A(x) \cup (\cap) \omega_B(x))}, \\ q_A(x) \wedge\ (\vee) q_B(x). e^{j(\varphi_A(x) \cup (\cap) \varphi_B(x))}, \\ r_A(x) \wedge\ (\vee) r_B(x). e^{j(\omega_A(x) \cup (\cap) \omega_B(x))}, \end{cases}$$

where $\vee$ and $\wedge$ respectively denote the maximum and minimal operators.

3. Interval Valued Complex Neutrosophic Soft Set

The interval complex neutrosophic soft set(IV-CNSS) model, which is a combination of the IV-CNS and softset models, is presented in this section. As seen below, the formal description of this model and some definitions related to this model are:

Defintion 3.1 : Let U be an initial universe, E be a set of parameters under consideration, $A \subset E$ and IV-CNS(U) denotes the set of IV-CNS-subset of U. Then a pair $(\bar{S}, A)$ is called an interval-valued complex neutrosophic soft set in short (IV-CNSS) over U , where $\bar{S}$ is a mapping given by $\bar{S}: A \to IVCNS(U)$ such that $\bar{S}_A(x) = \{\langle\ a, T_{\bar{S}_a}(x), I_{\bar{S}_a}(x), F_{\bar{S}_a}(x)\ \rangle : x \in U, a \in A\ \}$. The IV-CNSSs model is defined over a universe of discourse U by three membership function its truth membership function $T_{\bar{S}_a}$,an indeterminate membership function $I_{\bar{S}_a}$, and falsehood membership function $F_{\bar{S}_a}$ as follows:

$$T_{\bar{S}_a}: A \to ICNS(U), T_{\bar{S}_a}(x) = t_{\bar{S}_a}(x). e^{j\alpha\omega_{\bar{S}_a}(x)}$$

$$I_{\bar{S}_a}: A \to ICNS(U), I_{\bar{S}_a}(x) = i_{\bar{S}_a}(x). e^{j\beta\psi_{\bar{\bar{S}}_a}(x)}$$

$F_{\bar{S}_a}: A \rightarrow ICNS(U), F_{\bar{S}_a}(x) = f_{\bar{S}_a}(x).e^{j\gamma\phi_{\bar{S}_a(X)}}$

In the above equations, the IV-CNS is a collection of complex neutrosophical interval sets,the function of the interval truth membership is $t_{\bar{S}_a}(x)$, $i_{\bar{S}_a}(x)$ is the interval indeterminate membership and $f_{\bar{S}_a}(x)$ is the interval falsehood membership function, while $e^{j\alpha\omega_{\bar{S}_a}(x)}$, $e^{j\beta\psi_{\bar{S}_a}(x)}$ and $e^{j\gamma\phi_{\bar{S}_a(X)}}$ are the corresponding interval-valued phase terms, respectively, with $j = \sqrt{-1}$.The scaling factors α, β and γ lie within the interval $(0,2\pi]$. This study implies that the values $\alpha, \beta, \gamma = 2\pi$. An interval-complex neutrosophic soft set can be written in set theoretical form as:

$$(\bar{S}, A) = \{a, \langle \frac{T_{\bar{S}_a}(x) = t_{\bar{S}_a}(x).e^{j\alpha\omega_{\bar{S}_a}(x)}, I_{\bar{S}_a}(x) = i_{\bar{S}_a}(x).e^{j\beta\psi_{\bar{S}_a}(x)}, F_{\bar{S}_a}(x) = f_{\bar{S}_a}(x).e^{j\gamma\phi_{\bar{S}_a(X)}}}{x} \rangle : x \in U, a \in A\}$$

In the above set theoretic form, the amplitude interval-valued terms $t_{\bar{S}_a}(x), i_{\bar{S}_a}(x)$ and $f_{\bar{S}_a}(x)$ can be further split as $t_{\bar{S}_a}(x) = [t^L{}_{\bar{S}_a}(x), t^U{}_{\bar{S}_a}(x)], i_{\bar{S}_a}(x) = [i^L{}_{\bar{S}_a}(x), i^U{}_{\bar{S}_a}(x)]$ and $f_{\bar{S}_a}(x) = [f^L{}_{\bar{S}_a}(x), f^U{}_{\bar{S}_a}(x)]$,where $t^L{}_{\bar{S}_a}(x), i^L{}_{\bar{S}_a}(x), f^L{}_{\bar{S}_a}(x)$ represent the lower bound, while $t^U{}_{\bar{S}_a}(x), i^U{}_{\bar{S}_a}(x), f^U{}_{\bar{S}_a}(x)$ represent the upper bound in each interval, respectively. likewise, for the phases: $\omega_{\bar{S}_a}(x) = [\omega^L{}_{\bar{S}_a}(x), \omega^U{}_{\bar{S}_a}(x)], \psi_{\bar{S}_a}(x) = [\psi^L{}_{\bar{S}_a}(x), \psi^U{}_{\bar{S}_a}(x)], \phi_{\bar{S}_a}(x) = [\phi^L{}_{\bar{S}_a}(x), \phi^U{}_{\bar{S}_a}(x)]$.

Example3.2 Let U be a set of developing countries in the area of West Asia (WA), considered to be a set of criteria that characterize the economic indicators of a nation, and $A = \{e_1, e_2, e_3, e_4\} \subset E$,where these sets are as defined below:

$$U = \{u_1 = Iraq, u_2 = Kingdom\ of\ Saudi\ Arabia, u_3 = Jordan, u_4 = UAE\}$$

$$E = \{e_1 = inflation\ rate, e_2 = population\ growth, e_3 = GDP\ growth\ rate, e_4 = unemployment\ rate, e_5 = export\ volume\}.$$

The IV-CNSS $\bar{S}_A(e_1), \bar{S}_A(e_2), \bar{S}_A(e_3)$ and $\bar{S}_A(e_4)$ are defined as:

$$\bar{S}_A(e_1) =$$

$$\{(\frac{[0.4,0.6].e^{j2\pi[0.5,0.6]}, [0.1,0.7].e^{j2\pi[0.1,0.3]}, [0.3,0.5].e^{j2\pi[0.8,0.9]}}{u_1}), (\frac{[0.2,0.4].e^{j2\pi[0.3,0.6]}, [0.1,0.1].e^{j2\pi[0.7,0.9]}, [0.5,0.9].e^{j2\pi[0.2,0.5]}}{u_2}),$$

$$(\frac{[0.3,0.4].e^{j2\pi[0.7,0.8]}, [0.6,0.7].e^{j2\pi[0.6,0.7]}, [0.2,0.6].e^{j2\pi[0.6,0.8]}}{u_3}), (\frac{[0,0.9].e^{j2\pi[0.9,1]}, [0.2,0.3].e^{j2\pi[0.7,0.8]}, [0.3,0.5].e^{j2\pi[0.4,0.5]}}{u_4})\}$$

$$\bar{S}_A(e_2) =$$

$$\{(\frac{[0.2,0.5].e^{j2\pi[0.4,0.7]}, [0.5,0.7].e^{j2\pi[0.2,0.3]}, [0.6,0.8].e^{j2\pi[0.6,0.9]}}{u_1}), (\frac{[0.1,0.3].e^{j2\pi[0.1,0.3]}, [0.2,0.7].e^{j2\pi[0.6,0.8]}, [0.4,0.7].e^{j2\pi[0.1,0.5]}}{u_2}),$$

$$(\frac{[0.6,0.8].e^{j2\pi[0.8,0.9]}, [0.4,0.6].e^{j2\pi[0.4,0.7]}, [0.1,0.4].e^{j2\pi[0.7,0.8]}}{u_3}), (\frac{[0.3,0.8].e^{j2\pi[0.7,0.9]}, [0,0.1].e^{j2\pi[0.7,0.7]}, [0.2,0.4].e^{j2\pi[0.6,0.8]}}{u_4})\}$$

$$\bar{S}_A(e_3) =$$

$$\{(\frac{[0.3,0.7].e^{j2\pi[0.5,0.6]},[0.1,0.7].e^{j2\pi[0.1,0.3]},[0.3,0.5].e^{j2\pi[0.8,0.9]}}{u_1}),(\frac{[0.4,0.4].e^{j2\pi[0.6,0.7]},[0.1,0.9].e^{j\pi[0.2,0.4]},[0.3,0.8].e^{j\pi[0.5,0.6]}}{u_2}),$$

$$(\frac{[0.37,0.64].e^{j\pi[0.47,0.50]},[0.36,0.57].e^{j\pi[0.64,0.7]},[0.28,0.66].e^{j\pi[0.16,0.20]}}{u_3}),(\frac{[0.15,0.52].e^{j\pi[0.1,0.2]},[0,0.5].e^{j\pi[0.6,0.7]},[0.3,0.3].e^{j\pi[0.6,0.7]}}{u_4})\}$$

$\bar{S}_A(e_4) =$

$$\{(\frac{[0.5,1].e^{j\pi[0.6,0.71]},[0.11,0.73].e^{j\pi[0.12,0.34]},[0.6,0.7].e^{j\pi[0.82,0.94]}}{u_1}),(\frac{[0.35,0.41].e^{j\pi[0.52,0.71]},[0.2,0.9].e^{j\pi[0.1,0.4]},[0.5,0.8].e^{j\pi[0.56,0.62]}}{u_2}),$$

$$(\frac{[0.21,0.63].e^{j\pi[0.41,0.55]},[0.31,0.52].e^{j\pi[0.11,0.67]},[0.17,0.49].e^{j\pi[0.26,0.40]}}{u_3}),(\frac{[0.22,0.42].e^{j\pi[0.2,0.35]},[0,3,0.65].e^{j\pi[0.36,0.88]},[0.4,0.6].e^{j\pi[0.74,0.91]}}{u_4})\}$$

Then a selection of IV-CNSS of the form can be written by the interval valued complex neutrosophic soft sets $(\bar{S}, A)$ such that $(\bar{S}, A) = \{\bar{S}_A(e_1), \bar{S}_A(e_2), \bar{S}_A(e_3), \bar{S}_A(e_4)\}$.

Definition 3.3 : Let $(\bar{S}, A) and\ (\bar{S}, B)$ be two IV-CNSSs over U. Then

$(\bar{S}, A)$ is said to be a subset of $(\bar{S}, B)$,denoted by $(\bar{S}, A) \subset (\bar{S}, B)$ iff $t^L{}_{\bar{s}_a}(x) \leq t^L{}_{\bar{s}_b}(x), i^L{}_{\bar{s}_a}(x) \leq i^L{}_{\bar{s}_b}(x)\ , f^L{}_{\bar{s}_a}(x) \leq f^L{}_{\bar{s}_b}(x)$ and $t^U{}_{\bar{s}_a}(x) \leq t^U{}_{\bar{s}_b}(x), i^U{}_{\bar{s}_a}(x) \leq t^U{}_{\bar{s}_b}(x), f^U{}_{\bar{s}_a}(x) \leq f^U{}_{\bar{s}_b}(x)$ for the amplitude terms, and $\omega^L{}_{\bar{s}_a}(x) \leq \omega^L{}_{\bar{s}_b}(x), \psi^L{}_{\bar{s}_a}(x) \leq \psi^L{}_{\bar{s}_b}(x), \phi^L{}_{\bar{s}_a}(x) \leq \phi^L{}_{\bar{s}_a}(x)$ and $\omega^U{}_{\bar{s}_a}(x) \leq \omega^U{}_{\bar{s}_b}(x), \psi^U{}_{\bar{s}_a}(x) \leq \psi^U{}_{\bar{s}_b}(x), \phi^U{}_{\bar{s}_a}(x) \leq \phi^U{}_{\bar{s}_b}(x)$ for the phase terms for all $x \in U$.

$(\bar{S}, A)$is said to be equal of $(\bar{S}, B)$,denoted by $(\bar{S}, A) = (\bar{S}, B)$ iff $t^L{}_{\bar{s}_a}(x) = t^L{}_{\bar{s}_b}(x), i^L{}_{\bar{s}_a}(x) = i^L{}_{\bar{s}_b}(x)\ , f^L{}_{\bar{s}_a}(x) = f^L{}_{\bar{s}_b}(x)$ and $t^U{}_{\bar{s}_a}(x) = t^U{}_{\bar{s}_b}(x), i^U{}_{\bar{s}_a}(x) = t^U{}_{\bar{s}_b}(x), f^U{}_{\bar{s}_a}(x) = f^U{}_{\bar{s}_b}(x)$ for the amplitude terms, and $\omega^L{}_{\bar{s}_a}(x) = \omega^L{}_{\bar{s}_b}(x), \psi^L{}_{\bar{s}_a}(x) = \psi^L{}_{\bar{s}_b}(x), \phi^L{}_{\bar{s}_a}(x) = \phi^L{}_{\bar{s}_b}(x)$ and $\omega^U{}_{\bar{s}_a}(x) = \omega^U{}_{\bar{s}_b}(x), \psi^U{}_{\bar{s}_a}(x) = \psi^U{}_{\bar{s}_b}(x), \phi^U{}_{\bar{s}_a}(x) = \phi^U{}_{\bar{s}_b}(x)$for the phase terms for all $x \in U$.

Definition 3.4: $(\bar{S}, A)$ is said to be a null IV-CNSS, denoted by $(\bar{S}, A)_\phi$if for all $x \in U, a \in A$,the amplitude and phase terms of the membership function are given by $t^L{}_{\bar{s}_a}(x) = t^U{}_{\bar{s}_a}(x), i^L{}_{\bar{s}_a}(x) = i^U{}_{\bar{s}_a}(x)\ , f^L{}_{\bar{s}_a}(x) = f^U{}_{\bar{s}_a}(x) = 0\ and\ \omega^L{}_{\bar{s}_a}(x) = \omega^U{}_{\bar{s}_a}(x), \psi^L{}_{\bar{s}_a}(x) = \psi^U{}_{\bar{s}_a}(x), \phi^L{}_{\bar{s}_a}(x) = \phi^U{}_{\bar{s}_a}(x) = 0\pi$,respectively

Definition 3.5: $(\bar{S}, A)$ is said to be an absolute IV-CNSS, denoted by $(\bar{S}, A)_\delta$if for all $x \in U$, the amplitude and phase terms of the membership function are given by

$t^L{}_{\bar{s}_a}(x) = t^U{}_{\bar{s}_a}(x), i^L{}_{\bar{s}_a}(x) = i^U{}_{\bar{s}_a}(x)\ , f^L{}_{\bar{s}_a}(x) = f^U{}_{\bar{s}_a}(x) = 1$ and $\omega^L{}_{\bar{s}_a}(x) = \omega^U{}_{\bar{s}_a}(x), \psi^L{}_{\bar{s}_a}(x) = \psi^U{}_{\bar{s}_a}(x), \phi^L{}_{\bar{s}_a}(x) = \phi^U{}_{\bar{s}_a}(x) = 2\pi$,respectively

The IV-CNSS effectively decreases to a crisp collection of the universe U in each of the cases defined in Definitions 3.4 and 3.5, i.e. $(\bar{S}, A)_{\Phi} = \left\{ \left(\frac{[0,0.].e^{j\pi[0.0]},[0.0].e^{j\pi[0,0]},[0,0].e^{j\pi[0,0]}}{x} \right) \right\} = \left\{ \left(\frac{0,0,0}{x} \right) \right\}.$

and $(\bar{S}, A)_{\delta} = \left\{ \left(\frac{[1,1].e^{j2\pi},[1.1].e^{j2\pi},[1,1].e^{j2\pi}}{x} \right) \right\} = \left\{ \left(\frac{1,1,1}{x} \right) \right\}.$

4. Set Theoretic Operations of Interval Valued Complex Neutrosophic Soft Set

The basic set of theoretical operations on IV-CNSS, namely the complement, union intersection, are described in this section.

Definition 4.1: Let $(\bar{S}, A) and\ (\bar{S}, B)$ be two IV-CNSSs over U. The union of $(\bar{S}, A)$and $(\bar{S}, B)$ is an IV-CNSS $(\bar{S}, C)$ where $= A \cup B$, $a \in A, b \in B$ and $x \in U$. to define the union we consider three cases :

Case 1: if $c \in A - B$.then

$$T_{\bar{S}_a}(x) = [inf t_{\bar{S}_a}(x), sup t_{\bar{S}_a}(x)].e^{j\alpha\omega_{\bar{S}_a}(x)}$$

$$I_{\bar{S}_a}(x) = [inf i_{\bar{S}_a}(x), sup i_{\bar{S}_a}(x)].e^{j\alpha\psi_{\bar{S}_a}(x)}$$

$$F_{\bar{S}_a}(x) = [inf f_{\bar{S}_a}(x), sup f_{\bar{S}_a}(x)].e^{j\alpha\phi_{\bar{S}_a}(x)}$$

Case 2: if $c \in B - A$.then

$$T_{\bar{S}_b}(x) = [inf t_{\bar{S}_b}(x), sup t_{\bar{S}_b}(x)].e^{j\alpha\omega_{\bar{S}_b}(x)}$$

$$I_{\bar{S}_b}(x) = [inf i_b(x), sup i_{\bar{S}_b}(x)].e^{j\alpha\psi_{\bar{S}_b}(x)}$$

$$F_{\bar{S}_b}(x) = [inf f_{\bar{S}_b}(x), sup f_{\bar{S}_b}(x)].e^{j\alpha\phi_{\bar{S}_b}(x)}$$

Case 3: if $c \in A \cap B$.then

$$T_{\bar{S}_C}(x) = [inf t_{\bar{S}_C}(x), sup t_{\bar{S}_C}(x)].e^{j\alpha\omega_{\bar{S}_C}(x)}$$

$$I_{\bar{S}_C}(x) = [inf i_{\bar{S}_C}(x), sup i_{\bar{S}_C}(x)].e^{j\alpha\psi_{\bar{S}_C}(x)}$$

$$F_{\bar{S}_C}(x) = [inf f_{\bar{S}_C}(x), sup f_{\bar{S}_C}(x)].e^{j\alpha\phi_{\bar{S}_C}(x)}$$

Where

$$inf t_{\bar{S}_C}(x) = \vee \left(inf t_{\bar{S}_A}(x), inf t_{\bar{S}_B}(x) \right), sup t_{\bar{S}_C}(x) = \vee \left(sup t_{\bar{S}_A}(x), sup t_{\bar{S}_B}(x) \right);$$

$$inf i_{\bar{S}_C}(x) = \wedge \left(inf i_{\bar{S}_A}(x), inf i_{\bar{S}_B}(x) \right), sup i_{\bar{S}_C}(x) = \wedge \left(sup i_{\bar{S}_A}(x), sup i_{\bar{S}_B}(x) \right);$$

$$inf f_{\bar{S}_C}(x) = \wedge \left(inf f_{\bar{S}_A}(x), inf f_{\bar{S}_B}(x) \right), sup i_{\bar{S}_C}(x) = \wedge \left(sup f_{\bar{S}_A}(x), sup f_{\bar{S}_B}(x) \right);$$

The union of the phase terms are the same as defined for the union of the amplitude terms . The symbols ∨,∧ represent respectively max and min operators.

Example 4.2. Let $U = \{u_1, u_2\}$ be a universe of discourse, $(\bar{S}, A) and\ (\bar{S}, B)$ be two interval complex neutrosophic soft sets such that *A={*e_1, e_2, e_3 *}, B={*e_2, e_3, e_4 } defined on *U* as follows:

$(\bar{S}, A) =$

$$\{e_1, \left(\frac{[0.1,0.8].e^{j2\pi[0.22,0.48]}, [0.4,0.8].e^{j2\pi[0.55,0.62]}, [0.37,0.46].e^{j2\pi[0.57,0.83]}}{u_1}\right), \left(\frac{[0.26,0.39].e^{j2\pi[0.53,0.86]}, [0.65,0.86].e^{j2\pi[0.43,0.61]}, [0.53,0.9].e^{j2\pi[0.2,0.5]}}{u_2}\right),$$

$$e_2, \left(\frac{[0.4,0.6].e^{j2\pi[0.5,0.6]}, [0.1,0.7].e^{j2\pi[0.1,0.3]}, [0.3,0.5].e^{j2\pi[0.8,0.9]}}{u_1}\right), \left(\frac{[0.2,0.4].e^{j2\pi[0.3,0.6]}, [0.1,0.1].e^{j2\pi[0.7,0.9]}, [0.5,0.9].e^{j2\pi[0.2,0.5]}}{u_2}\right)),$$

$$e_3, \left(\frac{[0.2,0.5].e^{j2\pi[0.4,0.7]}, [0.5,0.7].e^{j2\pi[0.2,0.3]}, [0.6,0.8].e^{2j\pi[0.6,0.9]}}{u_1}\right), \left(\frac{[0.1,0.3].e^{j2\pi[0.1,0.3]}, [0.2,0.7].e^{2j\pi[0.6,0.8]}, [0.4,0.7].e^{j2\pi[0.1,0.5]}}{u_2}\right)\}$$

$(\bar{S}, B) =$

$$\{e_2, \left(\frac{[0.3,0.7].e^{j2\pi[0.7,0.8]}, [0.4,0.9].e^{j2\pi[0.3,0.5]}, [0.6,0.8].e^{j2\pi[0.5,0.6]}}{u_1}\right), \left(\frac{[0.4,0.4].e^{j2\pi[0.6,0.7]}, [0.1,0.9].e^{j2\pi[0.2,0.4]}, [0.3,0.8].e^{j2\pi[0.5,0.6]}}{u_2}\right)$$

$$, e_3, \left(\frac{[0.3,0.7].e^{j2\pi[0.4,0.5]}, [0.3,0.5].e^{j2\pi[0.6,0.7]}, [0.2,0.6].e^{j2\pi[0.16,0.3]}}{u_1}\right), \left(\frac{[0.21,0.63].e^{j2\pi[0.41,0.55]}, [0.31,0.52].e^{j2\pi[0.11,0.67]}, [0.3,0.73].e^{j2\pi[0.2,0.58]}}{u_2}\right)$$

$$e_4, \left(\frac{[0.2,0.6].e^{j2\pi[0.6,0.7]}, [0.3,0.8].e^{j2\pi[0.2,0.4]}, [0.5,0.7].e^{j2\pi[0.4,0.5]}}{u_1}\right), \left(\frac{[0.5,0.6].e^{j2\pi[0.7,0.8]}, [0.2,0.8].e^{j2\pi[0.3,0.5]}, [0.4,0.9].e^{j2\pi[0.6,0.7]}}{u_2}\right)\}$$

Then the union between two IV-CNSS defined as:

$(\bar{S}, A) \cup (\bar{S}, B) = (\bar{S}, C) =$

$$\{e_1, \left(\frac{[0.1,0.8].e^{j2\pi[0.22,0.48]}, [0.4,0.8].e^{j2\pi[0.55,0.62]}, [0.37,0.46].e^{j2\pi[0.57,0.83]}}{u_1}\right), \left(\frac{[0.26,0.39].e^{j2\pi[0.53,0.86]}, [0.65,0.86].e^{j2\pi[0.43,0.61]}, [0.53,0.9].e^{j2\pi[0.2,0.5]}}{u_2}\right)),$$

$$\{(e_2, \left(\frac{[0.4,0.7].e^{j2\pi[0.7,0.8]}, [0.1,0.7].e^{j2\pi[0.1,0.3]}, [0.3,0.5].e^{j2\pi[0.5,0.6]}}{u_1}\right), \left(\frac{[0.4,0.4].e^{j2\pi[0.6,0.7]}, [0.1,0.1].e^{j2\pi[0.2,0.4]}, [0.4,0.91].e^{j2\pi[0.58,0.69]}}{u_2}\right)),$$

$$e_3, \left(\frac{[0.3,0.7].e^{j2\pi[0.4,0.7]}, [0.3,0.7].e^{j2\pi[0.6,0.7]}, [0.2,0.6].e^{j2\pi[0.16,0.3]}}{u_1}\right), \left(\frac{[0.21,0.63].e^{j2\pi[0.41,0.55]}, [0.2,0.52].e^{j2\pi[0.11,0.67]}, [0.3,0.8].e^{j2\pi[0.1,0.5]}}{u_2}\right),$$

$$e_4, \left(\frac{[0.2,0.6].e^{j2\pi[0.6,0.7]}, [0.3,0.8].e^{j2\pi[0.2,0.4]}, [0.5,0.7].e^{j2\pi[0.4,0.5]}}{u_1}\right), \left(\frac{[0.5,0.6].e^{j2\pi[0.7,0.8]}, [0.2,0.8].e^{j2\pi[0.3,0.5]}, [0.4,0.9].e^{j2\pi[0.6,0.7]}}{u_2}\right)\}$$

$where\ C = A \cup B.$

Definition 4.3. Let $(\bar{S}, A)$and $(\bar{S}, B)$ be two IV-CNSs over U. The intersection of $(\bar{S}, A)$and $(\bar{S}, B)$ is an IV-CNSS $(\bar{S}, C)$ where $C = A \cap B, a \in A, b \in B$ and $x \in U$. to define the intersection we consider three cases:

Case 1: if $c \in A - B$.then

$$T_{\bar{S}_a}(x) = \left[inf t_{\bar{S}_a}(x), sup t_{\bar{S}_a}(x)\right].e^{j\alpha\omega_{\bar{S}a}(x)}$$

$$I_{\bar{S}_a}(x) = \left[inf i_{\bar{S}_a}(x), sup i_{\bar{S}_a}(x)\right].e^{j\alpha\psi_{\bar{S}a}(x)}$$

$F_{\bar{S}_a}(x) = [inff_{\bar{S}_a}(x), supf_{\bar{S}_a}(x)].e^{j\alpha\phi_{\bar{S}_a}(x)}$

Case 2: if $c \in B - A$.then

$T_{\bar{S}_b}(x) = [inft_{\bar{S}_b}(x), supt_{\bar{S}_b}(x)].e^{j\alpha\omega_{\bar{S}_b}(x)}$

$I_{\bar{S}_b}(x) = [infi_b(x), supi_{\bar{S}_b}(x)].e^{j\alpha\psi_{\bar{S}_b}(x)}$

$F_{\bar{S}_b}(x) = [inff_{\bar{S}_b}(x), supf_{\bar{S}_b}(x)].e^{j\alpha\phi_{\bar{S}_b}(x)}$

Case 3: if $c \in A \cap B$.then

$T_{\bar{S}_C}(x) = [inft_{\bar{S}_C}(x), supt_{\bar{S}_C}(x)].e^{j\alpha\omega_{\bar{S}_C}(x)}$

$I_{\bar{S}_C}(x) = [infi_{\bar{S}_C}(x), supi_{\bar{S}_C}(x)].e^{j\alpha\psi_{\bar{S}_C}(x)}$

$F_{\bar{S}_C}(x) = [inff_{\bar{S}_C}(x), supf_{\bar{S}_C}(x)].e^{j\alpha\phi_{\bar{S}_C}(x)}$

Where

$inft_{\bar{S}_C}(x) = \wedge\left(inft_{\bar{S}_A}(x), inft_{\bar{S}_B}(x)\right), supt_{\bar{S}_C}(x) = \wedge\left(supt_{\bar{S}_A}(x), supt_{\bar{S}_B}(x)\right);$

$infi_{\bar{S}_C}(x) = \vee\left(infi_{\bar{S}_A}(x), infi_{\bar{S}_B}(x)\right), supi_{\bar{S}_C}(x) = \vee\left(supi_{\bar{S}_A}(x), supi_{\bar{S}_B}(x)\right);$

$inff_{\bar{S}_C}(x) = \vee\left(inff_{\bar{S}_A}(x), inff_{\bar{S}_B}(x)\right), supi_{\bar{S}_C}(x) = \vee\left(supf_{\bar{S}_A}(x), supf_{\bar{S}_B}(x)\right);$

The intersection of the phase terms are the same as defined for the intersection of the amplitude terms. The symbols ∨,∧ represent respectively max and min operators.

Example 4.4. As in Example 4.2, let $(\bar{S}, A)$and $(\bar{S}, B)$ be two two interval value complex neutrosophic soft sets . Then, $(\bar{S}, C)$ is given by the intersection of two interval value complex neutrosophic soft sets:

$(\bar{S}, A) \cap (\bar{S}, B) = (\bar{S}, C) =$

$\{e_1, (\frac{[0.1,0.8].e^{j2\pi[0.22,0.48]}, [0.4,0.8].e^{j2\pi[0.55,0.62]}, [0.37,0.46].e^{j2\pi[0.57,0.83]}}{u_1}), (\frac{[0.26,0.39].e^{j2\pi[0.53,0.86]}, [0.65,0.86].e^{j2\pi[0.43,0.61]}, [0.53,0.9].e^{j2\pi[0.2,0.5]}}{u_2}),$

$e_2, (\frac{[0.3,0.6].e^{j2\pi[0.5,0.6]}, [0.4,0.9].e^{j2\pi[0.3,0.5]}, [0.6,0.8].e^{j2\pi[0.8,0.9]}}{u_1}), (\frac{[0.2,0.4].e^{j2\pi[0.3,0.6]}, [0.1,0.9].e^{j2\pi[0.7,0.9]}, [0.5,0.9].e^{j2\pi[0.5,0.6]}}{u_2}),$

$e_3, (\frac{[0.2,0.5].e^{j2\pi[0.4,0.7]}, [0.5,0.7].e^{j2\pi[0.8,0.7]}, [0.2,0.8].e^{j2\pi[0.6,0.9]}}{u_1}), (\frac{[0.1,0.3].e^{j2\pi[0.1,0.3]}, [0.31,0.7].e^{j2\pi[0.6,0.8]}, [0.4,0.73].e^{j2\pi[0.2,0.58]}}{u_2}),$

$e_4, (\frac{[0.2,0.6].e^{j2\pi[0.6,0.7]}, [0.3,0.8].e^{j2\pi[0.2,0.4]}, [0.5,0.7].e^{j2\pi[0.4,0.5]}}{u_1}), (\frac{[0.5,0.6].e^{j2\pi[0.7,0.8]}, [0.2,0.8].e^{j2\pi[0.3,0.5]}, [0.4,0.9].e^{j2\pi[0.6,0.7]}}{u_2})\}$

$Where\ C = A \cap B$

Definition4.5. Let $(\bar{S}, A)$ be IV-CNSs over U. The complement of $(\bar{S}, A)$, denoted by $(\bar{S}, A)^c$ is as defined below:

$(\bar{S},A)^{c} = (\bar{S}^{c},A) = \{a,\langle \frac{T_{\bar{S}_{A^C}}(x)=t_{\bar{S}_{A^C}}(x).e^{j2\pi\omega_{\bar{S}_{A^C}}(x)}, I_{\bar{S}_{A^C}}(x)=i_{\bar{S}_{A^C}}(x).e^{j2\pi\psi_{\bar{S}_{A^C}}(x)}, F_{\bar{S}_{A^C}}(x)=f_{\bar{S}_{A^C}}(x).e^{j2\pi\phi_{\bar{S}_{A^C}}(x)}}{x}\rangle : x \in U, a \in A\}.$

Where $t_{\bar{S}_{A^c}}(x) = f_{\bar{S}_A}(x)$ and $\omega_{\bar{S}_{A^c}}(x) = 2\pi - \omega_{\bar{S}_A}(x)$. Similarly $i_{\bar{S}_{A^c}}(x) = \left(inf\, i_{\bar{S}_{A^c}}(x), sup\, i_{\bar{S}_{A^c}}(x)\right)$where $inf\, i_{\bar{S}_{A^c}}(x) = 1 - sup\, i_{\bar{S}_A}(x)$ and $sup\, i_{\bar{S}_{A^c}}(x) = 1 - inf\, i_{\bar{S}_A}(x)$,with phase term $\psi_{\bar{S}_{A^c}}(x) = 2\pi - \psi_{\bar{S}_A}(x)$ Also, $f_{\bar{S}_{A^c}}(x) = t_{\bar{S}_A}(x)$,while the phase term $\phi_{\bar{S}_{A^c}}(x) = 2\pi - \phi_{\bar{S}_A}(x)$.

Proposition 4.6. Let $(\bar{S},A)$ is a *IV-CNSs* over *U*, then, $((\bar{S},A)^C)^C = (\bar{S},A)$.

Proof.From Definition 4.5, we have

$(\bar{S},A)^{c} = (\bar{S}^{c},A) = \{a,\langle T_{\bar{S}_{A^C}}(x), I_{\bar{S}_{A^C}}(x), F_{\bar{S}_{A^C}}(x)\rangle : x \in U, a \in A\}.$

$=\{a,\langle t_{\bar{S}_{A^c}}(x).\, e^{j2\pi\omega_{\bar{S}_{A^c}}(x)}, i_{\bar{S}_{A^c}}(x).\, e^{j2\pi\psi_{\bar{S}_{A^c}}(x)}, f_{\bar{S}_{A^c}}(x).\, e^{j2\pi\phi_{\bar{S}_{A^c}}(x)}\rangle : x \in U, a \in A\}.$

$=\{a,\langle f_{\bar{S}_A}(x).\, e^{j2\pi(2\pi-\omega_{\bar{S}_A}(x))}, (inf\, i_{\bar{S}_{A^c}}(x), sup\, i_{\bar{S}_{A^c}}(x)).\, e^{j2\pi(2\pi-\psi_{\bar{S}_A}(x))}, t_{\bar{S}_A}(x).\, e^{j2\pi(2\pi-\phi_{\bar{S}_A}(x))}\rangle : x \in U, a \in A\}.$

$=\{a,\langle f_{\bar{S}_A}(x).\, e^{j2\pi(2\pi-\omega_{\bar{S}_A}(x))}, (1-sup\, i_{\bar{S}_A}(x), 1-inf\, i_{\bar{S}_A}(x)).\, e^{j2\pi(2\pi-\psi_{\bar{S}_A}(x))}, f_{\bar{S}_{A^c}}(x).\, e^{j2\pi(2\pi-\phi_{\bar{S}_A}(x))}\rangle : x \in U, a \in A\}.$

Thus

$((\bar{S},A)^{c})^{c}$

$=\{a,\langle f_{\bar{S}_{A^c}}(x).\, e^{j2\pi(2\pi-(2\pi-\omega_{\bar{S}_{A^c}}(x)))} (1-sup\, i_{\bar{S}_{A^c}}(x), 1-inf\, i_{\bar{S}_{A^c}}).\, e^{j2\pi(2\pi-\psi_{\bar{S}_{A^c}}(x))}, t_{\bar{S}_{A^c}}(x).\, e^{j2\pi(2\pi-(2\pi-\phi_{\bar{S}_{A^c}}(x)))}\rangle : x \in U, a \in A\}.$

$=\{a,\langle f_{\bar{S}_{A^c}}(x).\, e^{j2\pi(2\pi-(2\pi-(2\pi-\omega_{\bar{S}_A}(x)))} (1-(1-inf\, i_{\bar{S}_A}(x)), 1-(1-sup\, i_{\bar{S}_A}(x)).\, e^{j2\pi(2\pi-(2\pi-\psi_{\bar{S}_A}(x))}, t_{\bar{S}_{A^c}}(x).\, e^{j2\pi(2\pi-(2\pi-\phi_{\bar{S}_A}(x)}\rangle : x \in U, a \in A\}.$

$=\{a,\langle t_{\bar{S}_A}(x).\, e^{j2\pi\omega_{\bar{S}_A}(x)}, i_{\bar{S}_A}(x).\, e^{j2\pi\psi_{\bar{S}_A}(x)}, f_{\bar{S}_A}(x).\, e^{j2\pi\phi_A(x)}\rangle : x \in U, a \in A\}.$

$=(\bar{S},A).$

Example 4.7. Consider Example 1. The complement of $(\bar{S},A)$ is given by $(\bar{S},A)^{c} = \{\bar{S}_A{}^{c}(e_1), \bar{S}_A{}^{c}(e_2), \bar{S}_A{}^{c}(e_3), \bar{S}_A{}^{c}(e_4)\}$, we just give the complement to $\overline{S}_A{}^{c}(e_1)$below for the sake of brevity

$\bar{S}_A{}^{c}(e_1) =$

$\{(\frac{[0.3,0.5].\, e^{j2\pi[0.5,0.6]}, [0.3,0.9].\, e^{j2\pi[0.1,0.3]}, [0.4,0.6].\, e^{j2\pi[0.8,0.9]}}{u_1}), (\frac{[0.5,0.9].\, e^{j2\pi[0.3,0.6]}, [0.9,0.9].\, e^{j2\pi[0.7,0.9]}, [0.2,0.4].\, e^{j2\pi[0.2,0.5]}}{u_2}),$

$(\frac{[0.2,0.6].\, e^{j2\pi[0.7,0.8]}, [0.3,0.4].\, e^{j2\pi[0.6,0.7]}, [0.3,0.4].\, e^{j2\pi[0.6,0.8]}}{u_3}), (\frac{[0.3,0.5].\, e^{j2\pi[0.9,1]}, [0.7,0.8].\, e^{j2\pi[0.7,0.8]}, [0,0.9].\, e^{j2\pi[0.4,0.5]}}{u_4})\}$

Proposition 4.8. Let $(\bar{S},A)$, $(\bar{S},B)$ and $(\bar{S},C)$ be three interval complex neutrosophic soft sets over *U*. *Then:*

i. $(\bar{S}, A) \cup (\bar{S}, B) = (\bar{S}, B) \cup (\bar{S}, A)$ (*Commutative* law)

ii. $(\bar{S}, A) \cap (\bar{S}, B) = (\bar{S}, B) \cap (\bar{S}, A)$ (*Commutative* law)

iii. $(\bar{S}, A) \cup ((\bar{S}, B) \cup (\bar{S}, C)) = ((\bar{S}, A) \cup (\bar{S}, B)) \cup (\bar{S}, C)$ (*Associative* law)

iv. $(\bar{S}, A) \cap ((\bar{S}, B) \cap (\bar{S}, C)) = ((\bar{S}, A) \cap (\bar{S}, B)) \cap (\bar{S}, C)$ (*Associative* law)

v. $(\bar{S}, A) \cup ((\bar{S}, B) \cap (\bar{S}, C)) = ((\bar{S}, A) \cup (\bar{S}, B)) \cap ((\bar{S}, A)) \cup (\bar{S}, C))$ (*Distribution* law)

vi. $(\bar{S}, A) \cap ((\bar{S}, B) \cup (\bar{S}, C)) = ((\bar{S}, A) \cap (\bar{S}, B)) \cup ((\bar{S}, A)) \cap (\bar{S}, C))$ (*Distribution* law)

vii. $(\bar{S}, A) \cup ((\bar{S}, A) \cap (\bar{S}, B)) = (\bar{S}, A)$

viii. $(\bar{S}, A) \cap ((\bar{S}, A) \cup (\bar{S}, B)) = (\bar{S}, A)$

ix. $(\bar{S}, A) \cup (\bar{S}, B)^c = (\bar{S}, A)^c \cap (\bar{S}, B)^c$ (De *Morgan's* law)

x. $((\bar{S}, A) \cap (\bar{S}, B))^c = ((\bar{S}, A)^c \cup (\bar{S}, B))^c$ (De *Morgan's* law)

Proof: All of these assertions can be directly proven.

Theorem 4.9. Let $(\bar{S}, A)$ *and* $(\bar{S}, B)$be two interval complex neutrosophic soft set,Then The smallest one containing both $(\bar{S}, A)$ and $(\bar{S}, B)$is $(\bar{S}, A) \cup (\bar{S}, B)$.

Proof: Directly

Theorem 4.10. Let $(\bar{S}, A)$ *and* $(\bar{S}, B)$be two interval complex neutrosophic soft set,then the largest one present in both$(\bar{S}, A)$ and $(\bar{S}, B)$is $(\bar{S}, A) \cap (\bar{S}, B)$.

Proof: Directly

Theorem 4.11. Let $(\bar{S}, A)$ and $(\bar{S}, B)$. be two interval complex neutrosophic soft sets on U. Then, $(\bar{S}, A) \leq (\bar{S}, B)$iff $(\bar{S}, B)^c \leq (\bar{S}, A)^c$

Proof: Directly

Theorem 4.12. Let $\bar{P}$ be the power set of all interval complex neutrosophic soft set. Then, $(\bar{P},\cup,\cap)$ forms a distributive lattice.

Proof: Directly

5. Operational rules of operation Interval Valued Complex Neutrosophic Soft Sets

Let$(\bar{S},A)=\{a,([T^L{}_{\bar{s}_A},T^U{}_{\bar{s}_A}],[I^L{}_{\bar{s}_A},I^U{}_{\bar{s}_A}],[F^L{}_{\bar{s}_A},F^U{}_{\bar{s}_A}]),a\in A\}$.and$(\bar{S},B)=\{b,([T^L{}_{\bar{s}_B},T^U{}_{\bar{s}_B}],[I^L{}_{\bar{s}_B},I^U{}_{\bar{s}_B}],[F^L{}_{\bar{s}_B},F^U{}_{\bar{s}_B}]),b\in B\}$. be two interval valued complex neutrosophic soft sets over U which are defined by

$[T^L{}_{\bar{s}_A},T^U{}_{\bar{s}_A}]=[t^L{}_{\bar{s}_A}(x),t^U{}_{\bar{s}_A}(x)].e^{j\alpha[\omega^L{}_{\bar{s}_A}(x),\omega^U{}_{\bar{s}_A}(x)]}$, $[I^L{}_{\bar{s}_A},I^U{}_{\bar{s}_A}]=$

$[i^L{}_{\bar{s}_A}(x),i^U{}_{\bar{s}_A}(x)].e^{j\beta[\psi^L{}_{\bar{s}_A}(x),\psi^U{}_{\bar{s}_A}(x)]}$, $[F^L{}_{\bar{s}_A},F^U{}_{\bar{s}_A}]=$

$\left[f^L{}_{\bar{s}_A}(x),f^U{}_{\bar{s}_A}(x)\right].e^{j\gamma[\Phi^L{}_{\bar{s}_A}(x),\Phi^U{}_{\bar{s}_A}(x)]}$ and

$[T^L{}_{\bar{s}_B},T^U{}_{\bar{s}_B}]=[t^L{}_{\bar{s}_B}(x),t^U{}_{\bar{s}_B}(x)].e^{j\alpha[\omega^L{}_{\bar{s}_B}(x),\omega^U{}_{\bar{s}_B}(x)]}$, $[I^L{}_{\bar{s}_B},I^U{}_{\bar{s}_B}]=$

$[i^L{}_{\bar{s}_B}(x),i^U{}_{\bar{s}_B}(x)].e^{j\beta[\psi^L{}_{\bar{s}_B}(x),\psi^U{}_{\bar{s}_B}(x)]}$, $[F^L{}_{\bar{s}_B},F^U{}_{\bar{s}_B}]=$

$\left[f^L{}_{\bar{s}_B}(x),f^U{}_{\bar{s}_B}(x)\right].e^{j\gamma[\Phi^L{}_{\bar{s}_B}(x),\Phi^U{}_{\bar{s}_B}(x)]}$,

respectively. Then,some operational rules of IV-CNSSs as follows, are described:

(i) The product of $(\bar{S},A)$ and $(\bar{S},B)$. denoted as $(\bar{S},A)\times(\bar{S},B)$ is:
$\{\langle(a,b), T_{\bar{S}_{A\times B}}(x), I_{\bar{S}_{A\times B}}(x), F_{\bar{S}_{A\times B}}(x)\rangle:(a,b)\in A\times B\}$, where

$$T_{\bar{S}_{A\times B}}(x)=[t^L{}_{\bar{s}_A}(x).t^L{}_{\bar{s}_B}(x),t^U{}_{\bar{s}_A}(x).t^U{}_{\bar{s}_B}(x)].e^{j\alpha[\omega^L{}_{\bar{s}_{A\times B}}(x),\omega^U{}_{\bar{s}_{A\times B}}(x)]},$$

$$I_{\bar{S}_{A\times B}}(x)=[i^L{}_{\bar{s}_A}(x)+i^L{}_{\bar{s}_B}(x)-i^L{}_{\bar{s}_A}(x).i^L{}_{\bar{s}_B}(x),i^U{}_{\bar{s}_A}(x)+i^U{}_{\bar{s}_B}(x)-i^U{}_{\bar{s}_A}(x).i^U{}_{\bar{s}_B}(x)].e^{j\alpha[\psi^L{}_{\bar{s}_{A\times B}}(x),\psi^U{}_{\bar{s}_{A\times B}}(x)]},$$

$$F_{\bar{S}_{A\times B}}(x)=\left[f^L{}_{\bar{s}_A}(x)+f^L{}_{\bar{s}_B}(x)-f^L{}_{\bar{s}_A}(x).f^L{}_{\bar{s}_B}(x),f^U{}_{\bar{s}_A}(x)+f^U{}_{\bar{s}_B}(x)-f^U{}_{\bar{s}_A}(x).f^U{}_{\bar{s}_B}(x)\right].e^{j\alpha[\Phi^L{}_{\bar{s}_{A\times B}}(x),\Phi^U{}_{\bar{s}_{A\times B}}(x)]},$$

The product of phase terms is defined below:

$\omega^L{}_{\bar{s}_{A\times B}}(x)=\omega^L{}_{\bar{s}_A}(x)\omega^L{}_{\bar{s}_B}(x)$, $\omega^U{}_{\bar{s}_{A\times B}}(x)=\omega^U{}_{\bar{s}_A}(x)\omega^U{}_{\bar{s}_B}(x)$

$\psi^L{}_{\bar{s}_{A\times B}}(x)=\psi^L{}_{\bar{s}_A}(x)\psi^L{}_{\bar{s}_B}(x)$, $\psi^U{}_{\bar{s}_{A\times B}}(x)=\psi^U{}_{\bar{s}_A}(x)\psi^U{}_{\bar{s}_B}(x)$

$\Phi^L{}_{\bar{s}_{A\times B}}(x)=\Phi^L{}_{\bar{s}_A}(x)\Phi^L{}_{\bar{s}_B}(x)$, $\Phi^U{}_{\bar{s}_{A\times B}}(x)=\Phi^U{}_{\bar{s}_A}(x)\Phi^U{}_{\bar{s}_B}(x)$.

The addition of $(\bar{S},A)$ and $(\bar{S},B)$, denoted as $(\bar{S},A)+(\bar{S},B)$,is defined as :

$$T_{\bar{S}_{A+B}}(x)=[t^L{}_{\bar{s}_A}(x)+t^L{}_{\bar{s}_B}(x)-t^L{}_{\bar{s}_A}(x).t^L{}_{\bar{s}_B}(x),t^U{}_{\bar{s}_A}(x)+t^U{}_{\bar{s}_B}(x)-t^U{}_{\bar{s}_A}(x).t^U{}_{\bar{s}_B}(x)].e^{j\alpha[\omega^L{}_{\bar{s}_{A+B}}(x),\omega^U{}_{\bar{s}_{A+B}}(x)]}.$$

$$I_{\bar{S}_{A+B}}(x)=[i^L{}_{\bar{s}_A}(x).i^L{}_{\bar{s}_B}(x),i^U{}_{\bar{s}_A}(x).i^U{}_{\bar{s}_B}(x)].e^{j\alpha[\psi^L{}_{\bar{s}_{A+B}}(x),\psi^U{}_{\bar{s}_{A+B}}(x)]},$$

$$F_{\bar{S}_{A+B}}(x)=\left[f^L{}_{\bar{s}_A}(x).f^L{}_{\bar{s}_B}(x),f^U{}_{\bar{s}_A}(x).f^U{}_{\bar{s}_B}(x)\right].e^{j\alpha[\Phi^L{}_{\bar{s}_{A+B}}(x),\Phi^U{}_{\bar{s}_{A+B}}(x)]},$$

below is the addition of phase terms is defined :

$\omega^L{}_{\bar{s}_{A+B}}(x)= \omega^L{}_{\bar{s}_A}(x)+\omega^L{}_{\bar{s}_B}(x)$, $\omega^U{}_{\bar{s}_{A+B}}(x)= \omega^U{}_{\bar{s}_A}(x)+\omega^U{}_{\bar{s}_B}(x)$

$\psi^L{}_{\bar{s}_{A+B}}(x)= \psi^L{}_{\bar{s}_A}(x) + \psi^L{}_{\bar{s}_B}(x)$, $\psi^U{}_{\bar{s}_{A+B}}(x)= \psi^U{}_{\bar{s}_A}(x) + \psi^U{}_{\bar{s}_B}(x)$

$\Phi^L{}_{\bar{s}_{A+B}}(x)= \Phi^L{}_{\bar{s}_A}(x) + \Phi^L{}_{\bar{s}_B}(x)$, $\Phi^U{}_{\bar{s}_{A+B}}(x)= \Phi^U{}_{\bar{s}_A}(x) + \Phi^U{}_{\bar{s}_B}(x)$.

(iii) The scalar multiplication of $(\bar{S}, A)$ is an interval valued complex neutrosophic soft set denoted as $(\bar{S}, C)$ $=k(\bar{S}, A)$ and defined as:

$\{\langle a, T_{\bar{S}_c}(x), I_{\bar{S}_c}(x), F_{\bar{S}_c}(x)\rangle$: a$\in A\}$, where

$$T_{\bar{S}_C}(x) = \left[1-(1-t^L{}_{\bar{s}_C}(x))^k, 1-(1-t^U{}_{\bar{s}_A}(x))^k\right]. e^{j2\pi\left[\omega^L{}_{\bar{s}_C}(x), \omega^U{}_{\bar{s}_C}(x)\right]},$$

$$I_{\bar{S}_C}(x) = \left[(i^L{}_{\bar{s}_A}(x))^k, (i^U{}_{\bar{s}_A}(x))^k\right].\ e^{j2\pi\left[\psi^L{}_{\bar{s}_C}(x), \psi^U{}_{\bar{s}_C}(x)\right]},$$

$$F_{\bar{S}_C}(x) = \left[(f^L{}_{\bar{s}_A}(x))^k, (f^U{}_{\bar{s}_A}(x))^k\right].\ e^{j2\pi\left[\Phi^L{}_{\bar{s}_C}(x), \Phi^U{}_{\bar{s}_C}(x)\right]},$$

below is the scalar of phase terms defined :

$\omega^L{}_{\bar{s}_C}(x)= \omega^L{}_{\bar{s}_A}(x). k$, $\omega^U{}_{\bar{s}_C}(x)= \omega^U{}_{\bar{s}_A}(x). k$,

$\psi^L{}_{\bar{s}_C}(x)= \psi^L{}_{\bar{s}_A}(x). k$, $\psi^U{}_{\bar{s}_C}(x)= \psi^U{}_{\bar{s}_A}(x). k$,

$\Phi^L{}_{\bar{s}_C}(x)= \Phi^L{}_{\bar{s}_A}(x). k$, $\Phi^U{}_{\bar{s}_C}(x)= \Phi^U{}_{\bar{s}_A}(x). k$,

6. Interval Valued Complex Neutrosophic Soft Set Approach to Problem-Making Decisions

In this here section, by considering the following case, we introduce an application of IV-CNSSs to a decision-making problem.

Example 6.1. Assume that two kinds of a single commodity from a source must be compared by a merchant company and pick the most appropriate one. Assume that the current phase an expert view in two phases on these two categories of products: once before using the products and once again after reviewing a trial of one of the two types of products. Assume that the universe of the two alternatives consists of $U = \{u_1, u_2\}$ (the two product types) and $E = \{e_1, e_2, e_3\}$ is the set of attributes, where e_1 symbolize "easy to use", e_2 symbolize " functional " and e_3 symbolize "durable". The expert is now asked to decide on the most suitable choice based on the goals and constraints of setting up the IV-CNSS.

$$(\bar{S}, A) =$$

$$\left\{\left(e_1, \left(\left\{\left(\frac{[0.4,0.6]. e^{j2\pi[0.5,0.6]}, [0.1,0.7]. e^{j2\pi[0,8,0.9]}, [0.3,0,5]. e^{j2\pi[0.8,0.9]}}{u_1}\right), \left(\frac{[0.2,0.4]. e^{j2\pi[0.3,0.6]}, [0.1,0.1]. e^{j2\pi[0.7,0.9]}, [0.5,0.9]. e^{j2\pi[0.2,0.5]}}{u_2}\right)\right\}\right)\right),\right.$$

$$\left(e_2, \left(\left\{\left(\frac{[0.2,0.5]. e^{j2\pi[0.4,0.7]}, [0.5,0.7]. e^{j2\pi[0.5,0.6]}, [0.2,0.8]. e^{j2\pi[0.6,0.9]}}{u_1}\right), \left(\frac{[0.1,0.3]. e^{j2\pi[0.1,0.3]}, [0.31,0.7]. e^{j2\pi[0.6,0.8]}, [0.4,0.61]. e^{j2\pi[0.2,0.58]}}{u_2}\right)\right\}\right)\right),$$

$$\left.\left(e_3, \left(\left\{\left(\frac{[0.2,0.7]. e^{j2\pi[0.7,0.8]}, [0.4,0.9]. e^{j2\pi[0.3,0.5]}, [0.6,0.8]. e^{j2\pi[0.5,0.6]}}{u_1}\right), \left(\frac{[0.15,0.52]. e^{j2\pi[0.1,0.3]}, [0,0.5]. e^{j2\pi[0.6,0.8]}, [0.3,0.3]. e^{j2\pi[0.6,0.7]}}{u_2}\right)\right\}\right)\right)\right\}$$

The opinions of the experts in stage one reflect the amplitude of true membership, falsehood membership and indeterminate membership (before the products are used) in the IV-CNSS$(\bar{S}, A)$ above, while the terms of membership, non-membership and indeterminacy process reflect the opinions of the experts in the second phase (in accordance with having tried a selection of both of the product's two types). Thus, the amplitude term of the truth membership of the first phase and the phase term of the membership of the second phase form a complex-valued function of the truth membership of the IV-CNSS$(\bar{S}, A)$. Similarly, a complex-valued falsity membership function is generated by the non-membership amplitude term in phase one and the falsehood membership phase term in the second phase. In addition, the amplitude term of undecidedness in the first phase and the phase term of indeterminacy in the second phase form the complex-valued indeterminate membership function. Our problem now is to choose the most suitable product type for the merchant company. We use IV-CNSS $(\bar{S}, A)$ along with the proposed algorithm to disband this decision-making problem. Using a functional formula that allows fast computational decision-making without the need to perform guided operations on complex numbers, this algorithm converts interval complex neutrosophic soft values (I-CNSVs) to interval neutrosophic soft values (INSVs). This algorithm. After that we convert interval neutrosophic soft values (INSVs) to single neutrosophic soft values (SNSVs) by taking the arithmetic average of $T_{\bar{S}_A}(x), I_{\bar{S}_A}(x)$ and $F_{\bar{S}_A}(x)$ respectively. In this formula, we assign a weight to the amplitude terms (a weight to the expert's opinion previous to using the product) by multiplying the weight vector to each amplitude term. Similarly, by multiplying the weight vector for each phase term, we assign a weight to the phase terms (a weight to the opinion of the expert after using the product). Then, to obtain the IV-CNSVs that together reflect the views of the experts on both phases, We combine the values and phase terms of the weighted amplitude terms. After conducting these simple arithmetic for all membership functions of the IVCNSS $(\bar{S}, A)$, we then lead to the final decision using the single neutrosophic soft method.

Algorithm:

Step 1.Input the IV-CNSS $(\bar{S}, A)$,

Step 2.Convert IV-CNSS $(\bar{S}, A)$,to IVNSS (S,A) By gaining the values of the weighted aggregation of $T_{\bar{S}_A}(x), I_{\bar{S}_A}(x)$ and $F_{\bar{S}_A}(x)$, $\forall a \in A$ and $\forall x \in U$ as the following Formulas:

$$T_{\bar{S}_A}(x) = \left[w_1 t^L{}_{\bar{S}_A}(x) + w_2 \left(\frac{1}{2\pi}\right) \alpha\omega^L{}_{\bar{S}_A}(x)\, , w_1 t^U{}_{\bar{S}_A}(x) + w_2 \left(\frac{1}{2\pi}\right) \alpha\omega^U{}_{\bar{S}_A}(x)\right],$$

$$I_{\bar{S}_A}(x) = \left[w_1 i^L{}_{\bar{S}_A}(x) + w_2 \left(\frac{1}{2\pi}\right) \beta\psi^L{}_{\bar{S}_A}(x)\, , w_1 i^U{}_{\bar{S}_A}(x) + w_2 \left(\frac{1}{2\pi}\right) \beta\psi^U{}_{\bar{S}_A}(x)\right],$$

$$F_{\bar{S}_A}(x) = \left[w_1 f^L{}_{\bar{S}_A}(x) + w_2 \left(\frac{1}{2\pi}\right) \gamma\Phi^L{}_{\bar{S}_A}(x)\, , w_1 f^U{}_{\bar{S}_A}(x) + w_2 \left(\frac{1}{2\pi}\right) \gamma\Phi^U{}_{\bar{S}_A}(x)\right],$$

where $t^L{}_{\bar{S}_A}(x)$, $t^U{}_{\bar{S}_A}(x), i^L{}_{\bar{S}_A}(x), i^U{}_{\bar{S}_A}(x)$ and $f^L{}_{\bar{S}_A}(x), f^U{}_{\bar{S}_A}(x)$are the amplitude terms and $\omega^L{}_{\bar{S}_A}(x)$, $\omega^U{}_{\bar{S}_A}(x), \psi^L{}_{\bar{S}_A}(x), \psi^U{}_{\bar{S}_A}(x)$ and $\Phi^L{}_{\bar{S}_A}(x), \Phi^U{}_{\bar{S}_A}(x)$ are the phase terms in the the Interval Complex Neutrosophic Soft Set $(\bar{S}, A)$, respectively. $T_{\bar{S}_A}(x), I_{\bar{S}_A}(x)$ and $F_{\bar{S}_A}(x)$ are truth membership function ,anindeterminate membership function,and falsehood membership function in IV-NSS (S, A), respectively and w_1, w_2 the weights for the terms of the amplitude and phase terms, respectively, where $w_1, w_2 \in [0,1]$ and $w_1 + w_2 = 1$.

Step 3. Convert IVNSS (S,A) to SVNSS by taking the arithmetic average of $T_{\bar{S}_A}(x), I_{\bar{S}_A}(x)$ and $F_{\bar{S}_A}(x)$ respectively.

Step 4. Compute the comparison matrix of the SVNSS. Comparison matrix of SVNSS [40], is a matrix whose rows are labelled by the object names $u_1, u_2, \dots, u_n$ and the columns are labelled by the parameters $e_1, e_2, \dots, e_m$ The entries c_{ij} are calculated by $c_{ij} = \mathrm{a_{ij}} + (\mathrm{b_{ij}} - \mathrm{c_{ij}})$, where 'a' is the integer calculated as `how many times $T_{e_i}(u_j)$ exceeds or equal to$T_{e_i}(u_k)$, for $u_j \neq u_k, \forall\, u_j \in U$, 'b' is the integer calculated as `how many times $I_{e_i}(u_j)$ exceeds or equal to$I_{e_i}(u_k)$, for $u_J \neq u_k, \forall\, u_J \in U$, and 'c' is the integer calculated as `how many times $F_{e_i}(u_j)$ exceeds or equal to $I_{e_i}(u_k)$ ', for $u_i \neq u_k, \forall\, u_j \in U$.

Step 5.Calculate the scorec_i of u_i, $\forall i$. The score of an object u_i of c_i its calculated as $c_i = \sum_j c_{ij}$.

Step 6. The decision is to select u_i if $c_k = max_{u_i \in U} c_i$

Step 7. if we has more than one decision then any one of $u_i$could be the preferable choice.

Now, to change the form of the IV-CNSS $(\bar{\mathrm{S}}, \mathrm{A})$,to IV-NSS (S,A) we assume that the weight vectors are $w_1 = 0.6\ and\ w_2 = 0.4$. To illustrate this step, we calculate $T_{e_1}(u_1)$, $I_{e_1}(u_1)$ and $F_{e_1}(u_1)$, as shown below:

$$T_{e_1}(u_1) = \left[w_1 t^L{}_{\bar{s}_{e_1}}(u_1) + w_2\left(\frac{1}{2\pi}\right)\alpha\omega^L{}_{\bar{s}_{e_1}}(u_1)\,, w_1 t^U{}_{\bar{s}_{e_1}}(u_1) + w_2\left(\frac{1}{2\pi}\right)\alpha\omega^U{}_{\bar{s}_{e_1}}(u_1)\right]$$

$$=\left[0.6(0.4) +\ 0.4\left(\tfrac{1}{2\pi}\right)(2\pi)(0.5), 0.6(0.6) +\ 0.4\left(\tfrac{1}{2\pi}\right)(2\pi)(0.6)\right]$$

$$=[0.44, 0.6]$$

$$I_{e_1}(u_1) = \left[w_1 i^L{}_{\bar{s}_{e_1}}(u_1) + w_2\left(\frac{1}{2\pi}\right)\beta\ \psi^L{}_{\bar{s}_{e_1}}(u_1)\,, w_1 i^U{}_{\bar{s}_{e_1}}(u_1) + w_2\left(\frac{1}{2\pi}\right)\beta\ \psi^U{}_{\bar{s}_{e_1}}(u_1)\right]$$

$$=\left[0.6(0.1) +\ 0.4\left(\tfrac{1}{2\pi}\right)(2\pi)(0.8), 0.6(0.7) +\ 0.4\left(\tfrac{1}{2\pi}\right)(2\pi)(0.9)\right]$$

$$=[0.38, 0.78].$$

$$F_{e_1}(u_1) = \left[w_1 f^L{}_{\bar{s}_{e_1}}(u_1) + w_2\left(\frac{1}{2\pi}\right)\gamma\Phi^L{}_{\bar{s}_{e_1}}(u_1)\,, w_1 f^U{}_{\bar{s}_{e_1}}(u_1) + w_2\left(\frac{1}{2\pi}\right)\gamma\Phi^U{}_{\bar{s}_{e_1}}(u_1)\right]$$

$$= \left[0.6(0.3) +\ 0.4\left(\tfrac{1}{2\pi}\right)(2\pi)(0.8), 0.6(0.5) +\ 0.4\left(\tfrac{1}{2\pi}\right)(2\pi)(0.9)\right]$$

$$= [0.5, 0.66]$$

Then the IV-NSS

$$\left(\left[T^L{}_{\bar{s}_{e_1}}(u_1), T^U{}_{\bar{s}_{e_1}}(u_1)\right], \left[I^L{}_{\bar{s}_{e_1}}(u_1), I^U{}_{\bar{s}_{e_1}}(u_1)\right], \left[F^L{}_{\bar{s}_{e_1}}(u_1), F^U{}_{\bar{s}_{e_1}}(u_1)\right]\right)$$

$$= ([0.44, 0.6], [0.38, 0.78], [0.5, 0.66]).$$

we measure the other IV-NSSS in the same way. $\forall\, e_i \in A$ and $\forall\, u_j \in U$. as shown.Table 1.

Table1.values of IV-NSS

U	u_1	u_2
e_1	([0.44,0.6], [0.38, 0.78], [0.5,0.66])	([0.24,0.48], [0.34, 0.42], [0.38,0.74])
e_2	([0.28,0.58], [0.5,0.66], [0.36,0.84])	([0.10,0.30] ,[0.43,0.74] ,[0.32,0.59])
e_3	([0.4,0.74] ,[0.36,0.74] ,[0.68,0.72])	([0.13,0.43] ,[0.24,0.62] ,[0.42,0.46])

Now we convert the IV-NSVS to SVNSV by taking the arithmetic average of $T_{\bar{S}_A}(x), I_{\bar{S}_A}(x)$ and $F_{\bar{S}_A}(x)$ respectively as shown Table2

Table2 . values of SVNSS

U	u_1	u_2
e_1	(0.52,0.58,0.58)	(0.36,0.38 ,0.56)
e_2	(0.43,0.58,0.6)	(0.2,0.50,0.46)
e_3	(0.57,0.55,0.7)	(0.28 ,0.43 ,0.44)

Table 3. comparison matrix of the SVNSS

U	u_1	u_2
e_1	3	0
e_2	1	-1
e_3	0	2

Table 4: compute the score c_i

U	u_1	u_2
e_1	3	0
e_2	1	-1
e_3	0	2
Score(c_i)	4	1

Decision: The best option is to select u_1. Since $c_1 = max_{u_i \in U} c_i = u_1$. The expert advice therefore selects the form u_1 of this product as u_1 desirable alternative.

7. Conclusion

We established the concept of IV-CNSS by combining the two concepts of interval complex neutrosophic sets with soft sets. The basic operations on IV-CNSS, namely complement, subset, union, intersection operations, were defined. Subse-quently, the basic properties of these operations such as De Morgan's laws and other relevant laws pertain-ing to the concept of IV-CNSS were proven. Finally, a new algorithm is introduced and applied to the IV-CNSS model to solve a hypothetical decision-making problem, and its superiority and feasibility are further verified by comparison with other existing methods. This new extension will provide a significant addition to existing theories for handling indeterminacy, where time plays a vital role in the decision process, and spurs more developments of further research and pertinent applications. For further research, we intend to take into account unknown weight information to develop some real applications of IV-CNSS in other areas, where the phase term may represent other variables such as distance, speed, and temperature.

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Neutrosophication Functions and their Implementation by MATLAB Program

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Abstract: Neutrosophication is the process of converting crisp values into neutrosophic values, which is considered the first and basic step for any processing system that depends on the neutrosophic logical relationships and features, especially those that take into account indeterminacy values that result from ambiguity, noise, or inaccuracy. In this paper, we have presented a set of neutrosophication functions by modifying the functions used in fuzzy logic (trapezoid, triangle, gauss, bell-shaped, s-shaped, z-shaped) in a way that preserves the essence of the neutrosophic logic philosophy and the independence of truth, indeterminacy, and falsity values for each element of the neutrosophic set. Neutrosophication functions have also been implemented through the use of a suggested MATLAB code. It is possible through the proposed neutrosophication functions to build neutrosophic processing systems, especially digital image processing systems, by converting the crisp values of the pixels of the digital image to neutrosophic values using the proposed functions. Then, by building on the neutrosophic logic operations and the related researches, the new neutrosophic values are processed, after which they are returned to their crisp values through de-neutrosophication.

Keywords: Neutrosophication; trapezoid; triangle; gauss; bell-shaped; s-shaped; z-shaped.

1. Introduction

The proof of any mathematical matter depends mainly on making logical and mathematical steps on a set of data and hypotheses to reach the objective results. This importance prompts pure and applied mathematicians permanently and continuously to develop and infer logical relationships in accordance with the shape and features of the new and different groups of mathematical, descriptive and arithmetic values. In this context, neutrosophic logic was founded in 1995 by the American professor Florentin [1,2] to develop logical philosophy through the definition of the neutrosophic sets and the resulting definitions, consequences, and neutrosophic logical relationships [3-6].

What distinguishes the neutrosophic sets from other preceding sets, such as intuitionistic and n-hyperspherical fuzzy sets [2], is that they add an independent value: the degree of indeterminacy. Consequently, each element of the neutrosophic set is expressed by (T, I, F), where (T) degree of truth-membership, (I) degree of indeterminacy-membership, (F) degree of falsehood-membership. These three values are completely independent.

Importance of the neutrosophication functions comes from the fact that any neutrosophic data processing system [7-11] must start by converting the given values into neutrosophic values, using the neutrosophication functions.

Few researchers have made proposals for some of the neutrosophication functions. Broumi, Nagarajan, Bakali, and Talea (2019) [12] have introduced neutrosophic trapezoidal function and implementation using MATLAB program. Faruk Karaaslan (2018) [13] has studied neutrosophic gaussian function and its application about decision making. Chakraborty, Mondal, Ahmadian, Senu, Alam, and Salahshour (2018) [14] also have a study on neutrosophication and de-neutrosophication by neutrosophic triangular function, and their applications.

The previous researches have studied one function separately from the other functions and assumed that the value of truth-membership degree, indeterminacy-membership degree, and falsehood-membership degree was calculated according to one form of the neutrosophic functions.

This research is divided into 5 parts. Part 2 discusses preliminaries about the neutrosophic set. Part 3 presents new neutrosophication functions. Part 4 proposes MATLAB code for these neutrosophication functions. In part 5, we have concluded our research.

2. Neutrosophic Set [8]

Neutrosophic set was founded by Prof. Smarandache in 1995 and was published in 1998. It was an extension of many existing sets, such as spherical, intuitionistic, inconsistent intuitionistic, and q-rung orthopair fuzzy set. For any variable v in the neutrosophic set N, it is described by (t, i, f), where:

$t = T_N(v)$: Truth-membership function, for any v in the neutrosophic set N, where:
$T_N(v)\colon\ N \rightarrow\ ]0^-, 1^+[$

$i = I_N(V)$: Indeterminacy-membership function, for any v in the neutrosophic set N, where:
$I_N(v)\colon\ N \rightarrow\ ]0^-, 1^+[$

$f = F_N(v)$: Falsehood-membership function, for any v in the neutrosophic set N, where:
$F_N(v)\colon\ N \rightarrow\ ]0^-, 1^+[$

3. Neutrosophication Functions

The functions used to convert the crisp values into neutrosophic values are called neutrosophication functions. For each function fun [15], we distinguish two types in neutrosophic logic:

fun_0: function values starting from zero (down to up).
fun_1: function values starting from one (up to down).

The truth-membership, indeterminacy–membership and falsehood-membership functions take their forms from the proposed functions independently of each other, and they do not necessarily take the same form.

3.1. Neutrosophic trapezoidal function ($ntpf$)

The $ntpf$ is defined by specifying 5 parameters $(\alpha, \beta, \gamma, \delta, w)$ where:
$(\alpha, \beta, \gamma, \delta)$ are the vertices of the trapezoid.
(w) represents the height of the neutrosophic trapezoidal function.

Neutrosophic trapezoidal function is defined as:

$$ntpf_0(x, \alpha, \beta, \gamma, \delta, w) = \begin{cases} 0 & x \le \alpha \\ \frac{w(x-\alpha)}{\beta-\alpha} & \alpha \le x \le \beta \\ w & \beta \le x \le \gamma \\ \frac{w(\delta-x)}{\delta-\gamma} & \gamma \le x \le \delta \\ 0 & \delta \le x \end{cases} = \max\left(\min\left(\frac{w(x-\alpha)}{\beta-\alpha}, w, \frac{w(\delta-x)}{\delta-\gamma}\right), 0\right)$$

or

$$ntpf_1(x,\alpha,\beta,\gamma,\delta,w)=\begin{cases}1 & x\le\alpha\\ \dfrac{\beta-\alpha-w(x-\alpha)}{\beta-\alpha} & \alpha\le x\le\beta\\ 1-w & \beta\le x\le\gamma\\ \dfrac{\delta-\gamma-w(\delta-x)}{\delta-\gamma} & \gamma\le x\le\delta\\ 1 & \delta\le x\end{cases}=\min\left(\max\left(\frac{\beta-\alpha-w(x-\alpha)}{\beta-\alpha},1-w,\frac{\delta-\gamma-w(\delta-x)}{\delta-\gamma}\right),1\right)$$

Example 1. The diagrammatic representation of $ntpf_0(x,0.2,0.6,0.7,1,0.5)$ and $ntpf_1(x,0.3,0.5,0.8,0.9,0.8)$ is shown in figure 1.

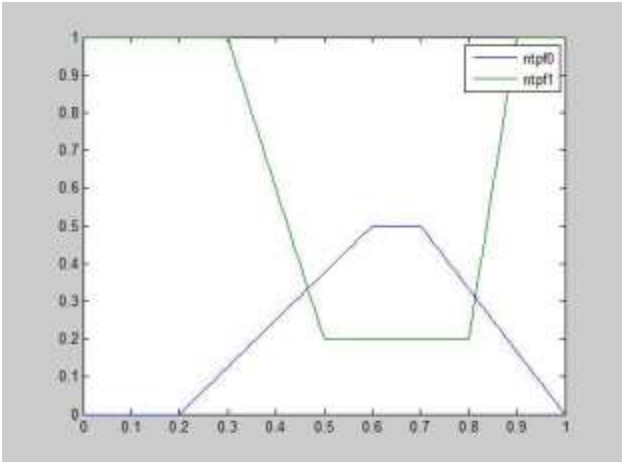

Figure 1. $ntpf_0$ and $ntpf_1$ for example 1.

3.2. Neutrosophic triangular function ($ntgf$)

The $ntgf$ is defined by specifying 4 parameters (α,β,γ,w) where:
(α,β,γ) are the vertices of the triangle.
(w) represents the height of the neutrosophic triangular function.
Neutrosophic triangular function is given as:

$$ntgf_0(x,\alpha,\beta,\gamma,w)=\begin{cases}0 & x\le\alpha\\ \dfrac{w(x-\alpha)}{\beta-\alpha} & \alpha\le x\le\beta\\ \dfrac{w(\gamma-x)}{\gamma-\beta} & \gamma\le x\le\delta\\ 0 & \delta\le x\end{cases}=\max\left(\min\left(\frac{w(x-\alpha)}{\beta-\alpha},\frac{w(\gamma-x)}{\gamma-\beta}\right),0\right)$$

or

$$ntgf_1(x,\alpha,\beta,\gamma,w)=\begin{cases}1 & x\le\alpha\\ \dfrac{\beta-\alpha-w(x-\alpha)}{\beta-\alpha} & \alpha\le x\le\beta\\ \dfrac{\gamma-\beta-w(\gamma-x)}{\gamma-\beta} & \beta\le x\le\gamma\\ 1 & \gamma\le x\end{cases}=\min\left(\max\left(\frac{\beta-\alpha-w(x-\alpha)}{\beta-\alpha},\frac{\gamma-\beta-w(\gamma-x)}{\gamma-\beta}\right),1\right)$$

Example 2. The graphic representation of $ntgf_0(x, 0.2, 0.6, 0.7, 0.9)$ and $ntgf_1(x, 0.3, 0.6, 1, 0.5)$ is shown in figure 2.

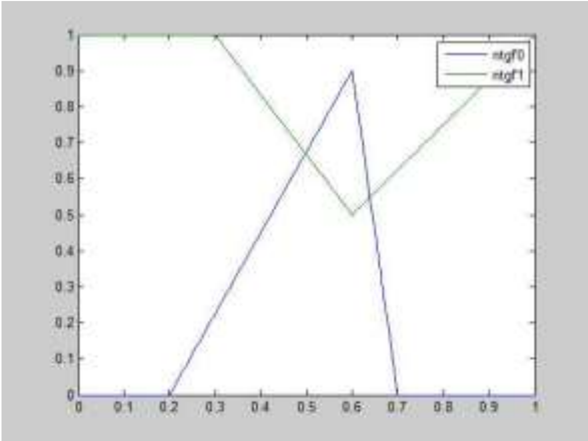

Figure 2. $ntgf_0$ and $ntgf_1$ for example 2.

3.3. Neutrosophic (S and Z)-shaped function ($nszf$)

The $nszf$ is defined by specifying 3 parameters (α, β, w) where:
(α, β) they control the start and end of the bend.
(w) represents the height of the neutrosophic (S and Z)-shaped function.
Neutrosophic (S and Z)-shaped function takes the form:

$$nszf_0(x, \alpha, \beta, w) = \begin{cases} 0 & x \leq \alpha \\ 2w\left(\frac{x-\alpha}{\beta-\alpha}\right)^2 & \alpha \leq x \leq \frac{\alpha+\beta}{2} \\ w - 2w\left(\frac{x-\beta}{\beta-\alpha}\right)^2 & \frac{\alpha+\beta}{2} \leq x \leq \beta \\ w & \beta \leq x \end{cases}$$

or

$$nszf_1(x, \alpha, \beta, w) = \begin{cases} 1 & x \leq \alpha \\ 1 - 2w\left(\frac{x-\alpha}{\beta-\alpha}\right)^2 & \alpha \leq x \leq \frac{\alpha+\beta}{2} \\ 1 - w + 2w\left(\frac{x-\beta}{\beta-\alpha}\right)^2 & \frac{\alpha+\beta}{2} \leq x \leq \beta \\ 1 - w & \beta \leq x \end{cases}$$

Example 3. The diagrammatic representation of $nszf_0(x, 0.3, 0.7, 0.8)$ and $nszf_1(x, 0.3, 0.7, 0.8)$ is shown in figure 3.

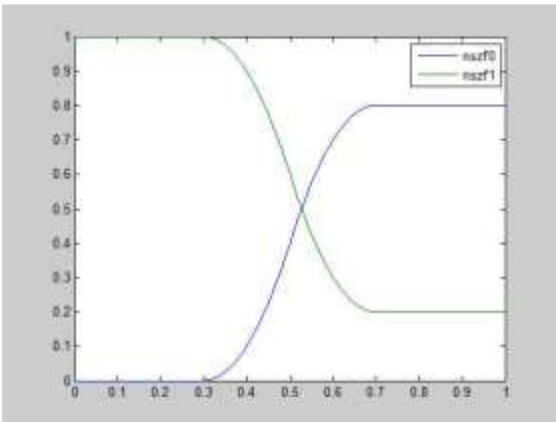

Figure 3. $nszf_0$ and $nszf_1$ for example 3.

3.4. Neutrosophic generalized bell-shaped function ($ngblf$)

The $ngblf$ is defined by specifying 4 parameters $(\alpha, \beta, \gamma, w)$ where:
(α) represents the width of the shape.
(β) is the intensity of the bend on the sides, whenever the value of β increases the bend becomes more intense.
(γ) is the center of the shape.
(w) represents the height of the neutrosophic generalized bell-shaped function.
Neutrosophic generalized bell-shaped function is given by:

$$ngblf_0(x, \alpha, \beta, \gamma, w) = \frac{w}{1 + \left|\frac{x-\gamma}{\alpha}\right|^{2\beta}}$$

or

$$ngblf_1(x, \alpha, \beta, \gamma, w) = \frac{1 + \left|\frac{x-\gamma}{\alpha}\right|^{2\beta} - w}{1 + \left|\frac{x-\gamma}{\alpha}\right|^{2\beta}}$$

Example 4. The graphic representation of $ngblf_0(x, 0.2,3, 0.6,1)$ and $ngblf_1(x, 0.3,7, 0.6,0.8)$ is shown in figure 4.

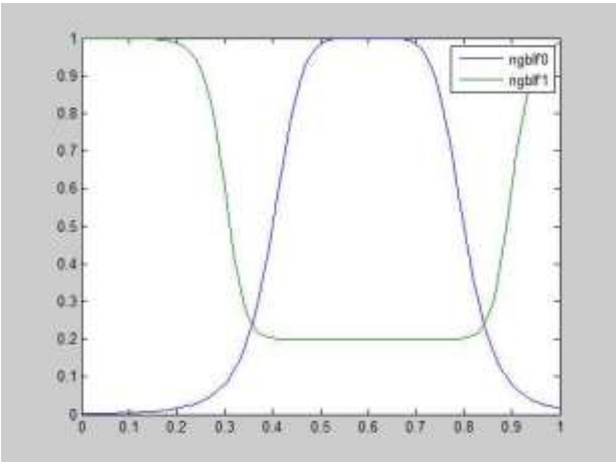

Figure 4. $ngblf_0$ and $ngblf_1$ for example 4.

3.5. Neutrosophic gaussian function ($ngsf$)

The $ngsf$ is defined by specifying 3 parameters (α, β, w) where:
(α) represents the standard deviation for shape.
(β) it is the center of the shape.
(w) represents the height of the Neutrosophic gaussian function.
Neutrosophic gaussian function is defined as:

$$ngsf_0(x, \alpha, \beta, w) = we^{\frac{-(x-\beta)^2}{2\alpha^2}}$$

or

$$ngsf_1(x, \alpha, \beta, w) = 1 - we^{\frac{-(x-\beta)^2}{2\alpha^2}}$$

Example 5. The diagrammatic representation of $ngsf_0(x, 0.1, 0.6,0.9)$ and $ngsf_1(x, 0.3, 0.6, 0.3)$ is shown in figure 5.

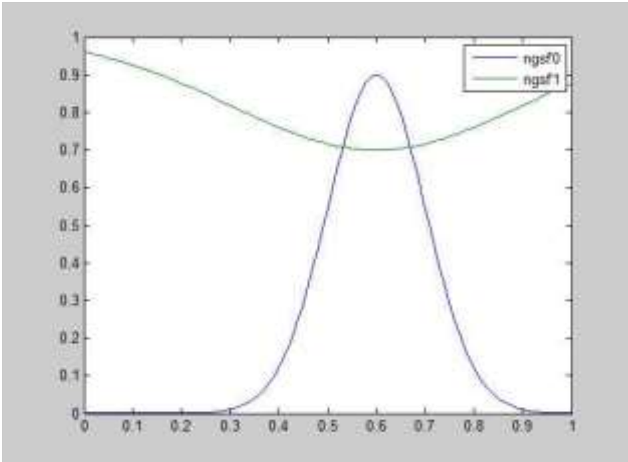

Figure 5. $ngsf_0$ and $ngsf_1$ for example 5.

3.6. Neutrosophic sigmoidal function ($nsmf$)

The $nsmf$ is defined by specifying 3 parameters (α, β, w) where:
(α) controls the width of the transition area.
(β) defines the center of the transition area.
(w) represents the height of the Neutrosophic sigmoidal function.
Neutrosophic sigmoidal function takes the form:

$$nsmf_0(x, \alpha, \beta, w) = \frac{w}{1 + e^{-\alpha(x-\beta)}}$$

or

$$nsmf_1(x, \alpha, \beta, w) = \frac{1 + e^{-\alpha(x-\beta)} - w}{1 + e^{-\alpha(x-\beta)}}$$

Example 6. The graphic representation of $nsmf_0(x, 15,0.5,1)$ and $nsmf_1(x, 15,0.5,0.4)$ is shown in figure 6.

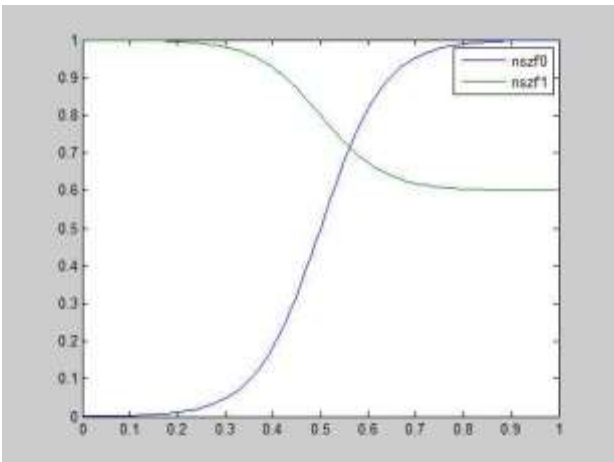

Figure 6. $nsmf_0$ and $nsmf_1$ for example 6.

4. Proposed MATLAB code to neutrosophication functions

In this section, neutrosophication functions have proposed using MATLAB program, a graphic representation has been given for the different membership values, and the MATLAB code has been designed as follows:

```
function [y,z,t]=nfun(x,tt,ii,ff)
y= feval(tt{1},x,tt);
z=feval(ii{1},x,ii);
t=feval(ff{1},x,ff);
plot(x,y,x,z,x,t);
legend('Truth-membership function','Indeterminacy–membership function','Falsehood-membership function');
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [y]=ntpf(x,tt)
y = trapmf(x,[tt{3} tt{4} tt{5} tt{6}])*tt{end};
if(tt{2}==1)
y=1-y;
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [y]=ntgf(x,tt)
y = trimf(x,[tt{3} tt{4} tt{5}])*tt{end};
if(tt{2}==1)
y=1-y;
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [y]=ngsf(x,tt)
y = gaussmf(x,[tt{3} tt{4}])*tt{end};
if(tt{2}==1)
y=1-y;
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [y]=ngblf(x,tt)
y = gbellmf(x,[tt{3} tt{4} tt{5}])*tt{end};
if(tt{2}==1)
y=1-y;
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [y]=nsmf(x,tt)
y = sigmf(x,[tt{3} tt{4}])*tt{end};
if(tt{2}==1)
y=1-y;
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [y]=nszf(x,tt)
y = smf(x,[tt{3} tt{4}])*tt{end};
if(tt{2}==1)
y=1-y;
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Example 7. The figure 7 represents the truth-membership function $ntpf_0(x, 0.3, 0.4,0.8,0.9,1)$, indeterminacy–membership function $ntgf_1(x, 0.2, 0.4, 1,0.6)$, and falsehood-membership function $ngsf_1(x, 0.1, 0.6, 0.7)$ by writing the code in the MATLAB program below:

```
x=0:0.01:1;
% {'fun name', 0≈fun0 or 1≈fun1, fun parameters}
tt={'ntpf',0,0.3, 0.4,0.8,0.9,1};
ii={'ntgf',1,0.2, 0.4, 1,0.6};
ff={'ngsf',1,0.1, 0.6, 0.7};
% [y,z,t]=nfun (x, truth, indeterminacy, falsehood)
[y,z,t]=nfun(x,tt,ii,ff);
```

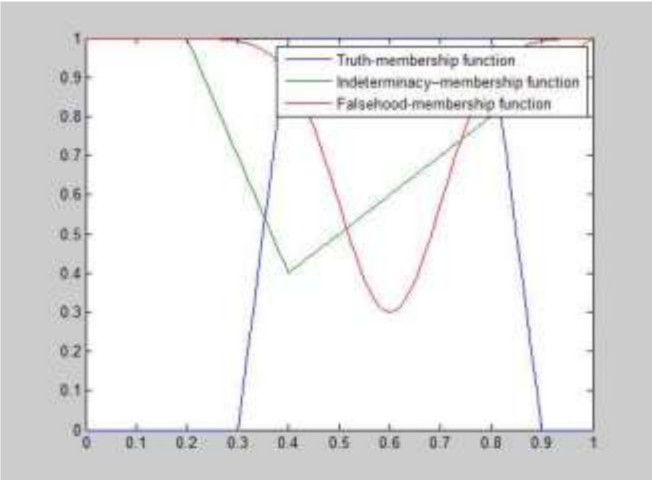

Figure 7. Represents neutrosophic functions in the example 7.

Example 8. The figure 8 represents the truth-membership function $ngblf_0(x, 0.2, 4,0.5,1)$, indeterminacy–membership function $nszf_1(x, 0.1, 0.7, 0.9)$, and falsehood-membership function $ntgf_1(x, 0.1, 0.5,0.9, 1)$ by writing the code in the MATLAB program below:

```
x=0:0.01:1;
tt={'ngblf',0,0.2, 4,0.5,1};
ii={'nszf',1,0.1, 0.7, 0.9};
ff={'ntgf',1,0.1, 0.5,0.9, 1};
[y,z,t]=nfun(x,tt,ii,ff);
```

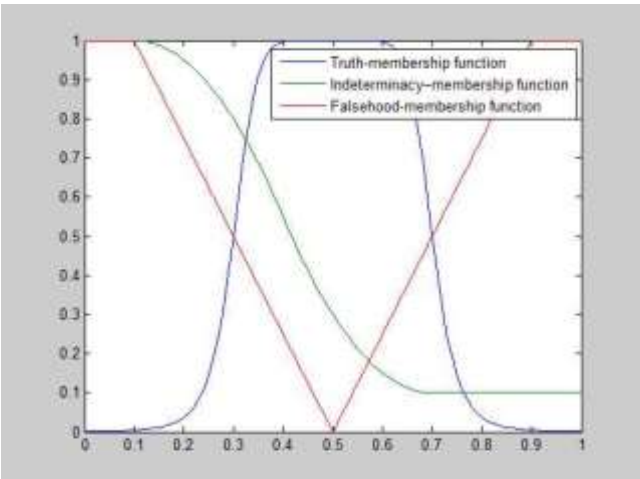

Figure 8. Represents neutrosophic functions in the example 8.

Example 9. The figure 9 represents the truth-membership function $ntpf_0(x, 15, 21, 26, 35, 1)$, indeterminacy–membership function $nsmf_0(x, 1, 35, 0.6)$, and falsehood-membership function $ngsf_1(x, 5, 24, 1)$ by writing the code in the MATLAB program below:

```
x=10:40;
tt={'ntpf',0,15, 21,26,35,1};
ii={'nsmf',0,1, 35,0.6};
ff={'ngsf',1,5, 24, 1};
[y,z,t]=nfun(x,tt,ii,ff);
```

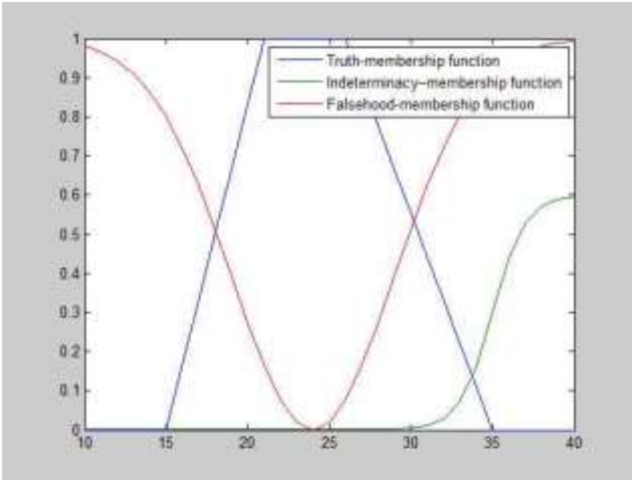

Figure 9. Represents neutrosophic functions in the example 9.

5. Conclusions

By taking advantage of the most important functions used in the different fuzzy processing systems, we introduced the neutrosophication functions in a way that preserves the properties and independence of the values of truth, indeterminacy, and falsity. These functions have been graphically represented using MATLAB by proposing a code for that.

Our current research is an important reference for writing papers related to neutrosophic processing systems by relying on the proposed functions, and this is what we will work on in the future in relation to digital image processing, in particular denoising digital images using neutrosophic logic.

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Neutrosophic Hypersoft Topological Spaces

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Abstract: Hypersoft sets have gained more importance as a generalization of soft sets and have been investigated for possible extensions in many fields of mathematics. The main objective of this paper is to introduce Fuzzy Hypersoft Topology and study some of its properties such as neighbourhood of fuzzy hypersoft set, interior hypersoft set and closure fuzzy hypersoft set. Fuzzy hypersoft topology is then extended to Intuitionistic Hypersoft topology, Neutrosophic Hypersoft topology and its basic properties are discussed.

Keywords: Fuzzy Hypersoft Set, Hypersoft Set, Fuzzy Hypersoft Topology, Intuitionistic Hypersoft Topology, Neutrosophic Hypersoft Topology, Interior, Closure.

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1. Introduction

Zadeh [1] in 1965 presented the idea of fuzzy set theory, which has a very important role in solving problems by providing a suitable way for the expression of vague concepts by having membership. Computer scientists and mathematicians have studied and developed fuzzy set theory with widened applications in fuzzy logic, fuzzy topology, fuzzy control systems, etc. Also theories such as fuzzy probability, soft and rough set theories are used to solve these problems. A new approach for handling uncertainty, the idea of soft theory was presented by Molodtsov [2] in 1999. Now, there is a rapid growth of soft theory with applications in many fields. Several basic notions of soft set theory were defined by Maji et al. [3] while his works were improved in [4-7]. A combination of fuzzy sets and soft sets, named as fuzzy soft set theory, was presented by Maji et al. [8].

The idea of soft sets was generalized into hypersoft sets by Smarandache [9] by transforming the argument function F into a multi-argument function. He also introduced many results on hypersoft sets. Saqlain et al. [10] utilized this notion and proposed a generalized TOPSIS method for decision making. Neutrosophic sets [17], from their very introduction, have seen many such extensions and have been very successful in applications [18-29, 48-50]. In 2019, Rana et al. [11] introduced Plithogenic Fuzzy Hypersoft Set (PFHS) in matrix form and defined some operations on PFHS. Single and multi-valued Neutrosophic Hypersoft set were proposed by Saqlain et al. [12], who also defined tangent similarity measure for single-valued sets and an application of the same in a decision making scenario. In an another effort, Saqlain et al. [13] also introduced aggregation operators for neutrosophic hypersoft sets. A recent development in this area of research is the introduction of basic operations on hypersoft sets in which hypersoft points in different fuzzy environments are also introduced [14].

Fuzzy topology, a collection of fuzzy sets fulfilling the axioms was defined by Chang [15] in 1968. Fuzzy set theory was applied into topology by Chang and many topological notions were introduced in fuzzy setting such as convergence and compactness [30-32]. Then Intuitionistic fuzzy topological spaces were introduced and were developed further into many new concepts as separation axioms, categorical property, connectedness [33, 34, 37-39]. Neutrosophic topological spaces were introduced by Salma et al. and further concepts as connectedness, semi closed sets and generalized closed sets were developed [40-44]. Olgun developed the concept of Pythagorean topological spaces and recently Pythagorean nano topological spaces were introduced and advanced into concepts such as weak open sets [35, 36, 45-47]. The notion of fuzzy soft topological structure was coined by Tanay et al. and was further enquired [16, 51, 52]. This notion was applied to the advanced sets as intuitionistic and neutrosophic soft sets thus developed as Intuitionistic and Neutrosophic soft topological spaces [53-59].

In this paper, we define the concept of 'Fuzzy Hypersoft Topology' with the fuzzy hypersoft sets and we define some basic notions. A logical extension of this topology would necessarily be Intuitionistic and Neutrosophic Hypersoft topologies. Hence we propose Intuitionisitc and Neutrosophic Hypersoft topology in this paper. Following this we describe the basic definitions and concepts in second section and the third section contains the introduction of the base fuzzy hypersoft topological spaces with few properties. Fourth and fifth sections contain the extension of fuzzy hypersoft topological spaces which are intuitionistic and neutrosophic hypersoft topological spaces along with basic properties.

2. Preliminaries

Definition 2.1

Let V be the universe, $P(V)$ the power set of V and $E_1, E_2, E_3 \dots E_m$ be the parameters which are pairwise disjoint. Let A_l be the non-empty subset of E_l for each $l = 1, 2, \dots m$. A hypersoft set is the pair $(\Theta, A_1 \times A_2 \times \dots \times A_m)$ where

$$\Theta: A_1 \times A_2 \times \dots \times A_m \to P(V).$$

Simply, we write the symbols $\mathbb{E}$ for $E_1 \times E_2 \times \dots \times E_m$, $\mathfrak{J}$ for $A_1 \times A_2 \times \dots \times A_m$ and $\boldsymbol{a}$ for an element of $\mathfrak{J}$.

Definition 2.2

Let the fuzzy universe be V, $\mathfrak{A}$ a subset of $\mathbb{E}$. Then $(\Theta, \mathfrak{J})$ is called

1. a null fuzzy hypersoft set if for each parameter $\boldsymbol{a} \in \mathfrak{J}$, $\Theta(\boldsymbol{a})$ is 0.
2. an absolute fuzzy hypersoft set if for each parameter $\boldsymbol{a} \in \mathfrak{J}$, $\Theta(\boldsymbol{a})$ is V.

Definition 2.3 [14]

Let $(\Theta, \mathfrak{J})$ and $(\vartheta, \mathfrak{B})$ be two fuzzy hypersoft (FH) sets over V. Then union of $(\Theta, \mathfrak{J})$ and $(\vartheta, \mathfrak{B})$ is $(\xi, \mathfrak{G}) = (\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})$ with $\mathfrak{G} = G_1 \times G_2 \times \dots \times G_n$ where $G_k = A_k \cup B_k$ for $k = 1, 2, \dots n$ and ξ is defined by

$$\xi(\boldsymbol{a}) = \begin{cases} \Theta(\boldsymbol{a}) & if\ \boldsymbol{a} \in \mathfrak{J} - \mathfrak{B} \\ \vartheta(\boldsymbol{a}) & if\ \boldsymbol{a} \in \mathfrak{B} - \mathfrak{J} \\ \Theta(\boldsymbol{a}) \cup \vartheta(\boldsymbol{a}) & if\ \boldsymbol{a} \in \mathfrak{J} \cup \mathfrak{B} \\ 0 & else \end{cases} \qquad \text{where } \boldsymbol{a} = (G_1, G_2, \dots G_n) \in \mathfrak{G}.$$

Definition 2.4 [14]

Let $(\Theta, \mathfrak{J})$ and $(\vartheta, \mathfrak{B})$ be two FH sets. The intersection is denoted by $(\xi, \mathfrak{G}) = (\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})$ where $\mathfrak{G} = G_1 \times G_2 \times \dots \times G_n$ where $G_k = A_k \cap B_k$ for $k = 1, 2, \dots n$.

$$\xi(\boldsymbol{a}) = \begin{cases} \Theta(\boldsymbol{a}) & if\ \boldsymbol{a} \in \mathfrak{J} - \mathfrak{B} \\ \vartheta(\boldsymbol{a}) & if\ \boldsymbol{a} \in \mathfrak{B} - \mathfrak{J} \\ \Theta(\boldsymbol{a}) \cap \vartheta(\boldsymbol{a}) & if\ \boldsymbol{a} \in \mathfrak{J} \cap \mathfrak{B} \end{cases} \qquad \text{where } \boldsymbol{a} = (G_1, G_2, \dots G_n) \in \mathfrak{G}.$$

Definition 2.5

Let $(\Theta, \mathfrak{J})$ and $(\vartheta, \mathfrak{B})$ be two FH sets. $(\Theta, \mathfrak{J})$ is called a FH subset of $(\vartheta, \mathfrak{B})$, if $\mathfrak{J} \subseteq \mathfrak{B}$ & $\Theta(\boldsymbol{a}) \subseteq \vartheta(\boldsymbol{a})$ for all $\boldsymbol{a} \in \mathfrak{J}$. We denote this by $(\Theta, \mathfrak{J}) \subseteq (\vartheta, \mathfrak{B})$ [14].

Definition 2.6

Let $(\Theta, \mathfrak{J})$ and $(\vartheta, \mathfrak{B})$ be two FH sets. $(\Theta, \mathfrak{J})$ & $(\vartheta, \mathfrak{B})$ are equal if and only if $(\Theta, \mathfrak{J}) \subseteq (\vartheta, \mathfrak{B})$ and $(\vartheta, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J})$ [14].

3. Fuzzy Hypersoft Topological Space

In this section, we define the concept of "Fuzzy Hypersoft Topology". Let $E_1, E_2, E_3 \dots E_n$ be the parameters of the universe V, the set of all fuzzy sets be $F(V)$, the collection of all FH sets over $V_{\mathbb{E}}$ (where $\mathbb{E} = E_1 \times E_2 \times E_3 \dots \times E_n$) be $\mathfrak{P}(V, \mathbb{E})$.

Definition 3.1

Let $(\varrho, \mathfrak{X})$ be an element of $\mathfrak{P}(V, \mathbb{E})$ (where $\mathfrak{X} = \mathrm{X}_1 \times \mathrm{X}_2 \times \mathrm{X}_3 \dots \times \mathrm{X}_n$ with each X_i is a subset of E_i $(i = 1, 2 \dots n)$, set of all fuzzy hypersoft (FH) subsets of $(\varrho, \mathfrak{X})$ be $P(\varrho, \mathfrak{X})$ and τ, a subcollection of $P(\varrho, \mathfrak{X})$.

(i) $\phi_{\mathfrak{X}}, (\varrho, \mathfrak{X}) \in \tau$
(ii) $(\Theta, \mathfrak{J}), (\vartheta, \mathfrak{B}) \in \tau \Longrightarrow (\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B}) \in \tau$
(iii) $\{(\Theta, \mathfrak{J})_l \mid l \in L\} \in \tau \Longrightarrow \bigcup_{l \in L} (\Theta, \mathfrak{J})_l \in \tau$

If the above axioms are satisfied then τ is fuzzy hypersoft topology (FHT) on $(\varrho, \mathfrak{X})$. $(\mathfrak{X}_\varrho, \tau)$ is called a fuzzy hypersoft topological space (FHTS). Every member of τ is called open fuzzy hypersoft set (OFHS). A fuzzy hypersoft set if called closed fuzzy hypersoft set (CFHS) if its complement is OFHS.

For example, $\{\phi_{\mathfrak{X}}, (\varrho, \mathfrak{X})\}$ and $P(\varrho, \mathfrak{X})$ are fuzzy hypersoft topology on $(\varrho, \mathfrak{X})$ and are called as indiscrete FHT and discrete FHT respectively.

Example 3.2

Let $V = \{x_1, x_2, x_3, x_4\}$ and the attributes be $E_1 = \{a_1, a_2\}$, $E_2 = \{a_3, a_4\}$ and $E_3 = \{a_5, a_6\}$. Then the fuzzy hypersoft set be

$$\left\{ \begin{matrix} \left((a_1, a_3, a_5), \left\{\frac{x_2}{0.4}, \frac{x_4}{0.6}\right\}\right), \left((a_1, a_3, a_6), \left\{\frac{x_1}{0.7}\right\}\right), \left((a_1, a_4, a_5), \left\{\frac{x_1}{0.4}, \frac{x_2}{0.3}\right\}\right), \left((a_1, a_4, a_6), \left\{\frac{x_1}{0.5}, \frac{x_3}{0.7}\right\}\right), \\ \left((a_2, a_3, a_5), \left\{\frac{x_2}{0.3}, \frac{x_3}{0.5}\right\}\right), \left((a_2, a_3, a_6), \left\{\frac{x_3}{0.8}\right\}\right), \left((a_2, a_4, a_5), \left\{\frac{x_4}{0.9}\right\}\right), \left((a_2, a_4, a_6), \left\{\frac{x_2}{0.6}\right\}\right) \end{matrix} \right\}$$

Let us consider this fuzzy hypersoft as $(\varrho, \mathfrak{X})$. Then the subfamily

$\tau = \{\phi_{\mathfrak{X}}, (\varrho, \mathfrak{X}),$

$\left\{\left((a_1, a_3, a_5), \left\{\frac{x_1}{0.3}, \frac{x_2}{0.6}\right\}\right), \left((a_2, a_3, a_5), \left\{\frac{x_2}{0.4}, \frac{x_3}{0.5}\right\}\right)\right\}, \left\{\left((a_1, a_3, a_5), \left\{\frac{x_2}{0.4}\right\}\right), \left((a_2, a_3, a_5), \left\{\frac{x_2}{0.3}, \frac{x_3}{0.5}\right\}\right)\right\},$

$$\left\{\begin{array}{c}\left((a_1,a_3,a_5),\left\{\frac{x_1}{0.3},\frac{x_2}{0.6},\frac{x_4}{0.6}\right\}\right),\left((a_1,a_4,a_6),\left\{\frac{x_1}{0.5},\frac{x_3}{0.7}\right\}\right),\left((a_1,a_3,a_6),\left\{\frac{x_1}{0.7}\right\}\right),\left((a_1,a_4,a_5),\left\{\frac{x_1}{0.4},\frac{x_2}{0.3}\right\}\right),\left((a_2,a_3,a_5),\left\{\frac{x_2}{0.4},\frac{x_3}{0.5}\right\}\right),\\ \left((a_2,a_3,a_6),\left\{\frac{x_3}{0.8}\right\}\right),\left((a_2,a_4,a_5),\left\{\frac{x_4}{0.9}\right\}\right),\left((a_2,a_4,a_6),\left\{\frac{x_2}{0.6}\right\}\right)\end{array}\right\}$$

of $P(\varrho,\mathfrak{X})$ is a FHT on $(\varrho,\mathfrak{X})$.

Definition 3.3

Let τ be a FHT on $(\varrho,\mathfrak{X})\in\mathfrak{P}(V,\mathbb{E})$ and $(\vartheta,\mathfrak{B})$ be a FH set in $P(\varrho,\mathfrak{X})$. A FH set $(\Theta,\mathfrak{J})$ in $P(\varrho,\mathfrak{X})$ is a neighbourhood of FH set of $(\vartheta,\mathfrak{B})$ if and only if there exists an OFHS $(\xi,\mathfrak{C})$ such that $(\vartheta,\mathfrak{B})\subset(\xi,\mathfrak{C})\subset(\Theta,\mathfrak{J})$.

Theorem 3.4

A FH set $(\Theta,\mathfrak{J})$ in $P(\varrho,\mathfrak{X})$ is an OFHS if and only if $(\Theta,\mathfrak{J})$ is a neighbourhood of each FH set $(\vartheta,\mathfrak{B})$ contained in $(\Theta,\mathfrak{J})$.

Proof:

Consider an OFHS $(\Theta,\mathfrak{J})$ and any FH set $(\vartheta,\mathfrak{B})$ confined in $(\Theta,\mathfrak{J})$. Thus we have $(\vartheta,\mathfrak{B})\subset(\Theta,\mathfrak{J})\subset(\Theta,\mathfrak{J})$. Implies that $(\theta,\mathfrak{A})$ is a neighbourhood of $(\vartheta,\mathfrak{B})$.

Let $(\theta,\mathfrak{A})$ be a neighbourhood of each FH set confined in it. Since $(\Theta,\mathfrak{J})\subset(\Theta,\mathfrak{J})$, there exists an OFHS $(\xi,\mathfrak{C})$ such that $(\Theta,\mathfrak{J})\subset(\xi,\mathfrak{C})\subset(\Theta,\mathfrak{J})$. Thus $(\Theta,\mathfrak{J})=(\xi,\mathfrak{C})$, $(\Theta,\mathfrak{J})$ is OFHS.

Definition 3.5

Let $(\mathfrak{X}_\varrho,\tau)$ is called a FHTS on $(\varrho,\mathfrak{X})$ and $(\Theta,\mathfrak{J})$ be a FH set in $P(\varrho,\mathfrak{X})$. The neighbourhood system of $(\Theta,\mathfrak{J})$ relative to τ is the collection of all neighbourhood of $(\Theta,\mathfrak{J})$ and denoted by $HN_{(\Theta,\mathfrak{J})}$.

Theorem 3.6

If $HN_{(\Theta,\mathfrak{J})}$ is the neighbourhood systems of FH set $(\Theta,\mathfrak{J})$. Then,

1. Finite intersection of member of $HN_{(\Theta,\mathfrak{J})}$ belongs to $HN_{(\Theta,\mathfrak{J})}$.
2. Each FH set which has a member of $HN_{(\Theta,\mathfrak{J})}$ belongs to $HN_{(\Theta,\mathfrak{J})}$.

Proof

1. $(\vartheta,\mathfrak{B})$ and $(\xi,\mathfrak{C})\in HN_{(\Theta,\mathfrak{J})}$ then there exists $(\vartheta',\mathfrak{B}'),(\xi',\mathfrak{C}')\in\tau$ such that

$$(\Theta,\mathfrak{J})\subset(\vartheta',\mathfrak{B}')\subset(\vartheta,\mathfrak{B})\text{ and }(\Theta,\mathfrak{J})\subset(\xi',\mathfrak{C}')\subset(\xi,\mathfrak{C}).$$

Since $(\vartheta',\mathfrak{B}')\cap(\xi',\mathfrak{C}')\in\tau$ we get $(\theta,\mathfrak{A})\subset(\vartheta',\mathfrak{B}')\cap(\xi',\mathfrak{C}')\subset(\vartheta,\mathfrak{B})\cap(\xi,\mathfrak{C})$

Hence $(\vartheta,\mathfrak{B})\cap(\xi,\mathfrak{C})$ belongs to $HN_{(\Theta,\mathfrak{J})}$.

2. Let $(\vartheta,\mathfrak{B})\in HN_{(\Theta,\mathfrak{J})}$ and $(\xi,\mathfrak{C})$ be a FH set having $(\vartheta,\mathfrak{B})$.

Since $(\vartheta,\mathfrak{B})\in HN_{(\Theta,\mathfrak{J})}$ there exists an OFHS containing $(\vartheta',\mathfrak{B}')$ such that $(\Theta,\mathfrak{J})\subset(\vartheta',\mathfrak{B}')\subset(\vartheta,\mathfrak{B})$ it follows that $(\Theta,\mathfrak{J})\subset(\vartheta',\mathfrak{B}')\subset(\xi,\mathfrak{C})$. Thus $(\xi,\mathfrak{C})$ belongs to $HN_{(\Theta,\mathfrak{J})}$.

Definition 3.7

Let $(\mathfrak{X}_\varrho,\tau)$ is called a FHTS and $(\Theta,\mathfrak{J}),(\vartheta,\mathfrak{B})$ be FH set in $P(\varrho,\mathfrak{X})$ such that $(\vartheta,\mathfrak{B})\subset(\Theta,\mathfrak{J})$. Then $(\vartheta,\mathfrak{B})$ is said to be an interior fuzzy hypersoft set (IFHS) of $(\Theta,\mathfrak{J})$ if and only if $(\Theta,\mathfrak{J})$ is a neighbourhood of $(\vartheta,\mathfrak{B})$.

The union of whole IFHS of $(\Theta,\mathfrak{J})$ is named the interior of $(\Theta,\mathfrak{J})$ and denoted as $(\Theta,\mathfrak{J})^\circ$.

Theorem 3.8

Let $(\mathfrak{X}_\varrho, \tau)$ is called a FHTS and $(\Theta, \mathfrak{J})$, a FH set in $P(\varrho, \mathfrak{X})$. Then,

i) $(\Theta, \mathfrak{J})^\circ$ is open and $(\Theta, \mathfrak{J})^\circ$ is the biggest OFHS confined in $(\Theta, \mathfrak{J})$.
ii) $(\Theta, \mathfrak{J})$ is OFHS iff $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^\circ$.

Proof

i) Since $(\Theta, \mathfrak{J})^\circ = \cup \{(\vartheta, \mathfrak{B})/(\Theta, \mathfrak{J})\ is\ a\ neighbourhood\ of\ (\vartheta, \mathfrak{B})\}$, $(\Theta, \mathfrak{J})^\circ$ is itself an IFHS of $(\Theta, \mathfrak{J})$. Then there exists an OFHS $(\xi, \mathfrak{C})$ such that $(\Theta, \mathfrak{J})^\circ \subset (\xi, \mathfrak{C}) \subset (\Theta, \mathfrak{J})$. $(\xi, \mathfrak{C})$ is an IFHS of $(\Theta, \mathfrak{J})$, hence $(\xi, \mathfrak{C}) \subset (\Theta, \mathfrak{J})^\circ$. Thus $(\Theta, \mathfrak{J})^\circ$ is the largest OFHS enclosed in $(\Theta, \mathfrak{J})$.

ii) Let $(\Theta, \mathfrak{J})$ be an OFHS. Since $(\Theta, \mathfrak{J})^\circ$ is the IFHS of $(\Theta, \mathfrak{J})$, we have $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^\circ$.

Conversely if $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^\circ$ then $(\Theta, \mathfrak{J})$ is OFHS.

Definition 3.9

Let $(\mathfrak{X}_\varrho, \tau_1)$ and $(\mathfrak{X}_\varrho, \tau_2)$ be two FHTS. If each $(\Theta, \mathfrak{J}) \in \tau_1$ is in τ_2 then τ_2 is called the FH finer than τ_1 (or) τ_1 is FH coarser than τ_2.

Definition 3.10

Let $(\mathfrak{X}_\varrho, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}) \in \mathfrak{P}(V, \mathbb{E})$. The fuzzy hypersoft closure (FHC) of $(\Theta, \mathfrak{J})$ is the intersection of all CFH sets that contains $(\Theta, \mathfrak{J})$ which is denoted by $\overline{(\Theta, \mathfrak{J})}$.

Thus, $\overline{(\Theta, \mathfrak{J})}$ is the smallest CFHS which has $(\Theta, \mathfrak{J})$ and $\overline{(\Theta, \mathfrak{J})}$ is CFHS.

Theorem 3.11

Let $(\mathfrak{X}_\varrho, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}), (\vartheta, \mathfrak{B}) \in \mathfrak{P}(V, \mathbb{E})$.
Then,

(i) $(\Theta, \mathfrak{J}) \subseteq \overline{(\Theta, \mathfrak{J})}$
(ii) $\overline{\overline{(\Theta, \mathfrak{J})}} = \overline{(\Theta, \mathfrak{J})}$
(iii) If $(\Theta, \mathfrak{J}) \subset (\vartheta, \mathfrak{B})$, then $\overline{(\Theta, \mathfrak{J})} \subset \overline{(\vartheta, \mathfrak{B})}$.
(iv) $(\Theta, \mathfrak{J})$ is a CFHS iff $(\Theta, \mathfrak{J}) = \overline{(\Theta, \mathfrak{J})}$.
(v) $\overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})} = \overline{(\Theta, \mathfrak{J})} \cup \overline{(\vartheta, \mathfrak{B})}$

Proof

From the definition of FHC, the proof of (i) to (iii) is attained.
(iv) Let $(\Theta, \mathfrak{J})$ be CFHS. By (i) $(\Theta, \mathfrak{J}) \subseteq \overline{(\Theta, \mathfrak{J})}$. Since $\overline{(\Theta, \mathfrak{J})}$ is the minutest CFHS which has $(\Theta, \mathfrak{J})$, then $\overline{(\Theta, \mathfrak{J})} \subseteq (\Theta, \mathfrak{J})$. Thus $(\Theta, \mathfrak{J}) = \overline{(\Theta, \mathfrak{J})}$.

Conversely let, $(\Theta, \mathfrak{J}) = \overline{(\Theta, \mathfrak{J})}$. Since $(\Theta, \mathfrak{J})$ is CFHS, then $(\Theta, \mathfrak{J})$ is also CFHS.

(v) By (iv) $\overline{(\Theta, \mathfrak{J})}$, $\overline{(\vartheta, \mathfrak{B})} \subseteq \overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$. So $\overline{(\Theta, \mathfrak{J})} \cup \overline{(\vartheta, \mathfrak{B})} \subseteq \overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$.

Conversely by (i), $(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B}) \subseteq \overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$.

Since $(\Theta, \mathfrak{J})$, $(\vartheta, \mathfrak{B})$ are FH sets and $\overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$ is the minutest CFHS which has $(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})$, then $\overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})} \subseteq \overline{(\Theta, \mathfrak{J})} \cup \overline{(\vartheta, \mathfrak{B})}$

Thus the equality is obtained.

Theorem 3.12

Let $(\mathfrak{X}_\varrho, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}), (\vartheta, \mathfrak{B}) \in \mathfrak{P}(V, \mathbb{E})$.
Then,

(i) $(\Theta, \mathfrak{J})^\circ \subseteq (\Theta, \mathfrak{J})$
(ii) $((\Theta, \mathfrak{J})^\circ)^\circ = (\Theta, \mathfrak{J})^\circ$
(iii) If $(\Theta, \mathfrak{J}) \subseteq (\vartheta, \mathfrak{B})$, then $(\Theta, \mathfrak{J})^\circ \subseteq (\vartheta, \mathfrak{B})^\circ$.
(iv) $(\Theta, \mathfrak{J})$ is OFHS iff $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^\circ$.
(v) $\left((\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})\right)^\circ = (\Theta, \mathfrak{J})^\circ \cap (\vartheta, \mathfrak{B})^\circ$.

Proof

(i) – (iii) are obvious from definition of interior

(iv) Let $(\Theta, \mathfrak{J})$ be a OFHS, by (i) $(\Theta, \mathfrak{J})^\circ \subseteq (\Theta, \mathfrak{J})$. Since $(\Theta, \mathfrak{J})^\circ$ is the largest OFHS that is contained in $(\Theta, \mathfrak{J})$, then $(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J})^\circ$. Thus $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^\circ$

Conversely, let $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^\circ$ since $(\Theta, \mathfrak{J})^\circ$ is OFHS, $(\Theta, \mathfrak{J})$ is also OFHS.

(v) $(\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}), (\vartheta, \mathfrak{B})$. Thus by (iii) $\left((\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})\right)^\circ \subseteq (\Theta, \mathfrak{J})^\circ \cap (\vartheta, \mathfrak{B})^\circ$.

Conversely by (i), $(\Theta, \mathfrak{J})^\circ \cap (\vartheta, \mathfrak{B})^\circ \subseteq (\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})$. Since $(\Theta, \mathfrak{J})^\circ$, $(\vartheta, \mathfrak{B})^\circ$ are OFHS & $\left((\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})\right)^\circ$ is the largest OFHS that has $(\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})$, then $(\Theta, \mathfrak{J})^\circ \cap (\vartheta, \mathfrak{B})^\circ \subseteq \left((\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})\right)^\circ$. Thus, the equality is achieved.

Definition 3.13

Let $(\mathfrak{X}_\varrho, \tau)$ be a FHTS and $\mathcal{B}$ be a subcollection of τ. If each element of τ can be written as the arbitrary union of few elements of $\mathcal{B}$, then $\mathcal{B}$ is called a fuzzy hypersoft basis (FHB) for the FHT τ.

Lemma 3.14

Let $(\mathfrak{X}_\varrho, \tau)$ be a FHTS and $\mathcal{B}$ be FHB for τ. Then τ is the collection of FH union of elements of $\mathcal{B}$.

Lemma 3.15

Let $(\mathfrak{X}_\varrho, \tau)$ and $(\mathfrak{X}_\varrho, \tau')$ be FHTS and $\mathcal{B}, \mathcal{B}'$ be FHB for τ and τ' respectively. If $\mathcal{B}' \subset \mathcal{B}$, then τ is FH finer than τ'.

Lemma 3.16

Let $\{(\vartheta^i, \mathfrak{B}_i)/i \in \mathrm{I}\}$ be a collection of FH sets corresponding to V, and $(\Theta, \mathfrak{J})$ be a FH over V. Then

(i) $\cup_{i\in \mathrm{I}} [(\Theta, \mathfrak{J}) \cap (\vartheta^i, \mathfrak{B}_i)] = (\Theta, \mathfrak{J}) \cap (\cup_{i\in \mathrm{I}} (\vartheta^i, \mathfrak{B}_i))$
(ii) $\cap_{i\in \mathrm{I}} [(\Theta, \mathfrak{J}) \cup (\vartheta^i, \mathfrak{B}_i)] = (\Theta, \mathfrak{J}) \cup (\cap_{i\in \mathrm{I}} (\vartheta^i, \mathfrak{B}_i))$

Proof

(i) Let $(\Theta, \mathfrak{J}) \cap (\vartheta^i, \mathfrak{B}_i) = (\xi, \mathfrak{C})$ where $\mathfrak{C} = \mathfrak{J} \cap \mathfrak{B}_i$. Then $\cup_{i\in \mathrm{I}} [(\Theta, \mathfrak{J}) \cap (\vartheta^i, \mathfrak{B}_i)] = (\xi', \mathfrak{C}')$ where $\mathfrak{C}' = \cup_{i\in \mathrm{I}} (\mathfrak{J} \cap \mathfrak{B}_i)$. let $\cup_{i\in \mathrm{I}} (\vartheta^i, \mathfrak{B}_i) = (\xi'', \mathfrak{C}'')$ where $\mathfrak{C}'' = \cup_{i\in \mathrm{I}} \mathfrak{B}_i$. Then $(\Theta, \mathfrak{J}) \cap \cup_{i\in \mathrm{I}} (\vartheta^i, \mathfrak{B}_i) = (\xi''', \mathfrak{C}''')$ where $\mathfrak{C}''' = (\mathfrak{J} \cap \mathfrak{C}'')$. Since $\mathfrak{J} \cap (\cup_{i\in \mathrm{I}} \mathfrak{B}) = \cup_{i\in \mathrm{I}} (\mathfrak{J} \cap \mathfrak{B}_i)$, we have

$$\mathfrak{C}' = \cup_{i\in \mathrm{I}} (\mathfrak{J} \cap \mathfrak{B}_i) \text{ and } \mathfrak{C}''' = \mathfrak{J} \cap (\cup_{i\in \mathrm{I}} \mathfrak{B}_i) = \cup_{i\in \mathrm{I}} (\mathfrak{J} \cap \mathfrak{B}_i).$$

Thus $(\Theta, \mathfrak{J}) \cap (\cup_{i\in I} (\vartheta^i, \mathfrak{B}_i)) = \cup_{i\in I} (\Theta, \mathfrak{J}) \cap (\vartheta^i, \mathfrak{B}_i)$.

(ii) Let $\cap_{i\in I} (\vartheta^i, \mathfrak{B}_i) = (\eta, \mathfrak{D})$ where $\mathfrak{D} = \cap_{i\in I} \mathfrak{B}_i$. Thus $(\Theta, \mathfrak{J}) \cup (\cap_{i\in I} (\vartheta^i, \mathfrak{B}_i)) = (\eta', \mathfrak{D}')$ where $\mathfrak{D}' = (\mathfrak{J} \cup \mathfrak{D})$. Now consider $(\Theta, \mathfrak{J}) \cup (\vartheta^i, \mathfrak{B}_i) = (\eta'', \mathfrak{D}'')$, where $\mathfrak{D}'' = (\mathfrak{J} \cup \mathfrak{B}_i)$. Then $\cap_{i\in I} ((\Theta, \mathfrak{J}) \cup (\vartheta^i, \mathfrak{B}_i)) = (\eta''', \mathfrak{D}''')$ where $\mathfrak{D}''' = \cap_{i\in I} (\mathfrak{D}'')$. Since, $\mathfrak{J} \cup (\cap_{i\in I} \mathfrak{B}) = \cap_{i\in I} (\mathfrak{J} \cup \mathfrak{B})$, we get

$$\mathfrak{D}' = (\mathfrak{J} \cup \mathfrak{D}) = \mathfrak{J} \cup (\cap_{i\in I} \mathfrak{B}_i) = \cap_{i\in I} (\mathfrak{J} \cup \mathfrak{B}_i) \text{ and } \mathfrak{D}''' = \cap_{i\in I} (\mathfrak{D}'') = \cap_{i\in I} (\mathfrak{J} \cup \mathfrak{B}).$$

Thus, $(\Theta, \mathfrak{J}) \cup (\cap_{i\in I} (\vartheta^i, \mathfrak{B}_i)) = \cap_{i\in I} ((\Theta, \mathfrak{J}) \cup (\vartheta^i, \mathfrak{B}_i))$.

Theorem 3.17

Let $(\mathfrak{X}_\varrho, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}) \epsilon P(\varrho, \mathfrak{X})$ then the collection $\tau_{(\Theta,\mathfrak{J})} = \{(\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B}) / (\vartheta, \mathfrak{B}) \in \tau\}$ is a FHT.

Proof

(i) Since $\phi_{\mathfrak{X}}, (\varrho, \mathfrak{X}) \in \tau$, $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J}) \cap (\varrho, \mathfrak{X})$ and $\phi_{\mathfrak{A}} = (\Theta, \mathfrak{J}) \cap \phi_{\mathfrak{X}}$, then $\phi_{\mathfrak{A}}, (\Theta, \mathfrak{J}) \in \tau_{(\Theta,\mathfrak{J})}$.

(ii) Consider $(\Theta_1, \mathfrak{J}_1), (\Theta_2\, \mathfrak{J}_2) \in \tau_{(\Theta,\mathfrak{J})}$. Then there exists $(\vartheta_i, \mathfrak{B}_i) \in \tau_{(\Theta,\mathfrak{J})}$ for each $i = 1,2$ such that $(\Theta_i, \mathfrak{J}_i) = (\Theta, \mathfrak{J}) \cap (\vartheta_i, \mathfrak{B}_i)$.

Thus, $(\Theta_1, \mathfrak{J}_1) \cap (\Theta_2, \mathfrak{J}_2) = [(\Theta, \mathfrak{J}) \cap (\vartheta_1, \mathfrak{B}_1)] \cap [(\Theta, \mathfrak{J}) \cap (\vartheta_2, \mathfrak{B}_2)]$

$= (\Theta, \mathfrak{J}) \cap [(\vartheta_1, \mathfrak{B}_1) \cap (\vartheta_2, \mathfrak{B}_2)]$

Since $[(\vartheta_1, \mathfrak{B}_1) \cap (\vartheta_2, \mathfrak{B}_2) \in \tau$, we have $(\Theta_1, \mathfrak{J}_1) \cap (\Theta_2, \mathfrak{J}_2) \in \tau_{(\Theta,\mathfrak{J})}$.

(iii) Let $\{(\vartheta, \mathfrak{B})_j / \mathrm{j} \in J\}$ be a subcollection of $\tau_{(\Theta,\mathfrak{J})}$. Then for each $j \in J$, there is a FH set $(\xi, \mathfrak{C})_j$ of τ such that $(\vartheta, \mathfrak{B})_j = (\Theta, \mathfrak{J}) \cap (\xi, \mathfrak{C})_j$.

Thus, $\cup_{j\in J} (\vartheta, \mathfrak{B})_j = \cup_{j\in J} ((\theta, \mathfrak{J}) \cap (\xi, \mathfrak{C})_j) = (\Theta, \mathfrak{J}) \cap (\cup_{j\in J} (\xi, \mathfrak{C})_j)$.

Since $\cup_{j\in J} (\xi, \mathfrak{C})_j \in \tau$, then $(\vartheta, \mathfrak{B})_j \in \tau_{(\Theta,\mathfrak{J})}$.

Definition 3.18

Let $(\mathfrak{X}_\varrho, \tau)$ be a FHTS and $(\Theta, \mathfrak{J}) \subset P(\varrho, \mathfrak{X})$. Then, the FHT $\tau_{(\Theta,\mathfrak{J})}$ as in Theorem 3.17 is called Fuzzy hypersoft subspace topology and $(\mathfrak{A}_\theta, \tau_{(\Theta,\mathfrak{J})})$ is called a fuzzy hypersoft subspace of $(\mathfrak{X}_\varrho, \tau)$.

4. Intuitionistic Hypersoft Topological Spaces

In this section, we define the concept of "Intuitionistic Hypersoft Topology". Let $E_1, E_2, E_3 \dots E_n$ be the parameters of the universe T, the set of all intuitionistic sets be $F(T)$, the collection of all intuitionistic hypersoft sets over $T_{\mathbb{E}}$ (where $\mathbb{E} = E_1 \times E_2 \times E_3 \dots \times E_n$) be $\mathfrak{P}(T, \mathbb{E})$.

Definition 4.1

Let $(\varrho, \mathfrak{H})$ be an element of $\mathfrak{P}(T, \mathbb{E})$ (where $\mathfrak{H} = \mathrm{H}_1 \times \mathrm{H} \times \mathrm{H}_3 \dots \times \mathrm{H}_\mathrm{n}$ with each Y_i is a subset of E_i $(i = 1, 2 \dots n)$, set of all intuitionistic hypersoft (IH) subsets of $(\varrho, \mathfrak{H})$ be $P(\varrho, \mathfrak{H})$ and τ, a subcollection of $P(\varrho, \mathfrak{H})$.

(i) $\phi_{\mathfrak{Y}}, (\varrho, \mathfrak{Y}) \in \tau$
(ii) $(\Theta, \mathfrak{J}), (\vartheta, \mathfrak{B}) \in \tau \Longrightarrow (\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B}) \in \tau$
(iii) $\{(\Theta, \mathfrak{J})_l \mid l \in L\} \in \tau \Longrightarrow \cup_{l\in L} (\Theta, \mathfrak{J})_l \in \tau$

If the above axioms are satisfied then τ is an intuitionistic hypersoft topology (IHT) on $(\varrho, \mathfrak{H})$ and $(\mathfrak{H}_\varrho, \tau)$ is called an intuitionistic hypersoft topological space (IHTS). Every member of τ is called an open intuitionistic hypersoft set (OIHS). An intuitionistic hypersoft set is called a closed intuitionistic hypersoft set (CIHS) if its complement is an OIHS.

Example 4.2

Let $T = \{y_1, y_2, y_3, y_4\}$ and the attributes be $E_1 = \{b_1, b_2\}$, $E_2 = \{b_3, b_4\}$ and $E_3 = \{b_5, b_6\}$. Then the intuitionitic hypersoft set be

$$\left\{\begin{array}{c}\left((b_1, b_3, b_5), \left\{\frac{y_2}{(0.4,0.3)}, \frac{y_4}{(0.6,0.2)}\right\}\right), \left((b_1, b_3, b_6), \left\{\frac{y_1}{(0.7,0.1)}\right\}\right), \left((b_1, b_4, b_5), \left\{\frac{y_1}{(0.4,0.4)}, \frac{y_2}{(0.3,0.2)}\right\}\right), \left((b_1, b_4, b_6), \left\{\frac{y_1}{(0.5,0.3)}, \frac{y_3}{(0.7,0.1)}\right\}\right), \\ \left((b_2, b_3, b_5), \left\{\frac{y_2}{(0.3,0.5)}, \frac{y_3}{(0.5,0.1)}\right\}\right), \left((b_2, b_3, b_6), \left\{\frac{y_3}{(0.8,01.)}\right\}\right), \left((b_2, b_4, b_5), \left\{\frac{y_4}{(0.9,0.1)}\right\}\right), \left((b_2, b_4, b_6), \left\{\frac{y_2}{(0.6,0.3)}\right\}\right)\end{array}\right\}$$

Let us consider this intuitionistic hypersoft set as $(\varrho, \mathfrak{H})$. Then the subfamily

$$\tau = \{\phi_{\mathfrak{H}}, (\varrho, \mathfrak{H}), \{\left((b_1, b_3, b_5), \left\{\frac{y_1}{(0.3,0.4)}, \frac{y_2}{(0.6,0.1)}\right\}\right), \left((b_2, b_3, b_5), \left\{\frac{y_2}{(0.4,0.3)}, \frac{y_3}{(0.5,0.3)}\right\}\right)\}, \{\left((b_1, b_3, b_5), \left\{\frac{y_2}{(0.4,0.3)}\right\}\right) \left((b_2, b_3, b_5), \left\{\frac{y_2}{(0.3,0.5)}, \frac{y_3}{(0.5,0.3)}\right\}\right)\}, \{\left((b_1, b_3, b_5), \left\{\frac{y_1}{(0.3,0.4)}, \frac{y_2}{(0.6,0.1)}, \frac{y_4}{(0.6,0.2)}\right\}\right), \left((b_1, b_4, b_6), \left\{\frac{y_1}{(0.5,0.3)}, \frac{y_3}{(0.7,0.1)}\right\}\right), \left((b_1, b_3, b_6), \left\{\frac{y_1}{(0.7,0.1)}\right\}\right), \left((b_1, b_4, b_5), \left\{\frac{y_1}{(0.4,0.4)}, \frac{y_2}{(0.3,0.2)}\right\}\right), \left((b_2, b_3, b_5), \left\{\frac{y_2}{(0.4,0.3)}, \frac{y_3}{(0.5,0.1)}\right\}\right), \left((b_2, b_3, b_6), \left\{\frac{y_3}{(0.8,0.1)}\right\}\right), \left((b_2, b_4, b_5), \left\{\frac{y_4}{(0.9,0.1)}\right\}\right), \left((b_2, b_4, b_6), \left\{\frac{y_2}{(0.6,0.3)}\right\}\right)\}\}$$

of $P(\varrho, \mathfrak{H})$ is a IHT on $(\varrho, \mathfrak{H})$.

Definition 4.3

Let τ be an IHT on $(\varrho, \mathfrak{H}) \in \mathfrak{P}(T, \mathbb{E})$ and $(\vartheta, \mathfrak{B})$ be an IH set in $P(\varrho, \mathfrak{H})$. An IH set $(\Theta, \mathfrak{J})$ in $P(\varrho, \mathfrak{H})$ is a neighbourhood of IH set of $(\vartheta, \mathfrak{B})$ if and only if there exists an OIHS $(\xi, \mathfrak{C})$ such that $(\vartheta, \mathfrak{B}) \subset (\xi, \mathfrak{C}) \subset (\Theta, \mathfrak{J})$.

Theorem 4.4

An IH set $(\Theta, \mathfrak{J})$ in $P(\varrho, \mathfrak{H})$ is an OIHS if and only if $(\Theta, \mathfrak{J})$ is a neighbourhood of each IH set $(\vartheta, \mathfrak{B})$ contained in $(\Theta, \mathfrak{J})$.

Proof:

Consider an OIHS $(\Theta, \mathfrak{J})$ and any IH set $(\vartheta, \mathfrak{B})$ confined in $(\Theta, \mathfrak{J})$. Thus we have $(\vartheta, \mathfrak{B}) \subset (\Theta, \mathfrak{J}) \subset (\Theta, \mathfrak{J})$. This implies that $(\theta, \mathfrak{A})$ is a neighbourhood of $(\vartheta, \mathfrak{B})$.

Let $(\theta, \mathfrak{A})$ be a neighbourhood of each IH set confined in it. Since $(\Theta, \mathfrak{J}) \subset (\Theta, \mathfrak{J})$, there exists an OIHS $(\xi, \mathfrak{C})$ such that $(\Theta, \mathfrak{J}) \subset (\xi, \mathfrak{C}) \subset (\Theta, \mathfrak{J})$. Thus $(\Theta, \mathfrak{J}) = (\xi, \mathfrak{C})$, $(\Theta, \mathfrak{J})$ is OIHS.

Definition 4.5

Let $(\mathfrak{H}_\varrho, \tau)$ be called an IHTS on $(\varrho, \mathfrak{H})$ and $(\Theta, \mathfrak{J})$ be an IH set in $P(\varrho, \mathfrak{H})$. The neighbourhood system of $(\Theta, \mathfrak{J})$ relative to τ is the collection of all neighbourhoods of $(\Theta, \mathfrak{J})$ and is denoted by $HNN_{(\Theta,\mathfrak{J})}$.

Theorem 4.6

If $HNN_{(\Theta,\mathfrak{J})}$ is the neighbourhood systems of IH set $(\Theta, \mathfrak{J})$. Then,

1. Finite intersection of member of $HNN_{(\Theta,\mathfrak{J})}$ belongs to $HNN_{(\Theta,\mathfrak{J})}$.
2. Each IH set which has a member of $HNN_{(\Theta,\mathfrak{J})}$ belongs to $HNN_{(\Theta,\mathfrak{J})}$.

Proof

1. $(\vartheta,\mathfrak{B})$ and $(\xi,\mathfrak{C}) \in HNN_{(\Theta,\mathfrak{J})}$ then there exists $(\vartheta',\mathfrak{B}'),(\xi',\mathfrak{C}') \in \tau$ such that

$$(\Theta,\mathfrak{J}) \subset (\vartheta',\mathfrak{B}') \subset (\vartheta,\mathfrak{B}) \text{ and } (\Theta,\mathfrak{J}) \subset (\xi',\mathfrak{C}') \subset (\xi,\mathfrak{C}).$$

Since $(\vartheta',\mathfrak{B}') \cap (\xi',\mathfrak{C}') \in \tau$ we get $(\theta,\mathfrak{A}) \subset (\vartheta',\mathfrak{B}') \cap (\xi',\mathfrak{C}') \subset (\vartheta,\mathfrak{B}) \cap (\xi,\mathfrak{C})$
Hence $(\vartheta,\mathfrak{B}) \cap (\xi,\mathfrak{C})$ belongs to $HNN_{(\Theta,\mathfrak{J})}$.

2. Let $(\vartheta,\mathfrak{B}) \in HNN_{(\Theta,\mathfrak{J})}$ and $(\xi,\mathfrak{C})$ be a IH set having $(\vartheta,\mathfrak{B})$.
Since $(\vartheta,\mathfrak{B}) \in HNN_{(\Theta,\mathfrak{J})}$ there exists an OIHS containing $(\vartheta',\mathfrak{B}')$ such that $(\Theta,\mathfrak{J}) \subset (\vartheta',\mathfrak{B}') \subset (\vartheta,\mathfrak{B})$ it follows that $(\Theta,\mathfrak{J}) \subset (\vartheta',\mathfrak{B}') \subset (\xi,\mathfrak{C})$. Thus $(\xi,\mathfrak{C})$ belongs to $HNN_{(\Theta,\mathfrak{J})}$.

Definition 4.7

Let $(\mathfrak{H}_\varrho,\tau)$ be an IHTS and $(\Theta,\mathfrak{J}),(\vartheta,\mathfrak{B})$ be an IH set in $P(\varrho,\mathfrak{H})$ such that $(\vartheta,\mathfrak{B}) \subset (\Theta,\mathfrak{J})$. Then $(\vartheta,\mathfrak{B})$ is said to be an interior intuitionistic hypersoft set (IIHS) of $(\Theta,\mathfrak{J})$ if and only if $(\Theta,\mathfrak{J})$ is a neighbourhood of $(\vartheta,\mathfrak{B})$.

The union of whole IIHS of $(\Theta,\mathfrak{J})$ is named the interior of $(\Theta,\mathfrak{J})$ and is denoted as $(\Theta,\mathfrak{J})^\circ$.

Theorem 4.8

Let $(\mathfrak{H}_\varrho,\tau)$ be an IHTS and $(\Theta,\mathfrak{J})$, an IH set in $P(\varrho,\mathfrak{H})$. Then,

i) $(\Theta,\mathfrak{J})^\circ$ is open and $(\Theta,\mathfrak{J})^\circ$ is the biggest OIHS confined in $(\Theta,\mathfrak{J})$.
ii) $(\Theta,\mathfrak{J})$ is OIHS iff $(\Theta,\mathfrak{J}) = (\Theta,\mathfrak{J})^\circ$.

Proof

i) Since $(\Theta,\mathfrak{J})^\circ = \cup\{(\vartheta,\mathfrak{B})/(\Theta,\mathfrak{J}) \text{ is a neighbourhood of } (\vartheta,\mathfrak{B})\}$, $(\Theta,\mathfrak{J})^\circ$ is itself an IIHS of $(\Theta,\mathfrak{J})$. Then there exists an OIHS $(\xi,\mathfrak{C})$ such that $(\Theta,\mathfrak{J})^\circ \subset (\xi,\mathfrak{C}) \subset (\Theta,\mathfrak{J})$. $(\xi,\mathfrak{C})$ is an IIHS of $(\Theta,\mathfrak{J})$, hence $(\xi,\mathfrak{C}) \subset (\Theta,\mathfrak{J})^\circ$. Thus $(\Theta,\mathfrak{J})^\circ$ is the largest OIHS enclosed in $(\Theta,\mathfrak{J})$.

ii) Let $(\Theta,\mathfrak{J})$ be an OIHS. Since $(\Theta,\mathfrak{J})^\circ$ is the IIHS of $(\Theta,\mathfrak{J})$, we have $(\Theta,\mathfrak{J}) = (\Theta,\mathfrak{J})^\circ$.

Conversely if $(\Theta,\mathfrak{J}) = (\Theta,\mathfrak{J})^\circ$ then $(\Theta,\mathfrak{J})$ is OIHS.

Definition 4.9

Let $(\mathfrak{H}_\varrho,\tau_1)$ and $(\mathfrak{H}_\varrho,\tau_2)$ be two IHTS. If each $(\Theta,\mathfrak{J}) \in \tau_1$ is in τ_2 then τ_2 is called the IH finer than τ_1 (or) τ_1 is IH coarser than τ_2.

Definition 4.10

Let $(\mathfrak{H}_\varrho,\tau)$ be a IHTS and $(\Theta,\mathfrak{J}) \in \mathfrak{P}(T,\mathbb{E})$. The intuitionistic hypersoft closure (IHC) of $(\Theta,\mathfrak{J})$ is the intersection of all CIH sets that contains $(\Theta,\mathfrak{J})$ which is denoted by $\overline{(\Theta,\mathfrak{J})}$.

Thus, $\overline{(\Theta,\mathfrak{J})}$ is the smallest CIHS which has $(\Theta,\mathfrak{J})$ and $\overline{(\Theta,\mathfrak{J})}$ is CIHS.

Theorem 4.11

Let $(\mathfrak{H}_\varrho,\tau)$ be an IHTS and $(\Theta,\mathfrak{J}),(\vartheta,\mathfrak{B}) \in \mathfrak{P}(T,\mathbb{E})$.
Then,

(i) $(\Theta, \mathfrak{J}) \subseteq \overline{(\Theta, \mathfrak{J})}$
(ii) $\overline{\overline{(\Theta, \mathfrak{J})}} = \overline{(\Theta, \mathfrak{J})}$
(iii) If $(\Theta, \mathfrak{J}) \subset (\vartheta, \mathfrak{B})$, then $\overline{(\Theta, \mathfrak{J})} \subset \overline{(\vartheta, \mathfrak{B})}$.
(iv) $(\Theta, \mathfrak{J})$ is a CIHS iff $(\Theta, \mathfrak{J}) = \overline{(\Theta, \mathfrak{J})}$.
(v) $\overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})} = \overline{(\Theta, \mathfrak{J})} \cup \overline{(\vartheta, \mathfrak{B})}$

Proof

From the definition of IHC, the proof of (i) to (iii) is attained.
(iv) Let $(\Theta, \mathfrak{J})$ be CIHS. By (i) $(\Theta, \mathfrak{J}) \subseteq \overline{(\Theta, \mathfrak{J})}$. Since $\overline{(\Theta, \mathfrak{J})}$ is the minutest CIHS which has $(\Theta, \mathfrak{J})$, then $\overline{(\Theta, \mathfrak{J})} \subseteq (\Theta, \mathfrak{J})$. Thus $(\Theta, \mathfrak{J}) = \overline{(\Theta, \mathfrak{J})}$.

Conversely let, $(\Theta, \mathfrak{J}) = \overline{(\Theta, \mathfrak{J})}$. Since $(\Theta, \mathfrak{J})$ is CIHS, then $(\Theta, \mathfrak{J})$ is also a CIHS.

(v) By (iv) $\overline{(\Theta, \mathfrak{J})}$, $\overline{(\vartheta, \mathfrak{B})} \subseteq \overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$. So $\overline{(\Theta, \mathfrak{J})} \cup \overline{(\vartheta, \mathfrak{B})} \subseteq \overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$.

Conversely by (i), $(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B}) \subseteq \overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$.

Since $(\Theta, \mathfrak{J})$, $(\vartheta, \mathfrak{B})$ are IH sets and $\overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$ is the minutest CIHS which has $(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})$, then $\overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})} \subseteq \overline{(\Theta, \mathfrak{J})} \cup \overline{(\vartheta, \mathfrak{B})}$

Thus the equality is obtained.

Theorem 4.12

Let $(\mathfrak{H}_{\varrho}, \tau)$ be an IHTS and $(\Theta, \mathfrak{J}), (\vartheta, \mathfrak{B}) \in \mathfrak{P}(T, \mathbb{E})$.
Then,
(i) $(\Theta, \mathfrak{J})^{\circ} \subseteq (\Theta, \mathfrak{J})$
(ii) $((\Theta, \mathfrak{J})^{\circ})^{\circ} = (\Theta, \mathfrak{J})^{\circ}$
(iii) If $(\Theta, \mathfrak{J}) \subseteq (\vartheta, \mathfrak{B})$, then $(\Theta, \mathfrak{J})^{\circ} \subseteq (\vartheta, \mathfrak{B})^{\circ}$.
(iv) $(\Theta, \mathfrak{J})$ is OIHS iff $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^{\circ}$.
(v) $\big((\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})\big)^{\circ} = (\Theta, \mathfrak{J})^{\circ} \cap (\vartheta, \mathfrak{B})^{\circ}$.

Proof

(i) – (iii) are obvious from the definition of interior

(iv) Let $(\Theta, \mathfrak{J})$ be a OIHS, by (i) $(\Theta, \mathfrak{J})^{\circ} \subseteq (\Theta, \mathfrak{J})$. Since $(\Theta, \mathfrak{J})^{\circ}$ is the largest OIHS that is contained in $(\Theta, \mathfrak{J})$, then $(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J})^{\circ}$. Thus $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^{\circ}$

Conversely, let $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^{\circ}$ since $(\Theta, \mathfrak{J})^{\circ}$ is OIHS, $(\Theta, \mathfrak{J})$ is also OIHS.

(v) $(\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B}) \subseteq (\Theta, \mathfrak{J}), (\vartheta, \mathfrak{B})$. Thus by (iii) $\big((\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})\big)^{\circ} \subseteq (\Theta, \mathfrak{J})^{\circ} \cap (\vartheta, \mathfrak{B})^{\circ}$.

Conversely by (i) $(\Theta, \mathfrak{J})^{\circ} \cap (\vartheta, \mathfrak{B})^{\circ} \subseteq (\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})$. Since $(\Theta, \mathfrak{J})^{\circ}$, $(\vartheta, \mathfrak{B})^{\circ}$ are OIHS & $\big((\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})\big)^{\circ}$ is the largest OIHS that has $(\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})$, then $(\Theta, \mathfrak{J})^{\circ} \cap (\vartheta, \mathfrak{B})^{\circ} \subseteq \big((\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})\big)^{\circ}$. Thus, the equality is achieved.

5. Neutrosophic Hypersoft Topological Spaces

In this section, we define the concept of "Neutrosophic Hypersoft Topology". Let $E_1, E_2, E_3 \ldots E_n$ be the parameters of the universe K, the set of all neutrosophic sets be $F(K)$, the collection of all neutrosophic hypersoft sets over $K_{\mathbb{E}}$ (where $\mathbb{E} = E_1 \times E_2 \times E_3 \ldots \times E_n$) be $\mathfrak{P}(K, \mathbb{E})$.

Definition 5.1

Let $(\varrho, \mathfrak{Y})$ be an element of $\mathfrak{P}(K, \mathbb{E})$ (where $\mathfrak{Y} = Y_1 \times Y_2 \times Y_3 \ldots \times Y_n$ with each Y_i is a subset of E_i $(i = 1, 2 \ldots n)$, set of all neutrosophic hypersoft (NH) subsets of $(\varrho, \mathfrak{Y})$ be $P(\varrho, \mathfrak{Y})$ and τ, a subcollection of $P(\varrho, \mathfrak{Y})$.

(i) $\phi_{\mathfrak{Y}}, (\varrho, \mathfrak{Y}) \in \tau$
(ii) $(\Theta, \mathfrak{J}), (\vartheta, \mathfrak{B}) \in \tau \Longrightarrow (\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B}) \in \tau$
(iii) $\{(\Theta, \mathfrak{J})_l \mid l \in L\} \in \tau \Longrightarrow \bigcup_{l \in L} (\Theta, \mathfrak{J})_l \in \tau$

If the above axioms are satisfied then τ is neutrosophic hypersoft topology (NHT) on $(\varrho, \mathfrak{Y})$. $(\mathfrak{Y}_\varrho, \tau)$ is called a neutrosophic hypersoft topological space (NHTS). Every member of τ is called open neutrosophic hypersoft set (ONHS). A neutrosophic hypersoft set if called closed fuzzy hypersoft set (CNHS) if its complement is ONHS.

Example 5.2

Let $K = \{z_1, z_2, z_3, z_4\}$ and the attributes be $E_1 = \{c_1, c_2\}$, $E_2 = \{c_3, c_4\}$ and $E_3 = \{c_5, c_6\}$. Then the neutrosophic hypersoft set be

$$\left\{\begin{array}{c}\left((c_1, c_3, c_5), \left\{\frac{z_2}{(0.4,0.3,0.4)}, \frac{z_4}{(0.6,0.2,0.4)}\right\}\right), \left((c_1, c_3, c_6), \left\{\frac{z_1}{(0.7,0.1,0.4)}\right\}\right), \left((c_1, c_4, c_5), \left\{\frac{z_1}{(0.4,0.4,0.6)}, \frac{z_2}{(0.3,0.2,0.8)}\right\}\right), \\ \left((c_1, c_4, c_6), \left\{\frac{z_1}{(0.5,0.3,0.4)}, \frac{z_3}{(0.7,0.1,0.6)}\right\}\right), \left((c_2, c_3, c_5), \left\{\frac{z_2}{(0.3,0.5,0.6)}, \frac{z_3}{(0.5,0.1,0.4)}\right\}\right), \\ , \left((c_2, c_3, c_6), \left\{\frac{z_3}{(0.8,0.1,0.6)}\right\}\right), \left((c_2, c_4, c_5), \left\{\frac{z_4}{(0.9,0.6,0.4)}\right\}\right), \left((c_2, c_4, c_6), \left\{\frac{z_2}{(0.6,0.3,0.7)}\right\}\right)\end{array}\right\}$$

Let us consider this neutrosophic hypersoft as $(\varrho, \mathfrak{Y})$. Then the subfamily

$$\tau = \{\phi_{\mathfrak{Y}}, (\varrho, \mathfrak{Y}), \left\{\left((c_1, c_3, c_5), \left\{\frac{z_1}{(0.3,0.4,0.5)}, \frac{z_2}{(0.6,0.1,0.7)}\right\}\right), \left((c_2, c_3, c_5), \left\{\frac{z_2}{(0.4,0.3,0.6)}, \frac{z_3}{(0.5,0.3,0.7)}\right\}\right)\right\}, \{\left((c_1, c_3, c_5), \left\{\frac{z_2}{(0.4,0.1,0.7)}\right\}\right)$$
$$\left((c_2, c_3, c_5), \left\{\frac{z_2}{(0.3,0.3,0.6)}, \frac{z_3}{(0.5,0.1,0.7)}\right\}\right)\}, \left((c_1, c_3, c_5), \left\{\frac{z_1}{(0.3,0.4,0.5)}, \frac{z_2}{(0.6,0.3,0.4)}, \frac{z_4}{(0.6,0.2,0.4)}\right\}\right), \left((c_1, c_4, c_6), \left\{\frac{z_1}{(0.5,0.3,0.4)}, \frac{z_3}{(0.7,0.1,0.6)}\right\}\right),$$
$$\left((c_1, c_3, c_6), \left\{\frac{z_1}{(0.7,0.1,0.4)}\right\}\right), \left((c_1, c_4, c_5), \left\{\frac{z_1}{(0.4,0.4,0.6)}, \frac{z_2}{(0.3,0.2,0.8)}\right\}\right), \left((c_2, c_3, c_5), \left\{\frac{z_2}{(0.4,0.5,0.6)}, \frac{z_3}{(0.5,0.3,0.4)}\right\}\right),$$
$$\left((c_2, c_3, c_6), \left\{\frac{z_3}{(0.8,0.1,0.6)}\right\}\right), \left((c_2, c_4, c_5), \left\{\frac{z_4}{(0.9,0.6,0.4)}\right\}\right), \left((c_2, c_4, c_6), \left\{\frac{z_2}{(0.6,0.3,0.7)}\right\}\right)\}\}$$

of $P(\varrho, \mathfrak{Y})$ is a NHT on $(\varrho, \mathfrak{Y})$.

Definition 5.3

Let τ be a NHT on $(\varrho, \mathfrak{Y}) \in \mathfrak{P}(K, \mathbb{E})$ and $(\vartheta, \mathfrak{B})$ be a NH set in $P(\varrho, \mathfrak{Y})$. A FH set $(\Theta, \mathfrak{J})$ in $P(\varrho, \mathfrak{Y})$ is a neighbourhood of NH set of $(\vartheta, \mathfrak{B})$ if and only if there exists an ONHS $(\xi, \mathfrak{C})$ such that $(\vartheta, \mathfrak{B}) \subset (\xi, \mathfrak{C}) \subset (\Theta, \mathfrak{J})$.

Theorem 5.4

A NH set $(\Theta, \mathfrak{J})$ in $P(\varrho, \mathfrak{Y})$ is an ONHS if and only if $(\Theta, \mathfrak{J})$ is a neighbourhood of each NH set $(\vartheta, \mathfrak{B})$ contained in $(\Theta, \mathfrak{J})$.

Proof:

Consider an ONHS $(\Theta, \mathfrak{J})$ and any NH set $(\vartheta, \mathfrak{B})$ confined in $(\Theta, \mathfrak{J})$. Thus we have $(\vartheta, \mathfrak{B}) \subset (\Theta, \mathfrak{J}) \subset (\Theta, \mathfrak{J})$. Implies that $(\theta, \mathfrak{A})$ is a neighbourhood of $(\vartheta, \mathfrak{B})$.

Let $(\theta, \mathfrak{A})$ be a neighbourhood of each NH set confined in it. Since $(\Theta, \mathfrak{J}) \subset (\Theta, \mathfrak{J})$, there exists an ONHS $(\xi, \mathfrak{C})$ such that $(\Theta, \mathfrak{J}) \subset (\xi, \mathfrak{C}) \subset (\Theta, \mathfrak{J})$. Thus $(\Theta, \mathfrak{J}) = (\xi, \mathfrak{C})$, $(\Theta, \mathfrak{J})$ is ONHS.

Definition 5.5

Let $(\mathfrak{Y}_{\varrho}, \tau)$ is called a NHTS on $(\varrho, \mathfrak{Y})$ and $(\Theta, \mathfrak{J})$ be a NH set in $P(\varrho, \mathfrak{Y})$. The neighbourhood system of $(\Theta, \mathfrak{J})$ relative to τ is the collection of all neighbourhood of $(\Theta, \mathfrak{J})$ and denoted by $HNN_{(\Theta,\mathfrak{J})}$.

Theorem 5.6

If $HNN_{(\Theta,\mathfrak{J})}$ is the neighbourhood systems of NH set $(\Theta, \mathfrak{J})$. Then,

1. Finite intersection of member of $HNN_{(\Theta,\mathfrak{J})}$ belongs to $HNN_{(\Theta,\mathfrak{J})}$.
2. Each NH set which has a member of $HNN_{(\Theta,\mathfrak{J})}$ belongs to $HNN_{(\Theta,\mathfrak{J})}$.

Proof

1. $(\vartheta, \mathfrak{B})$ and $(\xi, \mathfrak{C}) \in HNN_{(\Theta,\mathfrak{J})}$ then there exists $(\vartheta', \mathfrak{B}'), (\xi', \mathfrak{C}') \in \tau$ such that

$$(\Theta, \mathfrak{J}) \subset (\vartheta', \mathfrak{B}') \subset (\vartheta, \mathfrak{B}) \text{ and } (\Theta, \mathfrak{J}) \subset (\xi', \mathfrak{C}') \subset (\xi, \mathfrak{C}).$$

Since $(\vartheta', \mathfrak{B}') \cap (\xi', \mathfrak{C}') \in \tau$ we get $(\theta, \mathfrak{A}) \subset (\vartheta', \mathfrak{B}') \cap (\xi', \mathfrak{C}') \subset (\vartheta, \mathfrak{B}) \cap (\xi, \mathfrak{C})$

Hence $(\vartheta, \mathfrak{B}) \cap (\xi, \mathfrak{C})$ belongs to $HNN_{(\Theta,\mathfrak{J})}$.

2. Let $(\vartheta, \mathfrak{B}) \in HNN_{(\Theta,\mathfrak{J})}$ and $(\xi, \mathfrak{C})$ be a NH set having $(\vartheta, \mathfrak{B})$.

Since $(\vartheta, \mathfrak{B}) \in HNN_{(\Theta,\mathfrak{J})}$ there exists an ONHS containing $(\vartheta', \mathfrak{B}')$ such that $(\Theta, \mathfrak{J}) \subset (\vartheta', \mathfrak{B}') \subset (\vartheta, \mathfrak{B})$ it follows that $(\Theta, \mathfrak{J}) \subset (\vartheta', \mathfrak{B}') \subset (\xi, \mathfrak{C})$. Thus $(\xi, \mathfrak{C})$ belongs to $HNN_{(\Theta,\mathfrak{J})}$.

Definition 5.7

Let $(\mathfrak{Y}_{\varrho}, \tau)$ is called a NHTS and $(\Theta, \mathfrak{J}), (\vartheta, \mathfrak{B})$ be NH set in $P(\varrho, \mathfrak{Y})$ such that $(\vartheta, \mathfrak{B}) \subset (\Theta, \mathfrak{J})$. Then $(\vartheta, \mathfrak{B})$ is said to be an interior neutrosophic hypersoft set (INHS) of $(\Theta, \mathfrak{J})$ if and only if $(\Theta, \mathfrak{J})$ is a neighbourhood of $(\vartheta, \mathfrak{B})$.

The union of whole INHS of $(\Theta, \mathfrak{J})$ is named the interior of $(\Theta, \mathfrak{J})$ and denoted as $(\Theta, \mathfrak{J})^{\circ}$.

Theorem 5.8

Let $(\mathfrak{Y}_{\varrho}, \tau)$ is called a NHTS and $(\Theta, \mathfrak{J})$, a NH set in $P(\varrho, \mathfrak{Y})$. Then,

i) $(\Theta, \mathfrak{J})^{\circ}$ is open and $(\Theta, \mathfrak{J})^{\circ}$ is the biggest ONHS confined in $(\Theta, \mathfrak{J})$.
ii) $(\Theta, \mathfrak{J})$ is ONHS iff $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^{\circ}$.

Proof

i) Since $(\Theta, \mathfrak{J})^{\circ} = \cup \{(\vartheta, \mathfrak{B})/(\Theta, \mathfrak{J}) \text{ is a neighbourhood of } (\vartheta, \mathfrak{B})\}$, $(\Theta, \mathfrak{J})^{\circ}$ is itself an INHS of $(\Theta, \mathfrak{J})$. Then there exists an ONHS $(\xi, \mathfrak{C})$ such that $(\Theta, \mathfrak{J})^{\circ} \subset (\xi, \mathfrak{C}) \subset (\Theta, \mathfrak{J})$. $(\xi, \mathfrak{C})$ is an INHS of $(\Theta, \mathfrak{J})$, hence $(\xi, \mathfrak{C}) \subset (\Theta, \mathfrak{J})^{\circ}$. Thus $(\Theta, \mathfrak{J})^{\circ}$ is the largest ONHS enclosed in $(\Theta, \mathfrak{J})$.

ii) Let $(\Theta, \mathfrak{J})$ be an ONHS. Since $(\Theta, \mathfrak{J})^{\circ}$ is the INHS of $(\Theta, \mathfrak{J})$, we have $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^{\circ}$.

Conversely if $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^{\circ}$ then $(\Theta, \mathfrak{J})$ is ONHS.

Definition 5.9

Let $(\mathfrak{Y}_{\varrho}, \tau_1)$ and $(\mathfrak{Y}_{\varrho}, \tau_2)$ be two NHTS. If each $(\Theta, \mathfrak{J}) \in \tau_1$ is in τ_2 then τ_2 is called the NH finer than τ_1 (or) τ_1 is NH coarser than τ_2.

Definition 5.10

Let $(\mathfrak{Y}_{\varrho}, \tau)$ be a NHTS and $(\Theta, \mathfrak{J}) \in \mathfrak{P}(K, \mathbb{E})$. The neutrosophic hypersoft closure (NHC) of $(\Theta, \mathfrak{J})$ is the intersection of all CNH sets that contains $(\Theta, \mathfrak{J})$ which is denoted by $\overline{(\Theta, \mathfrak{J})}$.

Thus, $\overline{(\Theta, \mathfrak{J})}$ is the smallest CNHS which has $(\Theta, \mathfrak{J})$ and $\overline{(\Theta, \mathfrak{J})}$ is CNHS.

Theorem 5.11

Let $(\mathfrak{Y}_{\varrho}, \tau)$ be a NHTS and $(\Theta, \mathfrak{J}), (\vartheta, \mathfrak{B}) \in \mathfrak{P}(K, \mathbb{E})$.

Then,

(i) $(\Theta, \mathfrak{J}) \subseteq \overline{(\Theta, \mathfrak{J})}$

(ii) $\overline{\overline{(\Theta, \mathfrak{J})}} = \overline{(\Theta, \mathfrak{J})}$

(iii) If $(\Theta, \mathfrak{J}) \subset (\vartheta, \mathfrak{B})$, then $\overline{(\Theta, \mathfrak{J})} \subset \overline{(\vartheta, \mathfrak{B})}$.

(iv) $(\Theta, \mathfrak{J})$ is a CNHS iff $(\Theta, \mathfrak{J}) = \overline{(\Theta, \mathfrak{J})}$.

(v) $\overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})} = \overline{(\Theta, \mathfrak{J})} \cup \overline{(\vartheta, \mathfrak{B})}$

Proof

From the definition of NHC, the proof of (i) to (iii) is attained.

(iv) Let $(\Theta, \mathfrak{J})$ be CNHS. By (i) $(\Theta, \mathfrak{J}) \subseteq \overline{(\Theta, \mathfrak{J})}$. Since $\overline{(\Theta, \mathfrak{J})}$ is the minutest CNHS which has $(\Theta, \mathfrak{J})$, then $\overline{(\Theta, \mathfrak{J})} \subseteq (\Theta, \mathfrak{J})$. Thus $(\Theta, \mathfrak{J}) = \overline{(\Theta, \mathfrak{J})}$.

Conversely let, $(\Theta, \mathfrak{J}) = \overline{(\Theta, \mathfrak{J})}$. Since $(\Theta, \mathfrak{J})$ is CNHS, then $(\Theta, \mathfrak{J})$ is also CNHS.

(v) By (iv) $\overline{(\Theta, \mathfrak{J})}$, $\overline{(\vartheta, \mathfrak{B})} \subseteq \overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$. So $\overline{(\Theta, \mathfrak{J})} \cup \overline{(\vartheta, \mathfrak{B})} \subseteq \overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$.

Conversely by (i), $(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B}) \subseteq \overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$.

Since $(\Theta, \mathfrak{J})$, $(\vartheta, \mathfrak{B})$ are NH sets and $\overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})}$ is the minutest CNHS which has $(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})$, then $\overline{(\Theta, \mathfrak{J}) \cup (\vartheta, \mathfrak{B})} \subseteq \overline{(\Theta, \mathfrak{J})} \cup \overline{(\vartheta, \mathfrak{B})}$

Thus the equality is obtained.

Theorem 5.12

Let $(\mathfrak{Y}_{\varrho}, \tau)$ be a NHTS and $(\Theta, \mathfrak{J}), (\vartheta, \mathfrak{B}) \in \mathfrak{P}(K, \mathbb{E})$.

Then,

(i) $(\Theta, \mathfrak{J})^{\circ} \subseteq (\Theta, \mathfrak{J})$

(ii) $((\Theta, \mathfrak{J})^{\circ})^{\circ} = (\Theta, \mathfrak{J})^{\circ}$

(iii) If $(\Theta, \mathfrak{J}) \subseteq (\vartheta, \mathfrak{B})$, then $(\Theta, \mathfrak{J})^{\circ} \subseteq (\vartheta, \mathfrak{B})^{\circ}$.

(iv) $(\Theta, \mathfrak{J})$ is ONHS iff $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^{\circ}$.

(v) $\big((\Theta, \mathfrak{J}) \cap (\vartheta, \mathfrak{B})\big)^{\circ} = (\Theta, \mathfrak{J})^{\circ} \cap (\vartheta, \mathfrak{B})^{\circ}$.

Proof

(i) – (iii) are obvious from definition of interior

(iv) Let $(\Theta, \mathfrak{J})$ be a ONHS, by (i) $(\Theta, \mathfrak{J})^{\circ} \subseteq (\Theta, \mathfrak{J})$. Since $(\Theta, \mathfrak{J})^{\circ}$ is the largest ONHS that is contained in $(\Theta, \mathfrak{J})$, then $(\Theta, \mathfrak{J}) \subseteq (\Theta, \mathfrak{J})^{\circ}$. Thus $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^{\circ}$

Conversely, let $(\Theta, \mathfrak{J}) = (\Theta, \mathfrak{J})^{\circ}$ since $(\Theta, \mathfrak{J})^{\circ}$ is ONHS, $(\Theta, \mathfrak{J})$ is also ONHS.

(v) $(\Theta,\mathfrak{J}) \cap (\vartheta,\mathfrak{B}) \subseteq (\Theta,\mathfrak{J}),(\vartheta,\mathfrak{B})$. Thus by (iii) $\left((\Theta,\mathfrak{J}) \cap (\vartheta,\mathfrak{B})\right)^{\circ} \subseteq (\Theta,\mathfrak{J})^{\circ} \cap (\vartheta,\mathfrak{B})^{\circ}$.

Conversely by (i), $(\Theta,\mathfrak{J})^{\circ} \cap (\vartheta,\mathfrak{B})^{\circ} \subseteq (\Theta,\mathfrak{J}) \cap (\vartheta,\mathfrak{B})$. Since $(\Theta,\mathfrak{J})^{\circ}$, $(\vartheta,\mathfrak{B})^{\circ}$ are ONHS & $\left((\Theta,\mathfrak{J}) \cap (\vartheta,\mathfrak{B})\right)^{\circ}$ is the largest ONHS that has $(\Theta,\mathfrak{J}) \cap (\vartheta,\mathfrak{B})$, then $(\Theta,\mathfrak{J})^{\circ} \cap (\vartheta,\mathfrak{B})^{\circ} \subseteq \left((\Theta,\mathfrak{J}) \cap (\vartheta,\mathfrak{B})\right)^{\circ}$. Thus, the equality is achieved.

6. Conclusion

Herein we have defined fuzzy hypersoft topology and few basic properties have also been presented. In addition fuzzy hypersoft topology is extended to intuitionistic hypersoft, neutrosophic hypersoft topology along with some of its basic properties. In future, many properties of topological spaces can be extended to fuzzy hypersoft, intuitionistic and neutrosophic hypersoft topological spaces.

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New Neutrosophic Scale System Framework

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Abstract: Scaling system can be considered as range-base measurement system, it's a fatal tool used in all human activities in daily-bases, also, all business domains and sectors heavily use scaling systems in all business process specially in decisions-making as one of the main critical business activities, despite the fact that, there is no scientific base for calculate an unified scale system ranges, all provided scales or ranges are determined based on expert opinions', an enhanced scale system using single-valued neutrosophic set SVNS is offered that suggest a scientific methods for defining ranges in scaling systems, in addition, a new crisp value functions "De-neutrosophication" for converting both Simplified Neutrosophic Number SNN, and SVNS to them equivalent crisp values using distance measure based on Euclidean space are proposed, Finally, the offered framework and methods are implemented with numerical examples for best prove and validate of the framework and proposed methods.

Keywords: Neutrosophic; De-neutrosophic; Single-valued Neutrosophic Set SVNS; Scale system, Scoring System; Decisions-making

1. Introduction

Smarandache presented Neutrosophic Logic as a generalization of fuzzy logic considering Neutrosophic Set NS is a generalization of the intuitionistic set, classical set, and fuzzy set, where Neutrosophic uses every entity $< X >$ and its opposite or negation $< antiX >$ together with their neutralities $< neutX >$ in between them, therefore, the $< neutX >$ & $< antiX >$ together will considered as $< nonX >$, in neutrosophic logic a proposition has a degrees of truth (T), indeterminacy (I), falsity (F), where $(T), (I), (F)$ are standard or non-standard subsets of $^{-}]0,1[^{+}$ [1].

The Neutrosophic logic best fit in decision-making where its process mostly has a lot of vagueness, indeterminacies which is the typical case in real life decision-making process, therefore, using neutrosophic in decision-making activities provides decision-makers with a great flexibility to deal with indeterminacy and uncertainty, in addition, neutrosophic logic and its subfields has a lot of scientific implementations in numerous fields using the three neutrosophic logic's membership degrees (T) truth, (I) indeterminacy $and (F)$ falsity degree to express any system inputs' values in detailed way specially when the system inputs' values characterized with indeterminacy and uncertainty.

Measurement systems is a method of defining a measurement unit for best unify the scales, scaling systems is range-base measurement system, it's a critical tool used to classify measured items into ranges of values, each range has an equivalent qualitative values "Linguistic terms", though, there is no standard way for defining the ranges as ranges are determined based on expert opinions' such as, National Institute of Standards and Technology NIST [2], when performing risk assessments they uses five level scale, first level starting from 0% to 4% and name it “very low”, the second level started from 5% to 20% and name it “low”, the third level 21% to 79% as “moderate”, forth from 80% to 95% as “high”, and lastly from 96% to 100% as “very high”, while NIST uses different ranges in “Common Vulnerability Scoring System” [3] which firstly uses 10-base scale instead of 100-base scale, also uses different ranges, it was “very low” name it as “none” 0 %, “low” 1-39%, “moderate” 40-69%, “high” 70-89%, and lastly “very high” name it as “Critical” 90-100%, which clearly presenting same scale levels with different ranges, This research paper offers a scientific methods for defining ranges in scaling systems.

Many efforts done for calculate de-neutrosophication for SVNS using Entropy, cross-entropy, distance, similarity, score and accuracy functions which are very important in uncertainty environment while ranking neutrosophic sets and numbers, since entropy is typically developed to determining uncertain degree of information. Distance, similarity, score, accuracy and cross-entropy are mostly applied to calculate the level of similarity among two elements. The importance of these functions manifested of comparing or converting neutrosophic numbers and sets into a comparable crisp value, these functions are completely calculated based on the value of truth, falsity, and indeterminacy memberships [4].

Researchers made an attempt to present a neutrosophic 3D visualization for both SNN and SVNS using Euclidean space, in addition, new crisp value functions “De-neutrosophication” for converting both Simplified Neutrosophic Number SNN, and SVNS to them equivalent crisp values using similarity measure based on Euclidean distance are proposed, also the researchers propose a new Neutrosophic Scaling System algorithm, Finally, the proposed Neutrosophic Scaling System is applied to risk assessment case study.

The remining sections in this paper organized as follows: section two, represent a literature review about scaling system and some neutrosophic concepts used in the paper; Section three, contains some neutrosophic basic definitions are outlined; a proposed neutrosophic scaling system algorithm presented and two illustrative numerical examples are presented in section four; section five contains a conclusion followed by references.

2. Literature review

An overview of neutrosophic logic, Simplified Neutrosophic Number SNN, Single-Valued Neutrosophic Set SVNS, are discussed, in addition to evaluate some de-neutrosophication methods such as distance and similarity, also, concept of scale system is discussed.

Smarandache extend Neutrosophic logic as a branch of philosophy [5] that reviews the basis and scope of neutrality, neutrosophic was discussed by a lot of researchers and applied in a variety of businesses assisting in solving many challenges as a powerful scale in the selection [6], Multi-criteria decision making MCDM [7] [8] [9], achieving PERT in project management [10], exploring the influence of Internet of Things (IoT) and how IOT influence supply chain [11], a lot of studies propose an enhanced variety of aggregation operators [12]. Wen, et al, (2017) [13] offered a novel method to calculate the similarity between SVNSs, plus Jun and Shigui (2017) [14] offer distances, similarity and entropy methods for IVNS, Surapati and Kalyan (2015) [15] explain a rough cosine similarity calculation among two rough NS., said and Florentin (2014) [16] offer a novel cosine similarity among two IVNS based on Bhattacharya’s distance, Ye (2014) [17] suggest a few of aggregation operators, as well as a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator.

National Institute of Standards and Technology used five level Risk Assessment Scale in its special publication 800-30 “Guide for Conducting Risk Assessments” as standard scale where the percentages from 0% up to 4% refers to the linguistic scale of “Very Low” or lowest scale level, on the other hand they used the percentages from 96% up to 100% to refer to linguistic scale of “Very High” or highest scale level, all five levels of the qualitative risk scale values and its equivalent percentage ranges as proposed by NIST, nevertheless, NIST didn’t explain the scientific base for selecting this specific ranges for each Qualitative Values [2]

3. Preliminaries

In this section, the basic definitions related to NS, SVNS, absolute and empty NS, Simplified Neutrosophic Set SNS, SNN and them operations are outlined, in addition de-neutrosophication, score functions, similarity functions, and distance functions are evaluated and enhanced.

Definition 3.1. Neutrosophic Set:

Florentin Smarandache 1998 proposed neutrosophic logic and neutrosophic sets and coined the definition of "Neutrosophic Set" with three principles (membership, indeterminacy, and non-membership) [18], [7] Let $T_A(x), T_A(x), and\ F_A(x)$ be real standard or non-standard Statically subsets (sub) of]-0, 1+[, Let X is a universe of discourse, and M a set included in X, and x is an element from X is described with respect to the set A as $x(T_A(x), T_A(x), F_A(x))$ and belongs to A where x is $(t\%\ true)$ in the set, $(i\%\ indeterminate)$ or undefined in the set, and $(f\%\ false)$, considering that (t) changes in $T_A(x): X \to\ ]^-0,\ 1^+[$, (i) changes in $I_A(x): X \to\ ]^-0,\ 1^+[$, (f) changes in $F_A(x): X \to\ ]^-0,\ 1^+[$, without restriction in the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, and meets the condition of summation: $(^-0 \leq \mathrm{supT_A(x)} + \mathrm{supI_A(x)} + \mathrm{supF_A(x)} \leq 3^+)$

$$NS(A) = \{\langle x,\ T_A(x),\ I_A(x),\ F_A(x)\rangle | x \in X,\ T_A(x), I_A(x),\ F_A(x) \in\ ]^-0, 1^+[\ \} \quad (1)$$

Definition 3.2. Single-Valued Neutrosophic set (SVNS)

Wang et al. [19], presented "Single Valued Neutrosophic Set" (SVNS), as a subclass of the NS. which defined in Definition 3.1 and Simplified Neutrosophic Set SNS which defined in Definition 3.4 below, in consequence of that, SVNS is an instance of NS that can implemented in our life applications [20], [21], Let X be a universe of discourse, a SVNS A over X is an object with the form of $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}$, for the intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ refer to truth, indeterminacy, and falsity memberships degrees respectively of x to A, also, $T_A(x) \in [1,0]$, $I_A(x) \in [1,0]$ and $F_A(x) \in [1,0]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$, for X is discrete, a $SVNS\ A$ will stated as shown in formula (2), while X is continuous, a $SVNS\ A$ will stated as shown in formula (3).

$$\mathrm{SVNS}\ (A) = \sum_{i=1}^{n} \frac{\langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle}{x_i} \mid x_i \in X \quad (2)$$

$$SVNS\ (A) = \int_X \frac{\langle T_A(x), I_A(x), F_A(x)\rangle}{x} \mid x \in \mathrm{X} \quad (3)$$

Definition 3.3. Absolute and Empty Neutrosophic Set

Gayyar (2016) [22] defined two special cases for neutrosophic set which are the Null (Empty) neutrosophic set (0_N) and the absolute (universe) neutrosophic set (1_N), where Empty Neutrosophic Set has two forms $(0_N) =< x, 0,0,1 >\ x \in X$ and $(0_N) =< x, 0,1,1 >\ x \in X$, also the absolute neutrosophic set has two forms $(1_N) =< x, 1,1,0 >\ x \in X$, and $(1_N) =< x, 1,0,0 >\ x \in X$, which is not accepted where $< x, 0,0,1 >$ is not equal to $< x, 0,1,1 >$ and $< x, 0,1,1 >$ is not empty, on the other hand the $< x, 1,1,0 >$ is not equal to $< x, 1,0,0 >$ and $< x, 1,1,0 >$ is not universal set, Therefore, we propose that, "Empty Simplified Neutrosophic Number" can denoted by one form as shown in formula (4), and , "Absolute Simplified Neutrosophic Number" can denoted by one form as shown in formula (5) only.

$$0_N =< 0,0,1 >\ x \in X \quad (4)$$

$$1_N =< 1,0,0 >\ x \in X \quad (5)$$

Definition 3.4. Simplified Neutrosophic Set (SNS):

Ye, (2014) [17], SNS is an special case of NS, where the functions $T_A(x)$, $I_A(x), and\ F_A(x)$ represented as single points in the real standard [0,1] instead of subintervals / subsets in the real standard [0,1], that is $T_A(x) \in [1,0]$, $I_A(x) \in [1,0]$, and $F_A(x) \in [1,0]$. Therefore, $SNS\ A$ is represented by formula (6), with no limitation on the sum of $T_A(x), I_A(x),\ and\ F_A(x)$, satisfies the condition of: $0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3$.

$$SNS\ (A) = \{\langle x,\ T_A(x), I_A(x),\ F_A(x)\rangle | x \in X,\ T_A(x), I_A(x),\ F_A(x) \in]0,1[\ \} \tag{6}$$

Definition 3.5. Simplified Neutrosophic Number (SNN)

Considering SNS is a subclass of NS, Ye, (2014) [17] offer Simplified Neutrosophic Number (SNN) as a special case of SNS, in specific when X consist of one object of A, where $A = \{\langle\ T_A(x), I_A(x), F_A(x)\rangle | x \in X\}$ it named as SNN, for ease, SNN is presented as shown in formula (7),.

$$\text{SNN}\ (A) = \langle T_A, I_A, F_A\rangle \tag{7}$$

Definition 3.6. Cosine Similarity

Ye, (2014) [17], proposed a method to compare any SVNS with absolute SVNS built on the cosine similarity measure as shown in formula (8), that can be extended to SNN $x = (T, I, F)$ considering the absolute SNN = (1,0,0) as defined in formula (5),

$$COS(x) = \frac{T_x}{\sqrt{{T_x}^2 + {I_x}^2 + {F_x}^2}} \tag{8}$$

However, in some cases the formula (8) didn't represent the correct similarity for example: for $A = (0.1{,}0.1{,}0.1)$, $B = (0.9{,}0.9{,}0.9)$ and $K = (k, k, k)|1 \geq k > 0]$ where the three memberships has the same value then $COS(A) = COS(B) = COS(K) = 0.577350269$ using formula (8), also when falsity membership and indeterminacy membership are equal to zero formula (8) returns the similarity value of 1 regardless truth membership value, for $A = (0.1{,}0{,}0)$, $B = (0.9{,}0{,}0)$ and $K = (z, k, k)|z \in [0,1], k = 0]$, then $COS(A) = COS(B) = COS(K) = 1$, which is not accepted.

Definition 3.7. Kanika's similarity measure

Kanika, (2020) [23] propose a similarity measure $S_1(A, B)$ for SVNS, for $A = \langle x_i, T_A(x_i)\ , I_A(x_i), F_A(x_i)\rangle$, $B = \langle x_i, T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ where $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$, $T_B(x_i)$, $I_B(x_i)$, $F_B(x_i) \in [0,1]$, $x_i\ (i = 1, 2, \ldots, n)$ as shown in formula (9).

$$S_1(A, B) = 1 - \frac{1}{2n} \times \sum_{i}^{n} [|T_A(x_i) - T_B(x_i)| + Max\{|I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\ \}] \tag{9}$$

However, in some cases the formula (9) didn't return correct similarity value, for example: when using formula (9) to calculating the similarity between absolute SVNS $1_N = (1{,}0{,}0)$ and both $A = (0.5{,}0{,}0.2)$, $B = (0.4{,}0{,}0.1)$, $S_1(1_N, A) = S_1(1_N, B) = 0.65$, also for $A = (0.5{,}0.2{,}0.6)$, $B = (0.2{,}0.2{,}0.3)$, $S_1(1_N, A) = S_1(1_N, B) = 0.45$ which is not accepted.

Definition 3.8. Score Function

Nancy, et al (2016) [24] propose a score function $S_2(1_N, A)$ shown in formula (10), as an enhancement for $S_3(1_N, A)$ shown in formula (11) proposed by Şahin, (2014) [25], for $A = \langle x_i,\ T_A(x_i)\ , I_A(x_i), F_A(x_i)\rangle$ where $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i) \in [0,1]$, $x_i\ (i = 1, 2, \ldots, n)$, in case $T_A(x_i) + F_A(x_i) = 1$, Nancy, et al propose to use $S_3(1_N, A)$ shown in formula (11).

$$S_2(1_N, A) = \frac{1 + (T_A(x_i) - 2I_A(x_i) - F_A(x_i))(2 - T_A(x_i) - F_A(x_i))}{2} \quad (10)$$

$$S_3(1_N, A) = \frac{1 + T_A(x_i) - 2I_A(x_i) - F_A(x_i)}{2} \quad (11)$$

On the other hand, Both formulas (10) and (11) have some limitation in some cases for example: for $A = (0.4,0.9,0.5)$ both formulas return a negative similarity $= -0.545, -0.45$ respectively, in case of $T_A(x_i) + F_A(x_i) = 1, T_A(x_i) = 0, F_A(x_i) = 1$ both formulas return a negative similarity also, in addition, for $A = (0.4,0.4,0.4)$, formulas return $0.02, 0.1$, also for $A = (0.9,0.9,0.9)$ the formulas return $0.32, -0.4$ respectively, which are not accepted.

Definition 3.9. Euclidean-base similarity

Majumdar and Samanta (2014) [26], offer SVNS similarity formula for $A = \langle x_i, T_A(x_i), I_A(x_i), F_A(x_i)\rangle$, $B = \langle x_i, T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ where $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$, $T_B(x_i)$, $I_B(x_i)$, $F_B(x_i) \in [0,1]$, x_i $(i = 1, 2, \dots, n)$ as shown in formula (12).

$$S_4(A, B) = 1 - \frac{1}{3}(|T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)|) \quad (12)$$

Formula (12), has some drawbacks, such as for two SVNS $A = (0.5,0.2,0.6)$, $B = (0.2,0.2,0.3)$ which are two different SVNS but $S_4(1_N, A) = S_4(1_N, B) = 0.566666667$, which is not accepted for totally different SVNS, also for $A = (0.1,0,0)$, then $S_4(1_N, A) = 0.7$ which is not sound logical similarity value.

Ye, (2014) [27], extend the Euclidean distance measure by adding a weight for his method when measuring distance and similarity between SVNSs, for $A \; and \; B$, two SVNSs giving $SVNS\,(A) = \langle x_i, T_A(x_i), I_A(x_i), F_A(x_i)\rangle$, $SVNS\,(B) = \langle x_i, T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ where $T_A(x_i), I_A(x_i), F_A(x_i)$, $T_B(x_i)$, $I_B(x_i)$, $F_B(x_i) \in [0,1]$, consider the weight w_i $(i = 1, 2, \dots, n)$ of an object for x_i $(i = 1, 2, \dots, n)$, for $w_i \geq 0$ $(i = 1, 2, \dots, n)$ and $\sum_i^n w_i = 1$, single-valued neutrosophic weighted distance measure between $A, and \; B$ defined as shown in formula (13), which considered as a generic formula for calculating the distance using both Hamming and Euclidean distance methods, where, $p = 1$ in case of using Hamming distance and $p = 2$ in case of using Euclidean distance, also Ye, (2014) prove the relation distance and similarity are complementary where similarity $S_1(A, B) = 1 - d_p(A, B)$ and vice versa as shown in formula (14) [27].

$$d_p(A, B) = \sqrt[p]{\frac{\sum_i^n w_i([T_A(x_i) - T_B(x_i)]^P + [I_A(x_i) - I_B(x_i)]^P + [F_A(x_i) - F_B(x_i)]^P)}{3}} \; |p > 0 \quad (13)$$

$$S_1(A, B) = 1 - d_p(A, B) = 1 - \sqrt[p]{\frac{\sum_i^n w_i([T_A(x_i) - T_B(x_i)]^P + [I_A(x_i) - I_B(x_i)]^P + [F_A(x_i) - F_B(x_i)]^P)}{3}} \; | \; p > 0 \quad (14)$$

considering that distance $d_p(A, B)$ for $p > 0$ satisfies four properties first: $0 \leq d_p(A, B) \leq 1$; second: $d_p(A, B) = 0$ if and only if $A = B$; third: $d_p(A, B) = d_p(B, A)$; and forth property is: $If A \subseteq B \subseteq C$, for C is an SVNS in X, then $d_p(A, C) \geq d_p(A, B) \; and \; d_p(A, C) \geq d_p(B, C)$ [27], but formulas (13) and (14) have some limitation in some cases such as for when applying formula (14) for SVNS $A(x) = \{x, (0.40,0.65,0.60), (0.50,0.50,0.50), (0.40,0.65,0.60)\}$ it return = -0.005816418 which is not accepted, the proposed formula below overcome that shortage.

Definition 3.10. SNN and SVNS 3D visualization

Few effort paid in visualizing neutrosophic sets and numbers, Smarandache, el at (2019) and others [28] use Figure 1 to demonstrate the graphical visualization for neutrosophic environment, also this graph used as a part from Neutrosophic Sets and Systems journal's cover page.

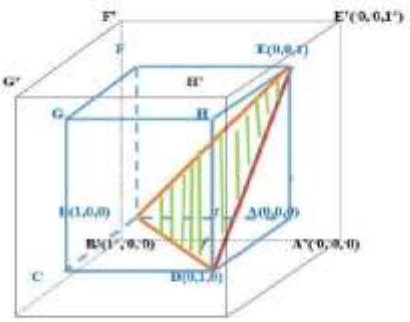

Figure 1 – neutrosophic graphical visualization [28]

Also, Garai et al, 2020 [29], use graph presentation shown in Figure 2 to represent for example SNVN $A = \langle((1,3,5,8),0.9),((1,2,6,8),0.3),((1,3,5,8),0.5)\rangle$.

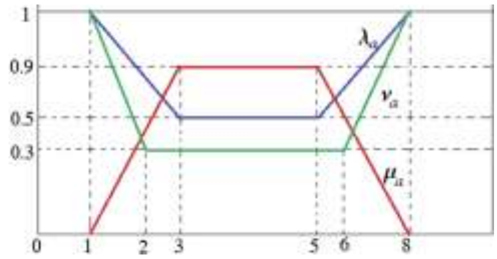

Figure 2 – Single-Valued neutrosophic number [29]

Meanwhile, Karaaslan & Hunu (2020) [30] present SVNN graphically as shown in Figure 3 which represent each truth, indeterminacy, and falsity memberships separately.

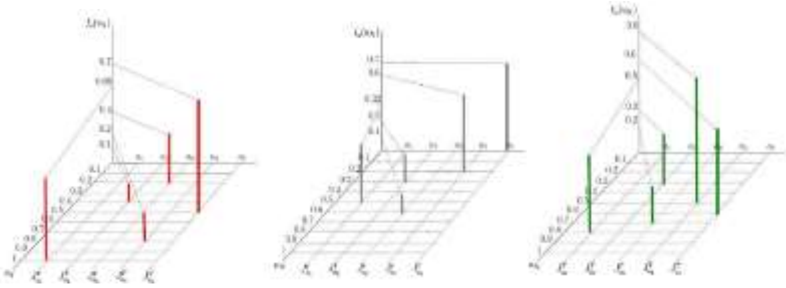

Figure 3 – type 2 SVNS graphical representation [30]

The researchers offer a graphical representation for simplified neutrosophic number SNN and single-valued neutrosophic set SVNS using 3-Dimentional Euclidean space as shown in Figure 4 below, where the empty SNN $0_N = (0,0,1)$ located in the origin point and the absolute SNN $1_N = (1,0,0)$ located in the top $T(x)$ axis, the SNN $A = (0.5,0.3,0.6)$ "an example" which presented in the graph with a "Red Point" using $T(x) = 0.5,\ I(x) = 0.3,\ F(x) = 0.6$.

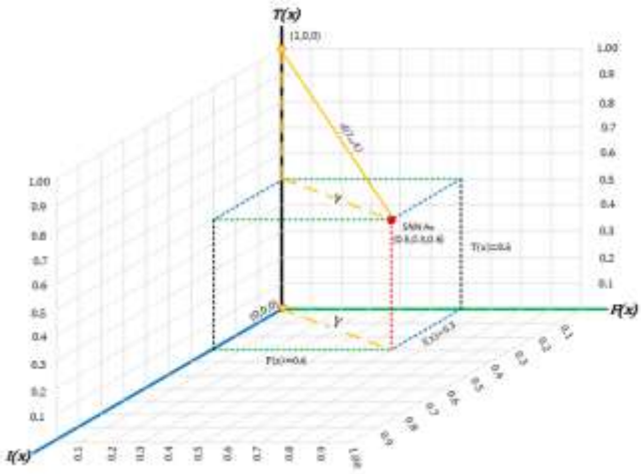

Figure 4 – SNN in 3-Dimentional Euclidean space

Extending the in 3-Dimentional visualization for SNN, Figure 5 below shows in 3-D visualization for two discrete SVNSs $A_i = \langle x_i, T_A(x_i)\, , I_A(x_i), F_A(x_i)\rangle$ and $B_i = \langle x_i, T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ where $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$, $T_B(x_i)$, $I_B(x_i)$, $F_B(x_i) \in [0,1]$, and x_i $(i = 1, 2,3,4)$ giving the value of each element in both SVNSs as the following, for A_i elements $A_1 = (0.8,0.3,0.8)$, $A_2 = (0.2,0.2,0.2)$, $A_3 = (0.5,0.3,0.5)$, $A_4 = (0.8,0.2,0.8)$ and for B_i elements $B_1 = (0.5,0.9,0.1)$, $B_2 = (0.7,0.7,0.4)$, $B_3 = (0.3,0.7,0.5)$, $B_4 = (0.2,1,1)$.

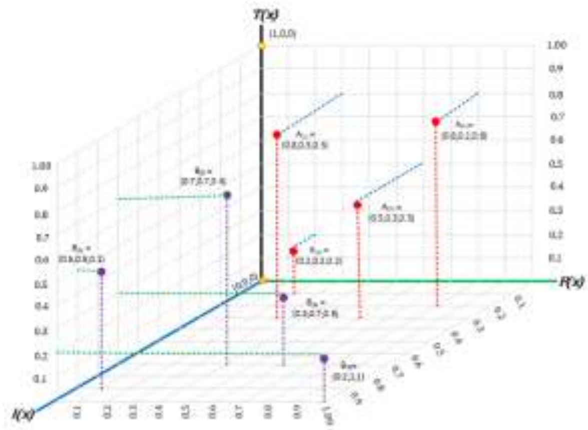

Figure 5 - Two SVNS in 3-Dimentional Euclidean space

Definition 3.11. SNN Euclidean distance

"Euclidean distance" or commanlly named as "Pythagorean distance" which is purely the straight-line distance between two points in the Euclidean space as shown in Figure 4 above, fomula (15) represent the Euclidean distance for $SNN\ (A)$ which refare to the straight-line distance between absolute $SNN\ 1_N = (1,0,0)$ and $SNN\ (A)$, where $d_5(1_n, A) = 0.931149915$ as a pure distance considring $0 \le d_5(1_n, A) \le \sqrt{3}$.

$$d_5(1_n, A) = \sqrt{T_A(x)^2 + I_A(x)^2 + F_A(x)^2} \quad (15)$$

Definition 3.12. Two SNN Euclidean distance

For generalization, it's clear from Figure 6 that, the Euclidean distance $d_6(A, B)$ between two SNNs A and B can be calculated using formula (16) considering that $0 \le d_6(A, B) \le \sqrt{3}$, for seek of normalizations formula (17) provided normalized Euclidean distance $d_7(A, B)$ between the SNN A and SNN B considering that $0 \le d_7(A, B) \le 1$, considering that Ye, (2014) prove the relation distance and similarity are complementary, therefore the normalized similarity $S_7(A,B)$ for normalized Euclidean distance $S_7(A,B) = \big(1 - d_7(A,B)\big) \times 100$ as shown in formula (18).

$$d_6(A,B) = \sqrt{|T_A(x_i) - T_B(x_i)|^2 + |I_A(x_i) - I_B(x_i)|^2 + |F_A(x_i) - F_B(x_i)|^2} \tag{16}$$

$$d_7(A,B) = \sqrt{\frac{|T_A(x_i) - T_B(x_i)|^2 + |I_A(x_i) - I_B(x_i)|^2 + |F_A(x_i) - F_B(x_i)|^2}{3}} \tag{17}$$

$$S_7(A,B) = \left(1 - \sqrt{\frac{|T_A(x_i) - T_B(x_i)|^2 + |I_A(x_i) - I_B(x_i)|^2 + |F_A(x_i) - F_B(x_i)|^2}{3}}\right) \times 100 \tag{18}$$

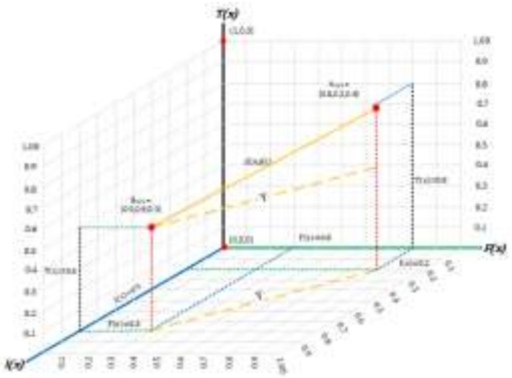

Figure 6 - Two SNN in 3-Dimentional Euclidean space

Definition 3.13. New SVNS distance and similarity measures

Figure 7 represents the Euclidean distance $d(A,B)$, between SVNSs A and B where $d_i(A_i,B_i) \mid (i = 1,2,3,4)$ represents the distance between each two elements in SVNS A_i and SVNS B_i, $d_8(A,B)$, formula (19) represents the Euclidean distance between SVNS A_i and SVNS B_i which extended from formula (16), where $0 \le d_8(A,B) \le \sqrt{3}$, for reaching normalizations, formula (21) provided normalized Euclidean distance $d_9(A,B)$ between SVNS A_i and SVNS B_i considring that $0 \le d_9(A,B) \le 1$, [27] where similarity equal 1- distance and vice versa so, $S_8(A,B) = \big(1 - d_8(A,B)\big) \times 100$ and $S_9(A,B) = \big(1 - d_9(A,B)\big) \times 100$ as shown in formulas (20) and (22) respectively.

$$d_8(A,B) = \sqrt{\textstyle\sum_i^n (|T_A(x_i) - T_B(x_i)|^2 + |I_A(x_i) - I_B(x_i)|^2 + |F_A(x_i) - F_B(x_i)|^2)} \tag{19}$$

$$S_8(A,B) = \left(1 - \sqrt{\sum_i^n (|T_A(x_i) - T_B(x_i)|^2 + |I_A(x_i) - I_B(x_i)|^2 + |F_A(x_i) - F_B(x_i)|^2)}\right) \times 100 \tag{20}$$

$$d_9(A,B) = \frac{\sum_i^n \sqrt{|T_A(x_i) - T_B(x_i)|^2 + |I_A(x_i) - I_B(x_i)|^2 + |F_A(x_i) - F_B(x_i)|^2}}{n\sqrt{3}} \tag{21}$$

$$S_9(A,B) = \left(1 - \frac{\sum_i^n \sqrt{|T_A(x_i) - T_B(x_i)|^2 + |I_A(x_i) - I_B(x_i)|^2 + |F_A(x_i) - F_B(x_i)|^2}}{n\sqrt{3}}\right) \times 100 \tag{22}$$

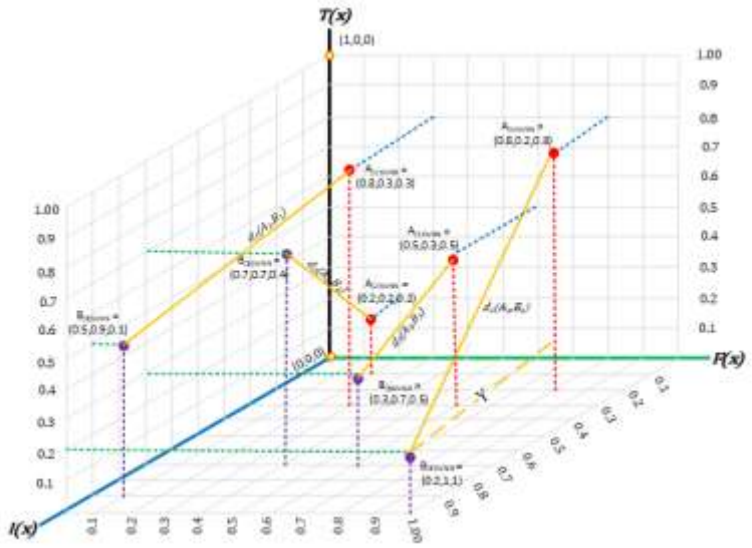

Figure 7 - Euclidean distance between SVNS A_i and SVNS B_i

4. Proposed Framework

in this research paper effort paid off to proposes a scaling system using Simplified Neutrosophic Number or Single-Valued neutrosophic set as elaborated in the below algorithm.

Neutrosophic scaling system algorithm:

Step 1: Create a sorted list of Qualitative terms "Linguistic terms", which will be used as final scaling system outputs, remarking that Linguistic terms shall be sorted either ascending or descending according to the purpose of scaling system, $\mathbb{N}$ represent the number of Linguistic terms as shown in formula (23).

$$\mathbb{N} = \text{Number of language terms} \tag{23}$$

Step 2: Business expert enter the SNN A or SVNS A value for each linguistic term, considering keeping Linguistic terms sorted "bad to good" or "good to bad".

Step 3: Using formula (24) to calculating the equivalent risk crisp value $\mathbb{Q}$ corresponding to each giving SNN using formula (18) similarity $S_7(1_n, A)$ or SVNS using formula (22) similarity $S_9(1_n, A)$ multiplied by number of Linguistic terms $\mathbb{N}$ calculated in formula (23), domain experts can override manually any of calculated equivalent crisp values $\mathbb{Q}$, in this case a modified flag must be added for each override/changed value, keeping in mind that modifying any equivalent crisp values must not changing the order of Linguistic terms.

$$\begin{cases} \mathbb{Q} = S_7(1_n, A) \times \mathbb{N} | \text{A is SNN} \\ \mathbb{Q} = S_9(1_n, A) \times \mathbb{N} | \text{A is SVNS} \end{cases} \tag{24}$$

Step 4: Build 2D Matrix with $\mathbb{N}$ rows and columns specified in Step 1: , then add Linguistic terms in the top row and first column with its corresponding equivalent crisp value $\mathbb{Q}$ and calculate the $matrix\ cells\ values$ by multiple the row value times column value.

Step 5: Convert all $cell\ value$ to $cell\ percentage$ using formula (25) by dividing each matrix cell value by maximum cell value squared, where maximum cell value squared equal $Max(\mathbb{Q})^2$ defined in Step 1: above.

$$cell\ percntage = \frac{cell\ value}{Max(\mathbb{Q})^2} \tag{25}$$

Step 6: Generate Strict risk assessment scale:

1. To determine maximum percentage value for each Linguistic term, look for intersected cells with same Linguistic term, considering these intersected cells as the maximum percentage value for Linguistic terms.
2. To determine minimum percentages values for each Linguistic term, use maximum percentage value for preceding Linguistic terms as minimum percentages values for Linguistic terms.
3. Domain expert can change the range boundary as appropriate.

Step 7: Generate Lenient risk assessment scale:

1. To determine minimum percentage value for each Linguistic term, look for intersected cells with same Linguistic term, considering these intersected cells as the minimum percentage value for Linguistic terms.
2. To determine maximum percentages values for each Linguistic term, use minimum percentage value for following Linguistic term as maximum percentages values for Linguistic terms and add 100% as a maximum for the highest Linguistic term.
3. Domain expert can change the range boundary as appropriate.

Neutrosophic risk assessment scale illustrative numerical example 1:

Step 1: Create a sorted list of qualitative terms "Linguistic terms", as shown in Table 1 $\mathbb{N} = 11$.

Table 1 qualitative value "Linguistic terms"

Linguistic terms	abbreviation
Extremely bad	EB
Very very bad	VVB
Very bad	VB
Bad	B
Medium bad	MB
Medium	M
Medium good	MG
Good	G
Very good	VG
Very very good	VVG
Extremely good	EG

Step 2: Enter the equivalent SNN value provided by business expert for each linguistic term, shown in Table 2

Table 2 Linguistic terms, Equivalent SNN

Linguistic terms bad to good	Linguistic terms good to bad	Equivalent SNN values
Extremely bad	Extremely good	(1,0,0)
Very very bad	Very very good	(0.9, 0.1, 0.1)
Very bad	Very good	(0.8,0.15,0.20)
Bad	Good	(0.70,0.25,0.30)
Medium bad	Medium good	(0.60,0.35,0.40)
Medium	Medium	(0.50,0.50,0.50)

Linguistic terms bad to good	Linguistic terms good to bad	Equivalent SNN values
Medium good	Medium bad	(0.40,0.65,0.60)
Good	Bad	(0.30,0.75,0.70)
Very good	Very bad	(0.20,0.85,0.80)
Very very good	Very very bad	(0.10,0.90,0.90)
Extremely good	Extremely bad	(0,1,1)

Step 3: Calculate the equivalent crisp value $\mathbb{Q}$ corresponding to each giving SNN Using formula (24) as shown in Table 3, noting that the Crisp Values of the linguistic term "Extremely good" was modified from 0 to 0.10 according to expert opinion and modified flag inserted.

Table 3 Linguistic terms, SNN, and its equivalent crisp values $\mathbb{Q}$

Linguistic value bad to good	Linguistic value good to bad	Equivalent SNN values	Calculated Crisp Values	Modified Crisp Values	Modified flag
Extremely bad	Extremely good	(1,0,0)	11	11.00	
Very very bad	Very very good	(0.9, 0.1, 0.1)	9.9	9.90	
Very bad	Very good	(0.8,0.15,0.20)	8.966735	8.97	
Bad	Good	(0.70,0.25,0.30)	7.872568	7.87	
Medium bad	Medium good	(0.60,0.35,0.40)	6.77537	6.78	
Medium	Medium	(0.50,0.50,0.50)	5.5	5.50	
Medium good	Medium bad	(0.40,0.65,0.60)	4.211714	4.21	
Good	Bad	(0.30,0.75,0.70)	3.112404	3.11	
Very good	Very bad	(0.20,0.85,0.80)	2.012926	2.01	
Very very good	Very very bad	(0.10,0.90,0.90)	1.1	1.10	
Extremely good	Extremely bad	(0,1,1)	0	0.10	*

Step 4: Build Two-dimensional Symmetric Matrix with $\mathbb{Q} = 11$ rows and columns then add Linguistic terms in the top row and first column as shown in Table 4 below, then calculate the matrix cells values by multiple the row value times column value, for example: cell(1,8) which are $(row, column)$ reflect is the intersection of row no 1: "EB as Extremely Bad" with the value of (11.00) and column no 8: "B as Bad" with value of (7.87), so the cell(1,8) value equal $7.87 \times 11.0 = 86.60$; Another example: the cell(5,3) which is the intersection of row no:5 "MB" with the value of (6.78) and column no:3 "VG" with value of (2.01), so cell(5.3) value equal $6.78 \times 2.01 = 13.64$, and so on for all matrix cells as shown in Table 4 below.

Table 4 Two-dimensional Symmetric Matrix value

Row No.	Col No.		1	2	3	4	5	6	7	8	9	10	11
	$\mathbb{Q}$		0.10	1.10	2.01	3.11	4.21	5.50	6.78	7.87	8.97	9.90	11.00
		term Prefix	EG	VVG	VG	G	MG	M	MB	B	VB	VVB	EB
1	11.00	EB	1.10	12.10	22.14	34.24	46.33	60.50	74.53	86.60	98.63	108.90	121.00
2	9.90	VVB	0.99	10.89	19.93	30.81	41.70	54.45	67.08	77.94	88.77	98.01	108.90
3	8.97	VB	0.90	9.86	18.05	27.91	37.77	49.32	60.75	70.59	80.40	88.77	98.63
4	7.87	B	0.79	8.66	15.85	24.50	33.16	43.30	53.34	61.98	70.59	77.94	86.60
5	6.78	MB	0.68	7.45	13.64	21.09	28.54	37.26	45.91	53.34	60.75	67.08	74.53
6	5.50	M	0.55	6.05	11.07	17.12	23.16	30.25	37.26	43.30	49.32	54.45	60.50
7	4.21	MG	0.42	4.63	8.48	13.11	17.74	23.16	28.54	33.16	37.77	41.70	46.33
8	3.11	G	0.31	3.42	6.27	9.69	13.11	17.12	21.09	24.50	27.91	30.81	34.24

Row No.	Col No.		1	2	3	4	5	6	7	8	9	10	11
	ℚ		0.10	1.10	2.01	3.11	4.21	5.50	6.78	7.87	8.97	9.90	11.00
		term Prefix	EG	VVG	VG	G	MG	M	MB	B	VB	VVB	EB
9	2.01	VG	0.20	2.21	4.05	6.27	8.48	11.07	13.64	15.85	18.05	19.93	22.14
10	1.10	VVG	0.11	1.21	2.21	3.42	4.63	6.05	7.45	8.66	9.86	10.89	12.10
11	0.10	EG	0.01	0.11	0.20	0.31	0.42	0.55	0.68	0.79	0.90	0.99	1.10

Step 5: Convert the matrix cells' value to percentage as shown in Table 5 using formula (25) where $\mathbb{Q} = 11$ and maximum cell value is $11^2 = 121$, so for example $cell(5,5)percntage = 28.54/121 = 23.58\%$another example the $cell(2,7)percntage = 67.08/121 = 55.43\%$, and so on for all matrix cells'.

Table 5 Two-dimensional Symmetric Matrix percentage

Row No.	Col No.		1	2	3	4	5	6	7	8	9	10	11
	ℚ		0.10	1.10	2.01	3.11	4.21	5.50	6.78	7.87	8.97	9.90	11.00
		term Prefix	EG	VVG	VG	G	MG	M	MB	B	VB	VVB	EB
1	11.00	EB	0.91%	10.00%	18.30%	28.29%	38.29%	50.00%	61.59%	71.57%	81.52%	90.00%	100%
2	9.90	VVB	0.82%	9.00%	16.47%	25.47%	34.46%	45.00%	55.43%	64.41%	73.36%	81.00%	90.00%
3	8.97	VB	0.74%	8.15%	14.92%	23.06%	31.21%	40.76%	50.21%	58.34%	66.45%	73.36%	81.52%
4	7.87	B	0.65%	7.16%	13.10%	20.25%	27.40%	35.78%	44.08%	51.22%	58.34%	64.41%	71.57%
5	6.78	MB	0.56%	6.16%	11.27%	17.43%	23.58%	30.80%	37.94%	44.08%	50.21%	55.43%	61.59%
6	5.50	M	0.45%	5.00%	9.15%	14.15%	19.14%	25.00%	30.80%	35.78%	40.76%	45.00%	50.00%
7	4.21	MG	0.35%	3.83%	7.01%	10.83%	14.66%	19.14%	23.58%	27.40%	31.21%	34.46%	38.29%
8	3.11	G	0.26%	2.83%	5.18%	8.01%	10.83%	14.15%	17.43%	20.25%	23.06%	25.47%	28.29%
9	2.01	VG	0.17%	1.83%	3.35%	5.18%	7.01%	9.15%	11.27%	13.10%	14.92%	16.47%	18.30%
10	1.10	VVG	0.09%	1.00%	1.83%	2.83%	3.83%	5.00%	6.16%	7.16%	8.15%	9.00%	10.00%
11	0.10	EG	0.01%	0.09%	0.17%	0.26%	0.35%	0.45%	0.56%	0.65%	0.74%	0.82%	0.91%

Step 6: Generate Strict risk assessment scale:

1. To determine maximum percentage value for each Linguistic term, highlight intersected cells with same Linguistic term as shown in Table 6, considering these intersected cells values as the maximum percentage value for Linguistic terms.

Table 6 two-dimensional Symmetric maximum value for category

Ling. Prefix	EG	VVG	VG	G	MG	M	MB	B	VB	VVB	EB
EB											100%
VVB										81.00%	
VB									66.45%		
B								51.22%			
MB							37.94%				
M						25.00%					
MG					14.66%						
G				8.01%							
VG			3.35%								
VVG		1.00%									
EG	0.01%										

2. To determine minimum percentages values for Linguistic terms, use maximum percentage value for preceding Linguistic terms as minimum percentages values for

Linguistic terms From the previous step minimum and maximum percentages values for each qualitative value and qualitative values rang have been determined as shown in Table 7, domain expert can change the range edge as appropriate.

Table 7 **Strict** Linguistic terms rang percentage

Linguistic Terms Good to bad	Linguistic Terms Bad to good	Min	Max	Strict Range
Extremely good	Extremely bad	81.00%	100.0%	>81.0% & <=100%
Very very good	Very very bad	66.45%	81.00%	>66.4% & <=81.0%
Very good	Very bad	51.22%	66.45%	>51.2% & <=66.4%
Good	Bad	37.94%	51.22%	>37.9% & <=51.2%
Medium good	Medium bad	25.00%	37.94%	>25.0% & <=37.9%
Medium	Medium	14.66%	25.00%	>14.7% & <=25.0%
Medium bad	Medium good	8.01%	14.66%	>8.0% & <=14.7%
Bad	Good	3.35%	8.01%	>3.3% & <=8.0%
Very bad	Very good	1.00%	3.35%	>1.0% & <=3.3%
Very very bad	Very very good	0.01%	1.00%	>0.01% & <=1.0%
Extremely bad	Extremely good	0.00%	0.01%	>0% & <=0.01%

Step 7: Generate Lenient risk assessment scale:

1. To determine minimum percentage value foreach Linguistic term, highlight intersected cells with same Linguistic term as shown in Table 6, considering these intersected cells values as the minimum percentage value for Linguistic terms.

Ling. Prefix	EG	VVG	VG	G	MG	M	MB	B	VB	VVB	EB
EB											100%
VVB										81.00%	
VB									66.45%		
B								51.22%			
MB							37.94%				
M						25.00%					
MG					14.66%						
G				8.01%							
VG			3.35%								
VVG		1.00%									
EG	0.01%										

2. To determine maximum percentages values for each Linguistic term, use minimum percentage value for following Linguistic term as maximum percentages values for Linguistic terms and add 100% as a maximum for the highest Linguistic term as shown Table 8.
3. Domain expert can change the range boundary as appropriate

Table 8 **Lenient** qualitative Values rang percentage

Linguistic Terms Good to bad	Linguistic Terms Bad to good	Min	Max	Lenient Range
Extremely good	Extremely bad	100%	100%	>=100.0%

Linguistic Terms Good to bad	Linguistic Terms Bad to good	Min	Max	Lenient Range
Very very good	Very very bad	81.00%	100%	>=81.00% & <100.0%
Very good	Very bad	66.45%	81.00%	>=66.45% & <81.0%
Good	Bad	51.22%	66.45%	>=51.22% & <66.4%
Medium good	Medium bad	37.94%	51.22%	>=37.94% & <51.2%
Medium	Medium	25.00%	37.94%	>=25.00% & <37.9%
Medium bad	Medium good	14.66%	25.00%	>=14.66% & <25.0%
Bad	Good	8.01%	14.66%	>=8.01% & <14.7%
Very bad	Very good	3.35%	8.01%	>=3.35% & <8.0%
Very very bad	Very very good	1.00%	3.35%	>=1.00% & <3.3%
Extremely bad	Extremely good	0.01%	1.00%	>=0% & <1.00%

Calculate risk assessment illustrative numerical example 2:

Step 1: This example aims to calculate risk assessment for a project has four 4 major risk areas named personnel quality, production equipment, work environment, and safety management; these areas contains 23 risk factors, Table 9 below contains list of risk categories and its risk factors.

Table 9 –Risks categories and factors

Risk Category	Factors (x_i)	Risk Factors (subcategory)
People quality	x_1	Education level
	x_2	Learner's time
	x_3	Age
	x_4	duration of service
	x_5	Worker density
	x_6	Body status
	x_7	Business period
Production equipment	x_8	Restrict dropping devices
	x_9	equipment design dependability
	x_{10}	equipment proper rate
	x_{11}	Protecting equipment dependability
	x_{12}	equipment flexibility
Environment	x_{13}	Heat
	x_{14}	Light
	x_{15}	humidity
	x_{16}	Environmental security dependability
	x_{17}	running surface efficiency
Safety management	x_{18}	Security system
	x_{19}	Safety society
	x_{20}	… feedback
	x_{21}	… assessment
	x_{22}	… cotching
	x_{23}	… checks

Step 2: In this case will use the linguistic terms and its equivalent "strict ranges" and "lenient ranges" previously calculated in Table 7 and Table 8 above, using sorted linguistics terms from "bad to good" as consolidated in Table 10.

Table 10 Linguistic terms, both Strict Range and Lenient Range

Linguistic Terms Bad to good	Strict Ranges	Lenient Ranges
Extremely bad	>81.0% & <=100%	>=100.0%
Very very bad	>66.4% & <=81.0%	>=81.00% & <100.0%
Very bad	>51.2% & <=66.4%	>=66.45% & <81.0%
Bad	>37.9% & <=51.2%	>=51.22% & <66.4%
Medium bad	>25.0% & <=37.9%	>=37.94% & <51.2%
Medium	>14.7% & <=25.0%	>=25.00% & <37.9%
Medium good	>8.0% & <=14.7%	>=14.66% & <25.0%
Good	>3.3% & <=8.0%	>=8.01% & <14.7%
Very good	>1.0% & <=3.3%	>=3.35% & <8.0%
Very very good	>0.01% & <=1.0%	>=1.00% & <3.3%
Extremely good	>0% & <=0.01%	>=0% & <1.00%

Step 3: Each risk factor x_i was evaluated by three experts E_n, each expert used even linguistics terms or SVNS to define the value of risk factors as shown in Table 11.

Table 11 Risk factors evaluation

x_i	E_1	E_2	E_3	SVNS $A(x_i)$
x_1	(0.8,0.15,0.20)	(0.60,0.35,0.40)	Risky	$A(x_1) = \{ x_1, (0.8,0.15,0.20), (0.60,0.35,0.40), (0.70,0.25,0.30)\}$
x_2	(0.60,0.35,0.40)	(0.50,0.50,0.50)	(0.70,0.25,0.30)	$A(x_2) = \{ x_2, (0.60,0.35,0.40), (0.50,0.50,0.50), (0.70,0.25,0.30)\}$
x_3	(0.40,0.65,0.60)	(0.50,0.50,0.50)	Medium low risky	$A(x_3) = \{ x_3, (0.40,0.65,0.60), (0.50,0.50,0.50), (0.40,0.65,0.60)\}$
x_4	(0.30,0.75,0.70)	(0.20,0.85,0.80)	(0.50,0.50,0.50)	$A(x_4) = \{ x_4, (0.30,0.75,0.70), (0.20,0.85,0.80), (0.50,0.50,0.50)\}$
x_5	(0.30,0.75,0.70)	Medium low risky	(0.30,0.75,0.70)	$A(x_5) = \{ x_5, (0.30,0.75,0.70), (0.40,0.65,0.60), (0.30,0.75,0.70)\}$
x_6	(0.60,0.35,0.40)	(0.8,0.15,0.20)	(0.60,0.35,0.40)	$A(x_6) = \{ x_6, (0.60,0.35,0.40), (0.8,0.15,0.20), (0.60,0.35,0.40)\}$
x_7	(0.50,0.50,0.50)	(0.70,0.25,0.30)	(0.50,0.50,0.50)	$A(x_7) = \{x_7, (0.50,0.50,0.50), (0.70,0.25,0.30), (0.50,0.50,0.50)\}$
x_8	(0.40,0.65,0.60)	(0.60,0.35,0.40)	Medium low risky	$A(x_8) = \{ x_8, (0.40,0.65,0.60), (0.60,0.35,0.40), (0.40,0.65,0.60)\}$
x_9	(0.30,0.75,0.70)	Medium risky	(0.30,0.75,0.70)	$A(x_9) = \{ x_9, (0.30,0.75,0.70), (0.50,0.50,0.50), (0.30,0.75,0.70)\}$
x_{10}	(0.8,0.15,0.20)	Risky	(0.20,0.85,0.80)	$A(x_{10}) = \{ x_{10}, (0.8,0.15,0.20), (0.70,0.25,0.30), (0.20,0.85,0.80)\}$
x_{11}	(0.70,0.25,0.30)	(0.60,0.35,0.40)	(0.60,0.35,0.40)	$A(x_{11}) = \{ x_{11}, (0.70,0.25,0.30), (0.60,0.35,0.40), (0.60,0.35,0.40)\}$
x_{12}	(0.60,0.35,0.40)	Medium risky	(0.50,0.50,0.50)	$A(x_{12}) = \{ x_{12}, (0.60,0.35,0.40), (0.50,0.50,0.50), (0.50,0.50,0.50)\}$
x_{13}	(0.50,0.50,0.50)	(0.40,0.65,0.60)	(0.40,0.65,0.60)	$A(x_{13}) = \{ x_{13}, (0.50,0.50,0.50), (0.40,0.65,0.60), (0.40,0.65,0.60)\}$
x_{14}	(0.50,0.50,0.50)	(0.70,0.25,0.30)	(0.30,0.75,0.70)	$A(x_{14}) = \{ x_{14}, (0.50,0.50,0.50), (0.70,0.25,0.30), (0.30,0.75,0.70)\}$

x_i	E_1	E_2	E_3	SVNS $A(x_i)$
x_{15}	(0.40,0.65,0.60)	(0.60,0.35,0.40)	(0.50,0.50,0.50)	$A(x_{15}) = \{x_{15}, (0.40,0.65,0.60), (0.60,0.35,0.40), (0.50,0.50,0.50)\}$
x_{16}	(0.30,0.75,0.70)	Medium risky	(0.40,0.65,0.60)	$A(x_{16}) = \{x_{16}, (0.30,0.75,0.70), (0.50,0.50,0.50), (0.40,0.65,0.60)\}$
x_{17}	(0.20,0.85,0.80)	(0.40,0.65,0.60)	(0.30,0.75,0.70)	$A(x_{17}) = \{x_{17}, (0.20,0.85,0.80), (0.40,0.65,0.60), (0.30,0.75,0.70)\}$
x_{18}	(0.8,0.15,0.20)	(0.70,0.25,0.30)	(0.20,0.85,0.80)	$A(x_{18}) = \{x_{18}, (0.8,0.15,0.20), (0.70,0.25,0.30), (0.20,0.85,0.80)\}$
x_{19}	(0.70,0.25,0.30)	(0.60,0.35,0.40)	Medium risky	$A(x_{19}) = \{x_{19}, (0.70,0.25,0.30), (0.60,0.35,0.40), (0.50,0.50,0.50)\}$
x_{20}	(0.60,0.35,0.40)	Medium risky	(0.40,0.65,0.60)	$A(x_{20}) = \{x_{20}, (0.60,0.35,0.40), (0.50,0.50,0.50), (0.40,0.65,0.60)\}$
x_{21}	Medium risky	(0.40,0.65,0.60)	(0.30,0.75,0.70)	$A(x_{21}) = \{x_{21}, (0.50,0.50,0.50), (0.40,0.65,0.60), (0.30,0.75,0.70)\}$
x_{22}	(0.20,0.85,0.80)	(0.30,0.75,0.70)	(0.40,0.65,0.60)	$A(x_{22}) = \{x_{22}, (0.20,0.85,0.80), (0.30,0.75,0.70), (0.40,0.65,0.60)\}$
x_{23}	(0.10,0.90,0.90)	(0.20,0.85,0.80)	(0.30,0.75,0.70)	$A(x_{23}) = \{x_{23}, (0.10,0.90,0.90), (0.20,0.85,0.80), (0.30,0.75,0.70)\}$

Step 4: Using formula (22) to calculate the crisp value for SVNS $A(x_i)$, results shown in Table 12, then used both Table 7 and Table 8 above to compare calculated crisp value for each SVNS $A(x_i)$ with risk ranges to select the equivalent risk level, result shown in Table 13 below.

Table 12 Risk factors and its crisp value

SVNS $A(x_i)$	Crisp value
$A(x_1) = \{x_1, (0.8,0.15,0.20), (0.60,0.35,0.40), (0.70,0.25,0.30)\}$	71.56%
$A(x_2) = \{x_2, (0.60,0.35,0.40), (0.50,0.50,0.50), (0.70,0.25,0.30)\}$	61.05%
$A(x_3) = \{x_3, (0.40,0.65,0.60), (0.50,0.50,0.50), (0.40,0.65,0.60)\}$	42.19%
$A(x_4) = \{x_4, (0.30,0.75,0.70), (0.20,0.85,0.80), (0.50,0.50,0.50)\}$	32.20%
$A(x_5) = \{x_5, (0.30,0.75,0.70), (0.40,0.65,0.60), (0.30,0.75,0.70)\}$	31.63%
$A(x_6) = \{x_6, (0.60,0.35,0.40), (0.8,0.15,0.20), (0.60,0.35,0.40)\}$	68.23%
$A(x_7) = \{x_7, (0.50,0.50,0.50), (0.70,0.25,0.30), (0.50,0.50,0.50)\}$	57.19%
$A(x_8) = \{x_8, (0.40,0.65,0.60), (0.60,0.35,0.40), (0.40,0.65,0.60)\}$	46.06%
$A(x_9) = \{x_9, (0.30,0.75,0.70), (0.50,0.50,0.50), (0.30,0.75,0.70)\}$	35.53%
$A(x_{10}) = \{x_{10}, (0.8,0.15,0.20), (0.70,0.25,0.30), (0.20,0.85,0.80)\}$	57.13%
$A(x_{11}) = \{x_{11}, (0.70,0.25,0.30), (0.60,0.35,0.40), (0.60,0.35,0.40)\}$	64.92%
$A(x_{12}) = \{x_{12}, (0.60,0.35,0.40), (0.50,0.50,0.50), (0.50,0.50,0.50)\}$	53.86%
$A(x_{13}) = \{x_{13}, (0.50,0.50,0.50), (0.40,0.65,0.60), (0.40,0.65,0.60)\}$	42.19%
$A(x_{14}) = \{x_{14}, (0.50,0.50,0.50), (0.70,0.25,0.30), (0.30,0.75,0.70)\}$	49.95%
$A(x_{15}) = \{x_{15}, (0.40,0.65,0.60), (0.60,0.35,0.40), (0.50,0.50,0.50)\}$	49.96%
$A(x_{16}) = \{x_{16}, (0.30,0.75,0.70), (0.50,0.50,0.50), (0.40,0.65,0.60)\}$	38.86%
$A(x_{17}) = \{x_{17}, (0.20,0.85,0.80), (0.40,0.65,0.60), (0.30,0.75,0.70)\}$	28.29%
$A(x_{18}) = \{x_{18}, (0.8,0.15,0.20), (0.70,0.25,0.30), (0.20,0.85,0.80)\}$	57.13%
$A(x_{19}) = \{x_{19}, (0.70,0.25,0.30), (0.60,0.35,0.40), (0.50,0.50,0.50)\}$	61.05%
$A(x_{20}) = \{x_{20}, (0.60,0.35,0.40), (0.50,0.50,0.50), (0.40,0.65,0.60)\}$	49.96%
$A(x_{21}) = \{x_{21}, (0.50,0.50,0.50), (0.40,0.65,0.60), (0.30,0.75,0.70)\}$	38.86%
$A(x_{22}) = \{x_{22}, (0.20,0.85,0.80), (0.30,0.75,0.70), (0.40,0.65,0.60)\}$	28.29%
$A(x_{23}) = \{x_{23}, (0.10,0.90,0.90), (0.20,0.85,0.80), (0.30,0.75,0.70)\}$	18.86%

Table 13 Risk factors and its equivalent risk level

$SVNS\ \boldsymbol{A(x_i)}$	Crisp value	Strict risk level	Lenient risk level
$A(x_1)$	71.56%	Very very bad	Very bad
$A(x_2)$	61.05%	Very bad	Bad
$A(x_3)$	42.19%	Bad	Medium bad
$A(x_4)$	32.20%	Medium bad	Medium
$A(x_5)$	31.63%	Medium bad	Medium
$A(x_6)$	68.23%	Very very bad	Very bad
$A(x_7)$	57.19%	Very bad	Bad
$A(x_8)$	46.06%	Bad	Medium bad
$A(x_9)$	35.53%	Medium bad	Medium
$A(x_{10})$	57.13%	Very bad	Bad
$A(x_{11})$	64.92%	Very bad	Bad
$A(x_{12})$	53.86%	Very bad	Bad
$A(x_{13})$	42.19%	Bad	Medium bad
$A(x_{14})$	49.95%	Bad	Medium bad
$A(x_{15})$	49.96%	Bad	Medium bad
$A(x_{16})$	38.86%	Bad	Medium bad
$A(x_{17})$	28.29%	Medium bad	Medium
$A(x_{18})$	57.13%	Very bad	Bad
$A(x_{19})$	61.05%	Very bad	Bad
$A(x_{20})$	49.96%	Bad	Medium bad
$A(x_{21})$	38.86%	Bad	Medium bad
$A(x_{22})$	28.29%	Medium bad	Medium
$A(x_{23})$	18.86%	Medium	Medium good

Step 5: After calculating the crisp values for each risk factor, risk assessment expert shall take the appropriate decisions.

5. Conclusion and future works:

In this research paper a neutrosophic 3D visualization for both SNN and SVNS was presented, in addition, some existing distance and similarity measure are validated and shortcoming are exposed, new crisp value functions "De-neutrosophication" for converting both Simplified Neutrosophic Number SNN, and Single-Valued Neutrosophic set SVNS to them equivalent crisp values using similarity measure based on Euclidean distance are proposed to overcome the exposed shortcoming, also a new Neutrosophic Scaling System algorithm is proposed, Finally, the proposed Neutrosophic Scaling System is applied to risk assessment case study.

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Implicative falling neutrosophic ideals of BCK-algebras

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Abstract: The notions of an implicative $(\in, \in)$-neutrosophic ideal and an implicative falling neutrosophic ideal are introduced, and several properties are investigated. Characterizations of an implicative $(\in, \in)$-neutrosophic ideal are considered, and relations between an implicative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal are discussed. Conditions for an $(\in, \in)$-neutrosophic ideal to be an implicative $(\in, \in)$-neutrosophic ideal are provided, and relations between an implicative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and an implicative falling neutrosophic ideal are studied. Conditions for a falling neutrosophic ideal to be implicative are provided. Relations between implicative falling neutrosophic ideal, commutative falling neutrosophic ideal and positive implicative falling neutrosophic ideal are discussed.

Keywords: neutrosophic random set; neutrosophic falling shadow; (positive implicative) $(\in, \in)$-neutrosophic ideal; (positive implicative) falling neutrosophic ideal; (commutative) $(\in, \in)$-neutrosophic ideal; (commutative) falling neutrosophic ideal; (implicative) $(\in, \in)$-neutrosophic ideal; (implicative) falling neutrosophic ideal.

1 Introduction

The fuzzy set was introduced by L.A. Zadeh in 1965, where each element had a degree of membership. As a generalization of fuzzy set, the intuitionistic fuzzy set on a universe X was introduced by K. Atanassov in 1983, where besides the degree of membership $\mu_A(x) \in [0, 1]$ of each element $x \in X$ to a set A there was considered a degree of non-membership $\nu_A(x) \in [0, 1]$, but such that $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. Neutrosophic set (NS) developed by Smarandache [19, 20, 21] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site http://fs.gallup.unm.edu/neutrosophy.htm. Jun et al. studied neutrosophic subalgebras/ideals in BCK/BCI-algebras based on neutrosophic points (see [1], [3], [7] [16] and [18]). It is a reasonable and convenient approach for the theoretical development and the practical applications of neutrosophic sets and neutrosophic logics. Jun et al. [10] introduced the notion of neutrosophic random set and neutrosophic falling shadow. Using these notions, they introduced the concept of falling neutrosophic subalgebra and falling neutrosophic ideal in BCK/BCI-algebras, and investigated related properties. They discussed relations between falling neutrosophic subalgebra and falling neutrosophic ideal, and established a characterization of falling neutrosophic ideal (see [9], [11], and [13]).Jun et al. [12] introduced the concepts of a commutative $(\in, \in)$-neutrosophic ideal and a commutative falling neutrosophic ideal, and investigate several properties. Bordbar et al. [2] introduced the concepts of a positive implicative $(\in, \in)$-neutrosophic ideal and a positive implicative falling neutrosophic ideal, and investigate several properties.

In this paper, we introduce the concepts of an implicative $(\in, \in)$-neutrosophic ideal and an implicative falling neutrosophic ideal, and investigate several properties. We obtain characterizations of an implicative $(\in, \in)$-neutrosophic ideal, and discuss relations between an implicative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal. We provide conditions for an $(\in, \in)$-neutrosophic ideal to be an implicative $(\in, \in)$-neutrosophic ideal, and consider relations between an implicative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and an implicative falling neutrosophic ideal. We give conditions for a falling neutrosophic ideal to be implicative. We consider relations between implicative falling neutrosophic ideal, commutative falling neutrosophic ideal and positive implicative falling neutrosophic ideal.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [5] and [6]).

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$
$$(\forall x, y, z \in X)\,(x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is said to be *positive implicative* if the following assertion is valid.

$$(\forall x, y, z \in X)\,((x * z) * (y * z) = (x * y) * z)\,. \tag{2.5}$$

A BCK-algebra X is said to be *implicative* if the following assertion is valid.

$$(\forall x, y \in X)\,(x = x * (y * x))\,. \tag{2.6}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.7}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{2.8}$$

A subset I of a BCK-algebra X is called a *commutative ideal* (see [15]) of X if it satisfies (2.7) and

$$(\forall x, y \in X)(\forall z \in I)\,((x * y) * z \in I \;\Rightarrow\; x * (y * (y * x)) \in I)\,. \tag{2.9}$$

Observe that every commutative ideal is an ideal, but the converse is not true (see [15]).

A subset I of a BCK-algebra X is called a *positive implicative ideal* (see [15]) of X if it satisfies (2.7) and

$$(\forall x, y, z \in X)(((x * y) * z \in I,\, y * z \in I \;\Rightarrow\; x * z \in I)\,. \tag{2.10}$$

Observe that every positive implicative ideal is an ideal, but the converse is not true (see [15]).

A subset I of a BCK-algebra X is called an *implicative ideal* (see [15]) of X if it satisfies (2.7) and

$$(\forall x, y, z \in X)((x * (y * x)) * z \in I,\, z \in I \;\Rightarrow\; x \in I)\,. \tag{2.11}$$

Observe that every implicative ideal is an ideal, but the converse is not true (see [15]).

We refer the reader to the books [4, 15] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [20]) is a structure of the form:

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \rightarrow [0, 1]$ is a truth membership function, $A_I : X \rightarrow [0, 1]$ is an indeterminate membership function, and $A_F : X \rightarrow [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A_\sim = (A_T, A_I, A_F)$ for the neutrosophic set

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$\begin{aligned} T_\in(A_\sim; \alpha) &:= \{x \in X \mid A_T(x) \geq \alpha\}, \\ I_\in(A_\sim; \beta) &:= \{x \in X \mid A_I(x) \geq \beta\}, \\ F_\in(A_\sim; \gamma) &:= \{x \in X \mid A_F(x) \leq \gamma\}. \end{aligned}$$

We say $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are *neutrosophic* $\in$*-subsets*.

A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$*-neutrosophic subalgebra* of X (see [7]) if the following assertions are valid.

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_\in(A_\sim; \alpha_x),\ y \in T_\in(A_\sim; \alpha_y) \ \Rightarrow\ x * y \in T_\in(A_\sim; \alpha_x \wedge \alpha_y), \\ x \in I_\in(A_\sim; \beta_x),\ y \in I_\in(A_\sim; \beta_y) \ \Rightarrow\ x * y \in I_\in(A_\sim; \beta_x \wedge \beta_y), \\ x \in F_\in(A_\sim; \gamma_x),\ y \in F_\in(A_\sim; \gamma_y) \ \Rightarrow\ x * y \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array} \right) \tag{2.12}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$*-neutrosophic ideal* of X (see [18]) if the following assertions are valid.

$$(\forall x \in X) \left(\begin{array}{l} x \in T_\in(A_\sim; \alpha_x) \ \Rightarrow\ 0 \in T_\in(A_\sim; \alpha_x) \\ x \in I_\in(A_\sim; \beta_x) \ \Rightarrow\ 0 \in I_\in(A_\sim; \beta_x) \\ x \in F_\in(A_\sim; \gamma_x) \ \Rightarrow\ 0 \in F_\in(A_\sim; \gamma_x) \end{array} \right) \tag{2.13}$$

and

$$(\forall x, y \in X) \left(\begin{array}{l} x * y \in T_\in(A_\sim; \alpha_x),\ y \in T_\in(A_\sim; \alpha_y) \ \Rightarrow\ x \in T_\in(A_\sim; \alpha_x \wedge \alpha_y) \\ x * y \in I_\in(A_\sim; \beta_x),\ y \in I_\in(A_\sim; \beta_y) \ \Rightarrow\ x \in I_\in(A_\sim; \beta_x \wedge \beta_y) \\ x * y \in F_\in(A_\sim; \gamma_x),\ y \in F_\in(A_\sim; \gamma_y) \ \Rightarrow\ x \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array} \right) \tag{2.14}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

In what follows, let X and $\mathcal{P}(X)$ denote a BCK/BCI-algebra and the power set of X, respectively, unless otherwise specified.

For each $x \in X$ and $D \in \mathcal{P}(X)$, let

$$\bar{x} := \{C \in \mathcal{P}(X) \mid x \in C\}, \tag{2.15}$$

and

$$\bar{D} := \{\bar{x} \mid x \in D\}. \tag{2.16}$$

An ordered pair $(\mathcal{P}(X), \mathcal{B})$ is said to be a *hyper-measurable structure* on X if $\mathcal{B}$ is a σ-field in $\mathcal{P}(X)$ and $\bar{X} \subseteq \mathcal{B}$.

Given a probability space $(\Omega, \mathcal{A}, P)$ and a hyper-measurable structure $(\mathcal{P}(X), \mathcal{B})$ on X, a *neutrosophic* *random set* on X (see [10]) is defined to be a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which ξ_T, ξ_I and ξ_F are mappings from Ω to $\mathcal{P}(X)$ which are $\mathcal{A}$-$\mathcal{B}$ measurables, that is,

$$(\forall C \in \mathcal{B}) \begin{pmatrix} \xi_T^{-1}(C) = \{\omega_T \in \Omega \mid \xi_T(\omega_T) \in C\} \in \mathcal{A} \\ \xi_I^{-1}(C) = \{\omega_I \in \Omega \mid \xi_I(\omega_I) \in C\} \in \mathcal{A} \\ \xi_F^{-1}(C) = \{\omega_F \in \Omega \mid \xi_F(\omega_F) \in C\} \in \mathcal{A} \end{pmatrix}. \tag{2.17}$$

Given a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, consider functions:

$$\begin{aligned} &\tilde{H}_T : X \to [0,1],\ x_T \mapsto P(\omega_T \mid x_T \in \xi_T(\omega_T)), \\ &\tilde{H}_I : X \to [0,1],\ x_I \mapsto P(\omega_I \mid x_I \in \xi_I(\omega_I)), \\ &\tilde{H}_F : X \to [0,1],\ x_F \mapsto 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)). \end{aligned}$$

Then $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a neutrosophic set on X, and we call it a *neutrosophic falling shadow* (see [10]) of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$, and $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *neutrosophic cloud* (see [10]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

For example, consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic set in X. Then a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\begin{aligned} &\xi_T : [0,1] \to \mathcal{P}(X), \alpha \mapsto T_\in(\tilde{H}; \alpha), \\ &\xi_I : [0,1] \to \mathcal{P}(X), \beta \mapsto I_\in(\tilde{H}; \beta), \\ &\xi_F : [0,1] \to \mathcal{P}(X), \gamma \mapsto F_\in(\tilde{H}; \gamma) \end{aligned}$$

is a neutrosophic random set and $\xi := (\xi_T, \xi_I, \xi_F)$ is a neutrosophic cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$. We will call $\xi := (\xi_T, \xi_I, \xi_F)$ defined above as the *neutrosophic cut-cloud* (see [10]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are subalgebras (resp., ideals) of X for all $\omega_T, \omega_I, \omega_F \in \Omega$, then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *falling neutrosophic subalgebra* (resp., *falling neutrosophic ideal*) of X (see [10]).

3 Implicative $(\in, \in)$-neutrosophic ideals

Definition 3.1. A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK-algebra X is called an *implicative* $(\in, \in)$*-neutrosophic ideal* of X if it satisfies the condition (2.13) and

$$\begin{aligned} &(x*(y*x))*z \in T_\in(A_\sim;\alpha_x),\ z \in T_\in(A_\sim;\alpha_y) \ \Rightarrow\ x \in T_\in(A_\sim;\alpha_x \wedge \alpha_y) \\ &(x*(y*x))*z \in I_\in(A_\sim;\beta_x),\ z \in I_\in(A_\sim;\beta_y) \ \Rightarrow\ x \in I_\in(A_\sim;\beta_x \wedge \beta_y) \\ &(x*(y*x))*z \in F_\in(A_\sim;\gamma_x),\ z \in F_\in(A_\sim;\gamma_y) \ \Rightarrow\ x \in F_\in(A_\sim;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

Example 3.2. Consider a set $X = \{0,1,2,3,4\}$ with the binary operation $*$ which is given in Table 1. Then

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

$(X; *, 0)$ is a BCK-algebra (see [15]). Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 2.

Table 2: Tabular representation of $A_\sim = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.6	0.1
1	0.7	0.6	0.1
2	0.7	0.6	0.1
3	0.5	0.2	0.6
4	0.3	0.4	0.9

Routine calculations show that $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X.

Theorem 3.3. *Every implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X is an $(\in, \in)$-neutrosophic ideal of X.*

Proof. It is clear by substituting x for y in (3.1) and using (2.1). □

Corollary 3.4. *Every implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X is an $(\in, \in)$-neutrosophic subalgebra of X.*

The converse of Theorem 3.3 is not true as seen in the following example.

Example 3.5. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 3. Then $(X; *, 0)$ is a BCK-algebra (see [15]). Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 4. It is routine to verify that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X, but it is not an implicative $(\in, \in)$-neutrosophic ideal of X since

$$(1 * (3 * 1)) * 2 = 0 \in T_\in(A_\sim; 0.6) \text{ and } 2 \in T_\in(A_\sim; 0.65)$$

but $1 \notin T_\in(A_\sim; 0.6 \wedge 0.65) = T_\in(A_\sim; 0.6)$, and/or

$$(1 * (3 * 1)) * 2 = 0 \in F_\in(A_\sim; 0.35) \text{ and } 2 \in F_\in(A_\sim; 0.45),$$

but $1 \notin F_\in(A_\sim; 0.45) = F_\in(A_\sim; 0.35 \vee 0.45)$.

Table 3: Cayley table for the binary operation “$*$”

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Table 4: Tabular representation of $A_\sim = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.8	0.3
1	0.5	0.6	0.5
2	0.7	0.4	0.4
3	0.5	0.2	0.9
4	0.5	0.2	0.9

Theorem 3.6. *For a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK-algebra X, the following are equivalent.*

(1) *The non-empty $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are implicative ideals of X for all $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$.*

(2) *$A_\sim = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X)\left(\ A_T(0) \geq A_T(x),\ A_I(0) \geq A_I(x),\ A_F(0) \leq A_F(x)\ \right) \tag{3.2}$$

and

$$(\forall x, y, z \in X)\left(\begin{array}{l} A_T(x) \geq A_T((x*(y*x))*z) \wedge A_T(z) \\ A_I(x) \geq A_I((x*(y*x))*z) \wedge A_I(z) \\ A_F(x) \leq A_F((x*(y*x))*z) \vee A_F(z) \end{array}\right) \tag{3.3}$$

Proof. Assume that the non-empty $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are implicative ideals of X for all $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$. If $A_T(0) < A_T(a)$ for some $a \in X$, then $a \in T_\in(A_\sim; A_T(a))$ and $0 \notin T_\in(A_\sim; A_T(a))$. This is a contradiction, and so $A_T(0) \geq A_T(x)$ for all $x \in X$. Similarly, $A_I(0) \geq A_I(x)$ for all $x \in X$. Suppose that $A_F(0) > A_F(a)$ for some $a \in X$. Then $a \in F_\in(A_\sim; A_F(a))$ and $0 \notin F_\in(A_\sim; A_F(a))$. This is a contradiction, and thus $A_F(0) \leq A_F(x)$ for all $x \in X$. Therefore (3.2) is valid. Assume that there exist $a, b, c \in X$ such that

$$A_T(a) < A_T((a*(b*a))*c) \wedge A_T(c).$$

Taking $\alpha := A_T((a*(b*a))*c) \wedge A_T(c)$ implies that $(a*(b*a))*c \in T_\in(A_\sim;\alpha)$ and $c \in T_\in(A_\sim;\alpha)$ but

$a \notin T_{\in}(A_{\sim};\alpha)$, which is a contradiction. Hence

$$A_T(x) \geq A_T((x*(y*x))*z) \wedge A_T(z)$$

for all $x, y, z \in X$. By the similar way, we can verify that

$$A_I(x) \geq A_I((x*(y*x))*z) \wedge A_I(z)$$

for all $x, y, z \in X$. Now suppose there are $x, y, z \in X$ such that

$$A_F(x) > A_F((x*(y*x))*z) \vee A_F(z) := \gamma.$$

Then $(x*(y*x))*z \in F_{\in}(A_{\sim};\gamma)$ and $z \in F_{\in}(A_{\sim};\gamma)$ but $x \notin F_{\in}(A_{\sim};\gamma)$, a contradiction. Thus

$$A_F(x) \leq A_F((x*(y*x))*z) \vee A_F(z)$$

for all $x, y, z \in X$.

Conversely, let $A_{\sim} = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying two conditions (3.2) and (3.3). Assume that $T_{\in}(A_{\sim};\alpha)$, $I_{\in}(A_{\sim};\beta)$ and $F_{\in}(A_{\sim};\gamma)$ are nonempty for $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Let $x \in T_{\in}(A_{\sim};\alpha)$, $a \in I_{\in}(A_{\sim};\beta)$ and $u \in F_{\in}(A_{\sim};\gamma)$ for $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Then $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(a) \geq \beta$, and $A_F(0) \leq A_F(u) \leq \gamma$ by (3.2). It follows that $0 \in T_{\in}(A_{\sim};\alpha)$, $0 \in I_{\in}(A_{\sim};\beta)$ and $0 \in F_{\in}(A_{\sim};\gamma)$. Let $a, b, c \in X$ be such that $(a*(b*a))*c \in T_{\in}(A_{\sim};\alpha)$ and $c \in T_{\in}(A_{\sim};\alpha)$ for $\alpha \in (0,1]$. Then

$$A_T(a) \geq A_T((a*(b*a))*c) \wedge A_T(c) \geq \alpha$$

by (3.3), and so $a \in T_{\in}(A_{\sim};\alpha)$. If $(x*(y*x))*z \in I_{\in}(A_{\sim};\beta)$ and $z \in I_{\in}(A_{\sim};\beta)$ for all $x, y, z \in X$ and $\beta \in (0,1]$, then $A_I((x*(y*x))*z) \geq \beta$ and $A_I(z) \geq \beta$. Hence the condition (3.3) implies that

$$A_I(x) \geq A_I((x*(y*x))*z) \wedge A_I(z) \geq \beta,$$

that is, $x \in I_{\in}(A_{\sim};\beta)$. Finally, suppose that $(x*(y*x))*z \in F_{\in}(A_{\sim};\gamma)$ and $z \in F_{\in}(A_{\sim};\gamma)$ for all $x, y, z \in X$ and $\gamma \in (0,1]$. Then $A_F((x*(y*x))*z) \leq \gamma$ and $A_F(z) \leq \gamma$, which imply from the condition (3.3) that

$$A_F(x) \leq A_F((x*(y*x))*z) \vee A_F(z) \leq \gamma.$$

Hence $x \in F_{\in}(A_{\sim};\gamma)$. Therefore the non-empty $\in$-subsets $T_{\in}(A_{\sim};\alpha)$, $I_{\in}(A_{\sim};\beta)$ and $F_{\in}(A_{\sim};\gamma)$ are implicative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. □

Theorem 3.7. *Let $A_{\sim} = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A_{\sim} = (A_T, A_I, A_F)$ is a implicative $(\in, \in)$-neutrosophic ideal of X if and only if the non-empty neutrosophic $\in$-subsets $T_{\in}(A_{\sim};\alpha)$, $I_{\in}(A_{\sim};\beta)$ and $F_{\in}(A_{\sim};\gamma)$ are implicative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$.*

Proof. Let $A_{\sim} = (A_T, A_I, A_F)$ be an implicative $(\in, \in)$-neutrosophic ideal of X and assume that $T_{\in}(A_{\sim};\alpha)$, $I_{\in}(A_{\sim};\beta)$ and $F_{\in}(A_{\sim};\gamma)$ are nonempty for $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Then there exist $x, y, z \in X$ such that $x \in T_{\in}(A_{\sim};\alpha)$, $y \in I_{\in}(A_{\sim};\beta)$ and $z \in F_{\in}(A_{\sim};\gamma)$. It follows from (2.13) that $0 \in T_{\in}(A_{\sim};\alpha)$, $0 \in I_{\in}(A_{\sim};\beta)$ and $0 \in F_{\in}(A_{\sim};\gamma)$. Let $x, y, z, a, b, c, u, v, w \in X$ be such that $(x*(y*x))*z \in T_{\in}(A_{\sim};\alpha)$, $z \in T_{\in}(A_{\sim};\alpha)$, $(a*(b*a))*c \in I_{\in}(A_{\sim};\beta)$, $c \in I_{\in}(A_{\sim};\beta)$, $(u*(v*u))*w \in F_{\in}(A_{\sim};\gamma)$ and $w \in F_{\in}(A_{\sim};\gamma)$. Then

$x \in T_\in(A_\sim;\alpha \wedge \alpha) = T_\in(A_\sim;\alpha)$, $a \in I_\in(A_\sim;\beta \wedge \beta) = I_\in(A_\sim;\beta)$, and $u \in F_\in(A_\sim;\gamma \vee \gamma) = F_\in(A_\sim;\gamma)$ by (3.1). Hence the non-empty neutrosophic $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are implicative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$.

Conversely, let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X for which $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are nonempty and are implicative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Obviously, (2.13) is valid. Let $x, y, z \in X$ and $\alpha_x, \alpha_y \in (0,1]$ be such that $(x*(y*x))*z \in T_\in(A_\sim;\alpha_x)$ and $z \in T_\in(A_\sim;\alpha_y)$. Then $(x*(y*x))*z \in T_\in(A_\sim;\alpha)$ and $z \in T_\in(A_\sim;\alpha)$ where $\alpha = \alpha_x \wedge \alpha_y$. Since $T_\in(A_\sim;\alpha)$ is an implicative ideal of X, it follows that $x \in T_\in(A_\sim;\alpha) = T_\in(A_\sim;\alpha_x \wedge \alpha_y)$. Similarly, if $(x*(y*x))*z \in I_\in(A_\sim;\beta_x)$ and $z \in I_\in(A_\sim;\beta_y)$ for all $x, y, z \in X$ and $\beta_x, \beta_y \in (0,1]$, then $x \in I_\in(A_\sim;\beta_x \wedge \beta_y)$. Now, suppose that $(x*(y*x))*z \in F_\in(A_\sim;\gamma_x)$ and $z \in F_\in(A_\sim;\gamma_y)$ for all $x, y, z \in X$ and $\gamma_x, \gamma_y \in [0,1)$. Then $(x*(y*x))*z \in F_\in(A_\sim;\gamma)$ and $z \in F_\in(A_\sim;\gamma)$ where $\gamma = \gamma_x \vee \gamma_y$. Hence $x \in F_\in(A_\sim;\gamma) = F_\in(A_\sim;\gamma_x \vee \gamma_y)$ since $F_\in(A_\sim;\gamma)$ is an implicative ideal of X. Therefore $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X. □

Corollary 3.8. *Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X if and only if it satisfies two conditions (3.2) and (3.3).*

We provide conditions for an $(\in, \in)$-neutrosophic ideal to be an implicative $(\in, \in)$-neutrosophic ideal.

Theorem 3.9. *If X is an implicative BCK-algebra, then every $(\in, \in)$-neutrosophic ideal is an implicative $(\in, \in)$-neutrosophic ideal.*

Proof. If X is an implicative BCK-algebra, then $x = x*(y*x)$ for all $x, y \in X$. Let $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X. Then

$$A_T(x) \geq A_T(x*z) \wedge A_T(z) \geq A_T((x*(y*x))*z) \wedge A_T(z),$$

$$A_I(x) \geq A_I(x*z) \wedge A_I(z) \geq A_I((x*(y*x))*z) \wedge A_I(z),$$

and

$$A_F(x) \leq A_F(x*z) \vee A_F(z) \leq A_F((x*(y*x))*z) \vee A_F(z)$$

for all $x, y, z \in X$. Therefore $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X by Corollary 3.8. □

Lemma 3.10 ([17]). *Every $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies the following assertion.*

$$(\forall x, y \in X)\left(x \leq y \;\Rightarrow \left\{\begin{array}{l} A_T(x) \geq A_T(y) \\ A_I(x) \geq A_I(y) \\ A_F(x) \leq A_F(y) \end{array}\right.\right). \tag{3.4}$$

Lemma 3.11 ([17]). *Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, the following assertions are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.*

(2) $A_\sim = (A_T, A_I, A_F)$ *satisfies the following assertions.*

$$(\forall x \in X)\left(\ A_T(0) \geq A_T(x),\, A_I(0) \geq A_I(x),\, A_F(0) \leq A_F(x)\ \right) \tag{3.5}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \geq A_T(x * y) \wedge A_T(y) \\ A_I(x) \geq A_I(x * y) \wedge A_I(y) \\ A_F(x) \leq A_F(x * y) \vee A_F(y) \end{array}\right) \tag{3.6}$$

Theorem 3.12. *Suppose that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. Then the following assertions are equivalent. Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK-algebra X, the following assertions are equivalent.*

(1) $A_\sim = (A_T, A_I, A_F)$ *is an implicative $(\in, \in)$-neutrosophic ideal of X.*

(2) $A_\sim = (A_T, A_I, A_F)$ *is an $(\in, \in)$-neutrosophic ideal of X satisfying the condition:*

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \geq A_T(x * (y * x)) \\ A_I(x) \geq A_I(x * (y * x)) \\ A_F(x) \leq A_F(x * (y * x)). \end{array}\right) \tag{3.7}$$

(3) $A_\sim = (A_T, A_I, A_F)$ *is an $(\in, \in)$-neutrosophic ideal of X satisfying the condition:*

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) = A_T(x * (y * x)) \\ A_I(x) = A_I(x * (y * x)) \\ A_F(x) = eA_F(x * (y * x)). \end{array}\right) \tag{3.8}$$

Proof. (1) ⇒ (2). Let $A_\sim = (A_T, A_I, A_F)$ be an implicative $(\in, \in)$-neutrosophic ideal of X. Then $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.3. Using (3.2) and (3.3) implies that

$$A_T(x) \geq A_T((x * (y * x)) * 0) \wedge A_T(0) = A_T(x * (y * x)) \wedge A_T(0) = A_T(x * (y * x)),$$

$$A_I(x) \geq A_I((x * (y * x)) * 0) \wedge A_I(0) = A_I(x * (y * x)) \wedge A_I(0) = A_I(x * (y * x))$$

and

$$A_F(x) \leq A_F((x * (y * x)) * 0) \vee A_F(0) = A_F(x * (y * x)) \vee A_F(0) = A_F(x * (y * x))$$

for all $x, y \in X$.

(2) ⇒ (3). Observe that $x * (y * x) \leq x$ for all $x, y \in X$. Using Lemma 3.10, we have $A_T(x) \leq A_T(x * (y * x))$, $A_I(x) \leq A_I(x * (y * x))$ and $A_F(x) \geq A_F(x * (y * x))$. It follows from (3.7) that $A_T(x) = A_T(x * (y * x))$, $A_I(x) = A_I(x * (y * x))$ and $A_F(x) = A_F(x * (y * x))$ for all $x, y \in X$.

(3) ⇒ (1). Let $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X satisfying the condition (3.8).

Then

$$\begin{aligned}A_T(x) &= A_T(x*(y*x)) \geq A_T((x*(y*x))*z) \wedge A_T(z),\\ A_I(x) &= A_I(x*(y*x)) \geq A_I((x*(y*x))*z) \wedge A_I(z),\\ A_F(x) &= A_F(x*(y*x)) \leq A_F((x*(y*x))*z) \vee A_F(z)\end{aligned}$$

for all $x, y, z \in X$ by (3.8) and (3.6). Therefore $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X. □

Lemma 3.13 ([14]). *Let I and A be ideals of a BCK-algebra X such that $I \subseteq A$. If I is an implicative ideal of X, then so is A.*

Theorem 3.14. *Let $A_\sim = (A_T, A_I, A_F)$ and $B_\sim = (B_T, B_I, B_F)$ be $(\in, \in)$-neutrosophic ideals of X such that $A_\sim \sqsubseteq B_\sim$, that is, $A_T(x) \leq B_T(x)$, $A_I(x) \leq B_I(x)$ and $A_F(x) \geq B_F(x)$ for all $x \in X$. If $A_\sim = (A_T, A_I, A_F)$ is implicative, then so is $B_\sim = (B_T, B_I, B_F)$.*

Proof. It is sufficient to show that the non-empty neutrosophic $\in$-subsets $T_\in(B_\sim; \alpha)$, $I_\in(B_\sim; \beta)$ and $F_\in(B_\sim; \gamma)$ are implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. If $x \in T_\in(A_\sim; \alpha)$, then $B_T(x) \geq A_T(x) \geq \alpha$ and so $T_\in(A_\sim; \alpha) \subseteq T_\in(B_\sim; \alpha)$. Similarly, $I_\in(A_\sim; \beta) \subseteq I_\in(B_\sim; \beta)$. If $x \in F_\in(A_\sim; \gamma)$, then $B_F(x) \leq A_F(x) \leq \gamma$ and thus $F_\in(A_\sim; \gamma) \subseteq F_\in(B_\sim; \gamma)$. Since $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X, it follows from Theorem 3.7 that $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are implicative ideals of X. Therefore $T_\in(B_\sim; \alpha)$, $I_\in(B_\sim; \beta)$ and $F_\in(B_\sim; \gamma)$ are implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, and hence $B_\sim = (B_T, B_I, B_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X. □

4 Implicative falling neutrosophic ideals

Definition 4.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on a BCK-algebra X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are implicative ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$, then the neutrosophic shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, that is,

$$\begin{aligned}\tilde{H}_T(x_T) &= P(\omega_T \mid x_T \in \xi_T(\omega_T)),\\ \tilde{H}_I(x_I) &= P(\omega_I \mid x_I \in \xi_I(\omega_I)),\\ \tilde{H}_F(x_F) &= 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F))\end{aligned} \tag{4.1}$$

is called an *implicative falling neutrosophic ideal* of X.

Example 4.2. Consider a set $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 5. Then $(X; *, 0)$ is a BCK-algebra (see [15]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in [0, 0.25),\\ \{0,1\} & \text{if } t \in [0.25, 0.55),\\ \{0,1,3\} & \text{if } t \in [0.55, 0.95),\\ X & \text{if } t \in [0.95, 1],\end{cases}$$

Table 5: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in [0, 0.45), \\ \{0,2\} & \text{if } t \in [0.45, 0.65), \\ \{0,2,3\} & \text{if } t \in [0.65, 0.95), \\ X & \text{if } t \in [0.95, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.9, 1], \\ \{0,3\} & \text{if } t \in (0.7, 0.9], \\ \{0,1,2\} & \text{if } t \in (0.5, 0.7], \\ \{0,1,3\} & \text{if } t \in (0.3, 0.5], \\ X & \text{if } t \in [0, 0.3]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are implicative ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is an implicative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.75 & \text{if } x = 1, \\ 0.05 & \text{if } x = 2, \\ 0.35 & \text{if } x = 3, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.05 & \text{if } x = 1, \\ 0.55 & \text{if } x = 2, \\ 0.35 & \text{if } x = 3, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.3 & \text{if } x = 1, \\ 0.5 & \text{if } x = 2, \\ 0.3 & \text{if } x = 3. \end{cases}$$

Given a probability space $(\Omega, \mathcal{A}, P)$, let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$. For $x \in X$, let

$$\Omega(x;\xi_T) := \{\omega_T \in \Omega \mid x \in \xi_T(\omega_T)\},$$
$$\Omega(x;\xi_I) := \{\omega_I \in \Omega \mid x \in \xi_I(\omega_I)\},$$
$$\Omega(x;\xi_F) := \{\omega_F \in \Omega \mid x \in \xi_F(\omega_F)\}.$$

Then $\Omega(x;\xi_T), \Omega(x;\xi_I), \Omega(x;\xi_F) \in \mathcal{A}$ (see [10]).

Proposition 4.3. *Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is an implicative falling neutrosophic ideal of X, then*

$$(\forall x, y, z \in X)\left(\begin{array}{l} \Omega((x*(y*x))*z;\xi_T) \cap \Omega(z;\xi_T) \subseteq \Omega(x;\xi_T) \\ \Omega((x*(y*x))*z;\xi_I) \cap \Omega(z;\xi_I) \subseteq \Omega(x;\xi_I) \\ \Omega((x*(y*x))*z;\xi_F) \cap \Omega(z;\xi_F) \subseteq \Omega(x;\xi_F) \end{array}\right), \tag{4.2}$$

$$(\forall x, y, z \in X)\left(\begin{array}{l} \Omega(x;\xi_T) \subseteq \Omega((x*(y*x))*z;\xi_T) \\ \Omega(x;\xi_I) \subseteq \Omega((x*(y*x))*z;\xi_I) \\ \Omega(x;\xi_F) \subseteq \Omega((x*(y*x))*z;\xi_F) \end{array}\right). \tag{4.3}$$

Proof. Let $\omega_T \in \Omega((x*(y*x))*z;\xi_T) \cap \Omega(z;\xi_T)$, $\omega_I \in \Omega((x*(y*x))*z;\xi_I) \cap \Omega(z;\xi_I)$ and $\omega_F \in \Omega((x*(y*x))*z;\xi_F) \cap \Omega(z;\xi_F)$ for all $x, y, z \in X$. Then
$(x*(y*x))*z \in \xi_T(\omega_T)$ and $z \in \xi_T(\omega_T)$,
$(x*(y*x))*z \in \xi_I(\omega_I)$ and $z \in \xi_I(\omega_I)$,
$(x*(y*x))*z \in \xi_F(\omega_F)$ and $z \in \xi_F(\omega_F)$.
Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are implicative ideals of X, it follows from (2.11) that $x \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$ and so that $\omega_T \in \Omega(x;\xi_T)$, $\omega_I \in \Omega(x;\xi_I)$ and $\omega_F \in \Omega(x;\xi_F)$. Hence (4.2) is valid. Now let $x, y, z \in X$ be such that $\omega_T \in \Omega(x;\xi_T)$, $\omega_I \in \Omega(x;\xi_I)$, and $\omega_F \in \Omega(x;\xi_F)$. Then $x \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$. Note that

$$\begin{aligned}((x*(y*x))*z)*x &= ((x*(y*x))*x)*z \\ &= ((x*x)*(y*x))*z = (0*(y*x))*z = 0*z = 0,\end{aligned}$$

and thus

$$((x*(y*x))*z)*x = 0 \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are implicative ideals and hence ideals of X, it follows that $(x*(y*x))*z \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$. Hence $\omega_T \in \Omega((x*(y*x))*z;\xi_T)$, $\omega_I \in \Omega((x*(y*x))*z;\xi_I)$, and $\omega_F \in \Omega((x*(y*x))*z;\xi_F)$. Therefore (4.3) is valid. □

Given a probability space $(\Omega, \mathcal{A}, P)$, let

$$\mathcal{F}(X) := \{f \mid f : \Omega \to X \text{ is a mapping}\}. \tag{4.4}$$

Define a binary operation $\circledast$ on $\mathcal{F}(X)$ as follows:

$$(\forall \omega \in \Omega)\,((f \circledast g)(\omega) = f(\omega) * g(\omega)) \tag{4.5}$$

for all $f, g \in \mathcal{F}(X)$. Then $(\mathcal{F}(X); \circledast, \theta)$ is a BCK/BCI-algebra (see [8]) where θ is given as follows:

$$\theta : \Omega \to X,\ \omega \mapsto 0.$$

For any subset A of X and $g_T, g_I, g_F \in \mathcal{F}(X)$, consider the followings:

$$\begin{aligned} A_T^g &:= \{\omega_T \in \Omega \mid g_T(\omega_T) \in A\}, \\ A_I^g &:= \{\omega_I \in \Omega \mid g_I(\omega_I) \in A\}, \\ A_F^g &:= \{\omega_F \in \Omega \mid g_F(\omega_F) \in A\} \end{aligned}$$

and

$$\begin{aligned} &\xi_T : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_T \mapsto \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in A\}, \\ &\xi_I : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_I \mapsto \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in A\}, \\ &\xi_F : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_F \mapsto \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in A\}. \end{aligned}$$

Then A_T^g, A_I^g, $A_F^g \in \mathcal{A}$ (see [10]).

Theorem 4.4. *If K is an implicative ideal of a BCK-algebra X, then*

$$\begin{aligned} \xi_T(\omega_T) &= \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in K\}, \\ \xi_I(\omega_I) &= \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in K\}, \\ \xi_F(\omega_F) &= \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in K\} \end{aligned}$$

are implicative ideals of $\mathcal{F}(X)$.

Proof. Assume that K is an implicative ideal of a BCK-algebra X. Since $\theta(\omega_T) = 0 \in K$, $\theta(\omega_I) = 0 \in K$ and $\theta(\omega_F) = 0 \in K$ for all $\omega_T, \omega_I, \omega_F \in \Omega$, we have

$$\theta \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Let $f_T, g_T, h_T \in \mathcal{F}(X)$ be such that $(f_T \circledast (g_T \circledast f_T)) \circledast h_T \in \xi_T(\omega_T)$ and $h_T \in \xi_T(\omega_T)$. Then

$$(f_T(\omega_T) * (g_T(\omega_T) * f_T(\omega_T))) * h_T(\omega_T) = ((f_T \circledast (g_T \circledast f_T)) \circledast h_T)(\omega_T) \in K$$

and $h_T(\omega_T) \in K$. Since K is an implicative ideal of X, it follows from (2.11) that $f_T(\omega_T) \in K$, that is, $f_T \in \xi_T(\omega_T)$. Hence $\xi_T(\omega_T)$ is an implicative ideal of $\mathcal{F}(X)$. Similarly, we can verify that $\xi_I(\omega_I)$ is an implicative ideal of $\mathcal{F}(X)$. Now, let $f_F, g_F, h_F \in \mathcal{F}(X)$ be such that $(f_F \circledast (g_F \circledast f_F)) \circledast h_F \in \xi_F(\omega_F)$ and $h_F \in \xi_F(\omega_F)$. Then

$$(f_F(\omega_F) * (g_F(\omega_F) * f_F(\omega_F))) * h_F(\omega_F) = ((f_F \circledast (g_F \circledast f_F)) \circledast h_F)(\omega_F) \in K$$

and $h_F(\omega_F) \in K$. Hence $f_F(\omega_F) \in K$, i.e., $f_F \in \xi_F(\omega_F)$. Therefore $\xi_F(\omega_F)$ is an implicative ideal of $\mathcal{F}(X)$. This completes the proof. □

Theorem 4.5. *If we consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, then every implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra is an implicative falling neutrosophic ideal.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be an implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X. Then $T_{\in}(\tilde{H}; \alpha)$, $I_{\in}(\tilde{H}; \beta)$ and $F_{\in}(\tilde{H}; \gamma)$ are implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ by Theorem 3.7. Hence a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\xi_T : [0,1] \to \mathcal{P}(X),\ \alpha \mapsto T_{\in}(\tilde{H}; \alpha),$$
$$\xi_I : [0,1] \to \mathcal{P}(X),\ \beta \mapsto I_{\in}(\tilde{H}; \beta),$$
$$\xi_F : [0,1] \to \mathcal{P}(X),\ \gamma \mapsto F_{\in}(\tilde{H}; \gamma)$$

is a neutrosophic cut-cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$, and so $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is an implicative falling neutrosophic ideal of X. □

The converse of Theorem 4.5 is not true as seen in the following example.

Example 4.6. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 6.

Table 6: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Then $(X; *, 0)$ is a BCK-algebra (see [15]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0,1\} & \text{if } t \in [0, 0.25), \\ \{0,2\} & \text{if } t \in [0.25, 0.55), \\ \{0,2,4\} & \text{if } t \in [0.55, 0.7), \\ \{0,1,2,3\} & \text{if } t \in [0.7, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0,2\} & \text{if } t \in [0, 0.28), \\ \{0,4\} & \text{if } t \in [0.28, 0.68), \\ \{0,1,2,3\} & \text{if } t \in [0.68, 1] \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.75, 1], \\ \{0,4\} & \text{if } t \in (0.63, 0.75], \\ \{0,2,4\} & \text{if } t \in (0.44, 0.63], \\ \{0,1,4\} & \text{if } t \in (0.23, 0.44], \\ \{0,1,2,3\} & \text{if } t \in [0, 0.23]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are implicative ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is an implicative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.55 & \text{if } x = 1, \\ 0.75 & \text{if } x = 2, \\ 0.3 & \text{if } x = 3, \\ 0.15 & \text{if } x = 4, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.32 & \text{if } x = 1, \\ 0.6 & \text{if } x = 2, \\ 0.32 & \text{if } x = 3, \\ 0.4 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.56 & \text{if } x = 1, \\ 0.58 & \text{if } x = 2, \\ 0.77 & \text{if } x = 3, \\ 0.48 & \text{if } x = 4. \end{cases}$$

If $\alpha \in [0, 0.55)$, then $T_\in(\tilde{H}_T; \alpha) = \{0, 1, 2\}$ is not an implicative ideal of X since

$$(3 * (2 * 3)) * 1 = (3 * 0) * 1 = 3 * 1 = 2 \in T_\in(\tilde{H}_T; \alpha)$$

and $1 \in T_\in(\tilde{H}_T; \alpha)$, but $3 \notin T_\in(\tilde{H}_T; \alpha)$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not an implicative $(\in, \in)$-neutrosophic ideal of X by Theorem 3.7.

We provide relations between a falling neutrosophic ideal and an implicative falling neutrosophic ideal .

Theorem 4.7. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is an implicative falling neutrosophic ideal of X, then it is a falling neutrosophic ideal of X.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be an implicative falling neutrosophic ideal of a BCK-algebra X. Then $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are implicative ideals of X, and so $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X. □

The following example shows that the converse of Theorem 4.7 is not true in general.

Example 4.8. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 7. Then $(X; *, 0)$ is a BCK-algebra (see [15]). Consider $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a

Table 7: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \ \ x \mapsto \begin{cases} \{0,3\} & \text{if } t \in [0, 0.37), \\ \{0,1,2,3\} & \text{if } t \in [0.37, 0.67), \\ \{0,1,2\} & \text{if } t \in [0.67, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \ \ x \mapsto \begin{cases} \{0,1,2\} & \text{if } t \in [0, 0.45), \\ \{0,1,2,4\} & \text{if } t \in [0.45, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \ \ x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.74, 1], \\ \{0,3\} & \text{if } t \in (0.66, 0.74], \\ \{0,1,2\} & \text{if } t \in (0.48, 0.66], \\ \{0,1,2,3\} & \text{if } t \in [0, 0.48]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a falling neutrosophic ideal of X. But it is not an implicative falling neutrosophic ideal of X because if $\alpha \in [0.67, 1]$, $\beta \in [0, 0.45)$ and $\gamma \in (0.66, 0.74]$, then $\xi_T(\alpha) = \{0,1,2\}$, $\xi_I(\beta) = \{0,1,2\}$ and $\xi_F(\gamma) = \{0,3\}$ are not implicative ideals of X respectively.

Since every ideal is implicative in an implicative BCK-algebra (see [15]), we have the following theorem.

Theorem 4.9. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on an implicative BCK-algebra. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is an implicative falling neutrosophic ideal of X.*

Corollary 4.10. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. For any BCK-algebra X which satisfies one of the following assertions*

$$(\forall x, y \in X)(y * (y * x) = (x * (x * y)) * (x * y)),$$
$$(\forall x, y \in X)((x * (x * y)) * (y * x) = y * (y * x)),$$
$$(\forall x, y \in X)((x * (x * y)) * (x * y) = (y * (y * x)) * (y * x)),$$

let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X.

If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is an implicative falling neutrosophic ideal of X.

Definition 4.11 ([12]). Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on a BCK-algebra X. Then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *commutative falling neutrosophic ideal* of X if $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$.

Definition 4.12 ([2]). Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on a BCK-algebra X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are positive implicative ideals of X for all ω_T, ω_I, $\omega_F \in \Omega$, then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, that is,

$$\begin{array}{l} \tilde{H}_T(x_T) = P(\omega_T \mid x_T \in \xi_T(\omega_T)), \\ \tilde{H}_I(x_I) = P(\omega_I \mid x_I \in \xi_I(\omega_I)), \\ \tilde{H}_F(x_F) = 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)) \end{array} \tag{4.6}$$

is called a *positive implicative falling neutrosophic ideal* of X.

Since every implicative ideal is both a commutative ideal and a positive implicative ideal in BCK-algebras (see [15]), the following theorem is straightforward.

Theorem 4.13. *Every implicative falling neutrosophic ideal is both a commutative falling neutrosophic ideal and a positive implicative falling neutrosophic ideal.*

The following example shows that there exist a commutative falling neutrosophic ideal and a positive implicative falling neutrosophic ideal which is not an implicative falling neutrosophic ideal.

Example 4.14. (1) Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ which is given in Example 3.2. Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,3\} & \text{if } t \in [0, 0.25), \\ \{0,4\} & \text{if } t \in [0.25, 0.55), \\ \{0,1,2\} & \text{if } t \in [0.55, 0.85), \\ \{0,3,4\} & \text{if } t \in [0.85, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,1,2\} & \text{if } t \in [0, 0.45), \\ \{0,1,2,3\} & \text{if } t \in [0.45, 0.75), \\ \{0,1,2,4\} & \text{if } t \in [0.75, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.9, 1], \\ \{0,3\} & \text{if } t \in (0.7, 0.9], \\ \{0,4\} & \text{if } t \in (0.5, 0.7], \\ \{0,1,2,3\} & \text{if } t \in (0.3, 0.5], \\ X & \text{if } t \in [0, 0.3]. \end{cases}$$

Then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a commutative falling neutrosophic ideal of X (see [12]). If $t \in [0.85, 1]$, then $\xi_T(t) = \{0, 3, 4\}$ is not an implicative ideal of X. Also, if $t \in (0.5, 0.7]$, then $\xi_F(t) = \{0, 4\}$ is not an implicative ideal of X. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not an implicative falling neutrosophic ideal of X.

(2) Let $X = \{0, 1, 2, 3\}$ be a set with the binary operation $*$ which is given in Table 8.

Table 8: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Then $(X; *, 0)$ is a BCK-algebra (see [15]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in [0, 0.35), \\ \{0, 2\} & \text{if } t \in [0.35, 0.55), \\ \{0, 1, 2\} & \text{if } t \in [0.55, 0.95), \\ X & \text{if } t \in [0.95, 1], \end{cases}$$

$$\xi_I : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0, 1\} & \text{if } t \in [0, 0.2), \\ \{0, 2\} & \text{if } t \in [0.2, 0.5), \\ \{0, 1, 2\} & \text{if } t \in [0.5, 0.9), \\ X & \text{if } t \in [0.9, 1], \end{cases}$$

and

$$\xi_F : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.95, 1], \\ \{0, 1\} & \text{if } t \in (0.6, 0.95], \\ \{0, 2\} & \text{if } t \in (0.4, 0.6], \\ \{0, 1, 2\} & \text{if } t \in (0.1, 0.4], \\ X & \text{if } t \in [0, 0.1]. \end{cases}$$

Then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a positive implicative falling neutrosophic ideal of X. If $t \in [0.35, 0.55)$, then $\xi_T(t) = \{0, 2\}$ is not an implicative ideal of X. If $t \in [0.2, 0.5)$, then $\xi_I(t) = \{0, 2\}$ is not an implicative ideal of X. Also, if $t \in (0.6, 0.95]$, then $\xi_F(t) = \{0, 1\}$ is not an implicative ideal of X. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not an implicative falling neutrosophic ideal of X.

The notions of a commutative falling neutrosophic ideal and a positive implicative falling neutrosophic ideal are independent, that is, a commutative falling neutrosophic ideal need not be a positive implicative falling neutrosophic ideal, and vice versa. In fact, the commutative falling neutrosophic ideal $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ in

Example 4.14(1) is not a positive implicative falling neutrosophic ideal. Also the positive implicative falling neutrosophic ideal $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ in Example 4.14(2) is not a commutative implicative falling neutrosophic ideal.

Theorem 4.15. *If the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is both a commutative implicative falling neutrosophic ideal and a positive implicative falling neutrosophic ideal, then it is an implicative falling neutrosophic ideal.*

Proof. It is straightforward because if any ideal is both commutative and position implicative, then it is implicative. □

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Neutrosophic $\mathcal{N}-$structures on strong Sheffer stroke non-associative MV-algebras

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Abstract. The aim of the study is to examine a neutrosophic $\mathcal{N}-$subalgebra, a neutrosophic $\mathcal{N}-$filter, level sets of these neutrosophic $\mathcal{N}-$structures and their properties on a strong Sheffer stroke non-associative MV-algebra. We show that the level set of neutrosophic $\mathcal{N}-$subalgebras on this algebra is its strong Sheffer stroke non-associative MV-subalgebra and vice versa. Then it is proved that the family of all neutrosophic $\mathcal{N}-$subalgebras of a strong Sheffer stroke non-associative MV-algebra forms a complete distributive lattice. By defining a neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke non-associative MV-algebra, it is presented that every neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke non-associative MV-algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse is generally not true, and some properties

Keywords: strong Sheffer stroke non-associative MV- algebra, filter, neutrosophic $\mathcal{N}-$subalgebra, neutrosophic $\mathcal{N}-$filter.

1. Introduction

The concept of fuzzy sets which has the truth (t) (membership) function was introduced by L. Zadeh [29]. Since a positive meaning of information is explained by means of fuzzy theory, researchers desire to deal with a negative meaning of information. Thus, Atanassov introduced intuitionistic fuzzy sets [2] which are fuzzy sets with the falsehood (f) (nonmembership) function. Then, Smarandache introduced neutrosophic sets which are intuitionistic fuzzy sets with the indeteminacy/neutrality (i) function [26,27]. Accordingly, neutrosophic sets are defined on three components: $(t, i, f) : (truth,\ indeteminacy,\ falsehood)$ [32]. Specially, many scientists applied neutrosophic sets to the algebraic structures such as BCK/BCI-algebras, BE-algebras and semigroups [3, 4, 11–16, 24, 28, 30, 31].

Sheffer stroke, which is also called the NAND operator in computer science, was firstly introduced by H. M. Sheffer [25]. Since any axioms and formulas in Boolean algebras can be written only by using this operation [17], Sheffer stroke can be applied to many logical algebras such as orthoimplication algebras [1], ortholattices [5], Hilbert algebras [18]- [19], BL-algebras [23], UP-algebras [20] and BG-algebras [21]. Therefore, it is easier to control a logical system consisting of Sheffer stroke itself. Moreover, C. C. Chang introduced MV-algebras which are algebraic counterparts of Lukasiewicz many-valued logic [9, 10]. Then Chajda et al. introduced and improved non-associative MV-algebras (briefly, NMV-algebras) which are generalizations of MV-algebras [7,8]. Also, non-associative MV-algebras with Sheffer stroke [6] and their filters [22] are presented.

Basic definitions and notions about strong Sheffer stroke non-associative MV-algebras, $\mathcal{N}-$functions and neutrosophic $\mathcal{N}-$structures defined by the $\mathcal{N}-$functions on a nonempty universe X are presented. Then the concepts of a neutrosophic $\mathcal{N}-$subalgebra and a $(a, b, c)-$level set defined by $\mathcal{N}-$functions are given on strong Sheffer stroke non-associative MV-algebras. It is shown that the $(a, b, c)-$level set of a neutrosophic $\mathcal{N}-$subalgebra defined by $\mathcal{N}-$functions on strong Sheffer stroke non-associative MV-algebras is its strong Sheffer stroke non-associative MV-subalgebra and the inverse is true. In fact, we state that the family of all neutrosophic $\mathcal{N}-$subalgebras of this algebraic structure forms a complete distributive lattice. Some properties of neutrosophic $\mathcal{N}-$subalgebras of strong Sheffer stroke non-associative MV-algebras are analyzed. Also, it is investigated the images of the sequence under $\mathcal{N}-$functions on a strong Sheffer stroke non-associative MV-algebra. Besides, we examine that the case which $\mathcal{N}-$functions defining a neutrosophic $\mathcal{N}-$subalgebra of a strong Sheffer stroke non-associative MV-algebra are constant. After defining a neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke non-associative MV-algebra by $\mathcal{N}-$functions, some features of $\mathcal{N}-$functions defining the neutrosophic $\mathcal{N}-$filter are studied. We propound that $(a, b, c)-$level set of a neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke non-associative MV-algebra is its filter and that the subsets defined by $\mathcal{N}-$functions on a strong Sheffer stroke non-associative MV-algebra must be its filters so that a neutrosophic $\mathcal{N}-$structure on this algebra is a neutrosophic $\mathcal{N}-$filter. It is stated that every neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke non-associative MV-algebra is its neutrosophic $\mathcal{N}-$subalgebra while the inverse is usually not valid. In addition, new subsets of a strong Sheffer stroke non-associative MV-algebra are described by the $\mathcal{N}-$functions and certain elements in the algebra. We show that these subsets are filters of a strong Sheffer stroke non-associative MV-algebra for its neutrosophic $\mathcal{N}-$filter but the inverse does not mostly hold.

2. Preliminaries

In this section, we give basic definitions and notions about strong Sheffer stroke non-associative MV-algebras (briefly, strong Sheffer stroke NMV-algebras) and neutrosophic $\mathcal{N}-$structures.

Definition 2.1. [5] Let $\mathcal{A} = \langle A, | \rangle$ be a groupoid. The operation $|$ is said to be a *Sheffer stroke operation* if it satisfies the following conditions:

$(S1)$ $x|y = y|x$,
$(S2)$ $(x|x)|(x|y) = x$,
$(S3)$ $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,
$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Definition 2.2. [6] A strong Sheffer stroke NMV-algebra is an algebra $(A, |, 1)$ of type $(2, 0)$ satisfying the identities for all $x, y, z \in A$:

$(n1)$ $x|y \approx y|x$,
$(n2)$ $x|0 \approx 1$,
$(n3)$ $(x|1)|1 \approx x$,
$(n4)$ $((x|1)|y)|y \approx ((y|1)|x)|x$,
$(n5)$ $(x|1)|((x|y)|1) \approx 1$,
$(n6)$ $x|(((((x|y)|y)|z)|z)|1) \approx 1$,

where 0 denotes the algebraic constant $1|1$.

Proposition 2.3. *[22] Let $(A, |, 1)$ be a strong Sheffer stroke NMV-algebra. Then the binary relation $\leq$ defined by*

$$x \leq y \text{ if and only if } x|(y|1) \approx 1$$

is a partial order on A. Hence, $(A, \leq)$ is a poset with the least element 0 and the greatest element 1.

Lemma 2.4. *[22] In a strong Sheffer stroke NMV-algebra $(A, |, 1)$, the following properties hold for all $x, y, z \in A$:*

(i) $x|(x|1) \approx 1$,
(ii) $x \leq y \Leftrightarrow y|1 \leq x|1$,
(iii) $y \leq x|(y|1)$,
(iv) $y|1 \leq x|y$,
(v) $x \leq (x|y)|y$,
(vi) $x \leq (((x|y)|y)|z)|z$,
(vii) $((x|y)|y)|y \approx x|y$,

$(viii)$ $x|1 \approx x|x$,
(ix) $x|(x|x) \approx 1$,
(x) $1|(x|x) \approx x$,
(xi) $x \leq y \Rightarrow y|z \leq x|z$,
(xii) $x|(y|1) \leq (y|(z|1))|((x|(z|1))|1)$,
$(xiii)$ $x|(y|1) \leq (z|(x|1))|((z|(y|1))|1)$,
(xiv) $x \leq y$ *and* $z \leq t$ *imply* $y|t \leq x|z$.

Definition 2.5. [22] A nonempty subset $F \subseteq A$ is called a filter of A if it satisfies the following properties:
$(S_f - 1)$ $1 \in F$,
$(S_f - 2)$ For all $x, y \in A$, $x|(y|1) \in F$ and $x \in F$ imply $y \in F$.

Lemma 2.6. *[22] A nonempty subset $F \subseteq A$ is a filter of A if and only if $1 \in F$ and $x \leq y$ and $x \in F$ imply $y \in F$.*

Definition 2.7. [11] $\mathcal{F}(X, [-1, 0])$ denotes the collection of functions from a set X to $[-1, 0]$ and a element of $\mathcal{F}(X, [-1, 0])$ is called a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}-$function on X). An $\mathcal{N}-$structure refers to an ordered pair (X, f) of a set X and an $\mathcal{N}-$function f on X.

Definition 2.8. [16] A neutrosophic $\mathcal{N}-$structure over a nonempty universe X is defined by

$$X_N := \frac{X}{(T_N, I_N, F_N)} = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X\},$$

where T_N, I_N and F_N are $\mathcal{N}-$function on X, called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively.

Every neutrosophic $\mathcal{N}-$structure X_N over X satisfies the condition

$$(\forall x \in X)(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0).$$

3. Neutrosophic $\mathcal{N}-$structures

In this section, we give neutrosophic $\mathcal{N}-$subalgebras and neutrosophic $\mathcal{N}-$filters on strong Sheffer stroke NMV-algebras. Unless indicated otherwise, A states a strong Sheffer stroke NMV-algebra.

Definition 3.1. A neutrosophic $\mathcal{N}-$subalgebra A_N on a strong Sheffer stroke NMV-algebra A is a neutrosophic $\mathcal{N}-$structure of A satisfying the conditions

$$\min\{T_N(x), T_N(y)\} \leq T_N(x|(y|1)),$$

$$\max\{I_N(x), I_N(y)\} \geq I_N(x|(y|1))$$

and

$$\max\{F_N(x), F_N(y)\} \geq F_N(x|(y|1)),$$

for all $x, y \in A$.

Example 3.2. Consider a strong Sheffer stroke NMV-algebra A in which the set $A = \{0, u, v, 1\}$ and the Sheffer operation $|$ on A has the following Cayley table:

TABLE 1

$\vert$	0	u	v	1
0	1	1	1	1
u	1	v	1	v
v	1	1	u	u
1	1	v	u	0

A neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{0}{(-0.79, -0.001, 0)}, \frac{u}{(-0.68, -0.72, -0.4)}, \frac{v}{(-0.68, -0.72, -0.4)}, \frac{1}{(0, -0.88, -1)}\}$$

on A is a neutrosophic $\mathcal{N}-$subalgebra of A.

Definition 3.3. Let A_N be a neutrosophic $\mathcal{N}-$structure on a strong Sheffer stroke NMV-algebra A and a, b, c be any elements of $[-1, 0]$ such that $-3 \leq a + b + c \leq 0$. For

$$T_N^a := \{x \in A : T_N(x) \geq a\},$$

$$I_N^b := \{x \in A : I_N(x) \leq b\}$$

and

$$F_N^c := \{x \in A : F_N(x) \leq c\},$$

the set

$$A_N(a, b, c) := \{x \in H : T_N(x) \geq a, I_N(x) \leq b \textit{ and } F_N(x) \leq c\}$$

is called the $(a, b, c)-$level set of A_N. Moreover,

$$A_N(a, b, c) = T_N^a \cap I_N^b \cap F_N^c.$$

Definition 3.4. [22] A subset B of a strong Sheffer stroke NMV-algebra A is called a strong Sheffer stroke NMV-subalgebra of A if 1 of A is in B and $(B, |, 1)$ forms a strong Sheffer stroke NMV-algebra. Clearly, A itself and $\{1\}$ are strong Sheffer stroke NMV-subalgebras of A.

Lemma 3.5. *Let B be a nonempty subset of a strong Sheffer stroke NMV-algebra A. Then B is a strong Sheffer stroke NMV-subalgebra of A if and only if $x|(y|1) \in B$, for all $x, y \in B$.*

Proof. Let B be a nonempty subset of a strong Sheffer stroke NMV-algebra A such that $x|(y|1) \in B$, for all $x, y \in B$. Then $1 \approx x|(x|1) \in B$ from Lemma 2.4 (i). Since $B \subseteq A$, $(B, |, 1)$ satisfies (n1)-(n6), for all $x, y, z \in B$. Thus, $(B, |, 1)$ is a strong Sheffer stroke NMV-subalgebra A.

Conversely, let B be a strong Sheffer stroke NMV-subalgebra of A. Since B states a strong Sheffer stroke NMV-algebra, it must be closed under the Sheffer operation $|$, that is, $x|y \in B$, for all $x, y \in B$. Hence, $x|(y|1) \in B$, for all $x, y \in B$. □

Theorem 3.6. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a strong Sheffer stroke NMV-algebra A and a, b, c be any elements in $[-1, 0]$ such that $-3 \leq a + b + c \leq 0$. If A_N is a neutrosophic $\mathcal{N}-$subalgebra of A, then the nonempty level set $A_N(a, b, c)$ of A_N is a subalgebra of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of A and x, y be any elements in $A_N(a, b, c)$, for $a, b, c \in [-1, 0]$ with $-3 \leq a + b + c \leq 0$. Then $T_N(x) \geq a, I_N(x) \leq b, F_N(x) \leq c$, $T_N(y) \geq a, I_N(y) \leq b$ and $F_N(y) \leq c$. Since

$$T_N(x|(y|1)) \geq \min\{T_N(x), T_N(y)\} \geq a,$$

$$I_N(x|(y|1))| \leq \max\{I_N(x), I_N(y)\} \leq b$$

and

$$F_N(x|(y|1)) \leq \max\{F_N(x), F_N(y)\} \leq c,$$

for all $x, y \in A$, it follows that $x|(y|1) \in T_N^a$, $x|(y|1) \in I_N^b$ and $x|(y|1) \in F_N^c$, which implies that $x|(y|1) \in T_N^a \cap I_N^b \cap F_N^c = A_N(a, b, c)$. Thus, $A_N(a, b, c)$ is a subalgebra of A by Lemma 3.5. □

Theorem 3.7. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a strong Sheffer stroke NMV-algebra A and T_N^a, I_N^b and F_N^c be subalgebras of A, for all $a, b, c \in [-1, 0]$ with $-3 \leq a + b + c \leq 0$. Then A_N is a neutrosophic $\mathcal{N}-$subalgebra of A.*

Proof. Let T_N^a, I_N^b and F_N^c be subalgebras of A, for all $a, b, c \in [-1, 0]$ with $-3 \leq a+b+c \leq 0$. Assume that x and y are any elements in A such that $u_1 = T_N(x|(y|1)) < \min\{T_N(x), T_N(y)\} = v_1$. If $a_0 = \frac{1}{2}(u_1 + v_1) \in [-1, 0)$, then $u_1 < a_0 < v_1$. So, $x, y \in T_N^{a_0}$ while $x|(y|1) \notin T_N^{a_0}$, which is a contradiction. Thus, $\min\{T_N(x), T_N(y)\} \leq T_N(x|(y|1))$, for all $x, y \in A$.

Suppose that x and y are any elements in A such that $u_2 = \max\{I_N(x), I_N(y)\} < I_N(x|(y|1)) = v_2$. If $b_0 = \frac{1}{2}(u_2 + v_2) \in [-1, 0)$, then $u_2 < b_0 < v_2$, which implies that $x, y \in I_N^{b_0}$ but $x|(y|1) \notin I_N^{b_0}$. This is a contradiction. Thus, $I_N(x|(y|1)) \leq \max\{I_N(x), I_N(y)\}$, for all $x, y \in A$.

Assume that x and y are any elements in A such that $v_3 = F_N(x|(y|1)) > \max\{F_N(x), F_N(y)\} = u_3$. If $c_0 = \frac{1}{2}(u_3 + v_3) \in [-1, 0)$, then $u_3 < c_0 < v_3$. Thus, $x, y \in F_N^{c_0}$ but $x|(y|1) \notin F_N^{c_0}$, which is a contradiction. Thereby, $\max\{F_N(x), F_N(y)\} \geq F_N(x|(y|1))$, for all $x, y \in A$.

Therefore, A_N is a neutrosophic $\mathcal{N}-$subalgebra of A. □

Theorem 3.8. *Let $\{A_{N_i} : i \in \mathbb{N}\}$ be a family of all neutrosophic $\mathcal{N}-$subalgebras of a strong Sheffer stroke NMV-algebra A. Then $\{A_{N_i} : i \in \mathbb{N}\}$ forms a complete distributive lattice.*

Proof. Let B be a nonempty subset of $\{A_{N_i} : i \in \mathbb{N}\}$. Since A_{N_i} is a neutrosophic $\mathcal{N}-$subalgebra of A, for all $A_{N_i} \in B$, it satisfies

$$\min\{T_N(x), T_N(y)\} \leq T_N(x|(y|1)),$$

$$I_N(x|(y|1)) \leq \max\{I_N(x), I_N(y)\}$$

and

$$F_N(x|(y|1)) \leq \max\{F_N(x), F_N(y)\},$$

for all $x, y \in A$. Then $\bigcap B$ satisfies these inequalities. Thus, $\bigcap B$ is a neutrosophic $\mathcal{N}-$subalgebra of A.

Let C be a family of all neutrosophic $\mathcal{N}-$subalgebras of A containing $\bigcup\{A_{N_i} : i \in \mathbb{N}\}$. Then $\bigcap C$ is also a neutrosophic $\mathcal{N}-$subalgebra of A.

If $\bigwedge_{i\in\mathbb{N}} A_{N_i} = \bigcap_{i\in\mathbb{N}} A_{N_i}$ and $\bigvee_{i\in\mathbb{N}} A_{N_i} = \bigcap C$, then $(\{A_{N_i} : i \in \mathbb{N}\}, \bigvee, \bigwedge)$ forms a complete lattice. Moreover, it is distibutive by the definitions of $\bigvee$ and $\bigwedge$. □

Lemma 3.9. *If a neutrosophic $\mathcal{N}-$structure A_N on a strong Sheffer stroke NMV-algebra A is a neutrosophic $\mathcal{N}-$subalgebra of A, then $T_N(x) \leq T_N(1)$, $I_N(x) \geq I_N(1)$ and $F_N(x) \geq F_N(1)$, for all $x \in A$.*

Proof. Let a neutrosophic $\mathcal{N}-$structure A_N on a strong Sheffer stroke NMV-algebra A be a neutrosophic $\mathcal{N}-$subalgebra of A. By substituting $[y := x]$ in the inequalities in Definition 3.1, it is obtained from Lemma 2.4 (i) that

$$T_N(x) = \min\{T_N(x), T_N(x)\} \leq T_N(x|(x|1)) = T_N(1),$$

$$I_N(1) = I_N(x|(x|1)) \leq \max\{I_N(x), I_N(x)\} = I_N(x)$$

and

$$F_N(1) = F_N(x|(x|1)) \leq \max\{F_N(x), F_N(x)\} = F_N(x),$$

for all $x \in H$. □

The inverse of Lemma 3.9 does not hold in general.

Example 3.10. Consider the strong Sheffer stroke NMV-algebra A in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{0}{(-0.8,-0.7,-0.02)}, \frac{u}{(-0.5,-0.4,-0.3)}, \frac{v}{(-0.2,-0.1,-0.11)}, \frac{1}{(0,-1,-0.6)}\}$$

on A is not a neutrosophic $\mathcal{N}-$subalgebra of A since

$$\max\{I_N(u), I_N(0)\} = \max\{-0.4,-0.7\} = -0.4 < -0.1 = I_N(v) = I_N(u|(0|1)).$$

Lemma 3.11. *Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of a strong Sheffer stroke NMV-algebra A. If there exists a sequence $\{a_n\}$ on A such that*

$$\lim_{n\longrightarrow\infty} T_N(a_n) = 0,\ \lim_{n\longrightarrow\infty} I_N(a_n) = -1 \text{ and } \lim_{n\longrightarrow\infty} F_N(a_n) = -1,$$

then

$$T_N(1) = 0,\ I_N(1) = -1 \text{ and } F_N(1) = -1.$$

Proof. Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of a strong Sheffer stroke NMV-algebra A. Suppose that there exists a sequence $\{a_n\}$ on A such that $\lim_{n\longrightarrow\infty} T_N(a_n) = 0$ and $\lim_{n\longrightarrow\infty} I_N(a_n) = -1 = \lim_{n\longrightarrow\infty} F_N(a_n)$. Since $T_N(a_n) \leq T_N(1)$, $I_N(a_n) \geq I_N(1)$ and $F_N(a_n) \geq F_N(1)$, for every $n \in \mathbb{N}$ from Lemma 3.9, it is obtained that

$$0 = \lim_{n\longrightarrow\infty} T_N(a_n) \leq \lim_{n\longrightarrow\infty} T_N(1) = T_N(1) \leq 0,$$

$$-1 \leq I_N(1) = \lim_{n\longrightarrow\infty} I_N(1) \leq \lim_{n\longrightarrow\infty} I_N(a_n) = -1$$

and

$$-1 \leq F_N(1) = \lim_{n\longrightarrow\infty} F_N(1) \leq \lim_{n\longrightarrow\infty} F_N(a_n) = -1.$$

Thus, $T_N(1) = 0$ and $I_N(1) = F_N(1) = -1$. □

Lemma 3.12. *A neutrosophic $\mathcal{N}-$subalgebra A_N of a strong Sheffer stroke NMV-algebra A satisfies $T_N(x) \leq T_N(x|(y|1))$, $I_N(x) \geq I_N(x|(y|1))$ and $F_N(x) \geq F_N(x|(y|1))$, for all $x, y \in A$ if and only if T_N, I_N and F_N are constant.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of a strong Sheffer stroke NMV-algebra A satisfying $T_N(x) \leq T_N(x|(y|1))$, $I_N(x) \geq I_N(x|(y|1))$ and $F_N(x) \geq F_N(x|(y|1))$, for all $x, y \in A$. Since $T_N(1) \leq T_N(1|(x|1)) = T_N((x|1)|1) = T_N(x)$, $I_N(1) \geq I_N(1|(x|1)) = I_N((x|1)|1) = I_N(x)$ and $F_N(1) \geq F_N(1|(x|1)) = F_N((x|1)|1) = F_N(x)$ from (n1) and (n3), it follows from Lemma 3.9 that $T_N(x) = T_N(1)$, $I_N(x) = I_N(1)$ and $F_N(x) = F_N(1)$, for all $x \in A$.

Conversely, every neutrosophic $\mathcal{N}-$subalgebra A_N of a strong Sheffer stroke NMV-algebra A satisfies $T_N(x) \leq T_N(x|(y|1))$, $I_N(x) \geq I_N(x|(y|1))$ and $F_N(x) \geq F_N(x|(y|1))$, for all $x, y \in A$ because T_N, I_N and F_N are constant. □

Definition 3.13. A neutrosophic $\mathcal{N}-$structure A_N on a strong Sheffer stroke NMV-algebra A is called a neutrosophic $\mathcal{N}-$filter of A if

$$\min\{T_N(x|(y|1)), T_N(x)\} \leq T_N(y) \leq T_N(1),$$

$$I_N(1) \leq I_N(y) \leq \max\{I_N(x|(y|1)), I_N(x)\}$$

and

$$F_N(1) \leq F_N(y) \leq \max\{F_N(x|(y|1)), F_N(x)\},$$

for all $x, y \in A$.

Example 3.14. Consider the strong Sheffer stroke NMV-algebra A in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{0}{(-0.23, -0.3, -0.01)}, \frac{u}{(-0.02, -0.98, -0.11)}, \frac{v}{(-0.23, -0.3, -0.01)}, \frac{1}{(-0.02, -0.98, -0.11)}\}$$

on A is a neutrosophic $\mathcal{N}-$filter of A.

Lemma 3.15. *Every a neutrosophic $\mathcal{N}-$filter A_N of a strong Sheffer stroke NMV-algebra A satisfies that $x \leq y$ implies $T_N(x) \leq T_N(y)$, $I_N(x) \geq I_N(y)$ and $F_N(x) \geq F_N(y)$, for all $x, y \in A$.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke NMV-algebra A and $x \leq y$. Then $x|(y|1) \approx 1$ from Proposition 2.3. Thus,

$$T_N(x) = \min\{T_N(1), T_N(x)\} = \min\{T_N(x|(y|1)), T_N(x)\} \leq T_N(y),$$

$$I_N(x) = \max\{I_N(1), I_N(x)\} = \max\{I_N(x|(y|1)), I_N(x)\} \geq I_N(y)$$

and

$$F_N(x) = \max\{F_N(1), F_N(x)\} = \max\{F_N(x|(y|1)), F_N(x)\} \geq F_N(y),$$

for any $x, y \in A$. □

The inverse of Lemma 3.15 is generally not true.

Example 3.16. Consider the neutrosophic $\mathcal{N}-$filter of A in Example 3.14. Then $v \nleq u$ when $-0.98 = I_N(u) \leq I_N(v) = -0.3$.

Lemma 3.17. *Let A_N be a neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke NMV-algebra A. Then*

$$T_N((x|(y|1))|(z|1)) \leq T_N((x|(z|1))|((y|(z|1))|1)),$$

$$I_N((x|(y|1))|(z|1)) \geq I_N((x|(z|1))|((y|(z|1))|1)), \quad (1)$$

and

$$F_N((x|(y|1))|(z|1)) \geq F_N((x|(z|1))|((y|(z|1))|1)),$$

for all $x, y, z \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke NMV-algebra A. Since $(x|(y|1))|(z|1) \leq y|(z|1) \leq (x|(z|1))|((y|(z|1))|1)$ from Lemma 2.4 (iii) and (xi), it follows from Lemma 3.15 that

$$T_N((x|(y|1))|(z|1)) \leq T_N((x|(z|1))|((y|(z|1))|1)),$$

$$I_N((x|(y|1))|(z|1)) \geq I_N((x|(z|1))|((y|(z|1))|1))$$

and

$$F_N((x|(y|1))|(z|1)) \geq F_N((x|(z|1))|((y|(z|1))|1)),$$

for all $x, y, z \in A$. □

The inverse of Lemma 3.17 does not usually hold.

Example 3.18. Consider the strong Sheffer stroke NMV-algebra A in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{0}{(-0.69, -0.12, 0)}, \frac{u}{(-0.58, -0.87, -0.22)}, \frac{v}{(-0.58, -0.87, -0.22)}, \frac{1}{(-0.14, -0.93, 0.96)}\}$$

on A satisfies the condition (1) in Lemma 3.17 but it is not a neutrosophic $\mathcal{N}-$filter of A since

$$\min\{T_N(u|(0|1)), T_N(u)\} = \min\{T_N(v), T_N(u)\} = -0.58 > -0.69 = T_N(0).$$

Lemma 3.19. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a strong Sheffer stroke NMV-algebra A and a, b, c be any elements of $[-1, 0]$ with $-3 \leq a + b + c \leq 0$. If A_N is a neutrosophic $\mathcal{N}-$filter of A, then the nonempty subset $A_N(a, b, c)$ is a filter of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke NMV-algebra A and $A_N(a, b, c) \neq \emptyset$ for $a, b, c \in [-1, 0]$ with $-3 \leq a + b + c \leq 0$. Since $a \leq T_N(x) \leq T_N(1), b \geq I_N(x) \geq I_N(1)$ and $c \geq F_N(x) \geq F_N(1)$, for all $x \in A_N(a, b, c)$, we have $1 \in A_N(a, b, c)$. Let $x|(y|1), x \in A_N(a, b, c)$. Then $a \leq T_N(x)$, $I_N(x) \leq b$, $F_N(x) \leq c$, $a \leq T_N(x|(y|1))$, $I_N(x|(y|1)) \leq b$ and $F_N(x|(y|1)) \leq c$. Since

$$a \leq \min\{T_N(x|(y|1)), T_N(x)\} \leq T_N(y),$$

$$I_N(y) \leq \max\{I_N(x|(y|1)), I_N(x)\} \leq b$$

and

$$F_N(y) \leq \max\{F_N(x|(y|1)), F_N(x)\} \leq c,$$

for all $x, y \in A$, it is obtained $y \in A_N(a, b, c)$. Hence, $A_N(a, b, c)$ is a filter of A. □

Theorem 3.20. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a strong Sheffer stroke NMV-algebra A and T_N^a, I_N^b, F_N^c be filters of A, for all $a, b, c \in [-1, 0]$ with $-3 \leq a+b+c \leq 0$. Then A_N is a neutrosophic $\mathcal{N}-$filter of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$structure on a strong Sheffer stroke NMV-algebra A and T_N^a, I_N^b, F_N^c be filters of A, for all $a, b, c \in [-1, 0]$ with $-3 \leq a+b+c \leq 0$. Assume that $T_N(1) < T_N(x_0)$, $I_N(y_0) < I_N(1)$ and $F_N(z_0) < F_N(1)$. If $a_0 = \frac{1}{2}(T_N(1) + T_N(x_0))$, $b_0 = \frac{1}{2}(I_N(1) + I_N(y_0))$ and $c_0 = \frac{1}{2}(F_N(1) + F_N(z_0))$ in $[-1, 0)$, then $T_N(1) < a_0 < T_N(x_0)$, $I_N(1) > b_0 > I_N(y_0)$ and $F_N(1) > c_0 > F_N(z_0)$. Thus, $1 \notin T_N^{a_0}, 1 \notin I_N^{b_0}$ and $1 \notin F_N^{c_0}$, which contradict with $(S_f - 1)$. Hence, $T_N(x) \leq T_N(1)$, $I_N(x) \geq I_N(1)$ and $F_N(x) \geq F_N(1)$, for all $x \in A$. Suppose that x_1, x_2, x_3, y_1, y_2 and y_3 are any elements of A such that

$$v_1 = T_N(y_1) < \min\{T_N(x_1|(y_1|1)), T_N(x_1)\} = u_1,$$

$$u_2 = \max\{I_N(x_2|(y_2|1)), I_N(x_2)\} < I_N(y_2) = v_2,$$

and

$$u_3 = \max\{F_N(x_3|(y_3|1)), F_N(x_3)\} < F_N(y_3) = v_3.$$

If $a' = \frac{1}{2}(u_1 + v_1)$, $b' = \frac{1}{2}(u_2 + v_2)$ and $c' = \frac{1}{2}(u_3 + v_3)$ in $[-1, 0)$, then $v_1 < a' < u_1$, $u_2 < b' < v_2$ and $u_3 < c' < v_3$. So, $y_1 \notin T_N^{a'}$, $y_2 \notin I_N^{b'}$ and $y_3 \notin F_N^{c'}$ when $x_1|(y_1|1), x_1 \in T_N^{a'}$, $x_2|(y_2|1), x_2 \in I_N^{b'}$ and $x_3|(y_3|1), x_3 \in F_N^{c'}$. This is a contradiction. Thereby,

$$\min\{T_N(x|(y|1)), T_N(x)\} \leq T_N(y),$$

$$I_N(y) \leq \max\{I_N(x|(y|1)), I_N(x)\}$$

and

$$F_N(y) \leq \max\{F_N(x|(y|1)), F_N(x)\},$$

for all $x, y \in A$. Therefore, A_N is a neutrosophic $\mathcal{N}-$filter of A. □

Lemma 3.21. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a strong Sheffer stroke NMV-algebra A. Then A_N is a neutrosophic $\mathcal{N}-$filter of A if and only if $z \leq y|(x|1)$ implies*

$$\min\{T_N(y), T_N(z)\} \leq T_N(x),$$

$$I_N(x) \leq \max\{I_N(y), I_N(z)\}$$

and

$$F_N(x) \leq \max\{F_N(y), F_N(z)\},$$

for all $x, y, z \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$filter of A and x, y and z be any elements of A such that $z \leq y|(x|1)$. Since $T_N(z) \leq T_N(y|(x|1))$, $I_N(z) \geq I_N(y|(x|1))$ and $F_N(z) \geq F_N(y|(x|1))$ from Lemma 3.15, it follows that

$$\min\{T_N(y), T_N(z)\} \leq \min\{T_N(y|(x|1)), T_N(y)\} \leq T_N(x),$$

$$I_N(x) \leq \max\{I_N(y|(x|1)), I_N(y)\} \leq \max\{I_N(y), I_N(z)\}$$

and

$$F_N(x) \leq \max\{F_N(y|(x|1)), F_N(y)\} \leq \max\{F_N(y), F_N(z)\},$$

for all $x, y, z \in A$.

Conversely, suppose that A_N is a neutrosophic $\mathcal{N}-$structure on A such that $z \leq y|(x|1)$ implies

$$\min\{T_N(y), T_N(z)\} \leq T_N(x),$$

$$I_N(x) \leq \max\{I_N(y), I_N(z)\}$$

and

$$F_N(x) \leq \max\{F_N(y), F_N(z)\},$$

for all $x, y, z \in A$. Since $x \leq 1 \approx x|0 \approx x|(1|1)$ from (n2), it is obtained that $T_N(x) \leq T_N(1)$, $I_N(1) \leq I_N(x)$ and $F_N(1) \leq F_N(x)$, for all $x \in A$. Since $x \leq (x|(y|1))|(y|1)$ from Lemma 2.4 (v), we have

$$\min\{T_N(x|(y|1)), T_N(x)\} \leq T_N(y),$$

$$I_N(y) \leq \max\{I_N(x|(y|1)), I_N(x)\}$$

and

$$F_N(y) \leq \max\{F_N(x|(y|1)), F_N(x)\},$$

for all $x, y \in A$. Hence, A_N is a neutrosophic $\mathcal{N}-$filter of A. □

Theorem 3.22. *Every neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke NMV-algebra A is a neutrosophic $\mathcal{N}-$subalgebra of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$filter of A. Since

$$\begin{aligned} \min\{T_N(x),T_N(y)\} &\leq \min\{T_N(1),T_N(y)\} \\ &= \min\{T_N(((y|1)|1)|(((y|1)|x)|1)),T_N(y)\} \\ &= \min\{T_N(y|((x|(y|1))|1)),T_N(y)\} \\ &\leq T_N(x|(y|1)), \end{aligned}$$

and similarly,

$$I_N(x|(y|1)) \leq \max\{I_N(x),I_N(y)\}$$

and

$$F_N(x|(y|1)) \leq \max\{F_N(x),F_N(y)\},$$

from (n1), (n3) and (n5), it follows that A_N is a neutrosophic $\mathcal{N}-$subalgebra of A. □

The inverse of Theorem 3.22 does not usually hold.

Example 3.23. The neutrosophic $\mathcal{N}-$subalgebra A_N of A in Example 3.2. Then it is not a neutrosophic $\mathcal{N}-$filter of A since $\min\{T_N(u|(0|1)),T_N(u)\} = \min\{T_N(v),T_N(u)\} = -0.68 > -0.79 = T_N(0)$.

Definition 3.24. Let A be a strong Sheffer stroke NMV-algebra. Define

$$A_N^{x_t} := \{x \in A : T_N(x_t) \leq T_N(x)\},$$

$$A_N^{x_i} := \{x \in A : I_N(x) \leq I_N(x_i)\}$$

and

$$A_N^{x_f} := \{x \in A : F_N(x) \leq F_N(x_f)\},$$

for all $x_t, x_i, x_f \in A$. Obviously, $x_t \in A_N^{x_t}, x_i \in A_N^{x_i}$ and $x_f \in A_N^{x_f}$.

Example 3.25. Consider the strong Sheffer stroke NMV-algebra A in Example 3.2. Let $T_N(0) = -0.113, T_N(u) = -0.12, T_N(v) = -0.13, T_N(1) = 0, I_N(0) = -0.21, I_N(u) = -0.22, I_N(v) = -0.23, I_N(1) = -1, F_N(0) = -0.31, F_N(u) = -0.32, F_N(v) = -0.33, F_N(1) = -0.34, x_t = u, x_i = v$ and $x_f = 0$. Then

$$A_N^{x_t} = \{x \in A : T_N(u) \leq T_N(x)\} = \{0,u,1\},$$

$$A_N^{x_i} = \{x \in A : I_N(x) \leq I_N(v)\} = \{v,1\}$$

and

$$A_N^{x_f} = \{x \in A : F_N(x) \leq F_N(0)\} = A.$$

Theorem 3.26. *Let x_t, x_i and x_f be any elements of a strong Sheffer stroke NMV-algebra A. If A_N is a neutrosophic $\mathcal{N}-$filter of A, then $A_N^{x_t}, A_N^{x_i}$ and $A_N^{x_f}$ are filters of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke NMV-algebra A. Since $T_N(x_t) \leq T_N(1)$, $I_N(1) \leq I_N(x_i)$ and $F_N(1) \leq F_N(x_f)$, for any $x_t, x_i, x_f \in A$, we have $1 \in A_N^{x_t}$, $1 \in A_N^{x_i}$ and $1 \in A_N^{x_f}$. Let $x_1|(y_1|1), x_1 \in A_N^{x_t}$, $x_2|(y_2|1), x_2 \in A_N^{x_i}$ and $x_3|(y_3|1), x_3 \in A_N^{x_f}$. Then $T_N(x_t) \leq T_N(x_1|(y_1|1))$, $T_N(x_t) \leq T_N(x_1)$, $I_N(x_2|(y_2|1)) \leq I_N(x_i)$, $I_N(x_2) \leq I_N(x_i)$ and $F_N(x_3|(y_3|1)) \leq F_N(x_f)$, $F_N(x_3) \leq F_N(x_f)$. Since

$$T_N(x_t) \leq \min\{T_N(x_1|(y_1|1)), T_N(x_1)\} \leq T_N(y_1),$$

$$I_N(y_2) \leq \max\{I_N(x_2|(y_2|1)), I_N(x_2)\} \leq I_N(x_i)$$

and

$$F_N(y_3) \leq \max\{F_N(x_3|(y_3|1)), F_N(x_3)\} \leq F_N(x_f),$$

we get $y_1 \in A_N^{x_t}$, $y_2 \in A_N^{x_i}$ and $y_3 \in A_N^{x_f}$. Thus, $A_N^{x_t}$, $A_N^{x_i}$ and $A_N^{x_f}$ are filters of A. □

Example 3.27. Consider the strong Sheffer stroke NMV-algebra A in Example 3.2. For a neutrosophic $\mathcal{N}-$filter

$$A_N = \{\frac{0}{(-0.32, -0.29, -0.07)}, \frac{u}{(-0.32, -0.29, -0.07)}, \frac{v}{(-0.1, -0.78, -0.17)}, \frac{1}{(-0.1, -0.78, -0.17)}\}$$

of A, $x_t = u$, $x_i = v$ and $x_f = 1 \in A$, the subsets

$$A_N^{x_t} = \{x \in A : T_N(u) \leq T_N(x)\} = A,$$

$$A_N^{x_i} = \{x \in A : I_N(x) \leq I_N(v)\} = \{v, 1\}$$

and

$$A_N^{x_f} = \{x \in A : F_N(x) \leq F_N(1)\} = \{v, 1\}$$

of A are filterss of A.

Theorem 3.28. *Let x_t, x_i and x_f be any elements of a strong Sheffer stroke NMV-algebra A and A_N be a neutrosophic $\mathcal{N}-$structure on A.*

(a) *If $A_N^{x_t}, A_N^{x_i}$ and $A_N^{x_f}$ are filters of A, then*

$$T_N(x) \leq \min\{T_N(y|(z|1)), T_N(y)\} \Rightarrow T_N(x) \leq T_N(z),$$

$$I_N(x) \geq \max\{I_N(y|(z|1)), I_N(y)\} \Rightarrow I_N(x) \geq I_N(z) \quad (2)$$

and

$$F_N(x) \geq \max\{F_N(y|(z|1)), F_N(y)\} \Rightarrow F_N(x) \geq F_N(z),$$

for all $x, y, z \in A$.

(b) *If A_N satisfies the condition (3.2) and*

$$T_N(x) \leq T_N(1), \quad I_N(1) \leq I_N(x) \quad and \quad F_N(1) \leq F_N(x), \quad for\ all\ x \in A, \tag{3}$$

then $A_N^{x_t}, A_N^{x_i}$ and $A_N^{x_f}$ are filters of A, for all $x_t \in T_N^{-1}$, $x_i \in I_N^{-1}$ and $x_f \in F_N^{-1}$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$structure on a strong Sheffer stroke NMV-algebra A.

(a) Let $A_N^{x_t}, A_N^{x_i}$ and $A_N^{x_f}$ be filters of A, for any $x_t, x_i, x_f \in A$, and x, y, z be any elements of A such that $T_N(x) \leq \min\{T_N(y|(z|1)), T_N(y)\}$, $I_N(x) \geq \max\{I_N(y|(z|1)), I_N(y)\}$ and $F_N(x) \geq \max\{F_N(y|(z|1)), F_N(y)\}$. Since $y|(z|1), y \in A_N^{x_t}$, $y|(z|1), y \in A_N^{x_i}$ and $y|(z|1), y \in A_N^{x_f}$, where $x_t = x_i = x_f = x$, it follows from $(S_f - 2)$ that $z \in A_N^{x_t}$, $z \in A_N^{x_i}$ and $z \in A_N^{x_f}$, where $x_t = x_i = x_f = x$. Thus, $T_N(x) \leq T_N(z)$, $I_N(z) \leq I_N(x)$ and $F_N(z) \leq F_N(x)$, for all $x, y, z \in A$.

(b) Let A_N be a neutrosophic $\mathcal{N}-$structure on A satisfying the conditions (2) and (3), for $x_t \in T_N^{-1}$, $x_i \in I_N^{-1}$ and $x_f \in F_N^{-1}$. Then $1 \in A_N^{x_t}$, $1 \in A_N^{x_i}$ and $1 \in A_N^{x_f}$ from the condition (3). Let $x_1|(y_1|1), x_1 \in X_N^{x_t}$, $x_2|(y_2|1), x_2 \in A_N^{x_i}$ and $x_3|(y_3|1), x_3 \in A_N^{x_f}$. Thus, $T_N(x_t) \leq T_N(x_1|(y_1|1))$, $T_N(x_t) \leq T_N(x_1)$, $I_N(x_2|(y_2|1)) \leq I_N(x_i)$, $I_N(x_2) \geq I_N(x_i)$ and $F_N(x_3|(y_3|1)) \leq F_N(x_f)$, $F_N(x_3) \leq F_N(x_f)$. Since

$$T_N(x_t) \leq \min\{T_N(x_1|(y_1|1)), T_N(x_1)\},$$

$$\max\{I_N(x_2|(y_2|1)), I_N(x_2)\} \leq I_N(x_i)$$

and

$$\max\{F_N(x_3|(y_3|1)), F_N(x_3)\} \leq F_N(x_f),$$

it follows from the condition (2) that $T_N(x_t) \leq T_N(y_1)$, $I_N(y_2) \leq I_N(x_i)$ and $F_N(y_3) \leq F_N(x_f)$. Hence, $y_1 \in A_N^{x_t}$, $y_2 \in A_N^{x_i}$ and $y_3 \in A_N^{x_f}$. Therefore, $A_N^{x_t}$, $A_N^{x_i}$ and $A_N^{x_f}$ are filters of A. □

Example 3.29. Consider the strong Sheffer stroke NMV-algebra A in Example 3.2. Let $T_N(0) = T_N(v) = -1, T_N(u) = T_N(1) = 0, I_N(0) = I_N(v) = 0, I_N(u) = I_N(1) = -1, F_N(0) = F_N(v) = -0.71, F_N(u) = F_N(1) = -0.5$. Then the filters

$$A_N^{x_t} = A, A_N^{x_i} = \{u.1\} \text{ and } A_N^{x_f} = A$$

of A satisfy the condition (2) in Theorem 3.28, for $x_t = v, x_i = u$ and $x_f = 1 \in A$.

Moreover, let

$$A_N = \{\frac{0}{(-0.99, 0, -0.01)}, \frac{u}{(-0.99, 0, -0.01)}, \frac{v}{(-0.99, 0, -0.01)}, \frac{1}{(0, -1, -1)}\}$$

be a neutrosophic $\mathcal{N}-$structure on A satisfying the conditions (2) and (3) in Theorem 3.28. Then the subsets

$$A_N^{x_t} = \{x \in A : T_N(1) \leq T_N(x)\} = \{1\},$$

$$A_N^{x_i} = \{x \in A : I_N(x) \leq I_N(0)\} = A$$

and

$$A_N^{x_f} = \{x \in A : F_N(x) \leq F_N(u)\} = A$$

of A are filters of A, where $x_t = 1, x_i = 0$ and $x_f = u \in A$.

4. Conclusion

In this study, neutrosophic $\mathcal{N}-$structures defined by $\mathcal{N}-$functions on strong Sheffer stroke NMV-algebras have been investigated. Basic definitions and notions about strong Sheffer stroke NMV-algebras and neutrosophic $\mathcal{N}-$structures defined by $\mathcal{N}-$functions on a nonempty universe X are presented and then a neutrosophic $\mathcal{N}-$subalgebra and a $(a, b, c)-$ level set of a neutrosophic $\mathcal{N}-$structure are defined by the help of $\mathcal{N}-$functions on strong Sheffer stroke NMV-algebras. It is shown that the $(a, b, c)-$level set of a neutrosophic $\mathcal{N}-$subalgebra of a strong Sheffer stroke NMV-algebra is its strong Sheffer stroke NMV-subalgebra and vice versa. Also, it is proved that the family of all neutrosophic $\mathcal{N}-$subalgebras of this algebraic structure forms a complete distributive lattice. It is illustrated that every neutrosophic $\mathcal{N}-$subalgebra of a strong Sheffer stroke NMV-algebra satisfies $T_N(x) \leq T_N(1)$, $I_N(1) \leq I_N(x)$ and $F_N(1) \leq F_N(x)$, for all elements x in this algebra but a neutrosophic $\mathcal{N}-$structure on a strong Sheffer stroke NMV-algebra satisfying this property is generally not its neutrosophic $\mathcal{N}-$subalgebra. Besides, it is interpreted the images of the sequence under $\mathcal{N}-$functions on a strong Sheffer stroke NMV-algebra. Moreover, it is stated the case which $\mathcal{N}-$functions determining a neutrosophic $\mathcal{N}-$subalgebra of a strong Sheffer stroke NMV-algebra are constant. Then a neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke NMV-algebra is defined via $\mathcal{N}-$functions and shown that the functions T_N, I_N and F_N defining the neutrosophic $\mathcal{N}-$filter satisfies $T_N(x) \leq T_N(y)$, $I_N(x) \geq I_N(y)$ and $F_N(x) \geq F_N(y)$ when $x \leq y$, but the inverse does not usually hold. It is demonstrated that $(a, b, c)-$level set of a neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke NMV-algebra is its filter. Indeed, it is given that the subsets defined by $\mathcal{N}-$functions on a strong Sheffer stroke NMV-algebra must be its filters so that a neutrosophic $\mathcal{N}-$structure on this algebra is a neutrosophic $\mathcal{N}-$filter. It is proved that every neutrosophic $\mathcal{N}-$filter of a strong Sheffer stroke NMV-algebra is its neutrosophic $\mathcal{N}-$subalgebra whereas the inverse is not true in general. Additionally, new three subsets $A_N^{x_t}, A_N^{x_i}$ and $A_N^{x_f}$ of a strong Sheffer stroke NMV-algebra are defined by $\mathcal{N}-$functions and any elements x_t, x_i and x_f of the algebra. We show that these subsets are filters of a strong Sheffer stroke NMV-algebra for its neutrosophic $\mathcal{N}-$filter but the inverse holds under special conditions.

In our future works, we wish to introduce new Sheffer stroke algebraic structures and investigate their neutrosophic $\mathcal{N}-$structures.

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An integrated model of Neutrosophic TOPSIS with application in Multi-Criteria Decision-Making Problem

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Abstract: Multi-criteria decision making (MCDM) is the technique of selecting the best alternative from multiple alternatives and multiple conditions. The technique for order preference by similarity to an ideal solution (TOPSIS) is a crucial practical technique for ranking and selecting different options by using a distance measure. In this article, we protract the fuzzy TOPSIS technique to neutrosophic fuzzy TOPSIS and prove the accuracy of the method by explaining the MCDM problem with single-valued neutrosophic information and use the method for supplier selection in the production industry. We hope that this article will promote future scientific research on numerous existing issues based on multi-criteria decision making.

Keywords: Neutrosophic set, Single valued Neutrosophic set, TOPSIS, MCDM

1. Introduction

We faced a lot of complications in different areas of life which contain vagueness such as engineering, economics, modeling, and medical diagnoses, etc. However, a general question is raised that in mathematical modeling how we can express and use the uncertainty. A lot of researchers in the world proposed and recommended different approaches to solve those problems that contain uncertainty. In decision-making problems, multiple attribute decision making (MADM) is the most essential part which provides us to find the most appropriate and extraordinary alternative. However, choosing the appropriate alternative is very difficult because of vague information in some cases. To overcome such situations, Zadeh developed the notion of fuzzy sets (FSs) [1] to solve those problems which contain uncertainty and vagueness. Fuzzy sets are like sets whose components have membership (Mem) degrees. In the classical set theory, the Mem degree of the elements in the set is checked in binary form according to the bivalent condition of whether the elements completely belong to the set. In contrast, the fuzzy set theory allows modern ratings of the Mem of elements in the set. This is represented by the Mem function, and the effective unit interval of the Mem function is [0, 1]. The fuzzy set is the generalization of the classical set because the indicator function of the classic set is a special case of the Mem function of the fuzzy set if the latter only takes the value 0 or 1. In the fuzzy set theory, the classical bivalent set is usually called the crisp set. Fuzzy set theory can be used in a wide range of fields with incomplete or imprecise information.

It is observed that in some cases circumstances cannot be handled by fuzzy sets, to overcome such types of situations Turksen [2] gave the idea of interval-valued fuzzy sets (IVFSs). In some cases, we must deliberate membership unbiassed as the non-membership values for the suitable representation of an object in uncertain and indeterminate conditions that could not be handled by FSs nor IVFSs. To overcome these difficulties Atanassov offered the concept of Intuitionistic fuzzy sets (IFSs) [3]. The theory which was presented by Atanassov only deals the insufficient data considering both the membership and non-membership values, but the intuitionistic fuzzy set theory cannot handle the incompatible and imprecise information. To deal with such incompatible and imprecise data Smarandache [4] extended the work of Atanassov IFSs and proposed a powerful tool comparative to FSs and IFSs to deal with indeterminate, incomplete, and inconsistent information's faced in real-life problems. Since the direct use of Neutrosophic sets (NSs) for TOPSIS is somewhat difficult. To apply the NSs, Wang et al. introduced a subclass of NSs known as single-valued Neutrosophic sets (SVNSs) in [5]. In [6] the author proposed a geometric interpretation by using NSs. Gulfam et al. [7] introduced a new distance formula for SVNSs and developed some new techniques under the Neutrosophic environment. The concept of a single-valued Neutrosophic soft expert set is proposed in [8] by combining the SVNSs and soft expert sets. To solve MCDM problems with single-valued Neutrosophic numbers (SVNNs) presented by Deli and Subas in [9], they constructed the concept of cut sets of SVNNs. On the base of the correlation of IFSs, the term correlation coefficient of SVNSs [10] introduced and proposed a decision-making method by using a weighted correlation coefficient or the weighted cosine similarity measure of SVNSs. In [11] the idea of simplified Neutrosophic sets introduced with some operational laws and aggregation operators such as real-life Neutrosophic weighted arithmetic average operator and weighted geometric average operator. They constructed an MCDM method based on proposed aggregation operators and cosine similarity measure for simplified neutrosophic sets. Sahin and Yiğider [12] extended the TOPSIS method to MCDM with a single-valued neutrosophic technique.

Hwang and Yoon [13] established TOPSIS to solve the general difficulties of DM. The TOPSIS method can effectively maintain the minimum distance from the ideal solution, thereby helping to select the finest choice. After the TOPSIS technique came out, some investigators utilized the TOPSIS technique for DM and protracted the TOPSIS technique to several other hybrid structures of FS. The most important determinant of current scientific research is to present an integrated model for neutrosophic TOPSIS to solve the MCDM problem. Chen & Hwang [14] extended the idea of the TOPSIS method and proposed a new TOPSIS model. The author uses the newly proposed decision-making method to solve uncertain data [15]. Zulqarnain et al. [16] utilized the TOPSIS method for the prediction of diabetic patients in medical diagnosis. They also utilized the TOPSIS extensions of different hybrid structures of FS [17–19] and used them for decision making. Pramanik et al. [21] established the TOPSIS to resolve the multi-attribute decision-making problem under a single-valued neutrosophic soft set expert scenario. Zulqarnain et al. [21] presented the generalized neutrosophic TOPSIS to solve the MCDM problem. Zulqarnain et al. [22] utilized fuzzy TOPSIS to solve the MCDM problem. Maji [23] proposed the concept of neutrosophic soft sets (NSSs) with some properties and operations. The authors studied NSSs and gave some new definitions on NSSs [24], they also gave the idea of neutrosophic soft matrices with some operations and proposed a decision-making method. Many researchers developed the decision-making models by using the NSSs reported in the literature [25–27]. Elhassouny and Smarandache [28] extended the work on a simplified TOPSIS method and by using single-valued Neutrosophic information they proposed Neutrosophic simplified TOPSIS method. The concept of single-valued neutrosophic cross-entropy measure introduced by Jun [29], he also constructed an MCDM method and claimed that this proposed method is more appropriate than previous methods for decision making.

Saha and Broumi [31] studied the interval-valued neutrosophic sets (IVNSs) and developed some new set-theoretic operations on IVNSs with their properties. The idea of an Interval-valued generalized single valued neutrosophic trapezoidal number (IVGSVTrN) was presented by Deli [32] with some operations and discussed their properties based on neutrosophic numbers. Hashim et al

[33], studied the vague set and interval neutrosophic set and established a new theory known as interval neutrosophic vague set (INVS), they also presented some operations for INVS with their properties and derived the properties by using numerical examples. Abdel basset et al. [34] applied TODIM and TOPSIS methods based on the best-worst method to increase the accuracy of evaluation under uncertainty according to the NSs. They also used the Plithogenic set theory to resolve the indeterminate information and evaluate the economic performance of manufacturing industries, they used the AHP method to find the weight vector of the financial ratios to achieve this goal after that they used the VIKOR and TOPSIS methods to utilize the companies ranking [35, 36]. Nabeeh et al. [37] utilized the integrating neutrosophic analytical hierarchy process (AHP) with the TOPSIS for personal selection. Nabeeh et al. [38] developed the AHP neutrosophic by merging the AHP and NS. Abdel-Basset et al. [39] merged the AHP, MCDM approach, and NS to handle the indefinite and irregularity in decision making. Abdel-Basset et al. [40] constructed the TOPSIS technique for type-2 neutrosophic numbers and utilized the presented approach for supplier selection. Abdel-Basset et al. [41] utilized the neutrosophic TOPSIS for the selection of medical instruments and many. Saqlain *et. al.* applied TOPSIS for the prediction of sports, and in MCDM problems [42-44].

The FS and IFS theories do not provide any information about the indeterminacy part of the object. Because the above work is considered to examine the environment of linear inequality between the degree of membership (MD) and the degree of non-membership (NMD) of the considered attributes. However, all existing studies only deal with the scenario by using MD and NMD of attributes. If any decision-maker considers the truthiness, falsity, and indeterminacy of any attribute of the alternatives, then clearly, we can see that it cannot be handled by the above-mentioned FS and IFS theories. To overcome the above limitations, Smarandache [4] proposed the NS to solve uncertain objects by considering the truthiness, falsity, and indeterminacy. In the following article, we explain some positive impacts of this research. The concentration of this study is to evaluate the best supplier for the production industry. This research is a very suitable illustration of Neutrosophic TOPSIS. A group of decision-makers chooses the best supplier for the production industry. The Neutrosophic TOPSIS method increases alternative performances based on the best and worst solutions. Classical TOPSIS uses clear techniques for language assessment, but due to the imprecision and ambiguity of language assessment, we propose neutrosophic TOPSIS. In this paper, we discuss the NSs and SVNSs with some operations. We presented the generalization of TOPSIS for the SVNSs and use the proposed method for supplier selection.

In Section 2, some basic definitions have been added, which will help us to design the structure of the current article. In section 3, we develop an integrated model to solve the MCDM problem under single-valued neutrosophic information. We also established the graphical and mathematical structure of the proposed TOPSIS approach. To ensure the validity of the developed methodology we presented a numerical illustration for supplier selection in the production industry in section 4.

2. Preliminaries

In this section, we remind some basic definitions such as NSs and SVNSs with some operations that will be used in the following sequel.

Neutrosophic Set (NS) [30]: Let X be a space of points and x be an arbitrary element of X. A neutrosophic set A in X is defined by a Truth-membership function $T_A(x)$, an Indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$ i.e.; $T_A(x), I_A(x), F_A(x): X \to]0^-, 1^+[$, and $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Single Valued Neutrosophic Sets [5]**:** Let E be a universe. An SVNS over E is an NS over E, but truthiness, indeterminacy, and falsity membership functions are defined

$T_A(x): X \rightarrow [0, 1]$, $I_A(x): X \rightarrow [0, 1]$, $F_A(x): X \rightarrow [0, 1]$, and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Multiplication of SVNS [11]: Let A = $\{\alpha_1, \alpha_2, \alpha_3\}$ and B = $\{\beta_1, \beta_2, \beta_3\}$ are two SVN numbers, then their multiplication is defined as follows $A \otimes B = (\alpha_1\beta_1, \alpha_2 + \beta_2 - \alpha_2\beta_2, \alpha_3 + \beta_3 - \alpha_3\beta_3)$.

3. Neutrosophic TOPSIS [11]

3. 1. Algorithm for Neutrosophic TOPSIS using SVNNs

To explain the procedure of Neutrosophic TOPSIS using SVNNs the following steps are followed. Let A = {A_1, A_2, A_3, …., A_m} be a set of alternatives and C = {C_1, C_2, C_3, …., C_n} be a set of evaluation criteria and *DM* be a set of "*l*" decision-makers as follows *DM* = {DM_1, DM_2, DM_3,…, DM_l}. In the form of linguistic variables, the importance of the evaluation criteria, DMs, and alternative ratings are given in Table 1.

Step 1: Computation of weights of the DMs

Let the SVN number for rating the k^{th} DM is denoted by

$D_k = (T_k^{dm}, I_k^{dm}, F_k^{dm})$

The weight of the k^{th} DM can be found by the following formula

$$\lambda_k = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}}{\sum_{k=1}^{l}\left(1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}\right)} \text{ ; where } \lambda_k \geq 0 \text{ and } \sum_{k=1}^{l}\lambda_k = 1 \quad (1)$$

Step 2: Computation of the Aggregated Neutrosophic Decision Matrix (ANDM)

The ANDM is given as follows

$$D = \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} = [r_{ij}]_{m\times n} \quad (2)$$

where r_{ij} can be defined as

$r_{ij} = (T_{ij}, I_{ij}, F_{ij}) = (T_{A_i}(x_j), I_{A_i}(x_j), F_{A_i}(x_j))$, where i = 1, 2, 3, …., m; j = 1, 2, 3, …., n

Therefore, ANDM written as follows

$$D = \begin{bmatrix} (T_{A_1}(x_1), I_{A_1}(x_1), F_{A_1}(x_1)) & (T_{A_1}(x_2), I_{A_1}(x_2), F_{A_1}(x_2)) & \cdots & (T_{A_1}(x_n), I_{A_1}(x_n), F_{A_1}(x_n)) \\ (T_{A_2}(x_1), I_{A_2}(x_1), F_{A_2}(x_1)) & (T_{A_2}(x_2), I_{A_2}(x_2), F_{A_2}(x_2)) & \cdots & (T_{A_2}(x_n), I_{A_2}(x_n), F_{A_2}(x_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{A_m}(x_1), I_{A_m}(x_1), F_{A_m}(x_1)) & (T_{A_m}(x_2), I_{A_m}(x_2), F_{A_m}(x_2)) & \cdots & (T_{A_m}(x_n), I_{A_m}(x_n), F_{A_m}(x_n)) \end{bmatrix}$$

rating for the i^{th} alternative w.r.t. the j^{th} criterion by the k^{th} *DM*

$$r_{ij}^{(k)} = (T_{ij}^{(k)}, I_{ij}^{(k)}, F_{ij}^{(k)})$$

For *DM* weights and alternative ratings r_{ij} can be calculated by using a single-valued neutrosophic weighted averaging operator (SVNWAO)

$$r_{ij} = \left[1 - \prod_{k=1}^{l}(1 - T_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_{ij}^{(k)})^{\lambda_k}\right] \quad (3)$$

Step 3: Computation of the weights for the criteria

Let an SVNN allocated to the criterion by X_j the k^{th} *DM* is denoted as

$$w_j^{(k)} = \left(T_j^{(k)}, I_j^{(k)}, F_j^{(k)}\right)$$

SVNWAO to compute the weights of the criteria is given as follows

$$w_j = \left[1 - \prod_{k=1}^{l}(1 - T_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_j^{(k)})^{\lambda_k}\right] \quad (4)$$

The aggregated weight for the criterion X_j is represented as

$$w_j = (T_j, I_j, F_j) \qquad j = 1, 2, 3, \ldots, n$$

$$W = [w_1, w_2, w_3, \ldots, w_n]^{Transpose}$$

Step 4: Computation of Aggregated Weighted Neutrosophic Decision Matrix (AWNDM)

The AWNDM is calculated as follows

$$R' = \begin{bmatrix} r'_{11} & r'_{12} & \cdots & r'_{1n} \\ r'_{21} & r'_{22} & \cdots & r'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r'_{m1} & r'_{m2} & \cdots & r'_{mn} \end{bmatrix} = [r'_{ij}]_{m\times n} \quad (5)$$

where $r'_{ij} = \left(T_{A_i.W}(x_j), I_{A_i.W}(x_j), F_{A_i.W}(x_j)\right)$ where $i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$.

Therefore, R' can be written as

$$R' = \begin{bmatrix} (T_{A_1.W}(x_1), I_{A_1.W}(x_1), F_{A_1.W}(x_1)) & (T_{A_1.W}(x_2), I_{A_1.W}(x_2), F_{A_1.W}(x_2)) & \cdots & (T_{A_1.W}(x_n), I_{A_1.W}(x_n), F_{A_1.W}(x_n)) \\ (T_{A_2.W}(x_1), I_{A_2.W}(x_1), F_{A_2.W}(x_1)) & (T_{A_2.W}(x_2), I_{A_2.W}(x_2), F_{A_2.W}(x_2)) & \cdots & (T_{A_2.W}(x_n), I_{A_2.W}(x_n), F_{A_2.W}(x_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{A_m.W}(x_1), I_{A_m.W}(x_1), F_{A_m.W}(x_1)) & (T_{A_m.W}(x_2), I_{A_m.W}(x_2), F_{A_m.W}(x_2)) & \cdots & (T_{A_m.W}(x_n), I_{A_m.W}(x_n), F_{A_m.W}(x_n)) \end{bmatrix}$$

To find $T_{A_i.W}(x_j)$, $I_{A_i.W}(x_j)$ and $F_{A_i.W}(x_j)$ we used

$$R \otimes W = \left\{\langle x, T_{A_i.W}(x)\rangle, \langle x, I_{A_i.W}(x)\rangle, \langle x, F_{A_i.W}(x)\rangle \,\middle|\, x \in X\right\} \quad (6)$$

The components of the product given as

$T_{A_i.W}(x) = T_{A_i}(x).\ T_j$

$I_{A_i.W}(x) = I_{A_i}(x) + I_j(x) - I_{A_i}(x) \times I_j(x)$

$F_{A_i.W}(x) = F_{A_i}(x) + F_j(x) - F_{A_i}(x) \times F_j(x)$

Step 5: Computation of Single Valued Neutrosophic Positive Ideal Solution (SVN-PIS) and Single Valued Neutrosophic Positive Ideal Solution (SVN-NIS)

Let J_1 be the benefit criteria and J_2 be the cost criteria. A^* be an SVN-PIS and A' be an SVN-NIS as follows

$A^* = (T_{A^*W}(x_j), I_{A^*W}(x_j), F_{A^*W}(x_j))$ and

$A' = (T_{A'W}(x_j), I_{A'W}(x_j), F_{A'W}(x_j))$

The components of SVN-PIS and SVN-NIS are following

$$T_{A^*W}(x_j) = \left(\left(\max_i T_{A_i.W}(x_j) \mid j \in j_1\right), \left(\min_i T_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

$$I_{A^*W}(x_j) = \left(\left(\min_i I_{A_i.W}(x_j) \mid j \in j_1\right), \left(\max_i I_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

$$F_{A^*W}(x_j) = \left(\left(\min_i F_{A_i.W}(x_j) \mid j \in j_1\right), \left(\max_i F_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

$$T_{A'W}(x_j) = \left(\left(\min_i T_{A_i.W}(x_j) \mid j \in j_1\right), \left(\max_i T_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

$$I_{A'W}(x_j) = \left(\left(\max_i I_{A_i.W}(x_j) \mid j \in j_1\right), \left(\min_i I_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

$$F_{A'W}(x_j) = \left(\left(\max_i F_{A_i.W}(x_j) \mid j \in j_1\right), \left(\min_i F_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

Step 6: Computation of Separation Measures

For the separation measures d^* and d', Normalized Euclidean Distance is used as given as

$$d_i^* = \left(\frac{1}{3n}\sum_{j=1}^{n}\left[\left(T_{A_i.W}(x_j) - T_{A^*W}(x_j)\right)^2 + \left(I_{A_i.W}(x_j) - I_{A^*W}(x_j)\right)^2 + \left(F_{A_i.W}(x_j) - F_{A^*W}(x_j)\right)^2\right]\right)^{0.5} \quad (7)$$

$$d_i' = \left(\frac{1}{3n}\sum_{j=1}^{n}\left[\left(T_{A_i.W}(x_j) - T_{A'W}(x_j)\right)^2 + \left(I_{A_i.W}(x_j) - I_{A'W}(x_j)\right)^2 + \left(F_{A_i.W}(x_j) - F_{A'W}(x_j)\right)^2\right]\right)^{0.5} \quad (8)$$

Step 7: Computation of Relative Closeness Coefficient (*RCC*)

The *RCC* of an alternative *Ai* w.r.t. the SVN-PIS A^* is computed as

$$RCCi = \frac{d_i'}{d_i' + d_i^*} \qquad \text{where } 0 \le RCCi \le 1 \quad (9)$$

Step 8: Ranking alternatives

After computation of *RCCi* for each alternative A_i, the rank of the alternatives presented in descending orders of *RCCi*.

The flow chart of the presented technique can be seen in Figure 1.

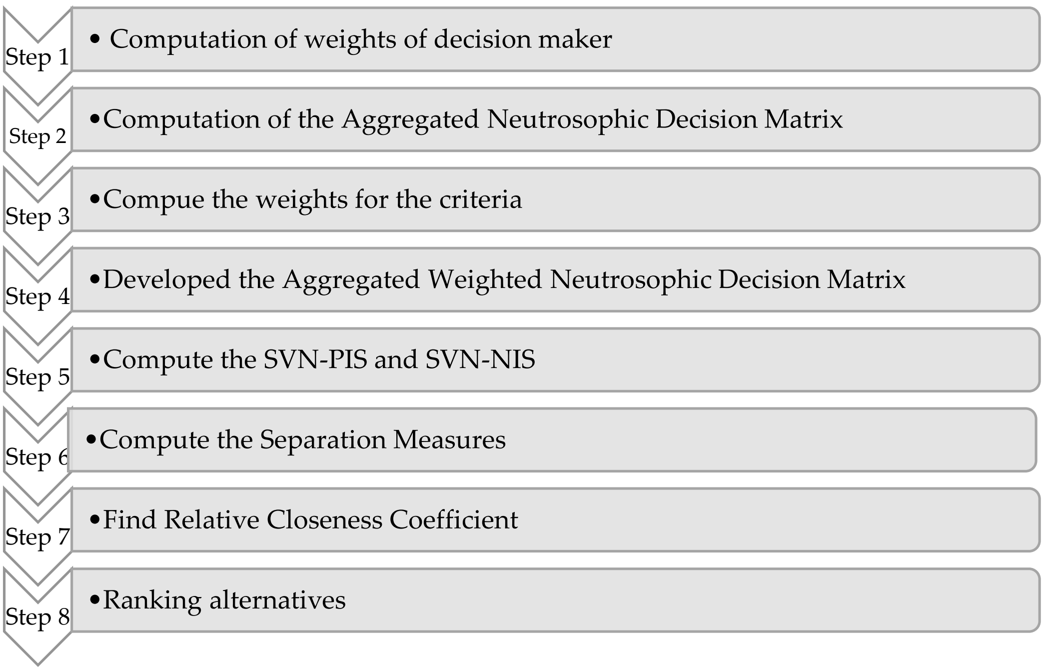

Figure 1: Flow chart of the presented approach

4. Application of Neutrosophic TOPSIS in decision making

A production industry wants to hire a supplier, for the selection of supplier managing director of the industry decides the criteria for supplier selection. The industry hires a team of decision-makers for the selection of the best supplier. Consider A = {A_i: *i* = 1, 2, 3, 4, 5} be a set of supplier and *DM* = {DM_1, DM_2, DM_3, DM_4} be a team of decision-makers (*l* = 4). The evaluation criteria (n = 5) for the selection of supplier given as follows,

$$C = \begin{cases} Benifit\ Criteria \\ Cost\ Criteria \end{cases} \quad j_1 = \begin{cases} X_1: & Delivery \\ X_2: & Quality \\ X_3: & Flexibility \\ X_4: & Service \end{cases} \quad j_2 = \{X_5 : Price$$

Calculations of the problem using the proposed SVN-TOPSIS for the importance of criteria and DMs SVN rating scale is given in the following Table

Table 1. Linguistic variables LV's for rating the importance of criteria and decision-makers

LVs	SVNNs
VI	(.90, .10, .10)
I	(.75, .25, .20)
M	(.50, .50, .50)
UI	(.35, .75, .80)
VUI	(.10, .90, .90)

Where VI, I, M, UI, VUI stand for very important, important, medium, unimportant, very unimportant respectively. The alternative ratings are given in the following table

Table 2. Alternative Ratings for Linguistic Variables

LVs	SVNNs
EG	(1.0, 0.0,0.0)
VVG	(.90, .10, .10)
VG	(.80, .15, .20)
G	(.70, .25, .30)
MG	(.60, .35, .40)
M	(.50, .50, .50)
MB	(.40, .65, .60)
B	(.30, .75, .70)
VB	(.20, .85, .80)
VVB	(.10, .90, .90)
EB	(0.0,1.0,1.0)

Where EG, VVG, VG, G, MG, M, MB, B, VB, VVB, EB are representing extremely good, very very good, very good, good, medium good, medium, medium bad, bad, very bad, very very bad, extremely bad respectively.

Step 1: Determine the weights of the DMs

By using Equation 1, weights for the *DMs* are calculated as follows:

$$\lambda_k = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}}{\sum_{k=1}^{l}\left(1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}\right)} \;;\; \lambda_k \geq 0 \text{ and } \sum_{k=1}^{l}\lambda_k = 1$$

$$\lambda_1 = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_1^{dm}(x)\right)^2+\left(I_1^{dm}(x)\right)^2+\left(F_1^{dm}(x)\right)^2\right\}\right]^{0.5}}{\sum_{k=1}^{l}\left(1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}\right)}$$

$$\lambda_1 = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_1^{dm}(x)\right)^2+\left(I_1^{dm}(x)\right)^2+\left(F_1^{dm}(x)\right)^2\right\}\right]^{0.5}}{\begin{array}{c}1-\left[\frac{1}{3}\left\{\left(1-T_1^{dm}(x)\right)^2+\left(I_1^{dm}(x)\right)^2+\left(F_1^{dm}(x)\right)^2\right\}\right]^{0.5}+1-\left[\frac{1}{3}\left\{\left(1-T_2^{dm}(x)\right)^2+\left(I_2^{dm}(x)\right)^2+\left(F_2^{dm}(x)\right)^2\right\}\right]^{0.5}+\\ 1-\left[\frac{1}{3}\left\{\left(1-T_3^{dm}(x)\right)^2+\left(I_3^{dm}(x)\right)^2+\left(F_3^{dm}(x)\right)^2\right\}\right]^{0.5}+1-\left[\frac{1}{3}\left\{\left(1-T_4^{dm}(x)\right)^2+\left(I_4^{dm}(x)\right)^2+\left(F_4^{dm}(x)\right)^2\right\}\right]^{0.5}\end{array}}$$

$$\lambda_1 = \frac{1-\left[\frac{1}{3}\{(1-0.9)^2+(0.10)^2+(0.10)^2\}\right]^{0.5}}{\begin{array}{c}1-\left[\frac{1}{3}\{(1-0.9)^2+(0.10)^2+(0.10)^2\}\right]^{0.5}+1-\left[\frac{1}{3}\{(1-0.75)^2+(0.25)^2+(0.20)^2\}\right]^{0.5}+\\ 1-\left[\frac{1}{3}\{(1-0.50)^2+(0.50)^2+(0.50)^2\}\right]^{0.5}+1-\left[\frac{1}{3}\{(1-0.35)^2+(0.75)^2+(0.80)^2\}\right]^{0.5}\end{array}}$$

$$\lambda_1 = \frac{0.9}{0.9+0.76548+0.5+0.26402}$$

$$\lambda_1 = \frac{0.9}{2.42950} = 0.37045$$

$\lambda_1 = 0.37045$

Similarly, we get the weights for the other decision-makers as follows

$$\lambda_2 = \frac{0.76548}{2.42950} = 0.31508$$

$\lambda_2 = 0.31508$

$$\lambda_3 = \frac{0.5}{2.42950} = 0.20580$$

$\lambda_3 = 0.20580$

$$\lambda_4 = \frac{0.26402}{2.42950} = 0.10867$$

$\lambda_4 = 0.10867$

The weights for *DMs* are given in the following Table

Table 3. Weights of Decision Makers

Criteria	Alternatives	Decision Makers			
		DM1	DM2	DM3	DM4
X_1	A_1	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)
		$r_{11}^{(1)} = (T_{11}^{(1)}, I_{11}^{(1)}, F_{11}^{(1)})$	$r_{11}^{(2)} = (T_{11}^{(2)}, I_{11}^{(2)}, F_{11}^{(2)})$	$r_{11}^{(3)} = (T_{11}^{(3)}, I_{11}^{(3)}, F_{11}^{(3)})$	$r_{11}^{(4)} = (T_{11}^{(4)}, I_{11}^{(4)}, F_{11}^{(4)}$
	A_2	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)
		$r_{21}^{(1)} = (T_{21}^{(1)}, I_{21}^{(1)}, F_{21}^{(1)})$	$r_{21}^{(2)} = (T_{21}^{(2)}, I_{21}^{(2)}, F_{21}^{(2)})$	$r_{21}^{(3)} = (T_{21}^{(3)}, I_{21}^{(3)}, F_{21}^{(3)})$	$r_{21}^{(4)} = (T_{21}^{(4)}, I_{21}^{(4)}, F_{21}^{(4)})$
	A_3	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)
		$r_{31}^{(1)} = (T_{31}^{(1)}, I_{31}^{(1)}, F_{31}^{(1)})$	$r_{31}^{(2)} = (T_{31}^{(2)}, I_{31}^{(2)}, F_{31}^{(2)})$	$r_{31}^{(3)} = (T_{31}^{(3)}, I_{31}^{(3)}, F_{31}^{(3)})$	$r_{31}^{(4)} = (T_{31}^{(4)}, I_{31}^{(4)}, F_{31}^{(4)})$
	A_4	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)
		$r_{41}^{(1)} = (T_{41}^{(1)}, I_{41}^{(1)}, F_{41}^{(1)})$	$r_{41}^{(2)} = (T_{41}^{(2)}, I_{41}^{(2)}, F_{41}^{(2)})$	$r_{41}^{(3)} = (T_{41}^{(3)}, I_{41}^{(3)}, F_{41}^{(3)})$	$r_{41}^{(4)} = (T_{41}^{(4)}, I_{41}^{(4)}, F_{41}^{(4)})$
	A_5	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)
		$r_{51}^{(1)} = (T_{51}^{(1)}, I_{51}^{(1)}, F_{51}^{(1)})$	$r_{51}^{(2)} = (T_{51}^{(2)}, I_{51}^{(2)}, F_{51}^{(2)})$	$r_{51}^{(3)} = (T_{51}^{(3)}, I_{51}^{(3)}, F_{51}^{(3)})$	$r_{51}^{(4)} = (T_{51}^{(4)}, I_{51}^{(4)}, F_{51}^{(4)})$
X_2	A_1	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)
		$r_{12}^{(1)} = (T_{12}^{(1)}, I_{12}^{(1)}, F_{12}^{(1)})$	$r_{12}^{(2)} = (T_{12}^{(2)}, I_{12}^{(2)}, F_{12}^{(2)})$	$r_{12}^{(3)} = (T_{12}^{(3)}, I_{12}^{(3)}, F_{12}^{(3)})$	$r_{12}^{(4)} = (T_{12}^{(4)}, I_{12}^{(4)}, F_{12}^{(4)})$
	A_2	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)
		$r_{22}^{(1)} = (T_{22}^{(1)}, I_{22}^{(1)}, F_{22}^{(1)})$	$r_{22}^{(2)} = (T_{22}^{(2)}, I_{22}^{(2)}, F_{22}^{(2)})$	$r_{22}^{(3)} = (T_{22}^{(3)}, I_{22}^{(3)}, F_{22}^{(3)})$	$r_{22}^{(4)} = (T_{22}^{(4)}, I_{22}^{(4)}, F_{22}^{(4)})$
	A_3	M (0.50,0.50,0.50)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)
		$r_{32}^{(1)} = (T_{32}^{(1)}, I_{32}^{(1)}, F_{32}^{(1)})$	$r_{32}^{(2)} = (T_{32}^{(2)}, I_{32}^{(2)}, F_{32}^{(2)})$	$r_{32}^{(3)} = (T_{32}^{(3)}, I_{32}^{(3)}, F_{32}^{(3)})$	$r_{32}^{(4)} = (T_{32}^{(4)}, I_{32}^{(4)}, F_{32}^{(4)})$
	A_4	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)
		$r_{42}^{(1)} = (T_{42}^{(1)}, I_{42}^{(1)}, F_{42}^{(1)})$	$r_{42}^{(2)} = (T_{42}^{(2)}, I_{42}^{(2)}, F_{42}^{(2)})$	$r_{42}^{(3)} = (T_{42}^{(3)}, I_{42}^{(3)}, F_{42}^{(3)})$	$r_{42}^{(4)} = (T_{42}^{(4)}, I_{42}^{(4)}, F_{42}^{(4)})$
	A_5	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)
		$r_{52}^{(1)} = (T_{52}^{(1)}, I_{52}^{(1)}, F_{52}^{(1)})$	$r_{52}^{(2)} = (T_{52}^{(2)}, I_{52}^{(2)}, F_{52}^{(2)})$	$r_{52}^{(3)} = (T_{52}^{(3)}, I_{52}^{(3)}, F_{52}^{(3)})$	$r_{52}^{(4)} = (T_{52}^{(4)}, I_{52}^{(4)}, F_{52}^{(4)})$
X_3	A_1	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)

		$r_{13}^{(1)} = (T_{13}^{(1)}, I_{13}^{(1)}, F_{13}^{(1)})$	$r_{13}^{(2)} = (T_{13}^{(2)}, I_{13}^{(2)}, F_{13}^{(2)})$	$r_{13}^{(3)} = (T_{13}^{(3)}, I_{13}^{(3)}, F_{13}^{(3)})$	$r_{13}^{(4)} = (T_{13}^{(4)}, I_{13}^{(4)}, F_{13}^{(4)})$
	*A*2	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)
		$r_{23}^{(1)} = (T_{23}^{(1)}, I_{23}^{(1)}, F_{23}^{(1)})$	$r_{23}^{(2)} = (T_{23}^{(2)}, I_{23}^{(2)}, F_{23}^{(2)})$	$r_{23}^{(3)} = (T_{23}^{(3)}, I_{23}^{(3)}, F_{23}^{(3)})$	$r_{23}^{(4)} = (T_{23}^{(4)}, I_{23}^{(4)}, F_{23}^{(4)})$
	*A*3	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)
		$r_{33}^{(1)} = (T_{33}^{(1)}, I_{33}^{(1)}, F_{33}^{(1)})$	$r_{33}^{(2)} = (T_{33}^{(2)}, I_{33}^{(2)}, F_{33}^{(2)})$	$r_{33}^{(3)} = (T_{33}^{(3)}, I_{33}^{(3)}, F_{33}^{(3)})$	$r_{33}^{(4)} = (T_{33}^{(4)}, I_{33}^{(4)}, F_{33}^{(4)})$
	*A*4	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)
		$r_{43}^{(1)} = (T_{43}^{(1)}, I_{43}^{(1)}, F_{43}^{(1)})$	$r_{43}^{(2)} = (T_{43}^{(2)}, I_{43}^{(2)}, F_{43}^{(2)})$	$r_{43}^{(3)} = (T_{43}^{(3)}, I_{43}^{(3)}, F_{43}^{(3)})$	$r_{43}^{(4)} = (T_{43}^{(4)}, I_{43}^{(4)}, F_{43}^{(4)})$
	*A*5	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)
		$r_{53}^{(1)} = (T_{53}^{(1)}, I_{53}^{(1)}, F_{53}^{(1)})$	$r_{53}^{(2)} = (T_{53}^{(2)}, I_{53}^{(2)}, F_{53}^{(2)})$	$r_{53}^{(3)} = (T_{53}^{(3)}, I_{53}^{(3)}, F_{53}^{(3)})$	$r_{53}^{(4)} = (T_{53}^{(4)}, I_{53}^{(4)}, F_{53}^{(4)})$
X4	*A*1	G (0.70,0.25,0.30)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)
		$r_{14}^{(1)} = (T_{14}^{(1)}, I_{14}^{(1)}, F_{14}^{(1)})$	$r_{14}^{(2)} = (T_{14}^{(2)}, I_{14}^{(2)}, F_{14}^{(2)})$	$r_{14}^{(3)} = (T_{14}^{(3)}, I_{14}^{(3)}, F_{14}^{(3)})$	$r_{14}^{(4)} = (T_{14}^{(4)}, I_{14}^{(4)}, F_{14}^{(4)})$
	*A*2	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)
		$r_{24}^{(1)} = (T_{24}^{(1)}, I_{24}^{(1)}, F_{24}^{(1)})$	$r_{24}^{(2)} = (T_{24}^{(2)}, I_{24}^{(2)}, F_{24}^{(2)})$	$r_{24}^{(3)} = (T_{24}^{(3)}, I_{24}^{(3)}, F_{24}^{(3)})$	$r_{24}^{(4)} = (T_{24}^{(4)}, I_{24}^{(4)}, F_{24}^{(4)})$
	*A*3	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)
		$r_{34}^{(1)} = (T_{34}^{(1)}, I_{34}^{(1)}, F_{34}^{(1)})$	$r_{34}^{(2)} = (T_{34}^{(2)}, I_{34}^{(2)}, F_{34}^{(2)})$	$r_{34}^{(3)} = (T_{34}^{(3)}, I_{34}^{(3)}, F_{34}^{(3)})$	$r_{34}^{(4)} = (T_{34}^{(4)}, I_{34}^{(4)}, F_{34}^{(4)})$
	*A*4	M (0.50,0.50,0.50)	MB (0.40,0.65,0.60)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)
		$r_{44}^{(1)} = (T_{44}^{(1)}, I_{44}^{(1)}, F_{44}^{(1)})$	$r_{44}^{(2)} = (T_{44}^{(2)}, I_{44}^{(2)}, F_{44}^{(2)})$	$r_{44}^{(3)} = (T_{44}^{(3)}, I_{44}^{(3)}, F_{44}^{(3)})$	$r_{44}^{(4)} = (T_{44}^{(4)}, I_{44}^{(4)}, F_{44}^{(4)})$
	*A*5	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)
		$r_{54}^{(1)} = (T_{54}^{(1)}, I_{54}^{(1)}, F_{54}^{(1)})$	$r_{54}^{(2)} = (T_{54}^{(2)}, I_{54}^{(2)}, F_{54}^{(2)})$	$r_{54}^{(3)} = (T_{54}^{(3)}, I_{54}^{(3)}, F_{54}^{(3)})$	$r_{54}^{(4)} = (T_{54}^{(4)}, I_{54}^{(4)}, F_{54}^{(4)})$
X5	*A*1	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)
		$r_{15}^{(1)} = (T_{15}^{(1)}, I_{15}^{(1)}, F_{15}^{(1)})$	$r_{15}^{(2)} = (T_{15}^{(2)}, I_{15}^{(2)}, F_{15}^{(2)})$	$r_{15}^{(3)} = (T_{15}^{(3)}, I_{15}^{(3)}, F_{15}^{(3)})$	$r_{15}^{(4)} = (T_{15}^{(4)}, I_{15}^{(4)}, F_{15}^{(4)})$
	*A*2	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)
		$r_{25}^{(1)} = (T_{25}^{(1)}, I_{25}^{(1)}, F_{25}^{(1)})$	$r_{25}^{(2)} = (T_{25}^{(2)}, I_{25}^{(2)}, F_{25}^{(2)})$	$r_{25}^{(3)} = (T_{25}^{(3)}, I_{25}^{(3)}, F_{25}^{(3)})$	$r_{25}^{(4)} = (T_{25}^{(4)}, I_{25}^{(4)}, F_{25}^{(4)})$
	*A*3	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)
		$r_{35}^{(1)} = (T_{35}^{(1)}, I_{35}^{(1)}, F_{35}^{(1)})$	$r_{35}^{(2)} = (T_{35}^{(2)}, I_{35}^{(2)}, F_{35}^{(2)})$	$r_{35}^{(3)} = (T_{35}^{(3)}, I_{35}^{(3)}, F_{35}^{(3)})$	$r_{35}^{(4)} = (T_{35}^{(4)}, I_{35}^{(4)}, F_{35}^{(4)})$
	*A*4	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)
		$r_{45}^{(1)} = (T_{45}^{(1)}, I_{45}^{(1)}, F_{45}^{(1)})$	$r_{45}^{(2)} = (T_{45}^{(2)}, I_{45}^{(2)}, F_{45}^{(2)})$	$r_{45}^{(3)} = (T_{45}^{(3)}, I_{45}^{(3)}, F_{45}^{(3)})$	$r_{45}^{(4)} = (T_{45}^{(4)}, I_{45}^{(4)}, F_{45}^{(4)})$
	*A*5	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)
		$r_{55}^{(1)} = (T_{55}^{(1)}, I_{55}^{(1)}, F_{55}^{(1)})$	$r_{55}^{(2)} = (T_{55}^{(2)}, I_{55}^{(2)}, F_{55}^{(2)})$	$r_{55}^{(3)} = (T_{55}^{(3)}, I_{55}^{(3)}, F_{55}^{(3)})$	$r_{55}^{(4)} = (T_{55}^{(4)}, I_{55}^{(4)}, F_{55}^{(4)})$

Table 4. Importance and Weights of Decision-Makers

	DM1	*DM2*	*DM3*	*DM4*
Linguistic	VI(0.90,0.10,0.10)	I (0.75,0.25,0.20)	M (0.50,0.50,0.50)	UI (0.35,0.75,0.80)
Variables	$(T_1^{dm}, I_1^{dm}, F_1^{dm})$	$(T_2^{dm}, I_2^{dm}, F_2^{dm})$	$(T_3^{dm}, I_3^{dm}, F_3^{dm})$	$(T_4^{dm}, I_4^{dm}, F_4^{dm})$
Weights	λ_{DM_1}= 0.37045	λ_{DM_2}= 0.31508	λ_{DM_3}= 0.20580	λ_{DM_4}= 0.10867

Step 2: Computation of Aggregated Single Valued Neutrosophic Decision Matrix (ASVNDM)

To find the ASVNDM not only the weights of the DMs, but the alternative ratings are also required. The alternative ratings, according to the *DMs* given in the following table.

Now by using Equation 3, alternative ratings $r_{ij}^{(k)}$ and the DM weights λ_k we get

$r_{ij}= \lambda_1 r_{ij}^{(1)} \oplus \lambda_2 r_{ij}^{(2)} \oplus \lambda_3 r_{ij}^{(3)} \oplus \cdots \oplus \lambda_l r_{ij}^{(l)}$

$r_{ij} = \left(1-\prod_{k=1}^{l}(1-T_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_{ij}^{(k)})^{\lambda_k}\right)$

where i = 1, 2, 3, 4, 5; j = 1, 2, 3, 4, 5 and (l = 4).

For $i = j = 1$ and $l = 4$

$r_{11}= \lambda_1 r_{11}^{(1)} \oplus \lambda_2 r_{11}^{(2)} \oplus \lambda_3 r_{11}^{(3)} \oplus \cdots \oplus \lambda_l r_{11}^{(l)}$

$r_{11} = \left(1-\prod_{k=1}^{4}(1-T_{11}^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(I_{11}^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(F_{11}^{(k)})^{\lambda_k}\right)$

r_{11} = (1- $(1-T_{11}^{(1)})^{\lambda_1}(1-T_{11}^{(2)})^{\lambda_2}(1-T_{11}^{(3)})^{\lambda_3}(1-T_{11}^{(4)})^{\lambda_4}$, $(I_{11}^{(1)})^{\lambda_1}(I_{11}^{(2)})^{\lambda_2}(I_{11}^{(3)})^{\lambda_3}(I_{11}^{(4)})^{\lambda_4}$, $(F_{11}^{(1)})^{\lambda_1}(F_{11}^{(2)})^{\lambda_2}(F_{11}^{(3)})^{\lambda_3}(F_{11}^{(4)})^{\lambda_4}$)

r_{11} = (1-($(1-0.8)^{0.37045}(1-0.6)^{0.31508}(1-0.8)^{0.20580}(1-0.7)^{0.10867}$),

($(0.15)^{0.37045}(0.35)^{0.31508}(0.15)^{0.20580}(0.25)^{0.10867}$) ,

($(0.20)^{0.37045}(0.40)^{0.31508}(0.20)^{0.20580}(0.30)^{0.10867}$))

r_{11} = (0.740, 0.207, 0.260)

Similarly, we can find other values

r_{21} = (0.711, 0.237, 0.289)

r_{31} = (0.593, 0.373, 0.407)

r_{41} = (0.661, 0.288, 0.339)

r_{51} = (0.706, 0.241, 0.294)

r_{12} = (0.682, 0.268, 0.318)

r_{22} = (0.676, 0.275, 0.324)

r_{32} = (0.681, 0.275, 0.324)

r_{42} = (0.619, 0.342, 0.381)

r_{52} = (0.695, 0.253, 0.305)

r_{13} = (0.505, 0.392, 0.429)

r_{23} = (0.773, 0.176, 0.227)

r_{33} = (0.603, 0.359, 0.397)

r_{43} = (0.661, 0.288, 0.339)

r_{53} = (0.693, 0.255, 0.307)

r_{14} = (0.605, 0.359, 0.395)

r_{24} = (0.748, 0.203, 0.252)

r_{34} = (0.600, 0.350, 0.400)

r_{44} = (0.542, 0.443, 0.458)

r_{54} = (0.693, 0.339, 0.307)

r_{15} = (0.614, 0.349, 0.386)

r_{25} = (0.697, 0.257, 0.303)

r_{35} = (0.656, 0.299, 0.344)

r_{45} = (0.548, 0.431, 0.452)

r_{55} = (0.768, 0.181, 0.232)

Table 5. Aggregated Single Valued Neutrosophic Decision Matrix D = $[r_{ij}]_{5\times4}$

	X_1	X_2	X_3	X_4	X_5
A_1	r_{11} = (0.740, 0.207, 0.260)	r_{12} = (0.682, 0.268, 0.318)	r_{13} = (0.505, 0.392, 0.429)	r_{14} = (0.605, 0.359, 0.395)	r_{15} = (0.614, 0.349, 0.386)
A_2	r_{21} = (0.711, 0.237, 0.289)	r_{22} = (0.676, 0.275, 0.324)	r_{23} = (0.773, 0.176, 0.227)	r_{24} = (0.748, 0.203, 0.252)	r_{25} = (0.697, 0.257, 0.303)

A_3	r_{31} = (0.593, 0.373, 0.407)	r_{32} = (0.681, 0.275, 0.324)	r_{33} = (0.603, 0.359, 0.397)	r_{34} = (0.600, 0.350, 0.400)	r_{35} = (0.656, 0.299, 0.344)
A_4	r_{41} = (0.661, 0.288, 0.339)	r_{42} = (0.619, 0.342, 0.381)	r_{43} = (0.661, 0.288, 0.339)	r_{43} = (0.661, 0.288, 0.339)	r_{45} = (0.548, 0.431, 0.452)
A_5	r_{51} = (0.706, 0.241, 0.294)	r_{52} = (0.695, 0.253, 0.305)	r_{53} = (0.693, 0.255, 0.307)	r_{54}= (0.693, 0.339, 0.307)	r_{55} = (0.768, 0.181, 0.232)

Step 3: Computation of the weights of the criteria

The individual weights given by each DM is given in Table 6.

Table 6. Weights of alternatives determined by the DMs $w_j^{(k)} = (T_j^{(k)}, I_j^{(k)}, F_j^{(k)})$

Criteria	*DM₁*	*DM₂*	*DM₃*	*DM₄*
X_1	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)	I (0.75,0.25,0.20)
(DELIVERY)	$w_1^{(1)} = (T_1^{(1)}, I_1^{(1)}, F_1^{(1)})$	$w_1^{(2)} = (T_1^{(2)}, I_1^{(2)}, F_1^{(2)})$	$w_1^{(3)} = (T_1^{(3)}, I_1^{(3)}, F_1^{(3)})$	$w_1^{(4)} = (T_1^{(4)}, I_1^{(4)}, F_1^{(4)})$
X_2	I (0.75,0.25,0.20)	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)	I (0.75,0.25,0.20)
(QUALITY)	$w_2^{(1)} = (T_2^{(1)}, I_2^{(1)}, F_2^{(1)})$	$w_2^{(2)} = (T_2^{(2)}, I_2^{(2)}, F_2^{(2)})$	$w_2^{(3)} = (T_2^{(3)}, I_2^{(3)}, F_2^{(3)})$	$w_2^{(4)} = (T_2^{(4)}, I_2^{(4)}, F_2^{(4)})$
X_3	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)	I (0.75,0.25,0.20)	VI (0.90,0.10,0.10)
(FLEXIBILITY)	$w_3^{(1)} = (T_3^{(1)}, I_3^{(1)}, F_3^{(1)})$	$w_3^{(2)} = (T_3^{(2)}, I_3^{(2)}, F_3^{(2)})$	$w_3^{(3)} = (T_3^{(3)}, I_3^{(3)}, F_3^{(3)})$	$w_3^{(4)} = (T_3^{(4)}, I_3^{(4)}, F_3^{(4)})$
X_4	I (0.75,0.25,0.20)	I (0.75,0.25,0.20)	M (0.50,0.50,0.50)	UI (0.35,0.75,0.80)
(SERVICE)	$w_4^{(1)} = (T_4^{(1)}, I_4^{(1)}, F_4^{(1)})$	$w_4^{(2)} = (T_4^{(2)}, I_4^{(2)}, F_4^{(2)})$	$w_4^{(3)} = (T_4^{(3)}, I_4^{(3)}, F_4^{(3)})$	$w_4^{(4)} = (T_4^{(4)}, I_4^{(4)}, F_4^{(4)})$
X_5	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)
(PRICE)	$w_5^{(1)} = (T_5^{(1)}, I_5^{(1)}, F_5^{(1)})$	$w_5^{(2)} = (T_5^{(2)}, I_5^{(2)}, F_5^{(2)})$	$w_5^{(3)} = (T_5^{(3)}, I_5^{(3)}, F_5^{(3)})$	$w_5^{(4)} = (T_5^{(4)}, I_5^{(4)}, F_5^{(4)})$

By using the values from Table 6, the aggregated criteria weights are calculated as follows

$w_j = (T_j, I_j, F_j) = \lambda_1 w_j^{(1)} \oplus \lambda_2 w_j^{(2)} \oplus \lambda_3 w_j^{(3)} \oplus \cdots \oplus \lambda_l w_j^{(l)}$

$w_j = (1-\prod_{k=1}^{l}(1-T_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_j^{(k)})^{\lambda_k})$ where j = 1, 2, 3, 4, 5 and (l = 4).

For j = 1 and l = 4

$w_1 = \lambda_1 w_1^{(1)} \oplus \lambda_2 w_1^{(2)} \oplus \lambda_3 w_1^{(3)} \oplus \lambda_4 w_1^{(4)}$

$w_1 = (1-\prod_{k=1}^{4}(1-T_1^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(I_1^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(F_1^{(k)})^{\lambda_k})$

$w_1 = (1- (1-T_1^{(1)})^{\lambda_1}(1-T_1^{(2)})^{\lambda_2}(1-T_1^{(3)})^{\lambda_3}(1-T_1^{(4)})^{\lambda_4},\ (I_1^{(1)})^{\lambda_1}(I_1^{(2)})^{\lambda_2}(I_1^{(3)})^{\lambda_3}(I_1^{(4)})^{\lambda_4},$
$(F_1^{(1)})^{\lambda_1}(F_1^{(2)})^{\lambda_2}(F_1^{(3)})^{\lambda_3}(F_1^{(4)})^{\lambda_4})$

$w_1 = (1-((1-0.9)^{0.37045}(1-0.9)^{0.31508}(1-0.9)^{0.20580}(1-0.75)^{0.10867}),$
$((0.10)^{0.37045}(0.10)^{0.31508}(0.10)^{0.20580}(0.25)^{0.10867})$,
$((0.10)^{0.37045}(0.10)^{0.31508}(0.10)^{0.20580}(0.20)^{0.10867}))$

r_{11} = (0.740, 0.207, 0.260)

$w_1 = (T_1, I_1, F_1)$ = (0.890, 0.110, 0.108)

Similarly, we can get other values

Therefore

$$W_{\{X_1,X_2,X_3,X_4\}} = \begin{bmatrix}(0.890, 0.110, 0.108)\\(0.641, 0.359, 0.322)\\(0.879, 0.121, 0.115)\\(0.680, 0.325, 0.281)\\(0.699, 0.301, 0.301)\end{bmatrix}^T$$

Step 4: Construction of Aggregated Weighted Single Valued Neutrosophic Decision Matrix (AWSVNDM)

After finding the weights of the criteria and the alternative ratings, the aggregated weighted single-valued neutrosophic ratings are calculated by using Equation 4 as follows:

$r'_{ij} = (T'_{ij},\ I'_{ij},\ rF'_{ij}) = (T_{A_i}(x).T_j,\ I_{A_i}(x) +\ I_j\ -\ I_{A_i}(x).I_j,\ F_{A_i}(x) +\ F_j\ -\ F_{A_i}(x).F_j)$

By using the above equation, we can get an aggregated weighted single-valued neutrosophic decision matrix.

Table 7. Aggregated Weighted Single Valued Neutrosophic Decision Matrix $R' = [r'_{ij}]_{5\times5}$

	X_1	X_2	X_3	X_4	X_5
A_1	r'_{11} = (0.659,0.294,0.340)	r'_{12} = (0.437,0.531,0.538)	r'_{13} = (0.444,0.466,0.495)	r'_{14} = (0.411,0.567,0.565)	r'_{15} = (0.429,0.545,0.571)
A_2	r'_{21} = (0.633,0.321,0.366)	r'_{22} = (0.433,0.535,0.542)	r'_{23} = (0.679,0.276,0.316)	r'_{24} = (0.509,0.462,0.462)	r'_{25} = (0.487,0.481,0.513)
A_3	r'_{31} = (0.528,0.442,0.471)	r'_{32} = (0.437,0.535,0.542)	r'_{33} = (0.530,0.437,0.466)	r'_{34} = (0.408,0.561,0.569)	r'_{35} = (0.459,0.510,0.541)
A_4	r'_{41} = (0.588,0.366,0.410)	r'_{42} = (0.397,0.578,0.580)	r'_{43} = (0.581,0.374,0.415)	r'_{44} = (0.037,0.624,0.610)	r'_{45} = (0.383,0.602,0.617)
A_5	r'_{51} = (0.628,0.324,0.3700	r'_{52} = (0.445,0.521,0.529)	r'_{53} = (0.609,0.345,0.387)	r'_{54} = (0.471,0.554,0.502)	r'_{55} = (0.537,0.428,0.463)

Step 5: Computation of SVN-PIS and SVN-NIS

Since Delivery, Quality, Flexibility, and Services are benefit criteria that is why they are in the set

$$J_1 = \{X_1, X_2, X_3, X_4\}$$

whereas Price being the cost criteria, so it is in the set $J_2 = \{X_2\}$ SVN-PIS and SVN-NIS are calculated as,

Table 8. SVN-PIS and SVN-NIS

SVN-PIS	SVN-NIS
$\boldsymbol{T_1^+}$ = max {0.659,0.633,0.528,0.588,0.628} = 0.659	T_1^- = min {0.659,0.633,0.528,0.588,0.628} = 0.528
$\boldsymbol{I_1^+}$ = min {0.294,0.321,0.442,0.366,0.324} = 0.294	I_1^- = max {0.294,0.321,0.442,0.366,0.324} = 0.442
$\boldsymbol{F_1^+}$ = min {0.340,0.366,0.471,0.410,0.370} = 0.340	F_1^- = max {0.340,0.366,0.471,0.410,0.370} = 0.471
$\boldsymbol{T_2^+}$ = max {0.437,0.433,0.437,0.397,0.445} = 0.445	T_2^- = min {0.437,0.433,0.437,0.397,0.445} = 0.397
$\boldsymbol{I_2^+}$ = min {0.531,0.535,0.535,0.578,0.521} = 0.521	I_2^- = max {0.531,0.535,0.535,0.578,0.521} = 0.578
$\boldsymbol{F_2^+}$ = min {0.538,0.542,0.542,0.580,0.529} = 0.529	F_2^- = max {0.538,0.542,0.542,0.580,0.529} = 0.580
$\boldsymbol{T_3^+}$= max {0.444,0.679,0.530,0.581,0.609} = 0.679	T_3^- = min {0.444,0.679,0.530,0.581,0.609} = 0.444
$\boldsymbol{I_3^+}$ = min {0.466,0.276,0.437,0.374,0.345} = 0.276	I_3^- = max {0.466,0.276,0.437,0.374,0.345} = 0.466
$\boldsymbol{F_3^+}$ = min {0.495,0.316,0.466,0.415,0.387} = 0.316	F_3^- = max {0.495,0.316,0.466,0.415,0.387} = 0.495
$\boldsymbol{T_4^+}$ = max {0.411,0.509,0.408,0.037,0.471} = 0.509	T_4^- = min {0.411,0.509,0.408,0.037,0.471} = 0.037
$\boldsymbol{I_4^+}$ = min {0.567,0.462,0.561,0.624,0.554} = 0.462	I_4^- = max {0.567,0.462,0.561,0.624,0.554} = 0.624
$\boldsymbol{F_4^+}$ = min {0.565,0.462,0.569,0.610,0.502} = 0.462	F_4^- = max {0.565,0.462,0.569,0.610,0.502} = 0.610
$\boldsymbol{T_5^+}$ = min {0.429,0.487,0.459,0.383,0.537} = 0.383	T_5^- = max {0.429,0.487,0.459,0.383,0.537} = 0.537
$\boldsymbol{I_5^+}$ = max {0.545,0.481,0.510,0.602,0.428} = 0.602	I_5^- = min {0.545,0.481,0.510,0.602,0.428} = 0.428
$\boldsymbol{F_5^+}$ = max {0.571,0.513,0.541,0.617,0.463} = 0.617	F_5^- = min {0.571,0.513,0.541,0.617,0.463} = 0.463

$$A^{+} = \begin{Bmatrix} (0.659, 0.294, 0.340), \\ (0.445, 0.521, 0.529), \\ (0.679, 0.276, 0.316), \\ (0.509, 0.462, 0.462), \\ (0.383, 0.602, 0.617) \end{Bmatrix} \qquad A^{-} = \begin{Bmatrix} (0.528, 0.442, 0.471), \\ (0.397, 0.578, 0.580), \\ (0.444, 0.466, 0.495), \\ (0.037, 0.624, 0.610), \\ (0.537, 0.428, 0.463) \end{Bmatrix}$$

Step 6: Computation of Separation Measures

Normalized Euclidean Distance Measure is used to find the negative and positive separation measures $\boldsymbol{d}^{+}$ and $\boldsymbol{d}^{-}$respectively by using Equation 7, 8. Now for the SVN-PIS, we use

$$d_i^{+} = \left(\frac{1}{3n}\sum_{j=1}^{n}\left[\left(T_{A_i.W}(x_j) - T_{A^*W}(x_j)\right)^2 + \left(I_{A_i.W}(x_j) - I_{A^*W}(x_j)\right)^2 + \left(F_{A_i.W}(x_j) - F_{A^*W}(x_j)\right)^2\right]\right)^{0.5}$$

For i = 1and n = 5

$$d_1^{+} = \left(\frac{1}{3(5)}\sum_{j=1}^{5}\left[\left(T_{A_1.W}(x_j) - T_{A^*W}(x_j)\right)^2 + \left(I_{A_1.W}(x_j) - I_{A^*W}(x_j)\right)^2 + \left(F_{A_1.W}(x_j) - F_{A^*W}(x_j)\right)^2\right]\right)^{0.5}$$

$$d_1^{+} = \left(\frac{1}{15}\begin{bmatrix} \left(T_{A_1.W}(X_1) - T_{A^*W}(X_1)\right)^2 + \left(I_{A_1.W}(X_1) - I_{A^*W}(X_1)\right)^2 + \left(F_{A_1.W}(X_1) - F_{A^*W}(X_1)\right)^2 + \\ \left(T_{A_1.W}(X_2) - T_{A^*W}(X_2)\right)^2 + \left(I_{A_1.W}(X_2) - I_{A^*W}(X_2)\right)^2 + \left(F_{A_1.W}(X_2) - F_{A^*W}(X_2)\right)^2 + \\ \left(T_{A_1.W}(X_3) - T_{A^*W}(X_3)\right)^2 + \left(I_{A_1.W}(X_3) - I_{A^*W}(X_3)\right)^2 + \left(F_{A_1.W}(X_3) - F_{A^*W}(X_3)\right)^2 + \\ \left(T_{A_1.W}(X_4) - T_{A^*W}(X_4)\right)^2 + \left(I_{A_1.W}(X_4) - I_{A^*W}(X_4)\right)^2 + \left(F_{A_1.W}(X_4) - F_{A^*W}(X_4)\right)^2 + \\ \left(T_{A_1.W}(X_5) - T_{A^*W}(X_5)\right)^2 + \left(I_{A_1.W}(X_5) - I_{A^*W}(X_5)\right)^2 + \left(F_{A_1.W}(X_5) - F_{A^*W}(X_5)\right)^2 \end{bmatrix}\right)^{0.5}$$

$$d_1^{+} = \left(\frac{1}{15}\begin{bmatrix} (0659 - 0.659)^2 + (0.294 - 0.294)^2 + (0.340 - 0.340)^2 + \\ (0.437 - 0.445)^2 + (0.531 - 0.521)^2 + (0.538 - 0.529)^2 + \\ (0.444 - 0.679)^2 + (0.466 - 0.276)^2 + (0.495 - 0.316)^2 + \\ (0.411 - 0.509)^2 + (0.567 - 0.462)^2 + (0.565 - 0.462)^2 + \\ (0.429 - 0.383)^2 + (0.545 - 0.602)^2 + (0.571 - 0.617)^2 \end{bmatrix}\right)^{0.5}$$

$$d_1^{+} = \left[\frac{1}{15}\,(0.000245 + 0.123366 + 0.031238 + 0.007481)\right]^{0.5}$$

d_1^{+}= 0.1040

Similarly, we can find other separation measures.

Step 7: Computation of Relative Closeness Coefficient (RCC)

The RCC is calculated by using Equation 9.

RCCi = $\frac{d_i'}{d_i' + d_i^*}$; i = 1, 2, 3, 4, 5

RCC1 = $\frac{d_1'}{d_1' + d_1^*} = \frac{0.127532}{0.127532+0.104029}$ = 0.551

RCC2 = 0.896

RCC3 = 0.505

RCC4 = 0.363

RCC5 = 0.757

The separation measure and the value of relative closeness coefficient (RCC) expressed in the following Figure 2.

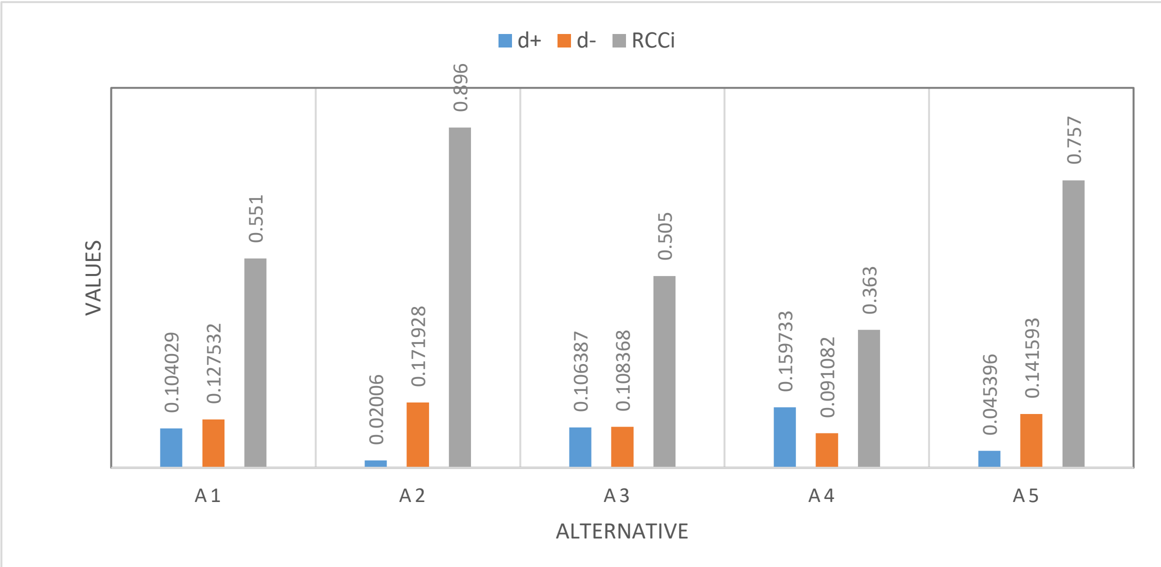

Figure 2. Separation measure and the RCC for each Alternative

Step 8: Ranking alternatives

From the above figure, we can see the RCC are ranked as follows

$RCC_2 > RCC_5 > RCC_1 > RCC_3 > RCC_4 \Rightarrow A_2 > A_5 > A_1 > A_3 > A_4$

By using the presented technique, we choose the best supplier for the production industry and observe that A_2 is the best alternative.

5. Conclusion

In this paper, we studied the neutrosophic set and SVNSs with some basic operations and developed the generalized neutrosophic TOPSIS by using single-valued neutrosophic numbers. By using crisp data, it is more difficult to solve decision-making problems in uncertain environments. Single valued neutrosophic sets can handle these limitations competently and provide the appropriate choice to decision-makers. We also developed the integrated model for neutrosophic TOPSIS. The closeness coefficient has been defined to compute the ranking of the alternatives by using an established approach under-considered environment. Moreover, for the justification of the proposed technique an illustrated example has been described for the selection of suppliers in the production industry. Consequently, relying upon the obtained results it can be confidently concluded that the proposed methodology indicates higher stability and usability for decision-makers in the DM process. Future research will surely concentrate upon presenting the TOPSIS technique based on correlation coefficient under-considered environment. The suggested approach can be applied to quite a lot of issues in real life, including the medical profession, robotics, artificial intelligence, pattern recognition, economics, etc.

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