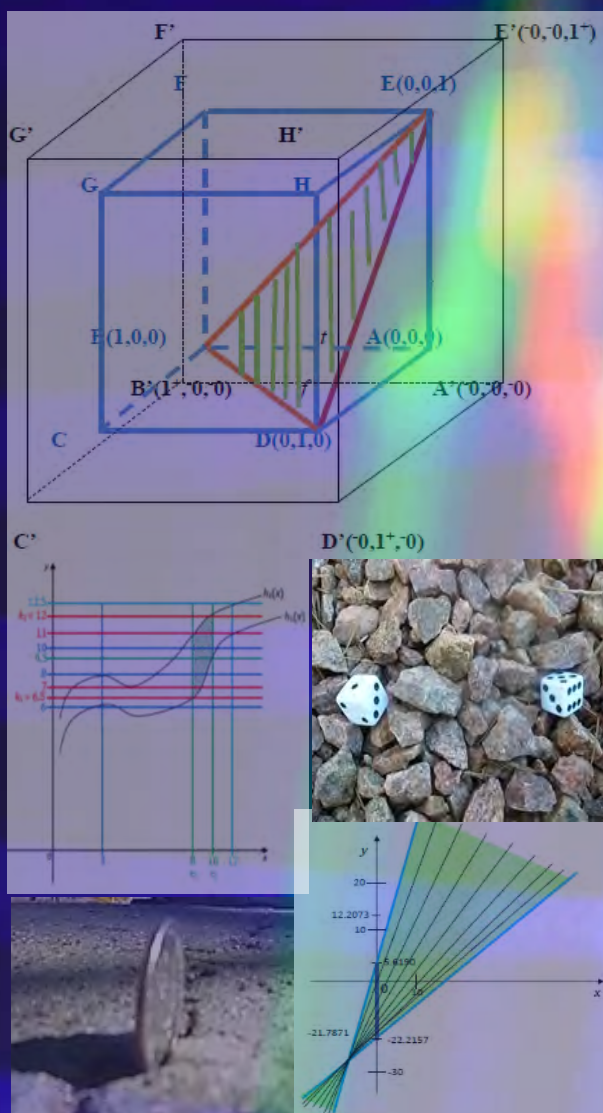


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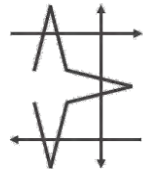
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$\langle A \rangle$ $\langle \text{neut}A \rangle$ $\langle \text{anti}A \rangle$

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea $\langle A \rangle$ together with its opposite or negation $\langle \text{anti}A \rangle$ and with their spectrum of neutralities $\langle \text{neut}A \rangle$ in between them (i.e. notions or ideas supporting neither $\langle A \rangle$ nor $\langle \text{anti}A \rangle$). The $\langle \text{neut}A \rangle$ and $\langle \text{anti}A \rangle$ ideas together are referred to as $\langle \text{non}A \rangle$.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on $\langle A \rangle$ and $\langle \text{anti}A \rangle$ only).

According to this theory every idea $\langle A \rangle$ tends to be neutralized and balanced by $\langle \text{anti}A \rangle$ and $\langle \text{non}A \rangle$ ideas - as a state of equilibrium.

In a classical way $\langle A \rangle$, $\langle \text{neut}A \rangle$, $\langle \text{anti}A \rangle$ are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that $\langle A \rangle$, $\langle \text{neut}A \rangle$, $\langle \text{anti}A \rangle$ (and $\langle \text{non}A \rangle$ of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]0, 1[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

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What distinguishes the neutrosophics from other fields is the $\langle \text{neut}A \rangle$, which means neither $\langle A \rangle$ nor $\langle \text{anti}A \rangle$.

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The Score, Accuracy, and Certainty Functions determine a Total Order on the Set of Neutrosophic Triplets (T, I, F)

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Abstract: In this paper we prove that the Single-Valued (and respectively Interval-Valued, as well as Subset-Valued) Score, Accuracy, and Certainty Functions determine a total order on the set of neutrosophic triplets (T, I, F) . This total order is needed in the neutrosophic decision-making applications.

Keywords: single-valued neutrosophic triplet numbers; single-valued neutrosophic score function; single-valued neutrosophic accuracy function; single-valued neutrosophic certainty function.

1. Introduction

We reveal the easiest to use single-valued neutrosophic score, accuracy, and certainty functions that exist in the literature and the algorithm how to use them all together. We present Xu and Da's Possibility Degree that an interval is greater than or equal to another interval, and we prove that this method is equivalent to the intervals' midpoints comparison. Also, Hong-yu Zhang et al.'s interval-valued neutrosophic score, accuracy, and certainty functions are listed, that we simplify these functions. Numerical examples are provided.

2. Single-Valued Neutrosophic Score, Accuracy, and Certainty Functions

We firstly present the most known and used in literature single-valued score, accuracy, and certainty functions.

Let M be the set of single-valued neutrosophic triplet numbers,

$$M = \{(T, I, F), \text{ where } T, I, F \in [0, 1], 0 \leq T + I + F \leq 3\}. \quad (1)$$

Let $N = (T, I, F) \in M$ be a generic single-valued neutrosophic triplet number. Then:

T = truth (or membership) represents the positive quality of N ;

I = indeterminacy represents a negative quality of N ,

hence $1 - I$ represents a positive quality of N ;

F = falsehood (or nonmembership) represents also a negative quality of N , hence $1 - F$ represents a positive quality of N .

We present the three most used and best functions in the literature:

2.1. The Single-Valued Neutrosophic Score Function

$$s: M \rightarrow [0, 1]$$

$$s(T, I, F) = \frac{T + (1 - I) + (1 - F)}{3} = \frac{2 + T - I - F}{3} \quad (2)$$

that represents the average of positiveness of the single-valued neutrosophic components T, I, F .

2.2. The Single-Valued Neutrosophic Accuracy Function

$$\begin{aligned} a: M &\rightarrow [-1, 1] \\ a(T, I, F) &= T - F \end{aligned} \quad (3)$$

2.3. The Single-Valued Neutrosophic Certainty Function

$$\begin{aligned} c: M &\rightarrow [0, 1] \\ c(T, I, F) &= T \end{aligned} \quad (4)$$

3. Algorithm for Ranking the Single-Valued Neutrosophic Triplets

Let (T_1, I_1, F_1) and (T_2, I_2, F_2) be two single-valued neutrosophic triplets from M , i.e. $T_1, I_1, F_1, T_2, I_2, F_2 \in [0, 1]$.

Apply the *Neutrosophic Score Function*.

1. If $s(T_1, I_1, F_1) > s(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) > (T_2, I_2, F_2)$.
2. If $s(T_1, I_1, F_1) < s(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) < (T_2, I_2, F_2)$.
3. If $s(T_1, I_1, F_1) = s(T_2, I_2, F_2)$, then apply the *Neutrosophic Accuracy Function*:
 - 3.1 If $a(T_1, I_1, F_1) > a(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) > (T_2, I_2, F_2)$.
 - 3.2 If $a(T_1, I_1, F_1) < a(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) < (T_2, I_2, F_2)$.
 - 3.3 If $a(T_1, I_1, F_1) = a(T_2, I_2, F_2)$, then apply the *Neutrosophic Certainty Function*.
 - 3.3.1 If $c(T_1, I_1, F_1) > c(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) > (T_2, I_2, F_2)$.
 - 3.3.2 If $c(T_1, I_1, F_1) < c(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) < (T_2, I_2, F_2)$.
 - 3.3.1 If $c(T_1, I_1, F_1) = c(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) \equiv (T_2, I_2, F_2)$, i.e. $T_1 = T_2, I_1 = I_2, F_1 = F_2$.

3.1. Theorem

We prove that the single-valued neutrosophic score, accuracy, and certainty functions all together form a *total order relationship* on M . Or:

for any two single-valued neutrosophic triplets (T_1, I_1, F_1) and (T_2, I_2, F_2) we have:

- a) Either $(T_1, I_1, F_1) > (T_2, I_2, F_2)$
- b) Or $(T_1, I_1, F_1) < (T_2, I_2, F_2)$
- c) Or $(T_1, I_1, F_1) \equiv (T_2, I_2, F_2)$, which means that $T_1 = T_2, I_1 = I_2, F_1 = F_2$.

Therefore, on the set of single-valued neutrosophic triplets $M = \{(T, I, F), \text{ with } T, I, F \in [0, 1], 0 \leq T + I + F \leq 3\}$, the score, accuracy, and certainty functions altogether form a *total order relationship*.

Proof.

Firstly we apply the score function.

The only problematic case is when we get equality:

$$s(T_1, I_1, F_1) = s(T_2, I_2, F_2). \quad (5)$$

That means:

$$\frac{2+T_1-I_1-F_1}{3} = \frac{2+T_2-I_2-F_2}{3}$$

$$\text{or } T_1 - I_1 - F_1 = T_2 - I_2 - F_2.$$

Secondly we apply the accuracy function.

Again the only problematic case is when we get equality:

$$a(T_1, I_1, F_1) = a(T_2, I_2, F_2) \text{ or } T_1 - F_1 = T_2 - F_2.$$

Thirdly, we apply the certainty function.

Similarly, the only problematic case may be when we get equality:

$$c(T_1, I_1, F_1) = c(T_2, I_2, F_2) \text{ or } T_1 = T_2.$$

For the most problematic case, we got the following linear algebraic system of 3 equations of 6 variables:

$$\begin{cases} T_1 - I_1 - F_1 = T_2 - I_2 - F_2 \\ T_1 - F_1 = T_2 - F_2 \\ T_1 = T_2 \end{cases}$$

Let's solve it.

Since $T_1 = T_2$, replacing this into the second equation we get $F_1 = F_2$.

Now, replacing both $T_1 = T_2$ and $F_1 = F_2$ into the first equation, we get $I_1 = I_2$.

Therefore the two neutrosophic triplets are identical: $(T_1, I_1, F_1) \equiv (T_2, I_2, F_2)$, i.e. equivalent (or equal), or $T_1 = T_2$, $I_1 = I_2$, and $F_1 = F_2$.

In conclusion, for any two single-valued neutrosophic triplets, either one is bigger than the other, or both are equal (identical).

4. Definition of Neutrosophic Negative Score Function

We have introduced in 2017 for the first time [1] the *Average Negative Quality Neutrosophic Function* of a single-valued neutrosophic triplet, defined as:

$$s^- : [0, 1]^3 \rightarrow [0, 1], s^-(t, i, f) = \frac{(1-t) + i + f}{3} = \frac{1-t+i+f}{3}. \quad (6)$$

4.1. Theorem

The average positive quality (score) neutrosophic function and the average negative quality neutrosophic function are complementary to each other, or

$$s^+(t, i, f) + s^-(t, i, f) = 1. \quad (7)$$

Proof.

$$s^+(t, i, f) + s^-(t, i, f) = \frac{2+t-i-f}{3} + \frac{1-t+i+f}{3} = 1. \quad (8)$$

The *Neutrosophic Accuracy Function* has been defined by:

$$h: [0, 1]^3 \rightarrow [-1, 1], h(t, i, f) = t - f. \quad (9)$$

We have also introduced [1] for the first time the *Extended Accuracy Neutrosophic Function*, defined as follows:

$$h_e: [0, 1]^3 \rightarrow [-2, 1], h_e(t, i, f) = t - i - f, \quad (10)$$

which varies on a range: from the worst negative quality (-2) [or minimum value], to the best positive quality (+1) [or maximum value].

4.2. Theorem

If $s(T_1, I_1, F_1) = s(T_2, I_2, F_2)$, $a(T_1, I_1, F_1) = a(T_2, I_2, F_2)$, and $c(T_1, I_1, F_1) = c(T_2, I_2, F_2)$,

then $T_1 = T_2$, $I_1 = I_2$, $F_1 = F_2$, or the two neutrosophic triplets are identical:

$$(T_1, I_1, F_1) \equiv (T_2, I_2, F_2).$$

Proof:

It results from the proof of Theorem 3.1.

5. Xu and Da's Possibility Degree

Xu and Da [3] have defined in 2002 the possibility degree $P(\cdot)$ that an interval is greater than another interval:

$$[a_1, a_2] \geq [b_1, b_2]$$

for $a_1, a_2, b_1, b_2 \in [0, 1]$ and $a_1 \leq a_2, b_1 \leq b_2$, in the following way:

$$P([a_1, a_2] \geq [b_1, b_2]) = \max \left\{ 1 - \max \left(\frac{b_2 - a_1}{a_2 - a_1 + b_2 - b_1}, 0 \right), 0 \right\},$$

where $a_2 - a_1 + b_2 - b_1 \neq 0$ (i.e. $a_2 \neq a_1$ or $b_2 \neq b_1$).

They proved the following:

5.1. Properties

- 1) $P([a_1, a_2] \geq [b_1, b_2]) \in [0, 1]$;
- 2) $P([a_1, a_2] \approx [b_1, b_2]) = 0.5$;
- 3) $P([a_1, a_2] \geq [b_1, b_2]) + P([b_1, b_2] \geq [a_1, a_2]) = 1$.

5.2. Example

Let $[0.4, 0.7]$ and $[0.3, 0.6]$ be two intervals.

Then,

$$\begin{aligned} P([0.4, 0.7] \geq [0.3, 0.6]) &= \max \left\{ 1 - \max \left(\frac{0.6 - 0.4}{0.7 - 0.4 + 0.6 - 0.3}, 0 \right), 0 \right\} = \max \left\{ 1 - \max \left(\frac{0.2}{0.6}, 0 \right), 0 \right\} \\ &= \max \left\{ 1 - \frac{0.2}{0.6}, 0 \right\} = \frac{0.4}{0.6} \approx 0.66 > 0.50, \end{aligned}$$

therefore $[0.4, 0.7] \geq [0.3, 0.6]$.

The opposite:

$$\begin{aligned} P([0.3, 0.6] \geq [0.4, 0.7]) &= \max \left\{ 1 - \max \left(\frac{0.7 - 0.3}{0.6 - 0.3 + 0.7 - 0.4}, 0 \right), 0 \right\} = \max \left\{ 1 - \max \left(\frac{0.4}{0.6}, 0 \right), 0 \right\} \\ &= \max \left\{ 1 - \frac{0.4}{0.6}, 0 \right\} = \frac{0.2}{0.6} \approx 0.33 < 0.50, \end{aligned}$$

therefore $[0.3, 0.6] \leq [0.4, 0.7]$.

We see that

$$P([0.4, 0.7] \geq [0.3, 0.6]) + P([0.3, 0.6] \geq [0.4, 0.7]) = \frac{0.4}{0.6} + \frac{0.2}{0.6} = 1.$$

Another method of ranking two intervals is the midpoint one.

6. Midpoint Method

Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two intervals included in or equal to $[0, 1]$, with

$m_A = (a_1 + a_2)/2$ and $m_B = (b_1 + b_2)/2$ the midpoints of A and respectively B . Then:

- 1) If $m_A < m_B$ then $A < B$.
- 2) If $m_A > m_B$ then $A > B$.
- 3) If $m_A = m_B$ then $A =_N B$, i.e. A is neutrosophically equal to B .

6.1. Example

1) We take the previous example,

where $A = [0.4, 0.7]$, and $m_A = \frac{0.4+0.7}{2} = 0.55$;

and $B = [0.3, 0.6]$, and $m_B = \frac{0.3+0.6}{2} = 0.45$.

Since $m_A = 0.55 > 0.45 = m_B$, we have $A > B$.

Let $C = [0.1, 0.7]$ and $D = [0.3, 0.5]$.

Then $m_C = \frac{0.1+0.7}{2} = 0.4$, and $m_D = \frac{0.3+0.5}{2} = 0.4$.

Since $m_C = m_D = 0.4$, we get $C =_N D$.

Let's verify the ranking relationship between C and D using Xu and Da's possibility degree method.

$$\begin{aligned} P([0.1, 0.7] \geq [0.3, 0.5]) &= \max \left\{ 1 - \max \left(\frac{0.5 - 0.1}{0.7 - 0.1 + 0.5 - 0.3}, 0 \right), 0 \right\} = \max \left\{ 1 - \max \left(\frac{0.4}{0.8}, 0 \right), 0 \right\} \\ &= \max \left\{ 1 - \frac{0.4}{0.8}, 0 \right\} = \max \left\{ \frac{0.4}{0.8}, 0 \right\} = 0.5; \end{aligned}$$

$$\text{and} \quad P([0.3, 0.5] \geq [0.1, 0.7]) = \max \left\{ 1 - \max \left(\frac{0.7 - 0.3}{0.5 - 0.3 + 0.7 - 0.1}, 0 \right), 0 \right\} = \max \left\{ 1 - \right.$$

$$\left. \max \left(\frac{0.4}{0.8}, 0 \right), 0 \right\} = \max \left\{ 1 - \frac{0.4}{0.8}, 0 \right\} = \max \left\{ \frac{0.4}{0.8}, 0 \right\} = 0.5;$$

thus, $[0.1, 0.7] =_N [0.3, 0.5]$.

6.2. Corollary

The possibility method for two intervals having the same midpoint gives always 0.5.

For example:

$$\begin{aligned} p([0.3, 0.5] \geq [0.2, 0.6]) &= \max \{ 1 - \max(((0.6-0.3) / (0.5-0.3 + 0.6-0.2)), 0), 0 \} = \\ &= \max \{ 1 - \max(((0.3) / (0.6)), 0), 0 \} = \max \{ 1 - \max(0.5, 0), 0 \} = 0.5. \end{aligned}$$

Similarly,

$$p([0.2, 0.6] \geq [0.3, 0.5]) = \max \{ 1 - \max(((0.5-0.2) / (0.6-0.2 + 0.5-0.3)), 0), 0 \} = 0.5.$$

Hence, none of the intervals $[0.3, 0.5]$ and $[0.2, 0.6]$ is bigger than the other.

Therefore, we may consider that the intervals $[0.3, 0.5] =_N [0.2, 0.6]$ are neutrosophically equal (or neutrosophically equivalent).

7. Normalized Hamming Distance between Two Intervals

Let's consider the *Normalized Hamming Distance* between two intervals $[a_1, a_2]$ and $[b_1, b_2]$

$$h : \text{int}([0, 1]) \times \text{int}([0, 1]) \rightarrow [0, 1]$$

defined as follows:

$$h([a_1, b_1], [a_2, b_2]) = \frac{1}{2}(|a_1 - b_1| + |a_2 - b_2|).$$

7.1. Theorem

7.1.1. The Normalized Hamming Distance between two intervals having the same midpoint and the negative-ideal interval $[0, 0]$ is the same.

7.1.2. The Normalized Hamming Distance between two intervals having the same midpoint and the positive-ideal interval $[1, 1]$ is also the same (Jun Ye [4, 5]).

Proof.

Let $A = [m - a, m + a]$ and $B = [m - b, m + b]$ be two intervals from $[0, 1]$, where $m - a, m + a, m - b, m + b, a, b, m \in [0, 1]$. A and B have the same midpoint m .

$$7.1.1. h([m - a, m + a], [0, 0]) = \frac{1}{2}(|m - a - 0| + |m + a - 0|) = \frac{1}{2}(m - a + m + a) = m, \text{ and}$$

$$h([m - b, m + b], [0, 0]) = \frac{1}{2}(|m - b - 0| + |m + b - 0|) = \frac{1}{2}(m - b + m + b) = m,$$

$$7.1.2. h([m - a, m + a], [1, 1]) = \frac{1}{2}(|m - a - 1| + |m + a - 1|) = \frac{1}{2}(1 - m + a + 1 - m - a) = 1 - m, \text{ and}$$

$$h([m - b, m + b], [1, 1]) = \frac{1}{2}(|m - b - 1| + |m + b - 1|) = \frac{1}{2}(1 - m + b + 1 - m - b) = 1 - m.$$

8. Xu and Da's Possibility Degree Method is equivalent to the Midpoint Method

We prove the following:

8.1. Theorem

The Xu and Da's Possibility Degree Method is equivalent to the Midpoint Method in ranking two intervals included in $[0, 1]$.

Proof.

Let A and B be two intervals included in $[0, 1]$. Without loss of generality, we write each interval in terms of each midpoint:

$$A = [m_1 - a, m_1 + a] \text{ and } B = [m_2 - b, m_2 + b],$$

where $m_1, m_2 \in [0, 1]$ are the midpoints of A and respectively B , and $a, b \in [0, 1]$, $A, B \subseteq [0, 1]$.

(For example, if $A = [0.4, 0.7]$, $m_A = \frac{0.4+0.7}{2} = 0.55$, $0.55-0.4=0.15$, then $A = [0.55 - 0.15, 0.55 + 0.15]$).

1) First case: $m_1 < m_2$. According to the Midpoint Method, we get $A < B$. Let's prove the same inequality results with the second method.

Let's apply Xu and Da's Possibility Degree Method:

$$\begin{aligned} P(A \geq B) &= P([m_1 - a, m_1 + a] \geq [m_2 - b, m_2 + b]) \\ &= \max \left\{ 1 - \max \left(\frac{(m_2 + b) - (m_1 - a)}{(m_1 + a) - (m_1 - a) + (m_2 + b) - (m_2 - b)}, 0 \right), 0 \right\} \\ &= \max \left\{ 1 - \max \left(\frac{m_2 - m_1 + a + b}{2a + 2b}, 0 \right), 0 \right\} \\ &= \max \left\{ 1 - \frac{m_2 - m_1 + a + b}{2a + 2b}, 0 \right\}, \text{ because } m_1 < m_2, \\ &= \max \left\{ \frac{2a + 2b - m_2 + m_1 - a - b}{2a + 2b}, 0 \right\} = \max \left\{ \frac{a + b + m_1 - m_2}{2a + 2b}, 0 \right\} \end{aligned}$$

i) If $a + b + m_1 - m_2 \leq 0$, then $p(A \geq B) = \max \left\{ \frac{a + b + m_1 - m_2}{2a + 2b}, 0 \right\} = 0$, hence $A < B$.

ii) If $a + b + m_1 - m_2 > 0$,

then $p(A \geq B) = \max \left\{ \frac{a + b + m_1 - m_2}{2a + 2b}, 0 \right\} = \frac{a + b + m_1 - m_2}{2a + 2b} > 0$.

We need to prove that $\frac{a+b+m_1-m_2}{2a+2b} < 0.5$,

or $a + b + m_1 - m_2 < 0.5(2a + 2b)$,

or $a + b + m_1 - m_2 < a + b$,

or $m_1 - m_2 < 0$,

or $m_1 < m_2$, which is true according to the first case assumption.

2) Second case: $m_1 = m_2$. According to the Midpoint Method, A is neutrosophically equal to B (we write $A =_N B$).

Let's prove that we get the same result with Xu and Da's Method.

Then $A = [m_1 - a, m_1 + a]$, and $B = [m_1 - b, m_1 + b]$.

Let's apply Xu and Da's Method:

$$\begin{aligned} P(A \geq B) &= \max \left\{ 1 - \max \left(\frac{(m_1 + b) - (m_1 - a)}{(m_1 + a) - (m_1 - a) + (m_1 + b) - (m_1 - b)}, 0 \right), 0 \right\} \\ &= \max \left\{ 1 - \max \left(\frac{a + b}{2a + 2b}, 0 \right), 0 \right\} = \max \left\{ 1 - \frac{1}{2}, 0 \right\} = 0.5 \end{aligned}$$

Similarly:

$$\begin{aligned} P(B \geq A) &= \max \left\{ 1 - \max \left(\frac{(m_1 + a) - (m_1 - b)}{(m_1 + b) - (m_1 - b) + (m_1 + a) - (m_1 - a)}, 0 \right), 0 \right\} \\ &= \max \left\{ 1 - \max \left(\frac{a + b}{2a + 2b}, 0 \right), 0 \right\} = 0.5 \end{aligned}$$

Therefore, again $A =_N B$.

3) If $m_1 > m_2$, according to the Midpoint Method, we get $A > B$.

Let's prove the same inequality using Xu and Da's Method.

$$\begin{aligned} P(A \geq B) &= P([m_1 - a, m_1 + a] \geq [m_2 - b, m_2 + b]) = \\ &= \max \left\{ 1 - \max \left(\frac{(m_2 + b) - (m_1 - a)}{(m_1 + a) - (m_1 - a) + (m_2 + b) - (m_2 - b)}, 0 \right), 0 \right\} \\ &= \max \left\{ 1 - \max \left(\frac{m_2 - m_1 + a + b}{2a + 2b}, 0 \right), 0 \right\} \end{aligned}$$

i) If $m_2 - m_1 + a + b \leq 0$, then $P(A \geq B) = \max\{1 - 0, 0\} = 1$, therefore $A > B$.

ii) If $m_2 - m_1 + a + b > 0$, then

$$P(A \geq B) = \max \left\{ 1 - \frac{m_2 - m_1 + a + b}{2a + 2b}, 0 \right\} = \frac{2a + 2b - m_2 + m_1 - a - b}{2a + 2b} = \frac{a + b + m_1 - m_2}{2a + 2b}$$

We need to prove that $\frac{a+b+m_1-m_2}{2a+2b} > 0.5$,

or $a + b + m_1 - m_2 > 0.5(2a + 2b)$

or $a + b + m_1 - m_2 > a + b$

or $m_1 - m_2 > 0$

or $m_1 > m_2$, which is true according to the third case. Thus $A > B$.

8.2. Consequence

All intervals, included in $[0, 1]$, with the same midpoint are considered neutrosophically equal.

$\mathcal{C}(m) = \{[m - a, m + a], \text{ where all } m, a, m - a, m + a \in [0, 1]\}$

represents the class of all neutrosophically equal intervals included in $[0, 1]$ whose midpoint is m .

- i) If $m = 0$ or $m = 1$, there is only one interval centered in 0, i.e. $[0, 0]$, and only one interval centered in 1, i.e. $[1, 1]$.
- ii) If $m \notin \{0, 1\}$, there are infinitely many intervals from $[0, 1]$, centered in m .

8.3. Consequence

Remarkably we can rank an interval $[a, b] \subseteq [0, 1]$ with respect to a number $n \in [0, 1]$ since the number may be transformed into an interval $[n, n]$ as well.

For example $[0.2, 0.8] > 0.4$ since the midpoint of $[0.2, 0.8]$ is 0.5, and the midpoint of $[0.4, 0.4] = 0.4$, hence $0.5 > 0.4$.

Similarly, $0.7 > (0.5, 0.8)$.

9. Interval (-Valued) Neutrosophic Score, Accuracy, and Certainty Functions

Let $T, I, F \subseteq [0, 1]$ be three open, semi-open / semi-closed, or closed intervals.

Let $T^L = \inf T$ and $T^U = \sup T$; $I^L = \inf I$ and $I^U = \sup I$; $F^L = \inf F$ and $F^U = \sup F$.

Let $T^L, T^U, I^L, I^U, F^L, F^U \in [0, 1]$, with $T^L \leq T^U, I^L \leq I^U, F^L \leq F^U$.

We consider all possible types of intervals: open (a, b) , semi-open / semi-closed $(a, b]$ and $[a, b)$, and closed $[a, b]$. For simplicity of notations, we are using only $[a, b]$, but we understand all types.

Then $A = ([T^L, T^U], [I^L, I^U], [F^L, F^U])$ is an Interval Neutrosophic Triplet.

T^L is the lower limit of the interval T ,

T^U is the upper limit of the interval T ,

and similarly for I^L, I^U , and F^L, F^U for the intervals I , and respectively F .

Hong-yu Zhang, Jian-qiang Wang, and Xiao-hong Chen [2] in 2014 defined the Interval Neutrosophic Score, Accuracy, and Certainty Functions as follows.

Let's consider $\text{int}([0, 1])$ the set of all (open, semi-open/semi-closed, or closed) intervals included in or equal to $[0, 1]$, where the abbreviation and index *int* stand for *interval*, and *Zhang* stands for Hong-yu Zhang, Jian-qiang Wang, and Xiao-hong Chen.

9.1. Zhang Interval Neutrosophic Score Function

$$s_{\text{int}}^{\text{Zhang}} : \{\text{int}([0, 1])\}^3 \rightarrow \text{int}([0, 1])$$

$$s_{\text{int}}^{\text{Zhang}}(A) = [T^L + 1 - I^U + 1 - F^U, T^U + 1 - I^L + 1 - F^L] \quad (11)$$

9.2. Zhang Interval Neutrosophic Accuracy Function

$$a_{\text{int}}^{\text{Zhang}} : \{\text{int}([0, 1])\}^3 \rightarrow \text{int}([0, 1])$$

$$a_{\text{int}}^{\text{Zhang}}(A) = [\min\{T^L - F^L, T^U - F^U\}, \max\{T^L - F^L, T^U - F^U\}] \quad (12)$$

9.3. Zhang Interval Neutrosophic Certainty Function

$$c_{\text{int}}^{\text{Zhang}} : \{\text{int}([0, 1])\}^3 \rightarrow \text{int}([0, 1])$$

$$c_{\text{int}}^{\text{Zhang}}(A) = [T^L, T^U] \quad (13)$$

9. New Interval Neutrosophic Score, Accuracy, and Certainty Functions

Since comparing/ranking two intervals is equivalent to comparing/ranking two members (i.e. the intervals' midpoints), we simplify Zhang Interval Neutrosophic Score (s_{int}^{Zhang}), Accuracy (a_{int}^{Zhang}), Certainty (c_{int}^{Zhang}) functions, as follows:

$$\begin{aligned}s_{int}^{FS} : \{\text{int}([0,1])\}^3 &\rightarrow [0,1] \\ a_{int}^{FS} : \{\text{int}([0,1])\}^3 &\rightarrow [-1,1] \\ c_{int}^{FS} : \{\text{int}([0,1])\}^3 &\rightarrow [0,1]\end{aligned}$$

where the upper index FS stands for our name's initials, in order to distinguish these new functions from the previous ones:

10.1. New Interval Neutrosophic Score Function

$$s_{int}^{FS}([T^L, T^U], [I^L, I^U], [F^L, F^U]) = \frac{T^L + T^U + (1 - I^L) + (1 - I^U) + (1 - F^L) + (1 - F^U)}{6} = \frac{4 + T^L + T^U - I^L - I^U - F^L - F^U}{6},$$

which means the average of six positiveness;

10.2. New Interval Neutrosophic Accuracy Function

$$a_{int}^{FS}([T^L, T^U], [I^L, I^U], [F^L, F^U]) = \frac{T^L + T^U - F^L - F^U}{2}, \text{ which means the average of differences}$$

between positiveness and negativeness;

10.3. New Interval Neutrosophic Certainty Function

$$c_{int}^{FS}([T^L, T^U], [I^L, I^U], [F^L, F^U]) = \frac{T^L + T^U}{2},$$

which means the average of two positivenesses.

10.4. Theorem

Let $\mathcal{M}_{int} = \{(T, I, F), \text{ where } T, I, F \subseteq [0, 1], \text{ and } T, I, F \text{ are intervals}\}$, be the set of interval neutrosophic triplets.

The New Interval Neutrosophic Score, Accuracy, and Certainty Functions determine a total order relationship on the set \mathcal{M}_{int} of Interval Neutrosophic Triplets.

Proof.

Let's assume we have two interval neutrosophic triplets:

$$P_1 = ([T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U]),$$

$$\text{and } P_2 = ([T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U]), \text{ both from } \mathcal{M}_{int}.$$

We have to prove that: either $P_1 > P_2$, or $P_1 < P_2$, or $P_1 = P_2$.

Apply the new interval neutrosophic score function (s_{int}^{FS}) to both of them:

$$\begin{aligned}s_{int}^{FS}(P_1) &= \frac{4 + T_1^L + T_1^U - I_1^L - I_1^U - F_1^L - F_1^U}{6} \\ s_{int}^{FS}(P_2) &= \frac{4 + T_2^L + T_2^U - I_2^L - I_2^U - F_2^L - F_2^U}{6}\end{aligned}$$

If $s_{\text{int}}^{FS}(P_1) > s_{\text{int}}^{FS}(P_2)$, then $P_1 > P_2$.

If $s_{\text{int}}^{FS}(P_1) < s_{\text{int}}^{FS}(P_2)$, then $P_1 < P_2$.

If $s_{\text{int}}^{FS}(P_1) = s_{\text{int}}^{FS}(P_2)$, then we get from equating the above two equalities that:

$$T_1^L + T_1^U - I_1^L - I_1^U - F_1^L - F_1^U = T_2^L + T_2^U - I_2^L - I_2^U - F_2^L - F_2^U$$

In this problematic case, we apply the new interval neutrosophic accuracy function

(a_{int}^{FS}) to both P_1 and P_2 , and we get:

$$a_{\text{int}}^{FS}(P_1) = \frac{T_1^L + T_1^U - F_1^L - F_1^U}{2}$$

$$a_{\text{int}}^{FS}(P_2) = \frac{T_2^L + T_2^U - F_2^L - F_2^U}{2}$$

If $a_{\text{int}}^{FS}(P_1) > a_{\text{int}}^{FS}(P_2)$, then $P_1 > P_2$.

If $a_{\text{int}}^{FS}(P_1) < a_{\text{int}}^{FS}(P_2)$, then $P_1 < P_2$.

If $a_{\text{int}}^{FS}(P_1) = a_{\text{int}}^{FS}(P_2)$, then we get from equating the two above equalities that:

$$T_1^L + T_1^U - F_1^L - F_1^U = T_2^L + T_2^U - F_2^L - F_2^U$$

Again, a problematic case, so we apply the new interval neutrosophic certainty function (c_{int}^{FS})

to both P_1 and P_2 , and we get:

$$c_{\text{int}}^{FS}(P_1) = T_1^L + T_1^U$$

$$c_{\text{int}}^{FS}(P_2) = T_2^L + T_2^U$$

If $c_{\text{int}}^{FS}(P_1) > c_{\text{int}}^{FS}(P_2)$, then $P_1 > P_2$.

If $c_{\text{int}}^{FS}(P_1) < c_{\text{int}}^{FS}(P_2)$, then $P_1 < P_2$.

If $c_{\text{int}}^{FS}(P_1) = c_{\text{int}}^{FS}(P_2)$, then we get:

$$T_1^L + T_1^U = T_2^L + T_2^U$$

We prove that in the last case we get:

$P_1 =_N P_2$ (or P_1 is neutrosophically equal to P_2).

We get the following linear algebraic system of 3 equations and 12 variables:

$$\begin{cases} T_1^L + T_1^U - I_1^L - I_1^U - F_1^L - F_1^U = T_2^L + T_2^U - I_2^L - I_2^U - F_2^L - F_2^U \\ T_1^L + T_1^U - F_1^L - F_1^U = T_2^L + T_2^U - F_2^L - F_2^U \\ T_1^L + T_1^U = T_2^L + T_2^U \end{cases}$$

Second equation minus the third equation gives us:

$$-F_1^L - F_1^U = -F_2^L - F_2^U, \text{ or } F_1^L + F_1^U = F_2^L + F_2^U.$$

First equation minus the second equation gives us:

$$-I_1^L - I_1^U = -I_2^L - I_2^U, \text{ or } I_1^L + I_1^U = I_2^L + I_2^U.$$

The previous system is now equivalent to the below system:

$$\begin{cases} T_1^L + T_1^U = T_2^L + T_2^U \\ I_1^L + I_1^U = I_2^L + I_2^U \\ F_1^L + F_1^U = F_2^L + F_2^U \end{cases} \Leftrightarrow \begin{cases} \frac{T_1^L + T_1^U}{2} = \frac{T_2^L + T_2^U}{2} \\ \frac{I_1^L + I_1^U}{2} = \frac{I_2^L + I_2^U}{2} \\ \frac{F_1^L + F_1^U}{2} = \frac{F_2^L + F_2^U}{2} \end{cases}$$

which means that:

i) the intervals $[T_1^L, T_1^U]$ and $[T_2^L, T_2^U]$ have the same midpoint, therefore they are neutrosophically equal.

ii) the intervals $[I_1^L, I_1^U]$ and $[I_2^L, I_2^U]$ have also the same midpoint, so they are neutrosophically equal.

iii) similarly, the intervals $[F_1^L, F_1^U]$ and $[F_2^L, F_2^U]$ have the same midpoint, and again they are neutrosophically equal.

Whence, the interval neutrosophic triplets P_1 and P_2 are neutrosophically equal, i.e. $P_1 =_N P_2$.

10.5. Theorem

Let's consider the ranking of intervals defined by Xu and Da, which is equivalent to the ranking of intervals' midpoints. Then, the algorithm by Hong-yu Zhang et al. for ranking the interval neutrosophic triplets is equivalent to our algorithm.

Proof

Let's consider two interval neutrosophic triplets, P_1 and $P_2 \in \mathcal{M}_{int}$,

$$P_1 = ([T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U]),$$

$$\text{and } P_2 = ([T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U]).$$

Let's rank them using both methods and prove we get the same results. We denote by s_{int}^{Zhang} ,

a_{int}^{Zhang} , c_{int}^{Zhang} , and s_{int}^{FS} , a_{int}^{FS} , c_{int}^{FS} the Interval Neutrosophic Score, Accuracy, and Certainty

Functions, by Hong-yu Zhang et al. and respectively by us.

Interval Neutrosophic Score Function

$$s_{int}^{Zhang}(P_1) = [T_1^L + 1 - I_1^U + 1 - F_1^U, T_1^U + 1 - I_1^L + 1 - F_1^L]$$

$$s_{int}^{Zhang}(P_2) = [T_2^L + 1 - I_2^U + 1 - F_2^U, T_2^U + 1 - I_2^L + 1 - F_2^L]$$

a) If $s_{int}^{Zhang}(P_1) > s_{int}^{Zhang}(P_2)$ then

the midpoint of the interval $s_{int}^{Zhang}(P_1) >$ midpoint of the interval $s_{int}^{Zhang}(P_2)$,

$$\text{or } \frac{T_1^L+1-I_1^U+1-F_1^U+T_1^U+1-I_1^L+1-F_1^L}{2} > \frac{T_2^L+1-I_2^U+1-F_2^U+T_2^U+1-I_2^L+1-F_2^L}{2},$$

$$\text{or } T_1^L + T_1^U - I_1^L - I_1^U - F_1^L - F_1^U > T_2^L + T_2^U - I_2^L - I_2^U - F_2^L - F_2^U,$$

$$\text{or } \frac{4+T_1^L+T_1^U-I_1^L-I_1^U-F_1^L-F_1^U}{6} > \frac{4+T_2^L+T_2^U-I_2^L-I_2^U-F_2^L-F_2^U}{6},$$

$$\text{or } s_{int}^{FS}(P_1) > s_{int}^{FS}(P_2).$$

b) If $s_{int}^{Zhang}(P_1) < s_{int}^{Zhang}(P_2)$, the proof is similar, we only replace the inequality symbol $>$ by $<$ into the above proof.

c) If $s_{int}^{Zhang}(P_1) = s_{int}^{Zhang}(P_2)$, the proof again is similar with the above, we only replace $>$ by $=$ into the above proof.

Interval Neutrosophic Accuracy Function

$$a_{int}^{Zhang}(P_1) = [\min\{T_1^L - F_1^L, T_1^U - F_1^U\}, \max\{T_1^L - F_1^L, T_1^U - F_1^U\}]$$

$$a_{int}^{Zhang}(P_2) = [\min\{T_2^L - F_2^L, T_2^U - F_2^U\}, \max\{T_2^L - F_2^L, T_2^U - F_2^U\}]$$

a) If $a_{int}^{Zhang}(P_1) > a_{int}^{Zhang}(P_2)$ then

the midpoint of the interval $a_{int}^{Hong}(P_1) >$ the midpoint of the interval $a_{int}^{Hong}(P_2)$,

$$\text{or } \frac{T_1^L-F_1^L+T_1^U-F_1^U}{2} > \frac{T_2^L-F_2^L+T_2^U-F_2^U}{2},$$

$$\text{or } a_{int}^{FS}(P_1) > a_{int}^{FS}(P_2).$$

b) Similarly, if $a_{int}^{Zhang}(P_1) < a_{int}^{Zhang}(P_2)$, just replacing $>$ by $<$ into the above proof.

c) Again, similarly if $a_{int}^{Zhang}(P_1) = a_{int}^{Zhang}(P_2)$, only replacing $>$ by $=$ into the above proof.

Interval Neutrosophic Certainty Function

$$c_{int}^{Zhang}(P_1) = [T_1^L, T_1^U]$$

$$c_{int}^{Zhang}(P_2) = [T_2^L, T_2^U]$$

a) If $c_{int}^{Zhang}(P_1) > c_{int}^{Zhang}(P_2)$ then

b) the midpoint of the interval $c_{int}^{Zhang}(P_1) >$ the midpoint of the interval $c_{int}^{Zhang}(P_2)$,

$$\text{or } \frac{T_1^L+T_1^U}{2} > \frac{T_2^L+T_2^U}{2}$$

$$\text{or } c_{int}^{FS}(P_1) > c_{int}^{FS}(P_2).$$

b) Similarly, if $c_{int}^{Zhang}(P_1) < c_{int}^{Zhang}(P_2)$, just replacing $>$ by $<$ into the above proof.

c) Again, if $c_{\text{int}}^{\text{Zhang}}(P_1) = c_{\text{int}}^{\text{Zhang}}(P_2)$, only replace $>$ by $=$ into the above proof.

Therefore, we proved that, for any interval neutrosophic triplet P ,

$$s_{\text{int}}^{\text{Zhang}}(P) \sim s_{\text{int}}^{\text{FS}}(P) \text{ where } \sim \text{ means equivalent;}$$

$$a_{\text{int}}^{\text{Zhang}}(P) \sim a_{\text{int}}^{\text{FS}}(P)$$

$$\text{and } c_{\text{int}}^{\text{Zhang}}(P) \sim c_{\text{int}}^{\text{FS}}(P).$$

11. Subset Neutrosophic Score, Accuracy, and Certainty Functions

Let $M_{\text{subset}} = \{(T_{\text{subset}}, I_{\text{subset}}, F_{\text{subset}})$, where the subsets $T_{\text{subset}}, I_{\text{subset}}, F_{\text{subset}} \subseteq [0, 1]\}$.

We approximate each subset by the smallest closed interval that includes it.

Let's denote:

$$T^L = \inf(T_{\text{subset}}) \text{ and } T^U = \sup(T_{\text{subset}}); \text{ therefore } T_{\text{subset}} \subseteq [T^L, T^U];$$

$$I^L = \inf(I_{\text{subset}}) \text{ and } I^U = \sup(I_{\text{subset}}); \text{ therefore } I_{\text{subset}} \subseteq [I^L, I^U];$$

$$F^L = \inf(F_{\text{subset}}) \text{ and } F^U = \sup(F_{\text{subset}}); \text{ therefore } F_{\text{subset}} \subseteq [F^L, F^U].$$

Then:

$$M_{\text{subset}} \approx \left\{ ([T^L, T^U], [I^L, I^U], [F^L, F^U]), \text{ where } T^L, T^U, I^L, I^U, F^L, F^U \in [0, 1], \right. \\ \left. \text{and } T^L \leq T^U, I^L \leq I^U, F^L \leq F^U \right\}$$

11.1. Definition of Subset Neutrosophic Score, Accuracy, and Certainty Functions

Then, the formulas for Subset Neutrosophic Score, Accuracy, and Certainty Functions will coincide with those for Interval Neutrosophic Score Accuracy, and Certainty Functions by Hong-yu Zhang, and respectively by us:

11.2. Theorem

Let N be the Interval Neutrosophic Triplet

$$N = ([T^L, T^U], [I^L, I^U], [F^L, F^U]),$$

$$\text{where } T^L \leq T^U, I^L \leq I^U, F^L \leq F^U,$$

$$\text{and all } [T^L, T^U], [I^L, I^U], [F^L, F^U] \subseteq [0, 1].$$

If each interval collapses to a single point, i.e.

$$T^L = T^U = T, \text{ then } [T^L, T^U] = [T, T] \equiv T \in [0, 1],$$

$$I^L = I^U = I, \text{ then } [I^L, I^U] = [I, I] \equiv I \in [0, 1],$$

$$F^L = F^U = F, \text{ then } [F^L, F^U] = [F, F] \equiv F \in [0, 1],$$

$$\text{then } s_{\text{int}}^{\text{FS}}(N) = s(N), a_{\text{int}}^{\text{FS}}(N) = a(N), \text{ and } c_{\text{int}}^{\text{FS}}(N) = c(N).$$

Proof

$$s_{\text{int}}^{\text{FS}}(N) = \frac{4 + T^L + T^U - I^L - I^U - F^L - F^U}{6} = \frac{4 + T + T - I - I - F - F}{6} = \frac{4 + 2T - 2I - 2F}{6}$$

$$= \frac{2 + T - I - F}{3} = s(N).$$

$$a_{\text{int}}^{\text{FS}}(N) = \frac{T^L + T^U - F^L - F^U}{2} = \frac{T + T - F - F}{2} = \frac{2(T - F)}{2} = a(N).$$

$$c_{int}^{FS}(N) = \frac{T^L + T^U}{2} = \frac{T + T}{2} = \frac{2T}{2} = c(N).$$

12. Conclusion

The most used and easy for ranking the Neutrosophic Triplets (T, I, F) are the following functions, that provide a total order:

Single-Valued Neutrosophic Score, Accuracy, and Certainty Functions:

$$s(T, I, F) = \frac{2 + T - I - F}{3}$$

$$a(T, I, F) = T - F$$

$$c(T, I, F) = T$$

Interval-Valued Neutrosophic Score, Accuracy, and Certainty Functions:

$$s_{int}^{FS}([T^L, T^U], [I^L, I^U], [F^L, F^U]) = \frac{4 + T^L + T^U - I^L - I^U - F^L - F^U}{6}$$

$$a_{int}^{FS}([T^L, T^U], [I^L, I^U], [F^L, F^U]) = \frac{T^L + T^U - F^L - F^U}{2}$$

$$c_{int}^{FS}([T^L, T^U], [I^L, I^U], [F^L, F^U]) = \frac{T^L + T^U}{2}$$

All these functions are very much used in decision-making applications.

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Introduction to NeutroHyperGroups

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Abstract. NeutroSophication and AntiSophication are processes through which NeutroAlgebraic and AntiAlgebraic structures can be generated from any classical structures. Given any classical structure with m operations (laws and axioms) where $m \geq 1$ we can generate $(2^m - 1)$ NeutroStructures and $(3^m - 2^m)$ AntiStructures. In this paper, we introduce for the first time the concept of NeutroHyperGroups. Specifically, we study a particular class of NeutroHyperGroups called $[2, 3]$ -NeutroHyperGroups and present their basic properties and several examples. It is shown that the intersection of two $[2, 3]$ -NeutroSubHyperGroups is not necessarily a $[2, 3]$ -NeutroSubHypergroup but their union may produce a $[2, 3]$ -NeutroSubhypergroup. Also, the quotient of a $[2, 3]$ -NeutroHyperGroup factored by a $[2, 3]$ -NeutroSubHyperGroup is shown to be a $[2, 3]$ -NeutroHyperGroup. Examples are provided to show that in the neutrosophic environment, $[2, 3]$ -NeutroHyperGroups are associated with dismutation reactions in some chemical reactions and biological processes.

Keywords: NeutroHyperGroup, NeutroSubHyperGroup, NeutroHyperGroupHomomorphism, NeutroHyperGroupIsomorphism.

1. Introduction

In 2013, Agboola and Davvaz established the connections between neutrosophic set and algebraic hyperstructures. In [1–3] they studied neutrosophic hypergroup, neutrosophic canonical hypergroup and neutrosophic hyperrings. Since then several neutrosophic algebraic structures have been studied and many results have been obtained and published. Recently, Ibrahim and Agboola in [13] studied Neutrosophic Hypernearings and presented some of their properties. In 2019, Florentin Smarandache in [20] presented the concept of NeutroAlgebraicStructures and AntiAlgebraicStructures which can be generated from classical algebraic structures through processes called NeutroSophication and AntiSophication respectively. He recalled, improved and extended several definitions and properties of these new structures in [19]. These new concepts have provided new methodologies for handling indeterminate,

incomplete and imprecise information and processes. The work of Smarandache in [20] was studied viz-a-viz the classical number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and \mathbb{C} by Agboola et al. in [4]. In [5, 6], Agboola formally presented the notions of NeutroGroups, NeutroSubgroups, NeutroRings, NeutroSubrings, NeutroIdeal, NeutroQuotientRings, and he established several properties of these structures and their substructures for the classes he considered. Recently, Rezaei and Smarandache in [17] introduced the concepts of Neutro-BE-algebras and Anti-BE-algebras and in [7, 12] Agboola and Ibrahim introduced the concept of NeutroVectorSpaces and AntiRings. The present paper will be concerned with the introduction of the concept of NeutroHyperGroups and presentations of their basic properties and examples. For more details on Neutrosophy and applications, the readers should see [8, 9, 14–16, 18, 22].

2. Preliminaries

In this section, we will give some definitions, examples and results that will be used in the sequel.

Definition 2.1. Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. The couple (H, \circ) is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad A \circ x = A \circ \{x\} \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

Definition 2.2. A hypergroupoid (H, \circ) is called a semihypergroup if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

A hypergroupoid (H, \circ) is called a quasihypergroup if for all a of H we have $a \circ H = H \circ a = H$. This condition is also called the reproduction axiom.

Definition 2.3. A hypergroupoid (H, \circ) which is both a semihypergroup and a quasi-hypergroup is called a hypergroup.

Definition 2.4. Let (H, \circ) and (H', \circ') be two hypergroupoids. A map $\phi : H \longrightarrow H'$, is called

- (1) an inclusion homomorphism if for all x, y of H , we have $\phi(x \circ y) \subseteq \phi(x) \circ' \phi(y)$;
- (2) a good homomorphism if for all x, y of H , we have $\phi(x \circ y) = \phi(x) \circ' \phi(y)$.

Definition 2.5. Let H be a non-empty set and let $+$ be a hyperoperation on H . The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

- (1) $x + y = y + x$, for all $x, y \in H$,
- (2) $x + (y + z) = (x + y) + z$, for all $x, y, z \in H$,
- (3) there exists a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, for all $x \in H$,
- (4) for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,
- (5) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in H$.

Definition 2.6. [21]

- (i) A classical operation is an operation well-defined for all the set's elements.
- (ii) A classical hyper-operation is a hyper-operation well-defined for all the set's elements.
- (iii) A neutro operation is an operation partially well-defined or partially indeterminate or partially outer defined on a given set.
- (iv) (Anti) Operation is an operation that is outer defined for all set's elements.
- (v) A classical law/axiom defined on a nonempty set is a law/axiom that is totally true (i.e., true for all set's elements).
- (vi) A NeutroLaw/NeutroAxiom defined on a nonempty set is a law/axiom that is true for some set's element [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the classical axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiAxiom.
- (vii) An AntiLaw/AntiAxiom defined on a nonempty set is a law/axiom that is false for all set's elements.
- (viii) NeutroHyperOperation is a hyper-operation partially well-defined, partially indeterminate, and partially outer-defined on a given set.
- (ix) AntiHyperOperation is a hyper-operation outer-defined for all set's elements.
- (x) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements).
- (xi) An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

Theorem 2.7. [17] Let \mathbb{U} be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of \mathbb{U} . If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras.

Definition 2.8. [Classical group] Let G be a nonempty set and let $*$: $G \times G \rightarrow G$ be a binary operation on G . The couple $(G, *)$ is called a classical group if the following conditions hold:

- (G1) $x * y \in G \forall x, y \in G$ [closure law].
- (G2) $x * (y * z) = (x * y) * z \forall x, y, z \in G$ [axiom of associativity].
- (G3) There exists $e \in G$ such that $x * e = e * x = x \forall x \in G$ [axiom of existence of neutral element].
- (G4) There exists $y \in G$ such that $x * y = y * x = e \forall x \in G$ [axiom of existence of inverse element] where e is the neutral element of G .

If in addition $\forall x, y \in G$, we have

- (G5) $x * y = y * x$, then $(G, *)$ is called an abelian group.

Definition 2.9. [NeutroSophication of the law and axioms of the classical group] [5]

- (NG1) There exist some duplets $(x, y), (u, v), (p, q), \in G$ such that $x * y \in G$ (inner-defined with degree of truth T) and $[u * v = \text{indeterminate (with degree of indeterminacy I) or } p * q \notin G \text{ (outer-defined/falsehood with degree of falsehood F)}]$ [NeutroClosureLaw].
- (NG2) There exist some triplets $(x, y, z), (p, q, r), (u, v, w) \in G$ such that $x * (y * z) = (x * y) * z$ (inner-defined with degree of truth T) and $[[p * (q * r)] \text{ or } [(p * q) * r] = \text{indeterminate (with degree of indeterminacy I) or } u * (v * w) \neq (u * v) * w \text{ (outer-defined/falsehood with degree of falsehood F)}]$ [NeutroAxiom of associativity (NeutroAssociativity)].
- (NG3) There exists an element $e \in G$ such that $x * e = e * x = x$ (inner-defined with degree of truth T) and $[[x * e] \text{ or } [e * x] = \text{indeterminate (with degree of indeterminacy I) or } x * e \neq x \neq e * x \text{ (outer-defined/falsehood with degree of falsehood F)}]$ for at least one $x \in G$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].
- (NG4) There exists an element $u \in G$ such that $x * u = u * x = e$ (inner-defined with degree of truth T) and $[[x * u] \text{ or } [u * x]] = \text{indeterminate (with degree of indeterminacy I) or } x * u \neq e \neq u * x \text{ (outer-defined/falsehood with degree of falsehood F)}]$ for at least one $x \in G$ [NeutroAxiom of existence of inverse element (NeutroInverseElement)] where e is a NeutroNeutralElement in G .
- (NG5) There exist some duplets $(x, y), (u, v), (p, q) \in G$ such that $x * y = y * x$ (inner-defined with degree of truth T) and $[[u * v] \text{ or } [v * u] = \text{indeterminate (with degree of indeterminacy I) or } p * q \neq q * p \text{ (outer-defined/falsehood with degree of falsehood F)}]$ [NeutroAxiom of commutativity (NeutroCommutativity)].

Definition 2.10. [AntiSophication of the law and axioms of the classical group] [5]

- (AG1) For all the duplets $(x, y) \in G$, $x * y \notin G$ [AntiClosureLaw].
- (AG2) For all the triplets $(x, y, z) \in G$, $x * (y * z) \neq (x * y) * z$ [AntiAxiom of associativity (AntiAssociativity)].
- (AG3) There does not exist an element $e \in G$ such that $x * e = e * x = x \forall x \in G$ [AntiAxiom of existence of neutral element (AntiNeutralElement)].
- (AG4) There does not exist $u \in G$ such that $x * u = u * x = e \forall x \in G$ [AntiAxiom of existence of inverse element (AntiInverseElement)] where e is an AntiNeutralElement in G .
- (AG5) For all the duplets $(x, y) \in G$, $x * y \neq y * x$ [AntiAxiom of commutativity (AntiCommutativity)].

Definition 2.11. [5] A NeutroGroup NG is an alternative to the classical group G that has at least one NeutroLaw or at least one of $\{NG1, NG2, NG3, NG4\}$ with no AntiLaw or AntiAxiom.

Definition 2.12. [5] An AntiGroup AG is an alternative to the classical group G that has at least one AntiLaw or at least one of $\{AG1, AG2, AG3, AG4\}$.

Definition 2.13. [5] A NeutroAbelianGroup NG is an alternative to the classical abelian group G that has at least one NeutroLaw or at least one of $\{NG1, NG2, NG3, NG4\}$ and $NG5$ with no AntiLaw or AntiAxiom.

Definition 2.14. [5] An AntiAbelianGroup AG is an alternative to the classical abelian group G that has at least one AntiLaw or at least one of $\{AG1, AG2, AG3, AG4\}$ and $AG5$.

Proposition 2.15. [5] Let $(G, *)$ be a finite or infinite classical non abelian group. Then:

- (i) there are 15 types of NeutroNonAbelianGroups,
- (ii) there are 65 types of AntiNonAbelianGroups.

Proposition 2.16. [5] Let $(G, *)$ be a finite or infinite classical abelian group. Then:

- (i) there are 31 types of NeutroAbelianGroups,
- (ii) there are 211 types of AntiAbelianGroups.

Definition 2.17. [5] Let $(NG, *)$ be a NeutroGroup. A nonempty subset NH of NG is called a NeutroSubgroup of NG if $(NH, *)$ is also a NeutroGroup of the same type as NG . If $(NH, *)$ is a NeutroGroup of a type different from that of NG , then NH will be called a QuasiNeutroSubgroup of NG .

Example 2.18. [5]

- (i) Let $NG = \mathbb{N} = \{1, 2, 3, 4, \dots\}$. Then $(NG, .)$ is a finite NeutroGroup where $"."$ is the binary operation of ordinary multiplication.
- (ii) Let $AG = \mathbb{Q}_+^*$ be the set of all irrational positive numbers. Then $(AG, *)$ is an infinite AntiGroup.
- (iii) Let $\mathbb{U} = \{a, b, c, d, e, f\}$ be a universe of discourse and let $AG = \{a, b, c, \}$ be a subset of \mathbb{U} . Let $*$ be a binary operation defined on AG as shown in the Cayley table below:

$*$	a	b	c
a	d	c	b
b	c	e	a
c	b	a	f

Then $(AG, *)$ is a finite AntiGroup.

3. Formulation of a NeutroHyperGroup

Definition 3.1. [Classical Hypergroup]

Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then (H, \circ) is a hypergroup if the following conditions hold:

- (H1) for all $x, y \in H$, $x \circ y \subseteq H$ (closure law),
- (H2) for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$ (associative axiom),

(H3) for all $x \in H$, $x \circ H = H \circ x = H$ (reproductive axiom).

Definition 3.2. [Neutrosophication of the law and axioms of the classical hypergroup]

- (NH1) There exist some duplets $(u, v), (x, y), (p, q) \in H$ such that $u \circ v \subseteq H$ (inner-defined with the degree of truth T) and $[x \circ y = \text{indeterminate (with the degree of indeterminacy I) or } p \circ q \not\subseteq H \text{ (outer-defined/falsehood with degree of falsehood F)}]$.
- (NH2) There exist some triplets $(u, v, w), (x, y, z), (p, q, r) \in H$ such that $u \circ (v \circ w) = (u \circ v) \circ w$ (inner-defined with the degree of truth T) and $[x \circ (y \circ z) \text{ or } (x \circ y) \circ z = \text{indeterminate (with the degree of indeterminacy I) or } (p \circ q) \circ r \neq p \circ (q \circ r) \text{ (outer-defined/falsehood with degree of falsehood F)}]$.
- (NH3) There exists at least a triplet $(u, v, x) \in H$ such that $u \circ H = H \circ u = H$ (inner-defined with the degree of truth T) and $[v \circ H \text{ or } H \circ v = \text{indeterminate (with the degree of indeterminacy I) or } x \circ H \neq H \neq H \circ x \text{ (outer-defined/falsehood with degree of falsehood F)}]$.

Definition 3.3. [AntiSophication of the law and axioms of the classical hypergroup]

- (AH1) $u \circ v \not\subseteq H \forall u, v \in H$ (anti closure law).
- (AH2) $u \circ (v \circ w) \neq (u \circ v) \circ w \forall u, v, w \in H$ (anti associative axiom)
- (AH3) $x \circ H \neq H$ and $H \circ x \neq H \forall x \in H$ (anti reproductive axiom).

Definition 3.4. A NeutroHyperGroup (NH, \circ) is an alternative to the classical hypergroup (H, \circ) that has a NeutroLaw or at least one of NH2 and NH3 with no Antilaw or AntiAxiom.

Definition 3.5. An AntiHyperGroup (AH, \circ) is an alternative to the classical hypergroup (H, \circ) that has an AntiLaw or at least one of AH2 and AH3.

Theorem 3.6. Let (H, \circ) be a classical hypergroup. Then,

- (1) there are 7 classes of NeutroHyperGroup.
- (2) there are 19 classes of AntiHyperGroup.

Proof. The proof follows easily from Theorem 2.7 . \square

Theorem 3.6 shows that there are 7 classes of NeutroHypergroups. The classes where NH1 – NH3 hold are called the trivial NeutroHyperGroups. Examples of NeutroHyperGroups in this class are presented below.

Example 3.7. Let $V = \{u, v, w, s, t, z\}$ be a universe of discourse and let $NH = \{v, w, s, z\}$ be a subset of V . Define on NH the binary Operation \circ as shown in the table below.

It can easily be deduced from the table that (NH, \circ) is a trivial NeutroGroup. The subset $NK = \{v, s\}$ of NH is also a trivial NeutroGroup and hence a NeutroSubgroup of NH .

\circ	v	w	s	z
v	s	t	z	v
w	z	u	v	s
s	v	w	s	z
z	z	u	v	s

Now, consider the NeutroSubgroup $NK = \{v, s\}$. Defined on NK a hyperoperation \star_{NH} as follows :

$$x \star_{NK} y = \begin{cases} x \circ NK \circ y = \{x \circ z \circ y : z \in NK\} \text{ if } x = y, \\ \{x, x \circ y\} \text{ if } x \neq y. \end{cases}$$

From this definition we construct the table below:

TABLE 1. Cayley table for the hyperoperation " \star_{NK} "

\star_{NK}	v	w	s	z
v	$\{v, z\}$	$\{t, v\}$	$\{v, z\}$	$\{v\}$
w	$\{w, z\}$	$\{t, u\}$	$\{v, w\}$	$\{s, w\}$
s	$\{s, v\}$	$\{s, w\}$	$\{s, z\}$	$\{s, z\}$
z	$\{z\}$	$\{u, z\}$	$\{v, z\}$	$\{s, v\}$

It can be seen from Table 1 that \star_{NK} satisfies :

- (1) NeutroClosureLaw (NH1) : Except for the composition

$v \star_{NK} w = \{t, v\}$, $w \star_{NK} w = \{t, u\}$ and $z \star_{NK} w = \{u, z\}$ which are false with 18.75% degree of falsehood, all other composition are true with 81.25% degree of truth.

- (2) NeutroAssociative (NH2) :

$$s \star_{NK} (v \star_{NK} v) = (s \star_{NK} v) \star_{NK} v = \{s, v, z\}.$$

$$s \star_{NK} (w \star_{NK} v) = \{s, w, z\} \text{ but } (s \star_{NK} w) \star_{NK} v = \{s, v, w, z\} \neq \{s, w, z\}.$$

- (3) NeutroReproductionAxiom (NH3) :

$$s \star_{NK} NH = NH \star_{NK} s = \{v, w, s, z\} = NH.$$

$$w \star_{NK} NH = NH \star_{NK} w = \{u, v, w, s, t, z\} \neq NH.$$

Hence, (NH, \star_{NK}) is a trivial NeutroHyperGroup.

Example 3.8. Let $V = \{u, v, w, s, t, z\}$ be a universe of discourse and let $NH = \{u, v, w, z\}$ be a subset of V . Define on NH the binary operation \circ as shown in the table below.

It can be shown from the table that (NH, \circ) is a NeutroGroup and the subset $NK = \{u, v\}$ of NH is a classical group with respect to \circ .

\circ	u	v	s	z
u	u	v	s	z
v	v	u	t	w
s	s	w	u	t
z	z	t	w	u

Now, defined on NH a hyperoperation \star_{NK} as follows :

$$x \star_{NK} y = \begin{cases} x \circ NK \circ y = \{x \circ z \circ y : z \in NK\} & \text{if } x \neq u, y \neq u \text{ and } x \neq y, \\ x \circ y & \text{if } x = y, \\ \{x, y\} & \text{otherwise.} \end{cases}$$

Note, if $x \circ y$ is indeterminate, we write $x \circ y = I$.

From this definition we construct the table below:

TABLE 2. Cayley table for the hyperoperation " \star_{NK} "

\star_{NK}	u	v	s	z
u	u	$\{u, v\}$	$\{u, s\}$	$\{u, z\}$
v	$\{u, v\}$	u	$\{s, t\}$	$\{w, z\}$
s	$\{u, s\}$	$\{w, s\}$	u	$\{v, t\}$
z	$\{u, z\}$	$\{t, I\}$	$\{w, I\}$	u

It can be seen from the table that \star_{NK} satisfies :

- (1) NeutroClosureLaw ($NH1$) : Except for the compositions

$v \star_{NK} s = \{s, t\}$, $v \star_{NH} z = \{w, z\}$, $s \star_{NK} v = \{w, s\}$ and $s \star_{NH} z = \{v, t\}$ which are false with 25.0% degree of falsehood, and the compositions $z \star_{NK} v = \{t, I\}$ and $z \star_{NK} s = \{w, I\}$ which are indeterminate with 12.5% degree of indeterminacy all other compositions are true with 62.5% degree of truth.

- (2) NeutroAssociative ($NH2$) :

$$u \star_{NK} (v \star_{NK} u) = (u \star_{NK} v) \star_{NK} u = \{u, v\}.$$

$$s \star_{NK} (u \star_{NK} v) = \{u, w, s\} \text{ but } (s \star_{NK} u) \star_{NK} v = \{u, v, w, s\} \neq \{u, w, s\}.$$

- (3) NeutroReproductionAxiom ($NH3$) :

$$u \star_{NK} NH = NH \star_{NK} s = \{u, v, s, z\} = NH.$$

$$z \star_{NK} NH = \{u, w, t, z, I\} \neq NH \text{ and } NH \star_{NK} z = \{u, v, w, t, z\} \neq NH.$$

Hence, (NH, \star_{NK}) is a trivial NeutroHyperGroup.

4. Study of a Class of NeutroHyperGroup

In this section, we are going to consider a particular class of NeutroHyperGroups (NH, \star) where

- (i) (NH, \star) is a classical hypergroupoid,
- (ii) the hypergroupoid (NH, \star) is a NeutroSemiHyperGroup and
- (iii) the hypergroupoid (NH, \star) is a NeutroQuasiHyperGroup.

We will refer to this class of NeutroHyperGroups as [2, 3]-NeutroHyperGroup (i.e., H2 and H3 of Definition 3.1 are NeutroAxioms).

Example 4.1. Let $NH = \{u, v, s, t\}$ be a non empty set and let $"."$ be a binary operations defined on NH as shown in the table below.

\cdot	u	v	s	t
u	v	t	s	u
v	v	u	t	u
s	s	t	v	u
t	u	u	u	u

Now consider the subset $NK = \{u, v\}$. Defined on NH a hyperoperation \star_{NK} as follows :

$$x \star_{NK} y = \begin{cases} x \cdot NK \cdot y = \{x \cdot z \cdot y : z \in NK\} & \text{if } x \neq y \text{ and } x, y \neq u, \\ x \cdot y & \text{otherwise.} \end{cases}$$

From this definition we construct the table below.

TABLE 3. Cayley table for the hyperoperation $"\star_{NK}"$

\star_{NK}	u	v	s	t
u	v	t	s	u
v	v	$\{u, t\}$	$\{s, t\}$	u
s	s	$\{u, t\}$	$\{u, v\}$	u
t	u	t	s	u

It can be seen from Table 3 that :

- (1) (NH, \star_{NK}) is a hypergroupoid.
- (2) \star_{NK} is NeutroAssociative, since

$$(t \star_{NK} v) \star_{NK} t = t \star_{NK} (v \star_{NK} t) = \{u\}.$$

$$(v \star_{NK} s) \star_{NK} t = \{u\} \text{ but } v \star_{NK} (s \star_{NK} t) = \{v\} \neq \{u\}.$$

Hence, the hypergroupoid (NH, \star_{NK}) is NeutroSemiHyperGroup.

(3) \star_{NK} satisfies NeutroReproductiveAxiom, since

$$s \star_{NK} NH = NH \star_{NK} s = \{u, v, s, t\} = NG.$$

$$v \star_{NK} NH = \{u, v, s, t\} \neq \{u, t\} = NH \star_{NK} v.$$

Hence, the hypergroupoid (NH, \star_{NK}) is a NeutroQuasiHyperGroup.

Accordingly, (NH, \star_{NK}) is a $[2, 3]$ -NeutroHyperGroup .

Example 4.2. Let $NH = \{\alpha, \beta, \gamma, \phi, \psi\}$ and let \star be a hyperoperation defined on NH as shown in the table below ;

TABLE 4. Cayley table for the hyperoperation " \star "

\star	α	β	γ	ϕ	ψ
α	α	α	α	α	α
β	α	$\{\gamma, \phi\}$	$\{\phi, \psi\}$	$\{\beta, \gamma\}$	$\{\alpha, \psi\}$
γ	α	$\{\alpha, \gamma\}$	$\{\alpha, \gamma\}$	γ	$\{\alpha, \gamma\}$
ϕ	α	$\{\alpha, \phi\}$	ϕ	$\{\alpha, \phi\}$	$\{\alpha, \phi\}$
ψ	α	$\{\gamma, \phi\}$	$\{\beta, \phi\}$	$\{\gamma, \psi\}$	$\{\alpha, \beta\}$

Then, (NH, \star) is a $[2, 3]$ -NeutroHyperGroup.

It can be seen from Table 4 that :

- (1) (NH, \star) is a hypergroupoid.
- (2) \star is NeutroAssociative, since

$$(\phi \star \beta) \star \psi = \phi \star (\beta \star \psi) = \{\alpha, \phi\}.$$

$$(\beta \star \phi) \star \gamma = \{\alpha, \gamma, \phi, \psi\} \text{ but } \beta \star (\phi \star \gamma) = \{\beta, \gamma\} \neq \{\alpha, \gamma, \phi, \psi\}.$$

Hence, the hypergroupoid (NH, \star) is NeutroSemiHyperGroup.

(3) \star satisfies NeutroReproductiveAxiom, since

$$\psi \star NH = NH \star \psi = \{\alpha, \beta, \gamma, \phi, \psi\} = NH.$$

$$\phi \star NH = \{\alpha, \phi\} \neq \{\alpha, \beta, \gamma, \phi, \psi\} = NH \star \phi.$$

Hence, the hypergroupoid (NH, \star) is a NeutroQuasiHyperGroup.

Accordingly, (NH, \star) is a $[2, 3]$ -NeutroHyperGroup.

Example 4.3. Let $NH = \{m, n, p, q\}$ and let \circ be a hyperoperation defined on NH as shown in table 5 below.

Then, (NH, \circ) is a $[2, 3]$ -NeutroHyperGroup. It can be seen from the Table 5 that :

- (1) (NH, \circ) is a Hypergroupoid.

TABLE 5. Cayley table for the hyperoperation " \circ "

\circ	m	n	p	q
m	m	$\{m, n\}$	$\{m, p\}$	$\{m, q\}$
n	$\{m, n\}$	p	n	$\{n, p\}$
p	p	$\{p, q\}$	p	$\{p, q\}$
q	q	$\{m, q\}$	q	$\{m, q\}$

(2) \circ is NeutroAssociative, since

$$(m \circ m) \circ n = m \circ (m \circ n) = \{m, n\}.$$

$$(m \circ n) \circ q = \{m, n, p, q\} \text{ but } m \circ (n \circ q) = \{m, n, p\} \neq \{m, n, p, q\}.$$

Hence, the hypergroupoid (NH, \circ) is NeutroSemiHyperGroup.

(3) \circ satisfies NeutroReproductiveAxiom, since

$$m \circ NH = NH \circ m = \{m, n, p, q\} = NH.$$

$$p \circ NH = \{p, q\} \neq \{m, n, p, q\} = NH \circ p.$$

Hence, the hypergroupoid (NH, \circ) is a NeutroQuasiHyperGroup.

Accordingly, (NH, \circ) is a $[2, 3]$ -NeutroHyperGroup.

Example 4.4. Let $NH = \{1, 2, 3, 4, 5, 6\}$ and let \star be a hyperoperation defined on NH as shown in the table below ;

TABLE 6. Cayley table for the hyperoperation " \star "

\star	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	$\{1, 2\}$	2	$\{2, 4\}$	$\{1, 2\}$	$\{2, 4\}$
3	1	3	$\{1, 3\}$	3	$\{1, 3\}$	$\{1, 3\}$
4	1	$\{1, 4\}$	4	$\{2, 4\}$	$\{1, 4\}$	$\{2, 4\}$
5	1	$\{3, 5\}$	$\{2, 5\}$	$\{5, 6\}$	$\{1, 2, 3, 5\}$	$\{2, 4, 5, 6\}$
6	1	$\{3, 6\}$	$\{4, 6\}$	$\{5, 6\}$	$\{1, 3, 4, 6\}$	$\{2, 4, 5, 6\}$

It can be shown from Table 6 that, (NH, \star) is a $[2, 3]$ -NeutroHyperGroup.

Proposition 4.5. Let (NH_1, \star_1) and (NH_2, \star_2) be any two $[2, 3]$ -NeutroHyperGroups. Let

$$NH_1 \times NH_2 = \{(v, k) : v \in NH_1 \text{ and } k \in NH_2\},$$

for $x = (v_1, k_1)$, $y = (v_2, k_2) \in NH_1 \times NH_2$ define :

$$x \star y = ((v_1 \star_1 v_2), (k_1 \star_2 k_2)).$$

Then $(NH_1 \times NH_2, \star)$ is a $[2, 3]$ -NeutroHyperGroup.

Proof. (1) Let $x = (v_1, k_1)$, $y = (v_2, k_2) \in NH_1 \times NH_2$, then

$$x \star y = (v_1, k_1) \star (v_2, k_2) = (v_1 \star_1 v_2, k_1 \star_2 k_2) \subseteq NH_1 \times NH_2. \quad \because v_1 \star_1 v_2 \subseteq NH_1 \text{ and } k_1 \star_2 k_2 \subseteq NH_2.$$

Hence, $(NH_1 \times NH_2, \star)$ is a hypergroupoid.

(2) There exists at least a triplet $((v_1, k_1), (v_2, k_2), (v_3, k_3)) \in NH_1 \times NH_2$ such that

$$\begin{aligned} ((v_1, k_1) \star (v_2, k_2)) \star (v_3, k_3) &= ((v_1 \star_1 v_2), (k_1 \star_2 k_2)) \star (v_3, k_3) \\ &= ((v_1 \star_1 v_2) \star_1 v_3, (k_1 \star_2 k_2) \star_2 k_3) \\ &= (v_1 \star_1 (v_2 \star_1 v_3), k_1 \star_2 (k_2 \star_2 k_3)) \quad \because NH2 \text{ holds in } NH_1 \text{ and } NH_2. \\ &= (v_1, k_1) \star ((v_2 \star_1 v_3), (k_2 \star_2 k_3)) \\ &= (v_1, k_1) \star ((v_2, k_2) \star (v_3, k_3)). \end{aligned}$$

Also, there exists at least a triplet $((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in NH_1 \times NH_2$ such that

$$\begin{aligned} ((a_1, b_1) \star (a_2, b_2)) \star (a_3, b_3) &= ((a_1 \star_1 a_2), (b_1 \star_2 b_2)) \star (a_3, b_3) \\ &= ((a_1 \star_1 a_2) \star_1 a_3, (b_1 \star_2 b_2) \star_2 b_3) \\ &\neq (a_1 \star_1 (a_2 \star_1 a_3), b_1 \star_2 (b_2 \star_2 b_3)) \quad \because NH2 \text{ holds in } NH_1 \text{ and } NH_2. \\ &= (a_1, b_1) \star ((a_2 \star_1 a_3), (b_2 \star_2 b_3)) \\ &= (a_1, b_1) \star ((a_2, b_2) \star (a_3, b_3)). \end{aligned}$$

Hence, $NH2$ holds in $NH_1 \times NH_2$.

(3) There exists at least a $(v, k) \in NH_1 \times NH_2$ with $v \in NH_1$ and $k \in NH_2$ such that

$$\begin{aligned} (v, k) \star NH_1 \times NH_2 &= (v, k) \star \{(v_1, k_1) : v_1 \in NH_1, k_1 \in NH_2\} \\ &= \{(v, k) \star (v_1, k_1) : v_1 \in NH_1, k_1 \in NH_2\} \\ &= \{((v \star_1 v_1), (k \star_2 k_1)) : v_1 \in NH_1, k_1 \in NH_2\} \\ &= (v \star_1 NH_1, k \star_2 NH_2) \\ &= (NH_1 \star_1 v, NH_2 \star_2 k) \quad (\because NH3 \text{ holds in } NH_1 \text{ and } NH_2) \\ &= NH_1 \times NH_2 \star (v, k) \\ &= NH_1 \times NH_2. \end{aligned}$$

Also, there exists at least a $(u, q) \in NH_1 \times NH_2$ with $u \in NH_1$ and $q \in NH_2$ such that

$$\begin{aligned} (u, q) \star NH_1 \times NH_2 &= (u, q) \star \{(v_1, k_1) : v_1 \in NH_1, k_1 \in NH_2\} \\ &= \{(u, q) \star (v_1, k_1) : v_1 \in NH_1, k_1 \in NH_2\} \\ &= \{((u \star_1 v_1), (q \star_2 k_1)) : v_1 \in NH_1, k_1 \in NH_2\} \\ &= (u \star_1 NH_1, q \star_2 NH_2) \\ &\neq (NH_1 \star_1 u, NH_2 \star_2 q) \quad (\because NH3 \text{ holds in } NH_1 \text{ and } NH_2) \\ &= NH_1 \times NH_2 \star (u, q). \end{aligned}$$

Accordingly, $(NH_1 \times NH_2, \star)$ is a $[2, 3]$ -NeutroHyperGroup. \square

Proposition 4.6. Let (NV, \star_1) be a $[2, 3]$ -NeutroHyperGroup and (H, \star_2) be any hypergroup. Let

$$NV \times H = \{(v, h) : v \in NV \text{ and } h \in H\},$$

for $x = (v_1, h_1)$, $y = (v_2, h_2) \in NV \times H$ define :

$$x \star y = ((v_1 \star_1 v_2), (h_1 \star_2 h_2)).$$

Then $(NV \times H, \star)$ is a $[2, 3]$ -NeutroHyperGroup.

Proof. The proof follows similar approach as the proof of 4.5 . \square

Definition 4.7. Let NH be a $[2, 3]$ –NeutroHyperGroup, a non-empty subset NK of NH is called a $[2, 3]$ –NeutroSubHyperGroup of NH if NK is itself a $[2, 3]$ –NeutroHyperGroup.

Example 4.8. Let (NH, \circ) be the $[2, 3]$ –NeutroHyperGroup defined in Example 4.3 and let $NK = \{m, p, q\}$ be a subset of NH . Let \circ be defined as shown in the table below.

TABLE 7. Cayley table for the hyperoperation " \circ "

\circ	m	p	q
m	m	$\{m, p\}$	$\{m, q\}$
p	p	p	$\{p, q\}$
q	q	q	$\{m, q\}$

It can be shown from Table 7 that (NK, \circ) is a $[2, 3]$ –NeutroHyperGroup. Then, (NK, \circ) is a $[2, 3]$ –NeutroSubHyperGroup of NH .

Example 4.9. Let (NH, \circ) be the $[2, 3]$ –NeutroHyperGroup defined in Example 4.4 and let $NK = \{1, 2, 3, 4\}$ and $NW = \{1, 2, 3, 5\}$ be subsets of NH .

Let \star be defined as shown in the tables below :

TABLE 8. Cayley table for the hyperoperation " \star "

\star	1	2	3	4
1	1	1	1	1
2	1	$\{1, 2\}$	2	$\{2, 4\}$
3	1	3	$\{1, 3\}$	3
4	1	$\{1, 4\}$	4	$\{2, 4\}$

TABLE 9. Cayley table for the hyperoperation " \star "

\star	1	2	3	5
1	1	1	1	1
2	1	$\{1, 2\}$	2	$\{1, 2\}$
3	1	3	$\{1, 3\}$	$\{1, 3\}$
5	1	$\{3, 5\}$	$\{2, 5\}$	$\{1, 2, 3, 5\}$

We can see from Tables 8 and 9 that (NK, \star) and (NW, \star) are $[2, 3]$ –NeutroHyperGroups. Then, (NK, \star) and (NW, \star) are $[2, 3]$ –NeutroSubHyperGroups of NH .

Now, consider the following:

- (1) $NK \cup NW = \{1, 2, 3, 4, 5\}$.
- (2) $NK \cap NW = \{1, 2, 3\}$.

It can be shown from Table 6 that $NK \cup NW$ is a $[2, 3]$ –NeutroSubHyperGroup of NR but $NK \cap NW$ is a non-trivial NeutroSemiHyperGroup of NH .

These observations are recorded in Remark 4.10

Remark 4.10. Let NK and NW be two $[2, 3]$ –NeutroSubHyperGroups of a $[2, 3]$ –NeutroHyperGroup NH . Then

- (1) $NK \cup NW$ can be a $[2, 3]$ – NeutroSubHyperGroup of NH .
- (2) $NK \cap NW$ is not necessarily a $[2, 3]$ – NeutroSubHyperGroup of NH .

Proposition 4.11. Let NH be a $[2, 3]$ – NeutroHyperGroup and let NW be a $[2, 3]$ – NeutroSubHyperGroup of NH . For $aNW, bNW \in NH/NW$ with $a, b \in NH$, let \star be a hyperoperation defined on NH/NW by

$$aNW \star bNW = \{cd \mid c \in aNW, d \in bNW\}.$$

Then, $(NH/NW, \star)$ is a $[2, 3]$ – NeutroHyperGroup which is known as a $[2, 3]$ – NeutroQuotientHyperGroup.

Proof. The proof of this Proposition will be by a constructed example as given in Example 4.12. \square

Example 4.12. Let (NH, \circ) be the $[2, 3]$ – NeutroHyperGroup defined in Example 4.4 and let NW be the $[2, 3]$ –NeutroSubHyperGroup of Example 4.9 . Then we have

$$NH/NW = \{NW, p \circ NW, n \circ NW, q \circ NW\}.$$

Define on NH/NW a hyperoperation \star as shown in the table below

TABLE 10. Cayley table for the hyperoperation " \star "

\star	NW	pNW	nNW	qNW
NW	$\{NW, pNW, qNW\}$	$\{NW, pNW, qNW\}$	$\{NW, pNW, nNW\}$	$\{NW, pNW, qNW\}$
pNW	$\{pNW, qNW\}$	$\{pNW, qNW\}$	$\{pNW, qNW\}$	$\{pNW, qNW\}$
nNW	$\{NW, pNW, nNW\}$	$\{pNW, nNW\}$	$\{NW, nNW\}$	$\{pNW, qNW, nNW\}$
qNW	$\{NW, qNW\}$	$\{NW, qNW\}$	$\{nNW, qNW\}$	qNW

Then, it can be seen from the table that :

- (1) $(NH/NW, \star)$ is a hypergroupoid.
- (2) there exists at least a triplet $(pNW, nNW, qNW) \in NH/NW$ such that

$$pNW \star (nNW \star qNW) = (pNW \star nNW) \star qNW = \{pNW, qNW\}.$$

And, there exists at least a triplet $(qNW, nNW, qNW) \in NH/NW$ such that

$$(qNW \star nNW) \star qNW = \{pNW, nNW, qNW\} \neq \{NW, nNW, qNW\} = qNW \star (nNW \star qNW).$$

- (3) there exists $nNW \in NH/NW$ such that

$$nNW \star NH/NW = NH/NW \star nNW = NH/NW.$$

And, there exists $pNW \in NH/NW$ such that

$$pNW \star NH/NW = \{pNW, qNW\} \neq \{NW, pNW, nNW, qNW\} = NH/NW \star pNW.$$

Accordingly, $(NH/NW, \star)$ is a $[2, 3]$ – NeutroHyperGroup.

Definition 4.13. Let (NH, \star) and (NW, \circ) be any two $[2, 3]$ – NeutroHyperGroups . The mapping

$$\phi : NH \longrightarrow NW$$

- (1) is called a NeutroHyperGroupHomomorphism if $\phi(a \star b) \subseteq \phi(a) \circ \phi(b)$ for at least a duplet $(x, y) \in NH$.
- (2) is called a good NeutroHyperGroupHomomorphism if $\phi(a \star b) = \phi(a) \circ \phi(b)$ for at least a duplet $(x, y) \in NH$.
- (3) is called NeutroHyperGroupIsomorphism if ϕ is a NeutroHyperGroupHomomorphism and ϕ^{-1} is also a NeutroHyperGroupHomomorphism.

Definition 4.14. Let (NH, \star) and (NW, \star) be any two $[2, 3]$ – NeutroHyperGroups with NeutroNeutralElements e_{NH} and e_{NW} respectively.

Let $\phi : NH \longrightarrow NW$ be a good NeutroHyperGroupHomomorphism.

The kernel of ϕ denoted by $NHKer\phi$ is defined as

$$NHKer\phi = \{x : \phi(x) = e_{NW}\}.$$

The image of ϕ denoted by $NHIm\phi$ is defined as

$$NHIm\phi = \{y \in NW : y = \phi(x) \text{ for at least one } x \in NH\}.$$

Example 4.15. Let (NK, \circ) be the $[2, 3]$ – NeutroHyperGroup of Example 4.8 and let

$$\phi : NK \times NK \longrightarrow P^*(NK)$$

be given by $\phi(k_1, k_2) = k_1 \circ k_2$ for all $k_1, k_2 \in NK$.

Then ϕ is a good NeutroHyperGroupHomomorphism.

We have $NHKer\phi = \{(m, m), (p, m), (p, p), (q, m), (q, p)\}$ and

$$NHIm\phi = \{m, p, q, \{m, p\}, \{m, q\}, \{p, q\}\}.$$

It can be shown that $NHKer\phi$ is a $[2, 3]$ – NeutroSubHyperGroup of $(NK \times NK, \circ)$ and $NHIm\phi$ is a $[2, 3]$ – NeutroSubHyperGroup of $(P^*(NK), \circ)$.

Proposition 4.16. Let (NH, \star) and (NW, \star) be any two $[2, 3]$ –NeutroHypergroups.

Let $\phi : NH \longrightarrow NW$ be a good NeutroHyperGroupHomomorphism. Then :

- (1) $NHKer\phi$ is a NeutroSubHyperGroup of NH .
- (2) $NHIm\phi$ is a NeutroSubHyperGroup of NW .

Proof. The proof follows from Example 4.15 . \square

5. Applications of [2, 3]– NeutroHyperGroups in Biological and Chemical Sciences

In [10], Davvaz et al. provided some examples of hyperstructures and weak hyperstructures associated with dismutation reactions. In what follows, we will provide examples to show that when dismutation reactions take place in the neutrosophic environment, they are associated with [2, 3]– NeutroHyperGroups.

Example 5.1. Let $NX = \{x_0 = Sn, x_2 = Sn^{2+}, x_4 = Sn^{4+}\}$ be a set of Tin in different oxidation state. Define on NX , a hyperoperation \star as shown in the table below, where \star is a comproportionation reaction (without energy). Then, it can be seen from Table 11 that :

TABLE 11. Cayley table for the hyperoperation " \star "

\star	x_0	x_2	x_4
x_0	x_0	$\{x_0, x_2\}$	x_2
x_2	$\{x_0, x_2\}$	x_2	$\{x_2, x_4\}$
x_4	x_2	$\{x_2, x_4\}$	x_4

- (1) (NX, \star) is a hypergroupoid.
- (2) For the triplet $(x_2, x_4, x_2) \in NX$, we have

$$x_2 \star (x_4 \star x_2) = (x_2 \star x_4) \star x_2 = \{x_2, x_4\}$$

and for the triplet $(x_0, x_2, x_4) \in NX$, we have have

$$(x_0 \star x_2) \star x_4 = \{x_2, x_4\} \neq \{x_0, x_2\} = x_0 \star (x_2 \star x_4).$$

- (3) For $x_2 \in NX$, we have

$$x_2 \star NX = NX \star x_2 = \{x_0, x_2, x_4\}$$

and for $x_4 \in NX$, we have

$$x_4 \star NX = NX \star x_4 = \{x_2, x_4\} \neq NX.$$

Accordingly, (NX, \star) is a [2, 3]– NeutroHyperGroup.

Example 5.2. Let $BG = \{a_0 = AA, a_1 = AS, a_3 = SS\}$ be a set of blood group. Define on BG , a hyperoperation \star as shown in the table below, where \star denote mating.

Then, it can be seen from table 12 that :

- (1) (BG, \star) is a hypergroupoid.

A Comproportionation is a chemical reaction where two reactants each containing the same element but with a different oxidation number, will give a product with oxidation number intermediate of the two reactant.

TABLE 12. Cayley table for the hyperoperation " \star "

\star	a_0	a_1	a_3
a_0	a_0	$\{a_0, a_1\}$	a_1
a_1	$\{a_0, a_1\}$	$\{a_0, a_1, a_3\}$	$\{a_1, a_3\}$
a_3	a_1	$\{a_1, a_3\}$	a_3

(2) For the triplet $(a_1, a_3, a_1) \in BG$, we obtain

$$a_1 \star (a_3 \star a_1) = (a_1 \star a_3) \star a_1 = \{a_0, a_1, a_3\}.$$

and for the triplet $(a_0, a_1, a_3) \in BG$, we obtain

$$(a_0 \star a_1) \star a_3 = \{a_1, a_3\} \neq \{a_0, a_1\} = a_0 \star (a_1 \star a_3).$$

(3) For an element $a_1 \in BG$, we obtain

$$a_1 \star BG = BG \star a_1 = \{a_0, a_1, a_3\} = BG$$

and for an element $a_3 \in BG$, we obtain

$$a_3 \star BG = BG \star a_3 = \{a_1, a_3\} \neq BG.$$

Accordingly, (BG, \star) is a $[2, 3]$ -NeutroHyperGroup.

Remark 5.3. It is evident from Examples 5.1 and 5.2 that in the neutrosophic environment, $[2, 3]$ -NeutroHyperGroups are associated with dismutation reactions in some chemical reactions and biological processes.

6. Conclusions

In this paper, we have for the first time introduced the concept of NeutroHyperGroups. Specifically, a class of NeutroHyperGroups called $[2, 3]$ -NeutroHyperGroup was investigated and some of their elementary properties and several examples were presented. It was shown that the intersection of two $[2, 3]$ -NeutroSubHyperGroups is not necessarily a $[2, 3]$ -NeutroSubHyperGroup but their union may produce a $[2, 3]$ -NeutroSubHyperGroup. Also, it was shown that the quotient of a $[2, 3]$ -NeutroHyperGroup factored by a $[2, 3]$ -NeutroSubHyperGroup is a $[2, 3]$ -NeutroHyperGroup. Examples were provided to show that in the neutrosophic environment, $[2, 3]$ -NeutroHyperGroups are associated with dismutation reactions in some chemical reactions and biological processes. In our future work, we hope to use the algebraic properties of NeutroHyperGroups to analyze some chemical reactions and biological processes.

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On NeutroNilpotentGroups

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Abstract. In this paper, we introduce the notion of commutator of two elements in a specific NeutroGroup. Then we define the notion of a NeutroNilpotentGroup and we study some of their properties. Moreover, we show that the intersection of two NeutroNilpotentGroups is a NeutroNilpotentGroup. Also, we show that the quotient of a NeutroNilpotentGroup is a NeutroNilpotentGroup. Specially, using NeutroHomomorphism we prove the NeutroNilpotency is closed with respect to homomorphic image.

Keywords: NeutroGroup; NeutroSubgroup; NeutroNilpotentGroup; NeutroQuotientGroup; NeutroGroup Homomorphism.

1. Introduction

One of the most important concepts in the study of groups is the notion of nilpotency [6]. Nilpotent groups arose in Galois theory, as well as in the classification of groups. By Galois theory, certain problems in field theory reduced to group theory. In [10, 11], Smarandache introduced the notions of NeutroDefined, AntiDefined laws, NeutroAxiom and AntiAxiom. Then in [9], he studied NeutroAlgebras and AntiAlgebras. Rezaei et al. in [5], proved that there are $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras in a classical algebra S with n operations and axioms all together, where $n \geq 1$. Agboola et al. in [1], studied NeutroGroups $(NG, *)$ where the law of composition and axioms defined on NG may either be only partially defined (partially true), or partially undefined (partially false), or totally undefined (totally false) with respect to $*$. Moreover, they considered three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element) to show the difference between groups and NeutroGroups. Also, in [3], Agboola studied NeutroRings by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and

NeuroDistributivity (multiplication over addition)). Scholars applied the notion of NeuroAxioms and NeuroLaw on Rings, Subgrings, Ideals, QuotientRings and Ring Homomorphism to present some new notions and several results are obtained (see [3], [7]). In this paper, we consider a class of NeuroGroups was introduced in [1], and define the notion of NeuroNilpotentGroups. Moreover, we investigate elementary properties of NeuroNilpotentGroups. Specially, we show that the intersection of two NeuroNilpotentGroups is a NeuroNilpotent-Group. Also, we prove the NeuroNilpotency is closed with respect to homomorphic image.

2. Preliminaries

We recall some basic definitions and results which are proposed by the pioneers of this subject.

Definition 2.1 ([8]). (i) A classical operation is well defined for all the set's elements.
(ii) A NeuroOperation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set.
(iii) A classical law/axiom defined on a nonempty set is totally true (i.e. true for all set's elements).
(iv) A NeuroLaw/NeuroAxiom (or NeutrosophicLaw/NeutrosophicAxiom) defined on a nonempty set is a law/axiom that is true for some set's elements (degree of truth (T)), indeterminate for other set's elements (degree of indeterminacy (I)), or false for the other set's elements (degree of falsehood (F)), where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the classical axiom.
(v) A NeuroAlgebra is an algebra that has at least one NeuroOperation or one NeuroAxiom (axiom that is true for some elements, indeterminate for other elements and false for other elements).

Definition 2.2 ([4]). For a nonempty set G and a binary operation $*$ on G the couple $(G, *)$ is called a classical group if the following conditions hold:

- (G1) $x * y \in G$ for all $x, y \in G$.
- (G2) $x * (y * z) = (x * y) * z$ for all $x, y, z \in G$.
- (G3) There exists $e \in G$ such that $x * e = e * x = x$ for all $x \in G$.
- (G4) There exists $y \in G$ such that $x * y = y * x = e$ for all $x \in G$, where e is the neutral element of G .

If for all $x, y \in G$, (G5) $x * y = y * x$, then $(G, *)$ is called an abelian group.

Note that $x * y$ will be written as xy for all $x, y \in G$.

Definition 2.3 ([6]). A group $(G, *)$ is called nilpotent if it has a central series, that is, a normal series $e = G_0 \leq G_1 \leq \dots \leq G_n = G$ such that G_{i+1}/G_i is contained in the center of G/G_i for all i . The length of a shortest central series of G is the nilpotent class of n .

Definition 2.4 ([6]). Let $(G, *)$ be a group and x_1, \dots, x_n be elements of G . Commutator of x_1 and x_2 is $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$. A commutator of weight $n \geq 2$ is defined by $[x_1, \dots, x_n] = [x_1, \dots, x_{n-1}, x_n]$, where by convention $[x_1] = x_1$.

A NeutroGroup is an alternative of a group that has either one NeutroOperation (partially well-defined, partially indeterminate and partially outer-defined), or at least one NeutroAxiom (NeutroAssociativity, NeutroNeutralElement or NeutroInverseElement) with no AntiOperation (is an operation outer-defined for all the set's elements (totally falsehood)) or AntiAxiom (is an axiom that is false for all set's elements). It is possible to define NeutroGroup in another way by considering only one NeutroAxiom or by considering two NeutroAxioms or etc.

Definition 2.5. Let NG be a nonempty set and $*$ be a binary operation on NG . The couple $(NG, *)$ is called a NeutroGroup if the following conditions are satisfied:

- (NG1) There exists some triplet $(x, y, z) \in NG$ such that $x*(y*z) = (x*y)*z$ and $u*(v*w) \neq (u*v)*w$ for some $(u, v, w) \in NG$ or there exists some $(r, s, t) \in NG$ such that $r*(s*t) = \text{indeterminate}$ or $(r*s)*t = \text{indeterminate}$ (NeutroAssociativity).
- (NG2) There exists at least an element $a \in NG$ that has a single neutral element i.e., we have $e \in NG$ such that $a*e = e*a = a$ and for $b \in NG$ there does not exist $e \in NG$ such that $b*e = e*b \neq b$ or there exists $e_1, e_2 \in NG$ such that $b*e_1 = e_1*b = b$ or $b*e_2 = e_2*b = b$ with $e_1 \neq e_2$ or there exists at least an element $c \in NG$ that there is $d \in NG$ such that $c*d = d*c = \text{indeterminate}$ (NeutroNeutralElement).
- (NG3) There exists an element $a \in NG$ that has an inverse $b \in NG$ w.r.t. a unit element $e \in NG$ i.e., $a*b = b*a = e$, or there exists at least one element $b \in NG$ that has two or more inverses $c, d \in NG$ w.r.t. some unit element $u \in NG$ i.e., $b*c = c*b = u$, $b*d = d*b = u$ or there exists at least one element $r \in NG$ that has one element $s \in NG$ such that $r*s = s*r = \text{indeterminate}$ (NeutroInverseElement).
- (NG4) There exists some duplet $(a, b) \in NG$ such that $a*b = b*a$ and there exists some duplet $(c, d) \in NG$ such that $c*d \neq d*c$, or there exists some $(r, s) \in NG$, $r*s = \text{indeterminate}$ or $s*r = \text{indeterminate}$, then $(NG, *)$ is called a NeutroAbelianGroup (NeutroAbelianGroup).

Example 2.6. Let $U = \{a, b, c, d, e, f\}$ be a universe of discourse and $NG = \{a, b, c, d\}$ be a subset of U . Define the operation $*_1$ on NG in table 1. Then $*_1$ is a NeutroLow since $c*_1 d = \text{indeterminate}$. Also,

$$a*_1(b*_1 c) = (a*_1 b)*_1 c \text{ and } c*_1(a*_1 c) = c*_1 d = \text{indeterminate}$$

TABLE 1. The table of NeutroGroup $(NG, *_1)$

$*_1$	a	b	c	d
a	b	c	d	a
b	c	d	a	c
c	d	a	b	?
d	a	b	?	a

Thus, $(NG, *_1)$ is a NeutroGroup.

Note that $x * y$ will be written as xy for all $x, y \in NG$.

Theorem 2.7 ([1]). Let $(NH, *)$ be a NeutroSubgroup of the NeutroGroup $(NG, *)$. The sets $(NG/NH)_l = \{xNH : x \in NG\}$ and $(NG/NH)_r = \{NHx : x \in NG\}$ are two NeutroGroups with operations \circ_l, \circ_r where for any $xNH, yNH \in (NG/NH)_l$, $NHx, NHy \in (NG/NH)_r$, $x, y \in NG$ we have

$$xNH \circ_l yNH = xyNH, \quad NHx \circ_r NHy = NHxy.$$

Definition 2.8 ([1]). Let $(NG, *)$ and (NK, \circ) be two NeutroGroups. The mapping $\varphi : NG \rightarrow NK$ is called a NeutroGroup Homomorphism if for every duplet $(x, y) \in G$, we have $\varphi(x * y) = \varphi(x) \circ \varphi(y)$.

In addition, if φ is a NeutroBijection, then φ is called a NeutroGroup Isomorphism. NeutroGroup Epimorphism, NeutroGroup Monomorphism, NeutroGroup Endomorphism are defined similarly.

Theorem 2.9 ([1]). Let $(NG, *)$ and (NK, \circ) be NeutroGroups and let e_{NG} and e_{NH} be NeutroNeutralElements in NG and NK respectively. Suppose that $\varphi : NG \rightarrow NK$ is a NeutroGroup Homomorphism. Then $\varphi(e_{NG}) = e_{NK}$.

From now on, NG is a NeutroGroup with tree NeutroAxioms (NeutroAssociativity, NeutroNeutralElement and NeutroInverseElement). Also, for all $x \in NG$, N_x and I_x represent the NeutroNeutralElement and the NeutroInverseElement respectively.

3. Some Results On NeutroNilpotentGroups

In this section, we introduce the notion of commutator of two elements in a NeutroGroup and study a new concept as *NeutroNilpotentGroups* and their properties are given.

Let x, y be elements of a NeutroGroup NG . The commutator of x, y , denoted by $[x, y]$, is the element $I_x I_y xy$, i.e., $[x, y] = I_x I_y xy$. If I_x or I_y does not exist, then put $I_x = x$ and $I_y = y$.

Also, for any $x, y_1, \dots, y_n \in NG$, define the commutator $[x, y_1, \dots, y_n]$ by $[x, y_1, \dots, y_n] = [[x, y_1, \dots, y_{n-1}], y_n]$.

TABLE 2. The table of NeutoNilpotentGroup $(NG, *_2)$

$*_2$	a	b	c	d
a	b	c	d	a
b	c	d	a	c
c	d	a	b	d
d	a	b	c	a

TABLE 3. The table of NeutoAbelianGroup $(NG, *_3)$

$*_3$	a	b	c	e
a	b	a	b	a
b	c	f	c	b
c	d	c	e	c
e	a	b	c	e

Definition 3.1. A NeutroGroup $(NG, *)$ is called *NeutroNilpotentGroup* if $Z_n(NG) = NG$ for some $n \in \mathbb{N}$, where

$$Z_n(NG) = \{x \in NG : [x, g_1, g_2, \dots, g_n] = N_z \text{ for at least one } g_1, \dots, g_n, z \in NG\}.$$

The smallest such n is called the NeutroNilpotency of NG .

Note that, if NG is a NeutroNilpotentGroup, then for any $x \in NG$ there exists at least one $g_1, \dots, g_n, z \in NG$ such that $[x, g_1, g_2, \dots, g_n] = N_z$.

Example 3.2. Let $U = \{a, b, c, d, e, f\}$ be a universe of discourse and $NG = \{a, b, c, d\}$ be a subset of U . Define the operation $*_2$ on NG in table 2. Since $[a, b] = d$, $[d, b] = d$, $[c, c] = d$ and $[b, b] = a$, we have $[c, d, b] = [b, b, b] = [a, b] = d = N_a$, $[d, b, b] = [d, b] = N_a$ and $[a, b, b] = [d, b] = N_a$. Therefore, NG is a NeutroNilpotentGroup of class 2.

Example 3.3. Let $U = \{a, b, c, d, e, f\}$ be a universe of discourse and let $NG = \{e, a, b, c\}$ be a subset of U . Define the operation $*_3$ on NG in table 3. Since $[a, b, a] = e = N_e$, $[b, a, e] = [e, e] = e$, $[c, e, c] = [c, c] = e$, $[e, a, a] = [b, a] = e$, we have NG is NeutroAbelianGroup and a NeutroNilpotentGroup of class 2.

In what follows we have a non Abelian NeutroNilpotentGroup.

Example 3.4. Let $U = \{a, b, c, d\}$ be a universe and $NG = \{a, b, c\}$ be a NeutroGroup by the Cayley table 4. Then $H = \{a, b\}$, by the operation $*_4$, is a NeutroSubgroup of NG (see [1]). Since $N_a = a$, $I_a = a$, N_b , I_b does not exist, we have $[a, a] = aaaa = a = N_a$ and $[b, b] = a = N_a$. Therefore, H is a NeutroNilpotentSubgroup that is not an AbelianNeutroGroup.

TABLE 4. The table of non Abelian NeutoAbelianGroup $(NG, *_4)$

$*_4$	a	b	c
a	a	c	b
b	c	a	c
c	a	c	d

TABLE 5. The table of NeutoSubgroup $(H, *_4)$

$*_4$	a	b
a	a	c
b	c	a

TABLE 6. The table of NeutoSubgroup $(NH, *_5)$ of $(NG, *_5)$

$*_5$	a	c	d
a	b	d	a
c	d	b	d
d	a	c	a

Theorem 3.5. Let NG and NK be two NeutroGroups. Then $Z_n(NG \times NK) = Z_n(NG) \times Z_n(NK)$. Moreover, $NG \times NK$ is a NeutroNilpotentGroup of class n if and only if NG and NK are NeutroNilpotentGroups of class n .

Proof. Assume $(x, y) \in Z_n(NG \times NK)$, $z \in NG$ and $t \in NK$. Then for some $(x_1, y_1), \dots, (x_n, y_n) \in NG \times NK$, we have

$$\begin{aligned}
 (N_z, N_t) &= [(x, y), (x_1, y_1), \dots, (x_n, y_n)] = ([x, x_1, \dots, x_n], [y, y_1, \dots, y_n]) \\
 &\Leftrightarrow [x, x_1, \dots, x_n] = N_z, [y, y_1, \dots, y_n] = N_t \\
 &\Leftrightarrow x \in Z_n(NG), y \in Z_n(NK) \\
 &\Leftrightarrow (x, y) \in Z_n(NG) \times Z_n(NK).
 \end{aligned}$$

Therefore, $Z_n(NG \times NK) = Z_n(NG) \times Z_n(NK)$.

Moreover, $NG \times NK$ is NeutroNilpotentGroup if and only if $Z_n(NG \times NK) = NG \times NK = Z_n(NG) \times Z_n(NK)$ if and only if NG and NK are NeutroNilpotentGroups. \square

In what follows we have a NeutoSubgroup that is not NeutroNilpotentGroup.

Example 3.6. Consider the NeutroGroup NG from Example 3.2. Define the operation $*_5$ on NG in table 6. Then $NH = \{a, c, d\}$ is a NeutoSubgroup of NG (see [1]). Since $[a, d] = a$, $[a, a]$ and $[a, c]$ does not exist, we get $[a, g_1, \dots, g_n]$ does not exist for any $g_1, \dots, g_n \in NH$, and so $a \notin Z_n(NH)$ i.e., NH is not NeutroNilpotentGroup.

Theorem 3.7. *Let NH be a NeutroSubgroup of the NeutroNilpotentGroup NG . Then NeutroQuotientGroups $(NG/NH)_l$ and $(NG/NH)_r$ are NeutroNilpotentGroups.*

Proof. Assume NH be a NeutroSubgroup of NG and $gH \in (NG/NH)_l$. Since NG is a NeutroNilpotentGroup, we have $[g, g_1, \dots, g_n] = N_z$ for some $g_1, \dots, g_n, z \in NG$, and so $[gNH, g_1NH, \dots, g_nNH] = [g, g_1, \dots, g_n]NH = N_zNH$. Since $(zNH) \circ_l (N_zNH) = (z * N_z)NH = zNH = (N_z)NH \circ_l zNH$, we get $(N_z)NH$ is a NeutroNaturalElement of $(NG/NH)_l$. Therefore, $(NG/NH)_l$ is a NeutroNilpotentGroup. Similarly, $(NG/NH)_r$ is a NeutroNilpotentGroup. \square

We recall that the intersection of two NeutroGroups is a NeutroGroup (see [1]). Now we have the following:

Theorem 3.8. *Let NG and NK be two NeutroNilpotentGroups. Then $NG \cap NK$ is a NeutroNilpotentGroup.*

Proof. Straightforward. \square

Theorem 3.9. *Let NH be a NeutroNilpotentSubgroup of a NeutroGroup NG and for all $x, t \in NG$ we have*

$$xNH = NH \Rightarrow x \in NH, \quad (N_t)NH = NH.$$

If $(NG/NH)_l$ is a NeutroNilpotentQuotientGroup, then NG is a NeutroNilpotentGroup.

Proof. Assume $(NG/NH)_l$ is NeutroNilpotentGroup of class n and NH is NeutroNilpotentGroup of class m . Then for any $xNH \in (NG/NH)_l$, there exist $g_1NH, \dots, g_nNH \in (NG/NH)_l$ such that $[xNH, g_1NH, \dots, g_nNH] = (N_z)NH$, where $z \in NG$. Then $[x, g_1, \dots, g_n]NH = (N_z)NH = NH$, and so $[x, g_1, \dots, g_n] \in NH$. Since NH is NeutroNilpotentGroup, we get there exist $k_1, \dots, k_m \in NH$ such that $[[x, g_1, \dots, g_n], k_1, \dots, k_m] = N_t$, for some $t \in NG$. Consequently, NG is NeutroNilpotentGroup of class $n + m$. \square

Theorem 3.10. *Let NH be a NeutroNilpotentSubgroup of a NeutroGroup NG and for all $x, t \in NG$ we have*

$$NHx = NH \Rightarrow x \in NH, \quad NH(N_t) = NH.$$

If $(NG/NH)_r$ is a NeutroNilpotentQuotientGroup, then NG is a NeutroNilpotentGroup.

Proof. Similar to the proof of Theorem 3.9. \square

Theorem 3.11. *Every homomorphic image of a NeutroNilpotentGroup is NeutroNilpotentGroup.*

Proof. Assume NH be a NeutroSubgroup of a NeutroNilpotentGroup NG and e_1, e_2 be NeutroNeutralElements in NG and NH , respectively. Suppose that $\psi : NG \rightarrow NH$ is a NeutroGroup Epimorphism. Then for any $h \in NH$, there exists $x \in NG$ such that $h = \psi(x)$. Since NG is NeutroNilpotentGroup, for $x \in NG$, there exist $g_1, \dots, g_n \in NG$ such that $[x, g_1, \dots, g_n] = e_1$. Take $k_1 = \psi(g_1), \dots, k_n = \psi(g_n)$. Therefore, $[h, k_1, \dots, k_n] = \psi([x, g_1, \dots, g_n]) = \psi(e_1) = e_2$, and so NH is a NeutroNilpotentGroup. \square

4. Conclusion

In this paper, we defined a class of NeutroGroups, named NeutroNilpotentGroups, and their elementary properties were presented. The intersection of two NeutroSubgroups is not necessarily a NeutroSubgroup while their union is a NeutroSubgroup. We hope to study NeutroSolvabelGroups, NeutroEngelGroups in our future works.

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Neutrosophic θ -Closure Operator

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Abstract. The fundamental intent of this article is to develop the idea of neutrosophic θ -cluster point, neutrosophic θ -closure operator, neutrosophic $\epsilon\theta$ -neighbourhood in neutrosophic topological spaces. We characterize some types of functions like neutrosophic θ -continuous, neutrosophic strongly- θ -continuous, neutrosophic weakly continuous functions in terms of $\mathcal{N}\theta$ -closure operator are discussed. Further, neutrosophic regular space is also introduced.

Keywords: neutrosophic quasi coincident; $\mathcal{N}_\epsilon^q - nbd$; $\mathcal{N}_\epsilon^{\theta q} - nbd$; $\mathcal{N}\theta$ -cluster points; $\mathcal{N}\theta$ -closure; $\mathcal{N}\theta$ -closed set, \mathcal{N} Strongly θ -continuous; \mathcal{N} weakly-continuous.

1. Introduction

Fuzzy set theory is introduced and studied as a mathematical tool concern with uncertainties where every element had a "degree of membership, truth(t)", by Zadeh [28]. A fuzzy set is one where every element had a "degree of membership" which lies between 0 and 1. Atanassov [11] developed intuitionistic fuzzy set (IFS) as a generalization of fuzzy sets where besides, the "degree of non-membership" is assigned to each element. Both degrees belong to the interval [0,1] with the restriction that their sum is should not exceed 1. In IFS, the "degree of non-membership" depends on the "degree of membership".

Neutrality (i), "the degree of indeterminacy", as an independent notion, was proposed by F. Smarandache [26,27]. In addition he described neutrosophic set on "three components (t, f, i) = (truth, falsehood, indeterminacy)". In neutrosophic set respectively "degree of membership, indeterminacy and non-membership" assigned to every element and it lies between [0,1]*, non-standard unit interval. Unlike in IFS, where the uncertainty depends on both "degree of

membership” as well as ”non-membership”, here the uncertainty is independent of ”degree of membership and non-membership”. Neutrosophic sets are certainly too general than IFS as there are no restrictions between ”degree of membership, degree of indeterminacy and degree of membership”.

Neutrosophic notion have many applications in the fields of Information Systems, Artificial Intelligence, decision making and evaluating airline service quality [1–4]. As developments goes on, some researchers [5–9] have extended the idea of neutrosophic set into plithogenic set and applied it in MCDM, MADM and optimization technique supply chain based model. Salama et al [23] developed Neutrosophic topological space in 2012. This gave the way for investigation in terms of neutrosophic topology and its application in decision making problems. The properties of neutrosophic open sets, neutrosophic closed sets, neutrosophic interior operator and neutrosophic closure operator gave the way for applying neutrosophic topology. Researchers established the sets which are close to neutrosophic open sets as well as neutrosophic closed sets. Like this, Neutrosophic closed sets as well as Neutrosophic continuous mappings were developed in [24]. Arokiarani et al. [10] introduced neutrosophic semi-open (sequentially, pre-open as well as α -open) mappings and discussed their properties. R. Dhavaseelan et al. [12] introduced generalized neutrosophic closed sets. In [14, 15] the concept of neutrosophic generalized α -contra continuous along with neutrosophic Almost α -contra-continuous functions are introduced and studied their properties. Dhavaseelan et al. [16] presented the idea of neutrosophic α^m -continuity. Narmada Devi, et al. [20] presented the idea of Neutrosophic structure ring contra strong precontinuity. The notion of fuzzy θ -closure operator introduced in [19]. Hanafy et al [17] established the notion of intuitionistic fuzzy θ -closure operator and intuitionistic fuzzy weakly continuous functions.

The main contribution of the article is

- To establish the notion of neutrosophic θ -closure operator along with its properties in neutrosophic topological spaces.
- Neutrosophic θ -closed set is also defined using the operator defined.
- As application of this new notion, neutrosophic θ -continuous, neutrosophic strongly θ -continuous and neutrosophic weakly continuous functions are characterized in terms of neutrosophic θ -closure operator.
- At the end we have shown the relation between these neutrosophic continuous functions through implication diagram.

2. Preliminaries

Definition 2.1. [26, 27] For a nonempty fixed set N_X a neutrosophic set [in short, NS] K is an object of the form $K = \{\langle x, \mu_K(x), \sigma_K(x), \gamma_K(x) \rangle : x \in N_X\}$ where $\mu_K(x)$, $\sigma_K(x)$ and $\gamma_K(x)$ are real numbers in $[0, 1]$ such that $\mu_K(x) + \sigma_K(x) + \gamma_K(x) \leq 1$.
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$\gamma_K(x)$ respectively denotes the "degree of membership function ($\mu_K(x)$)", the "degree of indeterminacy ($\sigma_K(x)$)" as well as the "degree of nonmembership ($\gamma_K(x)$)" of each element $x \in N_X$ to the set K .

Remark 2.1. [26, 27]

- (1) A NS $K = \{\langle j, \mu_K(j), \sigma_K(j), \gamma_K(j) \rangle : j \in K\}$ can be recognized as an ordered triple $\langle \mu_K, \sigma_K, \gamma_K \rangle$ in $]0^-, 1^+[$ on N_X .
- (2) For convenience, we write $K = \langle \mu_K, \sigma_K, \gamma_K \rangle$ for the NS set $K = \{\langle j, \mu_K(j), \sigma_K(j), \gamma_K(j) \rangle : j \in N_X\}$.

Definition 2.2. [26, 27] Consider a nonempty set N_X along with NSs K as well as H in the form

$K = \{\langle j, \mu_K(j), \sigma_K(j), \gamma_K(j) \rangle : j \in N_X\}$, $H = \{\langle j, \mu_H(j), \sigma_H(j), \gamma_H(j) \rangle : j \in N_X\}$. Then

- (a) $K \subseteq H$ iff $\mu_K(j) \leq \mu_H(j)$, $\sigma_K(j) \leq \sigma_H(j)$ and $\gamma_K(j) \geq \gamma_H(j)$ for every $j \in N_X$;
- (b) $K = H$ iff $K \subseteq H$ and $H \subseteq K$;
- (c) $\bar{K} = \{\langle j, \gamma_K(j), \sigma_K(j), \mu_K(j) \rangle : j \in N_X\}$; [Complement of K]
- (d) $K \cap H = \{\langle j, \mu_K(j) \wedge \mu_H(j), \sigma_K(j) \wedge \sigma_H(j), \gamma_K(j) \vee \gamma_H(j) \rangle : j \in N_X\}$;
- (e) $K \cup H = \{\langle j, \mu_K(j) \vee \mu_H(j), \sigma_K(j) \vee \sigma_H(j), \gamma_K(j) \wedge \gamma_H(j) \rangle : j \in N_X\}$;
- (f) $]K = \{\langle j, \mu_K(j), \sigma_K(j), 1 - \mu_K(j) \rangle : j \in N_X\}$;
- (g) $\langle j \rangle K = \{\langle j, 1 - \gamma_K(j), \sigma_K(j), \gamma_K(j) \rangle : j \in N_X\}$.

Definition 2.3. [26, 27] Let $\{K_i : i \in J\}$ be any family of NSs in N_X . Then

- (a) $\bigcap K_i = \{\langle x, \wedge \mu_{K_i}(x), \wedge \sigma_{K_i}(x), \vee \gamma_{K_i}(x) \rangle : x \in N_X\}$;
- (b) $\bigcup K_i = \{\langle x, \vee \mu_{K_i}(x), \vee \sigma_{K_i}(x), \wedge \gamma_{K_i}(x) \rangle : x \in N_X\}$.

Since our main work is to construct the tools for generating neutrosophic topological spaces, so we present the NSs 0_N and 1_N in N_X as below:

Definition 2.4. [26, 27] $0_N = \{\langle x, 0, 0, 1 \rangle : x \in N_X\}$ and $1_N = \{\langle x, 1, 1, 0 \rangle : x \in N_X\}$.

Definition 2.5. [23] A neutrosophic topology (NT) on a nonempty set N_X is a collection Ω of NSs in N_X satisfy the axioms given below:

- (i) $0_N, 1_N \in \Omega$,
- (ii) $R_1 \cap R_2 \in \Omega$ for any $R_1, R_2 \in \Omega$,
- (iii) $\bigcup R_i \in \Omega$ for arbitrary collection $\{R_i \mid i \in \Lambda\} \subseteq \Omega$.

Here the ordered pair (N_X, Ω) or only N_X is termed as neutrosophic topological space (NTS) and each NS in Ω is known as neutrosophic open set (NOS). The complement \bar{R} of a NOS R in X is known as neutrosophic closed set (NCS) in N_X .

Definition 2.6. [13] Consider a NS, K in a NTS N_X . Then

$\mathcal{N}int(K) = \bigcup \{R \mid R \text{ is a NOS in } N_X \text{ also } R \subseteq K\}$ is referred as neutrosophic interior of K ;
 $\mathcal{N}cl(K) = \bigcap \{R \mid R \text{ is a NCS in } N_X \text{ with } R \supseteq K\}$ is referred as neutrosophic closure of K .

Definition 2.7. [12] Consider a nonempty set as N_X . Whenever t, i, f be "real standard or non standard" subsets of $]0^-, 1^+[$ then the NS $x_{t,i,f}$ is named as neutrosophic point (shortly, NP) in N_X given by

$$x_{t,i,f}(x_p) = \begin{cases} (t, i, f), & \text{whenever } x = x_p \\ (0, 0, 1), & \text{whenever } x \neq x_p \end{cases}$$

for $x_p \in N_X$ is called the support of $x_{t,i,f}$, where t denotes the "degree of membership", i the "degree of indeterminacy" and f is the "degree of non-membership" of $x_{t,i,f}$.

3. Neutrosophic θ -Closure Operator

Definition 3.1. (1) A NP $x_{(\alpha,\beta,\lambda)}$ in N_X is termed as quasi-coincident with the NS $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle\}$, represented as $x_{(\alpha,\beta,\lambda)}q\Lambda$ iff $\alpha + \mu_\Lambda > 1$, $\beta + \sigma_\Lambda > 1$ and $\lambda + \gamma_\Lambda < 1$.

(2) Consider $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle\}$ along with $\Gamma = \{\langle x, \mu_\Gamma(x), \sigma_\Gamma(x), \gamma_\Gamma(x) \rangle\}$ as NSs in N_X . Then Λ is said to be quasi-coincident with Γ , indicated as $\Lambda q\Gamma$ iff there exists an element $x \in N_X$ such that $\mu_\Lambda(x) + \mu_\Gamma(x) > 1$, $\sigma_\Lambda(x) + \sigma_\Gamma(x) > 1$ and $\gamma_\Lambda(x) + \gamma_\Gamma(x) < 1$.

The expression "not quasi-coincident" will be summarized as \tilde{q} .

Proposition 3.1. Let Λ and Γ be two NSs along with a NP $x_{(\alpha,\beta,\lambda)}$ in N_X . Then

- i) $\Lambda \tilde{q} \bar{\Gamma}$ iff $\Lambda \subseteq \Gamma$.
- ii) $\Lambda q \Gamma$ iff $\Lambda \not\subseteq \Gamma$.
- iii) $x_{(\alpha,\beta,\lambda)} \subseteq \Lambda$ iff $x_{(\alpha,\beta,\lambda)} \tilde{q} \bar{\Lambda}$
- iv) $x_{(\alpha,\beta,\lambda)} q \Lambda$ iff $x_{(\alpha,\beta,\lambda)} \not\subseteq \bar{\Lambda}$

Definition 3.2. Let $\mu : N_X \rightarrow N_Y$ be a function and $x_{(\alpha,\beta,\lambda)}$ be a NP in N_X . Then the preimage of $x_{(\alpha,\beta,\lambda)}$ under μ , designated as $\mu(x_{(\alpha,\beta,\lambda)})$ is defined by
 $\mu(x_{(\alpha,\beta,\lambda)}) = \{\langle y, \mu(x_p)_\alpha, \mu(x_p)_\beta, (1 - \mu(x_p)_{1-\lambda}) \rangle : y \in N_Y\}$.

Proposition 3.2. Let $f : N_X \rightarrow N_Y$ be a function and $x_{(\alpha,\beta,\lambda)}$ be a NP in N_X .

- i) $x_{(\alpha,\beta,\lambda)} q f^{-1}(\Gamma)$ if $f(x_{(\alpha,\beta,\lambda)}) q \Gamma$. for any NS Γ in N_Y .
- ii) $f(x_{(\alpha,\beta,\lambda)}) q f(\Lambda)$ if $x_{(\alpha,\beta,\lambda)} q \Lambda$ for any NS Λ in N_X

Definition 3.3. Let (X, Θ) be a NTS on N_X and $x_{(\alpha,\beta,\lambda)}$ be a NP in N_X . A NS Λ is called $\mathcal{N}eq - nbd$ of $x_{(\alpha,\beta,\lambda)}$, if there exists a neutrosophic open Γ in N_X such that $x_{(\alpha,\beta,\lambda)} q \Gamma$ and $\Gamma \subseteq \Lambda$. The family of all $\mathcal{N}eq - nbd$ of $x_{(\alpha,\beta,\lambda)}$ is indicated as $\mathcal{N}N_\epsilon^q(x_{(\alpha,\beta,\lambda)})$.

Definition 3.4. A NP $x_{(\alpha, \beta, \lambda)}$ is known as neutrosophic θ -cluster point ($\mathcal{N}\theta$ -cluster point, for short) of a NS Λ iff for each Γ in $\mathcal{N}\epsilon q - nbd$ of $x_{(\alpha, \beta, \lambda)}$ and $\mathcal{N}cl(\Gamma)q\Lambda$. The set of all $\mathcal{N}\theta$ -cluster points of Λ is named as neutrosophic θ closure of Λ and denoted by $\mathcal{N}cl_\theta$.

A NS Λ will be $\mathcal{N}\theta$ -closed set ($\mathcal{N}\theta$ CS for short) iff $\Lambda = \mathcal{N}cl_\theta(\Lambda)$. The complement of a $\mathcal{N}\theta$ -closed set is $\mathcal{N}\theta$ -open set (in short $\mathcal{N}\theta$ OS).

Proposition 3.3. Let (N_X, Θ) be a NTS and let Λ and Γ be two NSs in N_X . Then

- i) $\Lambda \subseteq \Gamma \Rightarrow \mathcal{N}cl_\theta(\Lambda) \subseteq \mathcal{N}cl_\theta(\Gamma)$
- ii) $\Lambda \cup \Gamma \Rightarrow \mathcal{N}cl_\theta(\Lambda) \cup \mathcal{N}cl_\theta(\Gamma)$
- iii) $\mathcal{N}int_\theta(\Lambda) = \overline{\mathcal{N}cl_\theta(\overline{\Lambda})}$

Definition 3.5. A NS Λ of a NTS N_X is named as $\mathcal{N}\epsilon\theta q - nbd$ of a NP $x_{(\alpha, \beta, \lambda)}$ if there arises a $\mathcal{N}\epsilon q - nbd$ Γ of $x_{(\alpha, \beta, \lambda)}$ such that $\mathcal{N}cl(\Gamma)\tilde{q}\Lambda$. The family of all $\mathcal{N}\epsilon\theta q - nbd$ of $x_{(\alpha, \beta, \lambda)}$ is represented as $\mathcal{N}N_\epsilon^{\theta q}(x_{(\alpha, \beta, \lambda)})$.

Remark 3.1. For any NS Λ in a NTS N_X , $\mathcal{N}cl(\Lambda) \subseteq \mathcal{N}cl_\theta(\Lambda)$.

Proposition 3.4. If Λ is a NOS in a NTS N_X , then $\mathcal{N}cl(\Lambda) = \mathcal{N}cl_\theta(\Lambda)$.

Proof. It is enough to prove $\mathcal{N}cl(\Lambda) \supseteq \mathcal{N}cl_\theta(\Lambda)$. Consider $x_{(\alpha, \beta, \lambda)}$ be a NP in N_X so as t $x_{(\alpha, \beta, \lambda)} \notin \mathcal{N}cl(\Lambda)$, then there exists $\Gamma \in \mathcal{N}N_\epsilon^q(x_{(\alpha, \beta, \lambda)})$ such that $\Gamma\tilde{q}\Lambda$ and hence $\Gamma \subseteq \overline{\Lambda}$. Then $\mathcal{N}cl(\Gamma) \subseteq \overline{\mathcal{N}int(\Lambda)} \subseteq \overline{\Lambda}$, as Λ is a NOS in N_X . Thus $\mathcal{N}cl(\Gamma)\tilde{q}\Lambda$ which implies $x_{(\alpha, \beta, \lambda)} \notin \mathcal{N}cl_\theta(\Lambda)$. Then $\mathcal{N}cl_\theta(\Lambda) \subseteq \mathcal{N}cl(\Lambda)$. Thus $\mathcal{N}cl(\Lambda) = \mathcal{N}cl_\theta(\Lambda)$. \square

Proposition 3.5. Let (N_X, Θ) be a NTS, the conditions are satisfied

- i) Finite union and arbitrary intersection of neutrosophic θ -closed sets in N_X is a $\mathcal{N}\theta$ CS.
- ii) For two neutrosophic sets Λ and Γ in N_X , if $\Lambda \subseteq \Gamma$, then $\mathcal{N}cl_\theta(\Lambda) \subseteq \mathcal{N}cl_\theta(\Gamma)$.
- iii) 0_N and 1_N are neutrosophic θ -closed sets.

Corollary 3.1. Let Λ be a NS in NTS N_X . $\mathcal{N}cl_\theta(\Lambda)$ is evidently NCS. The converse of the Corollary doesn't hold.

Example 3.1. For $N_X = \{k_1, k_2, k_3\}$ NSs Λ, Γ and K in N_X are defined as :

$$\Lambda = \langle x, (\frac{k_1}{0.6}, \frac{k_2}{0.6}, \frac{k_3}{0.2}), (\frac{k_1}{0.6}, \frac{k_2}{0.6}, \frac{k_3}{0.2}), (\frac{k_1}{0.3}, \frac{k_2}{0.4}, \frac{k_3}{0.1}) \rangle,$$

$$\Gamma = \langle x, (\frac{k_1}{0.4}, \frac{k_2}{0.3}, \frac{k_3}{0.3}), (\frac{k_1}{0.4}, \frac{k_2}{0.3}, \frac{k_3}{0.3}), (\frac{k_1}{0.5}, \frac{k_2}{0.6}, \frac{k_3}{0.3}) \rangle \text{ and}$$

$K = \langle x, (\frac{k_1}{0.3}, \frac{k_2}{0.3}, \frac{k_3}{0.1}), (\frac{k_1}{0.3}, \frac{k_2}{0.3}, \frac{k_3}{0.1}), (\frac{k_1}{0.6}, \frac{k_2}{0.7}, \frac{k_3}{0.1}) \rangle$. Then the family $\Theta = \{0_N, 1_N, \Lambda, \Gamma\}$ is NT on N_X . So, (N_X, Θ) is NTSs. Let $x_{(0.6, 0.6, 0.3)}(k_1)$ and $x_{(0.8, 0.8, 0.1)}(k_1)$ are neutrosophic points in N_X . Here

$x_{(0.6, 0.6, 0.3)}(k_1) \in \mathcal{N}cl_\theta(K)$, that is $x_{(0.6, 0.6, 0.3)}(k_1)q\Lambda \subseteq \Lambda$ and $\mathcal{N}cl(\Lambda) = 1_N q K$. Now $x_{(0.8, 0.8, 0.1)}(k_1) \notin \mathcal{N}cl_\theta(K)$, that is $x_{(0.8, 0.8, 0.1)}(k_1)q\Gamma$, $\mathcal{N}cl(\Gamma) = \overline{\Gamma}\tilde{q}K$. But $x_{(0.8, 0.8, 0.1)}(k_1) \in \mathcal{N}cl_\theta(x_{(0.6, 0.6, 0.3)}(k_1)) \subseteq \mathcal{N}cl_\theta(\mathcal{N}cl_\theta(K))$. Hence $\mathcal{N}cl_\theta(K)$ is not $\mathcal{N}\theta$ CS.

Proposition 3.6. A NS Λ is $\mathcal{N}\theta$ OS in NTS N_X iff for each NP $x_{(\alpha,\beta,\lambda)}$ in N_X with $x_{(\alpha,\beta,\lambda)}q\Lambda$, Λ is a $\mathcal{N}\epsilon q - nbd$ of $x_{(\alpha,\beta,\lambda)}$.

Proposition 3.7. For any NS Λ in a NTS (N_X, Θ) , $\mathcal{N}cl_\theta(\Lambda) = \cap\{\mathcal{N}cl_\theta(\Gamma) : \Gamma \in \Theta \text{ and } \Lambda \subseteq \Gamma\}$.

Proof. Obviously $\mathcal{N}cl_\theta(\Lambda) \subseteq \cap\{\mathcal{N}cl_\theta(\Gamma) : \Gamma \in \Theta \text{ and } \Lambda \subseteq \Gamma\}$.

Now, let $x_{(\alpha,\beta,\lambda)} \in \cap\{\mathcal{N}cl_\theta(\Gamma) : \Gamma \in \Theta \text{ and } \Lambda \subseteq \Gamma\}$, but $x_{(\alpha,\beta,\lambda)} \notin \mathcal{N}cl_\theta(\Lambda)$. Consequently there arises a $\mathcal{N}\epsilon q - nbd\eta$ of $x_{(\alpha,\beta,\lambda)}$ so that $\mathcal{N}cl(\eta)\tilde{q}\Lambda$ and hence by Proposition 3.1, $\Lambda \subseteq \overline{\mathcal{N}cl(\eta)}$. Then $x_{(\alpha,\beta,\lambda)} \in \mathcal{N}cl_\theta(\overline{\mathcal{N}cl(\eta)})$ and consequently, $\mathcal{N}cl(\eta)q\overline{\mathcal{N}cl(\eta)}$. Which is a contradiction. \square

Definition 3.6. A NTS N_X is named as neutrosophic regular (\mathcal{NRS} in short) iff for each $x_{(\alpha,\beta,\lambda)}$ in N_X and each $\mathcal{N}\epsilon q - nbd \eta$ of $x_{(\alpha,\beta,\lambda)}$, there arises $\mathcal{N}\epsilon q - nbd \Gamma$ of $x_{(\alpha,\beta,\lambda)}$ such that $\mathcal{N}cl(\Gamma) \subseteq \eta$.

Proposition 3.8. A NTS N_X is \mathcal{NRS} iff for each NS Λ in N_X , $\mathcal{N}cl(\Lambda) = \mathcal{N}cl_\theta(\Lambda)$.

Proof. Let N_X be a \mathcal{NRS} . It is true that $\mathcal{N}cl(\Lambda) \subseteq \mathcal{N}cl_\theta(\Lambda)$ for any NS Λ . Now, consider $x_{(\alpha,\beta,\lambda)}$ be NP in N_X with $x_{(\alpha,\beta,\lambda)} \in \mathcal{N}cl_\theta(\Lambda)$ and let Γ be a $\mathcal{N}\epsilon q - nbd$ of $x_{(\alpha,\beta,\lambda)}$. Then by \mathcal{NRS} N_X , there exists $\mathcal{N}\epsilon q - nbd \eta$ of $x_{(\alpha,\beta,\lambda)}$ such that $\mathcal{N}cl(\eta) \subseteq \Gamma$. Now, $x_{(\alpha,\beta,\lambda)} \in \mathcal{N}cl_\theta(\Lambda)$ implies $\mathcal{N}cl(\eta)q\Lambda$ implies $\Gamma q\Lambda$ implies $x_{(\alpha,\beta,\lambda)} \in \mathcal{N}cl(\Lambda)$. Hence $\mathcal{N}cl_\theta(\Lambda) \subseteq \mathcal{N}cl(\Lambda)$. Thus $\mathcal{N}cl(\Lambda) = \mathcal{N}cl_\theta(\Lambda)$.

Contrarily, let $x_{(\alpha,\beta,\lambda)}$ be a NP in N_X and Λ be a $\mathcal{N}\epsilon q - nbd$ of $x_{(\alpha,\beta,\lambda)}$. Thereupon $x_{(\alpha,\beta,\lambda)} \notin \overline{\Lambda} = \mathcal{N}cl(\overline{\Lambda}) = \mathcal{N}cl_\theta(\overline{\Lambda})$. Thus there exists a $\mathcal{N}\epsilon q - nbd \eta$ of $x_{(\alpha,\beta,\lambda)}$ such that $\mathcal{N}cl(\eta)\tilde{q}\overline{\Lambda}$ and then $\mathcal{N}cl(\eta) \subseteq \Lambda$. Hence N_X is \mathcal{NRS} . \square

4. Applications

Here we characterize some types of functions in terms of $\mathcal{N}\theta$ -closure operator as application. Using this operator, we characterize neutrosophic strongly- θ -continuous, neutrosophic weakly continuous functions.

Definition 4.1. A function $f : (N_X, \Theta) \rightarrow (N_Y, \Xi)$ is termed as neutrosophic strongly θ -continuous ($\mathcal{NStr}\theta$ -continuous, for short), if for each NP $x_{(\alpha,\beta,\lambda)}$ in N_X and $\Gamma \in \mathcal{NN}_\epsilon^q(f(x_{(\alpha,\beta,\lambda)}))$, there exists $\Lambda \in \mathcal{NN}_\epsilon^q(x_{(\alpha,\beta,\lambda)})$ such that $f(\mathcal{N}cl(\Lambda)) \subseteq \Gamma$.

Proposition 4.1. For a function $\mu : (N_X, \Theta) \rightarrow (N_Y, \Xi)$ the conditions are equivalent :

- i) μ is $\mathcal{NStr}\theta$ -continuous.
- ii) $\mu(\mathcal{N}cl_\theta(\Lambda)) \subseteq \mathcal{N}cl(\mu(\Lambda))$ for each NS $\Lambda \in N_Y$.
- iii) $\mathcal{N}cl_\theta(\mu^{-1}(\Gamma)) \subseteq \mu^{-1}(\mathcal{N}cl(\Gamma))$ for each NS $\Gamma \in N_Y$.

- iv) $\mu^{-1}(\Gamma)$ is a $\mathcal{N}\theta CS$ in N_X for each $\mathcal{N}CS$ $\Gamma \in N_Y$.
- v) $\mu^{-1}(\Gamma)$ is a $\mathcal{N}\theta OS$ in N_X for every $\mathcal{N}OS$ $\Gamma \in N_Y$.

Proof. $i) \Rightarrow ii)$ Let $x_{(\alpha, \beta, \lambda)} \in \mathcal{N}cl_{\theta}(\Lambda)$ and $\Omega \in \mathcal{N}N_{\epsilon}^q(\beta(x_{(\alpha, \beta, \lambda)}))$. By (i), there exists $\eta \in \mathcal{N}N_{\epsilon}^q(x_{(\alpha, \beta, \lambda)})$ such that $\beta(\mathcal{N}cl(\eta)) \subseteq \Omega$. Now, using Definition 3.4 and Proposition 3.1, we have $x_{(\alpha, \beta, \lambda)} \in \mathcal{N}cl_{\theta}(\Lambda) \Rightarrow \mathcal{N}cl(\eta)q\Lambda \Rightarrow \beta(\mathcal{N}cl(\eta)q\mu(\Lambda)) \Rightarrow \Omega q\mu(\Lambda) \Rightarrow \mu(x_{(\alpha, \beta, \lambda)}) \in \mathcal{N}cl(\mu(\Lambda)) \Rightarrow x_{(\alpha, \beta, \lambda)} \in \mu^{-1}(\mathcal{N}cl(\mu(\Lambda)))$. Hence $\mathcal{N}cl_{\theta}(\Lambda) \subseteq \mu^{-1}(\mathcal{N}cl(\mu(\Lambda)))$ and so $\mu(\mathcal{N}cl_{\theta}(\Lambda)) \subseteq \mathcal{N}cl(\beta(\Lambda))$

$ii) \Rightarrow iii)$ Is obvious by substituting $\Lambda = \mu^{-1}(\Lambda)$.

$iii) \Rightarrow iv)$ Take Γ be a $\mathcal{N}CS$ in N_Y . By (iii), we have $\mathcal{N}cl_{\theta}(\mu^{-1}(\Gamma)) \subseteq \beta^{-1}(\mathcal{N}cl(\Gamma)) = \mu^{-1}(\Gamma)$ which implies that $\mu^{-1}(\Gamma) = \mathcal{N}cl_{\theta}(\Gamma)$. Hence $\mu^{-1}(\Gamma)$ is a $\mathcal{N}\theta CS$ in N_X .

$iv) \Rightarrow v)$ Let $\bar{\Gamma}$ as a $\mathcal{N}OS$ in N_Y . By (iii), we have $\overline{\mathcal{N}cl_{\theta}(\mu^{-1}(\bar{\Gamma}))} \supseteq \overline{\mu^{-1}(\mathcal{N}cl(\bar{\Gamma}))} = \overline{\mu^{-1}(\bar{\Gamma})}$ which implies that $\overline{\mu^{-1}(\bar{\Gamma})} = \overline{\mathcal{N}cl_{\theta}(\bar{\Gamma})}$. Hence $\overline{\mu^{-1}(\bar{\Gamma})}$ is a $\mathcal{N}\theta OS$ in N_X .

$v) \Rightarrow i)$ Consider $x_{(\alpha, \beta, \lambda)}$ be a NP and $\Omega \in \mathcal{N}N_{\epsilon}^q(\beta(x_{(\alpha, \beta, \lambda)}))$. By (v), $\mu^{-1}(\Omega)$ is a $\mathcal{N}\theta OS$ in N_X . Now, using Proposition 3.1, we have $\mu(x_{(\alpha, \beta, \lambda)})q\Omega \Rightarrow x_{(\alpha, \beta, \lambda)}q\mu^{-1}(\Omega) \Rightarrow x_{(\alpha, \beta, \lambda)} \notin \overline{\mu^{-1}(\Omega)}$. Hence $\overline{\mu^{-1}(\Omega)}$ is a $\mathcal{N}\theta CS$, such that $x_{(\alpha, \beta, \lambda)} \notin \overline{\mu^{-1}(\Omega)}$. Then there exists $\eta \in \mathcal{N}N_{\epsilon}^q(\beta(x_{(\alpha, \beta, \lambda)}))$ such that $\mathcal{N}cl(\eta)\tilde{q}\mu^{-1}(\Omega)$ which implies that $\mu(\mathcal{N}cl(\eta)) \subseteq \Omega$. Hence μ is a $\mathcal{N}Str\theta$ -continuous.

□

Definition 4.2. A function $\beta : (N_X, \Theta) \rightarrow (Y, \Xi)$ is termed as neutrosophic weakly continuous [$\mathcal{N}w$ -continuous for short], iff for each $\mathcal{N}OS$ Λ in Y , $\beta^{-1}(\Lambda) \subseteq \mathcal{N}int(\beta^{-1}(\mathcal{N}cl(\Lambda)))$.

Proposition 4.2. Let $\beta : (N_X, \Theta) \rightarrow (N_Y, \Xi)$ be a function. Then for a NS Γ in N_Y , $\beta(\overline{\beta^{-1}(\Gamma)}) \subseteq \bar{\Gamma}$, wherein equality holds if β is surjective.

Proposition 4.3. Let \mathfrak{D} be a NS and $x_{(\alpha, \beta, \lambda)}$ be NP in a NTS (N_X, Θ) . Then the function $f : (N_X, \Theta) \rightarrow (N_Y, \Xi)$ if $x_{(\alpha, \beta, \lambda)} \in \mathfrak{D}$ then $f(x_{(\alpha, \beta, \lambda)}) \in f(\mathfrak{D})$.

Proposition 4.4. The successive results are equivalent for a function $\beta : (N_X, \Theta) \rightarrow (N_Y, \Xi)$:

- a) β is a $\mathcal{N}w$ -continuous.
- b) $\beta(\mathcal{N}cl(\mathfrak{D})) \subseteq \mathcal{N}cl_{\theta}(\beta(\mathfrak{D}))$ for each NS \mathfrak{D} in N_X .
- c) $\mathcal{N}cl(\beta^{-1}(\mathfrak{G})) \subseteq \beta^{-1}(\mathcal{N}cl_{\theta}(\mathfrak{G}))$ for each NS \mathfrak{G} in N_Y .
- d) $\mathcal{N}cl(\beta^{-1}(\mathfrak{G})) \subseteq \beta^{-1}(\mathcal{N}cl(\mathfrak{G}))$ for each NOS \mathfrak{G} in N_Y .

Proposition 4.5. Let $f : (N_X, \Theta) \rightarrow (N_Y, \Xi)$ be a $\mathcal{N}w$ -continuous function, then

- i) $f^{-1}(\Gamma)$ is a $\mathcal{N}CS$ in N_X , for every $\mathcal{N}\theta CS$ Γ in N_Y .
- ii) $f^{-1}(\Gamma)$ is a $\mathcal{N}OS$ in N_X , for each $\mathcal{N}\theta OS$ Γ in N_Y .

Definition 4.3. A function $\mu : (N_X, \Theta) \rightarrow (N_Y, \Xi)$ is known as neutrosophic θ continuous ($\mathcal{N}\theta$ -continuous, for short), iff for each NP $x_{(\alpha, \beta, \lambda)}$ in N_X and each $\Gamma \in \mathcal{N}_\epsilon^q(\mu(x_{(\alpha, \beta, \lambda)}))$, there arises $\Lambda \in \mathcal{N}_\epsilon^q(x_{(\alpha, \beta, \lambda)})$ so as $\mu(\mathcal{N}cl(\Lambda)) \subseteq \mathcal{N}cl(\Gamma)$.

Proposition 4.6. For $\mu : (N_X, \Theta) \rightarrow (N_Y, \Xi)$, the successive results are identical:

- a) μ is a $\mathcal{N}\theta$ -continuous.
- b) $\mu(\mathcal{N}cl_\theta(\mathfrak{D})) \subseteq \mathcal{N}cl_\theta(\mu(\mathfrak{D}))$ for each NS \mathfrak{D} in N_X .
- c) $\mathcal{N}cl_\theta(\mu^{-1}(\mathfrak{G})) \subseteq \mu^{-1}(\mathcal{N}cl_\theta(\mathfrak{G}))$ for every NS \mathfrak{G} in N_Y .
- d) $\mathcal{N}cl_\theta(\mu^{-1}(\mathfrak{G})) \subseteq \mu^{-1}(\mathcal{N}cl(\mathfrak{G}))$ for each NOS \mathfrak{G} in N_Y .

Remark 4.1. Based on the above results we have implication diagram as shown below.

$$\begin{array}{c} \mathcal{N}Str\text{-continuous} \implies \mathcal{N}\text{-continuous} \implies \mathcal{N}w\text{-continuous} \\ \Downarrow \\ \mathcal{N}\theta\text{-continuous} \end{array}$$

5. Conclusion

This research article presents and establishes the idea of neutrosophic θ -closure operator in neutrosophic topological spaces. Using this operator neutrosophic θ -closed set is defined. Some results are discussed and further more, as applications of these concepts, certain functions like neutrosophic θ -continuous, neutrosophic strongly θ -continuous together with neutrosophic weakly continuous are characterized in terms of neutrosophic θ -closure operator. Neutrosophic regular space is also introduced and characterized in terms of neutrosophic θ -closure operator. In future, using this operator, one can define the neutrosophic θ -generalized closed set and do the further interesting research.

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Neutrosophic $\underline{\mu}$ -Topological spaces

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Abstract. In this paper, the concept of neutrosophic $\underline{\mu}$ -topological spaces is introduced. We define and study the properties of neutrosophic $\underline{\mu}$ -open sets, $\underline{\mu}$ -closed sets, $\underline{\mu}$ -interior and $\underline{\mu}$ -closure. The set of all generalize neutrosophic pre-closed sets $GNPC(\tau)$ and the set of all neutrosophic α -open sets in a neutrosophic topological space (X, τ) can be considered as examples of generalized neutrosophic $\underline{\mu}$ -topological spaces. The concept of neutrosophic $\underline{\mu}$ -continuity is defined and we studied their properties. We define and study the properties of neutrosophic $\underline{\mu}$ -compact, $\underline{\mu}$ -Lindelöf and $\underline{\mu}$ -countably compact spaces. We prove that for a countable neutrosophic $\underline{\mu}$ -space X : $\underline{\mu}$ -countably compactness and $\underline{\mu}$ -compactness are equivalent. We give an example of a neutrosophic $\underline{\mu}$ -space X which has a neutrosophic countable $\underline{\mu}$ -base but it is not neutrosophic $\underline{\mu}$ -countably compact .

neutrosophic $\underline{\mu}$ -topological spaces; $\underline{\mu}$ -open; $\underline{\mu}$ -closed; $\underline{\mu}$ -interior; $\underline{\mu}$ -closure; generalize neutrosophic pre-closed sets; neutrosophic α -open sets; neutrosophic $\underline{\mu}$ -continuity; neutrosophic $\underline{\mu}$ -compact; neutrosophic $\underline{\mu}$ -Lindelöf; neutrosophic $\underline{\mu}$ -countably compact space.

1. Introduction

The fuzzy set was introduced by Zadeh [24] in 1965, where each element had a degree of membership. The intuitionistic fuzzy set (Ifs for short) on a universe X was introduced by K. Atanassov [10–12] in 1983 as a generalization of fuzzy set, where besides the degree of membership we have the degree of non- membership of each element. The concept of neutrosophic sets first introduced by Smarandache [19, 22] as a generalization of intuitionistic fuzzy sets, where we have the degree of membership, the degree of indeterminacy and the degree of non-membership of each element in X . After the introduction of the neutrosophic sets, neutrosophic set operations have been investigated. Many researchers have studied topology on neutrosophic sets, such as Smarandache [22], Lupianez [15, 16] and Salama [17]. The neutrosophic interior, neutrosophic closure, neutrosophic exterior, neutrosophic boundary and neutrosophic subspace can be found in [20]. Neutrosophy has many applications specially

in decision making, for more details about new trends of neutrosophic applications one can consult [1], [2], [3] and [4].

Definition 1.1. [19]: A *neutrosophic set* A on the universe of discourse X is defined as $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x) \rangle; x \in X\}$ where $\mu, \sigma, \nu : X \rightarrow]-0, 1^+[$ and $-0 \leq \mu(x) + \sigma(x) + \nu(x) \leq 3^+$.

The class of all neutrosophic set on X will be denoted by $\mathcal{N}(X)$. We will exhibit the basic neutrosophic operations definitions (union, intersection and complement. Since there are different definitions of neutrosophic operations, we will organize the existing definitions into two types, in each type these operation will be consistent and functional.

Definition 1.2. [18][*Neutrosophic sets operations of Type.I*] Let $A, A_\alpha, B \in \mathcal{N}(X)$ such that $\alpha \in \Delta$. Then we define the neutrosophic:

- (1) (*Inclusion*): $A \sqsubseteq B$ If $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
- (2) (*Equality*): $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.
- (3) (*Intersection*) $\bigcap_{\alpha \in \Delta} A_\alpha(x) = \{\langle x, \bigwedge_{\alpha \in \Delta} \mu_{A_\alpha}(x), \bigvee_{\alpha \in \Delta} \sigma_A(x), \bigvee_{\alpha \in \Delta} \nu_A(x) \rangle; x \in X\}$.
- (4) (*Union*) $\bigcup_{\alpha \in \Delta} A_\alpha(x) = \{\langle x, \bigvee_{\alpha \in \Delta} \mu_{A_\alpha}(x), \bigwedge_{\alpha \in \Delta} \sigma_A(x), \bigwedge_{\alpha \in \Delta} \nu_A(x) \rangle; x \in X\}$.
- (5) (*Complement*) $A^c = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x) \rangle; x \in X\}$
- (6) (*Universal set*) $1_X = \{\langle x, 1, 0, 0 \rangle; x \in X\}$; will be called the *neutrosophic universal set*.
- (7) (*Empty set*) $0_X = \{\langle x, 0, 1, 1 \rangle; x \in X\}$; will be called the *neutrosophic empty set*.

Definition 1.3. [18][*Neutrosophic sets operations of Type.II*] Let $A, A_\alpha, B \in \mathcal{N}(X)$ for every $\alpha \in \Delta$. Then we define the neutrosophic:

- (1) (*Inclusion*): $A \sqsubseteq B$ If $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
- (2) (*Equality*): $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.
- (3) (*Intersection*) $\bigcap_{\alpha \in \Delta} A_\alpha(x) = \{\langle x, \bigwedge_{\alpha \in \Delta} \mu_{A_\alpha}(x), \bigwedge_{\alpha \in \Delta} \sigma_A(x), \bigvee_{\alpha \in \Delta} \nu_A(x) \rangle; x \in X\}$.
- (4) (*Union*) $\bigcup_{\alpha \in \Delta} A_\alpha(x) = \{\langle x, \bigvee_{\alpha \in \Delta} \mu_{A_\alpha}(x), \bigvee_{\alpha \in \Delta} \sigma_A(x), \bigwedge_{\alpha \in \Delta} \nu_A(x) \rangle; x \in X\}$.
- (5) (*Complement*) $A^c = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x) \rangle; x \in X\}$
- (6) (*Universal set*) $1_X = \{\langle x, 1, 1, 0 \rangle; x \in X\}$; will be called the *neutrosophic universal set*.
- (7) (*Empty set*) $0_X = \{\langle x, 0, 0, 1 \rangle; x \in X\}$; will be called the *neutrosophic empty set*.

Proposition 1.4. [18] For any $A, B, C \in \mathcal{N}(X)$ we have:

- (1) $A \sqcap A = A$, $A \sqcup A = A$, $A \sqcap 0_X = 0_X$, $A \sqcup 0_X = A$, $A \sqcap 1_X = A$, $A \sqcup 1_X = 1_X$.
- (2) $A \sqcap (B \sqcap C) = (A \sqcap B) \sqcap C$ and $A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$.
- (3) $A \sqcap (\bigcup_{\alpha \in \Delta} A_\alpha) = \bigcup_{\alpha \in \Delta} (A \sqcap A_\alpha)$.
- (4) $A \sqcup (\bigcap_{\alpha \in \Delta} A_\alpha) = \bigcap_{\alpha \in \Delta} (A \sqcup A_\alpha)$.
- (5) $(A^c)^c = A$.

(6) De Morgan's law:

- (a) $(\bigcap_{\alpha \in \Delta} A_\alpha)^c = \bigcup_{\alpha \in \Delta} A_\alpha^c$.
- (b) $(\bigcup_{\alpha \in \Delta} A_\alpha)^c = \bigcap_{\alpha \in \Delta} A_\alpha^c$.

Definition 1.5. [18] [*Neutrosophic Topology*] Let $\tau \subseteq \mathcal{N}(X)$. Then τ is called a *neutrosophic topology* on X if

- (1) $0_X, 1_X \in \tau$.
- (2) The union of any number of neutrosophic sets in τ belongs to τ ,
- (3) The intersection of two neutrosophic sets in τ belongs to τ .

The pair (X, τ) is called a *neutrosophic topological space* over X . Moreover, the members of τ are said to be *neutrosophic open sets* in X . For any $A \in \mathcal{N}(X)$, If $A^c \in \tau$, then A is said to be *neutrosophic closed set* in X .

Definition 1.6. [20][*Neutrosophic interior*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then, the *neutrosophic interior* of A , denoted by $\text{int}(A)$ is the union of all neutrosophic open subsets of A .

Clearly that $\text{int}(A)$ is the biggest neutrosophic open set over X which containing A .

Theorem 1.7. [20] Let (X, τ) be a neutrosophic topological space over X and $A, B \in \mathcal{N}(X)$. Then

- (1) $\text{int}(1_X) = 1_X, \text{int}(0_X) = 0_X$ and $\text{int}(A) \subseteq A$.
- (2) $\text{int}(\text{int}(A)) = \text{int}(A)$.
- (3) $A \subseteq B$ implies $\text{int}(A) \subseteq \text{int}(B)$.
- (4) $\text{int}(A \sqcap B) = \text{int}(A) \sqcap \text{int}(B)$.

Definition 1.8. [20][*Neutrosophic closure*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then, the *neutrosophic closure* of A , denoted by $\text{cl}(A)$ is the intersection of all neutrosophic closed super sets of A .

Clearly, $\text{cl}(A)$ is the smallest neutrosophic closed set over X which contains A .

Theorem 1.9. [20] Let (X, τ) be a neutrosophic topological space over X and $A, B \in \mathcal{N}(X)$. Then,

- (1) $\text{cl}(1_X) = 1_X, \text{cl}(0_X) = 0_X$ and $A \subseteq \text{cl}(A)$.
- (2) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.
- (3) $A \subseteq B$ implies $\text{cl}(A) \subseteq \text{cl}(B)$.
- (4) $\text{cl}(A \sqcup B) = \text{cl}(A) \sqcup \text{cl}(B)$.

Definition 1.10. [5][*Neutrosophic pre-open and pre-closed*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then A is said to be *neutrosophic pre-open set* (NPOS), if $A \subseteq \text{Int}(\text{Cl}(A))$. The complement of a neutrosophic pre-open set is called *neutrosophic pre-closed set* (NPCS).

Definition 1.11. [6][*Neutrosophic α -open*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. A is said to be an α -open set, if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$. The set of all neutrosophic α -open sets in (X, τ) will be denoted by $N\alpha - O(\tau)$.

Definition 1.12. [5][*Neutrosophic pre-closure*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. The *neutrosophic pre-closure* of A , denoted by $pNCL(A)$ is the intersection of all neutrosophic pre-closed super sets of A .

Definition 1.13. [5][*Generalized Neutrosophic pre-closed sets*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. A is said to be a *neutrosophic generalized pre-closed set* (GNPCS) in (X, τ) if $pNCL(A) \subseteq B$ whenever $A \subseteq B$ and B is neutrosophic open. The set of all generalized neutrosophic pre-closed sets in (X, τ) will be denoted by $GNPC(\tau)$.

Theorem 1.14. [5, 6] Let (X, τ) be a neutrosophic topological space over X . Then

- (1) The union of any collection of α -open sets is an α -open set.
- (2) The union of any collection of GNPCs is GNPC.

The following is an improvement of a definition in [14] makes it suitable for type.I and type.II neutrosophic sets.

Definition 1.15. Let X and Y be two nonempty sets and $\Omega : X \rightarrow Y$ be any function. Then for any neutrosophic sets $A \in \mathcal{N}(X)$ and $B \in \mathcal{N}(Y)$ we have:

- (1) The *Type.I (Type.II) pre-image* of B under Ω , denoted by $\Omega^{-1}(B)$, is the Neutrosophic set in X defined by

$$\Omega^{-1}(B) = \{\langle x, \mu_B(\Omega(x)), \sigma_B(\Omega(x)), \nu_B(\Omega(x)) \rangle; x \in X\}$$

- (2) The *Type.I (Type.II) image* of A under Ω , denoted by $\Omega(A)$, is the Neutrosophic set in Y defined by

$$\Omega(A) = \{\langle y, \Omega(\mu_A)(y), \Omega(\sigma_A)(y), (1 - \Omega(1 - \nu_A))(y) \rangle; y \in Y\} \text{ where}$$

$$(\mu_A)(y) = \begin{cases} \sup_{x \in \Omega^{-1}(y)} \mu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 0 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases}$$

$$(\sigma_A)(y) = \begin{cases} \inf_{x \in \Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 1 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} \quad (\text{Type.I})$$

$$(\sigma_A)(y) = \begin{cases} \sup_{x \in \Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 0 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} \quad (\text{Type.II})$$

$$(1 - \Omega(1 - \nu_A))(y) = \begin{cases} \inf_{x \in \Omega^{-1}(y)} \nu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 1 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases}$$

For the sake of simplicity we write $\Omega - (\nu_A$ instead of $(1 - \Omega(1 - \nu_A))$.

Note that the only difference between Type.I and Type.II images lies in the definition of the image of σ and this is important to make sure both Type.I and Type.II neutrosophic functions satisfy the following proposition.

Proposition 1.16. [14] Let X and Y be two nonempty sets and $\Omega : X \rightarrow Y$ be any function. Let $A, A_\alpha \in \mathcal{N}(X)$ and $B, B_\alpha \in \mathcal{N}(Y)$. Then we have:

- (1) $A_1 \subseteq A_2 \Rightarrow \Omega(A_1) \subseteq \Omega(A_2)$.
- (2) $B_1 \subseteq B_2 \Rightarrow \Omega^{-1}(B_1) \subseteq \Omega^{-1}(B_2)$.
- (3) $A \subseteq \Omega^{-1}(\Omega(A))$ and equality holds if Ω is injective.
- (4) $\Omega(\Omega^{-1}(A)) \subseteq A$ and equality holds if Ω is surjective.
- (5) $\Omega(\sqcup_{\alpha \in \Delta} A_\alpha) = \sqcup_{\alpha \in \Delta} \Omega(A_\alpha)$.
- (6) $\Omega(\sqcap_{\alpha \in \Delta} A_\alpha) \subseteq \sqcap_{\alpha \in \Delta} \Omega(A_\alpha)$ and equality holds when Ω is injective.
- (7) $\Omega^{-1}(\sqcup_{\alpha \in \Delta} B_\alpha) = \sqcup_{\alpha \in \Delta} \Omega^{-1}(B_\alpha)$.
- (8) $\Omega^{-1}(\sqcap_{\alpha \in \Delta} B_\alpha) = \sqcap_{\alpha \in \Delta} \Omega^{-1}(B_\alpha)$.
- (9) $\Omega^{-1}(1_N) = 1_N, \Omega^{-1}(0_N) = 0_N$.
- (10) $\Omega(1_N) = 1_N$ and $\Omega(0_N) = 0_N$, whenever Ω is surjective.

Definition 1.17. Let X be a nonempty set and $0 < \alpha, \beta, \gamma < 1$. Then a neutrosophic set $A \in \mathcal{N}(X)$ is called:

- (1) A *neutrosophic point of Type.I* if and only if there exists $x \in X$ such that $A = \{\langle x, \alpha, \beta, \gamma \rangle\} \cup \{\langle \acute{x}, 0, 1, 1 \rangle; \acute{x} \neq x\}$.
- (2) A *neutrosophic point of Type.II* if $A = \{\langle x, \alpha, \beta, \gamma \rangle\} \cup \{\langle \acute{x}, 0, 0, 1 \rangle; \acute{x} \neq x\}$. Neutrosophic points will be denoted by $x_{\alpha, \beta, \gamma}$.

Now, we will exhibit some definitions and properties of μ -topological spaces. Á. Császár [13] introduced the notion of Generalized Topological Space (GTS). He also introduced the notion of Murad Arar and Saeid Jafari, Neutrosophic $\underline{\mu}$ -Topological spaces

$(\mu_1; \mu_2)$ -continuous function on GTS's. μ -compactness introduced in [23] and [21]. Countably μ -paracompact introduced and studied in [8]. Strongly Generalized neighborhood systems introduced and studies in [9].

Let X be a nonempty set. A collection μ of subsets of X is called a generalized topology on X and the pair (X, μ) is called a generalized topological space, if μ satisfies the following two conditions:

- (1) $\emptyset \in \mu$.
- (2) Any union of elements of μ belongs to μ .

Let $\beta \subseteq \exp(X)$ and $\emptyset \in \beta$. Then β is called a μ -base for μ if $\mu = \{\bigcup \beta'; \beta' \subset \beta\}$. We also say μ is generated by β . If β is countable, then it is said a countable μ -base. A generalized topological space (X, μ) is said to be strong if $X \in \mu$. A subset B of X is called μ -open (resp. μ -closed) if $B \in \mu$ (resp. if $X - B \in \mu$). The set of all μ -open sets containing a point $x \in X$ will be denoted by μ_x (i.e. $\mu_x = \{U \in \mu; x \in U\}$).

Definition 1.18. Let (X, μ_1) and (X, μ_2) be two μ -topological space. A function $f : (X, \mu_1) \rightarrow (X, \mu_2)$ is said to be (μ_1, μ_2) -continuous if and only if $f^{-1}(V) \in \mu_1$ whenever $V \in \mu_2$.

Definition 1.19. Let X be a generalized topological space and let \mathfrak{F} be a collection of subsets of X . Then \mathfrak{F} is said to be:

- (1) A μ -cover of X if $X = \bigcup \{U; U \in \mathfrak{F}\}$.
- (2) A μ -open cover of X if \mathfrak{F} is a μ -cover of X and $U \in \mu$ for every $U \in \mathfrak{F}$.

Definition 1.20. Let X be a generalized topological space and let \mathfrak{F} and \mathfrak{C} be μ -covers of X . Then \mathfrak{C} is said to be a μ -subcover of \mathfrak{F} , if $\mathfrak{C} \subseteq \mathfrak{F}$.

Definition 1.21. A generalized topological space X is said to be μ -compact (resp. μ -Lindelöf) if and only if every μ -open cover of X has a finite (resp. countable) μ -subcover.

The following theorem shows some differences between topological spaces and μ -topological spaces.

Theorem 1.22.

- (1) In μ -topological spaces $Int_\mu(A \cap B) = Int_\mu(A) \cap Int_\mu(B)$ is not satisfied where $Int_\mu(A)$ stands for interior of A .
- (2) In μ -topological spaces $Cl_\mu(A \cup B) = Cl_\mu(A) \cup Cl_\mu(B)$ is not satisfied where $Cl_\mu(A)$ stands for the closure of A in μ .

- (3) [7] There exists a μ -normal space with a countable μ -base which has a μ -open cover with no μ -open point-finite refinement.

2. Neutrosophic $\underline{\mu}$ -Topological Spaces

In the literature of generalized topological spaces the symbol μ is used to refer the μ -topology and in neutrosophic sets it is used to refer the membership function μ , so, to avoid ambiguity, we will use the underlined $\underline{\mu}$ to refer the $\underline{\mu}$ -topology and keep μ for the membership function in neutrosophic sets.

Definition 2.1 (*Neutrosophic $\underline{\mu}$ -Topology*). Let $\underline{\mu} \subseteq \mathcal{N}(X)$. Then $\underline{\mu}$ is called a *neutrosophic $\underline{\mu}$ -topology* on X if

- (1) $0_X \in \underline{\mu}$.
- (2) The union of any number of neutrosophic sets in $\underline{\mu}$ belongs to $\underline{\mu}$.

The pair $(X, \underline{\mu})$ is called a *neutrosophic $\underline{\mu}$ -topological space* over X . The members of $\underline{\mu}$ are said to be *neutrosophic $\underline{\mu}$ -open sets* in X . If $1_X \in \underline{\mu}$, then $(X, \underline{\mu})$ is called a *strong neutrosophic $\underline{\mu}$ -topological space*. For any $A \in \mathcal{N}(X)$, if $A^c \in \underline{\mu}$, then A is said to be *neutrosophic $\underline{\mu}$ -closed set* in X . Since there are two types of neutrosophic sets, a neutrosophic $\underline{\mu}$ -topology is said to be Type.I(Type.II) neutrosophic topology if its elements are treated as Type.I(Type.II) neutrosophic sets.

Example 2.2. Let $X = \{a, b, c\}$ and $A, B, C, \acute{C} \in \mathcal{N}(X)$ with:

$A = \{\langle a, 0.3, 0.5, 0.7 \rangle, \langle b, 0.3, 0.4, 1 \rangle\}$, $B = \{\langle a, 0.4, 0.7, 0.1 \rangle, \langle b, 0.2, 0.6, 0.9 \rangle\}$, $C = \{\langle a, 0.4, 0.5, 0.1 \rangle, \langle b, 0.3, 0.4, 0.9 \rangle\}$, $\acute{C} = \{\langle a, 0.4, 0.7, 0.1 \rangle, \langle b, 0.3, 0.6, 0.9 \rangle\}$. Then $\underline{\mu} = \{0_X, A, B, C\}$ is a Type.I neutrosophic $\underline{\mu}$ -topology and $\acute{\underline{\mu}} = \{1_X, 0_X, A, B, \acute{C}\}$ is a Type.II strong neutrosophic $\underline{\mu}$ -topology. Neither $\underline{\mu}$ nor $\acute{\underline{\mu}}$ is neutrosophic topology. Note that, in $(X, \underline{\mu})$, $A \sqcap B = \{\langle a, 0.3, 0.7, 0.7 \rangle, \langle b, 0.2, 0.6, 1 \rangle\}$ is not neutrosophic $\underline{\mu}$ -open (here we apply type.I intersection). And in $(X, \acute{\underline{\mu}})$ we have $A \sqcap B = \{\langle a, 0.3, 0.5, 0.7 \rangle, \langle b, 0.2, 0.4, 1 \rangle\}$ is not neutrosophic $\underline{\mu}$ -open (here we apply type.II intersection).

Most examples and theorems will be considered for Type.I neutrosophic sets, since the two types of neutrosophic sets have the same properties.

Definition 2.3 (*Neutrosophic $\underline{\mu}$ -interior*). Let $(X, \underline{\mu})$ be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then, the *neutrosophic $\underline{\mu}$ -interior* of A , denoted by $\text{int}_{\underline{\mu}}(A)$ is the union of all neutrosophic $\underline{\mu}$ -open subsets of A . Clearly $\text{int}_{\underline{\mu}}(A)$ is the biggest neutrosophic $\underline{\mu}$ -open set over X contained in A .

Theorem 2.4. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$ -topological space over X and $A, B \in \mathcal{N}(X)$. Then,

- (1) $\text{int}_{\underline{\mu}}(0_X) = 0_X$ and $\text{int}_{\underline{\mu}}(A) \subseteq A$.
- (2) $\text{int}_{\underline{\mu}}(1_X) = 1_X$ whenever $\underline{\mu}$ is a strong $\underline{\mu}$ -topology.
- (3) $\text{int}_{\underline{\mu}}(\text{int}_{\underline{\mu}}(A)) = \text{int}_{\underline{\mu}}(A)$.
- (4) $A \subseteq B$ implies $\text{int}_{\underline{\mu}}(A) \subseteq \text{int}_{\underline{\mu}}(B)$.
- (5) $\text{int}_{\underline{\mu}}(A) = A$ if and only if $A \in \underline{\mu}$.
- (6) If $A \subseteq B$, then $\text{int}_{\underline{\mu}}(A) \subseteq \text{int}_{\underline{\mu}}(B)$.
- (7) $\text{int}_{\underline{\mu}}(A \sqcap B) \subseteq \text{int}_{\underline{\mu}}(A) \sqcap \text{int}_{\underline{\mu}}(B)$. Equality does not hold, see Example 2.5.

Proof. We will establish a proof for (4) and (7).

(4) Since $A \subseteq B$, $\{U \in \underline{\mu}; U \subseteq A\} \subseteq \{U \in \underline{\mu}; U \subseteq B\}$. So that $\mu_{\text{int}_{\underline{\mu}}(A)}(x) = \sup\{\mu_U(x); U \in \underline{\mu}, U \subseteq A\} \leq \sup\{\mu_U(x); U \in \underline{\mu}, U \subseteq B\} = \mu_{\text{int}_{\underline{\mu}}(B)}(x)$, $\sigma_{\text{int}_{\underline{\mu}}(A)}(x) = \inf\{\sigma_U(x); U \in \underline{\mu}, U \subseteq A\} \geq \inf\{\sigma_U(x); U \in \underline{\mu}, U \subseteq B\} = \sigma_{\text{int}_{\underline{\mu}}(B)}(x)$, and $\nu_{\text{int}_{\underline{\mu}}(A)}(x) = \inf\{\nu_U(x); U \in \underline{\mu}, U \subseteq A\} \geq \inf\{\nu_U(x); U \in \underline{\mu}, U \subseteq B\} = \nu_{\text{int}_{\underline{\mu}}(B)}(x)$. Which means $\text{int}_{\underline{\mu}}(A) \subseteq \text{int}_{\underline{\mu}}(B)$.

(7) Since $A \sqcap B \subseteq A$ and B , $\text{int}_{\underline{\mu}}(A \sqcap B) \subseteq \text{int}_{\underline{\mu}}(A)$ and $\text{int}_{\underline{\mu}}(A \sqcap B) \subseteq \text{int}_{\underline{\mu}}(B)$ (by (4)), so we have $\text{int}_{\underline{\mu}}(A \sqcap B) \subseteq \text{int}_{\underline{\mu}}(A) \sqcap \text{int}_{\underline{\mu}}(B)$. \square

Example 2.5. Consider $(X, \underline{\mu})$ as in Example 2.2. Note that:

- (1) $\text{int}_{\underline{\mu}}(1_X) = 0_X \sqcup A \sqcup B \sqcup C = C \neq 1_X$.
- (2) Since $A \sqcap B = \{\langle a, 0.3, 0.7, 0.7 \rangle, \langle b, 0.2, 0.6, 1 \rangle\}$ and there is no neutrosophic $\underline{\mu}$ -open set in $\underline{\mu}$ contained in $A \sqcap B$ except 0_X , we have $\text{int}_{\underline{\mu}}(A \sqcap B) = 0_X$, and since $A, B \in \underline{\mu}$, $\text{int}_{\underline{\mu}}(A) \sqcap \text{int}_{\underline{\mu}}(B) = A \sqcap B \neq \text{int}_{\underline{\mu}}(A \sqcap B) = 0_X$.

Definition 2.6 (Neutrosophic $\underline{\mu}$ -closure). Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$ -topological space over X and $A \in \mathcal{N}(X)$. Then, the neutrosophic $\underline{\mu}$ -closure of A , denoted by $cl_{\underline{\mu}}(A)$, is the intersection of all neutrosophic $\underline{\mu}$ -closed super sets of A .

Clearly $cl_{\underline{\mu}}(A)$ is the smallest neutrosophic $\underline{\mu}$ -closed set over X which containing A .

Theorem 2.7. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$ -topological space over X and $A, B \in \mathcal{N}(X)$. Then,

- (1) $cl_{\underline{\mu}}(1_X) = 1_X$ and $A \subseteq cl_{\underline{\mu}}(A)$.
- (2) $cl_{\underline{\mu}}(0_X) = 0_X$ whenever $\underline{\mu}$ is a strong $\underline{\mu}$ -topology.
- (3) $cl_{\underline{\mu}}(cl_{\underline{\mu}}(A)) = cl_{\underline{\mu}}(A)$.
- (4) $A \subseteq B$ implies $cl_{\underline{\mu}}(A) \subseteq cl_{\underline{\mu}}(B)$.
- (5) A is $\underline{\mu}$ -closed if and only if $cl_{\underline{\mu}}(A) = A$.
- (6) $cl(A) \sqcup cl(B) \subseteq cl(A \sqcup B)$. The equality does not hold.

Example 2.8. Consider $(X, \underline{\mu})$ as in Example 2.2. The only $\underline{\mu}$ -closed sets in $(X, \underline{\mu})$ are:

- (1) $0_X^c = 1_X$.

$$(2) A^c = \{\langle a, 0.7, 0.5, 0.3 \rangle, \langle b, 1, 0.6, 0.3 \rangle\}.$$

$$(3) B^c = \{\langle a, 0.1, 0.3, 0.4 \rangle, \langle b, 0.9, 0.4, 0.2 \rangle\}.$$

$$(4) C^c = \{\langle a, 0.1, 0.5, 0.4 \rangle, \langle b, 0.9, 0.6, 0.3 \rangle\}$$

It is clear that $cl_{\underline{\mu}}(0_X) = 0_X^c \sqcap A^c \sqcap B^c \sqcap C^c = \{\langle a, 0.1, 0.5, 0.4 \rangle, \langle b, 0.9, 0.6, 0.3 \rangle\} \neq 0_X$. Let $H = A^c$ and $K = B^c$. Then $cl_{\underline{\mu}}(H) \sqcup cl_{\underline{\mu}}(K) = \{\langle a, 0.7, 0.3, 0.3 \rangle, \langle b, 1, 0.4, 0.2 \rangle\}$ and $cl_{\underline{\mu}}(H \sqcup K) = 1_X$, since the only neutrosophic $\underline{\mu}$ -closed set containing $H \sqcup K = cl_{\underline{\mu}}(H) \sqcup cl_{\underline{\mu}}(K)$ is 1_X .

The following theorem shows the importance of generalized neutrosophic $\underline{\mu}$ -topological spaces.

Theorem 2.9. *Let (X, τ) be a neutrosophic topological space over X . Then:*

- (1) *The set $N\alpha - O(\tau)$ of all neutrosophic α -open sets over (X, τ) is a strong neutrosophic $\underline{\mu}$ -topology over X .*
- (2) *The set $GNPC(\tau)$ of all neutrosophic pre-closed sets in (X, τ) is a strong neutrosophic $\underline{\mu}$ -topology over X .*

Proof. Easy! we just call Theorem 1.14. \square

Definition 2.10. Let $(X, \underline{\mu})$ and $(Y, \underline{\mu})$ be two neutrosophic $\underline{\mu}$ -topological spaces and let $\Omega : X \rightarrow Y$ be any function. Then Ω is said to be neutrosophic $(\underline{\mu}, \underline{\mu})$ -continuous if for any neutrosophic point $x_{\alpha, \beta, \gamma}$ and for any neutrosophic $\underline{\mu}$ -open set $V \in \tau$ such that $f(x_{\alpha, \beta, \gamma}) \in V$ there exists $U \in \tau$ such that $x_{\alpha, \beta, \gamma} \in U$ and $\Omega(U) \subseteq V$.

Theorem 2.11. *Let X and Y be two nonempty sets and $\Omega : X \rightarrow Y$ be any function. Let $x_{\alpha, \beta, \gamma}$ be a neutrosophic point in X . Then $\Omega(x_{\alpha, \beta, \gamma}) = \Omega(x)_{\alpha, \beta, \gamma}$; that is the image of a neutrosophic point is a neutrosophic point.*

Proof. We will prove it for Type.I and Type.II neutrosophic sets. Let $A = x_{\alpha, \beta, \gamma}$ and $\Omega(x) = y$. Then the Type.I (Type.II) image of A under Ω , denoted by $\Omega(A)$, is the Neutrosophic set:

$$\Omega(A) = \{\langle y, \Omega(\mu_A)(y), \Omega(\sigma_A)(y), (1 - \Omega(1 - \nu_A))(y) \rangle; y \in Y\}, \text{ where}$$

$$(\mu_A)(y) = \begin{cases} \sup_{x \in \Omega^{-1}(y)} \mu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 0 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} = \begin{cases} \alpha & \text{if } y = y \\ 0 & \text{if } y \neq y \end{cases}$$

$$(\sigma_A)(y) = \begin{cases} \inf_{x \in \Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 1 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} = \begin{cases} \beta & \text{if } y = y \\ 1 & \text{if } y \neq y \end{cases} \quad (Type.I)$$

$$(\sigma_A)(y) = \begin{cases} \sup_{x \in \Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 0 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} = \begin{cases} \beta & \text{if } y = \acute{y} \\ 0 & \text{if } y \neq \acute{y} \end{cases} \quad (\text{Type.II})$$

$$(1 - \Omega(1 - \nu_A))(y) = \begin{cases} \inf_{x \in \Omega^{-1}(y)} \nu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 1 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} = \begin{cases} \gamma & \text{if } y = \acute{y} \\ 1 & \text{if } y \neq \acute{y} \end{cases}$$

That is -in Type.I and Type.II neutrosophic sets- $\Omega(x_{\alpha,\beta,\gamma}) = \acute{y}_{\alpha,\beta,\gamma}$ where $\acute{y} = \Omega(x)$. \square

Definition 2.12. A neutrosophic point of type.I (type.II) $x_{\alpha,\beta,\gamma}$ is said to be in the neutrosophic set A -in symbols $x_{\alpha,\beta,\gamma} \in A$ - if and only if $\alpha < \mu_A(x), \beta > \sigma_A(x)$ and $\gamma > \nu_A(x)$ ($\alpha < \mu_A(x), \beta < \sigma_A(x)$ and $\gamma > \nu_A(x)$).

Lemma 2.13. Let $A \in \mathcal{N}(X)$ and suppose that for every $x_{\alpha,\beta,\gamma} \in A$ there exists a neutrosophic set $B(x_{\alpha,\beta,\gamma}) \in \mathcal{N}(X)$ such that $x_{\alpha,\beta,\gamma} \in B(x_{\alpha,\beta,\gamma}) \subseteq A$. Then $A = \sqcup \{B(x_{\alpha,\beta,\gamma}); x_{\alpha,\beta,\gamma} \in A\}$.

Proof. The proof will be established for Type.I. Set $H = \sqcup \{B(x_{\alpha,\beta,\gamma}); x_{\alpha,\beta,\gamma} \in A\}$. It suffices to show that $A \subseteq H$ and $H \subseteq A$. First note that for every $B(x_{\alpha,\beta,\gamma}) \subseteq A$ we have $\mu_{B(x_{\alpha,\beta,\gamma})}(x) \leq \mu_A(x)$, $\sigma_{B(x_{\alpha,\beta,\gamma})}(x) \geq \sigma_A(x)$ and $\nu_{B(x_{\alpha,\beta,\gamma})}(x) \geq \nu_A(x)$ for every $x \in X$. Let $x \in X$. Then $\mu_H(x) = \sup\{\mu_{B(x_{\alpha,\beta,\gamma})}(x); x_{\alpha,\beta,\gamma} \in A\} \leq \mu_A(x)$, $\sigma_H(x) = \inf\{\sigma_{B(x_{\alpha,\beta,\gamma})}(x); x_{\alpha,\beta,\gamma} \in A\} \geq \sigma_A(x)$, and $\nu_H(x) = \inf\{\nu_{B(x_{\alpha,\beta,\gamma})}(x); x_{\alpha,\beta,\gamma} \in A\} \geq \nu_A(x)$, this means $H \subseteq A$. To prove the converse, let $x \in X$ and let $\alpha_1 = \mu_A(x)$, $\beta_1 = \sigma_A(x)$, and $\gamma_1 = \nu_A(x)$. Consider the neutrosophic points $x_{\alpha,\beta,\gamma}$ such that $\alpha < \alpha_1$, $\beta > \beta_1$ and $\gamma > \gamma_1$. Then $x_{\alpha,\beta,\gamma} \in A$. Let $A_x = \cup\{B(x_{\alpha,\beta,\gamma}); \alpha < \alpha_1, \beta > \beta_1 \text{ and } \gamma > \gamma_1\}$. It is clear that $A_x \subseteq H$ so that $\mu_{A_x}(x) \leq \mu_H(x)$, $\sigma_{A_x}(x) \geq \sigma_H(x)$ and $\nu_{A_x}(x) \geq \nu_H(x)$. But $\mu_{A_x}(x) = \sup\{\mu_{A_{x_{\alpha,\beta,\gamma}}}(x); \alpha < \alpha_1, \beta > \beta_1, \gamma > \gamma_1\} = \alpha_1 = \mu_A(x)$, $\sigma_{A_x}(x) = \inf\{\sigma_{A_{x_{\alpha,\beta,\gamma}}}(x); \alpha < \alpha_1, \beta > \beta_1, \gamma > \gamma_1\} = \beta_1 = \sigma_A(x)$ and $\nu_{A_x}(x) = \sup\{\nu_{A_{x_{\alpha,\beta,\gamma}}}(x); \alpha < \alpha_1, \beta > \beta_1, \gamma > \gamma_1\} = \gamma_1 = \nu_A(x)$, which implies $\mu_A(x) \leq \mu_H(x)$, $\sigma_A \geq \sigma_H(x)$ and $\nu_A \geq \nu_H(x)$ or, equivalently, $A \subseteq H$. \square

Corollary 2.14. Let $(X, \underline{\mu})$ be a neutrosophic topological space over X and let $A \in \mathcal{N}(X)$. Then A is neutrosophic $\underline{\mu}$ -open in $(X, \underline{\mu})$ if and only if for every $x_{\alpha,\beta,\gamma} \in A$ there exists a neutrosophic $\underline{\mu}$ -open set $B(x_{\alpha,\beta,\gamma}) \in \underline{\mu}$ such that $x_{\alpha,\beta,\gamma} \in B(x_{\alpha,\beta,\gamma}) \subseteq A$.

Definition 2.15. Let $(X, \underline{\mu})$ be a neutrosophic topological space over X . A sub-collection $\mathcal{B} \subseteq \underline{\mu}$ is called a neutrosophic $\underline{\mu}$ -base for $\underline{\mu}$ if and only if for any $U \in \underline{\mu}$ there exists $\acute{\mathcal{B}} \subseteq \mathcal{B}$ such that $U = \sqcup \{B; B \in \acute{\mathcal{B}}\}$.

Corollary 2.16. Let $(X, \underline{\mu})$ be a neutrosophic topological space over X . Then a subcollection \mathcal{B} of $\underline{\mu}$ is a neutrosophic $\underline{\mu}$ -base for $\underline{\mu}$ if and only if for every $U \in \underline{\mu}$ and every $x_{\alpha, \beta, \gamma} \in U$ there exists $B \in \mathcal{B}$ such that $x_{\alpha, \beta, \gamma} \in B \subseteq U$.

Theorem 2.17. Let $(X, \underline{\mu})$ and $(Y, \underline{\mu})$ be two neutrosophic $\underline{\mu}$ -topological spaces and let $\Omega : X \rightarrow Y$ be any function. Then Ω is neutrosophic $(\underline{\mu}, \underline{\mu})$ -continuous if and only if $\Omega^{-1}(V)$ is a neutrosophic $\underline{\mu}$ -open set whenever V is a neutrosophic $\underline{\mu}$ -open set.

Proof. Suppose that Ω is neutrosophic $(\underline{\mu}, \underline{\mu})$ -continuous, V be a neutrosophic $\underline{\mu}$ -open set, and $x_{\alpha, \beta, \gamma} \in \Omega^{-1}(V)$. Then $\Omega(x_{\alpha, \beta, \gamma}) = \Omega(x)_{\alpha, \beta, \gamma} \in \Omega(\Omega^{-1}(V)) \subseteq V$ (we used theorem 1.16(4)). Since Ω is $(\underline{\mu}, \underline{\mu})$ -continuous, there exists a neutrosophic $\underline{\mu}$ -open set $V(x_{\alpha, \beta, \gamma})$ such that $x_{\alpha, \beta, \gamma} \in V(x_{\alpha, \beta, \gamma})$ and $\Omega(V(x_{\alpha, \beta, \gamma})) \subseteq V$, which implies, by theorem 1.16(3), $V(x_{\alpha, \beta, \gamma}) \subseteq \Omega^{-1}(\Omega(V(x_{\alpha, \beta, \gamma}))) \subseteq \Omega^{-1}(V)$, that is, by corollary 2.14, $\Omega^{-1}(V)$ is $\underline{\mu}$ -open. Conversely, suppose the condition of the theorem is true. To show that Ω is $(\underline{\mu}, \underline{\mu})$ -continuous let $x_{\alpha, \beta, \gamma}$ be a neutrosophic point in X and V is a neutrosophic $\underline{\mu}$ -open set such that $\Omega(x_{\alpha, \beta, \gamma}) \in V$. By the condition of the theorem, $\Omega^{-1}(V)$ is neutrosophic $\underline{\mu}$ -open set, and from theorem 1.16 (3) and (4) we have $x_{\alpha, \beta, \gamma} \in \Omega^{-1}(\Omega(x_{\alpha, \beta, \gamma})) \subseteq \Omega^{-1}(V)$, and $\Omega(\Omega^{-1}(V)) \subseteq V$, respectively. So we have $\Omega^{-1}(V)$ is neutrosophic $\underline{\mu}$ -open, $x_{\alpha, \beta, \gamma} \in \Omega^{-1}(V)$ and $\Omega(\Omega^{-1}(V)) \subseteq V$ which mean Ω is a neutrosophic $(\underline{\mu}, \underline{\mu})$ -continuous function. \square

Theorem 2.18. Let $(X, \underline{\mu})$ and $(Y, \underline{\mu})$ be two neutrosophic $\underline{\mu}$ -topological spaces, $\Omega : X \rightarrow Y$ be any function, and $\dot{\mathcal{B}}$ is a neutrosophic $\underline{\mu}$ -base for $\underline{\mu}$. Then Ω is neutrosophic $(\underline{\mu}, \underline{\mu})$ -continuous if and only if $\Omega^{-1}(V)$ is a neutrosophic $\underline{\mu}$ -open set for every $V \in \dot{\mathcal{B}}$.

Proof. \Rightarrow) Obvious!

\Leftarrow) Suppose that Ω satisfies the condition of the theorem, and let V be any neutrosophic $\underline{\mu}$ -open set. Since $\dot{\mathcal{B}}$ is a neutrosophic $\underline{\mu}$ -base for $\underline{\mu}$, there exists a sub-collection \mathcal{B}^* from $\dot{\mathcal{B}}$ such that $V = \sqcup \{B; B \in \mathcal{B}^*\}$. But $\Omega^{-1}(V) = \Omega^{-1}(\sqcup \{B; B \in \mathcal{B}^*\}) = \sqcup \{\Omega^{-1}(B); B \in \mathcal{B}^*\}$. Since $\Omega^{-1}(B)$ is neutrosophic $\underline{\mu}$ -open for every $B \in \mathcal{B}^*$, $\Omega^{-1}(V)$ is neutrosophic $\underline{\mu}$ -open, and so Ω is a neutrosophic $(\underline{\mu}, \underline{\mu})$ -continuous function. \square

Definition 2.19. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$ -topological space. A sub-collection $\mathcal{U} \subseteq \underline{\mu}$ is called a type.I (type.II) neutrosophic $\underline{\mu}$ -open cover of X , if $1_X = \sqcup \{U; U \in \mathcal{U}\}$.

Definition 2.20. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$ -topological space, and let \mathcal{U} be a neutrosophic $\underline{\mu}$ -open cover of X . A sub-collection $\dot{\mathcal{U}} \subseteq \mathcal{N}(X)$ is called a neutrosophic $\underline{\mu}$ -subcover of X from \mathcal{U} , if $\dot{\mathcal{U}}$ is a neutrosophic $\underline{\mu}$ -open cover of X and $\dot{\mathcal{U}} \subseteq \mathcal{U}$.

Corollary 2.21. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$ -topological space. A sub-collection $\mathcal{U} \subseteq \underline{\mu}$ is a $\underline{\mu}$ -open cover of X if and only if for every $x_{\alpha, \beta, \gamma}$ in X there exists $U \in \mathcal{U}$ such that $x_{\alpha, \beta, \gamma} \in U$.

Definition 2.22. A neutrosophic $\underline{\mu}$ -topological space $(X, \underline{\mu})$ is called *neutrosophic $\underline{\mu}$ -compact space* if every neutrosophic $\underline{\mu}$ -open cover of X from $\underline{\mu}$ has a finite neutrosophic $\underline{\mu}$ -subcover of X .

Theorem 2.23. Let $\Omega : (X, \underline{\mu}) \rightarrow (Y, \underline{\mu})$ be a neutrosophic $(\underline{\mu}, \underline{\mu})$ -continuous function. If $(X, \underline{\mu})$ is neutrosophic $\underline{\mu}$ -compact, then $(Y, \underline{\mu})$ is neutrosophic $\underline{\mu}$ -compact.

Proof. Let \mathcal{V} be a neutrosophic $\underline{\mu}$ -open cover of Y . Consider the collection $\mathcal{V}^{-1} = \{\Omega^{-1}(V); V \in \mathcal{V}\}$. Since Ω is neutrosophic $(\underline{\mu}, \underline{\mu})$ -continuous, $\mathcal{V}^{-1} \subseteq \underline{\mu}$. Set $A = \sqcup \{\Omega^{-1}(V); V \in \mathcal{V}\}$. To show that $A = 1_X$. But $A = \sqcup \{\Omega^{-1}(V); V \in \mathcal{V}\} = \Omega^{-1}(\sqcup \{V; V \in \mathcal{V}\}) = \Omega^{-1}(1_Y) = 1_X$ (we used Proposition 1.16, (9)); i.e. \mathcal{V}^{-1} is a neutrosophic $\underline{\mu}$ -open cover of X . Since X is neutrosophic $\underline{\mu}$ -compact space, \mathcal{V}^{-1} has a finite neutrosophic $\underline{\mu}$ -open sub-cover \mathcal{V}^{*-1} . Suppose that $\mathcal{V}^{*-1} = \{\Omega^{-1}(V_i); i = 1, 2, \dots, n\}$. Set $\mathcal{V}^* = \{V_i; i = 1, 2, \dots, n\}$. It is clear that $\mathcal{V}^* \subseteq \mathcal{V}$. Since Ω is surjective, $\Omega(\Omega^{-1}(V_i)) = V_i$ for every $i = 1, 2, \dots, n$, so we have $\sqcup \{V_i; i = 1, 2, \dots, n\} = \sqcup \{\Omega(\Omega^{-1}(V_i)); i = 1, 2, \dots, n\} = \Omega(\sqcup \{\Omega^{-1}(V_i); i = 1, 2, \dots, n\}) = \Omega(1_X) = 1_Y$, that is \mathcal{V}^* is a neutrosophic $\underline{\mu}$ -subcover of Y from \mathcal{V} . \square

Theorem 2.24. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$ -topological space, and \mathcal{B} be a neutrosophic $\underline{\mu}$ -base for $\underline{\mu}$. Then $(X, \underline{\mu})$ is neutrosophic $\underline{\mu}$ -compact if and only if every neutrosophic $\underline{\mu}$ -open cover of X from \mathcal{B} has a finite neutrosophic $\underline{\mu}$ -subcover.

Proof. \Rightarrow) Obvious!

\Leftarrow) Suppose that X satisfies the condition of the theorem. Let \mathcal{U} be a neutrosophic $\underline{\mu}$ -open cover of X . For every $U \in \mathcal{U}$ there exists $\mathcal{B}_U \subseteq \mathcal{B}$ such that $U = \sqcup \mathcal{B}_U$. Set $\mathcal{B}_1 = \{B; B \in \mathcal{B}_U, U \in \mathcal{U}\}$. It is clear that \mathcal{B}_1 is a neutrosophic $\underline{\mu}$ -open cover of X from \mathcal{B} , so it has a finite neutrosophic $\underline{\mu}$ -subcover \mathcal{B}_1^* . For every $B \in \mathcal{B}_1^*$ there exists $U_B \in \mathcal{U}$ such that $B \subseteq U_B$. Let $\mathcal{U}^* = \{U_B; B \in \mathcal{B}_1^*\}$. Since \mathcal{B}_1^* is a finite neutrosophic $\underline{\mu}$ -open cover of X , \mathcal{U}^* is a finite $\underline{\mu}$ -subcover of X from \mathcal{U} , and X is neutrosophic $\underline{\mu}$ -compact. \square

Definition 2.25. A neutrosophic $\underline{\mu}$ -topological space $(X, \underline{\mu})$ is called:

- (1) *neutrosophic $\underline{\mu}$ -Lindelöf space* if every neutrosophic $\underline{\mu}$ -open cover of X from $\underline{\mu}$ has a countable neutrosophic $\underline{\mu}$ -subcover of X .
- (2) *neutrosophic $\underline{\mu}$ -countably compact space* if every neutrosophic $\underline{\mu}$ -open countable cover of X from $\underline{\mu}$ has a finite neutrosophic $\underline{\mu}$ -subcover of X .

Theorem 2.26. *Every neutrosophic $\underline{\mu}$ -topological space with a countable neutrosophic $\underline{\mu}$ -base is neutrosophic $\underline{\mu}$ -Lindelöf.*

Proof. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$ -topological space with a countable neutrosophic $\underline{\mu}$ -base \mathcal{B} . Let \mathcal{U} be a neutrosophic $\underline{\mu}$ -open cover of X . For every $U \in \mathcal{U}$, there exists $\mathcal{B}_U \subseteq \mathcal{B}$ such that $U = \sqcup \mathcal{B}_U$. Let $\mathcal{B}^* = \cup \{\mathcal{B}_U; U \in \mathcal{U}\}$. Since \mathcal{U} is a neutrosophic $\underline{\mu}$ -open cover of X , \mathcal{B}^* is a neutrosophic $\underline{\mu}$ -open cover of X . And since $\mathcal{B}^* \subseteq \mathcal{B}$, \mathcal{B}^* is countable. We can write $\mathcal{B}^* = \{B_i; i = 1, 2, 3, \dots\}$. For every $i = 1, 2, 3, \dots$ pick a unique $U_i \in \mathcal{U}$ such that $B_i \subseteq U_i$. Let $\mathcal{U}^* = \{U_i; i = 1, 2, \dots\}$. Since \mathcal{B}^* is a neutrosophic $\underline{\mu}$ -open cover of X , \mathcal{U}^* is a neutrosophic $\underline{\mu}$ -open subcover of X from \mathcal{U} , and hence X is a neutrosophic $\underline{\mu}$ -Lindelöf space. \square

Theorem 2.27. *Every neutrosophic $\underline{\mu}$ -Lindelöf and $\underline{\mu}$ -countably compact space is $\underline{\mu}$ -compact.*

Proof. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$ -Lindelöf and $\underline{\mu}$ -countably compact space, and let \mathcal{U} be a neutrosophic $\underline{\mu}$ -open cover of X . Since X is neutrosophic $\underline{\mu}$ -Lindelöf, \mathcal{U} has a countable neutrosophic $\underline{\mu}$ -subcover (say \mathcal{U}_1) of X from \mathcal{U} . And since X is neutrosophic $\underline{\mu}$ -countably compact, \mathcal{U}_1 has a neutrosophic $\underline{\mu}$ -finite subcover, say \mathcal{U}_2 , from \mathcal{U}_1 . It is clear that \mathcal{U}_2 is a neutrosophic $\underline{\mu}$ -finite subcover of X from \mathcal{U} , that means $(X, \underline{\mu})$ is a neutrosophic $\underline{\mu}$ -compact. \square

Corollary 2.28. *Every neutrosophic $\underline{\mu}$ -countably compact space with a neutrosophic countable $\underline{\mu}$ -base is $\underline{\mu}$ -compact.*

Example 2.29. Let $X = \{a, b\}$ and $\beta = \{A_n; n = 1, 2, 3, \dots\}$ where $A_n = \{\langle x, 1 - \frac{1}{2n}, \frac{1}{2n}, \frac{1}{2n} \rangle; x \in X\}$. Consider the neutrosophic $\underline{\mu}$ -topology $\tau(\beta)$ generated by the neutrosophic $\underline{\mu}$ -base β . Since $\tau(\beta)$ has a countable base, $\tau(\beta)$ is neutrosophic $\underline{\mu}$ -Lindelöf. Note that $\tau(\beta)$ is strong neutrosophic $\underline{\mu}$ -topological space, since β covers X , actually:

$\sqcup \beta = \sqcup \{A_n; n = 1, 2, 3, \dots\} = \{\langle x, \bigvee_1^\infty 1 - \frac{1}{2n}, \bigwedge_1^\infty \frac{1}{2n}, \bigwedge_1^\infty \frac{1}{2n} \rangle; x \in X\} = \{\langle x, 1, 0, 0 \rangle; x \in X\} = 1_X$. Now, we will show that $\tau(\beta)$ is not neutrosophic $\underline{\mu}$ -countably paracompact (which implies it is not neutrosophic $\underline{\mu}$ -compact). By contrapositive, suppose X is neutrosophic $\underline{\mu}$ -countably paracompact. Then $\mathcal{U} = \beta$ is a countable neutrosophic $\underline{\mu}$ -open cover of X . Since we suppose X is neutrosophic $\underline{\mu}$ -countably paracompact, \mathcal{U} has a finite $\underline{\mu}$ -subcover, say $\mathcal{U}^* = \{A_{n_1}, A_{n_2}, \dots, A_{n_k}\}$. But $A_{n_1} \sqcup A_{n_2} \sqcup \dots \sqcup A_{n_k} = A_t$ where $t = \max\{n_1, n_2, \dots, n_k\}$, and $A_t = \{\langle x, 1 - \frac{1}{2t}, \frac{1}{2t}, \frac{1}{2t} \rangle; x \in X\} \neq 1_X$, a contradiction. So X is not neutrosophic $\underline{\mu}$ -countably paracompact and hence is not neutrosophic $\underline{\mu}$ -compact.

The following theorem shows that neutrosophic $\underline{\mu}$ -compact space and neutrosophic $\underline{\mu}$ -countably compact space are equivalent if X is countable, which is not true in topological spaces.

Theorem 2.30. *For every countable neutrosophic $\underline{\mu}$ -topological space X , the following two statements are equivalent:*

- (1) X is neutrosophic $\underline{\mu}$ -compact.
- (2) X is neutrosophic $\underline{\mu}$ -countably compact.

Proof. \Rightarrow) Obvious!

\Leftarrow) Suppose that X is a countable neutrosophic $\underline{\mu}$ -countably compact space, and let \mathcal{U} be a neutrosophic $\underline{\mu}$ -open cover of X . For every $x \in X$ we define the following three subsets of $[0, 1]$.

- (1) $D_{\mu}^x = \{\mu_A(x); A \in \mathcal{U}\}.$
- (2) $D_{\sigma}^x = \{\sigma_A(x); A \in \mathcal{U}\}.$
- (3) $D_{\nu}^x = \{\nu_A(x); A \in \mathcal{U}\}.$

Let D_1^x , D_2^x and D_3^x be three countable dense subsets of D_{μ}^x , D_{σ}^x and D_{ν}^x respectively in the usual sense (the usual topology on the unit interval). Since \mathcal{U} is a neutrosophic $\underline{\mu}$ -open cover of X , we have $\sup D_1^x = \sup D_{\mu}^x = 1$, $\inf D_2^x = \inf D_{\sigma}^x = 0$ and $\inf D_3^x = \inf D_{\nu}^x = 0$. Let $\mathcal{U}(x) = \{A \in \mathcal{U}; \mu_A(x) \in D_1^x, \sigma_A(x) \in D_2^x \text{ or } \nu_A(x) \in D_3^x\}$. It is clear that $\mathcal{U}(x)$ is countable. Let $\mathcal{U}^* = \cup\{\mathcal{U}(x); x \in X\}$. Since X is countable, \mathcal{U}^* is a countable sub-collection from \mathcal{U} . We will show that \mathcal{U}^* is a neutrosophic $\underline{\mu}$ -cover of X . Set $B = \sqcup \mathcal{U}^*$. For every $x \in X$ we have:

- (1) $\mu_B(x) = \vee\{\mu_A(x); A \in B\} \geq \vee\{\mu_A(x); A \in D_1^x\} = \sup D_1^x = 1.$
- (2) $\sigma_B(x) = \wedge\{\sigma_A(x); A \in B\} \geq \wedge\{\sigma_A(x); A \in D_1^x\} = \inf D_2^x = 0.$
- (3) $\nu_B(x) = \wedge\{\nu_A(x); A \in B\} \geq \wedge\{\nu_A(x); A \in D_1^x\} = \inf D_3^x = 0.$

Which implies that $B = 1_X$ and \mathcal{U}^* is a neutrosophic countable $\underline{\mu}$ -open cover. Since X is a neutrosophic $\underline{\mu}$ -countably compact space, \mathcal{U}^* has a finite subcover, that is X is compact. \square

Question 2.31. *Are neutrosophic $\underline{\mu}$ -compactness and neutrosophic $\underline{\mu}$ -countably compactness equivalent.*

3. Applications and further studies

All existing studies are about neutrosophic topological spaces and since Neutrosophic $\underline{\mu}$ -topological space is a generalization of neutrosophic topological spaces we can get more generalized results in Neutrosophic $\underline{\mu}$ -topological space that are true for neutrosophic topological spaces, see for example Theorem 2.30, and some previous notations about neutrosophic sets can be considered as examples of neutrosophic $\underline{\mu}$ -topological spaces, see Theorem 2.9 which shows the relationship between $\underline{\mu}$ -topological space and previous studies. In the future work we need to answer the question posted in this paper: Are neutrosophic $\underline{\mu}$ -compactness and

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neutrosophic $\underline{\mu}$ -countably compactness equivalent. Furthermore; many notations about neutrosophic $\underline{\mu}$ -topological spaces need to be studied for example, first and second countable spaces, neighborhood systems, the relation between the usual topology defined on the interval $[0,1]$ (which is the range of μ , σ and ν functions) and the neutrosophic $\underline{\mu}$ -topology defined on X .

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Single-valued neutrosophic \mathcal{N} -soft set and intertemporal single-valued neutrosophic \mathcal{N} -soft set to assess and pre-assess the mental health of students amidst *COVID-19*

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Abstract. **Aim** - Stress binds everyone as we face uncertainty in our lives. So, it is notable that we experience anxiety during this coronavirus disease (*COVID* – 19) pandemic context. When we try to handle stress for longer duration leads to chronic, and it can affect both physical and mental health. The scientific techniques to precisely pre-assess or assess mental health disorders are hardly available for the students. This paper intends to provide an explication to pre-assess or assess the mental health of the students amidst this pandemic. We present the notions of single-valued neutrosophic \mathcal{N} -soft set (*SVNNSS*) and the quasi-hyperbolic discounting intertemporal single-valued neutrosophic \mathcal{N} -soft set (*QHDISVNNSS*) to show the mental condition of the students. **Design**- We develop a four-phase method to pre-assess or assess the mental health disorders of students. In the initial phase, we present an outline to identify the students, parameters, and the psychosocial aspects of the students. Also, we provide the framework of positive and negative statements for each parameter, rating scales, and scoring norms. In the second phase, we execute case studies based on observation of the students and mention the values using neutrosophic numbers for each counseling session with no loss of information. Then we apply the concept of score function (*SF*) and weighted single-valued neutrosophic vector (*WSVNV*). In the third phase, we construct *SVNNSS* or *QHDISVNNSS* to access or pre-access the mental health of the students. Finally, we assess the scores of each student with the help of norms and predict mental health disorders. **Results**- Using *SVNNSS*, we can assess the mental health of the students and able to pre-assess the mental health of the students by using *QHDISVNNSS*. Hence, this result supports the psychiatrist or the counselor to focus on those with mental health conditions, as they are known to experience a higher level of emotional distress. **Contributions**- This study shows how the significance of the neutrosophic concept can be modified and implemented in the psychology field to determine the mental health of the students. **Implications**- As pointed out by the counselor and the therapist, the first step to self-care is to take care of our mental health. Here, in this study, we provide a solution to pre-assess and assess the mental health of the students by using these concepts. This method gives a valuable solution to the counselor or the therapist for analyzing the psychosocial aspects.

Keywords: single-valued neutrosophic \mathcal{N} -soft set; intertemporal single-valued neutrosophic \mathcal{N} -soft set; quasi-hyperbolic discounting function.

1. Introduction

World health organization (*WHO*) observes 10 October as world health day to understand the enhancing mental health issues. Mental health holds the emotional, psychological, and social wellness within us. Mental health issues have become one of the worlds major causes of the burden of chronic disease, and it frequently begins at an early age and can ruin lives, influencing families, peers, and societies. Students mental wellness is a subject of concern worldwide. The success of the student hinges on mental health. The socially acceptable conduct of student behavior depends on his mental health. Disturbance in his mental health creates a negative impact on the student as well to the community. Hence, mental health plays a significant role in a students life. On 11 March 2020, *WHO* conceded the spread of *COVID* – 19 to be a pandemic. When analyzing the inflation in *COVID* – 19, the only approach left to slow the spread of the infection is a complete lockdown. A study carried on over 8,000 people by YourDost [1], an online mental health site, found that college students are the most affected by *COVID* – 19. Because of the lockdown effect, many students undergo emotional stress, and there is a need to assess their mental health status. A recent survey conducted by *WHO* [2] in 130 countries from June to August 2020 showed that there is a disruption in mental health services in 93 percent of countries. The findings show that 89 percent of countries have national mental health and psychological support plans, but only 17 percent of them have funds allocated to implement those plans. They have also found that only 7 percent of countries have reported no service interruption, meaning that some disruption of service has occurred in 93 percent of countries. Based on the global burden of disease research work [3], around 792 million individuals have a mental illness. The representation of the global ratio is 10.7 percent, slightly over one in ten individuals. Hence, there's a need for a therapist to assess the students' mental aspect during this pandemic.

When most of the models apporportioned with fuzzy set (*FS*) [4] and intuitionistic *FS* (*IFS*) [5] to solve the problems of uncertainty situations. Smarandache [6] presented the concept of the neutrosophic set (*NS*), a combination of truth, indeterminate, and falsity membership values. Later, Wang [7] introduced single-valued *NS* (*SVNS*) to overcome the difficulties faced in *NS*. Maji [8] established the concept of a single-valued neutrosophic soft set (*SVNSS*) and its properties. During an uncertain condition, the indeterminate membership value plays a vital role in ranking the alternatives, and the domination of neutrosophic theory in various fields started from thereon. We highlight some of the recent works that have used neutrosophic theory in decision making problems. Abdel-Basset et al. [9] proposed the type 2 *SVNS* and Chinnadurai and Bobin, Applications to assess and pre-assess the mental health of students

defined some of its operational rules. Abdel-Basset et al. [10] presented a novel approach for estimating smart medical devices in a neutrosophic environment. Abdel-Basset et al. [11] proposed the concept of the analytical network method with *SVNS* for dealing with multi-criteria decision making (*MCDM*) problems. Abdel-Basset et al. [12] implemented a new technique for project selection in an ambiguous environment. Chinnadurai et al. [13] introduced the concept of a unique ranking for the alternatives using parameters. Chinnadurai and Bobin [14] presented a study to rank the attributes in *MCDM* problems using prospect theory. Sudan et al. [15] proposed a novel approach for stock value prediction based on real data. Nabeeh et al. [16] illustrated an ideal solution in ranking the personnel selection process by integrating neutrosophic and analytical hierarchy method. Mohana and Smarandache [17] solved *MCDM* problems using bipolar *SVNS* (*BSVNS*). Broumi et al. [18] introduced a new algorithm to find out the shortest path between each pair of nodes. Kumar et al. [19] presented shortest path problem using neutrosophic graph. Abdel-Basset [20] presented the framework for the professional candidate selection process using *BSVNS*. Abdel-Basset [21] proposed a concept to solve the supply chain problem by using the combination of the plithogenic set (*PS*) and the best-worst method. Abdel-Basset [22] integrated *PS* with different *MCDM* applications to assess the progress of manufacturing industries. Abdel-Basset [23] implemented smart product-service systems to process a large amount of information in *MCDM* problems. Abdel-Basset [24] presented a model to diagnose *COVID – 19* by using *PS* and computerized tomography scans. Rohini et al. [25] presented the concept of single-valued neutrosophic coloring. It is used widely in information technology, banking technology, psychology, sociology, and other fields where the indeterminacy occurs. Edward and Narmadhagnanam [26] developed a concept using rough -*SVNS* to diagnose the disease. Villamar et al. [27] analyzed Ecuador's gross domestic product by using a neutrosophic cognitive map. The domination of neutrosophic theories in various fields is clear from these research works.

Zadeh [4] proposed the notion of *FSs* to deal with the concept of vagueness. The thoughts inside the human brain for learning, understanding, and describing are naturally vague and imprecise. The boundaries of these concepts are not precisely defined. Therefore, the judging and rationalizing that develop from human brain also become uncertain. In the late 80s, amid criticism and controversy, *FSs* gained credibility in psychology [28]. Although psychologists have shown interest on *FS* theory concepts and fuzzy logic have been slow to take up the field. Rosch [29], Hersch and Caramazza [30], Rubin [31] and Oden [32] conducted experimental research using *FS* theory. Oden and Massaro [33] explained the perception theory by using a *FS*. Hesketh et al. [34] introduced the concept of fuzzy logic to study the thought processes which cannot fit into classical mathematical techniques. Broughton [35] insists that

typicality and FS helps in refining personality assessment tools and improving abnormal diagnosis. Horowitz and Malle [36] examined depression by using the fuzzy concept. Alliger et al. [37] applied the concept of the fuzzy approach in decision-making problems of personnel assessment and selection. Vasantha et al. [38] defined the concept of a single-valued refined NS . They analyzed the age group of 1 to 10 years to study the imaginative play in children. Hernandex et al. [39] presented the pedagogical validation through ladov technique. Nandita et al. [40] detailed the aspects of mental health and presented the details using soft computing and neuro-fuzzy techniques. Wang et al. [41] identified the various types of psychological dysfunctions in construction designs. They developed a fuzzy mapping to determine the influence of psychological disorders in the context of the time, cost, and quality of construction. Sanpreet [42] designed an expert system to aid the psychiatrists in assessing the mental health of the individuals. Sumathi and Poorna [43] presented the concept of machine learning techniques, Bayesian networks, and fuzzy clustering to study the mental health associated with children. Srivastava et al. [44] analyzed the aspects of psychological behavior by using fuzzy logic rules. Nuovo et al. [45] implemented a method to classify the mental retardation level. It is vital to select the best therapeutic medication and to ensure a quality of life that is sufficient for the particular condition of the patient. Chicaiza et al. [46] studied the state of emotional intelligence of the students. Since a high emotional intelligence guarantees a better future professional and higher quality learning.

Psychologists believe that the FS theory suffers mismatches with human perception, and lacks measurement foundations, from theoretical incoherence or paradoxes. Judgment and decision-making psychologists remained unconvinced that FS s could deliver something not already handled by subjective likelihood and utility. These manuscripts bring out the significance of the FS and other hybrid sets in analyzing personality assessment, diagnosis of disorders, and occupational counseling rather than using traditional set theory. Although the usage of the FS and other hybrid theory is clear in psychology, the preference of using it is not widespread. The psychiatrists are used to analyze scaled data with statistical techniques. They are always in the mindset to follow the traditional method of handling scale construction and classical test-theory. These conventional concepts have forced the psychiatrists to use scale construction rather than FS s and other hybrid sets. Also, most of the psychological study deals with questionnaires to study human behavior. In this process, we can never ignore the prejudice of the subject when the subjects express their thought process using a questionnaire. That's the reason when the information received by a questionnaire are imprecise since 'raw' values include hidden risks. Neutrosophic logic acts as a vital tool to deal with uncertainty. The reason for introducing the neutrosophic concept in the study of mental health is that much of the data received by the questionnaire is vague. Using neutrosophic information instead

of raw data has the advantage of reducing vagueness. In psychology, this concept offers an additional benefit and allows us to use vagueness measures to quantify the ambiguity associated with the prediction of mental health parameters. Hence, there is a need to define a novel set that is user friendly for the psychiatrists to assess the psychological behaviors of human beings.

Fatimah et al. [47] introduced the notion of \mathcal{N} -soft set (\mathcal{NSS}) with real-life illustrations. Later, Akram et al. coined the definition of fuzzy \mathcal{NSS} ($F\mathcal{NSS}$) [48] and hesitant \mathcal{NSS} ($H\mathcal{NSS}$) [49] by combining fuzzy and hesitancy sets with \mathcal{NSS} respectively. Kamaci and Petchimuthu [50] presented the concept of bipolar \mathcal{NSS} and its properties. Zhang et al. [51] studied the properties of Pythagorean fuzzy \mathcal{NSS} . Riaz et al. [52] detailed on neutrosophic \mathcal{NSS} along with their properties. The implementation of \mathcal{NSS} in various theories is evident from the above research works. But, we find there are some limitations when the combination of hybrid sets and \mathcal{NSS} happens and maybe insignificant when applied in the psychology field.

- i) We cannot accommodate the membership value of indeterminacy in $F\mathcal{NSS}$ and $H\mathcal{NSS}$.
- ii) We would like to refer the Example 2.5 in Akram et al. [53]. They decide the grading criteria based on the membership values in IFS and discard the non-membership values, assigned independently in IFS . Similarly, in Example 5.1, Riaz et al. [52] decides on the grading criteria (Table 21) based on the truth membership values in $SV\mathcal{NSS}$ and discard the indeterminacy and falsity membership values, assigned independently in $SV\mathcal{NSS}$. By discarding the non-membership values in IFS and the indeterminacy and falsity of membership values in $SV\mathcal{NSS}$, may restrict in analyzing the psychological aspects of human beings. This limitation may initiate a research gap in the psychological field.

In 1968, Phelps and Pollak [54] introduced the notion of the quasi-hyperbolic discounting function ($QHDF$). In 1997, David [55] coined the definition of $QHDF$ to capture the qualitative properties. Later, Peter and Botond [56] changed the notion introduced by David to deal with $QHDF$. Takanori [57] analyzed whether smoking status, including cigarette addiction, can be accurately predicted by two-time perception parameters. Nascimento [58] showed that fuzzy temporal logic expresses patterns of perception to interpret decision-making behaviors. Dou et al. [59] implemented a method using fuzzy temporal logic to forecast the passenger flow. Alnahhas and Alkhatib [60] supported a decision system to manage the crisis by combining fuzzy logic and temporal techniques. Alcantud and Torrecillas [61] introduced the intertemporal framework to fill the gap in the fuzzy soft set theory. Lie et al. [62] proposed an intertemporal hesitant fuzzy soft set and showed the significance of the set with $MCDM$ problems. Although the temporal logic plays a significant role in considering the ‘immediate effect’ from different parameters and sessions, the application of neutrosophic theory is still open for research.

On examining these manuscripts, we impersonate the following research scopes in a nutshell. Initially, during a pandemic, a change in environment is prevalent, which affects the psychosocial aspects substantially among the individuals. So, it is vital to analyze the mental health aspects of the students during this lockdown situations. Second, neutrosophic theory shows competence in decision making across all the fields. But, in psychology, there is still scope for enhancement. Third, the available standardized psychological tool for analyzing psychosocial behavior has limitations. A simple rating scale distribution cannot provide the exact risk level and prohibits the remedy process. Upon further analysis, we found that psychiatrists are comfortable in using raw data and rating scale criteria. So, a novel set that could handle the indeterminacy and the traditional method of assessing the human behaviors aids psychiatrist. Finally, the treatment process has many sessions to diagnose socially unacceptable behaviors. Hence, implementing intertemporal choice for capturing information is being preferred by the psychiatrists. The principal objectives of this manuscript are to overcome the mentioned research gap. i) to define a new set $SVNNSS$, by combining SF value of $SVNSS$ with NSS . This set enables us to use the SF of $SVNSS$, which represents the information independently in truth, indeterminate, and falsity. Later, with the help of a rating scale distribution, we relate the NSS to the corresponding SF value. ii) to define a new set of $QHDISVNNSS$, a combination of intertemporal $SVNNSS$ ($ISVNNSS$) with $QHDF$. This set enables us to record the intertemporal information and pre-assess the risk level associated with each session with the help of $QHDF$. We contemplate that these two novel sets will bridge the gap and aid the psychiatrist to use neutrosophic theory.

We organize the structure of this manuscript as below. Section 2 recalls existing definitions. Section 3 defines a new SF and $WSVNV$. Section 4 shows a comparison study between the proposed SF and existing SFs . Section 5 introduces the definition of $SVNNSS$. Section 6 provides the method, algorithm, and flowchart to assess the mental health of the students. Section 7 illustrates the case studies to assess the mental health of the students by using $SVNNSS$. Section 8 introduces the definition of $QHDISVNNSS$. Section 9 provides the method, algorithm, and flowchart to pre-assess the mental health of the students. Section 10 illustrates a case study by using $QHDISVNNSS$. Section 11 shows the significance and a comparison study of $QHDISVNNSS$ and finally, section 12 ends with limitations, conclusion and future works.

2. Preliminaries

In this section, we discuss some basic definitions, essential for understanding this manuscript. Let \mathcal{U} denote a universal set, \mathcal{P} a set of parameters, $\mathcal{E} \subseteq \mathcal{P}$ and $2^{\mathcal{U}}$ the power set of \mathcal{U} .

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Definition 2.1. [7] A single-valued neutrosophic set (*SVNS*) is represented as, $N = \{(u, T_N(u), I_N(u), F_N(u)) | u \in \mathcal{U}\}$, where $T_N(u) : \mathcal{U} \rightarrow [0, 1]$ represents truth-value, $I_N(u) : \mathcal{U} \rightarrow [0, 1]$ represents indeterminate-value and $F_N(u) : \mathcal{U} \rightarrow [0, 1]$ represents falsity-value with a condition $0 \leq T_N(u) + I_N(u) + F_N(u) \leq 3 \forall u \in \mathcal{U}$. Let $N^{\mathcal{U}}$ denote the collection of all SVNSs defined on \mathcal{U} .

Definition 2.2. [63] A pair (F, \mathcal{E}) is called a soft set (*SS*) over \mathcal{U} , F is a mapping given by $F : \mathcal{E} \rightarrow 2^{\mathcal{U}}$. Thus a *SS* is a parameterized family of subsets of \mathcal{U} .

Example 2.3. Let $\mathcal{U} = \{c_1, c_2, c_3\}$ be a set of clients with psychosocial conditions and $\mathcal{E} = \{p_1, p_2, p_3\}$ be the set of dimensions which stand for anxiety, depression and sleeping disorder respectively. A *SS* (F, \mathcal{E}) is a collection of subsets of \mathcal{U} , based on the description (Table 1).

TABLE 1. Representation of clients with psychosocial conditions in *SS* form

\mathcal{U}	anxiety(p_1)	depression(p_2)	sleeping disorder(p_3)
c_1	1	0	1
c_2	0	1	1
c_3	1	1	0

$$F(\text{anxiety}) = \{c_1, c_3\}, F(\text{depression}) = \{c_2, c_3\} \text{ and } F(\text{sleeping disorder}) = \{c_1, c_2\}.$$

Definition 2.4. [8] A single-valued neutrosophic soft set (*SVNSS*) over \mathcal{U} is defined as a pair (F, \mathcal{E}) , where $F : \mathcal{E} \rightarrow N^{\mathcal{U}}$. A *SVNSS* is represented as, $\tilde{N} = (F, \mathcal{E}) = \{(p, T_F(p)(u), I_F(p)(u), F_F(p)(u)) | u \in \mathcal{U} \text{ and } p \in \mathcal{E}\}$, where $T_F(p)(u)$, $I_F(p)(u)$, $F_F(p)(u) \in [0, 1]$, are the membership values of truth, indeterminacy and falsity respectively.

Example 2.5. Let \mathcal{U} and \mathcal{E} represent the same as in Example 2.3. A *SVNSS* (F, \mathcal{E}) describes the subset of clients with psychosocial conditions approximately in terms of membership values of truth, indeterminacy and falsity as in Table 2.

TABLE 2. Clients with psychosocial conditions in *SVNSS* form (F, \mathcal{E})

\mathcal{U}	anxiety(p_1)	depression(p_2)	sleeping disorder(p_3)
c_1	$\langle 0.55, 0.25, 0.45 \rangle$	$\langle 0.75, 0.55, 0.55 \rangle$	$\langle 0.90, 0.95, 0.20 \rangle$
c_2	$\langle 0.70, 0.45, 0.40 \rangle$	$\langle 0.35, 0.10, 0.40 \rangle$	$\langle 0.35, 0.45, 0.25 \rangle$
c_3	$\langle 0.85, 0.60, 0.15 \rangle$	$\langle 0.25, 0.35, 0.15 \rangle$	$\langle 0.50, 0.20, 0.60 \rangle$

Definition 2.6. [64] A *SVNSS* can be represented in matrix form as,

$$N^* = [n_{ij}] = \begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1n} \\ n_{21} & n_{22} & \cdots & n_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ n_{m1} & n_{m2} & \cdots & n_{mn} \end{bmatrix},$$

where $[n_{ij}] = \langle T_{ij}, I_{ij}, F_{ij} \rangle$; $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. N^* is an $m \times n$ single-valued neutrosophic soft matrix (*SVNSM*).

Example 2.7. The *SVNSM* for the Example 2.5 is as below:

$$N^* = \begin{bmatrix} \langle 0.55, 0.25, 0.45 \rangle & \langle 0.75, 0.55, 0.55 \rangle & \langle 0.90, 0.95, 0.20 \rangle \\ \langle 0.70, 0.45, 0.40 \rangle & \langle 0.35, 0.10, 0.40 \rangle & \langle 0.35, 0.45, 0.25 \rangle \\ \langle 0.85, 0.60, 0.15 \rangle & \langle 0.25, 0.35, 0.15 \rangle & \langle 0.50, 0.20, 0.60 \rangle \end{bmatrix}$$

Definition 2.8. [47] Let $\mathcal{G} = \{0, 1, \dots, \mathcal{N} - 1\}$ be a set of ordered grades, where $\mathcal{N} \in \{2, 3, \dots\}$. Then, $(F, \mathcal{E}, \mathcal{N})$ is a \mathcal{N} - soft set (*NSS*) on \mathcal{U} if $F : \mathcal{E} \longrightarrow 2^{\mathcal{U} \times \mathcal{G}}$ with the condition that for each $p \in \mathcal{E}$ there exists a unique $(u, g_p) \in \mathcal{U} \times \mathcal{G}$, such that $(u, g_p) \in F(p)$, $u \in \mathcal{U}$, $g_p \in \mathcal{G}$.

Definition 2.9. [56] In a T-horizon game, the quasi-hyperbolic discounting function (*QHDF*) for the period t 's is given as,

$$u(q_t) + \beta \sum_{i=1}^{T-t} \delta^i u(q_{t+i}),$$

with $\beta, \delta \in [0, 1]$ and represent the short-term and long-term discounting parameters.

Definition 2.10. Let $\tilde{N} = \langle T_{\tilde{N}}, I_{\tilde{N}}, F_{\tilde{N}} \rangle$ represent *SVNSS*. Then the framework of existing *SF* definitions are given in Table 3.

TABLE 3. Representation of existing *SFs*

Existing author details	<i>SFs</i>
Ridvan [65]	$R(\tilde{N}) = \frac{1+T_{\tilde{N}}-2(I_{\tilde{N}})-F_{\tilde{N}}}{2}$
Nancy and Garg [66]	$G(\tilde{N}) = \frac{1+(T_{\tilde{N}}-2(I_{\tilde{N}})-F_{\tilde{N}})(2-T_{\tilde{N}}-F_{\tilde{N}})}{2}$
Pal and Jana [67]	$P(\tilde{N}) = T_{\tilde{N}} + I_{\tilde{N}} + F_{\tilde{N}}$
Broumi et al. [68]	$B(\tilde{N}) = \frac{2+T_{\tilde{N}}-I_{\tilde{N}}-F_{\tilde{N}}}{3}$
Mondal and Pramanik [69]	$M(\tilde{N}) = \frac{1+T_{\tilde{N}}-F_{\tilde{N}}}{2}$
Peng et al. [70]	$P(\tilde{N}) = \frac{T_{\tilde{N}}+1-I_{\tilde{N}}+1-F_{\tilde{N}}}{3}$

3. Score function and weighted vector of neutrosophic

In this section, we introduce two new definitions to solve the case studies mentioned in sections 7 and 11. i) Score function (*SF*) of a *SVNSM* helps to integrate the neutrosophic number into a single real number to bring out the importance of truth, indeterminacy, and falsity membership values. ii) In *MCDM* problems, decision makers (*DMs*) always consider each parameter uniquely and also provide the weightage value based on their experiences. So, the weighted single-valued neutrosophic vector (*WSVNV*) provides an added advantage to the *DMs* to consider each criterion uniquely based on the selection of problem.

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Definition 3.1. Let $N^* = [n_{ij}] = \langle T_{N_{ij}^*}, I_{N_{ij}^*}, F_{N_{ij}^*} \rangle$. Then define the SF for the element n_{ij} as,

$$S(N^*) = [s_{ij}] = \frac{[T_{N_{ij}^*} + I_{N_{ij}^*}] - F_{N_{ij}^*}}{2} \quad \forall i, j.$$

Example 3.2. The SF values for the Example 2.3 is given below:

$$S(N^*) = \begin{bmatrix} 0.18 & 0.38 & 0.83 \\ 0.38 & 0.03 & 0.28 \\ 0.65 & 0.23 & 0.05 \end{bmatrix}$$

Definition 3.3. Let ζ be the collection of all SFs deduced from neutrosophic values and $M = \{s_1, s_2, \dots, s_l\}$ be a neutrosophic vector with components of ζ . Let $W = \{w_1, w_2, \dots, w_l\}$ be a weight vector associated with M . w_i can be considered as the significance attached to s_i ; $i = 1, 2, \dots, l$ with $w_i \in [0, 1]$, $\sum_{i=1}^l w_i = 1$. Then the $WSNVN$ corresponding to M and W denoted by WM is defined as, $WM = \{w_1 s_1, w_2 s_2, \dots, w_l s_l\}$.

Example 3.4. Let $W = (0.35, 0.35, 0.30)$ be the weight vector assigned to the parameters. Then the $WSNVN$ for the Example 3.2 is as below:

$$WS(N^*) = \begin{bmatrix} 0.06 & 0.13 & 0.25 \\ 0.13 & 0.01 & 0.08 \\ 0.23 & 0.08 & 0.02 \end{bmatrix}$$

4. Comparison of proposed score function with existing score functions

In this section, we compare and analyze existing SFs namely; Ridvan [65], Nancy and Garg [66], Pal and Jana [67], Broumi et al. [68], Mondal and Pramanik [69] and Peng et al. [70] with proposed SF to show the ranking constraints in neutrosophic environment. From Table 4, we infer that in some conditions, the existing SFs cannot rank the alternatives whereas the proposed SF can rank the alternatives in the best way.

5. Single-valued neutrosophic \mathcal{N} -soft set

In this section, we define the notion of single-valued neutrosophic \mathcal{N} -soft set and single-valued neutrosophic \mathcal{N} -soft matrix with suitable examples.

Definition 5.1. Let \mathcal{U} be the universal set and \mathcal{P} be a set of parameters, $\mathcal{E} \subseteq \mathcal{P}$. Let $\mathcal{G} = \{1, 2, \dots, \mathcal{N}\}$ be a set of rating scales, where $\mathcal{N} \geq 2$. Then the triple $(\psi, \mathcal{J}, \mathcal{N})$ is said to be a single-valued neutrosophic \mathcal{N} -soft set ($SVN\mathcal{N}SS$), where $\mathcal{J} = (F, \mathcal{E}, \mathcal{N})$ is a \mathcal{N} -soft set over \mathcal{U} and ψ maps every parameter in \mathcal{E} with a score function of $SVN\mathcal{N}SS$, $S(\tilde{N})$ over $F(p)$ which is clearly a subset of $\mathcal{U} \times \mathcal{G}$ and $p \in \mathcal{E}$. That is, for each parameter $p \in \mathcal{E}$, there exists a unique $(u, g_p) \in \mathcal{U} \times \mathcal{G}$ such that $(u, g_p) \in F(p)$, $u \in \mathcal{U}$, $g_p \in \mathcal{G}$ and $\langle (u, g_p), S(\tilde{N}) \rangle \in \psi(p)$ or $\tilde{N}(\mathcal{N}) = \psi(p)(u) = \langle g_p, S(\tilde{N}) \rangle$.

TABLE 4. Shows the ranking constraints in existing SFs

$SVNSSs$	SFs	Score values	Remarks
$\tilde{N}_1 = \langle 0.25, 0.35, 0.15 \rangle$	Ridvan [65]	$R(\tilde{N}_1) = R(\tilde{N}_2) = R(\tilde{N}_3) = 0.20$	$\tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3$
$\tilde{N}_2 = \langle 0.70, 0.45, 0.40 \rangle$			
$\tilde{N}_3 = \langle 0.45, 0.40, 0.25 \rangle$	Proposed	$S(\tilde{N}_1) = 0.22, S(\tilde{N}_2) = 0.37, S(\tilde{N}_3) = 0.30$	$\tilde{N}_2 > \tilde{N}_3 > \tilde{N}_1$
$\tilde{N}_1 = \langle 0.55, 0.25, 0.45 \rangle$	Nancy and Garg [66]	$G(\tilde{N}_1) = G(\tilde{N}_2) = G(\tilde{N}_3) = 0.30$	$\tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3$
$\tilde{N}_2 = \langle 0.50, 0.20, 0.50 \rangle$			
$\tilde{N}_3 = \langle 0.60, 0.23, 0.72 \rangle$	Proposed	$S(\tilde{N}_1) = 0.17, S(\tilde{N}_2) = 0.10, S(\tilde{N}_3) = 0.05$	$\tilde{N}_1 > \tilde{N}_2 > \tilde{N}_3$
$\tilde{N}_1 = \langle 0.40, 0.35, 0.45 \rangle$	Pal and Jana [67]	$P(\tilde{N}_1) = P(\tilde{N}_2) = P(\tilde{N}_3) = 1.20$	$\tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3$
$\tilde{N}_2 = \langle 0.35, 0.45, 0.40 \rangle$			
$\tilde{N}_3 = \langle 0.45, 0.60, 0.15 \rangle$	Proposed	$S(\tilde{N}_1) = 0.15, S(\tilde{N}_2) = 0.20, S(\tilde{N}_3) = 0.45$	$\tilde{N}_3 > \tilde{N}_2 > \tilde{N}_1$
$\tilde{N}_1 = \langle 0.35, 0.45, 0.40 \rangle$	Broumi et al. [68]	$B(\tilde{N}_1) = B(\tilde{N}_2) = B(\tilde{N}_3) = 0.50$	$\tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3$
$\tilde{N}_2 = \langle 0.20, 0.25, 0.45 \rangle$			
$\tilde{N}_3 = \langle 0.60, 0.55, 0.55 \rangle$	Proposed	$S(\tilde{N}_1) = 0.20, S(\tilde{N}_2) = 0.00, S(\tilde{N}_3) = 0.30$	$\tilde{N}_3 > \tilde{N}_1 > \tilde{N}_2$
$\tilde{N}_1 = \langle 0.55, 0.50, 0.45 \rangle$	Mondal and Pramanik [69]	$M(\tilde{N}_1) = M(\tilde{N}_2) = M(\tilde{N}_3) = 0.55$	$\tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3$
$\tilde{N}_2 = \langle 0.35, 0.35, 0.25 \rangle$			
$\tilde{N}_3 = \langle 0.60, 0.55, 0.50 \rangle$	Proposed	$S(\tilde{N}_1) = 0.30, S(\tilde{N}_2) = 0.22, S(\tilde{N}_3) = 0.32$	$\tilde{N}_3 > \tilde{N}_1 > \tilde{N}_2$
$\tilde{N}_1 = \langle 0.55, 0.50, 0.40 \rangle$	Peng et al. [70]	$P(\tilde{N}_1) = P(\tilde{N}_2) = P(\tilde{N}_3) = 0.55$	$\tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3$
$\tilde{N}_2 = \langle 0.55, 0.40, 0.50 \rangle$			
$\tilde{N}_3 = \langle 0.75, 0.55, 0.50 \rangle$	Proposed	$S(\tilde{N}_1) = 0.32, S(\tilde{N}_2) = 0.22, S(\tilde{N}_3) = 0.37$	$\tilde{N}_3 > \tilde{N}_1 > \tilde{N}_2$

Definition 5.2. Let $\mathcal{U} = \{u_1, u_2, \dots, u_m\}$ be the universal set. Let $\mathcal{P} = \{p_1, p_2, \dots, p_n\}$ be set of parameters and $\mathcal{G} = \{1, 2, \dots, \mathcal{N}\}$ be a set of rating scale. Then $SVNNSS(\psi, \mathcal{J}, \mathcal{N})$ can be expressed in matrix form as,

$$N^*(\mathcal{N}) = \begin{matrix} & \begin{matrix} p_1 & p_2 & \dots & p_n \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{matrix} & \begin{bmatrix} \langle g_{p_{11}}, s_{11} \rangle & \langle g_{p_{12}}, s_{12} \rangle & \dots & \langle g_{p_{1n}}, s_{1n} \rangle \\ \langle g_{p_{21}}, s_{21} \rangle & \langle g_{p_{22}}, s_{22} \rangle & \dots & \langle g_{p_{2n}}, s_{2n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle g_{p_{m1}}, s_{m1} \rangle & \langle g_{p_{m2}}, s_{m2} \rangle & \dots & \langle g_{p_{mn}}, s_{mn} \rangle \end{bmatrix} \end{matrix}$$

such that $N^*(\mathcal{N}) = \langle g_{p_{ij}}, s_{ij} \rangle$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Then $N^*(\mathcal{N})$ is called an $m \times n$ single-valued neutrosophic \mathcal{N} -soft matrix ($SVNNSM$) of the $SVNNSS(\psi, \mathcal{J}, \mathcal{N})$.

Example 5.3. Consider a scenario where a mental health counselor (MHC) observes the behavior of students to understand their mental health conditions and provides the values in $SVNSM$ as in Example 2.7. Let's assume the MHC considers a 5 point rating scale (5-soft set) for positive and negative statements with the rating scale distribution as in Tables 5 and 6, respectively. The MHC can amend the values in Tables 5 and 6 as per their needs. The positive statements denote socially acceptable behavior and the negative statements denote socially deviant or problematic behavior. Here, let's assume that the MHC constructs positive

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statements for the parameters p_1 and p_3 and for p_2 - negative statement. Then, we compute the $N^*(\mathcal{N})$ for Example 3.2 as below.

TABLE 5. Showing the rating scale details

Positive statement	Negative statement
5	1
4	2
3	3
2	4
1	5

TABLE 6. Showing the rating scale distribution

Positive statement	Negative statement	Score values
5	1	$0.8 \leq s_{ij} \leq 1.0$
4	2	$0.6 \leq s_{ij} < 0.8$
3	3	$0.3 \leq s_{ij} < 0.6$
2	4	$0.0 \leq s_{ij} < 0.3$
1	5	$-0.5 \leq s_{ij} < 0.0$

$$N^*(5) = \begin{bmatrix} \langle 2, 0.18 \rangle & \langle 3, 0.38 \rangle & \langle 5, 0.83 \rangle \\ \langle 3, 0.38 \rangle & \langle 4, 0.03 \rangle & \langle 2, 0.28 \rangle \\ \langle 4, 0.65 \rangle & \langle 4, 0.23 \rangle & \langle 2, 0.05 \rangle \end{bmatrix}$$

We shall show the significance of $N^*(\mathcal{N})$ in sections 6 and 7 with constructive examples.

6. To assess the mental health of students amidst *COVID-19* using *SVNN \mathcal{SM}*

COVID – 19 has caused the entire world to a lockdown situation. For the progress of the world, it is vital to understand the mental health and psychosocial concerns of the students amidst this pandemic. To deal with this, we construct the concept of *SVNN \mathcal{SM}* , which supports to assess the mental condition of the individuals. In this section, we put forward a method to assess the condition of students amidst *COVID* – 19 with an algorithm and flowchart. We explain the feasibility and validity of the application with real-life case studies in the following section.

Consider a scenario where an institution approaches the *MHC* and wishes to assess the mental health of its students amidst the pandemic, *COVID*–19. Let us assume that the *MHC* selects a partially standardized method like video conferencing or telephonic conversations to assess the students. Let $\mathcal{U} = \{s_1, s_2, \dots, s_m\}$ denote the set of students and $\mathcal{E} = \{p_1, p_2, \dots, p_n\}$ the set of parameters to assess the psychosocial conditions. Let us assume the *MHC* gets in touch with a team of psychiatrist experts and frames the following details namely; positive and negative statements for parameters, rating scales with distribution criteria as in Table Chinnadurai and Bobin, Applications to assess and pre-assess the mental health of students

6, weightage criteria to assess the parameters, scoring keys, and mental health norms as in Table 7. These details should be chosen wisely and with extra cautiousness since it plays a significant role in describing the risk level of the students. Also, we signify that high scoring students are under risk and require immediate attention or treatment.

The *MHC* based on each question, say $r = \{1, 2, \dots, k\}$ evaluates the students by considering the parameters and present the results in the form of neutrosophic matrices, N_r^* of order $m \times n$. Now, we have to assess the mental health of the student with the help of pre-determined scores and norms.

6.1. Methodology to assess the mental health of the students

Construct the *SVNSMs*, N_r^* , $r = \{1, 2, \dots, k\}$ for each positive or negative statement by observing or understanding the behavior of the student based on the parameters. Apply *SF* Definition 3.1, to the *SVNSMs* and represent the resultant matrices by $S(N_r^*)$, $r = \{1, 2, \dots, k\}$. If weightage criteria are to be considered for each parameter, then calculate $WS(N_r^*)$ by using Definition 3.3. Now compare the entries in each $S(N_r^*)$ or in $WS(N_r^*)$ matrix and construct the $N_r^*(\mathcal{N})$ as below by using Definition 5.2. Also, with the help of the framed rating scale distribution.

$$N_r^*(\mathcal{N}) = \begin{matrix} & \begin{matrix} p_1 & p_2 & \dots & p_n \end{matrix} \\ \begin{matrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{matrix} & \begin{bmatrix} \langle g_{p_{11}}^r, s_{11}^r \rangle & \langle g_{p_{12}}^r, s_{12}^r \rangle & \dots & \langle g_{p_{1n}}^r, s_{1n}^r \rangle \\ \langle g_{p_{21}}^r, s_{21}^r \rangle & \langle g_{p_{22}}^r, s_{22}^r \rangle & \dots & \langle g_{p_{2n}}^r, s_{2n}^r \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle g_{p_{m1}}^r, s_{m1}^r \rangle & \langle g_{p_{m2}}^r, s_{m2}^r \rangle & \dots & \langle g_{p_{mn}}^r, s_{mn}^r \rangle \end{bmatrix} \end{matrix},$$

where $r = \{1, 2, \dots, k\}$.

Determine the $N_+^*(\mathcal{N})$ matrix as below by adding the corresponding entries of $N_1^*(\mathcal{N}), N_2^*(\mathcal{N}), \dots, N_k^*(\mathcal{N})$ matrices.

$$N_+^*(\mathcal{N}) = \begin{matrix} & \begin{matrix} p_1 & p_2 & \dots & p_n \end{matrix} \\ \begin{matrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{matrix} & \begin{bmatrix} g_{p_{11}}^+ & g_{p_{12}}^+ & \dots & g_{p_{1n}}^+ \\ g_{p_{21}}^+ & g_{p_{22}}^+ & \dots & g_{p_{2n}}^+ \\ \vdots & \vdots & \ddots & \vdots \\ g_{p_{m1}}^+ & g_{p_{m2}}^+ & \dots & g_{p_{mn}}^+ \end{bmatrix} \begin{matrix} \sum_{j=1}^n g_{p_{1j}}^+ \\ \sum_{j=1}^n g_{p_{2j}}^+ \\ \vdots \\ \sum_{j=1}^n g_{p_{mj}}^+ \end{matrix} \end{matrix},$$

where

$$g_{p_{11}}^+ = \sum_{r=1}^k g_{p_{11}}^r, g_{p_{12}}^+ = \sum_{r=1}^k g_{p_{12}}^r \text{ and } g_{p_{1n}}^+ = \sum_{r=1}^k g_{p_{1n}}^r.$$

$$g_{p_{21}}^+ = \sum_{r=1}^k g_{p_{21}}^r, g_{p_{22}}^+ = \sum_{r=1}^k g_{p_{22}}^r \text{ and } g_{p_{2n}}^+ = \sum_{r=1}^k g_{p_{2n}}^r.$$

similarly,

$$g_{p_{m1}}^+ = \sum_{r=1}^k g_{p_{m1}}^r, g_{p_{m2}}^+ = \sum_{r=1}^k g_{p_{m2}}^r \text{ and } g_{p_{mn}}^+ = \sum_{r=1}^k g_{p_{mn}}^r.$$

Now assess the risk level for each parameter as well for the overall by using the level norms. If the student attains a low-risk level, then he/she does not require psychological treatment. If otherwise, then *MHC* should start the remedy process for the students who show a high-risk level towards psychosocial conditions.

TABLE 7. Shows the qualitative norm details

Parameter	Scores	Norms
p_1, p_4	1-13	low
	14-25	average (avg)
	26-35	high
p_2, p_3	1-15	low
	16-24	avg
	25-30	high
Total	1-56	low
	57-97	avg
	98-130	high

6.2. Algorithm to assess the mental health of students

The following steps facilitate the *MHC* to assess the mental health of students in a better way.

Step 1: *MHC* identifies the problem, selects the students and the parameters.

Step 2: *MHC* involves a psychiatrist to frame the required details namely; positive and negative statements, rating scale with distribution, scoring keys and risk level.

Step 3: Constructs N_i^* , where $i = \{1, 2, \dots, k\}$ matrices for each question by observing the behavior of the students.

Step 4: Evaluates SN_i^* and WSN_i^* by using Definition 3.1 and 3.3 respectively.

Step 5: Constructs $N_i^*(\mathcal{N})$ by comparing it with rating scale and distribution details.

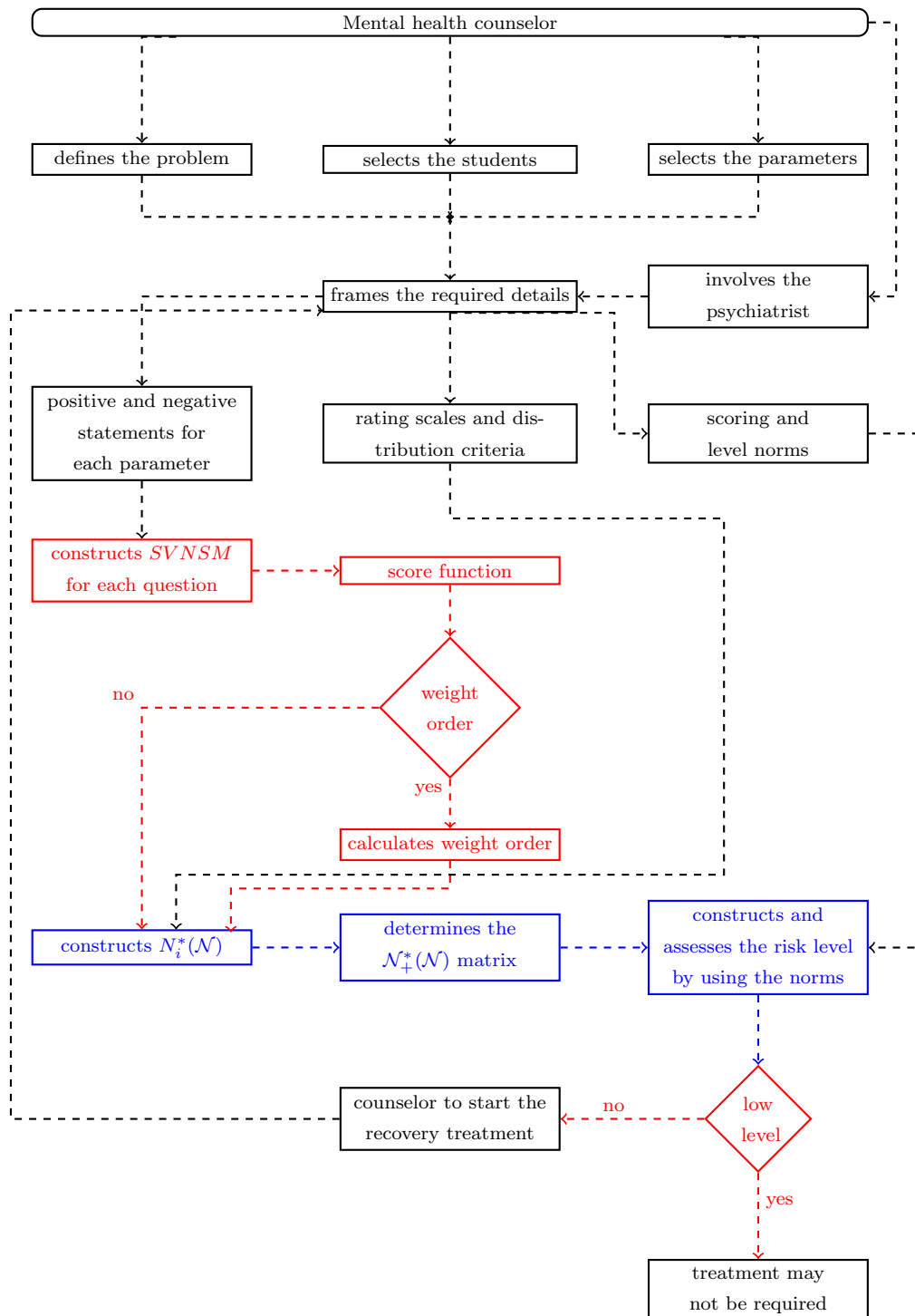
Step 6: Determines $N_+^*(\mathcal{N})$ matrix by summing the corresponding entries of $N_1^*(\mathcal{N}), N_2^*(\mathcal{N}), \dots, N_k^*(\mathcal{N})$ matrices.

Step 7: Tabulates and assesses the mental health risk level by using scoring keys and risk level norms.

Step 8: Start the treatment process, if the risk level is found to be high for the students.

6.3. Flowchart for single-valued neutrosophic \mathcal{N} -soft matrix

In this subsection, we depict the flow of the problem to assess the mental health of students. A step by step process is shown below to understand the nature and the complexity of the problem.



7. Case Studies using $SVNNSM$

In this section, we present two case studies with ill-structured problems faced by the students amidst *COVID* – 19. In the Case study Ia, the *MHC* tries to identify the students who show high-risk level towards mental illness and requires immediate attention. In this illustration, two students are at a high-risk level towards mental illness and require counseling or treatment to overcome the same. In the Case study Ib, *MHC* starts the process after counseling sessions for the students, who showed a high-risk level in the Case study Ia. After following the same method, we show that the two students are at low-risk levels and have shown progress towards the counseling or the treatment. In the Case study II, we discuss the same process by using *WSVNV* and show all the students are at an average-risk level towards overall mental health score.

7.1. Case study Ia

Let us assume an institution approaches a professional *MHC* to assess the mental health and psychosocial aspects of the students.

Step 1: Suppose that $\mathcal{U} = \{s_1, s_2, s_3, s_4\}$ be the set of students and $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$ be the set of parameters where p_1 = avoiding social activities (*ASA*), p_2 = thinking about suicide (*TAS*), p_3 = extreme mood changes (*EMC*) and p_4 = stress.

Step 2: The *MHC* in liaison with the psychiatrist frames seven questions for the parameters p_1 and p_4 and six questions for the parameters p_2 and p_3 . For the parameter p_1 , question numbers three and six are positive statements and others are negative statements. For the parameter p_2 , question number two is a negative statement and others are positive statements. For the parameter p_3 , question numbers five and six are positive statements and others are negative statements. Finally, for the parameter p_4 , question numbers one, two, and seven are positive statements, and others are negative statements. We provide the above information in a tabular form (Table 8).

TABLE 8. Shows the positive (*pos*) and negative (*neg*) statement details

	q_1	q_2	q_3	q_4	q_5	q_6	q_7
p_1	<i>neg</i>	<i>neg</i>	<i>pos</i>	<i>neg</i>	<i>neg</i>	<i>pos</i>	<i>neg</i>
p_2	<i>pos</i>	<i>neg</i>	<i>pos</i>	<i>pos</i>	<i>pos</i>	<i>pos</i>	-
p_3	<i>neg</i>	<i>neg</i>	<i>neg</i>	<i>neg</i>	<i>pos</i>	<i>pos</i>	-
p_4	<i>pos</i>	<i>pos</i>	<i>neg</i>	<i>neg</i>	<i>neg</i>	<i>neg</i>	<i>pos</i>

Let's assume a 5 point rating scale (5-soft set) for positive and negative statements as in Table 5, the rating scale distribution as in Table 6 and the norms for each parameter and overall parameters as in Table 7.

Step 3: The *MHC* observes or understands the behavior of each student based on the framed statements and records the values in neutrosophic form, N_i^* , where $i = \{1, 2, \dots, 7\}$ represent the number of questions, for each parameter. The highlighted values in the N_1^* , N_2^* , N_3^* , N_4^* , N_5^* , N_6^* and N_7^* matrices show the values of negative statements for easy understanding and scoring procedures.

$$N_1^* = \begin{array}{c} \begin{array}{c} p_1 \quad p_2 \quad p_3 \quad p_4 \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \end{array} \left[\begin{array}{cccc} \langle 0.90, 0.95, 0.05 \rangle & \langle 0.90, 0.80, 0.35 \rangle & \langle 0.88, 0.75, 0.12 \rangle & \langle 0.71, 0.25, 0.45 \rangle \\ \langle 0.45, 0.35, 0.20 \rangle & \langle 0.78, 0.68, 0.36 \rangle & \langle 0.45, 0.25, 0.35 \rangle & \langle 0.90, 0.82, 0.45 \rangle \\ \langle 0.24, 0.20, 0.05 \rangle & \langle 0.90, 0.15, 0.35 \rangle & \langle 0.89, 0.22, 0.32 \rangle & \langle 0.12, 0.25, 0.10 \rangle \\ \langle 0.45, 0.35, 0.55 \rangle & \langle 0.90, 0.85, 0.10 \rangle & \langle 0.25, 0.35, 0.25 \rangle & \langle 0.80, 0.85, 0.35 \rangle \end{array} \right]$$

$$N_2^* = \begin{array}{c} \begin{array}{c} p_1 \quad p_2 \quad p_3 \quad p_4 \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \end{array} \left[\begin{array}{cccc} \langle 0.98, 0.95, 0.10 \rangle & \langle 0.90, 0.95, 0.10 \rangle & \langle 0.90, 0.85, 0.12 \rangle & \langle 0.75, 0.25, 0.55 \rangle \\ \langle 0.35, 0.25, 0.35 \rangle & \langle 0.55, 0.30, 0.15 \rangle & \langle 0.60, 0.55, 0.58 \rangle & \langle 0.70, 0.88, 0.12 \rangle \\ \langle 0.98, 0.95, 0.10 \rangle & \langle 0.48, 0.32, 0.35 \rangle & \langle 0.78, 0.42, 0.45 \rangle & \langle 0.16, 0.12, 0.13 \rangle \\ \langle 0.55, 0.61, 0.23 \rangle & \langle 0.33, 0.20, 0.15 \rangle & \langle 0.10, 0.45, 0.50 \rangle & \langle 0.90, 0.75, 0.05 \rangle \end{array} \right]$$

$$N_3^* = \begin{array}{c} \begin{array}{c} p_1 \quad p_2 \quad p_3 \quad p_4 \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \end{array} \left[\begin{array}{cccc} \langle 0.80, 0.60, 0.10 \rangle & \langle 0.10, 0.20, 0.25 \rangle & \langle 0.91, 0.81, 0.10 \rangle & \langle 0.88, 0.78, 0.30 \rangle \\ \langle 0.75, 0.65, 0.10 \rangle & \langle 0.88, 0.91, 0.05 \rangle & \langle 0.25, 0.35, 0.33 \rangle & \langle 0.35, 0.55, 0.15 \rangle \\ \langle 0.80, 0.60, 0.10 \rangle & \langle 0.40, 0.50, 0.20 \rangle & \langle 0.45, 0.55, 0.25 \rangle & \langle 0.30, 0.20, 0.45 \rangle \\ \langle 0.89, 0.79, 0.10 \rangle & \langle 0.85, 0.75, 0.35 \rangle & \langle 0.15, 0.24, 0.10 \rangle & \langle 0.30, 0.20, 0.35 \rangle \end{array} \right]$$

$$N_4^* = \begin{array}{c} \begin{array}{c} p_1 \quad p_2 \quad p_3 \quad p_4 \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \end{array} \left[\begin{array}{cccc} \langle 0.90, 0.54, 0.10 \rangle & \langle 0.25, 0.45, 0.25 \rangle & \langle 0.75, 0.35, 0.16 \rangle & \langle 0.87, 0.67, 0.25 \rangle \\ \langle 0.25, 0.20, 0.10 \rangle & \langle 0.91, 0.88, 0.16 \rangle & \langle 0.55, 0.30, 0.45 \rangle & \langle 0.48, 0.57, 0.25 \rangle \\ \langle 0.90, 0.54, 0.10 \rangle & \langle 1.00, 1.00, 0.00 \rangle & \langle 0.10, 0.45, 0.20 \rangle & \langle 0.65, 0.45, 0.55 \rangle \\ \langle 0.15, 0.35, 0.45 \rangle & \langle 0.88, 0.78, 0.25 \rangle & \langle 0.25, 0.15, 0.10 \rangle & \langle 0.45, 0.65, 0.35 \rangle \end{array} \right]$$

$$N_5^* = \begin{array}{c} \begin{array}{c} p_1 \quad p_2 \quad p_3 \quad p_4 \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \end{array} \left[\begin{array}{cccc} \langle 0.79, 0.99, 0.10 \rangle & \langle 0.25, 0.75, 0.85 \rangle & \langle 0.75, 0.10, 0.25 \rangle & \langle 0.78, 0.86, 0.25 \rangle \\ \langle 0.25, 0.35, 0.40 \rangle & \langle 0.77, 0.66, 0.21 \rangle & \langle 0.92, 0.93, 0.22 \rangle & \langle 0.38, 0.48, 0.19 \rangle \\ \langle 0.79, 0.99, 0.10 \rangle & \langle 0.75, 0.65, 0.42 \rangle & \langle 0.90, 0.85, 0.10 \rangle & \langle 0.28, 0.25, 0.46 \rangle \\ \langle 0.35, 0.45, 0.55 \rangle & \langle 0.88, 0.77, 0.25 \rangle & \langle 0.85, 0.75, 0.15 \rangle & \langle 0.15, 0.35, 0.45 \rangle \end{array} \right]$$

$$N_6^* = \begin{array}{c} \begin{array}{c} p_1 \quad p_2 \quad p_3 \quad p_4 \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \end{array} \left[\begin{array}{cccc} \langle 0.35, 0.40, 0.30 \rangle & \langle 0.55, 0.45, 0.80 \rangle & \langle 0.74, 0.55, 0.09 \rangle & \langle 0.88, 0.78, 0.05 \rangle \\ \langle 0.30, 0.32, 0.21 \rangle & \langle 0.65, 0.75, 0.19 \rangle & \langle 0.55, 0.50, 0.40 \rangle & \langle 0.55, 0.15, 0.35 \rangle \\ \langle 0.35, 0.40, 0.30 \rangle & \langle 0.70, 0.50, 0.40 \rangle & \langle 0.65, 0.60, 0.50 \rangle & \langle 0.45, 0.20, 0.21 \rangle \\ \langle 0.79, 0.89, 0.27 \rangle & \langle 0.80, 0.75, 0.35 \rangle & \langle 0.85, 0.75, 0.40 \rangle & \langle 0.75, 0.55, 0.30 \rangle \end{array} \right]$$

Given that there are only six questions for p_2 and p_3 , we exclude these two parameters in N_7^* .

$$N_7^* = \begin{matrix} & p_1 & p_4 \\ s_1 & \langle 0.95, 0.85, 0.10 \rangle & \langle 0.27, 0.21, 0.09 \rangle \\ s_2 & \langle 0.20, 0.10, 0.05 \rangle & \langle 0.85, 0.75, 0.25 \rangle \\ s_3 & \langle 0.95, 0.85, 0.10 \rangle & \langle 0.80, 0.70, 0.10 \rangle \\ s_4 & \langle 0.45, 0.25, 0.35 \rangle & \langle 0.25, 0.85, 0.25 \rangle \end{matrix}$$

Step 4: By applying SF Definition 3.1, we get the following values in matrices form.

$$S(N_1^*) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & 0.900 & 0.675 & 0.755 & 0.255 \\ s_2 & 0.300 & 0.550 & 0.175 & 0.635 \\ s_3 & 0.195 & 0.350 & 0.395 & 0.135 \\ s_4 & 0.125 & 0.825 & 0.175 & 0.650 \end{matrix} \quad S(N_2^*) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & 0.915 & 0.875 & 0.815 & 0.225 \\ s_2 & 0.125 & 0.350 & 0.285 & 0.730 \\ s_3 & 0.915 & 0.225 & 0.375 & 0.074 \\ s_4 & 0.465 & 0.190 & 0.025 & 0.800 \end{matrix}$$

$$S(N_3^*) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & 0.650 & 0.025 & 0.810 & 0.680 \\ s_2 & 0.650 & 0.870 & 0.135 & 0.375 \\ s_3 & 0.650 & 0.350 & 0.375 & 0.025 \\ s_4 & 0.790 & 0.625 & 0.145 & 0.075 \end{matrix} \quad S(N_4^*) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & 0.670 & 0.225 & 0.470 & 0.645 \\ s_2 & 0.175 & 0.815 & 0.200 & 0.400 \\ s_3 & 0.670 & 1.000 & 0.175 & 0.275 \\ s_4 & 0.025 & 0.705 & 0.150 & 0.375 \end{matrix}$$

$$S(N_5^*) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & 0.840 & 0.075 & 0.300 & 0.695 \\ s_2 & 0.100 & 0.610 & 0.815 & 0.335 \\ s_3 & 0.840 & 0.490 & 0.825 & 0.035 \\ s_4 & 0.125 & 0.700 & 0.725 & 0.025 \end{matrix} \quad S(N_6^*) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & 0.225 & 0.100 & 0.600 & 0.805 \\ s_2 & 0.205 & 0.605 & 0.325 & 0.175 \\ s_3 & 0.225 & 0.400 & 0.375 & 0.220 \\ s_4 & 0.705 & 0.600 & 0.600 & 0.500 \end{matrix}$$

and

$$S(N_7^*) = \begin{matrix} & p_1 & p_4 \\ s_1 & 0.850 & 0.195 \\ s_2 & 0.125 & 0.675 \\ s_3 & 0.850 & 0.700 \\ s_4 & 0.175 & 0.425 \end{matrix}$$

Step 5: Now by comparing the score values with Table 6, rating scale distribution and by using Definition 5.2, we obtain the following values in matrices form.

$$N_1^*(5) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 1, 0.900 \rangle & \langle 4, 0.675 \rangle & \langle 2, 0.755 \rangle & \langle 2, 0.255 \rangle \\ s_2 & \langle 3, 0.300 \rangle & \langle 3, 0.550 \rangle & \langle 4, 0.175 \rangle & \langle 4, 0.635 \rangle \\ s_3 & \langle 4, 0.195 \rangle & \langle 3, 0.350 \rangle & \langle 3, 0.395 \rangle & \langle 2, 0.135 \rangle \\ s_4 & \langle 4, 0.125 \rangle & \langle 5, 0.825 \rangle & \langle 4, 0.175 \rangle & \langle 4, 0.650 \rangle \end{matrix}$$

$$N_2^*(5) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 1, 0.915 \rangle & \langle 1, 0.875 \rangle & \langle 1, 0.815 \rangle & \langle 2, 0.225 \rangle \\ s_2 & \langle 4, 0.125 \rangle & \langle 3, 0.350 \rangle & \langle 4, 0.285 \rangle & \langle 4, 0.730 \rangle \\ s_3 & \langle 1, 0.915 \rangle & \langle 4, 0.225 \rangle & \langle 3, 0.375 \rangle & \langle 2, 0.074 \rangle \\ s_4 & \langle 3, 0.465 \rangle & \langle 4, 0.190 \rangle & \langle 4, 0.025 \rangle & \langle 5, 0.800 \rangle \end{matrix}$$

$$N_3^*(5) = \begin{array}{c} \begin{array}{ccccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 4, 0.650 \rangle & \langle 2, 0.025 \rangle & \langle 1, 0.810 \rangle & \langle 2, 0.680 \rangle \\ s_2 & \langle 4, 0.650 \rangle & \langle 5, 0.870 \rangle & \langle 4, 0.135 \rangle & \langle 3, 0.375 \rangle \\ s_3 & \langle 4, 0.650 \rangle & \langle 3, 0.350 \rangle & \langle 3, 0.375 \rangle & \langle 4, 0.025 \rangle \\ s_4 & \langle 4, 0.790 \rangle & \langle 4, 0.625 \rangle & \langle 4, 0.145 \rangle & \langle 4, 0.075 \rangle \end{array} \end{array}$$

$$N_4^*(5) = \begin{array}{c} \begin{array}{ccccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 2, 0.670 \rangle & \langle 2, 0.225 \rangle & \langle 3, 0.470 \rangle & \langle 2, 0.645 \rangle \\ s_2 & \langle 4, 0.175 \rangle & \langle 5, 0.815 \rangle & \langle 4, 0.200 \rangle & \langle 3, 0.400 \rangle \\ s_3 & \langle 2, 0.670 \rangle & \langle 5, 1.000 \rangle & \langle 4, 0.175 \rangle & \langle 4, 0.275 \rangle \\ s_4 & \langle 4, 0.025 \rangle & \langle 4, 0.705 \rangle & \langle 4, 0.150 \rangle & \langle 3, 0.375 \rangle \end{array} \end{array}$$

$$N_5^*(5) = \begin{array}{c} \begin{array}{ccccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 1, 0.840 \rangle & \langle 2, 0.075 \rangle & \langle 3, 0.300 \rangle & \langle 2, 0.695 \rangle \\ s_2 & \langle 4, 0.100 \rangle & \langle 4, 0.610 \rangle & \langle 5, 0.815 \rangle & \langle 3, 0.335 \rangle \\ s_3 & \langle 1, 0.840 \rangle & \langle 3, 0.490 \rangle & \langle 5, 0.825 \rangle & \langle 4, 0.035 \rangle \\ s_4 & \langle 4, 0.125 \rangle & \langle 4, 0.700 \rangle & \langle 4, 0.725 \rangle & \langle 4, 0.025 \rangle \end{array} \end{array}$$

$$N_6^*(5) = \begin{array}{c} \begin{array}{ccccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 2, 0.225 \rangle & \langle 2, 0.100 \rangle & \langle 4, 0.600 \rangle & \langle 1, 0.805 \rangle \\ s_2 & \langle 2, 0.205 \rangle & \langle 4, 0.605 \rangle & \langle 3, 0.325 \rangle & \langle 4, 0.175 \rangle \\ s_3 & \langle 2, 0.225 \rangle & \langle 3, 0.400 \rangle & \langle 3, 0.375 \rangle & \langle 4, 0.220 \rangle \\ s_4 & \langle 4, 0.705 \rangle & \langle 4, 0.600 \rangle & \langle 4, 0.600 \rangle & \langle 3, 0.500 \rangle \end{array} \end{array} \text{ and}$$

$$N_7^*(5) = \begin{array}{c} \begin{array}{ccccc} & p_1 & p_4 \\ s_1 & \langle 1, 0.850 \rangle & \langle 2, 0.195 \rangle \\ s_2 & \langle 4, 0.125 \rangle & \langle 4, 0.675 \rangle \\ s_3 & \langle 1, 0.850 \rangle & \langle 4, 0.700 \rangle \\ s_4 & \langle 4, 0.175 \rangle & \langle 3, 0.425 \rangle \end{array} \end{array}$$

Step 6: Determine $N_+^*(5)$ matrix by summing the corresponding entries of $N_1^*(\mathcal{N}), N_2^*(\mathcal{N}), \dots, N_7^*(\mathcal{N})$ matrices.

$$N_+^*(5) = \begin{array}{c} \begin{array}{ccccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & 12 & 13 & 14 & 13 \\ s_2 & 25 & 24 & 24 & 25 \\ s_3 & 15 & 21 & 21 & 24 \\ s_4 & 27 & 25 & 24 & 26 \end{array} \end{array} \begin{array}{c} 52 \\ 98 \\ 81 \\ 102 \end{array}$$

Step 7: Tabulate the details as in Table 9 and assess the risk level of the students by using the norm details (Table 7).

Analysis: From Table 9, we suggest that for s_1 , the risk level is low for each parameter and also for the combined parameter scores, which signify that s_1 does not experience any mental illness and may not require any treatment from *MHC*. For s_2 , the risk level is average for each parameter and high for the combined parameter scores. Although the risk level is

TABLE 9. Shows students' mental health scores and levels for the parameters

s_i	p_1		p_2		p_3		p_4		$Total$	
	score	level	score	level	score	level	score	level	score	level
s_1	12	low	13	low	14	low	13	low	52	low
s_2	25	avg	24	avg	24	avg	25	avg	98	high
s_3	15	avg	21	avg	21	avg	24	avg	81	avg
s_4	27	high	25	high	24	avg	26	high	102	high

average for s_2 in each parameter, the total score is 98, which signifies that s_2 may require the help of the *MHC* or the psychiatrist to lower the risk level of mental illness. For s_3 , the risk level is average for each parameter and also for the combined parameter scores, which signify that s_3 may not experience any mental illness. For s_4 , excluding p_3 , the risk level is high for other parameters as well for the combined parameter scores, which signifies that s_4 requires aid from the *MHC* or the psychiatrist to lower the mental illness. Hence, in this study, we analyze the mental health of the students in a traditional method by using *SVNNS*.

7.2. Case study Ib

In this case study, we select the two students from case study 1a, who show high-risk level towards mental health. Let's assume that *MHC* records the details after the counseling or the treatment.

Step 1: Consider $\mathcal{U} = \{s_2, s_4\}$ be the set of students who are at high risk level towards mental health and $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$ be the same set of parameters as in earlier case.

Step 2: Let's assume that *MHC* provides positive and negative information as in Table 10. Let the rating scale distribution and the norms be as in Table 6 and 7.

TABLE 10. Shows the positive (*pos*) and negative (*neg*) statement details

	q_1	q_2	q_3	q_4	q_5
p_1	<i>pos</i>	<i>neg</i>	<i>pos</i>	<i>neg</i>	<i>pos</i>
p_2	<i>pos</i>	<i>pos</i>	<i>pos</i>	<i>pos</i>	<i>neg</i>
p_3	<i>neg</i>	<i>pos</i>	<i>pos</i>	<i>pos</i>	<i>pos</i>
p_4	<i>pos</i>	<i>pos</i>	<i>neg</i>	<i>pos</i>	<i>pos</i>

Step 3: The *MHC* observes the behavior of s_2 and s_4 based on the new set of framed statements and records the values in neutrosophic form, N_i^* , where $i = \{1, 2, \dots, 5\}$ represent the number of questions, for each parameter. The highlighted values in the N_1^* , N_2^* , N_3^* , N_4^* and N_5^* matrices show the values of negative statements.

$$N_1^* = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_2 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.35, 0.40, 0.25 \rangle \\ \langle 0.25, 0.35, 0.19 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.67, 0.78, 0.36 \rangle \\ \langle 0.65, 0.55, 0.15 \rangle \end{bmatrix} & \begin{bmatrix} \langle \mathbf{0.90}, \mathbf{0.85}, \mathbf{0.35} \rangle \\ \langle \mathbf{0.87}, \mathbf{0.88}, \mathbf{0.20} \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.85, 0.95, 0.40 \rangle \\ \langle 0.45, 0.55, 0.45 \rangle \end{bmatrix} \end{matrix}$$

$$N_2^* = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \langle 0.88, 0.98, 0.15 \rangle & \langle 0.20, 0.30, 0.10 \rangle & \langle 0.35, 0.45, 0.58 \rangle & \langle 0.41, 0.51, 0.12 \rangle \\ \langle 0.88, 0.75, 0.23 \rangle & \langle 0.35, 0.25, 0.15 \rangle & \langle 0.45, 0.55, 0.32 \rangle & \langle 0.25, 0.35, 0.05 \rangle \end{array}$$

$$N_3^* = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \langle 0.45, 0.55, 0.10 \rangle & \langle 0.55, 0.23, 0.10 \rangle & \langle 0.20, 0.30, 0.31 \rangle & \langle 0.35, 0.45, 0.10 \rangle \\ \langle 0.69, 0.79, 0.15 \rangle & \langle 0.66, 0.77, 0.24 \rangle & \langle 0.20, 0.34, 0.21 \rangle & \langle 0.88, 0.78, 0.22 \rangle \end{array}$$

$$N_4^* = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \langle 0.88, 0.78, 0.10 \rangle & \langle 0.34, 0.45, 0.10 \rangle & \langle 0.45, 0.32, 0.25 \rangle & \langle 0.34, 0.45, 0.10 \rangle \\ \langle 0.88, 0.91, 0.45 \rangle & \langle 0.45, 0.78, 0.35 \rangle & \langle 0.28, 0.48, 0.15 \rangle & \langle 0.45, 0.70, 0.30 \rangle \end{array}$$

$$N_5^* = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \langle 0.25, 0.35, 0.40 \rangle & \langle 0.38, 0.48, 0.19 \rangle & \langle 0.34, 0.45, 0.22 \rangle & \langle 0.77, 0.66, 0.21 \rangle \\ \langle 0.38, 0.45, 0.48 \rangle & \langle 0.35, 0.35, 0.40 \rangle & \langle 0.80, 0.75, 0.20 \rangle & \langle 0.78, 0.70, 0.25 \rangle \end{array}$$

Step 4: By applying *SF* Definition 3.1, we get the following values in matrices form.

$$S(N_1^*) = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ 0.250 & 0.545 & 0.700 & 0.700 \\ 0.205 & 0.525 & 0.775 & 0.275 \end{array} S(N_2^*) = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ 0.855 & 0.200 & 0.110 & 0.400 \\ 0.700 & 0.225 & 0.340 & 0.275 \end{array}$$

$$S(N_3^*) = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ 0.450 & 0.340 & 0.095 & 0.350 \\ 0.665 & 0.595 & 0.165 & 0.720 \end{array} S(N_4^*) = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ 0.780 & 0.345 & 0.260 & 0.345 \\ 0.670 & 0.440 & 0.305 & 0.425 \end{array}$$

$$S(N_5^*) = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ 0.100 & 0.335 & 0.285 & 0.610 \\ 0.175 & 0.150 & 0.675 & 0.615 \end{array}$$

Step 5: Now by comparing the score values with Table 6, rating scale distribution and by using Definition 5.2, we obtain the following values in matrices form.

$$N_1^*(5) = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \langle 2, 0.250 \rangle & \langle 3, 0.545 \rangle & \langle 2, 0.700 \rangle & \langle 4, 0.700 \rangle \\ \langle 2, 0.205 \rangle & \langle 3, 0.525 \rangle & \langle 2, 0.775 \rangle & \langle 2, 0.275 \rangle \end{array}$$

$$N_2^*(5) = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \langle 1, 0.855 \rangle & \langle 2, 0.200 \rangle & \langle 2, 0.110 \rangle & \langle 3, 0.400 \rangle \\ \langle 2, 0.700 \rangle & \langle 2, 0.225 \rangle & \langle 3, 0.340 \rangle & \langle 2, 0.275 \rangle \end{array}$$

$$N_3^*(5) = \begin{array}{c} s_2 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \langle 3, 0.450 \rangle & \langle 3, 0.340 \rangle & \langle 2, 0.095 \rangle & \langle 3, 0.350 \rangle \\ \langle 4, 0.665 \rangle & \langle 3, 0.595 \rangle & \langle 2, 0.165 \rangle & \langle 2, 0.720 \rangle \end{array}$$

$$N_4^*(5) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_2 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 2, 0.780 \rangle \\ \langle 2, 0.670 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.345 \rangle \\ \langle 3, 0.440 \rangle \end{bmatrix} & \begin{bmatrix} \langle 2, 0.260 \rangle \\ \langle 3, 0.305 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.345 \rangle \\ \langle 3, 0.425 \rangle \end{bmatrix} \end{matrix}$$

$$N_5^*(5) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_2 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 2, 0.100 \rangle \\ \langle 2, 0.175 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.335 \rangle \\ \langle 4, 0.150 \rangle \end{bmatrix} & \begin{bmatrix} \langle 2, 0.285 \rangle \\ \langle 4, 0.675 \rangle \end{bmatrix} & \begin{bmatrix} \langle 4, 0.610 \rangle \\ \langle 4, 0.615 \rangle \end{bmatrix} \end{matrix}$$

Step 6: Determine $N_+^*(5)$ matrix by summing the corresponding entries of $N_1^*(\mathcal{N}), N_2^*(\mathcal{N}), \dots, N_5^*(\mathcal{N})$ matrices.

$$N_+^*(5) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_2 \\ s_4 \end{matrix} & \begin{bmatrix} 10 & 14 & 10 & 17 \\ 12 & 15 & 14 & 13 \end{bmatrix} & \begin{matrix} 51 \\ 54 \end{matrix} \end{matrix}$$

Step 7: Tabulate the details as in Table 11 and assess the risk level of the two students.

TABLE 11. Shows students' mental health scores and levels for each parameter

s_i	p_1		p_2		p_3		p_4		$Total$	
	score	level	score	level	score	level	score	level	score	level
s_2	10	low	14	low	10	low	17	avg	51	low
s_4	12	low	15	low	14	low	13	low	54	low

Analysis: From Tables 9 and 11, we infer that for the parameter ASA , the student s_2 had an initial score of 25 with an average-risk level towards mental illness. After the remedy process, the student has attained a score of 10 with a low-risk for the same parameter. Likewise, for other parameters, TAS , EMC , and stress, in the initial stages, the scores are 24, 24, and 25, respectively, with an average-risk level. After the treatment, we find the scores are 14 and 10, with low-risk for the parameters TAS and EMC . For the parameter stress, the score is 17 and has attained an average-risk level. Similarly, the student s_4 showed a high risk with an initial score of 27 for the parameter ASA . After the treatment, a score of 12 with low-risk for the same parameter. Likewise, for other parameters, TAS , EMC , and stress, in the initial stages, the scores are 25, high risk, 24, average risk, and 26, high risk, respectively. After the treatment, we observe that for parameters TAS , EMC , and stress, the scores are 15, 14, and 13, with low-risk levels, respectively. Hence, we conclude that both the students have attained a low-level risk score of 51 and 54, respectively, towards mental illness and have responded well to the treatment.

7.3. Case study II

Let us consider the same example as in case study Ia. Let $W = (0.45, 0.15, 0.25, 0.15)$ be the weight vector assigned by the MHC to the parameters. Refer section 7.1, for steps 1 to 4 data information. In this section, we explain the method when MHC uses criteria weights.

Step 4: By applying $WSVNV$ Definition 3.3, we get the values in matrices as below.

$$\begin{aligned}
 WS(N_1^*) &= \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0.405 & 0.101 & 0.189 & 0.038 \\ 0.135 & 0.083 & 0.044 & 0.095 \\ 0.088 & 0.053 & 0.099 & 0.020 \\ 0.056 & 0.124 & 0.044 & 0.098 \end{bmatrix} \end{matrix} & WS(N_2^*) &= \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0.412 & 0.131 & 0.204 & 0.034 \\ 0.056 & 0.053 & 0.071 & 0.110 \\ 0.412 & 0.034 & 0.094 & 0.011 \\ 0.209 & 0.029 & 0.006 & 0.120 \end{bmatrix} \end{matrix} \\
 WS(N_3^*) &= \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0.293 & 0.004 & 0.203 & 0.102 \\ 0.293 & 0.131 & 0.034 & 0.056 \\ 0.293 & 0.053 & 0.094 & 0.004 \\ 0.356 & 0.094 & 0.036 & 0.011 \end{bmatrix} \end{matrix} & WS(N_4^*) &= \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0.302 & 0.034 & 0.118 & 0.097 \\ 0.079 & 0.122 & 0.050 & 0.060 \\ 0.302 & 0.150 & 0.044 & 0.041 \\ 0.011 & 0.106 & 0.038 & 0.056 \end{bmatrix} \end{matrix} \\
 WS(N_5^*) &= \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0.378 & 0.011 & 0.075 & 0.104 \\ 0.045 & 0.092 & 0.204 & 0.050 \\ 0.378 & 0.074 & 0.206 & 0.005 \\ 0.056 & 0.105 & 0.181 & 0.004 \end{bmatrix} \end{matrix} & WS(N_6^*) &= \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0.101 & 0.015 & 0.150 & 0.121 \\ 0.092 & 0.091 & 0.081 & 0.026 \\ 0.101 & 0.060 & 0.094 & 0.033 \\ 0.317 & 0.090 & 0.150 & 0.075 \end{bmatrix} \end{matrix}
 \end{aligned}$$

and

$$WS(N_7^*) = \begin{matrix} & p_1 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0.383 & 0.029 \\ 0.056 & 0.101 \\ 0.383 & 0.105 \\ 0.079 & 0.064 \end{bmatrix} \end{matrix}$$

Step 5: By comparing the $WSVNV$ values with Table 6, rating scale distribution and by using Definition 5.2, we obtain the following values in matrices form.

$$N_1^*(5) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 3, 0.405 \rangle & \langle 2, 0.101 \rangle & \langle 4, 0.189 \rangle & \langle 2, 0.038 \rangle \\ \langle 4, 0.135 \rangle & \langle 2, 0.083 \rangle & \langle 4, 0.044 \rangle & \langle 2, 0.095 \rangle \\ \langle 4, 0.088 \rangle & \langle 2, 0.053 \rangle & \langle 4, 0.099 \rangle & \langle 2, 0.020 \rangle \\ \langle 4, 0.056 \rangle & \langle 2, 0.124 \rangle & \langle 4, 0.044 \rangle & \langle 2, 0.098 \rangle \end{bmatrix} \end{matrix}$$

$$N_2^*(5) = \begin{array}{c} \begin{array}{cccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 3, 0.412 \rangle & \langle 4, 0.131 \rangle & \langle 4, 0.204 \rangle & \langle 2, 0.034 \rangle \\ s_2 & \langle 4, 0.056 \rangle & \langle 4, 0.053 \rangle & \langle 4, 0.071 \rangle & \langle 2, 0.110 \rangle \\ s_3 & \langle 3, 0.412 \rangle & \langle 4, 0.034 \rangle & \langle 4, 0.094 \rangle & \langle 2, 0.011 \rangle \\ s_4 & \langle 4, 0.209 \rangle & \langle 4, 0.029 \rangle & \langle 4, 0.006 \rangle & \langle 2, 0.120 \rangle \end{array} \end{array}$$

$$N_3^*(5) = \begin{array}{c} \begin{array}{cccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 2, 0.293 \rangle & \langle 2, 0.004 \rangle & \langle 4, 0.203 \rangle & \langle 4, 0.102 \rangle \\ s_2 & \langle 2, 0.293 \rangle & \langle 2, 0.131 \rangle & \langle 4, 0.034 \rangle & \langle 4, 0.056 \rangle \\ s_3 & \langle 2, 0.293 \rangle & \langle 2, 0.053 \rangle & \langle 4, 0.094 \rangle & \langle 4, 0.004 \rangle \\ s_4 & \langle 3, 0.356 \rangle & \langle 2, 0.094 \rangle & \langle 4, 0.036 \rangle & \langle 4, 0.011 \rangle \end{array} \end{array}$$

$$N_4^*(5) = \begin{array}{c} \begin{array}{cccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 3, 0.302 \rangle & \langle 2, 0.034 \rangle & \langle 4, 0.118 \rangle & \langle 4, 0.097 \rangle \\ s_2 & \langle 4, 0.079 \rangle & \langle 2, 0.122 \rangle & \langle 4, 0.050 \rangle & \langle 4, 0.060 \rangle \\ s_3 & \langle 3, 0.302 \rangle & \langle 2, 0.150 \rangle & \langle 4, 0.044 \rangle & \langle 4, 0.041 \rangle \\ s_4 & \langle 4, 0.011 \rangle & \langle 2, 0.106 \rangle & \langle 4, 0.038 \rangle & \langle 4, 0.056 \rangle \end{array} \end{array}$$

$$N_5^*(5) = \begin{array}{c} \begin{array}{cccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 3, 0.378 \rangle & \langle 2, 0.011 \rangle & \langle 2, 0.075 \rangle & \langle 4, 0.104 \rangle \\ s_2 & \langle 4, 0.045 \rangle & \langle 2, 0.092 \rangle & \langle 2, 0.204 \rangle & \langle 4, 0.050 \rangle \\ s_3 & \langle 3, 0.378 \rangle & \langle 2, 0.074 \rangle & \langle 2, 0.206 \rangle & \langle 4, 0.005 \rangle \\ s_4 & \langle 4, 0.056 \rangle & \langle 2, 0.105 \rangle & \langle 2, 0.181 \rangle & \langle 4, 0.004 \rangle \end{array} \end{array}$$

$$N_6^*(5) = \begin{array}{c} \begin{array}{cccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 2, 0.101 \rangle & \langle 2, 0.015 \rangle & \langle 2, 0.150 \rangle & \langle 4, 0.121 \rangle \\ s_2 & \langle 2, 0.092 \rangle & \langle 2, 0.091 \rangle & \langle 2, 0.081 \rangle & \langle 4, 0.026 \rangle \\ s_3 & \langle 2, 0.101 \rangle & \langle 2, 0.060 \rangle & \langle 2, 0.094 \rangle & \langle 4, 0.033 \rangle \\ s_4 & \langle 3, 0.317 \rangle & \langle 2, 0.090 \rangle & \langle 2, 0.150 \rangle & \langle 4, 0.075 \rangle \end{array} \end{array}$$

$$N_7^*(5) = \begin{array}{c} \begin{array}{cc} & p_1 & p_4 \\ s_1 & \langle 3, 0.383 \rangle & \langle 2, 0.029 \rangle \\ s_2 & \langle 4, 0.056 \rangle & \langle 2, 0.101 \rangle \\ s_3 & \langle 3, 0.383 \rangle & \langle 2, 0.105 \rangle \\ s_4 & \langle 4, 0.079 \rangle & \langle 2, 0.064 \rangle \end{array} \end{array}$$

Step 6: Determine $N_+^*(5)$ matrix by summing the corresponding entries of $N_1^*(5), N_2^*(5), \dots, N_7^*(5)$ matrices.

$$N_+^*(5) = \begin{array}{c} \begin{array}{cccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & 19 & 14 & 20 & 22 \end{array} \begin{array}{l} 75 \\ 80 \\ 76 \\ 82 \end{array} \end{array}$$

Step 7: Tabulate the details as in Table 12 and assess the risk level of the students.

Analysis: We observe from Table 12 that for the parameter, ASA , the mental health scores for the students, s_1 , s_2 , and s_3 are 19, 24, and 20 respectively, with an average-risk level. For Chinnadurai and Bobin, Applications to assess and pre-assess the mental health of students

TABLE 12. Shows students' mental health scores and levels for each parameter

s_i	p_1		p_2		p_3		p_4		$Total$	
	score	level	score	level	score	level	score	level	score	level
s_1	19	avg	14	low	20	avg	22	avg	75	avg
s_2	24	avg	14	low	20	avg	22	avg	80	avg
s_3	20	avg	14	low	20	avg	22	avg	76	avg
s_4	26	high	14	low	20	avg	22	avg	82	avg

s_4 , the score is 26 and at a high-risk may require a counseling session to lower the same. Similarly, for the parameter, TAS , the scores for all the students are 14 and at low-level risk. For the parameter, EMC , and stress, the scores are 20 and 22 with an average-risk level for all the students. The overall scores for the students are 75, 80, 76, and 82 show an average-risk level associated with mental illness.

8. Intertemporal single-valued neutrosophic \mathcal{N} -soft set

Definition 8.1. An intertemporal single-valued neutrosophic \mathcal{N} -soft set ($ISVNNSS$) is represented as a finite sequence of $SVNNSS$ over \mathcal{U} , and denoted by $\{(\psi^t, \mathcal{J}^t, \mathcal{N})\}_{t=k}^l$ for a session $k, l \in \mathbb{N}$ such that $(k \leq k' \leq l)$.

Definition 8.2. Let $\{(\psi^t, \mathcal{J}^t, \mathcal{N})\}_{t=k}^l$ for a session $k, l \in \mathbb{N}$ be an $ISVNNSS$, then the quasi-hyperbolic discounting intertemporal single-valued neutrosophic \mathcal{N} -soft set ($QHDISVNNSS$) computed from $\{(\psi^t, \mathcal{J}^t, \mathcal{N})\}_{t=k}^l$ at session k' , $(k \leq k' \leq l)$ is defined as,

$$\tilde{N}(\mathcal{N})_{k'} = \psi^{k'}(p)(u) = \left\langle g_p, \frac{1}{l - k' + 1} \left[S(\tilde{N})_{k'} + \beta \left(\sum_{t=1}^{l-k'} \delta^t \cdot S(\tilde{N})_{k'+t} \right) \right] \right\rangle,$$

where $\delta \in [0, 1]$ and $\beta \in [0, 1)$ are the long-term and short-term discounting parameters respectively and $S(\tilde{N})_{k'}$ and $S(\tilde{N})_{k'+t}$ are the SFs of $SVNNS$ for the session k' and $k' + t$ respectively.

Definition 8.3. Let $\mathcal{U} = \{u_1, u_2, \dots, u_m\}$ be the universal set. Let $\mathcal{P} = \{p_1, p_2, \dots, p_n\}$ be set of parameters and $\mathcal{G} = \{1, 2, \dots, \mathcal{N}\}$ be a set of rating scale. Then the $QHDISVNNSS$ computed from $\{(\psi^t, \mathcal{J}^t, \mathcal{N})\}_{t=k}^l$ at session k' , $(k \leq k' \leq l)$ is defined as,

$$N^*(\mathcal{N})_{k'} = [q_{ij}] = \begin{matrix} & \begin{matrix} p_1 & p_2 & \dots & p_n \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{matrix} & \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1n} \\ q_{21} & q_{22} & \dots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{m1} & q_{m2} & \dots & q_{mn} \end{bmatrix} \end{matrix},$$

such that

$$N^*(\mathcal{N})_{k'} = [q_{ij}] = \left\langle g_{p_{ij}}, \frac{1}{l - k' + 1} \left[(s_{ij})_{k'} + \beta \left(\sum_{t=1}^{l-k'} \delta^t \cdot (s_{ij})_{k'+t} \right) \right] \right\rangle,$$

$i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Then $N^*(\mathcal{N})_{k'}$ is called an $m \times n$ quasi-hyperbolic discounting intertemporal single-valued neutrosophic \mathcal{N} -soft matrix ($QHDISVNNSM$) of the $QHDISVNNSS \{(\psi^t, \mathcal{J}^t, \mathcal{N})\}_{t=k}^l$.

9. An application to pre-assess the mental health of students using $QHDISVNNSM$

Consider the MHC has planned for n counseling sessions in a phased manner to study and change the socially deviant behavior to socially acceptable behavior. To check the progress of the students, MHC would like to pre-assess the students after m sessions i.e., ($m < n$). Pre-assessment helps the MHC to understand the level of progress shown by the students in their behavior. To deal with this, we construct the concept of $QHDISVNNSM$, an algorithm, and a flowchart to pre-assess the mental health of the students.

Consider a scenario where the MHC wishes to assess the mental health of the students in a phased manner. Let $\mathcal{U} = \{s_1, s_2, \dots, s_m\}$ denote the set of students and $\mathcal{E} = \{p_1, p_2, \dots, p_n\}$ the set of parameters to assess the psychosocial conditions. Let us assume the MHC frames the following details namely; positive and negative statements for parameters, rating scales with distribution criteria (Table 13), weightage criteria to assess the parameters, scoring keys, and mental health norms (Table 14). The MHC based on each question, say $r = \{1, 2, \dots, h\}$ evaluates the students by considering the parameters and present the results in the form of neutrosophic matrices, $(N_r^*)_{k'}$ of order $m \times n$ for each session k' . Now, we have to pre-assess the mental health of the student with the help of pre-determined scores and norms.

TABLE 13. Shows the rating scale distribution

Positive statement	Negative statement	Score values
1	5	$0.8 \leq s_{ij} \leq 1.0$
2	4	$0.6 \leq s_{ij} < 0.8$
3	3	$0.3 \leq s_{ij} < 0.6$
4	2	$0.0 \leq s_{ij} < 0.3$
5	1	$-0.5 \leq s_{ij} < 0.0$

9.1. Methodology to pre-assess the mental health of the students

Construct the neutrosophic matrices $(N_r^*)_{k'}$, $r = \{1, 2, \dots, h\}$ for each positive or negative statement by observing the behavior of the student for each session. Apply SF Definition 3.1, to the neutrosophic matrices and represent the resultant matrices by $S(N_r^*)_{k'}$. If weightage

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TABLE 14. Shows the qualitative norm details

Parameter	Scores	Norms
p_1, p_2, p_3, p_4	1-5	low
	6-10	moderate (mod)
	11-15	borderline (bor)
	16-20	high
	21-25	very high (vh)
Total	1-20	low
	21-40	mod
	41-60	bor
	61-80	high
	81-100	vh

criteria are to be considered for each parameter, then calculate $WSVNV$ by using Definition 3.3. Now compare the entries in each $S(N_r^*)_{k'}$ matrix and construct the $N_r^*(\mathcal{N})_{k'}$ by using Definition 8.3 with the values of $\delta = 0.9$, $\beta = 0.5$ and by comparing the values with the framed rating scale distribution (Table 13). Determine the $N_+^*(\mathcal{N})_{k'}$ matrix by adding the corresponding entries of $N_1^*(\mathcal{N})_{k'}, N_2^*(\mathcal{N})_{k'}, \dots, N_k^*(\mathcal{N})_{k'}$ matrices. Now pre-assess the risk level for each parameter as well for the overall by using the level norms (Table 14). If the student attains a low/moderate-risk level in pre-assessment, then he/she responds to the treatment. If otherwise, then *MHC* should start an alternative remedy process for the students who show a high-risk level towards psychosocial conditions.

9.2. Algorithm to pre-assess the mental illness among the students

The following steps provide an insight to pre-assess the mental illness among the students.

Step 1: Identify the problem, select the students and the parameters.

Step 2: Involve a psychiatrist to frame the required details namely; positive and negative statements, rating scale with distribution, scoring keys and risk level.

Step 3: Construct $(N_r^*)_{k'}$, where $r = \{1, 2, \dots, h\}$ matrices for each question by observing the behavior of the students at k' session.

Step 4: Evaluate SF and $WSVNV$ by using Definition 3.1 and 3.3, respectively, and compute the risk-level analysis Table for the first session using Algorithm 6.2.

Step 5: Construct $N_r^*(\mathcal{N})_{k'}$ for the sessions by using Definition 8.3 and by comparing it with rating scale and distribution details.

Step 6: Determine $N_+^*(\mathcal{N})_{k'}$ matrix by summing the corresponding entries of $N_1^*(\mathcal{N})_{k'}, N_2^*(\mathcal{N})_{k'}, \dots, N_h^*(\mathcal{N})_{k'}$ matrices.

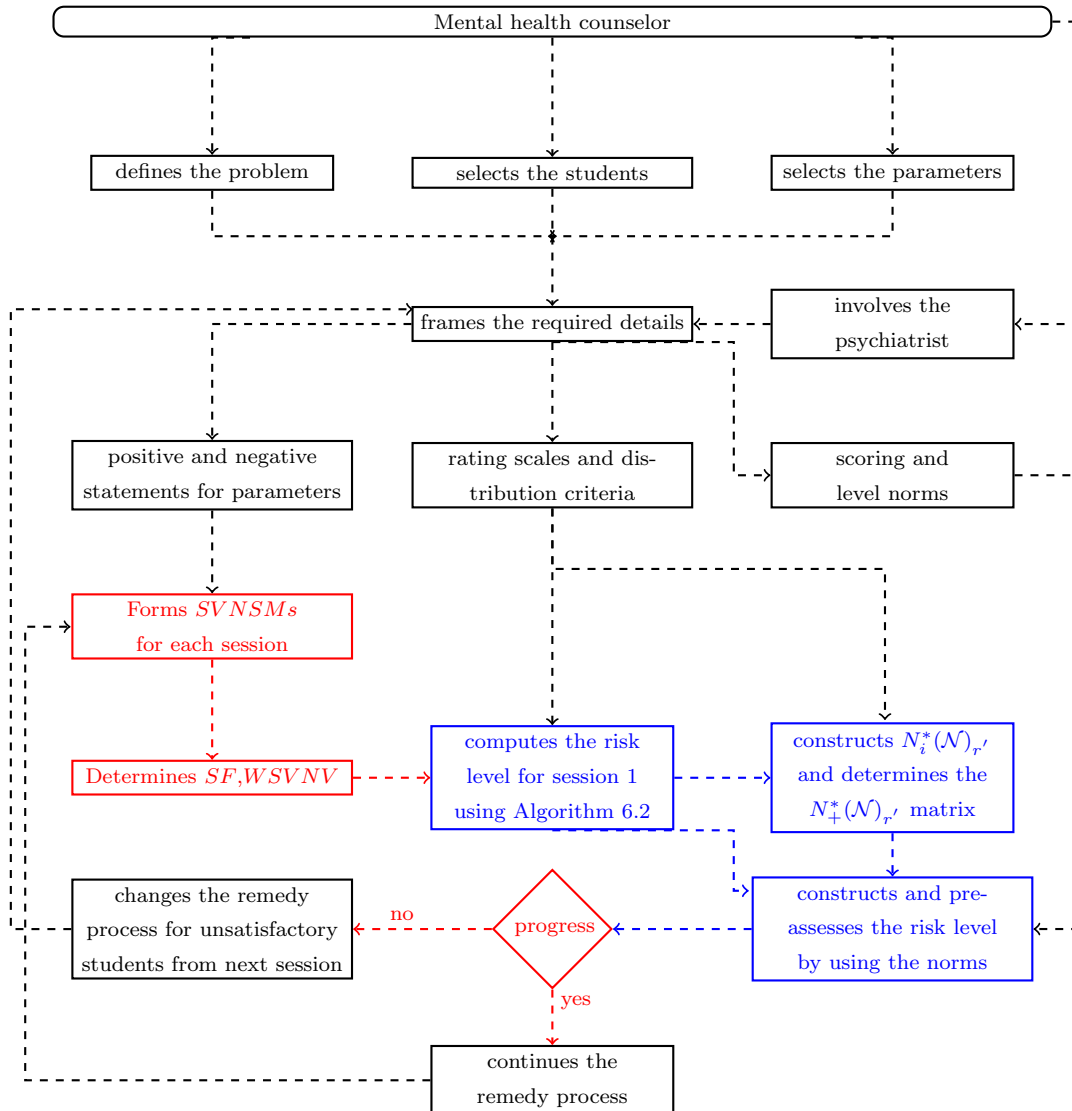
Step 7: Tabulate and pre-assess the mental health risk level by using the Table values determined in Step 4, and by using scoring keys and risk level norms.

Step 8: If the risk level is high for the students then the *MHC* to terminate the current

treatment and initiate an alternative treatment process from the next session.

9.3. Flowchart for intertemporal neutrosophic \mathcal{N} -soft matrix

In this subsection, we depict the flow of the problem to pre-assess the mental illness of students. A structured process is shown below to understand the nature of the problem.



10. Case study using QHDISVNN SM

The *MHC* or the psychiatrist might have to encounter multiple sessions to identify the mental health or the psychosocial behavior of the students. When there is a deviation in behavior, the *MHC* may find it difficult in which session the treatment or counseling failed to work for the students or could also be the students who did not follow the guidelines informed by the *MHC*. To overcome this gap, we present a method to pre-assess the mental illness of Chinnadurai and Bobin, Applications to assess and pre-assess the mental health of students

students with the information recorded during every session. Also, this method gives an insight into whether the *MHC* treatment or the counseling moves forward in the right direction.

Consider a scenario where the *MHC* observes the behavior of the students and records the information using *SVNSMs* for every lockdown session during the pandemic. Also, let the *MHC* compute the risk level for session 1 using Algorithm 6.2 to understand the risk level associated with the students. Let's assume that the students are at the beginning of the fourth lockdown session and the *MHC* would like to pre-assess the students and determine the risk level connected with the previous lockdown session. In the first case, let's consider the information from sessions 1 to 3, in the second, sessions 2 and 3, and the third, session 3.

Step 1: Suppose that $\mathcal{U} = \{s_1, s_2, s_3, s_4\}$ be the set of students who suffer from mental illness and $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$ be the set of parameters where p_1 = avoiding social activities (*ASA*), p_2 = thinking about suicide (*TAS*), p_3 = extreme mood changes (*EMC*) and p_4 = stress.

Step 2: Let's consider the *MHC* frames five positive questions for all the parameters across the three lockdown sessions. Let the rating scale distribution and level norms be as in Tables 13 and 14, respectively.

Step 3: Let *MHC* observes the behavior of each student based on the framed positive statements and provides the value in *SVNSMs* form, $(N_1^*)_1$, $(N_2^*)_1$, $(N_3^*)_1$, $(N_4^*)_1$ and $(N_5^*)_1$ for the first lockdown session.

$$(N_1^*)_1 = \begin{array}{c} \begin{array}{cccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 0.30, 0.32, 0.19 \rangle & \langle 0.40, 0.42, 0.35 \rangle & \langle 0.20, 0.30, 0.12 \rangle & \langle 0.40, 0.42, 0.05 \rangle \\ s_2 & \langle 0.35, 0.38, 0.20 \rangle & \langle 0.43, 0.45, 0.38 \rangle & \langle 0.30, 0.35, 0.13 \rangle & \langle 0.30, 0.35, 0.10 \rangle \\ s_3 & \langle 0.32, 0.35, 0.15 \rangle & \langle 0.50, 0.55, 0.40 \rangle & \langle 0.40, 0.45, 0.14 \rangle & \langle 0.20, 0.25, 0.05 \rangle \\ s_4 & \langle 0.23, 0.31, 0.22 \rangle & \langle 0.45, 0.50, 0.40 \rangle & \langle 0.20, 0.30, 0.05 \rangle & \langle 0.33, 0.43, 0.17 \rangle \end{array} \end{array}$$

$$(N_2^*)_1 = \begin{array}{c} \begin{array}{cccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 0.21, 0.24, 0.20 \rangle & \langle 0.40, 0.42, 0.15 \rangle & \langle 0.30, 0.32, 0.11 \rangle & \langle 0.25, 0.35, 0.06 \rangle \\ s_2 & \langle 0.25, 0.35, 0.20 \rangle & \langle 0.30, 0.34, 0.10 \rangle & \langle 0.40, 0.42, 0.13 \rangle & \langle 0.32, 0.45, 0.20 \rangle \\ s_3 & \langle 0.26, 0.30, 0.25 \rangle & \langle 0.20, 0.32, 0.10 \rangle & \langle 0.32, 0.35, 0.20 \rangle & \langle 0.40, 0.42, 0.15 \rangle \\ s_4 & \langle 0.30, 0.35, 0.23 \rangle & \langle 0.30, 0.40, 0.12 \rangle & \langle 0.33, 0.37, 0.07 \rangle & \langle 0.20, 0.32, 0.12 \rangle \end{array} \end{array}$$

$$(N_3^*)_1 = \begin{array}{c} \begin{array}{cccc} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \langle 0.28, 0.32, 0.17 \rangle & \langle 0.40, 0.42, 0.12 \rangle & \langle 0.32, 0.42, 0.14 \rangle & \langle 0.40, 0.42, 0.10 \rangle \\ s_2 & \langle 0.21, 0.31, 0.10 \rangle & \langle 0.35, 0.38, 0.10 \rangle & \langle 0.28, 0.32, 0.15 \rangle & \langle 0.37, 0.40, 0.15 \rangle \\ s_3 & \langle 0.30, 0.34, 0.16 \rangle & \langle 0.42, 0.45, 0.15 \rangle & \langle 0.25, 0.30, 0.16 \rangle & \langle 0.20, 0.25, 0.17 \rangle \\ s_4 & \langle 0.27, 0.32, 0.20 \rangle & \langle 0.30, 0.35, 0.25 \rangle & \langle 0.21, 0.31, 0.15 \rangle & \langle 0.30, 0.35, 0.25 \rangle \end{array} \end{array}$$

$$(N_4^*)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.23, 0.32, 0.15 \rangle \\ \langle 0.33, 0.35, 0.45 \rangle \\ \langle 0.31, 0.35, 0.20 \rangle \\ \langle 0.25, 0.28, 0.65 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.25, 0.30, 0.19 \rangle \\ \langle 0.26, 0.34, 0.55 \rangle \\ \langle 0.29, 0.32, 0.17 \rangle \\ \langle 0.30, 0.34, 0.55 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.30, 0.32, 0.21 \rangle \\ \langle 0.40, 0.42, 0.45 \rangle \\ \langle 0.35, 0.40, 0.20 \rangle \\ \langle 0.32, 0.35, 0.50 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.34, 0.38, 0.43 \rangle \\ \langle 0.35, 0.38, 0.44 \rangle \\ \langle 0.40, 0.43, 0.13 \rangle \\ \langle 0.32, 0.33, 0.24 \rangle \end{bmatrix} \end{matrix}$$

$$(N_5^*)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.40, 0.45, 0.15 \rangle \\ \langle 0.34, 0.35, 0.25 \rangle \\ \langle 0.20, 0.25, 0.12 \rangle \\ \langle 0.23, 0.24, 0.25 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.30, 0.34, 0.20 \rangle \\ \langle 0.32, 0.35, 0.30 \rangle \\ \langle 0.20, 0.30, 0.13 \rangle \\ \langle 0.30, 0.40, 0.23 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.34, 0.40, 0.30 \rangle \\ \langle 0.32, 0.35, 0.45 \rangle \\ \langle 0.20, 0.25, 0.20 \rangle \\ \langle 0.25, 0.30, 0.24 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.25, 0.28, 0.15 \rangle \\ \langle 0.35, 0.38, 0.25 \rangle \\ \langle 0.45, 0.48, 0.05 \rangle \\ \langle 0.50, 0.52, 0.34 \rangle \end{bmatrix} \end{matrix}$$

Now, compute the risk level analysis Table for session 1 using Algorithm 6.2 and with the help of norms (Table 14) understand the risk level associated with the students as in Table 15.

TABLE 15. Shows students' mental health scores and levels for session 1

s_i	p_1		p_2		p_3		p_4		$Total$	
	score	level	score	level	score	level	score	level	score	level
s_1	19	high	18	high	19	high	18	high	74	high
s_2	20	high	19	high	19	high	19	high	77	high
s_3	20	high	18	high	19	high	17	high	74	high
s_4	21	vh	20	high	19	high	19	high	79	high

Likewise, form $(N_1^*)_2$, $(N_2^*)_2$, $(N_3^*)_2$, $(N_4^*)_2$ and $(N_5^*)_2$ *SVNSMs* for the second lockdown session.

$$(N_1^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.90, 0.85, 0.15 \rangle \\ \langle 0.35, 0.40, 0.15 \rangle \\ \langle 0.82, 0.78, 0.25 \rangle \\ \langle 0.85, 0.35, 0.18 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.55, 0.45, 0.20 \rangle \\ \langle 0.55, 0.45, 0.15 \rangle \\ \langle 0.82, 0.77, 0.25 \rangle \\ \langle 0.85, 0.35, 0.18 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.40, 0.60, 0.20 \rangle \\ \langle 0.45, 0.40, 0.15 \rangle \\ \langle 0.82, 0.75, 0.25 \rangle \\ \langle 0.85, 0.35, 0.18 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.88, 0.78, 0.20 \rangle \\ \langle 0.35, 0.20, 0.15 \rangle \\ \langle 0.82, 0.30, 0.25 \rangle \\ \langle 0.85, 0.35, 0.18 \rangle \end{bmatrix} \end{matrix}$$

$$(N_2^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.79, 0.69, 0.20 \rangle \\ \langle 0.64, 0.60, 0.13 \rangle \\ \langle 0.80, 0.55, 0.20 \rangle \\ \langle 0.82, 0.34, 0.20 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.58, 0.48, 0.17 \rangle \\ \langle 0.23, 0.16, 0.16 \rangle \\ \langle 0.81, 0.76, 0.23 \rangle \\ \langle 0.82, 0.33, 0.15 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.55, 0.33, 0.24 \rangle \\ \langle 0.66, 0.22, 0.21 \rangle \\ \langle 0.77, 0.87, 0.24 \rangle \\ \langle 0.83, 0.31, 0.19 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.65, 0.32, 0.24 \rangle \\ \langle 0.77, 0.24, 0.20 \rangle \\ \langle 0.80, 0.66, 0.24 \rangle \\ \langle 0.82, 0.36, 0.25 \rangle \end{bmatrix} \end{matrix}$$

$$(N_3^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.83, 0.54, 0.22 \rangle \\ \langle 0.68, 0.23, 0.18 \rangle \\ \langle 0.85, 0.69, 0.22 \rangle \\ \langle 0.87, 0.31, 0.19 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.56, 0.32, 0.20 \rangle \\ \langle 0.64, 0.19, 0.15 \rangle \\ \langle 0.84, 0.79, 0.25 \rangle \\ \langle 0.85, 0.30, 0.18 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.68, 0.54, 0.33 \rangle \\ \langle 0.77, 0.20, 0.24 \rangle \\ \langle 0.72, 0.88, 0.18 \rangle \\ \langle 0.82, 0.35, 0.25 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.70, 0.65, 0.24 \rangle \\ \langle 0.80, 0.30, 0.18 \rangle \\ \langle 0.75, 0.65, 0.24 \rangle \\ \langle 0.81, 0.31, 0.19 \rangle \end{bmatrix} \end{matrix}$$

$$(N_4^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.55, 0.45, 0.25 \rangle \\ \langle 0.60, 0.25, 0.20 \rangle \\ \langle 0.45, 0.40, 0.15 \rangle \\ \langle 0.45, 0.52, 0.20 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.65, 0.55, 0.15 \rangle \\ \langle 0.75, 0.15, 0.20 \rangle \\ \langle 0.85, 0.54, 0.30 \rangle \\ \langle 0.80, 0.30, 0.25 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.58, 0.48, 0.22 \rangle \\ \langle 0.86, 0.22, 0.18 \rangle \\ \langle 0.88, 0.66, 0.24 \rangle \\ \langle 0.82, 0.34, 0.20 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.77, 0.67, 0.22 \rangle \\ \langle 0.68, 0.22, 0.16 \rangle \\ \langle 0.88, 0.58, 0.24 \rangle \\ \langle 0.84, 0.34, 0.20 \rangle \end{bmatrix} \end{matrix}$$

$$(N_5^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.88, 0.30, 0.20 \rangle \\ \langle 0.68, 0.40, 0.26 \rangle \\ \langle 0.88, 0.78, 0.25 \rangle \\ \langle 0.84, 0.38, 0.24 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.77, 0.30, 0.28 \rangle \\ \langle 0.67, 0.28, 0.15 \rangle \\ \langle 0.87, 0.89, 0.25 \rangle \\ \langle 0.46, 0.35, 0.12 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.55, 0.30, 0.22 \rangle \\ \langle 0.60, 0.20, 0.18 \rangle \\ \langle 0.77, 0.68, 0.30 \rangle \\ \langle 0.87, 0.35, 0.28 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.60, 0.34, 0.26 \rangle \\ \langle 0.54, 0.67, 0.28 \rangle \\ \langle 0.84, 0.76, 0.25 \rangle \\ \langle 0.85, 0.38, 0.18 \rangle \end{bmatrix} \end{matrix}$$

Similarly, form $(N_1^*)_3$, $(N_2^*)_3$, $(N_3^*)_3$, $(N_4^*)_3$ and $(N_5^*)_3$ SVN SMs for the third lockdown session.

$$(N_1^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.70, 0.80, 0.25 \rangle \\ \langle 0.80, 0.70, 0.20 \rangle \\ \langle 0.82, 0.75, 0.21 \rangle \\ \langle 0.90, 0.88, 0.10 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.92, 0.89, 0.25 \rangle \\ \langle 0.88, 0.90, 0.25 \rangle \\ \langle 0.84, 0.88, 0.10 \rangle \\ \langle 0.88, 0.84, 0.12 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.90, 0.85, 0.40 \rangle \\ \langle 0.91, 0.88, 0.05 \rangle \\ \langle 0.75, 0.72, 0.14 \rangle \\ \langle 0.77, 0.72, 0.10 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.90, 0.80, 0.20 \rangle \\ \langle 0.69, 0.59, 0.40 \rangle \\ \langle 0.85, 0.75, 0.10 \rangle \\ \langle 0.76, 0.78, 0.25 \rangle \end{bmatrix} \end{matrix}$$

$$(N_2^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.71, 0.81, 0.20 \rangle \\ \langle 0.55, 0.50, 0.25 \rangle \\ \langle 0.80, 0.85, 0.21 \rangle \\ \langle 0.85, 0.88, 0.22 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.88, 0.87, 0.22 \rangle \\ \langle 0.50, 0.60, 0.26 \rangle \\ \langle 0.95, 0.88, 0.32 \rangle \\ \langle 0.72, 0.75, 0.42 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.77, 0.87, 0.10 \rangle \\ \langle 0.57, 0.57, 0.21 \rangle \\ \langle 0.67, 0.65, 0.05 \rangle \\ \langle 0.77, 0.71, 0.10 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.88, 0.85, 0.10 \rangle \\ \langle 0.91, 0.80, 0.15 \rangle \\ \langle 0.88, 0.98, 0.05 \rangle \\ \langle 0.90, 0.88, 0.15 \rangle \end{bmatrix} \end{matrix}$$

$$(N_3^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.67, 0.72, 0.29 \rangle \\ \langle 0.85, 0.75, 0.30 \rangle \\ \langle 0.70, 0.80, 0.25 \rangle \\ \langle 0.88, 0.91, 0.21 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.82, 0.88, 0.30 \rangle \\ \langle 0.86, 0.77, 0.32 \rangle \\ \langle 0.72, 0.82, 0.28 \rangle \\ \langle 0.77, 0.88, 0.38 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.84, 0.90, 0.25 \rangle \\ \langle 0.88, 0.81, 0.30 \rangle \\ \langle 0.77, 0.80, 0.24 \rangle \\ \langle 0.79, 0.85, 0.34 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.85, 0.82, 0.20 \rangle \\ \langle 0.90, 0.85, 0.31 \rangle \\ \langle 0.85, 0.72, 0.21 \rangle \\ \langle 0.84, 0.81, 0.30 \rangle \end{bmatrix} \end{matrix}$$

$$(N_4^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.72, 0.75, 0.25 \rangle \\ \langle 0.95, 0.82, 0.31 \rangle \\ \langle 0.82, 0.78, 0.21 \rangle \\ \langle 0.72, 0.68, 0.24 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.91, 0.73, 0.21 \rangle \\ \langle 0.92, 0.84, 0.30 \rangle \\ \langle 0.88, 0.79, 0.23 \rangle \\ \langle 0.78, 0.69, 0.22 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.94, 0.75, 0.28 \rangle \\ \langle 0.92, 0.85, 0.21 \rangle \\ \langle 0.90, 0.79, 0.19 \rangle \\ \langle 0.75, 0.70, 0.22 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.93, 0.77, 0.20 \rangle \\ \langle 0.89, 0.88, 0.15 \rangle \\ \langle 0.72, 0.82, 0.10 \rangle \\ \langle 0.79, 0.77, 0.09 \rangle \end{bmatrix} \end{matrix}$$

$$(N_5^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 0.73, 0.76, 0.10 \rangle \\ \langle 0.95, 0.90, 0.15 \rangle \\ \langle 0.85, 0.80, 0.20 \rangle \\ \langle 0.80, 0.75, 0.25 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.93, 0.83, 0.25 \rangle \\ \langle 0.94, 0.92, 0.44 \rangle \\ \langle 0.88, 0.82, 0.21 \rangle \\ \langle 0.89, 0.72, 0.19 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.89, 0.80, 0.10 \rangle \\ \langle 0.88, 0.82, 0.17 \rangle \\ \langle 0.92, 0.55, 0.19 \rangle \\ \langle 0.94, 0.88, 0.25 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.85, 0.80, 0.25 \rangle \\ \langle 0.89, 0.85, 0.20 \rangle \\ \langle 0.79, 0.89, 0.15 \rangle \\ \langle 0.90, 0.85, 0.25 \rangle \end{bmatrix} \end{matrix}$$

Step 4: Apply *SF* Definition 3.1, to get the values in matrices form for session 1.

$$S(N_1^*)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.215 & 0.235 & 0.190 & 0.385 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.265 & 0.250 & 0.260 & 0.275 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.260 & 0.325 & 0.355 & 0.200 \end{bmatrix} \\ s_4 & \begin{bmatrix} 0.160 & 0.275 & 0.225 & 0.295 \end{bmatrix} \end{matrix} S(N_2^*)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.125 & 0.335 & 0.255 & 0.270 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.200 & 0.270 & 0.345 & 0.285 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.155 & 0.210 & 0.235 & 0.335 \end{bmatrix} \\ s_4 & \begin{bmatrix} 0.210 & 0.290 & 0.315 & 0.200 \end{bmatrix} \end{matrix}$$

$$S(N_3^*)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.215 & 0.350 & 0.300 & 0.360 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.210 & 0.315 & 0.225 & 0.310 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.240 & 0.360 & 0.195 & 0.140 \end{bmatrix} \\ s_4 & \begin{bmatrix} 0.195 & 0.200 & 0.185 & 0.200 \end{bmatrix} \end{matrix} S(N_4^*)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.200 & 0.180 & 0.205 & 0.145 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.115 & 0.025 & 0.185 & 0.145 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.230 & 0.220 & 0.275 & 0.350 \end{bmatrix} \\ s_4 & \begin{bmatrix} -0.060 & 0.045 & 0.085 & 0.205 \end{bmatrix} \end{matrix}$$

$$S(N_5^*)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.350 & 0.220 & 0.220 & 0.190 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.220 & 0.185 & 0.110 & 0.240 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.165 & 0.185 & 0.125 & 0.440 \end{bmatrix} \\ s_4 & \begin{bmatrix} 0.110 & 0.238 & 0.155 & 0.340 \end{bmatrix} \end{matrix}$$

Apply *SF* Definition 3.1, for session 2.

$$S(N_1^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.800 & 0.400 & 0.400 & 0.730 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.300 & 0.425 & 0.350 & 0.200 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.675 & 0.670 & 0.660 & 0.435 \end{bmatrix} \\ s_4 & \begin{bmatrix} 0.510 & 0.510 & 0.510 & 0.510 \end{bmatrix} \end{matrix} S(N_2^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.640 & 0.445 & 0.320 & 0.365 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.555 & 0.115 & 0.335 & 0.405 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.575 & 0.670 & 0.700 & 0.610 \end{bmatrix} \\ s_4 & \begin{bmatrix} 0.480 & 0.500 & 0.475 & 0.465 \end{bmatrix} \end{matrix}$$

$$S(N_3^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.575 & 0.340 & 0.445 & 0.555 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.365 & 0.340 & 0.366 & 0.460 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.660 & 0.690 & 0.710 & 0.580 \end{bmatrix} \\ s_4 & \begin{bmatrix} 0.495 & 0.485 & 0.460 & 0.465 \end{bmatrix} \end{matrix} S(N_4^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.375 & 0.525 & 0.420 & 0.610 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.325 & 0.350 & 0.450 & 0.370 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.350 & 0.545 & 0.650 & 0.610 \end{bmatrix} \\ s_4 & \begin{bmatrix} 0.385 & 0.425 & 0.480 & 0.490 \end{bmatrix} \end{matrix}$$

$$S(N_5^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.490 & 0.395 & 0.315 & 0.340 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.410 & 0.400 & 0.310 & 0.465 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.705 & 0.755 & 0.575 & 0.675 \end{bmatrix} \\ s_4 & \begin{bmatrix} 0.490 & 0.345 & 0.470 & 0.525 \end{bmatrix} \end{matrix}$$

Similarly, apply *SF* Definition 3.1, for session 3.

$$S(N_1^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.625 & 0.780 & 0.675 & 0.750 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.650 & 0.765 & 0.870 & 0.440 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.680 & 0.810 & 0.665 & 0.750 \end{bmatrix} \\ s_4 & \begin{bmatrix} 0.840 & 0.800 & 0.695 & 0.645 \end{bmatrix} \end{matrix} S(N_2^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ s_1 & \begin{bmatrix} 0.660 & 0.765 & 0.770 & 0.815 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0.400 & 0.420 & 0.465 & 0.780 \end{bmatrix} \\ s_3 & \begin{bmatrix} 0.720 & 0.755 & 0.635 & 0.905 \end{bmatrix} \\ s_4 & \begin{bmatrix} 0.755 & 0.525 & 0.690 & 0.815 \end{bmatrix} \end{matrix}$$

$$S(N_3^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0.550 & 0.700 & 0.745 & 0.735 \\ 0.650 & 0.655 & 0.695 & 0.720 \\ 0.625 & 0.630 & 0.665 & 0.680 \\ 0.790 & 0.635 & 0.650 & 0.675 \end{bmatrix} \end{matrix} S(N_4^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0.610 & 0.715 & 0.705 & 0.750 \\ 0.730 & 0.730 & 0.780 & 0.810 \\ 0.695 & 0.720 & 0.750 & 0.720 \\ 0.580 & 0.625 & 0.615 & 0.735 \end{bmatrix} \end{matrix}$$

$$S(N_5^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0.695 & 0.755 & 0.795 & 0.700 \\ 0.850 & 0.710 & 0.765 & 0.770 \\ 0.725 & 0.745 & 0.640 & 0.765 \\ 0.650 & 0.710 & 0.785 & 0.750 \end{bmatrix} \end{matrix}$$

Step 5: Let's consider the information from sessions 1 to 3 to pre-assess the mental illness of the students before the next lock-down session begins. By applying Definition 8.3, we get the following matrices. The $QHDISVNNSM$ at the beginning of session 1 is computed by

$$N_r^*(5)_1 = \left\langle g_{p_{ij}}, \frac{1}{3} \left[(s_{ij})_1 + 0.5 \left(\sum_{t=1}^2 0.9^t \cdot (s_{ij})_{1+t} \right) \right] \right\rangle,$$

$r = 1, 2, \dots, 5$, $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$.

$$N_1^*(5)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 4, 0.276 \rangle & \langle 4, 0.244 \rangle & \langle 4, 0.214 \rangle & \langle 3, 0.339 \rangle \\ \langle 4, 0.221 \rangle & \langle 4, 0.250 \rangle & \langle 4, 0.257 \rangle & \langle 4, 0.181 \rangle \\ \langle 4, 0.280 \rangle & \langle 3, 0.318 \rangle & \langle 3, 0.307 \rangle & \langle 4, 0.233 \rangle \\ \langle 4, 0.243 \rangle & \langle 4, 0.276 \rangle & \langle 4, 0.245 \rangle & \langle 4, 0.262 \rangle \end{bmatrix} \end{matrix}$$

$$N_2^*(5)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 4, 0.227 \rangle & \langle 4, 0.282 \rangle & \langle 4, 0.237 \rangle & \langle 4, 0.255 \rangle \\ \langle 4, 0.204 \rangle & \langle 4, 0.164 \rangle & \langle 4, 0.228 \rangle & \langle 4, 0.261 \rangle \\ \langle 4, 0.235 \rangle & \langle 4, 0.272 \rangle & \langle 4, 0.269 \rangle & \langle 3, 0.325 \rangle \\ \langle 4, 0.244 \rangle & \langle 4, 0.243 \rangle & \langle 4, 0.269 \rangle & \langle 4, 0.246 \rangle \end{bmatrix} \end{matrix}$$

$$N_3^*(5)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 4, 0.232 \rangle & \langle 4, 0.262 \rangle & \langle 4, 0.267 \rangle & \langle 3, 0.302 \rangle \\ \langle 4, 0.213 \rangle & \langle 4, 0.244 \rangle & \langle 4, 0.224 \rangle & \langle 4, 0.270 \rangle \\ \langle 4, 0.263 \rangle & \langle 3, 0.309 \rangle & \langle 4, 0.261 \rangle & \langle 4, 0.225 \rangle \\ \langle 4, 0.246 \rangle & \langle 4, 0.225 \rangle & \langle 4, 0.218 \rangle & \langle 4, 0.228 \rangle \end{bmatrix} \end{matrix}$$

$$N_4^*(5)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 4, 0.205 \rangle & \langle 4, 0.235 \rangle & \langle 4, 0.227 \rangle & \langle 4, 0.241 \rangle \\ \langle 4, 0.186 \rangle & \langle 4, 0.159 \rangle & \langle 4, 0.234 \rangle & \langle 4, 0.213 \rangle \\ \langle 4, 0.223 \rangle & \langle 4, 0.252 \rangle & \langle 4, 0.290 \rangle & \langle 3, 0.305 \rangle \\ \langle 4, 0.116 \rangle & \langle 4, 0.163 \rangle & \langle 4, 0.183 \rangle & \langle 4, 0.241 \rangle \end{bmatrix} \end{matrix}$$

$$N_5^*(5)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 4, 0.284 \rangle \\ \langle 4, 0.250 \rangle \\ \langle 4, 0.259 \rangle \\ \langle 4, 0.198 \rangle \end{bmatrix} & \begin{bmatrix} \langle 4, 0.235 \rangle \\ \langle 4, 0.218 \rangle \\ \langle 4, 0.275 \rangle \\ \langle 4, 0.227 \rangle \end{bmatrix} & \begin{bmatrix} \langle 4, 0.228 \rangle \\ \langle 4, 0.186 \rangle \\ \langle 4, 0.214 \rangle \\ \langle 4, 0.228 \rangle \end{bmatrix} & \begin{bmatrix} \langle 4, 0.209 \rangle \\ \langle 4, 0.254 \rangle \\ \langle 3, 0.351 \rangle \\ \langle 4, 0.293 \rangle \end{bmatrix} \end{matrix}$$

By applying Definition 8.3, we get the following matrices for sessions 2 to 3. The $QHDISVNN\mathcal{SM}$ at the beginning of session 2 is computed by

$$N_r^*(5)_2 = \left\langle g_{p_{ij}}, \frac{1}{2} [(s_{ij})_2 + 0.5(0.9^1).(s_{ij})_3] \right\rangle,$$

$r = 1, 2, \dots, 5$, $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$.

$$N_1^*(5)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 3, 0.541 \rangle \\ \langle 4, 0.296 \rangle \\ \langle 3, 0.491 \rangle \\ \langle 3, 0.444 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.376 \rangle \\ \langle 3, 0.385 \rangle \\ \langle 3, 0.517 \rangle \\ \langle 3, 0.435 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.352 \rangle \\ \langle 3, 0.371 \rangle \\ \langle 3, 0.480 \rangle \\ \langle 3, 0.411 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.534 \rangle \\ \langle 4, 0.199 \rangle \\ \langle 3, 0.386 \rangle \\ \langle 3, 0.400 \rangle \end{bmatrix} \end{matrix}$$

$$N_2^*(5)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 3, 0.469 \rangle \\ \langle 3, 0.368 \rangle \\ \langle 3, 0.450 \rangle \\ \langle 3, 0.410 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.395 \rangle \\ \langle 4, 0.152 \rangle \\ \langle 3, 0.505 \rangle \\ \langle 3, 0.368 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.333 \rangle \\ \langle 4, 0.272 \rangle \\ \langle 3, 0.493 \rangle \\ \langle 3, 0.393 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.366 \rangle \\ \langle 3, 0.378 \rangle \\ \langle 3, 0.509 \rangle \\ \langle 3, 0.416 \rangle \end{bmatrix} \end{matrix}$$

$$N_3^*(5)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 3, 0.411 \rangle \\ \langle 3, 0.329 \rangle \\ \langle 3, 0.471 \rangle \\ \langle 3, 0.425 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.328 \rangle \\ \langle 3, 0.317 \rangle \\ \langle 3, 0.487 \rangle \\ \langle 3, 0.385 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.390 \rangle \\ \langle 3, 0.339 \rangle \\ \langle 3, 0.505 \rangle \\ \langle 3, 0.376 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.443 \rangle \\ \langle 3, 0.392 \rangle \\ \langle 3, 0.443 \rangle \\ \langle 3, 0.384 \rangle \end{bmatrix} \end{matrix}$$

$$N_4^*(5)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 3, 0.325 \rangle \\ \langle 3, 0.327 \rangle \\ \langle 3, 0.331 \rangle \\ \langle 3, 0.323 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.423 \rangle \\ \langle 3, 0.339 \rangle \\ \langle 3, 0.435 \rangle \\ \langle 3, 0.353 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.369 \rangle \\ \langle 3, 0.401 \rangle \\ \langle 3, 0.494 \rangle \\ \langle 3, 0.378 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.474 \rangle \\ \langle 3, 0.367 \rangle \\ \langle 3, 0.467 \rangle \\ \langle 3, 0.410 \rangle \end{bmatrix} \end{matrix}$$

$$N_5^*(5)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} \langle 3, 0.401 \rangle \\ \langle 3, 0.396 \rangle \\ \langle 3, 0.516 \rangle \\ \langle 3, 0.391 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.367 \rangle \\ \langle 3, 0.360 \rangle \\ \langle 3, 0.545 \rangle \\ \langle 3, 0.332 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.336 \rangle \\ \langle 3, 0.327 \rangle \\ \langle 3, 0.432 \rangle \\ \langle 3, 0.412 \rangle \end{bmatrix} & \begin{bmatrix} \langle 3, 0.328 \rangle \\ \langle 3, 0.406 \rangle \\ \langle 3, 0.510 \rangle \\ \langle 3, 0.431 \rangle \end{bmatrix} \end{matrix}$$

By applying Definition 8.3, we get the following matrices for session 3. The $QHDISVNN\mathcal{SM}$ at the beginning of session 3 is computed by

$$N_r^*(5)_3 = \langle g_{p_{ij}}, (s_{ij})_3 \rangle,$$

$r = 1, 2, \dots, 5$, $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$.

$$N_1^*(5)_3 = \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \left[\begin{array}{cccc} \langle 2, 0.625 \rangle & \langle 2, 0.780 \rangle & \langle 2, 0.675 \rangle & \langle 2, 0.750 \rangle \\ \langle 2, 0.650 \rangle & \langle 2, 0.765 \rangle & \langle 1, 0.870 \rangle & \langle 3, 0.440 \rangle \\ \langle 2, 0.680 \rangle & \langle 1, 0.810 \rangle & \langle 2, 0.665 \rangle & \langle 2, 0.750 \rangle \\ \langle 1, 0.840 \rangle & \langle 1, 0.800 \rangle & \langle 2, 0.695 \rangle & \langle 2, 0.645 \rangle \end{array} \right] \end{array}$$

$$N_2^*(5)_3 = \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \left[\begin{array}{cccc} \langle 2, 0.660 \rangle & \langle 2, 0.765 \rangle & \langle 2, 0.770 \rangle & \langle 1, 0.815 \rangle \\ \langle 3, 0.400 \rangle & \langle 3, 0.420 \rangle & \langle 3, 0.465 \rangle & \langle 2, 0.780 \rangle \\ \langle 2, 0.720 \rangle & \langle 2, 0.755 \rangle & \langle 2, 0.635 \rangle & \langle 1, 0.905 \rangle \\ \langle 2, 0.755 \rangle & \langle 3, 0.525 \rangle & \langle 2, 0.690 \rangle & \langle 1, 0.815 \rangle \end{array} \right] \end{array}$$

$$N_3^*(5)_3 = \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \left[\begin{array}{cccc} \langle 3, 0.550 \rangle & \langle 2, 0.700 \rangle & \langle 2, 0.745 \rangle & \langle 2, 0.735 \rangle \\ \langle 2, 0.650 \rangle & \langle 2, 0.655 \rangle & \langle 2, 0.695 \rangle & \langle 2, 0.720 \rangle \\ \langle 2, 0.625 \rangle & \langle 2, 0.630 \rangle & \langle 2, 0.665 \rangle & \langle 2, 0.680 \rangle \\ \langle 2, 0.790 \rangle & \langle 2, 0.635 \rangle & \langle 2, 0.650 \rangle & \langle 2, 0.675 \rangle \end{array} \right] \end{array}$$

$$N_4^*(5)_3 = \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \left[\begin{array}{cccc} \langle 2, 0.610 \rangle & \langle 2, 0.715 \rangle & \langle 2, 0.705 \rangle & \langle 2, 0.750 \rangle \\ \langle 2, 0.730 \rangle & \langle 2, 0.730 \rangle & \langle 2, 0.780 \rangle & \langle 1, 0.810 \rangle \\ \langle 2, 0.695 \rangle & \langle 2, 0.720 \rangle & \langle 2, 0.750 \rangle & \langle 2, 0.720 \rangle \\ \langle 3, 0.580 \rangle & \langle 2, 0.625 \rangle & \langle 2, 0.615 \rangle & \langle 2, 0.735 \rangle \end{array} \right] \end{array}$$

$$N_5^*(5)_3 = \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \left[\begin{array}{cccc} \langle 2, 0.695 \rangle & \langle 2, 0.755 \rangle & \langle 2, 0.795 \rangle & \langle 2, 0.700 \rangle \\ \langle 1, 0.850 \rangle & \langle 2, 0.710 \rangle & \langle 2, 0.765 \rangle & \langle 2, 0.770 \rangle \\ \langle 2, 0.725 \rangle & \langle 2, 0.745 \rangle & \langle 2, 0.640 \rangle & \langle 2, 0.765 \rangle \\ \langle 2, 0.650 \rangle & \langle 2, 0.710 \rangle & \langle 2, 0.785 \rangle & \langle 2, 0.750 \rangle \end{array} \right] \end{array}$$

Step 6: Determine $N_+^*(5)_1$ matrix by summing the corresponding entries of $N_1^*(5)_1, N_2^*(5)_1, \dots, N_5^*(5)_1$ matrices.

$$N_+^*(5)_1 = \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \left[\begin{array}{cccc} 20 & 20 & 20 & 18 \\ 20 & 20 & 20 & 20 \\ 20 & 18 & 19 & 17 \\ 20 & 20 & 20 & 20 \end{array} \right] \begin{array}{c} 78 \\ 80 \\ 74 \\ 80 \end{array} \end{array}$$

For $N_+^*(5)_2$ matrix by summing the corresponding entries of $N_1^*(5)_2, N_2^*(5)_2, \dots, N_5^*(5)_2$ matrices.

$$N_+^*(5)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 15 & 15 & 15 & 15 \\ 16 & 16 & 16 & 16 \\ 15 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \end{bmatrix} & \begin{matrix} 60 \\ 64 \\ 60 \\ 60 \end{matrix} \end{matrix}$$

Similarly, form $N_+^*(5)_3$ matrix by summing the corresponding entries of $N_1^*(5)_3, N_2^*(5)_3, \dots, N_5^*(5)_3$ matrices.

$$N_+^*(5)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 11 & 10 & 10 & 9 \\ 10 & 11 & 10 & 10 \\ 10 & 9 & 10 & 9 \\ 10 & 10 & 10 & 9 \end{bmatrix} & \begin{matrix} 40 \\ 41 \\ 38 \\ 39 \end{matrix} \end{matrix}$$

Step 7: Tabulate the details as in Table 16 and pre-assess the risk level of the students during the lockdown sessions 1 to 3 .

TABLE 16. Pre-assessing students' mental illness from sessions 1 to 3

s_i	p_1		p_2		p_3		p_4		$Total$	
	score	level	score	level	score	level	score	level	score	level
s_1	20	high	20	high	20	high	18	high	78	high
s_2	20	high	20	high	20	high	20	high	80	high
s_3	20	high	18	high	19	high	17	high	74	high
s_4	20	high	20	high	20	high	20	high	80	high

Tabulate as in Table 17 and pre-assess the risk level of the students during the lock-down sessions 2 to 3.

TABLE 17. Pre-assessing students' mental health illness from sessions 2 and 3

s_i	p_1		p_2		p_3		p_4		$Total$	
	score	level	score	level	score	level	score	level	score	level
s_1	15	bor	15	bor	15	bor	15	bor	60	bor
s_2	16	high	16	high	16	high	16	high	64	high
s_3	15	bor	15	bor	15	bor	15	bor	60	bor
s_4	15	bor	15	bor	15	bor	15	bor	60	bor

Similarly, tabulate the details as in Table 18 and pre-assess the risk level of the students for the lock-down session 3.

Analysis: When we examine the total (last column) in Table 15, we understand that everyone in the group establishes a high risk-level towards mental health. Similarly, when we analyze the total data in Table 16, the students show a high risk-level in sessions 1 to 3. The reason is that $QHDF$ enhances the effect of value in the first session and decreases the

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TABLE 18. Pre-assessing students' mental health illness for session 3

s_i	p_1		p_2		p_3		p_4		$Total$	
	score	level	score	level	score	level	score	level	score	level
s_1	11	bor	10	mod	10	mod	9	mod	40	mod
s_2	10	mod	11	bor	10	mod	10	mod	41	bor
s_3	10	mod	9	mod	10	mod	9	mod	38	mod
s_4	10	mod	10	mod	10	mod	9	mod	39	mod

influence of value in the second and third sessions. Likewise, when we analyze the total data in Table 17, except s_2 , all others show a borderline risk of mental illness in sessions 2 and 3. The student s_2 is yet to subdue from high risk-level. The reason is the same as before. *QHDF* enhances the effect of value in the second session and decreases the influence of value in the third session. Hence, from the above observation, we conclude that in session 2, except for the student s_2 , all other students are in the borderline stage, and s_2 still shows a high-risk level towards mental health illness. Also, we state that the students s_1 , s_3 , and s_4 have reached the borderline stage from high-level mental health illness. On a similar note, when we analyze the total data in Table 18, except s_2 , all others show a moderate risk of mental illness in session 3. We infer from the data that students s_1 , s_3 , and s_4 have reached the borderline stage (session 3) from high-level mental health illness (session 1). This method helps the psychiatrist to understand the risk level during a longitudinal study when there are n counseling sessions. Also, this result provokes to follow an alternative remedy process to lower the risk level from the forthcoming session for the student s_2 .

11. Comparison and Significance of *QHDISVNN \mathcal{S} M*

This section will focus on the significance of *QHDISVNN \mathcal{S} M*. Since, this method is new and cannot be compared with existing methods, we choose a simple average method to show the superiority of *QHDISVNN \mathcal{S} M*.

Consider a scenario where the students have to undergo a total of four counseling sessions. Let's assume that *MHC* would like to pre-assess the mental health of the students before the completion of the last session. Here, the *MHC* wishes to pre-assess the students after the completion of the third session. The rating scale distribution and norms be as in Tables 19 and 20.

Step 1: Suppose that $\mathcal{U} = \{s_1, s_2\}$ be the set of students who suffer from mental illness and $\mathcal{P} = \{p_1, p_2, p_3, p_4, p_5\}$ be the set of parameters. Let *MHC* observes the behavior of each student based on a framed positive statement and provide the values in *SVNSMs* form, $(N_1^*)_1$ for the first session.

TABLE 19. Shows the rating scale distribution

Positive statement	Negative statement	Score values
5	1	$0.8 \leq s_{ij} \leq 1.0$
4	2	$0.6 \leq s_{ij} < 0.8$
3	3	$0.3 \leq s_{ij} < 0.6$
2	4	$0.0 \leq s_{ij} < 0.3$
1	5	$-0.5 \leq s_{ij} < 0.0$

TABLE 20. Shows the qualitative norm details

Parameter	Scores	Norms
p_1, p_2, p_3, p_4, p_5	1	low
	2-3	average (avg)
	4-5	high
Total	1-15	low
	16-20	avg
	21-25	high

$$(N_1^*)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \begin{matrix} s_1 \\ s_2 \end{matrix} & \begin{bmatrix} \langle 0.98, 0.76, 0.05 \rangle \\ \langle 0.92, 0.95, 0.17 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.88, 0.77, 0.06 \rangle \\ \langle 0.80, 0.95, 0.12 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.83, 0.69, 0.10 \rangle \\ \langle 0.90, 0.95, 0.25 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.78, 0.88, 0.43 \rangle \\ \langle 0.87, 0.83, 0.24 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.78, 0.83, 0.15 \rangle \\ \langle 0.88, 0.86, 0.34 \rangle \end{bmatrix} \end{matrix}$$

The *MHC* provides the value in *SVNSMs* form, $(N_1^*)_2$ for the second session.

$$(N_1^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \begin{matrix} s_1 \\ s_2 \end{matrix} & \begin{bmatrix} \langle 0.90, 0.80, 0.20 \rangle \\ \langle 0.76, 0.78, 0.25 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.88, 0.85, 0.10 \rangle \\ \langle 0.90, 0.88, 0.15 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.85, 0.82, 0.20 \rangle \\ \langle 0.84, 0.81, 0.30 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.93, 0.77, 0.20 \rangle \\ \langle 0.79, 0.77, 0.09 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.85, 0.80, 0.25 \rangle \\ \langle 0.90, 0.85, 0.25 \rangle \end{bmatrix} \end{matrix}$$

The *MHC* provides the value in *SVNSMs* form, $(N_1^*)_3$ for the third session.

$$(N_1^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \begin{matrix} s_1 \\ s_2 \end{matrix} & \begin{bmatrix} \langle 0.88, 0.78, 0.20 \rangle \\ \langle 0.85, 0.35, 0.18 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.65, 0.32, 0.24 \rangle \\ \langle 0.82, 0.36, 0.25 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.70, 0.65, 0.24 \rangle \\ \langle 0.81, 0.31, 0.19 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.77, 0.67, 0.22 \rangle \\ \langle 0.84, 0.34, 0.20 \rangle \end{bmatrix} & \begin{bmatrix} \langle 0.60, 0.34, 0.26 \rangle \\ \langle 0.85, 0.38, 0.18 \rangle \end{bmatrix} \end{matrix}$$

Step 2: Apply *SF* Definition 3.1, to get $S(N_1^*)_1$, $S(N_1^*)_2$ and $S(N_1^*)_3$ in matrices form for sessions 1, 2 and 3.

$$S(N_1^*)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \begin{matrix} s_1 \\ s_2 \end{matrix} & \begin{bmatrix} 0.845 & 0.795 & 0.710 & 0.615 & 0.730 \\ 0.850 & 0.815 & 0.800 & 0.730 & 0.700 \end{bmatrix} \end{matrix}$$

$$S(N_1^*)_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \begin{matrix} s_1 \\ s_2 \end{matrix} & \begin{bmatrix} 0.750 & 0.815 & 0.735 & 0.750 & 0.700 \\ 0.645 & 0.815 & 0.675 & 0.735 & 0.750 \end{bmatrix} \end{matrix}$$

$$S(N_1^*)_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \begin{matrix} s_1 \\ s_2 \end{matrix} & \begin{bmatrix} 0.730 & 0.365 & 0.555 & 0.610 & 0.340 \\ 0.510 & 0.465 & 0.465 & 0.490 & 0.525 \end{bmatrix} \end{matrix}$$

Step 3: Now, let's construct the \mathcal{AN}_1^* matrices by computing the average of each corresponding entries in $S(N_1^*)_1$, $S(N_1^*)_2$ and $S(N_1^*)_3$ matrices.

$$\mathcal{AN}_1^* = \begin{matrix} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \begin{matrix} s_1 \\ s_2 \end{matrix} & \begin{bmatrix} 0.775 & 0.658 & 0.667 & 0.658 & 0.590 \\ 0.668 & 0.698 & 0.647 & 0.652 & 0.658 \end{bmatrix} \end{matrix}$$

Step 4: Apply Definition 5.3, to determine the rating scale for each entry and represent the resultant matrix as $\mathcal{AN}_1^*(5)$.

$$\mathcal{AN}_1^*(5) = \begin{matrix} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \begin{matrix} s_1 \\ s_2 \end{matrix} & \begin{bmatrix} \langle 4, 0.775 \rangle & \langle 4, 0.658 \rangle & \langle 4, 0.667 \rangle & \langle 4, 0.658 \rangle & \langle 3, 0.590 \rangle \\ \langle 4, 0.668 \rangle & \langle 4, 0.698 \rangle & \langle 4, 0.647 \rangle & \langle 4, 0.652 \rangle & \langle 4, 0.658 \rangle \end{bmatrix} \end{matrix}$$

Step 5: Tabulate the details as in Table 21 and pre-assess the risk level of the students during the sessions 1 to 3 by using the norms (Table 20).

TABLE 21. Pre-assessing students' mental health scores and levels by taking average of sessions 1 to 3

s_i	p_1		p_2		p_3		p_4		p_5		$Total$	
	score	level	score	level	score	level	score	level	score	level	score	level
s_1	4	high	4	high	4	high	4	high	3	avg	19	avg
s_2	4	high	4	high	4	high	4	high	4	high	20	avg

Now, let's compute the $QHDISVNN\mathcal{SM}$ at the beginning of session 1. Steps 1 and 2 remain the same as above.

Step 3: By applying $QHDISVNN\mathcal{SM}$ Definition 8.3, for the corresponding entries in $S(N_1^*)_1$, $S(N_1^*)_2$ and $S(N_1^*)_3$ matrices, we get $N_1^*(5)_1$ matrix for sessions 1 to 3.

$$N_1^*(5)_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \begin{matrix} s_1 \\ s_2 \end{matrix} & \begin{bmatrix} \langle 3, 0.493 \rangle & \langle 3, 0.437 \rangle & \langle 3, 0.422 \rangle & \langle 3, 0.400 \rangle & \langle 3, 0.394 \rangle \\ \langle 3, 0.449 \rangle & \langle 3, 0.457 \rangle & \langle 3, 0.431 \rangle & \langle 3, 0.420 \rangle & \langle 3, 0.417 \rangle \end{bmatrix} \end{matrix}$$

Step 4: Tabulate the details as in Table 22 and pre-assess the risk level of the students during the sessions 1 to 3 .

TABLE 22. Pre-assessing students' mental health scores and levels by using *QHDISVNN \mathcal{S} M* from sessions 1 to 3

s_i	p_1		p_2		p_3		p_4		p_5		$Total$	
	score	level	score	level	score	level	score	level	score	level	score	level
s_1	3	avg	3	avg	3	avg	3	avg	3	avg	15	low
s_2	3	avg	3	avg	3	avg	3	avg	3	avg	15	low

Analysis: From Tables 21 and 22, we infer that the risk levels are different for the same values. When we use a simple average approach, the overall risk levels for the students are high. Similarly, when we implement the *QHDISVNN \mathcal{S} M*, the risk levels are low for the students. The reason being, in the former approach, we derive the average of the *SF* values, whereas, in the latter method, the computation approach is intermittent. Hence, the risk levels are different in each of the discussed methods. The *MHC* may take a discussion based on the results of immediate effect. Thus the *QHDISVNN \mathcal{S} M* proves to be significant than the simple average approach.

12. Limitations, Conclusion and Future works

The following are the limitations of the proposed research work: i) May require a qualified mental health counselor, therapist, psychiatrist to execute the case studies. ii) When we involve over one psychiatrist in examining the students, the risk of understanding the uncertainty information may lead to different remedy process. iii) Negative preferences for psychological applications. iv) There may be areas of ambiguity that test results do not reflect, even after comprehensive research because of students cautiousness.

Smarandache [71] presented the concept of neutrosophic to determine the vagueness associated with actions, memory, and temperaments of humans. Christianto and Smarandache [72] analyzed cultural psychology as one of the seven philosophical aspects by using neutrosophic theory. To find the hidden patterns in psychological models, Farahani et al. [73] developed a case study on mental health disorders. They compared the combined overlap block of fuzzy cognitive maps and neutrosophic cognitive maps to find out the hidden patterns. In most of the current psychological applications, we come across only a limited range of neutrosophic theoretical principles and methods. Most of such applications merely use membership classes, usually in combination with prototypes and product similarity measures. We have a scarcity of neutrosophic theories in the psychology field but may soon find a wide range of ways to make use of neutrosophic constructs in their pursuits. There are situations where psychologists appeal to vagueness have not progressed far beyond the theoretical level. In this study, we provide a suitable workaround to two critical issues, which represent a barrier for the domination of neutrosophic theory in psychology. i) Most of the psychological studies deal with Chinnadurai and Bobin, Applications to assess and pre-assess the mental health of students

questionnaires, and psychiatrists would like to follow the traditional method of handling scale construction and classical test-theory to access the conditions. But, considering the ambiguity conditions, it is not advisable to capture the information with raw data to analyze the vagueness associated with psychological aspects. ii) Also, psychiatrists would like to record the data and analyze the change in behavior based on the treatment given for each session. We present solutions to these arguments by using a blend of *SVNSS*, *NSS*, and *QHDF*. By applying the concept of *SVNNSS* and *QHDSVNNSS*, we can easily relate these theoretical theories to the neutrosophic group. These concepts support the psychiatrists to capture the information using neutrosophic and follow the rating method. *SVNNSS* helps the psychiatrists to use their traditional scoring method (positive and negative scoring keys). During the decision-making process, we consider the immediate influence of human action to decide on the consequences more accurately.

We may extend these notions to other fuzzy hybrid sets and determine the importance of the same with a real-life case study. Also, we may prepare a questionnaire with the support of a pilot study and try to pre-assess or assess the students psychosocial behavior during the pandemic.

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On Neutrosophic Mapping

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Abstract. In this paper some elementary types of neutrosophic functions and their inverse functions are defined based on Smarandache's definition. Also composition of two neutrosophic functions is introduced and some elementary theorems on them are developed.

Keywords: Neutro function; Neutro inverse function; Neutro composition mapping

1. Introduction

The importance of neutrosophic set invented by Smarandache [1] is increasing rapidly due to its characteristic inherent in its definition itself. It has acquired in latest years in extensive applicational areas [2–9]. Recently it is applied in professional selection [10], supply chain problem [11], evaluation of manufacturer industry [12], smart product-service system [13] (Mohammed Abdel Basset et al., 2020). Even neutrosophic set is included in the research-arena of algebra [14–16], calculus [17], topology [18–22] etc.

All laws in the world are not deterministic. So, the axioms need to be more flexible to cope up with our dynamic world. Neutrosophic algebraic and N algebraic structure, notion of groups, N-groups, semigroups, N-semigroups, N-loops etc. were discussed in [23] by Kandasamy et al. The author (Kandasamy et al.) also studied about neutrosophic rings [24]. Popular research papers are on neutrosophic groups, subgroups [25], neutrosophic rings [27, 28], hyper groups [26] and hyper rings [28]. In the context of neutrosophic theory N-bi-ideal [29] was discussed in semigroup (Porselvi et al.). In BCI/BCK-algebras BMBJ-neutrosophic subalgebras as well as their related properties were studied in [30] by H. Bordbar et al. G.R. Rezaei et al. [31] investigated about neutrosophic quadruple a-ideal. The idea of neutrosophic lattice ideals and LI-ideals [32] was introduced by Rajab Ali Borzooei et al.

In this way in algebra on various topics there are different types of researches based on neutrosophic theory. But on mapping, which has an significant role, a few research are there. In classical algebra the axioms characterized on a set are well defined. Yet there are various circumstances in science and in space of information with a maxim characterized on a set where algebraic axioms are partially followed. Herein lies the importance of neutro axioms. Smarandache introduced neutroalgebraic, antialgebraic structure in [33] and discussed the importance of neutro-axiom. Also the author [34] extended neutro algebra as a generalization of partial algebra and defined neutrosophic function. In this article first time different types of netro functions- one-one, onto, bijective neutro mapping, their composition and inverse neutro mapping are defined. Based on these definitions, some elementary theorems are also developed.

2. Preliminaries

2.1. Neutrosophic set [1]

Let U be an universe of discourse, then the neutrosophic set A is defined as $A = \{\langle x : T_A(x), I_A(x), F_A(x) \rangle, x \in U\}$, where the functions $T, I, F: U \rightarrow]-0, 1^+[$ define respectively the degree of membership (or Truth), the degree of indeterminacy and the degree of non-membership (or falsehood) of the element $x \in U$ to the set A with the condition $-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

2.2. Neutro Function [34]

A function $f : X \rightarrow Y$ is called a Neutro function if it has elements in X for which the function is well-defined, elements in X for which the function is indeterminate, and elements in X for which the function is outer-defined.

3. Physical implication of the research work

In this section let us consider some practical examples which indicate the implication of the proposed discussion.

Example 1. In case of radio active decaying element, among the disintegrating atoms of radio active element, some may have only a short existence, while others may remain unchanged for a long time- why we don't know. The disintegration of a particular atom is a chance incident, however, only half life period can be conveniently represented. Now if we want to make a relation between atoms of an radioactive element and its decay within half life period, mapping can not be defined rather a neutro function can be established.

Example 2. Suppose $(ct, c/t)$ denotes the position of a moving particle at any time t . Then the mapping can not be defined from $[0, T]$ (representing time interval) to its corresponding position. Because at $t = 0$, it is undefined; though it is a neutro mapping.

Example 3. Suppose a coin is tossed. Let us think a rule of correspondence $f(Head) = 1$, $f(Tail) = 0$. But it may happen that the coin is stucked at a split and it is neither head nor tail. Here also mapping is not defined., however neutro function is accepted.

In this way there are so many practical events where functions are not defined to give a rule for making relations, whereas neutro functions can be defined. Now based on the concepts of neutro function's definition and their variation, the idea of different types of functions and their elementary properties are introduced.

4. some proposed basic definitions

4.1. Neutro one-one function

A function $f : X \rightarrow Y$ is called a neutro one-one function if for each well defined pair of distinct elements of X , their f images are distinct or it has elements in X for which the function is undefined.

Consider $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $X = \{1, 2, 3, 4, 5, 6\}$, $Y = \{7, 8, 9, 10, 11\}$, $f(1) = 7$, $f(2) = 8$, $f(3) = 9$, $f(4) = \text{undefined}$, $f(5) = 10$, $f(6) = 11$. Then f is not a function but its a neutro one to one mapping.

Example 2., example 3. discussed in section 3 are the practical example of neutro one-one mapping.

4.2. Neutro onto function

A neutro mapping $f : X \rightarrow Y$ is said to be a neutro onto if for any element $y \in Y$ it is confirmed that every $y \in Y$ has its pre-image but exactly which one that may not be determined.

4.2.1. Example

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $X = \{1, 2, 3, 4, 5, 6\}$, $Y = \{7, 8, 9\}$, $f(1) = 7$, $f(2) = 8$, $f(3) = 8$, $f(\text{some numbers greater than } 4) = 9$.

Example 1. in sec. 3 is a neutro onto mapping. Because only confirmation is within half life period half of the radioactive quantity will decay, but it is not predictable particular which atom will decay.

4.3. Neutro bijection mapping

A neutro mapping $f : A \rightarrow B$ is said to be neutro bijective if f is both neutro one to one and neutro onto.

4.3.1. Example

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $X = \{1, 2, 3, 4, 5, 6\}$, $Y = \{7, 8, 9, 10, 11\}$, $f(1) = 7, f(2) = 8, f(3) = 9, f(4) = \text{undefined}, f(5) = 10, f(6) = 11$. Clearly f is a neutro bijective mapping.

Consider example 3. (sec. 3). If we define the correspondence from position of the coin after toss to the set $\{0, 1\}$. Then it's a neutro bijective mapping.

4.3.2. Neither one-one nor onto

Let $U = \{1, 2, 3, \dots, 9\}$, $X = \{1, 2, 3, 4, 5, 6\}$, $Y = \{7, 8, 9, 10, 11\}$, $f(1) = 7, f(2) = 7, f(3) = 9 \text{ or } 10 \text{ or } 11$. f is not one-one as images of 1, 2 are same (7). It is not onto also because it is not confirmed that 9, 10 and 11 have a pre-image.

Theorem 4.1. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be both neutro one-one mapping, then the composite mapping $gof : A \rightarrow C$ is neutro one-one.

Proof. Clearly, the undefined points does not affect the theorem. Consider two elements x_1, x_2 in A where f is well defined. Then clearly for $x_1 \neq x_2$, $gof(x_1) \neq gof(x_2)$. \square

Theorem 4.2. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two neutro mapping such that $gof : A \rightarrow C$ is neutro one-one, then f is neutro one-one.

Proof. If possible let f be not neutro one-one. Then there exists two elements x and y in A such that for $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$. So, $gof(x_1) \neq gof(x_2)$, which is a contradiction. So, f is neutro one to one. \square

Theorem 4.3. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be both neutro onto mapping, then the composite mapping $gof : A \rightarrow C$ is neutro onto.

Proof. Since g is onto, for any c in C , $g(\text{some element } b \text{ in } B) = c$ and in the same way $f(\text{some } a \text{ in } A) = b$. So, $gof(a) = c$. \square

Theorem 4.4. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two neutro mapping then $gof : A \rightarrow C$ is neutro onto then g is also neutro onto.

Proof. Let c be an element in C , then there exist some element a in A such that $gof(a) = c$, i.e., $g(b) = c$, for some b in B . Hence g is neutro onto. \square

4.4. Neutrosophic identity mapping

A mapping $f : A \rightarrow A$ is said to be the neutrosophic identity mapping on A if $f(x) = x$ at each elements where the function is well defined.

According to the definition of neutrosophic identity mapping the concept of inverse neutro mapping is developed:

4.5. Inverse Neutrosophic mapping

Let $f : A \rightarrow B$ be a neutrosophic mapping. If there exists a neutrosophic mapping $g : B \rightarrow A$, such that $gof = I_A$, then g is said to be left neutro inverse of f .

If there exists a mapping $h : B \rightarrow A$ such that $foh = I_B$, h is said to be right neutrosophic inverse of f .

Example 1: Take two neutrosophic mapping such that $f : R \rightarrow [-1, 1]$ by $f(x) = \sin x$ and $g : [-1, 1] \rightarrow R$ by $g(x) = \sin^{-1} x$, then $fog(x) = x$, but $gof(x)$ does not always give the value x . For example, for $x = \pi/2$, $gof(x) = (4n+1)\pi/2$, $n = 0, 1, 2, \dots$. So, g is the right neutrosophic inverse of f but not left neutrosophic inverse.

4.6. Neutrosophic inverse

A function g is called neutrosophic inverse of f if $gof = fog = I$.

Consider the mappings f and g from $R \rightarrow R$ by $f(x) = 1/x$ and $g(x) = 1/x$, then $fog = gof = I$. Here g is the neutrosophic inverse of f .

Theorem 4.5. *A neutro mapping $f : A \rightarrow B$ is invertible if and only if f is neutro one-one.*

Proof. Let $f : A \rightarrow B$ be a neutrosophic invertible, then there exists $g : B \rightarrow A$ such that $gof = I_A$. Clearly, I_A is neutro one to one mapping. So, f is neutro one to one.

Conversely, let f is neutro one-one mapping. Let $b \in f(A)$, since f is one-one, there exist $a \in A$ such that $f(a) = b$. Define a neutro mapping $g : B \rightarrow A$ such that $g(b) = a$. Then $gof(a) = g(b) = a$ and $fog(b) = f(a) = b$. Hence the proof. \square

Theorem 4.6. *Let f is a neutrosophic mapping and $f(P) \subseteq f(Q)$. Then $P \subseteq Q$.*

Proof.

Let $f(P) = \{(T_{f_P}(x), I_{f_P}(x), F_{f_P}(x)) : x \in P\}$ and $f(Q) = \{(T_{f_Q}(x), I_{f_Q}(x), F_{f_Q}(x)) : x \in Q\}$. Since $f(P) \subseteq f(Q)$, $T_{f_P}(x) \leq T_{f_Q}(x)$, $I_{f_P}(x) \geq I_{f_Q}(x)$, $F_{f_P}(x) \geq F_{f_Q}(x)$.

Evidently, the existence of x in P is less than Q , otherwise it would violate $f(P) \subseteq f(Q)$. \square

Theorem 4.7. *Let f is a onto neutrosophic mapping and $f^{-1}(P) \subseteq f^{-1}(Q)$, then $P \subseteq Q$.*

Proof. For an element y , $T_{f_P^{-1}}(y) \leq T_{f_Q^{-1}}(y)$, $I_{f_P^{-1}}(y) \geq I_{f_Q^{-1}}(y)$, $F_{f_P^{-1}}(y) \geq F_{f_Q^{-1}}(y)$, where, $T_{f_P^{-1}}(y)$, $F_{f_P^{-1}}(y)$ and $I_{f_P^{-1}}(y)$ denotes the truth, falsity and indeterminacy degree of belongingness of $f^{-1}(y)$ in f_P^{-1} respectively.

It assures that the belongingness of y in P less than Q , otherwise it would violate the above relations. \square

Remark: To develop any theorem on neutro algebra, the necessary definitions should be initially defined. All the theories are true only on the basis of the proposed definition.

5. Conclusion

In this paper several types of neutro mappings similar to classical algebra are defined. Also some elementary theorems are established following the proposed definitions. Since in science and technology the laws that describe them can hardly be rigorously defined, the neutro mapping based on neutro axioms will keep its incredible utilities.

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On Neutrosophic Triplet quasi–dislocated-b-metric space

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Abstract: The concept of neutrosophic triplet firstly introduced by F. Smarandache and M. Ali [28]. This notion (neutrosophic triplet) is a group of three elements that satisfy certain properties with some binary operation. These neutrosophic triplets highly depends on the proposed binary operation. In this article, we make some observations concerning Neutrosophic triplet metric space (NTMS), Neutrosophic triplet partial metric space (NTPMS), Neutrosophic triplet-b-metric space (NT-b-MS) introduced by Sahin et al. [18-20] and put our observation on the definitions defined in these articles. Moreover, inspired by Ur Rahaman [17] and Sahin et al. [18-20] further we define a new topological construction named as Neutrosophic Triplet quasi–dislocated-b-metric space (NT-qdb-MS) and study some properties of NT-qdb-MS. Furthermore using this construction, we establish some fixed point theorems in the context of NT-qdb-MS using graph. For the validity of our results, we also provide an example.

Keywords: Neutrosophic triplet group, neutrosophic triplet metric space, neutrosophic triplet partial metric space, neutrosophic triplet quasi–dislocated-b-metric space, fixed point, graph.

Abbreviations:

1. NTS – neutrosophic triplet set
2. NTG - neutrosophic triplet group
3. NTMS- neutrosophic triplet metric space
4. NTM- neutrosophic triplet metric
5. NTPMS- neutrosophic triplet partial metric space
6. NTPM- neutrosophic triplet partial metric
7. NT-b-MS- neutrosophic triplet– b-metric space
7. NT-qdb-MS- neutrosophic triplet quasi–dislocated-b-metric space
8. NT-qdb-M- neutrosophic triplet quasi–dislocated-b-metric

1. Introduction and Preliminaries

Concept of fuzzy sets were introduced by Zadeh [29] to deal the problem of uncertainty existing in real-world. Since its initiation, as a generalization of it, interval valued fuzzy set [13] and intuitionistic fuzzy set [24] have come into sight. These extensions can deal with uncertain real-world problems but it does not cope with indeterminate data. Thus, in order to cope with these uncertainties, Smarandache began to use the non-

standard analysis and proposed the term "neutrosophic" which means knowledge of neutral thought, and this neutral represents the main distinction between "fuzzy" and "intuitionistic fuzzy" set. Neutrosophy is a new subsidiary of philosophy which is initiated by Florentin Smarandache [27]. The concept of neutrosophic logic was first studied by Florentin Smarandache in 1995. Neutrosophic set is a stereotype of interval valued fuzzy set [13], intuitionistic fuzzy set [24], fuzzy sets [29] and classical sets which is used to handle problems issues containing inconsistent, indeterminate, falsity and imprecise data.

In the concept of neutrosophic logic and neutrosophic sets, there is T degree of membership, I degree of undeterminacy and F degree of non-membership. These degrees are defined independently of each other in neutrosophic logic and neutrosophic sets whereas these degrees are defined dependently of each other in intuitionistic fuzzy logic and intuitionistic fuzzy set. Thus, neutrosophic set is an extension of fuzzy and intuitionistic fuzzy set. Many authors have worked in neutrosophic theory for more details see [1-6, 9, 21,23, 24-26]. Furthermore, Smarandache and Ali deliberated neutrosophic triplet theory particularly NTG's in [28, 12, 30]. Later on, neutrosophic triplet theory has been studied with fixed point theory in [19, 20].

Moreover, a new direction in the theory of fixed point was recently given by Jachymski [11] and gave some generalization of the Banach contraction principle for mapping on a metric spaces endowed with a graph in 2007. Jachymski [11] generalized and unified the results existing in the literature using the languages of graph theory and opened an avenue for further development of fixed point theory in this direction. His work is considered as a reference in this domain. The fixed point theory with graph is a very curious way in the field of research and have wide number of applications in other fields. Motivated by the remarkable work of Jachymski [11], a lot of work in fixed point theory with graph have been done by several authors in various spaces with various contractive conditions, see in [7,8,16,22] and etc.

Sahin et al. [18-20] proposed the NT-b-MS, NTMS, NTPMS respectively by combining the fixed point theory with neutrosophy which is a new interesting approach in this direction. But in their paper [19] (pp. 699), according to their Definition 4.1 of NTMS, Example 4.3 doesn't support the definition 4.1. For this we give a counter Example 2.1 in this paper in Section 2. Also, in 2018 Sahin et al.[20] introduced the concept of NTPMS but we get the disparity of their Definition 4. with Example 1. in [20] (pp. 3). For this, we demonstrate a counter Example 2.2 in Section 2. Also, in their paper [20] (pp. 5) in Theorem 4, we can't write inequality (8)

i.e,

$$p_n(\kappa_n, \kappa_k) \leq p_n(\kappa_n, \kappa_k * \text{neut}(\kappa_{n-1}))$$

for any arbitrary element κ_{n-1} , since in Definition 8 [20] (pp. 5) they assumed that there exist any element c (any one) in A such that

$$p_n(a, b) \leq p_n(a, b * \text{neut}(c)) \text{ for all } a, b \text{ in } A.$$

If they assumed condition (i) of Definition 8 in [20](pp.5) for all 'c' elements then all these properties of Definition 4 in [20] (pp.3) becomes properties of the partial metric space. Therefore, Theorem 4 in [20] (pp. 5) becomes the existing result in partial metric space. But Sahin et al.[18] again redefined their Definition 4.1[19] of NTMS and Definition 4. of NTPMS which is in corrected form. Here, we also discussed what is the difference between taking "any element" or "atleast one element" in triangular inequality of NTMS and NTPMS.

Recently, Ur Rahaman[17] in 2015, introduced the topological properties of dislocated-quasi-b-metric space and proved some fixed point theorems. Motivated by Smarandache and Ali[28], Sahin et al.[18-20] and Ur Rahaman[17], we define a new topological structure NT-qdb-MS which is different from classical quasi-dislocated-b-metric space. A great benefit of NT-qdb-MS is that it gives a new space structure to those structures which are not quasi-dislocated-b-metric space with respect to some functions that not satisfy triangular inequality for all x, y, z since we don't need to verify the triangular inequality for all x, y, z in NT-qdb-MS as we can see in Definition 2.3. defined in Section 2. Triangular inequality in NT-qdb-MS is much weaker assumption as compare to the triangular inequality in quasi-dislocated-b-metric space. We also studied some interesting properties of this newly born structure. At the end, we obtained some fixed point results such as famous Banach fixed theorem(generalized version) and Kannan fixed point theorem inspired by [7,10,14] in this topological structure and provided an example to explain the results.

Now, we call some basic definitions from neutrosophic triplet theory following as:

Definition 1.1 [28]. Let S be a non-empty set with a binary operation \diamond then it is called a **NTS** if for any $s \in S$, there exist a neutral of s in S denoted by $\text{neut}(s)$ different from classical algebraic unitary element and also there exist antineutral of s in S named as $\text{anti}(s)$ such that

$$s \diamond \text{neut}(s) = \text{neut}(s) \diamond s = s \quad \text{and} \quad s \diamond \text{anti}(s) = \text{anti}(s) \diamond s = \text{neut}(s)$$

The triplet $(s, \text{neut}(s), \text{anti}(s))$ represents neutrosophic triplet. For the same element, there may be more neutrals to it $\text{neut}(s)$'s and more opposite of it $\text{anti}(s)$'s.

Definition 1.2 [28]. A non-empty set S with binary operation \diamond is called a **NTG** if it satisfies following properties:

- (i) $s_1 \diamond s_2 \in S$ for all $s_1, s_2 \in S$;
- (ii) $(s_1 \diamond s_2) \diamond s_3 = s_1 \diamond (s_2 \diamond s_3)$ for all $s_1, s_2, s_3 \in S$;
- (iii) for each $s_1 \in S$, there exist a neutral of s_1 in S denoted by $\text{neut}(s_1)$ different from classical algebraic unitary element such that

$$s_1 \diamond \text{neut}(s_1) = \text{neut}(s_1) \diamond s_1 = s_1$$

- (iv) and for each $s_1 \in S$, there exist anti-neutral of s_1 in S named as $\text{anti}(s_1)$ such that

$$s_1 \diamond \text{anti}(s_1) = \text{anti}(s_1) \diamond s_1 = \text{neut}(s_1).$$

Definition 1.3 [7]: A mapping $\varphi: R^+ \rightarrow R^+$ is said to be comparison function if it satisfies:

1. φ is monotonic increasing;
2. The sequence $\{\varphi^n(t)\}$ converges to zero for all $t \in R^+$.

2. Neutrosophic triplet quasi-dislocated-b-metric space and Revised definition of NTMS and NTPMS

In this section, first we define the revised definition of NTMS and NTPMS then we define NT-qdb-MS and its properties.

Definition 2.1 [19]. Let (M, \diamond) be a NTS with $x \diamond y \in M$ for all x, y in M . A mapping $d: M \times M \rightarrow [0, \infty)$ is called a NTM on M if satisfying the following properties for all $x, z \in M$,

- (i) $d(x, z) \geq 0$;
- (ii) If $x = z$ then $d(x, z) = 0$;
- (iii) $d(x, z) = d(z, x)$;
- (iv) If there exist at least one element y in M different from x and z in M such that

$$d(x, z) \leq d(x, z \diamond \text{neut}(y)) \quad \text{then} \quad d(x, z \diamond \text{neut}(y)) \leq d(x, y) + d(y, z).$$

The space $((M, \diamond), d)$ is known as NTMS.

Remark 2.1. In metric space, we have to verify the triangular inequality for all $x, y, z \in M$ and it is much stronger assumption as compare to the triangular inequality in NTMS since we don't need to verify the triangular inequality for all $x, y, z \in M$ in NTMS. In fact, we have to verify it for at least one element y in M different from x and z in M such that

$$d(x, z) \leq d(x, z \diamond \text{neut}(y)) \quad \text{then} \quad d(x, z \diamond \text{neut}(y)) \leq d(x, y) + d(y, z) \quad \text{and it implies} \\ d(x, z) \leq d(x, y) + d(y, z).$$

A big advantage of neutrosophic triplet metric space is that it gives a new space structure to those structures which are not metric spaces with respect to some functions that not satisfy triangular inequality for all $x, y, z \in M$.

Example 2.1. Let $M = \{1, 2\}$ and the power set of M , $\mathcal{P}(M) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ together with binary operation $\diamond = \cup$ form a NTS where $\text{neut}(I) = I$ and $\text{anti}(I) = I$, for all $I \in \mathcal{P}(M)$. Define a function $d: \mathcal{P}(M) \times \mathcal{P}(M) \rightarrow [0, \infty]$ such that

$$d(I, K) = |n(I) - n(K)| \quad \text{where } n(I) \text{ denotes the cardinality of } I.$$

Clearly (i), (ii) and (iii) of Definition 4.1 in [19] are satisfied.

Now, we will see condition (iv).

Take, $I = \{1\}$, $E = \{2\}$ and $K = \{1\}$

$$\begin{aligned} 0 = d(I, E) &\leq d(I, E \diamond \text{neut}(K)) \\ &= d(I, E \cup K) \\ &= |n(I) - n(E \cup K)| = 1 \end{aligned}$$

But $1 = d(I, E \diamond \text{neut}(K)) \not\leq d(I, K) + d(K, E) = 0$.

This shows that there exist an element $y \in \mathcal{P}(M)$ such that

$$d(x, z) \leq d(x, z \diamond \text{neut}(y))$$

but $d(x, z \diamond \text{neut}(y)) \leq d(x, y) + d(y, z)$ doesn't hold which contradict Example 4.3. provided in [19] (pp.699),

i.e, Example 4.3[19] doesn't satisfy the triangular inequality of Definition 4.1 in [19].

Remark 2.2. According to above Definition 2.1, Example 4.3 in [19] is accurate.

Definition 2.2 [20]. Let (M, \diamond) be a NTS with $x \diamond y \in M$ for all x, y in M . A NTPM is a mapping $P_N: M \times M \rightarrow [0, \infty)$ such that for all $x, z \in M$,

- (i) $P_N(\kappa, \kappa) \leq P_N(\kappa, \mathfrak{z})$;
- (ii) If $P_N(\kappa, \mathfrak{z}) = P_N(\kappa, \kappa) = P_N(\mathfrak{z}, \mathfrak{z})$ then $\kappa = \mathfrak{z}$;
- (iii) $P_N(\kappa, \mathfrak{z}) = P_N(\mathfrak{z}, \kappa)$;
- (iv) If there exist atleast one element γ' in M different from κ and \mathfrak{z} in M such that

$$P_N(\kappa, \mathfrak{z}) \leq P_N(\kappa, \mathfrak{z} \diamond \text{neut}(\gamma')) \text{ then } P_N(\kappa, \mathfrak{z} \diamond \text{neut}(\gamma')) \leq P_N(\kappa, \gamma') + P_N(\gamma', \mathfrak{z}) - P_N(\gamma', \gamma').$$
 The space $((M, \diamond), P_N)$ is known as NTPMS.

Remark 2.3. Concept of NTPMS(NTMS) is different from partial metric space(metric space respectively), neither of them is generalization of each other. First we see that, Is PMS implies NTPMS?. For this, we have to identify triangular inequality i.e, (iv) condition of NTPMS since all other conditions of NTPMS are satisfied by PMS.

If $P_N(\kappa, \mathfrak{z}) \leq P_N(\kappa, \gamma') + P_N(\gamma', \mathfrak{z}) - P_N(\gamma', \gamma')$ and $P_N(\kappa, \mathfrak{z}) \leq P_N(\kappa, \mathfrak{z} \diamond \text{neut}(\gamma'))$ then $P_N(\kappa, \mathfrak{z} \diamond \text{neut}(\gamma')) \leq P_N(\kappa, \gamma') + P_N(\gamma', \mathfrak{z}) - P_N(\gamma', \gamma')$ for atleast one element γ' , which is not possible always, if this is possible then (iv) is meaningless.

Clearly, NTPMS doesn't implies PMS.

If we take assumption (iv) in Definition 2.2 for any element γ in M as defined by Sahin et al. [20] (in Definition 4.[20] (pp. 3)) then examples which we have constructed not form NTPMS and it is difficult to find the examples for NTPMS which satisfy the properties of NTPMS defined by Sahin et al. in [20] since it is much stronger assumption as compare to assumption (iv) in Definition 2.2 and triangular inequality of partial metric space.

Moreover, we take here element γ' in M is different from κ and \mathfrak{z} since there exist always an element $\gamma' = \mathfrak{z}$ for all $\kappa, \mathfrak{z} \in M$ such that

$P_N(\kappa, \mathfrak{z}) \leq P_N(\kappa, \mathfrak{z} \diamond \text{neut}(\mathfrak{z})) = P_N(\kappa, \mathfrak{z})$ then $P_N(\kappa, \mathfrak{z} \diamond \text{neut}(\mathfrak{z})) \leq P_N(\kappa, \mathfrak{z}) + P_N(\mathfrak{z}, \mathfrak{z}) - P_N(\mathfrak{z}, \mathfrak{z})$ and the property (iv) becomes meaningless.

Counter example 2.2. Let M be any set, $\mathcal{P}(M)$ be the power set of M with binary operation $\diamond = \cup$ then $(\mathcal{P}(M), \cup)$ is a NTS where $\text{neut}(I) = I$ and $\text{anti}(I) = I$, for all $I \in \mathcal{P}(M)$.

Define a map $P_N : \mathcal{P}(M) \times \mathcal{P}(M) \rightarrow [0, \infty)$ such that

$$P_N(I, K) = \max \{n(I), n(K)\} \text{ where } n(I) \text{ denotes the cardinality of } I.$$

Condition (i), (ii) and (iii) are easy to verified of Definition 4. [20] (pp.3). Here we see condition (iv).

If we take the sets, I, K, L in $\mathcal{P}(M)$ such that

$$n(I) = 25, \quad n(K) = 22, \quad n(L) = 10 \text{ and } n(K \cap L) = 4.$$

Now,

$$P_N(I, L) \leq P_N(I, L \diamond \text{neut}(K)) = P_N(I, L \cup K)$$

$$\text{i.e., } \max \{n(I), n(L)\} \leq \max \{n(I), n(L) + n(K) - n(L \cap K)\}$$

$$\text{i.e., } 25 \leq 28$$

which is true.

$$\text{But } P_N(I, L \diamond \text{neut}(K)) \not\leq P_N(I, K) + P_N(K, L) - P_N(K, K)$$

$$\text{since } P_N(I, L \cup K) = 28 \text{ and } P_N(I, K) + P_N(K, L) - P_N(K, K) = 25 + 22 - 22 = 25.$$

Thus the condition (iv) of Definition 4. in [20] doesn't satisfied. Hence $(\mathcal{P}(M), \cup)$ is not a NTPMS according to Definition 4. [20] but it becomes a NTPMS according to Definition 2.2 as defined above. Since for any elements $\kappa = I \neq \emptyset$, $\gamma = K \neq \emptyset$, there exist $\mathfrak{z} = \emptyset$ such that

$$P_N(I, K) \leq P_N(I, K \diamond \text{neut}(\emptyset)) = P_N(I, K)$$

$$\text{then } P_N(I, K \diamond \text{neut}(\emptyset)) \leq P_N(I, \emptyset) + P_N(\emptyset, K) - P_N(\emptyset, \emptyset)$$

and for $\kappa = I \neq \emptyset$, $\gamma = K = \emptyset$, there exist $\mathfrak{z} = L$ in $\mathcal{P}(M)$ different from \emptyset , I such that

$$P_N(I, K) \leq P_N(I, K \diamond \text{neut}(L)) = P_N(I, K \cup L) = P_N(I, L)$$

$$\text{then } P_N(I, K \diamond \text{neut}(L)) \leq P_N(I, L) + P_N(L, K) - P_N(L, L).$$

Example 2.3. Let $M = \{0, 4, 8, 9\}$ be a NTG together binary operation $\diamond =$ multiplication modulo 12 in (Z_{12}, \times) . Neutrosophic triplet are:

$(0, 0, 0)$, $(4, 4, 4)$, $(8, 4, 8)$, $(9, 9, 9)$ where $(\kappa, \gamma, \mathfrak{z})$ denote here, $\kappa \in M$ be any element, $\gamma = \text{neut}(\kappa)$ and $\mathfrak{z} = \text{anti}(\kappa)$.

Now, we define a map $P_N: M \times M \rightarrow [0, \infty)$ such that

$$P_N(\kappa, \gamma) = \max\{\kappa, \gamma\} \text{ for all } \kappa, \gamma \in M$$

Clearly, conditions (i), (ii) and (iii) are satisfied. Now, we identify condition (iv) for all $\kappa, \gamma, \mathfrak{z} \in M$;

For $\kappa = 4$, $\gamma = 0 \exists \mathfrak{z} = 8$ such that

$$4 = P_N(\kappa, \gamma) \leq P_N(\kappa, \gamma \diamond \text{neut}(\mathfrak{z})) = P_N(\kappa, 0) = 4 \text{ then}$$

$$4 = P_N(\kappa, \gamma \diamond \text{neut}(\mathfrak{z})) \leq P_N(\kappa, \mathfrak{z}) + P_N(\mathfrak{z}, \gamma) - P_N(\mathfrak{z}, \mathfrak{z}) = 8 + 8 - 8 = 8.$$

For $\kappa = 8$, $\gamma = 0 \exists \mathfrak{z} = 4$ such that

$$8 = P_N(\kappa, \gamma) \leq P_N(\kappa, \gamma \diamond \text{neut}(\mathfrak{z})) = P_N(\kappa, 0) = 8$$

$$\text{then } 8 = P_N(\kappa, \gamma \diamond \text{neut}(\mathfrak{z})) \leq P_N(\kappa, \mathfrak{z}) + P_N(\mathfrak{z}, \gamma) - P_N(\mathfrak{z}, \mathfrak{z}) = 8 + 4 - 4 = 8.$$

Similarly, for $\kappa = 9$, $\gamma = 0 \exists \mathfrak{z} = 4$, for $\kappa = 8$, $\gamma = 4 \exists \mathfrak{z} = 9$, for $\kappa = 9$, $\gamma = 4 \exists \mathfrak{z} = 8$, for $\kappa = 9$, $\gamma = 8 \exists \mathfrak{z} = 4$, for $\kappa = 0$, $\gamma = 4 \exists \mathfrak{z} = 8$ and for $\kappa = 0$, $\gamma = 8 \exists \mathfrak{z} = 4$ such that

$$P_N(\kappa, \gamma) \leq P_N(\kappa, \gamma \diamond \text{neut}(\mathfrak{z})) \text{ then } P_N(\kappa, \gamma \diamond \text{neut}(\mathfrak{z})) \leq P_N(\kappa, \mathfrak{z}) + P_N(\mathfrak{z}, \gamma) - P_N(\mathfrak{z}, \mathfrak{z}).$$

But for $\kappa = 0$, $\gamma = 9$, for $\kappa = 4$, $\gamma = 8$, for $\kappa = 4$, $\gamma = 9$ and for $\kappa = 8$, $\gamma = 9$ there does not exist a different element \mathfrak{z} such that $P_N(\kappa, \gamma) \leq P_N(\kappa, \gamma \diamond \text{neut}(\mathfrak{z}))$ hold so, we will not see

$$P_N(\kappa, \gamma \diamond \text{neut}(\mathfrak{z})) \leq P_N(\kappa, \mathfrak{z}) + P_N(\mathfrak{z}, \gamma) - P_N(\mathfrak{z}, \mathfrak{z}).$$

Hence $((M, \diamond), P_N)$ is a NTPMS.

Definition 2.3. Let (M, \diamond) be a NTS with $\kappa \diamond \gamma \in M$ for all κ, γ in M . Then a NT-qdb-M is a mapping $N_{qdb}: M \times M \rightarrow [0, \infty)$ such that for all $\kappa, \gamma \in M$,

$$(N_{qdb}1.) \quad N_{qdb}(\kappa, \gamma) = N_{qdb}(\gamma, \kappa) = 0 \text{ implies } \kappa = \gamma$$

$$(N_{qdb}2.) \quad \text{If there exist atleast one element } \mathfrak{z} \text{ in } M \setminus \{\kappa, \gamma\} \text{ such that}$$

$$N_{qdb}(\kappa, \gamma) \leq N_{qdb}(\kappa, \gamma \diamond \text{neut}(\mathfrak{z})) \text{ then } N_{qdb}(\kappa, \gamma \diamond \text{neut}(\mathfrak{z})) \leq s[N_{qdb}(\kappa, \mathfrak{z}) + N_{qdb}(\mathfrak{z}, \gamma)]$$

where $s \geq 1$ be a real number.

The space $((M, \diamond), N_{qdb})$ is known as NT-qdb-MS.

Remark 2.4. Concept of NT-qdb-MS is different from dislocated-quasi-b-metric space. Here,

$N_{qdb}(\kappa, \gamma) \neq N_{qdb}(\gamma, \kappa)$ and $N_{qdb}(\kappa, \kappa) = 0$ may not be possible. For $s=1$, the space NT-qdb-MS $((M, \diamond), N_{qdb})$ becomes neutrosophic triplet quasi dislocated metric space.

Example 2.4. Let $M = \{0, 2, 3, 4\}$ be a NTG together binary operation $\diamond = \otimes_6$ in (Z_6, \times) . Neutrosophic triplet are:

$(0, 0, 0), (2, 4, 2), (3, 3, 3), (4, 4, 4)$ where $(\kappa, \gamma, \mathfrak{z})$ denote here, $\kappa \in M$ be any element, $\gamma = \text{neut}(\kappa)$ and $\mathfrak{z} = \text{anti}(\kappa)$.

Now, we define a map $N_{qdb} : M \times M \rightarrow [0, \infty)$ such that

$$\begin{aligned} N_{qdb}(0, 0) &= 0, & N_{qdb}(0, 2) &= 4, & N_{qdb}(0, 3) &= 9, & N_{qdb}(0, 4) &= 16, \\ N_{qdb}(2, 2) &= 4, & N_{qdb}(3, 3) &= 9, & N_{qdb}(4, 4) &= 16, & N_{qdb}(2, 0) &= 8, \\ N_{qdb}(3, 0) &= 18, & N_{qdb}(4, 0) &= 32, & N_{qdb}(2, 3) &= 5, & N_{qdb}(3, 2) &= 10, \\ N_{qdb}(2, 4) &= 8, & N_{qdb}(4, 2) &= 20, & N_{qdb}(3, 4) &= 10, & N_{qdb}(4, 3) &= 17 \end{aligned}$$

Next, we identify the conditions (Nqdb1.) and (Nqdb2.) of NT-qdb-MS.

(Nqdb1.) $N_{qdb}(\kappa, \gamma) = N_{qdb}(\gamma, \kappa) = 0$ for only $\kappa = \gamma = 0$ implies $\kappa = \gamma$.

(Nqdb2.) Take $s = \frac{3}{2}$.

For $\kappa = 2, \gamma = 0 \exists \mathfrak{z} = 3$ in M such that

$$\begin{aligned} N_{qdb}(2, 0) &\leq N_{qdb}(2, 0 \otimes_6 \text{neut}(3)) = N_{qdb}(2, 0) = 8 \text{ then} \\ 8 &= N_{qdb}(2, 0 \otimes_6 \text{neut}(3)) \leq s[N_{qdb}(2, 3) + N_{qdb}(3, 0)] = s[5 + 18] = 23s. \end{aligned}$$

For $\kappa = 3, \gamma = 0 \exists \mathfrak{z} = 2$ in M such that

$$\begin{aligned} N_{qdb}(3, 0) &\leq N_{qdb}(3, 0 \otimes_6 \text{neut}(2)) = N_{qdb}(3, 0) = 18 \text{ then} \\ 18 &= N_{qdb}(3, 0 \otimes_6 \text{neut}(2)) \leq s[N_{qdb}(3, 2) + N_{qdb}(2, 0)] = s[10 + 8] = 18s. \end{aligned}$$

For $\kappa = 4, \gamma = 0 \exists \mathfrak{z} = 2$ in M such that

$$\begin{aligned} N_{qdb}(4, 0) &\leq N_{qdb}(4, 0 \otimes_6 \text{neut}(2)) = N_{qdb}(4, 0) = 32 \text{ then} \\ 32 &= N_{qdb}(4, 0 \otimes_6 \text{neut}(2)) \leq s[N_{qdb}(4, 2) + N_{qdb}(2, 0)] = s[20 + 8] = 28s. \end{aligned}$$

For $\kappa = 2, \gamma = 3 \exists \mathfrak{z} = 0$ in M such that

$$\begin{aligned} 5 &= N_{qdb}(2, 3) \leq N_{qdb}(2, 3 \otimes_6 \text{neut}(0)) = N_{qdb}(2, 0) = 8 \text{ then} \\ 8 &= N_{qdb}(2, 3 \otimes_6 \text{neut}(0)) \leq s[N_{qdb}(2, 0) + N_{qdb}(0, 3)] = s[8 + 9] = 17s. \end{aligned}$$

For $\kappa = 2, \gamma = 4 \exists \mathfrak{z} = 0$ in M such that

$$\begin{aligned} 8 &= N_{qdb}(2, 4) \leq N_{qdb}(2, 4 \otimes_6 \text{neut}(0)) = N_{qdb}(2, 0) = 8 \text{ then} \\ 8 &= N_{qdb}(2, 4 \otimes_6 \text{neut}(0)) \leq s[N_{qdb}(2, 0) + N_{qdb}(0, 4)] = s[8 + 16] = 24s. \end{aligned}$$

For $\kappa = 3, \gamma = 4 \exists \mathfrak{z} = 0$ in M such that

$$10 = N_{qdb}(3, 4) \leq N_{qdb}(3, 4 \otimes_6 \text{neut}(0)) = N_{qdb}(3, 0) = 18 \text{ then}$$

$$18 = N_{qdb}(3, 4 \otimes_6 \text{neut}(0)) \leq s[N_{qdb}(3, 0) + N_{qdb}(0, 4)] = s[18 + 16] = 34s.$$

Similarly, for $x = 0, y = 2 \exists z = 4$, for $x = 0, y = 4 \exists z = 2$, for $x = 3, y = 2 \exists z = 0$, for $x = 4, y = 2 \exists z = 0$ and for $x = 4, y = 3 \exists z = 0$ such that

$$N_{qdb}(x, y) \leq N_{qdb}(x, y \diamond \text{neut}(z)) \text{ then } N_{qdb}(x, y \diamond \text{neut}(z)) \leq s[N_{qdb}(x, z) + N_{qdb}(z, y)].$$

See for $x = 0, y = 3, N_{qdb}(0, 3) \not\leq N_{qdb}(0, 3 \otimes_6 \text{neut}(z))$ for any element $z \in M \setminus \{x, y\}$

so, we will not see $N_{qdb}(x, y \diamond \text{neut}(z)) \leq s[N_{qdb}(x, z) + N_{qdb}(z, y)]$.

Hence $((M, \diamond), N_{qdb})$ is a NT-qdb-MS.

Example 2.5. Let M be any infinite set, $\mathcal{P}(M)$ be the power set of M with binary operation $\diamond = \cup$ then $(\mathcal{P}(M), \cup)$ is a NTS where $\text{neut}(I) = I$ and $\text{anti}(I) = I$, for all $I \in \mathcal{P}(M)$.

Define a map $N_{qdb}: \mathcal{P}(M) \times \mathcal{P}(M) \rightarrow [0, \infty)$ such that

$$N_{qdb}(I, K) = |n(I) - n(K)|^2 + |n(I)|^2 \text{ where } n(I) \text{ denotes the cardinality of } I.$$

$$(N_{qdb}1.) \quad N_{qdb}(I, K) = N_{qdb}(K, I) = 0$$

$$\text{implies that } |n(I) - n(K)|^2 + |n(I)|^2 = |n(I) - n(K)|^2 + |n(K)|^2 = 0$$

$$\text{i.e., } |n(I)|^2 = |n(K)|^2 = 0$$

$$\text{or } n(I) = n(K) = 0 \text{ implies } I = K = \emptyset.$$

($N_{qdb}2.$) For any sets $I \neq \emptyset, K \neq \emptyset$ in $\mathcal{P}(M)$ there exist set \emptyset in $\mathcal{P}(M)$ such that

$$N_{qdb}(I, K) = N_{qdb}(I, K \diamond \text{neut}(\emptyset)) = N_{qdb}(I, K)$$

$$\text{then } |n(I) - n(K)|^2 + |n(I)|^2 = N_{qdb}(I, K \diamond \text{neut}(\emptyset)) \leq s[N_{qdb}(I, \emptyset) + N_{qdb}(\emptyset, K)]$$

$$= s[|n(I) - n(\emptyset)|^2 + |n(I)|^2 + |n(\emptyset) - n(K)|^2 + |n(\emptyset)|^2]$$

$$= s[2|n(I)|^2 + |n(K)|^2]$$

and for $x = I \neq \emptyset, y = K = \emptyset$, we have

$$2|n(I)|^2 = N_{qdb}(I, \emptyset) \leq N_{qdb}(I, \emptyset \cup L) = N_{qdb}(I, L)$$

$$= |n(I) - n(L)|^2 + |n(I)|^2 \text{ for any } L \in \mathcal{P}(M) \setminus \{I, K\} \text{ with } n(L) \geq 2n(I)$$

implies

$$N_{qdb}(I, L) = N_{qdb}(I, \emptyset \cup L) \leq N_{qdb}(I, L) + N_{qdb}(L, \emptyset)$$

$$\text{Also, } |n(I)|^2 = N_{qdb}(\emptyset, I) \leq N_{qdb}(\emptyset, I \cup L) = |n(I \cup L)|^2 \text{ for any } L \in \mathcal{P}(M) \setminus \{I, \emptyset\} \text{ with } n(L) \geq n(I).$$

$$\text{implies } |n(I \cup L)|^2 = N_{qdb}(\emptyset, I \cup L)$$

$$\leq s[N_{qdb}(\emptyset, L) + N_{qdb}(L, I)]$$

$$= s[|n(L)|^2 + |n(L) - n(I)|^2 + |n(L)|^2] \text{ for } s = 2.$$

Hence, $(\mathcal{P}(M), \cup)$ is a NT-qdb-MS.

Remark 2.5. If M is finite set then it is also NT-qdb-MS.

$$\text{Since for } I = M \text{ itself, } |n(M)|^2 = N_{qdb}(\emptyset, M) \leq N_{qdb}(\emptyset, M \cup L) = |n(M \cup L)|^2$$

$$\text{for any } L \in \mathcal{P}(M) \setminus \{M, \emptyset\} \text{ with } n(L) = n(M) - 1.$$

$$\begin{aligned}
\text{implies } |n(M \cup E)|^2 &= N_{qdb}(\emptyset, M \cup E) \\
&\leq s[N_{qdb}(\emptyset, E) + N_{qdb}(E, M)] \\
&= s[|n(E)|^2 + |n(E) - n(M)|^2 + |n(E)|^2] \text{ for } s = 2.
\end{aligned}$$

Let $((M, \diamond), N_{qdb})$ be NT-qdb-MS and $G = (V, E)$ is a reflexive digraph where the vertex set $V(G) = M$ and the set $E(G)$ of its edges contains no parallel edges. By G^{-1} means that the graph obtained from G by reversing the direction of edges. By disregarding the direction of edges of G , we acquire \tilde{G} , the undirected graph from G . So, $E(\tilde{G}) = E(G) \cup E(G^{-1})$. Literally, it will be better appropriate for us to approach \tilde{G} as a digraph for which the set of its edges is symmetric.

Definition 2.4. Let $((M, \diamond), N_{qdb})$ be NT-qdb-MS with graph G then

- (1.) A sequence $\{\kappa_n\}$ in $((M, \diamond), N_{qdb})$ is said to be **0-Convergent** to $\kappa \in M$ if for each $\varepsilon > 0$, there exist a positive integer $n_0 > 0$ such that $N_{qdb}(\kappa_n, \kappa) < \varepsilon$ and $N_{qdb}(\kappa, \kappa_n) < \varepsilon$ for all $n \geq n_0$.
- (2.) The sequence $\{\kappa_n\}$ in $((M, \diamond), N_{qdb})$ is said to be **0-Cauchy** if for each $\varepsilon > 0$, there exist a positive integer $n_0 > 0$ such that $N_{qdb}(\kappa_n, \kappa_m) < \varepsilon$ and $N_{qdb}(\kappa_m, \kappa_n) < \varepsilon$ for all $n, m \geq n_0$.
- (3.) $((M, \diamond), N_{qdb})$ is said to be **complete** if every **0-Cauchy sequence** $\{\kappa_n\}$ in M converges to a point y in M .
- (4.) Mapping $f: M \rightarrow M$ is said to be **continuous** at $\kappa \in M$, if for each $\varepsilon > 0$, there exist a $\delta > 0$ such that whenever $N_{qdb}(\kappa, \mathfrak{z}) < \delta$ and $N_{qdb}(\mathfrak{z}, \kappa) < \delta$ implies $N_{qdb}(f\kappa, f\mathfrak{z}) < \varepsilon$ and $N_{qdb}(f\mathfrak{z}, f\kappa) < \varepsilon$ for all $\mathfrak{z} \in M$.

Sequentially continuous if whenever $\kappa_n \rightarrow \kappa$ then $f\kappa_n \rightarrow f\kappa$ as $n \rightarrow \infty$

$$\text{i.e., if } \lim_{n \rightarrow \infty} N_{qdb}(\kappa_n, \kappa) = 0 \text{ and } \lim_{n \rightarrow \infty} N_{qdb}(\kappa, \kappa_n) = 0$$

$$\text{then } \lim_{n \rightarrow \infty} N_{qdb}(f\kappa_n, f\kappa) = 0 \text{ and } \lim_{n \rightarrow \infty} N_{qdb}(f\kappa, f\kappa_n) = 0$$

- (5.) A mapping $f: M \rightarrow M$ is called **G-continuous** if given $\kappa \in M$ and a sequence $(\kappa_n)_{n \in \mathbb{N}}$, $\kappa_n \rightarrow \kappa$ as $n \rightarrow \infty$ and $(\kappa_n, \kappa_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ imply $f\kappa_n \rightarrow f\kappa$ as $n \rightarrow \infty$.

Remark 2.6. Here, a convergent sequence in NT-qdb-MS may not be Cauchy sequence and need not necessary limit of the sequence is unique. Also, a constant sequence need not be convergent. For instance, we can see in Example 2.4, the sequence $\{2, 2, 2, 2, \dots\}$ is not a convergent sequence. In fact, it is not a Cauchy sequence.

3. Main results

In this component, we shall obtain some fixed point results in context of complete NT-qdb-MS by proving Lemma 3.1, Lemma 3.2. and present an example in the support of obtained results.

Lemma 3.1: Let $\{\kappa_n\}$ be a 0-convergent sequence converges to \mathfrak{z} in $((M, \diamond), N_{qdb})$ be NT-qdb-MS and $N_{qdb}(\kappa_n, \kappa_m) \leq N_{qdb}(\kappa_n, \kappa_m \diamond \text{neut}(\mathfrak{z}))$ for all n, m then $\{\kappa_n\}$ is a 0-Cauchy sequence in NT-qdb-MS.

Proof. Since $\{\kappa_n\}$ be a 0-convergent sequence and it converges to \mathfrak{z} in M . Therefore, for given $\varepsilon > 0$ there exist a positive integer $k > 0$ such that

$$N_{qdb}(\kappa_n, \mathfrak{z}) < \varepsilon/2s \text{ and } N_{qdb}(\mathfrak{z}, \kappa_n) < \varepsilon/2s \text{ for all } n \geq k$$

By given assumption and triangle inequality (NTqdb2.), for all $n, m \geq k$, we have

$$N_{qdb}(\kappa_n, \kappa_m) \leq N_{qdb}(\kappa_n, \kappa_m \diamond \text{neut}(\mathfrak{z})) \leq s[N_{qdb}(\kappa_n, \mathfrak{z}) + N_{qdb}(\mathfrak{z}, \kappa_m)] < s[\varepsilon/2s + \varepsilon/2s] = \varepsilon$$

$$\text{i.e., } N_{qdb}(\kappa_n, \kappa_m) < \varepsilon$$

$$\text{Similarly, } N_{qdb}(\kappa_m, \kappa_n) < \varepsilon$$

Thus, $\{\kappa_n\}$ is a 0-Cauchy sequence in NT-qdb-MS.

Lemma 3.2. Let $\{\kappa_n\}$ be a 0-convergent sequence in NT-qdb-MS say $((M, \diamond), N_{qdb})$, let it converges to κ and γ in M . Assume that $N_{qdb}(\kappa, \gamma) \leq N_{qdb}(\kappa, \gamma \diamond \text{neut}(\kappa_k))$ and $N_{qdb}(\gamma, \kappa) \leq N_{qdb}(\gamma, \kappa \diamond \text{neut}(\kappa_k))$ for any $k \in \mathbb{N}$ then limit of the sequence $\{\kappa_n\}$ is unique.

Proof: By the assumption,

$$N_{qdb}(\kappa, \gamma) \leq N_{qdb}(\kappa, \gamma \diamond \text{neut}(\kappa_m))$$

and by (NTqdb2)

$$N_{qdb}(\kappa, \gamma \diamond \text{neut}(\kappa_m)) \leq s [N_{qdb}(\kappa, \kappa_m) + N_{qdb}(\kappa_m, \gamma)]$$

Since $\{\kappa_n\}$ is a 0-convergent sequence and converges to κ and γ , so right hand side of above equation tends to zero as $m \rightarrow \infty$ that is, we have

$$N_{qdb}(\kappa, \gamma) = 0.$$

Similarly, we can show that

$$N_{qdb}(\gamma, \kappa) = 0$$

Hence, by (NTqdb1), $\kappa = \gamma$ which completes the proof.

Theorem 3.1. Let $((M, \diamond), N_{qdb})$ be a complete NT-qdb-MS with graph G and coefficient $s \geq 1$. Let $T: M \rightarrow M$ be a G -continuous mapping satisfying

$$N_{qdb}(T\kappa, T\gamma) \leq \varphi(N_{qdb}(\kappa, \gamma)) \quad \text{for all } \kappa, \gamma \in E(\tilde{G}) \quad (1)$$

where φ is a comparison function, with the following properties:

(a) for a set $O(\kappa) = \{\kappa, T\kappa, T^2\kappa, T^3\kappa, \dots\}$, assume that $(T^n\kappa, T^m\kappa) \in E(\tilde{G})$ for all n, m and

$$N_{qdb}(T^n\kappa, T^m\kappa) \leq N_{qdb}(T^n\kappa, T^m\kappa \diamond \text{neut}(v)) \quad \text{for any } v \in M \text{ and for all } n, m.$$

(b) If sequence $\{\kappa_n\}$ be converges to κ and γ in M then $N_{qdb}(\kappa, \gamma) \leq N_{qdb}(\kappa, \gamma \diamond \text{neut}(\kappa_k))$ and

$$N_{qdb}(\gamma, \kappa) \leq N_{qdb}(\gamma, \kappa \diamond \text{neut}(\kappa_k)) \quad \text{for any } k \in \mathbb{N}.$$

In addition, if κ and κ^* are two fixed points with $(\kappa, \kappa^*) \in E(\tilde{G})$ then T has a unique fixed point.

Proof. Take $\kappa_0 \in M$ be any arbitrary point but it's fixed. Define a iterative sequence in M as follows:

$$\kappa_n = T\kappa_{n-1} \quad \text{where } n = 1, 2, 3, 4, \dots$$

$$\text{i.e., } \kappa_1 = T\kappa_0, \kappa_2 = T\kappa_1, \kappa_3 = T\kappa_2, \dots$$

If we assume that $\kappa_{n+1} = \kappa_n$ for some $n \in \mathbb{Z}^+$, then it follows that $\kappa_n = \kappa_{n+1} = T\kappa_n$. So, κ_n is fixed point and proof is finished. Therefore, we assume that

$$\kappa_n \neq \kappa_{n+1} \quad \text{for each } n \in \mathbb{Z}^+.$$

We claim that $\{\kappa_n\}$ is a 0-Cauchy sequence in M . Now, for $\kappa = \kappa_n$, $\gamma = \kappa_{n+1}$ with assumption (a), contractive condition (1) becomes,

$$N_{qdb}(\kappa_n, \kappa_{n+1}) = N_{qdb}(T\kappa_{n-1}, T\kappa_n) \leq \varphi(N_{qdb}(\kappa_{n-1}, \kappa_n))$$

$$\text{and } N_{qdb}(\kappa_{n-1}, \kappa_n) = N_{qdb}(T\kappa_{n-2}, T\kappa_{n-1}) \leq \varphi(N_{qdb}(\kappa_{n-2}, \kappa_{n-1}))$$

By assumption of φ ,

$$\varphi(N_{qdb}(\kappa_{n-1}, \kappa_n)) \leq \varphi^2(N_{qdb}(\kappa_{n-2}, \kappa_{n-1}))$$

Computing repeatedly in this way, we obtain

$$\begin{aligned} N_{qdb}(\kappa_n, \kappa_{n+1}) &\leq \varphi(N_{qdb}(\kappa_{n-1}, \kappa_n)) \\ &\leq \varphi^2(N_{qdb}(\kappa_{n-2}, \kappa_{n-1})) \\ &\leq \varphi^3(N_{qdb}(\kappa_{n-3}, \kappa_{n-2})) \end{aligned}$$

$$\dots\dots\dots \leq \varphi^n(N_{qdb}(\mathcal{X}_0, \mathcal{X}_1)) \quad (2)$$

Proceeding in similar way, we can obtain

$$N_{qdb}(\mathcal{X}_{n+1}, \mathcal{X}_n) \leq \varphi^n(N_{qdb}(\mathcal{X}_1, \mathcal{X}_0)) \quad (3)$$

If $N_{qdb}(\mathcal{X}_0, \mathcal{X}_1) = 0$ and $N_{qdb}(\mathcal{X}_1, \mathcal{X}_0) = 0$ then $\mathcal{X}_0 = T\mathcal{X}_0$ i.e., \mathcal{X}_0 is a fixed point. Therefore, assume that $N_{qdb}(\mathcal{X}_0, \mathcal{X}_1) > 0$ and $N_{qdb}(\mathcal{X}_1, \mathcal{X}_0) > 0$

To prove $\{\mathcal{X}_n\}$ is a 0-Cauchy sequence, consider $m > n$ and using (a), (Nqdb2.)

$$N_{qdb}(\mathcal{X}_n, \mathcal{X}_m) \leq s N_{qdb}(\mathcal{X}_n, \mathcal{X}_{n+1}) + s^2 N_{qdb}(\mathcal{X}_{n+1}, \mathcal{X}_{n+2}) + s^3 N_{qdb}(\mathcal{X}_{n+2}, \mathcal{X}_{n+3}) + \dots \quad (4)$$

Using (2), (4) becomes

$$N_{qdb}(\mathcal{X}_n, \mathcal{X}_m) \leq s \varphi^n(N_{qdb}(\mathcal{X}_0, \mathcal{X}_1)) + s^2 \varphi^{n+1}(N_{qdb}(\mathcal{X}_0, \mathcal{X}_1)) + s^3 \varphi^{n+2}(N_{qdb}(\mathcal{X}_0, \mathcal{X}_1)) + \dots$$

By the definition of function φ and letting $n, m \rightarrow \infty$, we have

$$\lim_{n, m \rightarrow \infty} N_{qdb}(\mathcal{X}_n, \mathcal{X}_m) = 0$$

Similarly, we can show that,

$$\lim_{n, m \rightarrow \infty} N_{qdb}(\mathcal{X}_m, \mathcal{X}_n) = 0$$

which shows that $\{\mathcal{X}_n\}$ is a 0-Cauchy sequence in M . Since M is complete NT-qdb-MS, there exist $p \in M$ such that $\mathcal{X}_n \rightarrow p$ as $n \rightarrow \infty$. Now, here we will show that p is a fixed point in M .

As $\mathcal{X}_n \rightarrow p$ as $n \rightarrow \infty$ and using G-continuity of T , it follows that

$$\lim_{n \rightarrow \infty} T\mathcal{X}_n = Tp$$

and we can write above equation as

$$\lim_{n \rightarrow \infty} \mathcal{X}_{n+1} = Tp$$

Thus, p is a fixed point in M by using assumption (b) and Lemma 3.2.

Now, we want to show that p is a unique fixed point. For this, suppose p^* be another fixed point.

Consider, $N_{qdb}(p, p^*) = N_{qdb}(Tp, Tp^*) \leq \varphi(N_{qdb}(p, p^*))$ by using (1).

By assumption of φ , above inequality implies that $N_{qdb}(p, p^*) = 0$, also $N_{qdb}(p^*, p) = 0$ by following same process as we have done above. Hence by (NTqdb1.) $p = p^*$.

Corollary 3.1: Let $((M, \diamond), N_{qdb})$ be a complete Neutrosophic quasi-dislocated-b-metric space with coefficient $s \geq 1$. Let $T: M \rightarrow M$ be a G-continuous mapping satisfying

$$N_{qdb}(T\mathcal{X}, T\mathcal{Y}) \leq \alpha N_{qdb}(\mathcal{X}, \mathcal{Y}) \text{ for all } \mathcal{X}, \mathcal{Y} \in E(\tilde{G})$$

where $\alpha \in (0, 1)$; with the following properties:

(a) for a set $O(\mathcal{X}) = \{\mathcal{X}, T\mathcal{X}, T^2\mathcal{X}, T^3\mathcal{X}, \dots\}$, assume that $(T^n\mathcal{X}, T^m\mathcal{X}) \in E(\tilde{G})$ for all n, m and

$$N_{qdb}(T^n\mathcal{X}, T^m\mathcal{X}) \leq N_{qdb}(T^n\mathcal{X}, T^m\mathcal{X} \diamond \text{neut}(v)) \text{ for any } v \in M \text{ and for all } n, m.$$

(b) If above sequence $\{\mathcal{X}_n\}$ be converges to \mathcal{X} and \mathcal{Y} in M and suppose

$$N_{qdb}(\mathcal{X}, \mathcal{Y}) \leq N_{qdb}(\mathcal{X}, \mathcal{Y} \diamond \text{neut}(\mathcal{X}_k)) \text{ and } N_{qdb}(\mathcal{Y}, \mathcal{X}) \leq N_{qdb}(\mathcal{Y}, \mathcal{X} \diamond \text{neut}(\mathcal{X}_k)) \text{ for any } k \in \mathbb{N}.$$

In addition, if \mathcal{X} and \mathcal{X}^* are two fixed points with $(\mathcal{X}, \mathcal{X}^*) \in E(\tilde{G})$ then T has a unique fixed point.

Theorem 3.2: Let $((M, \diamond), N_{qdb})$ be a complete NT-qdb-MS with graph G (need not be reflexive) and coefficient $s \geq 1$. Let $T: M \rightarrow M$ be a G-continuous mapping satisfying

$$N_{qdb}(Tx, Ty) \leq \mu [N_{qdb}(x, Tx) + N_{qdb}(y, Ty)] \quad \text{for all } x, y \in E(\tilde{G}) \quad (5)$$

where $0 < \mu < \frac{1}{s+1}$, with the following properties:

(a) for a sequence $x_{n+1} = Tx_n$, $n \in \mathbb{N}$ and $x_1 = x \in M$, assume that $(x_n, x_m) \in E(\tilde{G})$ for all $n, m \in \mathbb{N}$ and $N_{qdb}(x_n, x_m) \leq N_{qdb}(x_n, x_m \diamond \text{neut}(x_k))$ for all n, m, k except some first finite few terms.

(b) If the above sequence $\{x_n\}$ converges to x and y in M then $N_{qdb}(x, y) \leq N_{qdb}(x, y \diamond \text{neut}(x_k))$ and

$$N_{qdb}(y, x) \leq N_{qdb}(y, x \diamond \text{neut}(x_k)) \quad \text{for any } k \in \mathbb{N}.$$

Furthermore, if x and x^* are two fixed points with $(x, x^*) \in E(\tilde{G})$ then T has a unique fixed point.

Proof. Let $x \in M$ be any arbitrary point but it's fixed. Construct an iteration sequence in M as follows:

$$x, Tx, T^2x, T^3x, \dots$$

i.e., $x = x_1$ and $x_{n+1} = Tx_n = T^n x$ where $n \in \mathbb{N}$.

If $x_{n+1} = x_n$ for some $n \in \mathbb{Z}^+$, then it follows that $x_n = x_{n+1} = Tx_n$ then x_n is fixed point which completes the proof. Therefore, we suppose that $x_n \neq x_{n+1}$ for all n .

For $x = x$, $y = Tx$ and using (a), (5) becomes

$$N_{qdb}(Tx, T^2x) \leq \mu [N_{qdb}(x, Tx) + N_{qdb}(Tx, T^2x)]$$

$$\text{i.e., } (1 - \mu) N_{qdb}(Tx, T^2x) \leq \mu N_{qdb}(x, Tx)$$

$$\text{or } N_{qdb}(Tx, T^2x) \leq \frac{\mu}{1-\mu} N_{qdb}(x, Tx).$$

Again, for $x = Tx$, $y = T^2x$, (5) becomes

$$N_{qdb}(T^2x, T^3x) \leq \mu [N_{qdb}(Tx, T^2x) + N_{qdb}(T^2x, T^3x)]$$

$$N_{qdb}(T^2x, T^3x) \leq \frac{\mu}{1-\mu} N_{qdb}(Tx, T^2x) \leq \left(\frac{\mu}{1-\mu}\right)^2 N_{qdb}(x, Tx) = t^2 N_{qdb}(x, Tx) \text{ where } t = \frac{\mu}{1-\mu}.$$

Continuing in this way, we obtain

$$N_{qdb}(T^n x, T^{n+1} x) \leq t^n N_{qdb}(x, Tx) \quad (6)$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty, \text{ since } t \in [0, 1).$$

That is,

$$\lim_{n \rightarrow \infty} N_{qdb}(T^n x, T^{n+1} x) = 0 \quad \text{i.e.,} \quad \lim_{n \rightarrow \infty} N_{qdb}(x_{n+1}, x_{n+2}) = 0.$$

Next, it is desirable to show that $\{x_n\}$ is a 0-Cauchy sequence in M . For this, we take m, n are positive integers such that $m > n$ then by using definition of NT-qdb-MS and condition (a), (6), we have

$$\begin{aligned} N_{qdb}(x_n, x_m) &\leq s [N_{qdb}(x_n, x_{n+1}) + N_{qdb}(x_{n+1}, x_m)] \\ &\leq s N_{qdb}(x_n, x_{n+1}) + s^2 N_{qdb}(x_{n+1}, x_{n+2}) + s^3 N_{qdb}(x_{n+2}, x_{n+3}) + \dots + s^{m-n} N_{qdb}(x_{m-1}, x_m) \\ &\leq s t^{n-1} N_{qdb}(x, Tx) + s^2 t^n N_{qdb}(x, Tx) + s^3 t^{n+1} N_{qdb}(x, Tx) + \dots + s^{m-n} t^{m-2} N_{qdb}(x, Tx) \\ &\leq \frac{st^{n-1}}{1-st} N_{qdb}(x, Tx). \end{aligned}$$

Taking limit as $n, m \rightarrow \infty$, we have

$$\lim_{n, m \rightarrow \infty} N_{qdb}(x_n, x_m) = 0.$$

In similar way, we can obtain

$$\lim_{n, m \rightarrow \infty} N_{qdb}(x_m, x_n) = 0.$$

Thus, $\{\kappa_n\}$ is a 0-cauchy sequence in complete NT-qdb-MS. By definition of NT-qdb-MS, there exist a $\mathfrak{z} \in M$ such that $\kappa_n \rightarrow \mathfrak{z}$ as $n \rightarrow \infty$ that is

$$\lim_{n \rightarrow \infty} N_{\text{qdb}}(\kappa_n, \mathfrak{z}) = 0 = \lim_{n \rightarrow \infty} N_{\text{qdb}}(\mathfrak{z}, \kappa_n).$$

By using G-continuity of T and (a), it follow that

$$T(\lim_{n \rightarrow \infty} \kappa_{n+1}) = T\mathfrak{z} = \lim_{n \rightarrow \infty} T^{n+1}\kappa.$$

Now,

$$\begin{aligned} N_{\text{qdb}}(T\mathfrak{z}, \mathfrak{z}) &\leq s[N_{\text{qdb}}(T\mathfrak{z}, T^{n+1}\kappa) + N_{\text{qdb}}(T^{n+1}\kappa, \mathfrak{z})] \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

implies that $T\mathfrak{z} = \mathfrak{z}$ and also $N_{\text{qdb}}(\mathfrak{z}, \mathfrak{z}) = 0$.

Hence, \mathfrak{z} is a fixed point of T. For uniqueness, we assume that \mathfrak{z}^* be another fixed point of T different from \mathfrak{z} . Clearly, also $N_{\text{qdb}}(\mathfrak{z}^*, \mathfrak{z}^*) = 0$ by the above observation.

By the contractive condition (5), we have

$$\begin{aligned} N_{\text{qdb}}(\mathfrak{z}, \mathfrak{z}^*) &= N_{\text{qdb}}(T\mathfrak{z}, T\mathfrak{z}^*) \leq \mu [N_{\text{qdb}}(\mathfrak{z}, T\mathfrak{z}) + N_{\text{qdb}}(\mathfrak{z}^*, T\mathfrak{z}^*)] \\ &\leq \mu [N_{\text{qdb}}(\mathfrak{z}, \mathfrak{z}) + N_{\text{qdb}}(\mathfrak{z}^*, \mathfrak{z}^*)] \end{aligned}$$

implies that $N_{\text{qdb}}(\mathfrak{z}, \mathfrak{z}^*) = 0$ since $N_{\text{qdb}}(\mathfrak{z}, \mathfrak{z}) = 0$ and $N_{\text{qdb}}(\mathfrak{z}^*, \mathfrak{z}^*) = 0$ and which gives that $\mathfrak{z} = \mathfrak{z}^*$.

Thus, \mathfrak{z} is the unique fixed point of T.

Example 3.1: Let $M = \{0, 2, 3, 4\}$ be a NTG together binary operation $\diamond = \otimes_6$ in (Z_6, \times) . Neutrosophic triplet are:

$(0, 0, 0), (2, 4, 2), (3, 3, 3), (4, 4, 4)$ where $(\kappa, \gamma, \mathfrak{z})$ denote here, $\kappa \in M$ be any element, $\gamma = \text{neut}(\kappa)$ and $\mathfrak{z} = \text{anti}(\kappa)$. Now, we define a map $N_{\text{qdb}} : M \times M \rightarrow [0, \infty)$ such that

$$\begin{aligned} N_{\text{qdb}}(0, 0) &= 0, & N_{\text{qdb}}(0, 2) &= 4, & N_{\text{qdb}}(0, 3) &= 9, & N_{\text{qdb}}(0, 4) &= 16, \\ N_{\text{qdb}}(2, 2) &= 4, & N_{\text{qdb}}(3, 3) &= 9, & N_{\text{qdb}}(4, 4) &= 16, & N_{\text{qdb}}(2, 0) &= 8, \\ N_{\text{qdb}}(3, 0) &= 18, & N_{\text{qdb}}(4, 0) &= 32, & N_{\text{qdb}}(2, 3) &= 5, & N_{\text{qdb}}(3, 2) &= 10, \\ N_{\text{qdb}}(2, 4) &= 8, & N_{\text{qdb}}(4, 2) &= 20, & N_{\text{qdb}}(3, 4) &= 10, & N_{\text{qdb}}(4, 3) &= 17. \end{aligned}$$

Hence, $((M, \diamond), N_{\text{qdb}})$ is a NT-qdb-MS with coefficient $s \geq 1$, as we have proved in Example 2.4. and also it is complete since $\{0, 0, 0, 0, \dots\}$ is only Cauchy sequence which converges in M.

A mapping $T: M \rightarrow M$ defined as $T0=0, T2=3, T3=0$ and $T4=0$ and a graph $G = (V, E)$ defined as $V(G) = M$ and $E(G) = \{(0, 0), (3, 3), (4, 4), (0, 3), (0, 4), (2, 4), (3, 4), (3, 0), (4, 0), (3, 2), (4, 2), (4, 3)\}$.

Now, we have the following cases to identify the contractive condition for all $\kappa, \gamma \in E(\tilde{G})$ as:

Case I: for $\kappa = 0$ and $\gamma = 3$

$$0 = N_{\text{qdb}}(T0, T3) \leq \mu[N_{\text{qdb}}(0, T0) + N_{\text{qdb}}(3, T3)] = 18 \mu.$$

Case II: for $\kappa = 0$ and $\gamma = 4$

$$0 = N_{\text{qdb}}(T0, T4) \leq \mu[N_{\text{qdb}}(0, T0) + N_{\text{qdb}}(4, T4)] = 32 \mu.$$

Case III: for $\kappa = 2$ and $\gamma = 4$

$$18 = N_{qdb}(T2, T4) \leq \mu[N_{qdb}(2, T2)) + N_{qdb}(4, T4)] = 37 \mu.$$

Case IV: for $\kappa=3$ and $\gamma=4$

$$0 = N_{qdb}(T3, T4) \leq \mu[N_{qdb}(3, T3)) + N_{qdb}(4, T4)] = 50 \mu.$$

Case V: for $\kappa=3$ and $\gamma=0$

$$0 = N_{qdb}(T3, T0) \leq \mu[N_{qdb}(3, T3)) + N_{qdb}(0, T0)] = 18\mu.$$

Case VI: for $\kappa=4$ and $\gamma=0$

$$0 = N_{qdb}(T4, T0) \leq \mu[N_{qdb}(4, T4)) + N_{qdb}(0, T0)] = 32 \mu.$$

Case VII: for $\kappa=3$ and $\gamma=2$

$$9 = N_{qdb}(T3, T2) \leq \mu[N_{qdb}(3, T3)) + N_{qdb}(2, T2)] = 21 \mu$$

Case VIII: for $\kappa=4$ and $\gamma=2$

$$9 = N_{qdb}(T4, T2) \leq \mu[N_{qdb}(4, T4)) + N_{qdb}(2, T2)] = 37 \mu.$$

Case IX: for $\kappa=4$ and $\gamma=3$

$$0 = N_{qdb}(T4, T3) \leq \mu[N_{qdb}(4, T4)) + N_{qdb}(3, T3)] = 50 \mu.$$

Similarly, for $\kappa=0$ and $\gamma=0$, for $\kappa=3$ and $\gamma=3$ and for $\kappa=4$ and $\gamma=4$

This shows that contractive condition of Theorem 3.2 is satisfied for $\mu = \frac{18}{37}$ for all $\kappa, \gamma \in E(\tilde{G})$. Thus all

the conditions of Theorem 3.2 are satisfied. Therefore, 0 is a unique fixed point.

Conclusion: In this article, we reformulated the definition of NTMS and NTPMS and presented counter examples for dissimilarity of definitions with examples in [18,19]. Also, we established a new space NT-qdb-MS which is the generalization of the spaces established by Sahin et al. in [18,19]. Also, we studied some of their properties. Concept of NT-qdb-MS is absolutely different from classical quasi-dislocated b- metric space. The significance of NT-qdb-MS is that it provides a different space structure to those structures which are not quasi-dislocated-b- metric space with respect to some functions that not satisfy triangular inequality for all κ, γ, z . Finally, we proved generalize version of Banach fixed theorem and Kannan fixed point theorem in the framework of NT-qdb-MS with an example.

Open problems:

1. Can we also prove name theorems such as Chatterjee, Sehgal, Hardy and Rogers, Ciric, Meir-Keeler, F-contraction fixed point theorems in NTMS, NTPMS or NT-qdb-MS?
2. Can we extend the Theorem 3.1 and 3.2 for more than one mapping?

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Pairwise Neutrosophic- b -Open Set in Neutrosophic Bitopological Spaces

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Abstract: In this paper we introduce the notion of neutrosophic- b -open set, pairwise neutrosophic- b -open set in neutrosophic bitopological spaces. We have investigated some of their basic properties and established relation between the other existing notions.

Keywords: *Neutrosophic set; Neutrosophic topology; Neutrosophic bitopology; Neutrosophic- b -open set.*

1. Introduction

Smarandache (1998) introduced the notion of neutrosophic set as a generalization of intuitionistic fuzzy set. The concept of neutrosophic topological space was introduced by Salama and Alblowi (2012a). Salama and Alblowi (2012b) introduced the concept of generalized *neutrosophic set* and generalized neutrosophic topological space. Thereafter Ozturk and Ozkan (2019) introduce the concept of neutrosophic bitopological space. The concept of b -open sets in topological space was introduced by Andrijevic (1996). Ebenanjar, Immaculate, and Wilfred (2018) introduced neutrosophic b -open sets in neutrosophic topological spaces. Thangavelu and Thamizharsi (2011) introduce the concept of b_i -open sets in bitopological spaces. In this paper, we introduce the notion of pairwise neutrosophic b -open set in neutrosophic bitopological spaces.

2. Preliminaries and some properties

Definition 2.1. [Smarandache, 2005] Let X be a non-empty set. Then H , a neutrosophic set (NS in short) over X is denoted as follows:

$H = \{(y, T_H(y), I_H(y), F_H(y)) : y \in X \text{ and } T_H(y), I_H(y), F_H(y) \in]0, 1^*[, \text{ where } T_H(y), I_H(y) \text{ and } F_H(y) \text{ are the degree of truthness, indeterminacy and falseness respectively.}$

There is no restriction on the sum of $T_G(y)$, $F_G(y)$ and $I_G(y)$, so

$$0 \leq T_H(y) + I_H(y) + F_H(y) \leq 3^*.$$

Definition 2.2. [Smarandache, 2005] Let $H = \{(y, T_H(y), I_H(y), F_H(y)) : y \in X\}$ be a neutrosophic set over X . Then the complement of H is defined by $H^c = \{(y, 1-T_H(y), 1-I_H(y), 1-F_H(y)) : y \in X\}$.

Definition 2.3. [Smarandache, 2005] A neutrosophic set $H = \{(y, T_H(y), I_H(y), F_H(y)) : y \in X\}$ is contained in the other neutrosophic set $K = \{(y, T_K(y), I_K(y), F_K(y)) : y \in X\}$ (i.e. $H \subseteq K$) if and only if $T_H(y) \leq T_K(y)$, $I_H(y) \geq I_K(y)$, $F_H(y) \geq F_K(y)$, for each $y \in X$.

Definition 2.4. [Smarandache, 2005] If $H = \{(y, T_H(y), I_H(y), F_H(y)) : y \in X\}$ and $K = \{(y, T_K(y), I_K(y), F_K(y)) : y \in X\}$ are any two neutrosophic sets over X , then $H \cup K$ and $H \cap K$ is defined by
 $H \cup K = \{(y, T_H(y) \vee T_K(y), I_H(y) \wedge I_K(y), F_H(y) \wedge F_K(y)) : y \in X\};$
 $H \cap K = \{(y, T_H(y) \wedge T_K(y), I_H(y) \vee I_K(y), F_H(y) \vee F_K(y)) : y \in X\}.$

Here we can construct two neutrosophic set 0_N and 1_N over X as follows:

- 1) $0_N = \{(y, 0, 0, 1) : y \in X\};$
- 2) $1_N = \{(y, 1, 0, 0) : y \in X\}.$

The neutrosophic set 0_N is known as neutrosophic null set and neutrosophic set 1_N is known as neutrosophic whole set over X . Also, 0_N and 1_N over X have three other types of representation too. Clearly, $0_N \subseteq 1_N$.

The neutrosophic topological space is defined as follows:

Definition 2.5. [Salama & Alblowi, 2012a] Let X be a non-empty fixed set and τ be the family of some NSs over X . Then τ is said to be a neutrosophic topology (NT in short) on X if the following properties holds:

1. $0_N, 1_N \in \tau$,
2. $T_1, T_2 \in \tau \Rightarrow T_1 \cap T_2 \in \tau$,
3. $\cup_{i \in \Delta} T_i \in \tau$, for every $\{T_i : i \in \Delta\} \subseteq \tau$.

Then the pair (X, τ) is called a neutrosophic topological space (NTS in short). The members of τ are called neutrosophic-open set (NOS in short). A NS D is called a neutrosophic-closed set (NCS in short) in (X, τ) if and only if D^c is a neutrosophic-open set.

Example 2.1. Let $X = \{z_1, z_2\}$ and let

$$G = \{(z_1, 0.6, 0.5, 0.3), (z_2, 0.6, 0.7, 0.3) : z_1, z_2 \in X\}$$

$$H = \{(z_1, 0.5, 0.6, 0.8), (z_2, 0.4, 0.9, 0.8) : z_1, z_2 \in X\}$$

$$K = \{(z_1, 0.6, 0.6, 0.3), (z_2, 0.4, 0.8, 0.6) : z_1, z_2 \in X\}$$

be three NSs over X . Then clearly the family $\tau = \{0_N, 1_N, G, H, K\}$ is a NT on X .

Example 2.2. Let $X = \{z_1, z_2, z_3\}$ and let

$$L = \{(z_1, 0.6, 0.7, 0.4), (z_2, 0.5, 0.6, 0.8), (z_3, 0.5, 0.5, 0.4) : z_1, z_2, z_3 \in X\}$$

$$K = \{(z_1, 0.4, 0.9, 0.8), (z_2, 0.3, 0.7, 0.8), (z_3, 0.4, 0.6, 0.8) : z_1, z_2, z_3 \in X\}$$

$$J = \{(z_1, 0.4, 0.9, 0.9), (z_2, 0.2, 0.8, 0.9), (z_3, 0.3, 0.5, 0.8) : z_1, z_2, z_3 \in X\}$$

be three NSs over X .

Here the collection $\tau = \{0_N, 1_N, L, K, J\}$ is not a *neutrosophic topology* on X because $K \cap J \notin \tau$.

Definition 2.5. [Salama & Alblowi, 2012a] Let (X, τ) be a NTS and H be a NS over X . The neutrosophic-interior (in short N_{int}) and neutrosophic-closure (in short N_{cl}) of H are defined by

$$N_{int}(H) = \cup \{P : P \text{ is an NOS in } X \text{ and } P \subseteq H\};$$

$$N_{cl}(H) = \cap \{Q : Q \text{ is an NCS in } X \text{ and } H \subseteq Q\}.$$

Proposition 2.1. [Salama & Alblowi, 2012a] Let C, D are two neutrosophic subsets of (X, τ) . Then the following properties hold:

- 1) $C \subseteq N_{cl}(C)$;
- 2) $N_{int}(C) \subseteq C$;
- 3) $N_{int}(C) \subseteq N_{cl}(C)$;
- 4) $C \subseteq D \Rightarrow N_{int}(C) \subseteq N_{int}(D)$;
- 5) $C \subseteq D \Rightarrow N_{cl}(C) \subseteq N_{cl}(D)$;
- 6) $N_{cl}(0_N) = 0_N$;
- 7) $N_{int}(1_N) = 1_N$;
- 8) $N_{cl}(C \cup D) = N_{cl}(C) \cup N_{cl}(D)$;
- 9) $N_{int}(C \cup D) \supseteq N_{int}(C) \cup N_{int}(D)$;
- 10) $N_{int}(C \cap D) = N_{int}(C) \cap N_{int}(D)$;
- 11) $N_{cl}(C \cap D) \subseteq N_{cl}(C) \cap N_{cl}(D)$;
- 12) C is neutrosophic closed if and only if $N_{cl}(C) = C$;
- 13) C is neutrosophic open if and only if $N_{int}(C) = C$.

The neutrosophic bitopological space is defined as follows:

Definition 2.6. [Ozturk & Ozkan, 2019] Assume that (X, τ_1) and (X, τ_2) be two different NTSs. Then the triplet (X, τ_1, τ_2) is called a *neutrosophic bitopological space* (NBTS in short).

Example 2.3. Let $X = \{z_1, z_2\}$ and let

$$U_1 = \{(z_1, 0.6, 0.5, 0.4), (z_2, 0.8, 0.7, 0.6) : z_1, z_2 \in X\},$$

$$U_2 = \{(z_1, 0.4, 0.6, 0.5), (z_2, 0.7, 0.8, 0.8) : z_1, z_2 \in X\},$$

$$U_3 = \{(z_1, 0.4, 0.6, 0.8), (z_2, 0.6, 0.9, 0.8) : z_1, z_2 \in X\},$$

$$U_4 = \{(z_1, 0.6, 0.8, 0.7), (z_2, 0.4, 0.6, 0.7) : z_1, z_2 \in X\},$$

$$U_5 = \{(z_1, 0.8, 0.4, 0.5), (z_2, 0.6, 0.4, 0.5) : z_1, z_2 \in X\},$$

$$U_6 = \{(z_1, 0.7, 0.5, 0.6), (z_2, 0.6, 0.5, 0.5) : z_1, z_2 \in X\}$$

are six NSs over X . Then clearly $\tau_1 = \{0_N, 1_N, U_1, U_2, U_3\}$ and $\tau_2 = \{0_N, 1_N, U_4, U_5, U_6\}$ are two different NTs on X . So the triplet (X, τ_1, τ_2) is a *neutrosophic bitopological space*.

Definition 2.7. [Ozturk & Ozkan, 2019] Let (X, τ_1, τ_2) be a neutrosophic bitopological space. Then H , a neutrosophic set over X is called a pairwise open set in (X, τ_1, τ_2) if there exist an open set G_1 in τ_1 and an open set G_2 in τ_2 such that $H = G_1 \cup G_2$.

Remark 2.2. Let G be a neutrosophic subset of a neutrosophic bitopological space (X, τ_1, τ_2) . Then we shall use the following notations:

- 1) $N_{cl}^i(G) = \tau_i$ -neutrosophic-closure of G ($i=1, 2$);
- 2) $N_{int}^i(G) = \tau_i$ -neutrosophic-interior of G ($i=1, 2$).

3. τ_i -neutrosophic- b -open set:

Definition 3.1. Let (X, τ_1, τ_2) be a neutrosophic bitopological space. Then P , a NS over X is called

- 1) τ_i -neutrosophic-semi-open if and only if $P \subseteq N_{cl}^i N_{int}^i(P)$;
- 2) τ_i -neutrosophic-pre-open if and only if $P \subseteq N_{int}^i N_{cl}^i(P)$;
- 3) τ_i -neutrosophic- b -open if and only if $P \subseteq N_{cl}^i N_{int}^i(P) \cup N_{int}^i N_{cl}^i(P)$.

Remark 3.1. In a neutrosophic bitopological space (X, τ_1, τ_2) , a NS P over X is called a τ_i -neutrosophic- b -closed set if and only if its complement is τ_i -neutrosophic- b -open set.

We formulate the following results based on the above definitions.

Proposition 3.1. In a neutrosophic bitopological space (X, τ_1, τ_2) , if P is τ_i -neutrosophic-semi-open (τ_i -neutrosophic-pre-open), then P is τ_i -neutrosophic- b -open.

Proposition 3.2. In a neutrosophic bitopological space (X, τ_1, τ_2) , the union of two τ_i -neutrosophic- b -open set is a τ_i -neutrosophic- b -open set.

4. τ_{ij} -neutrosophic- b -open set:

Definition 4.1. Assume that (X, τ_1, τ_2) be a neutrosophic bitopological space. Then P , a NS over X is called

- 1) τ_{ij} -neutrosophic-semi-open if and only if $P \subseteq N_{cl}^i N_{int}^j(P)$;
- 2) τ_{ij} -neutrosophic-pre-open if and only if $P \subseteq N_{int}^j N_{cl}^i(P)$;
- 3) τ_{ij} -neutrosophic- b -open if and only if $P \subseteq N_{cl}^i N_{int}^j(P) \cup N_{int}^j N_{cl}^i(P)$.

Remark 4.1. A neutrosophic set P over X is called a τ_{ij} -neutrosophic- b -closed set if and only if P^c (complement of P) is τ_{ij} -neutrosophic- b -open set in (X, τ_1, τ_2) .

Definition 4.2. Assume that (X, τ_1, τ_2) be a neutrosophic bitopological space. Then a neutrosophic set G over X is said to be a

- 1) τ_{ij} -neutrosophic- p -set if and only if $N_{cl}^i N_{int}^j(G) \subseteq N_{int}^i N_{cl}^j(G)$;

- 2) Contra τ_{ij} -neutrosophic- p -set if and only if $N_{cl}^j N_{int}^i(G) \subseteq N_{int}^i N_{cl}^j(G)$;
- 3) τ_{ij} -neutrosophic- q -set if and only if $N_{int}^j N_{cl}^i(G) \subseteq N_{cl}^i N_{int}^j(G)$.
- 4) Contra τ_{ij} -neutrosophic- q -set if and only if $N_{int}^i N_{cl}^j(G) \subseteq N_{cl}^i N_{int}^j(G)$.

Theorem 4.1. In a neutrosophic bitopological space (X, τ_1, τ_2) ,

- 1) if G is τ_i -neutrosophic-closed and τ_{ij} -neutrosophic-pre-open then G is τ_{ij} -neutrosophic-semi-open.
- 2) if G is τ_j -neutrosophic-open and τ_{ij} -neutrosophic-semi-open then G is τ_{ij} -neutrosophic-pre-open.

Proof:

- 1) Let (X, τ_1, τ_2) be a neutrosophic bitopological space and G be a neutrosophic set over X , which is both τ_i -neutrosophic-closed and τ_{ij} -neutrosophic-pre-open. So, we have

$$G = N_{cl}^i(G) \quad \dots\dots\dots(1)$$

$$\text{and } G \subseteq N_{int}^j N_{cl}^i(G) \quad \dots\dots\dots(2)$$

From eq (2) we have $G \subseteq N_{int}^j N_{cl}^i(G)$

$$= N_{int}^j(G) \quad [\text{by eq (1)}]$$

$$\Rightarrow G \subseteq N_{int}^j(G) \subseteq N_{cl}^i N_{int}^j(G)$$

$$\Rightarrow G \subseteq N_{cl}^i N_{int}^j(G)$$

Hence, G is a τ_{ij} -neutrosophic-semi-open set in (X, τ_1, τ_2) .

- 2) Let (X, τ_1, τ_2) be a neutrosophic bitopological space and G be a NS over X , which is both τ_j -neutrosophic-open and τ_{ij} -neutrosophic-semi-open. So, we have

$$G = N_{int}^j(G) \quad \dots\dots\dots(3)$$

$$\text{and } G \subseteq N_{cl}^i N_{int}^j(G) \quad \dots\dots\dots(4)$$

From eq (4) we have

$$G \subseteq N_{cl}^i N_{int}^j(G)$$

$$= N_{cl}^i(G) \quad [\text{by eq (3)}]$$

$$\Rightarrow G \subseteq N_{cl}^i(G)$$

$$\Rightarrow N_{int}^j(G) \subseteq N_{int}^j N_{cl}^i(G)$$

$$\Rightarrow G = N_{int}^j(G) \subseteq N_{int}^j N_{cl}^i(G) \quad [\text{since } G = N_{int}^j(G)]$$

$$\Rightarrow G \subseteq N_{int}^j N_{cl}^i(G)$$

Hence, G is a τ_{ij} -neutrosophic-pre-open set in (X, τ_1, τ_2) .

Theorem 4.2. Let (X, τ_1, τ_2) be a neutrosophic bitopological space. If A is τ_{ij} -neutrosophic-semi-open (τ_{ij} -neutrosophic-pre-open), then A is τ_{ij} -neutrosophic- b -open.

Proof: Let us assume that A is τ_{ij} -neutrosophic-semi-open set in a neutrosophic bitopological space (X, τ_1, τ_2) . Then $A \subseteq N_{cl}^i N_{int}^j(A)$.

Now, $A \subseteq N_{cl}^i N_{int}^j(A)$

$$\Rightarrow A \subseteq N_{cl}^i N_{int}^j(A) \cup N_{int}^j N_{cl}^i(A).$$

Therefore, A is τ_{ij} -neutrosophic- b -open in (X, τ_1, τ_2) .

Similarly, we can show that if A is τ_{ij} -neutrosophic-pre-open set in (X, τ_1, τ_2) then it is τ_{ij} -neutrosophic- b -open.

Theorem 4.3. Let (X, τ_1, τ_2) be a neutrosophic bitopological space.

- 1) If A is τ_{ij} -neutrosophic- b -open, contra τ_{ji} -neutrosophic- p -set then A is τ_{ij} -neutrosophic-pre-open set;
- 2) If A is τ_{ij} -neutrosophic- b -open, contra τ_{ij} -neutrosophic- q -set then A is τ_{ij} -neutrosophic-semi-open set;
- 3) If A is τ_{ij} -neutrosophic- b -open, τ_{ij} -neutrosophic- p -set and contra τ_{ji} -neutrosophic- q -set then A is τ_{ji} -neutrosophic- b -open set;
- 4) If A is τ_{ij} -neutrosophic- q -set (τ_{ij} -neutrosophic- p -set) then A^c is contra τ_{ji} -neutrosophic- p -set (contra τ_{ij} -neutrosophic- q -set).

Proof:

- 1) Let A be both τ_{ij} -neutrosophic- b -open and contra τ_{ji} -neutrosophic- p -set in a neutrosophic bitopological space (X, τ_1, τ_2) .

$$\text{Then, we have } A \subseteq N_{cl}^i N_{int}^j(A) \cup N_{int}^j N_{cl}^i(A) \quad \dots\dots\dots(5)$$

$$\text{and } N_{cl}^i N_{int}^j(A) \subseteq N_{int}^j N_{cl}^i(A) \quad \dots\dots\dots(6)$$

From eqs (5) & (6) we get

$$\begin{aligned} A &\subseteq N_{cl}^i N_{int}^j(A) \cup N_{int}^j N_{cl}^i(A) \\ &\subseteq N_{int}^j N_{cl}^i(A) \cup N_{int}^j N_{cl}^i(A) \\ &= N_{int}^j N_{cl}^i(A) \\ \Rightarrow A &\subseteq N_{int}^j N_{cl}^i(A) \end{aligned}$$

Therefore, A is τ_{ij} -neutrosophic-pre-open set in (X, τ_1, τ_2) .

- 2) The proof is analogous to the proof of part (1), so omitted.
- 3) Let A be τ_{ij} -neutrosophic- b -open, τ_{ij} -neutrosophic- p -set and contra τ_{ji} -neutrosophic- q -set in a neutrosophic bitopological space (X, τ_1, τ_2) . Then we have

$$A \subseteq N_{cl}^i N_{int}^j(A) \cup N_{int}^j N_{cl}^i(A), \quad \dots\dots\dots(7)$$

$$N_{cl}^i N_{int}^j(A) \subseteq N_{int}^j N_{cl}^i(A) \quad \dots\dots\dots(8)$$

$$\text{and } N_{int}^j N_{cl}^i(A) \subseteq N_{cl}^i N_{int}^j(A) \quad \dots\dots\dots(9)$$

$$\begin{aligned} \text{From eq (7) we get } A &\subseteq N_{cl}^i N_{int}^j(A) \cup N_{int}^j N_{cl}^i(A) \\ &\subseteq N_{int}^j N_{cl}^i(A) \cup N_{cl}^i N_{int}^j(A) \quad [\text{by eqs (8) \& (9)}] \\ \Rightarrow A &\subseteq N_{int}^j N_{cl}^i(A) \cup N_{cl}^i N_{int}^j(A) \end{aligned}$$

Therefore, A is τ_{ji} -neutrosophic- b -open set in (X, τ_1, τ_2) .

Theorem 4.4. In a neutrosophic bitopological space (X, τ_1, τ_2)

- 1) if A is τ_{ij} -neutrosophic-semi-open and τ_{ij} -neutrosophic- p -set then A is τ_{ji} -neutrosophic-pre-open;
- 2) If A is τ_{ji} -neutrosophic-semi-open and contra τ_{ji} -neutrosophic- p -set then A is τ_{ji} -neutrosophic-pre-open.

Proof:

- 1) Let (X, τ_1, τ_2) be a neutrosophic bitopological space and A is both τ_{ij} -neutrosophic-semi-open and τ_{ji} -neutrosophic- p -set.

Since, A is τ_{ij} -neutrosophic-semi-open, so we have

$$A \subseteq N_{cl}^i N_{int}^j(A) \quad \dots\dots\dots(10)$$

Since, A is τ_{ji} -neutrosophic- p -set, so

$$N_{cl}^i N_{int}^j(A) \subseteq N_{int}^i N_{cl}^j(A) \quad \dots\dots\dots(11)$$

From eqs (10) & (11), we've got

$$A \subseteq N_{int}^i N_{cl}^j(A).$$

Hence, A is τ_{ji} -neutrosophic-pre-open in (X, τ_1, τ_2) .

- 2) The proof is analogous to the proof of the first part, so omitted.

Theorem 4.5. Let (X, τ_1, τ_2) be an neutrosophic bitopological space.

- 1) If A is τ_{ij} -neutrosophic- p -set and τ_{ji} -neutrosophic- q -set then $N_{cl}^i N_{int}^j(A) \subseteq N_{cl}^j N_{int}^i(A)$;
- 2) If A is contra τ_{ij} -neutrosophic- p -set and contra τ_{ji} -neutrosophic- q -set then $N_{cl}^j N_{int}^i(A) \subseteq N_{cl}^i N_{int}^j(A)$.

Proof:

- 1) Let (X, τ_1, τ_2) be a neutrosophic bitopological space and A be both τ_{ij} -neutrosophic- p -set and τ_{ji} -neutrosophic- q -set. Then, we have

$$N_{cl}^i N_{int}^j(A) \subseteq N_{int}^i N_{cl}^j(A) \quad \dots\dots\dots(12)$$

$$\text{and } N_{int}^i N_{cl}^j(A) \subseteq N_{cl}^j N_{int}^i(A) \quad \dots\dots\dots(13)$$

From eqs (12) & (13), we get

$$N_{cl}^i N_{int}^j(A) \subseteq N_{cl}^j N_{int}^i(A).$$

- 2) Let (X, τ_1, τ_2) be a neutrosophic bitopological space and A be both contra τ_{ij} -neutrosophic- p -set and contra τ_{ji} -neutrosophic- q -set. Then, we have

$$N_{cl}^j N_{int}^i(A) \subseteq N_{int}^i N_{cl}^j(A) \quad \dots\dots\dots(14)$$

$$N_{int}^i N_{cl}^j(A) \subseteq N_{cl}^i N_{int}^j(A) \quad \dots\dots\dots(15)$$

From eqs (14) & (15), we get

$$N_{cl}^j N_{int}^i(A) \subseteq N_{cl}^i N_{int}^j(A).$$

5. Pairwise τ_{ij} - b -open:

Definition 5.1. A neutrosophic set H is said to be pairwise τ_{ij} -neutrosophic-semi-open set (pairwise τ_{ij} -neutrosophic-pre-open set) in a neutrosophic bitopological space (X, τ_1, τ_2) if $H=K \cup L$, where K is a τ_{ij} -neutrosophic-semi-open set (τ_{ij} -neutrosophic-pre-open set) and L is a τ_{ji} -neutrosophic-semi-open set (τ_{ji} -neutrosophic-pre-open set) in (X, τ_1, τ_2) .

Definition 5.2. A neutrosophic set H is said to be pairwise τ_{ij} -neutrosophic- b -open set in a neutrosophic bitopological space (X, τ_1, τ_2) if $H=K \cup L$, where K is a τ_{ij} -neutrosophic- b -open set and L is a τ_{ji} -neutrosophic- b -open set in (X, τ_1, τ_2) .

Theorem 5.1. The union of two pairwise τ_{ij} -neutrosophic- b -open set in a neutrosophic bitopological space (X, τ_1, τ_2) is again a pairwise τ_{ij} -neutrosophic- b -open set.

Proof: Let A, B be two pairwise τ_{ij} -neutrosophic- b -open set in a neutrosophic bitopological space (X, τ_1, τ_2) . Then there exists two τ_{ij} -neutrosophic- b -open set G_1, G_2 and two τ_{ji} -neutrosophic- b -open set H_1, H_2 such that $A= G_1 \cup H_1$ and $B= G_2 \cup H_2$.

Since, G_1, G_2 are τ_{ij} -neutrosophic- b -open set so

$$G_1 \subseteq N_{cl}^i N_{int}^j(G_1) \cup N_{int}^j N_{cl}^i(G_1) \quad \dots\dots\dots(16)$$

$$\text{and } G_2 \subseteq N_{cl}^i N_{int}^j(G_2) \cup N_{int}^j N_{cl}^i(G_2) \quad \dots\dots\dots(17)$$

Since, H_1, H_2 are τ_{ji} -neutrosophic- b -open set so

$$H_1 \subseteq N_{cl}^j N_{int}^i(H_1) \cup N_{int}^i N_{cl}^j(H_1) \quad \dots\dots\dots(18)$$

$$\text{and } H_2 \subseteq N_{cl}^j N_{int}^i(H_2) \cup N_{int}^i N_{cl}^j(H_2) \quad \dots\dots\dots(19)$$

Now, we have

$$\begin{aligned} G_1 \cup G_2 &\subseteq N_{cl}^i N_{int}^j(G_1) \cup N_{int}^j N_{cl}^i(G_1) \cup N_{cl}^i N_{int}^j(G_2) \cup N_{int}^j N_{cl}^i(G_2) \quad [\text{using eqs (16) \& (17)}] \\ &= N_{cl}^i N_{int}^j(G_1) \cup N_{cl}^i N_{int}^j(G_2) \cup N_{int}^j N_{cl}^i(G_1) \cup N_{int}^j N_{cl}^i(G_2) \\ &\subseteq N_{cl}^i (N_{int}^j(G_1) \cup N_{int}^j(G_2)) \cup N_{int}^j (N_{cl}^i(G_1) \cup N_{cl}^i(G_2)) \\ &\subseteq N_{cl}^i (N_{int}^j(G_1 \cup G_2)) \cup N_{int}^j (N_{cl}^i(G_1 \cup G_2)) \end{aligned}$$

$\Rightarrow G_1 \cup G_2$ is a τ_{ij} -neutrosophic- b -open set.

Further, we have

$$\begin{aligned} H_1 \cup H_2 &\subseteq N_{cl}^j N_{int}^i(H_1) \cup N_{int}^i N_{cl}^j(H_1) \cup N_{cl}^j N_{int}^i(H_2) \cup N_{int}^i N_{cl}^j(H_2) \quad [\text{using eqs (18) \& (19)}] \\ &= N_{cl}^j N_{int}^i(H_1) \cup N_{cl}^j N_{int}^i(H_2) \cup N_{int}^i N_{cl}^j(H_1) \cup N_{int}^i N_{cl}^j(H_2) \\ &\subseteq N_{cl}^j (N_{int}^i(H_1) \cup N_{int}^i(H_2)) \cup N_{int}^i (N_{cl}^j(H_1) \cup N_{cl}^j(H_2)) \\ &\subseteq N_{cl}^j (N_{int}^i(H_1 \cup H_2)) \cup N_{int}^i (N_{cl}^j(H_1 \cup H_2)) \end{aligned}$$

$\Rightarrow H_1 \cup H_2$ is a τ_{ji} -neutrosophic- b -open set.

Hence, $A \cup B = (G_1 \cup H_1) \cup (G_2 \cup H_2) = (G_1 \cup G_2) \cup (H_1 \cup H_2) = G \cup H$.

Therefore there exists a τ_{ij} -neutrosophic- b -open set $G=(G_1 \cup G_2)$ and a τ_{ji} -neutrosophic- b -open set $H=(H_1 \cup H_2)$ such that $A \cup B= G \cup H$. Hence $A \cup B$ is a pairwise τ_{ij} -neutrosophic- b -open set. Thus the

union of two pairwise τ_{ij} -neutrosophic- b -open set in a neutrosophic bitopological space (X, τ_1, τ_2) is again a pairwise τ_{ij} -neutrosophic- b -open set.

Theorem 5.2. In a neutrosophic bitopological space (X, τ_1, τ_2) , every pairwise τ_{ij} -neutrosophic-semi open set (pairwise τ_{ij} -neutrosophic-pre-open set) is a pairwise τ_{ij} -neutrosophic- b -open set.

Proof: Let G be a pairwise τ_{ij} -neutrosophic-semi-open set (pairwise τ_{ij} -neutrosophic-pre-open set). Then there exist a τ_{ij} -neutrosophic-semi-open set A (τ_{ij} -neutrosophic-pre-open set A) and a τ_{ji} -neutrosophic-semi-open set B (τ_{ji} -neutrosophic-pre-open set B) such that $G=A \cup B$.

In theorem 4.2, it is clearly shown that every τ_{ij} -neutrosophic-semi-open set (τ_{ij} -neutrosophic-pre-open set) is a τ_{ij} -neutrosophic- b -open set and every τ_{ji} -neutrosophic-semi-open set (τ_{ji} -neutrosophic-pre-open set) is a τ_{ji} -neutrosophic- b -open set. So A be τ_{ij} -neutrosophic- b -open and B be τ_{ji} -neutrosophic- b -open set. Therefore, there exist a τ_{ij} -neutrosophic- b -open set A and a τ_{ji} -neutrosophic- b -open set B such that $G=A \cup B$. Hence, G is a pairwise τ_{ij} -neutrosophic- b -open set. Thus every pairwise τ_{ij} -neutrosophic-semi-open set (pairwise τ_{ij} -neutrosophic-pre-open set) is a pairwise τ_{ij} -neutrosophic- b -open set.

6. Conclusion

In this article, we studied neutrosophic- b -open set, pairwise neutrosophic- b -open set in neutrosophic bitopological spaces and investigate their basic properties. By defining neutrosophic- b -open set, pairwise neutrosophic- b -open set, we prove some theorems on neutrosophic bitopological spaces and some examples are given.

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Decomposition of Single-Valued Neutrosophic Ideal Continuity via Fuzzy Idealization

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Abstract: The aim of this paper is to introduce various types of r -single-valued neutrosophic open sets based on the single-valued neutrosophic ideals in Šostak Sense. Different mappings of single-valued continuity and ideal continuity based on the r -single-valued neutrosophic ideal openness are defined and many implications between them are investigated with counterexamples illustrated.

Keywords: r -single-valued neutrosophic open set; r -single-valued neutrosophic ideal closed set; single-valued neutrosophic continuous mappings and single-valued neutrosophic ideal continuous mappings.

1. Introduction

In the classic text, Kuratowski [1] dealt with the genesis of the concept of ideal in general topological spaces. This area of study is approached by many others and hence some sorts of ideals arise as one goes further in mathematics such as the ideal of finite subsets of \mathfrak{X} , the ideal of nowhere dense sets and ideal of meager sets. Many topologists introduced distinct types of operators as regards ideals, compatibility property, compactness module an ideal and other concedes. Vaidyanathaswamy [2] introduced the concept of local function of \tilde{f} in relation to $\tilde{\tau}$. The notion of fuzzy ideal and the concept of fuzzy local function of \tilde{f} with respect to $\tilde{\tau}$ had been introduced and examined by Sarkar [3]. Besides, the notion of compatibility of fuzzy ideals with fuzzy topologies had been introduced and studied by Sarkar. In [4], Šostak initiated a new definition of fuzzy topology, which is termed "fuzzy topology in Šostak sense", as an extension of both crisp topology and Chang's fuzzy topology, in the logic that not only the objects are fuzzified, but also the axiomatics. Šostak [5-7] presented some rules and explained how such an extension can be realized. Saber et al [8] familiarized and considered the notion of fuzzy ideal and the concept of fuzzy local function of \tilde{f} in respect of $\tilde{\tau}$ in Šostak sense. Saber et al [9-13] provided several rules and displayed how such an extension can be acquired.

Thus, Smarandache [14] generalizes almost all the existing logics like, fuzzy logic, intuitionistic fuzzy logic etc. After this, many researchers used neutrosophic sets and logic in topological spaces, such as Das et al. [15], Fatimah et al. [16], Riaz et al. [17], Porselvi et al. [18], Singh et al. [19]. In recent times, Abdel-Basset et al. have studied a novel neutrosophic approach [20-23] in many areas, in other words, information and communication technology. In the meantime, Salama et al. [24, 25] investigated the notions of generalized neutrosophic set (\mathcal{NS}) and Intuitionistic neutrosophic set (\mathcal{IFS}). Respectively, Hur et al [26, 27] brought to light classifications neutrosophic H-set (\mathcal{NHS}) and (\mathcal{NCS}) as well as neutrosophic crisp as they scrutinized them in a universe topological position. Still,

Salama and Alblowi [28] displayed neutrosophic topology in as much as they claimed a number of its features. Wang et al [29], among many others, shaped the single-valued neutrosophic set concept. Presently, Kim et al grappled with a neutrosophic partition single-value, neutrosophic equivalence relation single-value and neutrosophic relation single-value. The notion of single-valued neutrosophic ideal, single-valued neutrosophic ideal open local function and single-valued neutrosophic ideal open compatible are explored in (2020) by Saber et al [30, 31].

This paper is arranged as follows. Preliminaries of single-value neutrosophic sets and single-valued neutrosophic topology are reviewed in Section 2. In Section 3 and 4, we obtained very important relevant topics and results such as single-valued neutrosophic ideal closed sets in Šostak sense and single-valued neutrosophic ideal continuous (\mathcal{SVNI} – continuous) mappings, single-valued neutrosophic continuous (\mathcal{SVN} – continuous) mappings and investigated several characterizations of these crucial topics and ideas. These mappings are obviously considered to be generalizations of fuzzy ideal continuous mappings, introduced by Saber et al [32] In Section 5, we obtained very important relevant topics and results such as single-valued neutrosophic ideal closed sets ($r\text{-SVNSO}$) in Šostak sense and \mathcal{SVNI} – continuous. We have arrived to notable definitions theorems, and counter examples in detailed analysis to examine some of their substantial characteristics and to explore the best results and imports. We can safely claim that diverse decisive concepts in single-valued neutrosophic topology were established and generalized in this article. Distinct aspects like continuous and ideal continuous which have a major effect on the overall topology's notions were also considered.

Original aspects and credits of this article juxtaposed to pertinent recent research on groups related to it are very worthwhile. This study deals with continuous and ideal continuous of single-valued neutrosophic topological spaces (\mathcal{SVNTS}) in Šostak sense. The great import of this study is the introduction of the concept of r -single-valued neutrosophic open ($r\text{-SVNSO}$). The researchers secure some of its basic properties. Moreover, as an application, we give a multicriteria decision making for the combining effects of certain enzymes on chosen DNA.

2. Preliminaries

Here, in this section, we consider the fundamental concepts of single valued neutrosophic sets (briefly, \mathcal{SVNS}), single valued neutrosophic topological spaces (briefly, \mathcal{SVNTS}) and single-valued neutrosophic ideals (briefly, \mathcal{SVNI}). Although Section 2 is considered as a background for the material included in this paper.

Definition 2.1 [33] Suppose that $\tilde{\mathcal{X}}$ is a non empty set, then $\mathcal{S} = \{(\omega, \tilde{\gamma}_s, \tilde{\eta}_s, \tilde{\mu}_s) : \omega \in \tilde{\mathcal{X}}\}$, is called a neutrosophic set (briefly, \mathcal{NS}) in $\tilde{\mathcal{X}}$, where, $\tilde{\mu}_s, \tilde{\eta}_s, \tilde{\gamma}_s$ and the degree of non-membership (namely $\tilde{\mu}_s(\omega)$), the degree of indeterminacy (namely $\tilde{\eta}_s(\omega)$), and the degree of membership (namely $\tilde{\gamma}_s(\omega)$), for all $\omega \in \tilde{\mathcal{X}}$ to the set \mathcal{S} .

A neutrosophic set $\mathcal{S} = \langle \omega, \tilde{\gamma}_s, \tilde{\eta}_s, \tilde{\mu}_s : \omega \in \tilde{\mathcal{X}} \rangle$, can be identified as $\langle \tilde{\gamma}_s, \tilde{\eta}_s, \tilde{\mu}_s \rangle$ in $]^{-0}, 1^+]$ in $\tilde{\mathcal{X}}$.

Definition 2.2 [35] Suppose that \mathcal{S} and \mathcal{E} are \mathcal{NS} 's of the form $\mathcal{S} = \{(\omega, \tilde{\gamma}_s, \tilde{\eta}_s, \tilde{\mu}_s) : \omega \in \tilde{\mathcal{X}}\}$ and $\mathcal{E} = \{(\omega, \tilde{\gamma}_e, \tilde{\eta}_e, \tilde{\mu}_e) : \omega \in \tilde{\mathcal{X}}\}$. Then, $\mathcal{S} \subseteq \mathcal{E}$, iff for every $\omega \in \tilde{\mathcal{X}}$.

$$\inf \tilde{\eta}_s(\omega) \geq \inf \tilde{\eta}_e(\omega), \quad \inf \tilde{\mu}_s(\omega) \geq \inf \tilde{\mu}_e(\omega) \text{ and } \inf \tilde{\gamma}_e(\omega) \leq \inf \tilde{\gamma}_s(\omega), \\ \sup \tilde{\eta}_s(\omega) \geq \sup \tilde{\eta}_e(\omega), \quad \sup \tilde{\mu}_s(\omega) \geq \sup \tilde{\mu}_e(\omega) \text{ and } \sup \tilde{\gamma}_s(\omega) \leq \sup \tilde{\gamma}_e(\omega).$$

Definition 2.3 [29] Suppose that \mathcal{S} is a space of points (objects) with a generic element in $\tilde{\mathfrak{X}}$ denoted by ω . Then, \mathcal{S} is called a single-valued neutrosophic set (briefly, \mathcal{SVNS}) in $\tilde{\mathfrak{X}}$, if \mathcal{S} has the form $\mathcal{S} = \langle \tilde{\gamma}_{\mathcal{S}}, \tilde{\eta}_{\mathcal{S}}, \tilde{\mu}_{\mathcal{S}} \rangle$, where $\tilde{\gamma}_{\mathcal{S}}, \tilde{\eta}_{\mathcal{S}}, \tilde{\mu}_{\mathcal{S}}: \tilde{\mathfrak{X}} \rightarrow [0,1]$.

In this case, $\tilde{\gamma}_{\mathcal{S}}, \tilde{\eta}_{\mathcal{S}}, \tilde{\mu}_{\mathcal{S}}$ are called truth-membership mapping, indeterminacy-membership mapping, falsity-membership mapping, respectively, and we will denote the set of all \mathcal{SVNS} 's in $\tilde{\mathfrak{X}}$ as $I^{\tilde{\mathfrak{X}}}$.

Moreover, we will refer to the Null (empty) \mathcal{SVNS} (resp. the absolute (universe) \mathcal{SVNS}) in $\tilde{\mathfrak{X}}$ as 0_N (resp. 1_N) and define by $0_N = \langle 0,1,1 \rangle$ (resp. $1_N = \langle 1,0,0 \rangle$) for each $\omega \in \tilde{\mathfrak{X}}$.

Definition 2.4 [29]. Let $\mathcal{S} = \{ \langle \omega, \tilde{\gamma}_{\mathcal{S}}, \tilde{\eta}_{\mathcal{S}}, \tilde{\mu}_{\mathcal{S}} \rangle: \omega \in \tilde{\mathfrak{X}} \}$ be an \mathcal{SVNS} on $\tilde{\mathfrak{X}}$. The complement of the set \mathcal{S} (in sort, \mathcal{S}^c) maybe defined as, for all $\omega \in \tilde{\mathfrak{X}}$

$$\tilde{\gamma}_{\mathcal{S}^c}(\omega) = \tilde{\mu}_{\mathcal{S}}(\omega), \quad \tilde{\eta}_{\mathcal{S}^c}(\omega) = 1 - \tilde{\eta}_{\mathcal{S}}(\omega) \text{ and } \tilde{\mu}_{\mathcal{S}^c}(\omega) = \tilde{\gamma}_{\mathcal{S}}(\omega).$$

Definition 2.5 [34]. Let $\mathcal{S}, \mathcal{E} \in \mathcal{SVNS}(\tilde{\mathfrak{X}})$. Then,

1. $\mathcal{S} \subseteq \mathcal{E}$, if, for every $\omega \in \tilde{\mathfrak{X}}$,

$$\tilde{\eta}_{\mathcal{S}}(\omega) \geq \tilde{\eta}_{\mathcal{E}}(\omega), \quad \tilde{\mu}_{\mathcal{S}}(\omega) \geq \tilde{\mu}_{\mathcal{E}}(\omega) \text{ and } \tilde{\gamma}_{\mathcal{S}}(\omega) \leq \tilde{\gamma}_{\mathcal{E}}(\omega),$$

2. we say $\mathcal{S} = \mathcal{E}$ if $\mathcal{S} \subseteq \mathcal{E}$ and $\mathcal{S} \supseteq \mathcal{E}$.

Definition 2.6 [35]. Let $\mathcal{S}, \mathcal{E} \in \mathcal{SVNS}(\tilde{\mathfrak{X}})$. Then,

1. $\mathcal{S} \cap \mathcal{E}$ is a \mathcal{SVNS} in $\tilde{\mathfrak{X}}$ defined as:

$$\mathcal{S} \cap \mathcal{E} = (\tilde{\gamma}_{\mathcal{S}} \cap \tilde{\gamma}_{\mathcal{E}}, \tilde{\eta}_{\mathcal{S}} \cup \tilde{\eta}_{\mathcal{E}}, \tilde{\mu}_{\mathcal{S}} \cup \tilde{\mu}_{\mathcal{E}}).$$

Where, $(\tilde{\mu}_{\mathcal{S}} \cup \tilde{\mu}_{\mathcal{E}})(\omega) = \tilde{\mu}_{\mathcal{S}}(\omega) \cup \tilde{\mu}_{\mathcal{E}}(\omega)$ and $(\tilde{\gamma}_{\mathcal{S}} \cap \tilde{\gamma}_{\mathcal{E}})(\omega) = \tilde{\gamma}_{\mathcal{S}}(\omega) \cap \tilde{\gamma}_{\mathcal{E}}(\omega)$, for all $\omega \in \tilde{\mathfrak{X}}$.

2. $\mathcal{S} \cup \mathcal{E}$ is an \mathcal{SVNS} on $\tilde{\mathfrak{X}}$ defined as:

$$\mathcal{S} \cup \mathcal{E} = (\tilde{\gamma}_{\mathcal{S}} \cup \tilde{\gamma}_{\mathcal{E}}, \tilde{\eta}_{\mathcal{S}} \cap \tilde{\eta}_{\mathcal{E}}, \tilde{\mu}_{\mathcal{S}} \cap \tilde{\mu}_{\mathcal{E}}).$$

Definition 2.7 [28] Let $\mathcal{S} \in \mathcal{SVNS}(\tilde{\mathfrak{X}})$. Then.

1. The intersection of $\{\mathcal{S}_j: j \in \Delta\}$ (briefly, $\bigcap_{j \in \Delta} \mathcal{S}_j$) is \mathcal{SVNS} over $\tilde{\mathfrak{X}}$ defined as: for all $\omega \in \tilde{\mathfrak{X}}$,

$$\left(\bigcap_{j \in \Delta} \mathcal{S}_j \right) (\omega) = \left(\bigcap_{j \in \Delta} \tilde{\gamma}_{\mathcal{S}_j}(\omega), \bigcup_{j \in \Delta} \tilde{\eta}_{\mathcal{S}_j}(\omega), \bigcup_{j \in \Delta} \tilde{\mu}_{\mathcal{S}_j}(\omega) \right).$$

2. The union of $\{\mathcal{S}_j: j \in \Delta\}$ (briefly, $\bigcup_{j \in \Delta} \mathcal{S}_j$) is \mathcal{SVNS} over $\tilde{\mathfrak{X}}$ defined as: for all $\{\mathcal{S}_j: j \in \Delta\}$,

$$\left(\bigcup_{j \in \Delta} \mathcal{S}_j\right)(\omega) = \left(\bigcup_{j \in \Delta} \tilde{\gamma}_{\mathcal{S}_j}(\omega), \bigcap_{j \in \Delta} \tilde{\eta}_{\mathcal{S}_j}(\omega), \bigcap_{j \in \Delta} \tilde{\mu}_{\mathcal{S}_j}(\omega)\right).$$

Definition 2.8 [30]. Suppose that $t, s, k \in I_0$ and $s + t + k \leq 3$. A single-valued neutrosophic point (briefly, \mathcal{SVNP}) $x_{s,t,k}$ of $\tilde{\mathfrak{X}}$ is the \mathcal{SVNS} in $I^{\tilde{\mathfrak{X}}}$ for every $\omega \in \mathcal{S}$, defined by

$$x_{s,t,k}(\omega) = \begin{cases} (s, t, k), & \text{if } x = \omega, \\ (0, 1, 1), & \text{if } x \neq \omega. \end{cases}$$

A \mathcal{SVNP} $x_{s,t,k}$ is said to belong to a \mathcal{SVNS} $\mathcal{S} = \{(\omega, \tilde{\gamma}_{\mathcal{S}}, \tilde{\eta}_{\mathcal{S}}, \tilde{\mu}_{\mathcal{S}}) : \omega \in \tilde{\mathfrak{X}}\} \in I^{\tilde{\mathfrak{X}}}$, (notion: $x_{s,t,p} \in \mathcal{S}$ iff $s < \tilde{\gamma}_{\mathcal{S}}$, $t \geq \tilde{\eta}_{\mathcal{S}}$ and $k \geq \tilde{\mu}_{\mathcal{S}}$), and the set off all \mathcal{SVNP} in $\tilde{\mathfrak{X}}$ denoted by $\mathcal{SVNP}(\tilde{\mathfrak{X}})$.

Definition 2.9 [36] Suppose that $(\tilde{\tau}^{\tilde{\gamma}}, \tilde{\tau}^{\tilde{\eta}}, \tilde{\tau}^{\tilde{\mu}})$ be the collection of \mathcal{SVNS} s over $\tilde{\mathfrak{X}}$; then $(\tilde{\tau}^{\tilde{\gamma}}, \tilde{\tau}^{\tilde{\eta}}, \tilde{\tau}^{\tilde{\mu}})$ is called \mathcal{SVNT} on $\tilde{\mathfrak{X}}$ if $(\tilde{\tau}^{\tilde{\gamma}}, \tilde{\tau}^{\tilde{\eta}}, \tilde{\tau}^{\tilde{\mu}})$ satisfies the following axioms:

1. $\tilde{\tau}^{\tilde{\gamma}}(\underline{0}) = \tilde{\tau}^{\tilde{\gamma}}(\underline{1}) = 1$ and $\tilde{\tau}^{\tilde{\eta}}(\underline{0}) = \tilde{\tau}^{\tilde{\eta}}(\underline{1}) = \tilde{\tau}^{\tilde{\mu}}(\underline{0}) = \tilde{\tau}^{\tilde{\mu}}(\underline{1}) = 0$,
2. $\tau^{\tilde{\gamma}}(\mathcal{S} \cap \mathcal{E}) \geq \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}) \cap \tilde{\tau}^{\tilde{\gamma}}(\mathcal{E})$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S} \cap \mathcal{E}) \leq \tau^{\tilde{\eta}}(\mathcal{S}) \cup \tilde{\tau}^{\tilde{\eta}}(\mathcal{E})$ and $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S} \cap \mathcal{E}) \leq \tau^{\tilde{\mu}}(\mathcal{S}) \cup \tilde{\tau}^{\tilde{\mu}}(\mathcal{E})$, for every $\mathcal{S}, \mathcal{E} \in I^{\tilde{\mathfrak{X}}}$,
3. $\tilde{\tau}^{\tilde{\gamma}}(\bigcup_{j \in \Delta} \mathcal{S}_j) \geq \bigcap_{j \in \Delta} \tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}_j)$, $\tilde{\tau}^{\tilde{\eta}}(\bigcup_{j \in \Delta} \mathcal{S}_j) \leq \bigcup_{j \in \Delta} \tau^{\tilde{\eta}}(\mathcal{S}_j)$ and $\tilde{\tau}^{\tilde{\mu}}(\bigcup_{j \in \Delta} \mathcal{S}_j) \leq \bigcup_{j \in \Delta} \tau^{\tilde{\mu}}(\mathcal{S}_j)$, for every $\{\mathcal{S}_j, j \in \Delta\} \in I^{\tilde{\mathfrak{X}}}$.

The triplet $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}}, \tilde{\tau}^{\tilde{\eta}}, \tilde{\tau}^{\tilde{\mu}})$ is called \mathcal{SVNTS} , where $\tilde{\tau}^{\tilde{\gamma}}, \tilde{\tau}^{\tilde{\eta}}, \tilde{\tau}^{\tilde{\mu}} : I^{\tilde{\mathfrak{X}}} \rightarrow I$. Occasionally, we will write $\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$ for $(\tilde{\tau}^{\tilde{\gamma}}, \tilde{\tau}^{\tilde{\eta}}, \tilde{\tau}^{\tilde{\mu}})$ and it will cause no ambiguity.

Theorem 2.10 [30] Let $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be an \mathcal{SVNTS} . Then, for all $\mathcal{S} \in I^{\tilde{\mathfrak{X}}}$ and $r \in I_0$, we can define operator $C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}} : I^{\tilde{\mathfrak{X}}} \times I_0 \rightarrow I^{\tilde{\mathfrak{X}}}$ as follows:

$$C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, r) = \cap\{\mathcal{E} \in I^{\tilde{\mathfrak{X}}} : \mathcal{S} \leq \mathcal{E}, \tilde{\tau}^{\tilde{\gamma}}(\underline{1} - \mathcal{E}) \geq r, \tilde{\tau}^{\tilde{\eta}}(\underline{1} - \mathcal{E}) \leq 1 - r, \tilde{\tau}^{\tilde{\mu}}(\underline{1} - \mathcal{E}) \leq 1 - r\}.$$

Then, $(\tilde{\mathfrak{X}}, C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}})$ is an \mathcal{SVNCS} .

Definition 2.11 [30] A map $\tilde{\mathcal{J}}^{\tilde{\gamma}}, \tilde{\mathcal{J}}^{\tilde{\eta}}, \tilde{\mathcal{J}}^{\tilde{\mu}} : I^{\tilde{\mathfrak{X}}} \rightarrow I$ is called \mathcal{SVNI} on $\tilde{\mathfrak{X}}$ if it satisfies the following three conditions:

1. $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\underline{0}) = \tilde{\mathcal{J}}^{\tilde{\mu}}(\underline{0}) = 0$ and $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\underline{0}) = 1$,
2. If $\mathcal{S} \leq \mathcal{E}$ then $\tilde{\mathcal{J}}^{\tilde{\eta}}(\mathcal{E}) \geq \tilde{\mathcal{J}}^{\tilde{\eta}}(\mathcal{S})$, $\tilde{\mathcal{J}}^{\tilde{\mu}}(\mathcal{E}) \geq \tilde{\mathcal{J}}^{\tilde{\mu}}(\mathcal{S})$ and $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\mathcal{E}) \leq \tilde{\mathcal{J}}^{\tilde{\gamma}}(\mathcal{S})$, for all $\mathcal{S}, \mathcal{E} \in I^{\tilde{\mathfrak{X}}}$.
3. $\tilde{\mathcal{J}}^{\tilde{\eta}}(\mathcal{S} \cup \mathcal{E}) \leq \tilde{\mathcal{J}}^{\tilde{\eta}}(\mathcal{E}) \cup \tilde{\mathcal{J}}^{\tilde{\eta}}(\mathcal{S})$, $\tilde{\mathcal{J}}^{\tilde{\mu}}(\mathcal{S} \cup \mathcal{E}) \leq \tilde{\mathcal{J}}^{\tilde{\mu}}(\mathcal{S}) \cup \tilde{\mathcal{J}}^{\tilde{\mu}}(\mathcal{E})$ and $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\mathcal{S} \cup \mathcal{E}) \geq \tilde{\mathcal{J}}^{\tilde{\gamma}}(\mathcal{S}) \cap \tilde{\mathcal{J}}^{\tilde{\gamma}}(\mathcal{E})$, for all $\mathcal{S}, \mathcal{E} \in I^{\tilde{\mathfrak{X}}}$.

The triple $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is called a single-valued neutrosophic ideal topological space (briefly, \mathcal{SVNITS}).

Definition 2.12 [30] Let $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be a \mathcal{SVNTS} for each $\mathcal{S} \in I^{\tilde{\mathfrak{X}}}$. Then the single-valued neutrosophic ideal open local function $\mathcal{S}_r^*(\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathcal{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ of \mathcal{S} is the union of all single-valued neutrosophic points $x_{s,t,p}$ such that if $\mathcal{E} \in Q_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(x_{s,t,p}, r)$ and $\tilde{\mathcal{J}}^{\tilde{\gamma}}(\mathcal{D}) \geq r$, $\tilde{\mathcal{J}}^{\tilde{\eta}}(\mathcal{D}) \leq 1 - r$, $\tilde{\mathcal{J}}^{\tilde{\mu}}(\mathcal{D}) \leq$

$1 - r$, then there is at least one $\omega \in \tilde{\mathfrak{X}}$ for which $\tilde{\gamma}_{\mathcal{E}}(\omega) + \tilde{\gamma}_{\mathcal{S}}(\omega) - 1 > \tilde{\gamma}_{\mathcal{D}}(\omega)$, $\tilde{\eta}_{\mathcal{E}}(\omega) + \tilde{\eta}_{\mathcal{S}}(\omega) - 1 \leq \tilde{\eta}_{\mathcal{D}}(\omega)$ and $\tilde{\mu}_{\mathcal{E}}(\omega) + \tilde{\mu}_{\mathcal{S}}(\omega) - 1 \leq \tilde{\mu}_{\mathcal{D}}(\omega)$.

Occasionally, we will write \mathcal{S}_r^* for $\mathcal{S}_r^*(\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ and it will be no ambiguity.

Remark 2.13 [30] Suppose that $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is an \mathcal{SVNITS} and $\mathcal{S} \in I^{\tilde{\mathfrak{X}}}$. Then we obtain;

$$Cl_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(\mathcal{S}, r) = \mathcal{S} \cup \mathcal{S}_r^*, \quad int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(\mathcal{S}, r) = \mathcal{S} \wedge [((\mathcal{H}^c)_r^*)^c].$$

Theorem 2.14 [30] Let $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be a \mathcal{SVNITS} and $\tilde{\gamma}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$ be a \mathcal{SVNI} on $\tilde{\mathfrak{X}}$. Then

1. If $\mathcal{S} \leq \mathcal{E}$, then $\mathcal{S}_r^* \leq \mathcal{E}_r^*$;
2. If $\tilde{\gamma}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}} \leq \tilde{\gamma}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$, $\tilde{\gamma}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}} \geq \tilde{\gamma}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$ and $\tilde{\gamma}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}} \geq \tilde{\gamma}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$, then $\mathcal{S}_r^*(\tilde{\gamma}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \geq \mathcal{S}_r^*(\tilde{\gamma}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$,
3. $\mathcal{S}_r^* = C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_r^*, r) \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, r)$,
4. $(\mathcal{S}_r^*)_r \leq \mathcal{S}_r^*$,
5. $(\mathcal{S}_r^* \vee \mathcal{E}_r^*) = (\mathcal{S} \vee \mathcal{E})_r^*$,
6. If $\tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}) \geq r$, $\tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}) \leq 1 - r$, and $\tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}) \leq 1 - r$ then $(\mathcal{S} \vee \mathcal{E})_r^* = \mathcal{S}_r^* \vee \mathcal{E}_r^* = \mathcal{S}_r^*$,
7. If $\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}) \geq r$, $\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}) \leq 1 - r$, and $\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}) \leq 1 - r$, then $(\mathcal{E} \wedge \mathcal{S}_r^*) \leq (\mathcal{E} \wedge \mathcal{S})_r^*$,
8. $(\mathcal{S}_r^* \wedge \mathcal{E}_r^*) \geq (\mathcal{S} \wedge \mathcal{E})_r^*$.

3. Single-Valued Neutrosophic Ideal Closed Sets in Šostak Sense

The aim of this section is to define the r -single-valued neutrosophic ideal open (briefly, r -SVNIO), r -single valued neutrosophic semi-open (briefly, r -SVNSO), r -single-valued neutrosophic β -open (briefly, r -SVN β O) and r -single-valued neutrosophic pre-open sets (briefly, r -SVNPO) in the sense of Šostak.

Definition 3.1. A single-valued neutrosophic set \mathcal{S} of an \mathcal{SVNITS} $(\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is called:

1. r -SVNIO if $\mathcal{S} \leq int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_r^*, r)$, for $r \in I_0$,
2. r -SVNSO if $\mathcal{S} \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, r), r)$, for every $r \in I_0$,
3. r -SVN β O if for every $r \in I_0$ $\mathcal{S} \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, r), r), r)$,
4. r -SVNPO if $\mathcal{S} \leq int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, r), r)$, for every $r \in I_0$.

The complement of r -SVNIO (resp. r -SVNSO, r -SVN β O, r -SVNPO) is called r -SVNIC (resp. r -SVNSC, r -SVN β C, r -SVNPC).

Remark 3.2. r -SVNO and r -SVNIO are independent notions

Example 3.3. Let $\tilde{\mathfrak{X}} = \{a, b\}$. Define $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3 \in I^{\tilde{\mathfrak{X}}}$ as follows:

$$\begin{aligned} \mathcal{E}_1 &= \langle (0 \cdot 5, 0 \cdot 5), (0 \cdot 5, 0 \cdot 5), (0 \cdot 5, 0 \cdot 5) \rangle, \quad \mathcal{E}_2 = \langle (0 \cdot 4, 0 \cdot 3), (0 \cdot 4, 0 \cdot 1), (0 \cdot 1, 0 \cdot 2) \rangle, \\ \mathcal{E}_3 &= \langle (0 \cdot 1, 0 \cdot 3), (0 \cdot 4, 0 \cdot 1), (0 \cdot 5, 0 \cdot 4) \rangle, \quad \mathcal{D}_1 = \langle (0 \cdot 4, 0 \cdot 4), (0 \cdot 4, 0 \cdot 3), (0 \cdot 2, 0 \cdot 2) \rangle, \\ \mathcal{D}_2 &= \langle (0 \cdot 2, 0 \cdot 2), (0 \cdot 2, 0 \cdot 2), (0 \cdot 1, 0 \cdot 1) \rangle, \quad \mathcal{D}_3 = \langle (0 \cdot 1, 0 \cdot 1), (0 \cdot 1, 0 \cdot 1), (0 \cdot 1, 0 \cdot 1) \rangle, \end{aligned}$$

Define $\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}} : I^{\tilde{\mathfrak{X}}} \rightarrow I$ as follows:

$$\begin{aligned}
\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}) &= \begin{cases} 1, & \text{if } \mathcal{S} = (0,1,1), \\ 1, & \text{if } \mathcal{S} = (1,0,0), \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_1, \end{cases} & \tilde{\gamma}^{\tilde{\gamma}}(\mathcal{S}) &= \begin{cases} 1, & \text{if } \mathcal{S} = (0,1,1), \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{D}_1, \\ \frac{2}{3}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{D}_1, \end{cases} \\
\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}) &= \begin{cases} 0, & \text{if } \mathcal{S} = (0,1,1), \\ 0, & \text{if } \mathcal{S} = (1,0,0), \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_2, \end{cases} & \tilde{\gamma}^{\tilde{\eta}}(\mathcal{S}) &= \begin{cases} 0, & \text{if } \mathcal{S} = (0,1,1), \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{D}_2, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{D}_2, \end{cases} \\
\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}) &= \begin{cases} 0, & \text{if } \mathcal{S} = (0,1,1), \\ 0, & \text{if } \mathcal{S} = (1,0,0), \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_3, \end{cases} & \tilde{\gamma}^{\tilde{\mu}}(\mathcal{S}) &= \begin{cases} 0, & \text{if } \mathcal{S} = (0,1,1), \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{D}_2, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{D}_2. \end{cases}
\end{aligned}$$

Then, \mathcal{E}_1 is $\frac{1}{2}$ -SVNIO but $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}_1) = \frac{1}{2}$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{E}_1) = 1 \not\leq \frac{1}{2}$ and $\tilde{\tau}^{\tilde{\mu}}(\mathcal{E}_1) = 1 \not\leq \frac{1}{2}$ is not $\frac{1}{2}$ -SVNO.

Lemma 3.4. Let $(\tilde{\mathcal{I}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be a SVNITS. Then,

1. any union of **r-SVNIO** sets is **r-SVNIO**,
2. any intersection of **r-SVNIC** sets is **r-SVNIC**.

Proof

1. Let $\{\mathcal{S}_j, j \in \Delta\}$ is a family of **r-SVNIOs**. Then, we obtain, $\mathcal{S}_j \leq \text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}((\mathcal{S}_r^*)_j, r)$, and hence for each $\omega \in \tilde{\mathcal{I}}$,

$$\bigvee_{j \in \Delta} \tilde{\gamma}_{\mathcal{S}_j}(\omega) \leq \bigvee_{j \in \Delta} \tilde{\gamma}_{\text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}}(\mathcal{S}_r^*, r)(\omega) \leq \tilde{\gamma}_{\text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}}(\bigvee_{j \in \Delta} (\mathcal{S}_j)_r^*, r)(\omega) \leq \tilde{\gamma}_{\text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}}((\bigvee_{j \in \Delta} \mathcal{S}_j)_r^*, r)(\omega),$$

$$\bigvee_{j \in \Delta} \tilde{\eta}_{\mathcal{S}_j}(\omega) \geq \bigvee_{j \in \Delta} \tilde{\eta}_{\text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}}(\mathcal{S}_r^*, r)(\omega) \geq \tilde{\eta}_{\text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}}(\bigvee_{j \in \Delta} (\mathcal{S}_j)_r^*, r)(\omega) \geq \tilde{\eta}_{\text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}}((\bigvee_{j \in \Delta} \mathcal{S}_j)_r^*, r)(\omega),$$

$$\bigvee_{j \in \Delta} \tilde{\mu}_{\mathcal{S}_j}(\omega) \geq \bigvee_{j \in \Delta} \tilde{\mu}_{\text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}}(\mathcal{S}_r^*, r)(\omega) \geq \tilde{\mu}_{\text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}}(\bigvee_{j \in \Delta} (\mathcal{S}_j)_r^*, r)(\omega) \geq \tilde{\mu}_{\text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}}((\bigvee_{j \in \Delta} \mathcal{S}_j)_r^*, r)(\omega).$$

Therefore, $\bigvee_{j \in \Delta} \mathcal{S}_j \leq \text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\bigvee_{j \in \Delta} \mathcal{S}_j, r)$ Hence, $\bigvee_{j \in \Delta} \mathcal{S}_j$ is **r-SVNIO**.

2. Similarly to (1).

Proposition 3.5. Suppose that $(\tilde{\mathcal{I}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is a SVNITS. Then,

1. If \mathcal{S} is **r-SVNIO**, $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{E}) \leq 1 - r$ and $\tilde{\tau}^{\tilde{\mu}}(\mathcal{E}) \leq 1 - r$, then, $\mathcal{S} \cap \mathcal{E}$ is **r-SVNIO**.
2. If \mathcal{S} is **r-SVNIC**, $\tilde{\tau}^{\tilde{\gamma}}(\underline{1} - \mathcal{E}) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\underline{1} - \mathcal{E}) \leq 1 - r$ and $\tilde{\tau}^{\tilde{\mu}}(\underline{1} - \mathcal{E}) \leq 1 - r$, then, $\mathcal{S} \cup \mathcal{E}$ is **r-SVNIC**.
3. If \mathcal{S} is both **r-SVNIO** and **r-SVNSC** sets, then $\mathcal{S} = \text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_r^*, r)$.
4. If \mathcal{S} is **r-SVNIO** and $\mathcal{S} \leq \mathcal{E} \leq \mathcal{C}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, r)$, then \mathcal{E} is an **r-SVN β O** set.
5. $\mathcal{S} \cap \text{int}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}_r^*, r)$ is an **r-SVNIO** set.
6. If \mathcal{S} is **r-SVNIO**, then $\mathcal{S} \cap \mathcal{C}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}, r) \leq (\mathcal{S} \cap \mathcal{E})_r^*$, for every \mathcal{E} is **r-SVNSO**.
7. If \mathcal{S} is **r-SVNIO**, $\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{R}^c) \geq r$, $\tilde{\tau}^{\tilde{\eta}\tilde{\mu}}(\mathcal{R}^c) \leq 1 - r$ and $\tilde{\tau}^{\tilde{\mu}\tilde{\eta}\tilde{\mu}}(\mathcal{R}^c) \leq 1 - r$, then

$$\text{int}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) = \text{int}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r).$$

8. If \mathcal{S} is r-SVNIC, then $\mathcal{S} \geq (\text{int}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r))^*_r$.

Proof.

1. Since \mathcal{S} is r-SVNIO and $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{E}) \leq 1 - r$, $\tilde{\tau}^{\tilde{\mu}}(\mathcal{E}) \leq 1 - r$, for each $\omega \in \tilde{\mathfrak{I}}$,

$$\tilde{\gamma}_{\mathcal{E} \wedge \mathcal{S}}(v) \leq \tilde{\gamma}_{\mathcal{E} \wedge \text{int}_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r)}(\omega) \leq \tilde{\gamma}_{\text{int}_{\tilde{\tau}\tilde{\gamma}}((\mathcal{E} \wedge \mathcal{S}_r^*), r)}(\omega) \leq \tilde{\gamma}_{\text{int}_{\tilde{\tau}\tilde{\gamma}}((\mathcal{E} \wedge \mathcal{S})_r^*, r)}(\omega).$$

$$\tilde{\eta}_{\mathcal{E} \wedge \mathcal{S}}(\omega) \geq \tilde{\eta}_{\mathcal{E} \wedge \text{int}_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r)}(\omega) \geq \tilde{\eta}_{\text{int}_{\tilde{\tau}\tilde{\eta}}((\mathcal{E} \wedge \mathcal{S}_r^*), r)}(\omega) \geq \tilde{\eta}_{\text{int}_{\tilde{\tau}\tilde{\eta}}((\mathcal{E} \wedge \mathcal{S})_r^*, r)}(\omega).$$

$$\tilde{\mu}_{\mathcal{E} \wedge \mathcal{S}}(\omega) \geq \tilde{\mu}_{\mathcal{E} \wedge \text{int}_{\tilde{\tau}\tilde{\mu}}(\mathcal{S}_r^*, r)}(\omega) \geq \tilde{\mu}_{\text{int}_{\tilde{\tau}\tilde{\mu}}((\mathcal{E} \wedge \mathcal{S}_r^*), r)}(\omega) \geq \tilde{\mu}_{\text{int}_{\tilde{\tau}\tilde{\mu}}((\mathcal{E} \wedge \mathcal{S})_r^*, r)}(\omega)$$

Thus $\mathcal{E} \wedge \mathcal{S} \leq \text{int}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}((\mathcal{E} \wedge \mathcal{S})_r^*, r)$. Hence, $\mathcal{E} \wedge \mathcal{S}$ is an r-SVNIO set.

2. It is easily proved by the same manner.

3. Since \mathcal{S} is both r-SVNIO and r-SVNSC, then for each $\omega \in \tilde{\mathfrak{I}}$ and for each $\omega \in \tilde{\mathfrak{I}}$ (Theorem 2.14.(3)), we have

$$\tilde{\gamma}_{\mathcal{S}}(\omega) \leq \tilde{\gamma}_{\text{int}_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r)}(\omega) \leq \tilde{\gamma}_{\text{int}_{\tilde{\tau}\tilde{\gamma}}(c_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}, r), r)}(\omega) \leq \tilde{\gamma}_{\mathcal{S}}(\omega),$$

$$\tilde{\eta}_{\mathcal{S}}(\omega) \geq \tilde{\eta}_{\text{int}_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r)}(\omega) \geq \tilde{\eta}_{\text{int}_{\tilde{\tau}\tilde{\eta}}(c_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}, r), r)}(\omega) \geq \tilde{\eta}_{\mathcal{S}}(\omega),$$

$$\tilde{\mu}_{\mathcal{S}}(\omega) \geq \tilde{\eta}_{\text{int}_{\tilde{\tau}\tilde{\mu}}(\mathcal{S}_r^*, r)}(\omega) \geq \tilde{\mu}_{\text{int}_{\tilde{\tau}\tilde{\mu}}(c_{\tilde{\tau}\tilde{\mu}}(\mathcal{S}, r), r)}(\omega) \geq \tilde{\mu}_{\mathcal{S}}(\omega).$$

Thus, $\mathcal{S} = \text{int}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r)$.

4. Similarly to (3).

5. Since, $\text{int}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) = \mathcal{S}_r^* \cap \text{int}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r)$, for each $\omega \in \tilde{\mathfrak{I}}$ and as we obtained by Theorem 2.14(7), such that for each $\omega \in \tilde{\mathfrak{I}}$. Then we have,

$$\tilde{\gamma}_{\text{int}_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r)}(\omega) \leq \tilde{\gamma}_{(\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r))_r^*}(\omega), \quad \tilde{\eta}_{\text{int}_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r)}(\omega) \geq \tilde{\eta}_{(\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r))_r^*}(\omega),$$

$$\tilde{\mu}_{\text{int}_{\tilde{\tau}\tilde{\mu}}(\mathcal{S}_r^*, r)}(\omega) \geq \tilde{\mu}_{(\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\mu}}(\mathcal{S}_r^*, r))_r^*}(\omega).$$

Thus,

$$\tilde{\gamma}_{\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r)}(\omega) \leq \tilde{\gamma}_{\mathcal{R} \cap \text{int}_{\tilde{\tau}\tilde{\gamma}}((\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r))_r^*), r)}(\omega) \leq \tilde{\gamma}_{\text{int}_{\tilde{\tau}\tilde{\gamma}}((\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r))_r^*), r)}(\omega),$$

$$\tilde{\eta}_{\mathcal{R} \cap \text{int}_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r)}(\omega) \geq \tilde{\eta}_{\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\eta}}((\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r))_r^*), r)}(\omega) \geq \tilde{\eta}_{\text{int}_{\tilde{\tau}\tilde{\eta}}((\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r))_r^*), r)}(\omega),$$

$$\tilde{\mu}_{\mathcal{R} \cap \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*)}(\omega) \geq \tilde{\mu}_{\mathcal{R} \cap \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}((\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*))_{r,r}^*)}(\omega) \geq \tilde{\mu}_{\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}((\mathcal{S} \wedge \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*))_{r,r}^*)}(\omega).$$

Hence, $\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) \leq \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}((\mathcal{S} \wedge \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*))_{r,r}^*)$. Therefore, $\mathcal{S} \cap \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r)$ is **r-SVNIO**.

6. Let \mathcal{E} be **r-SVNSO**. Then $C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r) = C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r), r)$ and by Theorem 2.14(3,7), for each $\omega \in \tilde{\mathfrak{I}}$ we have,

$$\begin{aligned} \tilde{\gamma}_{\mathcal{S} \wedge C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r)} &\leq \tilde{\gamma}_{\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) \wedge C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r), r)} \\ &\leq \tilde{\gamma}_{C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^* \wedge \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}((\mathcal{E}, r), r))} \\ &\leq \tilde{\gamma}_{C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}((\mathcal{S} \wedge \mathcal{E})_{r,r}^*)} = \tilde{\gamma}_{(\mathcal{S} \wedge \mathcal{E})_{r,r}^*}, \end{aligned}$$

$$\begin{aligned} \tilde{\eta}_{\mathcal{S} \wedge C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r)} &\geq \tilde{\eta}_{\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) \wedge C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r), r)} \\ &\geq \tilde{\eta}_{C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^* \wedge \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}((\mathcal{E}, r), r))} \\ &\geq \tilde{\eta}_{C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}((\mathcal{S} \wedge \mathcal{E})_{r,r}^*)} = \tilde{\eta}_{(\mathcal{S} \wedge \mathcal{E})_{r,r}^*}, \end{aligned}$$

$$\begin{aligned} \tilde{\mu}_{\mathcal{S} \wedge C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r)} &\geq \tilde{\mu}_{\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) \wedge C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r), r)} \\ &\geq \tilde{\mu}_{C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^* \wedge \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}((\mathcal{E}, r), r))} \\ &\geq \tilde{\mu}_{C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}((\mathcal{S} \wedge \mathcal{E})_{r,r}^*)} = \tilde{\mu}_{(\mathcal{S} \wedge \mathcal{E})_{r,r}^*}. \end{aligned}$$

7. Similarly to (6).

8. Let \mathcal{S} be **r-SVNIC**. Then, $\mathcal{S}^c \leq \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}((\mathcal{S}^c)_{r,r}^*)$. Since, $\mathcal{S}_r^* \leq C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r)$, by Theorem 2.14(3),

$$\begin{aligned} \mathcal{S}^c &\leq \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}((\mathcal{S}_r^*)^c, r) \\ &\leq \text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r), r) \\ &= (C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r), r))^c. \end{aligned}$$

Then, $C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r), r) \leq \mathcal{S}$. Thus, $\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r)_{r,r}^* \leq C_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\text{int}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r), r) \leq \mathcal{S}$.

Theorem 3.6. Suppose that $(\tilde{\mathfrak{I}}, \tilde{\tau}\tilde{\eta}\tilde{\mu}, \tilde{\gamma}\tilde{\eta}\tilde{\mu})$ is a **SVNITS**, for each $\mathcal{S}, \mathcal{E} \in I^{\tilde{\mathfrak{I}}}$. Define the operator $\mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}: I^{\tilde{\mathfrak{I}}} \times I_0 \rightarrow I$ as follows:

$$\mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) = \bigcap \{ \mathcal{E} \in I^{\tilde{\mathfrak{I}}} \mid \mathcal{S} \leq \mathcal{E}, \mathcal{E} \text{ is } r\text{-SVNIC set} \}.$$

Then, for each $r \in I_0$ the operator $\mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}$ satisfies the following conditions:

1. $\mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\langle 0, 1, 1 \rangle, r) = \langle 0, 1, 1 \rangle,$
2. $\mathcal{S} \leq \mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r),$
3. $\mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) \vee \mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r) \leq \mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S} \vee \mathcal{E}, r),$
4. $\mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r), r) = \mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r),$
5. \mathcal{S} is **r-SVNIC**, iff $\mathcal{S} = \mathcal{IC}_{\tilde{\tau}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r),$

6. If $C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r)$ is r -SVNIC, then $C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{I}C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r), r) = \mathcal{I}C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r), r) = C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r)$.

Proof. It is trivial.

Theorem 3.7. Let $(\tilde{\mathcal{I}}, \tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}, \tilde{\gamma}\tilde{\eta}\tilde{\mu})$ be a \mathcal{SVNITS} , for each $\mathcal{S}, \mathcal{E} \in I^{\tilde{\mathcal{I}}}$, we define the operator $Jint_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}: I^{\tilde{\mathcal{I}}} \times I_0 \rightarrow I$ as follows:

$$Jint_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}r) = \cap \{\mathcal{E} \in I^{\tilde{\mathcal{I}}} | \mathcal{S} \leq \mathcal{E}, \mathcal{E} \text{ is } r\text{-SVNIO set}\}.$$

Then

1. $Jint_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}^c, r) = (\mathcal{I}C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r))^c$,
2. $Jint_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) \leq \mathcal{S} \leq \mathcal{I}C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r)$,
3. \mathcal{S} is r -SVNIO iff $Jint_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) = \mathcal{S}$,
4. $Jint_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) = \mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r)$,
5. $Jint_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) = \langle 0, 1, 1 \rangle$ if and only if $int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) = \langle 0, 1, 1 \rangle$,

Proof.

(1), (2) and (3) are trivial form the definition of $Jint_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$ and $\mathcal{I}C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}$.

(4) By Theorem 2.14(7), we have

$$\tilde{\gamma}_{int_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r)} = \tilde{\gamma}_{\mathcal{S}_r^* \wedge int_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r)} \leq \tilde{\gamma}_{(\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r))_r^*},$$

$$\tilde{\eta}_{int_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r)} = \tilde{\eta}_{\mathcal{S}_r^* \wedge int_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r)} \geq \tilde{\eta}_{(\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r))_r^*},$$

$$\tilde{\mu}_{int_{\tilde{\tau}\tilde{\mu}}(\mathcal{S}_r^*, r)} = \tilde{\mu}_{\mathcal{S}_r^* \wedge int_{\tilde{\tau}\tilde{\mu}}(\mathcal{S}_r^*, r)} \geq \tilde{\mu}_{(\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\mu}}(\mathcal{S}_r^*, r))_r^*}.$$

This implies that

$$\tilde{\gamma}_{\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r)} \leq \tilde{\gamma}_{\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}}((\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r))_r^*)} \leq \tilde{\gamma}_{int_{\tilde{\tau}\tilde{\gamma}}((\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}}(\mathcal{S}_r^*, r))_r^*)},$$

$$\tilde{\eta}_{\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r)} \geq \tilde{\eta}_{\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\eta}}((\mathcal{S}_r^*, r))_r^*} \geq \tilde{\eta}_{int_{\tilde{\tau}\tilde{\eta}}((\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\eta}}(\mathcal{S}_r^*, r))_r^*)},$$

$$\tilde{\mu}_{\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\mu}}(\mathcal{S}_r^*, r)} \geq \tilde{\mu}_{\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\mu}}((\mathcal{S}_r^*, r))_r^*} \geq \tilde{\mu}_{int_{\tilde{\tau}\tilde{\mu}}((\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\mu}}(\mathcal{S}_r^*, r))_r^*)}.$$

Thus, $\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r)$ is r -SVNIO, then $\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) \leq Jint_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r)$.

For each \mathcal{E} is r -SVNIO set and $\mathcal{E} \leq \mathcal{S}$ then by Theorem 2.14(1), we have $\mathcal{E}_r^* \leq \mathcal{S}_r^*$, and so, $int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}_r^*, r) \leq int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r)$,

$$\mathcal{E} \leq \mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}_r^*, r) \leq \mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r).$$

Thus, $Jint_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) \leq \mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r)$.

(5) Let $Jint_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) = \langle 0, 1, 1 \rangle$. Then, $\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) = \langle 0, 1, 1 \rangle$, implies that $C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S} \wedge int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r), r) = \langle 0, 1, 1 \rangle$ and $C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) \wedge int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) = \langle 0, 1, 1 \rangle$, by Theorem 2.14(3), $C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) \wedge$

$$\text{int}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) = \text{int}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) = \langle 0, 1, 1 \rangle.$$

On the other hand, let $\text{int}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) = \langle 0, 1, 1 \rangle$. Then $\mathcal{S} \wedge \text{int}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) = \langle 0, 1, 1 \rangle$. Hence, by (2), $\text{Jint}_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r) = \langle 0, 1, 1 \rangle$.

4. Single-Valued Neutrosophic Ideal Continuous Mappings

We introduce the notions of single-valued neutrosophic continuous (briefly, \mathcal{SVN} -continuous) (resp. single-valued neutrosophic ideal continuous (briefly, \mathcal{SVNI} -continuous), single-valued neutrosophic ideal-open (briefly, \mathcal{SVNI} -open), single-valued neutrosophic \mathcal{I} -closed (briefly, \mathcal{SVNI} -closed), single-valued neutrosophic pre continuous (briefly, \mathcal{SVNP} -continuous)) mappings. Also, we obtain new decompositions of \mathcal{SVN} -continuous in \mathcal{SVNITS} in Šostak Sense.

Definition 4.1. Suppose that $f: (\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\eta}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\tilde{\mathfrak{Y}}, \tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\eta}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is a mapping and $r \in I_0$. Then, f is called: \mathcal{SVNI} -continuous iff $f^{-1}(\mathcal{S})$ is r - \mathbf{SVNI} in $\tilde{\mathfrak{X}}$ for every $\mathcal{E} \in I^{\tilde{\mathfrak{Y}}}$, $\tilde{\sigma}^{\tilde{\gamma}}(\mathcal{S}) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{S}) \leq 1 - r$, $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{S}) \leq 1 - r$,

Definition 4.2. Suppose that $f: (\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\tilde{\mathfrak{Y}}, \tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\eta}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is a mapping and $r \in I_0$. Then, f is said to be:

1. \mathcal{SVNI} -open iff $f(\mathcal{S})$ is r - \mathbf{SVNI} in $\tilde{\mathfrak{Y}}$ for every $\mathcal{E} \in I^{\tilde{\mathfrak{X}}}$, $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}) \leq 1 - r$ and $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}) \leq 1 - r$,
2. \mathcal{SVNI} -closed iff $f(\mathcal{S})$ is r - \mathbf{SVNI} in $\tilde{\mathfrak{Y}}$ for every $\mathcal{E} \in I^{\tilde{\mathfrak{X}}}$, $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1 - r$ and $\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1 - r$.

Definition 4.3. Suppose that $f: (\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\tilde{\mathfrak{Y}}, \tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\eta}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is a mapping and $r \in I_0$. Then, f is called:

1. \mathcal{SVN} -continuous iff $f^{-1}(\mathcal{S})$ is r - \mathbf{SVNO} in $\tilde{\mathfrak{X}}$ for every $\mathcal{E} \in I^{\tilde{\mathfrak{Y}}}$, $\tilde{\sigma}^{\tilde{\gamma}}(\mathcal{S}) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{S}) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{S}) \leq 1 - r$,
2. \mathcal{SVNP} -continuous iff $f^{-1}(\mathcal{S})$ is r - \mathbf{SVNPO} in $\tilde{\mathfrak{X}}$ for every $\mathcal{E} \in I^{\tilde{\mathfrak{Y}}}$, $\tilde{\sigma}^{\tilde{\gamma}}(\mathcal{S}) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{S}) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{S}) \leq 1 - r$.

Remark 4.4.

1. \mathcal{SVNI} -continuous \Rightarrow \mathcal{SVNP} -continuous,
2. \mathcal{SVNL} -continuous and \mathcal{SVN} -continuous are independent.

Example 4.5. Suppose that $\tilde{\mathfrak{X}} = \{a, b\}$. Define $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in I^{\tilde{\mathfrak{X}}}$ as follows:

$$\begin{aligned} \mathcal{E}_1 &= \langle (0 \cdot 5, 0 \cdot 4), (0 \cdot 5, 0 \cdot 5), (0 \cdot 9, 0 \cdot 6) \rangle, \quad \mathcal{E}_2 = \langle (0 \cdot 4, 0 \cdot 4), (0 \cdot 1, 0 \cdot 1), (0 \cdot 1, 0 \cdot 1) \rangle, \\ \mathcal{E}_3 &= \langle (0 \cdot 3, 0 \cdot 1), (0 \cdot 1, 0 \cdot 1), (0 \cdot 1, 0 \cdot 4) \rangle, \quad \mathcal{C}_1 = \langle (0 \cdot 4, 0 \cdot 5), (0 \cdot 5, 0 \cdot 5), (0 \cdot 6, 0 \cdot 9) \rangle, \\ \mathcal{C}_2 &= \langle (0 \cdot 2, 0 \cdot 2), (0 \cdot 2, 0 \cdot 2), (0 \cdot 1, 0 \cdot 1) \rangle, \quad \mathcal{C}_3 = \langle (0 \cdot 1, 0 \cdot 1), (0 \cdot 1, 0 \cdot 1), (0 \cdot 1, 0 \cdot 1) \rangle. \end{aligned}$$

Define $\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\eta}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}} : I^{\tilde{\mathfrak{X}}} \rightarrow I$ as follows:

$$\begin{aligned}
\tilde{\tau}^{\tilde{\nu}}(\mathcal{S}) &= \begin{cases} 1, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ 1, & \text{if } \mathcal{S} = \langle 1,1,0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_1, \end{cases} & \tilde{\sigma}^{\tilde{\nu}}(\mathcal{S}) &= \begin{cases} 1, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ 1, & \text{if } \mathcal{S} = \langle 1,1,0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_1, \end{cases} \\
\tilde{\tau}^{\tilde{\eta}}(\mathcal{S}) &= \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1,1,0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_2, \end{cases} & \tilde{\sigma}^{\tilde{\eta}}(\mathcal{S}) &= \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1,1,0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_2, \end{cases} \\
\tilde{\tau}^{\tilde{\mu}}(\mathcal{S}) &= \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1,1,0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_3, \end{cases} & \tilde{\sigma}^{\tilde{\mu}}(\mathcal{S}) &= \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1,1,0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_3, \end{cases} \\
\tilde{\jmath}^{\tilde{\nu}}(\mathcal{S}) &= \begin{cases} 1, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_1, \\ \frac{2}{3}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{C}_1, \end{cases} & \tilde{\jmath}^{\tilde{\eta}}(\mathcal{S}) &= \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_2, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{C}_2 \end{cases} \\
& & \tilde{\jmath}^{\tilde{\mu}}(\mathcal{S}) &= \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_3, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{C}_3. \end{cases}
\end{aligned}$$

Define $f: (\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\nu}\tilde{\eta}\tilde{\mu}}, \tilde{\jmath}^{\tilde{\nu}\tilde{\eta}\tilde{\mu}}) \rightarrow (\tilde{\mathfrak{Y}}, \tilde{\sigma}^{\tilde{\nu}\tilde{\eta}\tilde{\mu}})$ as follows $f(a) = b$ and $f(b) = a$. If $\tilde{\jmath}^{\tilde{\nu}}(\mathcal{C}_1) \geq \frac{1}{2}$, $\tilde{\jmath}^{\tilde{\eta}}(\mathcal{C}_1) \leq 1 - \frac{1}{2}$ and $\tilde{\jmath}^{\tilde{\mu}}(\mathcal{C}_1) \leq 1 - \frac{1}{2}$. Then $f^{-1}(\mathcal{C}_1) = \langle (0 \cdot 5, 0 \cdot 4), (0 \cdot 5, 0 \cdot 5), (0 \cdot 9, 0 \cdot 6) \rangle$ is $\frac{1}{2}$ -SVNO in $\tilde{\mathfrak{X}}$. Thus, f is \mathcal{SVN} -continuous. However, it is not \mathcal{SVNJ} -continuous.

Theorem 4.6. Let $f: (\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\nu}\tilde{\eta}\tilde{\mu}}, \tilde{\jmath}^{\tilde{\nu}\tilde{\eta}\tilde{\mu}}) \rightarrow (\tilde{\mathfrak{Y}}, \tilde{\sigma}^{\tilde{\nu}\tilde{\eta}\tilde{\mu}})$ be a mapping and $r \in I_0$. Then the following are equivalent.

1. f is \mathcal{SVNJ} -continuous.
2. For any $x_{s,t,k} \in \mathcal{SVNP}(\tilde{\mathfrak{X}})$, $\tilde{\jmath}^{\tilde{\nu}}(\mathcal{S}) \geq r$, $\tilde{\jmath}^{\tilde{\eta}}(\mathcal{S}) \leq 1 - r$ and $\tilde{\jmath}^{\tilde{\mu}}(\mathcal{S}) \leq 1 - r$ containing $f(x_{s,t,k})$, there exists r -SVNIO set \mathcal{E} such that $x_{s,t,k} \in \mathcal{E}$, $f(\mathcal{E}) \leq \mathcal{S}$.
3. For any $\tilde{\sigma}^{\tilde{\nu}}(\mathcal{S}^c) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1 - r$, $f^{-1}(\mathcal{S})$ is r -SVIC set.
4. $C_{\tilde{\tau}^{\tilde{\nu}\tilde{\eta}\tilde{\mu}}}((f^{-1}(\mathcal{S}))^*_r, r) \leq f^{-1}(C_{\tilde{\sigma}^{\tilde{\nu}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}), r)$, for any $\mathcal{S} \in I^{\tilde{\mathfrak{Y}}}$.
5. $f(C_{\tilde{\tau}^{\tilde{\nu}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}^*_r, r)) \leq C_{\tilde{\sigma}^{\tilde{\nu}\tilde{\eta}\tilde{\mu}}}(f(\mathcal{E}), r)$ for any and $\mathcal{E} \in I^{\tilde{\mathfrak{X}}}$.

Proof.

(1) \Rightarrow (2): For any $x_{s,t,k} \in \mathcal{SVNP}(\tilde{\mathfrak{X}})$, $\tilde{\sigma}^{\tilde{\nu}}(\mathcal{S}) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{S}) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{S}) \leq 1 - r$. By \mathcal{SVNJ} -continuity of f we have $\mathcal{E} = f^{-1}(\mathcal{S})$ is an r -SVNIO set and for any $\omega \in \tilde{\mathfrak{X}}$

$$s < \tilde{\gamma}_{f^{-1}(\mathcal{S})}(\omega) = \tilde{\gamma}_{\mathcal{E}}(\omega), \quad t \geq \tilde{\eta}_{f^{-1}(\mathcal{S})}(\omega) = \tilde{\eta}_{\mathcal{E}}(\omega), \quad k \geq \tilde{\mu}_{f^{-1}(\mathcal{S})}(\omega) = \tilde{\mu}_{\mathcal{E}}(\omega).$$

Hence, $f(\mathcal{E}) \leq \mathcal{S}$.

(2) \Rightarrow (3): Suppose that $\tilde{\mathfrak{S}}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\mathfrak{S}}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1 - r$, $\tilde{\mathfrak{S}}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1 - r$ and $x_{s,t,k} \in f^{-1}(\underline{1} - \mathcal{S})$, by (2). There exists r-SVNIO set $\mathcal{E} \in I^{\tilde{\mathfrak{I}}}$ and $x_{s,t,k} \in \mathcal{E}$ such that $f(\mathcal{E}) \leq \underline{1} - \mathcal{S}$. Hence, for any $\omega \in \tilde{\mathfrak{I}}$

$$s < \tilde{\gamma}_{\mathcal{E}}(\omega) \leq \tilde{\gamma}_{int_{\tilde{\gamma}}(\mathcal{E}_r^*, r)}(\omega) \leq \tilde{\gamma}_{int_{\tilde{\gamma}}((f^{-1}(\mathcal{S}^c))_r^*, r)}(\omega),$$

$$t \geq \tilde{\eta}_{\mathcal{E}}(\omega) \geq \tilde{\eta}_{int_{\tilde{\eta}}(\mathcal{E}_r^*, r)}(\omega) \geq \tilde{\eta}_{int_{\tilde{\eta}}((f^{-1}(\mathcal{S}^c))_r^*, r)}(\omega),$$

$$k \geq \tilde{\mu}_{\mathcal{E}}(\omega) \geq \tilde{\mu}_{int_{\tilde{\mu}}(\mathcal{E}_r^*, r)}(\omega) \geq \tilde{\mu}_{int_{\tilde{\mu}}((f^{-1}(\mathcal{S}^c))_r^*, r)}(\omega).$$

Hence $f^{-1}(\mathcal{S}^c) \leq int_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}((f^{-1}(\mathcal{S}^c))_r^*, r)$. Then $f^{-1}(\mathcal{S}^c) = (f^{-1}(\mathcal{S}))^c$ is r-SVNIO set in $\tilde{\mathfrak{I}}$. Thus, $f^{-1}(\mathcal{S})$ is r-SVNIC set in $\tilde{\mathfrak{I}}$.

(3) \Rightarrow (4): For any $\mathcal{S} \in I^{\tilde{\mathfrak{I}}}$ and $r \in I_0$, since $\tilde{\sigma}^{\tilde{\gamma}}((C_{\tilde{\sigma}\tilde{\gamma}}(\mathcal{S}, r))^c) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}((C_{\tilde{\sigma}\tilde{\eta}}(\mathcal{S}, r))^c) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}((C_{\tilde{\sigma}\tilde{\mu}}(\mathcal{S}, r))^c) \leq 1 - r$, by (3), we have $f^{-1}(C_{\tilde{\sigma}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r))$ is r-SVNIC set. Hence,

$$f^{-1}(C_{\tilde{\sigma}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r)) \geq C_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}((f^{-1}(C_{\tilde{\sigma}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}, r)))_r^*, r) \geq C_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}((f^{-1}(\mathcal{S}, r))_r^*, r),$$

(4) \Rightarrow (5): For any $\mathcal{E} \in I^{\tilde{\mathfrak{I}}}$ and $r \in I_0$. Put $f(\mathcal{E}) = \mathcal{S}$. By (4), we have,

$$C_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}_r^*, r) \leq C_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}((f^{-1}(f(\mathcal{E})))_r^*, r) \leq f^{-1}(C_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f(\mathcal{E}), r)).$$

It implies $f(C_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}_r^*, r)) \leq C_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f(\mathcal{E}), r)$.

(5) \Rightarrow (1): Let $\tilde{\sigma}^{\tilde{\gamma}}(\mathcal{E}) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{E}) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{E}) \leq 1 - r$. Then by (5) and Theorem 2.14(3), we have,

$$f(C_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}((f^{-1}(\mathcal{E}^c))_r^*, r)) \leq C_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f(f^{-1}(\mathcal{E}^c)), r) \leq C_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}^c, r) = \mathcal{E}^c.$$

Therefore, $C_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}((f^{-1}(\mathcal{E}^c))_r^*, r) \leq f^{-1}(\mathcal{E}^c)$. This show that $f^{-1}(\mathcal{E})$ is r-SVNIO set. Thus, \mathcal{SVNJ} -continuous.

Theorem 4.7. Suppose that $f: (\tilde{\mathfrak{I}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mathfrak{J}}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\tilde{\mathfrak{J}}, \tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is \mathcal{SVNJ} -continuous for all $\mathcal{S} \in I^{\tilde{\mathfrak{I}}}$. $\mathcal{E} \in I^{\tilde{\mathfrak{I}}}$ and $r \in I_0$. Then the following are holds:

1. $(int_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f^{-1}(\mathcal{S}), r))_r^* \leq f^{-1}(\mathcal{S}_r^*)$, for every [r-single valued neutrosophic \star -dense-in-itself $\mathcal{S} \leq \mathcal{S}_r^*$].
2. $f(int_{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r))_r^* \leq (f(\mathcal{E}))_r^*$, for every [r-single valued neutrosophic \star -prefect $(\mathcal{E}_r^* = \mathcal{E})$].

Proof.

1. For every $\mathcal{S} \in I^{\tilde{\mathfrak{S}}}$ by Theorem 2.14(3), we obtain $C_{\tilde{\sigma}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r) = \mathcal{S}_r^*$, this implies that, $\tilde{\sigma}^{\tilde{\gamma}}((\mathcal{S}_r^*)^c) \geq r$, $\tilde{\tau}^{\tilde{\sigma}}((\mathcal{S}_r^*)^c) \geq r$ and $\tilde{\sigma}^{\tilde{\mu}}((\mathcal{S}_r^*)^c) \geq r$. Then by Theorem 4.6(3), we have $f^{-1}(\mathcal{S}_r^*)$ is r-SVNIC set in $\tilde{\mathfrak{X}}$. Thus, by using Proposition 3.5(8), we have $f^{-1}(\mathcal{S}_r^*) \geq (int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f^{-1}(\mathcal{S}_r^*), r))^*$. Hence,

$$f^{-1}(\mathcal{S}_r^*) \geq (int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f^{-1}(\mathcal{S}_r^*), r))^* \geq (int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f^{-1}(\mathcal{S}), r))^*.$$

2. For every $\mathcal{E} \in I^{\tilde{\mathfrak{X}}}$ and $r \in I_0$, Put $\mathcal{S} = f(\mathcal{E})$ from (2). Then

$$f^{-1}((f(\mathcal{E}))^*) \geq (int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f^{-1}(f(\mathcal{E})), r))^* \geq (int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r))^*.$$

It implies $f(int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r))^* \leq (f(\mathcal{E}))^*_r$.

Theorem 4.8. Suppose that $f: (\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\tilde{\mathfrak{Y}}, \tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is a \mathcal{SVNJ} -continuous for each $r \in I_0$. Then,

1. $f(int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}^*(\mathcal{S}, r), r)) \leq C_{\tilde{\sigma}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f(\mathcal{S}), r)$, for each r-SVNIO $\mathcal{S} \in I^{\tilde{\mathfrak{X}}}$.
2. $int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}^*(f^{-1}(\mathcal{E}), r), r) \leq f^{-1}(C_{\tilde{\sigma}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r))$, for each [r-r-single valued neutrosophic \star -dense-in-itself $\mathcal{E} \in I^{\tilde{\mathfrak{Y}}}$].

Proof.

1. Let $\mathcal{S} \in I^{\tilde{\mathfrak{X}}}$ be a r-SVNPO. Then $\mathcal{S} \leq int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r)$. Hence, by Theorem 4.6(5), we obtain

$$\begin{aligned} f\left(int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}^*(\mathcal{S}, r), r)\right) &\leq f(int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}^*(int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r), r), r)) \\ &\leq f(int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}^*(\mathcal{S}_r^*, r), r)) \\ &\leq f(C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{S}_r^*, r)) \leq C_{\tilde{\sigma}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f(\mathcal{S}), r), \end{aligned}$$

1. Let $\mathcal{E} \in I^{\tilde{\mathfrak{Y}}}$ be r-r-single valued neutrosophic \star -dense-in-itself. Then $\mathcal{E} \leq \mathcal{E}_r^*$. By Theorem 4.6(4), we obtain,

$$\begin{aligned} int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}^*(f^{-1}(\mathcal{E}), r), r) &\leq int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f^{-1}(\mathcal{E}), r), r) \\ &\leq int_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f^{-1}(\mathcal{E}_r^*), r), r) \\ &\leq C_{\tilde{\tau}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(f^{-1}(\mathcal{E}_r^*), r) \leq f^{-1}(C_{\tilde{\sigma}\tilde{\gamma}\tilde{\eta}\tilde{\mu}}(\mathcal{E}, r)). \end{aligned}$$

Theorem 4.9. A mapping $f: (\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\tilde{\mathfrak{Y}}, \tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\gamma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is \mathcal{SVNJ} -open iff for every $\mathcal{S} \in I^{\tilde{\mathfrak{X}}}$ and for each $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}) \geq r$, $\tilde{\tau}^{\tilde{\eta}}(\mathcal{E}) \leq 1 - r$ and $\tilde{\tau}^{\tilde{\mu}}(\mathcal{E}) \leq 1 - r$ such that $f^{-1}(\mathcal{S}) \leq \mathcal{E}$, there exists $\mathcal{D} \in I^{\tilde{\mathfrak{X}}}$ is r-SVNIC set containing \mathcal{S} such that $f^{-1}(\mathcal{D}) \leq \mathcal{E}$.

Proof. Obvious.

Theorem 4.10. If $f: (\mathfrak{L}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\mathfrak{J}, \tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\jmath}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is \mathcal{SVNJ} -open, then the following properties are holds:

1. $f^{-1}(C_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(\mathcal{S}, r), r)) \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}), r)$ for all $\tilde{\sigma}^{\tilde{\gamma}}(\mathcal{S}) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{S}) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{S}) \leq 1 - r$.
2. $f^{-1}(Cl_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(\mathcal{E}, r)) \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{E}), r)$ for all $\tilde{\sigma}^{\tilde{\gamma}}(\mathcal{E}) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{E}) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{E}) \leq 1 - r$.

Proof.

1. Since,

$\tilde{\tau}^{\tilde{\gamma}}((C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}), r))^c) \geq r$, $\tilde{\tau}^{\tilde{\gamma}}((C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}), r))^c) \leq 1 - r$, $\tilde{\tau}^{\tilde{\gamma}}((C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}), r))^c) \leq 1 - r$ for each $\mathcal{S} \in I^{\tilde{\mathfrak{J}}}$. By Theorem 4.9, there exists $\mathcal{D} \in I^{\tilde{\mathfrak{J}}}$ is r - \mathcal{SVNIC} set containing \mathcal{S} such that $f^{-1}(\mathcal{D}) \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}), r)$. Since \mathcal{D}^c is r - \mathcal{SVNIO} set, $f^{-1}(\mathcal{D}^c) \leq f^{-1}(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*((\mathcal{D}^c)_r, r))$, we obtain,

$$\begin{aligned} (f^{-1}(\mathcal{D}))^c &\leq f^{-1}(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*((\mathcal{D}^c)_r, r)) \leq f^{-1}((int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(\mathcal{D}^c)_r, r)) \\ &\leq f^{-1}(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(Cl_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{D}^c, r), r), r)) \\ &\leq (f^{-1}(C_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{D}, r), r), r)))^c. \end{aligned}$$

Since $\mathcal{S} \leq \mathcal{D}$, we obtain,

$$\begin{aligned} f^{-1}(C_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(\mathcal{S}, r), r)) &\leq f^{-1}(C_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{D}, r), r), r)) \\ &\leq f^{-1}(\mathcal{D}) \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}), r). \end{aligned}$$

Hence, $f^{-1}(C_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(\mathcal{S}, r), r)) \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}), r)$.

2. For each $\tilde{\sigma}^{\tilde{\gamma}}(\mathcal{E}) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{E}) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{E}) \leq 1 - r$. By (1), we have

$$\begin{aligned} f^{-1}(Cl_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(\mathcal{E}, r)) &\leq f^{-1}(C_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}, r)) \leq f^{-1}(C_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}, r), r)) \\ &\leq f^{-1}(C_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(\mathcal{E}, r), r)) \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{E}), r). \end{aligned}$$

Theorem 4.11 below, is similarly proved as Theorem 4.10.

Theorem 4.11. If $f: (\mathfrak{L}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\mathfrak{J}, \tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\jmath}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is \mathcal{SVNJ} -closed, then the following properties are holds:

1. $f^{-1}(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(\mathcal{S}, r), r)) \leq C_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{S}), r)$, for each $\tilde{\sigma}^{\tilde{\gamma}}(\mathcal{S}) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{S}) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{S}) \leq 1 - r$.
2. $f^{-1}(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}^*(\mathcal{E}, r)) \leq int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{E}), r)$, for each $\tilde{\sigma}^{\tilde{\gamma}}(\mathcal{E}^c) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{E}^c) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{E}^c) \leq 1 - r$.

Theorem 4.12. The following hold for the mappings $f: (\mathfrak{T}, \tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{j}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\mathfrak{T}, \tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{j}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ and $g: (\mathfrak{T}, \tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{j}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\mathfrak{T}, \tilde{\tau}_3^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{j}_3^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$;

1. $g \circ f$ is \mathcal{SVNJ} -continuous if f is \mathcal{SVNJ} -continuous and g is \mathcal{SVN} -continuous,
2. $g \circ f$ is \mathcal{SVNP} -continuous if f is \mathcal{SVNP} -continuous and g is \mathcal{SVNJ} -continuous,
3. $g \circ f$ is \mathcal{SVNJ} -open if f and g is \mathcal{SVNJ} -open, f is surjective and $g(\mathcal{S}_r^*) \leq (g(\mathcal{E}))_r^*$ for each $\mathcal{S} \leq \mathcal{E}$.

Proof. Straightforward.

Remark 4.13. The composition of two \mathcal{SVNJ} -continuous mappings need not to be a \mathcal{SVNJ} -continuous.

Example 4.14. Suppose that $\mathfrak{T} = \{a, b\}$. Define $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in I^{\mathfrak{T}}$ as follows:

$$\mathcal{E}_1 = \langle (0 \cdot 4, 0 \cdot 4), (0 \cdot 4, 0 \cdot 4), (0 \cdot 4, 0 \cdot 4) \rangle, \quad \mathcal{E}_2 = \langle (0 \cdot 3, 0 \cdot 3), (0 \cdot 3, 0 \cdot 3), (0 \cdot 3, 0 \cdot 3) \rangle,$$

$$\mathcal{E}_3 = \langle (0 \cdot 2, 0 \cdot 2), (0 \cdot 2, 0 \cdot 2), (0 \cdot 2, 0 \cdot 2) \rangle, \quad \mathcal{C}_1 = \langle (0 \cdot 2, 0 \cdot 2), (0 \cdot 2, 0 \cdot 2), (0 \cdot 2, 0 \cdot 2) \rangle,$$

$$\mathcal{C}_2 = \langle (0 \cdot 1, 0 \cdot 1), (0 \cdot 1, 0 \cdot 1), (0 \cdot 1, 0 \cdot 1) \rangle,$$

Define $\tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\tau}_3^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{j}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{j}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}: I^{\mathfrak{T}} \rightarrow I$ as follows:

$$\tilde{\tau}_1^{\tilde{\gamma}}(\mathcal{S}) = \begin{cases} 1, & \text{if } \mathcal{S} = \langle 0, 1, 1 \rangle, \\ 1, & \text{if } \mathcal{S} = \langle 1, 1, 0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_1, \end{cases} \quad \tilde{\tau}_2^{\tilde{\gamma}}(\mathcal{S}) = \begin{cases} 1, & \text{if } \mathcal{S} = \langle 0, 1, 1 \rangle, \\ 1, & \text{if } \mathcal{S} = \langle 1, 1, 0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_2, \end{cases}$$

$$\tilde{\tau}_1^{\tilde{\eta}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0, 1, 1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1, 1, 0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_1, \end{cases} \quad \tilde{\tau}_2^{\tilde{\eta}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0, 1, 1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1, 1, 0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_2, \end{cases}$$

$$\tilde{\tau}_1^{\tilde{\mu}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0, 1, 1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1, 1, 0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_1, \end{cases} \quad \tilde{\tau}_2^{\tilde{\mu}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0, 1, 1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1, 1, 0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_2, \end{cases}$$

$$\tilde{\tau}_3^{\tilde{\gamma}}(\mathcal{S}) = \begin{cases} 1, & \text{if } \mathcal{S} = \langle 0, 1, 1 \rangle, \\ 1, & \text{if } \mathcal{S} = \langle 1, 1, 0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_3, \end{cases} \quad \tilde{\tau}_2^{\tilde{\gamma}}(\mathcal{S}) = \begin{cases} 1, & \text{if } \mathcal{S} = \langle 0, 1, 1 \rangle, \\ 1, & \text{if } \mathcal{S} = \langle 1, 1, 0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_2, \end{cases}$$

$$\tilde{\tau}_3^{\tilde{\eta}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0, 1, 1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1, 1, 0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_3, \end{cases} \quad \tilde{\tau}_2^{\tilde{\eta}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0, 1, 1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1, 1, 0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_2, \end{cases}$$

$$\tilde{\tau}_3^{\tilde{\mu}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1,1,0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_3, \end{cases} \quad \tilde{\tau}_2^{\tilde{\mu}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ 0, & \text{if } \mathcal{S} = \langle 1,1,0 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{E}_2, \end{cases}$$

$$\tilde{j}_1^{\tilde{\gamma}}(\mathcal{S}) = \begin{cases} 1, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_1, \\ \frac{2}{3}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{C}_1, \end{cases} \quad \tilde{j}_2^{\tilde{\gamma}}(\mathcal{S}) = \begin{cases} 1, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_1, \\ \frac{2}{3}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{C}_2, \end{cases}$$

$$\tilde{j}_1^{\tilde{\eta}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_2, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{C}_1 \end{cases} \quad \tilde{j}_2^{\tilde{\eta}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_2, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{C}_2 \end{cases}$$

$$\tilde{j}_1^{\tilde{\mu}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_3, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{C}_1. \end{cases} \quad \tilde{j}_2^{\tilde{\mu}}(\mathcal{S}) = \begin{cases} 0, & \text{if } \mathcal{S} = \langle 0,1,1 \rangle, \\ \frac{1}{2}, & \text{if } \mathcal{S} = \mathcal{C}_3, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{S} < \mathcal{C}_2. \end{cases}$$

The identity mappings $id_X: (X, \tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{j}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (Y, \tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ and $id_X: (Y, \tilde{\tau}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{j}_2^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (Z, \tilde{\tau}_3^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ are \mathcal{SVNI} -continuous. But the identity mapping $id_X: (X, \tilde{\tau}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{j}_1^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (Z, \tilde{\tau}_3^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ is not \mathcal{SVNI} -continuous because, $\tilde{\tau}_3^{\tilde{\gamma}}(\mathcal{E}_3) \geq \frac{1}{2}$, $\tilde{\tau}_3^{\tilde{\eta}}(\mathcal{E}_3) \leq 1 - \frac{1}{2}$, $\tilde{\tau}_1^{\tilde{\mu}}(\mathcal{E}_3) \leq 1 - \frac{1}{2}$ and $(\mathcal{E}_3)_{\frac{1}{2}}^* = \langle 0,1,1 \rangle$ and $(\mathcal{E}_3)_{\frac{1}{2}}^* \not\leq int_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}((\mathcal{E}_3)_{\frac{1}{2}}^*, \frac{1}{2})$.

Propositions (4.13) and (4.14) are similarly proved from Theorems (4.6) and (4.8), respectively.

Proposition 4.13. Let $f: (\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{j}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\tilde{\mathfrak{Y}}, \tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be a mapping. Then, following statements are equivalent.

1. f is \mathcal{SVNI} -continuous.
2. $f^{-1}(\mathcal{S})$ is r - \mathbf{SVNIC} for each $\tilde{\sigma}^{\tilde{\gamma}}(\mathcal{S}^c) \geq r$, $\tilde{\sigma}^{\tilde{\eta}}(\mathcal{S}^c) \leq 1 - r$ and $\tilde{\sigma}^{\tilde{\mu}}(\mathcal{S}^c) \leq 1 - r$,
3. $f(\mathcal{IC}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, r)) \leq \mathcal{C}_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f(\mathcal{S}), r)$, for each $r \in I_0$ and $\mathcal{S} \in I^{\tilde{\mathfrak{X}}}$,
4. $\mathcal{IC}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{E}, r)) \leq f^{-1}(\mathcal{C}_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}, r))$, for each $r \in I_0$ and $\mathcal{E} \in I^{\tilde{\mathfrak{Y}}}$,
5. $f^{-1}(int_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{E}, r)) \leq Jint_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f^{-1}(\mathcal{E}, r))$, for each $r \in I_0$ and $\mathcal{E} \in I^{\tilde{\mathfrak{Y}}}$.

Proof. Obvious.

Proposition 4.14. Let $f: (\tilde{\mathfrak{X}}, \tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}) \rightarrow (\tilde{\mathfrak{Y}}, \tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}, \tilde{\mu}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}})$ be a mapping. Then, the following statements are hold:

1. f is called \mathcal{SVNJ} -closed.
2. $f(\mathcal{JC}_{\tilde{\tau}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(\mathcal{S}, r)) \leq \mathcal{C}_{\tilde{\sigma}^{\tilde{\gamma}\tilde{\eta}\tilde{\mu}}}(f(\mathcal{S}), r)$, for each $r \in I_0$ and $\mathcal{S} \in I^{\tilde{\mathfrak{X}}}$,
3. for any $\mathcal{S} \in I^{\tilde{\mathfrak{X}}}$ and $\tilde{\tau}^{\tilde{\gamma}}(\mathcal{E}) \geq r$, $\tilde{\eta}^{\tilde{\gamma}}(\mathcal{E}) \leq 1 - r$ and $\tilde{\mu}^{\tilde{\gamma}}(\mathcal{E}) \leq 1 - r$ such that $f^{-1}(\mathcal{S}) \leq \mathcal{E}$, there exists a r -SVNIO set $\mathcal{D} \in I^{\tilde{\mathfrak{X}}}$ with $\mathcal{S} \leq \mathcal{D}$ such that $f^{-1}(\mathcal{D}) \leq \mathcal{E}$.

Proof. Obvious.

6. Conclusions

In this paper, the author has made a study of the r -single-valued neutrosophic ideal open (r -SVNIO), the idea of r -single-valued neutrosophic β -open (r -SVN β O) and r -single-valued neutrosophic pre-open sets (r -SVNPO) in the sense of Šostak, which are different from the study taken so far and obtained some of their basic properties. Next, the concepts of a single-valued neutrosophic continuous (resp. single-valued neutrosophic ideal continuous, single-valued neutrosophic \mathcal{I} -open, single-valued neutrosophic \mathcal{I} -closed, single-valued neutrosophic pre continuous) mappings were introduced and studied and too obtained new decompositions of \mathcal{SVN} -continuous in \mathcal{SVNITS} in Šostak Sense.

Discussion for Further Works:

The theory can be extended in the following natural ways. One may study the properties of single-valued neutrosophic metric topological spaces using the concept of basis defined in this paper;

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A Study on Neutrosophic Bitopological Group

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Abstract. In this paper we try to introduce neutrosophic bitopological group. We try to investigate some new definition and properties of neutrosophic bitopological group.

Keywords: Neutrosophic Group; Neutrosophic Topological Group; Neutrosophic Bitopological Group.

1. Introduction

In 1965, Zadeh [1] defined the concept of fuzzy set (FS). With the help of FS, defined the concept of membership function and explained the idea of uncertainty. In 1986, Atanassov [4] generalised the concept of FS and introduced the degree of non-membership as an independent component and proposed the intuitionistics fuzzy set (IFS). After that many researchers defined various new concepts on generalisation of FS. Smarandache [2, 3] introduced the degree of indeterminacy as independent component and discovered the neutrosophic set (NS).

After the generalisation of FS many researchers have applied the generalisation of fuzzy set theory in many branches of Science and Technology. Chang [6] introduced the fuzzy topology space (FTS). Coker [7] defined the concept of Intuitionistic fuzzy topological space (IFTS). In 1963, Kelly [5] defined the study of bitopological spaces. Kandil et al [9] discussed on fuzzy bitopological space (FBTS). Lee et al [8] discussion on some properties of Intuitionistic fuzzy bitopological space (IFBTS).

In 2012, Salama and Alblowi [18] introduced the concept of neutrosophic set (NS) and neutrosophic topological space (NTS) and in 2018, Riad K. Al-Hamido, [28, 29] defined the concept of Neutrosophic Crisp Bi-Topological Spaces and Crisp Tri-Topological Spaces. Narmada Devi R. et al [27] discussed on separation axioms in an ordered neutrosophic bitopological space (NBTS). Ozturk and Ozkan discussion on neutrosophic bitopological spaces.

In 2015, Sumathi and Arockiarani [10], defined the fuzzy neutrosophic group (FNG). After defining neutrosophic group (NG) Sumathi and Arockiarani [22] defined the concept of topological group structure of neutrosophic set. Currently, neutrosophic set has been applying by researchers in various field of Science and Technology as a tool for getting more appropriate result. Abdel-Basset et al. [12] has applied neutrosophic as a tool on group discussion making framework. Abdel-Basset et al. [13] done on work in solving chain problem using base-worst method based on novel plithogenic model. Smarandache [39] extended neutrosophic set to neutrosophic Overset, Neutrosophic Underset and Neutrosophic Offset. Smarandache [39], in 2016, introduced Neutrosophic Tripolar Set, Neutrosophic Multipolar Set, Neutrosophic Tripolar Graph and Neutrosophic Multipolar Graph. Salama [40] studied some properties of topological space of rough sets with tools for data mining. Broumi et al. [41] introduced the concept of rough neutrosophic sets. Parimala et al. [42] studied on $\alpha\omega$ -closed sets and its connectedness in neutrosophic topological spaces. Pamucar and Bozanic [30] (2019) used single-valued neutrosophic sets to propose projection-based multi-attributive border approximation area comparison (MABAC) method. In 2018, Liu et al. [31] studied on new extension of decision-making trial and evaluation laboratory method (DEMATEL). Guo et al. [32] (2017) extended the rough set model to neutrosophic environment and used to multi-attribute decision making (MADM) problem. Nie et al. (2017) [33] studied the Weighted Aggregates Sum Product Assessment (WASPAS) method in the context of interval neutrosophic sets (INS). Ye (2016) [34] introduced interval neutrosophic hesitant fuzzy set (INHFS). Pamucar et al. (2018, 2019) [35, 36] studied application of linguistic neutrosophic number. Karaaslan (2020) [37] introduced type-2 single valued neutrosophic set along with some distance measures. Neutrosophic number is used by Maiti et al. (2019) [38] to solve multi-objective linear programming problem. Abdel-Basset et al. [24] recently studied integrated plithogenic MCDM approach for financial performance evaluation of manufacturing industries. Abdel-Basset et al. [26] observed on a novel framework to evaluate innovation value proposition for smart product-service systems.

NS is used to control uncertainty by using truth membership function, indeterminacy membership function and falsity membership function. Whereas FS is used to control uncertainty by using membership function only. NS is used indeterminacy as an independent measure of the membership and non-membership function. As a result, NS is considered as a generalization of FS and intuitionistic fuzzy set (IFS) and shows more better result. NS is more necessary to manage the real-life information which are uncertain and inconsistent in nature. In various problem FS and IFS can not completely assured due to in exact inconsistent characteristic. Therefore, NS shows more rational to design the membership function. By observing this we

are going to do our research and try to study neutrosophic bitopological group (NBTG) by using NS and try to prove some of their properties.

2. Preliminaries

2.1. Definition:[18]

A NS A on a universe of discourse X is defined as $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x) \rangle : x \in X\}$, where $\mathcal{T}, \mathcal{I}, \mathcal{F} : X \rightarrow [0, 1]$. Note that $0 \leq \mathcal{T}_A(x) + \mathcal{I}_A(x) + \mathcal{F}_A(x) \leq 3$.

2.2. Definition:[21, 18]

The complement of NS A is denoted by A^c and is defined as $A^c(x) = \{\langle x, \mathcal{T}_{A^c}(x) = \mathcal{F}_A(x), \mathcal{I}_{A^c}(x) = 1 - \mathcal{I}_A(x), \mathcal{F}_{A^c}(x) = \mathcal{T}_A(x) \rangle : x \in X\}$.

2.3. Definition:[21, 18]

Let $X \neq \phi$ and $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x) \rangle : x \in X\}$, $B = \{\langle x, \mathcal{T}_B(x), \mathcal{I}_B(x), \mathcal{F}_B(x) \rangle : x \in X\}$, are NSs. Then

- (i) $A \wedge B = \{\langle x, \min(\mathcal{T}_A(x), \mathcal{T}_B(x)), \min(\mathcal{I}_A(x), \mathcal{I}_B(x)), \max(\mathcal{F}_A(x), \mathcal{F}_B(x)) \rangle : x \in X\}$
- (ii) $A \vee B = \{\langle x, \max(\mathcal{T}_A(x), \mathcal{T}_B(x)), \max(\mathcal{I}_A(x), \mathcal{I}_B(x)), \min(\mathcal{F}_A(x), \mathcal{F}_B(x)) \rangle : x \in X\}$
- (iii) $A \leq B$ if for each $x \in X$, $\mathcal{T}_A(x) \leq \mathcal{T}_B(x)$, $\mathcal{I}_A(x) \leq \mathcal{I}_B(x)$, $\mathcal{F}_A(x) \geq \mathcal{F}_B(x)$.

2.4. Definition:[21]

Let X and Y be two non empty sets and let ϕ be a function from a set X to a set Y . Let $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x) \rangle : x \in X\}$, $B = \{\langle y, \mathcal{T}_B(y), \mathcal{I}_B(y), \mathcal{F}_B(y) \rangle : y \in Y\}$ be NS in X and Y . Then

- (i) $\phi^{-1}(B)$, the preimage of B under ϕ is the NS in X defined by

$$\phi^{-1}(B) = \{\langle x, \phi^{-1}(\mathcal{T}_B)(x), \phi^{-1}(\mathcal{I}_B)(x), \phi^{-1}(\mathcal{F}_B)(x) \rangle : x \in X\}$$

where for all $x \in X$, $\phi^{-1}(\mathcal{T}_B)(x) = \mathcal{T}_B(\phi(x))$, $\phi^{-1}(\mathcal{I}_B)(x) = \mathcal{I}_B(\phi(x))$, $\phi^{-1}(\mathcal{F}_B)(x) = \mathcal{F}_B(\phi(x))$.

- (ii) The image of A under ϕ denoted by $\phi(A)$ is a NS in Y defined by

$\phi(A) = (\phi(\mathcal{T}_A), \phi(\mathcal{I}_A), \phi(\mathcal{F}_A))$, where for each $u \in Y$,

$$\phi(\mathcal{T}_A)(u) = \begin{cases} \bigvee_{x \in \phi^{-1}(u)} \mathcal{T}_A(x), & \text{if } \phi^{-1}(u) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\phi(\mathcal{I}_A)(u) = \begin{cases} \bigvee_{x \in \phi^{-1}(u)} \mathcal{I}_A(x), & \text{if } \phi^{-1}(u) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\phi(\mathcal{F}_A)(u) = \begin{cases} \bigvee_{x \in \phi^{-1}(u)} \mathcal{F}_A(x), & \text{if } \phi^{-1}(u) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}.$$

2.5. Definition:[19]

Let $\alpha, \beta, \gamma \in [0, 1]$ and $\alpha + \beta + \gamma \leq 3$. A neutrosophic point $x_{(\alpha, \beta, \gamma)}$ of X is the NS in X defined by

$$x_{(\alpha, \beta, \gamma)}(u) = \begin{cases} (\alpha, \beta, \gamma), & \text{if } x = u \\ (0, 0, 1), & \text{if } x \neq u \end{cases}; \text{ for each } u \in X.$$

A neutrosophic point is said to belong to a NS $A = \{ \langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x) \rangle : x \in X \}$ in X denoted by $x_{(\alpha, \beta, \gamma)} \in A$ if $\alpha \leq \mathcal{T}_A(x)$, $\beta \leq \mathcal{I}_A(x)$ and $\gamma \geq \mathcal{F}_A(x)$.

3. Neutrosophic Group

3.1. Definition:[10]

Let (X, \circ) be a group and let A be a neutrosophic group (NG) in X . Then A is said to be a NG in X if it satisfies the following conditions:

- (i) $\mathcal{T}_A(xy) \geq \mathcal{T}_A(x) \wedge \mathcal{T}_A(y)$, $\mathcal{I}_A(xy) \geq \mathcal{I}_A(x) \wedge \mathcal{I}_A(y)$ and $\mathcal{F}_A(xy) \leq \mathcal{F}_A(x) \vee \mathcal{F}_A(y)$,
- (ii) $\mathcal{T}_A(x^{-1}) \geq \mathcal{T}_A(x)$, $\mathcal{I}_A(x^{-1}) \geq \mathcal{I}_A(x)$, and $\mathcal{F}_A(x^{-1}) \leq \mathcal{F}_A(x)$.

3.2. Definition:[22]

Let X be a group and let \mathcal{G} be NG in X and e be the identity of X . We define the NS \mathcal{G}_e by

$$\mathcal{G}_e = \{ x \in X : \mathcal{T}_{\mathcal{G}}(x) = \mathcal{T}_{\mathcal{G}}(e), \mathcal{I}_{\mathcal{G}}(x) = \mathcal{I}_{\mathcal{G}}(e), \mathcal{F}_{\mathcal{G}}(x) = \mathcal{F}_{\mathcal{G}}(e) \}.$$

We note for a NG \mathcal{G} in a group X , for every $x \in X : \mathcal{T}_{\mathcal{G}}(x^{-1}) = \mathcal{T}_{\mathcal{G}}(x)$, $\mathcal{I}_{\mathcal{G}}(x^{-1}) = \mathcal{I}_{\mathcal{G}}(x)$ and $\mathcal{F}_{\mathcal{G}}(x^{-1}) = \mathcal{F}_{\mathcal{G}}(x)$. Also for the identity e of the group $X : \mathcal{T}_{\mathcal{G}}(e) \geq \mathcal{T}_{\mathcal{G}}(x)$, $\mathcal{I}_{\mathcal{G}}(e) \geq \mathcal{I}_{\mathcal{G}}(x)$, and $\mathcal{F}_{\mathcal{G}}(e) \leq \mathcal{F}_{\mathcal{G}}(x)$.

3.3. Proposition:

Let \mathcal{G} be a NG in a group X . Then for all $x, y \in X$,

- (1) $\mathcal{T}_{\mathcal{G}}(xy^{-1}) = \mathcal{T}_{\mathcal{G}}(e) \Rightarrow \mathcal{T}_{\mathcal{G}}(x) = \mathcal{T}_{\mathcal{G}}(y)$
- (2) $\mathcal{I}_{\mathcal{G}}(xy^{-1}) = \mathcal{I}_{\mathcal{G}}(e) \Rightarrow \mathcal{I}_{\mathcal{G}}(x) = \mathcal{I}_{\mathcal{G}}(y)$
- (3) $\mathcal{F}_{\mathcal{G}}(xy^{-1}) = \mathcal{F}_{\mathcal{G}}(e) \Rightarrow \mathcal{F}_{\mathcal{G}}(x) = \mathcal{F}_{\mathcal{G}}(y)$

3.4. Proposition:

Let X be a group. Then the following statements are equivalent;

- (i) \mathcal{G} is neutrosophic group in X .

- (ii) For all $x, y \in X$, $\mathcal{T}_{\mathcal{G}}(xy^{-1}) \geq \mathcal{T}_{\mathcal{G}}(x) \wedge \mathcal{T}_{\mathcal{G}}(y)$, $\mathcal{I}_{\mathcal{G}}(xy^{-1}) \geq \mathcal{I}_{\mathcal{G}}(x) \wedge \mathcal{I}_{\mathcal{G}}(y)$, $\mathcal{F}_{\mathcal{G}}(xy^{-1}) \leq \mathcal{F}_{\mathcal{G}}(x) \vee \mathcal{F}_{\mathcal{G}}(y)$.

3.5. Definition:[10]

Let $\phi : X \rightarrow Y$ be a group homomorphism and let A be a NG in a group X . Then A is said to be neutrosophic-invariant if for any $x, y \in X$, $\mathcal{T}_A(x) = \mathcal{T}_A(y)$, $\mathcal{I}_A(x) = \mathcal{I}_A(y)$ and $\mathcal{F}_A(x) = \mathcal{F}_A(y)$. It is clear that if A is neutrosophic invariant then $f(A) \in \text{NG}(Y)$. For each $A \in \text{neutrosophic group}(X)$, let $X_A = \{x \in X : \mathcal{T}_A(x) = \mathcal{T}_A(e), \mathcal{I}_A(x) = \mathcal{I}_A(e), \mathcal{F}_A(x) = \mathcal{F}_A(e)\}$. Then it is clear that X_A is a subgroup of X . For each $a \in X$, let $r_a : X \rightarrow X$ and $l_a : X \rightarrow X$ be the right and left translations of X into itself, defined by $r_a(x) = xa$ and $l_a(x) = ax$, respectively for each $x \in X$.

3.6. Definition:[18]

Let X be a non empty set and \mathfrak{A} neutrosophic topology is a family \mathfrak{T} of neutrosophic subsets of X satisfying the following axioms:

- (i) $0_A, 1_A \in \mathfrak{T}$
- (ii) $G_1 \cap G_2 \in \mathfrak{T}$ for any $G_1, G_2 \in \mathfrak{T}$
- (iii) $\bigcup G_i \forall \{G_i : i \in J\} \subseteq \mathfrak{T}$

In this case the pair (X, \mathfrak{T}) is called a neutrosophic topological space (NTS) and any neutrosophic set in \mathfrak{T} is known as neutrosophic open set. The elements of \mathfrak{T} are called open neutrosophic sets, a neutrosophic set F is neutrosophic closed set if and only if it $C(F)$ is neutrosophic open set.

3.7. Definition:[19]

Let (X, \mathfrak{T}) be a NTS and A be a NS in X . Then the induced neutrosophic topology on A is the collection of NSs in A which are the intersection of neutrosophic open sets in X with A . Then the pair (A, \mathfrak{T}_A) is called a neutrosophic subspace of (X, \mathfrak{T}) . The induced neutrosophic topology is denoted by \mathfrak{T}_A .

4. Neutrosophic Continuity

It is known by [4] that $f : (X, \mathfrak{T}_X) \rightarrow (Y, \mathfrak{T}_Y)$ is neutrosophic continuous if the preimage of each neutrosophic open set in Y is neutrosophic open set in X .

4.1. Theorem:

Let (X, \mathfrak{T}_X) and (Y, \mathfrak{T}_Y) be two NTGs and $f : (X, \mathfrak{T}_X) \rightarrow (Y, \mathfrak{T}_Y)$ be a mapping, then f is neutrosophic continuous if and only if f is neutrosophic continuous at neutrosophic point

$x_{(\alpha,\beta,\gamma)}$, for each $x \in X$.

5. Neutrosophic Bitopological Spaces

5.1. Definition:[23]

Let (X, \mathfrak{T}_1) and (X, \mathfrak{T}_2) be the two neutrosophic topologies on X . Then $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ is called a neutrosophic bitopological space (In short NBTS).

5.2. Definition:[23]

Let $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ be a NBTS. A NS $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x) \rangle : x \in X\}$ over X is said to a pairwise neutrosophic open set in $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ if there exist a NS $A_1 = \{\langle x, \mathcal{T}_{A_1}(x), \mathcal{I}_{A_1}(x), \mathcal{F}_{A_1}(x) \rangle : x \in X\}$ in \mathfrak{T}_1 and a NS $A_2 = \{\langle x, \mathcal{T}_{A_2}(x), \mathcal{I}_{A_2}(x), \mathcal{F}_{A_2}(x) \rangle : x \in X\}$ in \mathfrak{T}_2 such that $A = A_1 \cup A_2 = \{\langle x, \min(\mathcal{T}_{A_1}(x), \mathcal{T}_{A_2}(x)), \min(\mathcal{I}_{A_1}(x), \mathcal{I}_{A_2}(x)), \max(\mathcal{F}_{A_1}(x), \mathcal{F}_{A_2}(x)) \rangle : x \in X\}$.

6. Neutrosophic Topological Groups

6.1. Definition:[22]

Let X be a group and \mathcal{G} be a NG on X . Let $\mathfrak{T}^{\mathcal{G}}$ be a neutrosophic topology on \mathcal{G} then $(\mathcal{G}, \mathfrak{T}^{\mathcal{G}})$ is said to be neutrosophic topological group (In short NTG) if the following conditions are satisfied:

- (i) The mapping $\psi : (\mathcal{G}, \mathfrak{T}^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}^{\mathcal{G}})$ defined by $\psi(x, y) = xy$, for all $x, y \in X$, is relatively neutrosophic continuous.
- (ii) The mapping $\mu : (\mathcal{G}, \mathfrak{T}^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}^{\mathcal{G}})$ defined by $\mu(x) = x^{-1}$, for all $x \in X$, is relatively neutrosophic continuous.

6.2. Definition:[18]

Let X be a group and U, V be two NSs in X . We define the product UV of NS U, V and the inverse V^{-1} of V as follows:

$$UV(x) = \{\langle x, \mathcal{T}_{UV}(x), \mathcal{I}_{UV}(x), \mathcal{F}_{UV}(x) \rangle : x \in X\}$$

where

$$\mathcal{T}_{UV}(x) = \sup\{\min\{\mathcal{T}_U(x_1), \mathcal{T}_V(x_1)\}\}$$

$$\mathcal{I}_{UV}(x) = \sup\{\min\{\mathcal{I}_U(x_1), \mathcal{I}_V(x_1)\}\}$$

$$\mathcal{F}_{UV}(x) = \sup\{\min\{\mathcal{F}_U(x_1), \mathcal{F}_V(x_1)\}\}$$

where $x = x_1.x_2$ and for $V = \{\langle x, \mathcal{T}_V(x), \mathcal{I}_V(x), \mathcal{F}_V(x) \rangle : x \in X\}$, we have

$$V^{-1} = \{ \langle x, \mathcal{T}_V(x^{-1}), \mathcal{I}_V(x^{-1}), \mathcal{F}_V(x^{-1}) \rangle : x \in X \}.$$

7. Main Results:

7.1. Definition:

Let X be a group and \mathcal{G} be a NG on X . Let $\mathfrak{T}_1^{\mathcal{G}}, \mathfrak{T}_2^{\mathcal{G}}$ be two neutrosophic topologies on \mathcal{G} then $(\mathcal{G}, \mathfrak{T}_1^{\mathcal{G}}, \mathfrak{T}_2^{\mathcal{G}})$ is said to be neutrosophic bitopological group (In short NBTG) if the following conditions hold good:

- (i) The mapping $\psi : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ defined by $\psi(x, y) = xy$, for all $x, y \in X$, is relatively neutrosophic i -continuous for each $i=1, 2$.
- (ii) The mapping $\mu : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ defined by $\mu(x) = x^{-1}$, for all $x \in X$, is relatively neutrosophic i -continuous for each $i=1, 2$.

7.2. Definition:

Let \mathcal{G} be a NG of a group X . Then for fixed $a \in X$, the left translation $l_a : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ for each $i = 1, 2$; is defined by $l_a(x) = ax$, for all $x \in X$, where $ax = \{ \langle a, \mathcal{T}_i^{\mathcal{G}}(ax), \mathcal{I}_i^{\mathcal{G}}(ax), \mathcal{F}_i^{\mathcal{G}}(ax) \rangle : x \in X \}$ for each $i = 1, 2$.

Similarly, the right translation $r_a : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ for each $i = 1, 2$; is defined by $r_a(x) = xa$, for all $x \in X$, where $ax = \{ \langle a, \mathcal{T}_i^{\mathcal{G}}(xa), \mathcal{I}_i^{\mathcal{G}}(xa), \mathcal{F}_i^{\mathcal{G}}(xa) \rangle : x \in X \}$ for each $i = 1, 2$.

7.3. Lemma:

Let X be a group with NBTG \mathcal{G} in X with two neutrosophic topologies $\mathfrak{T}_1, \mathfrak{T}_2$. Then for each $a \in \mathcal{G}_e$, the translation l_a and r_a are relatively neutrosophic homeomorphism of $(\mathcal{G}, \mathfrak{T}_1^{\mathcal{G}}, \mathfrak{T}_2^{\mathcal{G}})$ into itself.

Proof: From Proposition 3.11 [10], we have $l_a[\mathcal{G}] = \mathcal{G}$ and $r_a[\mathcal{G}] = \mathcal{G}$, for all $a \in \mathcal{G}_e$ and let $h : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$, for each $i = 1, 2$; defined by $h(x) = (a, x)$ for each $x \in X$. Then $r_a : \psi \circ h$. Since $a \in \mathcal{G}_e$, $\mathcal{T}_i^{\mathcal{G}}(a) = \mathcal{T}_i^{\mathcal{G}}(e)$, $\mathcal{I}_i^{\mathcal{G}}(a) = \mathcal{I}_i^{\mathcal{G}}(e)$, and $\mathcal{F}_i^{\mathcal{G}}(a) = \mathcal{F}_i^{\mathcal{G}}(e)$, for each $i = 1, 2$. Thus $\mathcal{T}_i^{\mathcal{G}}(a) \geq \mathcal{T}_i^{\mathcal{G}}(x)$, $\mathcal{I}_i^{\mathcal{G}}(a) \geq \mathcal{I}_i^{\mathcal{G}}(x)$, and $\mathcal{F}_i^{\mathcal{G}}(a) \leq \mathcal{F}_i^{\mathcal{G}}(x)$, for each $x \in X$. It follows from proposition 3.34 [11] that $\phi : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is relatively neutrosophic i -continuous for each $i = 1, 2$. By the hypothesis ψ is relatively neutrosophic i -continuous for each $i = 1, 2$. So r_a is relatively neutrosophic i -continuous for each $i = 1, 2$. Moreover $r_a^{-1} = r_{a^{-1}}$. Similarly we are shown the relatively neutrosophic i -continuous for each $i = 1, 2$ of $l_a^{-1} = l_{a^{-1}}$.

7.4. Theorem:

Let \mathcal{G} be a NBTG on X with $\mathfrak{T}_1, \mathfrak{T}_2$ two neutrosophic topologies. Let U be a neutrosophic open set of $(\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ for each $i=1, 2$ and $x \in \mathcal{G}_e$, then xU and Ux are neutrosophic open set.

Proof: Since U is neutrosophic open set of \mathcal{G} and $x \in \mathcal{G}_e$, $\lambda : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is neutrosophic homeomorphism for each $i=1, 2$. This implies that $l_x(U) = xU$ is neutrosophic open set in \mathcal{G} . Similarly Ux is neutrosophic open set in \mathcal{G} .

7.5. Lemma:

Let X be a group and let \mathcal{G} be NBTG in X . Then

- (i) The inverse function $\phi : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ defined by $\phi(x) = x^{-1}$, for all $x \in X$ is relatively neutrosophic i -continuous homeomorphism for each $i=1, 2$.
- (ii) The inner automorphism $\lambda : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ defined by $\lambda(g) = aga^{-1} = \{\langle g, \mathcal{T}_i^{\mathcal{G}}(aga^{-1}), \mathcal{I}_i^{\mathcal{G}}(aga^{-1}), \mathcal{F}_i^{\mathcal{G}}(aga^{-1}) \rangle\}$, where $g \in X$ and $a \in \mathcal{G}_e$ is relatively neutrosophic homeomorphism for each $i=1, 2$.

Proof: (i) Clearly ϕ is one-to-one. Since $\phi(\mathcal{G}) = \{\langle x, \phi(\mathcal{T}_i^{\mathcal{G}}(x)), \phi(\mathcal{I}_i^{\mathcal{G}}(x)), \phi(\mathcal{F}_i^{\mathcal{G}}(x)) \rangle : x \in \mathcal{G}\}$ for each $i=1, 2$ where

$$\begin{aligned} \phi(\mathcal{T}_i^{\mathcal{G}}(x)) &= \begin{cases} \bigvee_{y \in \phi^{-1}(x)} \mathcal{T}_i^{\mathcal{G}}(y), & \text{if } \phi^{-1}(x) \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathcal{T}_i^{\mathcal{G}}(x^{-1}), & \text{if } \phi^{-1}(x) \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathcal{T}_i^{\mathcal{G}}(x), & \text{if } \phi^{-1}(x) \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Also, $\phi(\mathcal{I}_i^{\mathcal{G}}(x)) = \mathcal{I}_i^{\mathcal{G}}(x)$ and $\phi(\mathcal{F}_i^{\mathcal{G}}(x)) = \mathcal{F}_i^{\mathcal{G}}(x)$

Thus $\phi(\mathcal{G}) = \{\langle x, \mathcal{T}_i^{\mathcal{G}}(x), \mathcal{I}_i^{\mathcal{G}}(x), \mathcal{F}_i^{\mathcal{G}}(x) \rangle : x \in \mathcal{G}\}$, for each $i=1, 2$. Also ϕ is neutrosophic i -continuous for each $i=1, 2$ by definition because $(\mathcal{G}, \mathfrak{T}_1^{\mathcal{G}}, \mathfrak{T}_2^{\mathcal{G}})$ is NBTG. Since $\phi^{-1}(x) = x^{-1}$ is relatively neutrosophic i -continuous for each $i=1, 2$. Hence for every $x \in X$, ϕ is relatively neutrosophic open. Thus ϕ is relatively neutrosophic homeomorphism.

(ii) Since r_a and l_a are relatively neutrosophic homeomorphism and $r_a^{-1} = r_{a^{-1}}$. The inner automorphism λ is a composition $r_{a^{-1}}$ and l . Hence λ is a relative neutrosophic homeomorphism.

7.6. Theorem:

Let \mathcal{G} be a NBTG in a group X and e be the identity of X . If $a \in \mathcal{G}_e$ and N is a neighbourhood of e such that $\mathcal{T}_i^N(e) = 1$, $\mathcal{I}_i^N(e) = 1$, $\mathcal{F}_i^N(e) = 0$ for each $i=1, 2$ then aN is a nbd of a such that $aN(a) = 1_N$.

Proof: Since N is a nbd of e such that $\mathcal{T}_i^N(e) = 1$, $\mathcal{I}_i^N(e) = 1$, $\mathcal{F}_i^N(e) = 0$ for each $i=1, 2$; there exists a neutrosophic open set U such that $U \subseteq N$ and $\mathcal{T}_i^U(e) = \mathcal{T}_i^N(e) = 1$, $\mathcal{I}_i^U(e) = \mathcal{I}_i^N(e) = 1$,

$\mathcal{F}_i^U(e) = \mathcal{F}_i^N(e) = 0$, for each $i=1, 2$. Let $l_a : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ be a left translation defined by $l_a(g) = ag$, for each $g \in X$ and $i=1, 2$. Then l_a is neutrosophic homeomorphism. Then aU is a neutrosophic open set. Now,

$$\begin{aligned} aU(a) &= \{\langle a, \mathcal{T}_i^{aU}(a), \mathcal{I}_i^{aU}(a), \mathcal{F}_i^{aU}(a) \rangle\}, \text{ for each } i = 1, 2. \\ &= \{\langle a, \mathcal{T}_i^U(aa^{-1}), \mathcal{I}_i^U(aa^{-1}), \mathcal{F}_i^U(aa^{-1}) \rangle\} \\ &= \{\langle a, \mathcal{T}_i^U(e), \mathcal{I}_i^U(e), \mathcal{F}_i^U(e) \rangle\} \\ &= \{\langle a, 1, 1, 0 \rangle\} \end{aligned}$$

$$\begin{aligned} \text{Also, } aN(x) &= \{\langle x, \mathcal{T}_i^{aN}(x), \mathcal{I}_i^{aN}(x), \mathcal{F}_i^{aN}(x) \rangle : x \in X\}, \text{ for each } i = 1, 2. \\ &= \{\langle x, \mathcal{T}_i^N(a^{-1}x), \mathcal{I}_i^N(a^{-1}x), \mathcal{F}_i^N(a^{-1}x) \rangle : x \in X\} \\ &\geq \{\langle x, \mathcal{T}_i^U(a^{-1}x), \mathcal{I}_i^U(a^{-1}x), \mathcal{F}_i^U(a^{-1}x) \rangle : x \in X\} \\ &= \{\langle x, \mathcal{T}_i^{aU}(x), \mathcal{I}_i^{aU}(x), \mathcal{F}_i^{aU}(x) \rangle\} \\ &= aU(x) \end{aligned}$$

$$aN(x) \geq aU(x); \text{ for each } x \in X.$$

$$\begin{aligned} \text{and } aN(a) &= \{\langle a, \mathcal{T}_i^{aN}(a), \mathcal{I}_i^{aN}(a), \mathcal{F}_i^{aN}(a) \rangle\}, \text{ for each } i = 1, 2. \\ &= \{\langle a, \mathcal{T}_i^N(aa^{-1}), \mathcal{I}_i^N(aa^{-1}), \mathcal{F}_i^N(aa^{-1}) \rangle\} \\ &= \{\langle a, \mathcal{T}_i^N(e), \mathcal{I}_i^N(e), \mathcal{F}_i^N(e) \rangle\} \\ &= \{\langle a, 1, 1, 0 \rangle\} \\ \Rightarrow aN(a) &= \{\langle a, 1, 1, 0 \rangle\} \end{aligned}$$

Thus, there exist a neutrosophic open set aU such that $aU \subseteq aN$ and $aU(a) = aN(a) = \{\langle a, 1, 1, 0 \rangle\}$.

7.7. Proposition:

Let X be a group and \mathcal{G} be a NBTG on X with $\mathfrak{T}_1, \mathfrak{T}_2$ two neutrosophic topologies. Let $\lambda : X \times X \rightarrow X$ be the function defined by $\lambda(g, h) = gh^{-1}$ for any $g, h \in X$. Then \mathcal{G} is a NBTG in X iff the function $\lambda : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is relatively neutrosophic i -continuous for each $i=1, 2$.

Proof: The function $\mu : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is neutrosophic relatively i -continuous for each $i=1, 2$; by the corollary to Proposition 3.28 [11]. Also since \mathcal{G} is a NBTG in X by the Definition [7.1] $\psi : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is relatively neutrosophic i -continuous for each $i=1, 2$. Then $\beta : \psi \circ \mu : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is relatively neutrosophic i -continuous for each $i=1, 2$.

Conversely, let $\beta : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is relatively neutrosophic i -continuous for each $i=1, 2$. If e is the identity element of X , then $\mathcal{T}_i^{\mathcal{G}}(e) \geq \mathcal{T}_i^{\mathcal{G}}(g)$, $\mathcal{I}_i^{\mathcal{G}}(e) \geq \mathcal{I}_i^{\mathcal{G}}(g)$ and $\mathcal{F}_i^{\mathcal{G}}(e) \leq \mathcal{F}_i^{\mathcal{G}}(g)$ for all $g \in X$. By the Proposition 3.34 [11], the function $\phi : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ defined by $\phi(h) = (e, h)$ is relatively neutrosophic i -continuous for each $i=1, 2$. Thus the function $\alpha = \beta \circ \phi : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is relatively neutrosophic i -continuous for each $i=1, 2$. The function $\mu : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is relatively neutrosophic i -continuous for each $i=1, 2$ by the corollary to Proposition 3.28 [11]. Thus $\psi = \beta \circ \mu : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is relatively neutrosophic i -continuous for each $i=1, 2$. Therefore \mathcal{G} is a NBTG in X .

7.8. Proposition:

Let $\phi : X \rightarrow Y$ be a group homomorphism and $\mathfrak{T}_1, \mathfrak{T}_2$ and $\mathcal{U}_1, \mathcal{U}_2$ be the neutrosophic topologies on X and Y respectively, where \mathfrak{T}_i is the inverse image of \mathcal{U}_i under ϕ and let \mathcal{G} be a NBTG in Y . Then the inverse image $\phi^{-1}(\mathcal{G})$ of \mathcal{G} is a NBTG in X .

Proof: Consider the function $\alpha : X \times X \rightarrow X$ defined by $\alpha(g_1, g_2) = g_1 g_2^{-1}$ for any $g_1, g_2 \in X$. We have to prove that the function $\alpha : (\phi^{-1}(\mathcal{G}), \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})}) \times (\phi^{-1}(\mathcal{G}), \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})}) \rightarrow (\phi^{-1}(\mathcal{G}), \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})})$ is relatively neutrosophic i -continuous for each $i=1, 2$. Since \mathfrak{T}_i is the inverse image of \mathcal{U}_i under ϕ , $\phi : (X, \mathfrak{T}_i) \rightarrow (X, \mathcal{U}_i)$ is the neutrosophic i -continuous for each $i=1, 2$. Also, $\phi(\phi^{-1}(\mathcal{G})) \subset \mathcal{G}$. By Proposition 3.9 [11], $\phi : (\phi^{-1}(\mathcal{G}), \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})}) \rightarrow (\mathcal{G}, \mathcal{U}_i^{\mathcal{G}})$ is relatively neutrosophic i -continuous for each $i=1, 2$. Let $U = \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})}$. Then there exist a $V = \mathcal{U}_i^{\mathcal{G}}$ such that $\phi^{-1}(V) = U$. Let $(g_1, g_2) \in X \times X$. Then

$$\begin{aligned} \mathcal{T}_i^{\alpha^{-1}(U)}(g_1, g_2) &= \alpha^{-1}(\mathcal{T}_i^U)(g_1, g_2) = \mathcal{T}_i^U(\alpha(g_1, g_2)) = \mathcal{T}_i^U(g_1, g_2^{-1}), \text{ for each } i = 1, 2. \\ &= \mathcal{T}_i^{\phi^{-1}(V)}(g_1, g_2^{-1}) = \phi_{(\mathcal{T}_i^V)}^{-1}(g_1, g_2^{-1}) = \mathcal{T}_i^V(\phi(g_1, g_2^{-1})) \\ &= \mathcal{T}_i^V(\phi(g_1), \phi(g_2^{-1})) = \mathcal{T}_i^V(\phi(g_1), (\phi(g_2))^{-1}) \end{aligned}$$

$$\text{Thus } \mathcal{T}_i^{\alpha^{-1}(U)}(g_1, g_2) = \mathcal{T}_i^V(\phi(g_1), (\phi(g_2))^{-1})$$

Similarly we have $\mathcal{I}_i^{\alpha^{-1}(U)}(g_1, g_2) = \mathcal{I}_i^V(\phi(g_1), (\phi(g_2))^{-1})$ and $\mathcal{F}_i^{\alpha^{-1}(U)}(g_1, g_2) = \mathcal{F}_i^V(\phi(g_1), (\phi(g_2))^{-1})$ for each $i=1, 2$. By the hypothesis, the function $\beta : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ given by $\beta(h_1, h_2) = h_1 h_2^{-1}$ for any $h_1, h_2 \in Y$ is relatively neutrosophic i -continuous for each $i=1, 2$. By corollary to the Proposition 3.28 [11] the product function $\phi \times \phi : (\phi^{-1}(\mathcal{G}), \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})}) \times (\phi^{-1}(\mathcal{G}), \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is the neutrosophic i -continuous for each $i=1, 2$. Now, let $(g_1, g_2) \in X \times X$. Then

$$\mathcal{T}_i^V(\phi(g_1), (\phi(g_2))^{-1}) = \mathcal{T}_i^{\beta^{-1}(V)}(\phi(g_1), \phi(g_2)) = \mathcal{T}_i^{(\phi \times \phi)^{-1}(\beta^{-1}(V))}(g_1, g_2),$$

$$\begin{aligned}\mathcal{I}_i^V\left(\phi(g_1), (\phi(g_2))^{-1}\right) &= \mathcal{I}_i^{\beta^{-1}(V)}(\phi(g_1), \phi(g_2)) = \mathcal{I}_i^{(\phi \times \phi)^{-1}(\beta^{-1}(V))}(g_1, g_2) \\ \text{and } \mathcal{F}_i^V\left(\phi(g_1), (\phi(g_2))^{-1}\right) &= \mathcal{F}_i^{\beta^{-1}(V)}(\phi(g_1), \phi(g_2)) = \mathcal{F}_i^{(\phi \times \phi)^{-1}(\beta^{-1}(V))}(g_1, g_2) \\ &\text{for each } i = 1, 2.\end{aligned}$$

$$\begin{aligned}\text{Thus } \alpha^{-1}(U) \cap \left(\phi^{-1}(\mathcal{G}) \times \phi^{-1}(\mathcal{G})\right) &= (\phi \times \phi)^{-1}(\beta^{-1}(V)) \cap \left(\phi^{-1}(\mathcal{G}) \times \phi^{-1}(\mathcal{G})\right) \\ &= [\beta \circ (\phi \times \phi)]^{-1}(V) \cap \left(\phi^{-1}(\mathcal{G}) \times \phi^{-1}(\mathcal{G})\right).\end{aligned}$$

So $\alpha^{-1}(U) \cap \left(\phi^{-1}(\mathcal{G}) \times \phi^{-1}(\mathcal{G})\right) \in \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})} \times \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})}$, i.e., $\alpha : \left(\phi^{-1}(\mathcal{G}), \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})}\right) \times \left(\phi^{-1}(\mathcal{G}), \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})}\right) \rightarrow \left(\phi^{-1}(\mathcal{G}), \mathfrak{T}_i^{\phi^{-1}(\mathcal{G})}\right)$ is a relatively neutrosophic i -continuous for each $i=1,2$. By Result 3.9 [10], $\phi^{-1}(\mathcal{G})$ is neutrosophic group in X . Hence by Proposition [7.7], $\phi^{-1}(\mathcal{G})$ is NBTG in X .

7.9. Proposition:

Let $\phi : X \rightarrow Y$ be a group homomorphism. Let $\mathfrak{T}_1, \mathfrak{T}_2$ and $\mathcal{U}_1, \mathcal{U}_2$ be the neutrosophic topologies on X and Y respectively, where \mathcal{U}_i is the image under ϕ and of \mathfrak{T}_i , for each $i=1, 2$; and let \mathcal{G} be a NBTG in X . If \mathcal{G} is the neutrosophic invariant, then the image $\phi(\mathcal{G})$ of \mathcal{G} is a NBTG in Y .

Proof: Consider the function $\beta : Y \rightarrow Y$ defined by $\beta(h_1, h_2) = h_1 h_2^{-1}$ for any $h_1, h_2 \in Y$. We have to prove that the function $\beta : \left(\phi(\mathcal{G}), \mathcal{U}_i^{\phi(\mathcal{G})}\right) \times \left(\phi(\mathcal{G}), \mathcal{U}_i^{\phi(\mathcal{G})}\right) \rightarrow \left(\phi(\mathcal{G}), \mathcal{U}_i^{\phi(\mathcal{G})}\right)$ is a relatively neutrosophic i -continuous for each $i=1, 2$. Suppose \mathcal{G} is a neutrosophic invariant. By the Definition 3.2, $\phi(\mathcal{G})$ is a neutrosophic group in Y . Let $U \in \mathfrak{T}_i$. Also $U \subset \phi^{-1}(\phi(U))$. Then there exist a family $\{U_\lambda\}_{\lambda \in \Lambda} \subset \mathfrak{T}_i$ such that $\phi^{-1}(\phi(U)) = \bigcup_{\alpha \in \Lambda} U_\alpha$. So $\phi^{-1}(\phi(U)) \in \mathfrak{T}_i$. Since \mathcal{U}_i is the image of \mathfrak{T}_i under ϕ , $\phi(U) \in \mathcal{U}_i$, for each $i=1, 2$. So ϕ is neutrosophic i -open. Now, let $U \in \mathfrak{T}_i^{\mathcal{G}}$. Then there exist a $U = U_1 \cap \mathcal{G}$. Since \mathcal{G} is neutrosophic invariant, by Proposition 3.12 [10], $\phi(U) = \phi(U_1) \cap \phi(\mathcal{G})$. Since ϕ is neutrosophic i -open, $\phi(U_1) = \mathfrak{T}_i$, for each $i=1,2$. Then $\phi(U) \in \mathcal{U}_i^{\phi(\mathcal{G})}$, for each $i=1,2$. Thus $\phi : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow \left(\phi(\mathcal{G}), \mathcal{U}_i^{\phi(\mathcal{G})}\right)$ is relatively neutrosophic i -open for each $i=1,2$. By Proposition 3.31 [11], the product function $(\phi \times \phi) : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow \left(\phi(\mathcal{G}), \mathcal{U}_i^{\phi(\mathcal{G})}\right)$ is relative neutrosophic i -open for each $i=1,2$.

Let $V \in \mathcal{U}_i^{\phi(\mathcal{G})}$ and let $(g_1, g_2) \in X \times X$. Then

$$\begin{aligned}\mathcal{T}_i^{\beta \circ (\phi \times \phi)^{-1}(V)}(g_1, g_2) &= [\beta \circ (\phi \times \phi)]^{-1}(\mathcal{T}_i^V)(g_1, g_2), \text{ for each } i = 1, 2. \\ &= \mathcal{T}_i^V[\beta \circ (\phi \times \phi)](g_1, g_2) = \mathcal{T}_i^V(\phi(g_1), \phi(g_2)) \\ &= \mathcal{T}_i^V(\phi(g_1), (\phi(g_2))^{-1}) = \mathcal{T}_i^V(\phi(g_1), \phi(g_2^{-1})) [\text{Since } \phi \text{ is homomorphism}] \\ &= \mathcal{T}_i^V(\phi(g_1 g_2^{-1})) = \mathcal{T}_i^V \phi(\alpha(g_1, g_2)) = \mathcal{T}_i^V(\phi \circ \alpha(g_1, g_2))\end{aligned}$$

$$= (\phi \circ \alpha)^{-1}(\mathcal{T}_i^V(g_1, g_2)) = \mathcal{T}_i^{(\phi \circ \alpha)^{-1}(V)}(g_1, g_2),$$

where $\alpha : X \times X \rightarrow X$ is the mapping given by $\alpha(g_1, g_2) = g_1 g_2^{-1}$ for each $(g_1, g_2) \in X \times X$. Thus $\mathcal{T}_i^{[\beta \circ (\phi \times \phi)]^{-1}(V)} = \mathcal{T}_i^{(\phi \circ \alpha)^{-1}(V)}$, $\mathcal{T}_i^{(\phi \times \phi)^{-1}[\beta^{-1}(V)]} = \mathcal{T}_i^{\alpha^{-1}(\phi^{-1}(V))}$. Similarly $\mathcal{I}_i^{(\phi \times \phi)^{-1}[\beta^{-1}(V)]} = \mathcal{I}_i^{\alpha^{-1}(\phi^{-1}(V))}$ and $\mathcal{F}_i^{(\phi \times \phi)^{-1}[\beta^{-1}(V)]} = \mathcal{F}_i^{\alpha^{-1}(\phi^{-1}(V))}$. So $(\phi \times \phi)^{-1}[\beta^{-1}(V)] = \alpha^{-1}(\phi^{-1}(V))$. Since \mathcal{G} is NBTG in X , $\alpha : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$ is relatively neutrosophic i -continuous for each $i=1, 2$. Since \mathcal{U}_i is the image of \mathfrak{T}_i under ϕ , $\phi : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}})$, $\mathfrak{T}_i^{\phi(\mathcal{G})}$ is relatively neutrosophic i -continuous for each $i=1, 2$. Then $(\phi \times \phi) : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\phi(\mathcal{G}), \mathcal{U}_i^{\phi(\mathcal{G})}) \times (\phi(\mathcal{G}), \mathcal{U}_i^{\phi(\mathcal{G})})$ is relatively neutrosophic i -continuous for each $i=1, 2$. Thus $(\phi \times \phi) \circ \beta : (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \times (\mathcal{G}, \mathfrak{T}_i^{\mathcal{G}}) \rightarrow (\phi(\mathcal{G}), \mathcal{U}_i^{\phi(\mathcal{G})})$ is relatively neutrosophic i -continuous for each $i=1, 2$. Since \mathcal{G} is neutrosophic invariant, $(\phi \times \phi)^{-1}[\beta^{-1}(V) \cap (\phi(\mathcal{G}) \times \phi(\mathcal{G}))] = (\phi \times \phi)^{-1}[\beta^{-1}(V)] \cap (\mathcal{G} \times \mathcal{G})$.

So $(\phi \times \phi)^{-1}[\beta^{-1}(V) \cap (\phi(\mathcal{G}) \times \phi(\mathcal{G}))] \in \mathfrak{T}_i^{\mathcal{G}} \times \mathfrak{T}_i^{\mathcal{G}}$. Since $(\phi \times \phi)$ is relatively neutrosophic i -open for each $i=1, 2$; $(\phi \times \phi)(\phi \times \phi)^{-1}[\beta^{-1}(V) \cap (\phi(\mathcal{G}) \times \phi(\mathcal{G}))] \in \mathcal{U}_i^{\phi(\mathcal{G})} \times \mathcal{U}_i^{\phi(\mathcal{G})}$ for each $i=1, 2$. But $(\phi \times \phi)(\phi \times \phi)^{-1}[\beta^{-1}(V) \cap (\phi(\mathcal{G}) \times \phi(\mathcal{G}))] = \beta^{-1}(V) \cap (\phi(\mathcal{G}) \times \phi(\mathcal{G}))$. So $\beta^{-1}(V) \cap (\phi(\mathcal{G}) \times \phi(\mathcal{G})) \in \mathcal{U}_i^{\phi(\mathcal{G})} \times \mathcal{U}_i^{\phi(\mathcal{G})}$ for each $i=1, 2$. Hence $\phi(\mathcal{G})$ is a NBTG in Y .

7.10. Proposition:

Let X be a group and let \mathcal{G} be a NBTG in X with $\mathfrak{T}_1, \mathfrak{T}_2$ two neutrosophic topologies. N a normal subgroup of X and let f be the canonical homomorphism of X onto the quotient group X/N . If \mathcal{G} is constant on N , then \mathcal{G} is f invariant.

Proof: Suppose $f(x_1) = f(x_2)$ for any $x_1, x_2 \in N$. Then $x_1 N = x_2 N$. Thus there exist $k_1, k_2 \in N$ such that $x_1 k_1 = x_2 k_2$. Since \mathcal{G} is a constant on N , $\mathcal{T}_i^{\mathcal{G}}(x) = \mathcal{T}_i^{\mathcal{G}}(e)$, $\mathcal{I}_i^{\mathcal{G}}(x) = \mathcal{I}_i^{\mathcal{G}}(e)$ and $\mathcal{F}_i^{\mathcal{G}}(x) = \mathcal{F}_i^{\mathcal{G}}(e)$ for each $i=1, 2$ and $x \in X$. Then

$$\begin{aligned} \mathcal{T}_i^{\mathcal{G}}(x_1) &= \mathcal{T}_i^{\mathcal{G}}(x_2 k_2 k_1^{-1}) \geq \mathcal{T}_i^{\mathcal{G}}(x_2) \wedge \mathcal{T}_i^{\mathcal{G}}(k_2 k_1^{-1}) \\ &= \mathcal{T}_i^{\mathcal{G}}(x_2) \wedge \mathcal{T}_i^{\mathcal{G}}(e) (k_2 k_1^{-1} \in N) \\ &= \mathcal{T}_i^{\mathcal{G}}(x_2) \end{aligned}$$

$$\text{i.e., } \mathcal{T}_i^{\mathcal{G}}(x_1) \geq \mathcal{T}_i^{\mathcal{G}}(x_2).$$

Similarly, we get $\mathcal{T}_i^{\mathcal{G}}(x_2) \geq \mathcal{T}_i^{\mathcal{G}}(x_1)$. Thus $\mathcal{T}_i^{\mathcal{G}}(x_1) = \mathcal{T}_i^{\mathcal{G}}(x_2)$. Similarly we can show that $\mathcal{I}_i^{\mathcal{G}}(x_1) = \mathcal{I}_i^{\mathcal{G}}(x_2)$ and $\mathcal{F}_i^{\mathcal{G}}(x_1) = \mathcal{F}_i^{\mathcal{G}}(x_2)$. Hence \mathcal{G} is f invariant.

8. Conclusion:

The main characteristic of NS is that NS can deal with imprecises as well as inconsistent information which is very helpful to handle the various real-life application. By observing this we have studied NBTG on the basis of NS, so that we can deal with various problem of topological group with respect to NS. In this study we have introduced some new definition of NBTG. We investigated some properties and proved some propositions on NBTG. We hope our work will help in further study of generalised NBTG and also for study of neutrosophic almost topological group and neutrosophic almost bitopological group.

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On New Types of Weakly Neutrosophic Crisp Continuity

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Abstract: The article processes the conceptualizations of neutrosophic crisp α -open and neutrosophic crisp semi- α -open sets to define some new types of weakly "neutrosophic crisp continuity" essentially, neutrosophic crisp α^* -continuous, neutrosophic crisp α^{**} -continuous, neutrosophic crisp semi- α -continuous, neutrosophic crisp semi- α^* -continuous and neutrosophic crisp semi- α^{**} -continuous functions. Also, we shall explain the relationships between these types of weakly neutrosophic crisp continuity and the concepts of neutrosophic crisp continuity.

Keywords: Neutrosophic crisp α^* -continuous, neutrosophic crisp α^{**} -continuous, neutrosophic crisp semi- α -continuous, neutrosophic crisp semi- α^* -continuous, and neutrosophic crisp semi- α^{**} -continuous functions.

1. Introduction

In 2014, Salama et al. [1] performed the abstraction of neutrosophic crisp topological space (concisely, *NCTS*). Al-Hamido et al. [2] submitted the intellect of neutrosophic crisp semi- α -closed sets in *NCTS*s. Abdel-Basset et al. [3-8] gave a novel neutrosophic approach. Maheswari et al. [9] presented gb-closed sets and gb-continuity in aspects of the neutrosophic theory. Banupriya et al. [10] investigated the notion of α gs continuity and α gs irresolute maps in the sense of neutrosophic view. In [11], Dhavaseelan et al. exhibited the theme of neutrosophic α^m -continuity. Al-Hamido et al. [15] introduced neutrosophic crisp topology via N-topology. Imran et al. [16] introduced and studied the thought of neutrosophic generalized alpha generalized continuity. Hanif PAGE et al. [17] presented neutrosophic generalized homeomorphism. This paper aspires to lay on new types of weakly neutrosophic crisp continuity, for instance, neutrosophic crisp α^* -continuous, neutrosophic crisp α^{**} -continuous, neutrosophic crisp semi- α -continuous, neutrosophic crisp semi- α^* -continuous and neutrosophic crisp semi- α^{**} -continuous functions. Likewise, we shall explain the relationships between these types of weakly neutrosophic crisp continuity and the concepts of neutrosophic crisp continuity.

2. Preliminaries

For the whole of the disquisition, (\mathbb{X}, I_1) , (\mathbb{Y}, I_2) , and (\mathbb{Z}, I_3) (merely \mathbb{X} , \mathbb{Y} , and \mathbb{Z}) habitually intend *NCTS*s. Let \mathcal{C} be a neutrosophic crisp set (shortly, *NCS*) in *NCTS* (\mathbb{X}, I_1) and denote its complement by \mathcal{C}^c . Indicate the neutrosophic crisp open set as *NC-OS*, and the neutrosophic crisp closed set (its complement) as *NC-CS* in *NCTS* (\mathbb{X}, I_1) . Additionally, we refer to the neutrosophic crisp closure and neutrosophic crisp interior of \mathcal{C} via $NCcl(\mathcal{C})$ and $NCint(\mathcal{C})$, correspondingly.

Definition 2.1 [1]: Assume that nonempty particular understudy space \mathbb{X} has mutually disjoint subsets $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 . A NCS \mathcal{C} with form $\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \rangle$ is called an object.

Definition 2.2: For any NCS \mathcal{C} in $NCTS(\mathbb{X}, I_1)$, we have

- (i) if $\mathcal{C} \subseteq NCint(NCcl(NCint(\mathcal{C})))$, then it is called a neutrosophic crisp α -open set and symbolize by $NC\alpha$ -OS. Furthermore, its complement is named neutrosophic crisp α -closed set and signified by $NC\alpha$ -CS. Likewise, we reveal the collection consisting of all $NC\alpha$ -OSs in \mathbb{X} with $NC\alpha O(\mathbb{X})$. [12]
- (ii) if $\mathcal{C} \subseteq NCcl(NCint(NCcl(NCint(\mathcal{C}))))$, then it is said to be a neutrosophic crisp semi- α -open set and indicated via $NCS\alpha$ -OS. Moreover, its complement is known as a neutrosophic crisp semi- α -closed set and referred with $NCS\alpha$ -CS. Besides, we mentioned the collection of all $NCS\alpha$ -OSs in \mathbb{X} through $NCS\alpha O(\mathbb{X})$. [2]

Proposition 2.3 [12]: For any NCS \mathcal{C} in $NCTS(\mathbb{X}, I_1)$, then $\mathcal{C} \in NC\alpha O(\mathbb{X})$ iff we have at least a NC -OS \mathcal{D} satisfying $\mathcal{D} \subseteq \mathcal{C} \subseteq NCint(NCcl(\mathcal{D}))$.

Proposition 2.4 [14]: Every NC -OS is a $NC\alpha$ -OS, but the opposite is not valid in general.

Proposition 2.5 [2]: In a $NCTS(\mathbb{X}, I_1)$, the next assertions stand, but not vice versa:

- (i) All NC -OSs are $NCS\alpha$ -OSs.
- (ii) All $NC\alpha$ -OSs are $NCS\alpha$ -OSs.

Definition 2.6 [1]: Let $\eta: (\mathbb{X}, I_1) \rightarrow (\mathbb{Y}, I_2)$ be a function, we called it a neutrosophic crisp continuous and denoted by NC -continuous iff for all NC -OSs \mathcal{D} from \mathbb{Y} , then its inverse image $\eta^{-1}(\mathcal{D})$ is a NC -OS from \mathbb{X} .

Theorem 2.7 [1]: A function $\eta: (\mathbb{X}, I_1) \rightarrow (\mathbb{Y}, I_2)$ is NC -continuous iff $\eta^{-1}(NCint(\mathcal{D})) \subseteq NCint(\eta^{-1}(\mathcal{D}))$ for every $\mathcal{D} \subseteq \mathbb{Y}$.

Definition 2.8 [1]: Let $\eta: (\mathbb{X}, I_1) \rightarrow (\mathbb{Y}, I_2)$ be a function, we named it a neutrosophic crisp open and indicated via NC -open iff for all NC -OSs \mathcal{C} from \mathbb{X} , then its image $\eta(\mathcal{C})$ is a NC -OS from \mathbb{Y} .

Definition 2.9 [13]: Let $\eta: (\mathbb{X}, I_1) \rightarrow (\mathbb{Y}, I_2)$ be a function, we said it a neutrosophic crisp α -continuous and referred through $NC\alpha$ -continuous iff for all NC -OSs \mathcal{D} from \mathbb{Y} , then its inverse image $\eta^{-1}(\mathcal{D})$ is a $NC\alpha$ -OS from \mathbb{X} .

Proposition 2.10 [14]: Every NC -continuous function is a $NC\alpha$ -continuous, but the opposite is not valid in general.

3. Weakly Neutrosophic Crisp Continuity Functions

Definition 3.1: Let $\eta: (\mathbb{X}, I_1) \rightarrow (\mathbb{Y}, I_2)$ be a function, we call it as

- (i) a neutrosophic crisp α^* -continuous and denoted by $NC\alpha^*$ -continuous iff for all $NC\alpha$ -OSs \mathcal{D} from \mathbb{Y} , then its inverse image $\eta^{-1}(\mathcal{D})$ is a $NC\alpha$ -OS from \mathbb{X} .
- (ii) a neutrosophic crisp α^{**} -continuous and indicated via $NC\alpha^{**}$ -continuous iff for all $NC\alpha$ -OS \mathcal{D} from \mathbb{Y} , then its inverse image $\eta^{-1}(\mathcal{D})$ is a NC -OS from \mathbb{X} .

Definition 3.2: Let $\eta: (\mathbb{X}, I_1) \rightarrow (\mathbb{Y}, I_2)$ be a function, we named it as

- (i) a neutrosophic crisp semi- α -continuous and referred through $NCS\alpha$ -continuous iff for all NC -OSs \mathcal{D} from \mathbb{Y} , then its inverse image $\eta^{-1}(\mathcal{D})$ is a $NCS\alpha$ -OS from \mathbb{X} .
- (ii) a neutrosophic crisp semi- α^* -continuous and symbolize by $NCS\alpha^*$ -continuous iff for all $NCS\alpha$ -OSs \mathcal{D} from \mathbb{Y} , then its inverse image $\eta^{-1}(\mathcal{D})$ is a $NCS\alpha$ -OS from \mathbb{X} .
- (iii) a neutrosophic crisp semi- α^{**} -continuous and signified via $NCS\alpha^{**}$ -continuous iff for all $NCS\alpha$ -OSs \mathcal{D} from \mathbb{Y} , then its inverse image $\eta^{-1}(\mathcal{D})$ is a NC -OS from \mathbb{X} .

Theorem 3.3: Let $\eta: (\mathbb{X}, I_1) \rightarrow (\mathbb{Y}, I_2)$ be a function, then the next declarations are same:

- (i) η is a $NCS\alpha$ -continuous.
- (ii) its inverse image of each NC -CS from \mathbb{Y} is $NCS\alpha$ -CS from \mathbb{X} .
- (iii) $\eta(NCint(NCcl(NCint(NCcl(\mathcal{C})))) \subseteq NCcl(\eta(\mathcal{C}))$, for each $\mathcal{C} \in \mathbb{X}$.
- (iv) $NCint(NCcl(NCint(NCcl(\eta^{-1}(\mathcal{D})))) \subseteq \eta^{-1}(NCcl(\mathcal{D}))$, for each $\mathcal{D} \in \mathbb{Y}$.

Proof:

[(i) \Rightarrow (ii)] Suppose \mathcal{D} is a NC -CS from \mathbb{Y} . This implies that \mathcal{D}^c stands a NC -OS. Hence $\eta^{-1}(\mathcal{D}^c)$ is a $NCS\alpha$ -OS from \mathbb{X} . In other words, $(\eta^{-1}(\mathcal{D}))^c$ stands a $NCS\alpha$ -OS from \mathbb{X} . Thus $\eta^{-1}(\mathcal{D})$ is a $NCS\alpha$ -CS in \mathbb{X} .

[(ii) \Rightarrow (iii)] Let $\mathcal{C} \in \mathbb{X}$, then $NCcl(\eta(\mathcal{C}))$ stays a NC -CS from \mathbb{Y} . Hence $\eta^{-1}(NCcl(\eta(\mathcal{C})))$ is $NCS\alpha$ -CS in \mathbb{X} . Thus we have $\eta^{-1}(NCcl(\eta(\mathcal{C}))) \supseteq NCint(NCcl(NCint(NCcl(\eta^{-1}(NCcl(\eta(\mathcal{C})))))) \supseteq NCint(NCcl(NCint(NCcl(\mathcal{C}))))$.

Or $NCcl(\eta(\mathcal{C})) \supseteq \eta(NCint(NCcl(NCint(NCcl(\mathcal{C}))))$.

[(iii) \Rightarrow (iv)] Since $\mathcal{D} \in \mathbb{Y}, \eta^{-1}(\mathcal{D}) \in \mathbb{X}$. So, we have by our hypothesis the corresponding notation $NCint(NCcl(NCint(NCcl(\eta^{-1}(\mathcal{D})))) \subseteq NCcl(\eta(\eta^{-1}(\mathcal{D}))) \subseteq NCcl(\mathcal{D})$, and that leads us to this fact $NCint(NCcl(NCint(NCcl(\eta^{-1}(\mathcal{D})))) \subseteq \eta^{-1}(NCcl(\mathcal{D}))$.

[(iv) \Rightarrow (i)] Let \mathcal{D} be a NC -OS of \mathbb{Y} . Let $\mathcal{C} = \mathcal{D}^c$ and $\mathcal{D} = \eta^{-1}(\mathcal{C})$ by (iii) we have $NCint(NCcl(NCint(NCcl(\eta^{-1}(\mathcal{C})))) \subseteq NCcl(\mathcal{C}) = \mathcal{C}$.

That is $NCint(NCcl(NCint(NCcl(\eta^{-1}(\mathcal{D}^c)))) \subseteq \eta^{-1}(\mathcal{D}^c)$. Or $NCint(NCcl(NCint(NCcl(\eta^{-1}(\mathcal{D})))) \supseteq \eta^{-1}(\mathcal{D})$. Hence $\eta^{-1}(\mathcal{D})$ is a $NCS\alpha$ -OS in \mathbb{X} and thus η be there a $NCS\alpha$ -continuous. ■

Proposition 3.4:

- (i) all NC -continuous functions are $NCS\alpha$ -continuous, but the opposite is not valid in general.
- (ii) all $NC\alpha$ -continuous functions are $NCS\alpha$ -continuous, but the opposite is not exact in general.

Proof:

(i) Suppose $\eta: (\mathbb{X}, I_1) \rightarrow (\mathbb{Y}, I_2)$ is a NC -continuous function, and \mathcal{D} be a NC -OS from \mathbb{Y} . Next $\eta^{-1}(\mathcal{D})$ remains a NC -OS from \mathbb{X} . Since any NC -OS is a $NCS\alpha$ -OS, $\eta^{-1}(\mathcal{D})$ stays a $NCS\alpha$ -OS from \mathbb{X} . Thus η exists a $NCS\alpha$ -continuous function.

(ii) Let $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ be a $NC\alpha$ -continuous function and \mathcal{D} be a NC -OS from \mathbb{Y} . Subsequently $\eta^{-1}(\mathcal{D})$ happens a $NC\alpha$ -OS from \mathbb{X} . Since any $NC\alpha$ -OS is $NCS\alpha$ -OS, $\eta^{-1}(\mathcal{D})$ stays a $NCS\alpha$ -OS from \mathbb{X} . Thus η is a $NCS\alpha$ -continuous function. ■

Example 3.5: Suppose $\mathbb{X} = \{p, q, r, s\}$ and $\mathbb{Y} = \{u, v, w\}$. Then $\Gamma_1 = \{\phi_N, \mathbb{X}_N\} \cup \{\langle\{p\}, \phi, \phi\rangle\}$ and $\Gamma_2 = \{\phi_N, \mathbb{Y}_N\} \cup \{\langle\{u\}, \phi, \phi\rangle\}$ be neutrosophic crisp topologies (shortly, $NCTs$) on \mathbb{X} and \mathbb{Y} , correspondingly. Define the function $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ via $\eta(\langle\{p\}, \phi, \phi\rangle) = \eta(\langle\{q\}, \phi, \phi\rangle) = \langle\{u\}, \phi, \phi\rangle$, $\eta(\langle\{r\}, \phi, \phi\rangle) = \langle\{v\}, \phi, \phi\rangle$, $\eta(\langle\{s\}, \phi, \phi\rangle) = \langle\{w\}, \phi, \phi\rangle$. Then η is a $NC\alpha$ -continuous function but not NC -continuous since $\langle\{u\}, \phi, \phi\rangle$ is NC -OS but $\eta^{-1}(\langle\{u\}, \phi, \phi\rangle) = \langle\{p, q\}, \phi, \phi\rangle$ which is not NC -OS in \mathbb{X} . Also, η is a $NCS\alpha$ -continuous function but not NC -continuous, since $\langle\{u\}, \phi, \phi\rangle$ is NC -OS in \mathbb{Y} but $\eta^{-1}(\langle\{u\}, \phi, \phi\rangle) = \langle\{p, q\}, \phi, \phi\rangle$ is not NC -OS from \mathbb{X} .

Example 3.6: Suppose $\mathbb{X} = \{p, q, r\}$. Then $\Gamma = \{\phi_N, \mathbb{X}_N\} \cup \{\langle\{p\}, \phi, \phi\rangle, \langle\{q\}, \phi, \phi\rangle, \langle\{p, q\}, \phi, \phi\rangle\}$ be a NCT on \mathbb{X} .

Define the function $\eta: (\mathbb{X}, \Gamma) \rightarrow (\mathbb{X}, \Gamma)$ by $\eta(\langle\{p\}, \phi, \phi\rangle) = \langle\{p\}, \phi, \phi\rangle$, $\eta(\langle\{q\}, \phi, \phi\rangle) = \eta(\langle\{r\}, \phi, \phi\rangle) = \langle\{q\}, \phi, \phi\rangle$. It is easily seen that η is a $NCS\alpha$ -continuous function but not $NC\alpha$ -continuous, since $\langle\{q\}, \phi, \phi\rangle$ is NC -OS in \mathbb{X} but $\eta^{-1}(\langle\{q\}, \phi, \phi\rangle) = \langle\{q, r\}, \phi, \phi\rangle$ is not $NC\alpha$ -OS in \mathbb{X} .

Remark 3.7: The concepts of NC -continuity and $NC\alpha^*$ -continuity are independent, for examples.

Example 3.8: Suppose $\mathbb{X} = \{p, q, r, s\}$ and $\mathbb{Y} = \{u, v, w\}$. Then

$\Gamma_1 = \{\phi_N, \mathbb{X}_N\} \cup \{\langle\{p\}, \phi, \phi\rangle, \langle\{q, r\}, \phi, \phi\rangle, \langle\{p, q, r\}, \phi, \phi\rangle\}$ and $\Gamma_2 = \{\phi_N, \mathbb{Y}_N\} \cup \{\langle\{u\}, \phi, \phi\rangle\}$ be $NCTs$ on \mathbb{X} and \mathbb{Y} , correspondingly. Define the function $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ via $\eta(\langle\{p\}, \phi, \phi\rangle) = \langle\{u\}, \phi, \phi\rangle$, $\eta(\langle\{q\}, \phi, \phi\rangle) = \langle\{v\}, \phi, \phi\rangle$, $\eta(\langle\{r\}, \phi, \phi\rangle) = \eta(\langle\{s\}, \phi, \phi\rangle) = \langle\{w\}, \phi, \phi\rangle$. Then η is a NC -continuous function but not $NC\alpha^*$ -continuous, since $\langle\{u, v\}, \phi, \phi\rangle$ is $NC\alpha$ -OS in \mathbb{Y} but $\eta^{-1}(\langle\{u, v\}, \phi, \phi\rangle) = \langle\{p, q\}, \phi, \phi\rangle$ is not $NC\alpha$ -OS in \mathbb{X} .

Example 3.9: Assume $\mathbb{X} = \{p, q, r, s\}$ and $\mathbb{Y} = \{u, v, w\}$. Then $\Gamma_1 = \{\phi_N, \mathbb{X}_N\} \cup \{\langle\{p\}, \phi, \phi\rangle\}$ and $\Gamma_2 = \{\phi_N, \mathbb{Y}_N\} \cup \{\langle\{u\}, \phi, \phi\rangle\}$ be $NCTs$ on \mathbb{X} and \mathbb{Y} , correspondingly. Define the function $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ via $\eta(\langle\{p\}, \phi, \phi\rangle) = \eta(\langle\{q\}, \phi, \phi\rangle) = \langle\{u\}, \phi, \phi\rangle$, $\eta(\langle\{r\}, \phi, \phi\rangle) = \langle\{v\}, \phi, \phi\rangle$, $\eta(\langle\{s\}, \phi, \phi\rangle) = \langle\{w\}, \phi, \phi\rangle$. Then η is a $NC\alpha^*$ -continuous function but not NC -continuous, since $\langle\{u\}, \phi, \phi\rangle$ is NC -OS in \mathbb{Y} , but $\eta^{-1}(\langle\{u\}, \phi, \phi\rangle) = \langle\{p, q\}, \phi, \phi\rangle$ is not NC -OS in \mathbb{X} .

Theorem 3.10:

- (i) If a function $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ is NC -open, NC -continuous, and bijective, then η is a $NC\alpha^*$ -continuous.
- (ii) A function $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ is $NC\alpha^*$ -continuous iff $\eta: (\mathbb{X}, NC\alpha O(\mathbb{X})) \rightarrow (\mathbb{Y}, NC\alpha O(\mathbb{Y}))$ is a NC -continuous.

Proof:

- (i) Let $\mathcal{D} \in NC\alpha O(\mathbb{Y})$, to prove that $\eta^{-1}(\mathcal{D}) \in NC\alpha O(\mathbb{X})$, i.e., $\eta^{-1}(\mathcal{D}) \subseteq NCint(NCcl(NCint(\eta^{-1}(\mathcal{D}))))$. Let $r \in \eta^{-1}(\mathcal{D}) \Rightarrow \eta(r) \in \mathcal{D}$. Hence $\eta(r) \in NCint(NCcl(NCint(\mathcal{D})))$ (since $\mathcal{D} \in NC\alpha O(\mathbb{Y})$). Therefore, at least NC -OS \mathcal{H} from \mathbb{Y} where $\eta(r) \in \mathcal{H} \subseteq NCcl(NCint(\mathcal{D}))$. Then $r \in \eta^{-1}(\mathcal{H}) \subseteq$

$\eta^{-1}(NCcl(NCint(\mathcal{D})))$, but $\eta^{-1}(NCcl(NCint(\mathcal{D}))) \subseteq NCcl(\eta^{-1}(NCint(\mathcal{D})))$ (since η^{-1} is a NC -continuous, which is equivalent to η is a NC -open and bijective). Then $r \in \eta^{-1}(\mathcal{H}) \subseteq NCcl(\eta^{-1}(NCint(\mathcal{D})))$. Hence $r \in \eta^{-1}(\mathcal{H}) \subseteq NCcl(\eta^{-1}(NCint(\mathcal{D}))) \subseteq NCcl(NCint(\eta^{-1}(\mathcal{D})))$ (since η is a NC -continuous). Hence $r \in \eta^{-1}(\mathcal{H}) \subseteq NCcl(NCint(\eta^{-1}(\mathcal{D})))$, but $\eta^{-1}(\mathcal{H})$ remains a NC -OS from \mathbb{X} (because η be present a NC -continuous). Therefore, $r \in NCint(NCcl(NCint(\eta^{-1}(\mathcal{D}))))$.

Hence $\eta^{-1}(\mathcal{D}) \subseteq NCint(NCcl(NCint(\eta^{-1}(\mathcal{D})))) \Rightarrow \eta^{-1}(\mathcal{D}) \in NC\alpha O(\mathbb{X}) \Rightarrow \eta$ is a $NC\alpha^*$ -continuous.

(ii) The proof of (ii) is easily. ■

Theorem 3.11: A function $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ is a $NCS\alpha^*$ -continuous iff $\eta: (\mathbb{X}, NCS\alpha O(\mathbb{X})) \rightarrow (\mathbb{Y}, NCS\alpha O(\mathbb{Y}))$ is a NC -continuous.

Proof: Obvious. ■

Remark 3.12: The concepts of NC -continuity and $NCS\alpha^*$ -continuity are independent, for examples.

Example 3.13: Suppose $\mathbb{X} = \{p, q, r, s\}$ and $\mathbb{Y} = \{u, v, w\}$.

Then $\Gamma_1 = \{\phi_N, \mathbb{X}_N\} \cup \{\langle\{p\}, \phi, \phi\rangle, \langle\{q, r\}, \phi, \phi\rangle, \langle\{p, q, r\}, \phi, \phi\rangle\}$ and $\Gamma_2 = \{\phi_N, \mathbb{Y}_N\} \cup \{\langle\{u\}, \phi, \phi\rangle\}$ be NCT s on \mathbb{X} and \mathbb{Y} , correspondingly. Define the function $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ via $\eta(\langle\{p\}, \phi, \phi\rangle) = \langle\{u\}, \phi, \phi\rangle, \eta(\langle\{q\}, \phi, \phi\rangle) = \langle\{v\}, \phi, \phi\rangle, \eta(\langle\{r\}, \phi, \phi\rangle) = \eta(\langle\{s\}, \phi, \phi\rangle) = \langle\{w\}, \phi, \phi\rangle$. It is easily seen that η is a NC -continuous function but not $NCS\alpha^*$ -continuous, since $\langle\{u, v\}, \phi, \phi\rangle$ is $NCS\alpha$ -OS in \mathbb{Y} but $\eta^{-1}(\langle\{u, v\}, \phi, \phi\rangle) = \langle\{p, q\}, \phi, \phi\rangle$ is not $NCS\alpha$ -OS in \mathbb{X} .

Example 3.14: Assume $\mathbb{X} = \{p, q, r, s\}$ and $\mathbb{Y} = \{u, v, w\}$. Then $\Gamma_1 = \{\phi_N, \mathbb{X}_N\} \cup \{\langle\{p\}, \phi, \phi\rangle\}$ and $\Gamma_2 = \{\phi_N, \mathbb{Y}_N\} \cup \{\langle\{u\}, \phi, \phi\rangle\}$ be NCT s on \mathbb{X} and \mathbb{Y} , correspondingly.

Define the function $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ via $\eta(\langle\{p\}, \phi, \phi\rangle) = \eta(\langle\{q\}, \phi, \phi\rangle) = \langle\{u\}, \phi, \phi\rangle, \eta(\langle\{r\}, \phi, \phi\rangle) = \langle\{v\}, \phi, \phi\rangle, \eta(\langle\{s\}, \phi, \phi\rangle) = \langle\{w\}, \phi, \phi\rangle$. Then η is a $NCS\alpha^*$ -continuous function but not NC -continuous, since $\langle\{u\}, \phi, \phi\rangle$ is NC -OS in \mathbb{Y} , but $\eta^{-1}(\langle\{u\}, \phi, \phi\rangle) = \langle\{p, q\}, \phi, \phi\rangle$ is not NC -OS in \mathbb{X} .

Proposition 3.15: Every $NC\alpha^*$ -continuous function is a $NC\alpha$ -continuous and $NCS\alpha$ -continuous; however, the reverse generally is not valid.

Proof: Assume $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ is a $NC\alpha^*$ -continuous function and let \mathcal{D} be any NC -OS from \mathbb{Y} . Then we have \mathcal{D} as a $NC\alpha$ -OS from \mathbb{Y} [from proposition 2.4]. Since η is a $NC\alpha^*$ -continuous, then $\eta^{-1}(\mathcal{D})$ considers a $NC\alpha$ -OS from \mathbb{X} . Thus, η stands a $NC\alpha$ -continuous. Also, η is a $NCS\alpha$ -continuous. ■

Example 3.16: Let $\mathbb{X} = \{p, q, r, s\}$.

Then $\Gamma = \{\phi_N, \mathbb{X}_N\} \cup \{\langle\{p\}, \phi, \phi\rangle, \langle\{q\}, \phi, \phi\rangle, \langle\{p, q\}, \phi, \phi\rangle, \langle\{p, q, r\}, \phi, \phi\rangle\}$ be a NCT on \mathbb{X} . Define the function $\eta: (\mathbb{X}, \Gamma) \rightarrow (\mathbb{X}, \Gamma)$ by $\eta(\langle\{p\}, \phi, \phi\rangle) = \langle\{p\}, \phi, \phi\rangle, \eta(\langle\{q\}, \phi, \phi\rangle) = \eta(\langle\{r\}, \phi, \phi\rangle) = \langle\{s\}, \phi, \phi\rangle, \eta(\langle\{s\}, \phi, \phi\rangle) = \langle\{r\}, \phi, \phi\rangle$. It is easily seen that η is a $NC\alpha$ -continuous function but not $NC\alpha^*$ -continuous, since $\langle\{p, q, r\}, \phi, \phi\rangle$ is $NC\alpha$ -OS in \mathbb{X} , but $\eta^{-1}(\langle\{p, q, r\}, \phi, \phi\rangle) = \langle\{p, s\}, \phi, \phi\rangle$ is not $NC\alpha$ -OS in \mathbb{X} .

Example 3.17: Let $\mathbb{X} = \{p, q, r\}$. Then $\Gamma = \{\phi_N, \mathbb{X}_N\} \cup \{\langle\{p\}, \phi, \phi\rangle, \langle\{q\}, \phi, \phi\rangle, \langle\{p, q\}, \phi, \phi\rangle\}$ be a NCT on \mathbb{X} . Define a function $\eta: (\mathbb{X}, \Gamma) \rightarrow (\mathbb{X}, \Gamma)$ by $\eta(\langle\{p\}, \phi, \phi\rangle) = \langle\{p\}, \phi, \phi\rangle, \eta(\langle\{q\}, \phi, \phi\rangle) = \eta(\langle\{r\}, \phi, \phi\rangle) = \langle\{q\}, \phi, \phi\rangle$. It is easily seen that η is a $NCS\alpha$ -continuous function but not $NC\alpha^*$ -continuous, since $\langle\{q\}, \phi, \phi\rangle$ is $NC\alpha$ -OS in \mathbb{X} , but $\eta^{-1}(\langle\{q\}, \phi, \phi\rangle) = \langle\{q, r\}, \phi, \phi\rangle$ is not $NC\alpha$ -OS in \mathbb{X} .

Definition 3.18: A function $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ is called \mathcal{M} -function iff $\eta^{-1}(NCint(NCcl(\mathcal{D}))) \subseteq NCint(NCcl(\eta^{-1}(\mathcal{D})))$, for every $NC\alpha$ -OS \mathcal{D} from \mathbb{Y} .

Theorem 3.19: If $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ is a $NC\alpha$ -continuous function and \mathcal{M} -function, then η is a $NC\alpha^*$ -continuous.

Proof: Let \mathcal{C} be any $NC\alpha$ -OS of \mathbb{Y} , then we have at least a NC -OS \mathcal{D} from \mathbb{Y} where $\mathcal{D} \subseteq \mathcal{C} \subseteq NCint(NCcl(\mathcal{D}))$. Since η is \mathcal{M} -function, we have $\eta^{-1}(\mathcal{D}) \subseteq \eta^{-1}(\mathcal{C}) \subseteq \eta^{-1}(NCint(NCcl(\mathcal{D}))) \subseteq NCint(NCcl(\eta^{-1}(\mathcal{D})))$. By proposition 2.3, we have $\eta^{-1}(\mathcal{C})$ is a $NC\alpha$ -OS. Hence, η is a $NC\alpha^*$ -continuous. ■

Remark 3.20: The concepts of $NC\alpha^*$ -continuity and $NCS\alpha^*$ -continuity are independent as the following examples show.

Example 3.21: Assume $\mathbb{X} = \{p, q, r, s\}$ and $\mathbb{Y} = \{u, v, w\}$.

Then $\Gamma_1 = \{\phi_N, \mathbb{X}_N\} \cup \{\langle\{p\}, \phi, \phi\rangle, \langle\{q\}, \phi, \phi\rangle, \langle\{p, q\}, \phi, \phi\rangle, \langle\{p, q, r\}, \phi, \phi\rangle\}$ and $\Gamma_2 = \{\phi_N, \mathbb{Y}_N\} \cup \{\langle\{u\}, \phi, \phi\rangle, \langle\{v\}, \phi, \phi\rangle, \langle\{u, v\}, \phi, \phi\rangle\}$ be NCT s on \mathbb{X} and \mathbb{Y} , correspondingly. Define the function $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ via $\eta(\langle\{p\}, \phi, \phi\rangle) = \eta(\langle\{s\}, \phi, \phi\rangle) = \langle\{v\}, \phi, \phi\rangle, \eta(\langle\{r\}, \phi, \phi\rangle) = \langle\{w\}, \phi, \phi\rangle$ and $\eta(\langle\{q\}, \phi, \phi\rangle) = \langle\{u\}, \phi, \phi\rangle$. It is easily seen that η is a $NCS\alpha^*$ -continuous function but not $NC\alpha^*$ -continuous, since $\langle\{v\}, \phi, \phi\rangle$ is $NC\alpha$ -OS in \mathbb{Y} but $\eta^{-1}(\langle\{v\}, \phi, \phi\rangle) = \langle\{p, s\}, \phi, \phi\rangle$ is not $NC\alpha$ -OS in \mathbb{X} .

Example 3.22: Suppose $\mathbb{X} = \{p, q, r, s\}$.

Then $\Gamma = \{\phi_N, \mathbb{X}_N\} \cup \{\langle\{p\}, \phi, \phi\rangle, \langle\{q\}, \phi, \phi\rangle, \langle\{p, q\}, \phi, \phi\rangle, \langle\{p, q, r\}, \phi, \phi\rangle\}$ be a NCT on \mathbb{X} . Define the function $\eta: (\mathbb{X}, \Gamma) \rightarrow (\mathbb{X}, \Gamma)$ via $\eta(\langle\{p\}, \phi, \phi\rangle) = \eta(\langle\{q\}, \phi, \phi\rangle) = \langle\{q\}, \phi, \phi\rangle, \eta(\langle\{r\}, \phi, \phi\rangle) = \langle\{s\}, \phi, \phi\rangle, \eta(\langle\{s\}, \phi, \phi\rangle) = \langle\{r\}, \phi, \phi\rangle$. It is easily seen that η is a $NC\alpha^*$ -continuous function but not $NCS\alpha^*$ -continuous, since $\langle\{p, r\}, \phi, \phi\rangle$ is $NCS\alpha$ -OS in \mathbb{X} , but $\eta^{-1}(\langle\{p, r\}, \phi, \phi\rangle) = \langle\{s\}, \phi, \phi\rangle$ is not $NCS\alpha$ -OS in \mathbb{X} .

Theorem 3.23: If a function $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ is $NC\alpha^*$ -continuous, NC -open and bijective, then it is $NCS\alpha^*$ -continuous.

Proof: Let $\eta: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ be a $NC\alpha^*$ -continuous, NC -open and bijective. Let \mathcal{D} be a $NCS\alpha$ -OS in \mathbb{Y} . Then we have at least a $NC\alpha$ -OS say \mathcal{P} where $\mathcal{P} \subseteq \mathcal{D} \subseteq NCcl(\mathcal{P})$. Therefore $\eta^{-1}(\mathcal{P}) \subseteq \eta^{-1}(\mathcal{D}) \subseteq \eta^{-1}(NCcl(\mathcal{P})) \subseteq NCcl(\eta^{-1}(\mathcal{P}))$ (since η is a NC -open), but $\eta^{-1}(\mathcal{P}) \in NC\alpha O(\mathbb{X})$ (since η is a $NC\alpha^*$ -continuous). Hence $\eta^{-1}(\mathcal{P}) \subseteq \eta^{-1}(\mathcal{D}) \subseteq NCcl(\eta^{-1}(\mathcal{P}))$. Thus, $\eta^{-1}(\mathcal{D}) \in NCS\alpha O(\mathbb{X})$. Therefore, η is a $NCS\alpha^*$ -continuous. ■

Remark 3.24: Let $\eta_1: (\mathbb{X}, \Gamma_1) \rightarrow (\mathbb{Y}, \Gamma_2)$ and $\eta_2: (\mathbb{Y}, \Gamma_2) \rightarrow (\mathbb{Z}, \Gamma_3)$ be two functions, then:

- (i) If η_1 and η_2 are $NC \alpha$ -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ need not to be a $NC \alpha$ -continuous.
- (ii) If η_1 and η_2 are $NCS \alpha$ -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ need not to be a $NCS \alpha$ -continuous.

Theorem 3.25: Let $\eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Y}, I_2)$ and $\eta_2: (\mathbb{Y}, I_2) \rightarrow (\mathbb{Z}, I_3)$ be two functions, then:

- (i) If η_1 is $NC \alpha$ -continuous and η_2 is NC -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NC \alpha$ -continuous.
- (ii) If η_1 is $NC \alpha^*$ -continuous and η_2 is $NC \alpha$ -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NC \alpha$ -continuous.
- (iii) If η_1 and η_2 are $NC \alpha^*$ -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NC \alpha^*$ -continuous.
- (iv) If η_1 and η_2 are $NCS \alpha^*$ -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NCS \alpha^*$ -continuous.
- (v) If η_1 and η_2 are $NC \alpha^{**}$ -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NC \alpha^{**}$ -continuous.
- (vi) If η_1 and η_2 are $NCS \alpha^{**}$ -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NCS \alpha^{**}$ -continuous.
- (vii) If η_1 is $NC \alpha^{**}$ -continuous and η_2 is $NC \alpha^*$ -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NC \alpha^{**}$ -continuous.
- (viii) If η_1 is $NC \alpha^{**}$ -continuous and η_2 is $NC \alpha$ -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a NC -continuous.
- (ix) If η_1 is $NC \alpha$ -continuous and η_2 is $NC \alpha^{**}$ -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NC \alpha^*$ -continuous.
- (x) If η_1 is NC -continuous and η_2 is $NC \alpha^{**}$ -continuous, then $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NC \alpha^{**}$ -continuous.

Proof:

- (i) Assume \mathcal{F} considers a NC -OS from \mathbb{Z} . Since η_2 is a NC -continuous, $\eta_2^{-1}(\mathcal{F})$ is a $NC \alpha$ -OS in \mathbb{Y} . Since η_1 is a $NC \alpha$ -continuous, $\eta_1^{-1}(\eta_2^{-1}(\mathcal{F})) = (\eta_2 \circ \eta_1)^{-1}(\mathcal{F})$ is a $NC \alpha$ -OS in \mathbb{X} . Thus, $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ exists a $NC \alpha$ -continuous.
- (ii) Let \mathcal{F} be a NC -OS in \mathbb{Z} . Subsequently η_2 stands a $NC \alpha$ -continuous, and $\eta_2^{-1}(\mathcal{F})$ stays a $NC \alpha$ -OS from \mathbb{Y} . Since η_1 is a $NC \alpha^*$ -continuous, $\eta_1^{-1}(\eta_2^{-1}(\mathcal{F})) = (\eta_2 \circ \eta_1)^{-1}(\mathcal{F})$ is a $NC \alpha$ -OS in \mathbb{X} . Thus, $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NC \alpha$ -continuous.
- (iii) Let \mathcal{F} be a $NC \alpha$ -OS in \mathbb{Z} . Since η_2 is a $NC \alpha^*$ -continuous, $\eta_2^{-1}(\mathcal{F})$ is a $NC \alpha$ -OS in \mathbb{Y} . Since η_1 is a $NC \alpha^*$ -continuous, $\eta_1^{-1}(\eta_2^{-1}(\mathcal{F})) = (\eta_2 \circ \eta_1)^{-1}(\mathcal{F})$ is a $NC \alpha$ -OS in \mathbb{X} . Thus, $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NC \alpha^*$ -continuous.
- (iv) Let \mathcal{F} be a $NCS \alpha$ -OS in \mathbb{Z} . Since η_2 is a $NCS \alpha^*$ -continuous, $\eta_2^{-1}(\mathcal{F})$ is a $NCS \alpha$ -OS in \mathbb{Y} . Since η_1 is a $NCS \alpha^*$ -continuous, $\eta_1^{-1}(\eta_2^{-1}(\mathcal{F})) = (\eta_2 \circ \eta_1)^{-1}(\mathcal{F})$ is a $NCS \alpha$ -OS in \mathbb{X} . Thus, $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NCS \alpha^*$ -continuous.
- (v) Let \mathcal{F} be a $NC \alpha$ -OS in \mathbb{Z} . Since η_2 is a $NC \alpha^{**}$ -continuous, $\eta_2^{-1}(\mathcal{F})$ is a NC -OS in \mathbb{Y} . Since any NC -OS is a $NC \alpha$ -OS, $\eta_2^{-1}(\mathcal{F})$ is a $NC \alpha$ -OS in \mathbb{Y} . Since η_1 is a $NC \alpha^{**}$ -continuous, $\eta_1^{-1}(\eta_2^{-1}(\mathcal{F})) = (\eta_2 \circ \eta_1)^{-1}(\mathcal{F})$ is a NC -OS in \mathbb{X} . Thus, $\eta_2 \circ \eta_1: (\mathbb{X}, I_1) \rightarrow (\mathbb{Z}, I_3)$ is a $NC \alpha^{**}$ -continuous. The proof is obvious for others. ■

Remark 3.26: The next figure describes the relationship between various classes of weakly NC -continuous functions:

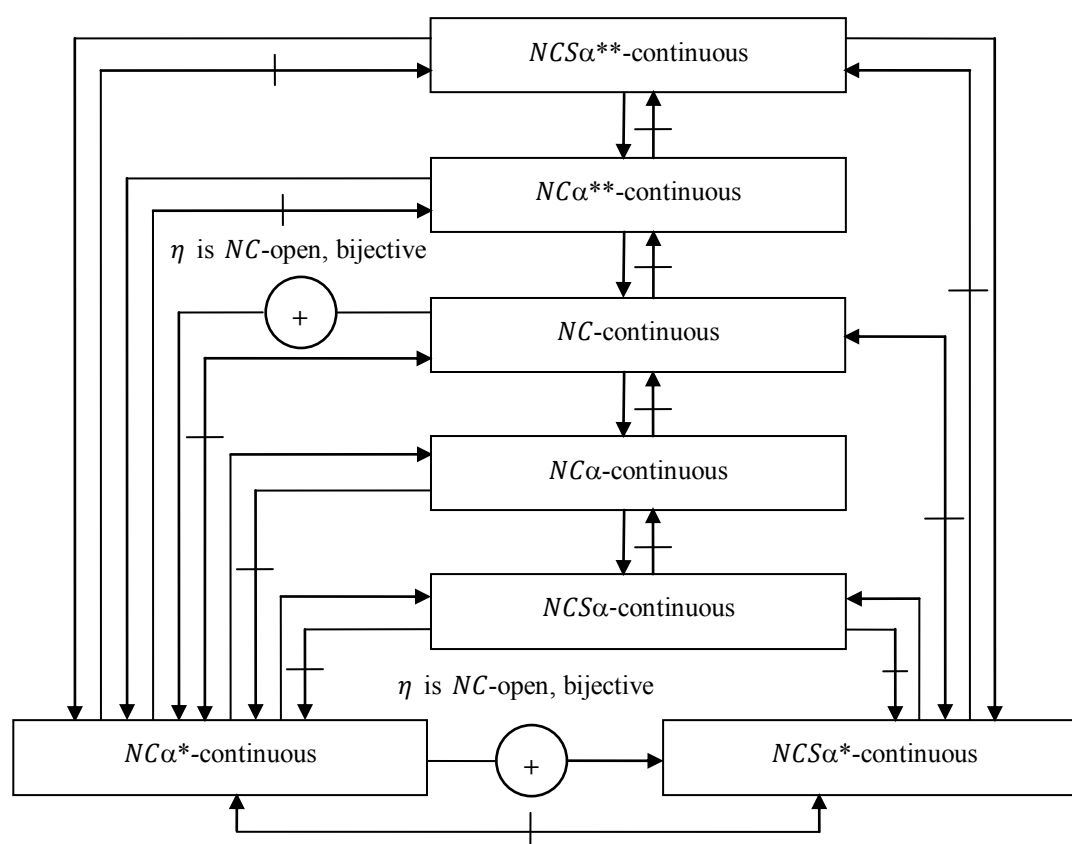


Fig. 1

4. Conclusion

We shall use the concepts of $NC\alpha$ -OS and $NCS\alpha$ -CS to define several new types of weakly NC -continuity such as; $NC\alpha^*$ -continuous, $NC\alpha^{**}$ -continuous, $NCS\alpha$ -continuous, $NCS\alpha^*$ -continuous and $NCS\alpha^{**}$ -continuous functions. The neutrosophic crisp α -open and neutrosophic crisp semi- α -open sets can be used to derive some new types of weakly NC -open (NC -closed) functions.

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A Novel Approach: Neutro-Spot Topology and Its Supra Topology With Separation Axioms and Computing the Impact on COVID-19

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Abstract: Neutrosophic soft set is a parametric set of uncertainty, whereas the neutrosophic soft point is an exceptional type of it which used highly to explore the separation axioms. In this study, the impression of neutrosophic soft topological space is stretched to a new topology which contains neutrosophic soft points as its elements and named as neutro-spot topological space. One more topology is defined on the complement of neutrosophic soft points which satisfies the condition of supra topological space and named as neutro-supra spot topological space. Also, defined the notion of interior and closure, and are approached in a different way, along with the concept of subspace topology of such topological spaces. Some related properties have been proved and disproved with counterexamples. Moreover, the approach to separation axioms in such spaces has been presented with descriptive examples. The current epidemic situation discussed as a real life application in decision making problem to detect the major impact of COVID-19 and recover them quickly. The affected people investigated by the doctors according to their symptoms and other medical issues. The process of solving specified in the algorithm and the estimation formula stated for calculation. The appropriate treatment is provided for affected people as per the estimated value.

Keywords: Neutro-spot topological space; neutro-spot absolute interior; neutro-supra spot topological space; neutro-supra spot absolute closure; neutro-spot subspace topological space; neutro-supra spot subspace topological space; neutro-spot $T_{i=0,1,2}$ -spaces and neutro-supra spot $T_{i=0,1,2}$ -spaces; decision making problem on COVID-19.

1. Introduction

The values of three independent membership degrees such as truth, falsity, and indeterminacy, consigned to each element of a set which characterized to neutrosophic set (NS) as originated by (1998) Smarandache [16, 17], which is a simplification of a fuzzy set (FS) defined by (1965) Zadeh [36], and intuitionistic fuzzy set (IFS) created by (1986) Atanassov [34]. It turns out to be a valuable mathematical utensil to examine formless, faulty, unclear data. In recent years many researchers have further expanded and developed the theory and application of NSs [1, 10, 12-15]. Also, (2017) Smarandache [18] originated a new trend set called plithogenic set (PS) and others developed [4, 8, 11].

A soft set (SS) is a study of parameterization of vagueness introduced by (1999) Molodtsov [32]. Later (2013) Maji [30] pooled two-hybrid sets NS and SS, and urbanized its construction as neutrosophic soft sets (NSSs). This type of set is extended by different researchers [2, 7, 9] and its application in decision making (DM) problems [3]. Topology plays a vital role among all these types of sets such as fuzzy topology (FT), soft topology (ST), fuzzy soft topology (FST), neutrosophic topology (NT) [31], etc. Likewise, (2017) Bera & Mahapatra [26, 6] created the conception of neutrosophic soft topological spaces (NSTSs) and others [19, 20, 24]. Neutrosophic soft point (NSP) is a special type of NSS, which gives rise to the concept of separation axioms on NSTS [23, 25], and their application to DM was considered in [21, 22, 28]. Mashhour et al. [35] (1983) diminished the conditions of general topology, termed as supra topological spaces (SpTSs). The real-life application of SpTS is applied and defined on various sets such as FS [33], SS [29, 27], NS [5], and so on.

The major contribution of this work is to initiate a topology on NSPs, whose open sets defines the concept of the interior on it. The complement of NSP is defined and named as neutrosophic soft whole set. Such type of sets obeys the condition of supra topology, and so generated new types of supra topology, whose open sets defines the concept of closure on it. Some essential definitions and remarkable properties are studied with appropriate examples. On the other hand, the impact of separation axioms is examined in both the topologies with suitable examples. The recent wide-ranging problem extended as a major application to provide appropriate treatment for COVID-19 patients. Those people are under investigation as stated by its symptoms and other medical issues. Also, provided the solving procedure and formulae for calculation. The main aim of this DM problem to recover them speedily via proper treatment.

This study is prearranged as follows. Some important definitions related to the study are presented in part 2. Part 3 introduces the definition of neutro-spot topology, its interior, its subspace topology with fundamental properties, and related examples. Part 4 introduces the definition of neutro-supra spot topology, its closure, its subspace topology with fundamental properties, and related examples. Part 5 extends this study to separation axioms on both the topologies with explanatory examples. Part 6 solves the DM problem to detect the impact of COVID-19. The algorithm and formulae are presented to find the final result and provided proper treatment for them. At last, concluded with few ideas for upcoming work in part 7.

2. Preliminaries

In this part, some essential definitions connected to this work are pointed.

Definition 2.1 [26] Let V be an initial universe set, E be a set of parameters, and P, Q is any two NSSs over (V, E) . Then a NSS P over V is a set defined by a set-valued function $f^{(E)}P$ representing a mapping $f^{(E)}P: E \rightarrow NS(V)$ where $f^{(E)}P$ is called the approximate function of the NSS P and $NS(V)$ is a family of NS over V .

$$P = f^{(E)}P = \{ \langle e, \langle v, T_{f^{(e)}P}(v), I_{f^{(e)}P}(v), F_{f^{(e)}P}(v) \rangle \rangle : v \in V \} : e \in E \}.$$

Definition 2.2 [24] A NSS P over (V, E) is said to be null NSS if $T_{f^{(e)}P}(v) = 0$, $I_{f^{(e)}P}(v) = 0$, $F_{f^{(e)}P}(v) = 1$, $\forall e \in E$, $\forall v \in V$. It is denoted by ϕ_t .

A NSS P over (V, E) is said to be absolute NSS if $T_{f^{(e)}P}(v) = 1$, $I_{f^{(e)}P}(v) = 1$, $F_{f^{(e)}P}(v) = 0$, $\forall e \in E$, $\forall v \in V$. It is denoted by 1_t .

Clearly, $(\phi_t)^c = 1_t$ and $(1_t)^c = \phi_t$.

Definition 2.3 [25] Let V be a universe and E be a set of parameters. Let K be a NSS over (V, E) . Let e be an element of E and let

$$K(e') = f^{(E)}K(e') = \begin{cases} (\alpha, \beta, \gamma), & \text{if } e' = e \\ (0, 0, 1), & \text{if } e' \neq e \end{cases} \quad \text{for all } e' \in E.$$

Then $K(e')$ is called NSP over (V, E) and $K = \bigcup_{e' \in E} K(e')$. That is, a NSS is the union of its NSPs.

3. Neutro-Spot Topology

This part defines the neutro-spot topology, neutro-spot absolute interior, and its subspace topology with some properties and examples.

Definition 3.1 Let V be a universe and E be a set of parameters. Let $NSP(V, E)$ be the family of all NSPs over V . Then $\tau_t \subset NSP(V, E)$ is said to be neutro-spot topology (NSPT) over (V, E) if it satisfies the following conditions

(i) $\phi_t, 1_t \in \tau_t$.

(ii) the finite intersection of NSPs in τ_t belongs to τ_t .

The trio (V, τ_t, E) is said to be neutro-spot topological space (NSPTS) over (V, E) . Elements in τ_t are called neutro-spot open sets (NSPOSs).

Example 3.2 A survey is taken based on the characteristics of houses owned by the people living in a slum area. Let $V = \{p_1, p_2, p_3\}$ be the set of sample people living in different areas in the slum and $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters of houses, where $e_1 = \text{neat}$, $e_2 = \text{beautiful}$, $e_3 = \text{compact}$, and $e_4 = \text{large}$. Let $\tau_t = \{\phi_t, 1_t, K_1, K_2, K_3\}$. According to the survey, the results have proceeded in the form of NSPs K_1, K_2, K_3 over V as follows:

$$K_1 = \begin{cases} f^{(e_1)}K_1 = \{<p_1, (0, 0, 1)>, <p_2, (0, 0, 1)>, <p_3, (0, 0, 1)>\} \\ f^{(e_2)}K_1 = \{<p_1, (0, 0, 1)>, <p_2, (0, 0, 1)>, <p_3, (0, 0, 1)>\} \\ f^{(e_3)}K_1 = \{<p_1, (0, 0, 1)>, <p_2, (.3, .4, .2)>, <p_3, (0, 0, 1)>\} \\ f^{(e_4)}K_1 = \{<p_1, (0, 0, 1)>, <p_2, (0, 0, 1)>, <p_3, (0, 0, 1)>\} \end{cases} ;$$

$$K_2 = \begin{cases} f^{(e_1)}K_2 = \{<p_1, (0, 0, 1)>, <p_2, (0, 0, 1)>, <p_3, (0, 0, 1)>\} \\ f^{(e_2)}K_2 = \{<p_1, (0, 0, 1)>, <p_2, (0, 0, 1)>, <p_3, (0, 0, 1)>\} \\ f^{(e_3)}K_2 = \{<p_1, (.5, .2, .6)>, <p_2, (0, 0, 1)>, <p_3, (0, 0, 1)>\} \\ f^{(e_4)}K_2 = \{<p_1, (0, 0, 1)>, <p_2, (0, 0, 1)>, <p_3, (0, 0, 1)>\} \end{cases} \text{ and}$$

$$K_3 = \begin{cases} f^{(e_1)}K_3 = \{<p_1, (0, 0, 1)>, <p_2, (0, 0, 1)>, <p_3, (0, 0, 1)>\} \\ f^{(e_2)}K_3 = \{<p_1, (0, 0, 1)>, <p_2, (0, 0, 1)>, <p_3, (0, 0, 1)>\} \\ f^{(e_3)}K_3 = \{<p_1, (0, 0, 1)>, <p_2, (0, 0, 1)>, <p_3, (.1, .2, .3)>\} \\ f^{(e_4)}K_3 = \{<p_1, (0, 0, 1)>, <p_2, (0, 0, 1)>, <p_3, (0, 0, 1)>\} \end{cases} .$$

Here $\phi_t \cap K_1 = \phi_t$, $\phi_t \cap K_2 = \phi_t$, $\phi_t \cap K_3 = \phi_t$, $1_t \cap K_1 = K_1$, $1_t \cap K_2 = K_2$, $1_t \cap K_3 = K_3$, $K_1 \cap K_2 = \phi_t$, $K_1 \cap K_3 = \phi_t$, $K_2 \cap K_3 = \phi_t$.

Then K_1, K_2 and K_3 are NSPOSs.

Thus (V, τ_t, E) is a NSPTS over (V, E) .

Proposition 3.3 Let (V, τ_{t1}, E) and (V, τ_{t2}, E) be two NSPTSs over (V, E) . Then $(V, \tau_{t1} \cap \tau_{t2}, E)$ is also a NSPTS over (V, E) .

Proof. Let (V, τ_{t1}, E) and (V, τ_{t2}, E) be two NSPTSs over (V, E) .

(i) Obviously, $\phi, 1_t \in \tau_{t1} \cap \tau_{t2}$.

(ii) Let $K_1, K_2 \in \tau_{t1} \cap \tau_{t2}$.

Then $K_1, K_2 \in \tau_{t1}$ and $K_1, K_2 \in \tau_{t2}$.

$\Rightarrow K_1 \cap K_2 \in \tau_{t1}$ and $K_1 \cap K_2 \in \tau_{t2}$.

$\Rightarrow K_1 \cap K_2 \in \tau_{t1} \cap \tau_{t2}$.

Thus $(V, \tau_{t1} \cap \tau_{t2}, E)$ is a NSPTS over (V, E) .

Remark 3.4 Let (V, τ_{t1}, E) and (V, τ_{t2}, E) be two NSPTSs over (V, E) . Then $(V, \tau_{t1} \cup \tau_{t2}, E)$ is not NSPTS over (V, E) .

Example 3.5 Let $V = \{g_1, g_2, g_3\}$, $E = \{e_1, e_2, e_3, e_4\}$. Let $\tau_{t1} = \{\phi, 1_t, K\}$ and $\tau_{t2} = \{\phi, 1_t, L\}$ where the NSPs K and L over V are defined as

$$K = \begin{cases} f^{(e_1)}K = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >, < g_3, (0, 0, 1) >\} \\ f^{(e_2)}K = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >, < g_3, (0, 0, 1) >\} \\ f^{(e_3)}K = \{< g_1, (0, 0, 1) >, < g_2, (.7, .6, .2) >, < g_3, (0, 0, 1) >\} \\ f^{(e_4)}K = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >, < g_3, (0, 0, 1) >\} \end{cases} \quad \text{and}$$

$$L = \begin{cases} f^{(e_1)}L = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >, < g_3, (0, 0, 1) >\} \\ f^{(e_2)}L = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >, < g_3, (0, 0, 1) >\} \\ f^{(e_3)}L = \{< g_1, (0, 0, 1) >, < g_2, (.3, .9, .5) >, < g_3, (0, 0, 1) >\} \\ f^{(e_4)}L = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >, < g_3, (0, 0, 1) >\} \end{cases}.$$

Then $\tau_{t1} \cup \tau_{t2} = \{\phi, 1_t, K, L\}$ is not an NSPTS over (V, E) , since $K \cap L \notin \tau_{t1} \cup \tau_{t2}$.

Thus $(V, \tau_{t1} \cup \tau_{t2}, E)$ is not NSPTS over (V, E) .

Proposition 3.6 Let K and L be any two NSPs on NSPTS (V, τ_t, E) over (V, E) . Then

$$(i) (K \cup L)^c = K^c \cap L^c.$$

$$(ii) (K \cap L)^c = K^c \cup L^c.$$

Proof. Straight forward.

Proposition 3.7 Let (V, τ_t, E) be a NSPTS over (V, E) and

$$\tau_t = \{K_i : K_i \in \text{NSP}(V, E)\} = \left\langle e, f^{(E)}K : K_i \in \text{NSP}(V, E) \right\rangle$$

where $f^{(E)}K = \left\langle v, T_{f^{(e)}K}(v), I_{f^{(e)}K}(v), F_{f^{(e)}K}(v) : v \in V, e \in E \right\rangle$.

Define

$$\tau_{t1} = \left\{ T_{f^{(e)}K}(V) \right\}_{e \in E},$$

$$\tau_{t2} = \left\{ I_{f^{(e)}K}(V) \right\}_{e \in E} \quad \text{and}$$

$$\tau_{t3} = \left\{ F_{f^{(e)}K}(V) \right\}_{e \in E}.$$

Then τ_{t1}, τ_{t2} and τ_{t3} are FSTs on (V, E) .

Proof. Let (V, τ_t, E) be a NSPTS over (V, E) .

(i) Since $\phi, 1_t \in \tau_t$,

$$\Rightarrow 0, 1 \in \tau_{t1}, 0, 1 \in \tau_{t2}, 1, 0 \in \tau_{t3}.$$

(ii) Let $K_1, K_2 \in \tau_t$. Then $K_1 \cap K_2 \in \tau_t$.

That is,

$$K_1 \cap K_2 = \left\{ \left\langle \min \left[T_{f^{(e)K_1}}(V), T_{f^{(e)K_2}}(V) \right]_{e \in E}, \min \left[I_{f^{(e)K_1}}(V), I_{f^{(e)K_2}}(V) \right]_{e \in E}, \max \left[F_{f^{(e)K_1}}(V), F_{f^{(e)K_2}}(V) \right]_{e \in E} \right\rangle \right\} \in \tau_t.$$

Thus

$$\begin{aligned} \left\{ \min \left[T_{f^{(e)K_1}}(V), T_{f^{(e)K_2}}(V) \right]_{e \in E} \right\} &\in \tau_{t1}, \\ \left\{ \min \left[I_{f^{(e)K_1}}(V), I_{f^{(e)K_2}}(V) \right]_{e \in E} \right\} &\in \tau_{t2} \text{ and} \\ \left\{ \min \left[F_{f^{(e)K_1}}(V), F_{f^{(e)K_2}}(V) \right]_{e \in E}^c \right\} &\in \tau_{t3}. \end{aligned}$$

(iii) Let $K_i \in \tau_t$, where $i \in I$. Then $\bigcup_{i \in I} K_i \in \tau_t$.

That is,

$$\bigcup_{i \in I} K_i = \left\{ \left\langle \max \left[T_{f^{(e)K_i}}(V) \right]_{e \in E}, \max \left[I_{f^{(e)K_i}}(V) \right]_{e \in E}, \min \left[F_{f^{(e)K_i}}(V) \right]_{e \in E} \right\rangle \right\}_{i \in I} \in \tau_t.$$

Thus

$$\begin{aligned} \left\{ \max \left[T_{f^{(e)K_i}}(V) \right]_{e \in E} \right\}_{i \in I} &\in \tau_{t1}, \\ \left\{ \max \left[I_{f^{(e)K_i}}(V) \right]_{e \in E} \right\}_{i \in I} &\in \tau_{t2} \text{ and} \\ \left\{ \max \left[F_{f^{(e)K_i}}(V) \right]_{e \in E}^c \right\}_{i \in I} &\in \tau_{t3}. \end{aligned}$$

Hence τ_{t1}, τ_{t2} and τ_{t3} are FSTs on (V, E) .

Remark 3.8 The following example illustrates that the converse of Proposition 3.7 is not true.

Example 3.9 Let $V = \{g_1, g_2, g_3\}$, $E = \{e_1, e_2, e_3, e_4\}$. Let $\tau_t = \{\phi, 1_t, K_1, K_2\}$ where the NSPs K_1 and K_2 over V are defined as

$$\begin{aligned} K_1 &= \begin{cases} f^{(e_1)}K_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)}K_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_3)}K_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (.7, .6, .2) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_4)}K_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases} \text{ and} \\ K_2 &= \begin{cases} f^{(e_1)}K_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)}K_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_3)}K_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (.3, .9, .5) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_4)}K_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases}. \end{aligned}$$

Then

$$\begin{aligned} \tau_{t1} &= \left\{ \left[T_{f^{(e)\phi_t}}(V), T_{f^{(e)1_t}}(V), T_{f^{(e)K_1}}(V), T_{f^{(e)K_2}}(V) \right]_{e \in E} \right\}, \\ \tau_{t2} &= \left\{ \left[I_{f^{(e)\phi_t}}(V), I_{f^{(e)1_t}}(V), I_{f^{(e)K_1}}(V), I_{f^{(e)K_2}}(V) \right]_{e \in E} \right\} \text{ and} \\ \tau_{t3} &= \left\{ \left[F_{f^{(e)\phi_t}}(V), F_{f^{(e)1_t}}(V), F_{f^{(e)K_1}}(V), F_{f^{(e)K_2}}(V) \right]_{e \in E} \right\} \end{aligned}$$

are FSTs on (V, E) , where

$$\begin{aligned} \tau_{t1} &= \{ \langle e_1, (0, 0, 0), (1, 1, 1), (0, 0, 0), (0, 0, 0) \rangle, \langle e_2, (0, 0, 0), (1, 1, 1), (0, 0, 0), (0, 0, 0) \rangle, \\ &\quad \langle e_3, (0, 0, 0), (1, 1, 1), (0, .7, 0), (0, .3, 0) \rangle, \langle e_4, (0, 0, 0), (1, 1, 1), (0, 0, 0), (0, 0, 0) \rangle \} \end{aligned}$$

and so on.

Thus $\tau_t = \{\phi, 1_t, K_1, K_2\}$ is not NSPT over (V, E) , since $K_1 \cap K_2 \notin \tau_t$.

Proposition 3.10 Let (V, τ_t, E) be a NSPTS over (V, E) and

$$\tau_t = \{K_i : K_i \in \text{NSP}(V, E)\} = \left\{ \langle e, f^{(E)}K \rangle : K_i \in \text{NSP}(V, E) \right\}$$

where $f^{(E)}K = \left\{ \langle v, T_{f^{(e)}K}(v), I_{f^{(e)}K}(v), F_{f^{(e)}K}(v) \rangle : v \in V, e \in E \right\}$.

Define

$$\begin{aligned}\tau_{t1} &= \left\{ T_{f^{(e)}K}(V) \right\}_{e \in E}, \\ \tau_{t2} &= \left\{ I_{f^{(e)}K}(V) \right\}_{e \in E} \text{ and} \\ \tau_{t3} &= \left\{ F_{f^{(e)}K}(V) \right\}_{e \in E} \text{ as FSTs on } (V, E).\end{aligned}$$

Then $\tau_{t1} \cup \tau_{t3}$ and $\tau_{t2} \cup \tau_{t3}$ are not FSTs on (V, E) .

Proposition 3.10 is illustrated by the following example.

Example 3.11 Let $V = \{g_1, g_2, g_3\}$, $E = \{e_1, e_2, e_3, e_4\}$. Let $\tau_t = \{\phi_t, 1_t, K_1, K_2, K_3\}$ where the NSPs K_1, K_2, K_3 over V are defined as

$$\begin{aligned}K_1 &= \begin{cases} f^{(e_1)}K_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)}K_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (.7, .6, .2) \rangle \} \\ f^{(e_3)}K_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_4)}K_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases} ; \\ K_2 &= \begin{cases} f^{(e_1)}K_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)}K_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (.8, .7, .5) \rangle \} \\ f^{(e_3)}K_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_4)}K_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases} \text{ and} \\ K_3 &= \begin{cases} f^{(e_1)}K_3 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)}K_3 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (.7, .6, .5) \rangle \} \\ f^{(e_3)}K_3 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_4)}K_3 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases} .\end{aligned}$$

Here $\phi_t \cap K_1 = \phi_t$, $\phi_t \cap K_2 = \phi_t$, $\phi_t \cap K_3 = \phi_t$, $1_t \cap K_1 = K_1$, $1_t \cap K_2 = K_2$, $1_t \cap K_3 = K_3$, $K_1 \cap K_2 = K_3$, $K_1 \cap K_3 = K_3$, $K_2 \cap K_3 = K_3$.

Then K_1, K_2 and K_3 are NSPOSs.

Thus (V, τ_t, E) is a NSPTS over (V, E) .

Then

$$\begin{aligned}\tau_{t1} &= \left\{ \langle e_1, (0, 0, 0), (1, 1, 1), (0, 0, 0), (0, 0, 0), (0, 0, 0) \rangle, \langle e_2, (0, 0, 0), (1, 1, 1), (0, 0, .7), (0, 0, .8), (0, 0, .7) \rangle, \right. \\ &\quad \left. \langle e_3, (0, 0, 0), (1, 1, 1), (0, 0, 0), (0, 0, 0), (0, 0, 0) \rangle, \langle e_4, (0, 0, 0), (1, 1, 1), (0, 0, 0), (0, 0, 0), (0, 0, 0) \rangle \right\} ,\end{aligned}$$

and so on, are FSTs on (V, E) .

But, $\tau_{t1} \cup \tau_{t3}$ and $\tau_{t2} \cup \tau_{t3}$ are not FSTs on (V, E) .

Proposition 3.12 Let (V, τ_t, E) be a NSPTS over (V, E) . Then

$$\begin{aligned}{}^e\tau_{t1} &= \left\{ T_{f^{(e)}K}(V) : K \in \tau_t \right\}, \\ {}^e\tau_{t2} &= \left\{ I_{f^{(e)}K}(V) : K \in \tau_t \right\} \text{ and} \\ {}^e\tau_{t3} &= \left\{ F_{f^{(e)}K}(V) : K \in \tau_t \right\}\end{aligned}$$

for each $e \in E$, define FTs on (V, E) .

Proof. Follows from Proposition 3.7.

Remark 3.13 The following example illustrates that the converse of Proposition 3.12 is not true.

Example 3.14 Consider Example 3.9.

Then

$$\begin{aligned} {}^{e_1}\tau_{t1} &= \{T_{f^{(e_1)\phi_t}}(V), T_{f^{(e_1)1_t}}(V), T_{f^{(e_1)K_1}}(V), T_{f^{(e_1)K_2}}(V)\}, \\ {}^{e_1}\tau_{t2} &= \{I_{f^{(e_1)\phi_t}}(V), I_{f^{(e_1)1_t}}(V), I_{f^{(e_1)K_1}}(V), I_{f^{(e_1)K_2}}(V)\} \text{ and} \\ {}^{e_1}\tau_{t3} &= \{F_{f^{(e_1)\phi_t}}(V), F_{f^{(e_1)1_t}}(V), F_{f^{(e_1)K_1}}(V), F_{f^{(e_1)K_2}}(V)\} \end{aligned}$$

are FTs on V , where

$${}^{e_1}\tau_{t1} = \{(0, 0, 0), (1, 1, 1), (0, 0, 0), (0, 0, 0)\} \text{ and so on.}$$

Consequently, $\{{}^{e_2}\tau_{t1}, {}^{e_2}\tau_{t2}, {}^{e_2}\tau_{t3}\}$ and $\{{}^{e_3}\tau_{t1}, {}^{e_3}\tau_{t2}, {}^{e_3}\tau_{t3}\}$ are fuzzy tritopologies on V .

Thus $\tau_t = \{\phi_t, 1_t, K_1, K_2\}$ is not NSPT over (V, E) , since $K_1 \cap K_2 \notin \tau_t$.

Definition 3.15 Let (V, τ_t, E) be a NSPTS over (V, E) and $L \in \text{NSP}(V, E)$ be any NSP. Then the neutro-spot absolute interior of L is denoted by \tilde{L}_o and defined as

(i) $\tilde{L}_o = \{K : K \in \text{NSPOS} \& K \subseteq L\}$ i.e., the union of all neutro-spot open subsets of L .

(ii)

$$\tilde{L}_o = \left\{ \left(\tilde{T}_{\circ f^{(e)L}}, \tilde{I}_{\circ f^{(e)L}}, \tilde{F}_{\circ f^{(e)L}} \right)_{e \in E} \right\} = \left\{ \left(\max_i T_{f^{(e)K_i}}, \max_i I_{f^{(e)K_i}}, \min_i F_{f^{(e)K_i}} \right)_{e \in E} : K_i \in \tau_t \& f^{(e)K_i} \subseteq f^{(e)L} \right\}$$

Example 3.16 Let $V = \{g_1, g_2\}$, $E = \{e_1, e_2, e_3\}$. Let $\tau_t = \{\phi_t, 1_t, K_1, K_2, K_3\}$ where the NSPs K_1, K_2, K_3 over V are defined as

$$K_1 = \begin{cases} f^{(e_1)K_1} = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >\} \\ f^{(e_2)K_1} = \{< g_1, (.9, .4, .2) >, < g_2, (0, 0, 1) >\} \\ f^{(e_3)K_1} = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >\} \end{cases};$$

$$K_2 = \begin{cases} f^{(e_1)K_2} = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >\} \\ f^{(e_2)K_2} = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >\} \\ f^{(e_3)K_2} = \{< g_1, (0, 0, 1) >, < g_2, (.6, .1, .8) >\} \end{cases} \text{ and}$$

$$K_3 = \begin{cases} f^{(e_1)K_3} = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >\} \\ f^{(e_2)K_3} = \{< g_1, (.4, .3, .5) >, < g_2, (0, 0, 1) >\} \\ f^{(e_3)K_3} = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >\} \end{cases}.$$

Here $\phi_t \cap K_1 = \phi_t$, $\phi_t \cap K_2 = \phi_t$, $\phi_t \cap K_3 = \phi_t$, $1_t \cap K_1 = K_1$, $1_t \cap K_2 = K_2$, $1_t \cap K_3 = K_3$, $K_1 \cap K_2 = \phi_t$, $K_1 \cap K_3 = K_3$, $K_2 \cap K_3 = \phi_t$.

Then K_1, K_2 and K_3 are NSPOSs.

Thus (V, τ_t, E) is a NSPTS over (V, E) .

Let $L \in \text{NSP}(V, E)$ be any NSP defined as

$$L = \begin{cases} f^{(e_1)L} = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >\} \\ f^{(e_2)L} = \{< g_1, (.9, .5, .1) >, < g_2, (0, 0, 1) >\} \\ f^{(e_3)L} = \{< g_1, (0, 0, 1) >, < g_2, (0, 0, 1) >\} \end{cases}.$$

Then $\phi_t, K_1, K_3 \subseteq L$.

Thus $\tilde{L}_o = \phi_t \cup K_1 \cup K_3 = K_1$.

Also,

$$\begin{aligned} f^{(e_1)}(K_1), f^{(e_1)}(K_2), f^{(e_1)}(K_3) &\subseteq f^{(e_1)}(L), \\ f^{(e_2)}(K_1), f^{(e_2)}(K_2), f^{(e_2)}(K_3) &\subseteq f^{(e_2)}(L) \text{ and} \\ f^{(e_3)}(K_1), f^{(e_3)}(K_3) &\subseteq f^{(e_3)}(L). \end{aligned}$$

Then

$$\begin{aligned} (\tilde{T}_{of^{(e_1)}(L)}, \tilde{I}_{of^{(e_1)}(L)}, \tilde{F}_{of^{(e_1)}(L)}) &= \{< g_1, (0,0,1) >, < g_2, (0,0,1) >\}, \\ (\tilde{T}_{of^{(e_2)}(L)}, \tilde{I}_{of^{(e_2)}(L)}, \tilde{F}_{of^{(e_2)}(L)}) &= \{< g_1, (.9,.4,.2) >, < g_2, (0,0,1) >\} \text{ and} \\ (\tilde{T}_{of^{(e_3)}(L)}, \tilde{I}_{of^{(e_3)}(L)}, \tilde{F}_{of^{(e_3)}(L)}) &= \{< g_1, (0,0,1) >, < g_2, (0,0,1) >\}. \end{aligned}$$

Thus $\{(\tilde{T}_{of^{(e)}(L)}, \tilde{I}_{of^{(e)}(L)}, \tilde{F}_{of^{(e)}(L)})\}_{e \in E} = K_1$.

Theorem 3.17 Let (V, τ_t, E) be a NSPTS over (V, E) and $L, S \in NSP(V, E)$ be any two NSPs.

Then,

- (i) $\tilde{L}_o \subseteq L$ and \tilde{L}_o is the largest NSPOS.
- (ii) $L \subseteq S \Rightarrow \tilde{L}_o \subseteq \tilde{S}_o$.
- (iii) \tilde{L}_o is a NSPOS i.e., $\tilde{L}_o \in \tau_t$.
- (iv) L is a NSPOS iff $\tilde{L}_o = L$.
- (v) $(\tilde{L}_o)_o = \tilde{L}_o$.
- (vi) $(\tilde{\phi}_o)_o = \tilde{\phi}_o$ and $(\tilde{l}_o)_o = \tilde{l}_o$.
- (vii) $(\tilde{L} \cap \tilde{S})_o = \tilde{L}_o \cap \tilde{S}_o$.
- (ix) $\tilde{L}_o \cup \tilde{S}_o \subseteq (\tilde{L} \cup \tilde{S})_o$.

Proof. Let (V, τ_t, E) be a NSPTS over (V, E) and $L, S \in NSP(V, E)$ be any two NSPs.

(i) Follows from Definition 3.15.

(ii) Let $L \subseteq S$. Then $\tilde{L}_o \subseteq L \subseteq S \Rightarrow \tilde{L}_o \subseteq S$ and $\tilde{S}_o \subseteq S$.

Since \tilde{S}_o is the largest NSPOS contained in S , hence $\tilde{L}_o \subseteq \tilde{S}_o$.

(iii) Follows from Definition 3.15.

(iv) Let L be a NSPOS. Then \tilde{L}_o is the largest NSPOS which contained in L is equal to L .

Hence $\tilde{L}_o = L$.

Conversely, assume that $\tilde{L}_o = L$.

By (iii), $\tilde{L}_o \in \tau_t$.

Then L is a NSPOS.

(v) Let $\tilde{L}_o = K$. Then $K \in \tau_t$ iff $\tilde{K}_o = K$.

Thus $(\tilde{L}_o)_o = \tilde{L}_o$.

(vi) Since $\phi_t, l_t \in \tau_t$ and by (iv), hence $(\tilde{\phi}_o)_o = \tilde{\phi}_o$ and $(\tilde{l}_o)_o = \tilde{l}_o$.

(vii) $L \cap S \subseteq L$ and $L \cap S \subseteq S$.

$\Rightarrow (\tilde{L} \cap \tilde{S})_o \subseteq \tilde{L}_o$ and $(\tilde{L} \cap \tilde{S})_o \subseteq \tilde{S}_o$.

$\Rightarrow (\tilde{L} \cap \tilde{S})_o \subseteq \tilde{L}_o \cap \tilde{S}_o$.

Also, $\tilde{L}_o \subseteq L$ and $\tilde{P}_o \subseteq P$.

Then $\tilde{L}_o \cap \tilde{S}_o \subseteq L \cap S$.

Since $(\tilde{L} \cap \tilde{S})_o \subseteq L \cap S$ and it is the largest NSPOS, then $\tilde{L}_o \cap \tilde{S}_o \subseteq (\tilde{L} \cap \tilde{S})_o$.

Hence $(\tilde{L} \cap \tilde{S})_o = \tilde{L}_o \cap \tilde{S}_o$.

(vii) $L \subseteq L \cup S$ and $S \subseteq L \cup S$.

$\Rightarrow \tilde{L}_o \subseteq (\tilde{L} \cup \tilde{S})_o$ and $\tilde{S}_o \subseteq (\tilde{L} \cup \tilde{S})_o$.

$\Rightarrow \tilde{L}_o \cup \tilde{S}_o \subseteq (\tilde{L} \cup \tilde{S})_o$.

Definition 3.18 Let (V, τ_t, E) be a NSPTS over (V, E) where τ_t is a NSPT over (V, E) and $L \in NSP(V, E)$ be any NSP. Then the subspace topology on NSPTS is denoted by τ_t^L and defined as $\tau_t^L = \{L \cap K_i : K_i \in \tau_t\}$ and $\phi_t, 1_t \in \tau_t^L$. Thus (V, τ_t^L, E) is a neutro-spot subspace topological space (NSPSTS) of (V, τ_t, E) , where τ_t^L is also a NSPT over (V, E) .

Example 3.19 Let $V = \{g_1, g_2, g_3\}$, $E = \{e_1, e_2\}$. Let $\tau_t = \{\phi_t, 1_t, K_1, K_2, K_3\}$ where the NSPs K_1, K_2, K_3 over V are defined as

$$K_1 = \begin{cases} f^{(e_1)} K_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)} K_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (.1, .4, .5) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases} ;$$

$$K_2 = \begin{cases} f^{(e_1)} K_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)} K_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (.4, .7, .8) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases} \text{ and}$$

$$K_3 = \begin{cases} f^{(e_1)} K_3 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)} K_3 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (.1, .4, .8) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases} .$$

Here $\phi_t \cap K_1 = \phi_t$, $\phi_t \cap K_2 = \phi_t$, $\phi_t \cap K_3 = \phi_t$, $1_t \cap K_1 = K_1$, $1_t \cap K_2 = K_2$, $1_t \cap K_3 = K_3$, $K_1 \cap K_2 = K_3$, $K_1 \cap K_3 = K_3$, $K_2 \cap K_3 = K_3$.

Then K_1, K_2 and K_3 are NSPOs.

Thus (V, τ_t, E) is a NSPTS over (V, E) .

Define $L_1 \in NSP(V, E)$ as

$$L_1 = \begin{cases} f^{(e_1)} L_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)} L_1 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (.7, .3, .6) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases} .$$

Thus we denote $L_1 \cap \phi_t = \phi_t$, $L_1 \cap 1_t = L_1$, $L_1 \cap K_1 = L_2$, $L_1 \cap K_2 = L_3$, $L_1 \cap K_3 = L_4$.

Then the NSPs L_2, L_3, L_4 over V are defined as

$$L_2 = \begin{cases} f^{(e_1)} L_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)} L_2 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (.1, .3, .6) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases} ;$$

$$L_3 = \begin{cases} f^{(e_1)} L_3 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)} L_3 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (.4, .3, .8) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases} \text{ and}$$

$$L_4 = \begin{cases} f^{(e_1)} L_4 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (0, 0, 1) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_2)} L_4 = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (.1, .3, .8) \rangle, \langle g_3, (0, 0, 1) \rangle \} \end{cases} .$$

Then $\tau_t^L = \{\phi_t, 1_t, L_1, L_2, L_3, L_4\}$ is a NSPST over (V, E) .

Thus (V, τ_t^L, E) is a NSPSTS of (V, τ_t, E) .

4. Neutro-Supra Spot Topology

This part defines the neutro-supra spot topology, neutro-supra spot absolute closure, and its subspace topology with some properties and examples.

Definition 4.1 Let V be a universe and E be a set of parameters. Then the complement of NSP is said to be neutrosophic soft whole set (NSWS) over (V, E) .

Definition 4.2 Let V be a universe and E be a set of parameters. Let $NSWS(V, E)$ be the family of all neutrosophic soft whole sets (NSWSs) over V . Then $(\tau_t)' \subset NSWS(V, E)$ is said to be neutro-supra spot topology (NSSPT) on V if it satisfies the following conditions

(i) $\phi_t, 1_t \in (\tau_t)'$.

(ii) the arbitrary union of NSWSs in $(\tau_t)'$ belongs to $(\tau_t)'$.

Then $(\tau_t)'$ is said to be a neutro-supra spot topology (NSSPT) over (V, E) , and the trio $(V, (\tau_t)', E)$ is said to be a neutro-supra spot topological spaces (NSSPTSs) over (V, E) . Elements in $(\tau_t)'$ are called neutro-supra spot open sets (NSSPOSs).

Example 4.3 Consider Example 3.2. Here $(\tau_t)' = \{\phi_t, 1_t, (K_1)^c, (K_2)^c, (K_3)^c\}$.

Then NSWSs $(K_1)^c, (K_2)^c, (K_3)^c$ over V are defined as

$$(K_1)^c = \begin{cases} f^{(e_1)}(K_1)^c = \{ \langle p_1, (1,1,0) \rangle, \langle p_2, (1,1,0) \rangle, \langle p_3, (1,1,0) \rangle \} \\ f^{(e_2)}(K_1)^c = \{ \langle p_1, (1,1,0) \rangle, \langle p_2, (1,1,0) \rangle, \langle p_3, (1,1,0) \rangle \} \\ f^{(e_3)}(K_1)^c = \{ \langle p_1, (1,1,0) \rangle, \langle p_2, (.2, .6, .3) \rangle, \langle p_3, (1,1,0) \rangle \} \\ f^{(e_4)}(K_1)^c = \{ \langle p_1, (1,1,0) \rangle, \langle p_2, (1,1,0) \rangle, \langle p_3, (1,1,0) \rangle \} \end{cases} ;$$

$$(K_2)^c = \begin{cases} f^{(e_1)}(K_2)^c = \{ \langle p_1, (1,1,0) \rangle, \langle p_2, (1,1,0) \rangle, \langle p_3, (1,1,0) \rangle \} \\ f^{(e_2)}(K_2)^c = \{ \langle p_1, (1,1,0) \rangle, \langle p_2, (1,1,0) \rangle, \langle p_3, (1,1,0) \rangle \} \\ f^{(e_3)}(K_2)^c = \{ \langle p_1, (.6, .8, .5) \rangle, \langle p_2, (1,1,0) \rangle, \langle p_3, (1,1,0) \rangle \} \\ f^{(e_4)}(K_2)^c = \{ \langle p_1, (1,1,0) \rangle, \langle p_2, (1,1,0) \rangle, \langle p_3, (1,1,0) \rangle \} \end{cases} \text{ and}$$

$$(K_3)^c = \begin{cases} f^{(e_1)}(K_3)^c = \{ \langle p_1, (1,1,0) \rangle, \langle p_2, (1,1,0) \rangle, \langle p_3, (1,1,0) \rangle \} \\ f^{(e_2)}(K_3)^c = \{ \langle p_1, (1,1,0) \rangle, \langle p_2, (1,1,0) \rangle, \langle p_3, (1,1,0) \rangle \} \\ f^{(e_3)}(K_3)^c = \{ \langle p_1, (1,1,0) \rangle, \langle p_2, (1,1,0) \rangle, \langle p_3, (.3, .8, .1) \rangle \} \\ f^{(e_4)}(K_3)^c = \{ \langle p_1, (1,1,0) \rangle, \langle p_2, (1,1,0) \rangle, \langle p_3, (1,1,0) \rangle \} \end{cases} .$$

Here $\phi_t \cup (K_1)^c = (K_1)^c$, $\phi_t \cup (K_2)^c = (K_2)^c$, $\phi_t \cup (K_3)^c = (K_3)^c$, $1_t \cup (K_1)^c = 1_t$, $1_t \cup (K_2)^c = 1_t$, $1_t \cup (K_3)^c = 1_t$, $(K_1)^c \cup (K_2)^c = 1_t$, $(K_1)^c \cup (K_3)^c = 1_t$, $(K_2)^c \cup (K_3)^c = 1_t$.

Then $(K_1)^c, (K_2)^c$ and $(K_3)^c$ are NSSPOSs.

Thus $(V, (\tau_t)', E)$ is a NSSPTS over (V, E) .

Proposition 4.4 Let $(V, (\tau_{t1})', E)$ and $(V, (\tau_{t2})', E)$ be two NSSPTSs over (V, E) . Then $(V, (\tau_{t1})' \cap (\tau_{t2})', E)$ is also a NSSPTS over (V, E) .

Proof. Let $(V, (\tau_{t1})', E)$ and $(V, (\tau_{t2})', E)$ be two NSSPTSs over (V, E) .

(i) Obviously, $\phi_t, 1_t \in (\tau_{t1})' \cap (\tau_{t2})'$.

(ii) Let $\{K_i \in NSW(S(V, E) : i \in I\} \in (\tau_{t1})' \cap (\tau_{t2})'$.

Then $\{K_i\} \in (\tau_{t1})'$ and $\{K_i\} \in (\tau_{t2})'$.

$$\Rightarrow \bigcup_{i \in I} K_i \in (\tau_{t1})' \text{ and } \bigcup_{i \in I} K_i \in (\tau_{t2})'.$$

$$\Rightarrow \bigcup_{i \in I} K_i \in (\tau_{t1})' \cap (\tau_{t2})'.$$

Thus $(V, (\tau_{t1})' \cap (\tau_{t2})', E)$ is a NSSPTS over (V, E) .

Remark 4.5 Let $(V, (\tau_{t1})', E)$ and $(V, (\tau_{t2})', E)$ be two NSSPTSs over (V, E) . Then $(V, (\tau_{t1})' \cup (\tau_{t2})', E)$ is not NSSPTS over (V, E) .

Example 4.6 Consider Example 3.5. Here $(\tau_{t1})' = \{\phi_t, 1_t, K^c\}$ and $\tau_{t2} = \{\phi_t, 1_t, L^c\}$ where the NSWs K^c and L^c over V are defined as

$$K^c = \begin{cases} f^{(e_1)} K^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \\ f^{(e_2)} K^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \\ f^{(e_3)} K^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (2, 4, 7) \rangle, \langle g_3, (1, 1, 0) \rangle \} \\ f^{(e_4)} K^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \end{cases} \quad \text{and}$$

$$L^c = \begin{cases} f^{(e_1)} L^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \\ f^{(e_2)} L^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \\ f^{(e_3)} L^c = \{ \langle g_1, (0, 0, 1) \rangle, \langle g_2, (5, 1, 3) \rangle, \langle g_3, (0, 0, 1) \rangle \} \\ f^{(e_4)} L^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \end{cases}.$$

Then $(\tau_{t1})' \cup (\tau_{t2})' = \{\phi_t, 1_t, K^c, L^c\}$ is not an NSSPTS over (V, E) , since $K^c \cup L^c \notin (\tau_{t1})' \cup (\tau_{t2})'$.

Thus $(V, (\tau_{t1})' \cup (\tau_{t2})', E)$ is not NSSPTS over (V, E) .

Proposition 4.7 Let K and L be any two NSWs on NSSPTS $(V, (\tau_t)', E)$ over (V, E) . Then

$$(i) (K \cup L)^c = K^c \cap L^c.$$

$$(ii) (K \cap L)^c = K^c \cup L^c.$$

Proof. Straight forward.

Proposition 4.8 Let $(V, (\tau_t)', E)$ be a NSSPTS over (V, E) and

$$(\tau_t)' = \{K_i : K_i \in NSW(S(V, E))\} = \left\{ e, f^{(E)} K : K_i \in NSW(S(V, E)) \right\}$$

where $f^{(E)} K = \left\{ v, T_{f^{(e)} K}(v), I_{f^{(e)} K}(v), F_{f^{(e)} K}(v) : v \in V, e \in E \right\}$.

Define

$$\begin{aligned} (\tau_{t1})' &= \left\{ T_{f^{(e)} K}(V) \right\}_{e \in E}, \\ (\tau_{t2})' &= \left\{ I_{f^{(e)} K}(V) \right\}_{e \in E} \text{ and} \\ (\tau_{t3})' &= \left\{ F_{f^{(e)} K}(V) \right\}_{e \in E}. \end{aligned}$$

Then $(\tau_{t1})', (\tau_{t2})'$ and $(\tau_{t3})'$ are FSTs on (V, E) .

Proof. Let $(V, (\tau_t)', E)$ be a NSSPTS over (V, E) .

(i) Since $\phi_t, 1_t \in (\tau_t)'$,

$$\Rightarrow 0, 1 \in (\tau_{t1})', 0, 1 \in (\tau_{t2})', 1, 0 \in (\tau_{t3})'.$$

(ii) Let $\{K_i \in NSW(S(V, E)) : i \in I\} \in (\tau_t)'$.

Then $\bigcup_{i \in I} K_i \in (\tau_t)'$.

That is,

$$\bigcup_{i \in I} K_i = \left\{ \left\langle \sup_{v \in E} [T_{f^{(e)K_i}}(V)], \sup_{v \in E} [I_{f^{(e)K_i}}(V)], \inf_{v \in E} [F_{f^{(e)K_i}}(V)] \right\rangle \right\}_{i \in I} \in (\tau_t)'.$$

Thus

$$\left\{ \sup_{v \in E} [T_{f^{(e)K_i}}(V)] \right\}_{i \in I} \in (\tau_{t1})',$$

$$\left\{ \sup_{v \in E} [I_{f^{(e)K_i}}(V)] \right\}_{i \in I} \in (\tau_{t2})' \text{ and}$$

$$\left\{ \sup_{v \in E} [T_{f^{(e)K_i}}(V)]^c \right\}_{i \in I} \in (\tau_{t3})'.$$

(iii) Let $K_1, K_2 \in (\tau_t)'$.

Then $K_1 \cap K_2 \in (\tau_t)'$.

That is,

$$K_1 \cap K_2 = \left\{ \left\langle \min_{v \in E} [T_{f^{(e)K_1}}(V), T_{f^{(e)K_2}}(V)], \min_{v \in E} [I_{f^{(e)K_1}}(V), I_{f^{(e)K_2}}(V)], \max_{v \in E} [F_{f^{(e)K_1}}(V), F_{f^{(e)K_2}}(V)] \right\rangle \right\} \in (\tau_t)'.$$

Thus

$$\left\{ \min_{v \in E} [T_{f^{(e)K_1}}(V), T_{f^{(e)K_2}}(V)] \right\} \in (\tau_{t1})',$$

$$\left\{ \min_{v \in E} [I_{f^{(e)K_1}}(V), I_{f^{(e)K_2}}(V)] \right\} \in (\tau_{t2})' \text{ and}$$

$$\left\{ \min_{v \in E} [F_{f^{(e)K_1}}(V), F_{f^{(e)K_2}}(V)]^c \right\} \in (\tau_{t3})'.$$

Hence $(\tau_{t1})', (\tau_{t2})'$ and $(\tau_{t3})'$ are FSTs on (V, E) .

Remark 4.9 The following example illustrates that the converse of Proposition 4.8 is not true.

Example 4.10 Consider Example 3.9. Here $(\tau_t)' = \{\emptyset, 1_t, (K_1)^c, (K_2)^c\}$ where the NSWs $(K_1)^c$ and $(K_2)^c$ over V are defined as

$$(K_1)^c = \begin{cases} f^{(e_1)}(K_1)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_2)}(K_1)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_3)}(K_1)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (.2, .4, .7) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_4)}(K_1)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \end{cases} \text{ and}$$

$$(K_2)^c = \begin{cases} f^{(e_1)}(K_2)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_2)}(K_2)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_3)}(K_2)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (.5, .1, .3) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_4)}(K_2)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \end{cases}.$$

Then

$$(\tau_{t1})' = \left\{ T_{f^{(e)\emptyset}}(V), T_{f^{(e)1_t}}(V), T_{f^{(e)(K_1)^c}}(V), T_{f^{(e)(K_2)^c}}(V) \right\}_{v \in E},$$

$$(\tau_{t2})' = \left\{ I_{f^{(e)\emptyset}}(V), I_{f^{(e)1_t}}(V), I_{f^{(e)(K_1)^c}}(V), I_{f^{(e)(K_2)^c}}(V) \right\}_{v \in E} \text{ and}$$

$$(\tau_{t3})' = \left\{ \left[F_{f^{(e)}\phi_t}(V), F_{f^{(e)}1_t}(V), F_{f^{(e)}(K_1)^c}(V), F_{f^{(e)}(K_2)^c}(V) \right]_{e \in E} \right\} \text{ are FSTs on } (V, E),$$

where

$$(\tau_{t1})' = \left\{ \langle e_1, (0, 0, 0), (1, 1, 1), (1, 1, 1), (1, 1, 1) \rangle, \langle e_2, (0, 0, 0), (1, 1, 1), (1, 1, 1), (1, 1, 1) \rangle, \right. \\ \left. \langle e_3, (0, 0, 0), (1, 1, 1), (1, .2, 1), (1, .5, 1) \rangle, \langle e_4, (0, 0, 0), (1, 1, 1), (1, 1, 1), (1, 1, 1) \rangle \right\} \text{ and so on.}$$

Thus $(\tau_t)' = \{\phi_t, 1_t, (K_1)^c, (K_2)^c\}$ is not NSSPT over (V, E) , since $(K_1)' \cap (K_2)' \notin (\tau_t)'$.

Proposition 4.11 Let $(V, (\tau_t)', E)$ be a NSSPTS over (V, E) and

$$(\tau_t)' = \{K_i : K_i \in NSW(S(V, E))\} = \left\{ e, f^{(E)}K : K_i \in NSW(S(V, E)) \right\}$$

Where

$$f^{(E)}K = \left\{ v, T_{f^{(e)}K}(v), I_{f^{(e)}K}(v), F_{f^{(e)}K}(v) : v \in V, e \in E \right\}.$$

Define

$$(\tau_{t1})' = \left\{ T_{f^{(e)}K}(V) \right\}_{e \in E}, \\ (\tau_{t2})' = \left\{ I_{f^{(e)}K}(V) \right\}_{e \in E} \text{ and} \\ (\tau_{t3})' = \left\{ F_{f^{(e)}K}(V) \right\}_{e \in E} \text{ are FSTs on } (V, E).$$

Then $(\tau_{t1})' \cup (\tau_{t3})'$ and $(\tau_{t2})' \cup (\tau_{t3})'$ are not FSTs on (V, E) .

Proposition 4.11 is illustrated by the following example.

Example 4.12 Consider Example 3.11. Here $(\tau_t)' = \{\phi_t, 1_t, (K_1)^c, (K_2)^c, (K_3)^c\}$ where the NSWs $(K_1)^c, (K_2)^c, (K_3)^c$ over V are defined as

$$(K_1)^c = \begin{cases} f^{(e_1)}(K_1)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \\ f^{(e_2)}(K_1)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (.2, .4, .7) \rangle \} \\ f^{(e_3)}(K_1)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \\ f^{(e_4)}(K_1)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \end{cases} ;$$

$$(K_2)^c = \begin{cases} f^{(e_1)}(K_2)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \\ f^{(e_2)}(K_2)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (.5, .3, .8) \rangle \} \\ f^{(e_3)}(K_2)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \\ f^{(e_4)}(K_2)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \end{cases} \text{ and}$$

$$(K_3)^c = \begin{cases} f^{(e_1)}(K_3)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \\ f^{(e_2)}(K_3)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (.5, .4, .7) \rangle \} \\ f^{(e_3)}(K_3)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \\ f^{(e_4)}(K_3)^c = \{ \langle g_1, (1, 1, 0) \rangle, \langle g_2, (1, 1, 0) \rangle, \langle g_3, (1, 1, 0) \rangle \} \end{cases} .$$

Here $\phi_t \cup (K_1)^c = (K_1)^c$, $\phi_t \cup (K_2)^c = (K_2)^c$, $\phi_t \cup (K_3)^c = (K_3)^c$, $1_t \cup (K_1)^c = 1_t$, $1_t \cup (K_2)^c = 1_t$, $1_t \cup (K_3)^c = 1_t$, $(K_1)^c \cup (K_2)^c = (K_3)^c$, $(K_1)^c \cup (K_3)^c = (K_3)^c$, $(K_2)^c \cup (K_3)^c = (K_3)^c$.

Then $(K_1)^c, (K_2)^c$ and $(K_3)^c$ are NSSPOs.

Thus $(V, (\tau_t)', E)$ is a NSSPTS over (V, E) .

Then

$$(\tau_{i1})' = \langle \langle e_1, (0,0,0), (1,1,1), (1,1,1), (1,1,1), (1,1,1) \rangle, \langle e_2, (0,0,0), (1,1,1), (1,1,2), (1,1,5), (1,1,5) \rangle, \langle e_3, (0,0,0), (1,1,1), (1,1,1), (1,1,1), (1,1,1) \rangle, \langle e_4, (0,0,0), (1,1,1), (1,1,1), (1,1,1), (1,1,1) \rangle \rangle,$$

and so on, are FSTs on (V, E) .

But, $(\tau_{i1})' \cup (\tau_{i3})'$ and $(\tau_{i2})' \cup (\tau_{i3})'$ are not FSTs on (V, E) .

Proposition 4.13 Let $(V, (\tau_i)', E)$ be a NSSPTS over (V, E) . Then

$$\begin{aligned}({}^e \tau_{i1})' &= \left\{ T_{f^{(e)}_K}(V) : K \in (\tau_i)' \right\}, \\({}^e \tau_{i2})' &= \left\{ I_{f^{(e)}_K}(V) : K \in (\tau_i)' \right\} \text{ and} \\({}^e \tau_{i3})' &= \left\{ F_{f^{(e)}_K}(V) : K \in (\tau_i)' \right\}\end{aligned}$$

for each $e \in E$, define FTs on (V, E) .

Proof. Follows from Proposition 4.8.

Remark 4.14 The following example illustrates that the converse of Proposition 4.13 is not true.

Example 4.15 Consider Example 4.10.

Then

$$\begin{aligned}({}^{e_1} \tau_{i1})^c &= \left\{ T_{f^{(e_1)}_{\phi_i}}(V), T_{f^{(e_1)}_{1_i}}(V), T_{f^{(e_1)}_{(K_1)^c}}(V), T_{f^{(e_1)}_{(K_2)^c}}(V) \right\}, \\({}^{e_1} \tau_{i2})^c &= \left\{ I_{f^{(e_1)}_{\phi_i}}(V), I_{f^{(e_1)}_{1_i}}(V), I_{f^{(e_1)}_{(K_1)^c}}(V), I_{f^{(e_1)}_{(K_2)^c}}(V) \right\} \text{ and} \\({}^{e_1} \tau_{i3})^c &= \left\{ F_{f^{(e_1)}_{\phi_i}}(V), F_{f^{(e_1)}_{1_i}}(V), F_{f^{(e_1)}_{(K_1)^c}}(V), F_{f^{(e_1)}_{(K_2)^c}}(V) \right\} \text{ are FTs on } V,\end{aligned}$$

where

$$({}^{e_1} \tau_{i1})' = \{(0,0,0), (1,1,1), (1,1,1), (1,1,1)\} \text{ and so on.}$$

Consequently, $\{({}^{e_2} \tau_{i1})^c, ({}^{e_2} \tau_{i2})^c, ({}^{e_2} \tau_{i3})^c\}$ and $\{({}^{e_3} \tau_{i1})^c, ({}^{e_3} \tau_{i2})^c, ({}^{e_3} \tau_{i3})^c\}$ are fuzzy tritopologies on V .

Thus $(\tau_i)' = \{\phi, 1_i, (K_1)^c, (K_2)^c\}$ is not NSSPT over (V, E) , since $(K_1)' \cap (K_2)' \notin (\tau_i)'$.

Definition 4.16 Let $(V, (\tau_i)', E)$ be a NSSPTS over (V, E) and $L \in NSWS(V, E)$ be any NSWS. Then the neutro-supra spot absolute closure of L is denoted by \tilde{L}_- and defined as

(i) $\tilde{L}_- = \{K : K \in NSSPOS \& K \subseteq L\}$ i.e., the intersection of all neutro-supra spot open subsets of L .

(ii)

$$\tilde{L}_- = \left\{ \left(\tilde{T}_{-f^{(e)}_L}, \tilde{I}_{-f^{(e)}_L}, \tilde{F}_{-f^{(e)}_L} \right)_{e \in E} \right\} = \left\{ \left(\min_i T_{f^{(e)}_{K_i}}, \min_i I_{f^{(e)}_{K_i}}, \max_i F_{f^{(e)}_{K_i}} \right)_{e \in E} : K_i \in (\tau_i)' \& f^{(e)}_{K_i} \subseteq f^{(e)}_L \right\}.$$

Example 4.17 Consider Example 3.16. Here $(\tau_i)' = \{\phi, 1_i, (K_1)^c, (K_2)^c, (K_3)^c\}$ where the NSWSs $(K_1)^c, (K_2)^c, (K_3)^c$ over V are defined as

$$(K_1)^c = \begin{cases} f^{(e_1)}_{(K_1)^c} = \{< g_1, (1,1,0) >, < g_2, (1,1,0) >\} \\ f^{(e_2)}_{(K_1)^c} = \{< g_1, (.2, .6, .9) >, < g_2, (1,1,0) >\} \\ f^{(e_3)}_{(K_1)^c} = \{< g_1, (1,1,0) >, < g_2, (1,1,0) >\} \end{cases};$$

$$(K_2)^c = \begin{cases} f^{(e_1)}(K_2)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle \} \\ f^{(e_2)}(K_2)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle \} \\ f^{(e_3)}(K_2)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (.8, .9, .6) \rangle \} \end{cases} \quad \text{and}$$

$$(K_3)^c = \begin{cases} f^{(e_1)}(K_3)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle \} \\ f^{(e_2)}(K_3)^c = \{ \langle g_1, (.5, .7, .4) \rangle, \langle g_2, (1,1,0) \rangle \} \\ f^{(e_3)}(K_3)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle \} \end{cases} .$$

Here $\phi_t \cup (K_1)^c = (K_1)^c$, $\phi_t \cup (K_2)^c = (K_2)^c$, $\phi_t \cup (K_3)^c = (K_3)^c$, $1_t \cup (K_1)^c = 1_t$, $1_t \cup (K_2)^c = 1_t$, $1_t \cup (K_3)^c = 1_t$, $(K_1)^c \cup (K_2)^c = 1_t$, $(K_1)^c \cup (K_3)^c = (K_3)^c$, $(K_2)^c \cup (K_3)^c = 1_t$.

Then $(K_1)^c, (K_2)^c$ and $(K_3)^c$ are NSSPOSs.

Thus $(V, (\tau_t)', E)$ is a NSSPTS over (V, E) .

Let $L \in NSW(V, E)$ be any NSW defined as

$$L = \begin{cases} f^{(e_1)}(L) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle \} \\ f^{(e_2)}(L) = \{ \langle g_1, (.1, .5, .9) \rangle, \langle g_2, (1,1,0) \rangle \} \\ f^{(e_3)}(L) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle \} \end{cases} .$$

Then $1_t, (K_1)^c, (K_3)^c \supseteq L$.

Thus $\tilde{L}_- = 1_t \cup (K_1)^c \cup (K_3)^c = (K_1)^c$.

Also,

$$f^{(e_1)}(K_1)^c, f^{(e_1)}(K_2)^c, f^{(e_1)}(K_3)^c \supseteq f^{(e_1)}(L),$$

$$f^{(e_2)}(K_1)^c, f^{(e_2)}(K_2)^c, f^{(e_2)}(K_3)^c \supseteq f^{(e_2)}(L) \quad \text{and}$$

$$f^{(e_3)}(K_1)^c, f^{(e_3)}(K_3)^c \supseteq f^{(e_3)}(L).$$

Then

$$(\tilde{T}_{-f^{(e_1)}(L)}, \tilde{I}_{-f^{(e_1)}(L)}, \tilde{F}_{-f^{(e_1)}(L)}) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle \},$$

$$(\tilde{T}_{-f^{(e_2)}(L)}, \tilde{I}_{-f^{(e_2)}(L)}, \tilde{F}_{-f^{(e_2)}(L)}) = \{ \langle g_1, (.2, .6, .9) \rangle, \langle g_2, (1,1,0) \rangle \} \quad \text{and}$$

$$(\tilde{T}_{-f^{(e_3)}(L)}, \tilde{I}_{-f^{(e_3)}(L)}, \tilde{F}_{-f^{(e_3)}(L)}) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle \} .$$

Thus $\left\{ (\tilde{T}_{-f^{(e)}(L)}, \tilde{I}_{-f^{(e)}(L)}, \tilde{F}_{-f^{(e)}(L)}) \right\}_{e \in E} = (K_1)^c$.

Theorem 4.18 Let $(V, (\tau_t)', E)$ be a NSSPTS over (V, E) and $L, S \in NSW(V, E)$ be any two NSWs.

Then,

(i) $L \subseteq \tilde{L}_-$ and \tilde{L}_- is the smallest NSSPOS.

(ii) $L \subseteq S \Rightarrow \tilde{L}_- \subseteq \tilde{S}_-$.

(iii) \tilde{L}_- is a NSSPOS i.e., $\tilde{L}_- \in (\tau_t)^c$.

(iv) L is a NSSPOS iff $\tilde{L}_- = L$.

(v) $(\tilde{L}_-)_- = \tilde{L}_-$.

$$(vi) \quad (\tilde{\phi}_{t-})_- = \tilde{\phi}_{t-} \quad \text{and} \quad (\tilde{l}_{t-})_- = \tilde{l}_{t-}.$$

$$(vii) \quad (\tilde{L} \cup \tilde{S})_- = \tilde{L}_- \cup \tilde{S}_-.$$

$$(ix) \quad (\tilde{L} \cap \tilde{S})_- \subseteq \tilde{L}_- \cap \tilde{S}_-.$$

Proof. Let $(V, (\tau_t)', E)$ be a NSSPTS over (V, E) and $L, S \in NSWs(V, E)$ be any two NSWs.

(i) Follows from Definition 4.16.

(ii) Let $L \subseteq S$. Then $L \subseteq \tilde{L}_-$ and $P \subseteq \tilde{S}_- \Rightarrow L \subseteq S \subseteq \tilde{S}_- \Rightarrow L \subseteq \tilde{S}_-$.

Since \tilde{L}_- is the smallest NSSPOS containing L , hence $\tilde{L}_- \subseteq \tilde{S}_-$.

(iii) Follows from Definition 4.16.

(iv) Let L be a NSSPOS. Then \tilde{L}_- is the smallest NSSPOS which containing L is equal to L .

Hence $\tilde{L}_- = L$.

Conversely, assume that $\tilde{L}_- = L$.

By (iii), $\tilde{L}_- \in (\tau_t)^c$.

Then L is a NSSPOS.

(v) Let $\tilde{L}_- = K$. Then $K \in (\tau_t)^c$ iff $\tilde{K}_- = K$.

Thus $(\tilde{L}_-)_- = \tilde{L}_-$.

(vi) Since $\phi_t, l_t \in (\tau_t)^c$ and by (iv), hence $(\tilde{\phi}_{t-})_- = \tilde{\phi}_{t-}$ and $(\tilde{l}_{t-})_- = \tilde{l}_{t-}$.

(vii) $L \subseteq L \cup S$ and $S \subseteq L \cup S$.

$$\Rightarrow \tilde{L}_- \subseteq (\tilde{L} \cup \tilde{S})_- \quad \text{and} \quad \tilde{S}_- \subseteq (\tilde{L} \cup \tilde{S})_-.$$

$$\Rightarrow \tilde{L}_- \cup \tilde{S}_- \subseteq (\tilde{L} \cup \tilde{S})_-.$$

Also, $L \subseteq \tilde{L}_-$ and $S \subseteq \tilde{S}_-$.

Then $L \cup S \subseteq \tilde{L}_- \cup \tilde{S}_-$.

Since $L \cup S \subseteq (\tilde{L} \cup \tilde{S})_-$ and it is the smallest NSSPOS, then $(\tilde{L} \cup \tilde{S})_- \subseteq \tilde{L}_- \cup \tilde{S}_-$.

Hence $(\tilde{L} \cup \tilde{S})_- = \tilde{L}_- \cup \tilde{S}_-$.

(vii) $L \cap S \subseteq L$ and $L \cap S \subseteq S$.

$$\Rightarrow (\tilde{L} \cap \tilde{S})_- \subseteq \tilde{L}_- \quad \text{and} \quad (\tilde{L} \cap \tilde{S})_- \subseteq \tilde{S}_-.$$

$$\Rightarrow (\tilde{L} \cap \tilde{S})_- \subseteq \tilde{L}_- \cap \tilde{S}_-.$$

Definition 4.19 Let $(V, (\tau_t)', E)$ be a NSSPTS over (V, E) where $(\tau_t)'$ is a NSSPT over (V, E) and $L \in NSWs(V, E)$ be any NSW. Then the subspace topology on NSSPTS is denoted by $(\tau_t^L)'$ and defined as $(\tau_t^L)' = \{L \cap K_i : K_i \in (\tau_t)'\}$ and $\phi_t, l_t \in (\tau_t^L)'$. Thus $(V, (\tau_t^L)', E)$ is a neutro-supra spot subspace topological space (NSSPSTS) of $(V, (\tau_t)', E)$, where $(\tau_t^L)'$ is also a NSSPT over (V, E) .

Example 4.20 Consider Example 3.19. Here $(\tau_t)' = \{\phi_t, l_t, (K_1)^c, (K_2)^c, (K_3)^c\}$ where the NSWs $(K_1)^c, (K_2)^c, (K_3)^c$ over V are defined as

$$(K_1)^c = \left\{ \begin{aligned} f^{(e_1)}(K_1)^c &= \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_2)}(K_1)^c &= \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (.5, .6, .1) \rangle, \langle g_3, (1,1,0) \rangle \} \end{aligned} \right. ;$$

$$(K_2)^c = \left\{ \begin{aligned} f^{(e_1)}(K_2)^c &= \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_2)}(K_2)^c &= \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (.8, .3, .4) \rangle, \langle g_3, (1,1,0) \rangle \} \end{aligned} \right. \quad \text{and}$$

$$(K_3)^c = \begin{cases} f^{(e_1)}(K_3)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_2)}(K_3)^c = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (.8, .6, .1) \rangle, \langle g_3, (1,1,0) \rangle \} \end{cases}.$$

Here $\phi_i \cup (K_1)^c = (K_1)^c$, $\phi_i \cup (K_2)^c = (K_2)^c$, $\phi_i \cup (K_3)^c = (K_3)^c$, $1_i \cup (K_1)^c = 1_i$, $1_i \cup (K_2)^c = 1_i$, $1_i \cup (K_3)^c = 1_i$, $(K_1)^c \cup (K_2)^c = (K_3)^c$, $(K_1)^c \cup (K_3)^c = (K_3)^c$, $(K_2)^c \cup (K_3)^c = (K_3)^c$.

Then $(K_1)^c, (K_2)^c$ and $(K_3)^c$ are NSSPOSs.

Thus $(V, (\tau_i)', E)$ is a NSSPTS over (V, E) .

Define $L_1 \in NSW(S(V, E))$ as

$$L_1 = \begin{cases} f^{(e_1)}(L_1) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_2)}(L_1) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (.6, .7, .7) \rangle, \langle g_3, (1,1,0) \rangle \} \end{cases}.$$

Thus we denote $L_1 \cap \phi_i = \phi_i$, $L_1 \cap 1_i = L_1$, $L_1 \cap (K_1)^c = L_2$, $L_1 \cap (K_2)^c = L_3$, $L_1 \cap (K_3)^c = L_4$.

Then the NSWs L_2, L_3, L_4 over V is defined as

$$L_2 = \begin{cases} f^{(e_1)}(L_2) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_2)}(L_2) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (.5, .6, .7) \rangle, \langle g_3, (1,1,0) \rangle \} \end{cases};$$

$$L_3 = \begin{cases} f^{(e_1)}(L_3) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_2)}(L_3) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (.6, .3, .7) \rangle, \langle g_3, (1,1,0) \rangle \} \end{cases} \text{ and}$$

$$L_4 = \begin{cases} f^{(e_1)}(L_4) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (1,1,0) \rangle, \langle g_3, (1,1,0) \rangle \} \\ f^{(e_2)}(L_4) = \{ \langle g_1, (1,1,0) \rangle, \langle g_2, (.6, .6, .7) \rangle, \langle g_3, (1,1,0) \rangle \} \end{cases}.$$

Then $(\tau_i^L)' = \{\phi_i, 1_i, L_1, L_2, L_3, L_4\}$ is a NSSPST over (V, E) .

Thus $(V, (\tau_i^L)', E)$ is a NSSPSTS of $(V, (\tau_i)', E)$.

5. Separation Axioms

This part is split into two parts as separation axioms on NSPTS and NSSPTS are defined with examples.

5.1. Separation Axioms on NSPTS

Definition 5.1.1 Let (V, τ_i, E) be a NSPTS over (V, E) where τ_i is a NSPT over (V, E) . Let p and q be any distinct NSPs. If there exists NSPOSs R and S such that

$$p \in R \text{ and } p \cap S = \phi_i \quad \text{or}$$

$$q \in S \text{ and } q \cap R = \phi_i,$$

Then (V, τ_i, E) is said to be a neutro-spot T_0 -space.

Definition 5.1.2 Let (V, τ_i, E) be a NSPTS over (V, E) where τ_i is a NSPT over (V, E) . Let p and q be any distinct NSPs. If there exists NSPOSs R and S such that

$$p \in R \text{ and } p \cap S = \phi_i \quad \text{and}$$

$$q \in S \text{ and } q \cap R = \phi_i,$$

Then (V, τ_t, E) is said to be a neutro-spot T_1 -space.

Definition 5.1.3 Let (V, τ_t, E) be a NSPTS over (V, E) where τ_t is a NSPT over (V, E) . Let p and q be any distinct NSPs. If there exists NSPOSS R and S such that

$$p \in R, \quad q \in S \quad \text{and} \quad R \cap S = \phi_t,$$

Then (V, τ_t, E) is said to be a neutro-spot T_2 -space.

Theorem 5.1.4 Let (V, τ_t, E) be a NSPTS over (V, E) . Every neutro-spot T_2 -space is also a neutro-spot T_1 -space and every neutro-spot T_1 -space is also a neutro-spot T_0 -space.

Proof. Follows from Definitions 5.1.1, 5.1.2, and 5.1.3.

Example 5.1.5 Let $V = \{v_1, v_2\}$, $E = \{e_1, e_2\}$. Let $\tau_t = \{\phi_t, 1_t, R_1, R_2, R_3, R_4\}$ where the NSPs R_1, R_2, R_3, R_4 over V are defined as

$$R_1 = \begin{cases} f^{(e_1)} R_1 = \{<v_1, (0, 0, 1)>, <v_2, (0, 0, 1)>\} \\ f^{(e_2)} R_1 = \{<v_1, (.7, .4, .2)>, <v_2, (0, 0, 1)>\} \end{cases} ;$$

$$R_2 = \begin{cases} f^{(e_1)} R_2 = \{<v_1, (0, 0, 1)>, <v_2, (0, 0, 1)>\} \\ f^{(e_2)} R_2 = \{<v_1, (.3, .5, .6)>, <v_2, (0, 0, 1)>\} \end{cases} ;$$

$$R_3 = \begin{cases} f^{(e_1)} R_3 = \{<v_1, (0, 0, 1)>, <v_2, (0, 0, 1)>\} \\ f^{(e_2)} R_3 = \{<v_1, (.3, .4, .6)>, <v_2, (0, 0, 1)>\} \end{cases} \quad \text{and}$$

$$R_4 = \begin{cases} f^{(e_1)} R_4 = \{<v_1, (0, 0, 1)>, <v_2, (0, 0, 1)>\} \\ f^{(e_2)} R_4 = \{<v_1, (0, 0, 1)>, <v_2, (.8, .4, .5)>\} \end{cases}.$$

Here $\phi_t \cap R_1 = \phi_t$, $\phi_t \cap R_2 = \phi_t$, $\phi_t \cap R_3 = \phi_t$, $\phi_t \cap R_4 = \phi_t$, $1_t \cap R_1 = R_1$, $1_t \cap R_2 = R_2$, $1_t \cap R_3 = R_3$, $1_t \cap R_4 = R_4$, $R_1 \cap R_2 = R_3$, $R_1 \cap R_3 = R_3$, $R_1 \cap R_4 = \phi_t$, $R_2 \cap R_3 = R_3$, $R_2 \cap R_4 = \phi_t$, $R_3 \cap R_4 = \phi_t$.

Then R_1, R_2, R_3 and R_4 are NSPOSSs.

Thus (V, τ_t, E) is a NSPTS over (V, E) .

Let p and q be any distinct NSPs which are defined as

$$p = \begin{cases} f^{(e_1)} p = \{<v_1, (0, 0, 1)>, <v_2, (0, 0, 1)>\} \\ f^{(e_2)} p = \{<v_1, (.1, .4, .7)>, <v_2, (0, 0, 1)>\} \end{cases} \quad \text{and}$$

$$q = \begin{cases} f^{(e_1)} q = \{<v_1, (0, 0, 1)>, <v_2, (0, 0, 1)>\} \\ f^{(e_2)} q = \{<v_1, (0, 0, 1)>, <v_2, (.2, .1, .5)>\} \end{cases}.$$

Hence (V, τ_t, E) is a neutro-spot T_2 -space, also a neutro-spot T_1 -space and a neutro-spot T_0 -space.

5.2. Separation Axioms on NSSPTS

Definition 5.2.1 Let $(V, (\tau_t)', E)$ be a NSSPTS over (V, E) where $(\tau_t)'$ is a NSSPT over (V, E) . Let p and q be any distinct NSPs. If there exists NSSPOSS R and S such that

$$p \in R \quad \text{and} \quad p \cap \cong \phi_t \quad \text{or}$$

$$q \in S \text{ and } q \cap R \cong \phi_t,$$

Then $(V, (\tau_t)', E)$ is said to be a neutro-supra spot T_0 -space.

Definition 5.2.2 Let $(V, (\tau_t)', E)$ be a NSSPTS over (V, E) where $(\tau_t)'$ is a NSSPT over (V, E) . Let p and q be any distinct NSPs. If there exists NSSPOs R and S such that

$$p \in R \text{ and } p \cap S \cong \phi_t \text{ and}$$

$$q \in S \text{ and } q \cap R \cong \phi_t,$$

Then $(V, (\tau_t)', E)$ is said to be a neutro-supra spot T_1 -space.

Definition 5.2.3 Let $(V, (\tau_t)', E)$ be a NSSPTS over (V, E) where $(\tau_t)'$ is a NSSPT over (V, E) . Let p and q be any distinct NSPs. If there exists NSSPOs R and S such that

$$p \in R, q \in S \text{ and } R \cap S \cong \phi_t,$$

Then $(V, (\tau_t)', E)$ is said to be a neutro-supra spot T_2 -space.

Theorem 5.2.4 Let $(V, (\tau_t)', E)$ be a NSSPTS over (V, E) . Every neutro-supra spot T_2 -space is also a neutro-supra spot T_1 -space and every neutro-supra spot T_1 -space is also a neutro-supra spot T_0 -space.

Proof. Follows from Definitions 5.2.1, 5.2.2, and 5.2.3.

Example 5.2.5 Consider Example 5.1.5. Here $(\tau_t)' = \{\phi_t, 1_t, (R_1)^c, (R_2)^c, (R_3)^c, (R_4)^c\}$ where the NSWs $(R_1)^c, (R_2)^c, (R_3)^c, (R_4)^c$ over V is defined as

$$(R_1)^c = \begin{cases} f^{(e_1)}(R_1)^c = \{<v_1, (1, 1, 0)>, <v_2, (1, 1, 0)>\} \\ f^{(e_2)}(R_1)^c = \{<v_1, (.2, .6, .7)>, <v_2, (1, 1, 0)>\} \end{cases};$$

$$(R_2)^c = \begin{cases} f^{(e_1)}(R_2)^c = \{<v_1, (1, 1, 0)>, <v_2, (1, 1, 0)>\} \\ f^{(e_2)}(R_2)^c = \{<v_1, (.6, .5, .3)>, <v_2, (1, 1, 0)>\} \end{cases};$$

$$(R_3)^c = \begin{cases} f^{(e_1)}(R_3)^c = \{<v_1, (1, 1, 0)>, <v_2, (1, 1, 0)>\} \\ f^{(e_2)}(R_3)^c = \{<v_1, (.6, .6, .3)>, <v_2, (1, 1, 0)>\} \end{cases} \text{ and}$$

$$(R_4)^c = \begin{cases} f^{(e_1)}(R_4)^c = \{<v_1, (1, 1, 0)>, <v_2, (1, 1, 0)>\} \\ f^{(e_2)}(R_4)^c = \{<v_1, (1, 1, 0)>, <v_2, (.5, .6, .8)>\} \end{cases}.$$

Here $\phi_t \cup (R_1)^c = (R_1)^c$, $\phi_t \cup (R_2)^c = (R_2)^c$, $\phi_t \cup (R_3)^c = (R_3)^c$, $\phi_t \cup (R_4)^c = (R_4)^c$, $1_t \cup (R_1)^c = 1_t$, $1_t \cup (R_2)^c = 1_t$, $1_t \cup (R_3)^c = 1_t$, $1_t \cup (R_4)^c = 1_t$, $(R_1)^c \cup (R_2)^c = (R_3)^c$, $(R_1)^c \cup (R_3)^c = (R_3)^c$, $(R_1)^c \cup (R_4)^c = 1_t$, $(R_2)^c \cup (R_3)^c = (R_3)^c$, $(R_2)^c \cup (R_4)^c = 1_t$, $(R_3)^c \cup (R_4)^c = 1_t$.

Then $(R_1)^c, (R_2)^c, (R_3)^c$ and $(R_4)^c$ are NSSPOs.

Thus $(V, (\tau_t)', E)$ is a NSSPTS over (V, E) .

Let p and q be any distinct NSPs which are defined as

$$p = \begin{cases} f^{(e_1)}(p) = \{<v_1, (0, 0, 1)>, <v_2, (0, 0, 1)>\} \\ f^{(e_2)}(p) = \{<v_1, (.001, .002, .987)>, <v_2, (0, 0, 1)>\} \end{cases} \text{ and}$$

$$q = \begin{cases} f^{(e_1)}(q) = \{<v_1, (0, 0, 1)>, <v_2, (0, 0, 1)>\} \\ f^{(e_2)}(q) = \{<v_1, (0, 0, 1)>, <v_2, (.002, .001, .998)>\} \end{cases}.$$

Hence $(V, (\tau_i)', E)$ is a neutro-supra spot T_2 -space, also a neutro-supra spot T_1 -space and a neutro-supra spot T_0 -space.

6. DM Problem to Detect the Impact on COVID-19

In this part, the DM problem explains the COVID-19 situation and detected its impact on corona virus patients to undergoing exact treatment for them according to their medical report. The process of evaluation is pointed out in the algorithm and formula defined for computing the result.

Definition 6.1 Let $(V, \tau_{i1}, E1)$ and $(V, \tau_{i2}, E2)$ be two NSPTSs over $(V, E1)$ and $(V, E2)$, respectively where V is the set of risk factors and $E1, E2$ are two different parametric sets of medical issues. Let M be a corona virus patient, were $^{E1}M \in \tau_{i1}$ and $^{E2}M \in \tau_{i2}$ are two NSPs.

Then for each $v \in V$, the Risk State Value (RSV) of $M(v)$ is given as:

$$RSV[M(v)^{(E1, E2)}] = \left\| \left(\frac{A-C}{2} \right) \times \left(1 - \frac{B}{2} \right) \right\|, \quad (6.1.1)$$

where

$$\begin{aligned} A &= \sum_{i \in I} T_{f(s_i)M}(v) + \sum_{j \in I} T_{f(r_j)M}(v), \\ B &= \sum_{i \in I} I_{f(s_i)M}(v) + \sum_{j \in I} I_{f(r_j)M}(v), \\ C &= \sum_{i \in I} F_{f(s_i)M}(v) + \sum_{j \in I} F_{f(r_j)M}(v) \end{aligned}$$

and for all $s_i \in E1$, $r_j \in E2$.

Then $\forall v \in V$, the Total Risk State Value (TRSV) of M is given as:

$$TRSV(M) = \sum_n \left(RSV[M(v_n)^{(E1, E2)}] \right), \quad (6.1.2)$$

Algorithm

Step 1: List the set of risk factors $v \in V$.

Step 2: List two different parametric sets, say $E1$ and $E2$, where $E1$ represents the symptoms of COVID-19 and $E2$ represents the pre-medical issues.

Step 3: Pick out the people affected by COVID-19, say M .

Step 4: Go through the medical status of each patients.

Step 5: Test their corona virus symptoms ($E1$) and categories its risk factors (V).

Step 6: Collect those data in the form of NSPs.

Step 7: Define a NSPT τ_{i1} and so $(V, \tau_{i1}, E1)$ is a NSPTS over $(V, E1)$.

Step 8: Check the pre-medical issues of each patient ($E2$) and categories its risk factors (V).

Step 9: Collect those data in the form of NSPs, which satisfies the condition of NSPT τ_{i2} .

Step 10: Define a NSPTS $(V, \tau_{i2}, E2)$ over $(V, E2)$.

Step 11: Use the formula 6.1.1 to calculate the RSV of $M(v)$, for each $v \in V$, and tabulate it.

Step 12: Use the formula 6.1.2 to calculate the TRSV of M , for all $v \in V$, and tabulate it.

Step 13: (i) If $\text{TRSV}(M)$ is greater than 2.5, then M undergoes ventilation treatment.

(ii) If $\text{TRSV}(M)$ lies between 1.5 and 2.5, then M needs to be hospitalized.

(iii) If $\text{TRSV}(M)$ is less than 1.5, then M should be self-isolated at home.

Step 14: Step 13 is also concluded according to the risk factor V .

Step 15: If two or more patients have the same TRSV , then each patient required the same treatment.

Problem 6.2 The survey on COVID-19 patients tested in a particular area. Its low-level risk scenarios are talking to someone face to face, walking, jogging, cycling, etc., The medium level risk scenarios are grocery shopping. The high-level risk scenarios are restaurants, public bathrooms, indoor spaces, and common areas. The very high-level risk scenarios are schools, colleges, parties, weddings, cinemas, and workplaces. The people who are affected by COVID-19 are tested by the doctors according to their symptoms and are under investigation. Also, considered other medical issues. They are categorized with some risk factors. Our problem is to recover them quickly by giving appropriate treatment for those affected people.

1. Let $V = \{v_1, v_2, v_3, v_4\}$ be the set of risk factors, where v_1 – low risk, v_2 – medium risk, v_3 – high risk and v_4 – very high risk.

2. Let $E1 = \{s_1, s_2, s_3, s_4\}$ and $E2 = \{r_1, r_2, r_3, r_4\}$ be two different parametric sets. The set $E1$ represents the symptoms of COVID-19 such as s_1 – fever, s_2 – dry cough, s_3 – chest pain, and s_4 – shortness of breath. The set $E2$ represents pre-medical issues such as r_1 – diabetes, r_2 – blood pressure, r_3 – cardiac diseases, and r_4 – respiratory diseases.

3. Let M_1, M_2, M_3 and M_4 be the people affected by COVID-19.

4. Go through the medical status of each patient.

5. First test their corona virus symptoms ($E1$) and categories its risk factors (V).

6. Those data are collected in the form of NSPs, are as follows:

$$\begin{aligned}
 {}^{E1}M_1 &= \begin{cases} f^{(s_1)}M_1 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(s_2)}M_1 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(s_3)}M_1 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(s_4)}M_1 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (.6, .5, .2) \rangle \} \end{cases} ; \\
 {}^{E1}M_2 &= \begin{cases} f^{(s_1)}M_2 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (.5, .2, .6) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(s_2)}M_2 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(s_3)}M_2 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(s_4)}M_2 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \end{cases} ; \\
 {}^{E1}M_3 &= \begin{cases} f^{(s_1)}M_3 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(s_2)}M_3 = \{ \langle v_1, (.3, .2, .8) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(s_3)}M_3 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(s_4)}M_3 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \end{cases} \text{ and} \\
 {}^{E1}M_4 &= \begin{cases} f^{(s_1)}M_4 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(s_2)}M_4 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(s_3)}M_4 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (.8, .6, .3) \rangle \} \\ f^{(s_4)}M_4 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \end{cases} .
 \end{aligned}$$

7. Then $\tau_{t1} = \{ {}^{E1}M_1, {}^{E1}M_2, {}^{E1}M_3, {}^{E1}M_4 \}$ is a NSPT.

Thus $(V, \tau_{t1}, E1)$ is a NSPTS over $(V, E1)$.

8. Then, check the pre-medical issues of each patients ($E2$) and categories its risk factors (V).

9. Those data are collected in the form of NSPs, are as follows:

$$\begin{aligned}
 {}^{E2}M_1 &= \begin{cases} f^{(r_1)}M_1 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_2)}M_1 = \{ \langle v_1, (.6, .7, .4) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_3)}M_1 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_4)}M_1 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \end{cases} ; \\
 {}^{E2}M_2 &= \begin{cases} f^{(r_1)}M_2 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (.4, .3, .4) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_2)}M_2 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_3)}M_2 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_4)}M_2 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \end{cases} ; \\
 {}^{E2}M_3 &= \begin{cases} f^{(r_1)}M_3 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_2)}M_3 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_3)}M_3 = \{ \langle v_1, (.4, .4, .2) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_4)}M_3 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \end{cases} \text{ and} \\
 {}^{E2}M_4 &= \begin{cases} f^{(r_1)}M_4 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_2)}M_4 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_3)}M_4 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (0, 0, 1) \rangle, \langle v_4, (0, 0, 1) \rangle \} \\ f^{(r_4)}M_4 = \{ \langle v_1, (0, 0, 1) \rangle, \langle v_2, (0, 0, 1) \rangle, \langle v_3, (.7, .5, .3) \rangle, \langle v_4, (0, 0, 1) \rangle \} \end{cases} .
 \end{aligned}$$

10. Then $\tau_{i2} = \{{}^{E2}M_1, {}^{E2}M_2, {}^{E2}M_3, {}^{E2}M_4\}$ is a NSPT.

Thus $(V, \tau_{i2}, E2)$ is a NSPTS over $(V, E2)$.

11. By using the formula 6.1.1, the RSV of $M(v)$ are calculated, for each $v \in V$.

These values are tabulated in the following table.

Table 6.2.1. RSV Table

	v1	v2	v3	v4
M_1	1.755	0	0	1.43
M_2	0	1.75	0	0
M_3	1.625	0	0	0
M_4	0	0	1.59	1.855

12. By using the formula 6.1.2, the TRSV of M are calculated, for all $v \in V$.

These values are tabulated in the following table.

Table 6.2.2. TRSV Table

	v1	v2	v3	v4	TRSV
M_1	1.755	0	0	1.43	3.185
M_2	0	1.75	0	0	1.75
M_3	1.625	0	0	0	1.625
M_4	0	0	1.59	1.855	3.445

13. Here $\text{TRSV}(M_1) = 3.185$ and $\text{TRSV}(M_4) = 3.445$, which are greater than 2.5.

Thus they both are at a very high-risk stage.

So, they each should be treated in ventilation for quick recovery.

Next, $\text{TRSV}(M_2) = 1.75$ and $\text{TRSV}(M_3) = 1.625$, which are greater than 1.5.

Here M_2 is under medium risk stage, and so need to be hospitalized.

Even though the TRSV of M_3 lies between 1.5 and 2.5, M_3 is under the low-risk stage.

So, M_3 should be self-isolated at home itself.

7. Conclusions

In this paper, NSPTS and NSSPTS are introduced and defined a subspace topology on them. Along with its absolute interior of NSPT and absolute closure of NSSPT are also defined. Few properties are examined with illustrative examples. This study extended to introduce a concept of separation axioms of NSPTS and NSSPTS are defined as neutro-spot $T_{i=0,1,2}$ -spaces and neutro-supra spot $T_{i=0,1,2}$ -spaces respectively with related examples. Additionally, the DM problem explains the COVID-19 situation and detected its impact on corona virus patients to undergoing exact treatment for them according to their medical report. The process of evaluation is pointed out in the algorithm and formula defined for computing the result. The appropriate treatment is provided for affected people as per the estimated value. Some more practical applications of such types of topologies can be explored for future work. Many more sets like open sets, closed sets, rough sets, crisp sets, etc., can be developed on NSPTS and NSSPTS. Later these concepts will step ahead on multi-criteria DM problems by upcoming researchers.

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A Mathematical Model to Analyze the Role of Uncertain and Indeterminate Factors in the Spread of Pandemics like COVID-19 Using Neutrosophy: A Case Study of India

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Abstract: COVID-19 has been declared as pandemic by WHO. This disease is caused by severe acute respiratory syndrome coronavirus 2 (SARS-CoV2). The spread of this virus from Wuhan, China to the rest of the world has led to increasing concerns as it is considered a threat to mankind. The present work in situation analysis using neutrosophy identifies factors which should be taken care of so that the spread of this virus can be contained. Known factors as put forward by experts together with indeterminate factors which are still not taken into consideration are used to model the situation. This work using neutrosophic cognitive maps proves mathematically how these indeterminate and unknown factors are responsible for the spread of COVID-19 in India. The Neutrosophic Cognitive Map (NCM) is an enhanced version of Fuzzy Cognitive Map (FCM) since it takes into account the concept of indeterminacy and uncertainty, which is not addressed by FCMs. The obtained results not only show the importance of determinate factors (measures taken by the government bodies) in containing the spread of this deadly virus but also show how indeterminate and uncertain factors such as poverty, negligence, ineffectual healthcare, age, immunity, unscientific practices on religious grounds and illiteracy play a vital role in this regard. This model, in general, helps policymakers and government agencies to represent any complicated situation mathematically thereby helping them in responding appropriately and forming new policies for dealing with such pandemics. This work also signifies how neutrosophic theories can be applied in the analysis of the situation for dealing with real life problems.

Keywords: SARS-CoV2, COVID-19, Neutrosophy, Neutrosophic Cognitive Maps, Situation Analysis.

1. Introduction

A medical professional knows how the word, "SARS-CoV2", has changed its meaning in the past 7 months [21] from merely being a virus that used to cause "common cold" to the present time where it has killed more than 9,70,238 people all over the world till September 22, 2020 [22]. The novel Corona Virus is said to have its origin in Wuhan, China wherein the first ever case was reported and soon it became a pandemic that has led the world to take it even more seriously. Later on, World Health Organization (WHO) also declared it as a pandemic [7][10]. The non-availability of a vaccine has caused its widespread all over the world. Since no standard treatment protocol is available till date; the only preventive measures which are considered by the government in India

are lockdown, social distancing, covering of body parts, home quarantine and making testing kits available to contain the spread of this highly contagious virus. These are the safeguard measures which are taken in most places irrespective of its demography, population density and economy. India adopted similar measures, as is evident from the statement of the Prime Minister Mr. Narendra Modi given on the night of 24th March 2020, announcing complete lockdown in the entire country. Later the lockdown measures were revised many times as a result of which India has entered in Lockdown 4.0 [23]. Undoubtedly, these measures are helpful to certain extent but the question is, “Are these measures enough in the Indian context?” India comes in the category of developing nations with the second largest population. Further, most of the people in India are poor, illiterate and unaware of dealing with such pandemics. When asked about these factors, experts say that poverty and unawareness may kill more people in India than the novel corona virus during lockdown [8] [12] [13]. Moreover, when it comes to India there are other factors, which need to be considered as explained by different experts. These factors may be immunity [6], age [1] [11], ineffectual healthcare, which may include lack of trained staff, unavailability of PPE & testing centers [14] [15], and unscientific practices on religious grounds as well [9] [10]. These factors are termed as indeterminate and uncertain throughout this research work. The major policies which are being implemented to contain the spread of SARS-CoV2 in India seems to be more appropriate to the richer section of the society since they have most of the facilities which helps in abiding by the rules. But there exists a large section of society that cannot afford such facilities and hence find it challenging to abide by the rules. This is the reason they are adversely affected in such circumstances.



Figure 1(a) USA (Source: New York Times)



Figure 2(b) Italy (Source: The Guardian)



Figure 3(c) Spain (Source: Aljazeera)



Figure 4(d) Germany (Source: BBC)

Figure 5 Scenes from different countries where complete lockdown is imposed as a measure to curb the spread of COVID-19

Figure 1 shows the pictures from some prominent countries where this SARS-CoV2 has badly affected the population and as a preventive measure they have announced complete lockdown. These pictures demonstrate that the action is serving its purpose. Figure 2 depicts a picture of India during lockdown that presents a contrasting view from rest of the world. This has not happened only once, instead such scenarios have been a regular phenomenon in India during lockdown.



Figure 6(a) People waiting for transport



Figure 7(b) People walking on foot to their homes

Figure 8 Scenes from India where complete lockdown is imposed as a measure to curb the spread of COVID-19

(Source: Aljazeera)

The above pictures from India show a totally different scenario. There was a large number of people on roads due to this lockdown. The poor section of the society is more considerate about losing their livelihood in this SARS-CoV2 than their lives, as they are afraid of the fact that starvation may kill them before COVID-19 does. The statistics show that the total number of cases in India have increased rapidly and it took the form of a pandemic, which is evident from the figures mentioned. The number of reported cases were 15,000 on April 19, 2020. Just after 10 days, it reached 30,000 mark, whereas it crossed 40,000 within next four days. This trend continued and in September 5, 2020 it reached 40, 20,239 cases. After 10 days, it crossed 50, 18,034 mark; while on September 22, 2020 it was recorded to be 55, 74,096 [28]. The statistics and the pictures discussed above motivate the analysis of situation prevailing in India due to COVID-19 so that more factors could be identified on which the government should work so that a large section of the society could be saved from this and such other deadly diseases. Some of the factors are said to be determinate since the Indian Government is currently working on it. Other factors, which are not considered yet, are termed as indeterminate and uncertain in this study. These factors must be addressed and situation needs to be modelled mathematically so that it gives clear, concise and optimal results. If modelled correctly using all the factors whether determinate or indeterminate, it is supposed to aid agencies to work at root level so that no disease turns into a pandemic. Since there are many mathematical theories for modelling determinate and certain events but indeterminate and uncertain events are still not addressed effectively. Neutrosophy is the field of study, which provides a way to address these factors [4]. Some researches on COVID-19 using neutrosophy have been reported [30] [31] [32] [34]. The recent works in neutrosophic theories are undertaken by well-known researchers in [35] [36].

This work aims to analyze all known, indeterminate and uncertain factors which may lead to the spread of COVID-19 in India. In this research, it has been attempted to model the situation in India using neutrosophic cognitive maps [3], which shows how these indeterminate and uncertain factors pose a serious threat as compared to that of the known factors. Neutrosophy is applied because it is extended version of fuzzy logic and covers all aspects of decision making [29] [33]. If modelled correctly considering all the factors whether determinate, indeterminate or uncertain it

would help government and organizations to take appropriate measures beforehand so that such a situation does not arise in the future.

The rest of the paper is divided into three sections. Section 1, as discussed above, deals with the background of this study followed by section 2, which models situation of COVID-19 in India using neutrosophic cognitive maps. Section 3 presents the interpretation of results while section 4 concludes the paper.

2. Materials and Methods

Situation analysis is an important aspect of our daily lives [24]. In situation analysis an agent analyzing the situation takes into account several factors to reach at a conclusion and take decisions accordingly [2]. These aspects which agent considers are totally based on the data that he/she collects from various sources including field experts. Since data is not always certain and known; some collected data is uncertain, indeterminate and not known. There is no denial of the fact that this data also captures some part of the information. Earlier there was no appropriate tool to deal with this type of data, but with the emergence of neutrosophic theory by Florentin Smarandache [3] [4], such data can be modeled and analyzed mathematically. This theory helps in analyzing the situation more appropriately and most accurate conclusions may be drawn by the agent, thereby helping in making optimal decisions. The present scenario around the world shows how this pandemic of COVID-19 has put a threat to mankind and if this situation is not analyzed critically then it could lead to a major disaster. There are various aspects, which are known to the agents who are analyzing the data in order to control this pandemic. On the other hand the indeterminate, uncertain and unknown aspects are not considered. If these factors are considered too, it would lead to better results. For this reason situation is modeled using neutrosophic cognitive maps [3]. Some basic concepts of neutrosophy, as given by Florentin Smarandache, is given below in order to carry out the mathematical work.

Definition 1. Let $\mathbf{N} = \{(\mathbf{T}, \mathbf{I}, \mathbf{F}) : \mathbf{T}, \mathbf{I}, \mathbf{F} \in (0, 1)\}$ be a neutrosophic set. Let $\mathbf{m} : \mathbf{P} \rightarrow \mathbf{N}$ is a mapping of a group of propositional formulas into \mathbf{N} , i.e., each sentence $\mathbf{p} \in \mathbf{P}$ is associated to a value in \mathbf{N} , as it is given in the Equation 1, meaning that \mathbf{p} is $\mathbf{T}\%$ true, $\mathbf{I}\%$ indeterminate and $\mathbf{F}\%$ false.

$$\mathbf{m}(\mathbf{p}) = (\mathbf{T}, \mathbf{I}, \mathbf{F}) \quad (1)$$

Hence, the neutrosophic logic is a generalization of fuzzy logic, based on the concept of neutrosophy according to [25].

Definition 2. A Neutrosophic matrix is a matrix $\mathbf{M} = [\mathbf{a}_{ij}]_{ij}$ where $\mathbf{i} = 1, 2, 3, \dots, \mathbf{m}$ and $\mathbf{j} = 1, 2, 3, \dots, \mathbf{n}$ such that each $\mathbf{a}_{ij} \in \mathbf{K}(\mathbf{I})$ where $\mathbf{K}(\mathbf{I})$ is a neutrosophic ring [3]. An example of neutrosophic matrix is given below. Suppose each element of matrix is represented by $\mathbf{a} + \mathbf{bI}$ where \mathbf{a} and \mathbf{b} are real numbers and \mathbf{I} is a factor of indeterminacy.

For Example:

$$\begin{pmatrix} -1 & 1 & 5\mathbf{I} \\ 1 & 4 & 7 \end{pmatrix} \begin{pmatrix} 1 & 9\mathbf{I} & 6 \\ 0 & 1 & 0 \\ -4 & 7 & 5 \end{pmatrix} = \begin{pmatrix} -21\mathbf{I} & 27\mathbf{I} & -6 + 25\mathbf{I} \\ -28 + \mathbf{I} & 49 + 13\mathbf{I} & 35 + 6\mathbf{I} \end{pmatrix}$$

Definition 3. A neutrosophic graph is a graph in which there exists an indeterminate node or an indeterminate edge with determinate edges. Now taking reference from the Definition 2 above, we can conclude that when $\mathbf{a}_{ij} = \mathbf{0}$ it means there is no connection between nodes \mathbf{i} and \mathbf{j} , $\mathbf{a}_{ij} = \mathbf{1}$ means there is a connection between nodes \mathbf{i} and \mathbf{j} and $\mathbf{a}_{ij} = \mathbf{I}$ means that connection is indeterminate (unknown).

Definition 4. Cognitive maps are cause-effect networks, with nodes representing concepts articulated by individuals, and directional linkages capturing causal dependencies [26].

Definition 5. A Neutrosophic Cognitive Map is a directed graph with nodes as events or policies and causalities or relationship as determinate and indeterminate edges.

3. Modeling COVID-19 Situation in India

For modeling the situation of COVID-19 in India, all factors must be taken into account. These include those being put forward by the government along with the factors which are not taken into consideration yet. The factors which seem to be less critical to the Indian Government are poverty, illiteracy, age group distribution, and unscientific practices on religious grounds which are prevalent in all over India, negligence by the most of the people including government & medical staff, immunity of people and ineffectual healthcare in India. These factors need to be considered otherwise the spread of COVID-19 cannot be controlled in India. Most of the countries, like Italy, Germany, Spain, France and USA have announced lockdown so that there is no community transmission of this deadly virus among people. These countries noticed desired effect of the lockdown since these are well established on all the grounds which have been termed as indeterminate factors in our research. According to UNESCO statistics, 25.63% of the total Indian population is illiterate [27]. Literacy plays a vital role in adhering to Government instructions and understanding of the danger which is recently posed by COVID-19 in India. Figure 3 below shows the comparison of literacy rates of different countries, which have announced the complete lockdown to protect its population from COVID-19.

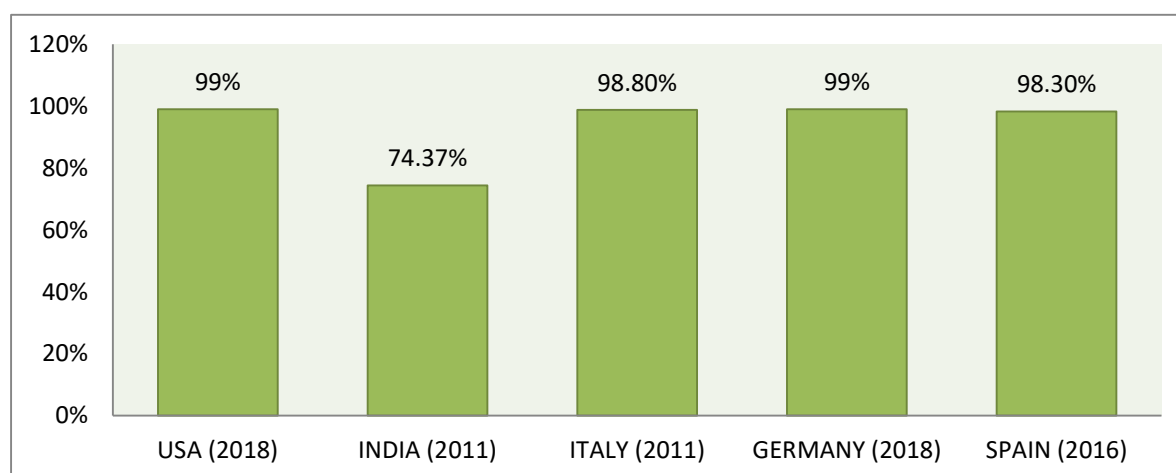


Figure 9 Countries with respective literacy rate

Figure 3 shows that the literacy rate in India is only 74.37% which is less than all other mentioned nations. Next is the poverty and negligence as mentioned in [8]. Since complete lockdown means that no one can go out except persons involved in essential services, then what about the daily wage workers and poor people who go out every single day in search of food and alms? This section of Indian society will die either of COVID-19 or due to lack of food; it would be shocking to know that they prefer the first one. Hence, the poor section of the society seems to be an important factor [12] [13] that is to be taken in consideration while taking measures to stop the spread of any deadly disease. A lot of people who are poor cannot abstain themselves from going out despite the fact that government is trying to support all such people. But, the help cannot be provided to all at the same time. Figure 4 shows the section of society living below poverty line in the countries where complete lockdown is ordered. 21.90% of the Indian population lives below poverty line; it means that this section would surely be affected by the government measures to contain the spread of any pandemic

of such nature. Though the population living below the poverty line in Italy is more than India i.e. 29.90% but compared to its total population this is a negligible amount that could easily be handled and managed. This is undoubtedly an indeterminate and uncertain factor in the Indian context that could lead to the failure of government actions taken to contain the spread of pandemic.

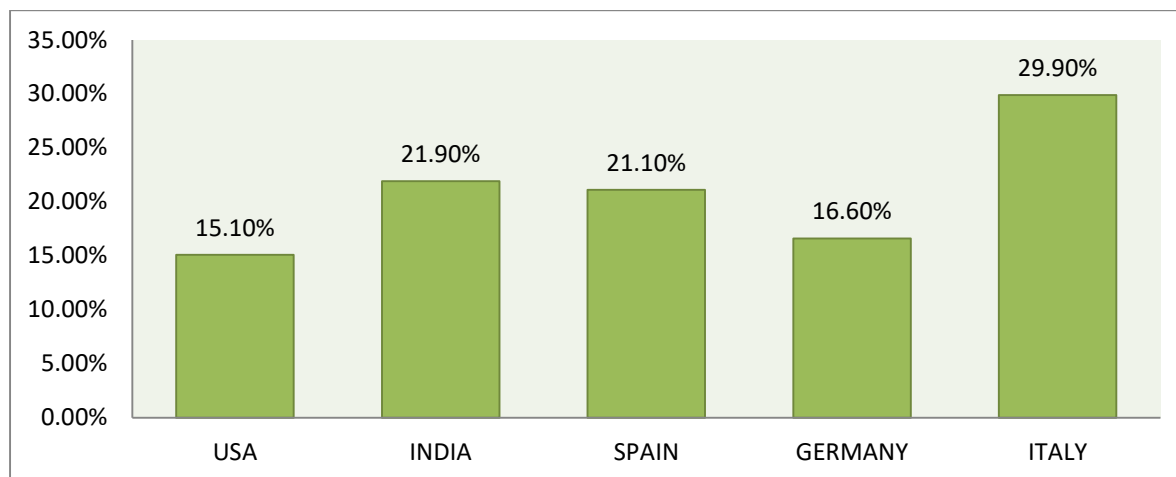


Figure 10 Population living below poverty line in different countries from Source [16]

The age group has emerged as one of the critical factors to date. The mortality rate due to COVID-19 is mostly based on the age. Though a large section of the society may get infected by this virus, out of these the people with weak immunity are most adversely affected. The report by India Spend [19] says that out of 100000 people 122 die due to ineffectual health care in India. According to Hindustan Times [15] dated April 04, 2020 an article titled as "Covid-19 is a wake-up call, Governments need to invest in healthcare" throws some light on the ongoing health conditions in India. As per the current scenario Italy has reported 35,813 numbers of deaths till date and most of the people who died are aged persons with weak immunity. Data in Figure 5 shows the total number of deaths in the mentioned countries according to their ages. Accordingly, Figure 6 presents the elderly population in the countries. These figures show that elderly population aged above 80 comprise 14.8% of total deaths, caused due to pandemic. In similar fashion in the age group of 70-79 and 60-69, 8% and 3.6% of the people have died, respectively. These age groups having weak immunity are considered more prone to deadly diseases. Figure 7 shows Italy (22.7%) has the largest number of the elderly population, followed by the USA (16%) having most extensive deaths due to COVID-19 on record. This shows how immunity and age factor may play a critical role in reducing the spread of any deadly virus like COVID-19 in India. This factor though indeterminate, must be taken care of since it includes elderly citizens having weak immunity and other chronic diseases. In India, it may be a critical factor to be considered so that the number of deaths due to pandemic could be reduced effectively.

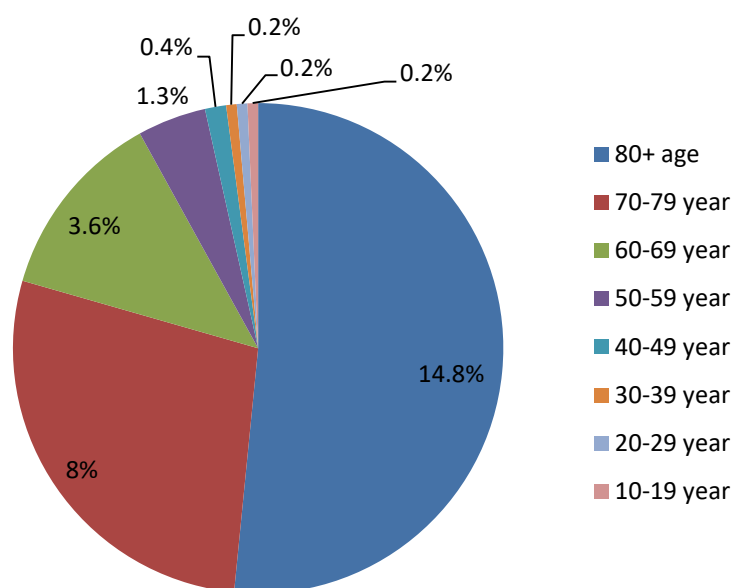


Figure 11 Death rate according to age in different countries

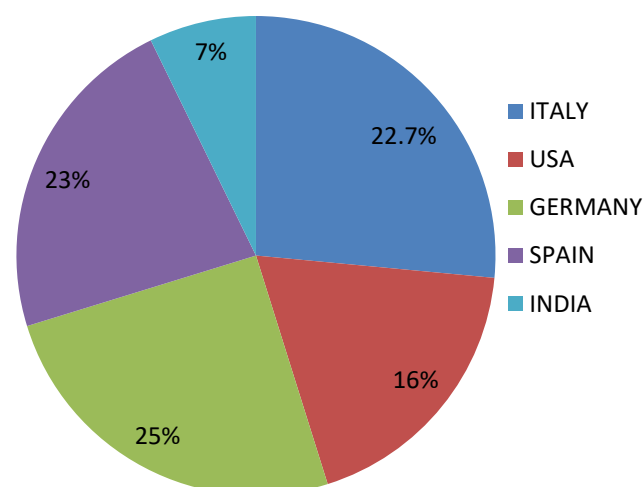


Figure 12 Elderly population in different countries

From Source [17]

All the mentioned determinate, indeterminate and uncertain factors in the Indian context are shown in Figure 7. Two categories of factors are mentioned. One which is being considered by the government of India and are given top priorities whereas second one comprises less considered factors which need further attention because the later may work on root level to fight the pandemic like COVID-19 in India. For the convenience of effective modelling of the situation using cognitive maps and later their representation in the adjacency matrix, all the factors discusses above are represented using abbreviations. These abbreviations are as follows:

DFL1 = Determinate Factor Lockdown

DFS2 = Determinate Factor Social Distancing

DFW3 = Determinate Factor Washing Hands & Wearing Mask

DFH4 = Determinate Factor Quarantine

DFA5 = Determinate Factor Availability of Ventilators & Testing Kits

NDFP1 = Non – Determinate Factor Poverty

NDFI2 = Non – Determinate Factor Illiteracy

NDFA3 = Non – Determinate Factor Age

NDFR4 = Non – Determinate Factor Unscientific Practices on Religious Ground

NDFN5 = Non – Determinate Factor Negligence

NDFI6 = Non – Determinate Factor Immunity

NDFI7 = Non – Determinate Factor Ineffectual Healthcare

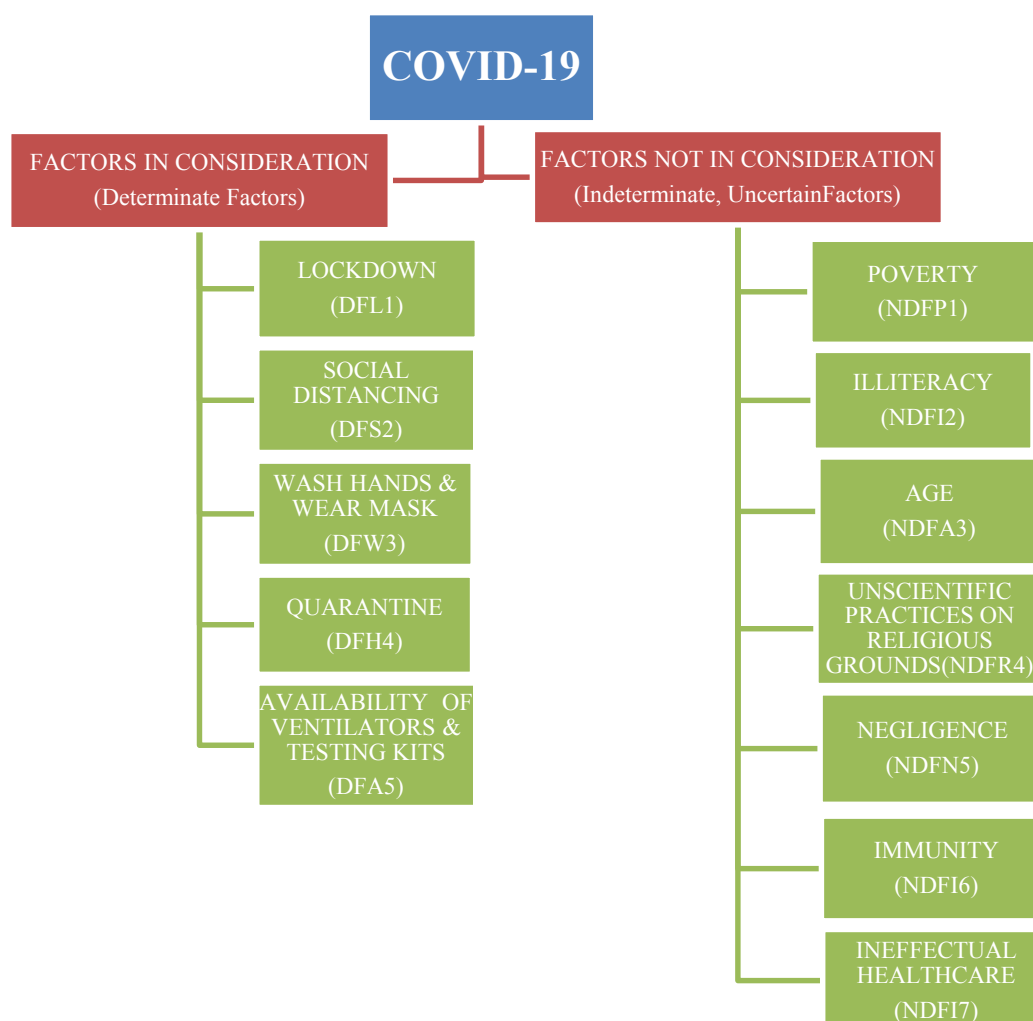


Figure 13 Determinate & Indeterminate Factors in consideration to avoid COVID-19 in India

4. Modeling Using Neutrosophic Cognitive maps

Fuzzy Cognitive Maps (FCMs) used earlier for modeling the situation were not that much effective as Neutrosophic Cognitive Maps (NCMs) since it does not have provision to represent uncertainty related to real life situations [37] [5]. The study in [24] compares FCMs with NCMs and shows the effectiveness of NCMs in modelling the situation. FCMs do not assign any weightage to indeterminate and uncertain factors; it simply neglects these factors as a result of which the results obtained by the agents for drawing the conclusion and making policies seem to be inappropriate. However, modeling of the situation using NCMs (Definition 3, 4, 5) is effective because it considers all the determinate or indeterminate factors [38]. For modeling situation using NCMs the determinate edges are given a weightage of '1' which means the factor is certainly having an effect on something under consideration, whereas '0' represents the absence of relationship among factors. On the other hand indeterminate relations are represented by the edge with weightage of 'I'. The modelling of the situation of COVID-19 in India considering all the factors is shown in Figure 8. The dotted edges represent indeterminate relations among factors. The figure clearly shows how factors whether determinate or indeterminate are related to spreading of COVID-19. It also shows how certain and determinate factors are affected by the uncertain and indeterminate factors.

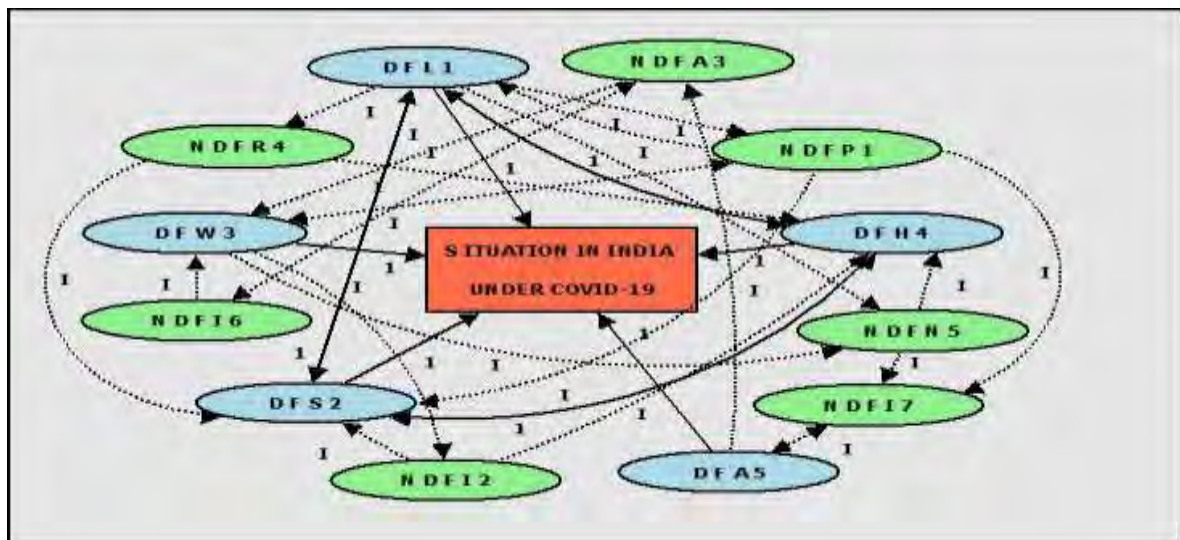


Figure 14 Neutrosophic Cognitive Map representing situation of COVID-19 in India

Now based on neutrosophic cognitive map shown in Figure 8, the obtained neutrosophic adjacency matrix (Definition 2) is shown in Figure 9.

Variables	COV	DFL1	DFS2	DFW3	DFH4	DFA5	NDFP1	NDFI2	NDFA3	NDFR4	NDFN5	NDFI6	NDFI7
COV	0	1	1	1	1	1	0	0	0	0	0	0	0
DFL1	1	0	1	0	1	0	I	0	0	I	I	0	0
DFS2	1	1	0	0	1	0	0	0	0	I	0	0	0
DFW3	1	0	0	0	0	0	I	I	0	0	I	0	0
DFH4	1	1	1	0	0	0	0	0	0	0	0	0	0
DFA5	1	0	0	0	0	0	0	0	I	0	0	0	I
NDFP1	0	I	I	I	0	0	0	0	0	0	0	0	I
NDFI2	0	0	I	0	I	0	0	0	0	0	0	0	0
NDFA3	0	0	0	I	0	0	0	0	0	0	0	I	0
NDFR4	0	0	I	0	I	0	0	0	0	0	0	0	0
NDFN5	0	0	0	0	I	0	0	0	0	0	0	0	I
NDFI6	0	0	I	I	0	0	0	0	I	0	0	0	0
NDFI7	0	0	0	0	0	I	0	0	0	0	0	0	0

Figure 15 Neutrosophic Adjacency Matrix obtained from cognitive mappings

The neutrosophic adjacency matrix is now evaluated using mathematical matrix calculations to know the effect of factors on spread of COVID-19 in India. The situation of COVID-19 in India is taken as ON state. Let this ON state vector be represented as $\mathbf{X}_1 = (1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$. This state vector is given as input to determine the effect of \mathbf{X}_1 on the combined system i.e. $\mathbf{X}_1 \mathbf{N}(\mathbf{E})$. The symbol \rightarrow denotes that the resultant vector is updated and threshold. The following calculation is carried out till a constant state vector is obtained. It is also referred as limit cycle.

$$X_1N(E) = (0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \rightarrow (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) = X_2$$

$$X_2N(E) = (5 \ 3 \ 3 \ 1 \ 3 \ 1 \ 2I \ I \ I \ 2I \ 2I \ 0 \ I) \rightarrow (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ I \ I \ I \ I \ I \ 0 \ I) = X_3$$

$$X_3N(E) = (5 \ I^2 + 3 \ 3I^2 + 3 \ 2I^2 + 1 \ 3I^2 + 3 \ 1 \ 2I \ I \ I \ 2I \ 2I \ I^2 \ 2I^2 + I) \rightarrow$$

$$(1 \ 1 \ 1 \ 1 \ 1 \ 1 \ I \ I \ I \ I \ I \ I) = X_4$$

$$X_4N(E) = (5 \ I^2 + 3 \ 4I^2 + 3 \ 3I^2 + 1 \ 3I^2 + 3 \ I^2 + 1 \ 2I \ I \ I^2 + I \ 2I \ 2I \ I^2 \ 2I^2 + I) \rightarrow$$

$$(1 \ 1 \ 1 \ 1 \ 1 \ 1 \ I \ I \ I \ I \ I \ I) = X_5$$

The above iteration is carried forward till a constant vector or limit cycle is acquired. This constant vector forms the basis for the interpretation of results. Notice that $X_4 = X_5$; a limit cycle is achieved so further iterations are not required. $X_5 = (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ I \ I \ I \ I \ I \ I)$ is a constant vector or fixed point or limit cycle. The significance of this limit cycle is utmost, since it shows a hidden pattern which is used in drawing inferences. These inferences show the joint effect of interacting knowledge. The current results obtained is $(1 \ 1 \ 1 \ 1 \ 1 \ 1 \ I \ I \ I \ I \ I \ I)$. Here '1' shows that the factors such as lockdown, social distancing, wash hands & wear mask, quarantine and availability of ventilators & testing kits are certain to stop the spread of this pandemic therefore regarded as determinate or certain factors. On the other hand 'I' in above vector shows that though the factors such as poverty, illiteracy, age group distribution, unscientific practices on religious grounds, negligence, immunity and ineffectual healthcare do not have direct influence on the spread of this virus but somehow they are of much importance as they affect determinate factors. If any of the mentioned factors either determinate or indeterminate are absent in current scenario the result will give '0' in its position. The results show that all factors somewhere or the other have influence on the spread of this pandemic since none has got entry '0' in its position in a limit cycle. This proves that the measures which are taken by government agencies may not serve its purpose if the factors termed as uncertain throughout this study are not taken care of. These uncertain factors one way or the other influence the known or certain factors. This may lead to the unsuccessful implementation of the measures taken by the government. Though the current method is one of the best to represent real life situations mathematically, it needs further enhancements for quantization of linguistic terms so that it includes the notion of indeterminacy together with truth and falsity of the statements.

5. Conclusions

The global spread of the pandemic COVID-19 has emerged as a threat to mankind. Since there is no vaccine or standard treatment protocol available for this disease till date, social distancing, lockdown, hotspots identification and isolation, etc. are being used as the most effective measures all over the world. This has resulted in flattening the peak by impeding its spread. At this juncture, it becomes all the more critical to identify, consider and study closely the factors contributing its wide spread. Some of the factors are well known and government is taking necessary steps to address these factors so that the spread of this virus can be stopped. Despite all these, there are certain factors which have not yet been taken into consideration and these may vary based on the demographic changes. These factors may be poverty, negligence, ineffectual healthcare and many more, which at this stage may be considered as indeterminate, uncertain and unknown factors. There is no denial of the fact that such factors may have their impacts to the overall spread of COVID-19. As the situation is emerging and more and more information is coming up from various sources all over the globe, it is enriching the understanding of various responsible factors. In this study known factors which are thought to curb the spread of the disease along with other factors which are indeterminate, uncertain and unknown are taken into account. These factors are based on the opinions of experts and other agencies. The influence of these factors on COVID-19 is represented and modeled mathematically using neutrosophic theory. This modeling is described graphically, and later conclusions are drawn using mathematical calculations. The results will highlight factors which are of utmost importance in curbing the spread of this pandemic. The resulting model would help policymakers in analyzing the situation more critically and formulating and prioritizing the policies to aid the government

agencies in handling the spread of this deadly disease. This would also help the government bodies working at the ground level in reducing the loss of precious lives of the citizens. Future work in this regard may include implementing and designing machine learning algorithms for carrying out the simulation using neutrosophic theories. Earlier proposed algorithms in machine learning for situation analysis might be combined with neutrosophic theories so that the output obtained could be validated with more optimized results.

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Direct product of Neutrosophic INK-Algebras

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Abstract: In this paper, first we define the notion direct product of neutrosophic sets in INK-algebras, neutrosophic set, neutrosophic INK-ideals, neutrosophic closed INK-ideals and direct product of neutrosophic INK-ideals in INK-algebras. We prove some theorems which show that there is some relation between these notions. Finally, we define the INK-subalgebra of INK-algebra and then we give related theorem about the relationship between their Images and direct product of neutrosophic INK-ideals.

Keywords: INK-algebra; neutrosophic set; direct product of neutrosophic INK-subalgebra; direct product of neutrosophic INK-ideal.

1. Introduction

In 1986, Atanassov Introduced the Intuitionistic fuzzy set and later intuitionistic fuzzy set was applied in BCI/BCK-algebra, Introduced by Imai and Iseki in the 1980s. Following this, various researchers published articles using the intuitionistic fuzzy set concept. In 2005, Smarandache invented the new notion of the neutrosophic set in 1998 and it is a common code from the intuitionistic fuzzy set [1-8] and [15-20]. This has been followed by a lot of researchers publishing various articles over the last few years. In [9], [10], [11], [13], [14] and [12] Kaviyarasu et. al published an article using the fuzzy concept set in INK-algebra and later in solve they neutrosophic set in INK-algebra. In this paper we have introduced a new code using two different neutrosophic sets called direct product of neutrosophic sets in INK-algebra. We are also examining the relationship between neutrosophic INK- subalgebra and neutrosophic INK-ideal and its conditions.

2. Preliminaries

Before we begin our study, we will give the definition and useful properties of INK-algebras.

Definition 2.1: An algebra $(X, *, 0)$ is called a INK-algebra if it satisfies the following conditions for any $a, b \in X$.

- i) $((a * b) * (a * c)) * (z * b) = 0$
- ii) $((a * c) * (b * c)) * (a * b) = 0$
- iii) $a * 0 = a$
- iv) $a * b = 0$ and $b * a = 0$ imply $a = b$.

where “*” is a binary operation and the “0” is a constant of X .

Definition 2.2: A non-empty subset S of a INK-algebra $(X, *, 0)$ is said to be a INK-subalgebra of X , if $a * b \in S$, whenever $a, b \in S$.

Definition 2.3: Let $(X, *, 0)$ be a INK-algebra. A nonempty subset I of X is called an ideal of X if it satisfies

- i) $0 \in I$
- ii) $a * b \in I$ and $b \in I$ imply $a \in I$ for all $a, b \in X$.

Definition 2.4: Let I be a non-empty subset of a INK- algebra X . Then I is called a INK-ideal of X , if

- i) $0 \in I$
- ii) $((c * a) * (c * b)) \in I$ and $b \in I$ imply $a \in I$ for all $a, b, c \in X$.

Definition 2.5: A neutrosophic set Λ in a nonempty set X is a structure of the form

$\Lambda = \{(X, \prec^T, \prec^I, \prec^F(a)) | a \in X\}$, where $\prec^T: X \rightarrow [0, 1]$ is a truth membership function $\prec^I: X \rightarrow [0, 1]$ is a indeterminate membership function and $\prec^F: X \rightarrow [0, 1]$ is a false membership function.

Definition 2.6: A neutrosophic set Λ in X is called a neutrosophic INK-subalgebra of X if it satisfies the following conditions, for all $a, b \in X$.

- i) $\prec^T(a * b) \geq \min\{\prec^T(a), \prec^T(b)\}$
- ii) $\prec^I(a * b) \leq \max\{\prec^I(a), \prec^I(b)\}$
- iii) $\prec^F(a * b) \geq \min\{\prec^F(a), \prec^F(b)\}$

Definition 2.7: A neutrosophic set Λ in X is called a neutrosophic ideal of X if it satisfies the following conditions, for all $a, b \in X$.

- i) $\prec^T(0) \geq \prec^T(a)$, $\prec^I(0) \leq \prec^I(a)$ and $\prec^F(0) \geq \prec^F(a)$
- ii) $\prec^T(a) \geq \min\{\prec^T(a * b), \prec^T(b)\}$
- iii) $\prec^I(a) \leq \max\{\prec^I(a * b), \prec^I(b)\}$
- iv) $\prec^F(a) \geq \min\{\prec^F(a * b), \prec^F(b)\}$.

Definition 2.8: A neutrosophic set Λ in X is called a neutrosophic INK-ideal of X if it satisfies the following conditions, for all $a, b, z \in X$.

- i) $\prec^T(0) \geq \prec^T(a)$, $\prec^I(0) \leq \prec^I(a)$ and $\prec^F(0) \geq \prec^F(a)$
- ii) $\prec^T(a) \geq \min\{\prec^T((c * a) * (c * b)), \prec^T(b)\}$
- iii) $\prec^I(a) \leq \max\{\prec^I((c * a) * (c * b)), \prec^I(b)\}$
- iv) $\prec^F(a) \geq \min\{\prec^F((c * a) * (c * b)), \prec^F(b)\}$.

3. Direct product of Neutrosophic INK-subalgebra and INK-ideal

Definition 3.1: Let Υ and Λ are two neutrosophic sets in INK-algebras X_1 and X_2 . The direct product of neutrosophic sets Υ and Λ is defined by $\Upsilon \times \Lambda = \langle \prec^T, \prec^I, \prec^F_{(\Upsilon \times \Lambda)} \rangle$ and defined by

- i) $\prec^T_{(\Upsilon \times \Lambda)}(a, b) = \min\{\prec^T_{\Upsilon}(a), \prec^T_{\Lambda}(b)\}$
- ii) $\prec^I_{(\Upsilon \times \Lambda)}(a, b) = \max\{\prec^I_{\Upsilon}(a), \prec^I_{\Lambda}(b)\}$
- iii) $\prec^F_{(\Upsilon \times \Lambda)}(a, b) = \min\{\prec^F_{\Upsilon}(a), \prec^F_{\Lambda}(b)\}$

For all $a, b \in X$.

Definition 3.2: A neutrosophic sets $\Upsilon \times \Lambda = \langle \prec^T, \prec^I, \prec^F_{(\Upsilon \times \Lambda)} \rangle$ of X_1 and X_2 is called direct product of neutrosophic INK-subalgebra of $X_1 \times X_2$, if

- i) $\prec^T_{(\Upsilon \times \Lambda)}((a_1, b_1) * (a_2, b_2)) \geq \min\{\prec^T_{(\Upsilon \times \Lambda)}(a_1, b_1), \prec^T_{(\Upsilon \times \Lambda)}(a_2, b_2)\}$
- ii) $\prec^I_{(\Upsilon \times \Lambda)}((a_1, b_1) * (a_2, b_2)) \leq \max\{\prec^I_{(\Upsilon \times \Lambda)}(a_1, b_1), \prec^I_{(\Upsilon \times \Lambda)}(a_2, b_2)\}$

iii) $\wedge^F_{(\gamma \times \lambda)}((a_1, b_1) * (a_2, b_2)) \geq \min \{ \wedge^F_{(\gamma \times \lambda)}(a_1, b_1), \wedge^F_{(\gamma \times \lambda)}(a_2, b_2) \}$
for all $a_i, b_i \in X$.

Definition 3.3: A neutrosophic sets $\gamma \times \lambda = \langle \wedge^T, \wedge^I, \wedge^F_{(\gamma \times \lambda)} \rangle$ of X_1 and X_2 is called direct product of neutrosophic INK-ideal of $X_1 \times X_2$, if

- i) $\wedge^T_{(\gamma \times \lambda)}(0, 0) \geq \wedge^T_{(\gamma \times \lambda)}(a_i, b_i)$
 - ii) $\wedge^I_{(\gamma \times \lambda)}(0, 0) \leq \wedge^I_{(\gamma \times \lambda)}(a_i, b_i)$
 - iii) $\wedge^F_{(\gamma \times \lambda)}(0, 0) \geq \wedge^F_{(\gamma \times \lambda)}(a_i, b_i)$
 - iv) $\wedge^T_{(\gamma \times \lambda)}(a_1, b_1) \geq \min \{ \wedge^T_{(\gamma \times \lambda)}(((a_3, b_3) * (a_1, b_1)) * ((a_3, b_3) * (a_2, b_2))), \wedge^T_{(\gamma \times \lambda)}(a_2, b_2) \}$
 - v) $\wedge^I_{(\gamma \times \lambda)}(a_1, b_1) \leq \max \{ \wedge^I_{(\gamma \times \lambda)}(((a_3, b_3) * (a_1, b_1)) * ((a_3, b_3) * (a_2, b_2))), \wedge^I_{(\gamma \times \lambda)}(a_2, b_2) \}$
 - vi) $\wedge^F_{(\gamma \times \lambda)}(a_1, b_1) \geq \min \{ \wedge^F_{(\gamma \times \lambda)}(((a_3, b_3) * (a_1, b_1)) * ((a_3, b_3) * (a_2, b_2))), \wedge^F_{(\gamma \times \lambda)}(a_2, b_2) \}$
- for all $a_i, b_i \in X$.

Definition 3.4: A neutrosophic sets $\gamma \times \lambda = \langle \wedge^T, \wedge^I, \wedge^F_{(\gamma \times \lambda)} \rangle$ of X_1 and X_2 is called direct product of neutrosophic closed INK-ideal of $X_1 \times X_2$, if it satisfies (Def 3.3 iv, v, vi) the following condition

- i) $\wedge^T_{(\gamma \times \lambda)}(0, 0) * (a_i, b_i) \geq \wedge^T_{(\gamma \times \lambda)}(a_i, b_i)$
- ii) $\wedge^I_{(\gamma \times \lambda)}(0, 0) * (a_i, b_i) \leq \wedge^I_{(\gamma \times \lambda)}(a_i, b_i)$
- iii) $\wedge^F_{(\gamma \times \lambda)}(0, 0) * (a_i, b_i) \geq \wedge^F_{(\gamma \times \lambda)}(a_i, b_i)$, for all $x, y \in X$.

Theorem 3.5: Let γ and λ be two neutrosophic INK-subalgebras of X_1 and X_2 . Then $\gamma \times \lambda = \langle \wedge^T, \wedge^I, \wedge^F_{(\gamma \times \lambda)} \rangle$ is a neutrosophic INK-subalgebra of $X_1 \times X_2$.

Proof. For any $(a_1, b_1), (a_2, b_2) \in X_1 \times X_2$. Then

$$\begin{aligned} \wedge^T_{(\gamma \times \lambda)}((a_1, b_1) * (a_2, b_2)) &= \wedge^T_{(\gamma \times \lambda)}((a_1 * a_2), (b_1 * b_2)) \\ &= \min \{ \wedge^T_{\gamma}((a_1 * a_2)), \wedge^T_{\lambda}(b_1 * b_2) \} \\ &\geq \min \{ \min \{ \wedge^T_{\gamma}(a_1), \wedge^T_{\gamma}(a_2) \}, \min \{ \wedge^T_{\lambda}(b_1), \wedge^T_{\lambda}(b_2) \} \} \\ &= \min \{ \min \{ \wedge^T_{\gamma}(a_1), \wedge^T_{\lambda}(b_1) \}, \min \{ \wedge^T_{\gamma}(a_2), \wedge^T_{\lambda}(b_2) \} \} \\ \wedge^T_{(\gamma \times \lambda)}((a_1, b_1) * (a_2, b_2)) &\geq \min \{ \wedge^T_{(\gamma \times \lambda)}(a_1, b_1), \wedge^T_{(\gamma \times \lambda)}(a_2, b_2) \}, \end{aligned}$$

$$\begin{aligned} \wedge^I_{(\gamma \times \lambda)}((a_1, b_1) * (a_2, b_2)) &= \wedge^I_{(\gamma \times \lambda)}((a_1 * a_2), (b_1 * b_2)) \\ &= \max \{ \wedge^I_{\gamma}((a_1 * a_2)), \wedge^I_{\lambda}(b_1 * b_2) \} \\ &\leq \max \{ \max \{ \wedge^I_{\gamma}(a_1), \wedge^I_{\gamma}(a_2) \}, \max \{ \wedge^I_{\lambda}(b_1), \wedge^I_{\lambda}(b_2) \} \} \\ &= \max \{ \max \{ \wedge^I_{\gamma}(a_1), \wedge^I_{\lambda}(b_1) \}, \max \{ \wedge^I_{\gamma}(a_2), \wedge^I_{\lambda}(b_2) \} \} \\ \wedge^I_{(\gamma \times \lambda)}((a_1, b_1) * (a_2, b_2)) &\leq \max \{ \wedge^I_{(\gamma \times \lambda)}(a_1, b_1), \wedge^I_{(\gamma \times \lambda)}(a_2, b_2) \} \end{aligned}$$

And

$$\begin{aligned} \wedge^F_{(\gamma \times \lambda)}((a_1, b_1) * (a_2, b_2)) &= \wedge^F_{(\gamma \times \lambda)}((a_1 * a_2), (b_1 * b_2)) \\ &= \min \{ \wedge^F_{\gamma}((a_1 * a_2)), \wedge^F_{\lambda}(b_1 * b_2) \} \\ &\geq \min \{ \min \{ \wedge^F_{\gamma}(a_1), \wedge^F_{\gamma}(a_2) \}, \min \{ \wedge^F_{\lambda}(b_1), \wedge^F_{\lambda}(b_2) \} \} \\ &= \min \{ \min \{ \wedge^F_{\gamma}(a_1), \wedge^F_{\lambda}(b_1) \}, \min \{ \wedge^F_{\gamma}(a_2), \wedge^F_{\lambda}(b_2) \} \} \\ \wedge^F_{(\gamma \times \lambda)}((a_1, b_1) * (a_2, b_2)) &\geq \min \{ \wedge^F_{(\gamma \times \lambda)}(a_1, b_1), \wedge^F_{(\gamma \times \lambda)}(a_2, b_2) \}. \end{aligned}$$

Hence, $\gamma \times \lambda = \langle \wedge^T, \wedge^I, \wedge^F_{(\gamma \times \lambda)} \rangle$ is a neutrosophic INK-subalgebra of $X_1 \times X_2$.

Theorem 3.6: Let γ and λ be two neutrosophic INK-ideals of X_1 and X_2 . Then $\gamma \times \lambda = \langle \wedge^T, \wedge^I, \wedge^F_{(\gamma \times \lambda)} \rangle$

is a neutrosophic INK-ideal of $X_1 \times X_2$.

Proof. For any (a_1, a_2, a_3) and $(b_1, b_2, b_3) \in X_1 \times X_2$.

$$\begin{aligned}\wedge^T_{(\gamma \times \lambda)}(0, 0) &= \min \{ \wedge^T_{\gamma}(0), \wedge^T_{\lambda}(0) \} \\ &\geq \min \{ \wedge^T_{\gamma}(a), \wedge^T_{\lambda}(b) \} \\ &= \wedge^T_{(\gamma \times \lambda)}(a, b), \\ \wedge^I_{(\gamma \times \lambda)}(0, 0) &= \max \{ \wedge^I_{\gamma}(0), \wedge^I_{\lambda}(0) \} \\ &\leq \max \{ \wedge^I_{\gamma}(a), \wedge^I_{\lambda}(b) \} \\ &= \wedge^I_{(\gamma \times \lambda)}(a, b),\end{aligned}$$

And

$$\begin{aligned}\wedge^F_{(\gamma \times \lambda)}(0, 0) &= \min \{ \wedge^F_{\gamma}(0), \wedge^F_{\lambda}(0) \} \\ &\geq \min \{ \wedge^F_{\gamma}(a), \wedge^F_{\lambda}(b) \} \\ &= \wedge^F_{(\gamma \times \lambda)}(a, b).\end{aligned}$$

Now (a_1, a_2, a_3) and $(b_1, b_2, b_3) \in X_1 \times X_2$.

$$\begin{aligned}\wedge^T_{(\gamma \times \lambda)}((a_1, b_1)) &= \min \{ \wedge^T_{\gamma}(a_1), \wedge^T_{\lambda}(b_1) \} \\ &\geq \min \{ \min \{ \wedge^T_{\gamma}((a_3 * a_1) * ((a_3 * a_2))), \wedge^T_{\gamma}(a_2) \}, \min \{ \wedge^T_{\lambda}((b_3 * b_1) * ((b_3 * b_2))), \wedge^T_{\lambda}(b_2) \} \} \\ &= \min \{ \min \{ \wedge^T_{\gamma}((a_3 * a_1) * ((a_3 * a_2))), \wedge^T_{\lambda}((b_3 * b_1) * ((b_3 * b_2))) \}, \min \{ \wedge^T_{\gamma}(a_2), \wedge^T_{\lambda}(b_2) \} \} \\ &= \min \{ \wedge^T_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1))), ((a_3, b_3) * ((a_2, b_2))), \wedge^T_{(\gamma \times \lambda)}(a_2, b_2) \}, \\ \wedge^I_{(\gamma \times \lambda)}((a_1, b_1)) &= \max \{ \wedge^I_{\gamma}(a_1), \wedge^I_{\lambda}(b_1) \} \\ &\leq \max \{ \max \{ \wedge^I_{\gamma}((a_3 * a_1) * ((a_3 * a_2))), \wedge^I_{\gamma}(a_2) \}, \max \{ \wedge^I_{\lambda}((b_3 * b_1) * ((b_3 * b_2))), \wedge^I_{\lambda}(b_2) \} \} \\ &= \max \{ \max \{ \wedge^I_{\gamma}((a_3 * a_1) * ((a_3 * a_2))), \wedge^I_{\lambda}((b_3 * b_1) * ((b_3 * b_2))) \}, \max \{ \wedge^I_{\gamma}(a_2), \wedge^I_{\lambda}(b_2) \} \} \\ &= \max \{ \wedge^I_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1))), ((a_3, b_3) * ((a_2, b_2))), \wedge^I_{(\gamma \times \lambda)}(a_2, b_2) \},\end{aligned}$$

And

$$\begin{aligned}\wedge^F_{(\gamma \times \lambda)}((a_1, b_1)) &= \min \{ \wedge^F_{\gamma}(a_1), \wedge^F_{\lambda}(b_1) \} \\ &\geq \min \{ \min \{ \wedge^F_{\gamma}((a_3 * a_1) * ((a_3 * a_2))), \wedge^F_{\gamma}(a_2) \}, \min \{ \wedge^F_{\lambda}((b_3 * b_1) * ((b_3 * b_2))), \wedge^F_{\lambda}(b_2) \} \} \\ &= \min \{ \min \{ \wedge^F_{\gamma}((a_3 * a_1) * ((a_3 * a_2))), \wedge^F_{\lambda}((b_3 * b_1) * ((b_3 * b_2))) \}, \min \{ \wedge^F_{\gamma}(a_2), \wedge^F_{\lambda}(b_2) \} \} \\ &= \min \{ \wedge^F_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1))), ((a_3, b_3) * ((a_2, b_2))), \wedge^F_{(\gamma \times \lambda)}(a_2, b_2) \}.\end{aligned}$$

Hence, $\gamma \times \lambda = \langle \wedge^T, \wedge^I, \wedge^F \rangle_{(\gamma \times \lambda)}$ is a neutrosophic INK-ideal of $X_1 \times X_2$.

Theorem 3.7: Let γ and λ be two neutrosophic closed INK-ideals of X_1 and X_2 . Then $\gamma \times \lambda = \langle \wedge^T, \wedge^I, \wedge^F \rangle_{(\gamma \times \lambda)}$ is a neutrosophic closed INK-ideal of $X_1 \times X_2$.

Proof. By using the theorem 3.6. $\gamma \times \lambda = \langle \wedge^T, \wedge^I, \wedge^F \rangle_{(\gamma \times \lambda)}$ is a neutrosophic INK-ideal of $X_1 \times X_2$.

Now for any $a, b \in X_1 \times X_2$, then

$$\begin{aligned}\wedge^T_{(\gamma \times \lambda)}(0, 0) * (a, b) &= \wedge^T_{(\gamma \times \lambda)}((0 * a), (0 * b)) \\ &= \min \{ \wedge^T_{\gamma}((0 * a), \wedge^T_{\lambda}(0 * b)) \} \\ &\geq \min \{ \wedge^T_{\gamma}(a), \wedge^T_{\lambda}(b) \}, \\ \wedge^I_{(\gamma \times \lambda)}(0, 0) * (a, b) &= \wedge^I_{(\gamma \times \lambda)}((0 * a), (0 * b)) \\ &= \max \{ \wedge^I_{\gamma}((0 * a), \wedge^I_{\lambda}(0 * b)) \} \\ &\leq \max \{ \wedge^I_{\gamma}(a), \wedge^I_{\lambda}(b) \}\end{aligned}$$

And

$$\begin{aligned}\wedge^F_{(\gamma \times \lambda)}(0, 0) * (a, b) &= \wedge^F_{(\gamma \times \lambda)}((0 * a), (0 * b)) \\ &= \min \{ \wedge^F_{\gamma}((0 * a), \wedge^F_{\lambda}(0 * b)) \} \\ &\geq \min \{ \wedge^F_{\gamma}(a), \wedge^F_{\lambda}(b) \},\end{aligned}$$

Hence, $\gamma \times \lambda = \langle \wedge^T, \cup^F \rangle_{(\gamma \times \lambda)}$ is a neutrosophic closed INK-ideal of $X_1 \times X_2$.

Theorem 3.8: Let γ and λ be two neutrosophic INK-ideals of X_1 and X_2 . Then

$\gamma \times \lambda = \langle \wedge^T, \cup^F \rangle_{(\gamma \times \lambda)}$ is a neutrosophic INK-ideal of $X_1 \times X_2$.

Proof. Since by theorem 3.7, $\gamma \times \lambda = \langle \wedge^T, \cup^F \rangle_{(\gamma \times \lambda)}$ is a neutrosophic INK-ideal of $X_1 \times X_2$. Then,

$$\begin{aligned} \wedge^T_{(\gamma \times \lambda)}(0, 0) &\geq \wedge^T_{(\gamma \times \lambda)}(a, b) \\ 1 - \wedge^T_{(\gamma \times \lambda)}(0, 0) &\leq 1 - \wedge^T_{(\gamma \times \lambda)}(a, b) \\ \bar{\wedge}^T_{(\gamma \times \lambda)}(0, 0) &\leq \bar{\wedge}^T_{(\gamma \times \lambda)}(a, b), \\ \wedge^I_{(\gamma \times \lambda)}(0, 0) &\leq \wedge^I_{(\gamma \times \lambda)}(a, b) \\ 1 - \wedge^I_{(\gamma \times \lambda)}(0, 0) &\leq 1 - \wedge^I_{(\gamma \times \lambda)}(a, b) \\ \bar{\wedge}^I_{(\gamma \times \lambda)}(0, 0) &\geq \bar{\wedge}^I_{(\gamma \times \lambda)}(a, b) \end{aligned}$$

And

$$\begin{aligned} \wedge^F_{(\gamma \times \lambda)}(0, 0) &\geq \wedge^F_{(\gamma \times \lambda)}(a, b) \\ 1 - \wedge^F_{(\gamma \times \lambda)}(0, 0) &\leq 1 - \wedge^F_{(\gamma \times \lambda)}(a, b) \\ \bar{\wedge}^F_{(\gamma \times \lambda)}(0, 0) &\leq \bar{\wedge}^F_{(\gamma \times \lambda)}(a, b). \end{aligned}$$

Now $(a_1, a_2, a_3), (b_1, b_2, b_3) \in X_1 \times X_2$.

$$\begin{aligned} \wedge^T_{(\gamma \times \lambda)}(a_1, b_1) &\geq \min \{ \wedge^T_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), \wedge^T_{(\gamma \times \lambda)}(a_2, b_2) \} \\ 1 - \wedge^T_{(\gamma \times \lambda)}(a_1, b_1) &\leq 1 - \min \{ \wedge^T_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), \wedge^T_{(\gamma \times \lambda)}(a_2, b_2) \} \\ \bar{\wedge}^T_{(\gamma \times \lambda)}(a, b) &\leq \max \{ 1 - \wedge^T_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), 1 - \wedge^T_{(\gamma \times \lambda)}(a_2, b_2) \} \\ \bar{\wedge}^T_{(\gamma \times \lambda)}(a, b) &\leq \max \{ \bar{\wedge}^T_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), \bar{\wedge}^T_{(\gamma \times \lambda)}(a_2, b_2) \} \end{aligned}$$

$$\begin{aligned} \wedge^I_{(\gamma \times \lambda)}(a_1, b_1) &\leq \max \{ \wedge^I_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), \wedge^I_{(\gamma \times \lambda)}(a_2, b_2) \} \\ 1 - \wedge^I_{(\gamma \times \lambda)}(a_1, b_1) &\leq 1 - \max \{ \wedge^I_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), \wedge^I_{(\gamma \times \lambda)}(a_2, b_2) \} \\ \bar{\wedge}^I_{(\gamma \times \lambda)}(a, b) &\geq \min \{ 1 - \wedge^I_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), 1 - \wedge^I_{(\gamma \times \lambda)}(a_2, b_2) \} \\ \bar{\wedge}^I_{(\gamma \times \lambda)}(a, b) &\geq \min \{ \bar{\wedge}^I_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), \bar{\wedge}^I_{(\gamma \times \lambda)}(a_2, b_2) \} \end{aligned}$$

$$\begin{aligned} \wedge^F_{(\gamma \times \lambda)}(a_1, b_1) &\geq \min \{ \wedge^F_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), \wedge^F_{(\gamma \times \lambda)}(a_2, b_2) \} \\ 1 - \wedge^F_{(\gamma \times \lambda)}(a_1, b_1) &\leq 1 - \min \{ \wedge^F_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), \wedge^F_{(\gamma \times \lambda)}(a_2, b_2) \} \\ \bar{\wedge}^F_{(\gamma \times \lambda)}(a, b) &\leq \max \{ 1 - \wedge^F_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), 1 - \wedge^F_{(\gamma \times \lambda)}(a_2, b_2) \} \\ \bar{\wedge}^F_{(\gamma \times \lambda)}(a, b) &\leq \max \{ \bar{\wedge}^F_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)) * ((a_3, b_3) * ((a_2, b_2))), \bar{\wedge}^F_{(\gamma \times \lambda)}(a_2, b_2) \}. \end{aligned}$$

Hence, $\gamma \times \lambda = \langle \wedge^T, \cup^F \rangle_{(\gamma \times \lambda)}$ is a neutrosophic INK-ideal of $X_1 \times X_2$.

Theorem 3.9: Let $\gamma \times \lambda = \langle \wedge^T, \cup^F \rangle_{(\gamma \times \lambda)}$ is a neutrosophic INK-ideal of $X_1 \times X_2$. Then

$(\gamma \times \lambda)^m = \langle \wedge^T, \cup^F \rangle_{(\gamma \times \lambda)^m}$ is a neutrosophic INK-ideal of $X_1 \times X_2$.

Proof. For any $(a, b) \in X_1 \times X_2$.

$$\begin{aligned} \text{Then } \wedge^T_{(\gamma \times \lambda)}(0, 0) &\geq \wedge^T_{(\gamma \times \lambda)}(a, b) \\ \{ \wedge^T_{(\gamma \times \lambda)}(0, 0) \}^m &\geq \{ \wedge^T_{(\gamma \times \lambda)}(a, b) \}^m \\ \wedge^T_{(\gamma \times \lambda)}(0, 0)^m &\geq \wedge^T_{(\gamma \times \lambda)}(a, b)^m \\ \wedge^T_{(\gamma \times \lambda)^m}(0, 0) &\geq \wedge^T_{(\gamma \times \lambda)^m}(a, b) \end{aligned}$$

$$\begin{aligned} \{ \wedge^I_{(\gamma \times \lambda)}(0, 0) \}^m &\leq \{ \wedge^I_{(\gamma \times \lambda)}(a, b) \}^m \\ \wedge^I_{(\gamma \times \lambda)}(0, 0)^m &\leq \wedge^I_{(\gamma \times \lambda)}(a, b)^m \end{aligned}$$

$$\wedge^l_{(\gamma \times \lambda)^m} (0, 0) \leq \wedge^l_{(\gamma \times \lambda)^m} (a, b)$$

And

$$\{\wedge^f_{(\gamma \times \lambda)} (0, 0)\}^m \geq \{\wedge^f_{(\gamma \times \lambda)} (a, b)\}^m$$

$$\wedge^f_{(\gamma \times \lambda)} (0, 0)^m \geq \wedge^f_{(\gamma \times \lambda)} (a, b)^m$$

$$\wedge^f_{(\gamma \times \lambda)^m} (0, 0) \geq \wedge^f_{(\gamma \times \lambda)^m} (a, b).$$

Now $(a_1, a_2, a_3), (b_1, b_2, b_3) \in X_1 \times X_2$.

$$\{\wedge^t_{(\gamma \times \lambda)} ((a_1, b_1))\}^m \geq \min \{\wedge^t_{(\gamma \times \lambda)} ((a_3, b_3) * ((a_1, b_1))), ((a_3, b_3) * ((a_2, b_2))), \wedge^t_{(\gamma \times \lambda)} (a_2, b_2)\}^m$$

$$\wedge^t_{(\gamma \times \lambda)} (a_1, b_1)^m \geq \min \{\wedge^t_{(\gamma \times \lambda)} (((a_3, b_3) * (a_1, b_1)) * ((a_3, b_3) * (a_2, b_2)))^m, \wedge^t_{(\gamma \times \lambda)} (a_2, b_2)^m\}$$

$$\wedge^t_{(\gamma \times \lambda)^m} (a_1, b_1) \geq \min \{\wedge^t_{(\gamma \times \lambda)^m} (((a_3, b_3) * (a_1, b_1)) * ((a_3, b_3) * (a_2, b_2))), \wedge^t_{(\gamma \times \lambda)^m} (a_2, b_2)\},$$

$$\{\wedge^l_{(\gamma \times \lambda)} ((a_1, b_1))\}^m \leq \max \{\wedge^l_{(\gamma \times \lambda)} ((a_3, b_3) * ((a_1, b_1))), ((a_3, b_3) * ((a_2, b_2))), \wedge^l_{(\gamma \times \lambda)} (a_2, b_2)\}^m$$

$$\wedge^l_{(\gamma \times \lambda)} (a_1, b_1)^m \leq \max \{\wedge^l_{(\gamma \times \lambda)} (((a_3, b_3) * (a_1, b_1)) * ((a_3, b_3) * (a_2, b_2)))^m, \wedge^l_{(\gamma \times \lambda)} (a_2, b_2)^m\}$$

$$\wedge^l_{(\gamma \times \lambda)^m} (a_1, b_1) \leq \max \{\wedge^l_{(\gamma \times \lambda)^m} (((a_3, b_3) * (a_1, b_1)) * ((a_3, b_3) * (a_2, b_2))), \wedge^l_{(\gamma \times \lambda)^m} (a_2, b_2)\}$$

And

$$\{\wedge^f_{(\gamma \times \lambda)} ((a_1, b_1))\}^m \geq \min \{\wedge^f_{(\gamma \times \lambda)} ((a_3, b_3) * ((a_1, b_1))), ((a_3, b_3) * ((a_2, b_2))), \wedge^f_{(\gamma \times \lambda)} (a_2, b_2)\}^m$$

$$\wedge^f_{(\gamma \times \lambda)} (a_1, b_1)^m \geq \min \{\wedge^f_{(\gamma \times \lambda)} (((a_3, b_3) * (a_1, b_1)) * ((a_3, b_3) * (a_2, b_2)))^m, \wedge^f_{(\gamma \times \lambda)} (a_2, b_2)^m\}$$

$$\wedge^f_{(\gamma \times \lambda)^m} (a_1, b_1) \geq \min \{\wedge^f_{(\gamma \times \lambda)^m} (((a_3, b_3) * (a_1, b_1)) * ((a_3, b_3) * (a_2, b_2))), \wedge^f_{(\gamma \times \lambda)^m} (a_2, b_2)\}.$$

Hence, $(\gamma \times \lambda)^m = \langle \wedge^t, \wedge^l, \wedge^f_{(\gamma \times \lambda)^m} \rangle$ is a neutrosophic INK-ideal of $X_1 \times X_2$.

Theorem 3.10: Let $\gamma \times \lambda = \langle \wedge^t, \wedge^l, \wedge^f_{(\gamma \times \lambda)} \rangle$ and $\gamma \times \Gamma = \langle \wedge^t, \wedge^l, \wedge^f_{(\gamma \times \Gamma)} \rangle$ is a neutrosophic INK-ideal of X_1 and X_2 . Then $(\gamma \times \lambda) \cap (\gamma \times \Gamma) = \langle \wedge^t, \wedge^l, \wedge^f_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)} \rangle$ is a neutrosophic INK-ideal of $X_1 \times X_2$.

Proof. For any $(a, b) \in X_1 \times X_2$.

$$\wedge^t_{(\gamma \times \lambda)} (0, 0) \geq \wedge^t_{(\gamma \times \lambda)} (a, b) \text{ and } \wedge^t_{(\gamma \times \lambda)} (0, 0) \geq \wedge^t_{(\gamma \times \Gamma)} (a, b)$$

$$\wedge^t_{(\gamma \times \lambda)} (0, 0), \wedge^t_{(\gamma \times \lambda)} (0, 0) \geq \wedge^t_{(\gamma \times \lambda)} (a, b), \wedge^t_{(\gamma \times \Gamma)} (a, b)$$

$$\min\{\wedge^t_{(\gamma \times \lambda)} (0, 0), \wedge^t_{(\gamma \times \lambda)} (0, 0)\} \geq \min\{\wedge^t_{(\gamma \times \lambda)} (a, b), \wedge^t_{(\gamma \times \Gamma)} (a, b)\}$$

$$\wedge^t_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)} (0, 0) \geq \wedge^t_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)} (a, b),$$

$$\wedge^l_{(\gamma \times \lambda)} (0, 0) \leq \wedge^l_{(\gamma \times \lambda)} (a, b) \text{ and } \wedge^l_{(\gamma \times \lambda)} (0, 0) \leq \wedge^l_{(\gamma \times \Gamma)} (a, b)$$

$$\wedge^l_{(\gamma \times \lambda)} (0, 0), \wedge^l_{(\gamma \times \lambda)} (0, 0) \leq \wedge^l_{(\gamma \times \lambda)} (a, b), \wedge^l_{(\gamma \times \Gamma)} (a, b)$$

$$\max\{\wedge^l_{(\gamma \times \lambda)} (0, 0), \wedge^l_{(\gamma \times \lambda)} (0, 0)\} \leq \max\{\wedge^l_{(\gamma \times \lambda)} (a, b), \wedge^l_{(\gamma \times \Gamma)} (a, b)\}$$

$$\wedge^l_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)} (0, 0) \leq \wedge^l_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)} (a, b),$$

$$\wedge^f_{(\gamma \times \lambda)} (0, 0) \geq \wedge^f_{(\gamma \times \lambda)} (a, b) \text{ and } \wedge^f_{(\gamma \times \lambda)} (0, 0) \geq \wedge^f_{(\gamma \times \Gamma)} (a, b)$$

$$\wedge^f_{(\gamma \times \lambda)} (0, 0), \wedge^f_{(\gamma \times \lambda)} (0, 0) \geq \wedge^f_{(\gamma \times \lambda)} (a, b), \wedge^f_{(\gamma \times \Gamma)} (a, b)$$

$$\min\{\wedge^f_{(\gamma \times \lambda)} (0, 0), \wedge^f_{(\gamma \times \lambda)} (0, 0)\} \geq \min\{\wedge^f_{(\gamma \times \lambda)} (a, b), \wedge^f_{(\gamma \times \Gamma)} (a, b)\}$$

$$\wedge^f_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)} (0, 0) \geq \wedge^f_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)} (a, b).$$

Now $(a_1, a_2, a_3), (b_1, b_2, b_3) \in X_1 \times X_2$.

$$\wedge^t_{(\gamma \times \lambda)} (a_1, b_1) = \min \{\wedge^t_{(\gamma \times \lambda)} ((a_3, b_3) * ((a_1, b_1))), ((a_3, b_3) * ((a_2, b_2))), \wedge^t_{(\gamma \times \lambda)} (a_2, b_2)\},$$

$$\wedge^t_{(\gamma \times \Gamma)} (a_1, b_1) = \min \{\wedge^t_{(\gamma \times \Gamma)} ((a_3, b_3) * ((a_1, b_1))), ((a_3, b_3) * ((a_2, b_2))), \wedge^t_{(\gamma \times \Gamma)} (a_2, b_2)\}.$$

$$\begin{aligned}
& \{\prec^T_{(\gamma \times \lambda)}((a_1, b_1), \prec^T_{(\gamma \times \Gamma)}(a_1, b_1))\} \\
& \geq \left\{ \min\{\prec^T_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)), ((a_3, b_3) * ((a_2, b_2))), \prec^T_{(\gamma \times \lambda)}(a_2, b_2)\} \right. \\
& \quad \left. \min\{\prec^T_{(\gamma \times \Gamma)}((a_3, b_3) * ((a_1, b_1)), ((a_3, b_3) * ((a_2, b_2))), \prec^T_{(\gamma \times \Gamma)}(a_2, b_2)\} \right\} \\
& \geq \min \left\{ \min\{\prec^T_{(\gamma \times \lambda)}((a_3, b_3) * (a_1, b_1)), ((a_3, b_3) * (a_2, b_2)), \prec^T_{(\gamma \times \Gamma)}((a_3, b_3) * (a_1, b_1)), ((a_3, b_3) * (a_2, b_2)), \right. \\
& \quad \left. \min\{\prec^T_{(\gamma \times \lambda)}(a_2, b_2), \prec^T_{(\gamma \times \Gamma)}(a_2, b_2)\} \right\} \\
& \prec^T_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)}(a_1, b_1) \\
& \geq \{\prec^T_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)}((a_3, b_3) * (a_1, b_1)), ((a_3, b_3) * (a_2, b_2)), \prec^T_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)}(a_2, b_2)\}, \\
& \prec^I_{(\gamma \times \lambda)}(a_1, b_1) = \max \{\prec^I_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)), ((a_3, b_3) * ((a_2, b_2))), \prec^I_{(\gamma \times \lambda)}(a_2, b_2)\}, \\
& \prec^I_{(\gamma \times \Gamma)}(a_1, b_1) = \max \{\prec^I_{(\gamma \times \Gamma)}((a_3, b_3) * ((a_1, b_1)), ((a_3, b_3) * ((a_2, b_2))), \prec^I_{(\gamma \times \Gamma)}(a_2, b_2)\}. \\
& \{\prec^I_{(\gamma \times \lambda)}((a_1, b_1), \prec^I_{(\gamma \times \Gamma)}(a_1, b_1))\} \\
& \leq \left\{ \max\{\prec^I_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)), ((a_3, b_3) * ((a_2, b_2))), \prec^I_{(\gamma \times \lambda)}(a_2, b_2)\} \right. \\
& \quad \left. \max\{\prec^I_{(\gamma \times \Gamma)}((a_3, b_3) * ((a_1, b_1)), ((a_3, b_3) * ((a_2, b_2))), \prec^I_{(\gamma \times \Gamma)}(a_2, b_2)\} \right\} \\
& \leq \max \left\{ \max\{\prec^I_{(\gamma \times \lambda)}((a_3, b_3) * (a_1, b_1)), ((a_3, b_3) * (a_2, b_2)), \prec^I_{(\gamma \times \Gamma)}((a_3, b_3) * (a_1, b_1)), ((a_3, b_3) * (a_2, b_2)), \right. \\
& \quad \left. \max\{\prec^I_{(\gamma \times \lambda)}(a_2, b_2), \prec^I_{(\gamma \times \Gamma)}(a_2, b_2)\} \right\} \\
& \prec^I_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)}(a_1, b_1) \\
& \leq \{\prec^I_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)}((a_3, b_3) * (a_1, b_1)), ((a_3, b_3) * (a_2, b_2)), \prec^I_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)}(a_2, b_2)\}
\end{aligned}$$

And

$$\begin{aligned}
& \prec^F_{(\gamma \times \lambda)}(a_1, b_1) = \min \{\prec^F_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)), ((a_3, b_3) * ((a_2, b_2))), \prec^F_{(\gamma \times \lambda)}(a_2, b_2)\}, \\
& \prec^F_{(\gamma \times \Gamma)}(a_1, b_1) = \min \{\prec^F_{(\gamma \times \Gamma)}((a_3, b_3) * ((a_1, b_1)), ((a_3, b_3) * ((a_2, b_2))), \prec^F_{(\gamma \times \Gamma)}(a_2, b_2)\}. \\
& \{\prec^F_{(\gamma \times \lambda)}((a_1, b_1), \prec^F_{(\gamma \times \Gamma)}(a_1, b_1))\} \\
& \geq \left\{ \min\{\prec^F_{(\gamma \times \lambda)}((a_3, b_3) * ((a_1, b_1)), ((a_3, b_3) * ((a_2, b_2))), \prec^F_{(\gamma \times \lambda)}(a_2, b_2)\} \right. \\
& \quad \left. \min\{\prec^F_{(\gamma \times \Gamma)}((a_3, b_3) * ((a_1, b_1)), ((a_3, b_3) * ((a_2, b_2))), \prec^F_{(\gamma \times \Gamma)}(a_2, b_2)\} \right\} \\
& \geq \min \left\{ \min\{\prec^F_{(\gamma \times \lambda)}((a_3, b_3) * (a_1, b_1)), ((a_3, b_3) * (a_2, b_2)), \prec^F_{(\gamma \times \Gamma)}((a_3, b_3) * (a_1, b_1)), ((a_3, b_3) * (a_2, b_2)), \right. \\
& \quad \left. \min\{\prec^F_{(\gamma \times \lambda)}(a_2, b_2), \prec^F_{(\gamma \times \Gamma)}(a_2, b_2)\} \right\} \\
& \prec^F_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)}(a_1, b_1) \\
& \geq \{\prec^F_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)}((a_3, b_3) * (a_1, b_1)), ((a_3, b_3) * (a_2, b_2)), \prec^F_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)}(a_2, b_2)\}.
\end{aligned}$$

Hence, $(\gamma \times \lambda) \cap (\gamma \times \Gamma) = \langle \prec^T, \prec^I, \prec^F \rangle_{(\gamma \times \lambda) \cap (\gamma \times \Gamma)}$ is a neutrosophic INK-ideal of $X_1 \times X_2$.

4. Conclusion

In this paper we applied the notion of direct product of neutrosophic set to INK-ideal of INK-algebra. We have introduced the direct product the concept of neutrosophic INK-algebra and a direct product of closed neutrosophic INK-ideal, and have investigated several properties. We have provided conditions for a direct product of neutrosophic set to be a direct product of neutrosophic INK-ideal in INK-algebra.

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Neutrosophic Simply Soft Open Set in Neutrosophic Soft Topological Space

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Abstract: In this paper, we introduce the notion of Neutrosophic Simply Soft Open (N^{SS-O}) set, Neutrosophic Simply Soft (N^{SS}) compact set in Neutrosophic Soft Topological Spaces (N^S-TS) and investigate several properties of it. Also, we furnish the proofs of some theorems associated with N^{SS} -compact spaces. Then, the notion of neutrosophic simply soft continuous (N^{SS} -continuous) mapping, N^{SS-O} mapping on an N^S-TS and its properties are developed here.

Keywords: neutrosophic simply soft open, neutrosophic simply soft closed, neutrosophic simply soft compact, neutrosophic simply soft continuous.

1. Introduction

Maji (2012; 2013) grounded the idea of Neutrosophic Soft Set (N^S-S) by combining Neutrosophic Set (NS) (Smarandache, 1998) and Soft Set (Molodtsov, 1999). The impact of NS and N^S-S has been reflected in their applicability in decision making (Smarandache & Pramanik, 2016; 2018; Mondal, Pramanik, & Giri, 2018a; 2018b; 2018c; Biswas, Pramanik, & Giri, 2014a; 2014b; 2019; Pramanik, Mallick, & Dasgupta, 2018; Dalapati et al., 2017; Pramanik, Dalapati, Alam, Smarandache, Roy, 2018; Das et al., 2019; Dey, Pramanik, & Giri, 2015; 2016a; 2016b; Karaaslan, 2015; Pramanik & Dalapati, 2016; Pramanik, Dey, & Giri, 2016; Jha et al., 2019). Broumi (2013) further studied N^S-S and proposed generalized N^S-S by combining generalized neutrosophic set (Salama and Alblowi, 2012a) and soft set (Molodtsov, 1999). Smarandache (2018) generalized the soft set to the hypersoft set and plithogenic hypersoft set.

Bera and Mahapatra (2017) defined the Neutrosophic Soft Topological Space (N^S-TS) and introduced separation axioms on N^S-TS by extending the Neutrosophic Topological Space (NTS) introduced by Salama and Alblowi (2012b). Aras, Ozturk, and Bayramov (2019) also studied the separation axioms on N^S-TS . Bera and Mahapatra (2018) presented connectedness and compactness on N^S-TS . Mukherjee, Datta, and Smarandache (2014) presented Interval Valued N^S-TS (IVN^S-TS).

Das and Pramanik (2020) recently presented neutrosophic b-open sets in NTS. Mehmood et al. (2020) presented neutrosophic soft α -open set in N^S -TS.

El Sayed, & Noaman (2013) presented the simply fuzzy generalized open and closed sets, simply fuzzy continuous mappings, simply fuzzy compactness, simply connectedness. In a neutrosophic soft set environment, these concepts have not been introduced.

Research gap: Investigations on Neutrosophic Simply Soft Open (N^{SS} -O) set in N^S -TS,

N^{SS} -continuous mapping, N^{SS} -O mapping, N^{SS} -compactness on an N^S -TS have not been reported in the literature.

Motivation: Since NS generalizes fuzzy set (Zadeh, 1965) and NS is more suitable to deal with uncertainty including inconsistency and indeterminacy, we get the motivation to extend the simply fuzzy set in a neutrosophic environment. To address the research gap, we introduce the N^{SS} -O set, N^{SS} -compactness on an N^S -TS.

The rest of the paper is designed as follows:

Section 2 recalls of some definitions, properties of N^S -S, N^S -T, and N^S -TS. Section 3 introduces N^{SS} -O set, N^{SS} -compactness, and proofs of some theorems, propositions on N^S -TS. Also, in this section, we develop the concept of N^{SS} -continuous mapping, N^{SS} -O mapping. Finally, Section 4 presents concluding remarks.

2. Preliminaries and some properties

Definition 2.1. Assume that W is a non-empty fixed set and P is a collection of parameters. Assume that $NS(W)$ denotes the set of all NSs over W . Then, for any $S \subseteq P$, a pair (N, S) is said to be an N^S -S (Maji, 2012) over W , where $N: S \rightarrow NS(W)$ is a mapping.

An N^S -S (N, S) is represented as follows:

$(N, S) = \{(f, \{(u, T_{N(f)}(u), I_{N(f)}(u), F_{N(f)}(u)) : u \in W\}) : f \in P\}$, where $T_{N(f)}(u)$, $I_{N(f)}(u)$, $F_{N(f)}(u)$ are the truth, indeterminacy, and falsity membership values of each u w.r.t. the parameter $f \in P$.

Example 2.1. Assume that $W = \{m_1, m_2, m_3\}$ is a set consisting of three mobiles and $P = \{f_1(\text{display}), f_2(\text{RAM}), f_3(\text{cost})\}$ be a set of parameters with respect to which the nature of mobile is described.

Let,

$N(f_1) = \{(m_1, 0.6, 0.5, 0.5), (m_2, 0.3, 0.8, 0.5), (m_3, 0.5, 0.3, 0.4)\}$,

$N(f_2) = \{(m_1, 0.7, 0.4, 0.6), (m_2, 0.6, 0.5, 0.4), (m_3, 0.7, 0.3, 0.3)\}$,

$N(f_3) = \{(m_1, 0.8, 0.5, 0.4), (m_2, 0.7, 0.8, 0.5), (m_3, 0.5, 0.3, 0.6)\}$.

Then $(N, P) = \{(f_1, N(f_1)), (f_2, N(f_2)), (f_3, N(f_3))\}$ is an N^S -S over W w.r.t the set P .

Definition 2.2. The complement of an N^S -S (N, P) (Maji, 2012) is denoted by (N^c, P) and is defined by $(N^c, P) = \{(f, \{(u, 1-T_{N(f)}(u), 1-I_{N(f)}(u), 1-F_{N(f)}(u)) : u \in W\}) : f \in P\}$.

Definition 2.3. Assume that (S_1, P) and (S_2, P) are any two N^S -Ss over W . Then (S_1, P) is said to be a neutrosophic soft subset (Maji, 2012) of (S_2, P) if $\forall f \in P$ and $\forall u \in W$, $T_{S_1(f)}(u) \leq T_{S_2(f)}(u)$, $I_{S_1(f)}(u)$

$\geq I_{S_2(f)}(u)$, and $F_{S_1(f)}(u) \geq F_{S_2(f)}(u)$. We write $(S_1, P) \subseteq (S_2, P)$. Then (S_2, P) is called the neutrosophic soft superset of (S_1, P) .

Definition 2.4. Assume that (S_1, P) and (S_2, P) be any two N^S -Ss over W . Then their union (Maji, 2012) is denoted by (H, P) , where $H = S_1 \cup S_2$ and is defined as:

$$(H, P) = \{(f, \{ (u, T_{H(f)}(u), I_{H(f)}(u), F_{H(f)}(u)) : u \in W \}) : f \in P\}, \text{ where } T_{H(f)}(u) = \max \{T_{S_1(f)}(u), T_{S_2(f)}(u)\}, I_{H(f)}(u) = \min \{I_{S_1(f)}(u) \text{ and } I_{S_2(f)}(u)\}, \text{ and } F_{H(f)}(u) = \min \{F_{S_1(f)}(u), F_{S_2(f)}(u)\}.$$

Definition 2.5. Assume that (S_1, P) and (S_2, P) are any two N^S -Ss over W . Then their intersection (Maji, 2012) is denoted by (H, P) , where $H = S_1 \cap S_2$ and is defined as:

$$(H, P) = \{(f, \{ (u, T_{H(f)}(u), I_{H(f)}(u), F_{H(f)}(u)) : u \in W \}) : f \in P\}, \text{ where } T_{H(f)}(u) = \min \{T_{S_1(f)}(u), T_{S_2(f)}(u)\}, I_{H(f)}(u) = \max \{I_{S_1(f)}(u) \text{ and } I_{S_2(f)}(u)\}, \text{ and } F_{H(f)}(u) = \max \{F_{S_1(f)}(u), F_{S_2(f)}(u)\}.$$

Definition 2.6. An N^S -S (S, P) over a non-empty set W is said to be a null N^S -S (Bera, & Mahapatra, 2017) if $T_{S(f)}(u) = 0$, $I_{S(f)}(u) = 1$, $F_{S(f)}(u) = 1 \forall u \in W$ w.r.t. the parameter $f \in P$. It is denoted by $0_{(S, P)}$.

Definition 2.7. An N^S -S (S, P) over a non-empty set W is called an absolute N^S -S (Bera, & Mahapatra, 2017) if $T_{S(f)}(u) = 1$, $I_{S(f)}(u) = 0$, $F_{S(f)}(u) = 0 \forall u \in W$ w.r.t. the parameter $f \in P$. It is denoted by $1_{(S, P)}$.

Clearly, $1^c_{(S, P)} = 0_{(S, P)}$ and $0^c_{(S, P)} = 1_{(S, P)}$.

Definition 2.8 Assume that N^S -S (W, P) be the collection of all N^S -Ss over W via parameters in P and $\tau \subseteq N^S$ -S (W, P) . Then τ is said to be an N^S -T (Bera, & Mahapatra, 2017) on (W, P) if the following axioms are satisfied.

- (i) $0_{(N, P)}, 1_{(N, P)} \in \tau$;
- (ii) $(R, P), (Q, P) \in \tau \Rightarrow (R \cap Q, P) \in \tau$
- (iii) $\{(Q_i, P) : i \in \Delta\} \subseteq \tau \Rightarrow (\cup_{i \in \Delta} Q_i, P) \in \tau$.

The triplet (W, P, τ) is said to be an N^S -TS. Every element of τ is called an N^S -O set. An N^S -S (S, P) is called a neutrosophic soft closed (N^S -C) set iff its complement (S^c, P) is an N^S -O set.

Definition 2.9. Assume that (W, P, τ) be an N^S -TS over (W, P) and (M, P) be an arbitrary element of N^S -S (W, P) . Then the neutrosophic soft interior ($N^{S_{int}}$) (Bera, & Mahapatra, 2017) and neutrosophic soft closure ($N^{S_{cl}}$) of (M, P) is defined as follows:

$$N^{S_{int}}(M, P) = \cup \{(Q, P) : (Q, P) \text{ is an } N^S\text{-O set in } W \text{ and } (Q, P) \subseteq (M, P)\}.$$

$$N^{S_{cl}}(M, P) = \cap \{(Q, P) : (Q, P) \text{ is an } N^S\text{-C set in } W \text{ and } (M, P) \subseteq (Q, P)\}.$$

Proposition 2.1. Assume that (W, P, τ) be an N^S -TS over (W, P) and $M, N \in N^S$ -S (W, P) . Then the following results holds:

- (i) $M \subseteq N \Rightarrow N^{S_{cl}}(M) \subseteq N^{S_{cl}}(N) \text{ \& } N^{S_{int}}(M) \subseteq N^{S_{int}}(N)$;
- (ii) $N^{S_{int}}(M) \subseteq M \subseteq N^{S_{cl}}(M)$;
- (iii) $N^{S_{int}}(0_{(N, P)}) = 0_{(N, P)} \text{ \& } N^{S_{cl}}(0_{(N, P)}) = 0_{(N, P)}$;
- (iv) $N^{S_{int}}(1_{(N, P)}) = 1_{(N, P)} \text{ \& } N^{S_{cl}}(1_{(N, P)}) = 1_{(N, P)}$;
- (v) $N^{S_{int}}N^{S_{int}}(M) = M \text{ \& } N^{S_{cl}}N^{S_{cl}}(M) = M$;
- (vi) $N^{S_{int}}(M \cap N) = N^{S_{int}}(M) \cap N^{S_{int}}(N)$;
- (vii) $N^{S_{int}}(M \cup N) \supseteq N^{S_{int}}(M) \cup N^{S_{int}}(N)$;

$$(viii) N^{S_{cl}}(M \cup N) = N^{S_{cl}}(M) \cup N^{S_{cl}}(N);$$

$$(ix) N^{S_{cl}}(M \cap N) \subseteq N^{S_{cl}}(M) \cap N^{S_{cl}}(N).$$

Proof. For proof see (Bera & Mahapatra, 2017).

Proposition 2.2. Assume that (X, E, τ) be an N^S -TS over (X, E) and $M \in NSS(X, E)$. Then the following results hold:

$$(i) (N^{S_{int}}(M))^c = N^{S_{cl}}(M^c);$$

$$(ii) (N^{S_{cl}}(M))^c = N^{S_{int}}(M^c).$$

Proof. For proof see (Bera & Mahapatra, 2017)

Definition 2.10. Assume that (W, P, τ) be an N^S -TS over (W, P) . Then a family $\{(Q_\alpha, P): \alpha \in \Delta\}$ of N^S -O sets in (W, P, τ) , is called an N^S -O cover (Bera & Mahapatra, 2018) of an N^S -S (Q, P) if $(Q, P) \subseteq \bigcup_{\alpha \in \Delta} (Q_\alpha, P)$.

Definition 2.11. An (W, P, τ) over (W, P) is said to be an N^S -compact set (Bera, & Mahapatra, 2018) if every N^S -O cover of W has a finite subcover.

3. Neutrosophic Simply Soft Open Set

Definition 3.1. Assume that (W, P, τ) be an N^S -TS over (W, P) . Then (Q, P) , a neutrosophic soft subset of (W, P, τ) is said to be a neutrosophic simply soft open (N^{SS} -O) set if $N^{S_{int}}N^{S_{cl}}(Q, P) \subseteq N^{S_{cl}}N^{S_{int}}(Q, P)$.

Definition 3.2. Assume that (W, P, τ) be an N^S -TS over (W, P) . Then (Q, P) , a neutrosophic soft subset of (W, P, τ) is said to be a neutrosophic simply soft closed (N^{SS} -C) set if its complement is an N^{SS} -O set in (W, P, τ) .

Theorem 3.3. In an N^S -TS (W, P, τ) , every N^S -O set is an N^{SS} -O set.

Proof. Assume that (Q, P) be an N^S -O set in an N^S -TS (W, P, τ) . Therefore $N^{S_{int}}(Q, P) = (Q, P)$.

Now, $(Q, P) \subseteq N^{S_{cl}}(Q, P)$. This implies $(Q, P) \subseteq N^{S_{cl}}N^{S_{int}}(Q, P)$.

Now $(Q, P) \subseteq N^{S_{cl}}N^{S_{int}}(Q, P)$

$$\Rightarrow N^{S_{cl}}(Q, P) \subseteq N^{S_{cl}}N^{S_{cl}}N^{S_{int}}(Q, P)$$

$$= N^{S_{cl}}N^{S_{int}}(Q, P) \quad [\text{since } N^{S_{cl}}N^{S_{int}}(Q, P) \text{ is an } N^S\text{-C set in } (W, P, \tau)]$$

$$\Rightarrow N^{S_{cl}}(Q, P) \subseteq N^{S_{cl}}N^{S_{int}}(Q, P) \quad (1)$$

$$\text{Again, } N^{S_{int}}N^{S_{cl}}(Q, P) \subseteq N^{S_{cl}}(Q, P) \quad (2)$$

From (1) and (2), we obtain,

$$N^{S_{int}}N^{S_{cl}}(Q, P) \subseteq N^{S_{cl}}N^{S_{int}}(Q, P).$$

Hence (Q, P) is an N^{SS} -O set in (W, P, τ) .

Definition 3.3. Assume that (W, P, τ) be an N^S -TS. Then the Neutrosophic Simply Soft interior ($N^{SS_{int}}$) and Neutrosophic Simply Soft closure ($N^{SS_{cl}}$) of a neutrosophic soft subset (Q, P) of (W, P, τ) is defined by

$$N^{SS_{int}}(M, P) = \bigcup \{(Q, P): (Q, P) \text{ is an } N^{SS}\text{-O set in } W \text{ and } (Q, P) \subseteq (M, P)\}.$$

$$N^{SS_{cl}}(M, P) = \bigcap \{(K, P): (K, P) \text{ is an } N^{SS}\text{-C set in } W \text{ and } (M, P) \subseteq (K, P)\}.$$

Definition 3.4. Assume that (W, P, τ) be an N^S -TS over (W, P) . Then a collection $\{(Q_\alpha, P): \alpha \in \Delta\}$ of N^{SS} -O sets in (W, P, τ) , is said to be an N^{SS} -O cover of an N^S -S (Q, P) if $(Q, P) \subseteq \bigcup_{\alpha \in \Delta} (Q_\alpha, P)$.

Definition 3.5. An N^S -TS (W, P, τ) over (W, P) is said to be an N^{SS} -compact space if every N^{SS} -O cover of W has a finite subcover.

Definition 3.6. A neutrosophic soft subset (K, P) of (W, P, τ) is said to be an N^{SS} -Compact set relative to W if every N^{SS} -O cover of (K, P) has a finite subcover.

Definition 3.6. A function $\psi: (W, P, \tau_1) \rightarrow (G, P, \tau_2)$ is called an N^{SS} -continuous function if for each N^S -O set (Z, P) in G , $\psi^{-1}(Z, P)$ is an N^{SS} -O set in W .

Definition 3.7. A function $\psi: (W, P, \tau_1) \rightarrow (G, P, \tau_2)$ is said to be an N^{SS} -O function if $\psi(K, P)$ is an N^{SS} -O set in G whenever (K, P) is an N^{SS} -O set in W .

Theorem 3.2. Every N^{SS} -C subset of an N^{SS} -compact space (W, P, τ) is an N^{SS} -compact set relative to W .

Proof. Assume that (W, P, τ) be an N^{SS} -compact space and (K, P) be an N^{SS} -C set in (W, P, τ) . Therefore (K^c, P) is an N^{SS} -O set in (W, P, τ) . Let $U = \{(U_i, P): i \in \Delta \text{ and } (U_i, P) \in N^{SS}\text{-O}(W)\}$ be an N^{SS} -O cover of (K, P) . Then $\mathcal{H} = \{(K^c, P)\} \cup U$ is an N^{SS} -O cover of W . Since W is an N^{SS} -compact space then it has a finite subcover say $\{(H_1, P), (H_2, P), (H_3, P), \dots, (H_n, P), (K^c, P)\}$. Then $\{(H_1, P), (H_2, P), (H_3, P), \dots, (H_n, P)\}$ is a neutrosophic finite simply soft open cover of (K, P) . Hence (K, P) is an N^{SS} -compact set relative to W .

Theorem 3.3. Every N^{SS} -compact space is a neutrosophic soft compact space.

Proof. Assume that (W, P, τ) is an N^{SS} -compact space. Suppose that (W, P, τ) is not an N^S -compact space. Therefore, there exists an N^S -O cover \mathfrak{R} (say) of W , which has no finite subcover. Since every N^S -O set is an N^{SS} -O set, so we have an N^{SS} -O cover \mathfrak{R} of W , which has no finite subcover. This contradicts our assumption. Hence (W, P, τ) must be an N^S -compact space.

Theorem 3.4. If $\psi: (W, P, \tau_1) \rightarrow (G, P, \tau_2)$ is an N^{SS} -O function and (G, P, τ_2) is an N^{SS} -compact space then (W, P, τ_1) is also an N^S -compact space.

Proof. Assume that $\psi: (W, P, \tau_1) \rightarrow (G, P, \tau_2)$ be an N^{SS} -O function and (G, P, τ_2) is an N^{SS} -compact space. Let $\mathcal{H} = \{(K_i, P): i \in \Delta \text{ and } (K_i, P) \in N^S\text{-O}(W)\}$ be an N^S -O cover of W . This implies that $\psi(\mathcal{H}) = \{\psi(K_i, P): i \in \Delta \text{ and } \psi(K_i, P) \in N^S\text{-O}(G)\}$ is an N^{SS} -O cover of G . Since (G, P, τ_2) is an N^{SS} -compact space, so there exists a finite subcover say $\{\psi(K_1, P), \psi(K_2, P), \dots, \psi(K_n, P)\}$ such that $M \subseteq \cup \{\psi(K_i, P): i = 1, 2, \dots, n\}$. This implies that $\{(K_1, P), (K_2, P), \dots, (K_n, P)\}$ is a finite subcover for W . Therefore (W, P, τ_1) is an N^S -compact space.

Theorem 3.5. If $\psi: (W, P, \tau_1) \rightarrow (G, P, \tau_2)$ is an N^S -continuous function then for each N^S -compact set (Q, P) relative to W , $\psi(Q, P)$ is an N^{SS} -compact set in (G, P, τ_2) .

Proof. Assume that $\psi: (W, P, \tau_1) \rightarrow (G, P, \tau_2)$ is an N^S -continuous function and (Q, P) is an N^{SS} -compact set relative to W . Let $\mathcal{H} = \{(H_i, P): i \in \Delta \text{ and } (H_i, P) \text{ be an } N^S\text{-O set in } G\}$ be an N^S -O cover of $\psi(Q, P)$. Therefore, by hypothesis $\psi^{-1}(\mathcal{H}) = \{\psi^{-1}(H_i, P): i \in \Delta \text{ and } \psi^{-1}(H_i, P) \text{ is an } N^S\text{-O set in } W\}$ is an N^S -O cover of $\psi^{-1}(\psi(Q, P)) = (Q, P)$. Since every N^S -O set is an N^{SS} -O set, so $\psi^{-1}(\mathcal{H}) = \{\psi^{-1}(H_i, P): i \in \Delta \text{ and } \psi^{-1}(H_i, P) \text{ is an } N^S\text{-O set in } W\}$ is an N^{SS} -O cover of (Q, P) . Since (Q, P) is an N^{SS} -compact set relative to

W , so there exists a finite subcover of (Q, P) say $\{\psi^{-1}(H_1, P), \psi^{-1}(H_2, P), \dots, \psi^{-1}(H_n, P)\}$ such that $(Q, P) \subseteq \cup_i \{\psi^{-1}(H_i, P): i = 1, 2, \dots, n\}$.

Now $(Q, P) \subseteq \cup_i \{\psi^{-1}(H_i, P): i = 1, 2, \dots, n\}$.

$\Rightarrow \psi(Q, P) \subseteq \cup_i \{\psi(\psi^{-1}(H_i, P)): i = 1, 2, \dots, n\} = \cup_i \{(H_i, P): i = 1, 2, \dots, n\}$.

Therefore there exist a finite subcover $\{(H_1, P), (H_2, P), \dots, (H_n, P)\}$ of $\psi(Q, P)$ such that $\psi(Q, P) \subseteq \cup_i \{(H_i, P): i = 1, 2, \dots, n\}$. Hence $\psi(Q, P)$ is an N^S -compact set relative to G .

Theorem 3.6. Every neutrosophic soft continuous function from an N^S -TS (W, P, τ_1) to another N^S -TS (G, P, τ_2) is an N^{SS} -continuous function.

Proof. Assume that $\psi: (W, P, \tau_1) \rightarrow (G, P, \tau_2)$ be a neutrosophic soft continuous function. Let (Q, P) be an N^S -O set in (G, P, τ_2) . Since ψ is a neutrosophic soft continuous function, so $\psi^{-1}(Q, P)$ is an N^S -O set in (W, P, τ_1) . Since every N^S -O set is an N^{SS} -O set, so $\psi^{-1}(Q, P)$ is an N^{SS} -O set in (G, P, τ_2) . Therefore $\psi^{-1}(Q, P)$ is an N^{SS} -O set in (G, P, τ_2) , whenever (Q, P) is an N^S -O set in (W, P, τ_1) . Hence $\psi: (W, P, \tau_1) \rightarrow (G, P, \tau_2)$ is an N^{SS} -continuous function.

Theorem 3.8. If $\psi: (W, P, \tau_1) \rightarrow (G, P, \tau_2)$ is an N^{SS} -continuous mapping and $\gamma: (G, P, \tau_2) \rightarrow (H, P, \tau_3)$ is a neutrosophic soft continuous mapping, then the composition mapping $\gamma \circ \psi: (W, P, \tau_1) \rightarrow (H, P, \tau_3)$ is an N^{SS} -continuous mapping.

Proof. Assume that (Q, P) is an N^S -O set in (H, P, τ_3) . Since $\gamma: (G, P, \tau_2) \rightarrow (H, P, \tau_3)$ is a neutrosophic soft continuous mapping, so $\gamma^{-1}(Q, P)$ is an N^S -O set in (G, P, τ_2) . Again since $\psi: (W, P, \tau_1) \rightarrow (G, P, \tau_2)$ is an N^{SS} -continuous mapping, so $\psi^{-1}(\gamma^{-1}(Q, P)) = (\gamma \circ \psi)^{-1}(Q, P)$ is an N^{SS} -O set in (W, P, τ_1) . Hence $(\gamma \circ \psi)^{-1}(Q, P)$ is an N^{SS} -O set in (W, P, τ_1) , whenever (Q, P) is an N^S -O set in (H, P, τ_3) . Therefore $\gamma \circ \psi: (W, P, \tau_1) \rightarrow (H, P, \tau_3)$ is an N^{SS} -continuous mapping.

4. Conclusions

In this article, we have introduced the N^{SS} -O cover, N^{SS} -compact set, in an N^S -TS. By defining N^{SS} -O cover, N^{SS} -compact set, we have proved some propositions, theorems on N^S -TS. In the future, we hope that based on these notions of neutrosophic simply soft compactness, many new investigations can be carried out. The proposed concepts can be explored in various neutrosophic hybrid sets such as rough neutrosophic set (Broumi, Smarandache, & Dhar, 2014), bipolar neutrosophic set (Deli, Ali, & Smarandache, 2015), etc.

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Bipolar quadripartitioned single valued neutrosophic rough set

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Abstract. Here bipolar quadripartitioned single valued neutrosophic rough (BQSVNR) set is introduced. Some basic set theoretic terminologies like constant BQSVNR set, subthood of two BQSVNR sets are shown. Algebraic operations like union, intersection and complement have also been defined. Different types of measure like similarity measure, quasi similarity measure and distance measures between two BQSVNR sets have been discussed with their properties. Again various measures of similarity namely distance based similarity measure, cosine similarity measure, membership function based similarity measure are introduced in this paper. A medical diagnosis problem has been solved using similarity measure at the end.

Keywords: SVN set, SVNR set, QSVNR set, BQSVNR set

1. Introduction

Neutrosophic set (NS) was introduced by Smarandache in 2005 [7] as a generalization of intuitionistic fuzzy set (IFS) [4]. Here in NS the indeterminacy factor is independent where as indeterminacy in IFS is dependent completely on the truth and falsity values. This is why NS's are more general in nature and can handle various types of data including incomplete, inconsistent and even para consistent data. Wang et. al [10] in 2010 has introduced a new version of NS called single valued neutrosophic set (SVN) which is much easier for the application than NS in solving physical problems. Currently the theory of NS has becomes a very successful and flourishing area of research and many researchers are doing research in different areas of both theory and application [22, 31, 32, 34–38]. In 2016 R. Chatterjee et. al [23] defined another new version of NS set called Quadripartitioned SVN (QSVN) set where the indeterminacy factor consists two divisions namely contradiction and ignorance. This QSVN set is expected to give better results and more realistic value as it characterizes the indeterminacy factor into two parts which is based on the notions of four valued logic of Belnap [2]. On the other hand

bipolar SVN set [19] is an identification of polarity. Thus bipolar concept which is very useful in many decision making concept as a large number of human decision making is based on double sided or bipolar judgement thinking on a positive side and negative side. Again Rough Set (RS) by Pawlak [3] is a well established technique to express imperfect information by employing vagueness to the boundary region of a set. RS theory has various applications in artificial intelligence and especially in machine learning [5, 6, 8, 9, 14, 22, 24].

Here the idea of BQSVNR set which is an further extension of the articles [17, 23] is introduced. In literature many types of NS exist together with various types of applications [10, 12, 16, 19, 21, 25–30, 33]. However we refer our readers to study NS theory [7], SVN theory [10], QSVN theory [23], RS theory [3], BRS Theory [24] for their convenience. In this manuscript we have defined BQSVNR set with it's various types of operations. Also various similarity measures of BQSVNR set are discussed. Later a uncertainty based real scientific problem has been worked out by using BQSVNR set model. Finally the future work related to our paper is given.

2. BQSVNR set

Throughout this paper we will consider all the definitions over $X \neq \phi$ together with an equivalence relation R and we will denote it by (X, R) . For the many other properties i.e. entropy, various types of similarity measures of a NS, SVN sets, BNS etc we refer our readers to follow any of the monograph say [7, 12, 18].

Definition 2.1. Suppose A be a BQSVN set in (X, R) with positive membership degrees $T^+(m), C^+(m), I^+(m), F^+(m)$ respectively and negative membership degrees $T^-(m), C^-(m), I^-(m), F^-(m)$ -respectively of an element $m \in X$. The lower and upper approximations of A in (X, R) denoted by $\underline{L}(A)$ and $\overline{L}(A)$ respectively are defined as follows:

$$\underline{L}(A) = \{ \langle m, T_{\underline{A}}^+(m), C_{\underline{A}}^+(m), I_{\underline{A}}^+(m), F_{\underline{A}}^+(m), T_{\underline{A}}^-(m), C_{\underline{A}}^-(m), I_{\underline{A}}^-(m), F_{\underline{A}}^-(m) \rangle | m \in [m]_R \subseteq X \}$$

$$\overline{L}(A) = \{ \langle m, T_{\overline{A}}^+(m), C_{\overline{A}}^+(m), I_{\overline{A}}^+(m), F_{\overline{A}}^+(m), T_{\overline{A}}^-(m), C_{\overline{A}}^-(m), I_{\overline{A}}^-(m), F_{\overline{A}}^-(m) \rangle | m \in [m]_R \subseteq X \},$$

where,

$$\begin{aligned}
 T_{\underline{A}}^+(m) &= \wedge_{m \in [m]_R} T_A^+(m), C_{\underline{A}}^+(m) = \wedge_{m \in [m]_R} C_A^+(m), I_{\underline{A}}^+(m) = \vee_{m \in [m]_R} I_A^+(m), \\
 F_{\underline{A}}^+(m) &= \vee_{m \in [m]_R} F_A^+(m), T_{\underline{A}}^-(m) = \vee_{m \in [m]_R} T_A^-(m), C_{\underline{A}}^-(m) = \vee_{m \in [m]_R} C_A^-(m), \\
 I_{\underline{A}}^-(m) &= \wedge_{m \in [m]_R} I_A^-(m), F_{\underline{A}}^-(m) = \wedge_{m \in [m]_R} F_A^-(m), T_{\overline{A}}^+(m) = \vee_{m \in [m]_R} T_A^+(m), \\
 C_{\overline{A}}^+(m) &= \vee_{m \in [m]_R} C_A^+(m), I_{\overline{A}}^+(m) = \wedge_{m \in [m]_R} I_A^+(m), F_{\overline{A}}^+(m) = \wedge_{m \in [m]_R} F_A^+(m), \\
 T_{\overline{A}}^-(m) &= \wedge_{m \in [m]_R} T_A^-(m), C_{\overline{A}}^-(m) = \wedge_{m \in [m]_R} C_A^-(m), I_{\overline{A}}^-(m) = \vee_{m \in [m]_R} I_A^-(m), \\
 F_{\overline{A}}^-(m) &= \vee_{m \in [m]_R} F_A^-(m),
 \end{aligned}$$

where $0 \leq T_{\underline{A}}^+(m) + C_{\underline{A}}^+(m) + I_{\underline{A}}^+(m) + F_{\underline{A}}^+(m) \leq 4$, $-4 \leq T_{\underline{A}}^-(m) + C_{\underline{A}}^-(m) + I_{\underline{A}}^-(m) + F_{\underline{A}}^-(m) \leq 0$, $0 \leq T_{\overline{A}}^+(m) + C_{\overline{A}}^+(m) + I_{\overline{A}}^+(m) + F_{\overline{A}}^+(m) \leq 4$, $-4 \leq T_{\overline{A}}^-(m) + C_{\overline{A}}^-(m) + I_{\overline{A}}^-(m) + F_{\overline{A}}^-(m) \leq 0$ and \vee, \wedge mean “max” and “min” operators respectively, $T_A(m), C_A(m), I_A(m), F_A(m)$ are the respective membership function of m w.r.t A . $\underline{L}(A)$ and $\overline{L}(A)$ are two bipolar QSVN sets in X . The pair $(\underline{L}(A), \overline{L}(A))$ is called BQSVNR set in (X, R) .

Example 2.2. Consider the case where four economists m_1, m_2, m_3, m_4 were asked to give their opinion on the statement “Rate of economical growth of India in 2020 will cross the rate of economic growth in the year 2019”. Each economists will give concern in terms of degree of agreement, agreement or disagreement both, neither agreement nor disagreement, disagreement together with positive and negative aspects respectively. Let R_1 be a set on U of all economists which may be considered as follows:

$$m, n \in U, mR_1n \text{ iff } m \text{ and } n$$

both belongs to same organization i.e. IMF, London school of economics etc.

The aggregate of their opinion can be very well expressed into the following equivalent class R_1 as following:

$$U/R_1 = \{\{m_1, m_2\}, \{m_3\}, \{m_4\}\}$$

We can develop a BQSVN set A on the basis of the economists opinion as follows:

$$\begin{aligned}
 A = \{ & (m_1, (0.8, 0.6, 0.2, 0.2, -0.4, -0.5, -0.3, -0.7)), \\
 & (m_2, (0.4, 0.6, 0.6, 0.8, -0.6, -0.5, -0.4, -0.8)), \\
 & (m_3, (0.5, 0.5, 0.7, 0.1, -0.8, -0.6, -0.4, -0.6)), \\
 & (m_4, (0.6, 0.7, 0.4, 0.1, -0.5, -0.3, -0.6, -0.4)) \}.
 \end{aligned}$$

Now by Definition 2.1, we have,

$$\begin{aligned}\underline{L}(A) &= \{(m_1, (0.4, 0.6, 0.6, 0.8, -0.4, -0.5, -0.4, -0.8)), \\ &\quad (m_2, (0.4, 0.6, 0.6, 0.8, -0.4, -0.5, -0.4, -0.8)), \\ &\quad (m_3, (0.5, 0.5, 0.7, 0.1, -0.8, -0.6, -0.4, -0.6)), \\ &\quad (m_4, (0.6, 0.7, 0.4, 0.1, -0.5, -0.3, -0.6, -0.4))\} \\ \overline{L}(A) &= \{(m_1, (0.8, 0.6, 0.2, 0.2, -0.6, -0.5, -0.3, -0.7)), \\ &\quad (m_2, (0.8, 0.6, 0.2, 0.2, -0.6, -0.5, -0.3, -0.7)), \\ &\quad (m_3, (0.5, 0.5, 0.7, 0.1, -0.8, -0.6, -0.4, -0.6)), \\ &\quad (m_4, (0.6, 0.7, 0.4, 0.1, -0.5, -0.3, -0.6, -0.4))\}\end{aligned}$$

Hence $(\underline{L}(A), \overline{L}(A))$ provides the rate of growth of India in 2020 in comparison with the rate of growth in 2019.

Definition 2.3. Suppose A be a BQSVN set in (X, R) . If

- (i) $\underline{L}(A) = \overline{L}(A)$, then $(\underline{L}(A), \overline{L}(A))$ is called constant BQSVNR set in (X, R) .
- (ii) $\forall m \in [m]_R \cap \underline{L}(A)$ (and $\overline{L}(A)$), $T_A^+(m) = 1 = C_A^+(m)$, $I_A^+(m) = F_A^+(m) = 0$, $T_A^-(m) = 0 = C_A^-(m)$, $I_A^-(m) = F_A^-(m) = 1$, then $(\underline{L}(A), \overline{L}(A))$ is called an unit BQSVNR set in (X, R) .
- (iii) $\forall m \in [m]_R \cap \underline{L}(A)$ (and $\overline{L}(A)$), $T_A^+(m) = 0 = C_A^+(m)$, $I_A^+(m) = F_A^+(m) = 1$, $T_A^-(m) = 1 = C_A^-(m)$, $I_A^-(m) = F_A^-(m) = 0$, then $(\underline{L}(A), \overline{L}(A))$ is called zero BQSVNR set in (X, R) and it is denoted by Φ .

Now some set-theoretic operations on BQSVNR set over (X, R) will be studied.

Definition 2.4. Consider $L(A) = (\underline{L}(A), \overline{L}(A))$ is a BQSVNR set in (X, R) . We define complement BQSVNR set $L^c(A)$ of $L(A)$ as $L^c(A) = ((\underline{L}(A))^c, (\overline{L}(A))^c)$, where

$$\begin{aligned}(\underline{L}(A))^c &= \{ \langle x, F_{\underline{A}}^+(m), 1 - I_{\underline{A}}^+(m), 1 - C_{\underline{A}}^+(m), T_{\underline{A}}^+(m), \\ &\quad F_{\underline{A}}^-(m), 1 - I_{\underline{A}}^-(m), 1 - C_{\underline{A}}^-(m), T_{\underline{A}}^-(m) \rangle | m \in [m]_R \subseteq X \} \\ (\overline{L}(A))^c &= \{ \langle x, F_{\overline{A}}^+(m), 1 - I_{\overline{A}}^+(m), 1 - C_{\overline{A}}^+(m), T_{\overline{A}}^+(m), \\ &\quad F_{\overline{A}}^-(m), 1 - I_{\overline{A}}^-(m), 1 - C_{\overline{A}}^-(m), T_{\overline{A}}^-(m) \rangle | m \in [m]_R \subseteq X \}\end{aligned}$$

Definition 2.5. Suppose $A = (\underline{L}(A), \overline{L}(A))$ and $B = (\underline{L}(B), \overline{L}(B))$ are two BQSVNR set over X . We say $A \subseteq B$ if $\underline{L}(A) \subseteq \underline{L}(B)$, $\overline{L}(A) \subseteq \overline{L}(B)$ i.e.

$$\begin{aligned}T_{\underline{A}}^+(m) &\leq T_{\underline{B}}^+(m), C_{\underline{A}}^+(m) \leq C_{\underline{B}}^+(m), I_{\underline{A}}^+(m) \geq I_{\underline{B}}^+(m), F_{\underline{A}}^+(m) \geq F_{\underline{B}}^+(m), T_{\underline{A}}^-(m) \geq \\ T_{\underline{B}}^-(m), C_{\underline{A}}^-(m) &\geq C_{\underline{B}}^-(m), I_{\underline{A}}^-(m) \leq I_{\underline{B}}^-(m), F_{\underline{A}}^-(m) \leq F_{\underline{B}}^-(m),\end{aligned}$$

$$T_A^+(m) \geq T_B^+(m), C_A^+(m) \geq C_B^+(m), I_A^+(m) \leq I_B^+(m), F_A^+(m) \leq F_B^+(m), T_A^-(m) \leq T_B^-(m), C_A^-(m) \leq C_B^-(m), I_A^-(m) \geq I_B^-(m), F_A^-(m) \geq F_B^-(m) \forall m \in [m]_R \subseteq X.$$

Definition 2.6. Suppose $A = (\underline{L}(A), \bar{L}(A))$ and $B = (\underline{L}(B), \bar{L}(B))$ are two BQSVNR set over X . Then the union of A, B i.e. $A \cup B = (\underline{L}(A) \cup \underline{L}(B), \bar{L}(A) \cup \bar{L}(B))$ is defined as: $T_{\underline{A}}^+(m) \vee T_{\underline{B}}^+(m), C_{\underline{A}}^+(m) \vee C_{\underline{B}}^+(m), I_{\underline{A}}^+(m) \wedge I_{\underline{B}}^+(m), F_{\underline{A}}^+(m) \wedge F_{\underline{B}}^+(m), T_{\underline{A}}^-(m) \wedge T_{\underline{B}}^-(m), C_{\underline{A}}^-(m) \wedge C_{\underline{B}}^-(m), I_{\underline{A}}^-(m) \vee I_{\underline{B}}^-(m), F_{\underline{A}}^-(m) \vee F_{\underline{B}}^-(m), T_{\bar{A}}^+(m) \wedge T_{\bar{B}}^+(m), C_{\bar{A}}^+(m) \wedge C_{\bar{B}}^+(m), I_{\bar{A}}^+(m) \vee I_{\bar{B}}^+(m), F_{\bar{A}}^+(m) \vee F_{\bar{B}}^+(m), T_{\bar{A}}^-(m) \vee T_{\bar{B}}^-(m), C_{\bar{A}}^-(m) \vee C_{\bar{B}}^-(m), I_{\bar{A}}^-(m) \wedge I_{\bar{B}}^-(m), F_{\bar{A}}^-(m) \wedge F_{\bar{B}}^-(m) \forall m \in [m]_R \subseteq X$.

Definition 2.7. Suppose $A = (\underline{L}(A), \bar{L}(A))$ and $B = (\underline{L}(B), \bar{L}(B))$ are two BQSVNR set over X . Then the intersection of A, B i.e. $A \cap B = (\underline{L}(A) \cap \underline{L}(B), \bar{L}(A) \cap \bar{L}(B))$ is defined as: $T_{\underline{A}}^+(m) \wedge T_{\underline{B}}^+(m), C_{\underline{A}}^+(m) \wedge C_{\underline{B}}^+(m), I_{\underline{A}}^+(m) \vee I_{\underline{B}}^+(m), F_{\underline{A}}^+(m) \vee F_{\underline{B}}^+(m), T_{\underline{A}}^-(m) \vee T_{\underline{B}}^-(m), C_{\underline{A}}^-(m) \vee C_{\underline{B}}^-(m), I_{\underline{A}}^-(m) \wedge I_{\underline{B}}^-(m), F_{\underline{A}}^-(m) \wedge F_{\underline{B}}^-(m), T_{\bar{A}}^+(m) \vee T_{\bar{B}}^+(m), C_{\bar{A}}^+(m) \vee C_{\bar{B}}^+(m), I_{\bar{A}}^+(m) \wedge I_{\bar{B}}^+(m), F_{\bar{A}}^+(m) \wedge F_{\bar{B}}^+(m), T_{\bar{A}}^-(m) \wedge T_{\bar{B}}^-(m), C_{\bar{A}}^-(m) \wedge C_{\bar{B}}^-(m), I_{\bar{A}}^-(m) \vee I_{\bar{B}}^-(m), F_{\bar{A}}^-(m) \vee F_{\bar{B}}^-(m) \forall m \in [m]_R \subseteq X$.

Proposition 2.8. Consider three BQSVNR sets $\Theta_1, \Theta_2, \Theta_3$ in (X, R) . Then for all for BQSVNR sets over X we have the following:

- (i) $\Theta_1 \cup \Theta_2 = \Theta_2 \cup \Theta_1$; $\Theta_1 \cap \Theta_2 = \Theta_2 \cap \Theta_1$.
- (ii) $\Theta_1 \cup (\Theta_2 \cup \Theta_3) = (\Theta_1 \cup \Theta_2) \cup \Theta_3$; $\Theta_1 \cap (\Theta_2 \cap \Theta_3) = (\Theta_1 \cap \Theta_2) \cap \Theta_3$
- (iii) $\Theta_3 \cap (\Theta_3 \cup \Theta_2) = \Theta_3$; $\Theta_3 \cup (\Theta_3 \cap \Theta_2) = \Theta_3$.
- (iv) $(\Theta_1^c)^c = \Theta_1$.
- (v) $(\Theta_1 \cup \Theta_2)^c = \Theta_1^c \cap \Theta_2^c$; $(\Theta_1 \cap \Theta_2)^c = \Theta_1^c \cup \Theta_2^c$
- (vi) $\Theta_1 \cup \Theta_1 = \Theta_1$; $\Theta_1 \cap \Theta_1 = \Theta_1$.

We omit the proof of the Proposition 2.8 as it is very straight forward.

3. Different similarity measures of BQSVNR sets

Consider an universal set $X \neq \phi$ and denote the set of BQSVNR set over (X, R) by $\mathcal{B}(X)$.

Definition 3.1. A mapping $s : \mathcal{B}(X) \times \mathcal{B}(X) \rightarrow [0, 1]$ is called a similarity measure iff for $W_1, W_2 \in \mathcal{B}(X)$,

- (i) $s(W_1, W_2) = s(W_2, W_1)$
- (ii) $0 \leq s(W_1, W_2) < 1$ and $s(W_1, W_2) = 1$ iff $W_1 = W_2$.
- (iii) for any $W_1, W_2, W_3 \in \mathcal{B}(X)$, $W_1 \subset W_2 \subset W_3$, $s(W_1, W_3) \leq s(W_1, W_2) \wedge s(W_2, W_3)$.

Although in Definition 3.1 the condition (iii) exists but some familiar similarity measure techniques i.e. weighted similarity measure, cosine similarity measures etc fail to satisfy it. On the other hand these similarity techniques has a wide application in real world discission

making problems. Thus it is essential to introduce a new definition of similarity measure say “Quasi Similarity Measure” which omits the condition (iii) of Definition 3.1.

Definition 3.2. Consider $\mathcal{B}(X)$, the set of BQSVNR sets over an universe X . Then a function $s' : \mathcal{B}(X) \times \mathcal{B}(X) \rightarrow [0, 1]$ is called a quasi similarity measure iff for $W_1, W_2 \in \mathcal{B}(X)$,

- (i) $s(W_1, W_2) = s(W_2, W_1)$
- (ii) $0 \leq s(W_1, W_2) < 1$ and $s(W_1, W_2) = 1$ iff $W_1 = W_2$.

3.1. Distance measures between two BQSVNR sets

Definition 3.3. A function $d_b : \mathcal{B}(X) \times \mathcal{B}(X) \rightarrow \mathbb{R}^+$ is called a distance measure for BQSVNR sets iff for $W_1, W_2, W_3 \in \mathcal{B}(X)$,

- (i) $d_b(W_1, W_2) = d_b(W_2, W_1)$
- (ii) $d_b(W_1, W_2) \geq 0$ and $d_b(W_1, W_2) = 0$ iff $W_1 = W_2$.
- (iii) $d_b(W_1, W_2) \leq d_b(W_1, W_3) + d_b(W_3, W_2)$.

Clearly d_b is a metric on $\mathcal{B}(X)$. Suppose $\Theta, \Gamma \in \mathcal{B}(X)$ over an universal set $X = \{x_1, x_2, \dots, x_n\}$.

Definition 3.4. The Hamming distance $h(\Theta, \Gamma)$ between two BQSVNR sets Θ and Γ is defined as

$$h(\Theta, \Gamma) = \min \{ \{h(\underline{\Theta}, \underline{\Gamma})\}, \{h(\overline{\Theta}, \overline{\Gamma})\} \} \text{ where,}$$

$$\begin{aligned} h(\underline{\Theta}, \underline{\Gamma}) &= \left\{ \sum_{j=1}^n |T_{\underline{\Theta}}^+(x_j) - T_{\underline{\Gamma}}^+(x_j)| + |C_{\underline{\Theta}}^+(x_j) - C_{\underline{\Gamma}}^+(x_j)| + |I_{\underline{\Theta}}^+(x_j) - I_{\underline{\Gamma}}^+(x_j)| + \right. \\ &\quad |F_{\underline{\Theta}}^+(x_j) - F_{\underline{\Gamma}}^+(x_j)| + |T_{\underline{\Theta}}^-(x_j) - T_{\underline{\Gamma}}^-(x_j)| + |C_{\underline{\Theta}}^-(x_j) - C_{\underline{\Gamma}}^-(x_j)| + \\ &\quad \left. |I_{\underline{\Theta}}^-(x_j) - I_{\underline{\Gamma}}^-(x_j)| + |F_{\underline{\Theta}}^-(x_j) - F_{\underline{\Gamma}}^-(x_j)| \right\} \\ h(\overline{\Theta}, \overline{\Gamma}) &= \left\{ \sum_{j=1}^n |T_{\overline{\Theta}}^+(x_j) - T_{\overline{\Gamma}}^+(x_j)| + |C_{\overline{\Theta}}^+(x_j) - C_{\overline{\Gamma}}^+(x_j)| + |I_{\overline{\Theta}}^+(x_j) - I_{\overline{\Gamma}}^+(x_j)| + \right. \\ &\quad |F_{\overline{\Theta}}^+(x_j) - F_{\overline{\Gamma}}^+(x_j)| + |T_{\overline{\Theta}}^-(x_j) - T_{\overline{\Gamma}}^-(x_j)| + |C_{\overline{\Theta}}^-(x_j) - C_{\overline{\Gamma}}^-(x_j)| + \\ &\quad \left. |I_{\overline{\Theta}}^-(x_j) - I_{\overline{\Gamma}}^-(x_j)| + |F_{\overline{\Theta}}^-(x_j) - F_{\overline{\Gamma}}^-(x_j)| \right\} \forall x_j \in X. \end{aligned}$$

Definition 3.5. The Normalized Hamming distance between Θ and Γ is defined as $h_N(\Theta, \Gamma) = \frac{1}{8n}(h(\Theta, \Gamma))$.

Definition 3.6. The Euclidean distance $E(\Theta, \Gamma)$ is defined as follows:

$$E(\Theta, \Gamma) = \min\{\{E(\underline{\Theta}, \underline{\Gamma})\}, \{E(\overline{\Theta}, \overline{\Gamma})\}\} \text{ where,}$$

$$E(\underline{\Theta}, \underline{\Gamma}) = \left\{ \sum_{j=1}^n |T_{\underline{\Theta}}^+(x_j) - T_{\underline{\Gamma}}^+(x_j)|^2 + |C_{\underline{\Theta}}^+(x_j) - C_{\underline{\Gamma}}^+(x_j)|^2 + |I_{\underline{\Theta}}^+(x_j) - I_{\underline{\Gamma}}^+(x_j)|^2 + \right. \\ \left. |F_{\underline{\Theta}}^+(x_j) - F_{\underline{\Gamma}}^+(x_j)|^2 + |T_{\underline{\Theta}}^-(x_j) - T_{\underline{\Gamma}}^-(x_j)|^2 + |C_{\underline{\Theta}}^-(x_j) - C_{\underline{\Gamma}}^-(x_j)|^2 + \right. \\ \left. |I_{\underline{\Theta}}^-(x_j) - I_{\underline{\Gamma}}^-(x_j)|^2 + |F_{\underline{\Theta}}^-(x_j) - F_{\underline{\Gamma}}^-(x_j)|^2 \right\}^{\frac{1}{2}}$$

$$E(\overline{\Theta}, \overline{\Gamma}) = \left\{ \sum_{j=1}^n |T_{\overline{\Theta}}^+(x_j) - T_{\overline{\Gamma}}^+(x_j)|^2 + |C_{\overline{\Theta}}^+(x_j) - C_{\overline{\Gamma}}^+(x_j)|^2 + |I_{\overline{\Theta}}^+(x_j) - I_{\overline{\Gamma}}^+(x_j)|^2 + \right. \\ \left. |F_{\overline{\Theta}}^+(x_j) - F_{\overline{\Gamma}}^+(x_j)|^2 + |T_{\overline{\Theta}}^-(x_j) - T_{\overline{\Gamma}}^-(x_j)|^2 + |C_{\overline{\Theta}}^-(x_j) - C_{\overline{\Gamma}}^-(x_j)|^2 + \right. \\ \left. |I_{\overline{\Theta}}^-(x_j) - I_{\overline{\Gamma}}^-(x_j)|^2 + |F_{\overline{\Theta}}^-(x_j) - F_{\overline{\Gamma}}^-(x_j)|^2 \right\}^{\frac{1}{2}} \forall x_j \in X.$$

Definition 3.7. The normalized Euclidean distance $Q(\Theta, \Gamma)$ is defined as follows:

$$Q(\Theta, \Gamma) = \frac{1}{2\sqrt{2n}} E(\Theta, \Gamma).$$

Gradually distance measurement process which gives an idea about the similarities between two BQSVNR sets becomes the main attraction among the researchers. Also different MCDM problems can be solved using similarity measures technique [20, 21]. On the other hand many mathematicians have used a variety of distance-based operators say induced weighted aggregation distance (IOWAD) operators, an extended version of common OWA operators to solve various problems [11, 13, 15]. However we will only concentrate only on the following distance oriented similarity measures.

3.2. Distance oriented similarity measure between two BQSVNR sets

Consider two BQSVNR set Θ_1, Θ_2 over $\mathcal{B}(X)$. Based on all previously defined distances two new similarity measures $S_1(\Theta_1, \Theta_2), S_2(\Theta_1, \Theta_2)$ for a pair of BQSVNR set Θ_1, Θ_2 can be defined:

$$S_1(\Theta_1, \Theta_2) = \frac{1}{1 + h(\Theta_1, \Theta_2)}, \quad S_2(\Theta_1, \Theta_2) = e^{-\alpha \cdot h(\Theta_1, \Theta_2)},$$

where $\alpha \in \mathbb{R}^+$ is the *steepness measure* of $S_2(\Theta_1, \Theta_2)$. In a similar way using Euclidian distance, we define another pair of similarity measure $S'_1(\Theta_1, \Theta_2), S'_2(\Theta_1, \Theta_2)$ as follows:

$$S'_1(\Theta_1, \Theta_2) = \frac{1}{1 + E(\Theta_1, \Theta_2)}, \quad S'_2(\Theta_1, \Theta_2) = e^{-\beta \cdot E(\Theta_1, \Theta_2)}.$$

where $\beta \in \mathbb{R}^+$ is the *steepness measure* of $S'_2(\Theta_1, \Theta_2)$. one can easily seen that $S_1(\Theta_1, \Theta_2), S_2(\Theta_1, \Theta_2), S'_1(\Theta_1, \Theta_2), S'_2(\Theta_1, \Theta_2)$ satisfies the axioms of Definitions 3.1 for two BQSVNR sets Θ_1, Θ_2 .

3.3. Cosine similarity measure of BQSVNR sets

Here we will discuss the cosine similarity measure, a basic similarity measure technique between two BQSVNR sets. To obtain this similarity measure we represent two BQSVNR sets as vectors. We illustrate our proposed new cosine similarity measure C_{BQSVNR} of BQSVNR sets as following:

Definition 3.8. Consider $A, B \in \mathcal{B}(X)$. For each $x_j \in X$, $j = 1, 2, \dots, n$, we define

$$C_{BQSVNR}(A, B) = \frac{1}{n} \sum_{j=1}^n \frac{S_1}{S_2 \cdot S_3}, \text{ where,}$$

$$S_1 = \partial T_A(x_j) \partial T_B(x_j) + \partial C_A(x_j) \partial C_B(x_j) + \partial I_A(x_j) \partial I_B(x_j) + \partial F_A(x_j) \partial F_B(x_j),$$

$$S_2 = \sqrt{\partial T_A(x_j)^2 + \partial C_A(x_j)^2 + \partial I_A(x_j)^2 + \partial F_A(x_j)^2},$$

$$S_3 = \sqrt{\partial T_B(x_j)^2 + \partial C_B(x_j)^2 + \partial I_B(x_j)^2 + \partial F_B(x_j)^2},$$

$$\text{where } \partial T_X^+(x_j) = \frac{T_X^+(x_j) + T_{\bar{X}}^+(x_j)}{2}, \partial T_X^-(x_j) = \frac{T_X^-(x_j) + T_{\bar{X}}^-(x_j)}{2}$$

$$\partial C_X^+(x_j) = \frac{C_X^+(x_j) + C_{\bar{X}}^+(x_j)}{2}, \partial C_X^-(x_j) = \frac{C_X^-(x_j) + C_{\bar{X}}^-(x_j)}{2}$$

$$\partial I_X^+(x_j) = \frac{I_X^+(x_j) + I_{\bar{X}}^+(x_j)}{2}, \partial I_X^-(x_j) = \frac{I_X^-(x_j) + I_{\bar{X}}^-(x_j)}{2}$$

$$\partial F_X^+(x_j) = \frac{F_X^+(x_j) + F_{\bar{X}}^+(x_j)}{2}, \partial F_X^-(x_j) = \frac{F_X^-(x_j) + F_{\bar{X}}^-(x_j)}{2}$$

$$\partial T_X(x_j) = \frac{\partial T_X^+(x_j) + \partial T_X^-(x_j)}{2}, \partial C_X(x_j) = \frac{\partial C_X^+(x_j) + \partial C_X^-(x_j)}{2}$$

$$\partial I_X(x_j) = \frac{\partial I_X^+(x_j) + \partial I_X^-(x_j)}{2}, \partial F_X(x_j) = \frac{\partial F_X^+(x_j) + \partial F_X^-(x_j)}{2},$$

where $X \in \{A, B\}$.

Theorem 3.9. $C_{BQSVNR}(A, B)$ is a similarity measure between two BQSVNR sets $A, B \in \mathcal{B}(X)$.

We omit the proof as it is very simple.

3.4. Similarity measures of BQSVNR sets based on membership values

Consider $A, B \in \mathcal{B}(X)$. For each $x_j \in X, j = 1, 2, \dots, n$ and for $k = 1, 2, \dots, 4$ define the functions $\underline{h}_k^+, \underline{h}_k^-, \overline{h}_k^+, \overline{h}_k^- : X \rightarrow [0, 1]$ respectively as

$$\begin{aligned}\underline{h}_1^\pm(x_j) &= |T_A^\pm(x_j) - T_B^\pm(x_j)|, \\ \underline{h}_2^\pm(x_j) &= |F_A^\pm(x_j) - F_B^\pm(x_j)|, \\ \underline{h}_3^\pm(x_j) &= \frac{1}{3}(h_1^\pm(x_j) + h_2^\pm(x_j) + |C_A^\pm(x_j) - C_B^\pm(x_j)|), \\ \underline{h}_4^\pm(x_j) &= |I_A^\pm(x_j) - I_B^\pm(x_j)|, \\ \overline{h}_1^\pm(x_j) &= |T_A^\pm(x_j) - T_B^\pm(x_j)|, \\ \overline{h}_2^\pm(x_j) &= |F_A^\pm(x_j) - F_B^\pm(x_j)|, \\ \overline{h}_3^\pm(x_j) &= \frac{1}{3}(h_1^\pm(x_j) + h_2^\pm(x_j) + |C_A^\pm(x_j) - C_B^\pm(x_j)|), \\ \overline{h}_4^\pm(x_j) &= |I_A^\pm(x_j) - I_B^\pm(x_j)|\end{aligned}$$

Now based on the above functions a new similarity measure $\tilde{S}(A, B)$ can be defined as follows:

$$\begin{aligned}\tilde{S}(A, B) &= 1 - \frac{1}{4n} \left[\sum_{j=1}^n \sum_{k=1}^4 \overline{h}_k^+(x_j) + \sum_{j=1}^n \sum_{k=1}^4 \overline{h}_k^-(x_j) + \sum_{j=1}^n \sum_{k=1}^4 \underline{h}_k^+(x_j) + \right. \\ &\quad \left. \sum_{j=1}^n \sum_{k=1}^4 \underline{h}_k^-(x_j) \right].\end{aligned}$$

The following theorem is obvious:

Theorem 3.10. $\tilde{S}(A, B)$ is a similarity measure between $A, B \in \mathcal{B}(X)$.

Proof. For a BQSVNR set all the positive membership values of $\underline{T}, \underline{C}, \underline{I}, \underline{F}$, $\overline{T}, \overline{C}, \overline{I}, \overline{F}$ lie between $[0, 1]$ and the negative membership values of $\underline{T}, \underline{C}, \underline{I}, \underline{F}$, $\overline{T}, \overline{C}, \overline{I}, \overline{F}$ lie between $[-1, 0]$. Among all these quantities, all has maximum value 1 and the minimum value -1 . As a result $0 \leq \tilde{S}(A, B) \leq 1$. Again $\tilde{S}(A, B) = 1$ implies that

$$\begin{aligned}T_A^+(x_j) &= T_B^+(x_j), C_A^+(x_j) = C_B^+(x_j), I_A^+(x_j) = I_B^+(x_j), F_A^+(x_j) = F_B^+(x_j), \\ T_A^-(x_j) &= T_B^-(x_j), C_A^-(x_j) = C_B^-(x_j), I_A^-(x_j) = I_B^-(x_j), F_A^-(x_j) = F_B^-(x_j), \\ T_A^+(x_j) &= T_B^+(x_j), C_A^+(x_j) = C_B^+(x_j), I_A^+(x_j) = I_B^+(x_j), F_A^+(x_j) = F_B^+(x_j), \\ T_A^-(x_j) &= T_B^-(x_j), C_A^-(x_j) = C_B^-(x_j), I_A^-(x_j) = I_B^-(x_j), F_A^-(x_j) = F_B^-(x_j) \\ \forall x_j &\in X.\end{aligned}$$

Lastly for $A, B, C \in \mathcal{B}(X)$ we suppose that $A \subseteq B \subseteq C$. Now by the Definition 2.5 we have $\forall x_j \in X, \forall j = 1, 2, \dots, 4$

$$\tilde{S}(A, C) < \tilde{S}(A, B) \wedge \tilde{S}(B, C).$$

Hence the result follows. \square

3.5. Weighted similarity measure

The weighted similarity measure between $A, B \in \mathcal{B}(X)$ is defined as follows:

$$S^w(A, B) = 1 - \frac{1}{4n} [\sum_{j=1}^n \sum_{k=1}^4 w_j \overline{h_k}^+(x_j) + \sum_{j=1}^n \sum_{k=1}^4 w_j \overline{h_k}^-(x_j) + \sum_{j=1}^n \sum_{k=1}^4 w_j \underline{h_k}^+(x_j) + \sum_{j=1}^n \sum_{k=1}^4 w_j \underline{h_k}^-(x_j)]^{\frac{1}{l}}.$$

where l is any integer defined to be the order of similarity, w_i are the weights corresponding with x_j , $j = 1, 2, \dots, n$ s.t. $\sum_{j=1}^n w_j = 1$. Using the same proof procedure of the Theorem 3.10, $S^w(A, B)$ is also a measure of similarity between the two BQSVNR sets $A, B \in \mathcal{B}(X)$.

4. An application of BQSVNR sets

By using a BQSVNR set model a real world medical diagnosis problem can be represented very well. To solve these type of medical problem similarity measure technique between two BQSVNR set is quite powerful procedure. Using these similarity measure technique anyone can detect whether a patient is being suffered with a disease or not. In between June to September it is seen that H1N1 virus spreads out rapidly in Kolkata and its sub-urban area of West Bengal India. The patient of these particular virus effected decease has primarily 4 symptoms, namely headache, high fever, cough, red rashes in the body. But in every patient the primary symptoms are not clearly visible. Also in many different viral infections these symptoms are common. The process of classification of patients by considering a variety of symptoms is a difficult task. Our similarity measurement technique which considers patients versus symptoms record provides an approximate way to treat a patient. The basic feature of our study considers only the positive as well as negative value of truth, ignorance, contradiction and false value respectively of each element of the BQSVNR sets.

Suppose P_2 and P_3 be two persons who are suspected to be infected by H1N1 virus. Let $D = \{\text{headache, high fever, cough, red rashes in the body}\}$ be a set of symptoms. Consider P_1 is a model patient who are infected by H1N1 virus. Our solution is to examine the condition of P_2, P_3 w.r.t. the symptoms of P_1 in BQSVNR environment. We have represented our problem

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as a BQSVNR set model as follows:

$$\begin{aligned}
 P_1 &= (\overline{N(P_1)}, \underline{N(P_1)}) \\
 &= \langle (0.6, 0.4, 0.2, 0.4, -0.4, -0.5, -0.1, -0.8), (0.8, 0.6, 0.7, 0.1, -0.4, -0.3, -0.4, -0.5) \rangle / x_1 \\
 &+ \langle (0.5, 0.5, 0.6, 0.4, -0.4, -0.3, -0.2, -0.9), (0.7, 0.4, 0.4, 0.1, -0.3, -0.2, -0.7, -0.6) \rangle / x_2 \\
 &+ \langle (0.7, 0.5, 0.6, 0.8, -0.5, -0.4, -0.8, -0.7), (0.6, 0.5, 0.5, 0.4, -0.2, -0.4, -0.6, -0.8) \rangle / x_3 \\
 &+ \langle (0.8, 0.6, 0.7, 0.1, -0.4, -0.2, -0.6, -0.3), (0.5, 0.6, 0.2, 0.3, -0.1, -0.4, -0.6, -0.9) \rangle / x_4. \\
 P_2 &= (\overline{N(P_2)}, \underline{N(P_2)}) \\
 &= \langle (0.4, 0.3, 0.3, 0.4, -0.5, -0.6, -0.1, -0.6), (0.4, 0.6, 0.6, 0.2, -0.4, -0.4, -0.4, -0.5) \rangle / x_1 \\
 &+ \langle (0.6, 0.5, 0.4, 0.3, -0.4, -0.3, -0.1, -0.8), (0.6, 0.6, 0.3, 0.2, -0.4, -0.2, -0.5, -0.4) \rangle / x_2 \\
 &+ \langle (0.8, 0.5, 0.5, 0.6, -0.6, -0.6, -0.6, -0.5), (0.6, 0.4, 0.5, 0.5, -0.3, -0.5, -0.4, -0.6) \rangle / x_3 \\
 &+ \langle (0.7, 0.7, 0.6, 0.2, -0.4, -0.3, -0.8, -0.3), (0.5, 0.6, 0.4, 0.5, -0.5, -0.7, -0.4, -0.2) \rangle / x_4. \\
 P_3 &= (\overline{N(P_3)}, \underline{N(P_3)}) \\
 &= \langle (0.7, 0.8, 0.9, 0.1, -0.4, -0.4, -0.5, -0.2), (0.6, 0.4, 0.7, 0.6, -0.5, -0.1, -0.9, -0.7) \rangle / x_1 \\
 &+ \langle (0.9, 0.2, 0.3, 0.7, -0.1, -0.4, -0.7, -0.8), (0.5, 0.6, 0.7, 0.2, -0.4, -0.4, -0.7, -0.6) \rangle / x_2 \\
 &+ \langle (0.1, 0.4, 0.8, 0.7, -0.4, -0.5, -0.5, -0.6), (0.4, 0.7, 0.4, 0.2, -0.5, -0.6, -0.4, -0.1) \rangle / x_3 \\
 &+ \langle (0.5, 0.6, 0.4, 0.8, -0.1, -0.4, -0.6, -0.9), (0.6, 0.4, 0.7, 0.4, -0.2, -0.6, -0.5, -0.2) \rangle / x_4.
 \end{aligned}$$

Now from Definition 3.4 to Definition 3.7 respectively, we have

$$\begin{aligned}
 h(P_1, P_2) &= 3.2, \quad h(P_1, P_3) = 7.1 \\
 h_N(P_1, P_2) &= 0.1, \quad h_N(P_1, P_3) = 0.845 \\
 E(P_1, P_2) &= 0.693, \quad E(P_1, P_3) = 1.609
 \end{aligned}$$

Now we calculate the following measures (as given by Section 3.2) between the pair of persons P_1, P_2 and P_1, P_3 as follows:

$$S'_1(P_1, P_2) = 0.591, \quad S'_1(P_1, P_3) = 0.621$$

Since any effected area the probability of infecting a healthy people by H1N1 virus is 80% [?] hence we have taken the steepness measure i.e. α, β as 0.8. From this we have,

$$S'_2(P_1, P_2) = 0.574, \quad S'_2(P_1, P_3) = 0.76$$

Since in between any two BQSVNR sets there must be similarity thus we restrict ourselves if the similarity measure is > 0.6 . Thus from the similarity measures S'_1, S'_2 we can conclude that the patient P_3 has a higher chance to be infected by H1N1 virus than the patient P_2 .

5. Conclusion

The fuzzy set theory (FST) [1] was introduced almost 55 years ago. After its invention, in next half a decade time, many generalizations of FST has been proposed such as intuitionistic fuzzy sets, interval valued fuzzy sets, hesitant fuzzy sets, bipolar fuzzy sets etc and also many other new theories like rough sets, soft sets, neutrosophic sets etc. has came into existence. The chief purpose of all these theories is to model real life situations under different uncertainties using available tools. But it is now a well established fact that no single theory is capable of modeling all different types of uncertainty. For example, fuzzy set can't model uncertainty due to incompleteness; intuitionistic fuzzy sets can't handle para consistent information, rough set is not suitable for handling situations with graded belongingness, soft set is not useful in modeling situations with vague boundaries. Therefore it is a common practice to combine two or more such sets to form a hybrid set. Hybrid set possesses the characteristics of more than one set and therefore has greater capabilities in handling uncertain situations. On the other hand four valued logic has multiple uses in many areas such as digital circuits and data transmission. The QSVN sets utilize the power of four valued logic in modeling uncertainty. We here introduced and investigated a new type of hybrid set called BQSVNR. The BQSVNR set is an extended version of QSVN set, bipolar set as well as rough set. It can handle uncertain situation arisen due to factors like fuzziness, incompleteness, vagueness, haziness, para compactness and bipolarity. Therefore our set is more capable of modeling uncertainty in a better way than any other existing set. In future, one can apply our newly developed hybrid set to model different real life problems. Also one may try to extend our set to bipolar multi-partitioned single valued neutrosophic rough sets and study its properties.

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Sumudu Transform for Solving Second Order Ordinary Differential Equation under Neutrosophic Initial Conditions

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Abstract: The ordinary differential equation of second order is being used in many engineering disciplines and sciences to model many real-life problems. These problems are mostly uncertain, vague and incomplete and thus they require some more advanced tool for modelling. Neutrosophic logic becomes the solution of all these kind of uncertain problems as it describe the conditions of uncertainty which occurs during the process of modelling on the basis of grade of membership of truth values, indeterminacy values and falsity values, that means it consider all the uncertain parameters on the basis of these degrees. In this research paper, we have considered the ordinary differential equation of second order with neutrosophic numbers as initial conditions of spring mass system is solved using Sumudu transform method which has advantage of unit preserving property over the well established Laplace Transform method. The solution obtained at various computational points by this method is shown in the form of table. Furthermore, the results obtained at different (α, β, γ) -cut and time values are also depicted graphically and are verified analytically by de-fuzzifying the data.

Keywords: Fuzzy numbers; Neutrosophic numbers; Neutrosophic triangular numbers; Strongly-generalized differentiability; Sumudu transform.

1. Introduction

In our daily lives we encounter many situations that are mostly vague, uncertain, ambiguous, incomplete, and inconsistent. With this limited and incomplete information, it becomes problematic to model and find the solution of the problem in a precise manner. To deal with these kinds of situations, L.A. Zadeh [1], in 1965 discovered the fuzzy set theory as the extension of classical set theory. This theory is more powerful than classical set theory in the sense that it considers uncertain environmental conditions as membership values, whereas classical set theory only studies true or false values and do not analyze any values between them. In real life situations, we often get information in the form of ambiguous words like good, very good, bad, and very bad, etc., but all these facts may differ from one person to another, because it is related to human thinking and hence depends on the human point of view. In fuzzy set theory these terms are known as linguistic terms and to these linguistic terms some membership grades are assigned according to their significance. All of these linguistic terms together with their membership degrees are written in the form of ordered pairs and finally fuzzy sets are formed using these ordered pairs. Sometimes, we have to deal with fuzzified numerical data also. For example, when we ask students how many hours do you self-study in a day? Then they use statements such as about 50 minutes a day, about 40 minutes or 50 minutes a day, or more than 50 minutes a day, etc. and these types of numbers are known as

fuzzy numbers. The set of real numbers act as a superset of these fuzzy numbers. These real values actually represent the grade of membership of a fuzzy set A , well defined over the universal set X and grade of membership $\mu_A(x) \in [0,1]$.

In some practical problems, considering only the membership value is not enough, it is also necessary to consider the non-membership value. The fuzzy sets are defined by considering the elements which considers only grade of membership of any information and grade of non-membership is not considered. Atanassov [2] studied in this direction and introduced another type of fuzzy set known as intuitionistic fuzzy set (IFS), which is the natural extension of fuzzy set and is more applicable in real life situation. Intuitionistic fuzzy sets considers as an extension of fuzzy sets, because it not only provide the information which belongs to the set but also which does not belong to the set. For example, suppose we want to collect the information of liking of any particular subject among students of class A and a questionnaire has been prepared for this purpose, which is distributed to the students so that they can fill it and then submit. The student can either fill the plus sign response to show the liking, minus sign to show dislikes or there is also one option to show nothing. In this way, for every student X , two responses are recorded, viz; $A(x)$ = number of acceptances/likes, $N(x)$ = number of non-acceptances/ dislikes. Another concept is also available in the world of uncertainty, which is known as Neutrosophic set theory, which studies the cause, description, and possibility of neutral thoughts. Neutrosophic sets deal with belongingness of truth values, indeterminate values and false values and it was first introduced by Florentin Smarandache [3]. In Neutrosophic logic, grade of membership of Truth values (T), Indeterminate values (I) and False values (F) has been defined within the non-standard interval $]^{-}0, 1^{+}[$. Neutrosophic set theory with non-standard interval works well in philosophical point of view. But practically when we deal with science and engineering problems, it is not possible to define data within this non-standard interval. To overcome this problem single valued Neutrosophic sets was defined by the researcher Wang et al. [4] by considering unit interval $[0,1]$ in its standard form. The values within this interval are called Neutrosophic numbers. Aal SIA et al. [5], Deli and Subas [6], Ye [7] and Chakraborty et al. [8], etc., defined different kind of Neutrosophic numbers. Abdel-Basset et al. [9-13] defined advanced Neutrosophic numbers and presented results on recent pandemic COVID-19, decision making problems, supply chain model, industrial and management problems. In this way, lots of work has been done for the development of the Neutrosophic set theory with applications in real life problems (see for instance [14-18]).

Neutrosophic logic becomes one of the important and valuable tools in almost all area of science and engineering to model various real life phenomenon using differential equations with uncertain and imprecise parameters. Fuzzy differential equations (FDE) were introduced by Dubois et al. [19-21], by considering only membership values. For defining FDE, fuzzy numbers and corresponding fuzzy functions were discovered by Chang et al. [22]. Further the concept of intuitionistic fuzzy differential equations [23-25] came into the existence containing both membership and non-membership values as its parameters. To study the solutions of these fuzzy differential equations, the necessity arises to understand the concept of derivatives in fuzzy environment. In this direction lots of work has been reported in the literature, such as differentials for fuzzy functions were discussed by Puri and Ralescu [26] and Fuzzy Calculus is studied by Goetschel and Voxman [27], etc. Fuzzy derivative concept is used in the solution of ordinary differential equations of first order with initial conditions by Seikkala and Kaleva [28-29]. Buckley and Feuring [30-31] have solved ordinary differential equation of n th order containing fuzzy initial conditions. In 2005, Bede and Gal [32] worked on fuzzy-valued function and defined generalized differentiability for that. Further he provided the solution of fuzzy differential equations with this generalized differentiability using the lower-upper representation of a fuzzy numbers. Using generalized Hukuhara derivative in 2009, Stefanini et al. [33] represent generalization of fuzzy interval valued function. In this way the advancement of the theory of differential equations in a fuzzy environment has taken place.

It is clear that fuzzy differential equations deal with uncertainty by considering only membership values and intuitionistic fuzzy differential equations which considers only membership and non-membership values but none of them considers indeterminacy. Thus, it was needed to develop neutrosophic differential equations theory to model all three values, i.e., membership, indeterminacy and non-membership. Smarandache introduced Neutrosophic Calculus [34] containing all the basic concepts such as limit, continuity, differentiability, important functions such as exponential and logarithmic, concept of differentials and integrals in the neutrosophic environment. This theory is the growing field and researchers started working in this area. One can find that the theory of neutrosophic differential equations is studied by Sumathi and Priya [35] in the year 2018 and also one recent paper of Sumathi and Sweety [36], which uses trapezoidal neutrosophic numbers for solving differential equations of first order having one independent variable.

One can find a rich literature based on the theory of fuzzy differential equations, which contains various tools for the solution of fuzzy differential equations. To find the appropriate solutions of fuzzy differential equations various analytical, semi-analytical and numerical schemes are suggested and used in the recent literature. Salahshour et al. [37] defined Laplace transform for a fuzzy differential equation and Mondal et al. [38-39] solved fuzzy differential equations using Laplace transform. In 2016, Tapaswini et al. [40] solved fuzzy differential equation using analytical method. Ahmadian et al. in 2018, [41] uses Runge-Kutta method of 4th order for the solution of fuzzified differential equation. Similarly, Sahni et al. [42] solved fuzzy differential equation of second order using trapezoidal Intuitionistic fuzzy numbers.

In 1993, Watugala [43] introduced a new transform, known as Sumudu transform, which is now being used as a tool for solving fuzzy differential equations. This transform have two important properties, viz; scale property and unit-preserving property, so that it cannot restore the new frequency domain and solves the differential equations. After that many fuzzified differential equations have been solved using this transform (see for instance [44]-[48]). It is needed to contribute more and more work for the development of the theory of Neutrosophic differential equations as it covers more real life situations. In this paper, we have attempted to solve ordinary differential equation of second order based on the spring mass system in a neutrosophic environment using Sumudu transform method. The solution is calculated at various levels of cut – points and time values. The results are shown graphically and further compared with the solution obtained by considering the crisp set values.

2. Preliminaries

Definition 2.1(Fuzzy set)[19]. Let X be any set which is non-empty. A fuzzy set M over the elements of the set X is defined as $= \{(x, \mu_M(x)) | x \in M, \mu_M(x) \in [0,1]\}$, where the symbol $\mu_M(x)$ denotes the grade of membership of the element $x \in M$.

Definition 2.2(Support)[19]. Let X is any Universal set. The crisp set formed from the set of all points in X having grade of membership which is not zero is called as the support of the fuzzy.

Definition 2.3(Core)[19]. The core related to fuzzy set M is defined as the set of all points of the Universal set, whose grade of membership is 1.

Definition: 2.4 (Convex set)[19]. If $X \in R$, a fuzzy set M is convex, if for $\mu_M(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_M(x_1), \mu_M(x_2))$, where $\lambda \in [0,1]$.

Definition 2.5(Fuzzy number)[19]. A fuzzy number is a generalization of the crisp number in which there is collection of possible values and not any single value and are connected to each other, where each of the possible value has its own weight between 0 and 1.

Definition 2.6(Alpha-cut)[19]. The α -level cut of the fuzzy set \widetilde{M} of X is a crisp set M_α that contains all the elements of X that have membership values in M greater than or equal to α , i.e., $M_\alpha = \{x: \mu_{\widetilde{M}}(x) \geq \alpha, x \in X, \alpha \in [0,1]\}$.

Definition 2.7(Triangular fuzzy number)[42]. A triangular fuzzy number A is a subset of fuzzy number in R with the following function defined as:

$$\mu_A(x) = \begin{cases} 0 & \text{for } x \leq p \\ \frac{x-p}{q-p} & \text{for } p \leq x \leq q \\ \frac{q-x}{r-x} & \text{for } q \leq x \leq r \\ 0 & \text{for } r \leq x \end{cases}$$

where $p \leq q \leq r$ and a triangular fuzzy number is denoted by $A_{Tr}((p, q, r))$.

Definition 2.8(Intuitionistic fuzzy number-IFS)[2]. Let U be a non empty Universal set. An intuitionistic fuzzy set is represented by $M = \{(x, \mu_M(x), \omega_M(x)) | x \in U\}$, where value $\mu_M(x)$ denotes the membership value of x in M , and value $\omega_M(x)$ denotes the non-membership value of x in M .

Definition 2.9(α, β)-cut[2]. The α, β -level set of the fuzzy set \widetilde{M} of X is a crisp set and $M_{\alpha, \beta}$ contains all the elements of X that have membership values in M greater than or equal to α and non-membership values in M greater than or equal to β , i.e., $M_{\alpha, \beta} = \{x: \mu_{\widetilde{M}}(x) \geq \alpha, x \in X, \alpha \in [0,1], x: \omega_{\widetilde{M}}(x) \geq \beta, x \in X, \beta \in [0,1]\}$

Definition 2.10 (Neutrosophic Set)[36]. Let U be a universal set. A neutrosophic set M on U is defined as $M = \{T_M(x), I_M(x), F_M(x): x \in U\}$, where $T_M(x), I_M(x), F_M(x): U \rightarrow]-0, 1+[$ represents the grade of membership values, grade of indeterminacy value, and grade of non-membership value respectively of the element $x \in U$, such that $-0 \leq T_M(x) + I_M(x) + F_M(x) \leq 3^+$.

Definition 2.11 (Single-Valued Neutrosophic Set (SVNS)) [36]. Let U be a universal set. Let M be any SVNS defined on the elements of U , then $M = \{T_M(x), I_M(x), F_M(x): x \in U\}$, where $T_M(x), I_M(x), F_M(x): U \rightarrow [0,1]$ represents the grade of membership, indeterminacy, and non-membership, respectively of the element $x \in U$.

Definition 2.12 (α, β, γ)-cut[36]. The (α, β, γ) -cut of neutrosophic set is denoted by $F(\alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in [0,1]$ and are fixed numbers, such that $\alpha + \beta + \gamma \leq 3$ and is defined as $F(\alpha, \beta, \gamma) = \{T_M(x), I_M(x), F_M(x): x \in U, T_M(x) \geq \alpha, I_M(x) \leq \beta, F_M(x) \leq \gamma\}$.

Definition 2.13 (Neutrosophic Number)[36]. A neutrosophic set M defined over the universal set of real numbers R is said to be neutrosophic number if it has the following properties:

- 1) M is normal: if there exist $x_0 \in R$, such that $T_M(x_0)=1$ ($I_M(x_0) = F_M(x_0)=0$)
- 2) M is convex set for the truth function $T_M(x)$, i.e., $T_M(\mu x_1 + (1-\mu)x_2) \geq \min(T_M(x_1), T_M(x_2))$, $\forall x_1, x_2 \in R$, $\mu \in [0, 1]$.
- 3) M is concave set for the indeterminacy function and false function $I_M(x)$ and $F_M(x)$, i.e.,
 $I_M(\mu x_1 + (1-\mu)x_2) \geq \max(I_M(x_1), I_M(x_2))$, $\forall x_1, x_2 \in R$, $\mu \in [0, 1]$,
 $F_M(\mu x_1 + (1-\mu)x_2) \geq \max(F_M(x_1), F_M(x_2))$, $\forall x_1, x_2 \in R$, $\mu \in [0, 1]$.

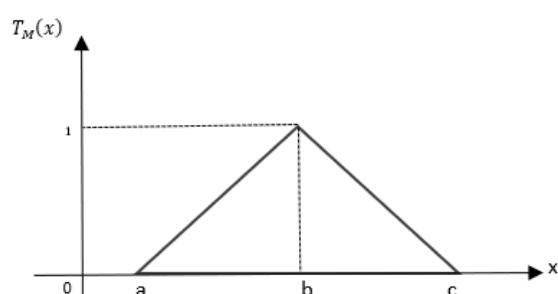
Definition 2.14 (Triangular Neutrosophic Number)[36]. A neutrosophic number in R is a superset of the triangular neutrosophic number M , having truth $T_M(x)$, indeterminacy $I_M(x)$ and false $F_M(x)$ membership function defined as

$$T_M(x) = \begin{cases} \left(\frac{x-a}{b-a}\right)u_m, & \text{for } a \leq x \leq b \\ u_m, & \text{for } x = b \\ \left(\frac{c-x}{c-b}\right)u_m, & \text{for } b \leq x \leq c \\ 0 & \text{otherwise} \end{cases}$$

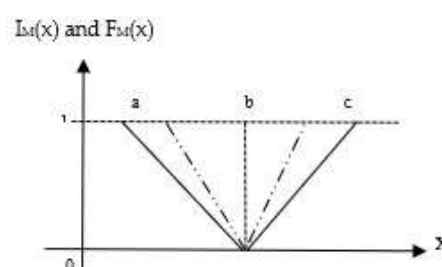
$$I_M(x) = \begin{cases} \left(\frac{b-x}{b-a}\right)v_m, & \text{for } a \leq x \leq b \\ v_m, & \text{for } x = b \\ \left(\frac{x-c}{c-b}\right)v_m, & \text{for } b \leq x \leq c \\ 1 & \text{otherwise} \end{cases}$$

$$F_M(x) = \begin{cases} \left(\frac{b-x}{b-a}\right)w_m, & \text{for } a \leq x \leq b \\ w_m, & \text{for } x = b \\ \left(\frac{x-c}{c-b}\right)w_m, & \text{for } b \leq x \leq c \\ 1 & \text{otherwise} \end{cases}$$

where $a \leq b \leq c$. A neutrosophic number in a triangular form is denoted by $M_{TN}((a, b, c); u_m, v_m, w_m)$. Here the truth membership function, i.e. $T_M(x)$ increases in a linear way for $x \in [a, b]$ and decreases in a linear form for $x \in [b, c]$. The inverse behaviour is seen for $I_M(x)$ and $F_M(x)$ for $x \in [a, b]$ and for $x \in [b, c]$, where $\alpha \in [0, u_m]$, $0 < u_m < 1$, $\beta \in [0, v_m]$, $0 < v_m < 1$, $\gamma \in [0, w_m]$, $0 < w_m < 1$.



(a) $T_M(x)$ as triangular form



(b) $I_M(x)$ and $F_M(x)$ as triangular form

Definition 2.15 (α, β, γ) -cut of a Triangular Neutrosophic Number [36].

The (α, β, γ) – cut of a triangular neutrosophic number $M_{TN}((a, b, c); u_m, v_m, w_m)$ is defined as follows:

$M_{(\alpha, \beta, \gamma)} = [M_1(\alpha), M_2(\alpha)]; [M_3(\beta), M_4(\beta)]; [M_5(\gamma), M_6(\gamma)]$, $0 \leq \alpha + \beta + \gamma \leq 3$, where,

$$M_{(\alpha, \beta, \gamma)} = [(a + \alpha(b - a))u_m, (c - \alpha(c - b))u_m], \\ [(b - \beta(b - a))v_m, (b + \beta(c - b))v_m], \\ [(b - \gamma(b - a))w_m, (b + \gamma(c - b))w_m].$$

Definition 2.16 (Differentiability) [36]. For a fuzzy valued function $f : (a, b) \rightarrow R$ at the point x_0 ,

the differentiability is defined as follows: $g'(x_0) = \lim_{h \rightarrow 0} \frac{g(x_0 + h) - g(x_0)}{h}$ and $g'(x_0)$ is D_1 -differentiable at x_0 . If $[g(x_0)]_\alpha = [g_1(x_0, \alpha), g_2(x_0, \alpha)]$ and $g'(x_0)$ is D_2 -differentiable at x_0 and if $[g(x_0)]_\alpha = [g_2(x_0, \alpha), g_1(x_0, \alpha)]$ for all $\alpha \in [0, 1]$.

Definition 2.17 (Generalized differentiability)[36]. The second-order derivative of a fuzzy value

function $g : (a, b) \rightarrow R$ at x_0 is defined as follows: $g''(x_0) = \lim_{h \rightarrow 0} \frac{g'(x_0 + h) - g'(x_0)}{h}$ and $g'(x_0)$ is

D_1 -differentiable at x_0 if

$$g''(x_0, \alpha) = \begin{cases} (g'_1(x_0, \alpha), g'_2(x_0, \alpha)) & \text{if } g \text{ is } D_1 - \text{differentiable on } (a, b) \\ (g'_2(x_0, \alpha), g'_1(x_0, \alpha)) & \text{if } g \text{ is } D_2 - \text{differentiable on } (a, b) \end{cases}$$

for all $\alpha \in [0, 1]$ and $g'(x_0)$ is D_2 -differentiable at x_0 if

$$g''(x_0, \alpha) = \begin{cases} (g'_2(x_0, \alpha), g'_1(x_0, \alpha)) & \text{if } g \text{ is } D_1 - \text{differentiable on } (a, b) \\ (g'_1(x_0, \alpha), g'_2(x_0, \alpha)) & \text{if } g \text{ is } D_2 - \text{differentiable on } (a, b) \end{cases}$$

for all $\alpha \in [0, 1]$.

3. Neutrosophic Sumudu Transform[NST]

Let $f(ut)$ is a neutrosophic valued function which is continuous. Suppose that $g(ut)e^{-t}$ be improper neutrosophic Riemann integrable on $[0, \infty)$ then $\int_0^\infty g(ut)e^{-t} dt$ is called neutrosophic Sumudu transform and it is defined as, $G(u) = S[g(t)] = \int_0^\infty g(ut)e^{-t} dt$, ($u \in [-\tau, \tau]$)

where variable u is used to factor the variable t in the argument of the neutrosophic valued function.

We have,

$g(t, r) = \{g_T(t, r), g_I(t, r), g_F(t, r)\}$, which are denoted in neutrosophic triangular form as

$$g_T(t, r) = \{\underline{g}_T(t, r), \overline{g}_T(t, r)\} g_I(t, r) = \{\underline{g}_I(t, r), \overline{g}_I(t, r)\} g_F(t, r) = \{\underline{g}_F(t, r), \overline{g}_F(t, r)\}$$

$$\int_0^\infty g_T(ut)e^{-t} dt = \left(\int_0^\infty \underline{g}_T(ut)e^{-t} dt, \int_0^\infty \overline{g}_T(ut)e^{-t} dt \right) \\ \int_0^\infty g_I(ut)e^{-t} dt = \left(\int_0^\infty \underline{g}_I(ut)e^{-t} dt, \int_0^\infty \overline{g}_I(ut)e^{-t} dt \right)$$

$$\int_0^\infty g_F(ut)e^{-t}dt = \left(\int_0^\infty \underline{g}_F(ut)e^{-t}dt, \int_0^\infty \overline{g}_F(ut)e^{-t}dt \right)$$

also using definition of classical Sumudu transform.

$$s[\underline{g}_T(t, r)] = \int_0^\infty \underline{g}_T(ut)e^{-t}dt$$

$$s[\overline{g}_T(t, r)] = \int_0^\infty \overline{g}_T(ut)e^{-t}dt,$$

$$s[\underline{g}_I(t, r)] = \int_0^\infty \underline{g}_I(ut)e^{-t}dt$$

$$s[\overline{g}_I(t, r)] = \int_0^\infty \overline{g}_I(ut)e^{-t}dt,$$

$$s[\underline{g}_F(t, r)] = \int_0^\infty \underline{g}_F(ut)e^{-t}dt$$

$$s[\overline{g}_F(t, r)] = \int_0^\infty \overline{g}_F(ut)e^{-t}dt,$$

then it follows $S[g(t)] = (s[\underline{g}(t, r)], s[\overline{g}(t, r)])$.

3.1 Some basic results on fuzzy differential equation using Sumudu transform

The following theorems are useful in our results:-

Theorem 3.1.1[44]. Let $g'(t)$ be a continuous neutrosophic valued function and $g(t)$ is the primitive of $g'(t)$ on $[0, \infty)$ then,

$$S[g'(t)] = \frac{S[g(t)]}{u} -_h \frac{g(0)}{u}, \text{ where } g \text{ is (a) differentiable or,}$$

$$S[g'(t)] = \frac{(-g(0))}{u} -_h \frac{(-S[g(t)])}{u}, \text{ where } g \text{ is (b) differentiable.}$$

where “ $-_h$ ” is notation of gh-differentiability.

Theorem 3.1.2[44]. Let $g(t), g'(t)$ be an continuous neutrosophic valued function on $[0, \infty)$ and that $g''(t)$ be piece wise continuous neutrosophic valued function on $[0, \infty)$ then,

$$S[g''(t)] = \frac{S[g(t)]}{u^2} -_h \frac{g(0)}{u} -_h \frac{g'(0)}{u}, \text{ where } g \text{ is (a) differentiable and } g' \text{ is (a) differentiable or}$$

$$S[g''(t)] = -\frac{g(0)}{u^2} -_h \left[-\frac{S[g(t)]}{u^2} \right] -_h \frac{g'(0)}{u}, \text{ where } g \text{ is (a) differentiable and } g' \text{ is (b) differentiable or}$$

$$S[g''(t)] = -\frac{g(0)}{u^2} -_h \left[-\frac{S[g(t)]}{u^2} \right] -_h \frac{g'(0)}{u}, \text{ where } g \text{ is (b) differentiable and } g' \text{ is (a) differentiable or}$$

$$S[g''(t)] = \frac{S[g(t)]}{u^2} -_h \frac{g(0)}{u^2} - \frac{g'(0)}{u}, \text{ where } g \text{ is (b) differentiable and } g' \text{ is (b) differentiable.}$$

where “ $-_h$ ” is notation of gh-differentiability.

Theorem 3.1.3. Let $g : R \rightarrow G(R)$ be a continuous neutrosophic valued function and denote

$$g_T(x) = [\underline{g}_{T\alpha}(x), \overline{g}_{T\alpha}(x)] \text{ for each } \alpha \in [0, 1],$$

$$g_I(x) = [\underline{g}_{I\beta}(x), \bar{g}_{I\beta}(x)] \text{ for each } \beta \in [0,1],$$

$$g_F(x) = [\underline{g}_{F\gamma}(x), \bar{g}_{F\gamma}(x)] \text{ for each } \gamma \in [0,1]. \text{ Then:}$$

$$(1) \text{ If } g_T \text{ is (1)-differentiable, then } \underline{g}_{T\alpha} \text{ and } \bar{g}_{T\alpha} \text{ are differentiable function and } f'(x) = [\underline{f}_{T\alpha}(x), \bar{f}_{T\alpha}(x)]$$

$$(2) \text{ If } g_T \text{ is (2)-differentiable, then } \underline{g}_{T\alpha} \text{ and } \bar{g}_{T\alpha} \text{ are differentiable function and } f'(x) = [\underline{f}_{T\alpha}(x), \bar{f}_{T\alpha}(x)]$$

$$(3) \text{ If } g_I \text{ is (1)-differentiable, then } \underline{g}_{I\beta} \text{ and } \bar{g}_{I\beta} \text{ are differentiable function and } f'(x) = [\underline{f}_{I\beta}(x), \bar{f}_{I\beta}(x)]$$

$$(4) \text{ If } g_I \text{ is (2)-differentiable, then } \underline{g}_{I\beta} \text{ and } \bar{g}_{I\beta} \text{ are differentiable function and } f'(x) = [\underline{f}_{I\beta}(x), \bar{f}_{I\beta}(x)]$$

$$(5) \text{ If } g_F \text{ is (1)-differentiable, then } \underline{g}_{F\gamma} \text{ and } \bar{g}_{F\gamma} \text{ are differentiable function and } f'(x) = [\underline{f}_{F\gamma}(x), \bar{f}_{F\gamma}(x)]$$

$$(6) \text{ If } g_F \text{ is (2)-differentiable, then } \underline{g}_{F\gamma} \text{ and } \bar{g}_{F\gamma} \text{ are differentiable function and } f'(x) = [\underline{f}_{F\gamma}(x), \bar{f}_{F\gamma}(x)]$$

3.2 Solution of General Second Order Ordinary Differential Equation in a Neutrosophic Environment using Sumudu Transform

Let us consider a general ordinary differential equation of second order given as follows:

$$y''(t) = f(t, y(t), y'(t)) \quad (1)$$

with the initial conditions $y(t_0) = y_0, y'(t_0) = z_0$, where $f : [t_0, P] \times R \rightarrow R$.

Suppose that initial values y_0 and z_0 are uncertain and are defined in terms of lower and upper bound of truth, indeterminacy and falsity, i.e. neutrosophic number.

Thus from equation no. 1, we have the following fuzzy initial value differential equation:

$$y''(t) = g(t, y(t), y'(t)), 0 \leq t \leq P$$

$$y_T(t_0) = y_0 = [\underline{y}_{T\alpha}(0), \bar{y}_{T\alpha}(0)], 0 < \alpha \leq 1, y'_T(t_0) = z_0 = [\underline{z}_{T\alpha}(0), \bar{z}_{T\alpha}(0)], 0 < \alpha \leq 1, \quad (2)$$

$$y_I(t_0) = y_0 = [\underline{y}_{I\beta}(0), \bar{y}_{I\beta}(0)], 0 < \beta \leq 1, y'_I(t_0) = z_0 = [\underline{z}_{I\beta}(0), \bar{z}_{I\beta}(0)], 0 < \beta \leq 1, \quad (3)$$

$$y_F(t_0) = y_0 = [\underline{y}_{F\gamma}(0), \bar{y}_{F\gamma}(0)], 0 < \gamma \leq 1, y'_F(t_0) = z_0 = [\underline{z}_{F\gamma}(0), \bar{z}_{F\gamma}(0)], 0 < \gamma \leq 1, \quad (4)$$

where $g: [t_0, P] \times F(R) \rightarrow F(R)$ is a function of continuous manner.

By applying Neutrosophic Sumudu Transform on given second order differential equation, we have

$$S[y''(t)] = s[g(t, y(t), y'(t))]$$

Case 1: Let $y''(t)$ is (1)-differentiable, and using the above theorem we have $y''(t) = [\underline{y}'_{\alpha}(t), \bar{y}'_{\alpha}(t)]$.

The differential equation is then reduced to the following:

$$\begin{aligned} \underline{y}'_{T\alpha}(t) &= \underline{g}_{T\alpha}(t, y(t), y'(t)), \underline{y}_{T\alpha}(t_0) = \underline{y}_{T\alpha}(0) \\ \bar{y}'_{T\alpha}(t) &= \bar{g}_{T\alpha}(t, y(t), y'(t)), \bar{y}_{T\alpha}(t_0) = \bar{y}_{T\alpha}(0) \\ \underline{y}'_{I\beta}(t) &= \underline{g}_{I\beta}(t, y(t), y'(t)), \underline{y}_{I\beta}(t_0) = \underline{y}_{I\beta}(0) \\ \bar{y}'_{I\beta}(t) &= \bar{g}_{I\beta}(t, y(t), y'(t)), \bar{y}_{I\beta}(t_0) = \bar{y}_{I\beta}(0) \\ \underline{y}'_{F\gamma}(t) &= \underline{g}_{F\gamma}(t, y(t), y'(t)), \underline{y}_{F\gamma}(t_0) = \underline{y}_{F\gamma}(0) \\ \bar{y}'_{F\gamma}(t) &= \bar{g}_{F\gamma}(t, y(t), y'(t)), \bar{y}_{F\gamma}(t_0) = \bar{y}_{F\gamma}(0) \end{aligned}$$

Using the Sumudu transform for solving, we get

$$s[y''(t)] = \frac{s[y(t)] - {}_h y(t_0) - {}_h u y'(t_0)}{u^2}.$$

The following six first order ordinary differential equations are developed, two for both Truth,

Indeterminacy and falsity and is defined as

$$\begin{aligned} S[\underline{g}_{T\alpha}(t, y(t), y'(t))] &= \frac{s[\underline{y}_{T\alpha}(t)] - {}_h \underline{y}_{T\alpha}(0) - {}_h u \underline{y}'_{T\alpha}(0)}{u^2} \\ S[\bar{g}_{T\alpha}(t, y(t), y'(t))] &= \frac{s[\bar{y}_{T\alpha}(t)] - {}_h \bar{y}_{T\alpha}(0) - {}_h u \bar{y}'_{T\alpha}(0)}{u^2} \\ S[\underline{g}_{I\beta}(t, y(t), y'(t))] &= \frac{s[\underline{y}_{I\beta}(t)] - {}_h \underline{y}_{I\beta}(0) - {}_h u \underline{y}'_{I\beta}(0)}{u^2} \\ S[\bar{g}_{I\beta}(t, y(t), y'(t))] &= \frac{s[\bar{y}_{I\beta}(t)] - {}_h \bar{y}_{I\beta}(0) - {}_h u \bar{y}'_{I\beta}(0)}{u^2} \\ S[\underline{g}_{F\gamma}(t, y(t), y'(t))] &= \frac{s[\underline{y}_{F\gamma}(t)] - {}_h \underline{y}_{F\gamma}(0) - {}_h u \underline{y}'_{F\gamma}(0)}{u^2} \\ S[\bar{g}_{F\gamma}(t, y(t), y'(t))] &= \frac{s[\bar{y}_{F\gamma}(t)] - {}_h \bar{y}_{F\gamma}(0) - {}_h u \bar{y}'_{F\gamma}(0)}{u^2} \end{aligned}$$

To solve this, we assume that

$$S[\underline{y}_{T\alpha}(t)] = L_{T\alpha}^1(u), S[\overline{y}_{T\alpha}(t)] = U_{T\alpha}^1(u),$$

$$S[\underline{y}_{I\beta}(t)] = L_{I\beta}^1(u), S[\overline{y}_{I\beta}(t)] = U_{I\beta}^1(u),$$

$$S[\underline{y}_{F\gamma}(t)] = L_{F\gamma}^1(u), S[\overline{y}_{F\gamma}(t)] = U_{F\gamma}^1(u)$$

where, $L_{T\alpha}^1(u), U_{T\alpha}^1(u), L_{I\beta}^1(u), U_{I\beta}^1(u), L_{F\gamma}^1(u), U_{F\gamma}^1(u)$ are solution of differential equations.

By using inverse neutrosophic Sumudu transform, we have

$$\underline{y}_{T\alpha}(t), \overline{y}_{T\alpha}(t), \underline{y}_{I\beta}(t), \overline{y}_{I\beta}(t), \underline{y}_{F\gamma}(t), \overline{y}_{F\gamma}(t)$$

and it follows that,

$$\underline{y}_{T\alpha}(t) = S^{-1}[L_{T\alpha}^1(u)], \overline{y}_{T\alpha}(t) = S^{-1}[U_{T\alpha}^1(u)]$$

$$\underline{y}_{I\beta}(t) = S^{-1}[L_{I\beta}^1(u)], \overline{y}_{I\beta}(t) = S^{-1}[U_{I\beta}^1(u)]$$

$$\underline{y}_{F\gamma}(t) = S^{-1}[L_{F\gamma}^1(u)], \overline{y}_{F\gamma}(t) = S^{-1}[U_{F\gamma}^1(u)]$$

Case 2: Let $y''(t)$ be (2)-differentiable, then from above theorem we have $y''(t) = [\underline{y}'_{\alpha}(t), \overline{y}'_{\alpha}(t)]$.

Reducing the second order ordinary differential equations into first order ordinary differential equation, we have the following differential equations to be solved,

$$\underline{y}'_{T\alpha}(t) = \underline{g}_{T\alpha}(t, y(t), y'(t)), \underline{y}_{T\alpha}(t_0) = \underline{y}_{T\alpha}(0)$$

$$\overline{y}'_{T\alpha}(t) = \overline{g}_{T\alpha}(t, y(t), y'(t)), \overline{y}_{T\alpha}(t_0) = \overline{y}_{T\alpha}(0)$$

$$\underline{y}'_{I\beta}(t) = \underline{g}_{I\beta}(t, y(t), y'(t)), \underline{y}_{I\beta}(t_0) = \underline{y}_{I\beta}(0)$$

$$\overline{y}'_{I\beta}(t) = \overline{g}_{I\beta}(t, y(t), y'(t)), \overline{y}_{I\beta}(t_0) = \overline{y}_{I\beta}(0)$$

$$\underline{y}'_{F\gamma}(t) = \underline{g}_{F\gamma}(t, y(t), y'(t)), \underline{y}_{F\gamma}(t_0) = \underline{y}_{F\gamma}(0)$$

$$\overline{y}'_{F\gamma}(t) = \overline{g}_{F\gamma}(t, y(t), y'(t)), \overline{y}_{F\gamma}(t_0) = \overline{y}_{F\gamma}(0)$$

$$\text{Using } [y''(t)] = \frac{(-y(t_0)) - {}_h(-S[y(t)] - {}_h u y'(t_0))}{u^2},$$

we get the following six first order ordinary differential equations as,

$$\begin{aligned} S[\underline{g}_{T\alpha}(t, y(t), y'(t))] &= \frac{(-\underline{y}_{T\alpha}(0)) - {}_h(-S[\underline{y}_{T\alpha}(t)] - {}_h u \underline{y}'_{T\alpha}(0))}{u^2} \\ S[\overline{g}_{T\alpha}(t, y(t), y'(t))] &= \frac{(-\overline{y}_{T\alpha}(0)) - {}_h(-S[\overline{y}_{T\alpha}(t)] - {}_h u \overline{y}'_{T\alpha}(0))}{u^2} \\ S[\underline{g}_{I\beta}(t, y(t), y'(t))] &= \frac{(-\underline{y}_{I\beta}(0)) - {}_h(-S[\underline{y}_{I\beta}(t)] - {}_h u \underline{y}'_{I\beta}(0))}{u^2} \end{aligned}$$

$$\begin{aligned}
S[\underline{g}_{I\beta}(t, y(t), y'(t))] &= \frac{\left(-\underline{y}_{I\beta}(0)\right) -_h \left(-s[\underline{y}_{I\beta}(t)]\right) -_h u \underline{y}'_{I\beta}(0)}{u^2} \\
S[\underline{g}_{F\gamma}(t, y(t), y'(t))] &= \frac{\left(-\underline{y}_{F\gamma}(0)\right) -_h \left(-s[\underline{y}_{F\gamma}(t)]\right) -_h u \underline{y}'_{F\gamma}(0)}{u^2} \\
S[\overline{g}_{F\gamma}(t, y(t), y'(t))] &= \frac{\left(-\overline{y}_{F\gamma}(0)\right) -_h \left(-s[\overline{y}_{F\gamma}(t)]\right) -_h u \overline{y}'_{F\gamma}(0)}{u^2}
\end{aligned}$$

To solve this, we assume that

$$\begin{aligned}
S[\underline{y}_{T\alpha}(t)] &= L_{T\alpha}^1(u), & S[\overline{y}_{T\alpha}(t)] &= U_{T\alpha}^1(u), \\
S[\underline{y}_{I\beta}(t)] &= L_{I\beta}^1(u), & S[\overline{y}_{I\beta}(t)] &= U_{I\beta}^1(u), \\
S[\underline{y}_{F\gamma}(t)] &= L_{F\gamma}^1(u), & S[\overline{y}_{F\gamma}(t)] &= U_{F\gamma}^1(u)
\end{aligned}$$

where, $L_{T\alpha}^1(u), U_{T\alpha}^1(u), L_{I\beta}^1(u), U_{I\beta}^1(u), L_{F\gamma}^1(u), U_{F\gamma}^1(u)$ are the Sumudu transform for Truth, Indeterminacy and falsity solution of equations.

By using inverse neutrosophic Sumudu transform, we have $\underline{y}_{T\alpha}(t), \overline{y}_{T\alpha}(t), \underline{y}_{I\beta}(t), \overline{y}_{I\beta}(t), \underline{y}_{F\gamma}(t), \overline{y}_{F\gamma}(t)$ and it follows:

$$\begin{aligned}
\underline{y}_{T\alpha}(t) &= S^{-1}[L_{T\alpha}^1(u)], \quad \overline{y}_{T\alpha}(t) = S^{-1}[U_{T\alpha}^1(u)] \\
\underline{y}_{I\beta}(t) &= S^{-1}[L_{I\beta}^1(u)], \quad \overline{y}_{I\beta}(t) = S^{-1}[U_{I\beta}^1(u)] \\
\underline{y}_{F\gamma}(t) &= S^{-1}[L_{F\gamma}^1(u)], \quad \overline{y}_{F\gamma}(t) = S^{-1}[U_{F\gamma}^1(u)]
\end{aligned}$$

3.3 Crisp and Fuzzy Solution for Ordinary differential equations of spring mass system

Consider an elastic string vertically tied to a rigid support. The object attached on the other side of an elastic string pulls the string downwards to a distance s from its usual length b . The place at which the string extended from its usual position is called as equilibrium position. Now, as the mass of the body pulls the spring downwards, the restoring force of the spring acts as a restraint and opposes the stretching of the spring. Mathematically, it can be written as $F = ks$ where $k > 0$ is the proportionality constant or spring constant, F is the restoring force and the s is the extension in the spring from its equilibrium position. Let ' m ' be the mass of the object and $y(t)$ represents the displacement, then according to Newton second law of motion, we have $= m \frac{d^2y}{dt^2}$. Suppose no forces are acting on the system, except the force of gravity, then the differential modeling such type of system can be written as

$$\begin{aligned}
m \frac{d^2y}{dt^2} &= \sum (\text{force acting on the system}) \\
&= -ks - ky + mg
\end{aligned}$$

At equilibrium $ks = mg$, so after calculation we get differential equation as,

$$m \frac{d^2y}{dt^2} + ky = 0$$

To solve such second order ordinary differential equations, we require two initial conditions $y(0) = p$ and $y'(0) = q$. Thus, the problem of finding the displacement $y(t)$ is reduced to solving the differential equations of the form,

$$m \frac{d^2 y}{dt^2} + ky = 0, \quad y(0) = p, \quad y'(0) = q.$$

If we consider the damping force $\left(n \frac{dy}{dt}\right)$ which is a function of velocity of the motion that helps to reduce the vibrations, then the above differential equation reduces to

$$m \frac{d^2 y}{dt^2} + n \frac{dy}{dt} + ky = 0, \quad y(0) = p, \quad y'(0) = q.$$

Solving the above initial value problem provides the displacement $y(t)$ in terms of constants m, k and n . The above differential equation can be solved by the method of substitution, which provides us the characteristic equation as

$$mr^2 + nr + k = 0,$$

with solutions of the auxiliary equation as $r_{1,2} = \frac{-n \pm \sqrt{n^2 - 4mk}}{2m}$.

The quantity inside the square root, i.e. $n^2 - 4mk$ classifies the solution into three cases:

Case 1. If $n^2 - 4mk > 0$, it is an over damped situation, as the proportionality constant k is very small as compared to damping coefficient n .

Case 2. If $n^2 - 4mk = 0$, the situation is critically damped and the resulting motion is oscillatory and the damping coefficient n slightly decreases.

Case 3. If $n^2 - 4mk < 0$, the behaviour of the motion is under damped and the value of spring constant is very large as compared to damping coefficient n .

4. Application

In this section a problem of spring mass system is considered. The differential equation formed for this system is solved in Neutrosophic environment and then compared it with crisp solution. It is shown here that, Neutrosophic environment includes differential equations with initial conditions containing parameters of belongingness, non-belongingness and indeterminacy, so that it provide more precise solution than crisp environment.

Problem Statement: A body of mass $8lb$ is tied to a spring of length $4ft$. At equilibrium position, the length of the spring has $6ft$. Let the damping force is defined as $F_R = 2dy/dt$ and the body is released from the equilibrium position with a down ward initial velocity of $1ft/s$, find the displacement $y(t)$ for any time t analytically and using fuzzy Sumudu transform.

Solution.

The differential equation of this spring-mass system is $my'' + ny' + ky = 0$, where the mass of the body is $m = 8/32 = 1/4$ slug, the spring constant with $8 = k \times 2$, so $k = 4\text{lb/ft}$. The resistive force $F_R = n \, dy/dt = 2dy/dt$ with the initial conditions $y(0) = 0$, $y'(0) = 1$.

After putting these values, we get $y''(t) + 8y'(t) + 16y(t) = 0$

Crisp solution

The analytical solutions is given by $y(t) = C_1 e^{-4t} + tC_2 e^{-4t}$. Using initial value $y(0) = 0$, $y'(0) = 1$, in the solution, we get $y(t) = te^{-4t}$. At $t = 0.1$, we get $y(0.1) = 0.067$.

Neutrosophic solution

Consider the Neutrosophic initial value problem, $y''(t) + 8y'(t) + 16y(t) = 0$

$$y_T(0) = [\alpha - 1, 1 - \alpha], \quad y'_T(0) = [\alpha, 1 - \alpha]$$

$$y_I(0) = [-0.5\beta, 0.5\beta], \quad y'_I(0) = [1 - 0.5\beta, 1 + 0.5\beta]$$

$$y_F(0) = [-0.2\gamma, 0.2\gamma], \quad y'_F(0) = [1 - 0.2\gamma, 1 + 0.2\gamma]$$

Using Neutrosophic Sumudu Transform, we get the expression in Truth, Indeterminacy and Falsity as,

$$\underline{y}_T(u, \alpha) = \left(\frac{(\alpha - 1)(1 + 8u)}{(1 + 4u)^2} + \frac{\alpha u}{(1 + 4u)^2} \right)$$

$$\bar{y}_T(u, \alpha) = \left(\frac{(1 - \alpha)(1 + 8u)}{(1 + 4u)^2} + \frac{(2 - \alpha)u}{(1 + 4u)^2} \right)$$

$$\underline{y}_I(u, \beta) = \left(\frac{-0.5\beta}{(1 + 4u)^2} - \frac{4\beta u}{(1 + 4u)^2} + \frac{(1 - 0.5\beta)u}{(1 + 4u)^2} \right)$$

$$\bar{y}_I(u, \beta) = \left(\frac{0.5\beta}{(1 + 4u)^2} + \frac{4\beta u}{(1 + 4u)^2} + \frac{(1 + 0.5\beta)u}{(1 + 4u)^2} \right)$$

$$\underline{y}_F(u, \gamma) = \left(\frac{-0.2\gamma}{(1 + 4u)^2} - \frac{1.6\gamma u}{(1 + 4u)^2} + \frac{(1 - 0.2\gamma)u}{(1 + 4u)^2} \right)$$

$$\bar{y}_F(u, \gamma) = \left(\frac{0.2\gamma}{(1 + 4u)^2} + \frac{1.6\gamma u}{(1 + 4u)^2} + \frac{(1 + 0.2\gamma)u}{(1 + 4u)^2} \right)$$

Now the solution as per lower and upper bound for truth value, indeterminacy value and false value respectively, are:

$$\underline{y}_T(t, \alpha) = ((\alpha - 1)e^{-4t} + (5\alpha - 4)te^{-4t})$$

$$\bar{y}_T(t, \alpha) = ((1 - \alpha)e^{-4t} + (6 - 5\alpha)te^{-4t})$$

$$\underline{y}_I(t, \beta) = ((-0.5\beta)e^{-4t} + (1 - 2.5\beta)te^{-4t})$$

$$\bar{y}_I(t, \beta) = ((0.5\beta)e^{-4t} + (1 + 2.5\beta)te^{-4t})$$

$$\underline{y}_F(t, \gamma) = ((-0.2\gamma)e^{-4t} + (1 - \gamma)te^{-4t})$$

$$\bar{y}_F(t, \gamma) = ((0.2\gamma)e^{-4t} + (1 + \gamma)te^{-4t})$$

4.1. Numerical Observation and Graphical Representation

The fuzzy differential equation is solved for different (α, β, γ) -cut values. For the solution the step size of 0.1 is considered. It is shown in the table 1 for $t = 0.1$. From table 1, it is observed that the for truth membership $y_T(t, \alpha)$, lower and upper bound both show an inverse behaviour, i.e one is

increasing and the other is decreasing. The crisp solution matches with the fuzzy solution at α – cut, with value 1.0. It is observed that for indeterminacy membership $y_I(t, \beta)$ and for False membership $y_F(t, \gamma)$, lower bound is decreasing and the upper bound is increasing at β, γ – cut, with value 0, the crisp solution matches with the fuzzy solution.

Table 1. The solutions for lower and upper bound at $t = 0.1$ and its comparison with the crisp solution.

(α, β, γ) -cut	$\underline{y}_T(t, \alpha)$	$\overline{y}_T(t, \alpha)$	$\underline{y}_I(t, \beta)$	$\overline{y}_I(t, \beta)$	$\underline{y}_F(t, \delta)$	$\overline{y}_F(t, \gamma)$	Exact solution
0	-0.938448	1.072510	0.067032	0.067032	0.0670300	0.0670320	0.067032
0.1	-0.83790	0.971964	0.016758	0.117306	0.0469224	0.0871416	...
0.2	-0.737352	0.871416	0.033516	0.167580	0.0268128	0.1072510	...
0.3	-0.636804	0.770868	-0.083790	0.217850	0.0067032	0.1273610	...
0.4	-0.536256	0.670303	-0.134064	0.268128	-0.0134060	0.1474700	...
0.5	-0.435708	0.569772	-0.184338	0.318402	-0.0335160	0.1675800	...
0.6	-0.335160	0.469224	-0.234612	0.368676	-0.0536256	0.1876902	...
0.7	-0.234612	0.368676	-0.284886	0.418958	-0.0737352	0.2077991	...
0.8	-0.134064	0.268128	-0.335160	0.469224	-0.0938448	0.2279091	...
0.9	-0.033516	0.167580	-0.385434	0.519498	-0.1139540	0.2480180	...
1	0.067032	0.067032	-0.435708	0.569772	-0.1340640	0.2681281	0.067032

In figure 1, the graph is plotted for different (α, β, γ) – cut values at $t = 0.1$. It is observed that as the α -cut values are increasing, the solution approaches to the exact solution and as the (β, γ) -cut values are decreasing the solution approaches to the exact solution.

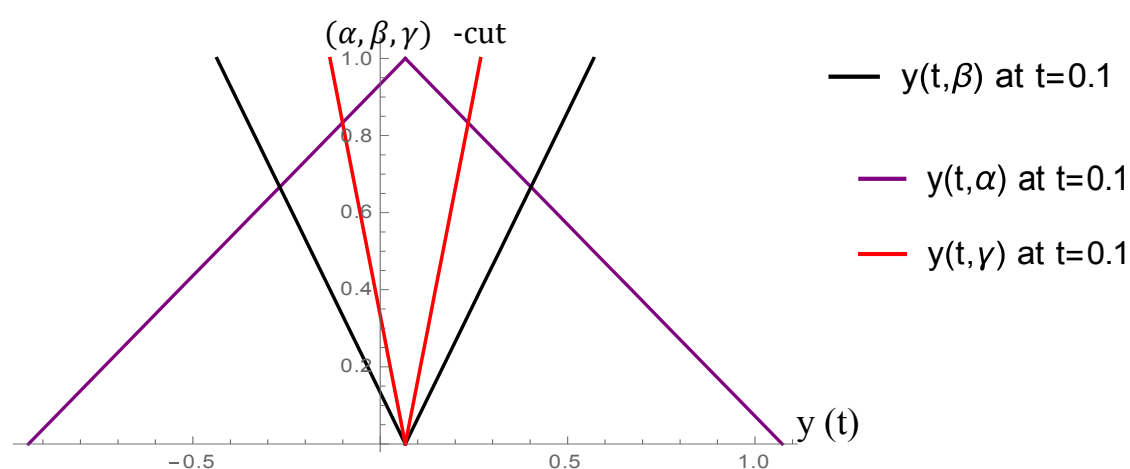


Figure 1. Graph for different values of (α, β, γ) -cuts and at time $t = 0.1$

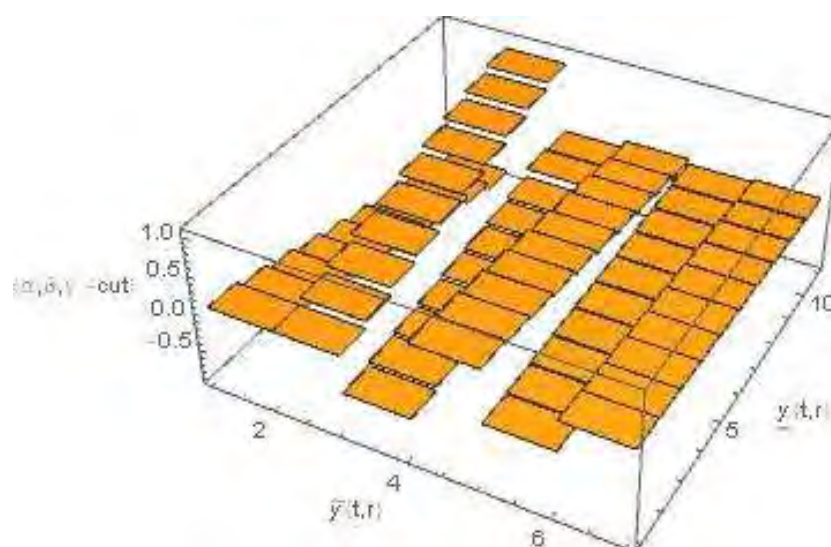


Figure 2. The lower and upper bounds for different values of (α, β, γ) - cut for time 0 to 0.5 .

From figure 2, we can observed that, the motion of the spring decreases with the increase in time for truth membership and motion of spring increases with the decreases in time for false and indermancy. We can further study the behaviour of motion of the spring under external force for different (α, β, γ) -cut values with varying time under neutrosophic initial values.

5. Conclusion and Future works

In this paper, the ordinary differential equation of mechanical spring mass system with neutrosophic initial conditions is solved using Sumudu transform method. The solution of neutrosophic environment obtained is compared with the crisp solution and is more generalized. The results are represented for different (α, β, γ) -cut values in table 1. The behavior is also depicted in the form of graphs, for different (α, β, γ) -cut values with varying time. This study helps in solving various other ordianry differential equations such as simultaneous differential equation, differential equation with variable coefficients under neutosophic environment. The solution of a differential equation helps in understanding the behaviour of physical systems under an uncertain environment. For non-linear differential equations, our intuition is that it cannot be applied.

Conflict of interest

The Authors have no conflict of interest.

Nomenclature and Symbols

$\mu_M(x)$	Fuzzy membership function of set M
M_α	The α - level set of the fuzzy set M
U	Universal set
$\mu_M(x)$	Fuzzy membership function of set M
$\omega_M(x)$	Fuzzy Non-membership function of set M

$M_{\alpha,\beta}$	The α, β - level set of the fuzzy set M
X	Crisp set
$T_M(x)$	Truth membership function of neutrosophic fuzzy set M
$I_M(x)$	Indeterminacy membership function of neutrosophic fuzzy set M
$F_M(x)$	False membership function of neutrosophic fuzzy set M
$F(\alpha, \beta, \gamma)$	(α, β, γ) - cut of neutrosophic set
$M(\alpha, \beta, \gamma)$	(α, β, γ) cut of a triangular neutrosophic number M_{TN}
$g(u)e^{-t}$	improper neutrosophic Riemann integrable
$\underline{g}(t, r)$	Lower bound of fuzzy membership
$\bar{g}(t, r)$	Upper bound of fuzzy membership
\neg_h	Generalized-differentiability
F_R	Resistive force
m	Mass of a body
k	Spring constant
\in	Belongs to
$A_{Tr} \langle (p, q, r) \rangle$	Triangular fuzzy number
$M_{TN} \langle (a, b, c); u_m, v_m, w_m \rangle$	Triangular neutrosophic number
$S[g(t)]$	Sumudu transform of function “g”
$L_{T\alpha}(u), U_{T\alpha}(u)$	Lower and upper bound solution of Sumudu transform with respect to α cut for Truth membership function of neutrosophic fuzzy set
$L_{T\beta}(u), U_{T\beta}(u)$	Lower and upper bound solution of Sumudu transform with respect to β -cut for Truth membership function of neutrosophic fuzzy set
$L_{T\gamma}(u), U_{T\gamma}(u)$	Lower and upper bound solution of Sumudu transform with respect to γ -cut for Truth membership function of neutrosophic fuzzy set

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Generalized Neutrosophic TOPSIS to Solve Multi-Criteria Decision-Making Problems

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Abstract: Multi-criteria decision making (MCDM) is the technique of selecting the best alternative from multiple alternatives and multiple conditions. The technique for order preference by similarity to an ideal solution (TOPSIS) is a crucial practical technique for ranking and selecting different options by using a distance measure. In this article, we protract the fuzzy TOPSIS technique to neutrosophic fuzzy TOPSIS, and prove the accuracy of the method by explaining the MCDM problem with single-value neutrosophic information, and use the method for supplier selection in the production industry. We hope that this article will promote future scientific research on numerous existence issues based on multi-criteria decision making.

Keywords: Neutrosophic set, Single valued Neutrosophic set, TOPSIS, MCDM

1. Introduction

We faced a lot of complications in different areas of life which contains vagueness such as engineering, economics, modeling, and medical diagnoses, etc. However, a general question is raised that in mathematical modeling how we can express and use the uncertainty. A lot of researchers in the world proposed and recommended different approaches to solve those problems that contain uncertainty. In decision-making problems, multiple attribute decision making (MADM) is the most essential part which provides us to find the most appropriate and extraordinary alternative. However, to choose the appropriate alternative is very difficult because of vague information in some cases. To overcome such situations, Zadeh developed the notion of fuzzy sets (FSs) [1] to solve those problems which contain uncertainty and vagueness. It is observed that in some cases circumstances cannot be handled by fuzzy sets, to overcome such types of situations Turksen [2] gave the idea of interval-valued fuzzy sets (IVFSs). In some cases, we must deliberate membership unbiased as the non- membership values for the suitable representation of an object in uncertain and indeterminate conditions that could not be handled by FSs nor IVFSs. To overcome these difficulties Atanassov offered the concept of Intuitionistic fuzzy sets (IFSs) [3]. The theory which was presented by Atanassov only deals the insufficient data considering both the membership and non-membership values, but the intuitionistic fuzzy set theory cannot handle the incompatible and imprecise information. To deal with such incompatible and imprecise data Smarandache [4] extended the work of Atanassov IFSs and proposed a powerful tool comparative to FSs and IFSs to deal with indeterminate, incomplete, and inconsistent information's which faced in real-life problems. Since the direct use of Neutrosophic sets (NSs) for TOPSIS is somewhat difficult. To apply the NSs, Wang et al. introduced a subclass of NSs known as single-valued Neutrosophic sets (SVNSs) in [5]. In [6] the author proposed a geometric interpretation by using NSs. Gulfam et al. [7] introduced a new distance formula for SVNSs and developed some new techniques under the Neutrosophic

environment. The concept of a single-valued Neutrosophic soft expert set proposed in [8] by combining the SVN_Ss and soft expert sets.

To solve MCDM problems with single-valued Neutrosophic numbers (SVNNs) presented by Deli and Subas in [9], they constructed the concept of cut sets of SVNNs. On the base of the correlation of IFSs, the term correlation coefficient of SVN_Ss [10] introduced and proposed a decision-making method by using a weighted correlation coefficient or the weighted cosine similarity measure of SVN_Ss. In [11] the idea of simplified Neutrosophic sets introduced with some operational laws and aggregation operators such as real-life Neutrosophic weighted arithmetic average operator and weighted geometric average operator. They constructed an MCDM method on the base of proposed aggregation operators and cosine similarity measure for simplified neutrosophic sets. Sahin and Yiğider [12] extended the TOPSIS method to MCDM with a single-valued neutrosophic technique.

The TOPSIS method is presented in [13] to solve multi-criteria decision problems with different choices. In [14], Chen & Hwang extended the idea of the TOPSIS method and proposed a new TOPSIS model. The author uses the newly proposed decision-making method to solve uncertain data [15]. In [16], the authors applied this method to the prediction of diabetic patients in medical diagnosis. In [17–19] the authors studied the soft set TOPSIS, fuzzy TOPSIS, and Intuitionistic Fuzzy TOPSIS respectively and used for decision making. In [20], for the solution of single-valued neutrosophic soft set expert based multi-attribute decision-making problems, the authors proposed the TOPSIS technique. Generalized fuzzy TOPSIS was given in [21,22] with accuracy function. Maji [23] proposed the concept of neutrosophic soft sets (NSSs) with some properties and operations. Authors studied NSSs and gave some new definitions on NSSs [24], they also gave the idea of neutrosophic soft matrices with some operations and proposed a decision-making method. Many researchers developed the decision-making models by using the NSSs reported in the literature [25–27]. Elhassouny and Smarandache [28] extended the work on a simplified TOPSIS method and by using single-valued Neutrosophic information they proposed Neutrosophic simplified TOPSIS method. Saqlain et.al [21] presented generalized neutrosophic TOPSIS using accuracy function for the neutrosophic hypersoft set environment. The concept of single-valued neutrosophic cross-entropy measure introduced by Jun [29], he also constructed an MCDM method and claimed that this proposed method is more appropriate than previous methods for decision making.

Saha and Broumi [31], studied the interval-valued neutrosophic sets (IVNSs) and developed some new set-theoretic operations on IVNSs with their properties. The idea of an Interval-valued generalized single valued neutrosophic trapezoidal number (IVGSVTrN) was presented by Deli [32] with some operations and discussed their properties based on neutrosophic numbers. Hashim et al [33], studied the vague set and interval neutrosophic set and established a new theory known as interval neutrosophic vague set (INVS), they also presented some operations for INVS with their properties and derived the properties by using numerical examples. In [34], Abdel basset et al. applied TODIM and TOPSIS methods based on the best-worst method to increase the accuracy of evaluation under uncertainty according to the NSs. They also used the plithogenic set theory to resolve the indeterminate information and evaluate the economic performance of manufacturing industries, they used the AHP method to find the weight vector of the financial ratios to achieve this goal after that they used the VIKOR and TOPSIS methods to utilize the companies ranking [35, 36].

In the following paragraph, we explain some positive impacts of this research. The concentration of this study is to evaluate the best supplier for the production industry. This research is a very suitable illustration of Neutrosophic TOPSIS. A group of decision-makers chooses the best supplier for the production industry. The Neutrosophic TOPSIS method increases alternative performances based on the best and worst solutions.

1.1 Motivation and Contribution

Classical TOPSIS uses clear techniques for language assessment, but due to the imprecision and ambiguity of language assessment, we propose neutrosophic TOPSIS. In this paper, we discuss the

NSs and SVNNS with some operations. We presented the generalization of TOPSIS for the SVNNS and use the proposed method for supplier selection.

1.2 Structure of Article

In Section 2, some basic definitions have been added, which will help the rest of this article. Section 3 consists of the main work of the article, which defines the neutrosophic TOPSIS algorithm. The application of the proposed method and calculations are presented in section 4 and finally, the conclusion draws in Section 5.

2. Preliminaries

In this section, we remind some basic definitions such as NSs and SVNNS with some operations that will be used in the following sequel.

Neutrosophic Set (NS) [30]: Let X be a space of points and x be an arbitrary element of X . A neutrosophic set A in X is defined by a Truth-membership function $T_A(x)$, an Indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$ i.e.; $T_A(x), I_A(x), F_A(x): X \rightarrow]0^-, 1^+[$, and $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Single Valued Neutrosophic Sets [5]: Let E be a universe. An SVNNS over E is an NS over E , but truthiness, indeterminacy, and falsity membership functions are defined

$T_A(x): X \rightarrow [0, 1]$, $I_A(x): X \rightarrow [0, 1]$, $F_A(x): X \rightarrow [0, 1]$, and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Multiplication of SVNNS [11]: Let $A = \{\alpha_1, \alpha_2, \alpha_3\}$ and $B = \{\beta_1, \beta_2, \beta_3\}$ are two SVN numbers, then their multiplication is defined as follows $A \otimes B = (\alpha_1\beta_1, \alpha_2 + \beta_2 - \alpha_2\beta_2, \alpha_3 + \beta_3 - \alpha_3\beta_3)$.

3. Neutrosophic TOPSIS [11]

3.1. Algorithm for Neutrosophic TOPSIS using SVNNS

To explain the procedure of Neutrosophic TOPSIS using SVNNS the following steps are followed. Let $A = \{A_1, A_2, A_3, \dots, A_m\}$ be a set of alternatives and $C = \{C_1, C_2, C_3, \dots, C_n\}$ be a set of evaluation criteria and DM be a set of "I" decision-makers as follows $DM = \{DM_1, DM_2, DM_3, \dots, DM_l\}$. In the form of linguistic variables, the importance of the evaluation criteria, DMs, and alternative ratings are given in Table 1.

Step 1: Computation of weights of the DMs

Let the SVN number for rating the k^{th} DM is denoted by

$$D_k = (T_k^{dm}, I_k^{dm}, F_k^{dm})$$

Weight of the k^{th} DM can be found by the following formula

$$\lambda_k = \frac{1 - \left[\frac{1}{3} \left\{ \left(1 - T_k^{dm}(x) \right)^2 + \left(I_k^{dm}(x) \right)^2 + \left(F_k^{dm}(x) \right)^2 \right\} \right]^{0.5}}{\sum_{k=1}^l \left(1 - \left[\frac{1}{3} \left\{ \left(1 - T_k^{dm}(x) \right)^2 + \left(I_k^{dm}(x) \right)^2 + \left(F_k^{dm}(x) \right)^2 \right\} \right]^{0.5} \right)} ; \text{ where } \lambda_k \geq 0 \text{ and } \sum_{k=1}^l \lambda_k = 1$$

Step 2: Computation of the Aggregated Neutrosophic Decision Matrix (ANDM)

The ANDM is given as follows

$$D = \begin{matrix} & A_1 & A_2 & \cdots & A_m \end{matrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} = [r_{ij}]_{m \times n}$$

where r_{ij} can be defined as

$$r_{ij} = (T_{ij}, I_{ij}, F_{ij}) = (T_{A_i}(x_j), I_{A_i}(x_j), F_{A_i}(x_j)), \text{ where } i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n$$

Therefore, ANDM written as follows

$$D = \begin{bmatrix} (T_{A_1}(x_1), I_{A_1}(x_1), F_{A_1}(x_1)) & (T_{A_1}(x_2), I_{A_1}(x_2), F_{A_1}(x_2)) & \cdots & (T_{A_1}(x_n), I_{A_1}(x_n), F_{A_1}(x_n)) \\ (T_{A_2}(x_1), I_{A_2}(x_1), F_{A_2}(x_1)) & (T_{A_2}(x_2), I_{A_2}(x_2), F_{A_2}(x_2)) & \cdots & (T_{A_2}(x_n), I_{A_2}(x_n), F_{A_2}(x_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{A_m}(x_1), I_{A_m}(x_1), F_{A_m}(x_1)) & (T_{A_m}(x_2), I_{A_m}(x_2), F_{A_m}(x_2)) & \cdots & (T_{A_m}(x_n), I_{A_m}(x_n), F_{A_m}(x_n)) \end{bmatrix}$$

rating for the i^{th} alternative w.r.t. the j^{th} criterion by the k^{th} DM

$$r_{ij}^{(k)} = (T_{ij}^{(k)}, I_{ij}^{(k)}, F_{ij}^{(k)})$$

For DM weights and alternative ratings r_{ij} can be calculated by using a single-valued neutrosophic weighted averaging operator (SVNWAOP)

$$r_{ij} = [1 - \prod_{k=1}^l (1 - T_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^l (I_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^l (F_{ij}^{(k)})^{\lambda_k}]$$

Step 3: Computation of the weights for the criteria

Let an SVN allocated to the criterion by X_j the k^{th} DM is denoted as

$$w_j^{(k)} = (T_j^{(k)}, I_j^{(k)}, F_j^{(k)})$$

SVNWAOP to compute the weights of the criteria is given as follows

$$w_j = [1 - \prod_{k=1}^l (1 - T_j^{(k)})^{\lambda_k}, \prod_{k=1}^l (I_j^{(k)})^{\lambda_k}, \prod_{k=1}^l (F_j^{(k)})^{\lambda_k}]$$

The aggregated weight for the criterion X_j is represented as

$$w_j = (T_j, I_j, F_j) \quad j = 1, 2, 3, \dots, n$$

$$W = [w_1, w_2, w_3, \dots, w_n]^{\text{Transpose}}$$

Step 4: Computation of Aggregated Weighted Neutrosophic Decision Matrix (AWNNDM)

The AWNNDM is calculated as follows

$$R' = \begin{matrix} & A_1 & A_2 & \cdots & A_m \end{matrix} \begin{bmatrix} r'_{11} & r'_{12} & \cdots & r'_{1n} \\ r'_{21} & r'_{22} & \cdots & r'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r'_{m1} & r'_{m2} & \cdots & r'_{mn} \end{bmatrix} = [r'_{ij}]_{m \times n}$$

where $r'_{ij} = (T_{A_i.W}(x_j), I_{A_i.W}(x_j), F_{A_i.W}(x_j))$ where $i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n$.

Therefore, R' can be written as

$$R' = \begin{bmatrix} (T_{A_1.W}(x_1), I_{A_1.W}(x_1), F_{A_1.W}(x_1)) & (T_{A_1.W}(x_2), I_{A_1.W}(x_2), F_{A_1.W}(x_2)) & \cdots & (T_{A_1.W}(x_n), I_{A_1.W}(x_n), F_{A_1.W}(x_n)) \\ (T_{A_2.W}(x_1), I_{A_2.W}(x_1), F_{A_2.W}(x_1)) & (T_{A_2.W}(x_2), I_{A_2.W}(x_2), F_{A_2.W}(x_2)) & \cdots & (T_{A_2.W}(x_n), I_{A_2.W}(x_n), F_{A_2.W}(x_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{A_m.W}(x_1), I_{A_m.W}(x_1), F_{A_m.W}(x_1)) & (T_{A_m.W}(x_2), I_{A_m.W}(x_2), F_{A_m.W}(x_2)) & \cdots & (T_{A_m.W}(x_n), I_{A_m.W}(x_n), F_{A_m.W}(x_n)) \end{bmatrix}$$

To find $T_{A_i.W}(x_j)$, $I_{A_i.W}(x_j)$ and $F_{A_i.W}(x_j)$ we used

$$R \otimes W = \{ \langle x, T_{A_i.W}(x) \rangle, \langle x, I_{A_i.W}(x) \rangle, \langle x, F_{A_i.W}(x) \rangle \mid x \in X \}$$

The components of the product given as

$$T_{A_i.W}(x) = T_{A_i}(x) \cdot T_j$$

$$I_{A_i.W}(x) = I_{A_i}(x) + I_j(x) - I_{A_i}(x) \times I_j(x)$$

$$F_{A_i.W}(x) = F_{A_i}(x) + F_j(x) - F_{A_i}(x) \times F_j(x)$$

Step 5: Computation of Single Valued Neutrosophic Positive Ideal Solution (SVN-PIS) and Single Valued Neutrosophic Positive Ideal Solution (SVN-NIS)

Let J_1 be the benefit criteria and J_2 be the cost criteria. A^* be an SVN-PIS and A' be an SVN-NIS as follows

$$A^* = (T_{A^*.W}(x_j), I_{A^*.W}(x_j), F_{A^*.W}(x_j)) \text{ and}$$

$$A' = (T_{A'.W}(x_j), I_{A'.W}(x_j), F_{A'.W}(x_j))$$

The components of SVN-PIS and SVN-NIS are following

$$T_{A^*.W}(x_j) = \left(\left(\max_i T_{A_i.W}(x_j) \mid j \in j_1 \right), \left(\min_i T_{A_i.W}(x_j) \mid j \in j_2 \right) \right)$$

$$I_{A^*.W}(x_j) = \left(\left(\min_i I_{A_i.W}(x_j) \mid j \in j_1 \right), \left(\max_i I_{A_i.W}(x_j) \mid j \in j_2 \right) \right)$$

$$F_{A^*.W}(x_j) = \left(\left(\min_i F_{A_i.W}(x_j) \mid j \in j_1 \right), \left(\max_i F_{A_i.W}(x_j) \mid j \in j_2 \right) \right)$$

$$T_{A'.W}(x_j) = \left(\left(\min_i T_{A_i.W}(x_j) \mid j \in j_1 \right), \left(\max_i T_{A_i.W}(x_j) \mid j \in j_2 \right) \right)$$

$$I_{A'.W}(x_j) = \left(\left(\max_i I_{A_i.W}(x_j) \mid j \in j_1 \right), \left(\min_i I_{A_i.W}(x_j) \mid j \in j_2 \right) \right)$$

$$F_{A'.W}(x_j) = \left(\left(\max_i F_{A_i.W}(x_j) \mid j \in j_1 \right), \left(\min_i F_{A_i.W}(x_j) \mid j \in j_2 \right) \right)$$

Step 6: Computation of Separation Measures

For the separation measures d^* and d' , Normalized Euclidean Distance is used as given as

$$d_i^* = \left(\frac{1}{3n} \sum_{j=1}^n \left[\left(T_{A_i.W}(x_j) - T_{A^*.W}(x_j) \right)^2 + \left(I_{A_i.W}(x_j) - I_{A^*.W}(x_j) \right)^2 + \left(F_{A_i.W}(x_j) - F_{A^*.W}(x_j) \right)^2 \right] \right)^{0.5}$$

$$d_i' = \left(\frac{1}{3n} \sum_{j=1}^n \left[\left(T_{A_i.W}(x_j) - T_{A'.W}(x_j) \right)^2 + \left(I_{A_i.W}(x_j) - I_{A'.W}(x_j) \right)^2 + \left(F_{A_i.W}(x_j) - F_{A'.W}(x_j) \right)^2 \right] \right)^{0.5}$$

Step 7: Computation of Relative Closeness Coefficient (RCC)

The RCC of an alternative A_i w.r.t. the SVN-PIS A^* is computed as

$$RCC_i = \frac{d_i'}{d_i' + d_i^*} \quad \text{where } 0 \leq RCC_i \leq 1$$

Step 8: Ranking alternatives

After computation of RCC_i for each alternative A_i , the rank of the alternatives presented in descending orders of RCC_i .

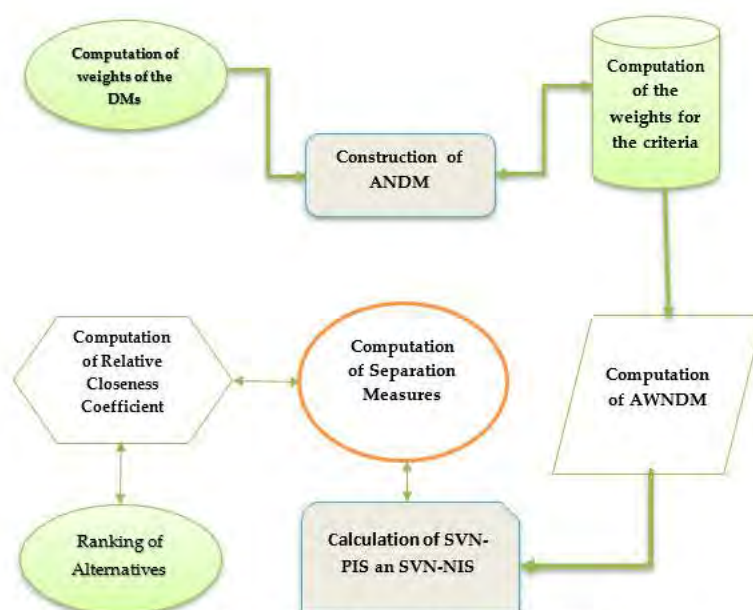


Figure 1 Algorithm of Proposed Neutrosophic TOPSIS

4. Application of Neutrosophic TOPSIS in decision making

A production industry wants to hire a supplier, for the selection of supplier managing director of the industry decides the criteria for supplier selection. The industry hires a team of decision-makers for the selection of the best supplier. Consider $A = \{A_i : i = 1, 2, 3, 4, 5\}$ be a set of supplier and $DM = \{DM_1, DM_2, DM_3, DM_4\}$ be a team of decision-makers ($l = 4$). The evaluation criteria ($n = 5$) for the selection of supplier given as follows,

$$C = \begin{cases} \text{Benefit Criteria} \\ \text{Cost Criteria} \end{cases} \quad j_1 = \begin{cases} X_1: \text{Delivery} \\ X_2: \text{Quality} \\ X_3: \text{Flexibility} \\ X_4: \text{Service} \end{cases} \quad j_2 = \{X_5: \text{Price}\}$$

Calculations of the problem using the proposed SVN-TOPSIS for the importance of criteria and DMs SVN rating scale is given in the following Table

Table 1. Linguistic variables LV's for rating the importance of criteria and decision-makers

LVs	SVNNs
VI	(.90, .10, .10)
I	(.75, .25, .20)
M	(.50, .50, .50)
UI	(.35, .75, .80)
VUI	(.10, .90, .90)

Where VI, I, M, UI, VUI stand for very important, important, medium, unimportant, very unimportant respectively. The alternative ratings are given in the following table

Table 2. Alternative Ratings for Linguistic Variables

LVs	SVNNs
EG	(1.0, 0.0, 0.0)

VVG	(.90, .10, .10)
VG	(.80, .15, .20)
G	(.70, .25, .30)
MG	(.60, .35, .40)
M	(.50, .50, .50)
MB	(.40, .65, .60)
B	(.30, .75, .70)
VB	(.20, .85, .80)
VVB	(.10, .90, .90)
EB	(0.0,1.0,1.0)

Where EG, VVG, VG, G, MG, M, MB, B, VB, VVB, EB are representing extremely good, very very good, very good, good, medium good, medium, medium bad, bad, very bad, very very bad, extremely bad respectively.

Step 1: Determine the weights of the DMs

Weights for the DMs are calculated as follows

$$\lambda_k = \frac{1 - \left[\frac{1}{3} \left\{ \left(1 - T_k^{dm}(x) \right)^2 + \left(I_k^{dm}(x) \right)^2 + \left(F_k^{dm}(x) \right)^2 \right\} \right]^{0.5}}{\sum_{k=1}^l \left(1 - \left[\frac{1}{3} \left\{ \left(1 - T_k^{dm}(x) \right)^2 + \left(I_k^{dm}(x) \right)^2 + \left(F_k^{dm}(x) \right)^2 \right\} \right]^{0.5} \right)} ; \lambda_k \geq 0 \text{ and } \sum_{k=1}^l \lambda_k = 1$$

$$\lambda_1 = \frac{1 - \left[\frac{1}{3} \left\{ \left(1 - T_1^{dm}(x) \right)^2 + \left(I_1^{dm}(x) \right)^2 + \left(F_1^{dm}(x) \right)^2 \right\} \right]^{0.5}}{\sum_{k=1}^l \left(1 - \left[\frac{1}{3} \left\{ \left(1 - T_k^{dm}(x) \right)^2 + \left(I_k^{dm}(x) \right)^2 + \left(F_k^{dm}(x) \right)^2 \right\} \right]^{0.5} \right)}$$

$$\lambda_1 = \frac{1 - \left[\frac{1}{3} \left\{ \left(1 - T_1^{dm}(x) \right)^2 + \left(I_1^{dm}(x) \right)^2 + \left(F_1^{dm}(x) \right)^2 \right\} \right]^{0.5}}{1 - \left[\frac{1}{3} \left\{ \left(1 - T_1^{dm}(x) \right)^2 + \left(I_1^{dm}(x) \right)^2 + \left(F_1^{dm}(x) \right)^2 \right\} \right]^{0.5} + 1 - \left[\frac{1}{3} \left\{ \left(1 - T_2^{dm}(x) \right)^2 + \left(I_2^{dm}(x) \right)^2 + \left(F_2^{dm}(x) \right)^2 \right\} \right]^{0.5} + 1 - \left[\frac{1}{3} \left\{ \left(1 - T_3^{dm}(x) \right)^2 + \left(I_3^{dm}(x) \right)^2 + \left(F_3^{dm}(x) \right)^2 \right\} \right]^{0.5} + 1 - \left[\frac{1}{3} \left\{ \left(1 - T_4^{dm}(x) \right)^2 + \left(I_4^{dm}(x) \right)^2 + \left(F_4^{dm}(x) \right)^2 \right\} \right]^{0.5}}$$

$$\lambda_1 = \frac{1 - \left[\frac{1}{3} \left\{ (1-0.9)^2 + (0.10)^2 + (0.10)^2 \right\} \right]^{0.5}}{1 - \left[\frac{1}{3} \left\{ (1-0.9)^2 + (0.10)^2 + (0.10)^2 \right\} \right]^{0.5} + 1 - \left[\frac{1}{3} \left\{ (1-0.75)^2 + (0.25)^2 + (0.20)^2 \right\} \right]^{0.5} + 1 - \left[\frac{1}{3} \left\{ (1-0.50)^2 + (0.50)^2 + (0.50)^2 \right\} \right]^{0.5} + 1 - \left[\frac{1}{3} \left\{ (1-0.35)^2 + (0.75)^2 + (0.80)^2 \right\} \right]^{0.5}}$$

$$\lambda_1 = \frac{0.9}{0.9+0.76548+0.5+0.26402}$$

$$\lambda_1 = \frac{0.9}{2.42950} = 0.37045$$

$$\lambda_1 = 0.37045$$

Similarly, we get the weights for the other decision-makers as follows

$$\lambda_2 = \frac{0.76548}{2.42950} = 0.31508$$

$$\lambda_2 = 0.31508$$

$$\lambda_3 = \frac{0.5}{2.42950} = 0.20580$$

$$\lambda_3 = 0.20580$$

$$\lambda_4 = \frac{0.26402}{2.42950} = 0.10867$$

$$\lambda_4 = 0.10867$$

The weights for DMs are given in the following Table

Table 3. Weights of Decision Makers

Criteria	Alternatives	Decision Makers			
		DM ₁	DM ₂	DM ₃	DM ₄
X ₁	A ₁	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)
		$r_{11}^{(1)} = (T_{11}^{(1)}, I_{11}^{(1)}, F_{11}^{(1)})$	$r_{11}^{(2)} = (T_{11}^{(2)}, I_{11}^{(2)}, F_{11}^{(2)})$	$r_{11}^{(3)} = (T_{11}^{(3)}, I_{11}^{(3)}, F_{11}^{(3)})$	$r_{11}^{(4)} = (T_{11}^{(4)}, I_{11}^{(4)}, F_{11}^{(4)})$
	A ₂	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)
		$r_{21}^{(1)} = (T_{21}^{(1)}, I_{21}^{(1)}, F_{21}^{(1)})$	$r_{21}^{(2)} = (T_{21}^{(2)}, I_{21}^{(2)}, F_{21}^{(2)})$	$r_{21}^{(3)} = (T_{21}^{(3)}, I_{21}^{(3)}, F_{21}^{(3)})$	$r_{21}^{(4)} = (T_{21}^{(4)}, I_{21}^{(4)}, F_{21}^{(4)})$
	A ₃	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)
		$r_{31}^{(1)} = (T_{31}^{(1)}, I_{31}^{(1)}, F_{31}^{(1)})$	$r_{31}^{(2)} = (T_{31}^{(2)}, I_{31}^{(2)}, F_{31}^{(2)})$	$r_{31}^{(3)} = (T_{31}^{(3)}, I_{31}^{(3)}, F_{31}^{(3)})$	$r_{31}^{(4)} = (T_{31}^{(4)}, I_{31}^{(4)}, F_{31}^{(4)})$
	A ₄	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)
		$r_{41}^{(1)} = (T_{41}^{(1)}, I_{41}^{(1)}, F_{41}^{(1)})$	$r_{41}^{(2)} = (T_{41}^{(2)}, I_{41}^{(2)}, F_{41}^{(2)})$	$r_{41}^{(3)} = (T_{41}^{(3)}, I_{41}^{(3)}, F_{41}^{(3)})$	$r_{41}^{(4)} = (T_{41}^{(4)}, I_{41}^{(4)}, F_{41}^{(4)})$
	A ₅	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)
		$r_{51}^{(1)} = (T_{51}^{(1)}, I_{51}^{(1)}, F_{51}^{(1)})$	$r_{51}^{(2)} = (T_{51}^{(2)}, I_{51}^{(2)}, F_{51}^{(2)})$	$r_{51}^{(3)} = (T_{51}^{(3)}, I_{51}^{(3)}, F_{51}^{(3)})$	$r_{51}^{(4)} = (T_{51}^{(4)}, I_{51}^{(4)}, F_{51}^{(4)})$
X ₂	A ₁	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)
		$r_{12}^{(1)} = (T_{12}^{(1)}, I_{12}^{(1)}, F_{12}^{(1)})$	$r_{12}^{(2)} = (T_{12}^{(2)}, I_{12}^{(2)}, F_{12}^{(2)})$	$r_{12}^{(3)} = (T_{12}^{(3)}, I_{12}^{(3)}, F_{12}^{(3)})$	$r_{12}^{(4)} = (T_{12}^{(4)}, I_{12}^{(4)}, F_{12}^{(4)})$
	A ₂	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)
		$r_{22}^{(1)} = (T_{22}^{(1)}, I_{22}^{(1)}, F_{22}^{(1)})$	$r_{22}^{(2)} = (T_{22}^{(2)}, I_{22}^{(2)}, F_{22}^{(2)})$	$r_{22}^{(3)} = (T_{22}^{(3)}, I_{22}^{(3)}, F_{22}^{(3)})$	$r_{22}^{(4)} = (T_{22}^{(4)}, I_{22}^{(4)}, F_{22}^{(4)})$
	A ₃	M (0.50,0.50,0.50)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)
		$r_{32}^{(1)} = (T_{32}^{(1)}, I_{32}^{(1)}, F_{32}^{(1)})$	$r_{32}^{(2)} = (T_{32}^{(2)}, I_{32}^{(2)}, F_{32}^{(2)})$	$r_{32}^{(3)} = (T_{32}^{(3)}, I_{32}^{(3)}, F_{32}^{(3)})$	$r_{32}^{(4)} = (T_{32}^{(4)}, I_{32}^{(4)}, F_{32}^{(4)})$
	A ₄	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)
		$r_{42}^{(1)} = (T_{42}^{(1)}, I_{42}^{(1)}, F_{42}^{(1)})$	$r_{42}^{(2)} = (T_{42}^{(2)}, I_{42}^{(2)}, F_{42}^{(2)})$	$r_{42}^{(3)} = (T_{42}^{(3)}, I_{42}^{(3)}, F_{42}^{(3)})$	$r_{42}^{(4)} = (T_{42}^{(4)}, I_{42}^{(4)}, F_{42}^{(4)})$
	A ₅	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)
		$r_{52}^{(1)} = (T_{52}^{(1)}, I_{52}^{(1)}, F_{52}^{(1)})$	$r_{52}^{(2)} = (T_{52}^{(2)}, I_{52}^{(2)}, F_{52}^{(2)})$	$r_{52}^{(3)} = (T_{52}^{(3)}, I_{52}^{(3)}, F_{52}^{(3)})$	$r_{52}^{(4)} = (T_{52}^{(4)}, I_{52}^{(4)}, F_{52}^{(4)})$
X ₃	A ₁	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)
		$r_{13}^{(1)} = (T_{13}^{(1)}, I_{13}^{(1)}, F_{13}^{(1)})$	$r_{13}^{(2)} = (T_{13}^{(2)}, I_{13}^{(2)}, F_{13}^{(2)})$	$r_{13}^{(3)} = (T_{13}^{(3)}, I_{13}^{(3)}, F_{13}^{(3)})$	$r_{13}^{(4)} = (T_{13}^{(4)}, I_{13}^{(4)}, F_{13}^{(4)})$
	A ₂	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)
		$r_{23}^{(1)} = (T_{23}^{(1)}, I_{23}^{(1)}, F_{23}^{(1)})$	$r_{23}^{(2)} = (T_{23}^{(2)}, I_{23}^{(2)}, F_{23}^{(2)})$	$r_{23}^{(3)} = (T_{23}^{(3)}, I_{23}^{(3)}, F_{23}^{(3)})$	$r_{23}^{(4)} = (T_{23}^{(4)}, I_{23}^{(4)}, F_{23}^{(4)})$
	A ₃	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)
		$r_{33}^{(1)} = (T_{33}^{(1)}, I_{33}^{(1)}, F_{33}^{(1)})$	$r_{33}^{(2)} = (T_{33}^{(2)}, I_{33}^{(2)}, F_{33}^{(2)})$	$r_{33}^{(3)} = (T_{33}^{(3)}, I_{33}^{(3)}, F_{33}^{(3)})$	$r_{33}^{(4)} = (T_{33}^{(4)}, I_{33}^{(4)}, F_{33}^{(4)})$
	A ₄	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)
		$r_{43}^{(1)} = (T_{43}^{(1)}, I_{43}^{(1)}, F_{43}^{(1)})$	$r_{43}^{(2)} = (T_{43}^{(2)}, I_{43}^{(2)}, F_{43}^{(2)})$	$r_{43}^{(3)} = (T_{43}^{(3)}, I_{43}^{(3)}, F_{43}^{(3)})$	$r_{43}^{(4)} = (T_{43}^{(4)}, I_{43}^{(4)}, F_{43}^{(4)})$
	A ₅	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)
		$r_{53}^{(1)} = (T_{53}^{(1)}, I_{53}^{(1)}, F_{53}^{(1)})$	$r_{53}^{(2)} = (T_{53}^{(2)}, I_{53}^{(2)}, F_{53}^{(2)})$	$r_{53}^{(3)} = (T_{53}^{(3)}, I_{53}^{(3)}, F_{53}^{(3)})$	$r_{53}^{(4)} = (T_{53}^{(4)}, I_{53}^{(4)}, F_{53}^{(4)})$
X ₄	A ₁	G (0.70,0.25,0.30)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)
		$r_{14}^{(1)} = (T_{14}^{(1)}, I_{14}^{(1)}, F_{14}^{(1)})$	$r_{14}^{(2)} = (T_{14}^{(2)}, I_{14}^{(2)}, F_{14}^{(2)})$	$r_{14}^{(3)} = (T_{14}^{(3)}, I_{14}^{(3)}, F_{14}^{(3)})$	$r_{14}^{(4)} = (T_{14}^{(4)}, I_{14}^{(4)}, F_{14}^{(4)})$
	A ₂	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)
		$r_{24}^{(1)} = (T_{24}^{(1)}, I_{24}^{(1)}, F_{24}^{(1)})$	$r_{24}^{(2)} = (T_{24}^{(2)}, I_{24}^{(2)}, F_{24}^{(2)})$	$r_{24}^{(3)} = (T_{24}^{(3)}, I_{24}^{(3)}, F_{24}^{(3)})$	$r_{24}^{(4)} = (T_{24}^{(4)}, I_{24}^{(4)}, F_{24}^{(4)})$

X_5	A_3	MG (0.60,0.35,0.40) $r_{34}^{(1)} = (T_{34}^{(1)}, I_{34}^{(1)}, F_{34}^{(1)})$	MG (0.60,0.35,0.40) $r_{34}^{(2)} = (T_{34}^{(2)}, I_{34}^{(2)}, F_{34}^{(2)})$	MG (0.60,0.35,0.40) $r_{34}^{(3)} = (T_{34}^{(3)}, I_{34}^{(3)}, F_{34}^{(3)})$	MG (0.60,0.35,0.40) $r_{34}^{(4)} = (T_{34}^{(4)}, I_{34}^{(4)}, F_{34}^{(4)})$
	A_4	M (0.50,0.50,0.50) $r_{44}^{(1)} = (T_{44}^{(1)}, I_{44}^{(1)}, F_{44}^{(1)})$	MB (0.40,0.65,0.60) $r_{44}^{(2)} = (T_{44}^{(2)}, I_{44}^{(2)}, F_{44}^{(2)})$	MG (0.60,0.35,0.40) $r_{44}^{(3)} = (T_{44}^{(3)}, I_{44}^{(3)}, F_{44}^{(3)})$	VG (0.80,0.15,0.20) $r_{44}^{(4)} = (T_{44}^{(4)}, I_{44}^{(4)}, F_{44}^{(4)})$
	A_5	MG (0.60,0.35,0.40) $r_{54}^{(1)} = (T_{54}^{(1)}, I_{54}^{(1)}, F_{54}^{(1)})$	G (0.70,0.25,0.30) $r_{54}^{(2)} = (T_{54}^{(2)}, I_{54}^{(2)}, F_{54}^{(2)})$	VG (0.80,0.15,0.20) $r_{54}^{(3)} = (T_{54}^{(3)}, I_{54}^{(3)}, F_{54}^{(3)})$	G (0.70,0.25,0.30) $r_{54}^{(4)} = (T_{54}^{(4)}, I_{54}^{(4)}, F_{54}^{(4)})$
	A_1	M (0.50,0.50,0.50) $r_{15}^{(1)} = (T_{15}^{(1)}, I_{15}^{(1)}, F_{15}^{(1)})$	MG (0.60,0.35,0.40) $r_{15}^{(2)} = (T_{15}^{(2)}, I_{15}^{(2)}, F_{15}^{(2)})$	VG (0.80,0.15,0.20) $r_{15}^{(3)} = (T_{15}^{(3)}, I_{15}^{(3)}, F_{15}^{(3)})$	M (0.50,0.50,0.50) $r_{15}^{(4)} = (T_{15}^{(4)}, I_{15}^{(4)}, F_{15}^{(4)})$
	A_2	VG (0.80,0.15,0.20) $r_{25}^{(1)} = (T_{25}^{(1)}, I_{25}^{(1)}, F_{25}^{(1)})$	M (0.50,0.50,0.50) $r_{25}^{(2)} = (T_{25}^{(2)}, I_{25}^{(2)}, F_{25}^{(2)})$	G (0.70,0.25,0.30) $r_{25}^{(3)} = (T_{25}^{(3)}, I_{25}^{(3)}, F_{25}^{(3)})$	G (0.70,0.25,0.30) $r_{25}^{(4)} = (T_{25}^{(4)}, I_{25}^{(4)}, F_{25}^{(4)})$
	A_3	G (0.70,0.25,0.30) $r_{35}^{(1)} = (T_{35}^{(1)}, I_{35}^{(1)}, F_{35}^{(1)})$	G (0.70,0.25,0.30) $r_{35}^{(2)} = (T_{35}^{(2)}, I_{35}^{(2)}, F_{35}^{(2)})$	M (0.50,0.50,0.50) $r_{35}^{(3)} = (T_{35}^{(3)}, I_{35}^{(3)}, F_{35}^{(3)})$	MG (0.60,0.35,0.40) $r_{35}^{(4)} = (T_{35}^{(4)}, I_{35}^{(4)}, F_{35}^{(4)})$
	A_4	M (0.50,0.50,0.50) $r_{45}^{(1)} = (T_{45}^{(1)}, I_{45}^{(1)}, F_{45}^{(1)})$	M (0.50,0.50,0.50) $r_{45}^{(2)} = (T_{45}^{(2)}, I_{45}^{(2)}, F_{45}^{(2)})$	MG (0.60,0.35,0.40) $r_{45}^{(3)} = (T_{45}^{(3)}, I_{45}^{(3)}, F_{45}^{(3)})$	G (0.70,0.25,0.30) $r_{45}^{(4)} = (T_{45}^{(4)}, I_{45}^{(4)}, F_{45}^{(4)})$
	A_5	G (0.70,0.25,0.30) $r_{55}^{(1)} = (T_{55}^{(1)}, I_{55}^{(1)}, F_{55}^{(1)})$	VG (0.80,0.15,0.20) $r_{55}^{(2)} = (T_{55}^{(2)}, I_{55}^{(2)}, F_{55}^{(2)})$	VG (0.80,0.15,0.20) $r_{55}^{(3)} = (T_{55}^{(3)}, I_{55}^{(3)}, F_{55}^{(3)})$	VG (0.80,0.15,0.20) $r_{55}^{(4)} = (T_{55}^{(4)}, I_{55}^{(4)}, F_{55}^{(4)})$

Table 4. Importance and Weights of Decision-Makers

	DM_1	DM_2	DM_3	DM_4
Linguistic Variables	VI (0.90,0.10,0.10) $(T_1^{dm}, I_1^{dm}, F_1^{dm})$	I (0.75,0.25,0.20) $(T_2^{dm}, I_2^{dm}, F_2^{dm})$	M (0.50,0.50,0.50) $(T_3^{dm}, I_3^{dm}, F_3^{dm})$	UI (0.35,0.75,0.80) $(T_4^{dm}, I_4^{dm}, F_4^{dm})$
Weights	$\lambda_{DM_1} = 0.37045$	$\lambda_{DM_2} = 0.31508$	$\lambda_{DM_3} = 0.20580$	$\lambda_{DM_4} = 0.10867$

Step 2: Computation of Aggregated Single Valued Neutrosophic Decision Matrix (ASVNDM)

To find the ASVNDM not only the weights of the DMs, but the alternative ratings are also required.

The alternative ratings, according to the DMs given in the following table.

Now by using the alternative ratings $r_{ij}^{(k)}$ and the DM weights λ_k we get

$$r_{ij} = \lambda_1 r_{ij}^{(1)} \oplus \lambda_2 r_{ij}^{(2)} \oplus \lambda_3 r_{ij}^{(3)} \oplus \dots \oplus \lambda_l r_{ij}^{(l)}$$

$$r_{ij} = (1 - \prod_{k=1}^l (1 - T_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^l (I_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^l (F_{ij}^{(k)})^{\lambda_k})$$

where $i = 1, 2, 3, 4, 5$; $j = 1, 2, 3, 4, 5$ and $(l = 4)$.

For $i = j = 1$ and $l = 4$

$$r_{11} = \lambda_1 r_{11}^{(1)} \oplus \lambda_2 r_{11}^{(2)} \oplus \lambda_3 r_{11}^{(3)} \oplus \dots \oplus \lambda_l r_{11}^{(l)}$$

$$r_{11} = (1 - \prod_{k=1}^4 (1 - T_{11}^{(k)})^{\lambda_k}, \prod_{k=1}^4 (I_{11}^{(k)})^{\lambda_k}, \prod_{k=1}^4 (F_{11}^{(k)})^{\lambda_k})$$

$$r_{11} = (1 - ((1 - T_{11}^{(1)})^{\lambda_1} (1 - T_{11}^{(2)})^{\lambda_2} (1 - T_{11}^{(3)})^{\lambda_3} (1 - T_{11}^{(4)})^{\lambda_4}), (I_{11}^{(1)})^{\lambda_1} (I_{11}^{(2)})^{\lambda_2} (I_{11}^{(3)})^{\lambda_3} (I_{11}^{(4)})^{\lambda_4}, (F_{11}^{(1)})^{\lambda_1} (F_{11}^{(2)})^{\lambda_2} (F_{11}^{(3)})^{\lambda_3} (F_{11}^{(4)})^{\lambda_4})$$

$$r_{11} = (1 - ((1 - 0.8)^{0.37045} (1 - 0.6)^{0.31508} (1 - 0.8)^{0.20580} (1 - 0.7)^{0.10867}), ((0.15)^{0.37045} (0.35)^{0.31508} (0.15)^{0.20580} (0.25)^{0.10867}), ((0.20)^{0.37045} (0.40)^{0.31508} (0.20)^{0.20580} (0.30)^{0.10867}))$$

$$r_{11} = (0.740, 0.207, 0.260)$$

Similarly, we can find other values

$$r_{21} = (0.711, 0.237, 0.289)$$

$r_{31} = (0.593, 0.373, 0.407)$
 $r_{41} = (0.661, 0.288, 0.339)$
 $r_{51} = (0.706, 0.241, 0.294)$
 $r_{12} = (0.682, 0.268, 0.318)$
 $r_{22} = (0.676, 0.275, 0.324)$
 $r_{32} = (0.681, 0.275, 0.324)$
 $r_{42} = (0.619, 0.342, 0.381)$
 $r_{52} = (0.695, 0.253, 0.305)$
 $r_{13} = (0.505, 0.392, 0.429)$
 $r_{23} = (0.773, 0.176, 0.227)$
 $r_{33} = (0.603, 0.359, 0.397)$
 $r_{43} = (0.661, 0.288, 0.339)$
 $r_{53} = (0.693, 0.255, 0.307)$
 $r_{14} = (0.605, 0.359, 0.395)$
 $r_{24} = (0.748, 0.203, 0.252)$
 $r_{34} = (0.600, 0.350, 0.400)$
 $r_{44} = (0.542, 0.443, 0.458)$
 $r_{54} = (0.693, 0.339, 0.307)$
 $r_{15} = (0.614, 0.349, 0.386)$
 $r_{25} = (0.697, 0.257, 0.303)$
 $r_{35} = (0.656, 0.299, 0.344)$
 $r_{45} = (0.548, 0.431, 0.452)$
 $r_{55} = (0.768, 0.181, 0.232)$

Table 5. Aggregated Single Valued Neutrosophic Decision Matrix $D = [r_{ij}]_{5 \times 4}$

	X_1	X_2	X_3	X_4	X_5
A_1	$r_{11} = (0.740, 0.207, 0.260)$	$r_{12} = (0.682, 0.268, 0.318)$	$r_{13} = (0.505, 0.392, 0.429)$	$r_{14} = (0.605, 0.359, 0.395)$	$r_{15} = (0.614, 0.349, 0.386)$
A_2	$r_{21} = (0.711, 0.237, 0.289)$	$r_{22} = (0.676, 0.275, 0.324)$	$r_{23} = (0.773, 0.176, 0.227)$	$r_{24} = (0.748, 0.203, 0.252)$	$r_{25} = (0.697, 0.257, 0.303)$
A_3	$r_{31} = (0.593, 0.373, 0.407)$	$r_{32} = (0.681, 0.275, 0.324)$	$r_{33} = (0.603, 0.359, 0.397)$	$r_{34} = (0.600, 0.350, 0.400)$	$r_{35} = (0.656, 0.299, 0.344)$
A_4	$r_{41} = (0.661, 0.288, 0.339)$	$r_{42} = (0.619, 0.342, 0.381)$	$r_{43} = (0.661, 0.288, 0.339)$	$r_{44} = (0.542, 0.443, 0.458)$	$r_{45} = (0.548, 0.431, 0.452)$
A_5	$r_{51} = (0.706, 0.241, 0.294)$	$r_{52} = (0.695, 0.253, 0.305)$	$r_{53} = (0.693, 0.255, 0.307)$	$r_{54} = (0.693, 0.339, 0.307)$	$r_{55} = (0.768, 0.181, 0.232)$

Step 3: Computation of the weights of the criteria

The individual weights given by each DM is given in Table 6.

Table 6. Weights of alternatives determined by the DMs $w_j^{(k)} = (T_j^{(k)}, I_j^{(k)}, F_j^{(k)})$

Criteria	DM_1	DM_2	DM_3	DM_4
X_1	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)	I (0.75,0.25,0.20)
(DELIVERY)	$w_1^{(1)} = (T_1^{(1)}, I_1^{(1)}, F_1^{(1)})$	$w_1^{(2)} = (T_1^{(2)}, I_1^{(2)}, F_1^{(2)})$	$w_1^{(3)} = (T_1^{(3)}, I_1^{(3)}, F_1^{(3)})$	$w_1^{(4)} = (T_1^{(4)}, I_1^{(4)}, F_1^{(4)})$

X_2	I (0.75,0.25,0.20)	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)	I (0.75,0.25,0.20)
(QUALITY)	$w_2^{(1)} = (T_2^{(1)}, I_2^{(1)}, F_2^{(1)})$	$w_2^{(2)} = (T_2^{(2)}, I_2^{(2)}, F_2^{(2)})$	$w_2^{(3)} = (T_2^{(3)}, I_2^{(3)}, F_2^{(3)})$	$w_2^{(4)} = (T_2^{(4)}, I_2^{(4)}, F_2^{(4)})$
X_3	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)	I (0.75,0.25,0.20)	VI (0.90,0.10,0.10)
(FLEXIBILITY)	$w_3^{(1)} = (T_3^{(1)}, I_3^{(1)}, F_3^{(1)})$	$w_3^{(2)} = (T_3^{(2)}, I_3^{(2)}, F_3^{(2)})$	$w_3^{(3)} = (T_3^{(3)}, I_3^{(3)}, F_3^{(3)})$	$w_3^{(4)} = (T_3^{(4)}, I_3^{(4)}, F_3^{(4)})$
X_4	I (0.75,0.25,0.20)	I (0.75,0.25,0.20)	M (0.50,0.50,0.50)	UI (0.35,0.75,0.80)
(SERVICE)	$w_4^{(1)} = (T_4^{(1)}, I_4^{(1)}, F_4^{(1)})$	$w_4^{(2)} = (T_4^{(2)}, I_4^{(2)}, F_4^{(2)})$	$w_4^{(3)} = (T_4^{(3)}, I_4^{(3)}, F_4^{(3)})$	$w_4^{(4)} = (T_4^{(4)}, I_4^{(4)}, F_4^{(4)})$
X_5	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)
(PRICE)	$w_5^{(1)} = (T_5^{(1)}, I_5^{(1)}, F_5^{(1)})$	$w_5^{(2)} = (T_5^{(2)}, I_5^{(2)}, F_5^{(2)})$	$w_5^{(3)} = (T_5^{(3)}, I_5^{(3)}, F_5^{(3)})$	$w_5^{(4)} = (T_5^{(4)}, I_5^{(4)}, F_5^{(4)})$

By using the values from Table 6, the aggregated criteria weights are calculated as follows

$$w_j = (T_j, I_j, F_j) = \lambda_1 w_j^{(1)} \oplus \lambda_2 w_j^{(2)} \oplus \lambda_3 w_j^{(3)} \oplus \dots \oplus \lambda_l w_j^{(l)}$$

$$w_j = (1 - \prod_{k=1}^l (1 - T_j^{(k)})^{\lambda_k}, \prod_{k=1}^l (I_j^{(k)})^{\lambda_k}, \prod_{k=1}^l (F_j^{(k)})^{\lambda_k}) \text{ where } j = 1, 2, 3, 4, 5 \text{ and } (l = 4).$$

For $j = 1$ and $l = 4$

$$w_1 = \lambda_1 w_1^{(1)} \oplus \lambda_2 w_1^{(2)} \oplus \lambda_3 w_1^{(3)} \oplus \lambda_4 w_1^{(4)}$$

$$w_1 = (1 - \prod_{k=1}^4 (1 - T_1^{(k)})^{\lambda_k}, \prod_{k=1}^4 (I_1^{(k)})^{\lambda_k}, \prod_{k=1}^4 (F_1^{(k)})^{\lambda_k})$$

$$w_1 = (1 - (1 - T_1^{(1)})^{\lambda_1} (1 - T_1^{(2)})^{\lambda_2} (1 - T_1^{(3)})^{\lambda_3} (1 - T_1^{(4)})^{\lambda_4}, (I_1^{(1)})^{\lambda_1} (I_1^{(2)})^{\lambda_2} (I_1^{(3)})^{\lambda_3} (I_1^{(4)})^{\lambda_4}, (F_1^{(1)})^{\lambda_1} (F_1^{(2)})^{\lambda_2} (F_1^{(3)})^{\lambda_3} (F_1^{(4)})^{\lambda_4})$$

$$w_1 = (1 - ((1 - 0.9)^{0.37045} (1 - 0.9)^{0.31508} (1 - 0.9)^{0.20580} (1 - 0.75)^{0.10867}),$$

$$((0.10)^{0.37045} (0.10)^{0.31508} (0.10)^{0.20580} (0.25)^{0.10867})$$

$$((0.10)^{0.37045} (0.10)^{0.31508} (0.10)^{0.20580} (0.20)^{0.10867}))$$

$$r_{11} = (0.740, 0.207, 0.260)$$

$$w_1 = (T_1, I_1, F_1) = (0.890, 0.110, 0.108)$$

Similarly, we can get other values

Therefore

$$W_{\{X_1, X_2, X_3, X_4\}} = \begin{bmatrix} (0.890, 0.110, 0.108)^T \\ (0.641, 0.359, 0.322) \\ (0.879, 0.121, 0.115) \\ (0.680, 0.325, 0.281) \\ (0.699, 0.301, 0.301) \end{bmatrix}$$

Step 4: Construction of Aggregated Weighted Single Valued Neutrosophic Decision Matrix (AWSVNDM)

After finding the weights of the criteria and the alternative ratings, the aggregated weighted single-valued neutrosophic ratings are calculated as follows

$$r'_{ij} = (T'_{ij}, I'_{ij}, rF'_{ij}) = (T_{A_i}(x).T_j, I_{A_i}(x) + I_j - I_{A_i}(x).I_j, F_{A_i}(x) + F_j - F_{A_i}(x).F_j)$$

By using the above equation, we can get an aggregated weighted single-valued neutrosophic decision matrix.

Table 7. Aggregated Weighted Single Valued Neutrosophic Decision Matrix $R' = [r'_{ij}]_{5 \times 5}$

	X_1	X_2	X_3	X_4	X_5
A_1	$r'_{11} =$ (0.659,0.294,0.340)	$r'_{12} =$ (0.437,0.531,0.538)	$r'_{13} =$ (0.444,0.466,0.495)	$r'_{14} =$ (0.411,0.567,0.565)	$r'_{15} =$ (0.429,0.545,0.571)

A₂	$r'_{21} =$ (0.633,0.321,0.366)	$r'_{22} =$ (0.433,0.535,0.542)	$r'_{23} =$ (0.679,0.276,0.316)	$r'_{24} =$ (0.509,0.462,0.462)	$r'_{25} =$ (0.487,0.481,0.513)
A₃	$r'_{31} =$ (0.528,0.442,0.471)	$r'_{32} =$ (0.437,0.535,0.542)	$r'_{33} =$ (0.530,0.437,0.466)	$r'_{34} =$ (0.408,0.561,0.569)	$r'_{35} =$ (0.459,0.510,0.541)
A₄	$r'_{41} =$ (0.588,0.366,0.410)	$r'_{42} =$ (0.397,0.578,0.580)	$r'_{43} =$ (0.581,0.374,0.415)	$r'_{44} =$ (0.037,0.624,0.610)	$r'_{45} =$ (0.383,0.602,0.617)
A₅	$r'_{51} =$ (0.628,0.324,0.3700)	$r'_{52} =$ (0.445,0.521,0.529)	$r'_{53} =$ (0.609,0.345,0.387)	$r'_{54} =$ (0.471,0.554,0.502)	$r'_{55} =$ (0.537,0.428,0.463)

Step 5: Computation of SVN-PIS and SVN-NIS

Since Delivery, Quality, Flexibility, and Services are benefit criteria that is why they are in the set

$$J_1 = \{X_1, X_2, X_3, X_4\}$$

whereas Price being the cost criteria, so it is in the set $J_2 = \{X_5\}$ SVN-PIS and SVN-NIS are calculated as,

Table 8. SVN-PIS and SVN-NIS

SVN-PIS	SVN-NIS
$T_1^+ = \max \{0.659, 0.633, 0.528, 0.588, 0.628\} = 0.659$	$T_1^- = \min \{0.659, 0.633, 0.528, 0.588, 0.628\} = 0.528$
$I_1^+ = \min \{0.294, 0.321, 0.442, 0.366, 0.324\} = 0.294$	$I_1^- = \max \{0.294, 0.321, 0.442, 0.366, 0.324\} = 0.442$
$F_1^+ = \min \{0.340, 0.366, 0.471, 0.410, 0.370\} = 0.340$	$F_1^- = \max \{0.340, 0.366, 0.471, 0.410, 0.370\} = 0.471$
$T_2^+ = \max \{0.437, 0.433, 0.437, 0.397, 0.445\} = 0.445$	$T_2^- = \min \{0.437, 0.433, 0.437, 0.397, 0.445\} = 0.397$
$I_2^+ = \min \{0.531, 0.535, 0.535, 0.578, 0.521\} = 0.521$	$I_2^- = \max \{0.531, 0.535, 0.535, 0.578, 0.521\} = 0.578$
$F_2^+ = \min \{0.538, 0.542, 0.542, 0.580, 0.529\} = 0.529$	$F_2^- = \max \{0.538, 0.542, 0.542, 0.580, 0.529\} = 0.580$
$T_3^+ = \max \{0.444, 0.679, 0.530, 0.581, 0.609\} = 0.679$	$T_3^- = \min \{0.444, 0.679, 0.530, 0.581, 0.609\} = 0.444$
$I_3^+ = \min \{0.466, 0.276, 0.437, 0.374, 0.345\} = 0.276$	$I_3^- = \max \{0.466, 0.276, 0.437, 0.374, 0.345\} = 0.466$
$F_3^+ = \min \{0.495, 0.316, 0.466, 0.415, 0.387\} = 0.316$	$F_3^- = \max \{0.495, 0.316, 0.466, 0.415, 0.387\} = 0.495$
$T_4^+ = \max \{0.411, 0.509, 0.408, 0.037, 0.471\} = 0.509$	$T_4^- = \min \{0.411, 0.509, 0.408, 0.037, 0.471\} = 0.037$
$I_4^+ = \min \{0.567, 0.462, 0.561, 0.624, 0.554\} = 0.462$	$I_4^- = \max \{0.567, 0.462, 0.561, 0.624, 0.554\} = 0.624$
$F_4^+ = \min \{0.565, 0.462, 0.569, 0.610, 0.502\} = 0.462$	$F_4^- = \max \{0.565, 0.462, 0.569, 0.610, 0.502\} = 0.610$
$T_5^+ = \min \{0.429, 0.487, 0.459, 0.383, 0.537\} = 0.383$	$T_5^- = \max \{0.429, 0.487, 0.459, 0.383, 0.537\} = 0.537$
$I_5^+ = \max \{0.545, 0.481, 0.510, 0.602, 0.428\} = 0.602$	$I_5^- = \min \{0.545, 0.481, 0.510, 0.602, 0.428\} = 0.428$
$F_5^+ = \max \{0.571, 0.513, 0.541, 0.617, 0.463\} = 0.617$	$F_5^- = \min \{0.571, 0.513, 0.541, 0.617, 0.463\} = 0.463$

$$A^+ = \left\{ \begin{array}{l} (0.659, 0.294, 0.340), \\ (0.445, 0.521, 0.529), \\ (0.679, 0.276, 0.316), \\ (0.509, 0.462, 0.462), \\ (0.383, 0.602, 0.617) \end{array} \right\}$$

$$A^- = \left\{ \begin{array}{l} (0.528, 0.442, 0.471), \\ (0.397, 0.578, 0.580), \\ (0.444, 0.466, 0.495), \\ (0.037, 0.624, 0.610), \\ (0.537, 0.428, 0.463) \end{array} \right\}$$

Step 6: Computation of Separation Measures

Normalized Euclidean Distance Measure is used to find the negative and positive separation measures d^+ and d^- respectively. Now for the SVN-PIS, we use

$$d_i^+ = \left(\frac{1}{3n} \sum_{j=1}^n \left[\left(T_{A_i.W}(x_j) - T_{A^*W}(x_j) \right)^2 + \left(I_{A_i.W}(x_j) - I_{A^*W}(x_j) \right)^2 + \left(F_{A_i.W}(x_j) - F_{A^*W}(x_j) \right)^2 \right] \right)^{0.5}$$

For $i = 1$ and $n = 5$

$$d_1^+ = \left(\frac{1}{3(5)} \sum_{j=1}^5 \left[\left(T_{A_1.W}(x_j) - T_{A^*W}(x_j) \right)^2 + \left(I_{A_1.W}(x_j) - I_{A^*W}(x_j) \right)^2 + \left(F_{A_1.W}(x_j) - F_{A^*W}(x_j) \right)^2 \right] \right)^{0.5}$$

$$d_1^+ = \left(\frac{1}{15} \left[\begin{aligned} &\left(T_{A_1.W}(X_1) - T_{A^*W}(X_1) \right)^2 + \left(I_{A_1.W}(X_1) - I_{A^*W}(X_1) \right)^2 + \left(F_{A_1.W}(X_1) - F_{A^*W}(X_1) \right)^2 + \\ &\left(T_{A_1.W}(X_2) - T_{A^*W}(X_2) \right)^2 + \left(I_{A_1.W}(X_2) - I_{A^*W}(X_2) \right)^2 + \left(F_{A_1.W}(X_2) - F_{A^*W}(X_2) \right)^2 + \\ &\left(T_{A_1.W}(X_3) - T_{A^*W}(X_3) \right)^2 + \left(I_{A_1.W}(X_3) - I_{A^*W}(X_3) \right)^2 + \left(F_{A_1.W}(X_3) - F_{A^*W}(X_3) \right)^2 + \\ &\left(T_{A_1.W}(X_4) - T_{A^*W}(X_4) \right)^2 + \left(I_{A_1.W}(X_4) - I_{A^*W}(X_4) \right)^2 + \left(F_{A_1.W}(X_4) - F_{A^*W}(X_4) \right)^2 + \\ &\left(T_{A_1.W}(X_5) - T_{A^*W}(X_5) \right)^2 + \left(I_{A_1.W}(X_5) - I_{A^*W}(X_5) \right)^2 + \left(F_{A_1.W}(X_5) - F_{A^*W}(X_5) \right)^2 \end{aligned} \right] \right)^{0.5}$$

$$d_1^+ = \left(\frac{1}{15} \left[\begin{aligned} &(0.659 - 0.659)^2 + (0.294 - 0.294)^2 + (0.340 - 0.340)^2 + \\ &(0.437 - 0.445)^2 + (0.531 - 0.521)^2 + (0.538 - 0.529)^2 + \\ &(0.444 - 0.679)^2 + (0.466 - 0.276)^2 + (0.495 - 0.316)^2 + \\ &(0.411 - 0.509)^2 + (0.567 - 0.462)^2 + (0.565 - 0.462)^2 + \\ &(0.429 - 0.383)^2 + (0.545 - 0.602)^2 + (0.571 - 0.617)^2 \end{aligned} \right] \right)^{0.5}$$

$$d_1^+ = \left[\frac{1}{15} (0.000245 + 0.123366 + 0.031238 + 0.007481) \right]^{0.5}$$

$$d_1^+ = 0.1040$$

Similarly, we can find other separation measures.

Step 7: Computation of Relative Closeness Coefficient (RCC)

The RCC is calculated by using

$$RCC_i = \frac{d_i'}{d_i' + d_i^*}; i = 1, 2, 3, 4, 5$$

$$RCC_1 = \frac{d_1'}{d_1' + d_1^*} = \frac{0.127532}{0.127532 + 0.104029} = 0.551$$

$$RCC_2 = 0.896$$

$$RCC_3 = 0.505$$

$$RCC_4 = 0.363$$

$$RCC_5 = 0.757$$

The separation measure and the value of relative closeness coefficient (RCC) expressed in the following figure.

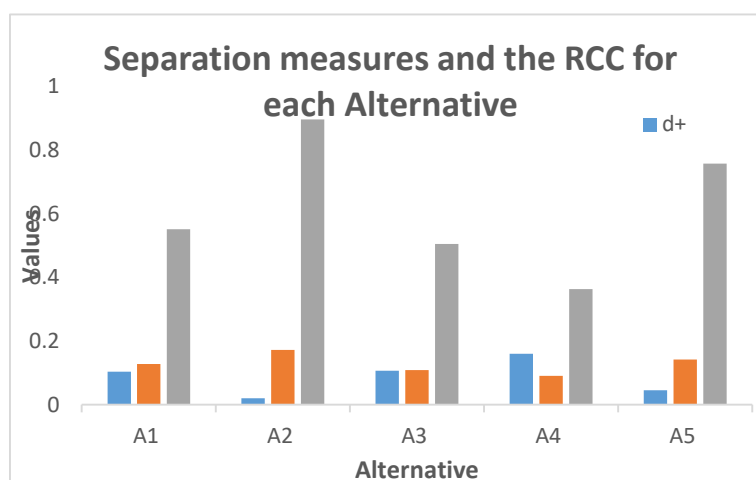


Figure 1. Separation measure and the RCC for each Alternative

Step 8: Ranking alternatives

From the above figure, we can see the RCC are ranked as follows

$$RCC_2 > RCC_5 > RCC_1 > RCC_3 > RCC_4 \Rightarrow A_2 > A_5 > A_1 > A_3 > A_4$$

By using the presented technique, we choose the best supplier for the production industry and observe that A₂ is the best alternative.

5. Conclusion

In this paper, we studied neutrosophic set and SVNss with some basic operations and developed the generalized neutrosophic TOPSIS by using single-valued neutrosophic numbers. By using crisp data, it is more difficult to solve decision-making problems under uncertain environments, to overcome such uncertainties single-valued neutrosophic sets are more appropriate. We also developed the graphical model for generalized neutrosophic TOPSIS. Finally, to show the validity of the proposed technique an illustrated example of the best supplier in the production industry is presented and observed that A₂ is the best supplier for the production industry. We consider this technique will be helpful in problem-solving and will expand the area of investigations for more accuracy in real-life issues.

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Neutrosophic Cubic Fuzzy Dombi Hamy Mean Operators with Application to Multi-Criteria Decision Making

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Abstract. The aim of the paper is to find most optimistic results from among uncertain information or vague data. We theoretically use the notion of neutrosophic cubic sets to create enhanced decision-making models for multi-criteria. The advantage of neutrosophic cubic sets is that it comprehends the knowledge of neutrosophic sets and interval valued neutrosophic sets. Aggregation operators are used to retrieve the core information from a collection of data. So, this research executes aggregation operators for neutrosophic cubic sets dynamically. In this paper we avail the aid of hamy mean and dombi operations to establish fuzzy dombi hamy mean aggregation operators for neutrosophic cubic sets. This paper also explains the algebraic sum and scalar multiplication operations. A decision making methodology has been generated to prove the necessity of the proposed operators. Finally an illustration is provided from a real life decision making situation.

Keywords: Fuzzy Sets; Neutrosophic Cubic Sets; Hamy Mean; Dombi Operations; Aggregations Operators; MCDM

1. Introduction

Every aspect of human life involves making decisions and most of the times human brain takes decision after processing information that comes to it in an incomplete and imprecise form. Hence we thrive to understand the fuzziness of the information in order to take decisions. To fulfil this need, in the year 1965 Zadeh's scientific studies revealed the concept of fuzzy

sets [1] and it is used to reduce fuzziness on difficult decision situations. The evolution of this notion of fuzzy sets with their operations has been presented in the literature [2–5]. Neutrosophic sets are a novel extended form of fuzzy sets and were initiated by F Smarandache [6]. Neutrosophic sets handle ambiguity using three membership types. Functions namely the functions of truthness, indeterminacy and falsehood membership provide a more general way of measuring vagueness. Further, Y.B. Jun et al. [7] and M. Ali et al. [8] effectively utilized the concept of cubic fuzzy sets to neutrosophic sets in order to introduce neutrosophic cubic fuzzy sets (NCFSSs) with some basic operations and NCFSSs deal with uncertain information in the form of intervals followed by single valued neutrosophic data. So this notion is a more general way to handle NSs. It is evident from the literature that the concept of decision making is one of the significant research areas in the field of neutrosophic sets. More recently, Ajay, D., et al. [9] used this notion with the help of weighted neutrosophic cubic fuzzy Bonferroni geometric mean aggregation operators.

In fuzzy mathematics aggregation operators play a significant role and it is more useful in aggregating knowledge that is involved in decision making systems. Various types of neutrosophic cubic aggregation operators available in the literature are, namely, Weighted Neutrosophic Cubic Cuzzy Bonferroni Geometric Mean ($WNCFBGM_w^{u,v}$) operator [9], Neutrosophic Cubic Dombi Weighted Arithmetic and Geometric Average (NCDWAA, NCDWGA) operators [10], Linguistic Neutrosophic Cubic Number Generalized Weighted Heronian Mean (LNCNGWHM) operator [11], Neutrosophic Cubic Einstein Weighted Geometric (NCEWG) operator [12], Neutrosophic Cubic Heronian Mean (NCHM) operator [13], Neutrosophic Cubic Einstein Ordered Weighted Geometric (NCEOWG) operator [14], New Operators on Interval Valued Neutrosophic Sets [15] and still there is a need for more efficient aggregation operators for huge underivable neutrosophic data.

Some of the measures on neutrosophic sets are utilized in decision making models. For example, Ajay D., et al. [16] introduced a new decision making approach based on bipolar neutrosophic similarity and entropy measures. Similarly, Lu, Z., et.al. [17] introduced neutrosophic cubic cosine similarity measure and applied in the field of multi criteria decision making. Abdel-Basset et al. [18] utilized bipolar neutrosophic sets to MCDM with Analytic hierarchy process (AHP) and Technique in order of preference by similarity to ideal solution (TOPSIS). Multi-objective optimisation on the basis of simple ratio (MOOSRA) method has been developed based on single valued triangular numbers which has been used to personnel selection [19].

One of the most recent generalization of neutrosophic sets is plithogenic set (PSs) which was introduced by Smarandache [20]. The elements of these PSs is characterized by one or more attributes, and each attribute may have many values. Using the notion of PSs,

best-worst method has been implemented in supply chain problem [21] and also plithogenic multi criteria decision making (MCDM) approach has been introduced based on neutrosophic AHP, TOPSIS and Vlse Kriterijumska Optimizacija Kompromisno Resenje (VIKOR) methods [22,23]. Moreover, Plithogenic n-super hypergraph is used in multi alternative decision making and it is a new perspective to deal with certain types of graphs for practical applications [24].

The available literature suggests that the decision making models are mainly focused on aggregation operators, rather than the studies on similarity measures. The introduction of neutrosophic set theory led to wide range of research areas like neutrosophic graph theory [25], Neutrosophic topology [26], etc. and more recently neutrosophic set theory has been used in finite automata [27]. Many real time applications exist under neutrosophic sets [28–35]. More recently on this pandemic, Health-Fog framework universal system has been introduced with the help of deep learning and neutrosophic classifiers to confront Covid-19 [36].

The Dombi and Hamy mean operators are efficient and flexible aggregation tools to handle information fusion in MCDM. Shi et al. [37] applied the dombi aggregation operators to neutrosophic cubic sets in order to make decision over uncertainty. They have investigated some of the properties of aggregation operators and illustrated a numerical problem in detail by changing the parameter values between 1 to 5. Recently, Liu et al. [38] combined the conventional Hamy mean to traditional power operator in interval valued neutrosophic sets and introduced interval valued power neutrosophic mean operators. However, it is clear that some research have been done on dombi and hamy mean operators separately and both of them have not yet been combined to formulate MCDM mechanism.

The main focus of this research is to find a new aggregation operator based on neutrosophic cubic sets with the help of combined dombi hamy mean operators. Two decision making methods are developed using score function, similarity measure and aggregation operators.

TABLE 1. Some notations with their descriptions

Notations	Descriptions
${}^*V, \check{v}_i$	Universal set, Element of *V
\mathbb{S}, \mathbb{N}	Fuzzy set, Neutrosophic set
$T_{\mathbb{N}}, I_{\mathbb{N}}, F_{\mathbb{N}}$	Truth, Indeterminacy and False membership functions
$\tilde{T}_{\mathbb{N}} = [T_{\mathbb{N}}^L, T_{\mathbb{N}}^U]$	Interval valued functions with respect to lower and upper bound

The framework of the rest of the paper is organized with five sections that follow. Section 2 addresses the basic definitions, operations and measures of similarity on neutrosophic cubic sets. The next section deals with hamy and dombi operations on neutrosophic cubic sets. Further, section 4 introduces neutrosophic cubic fuzzy dombi hamy mean aggregation operators

with weighted values. The section 5 of the paper describes algorithms of the proposed MCDM methods with suitable real life illustrations. Finally the conclusion of the research is made available in section 6.

2. Preliminaries of Neutrosophic Cubic Sets (NCSs) and their operations

In this section, we briefly review some basic concepts about NCSs, Dombi and Hamy Mean operators. Some of the notations description are given in table.1

Definition 2.1. [1] If \check{v} is a particular element of universe of discourse \check{V}^* , then a fuzzy set \mathbb{S} is defined by a fuzzy membership function $(\mu_{\mathbb{S}})$ which associates to each \check{v} a membership value in the closed unit interval of zero and one. i.e. $\mu_{\mathbb{S}}(\check{v}) : X \rightarrow [0, 1]$

Definition 2.2. [6] Let $\mathbb{N}_j = \left\{ (T_{\mathbb{N}_j}(\check{v}_i), I_{\mathbb{N}_j}(\check{v}_i), F_{\mathbb{N}_j}(\check{v}_i)) \mid \check{v}_i \in \check{V}^* \right\}$ be a neutrosophic set (Ns), where $\{T_{\mathbb{N}_j}(\check{v}_i), I_{\mathbb{N}_j}(\check{v}_i), F_{\mathbb{N}_j}(\check{v}_i) \in [0, 1]\}$ are called truth, indeterminacy and falsity functions, respectively. This can be represented by $\mathbb{N}_j = (T_{\mathbb{N}_j}, I_{\mathbb{N}_j}, F_{\mathbb{N}_j})$.

Definition 2.3. [6] Let $\mathbb{N}_j = \left\{ (\tilde{T}_{\mathbb{N}_j}(\check{v}_i), \tilde{I}_{\mathbb{N}_j}(\check{v}_i), \tilde{F}_{\mathbb{N}_j}(\check{v}_i)) \mid \check{v}_i \in \check{V}^* \right\}$ be an interval neutrosophic set in \check{V}^* , where $\{\tilde{T}_{\mathbb{N}_j}(\check{v}_i), \tilde{I}_{\mathbb{N}_j}(\check{v}_i), \tilde{F}_{\mathbb{N}_j}(\check{v}_i) \in [0, 1]\}$ is called truth, indeterminacy and falsity function in \check{V}^* , respectively. This can be represented by $\mathbb{N}_j = (\tilde{T}_{\mathbb{N}_j}, \tilde{I}_{\mathbb{N}_j}, \tilde{F}_{\mathbb{N}_j})$. For convenience, we denote $\mathbb{N}_j = (\tilde{T}_{\mathbb{N}_j}, \tilde{I}_{\mathbb{N}_j}, \tilde{F}_{\mathbb{N}_j})$ by $\mathbb{N}_j = (\tilde{T}_{\mathbb{N}_j} = [T_{\mathbb{N}_j}^L, T_{\mathbb{N}_j}^U], \tilde{I}_{\mathbb{N}_j} = [I_{\mathbb{N}_j}^L, I_{\mathbb{N}_j}^U], \tilde{F}_{\mathbb{N}_j} = [F_{\mathbb{N}_j}^L, F_{\mathbb{N}_j}^U])$.

Definition 2.4. [6] Let $\mathbb{N}_j = \left\{ (\tilde{T}_{\mathbb{N}_j}(\check{v}_i), \tilde{I}_{\mathbb{N}_j}(\check{v}_i), \tilde{F}_{\mathbb{N}_j}(\check{v}_i)) ; T_{\mathbb{N}_j}(\check{v}_i), I_{\mathbb{N}_j}(\check{v}_i), F_{\mathbb{N}_j}(\check{v}_i) \mid \check{v}_i \in \check{V}^* \right\}$ be a neutrosophic cubic sets in \check{V}^* , in which $\tilde{T}_{\mathbb{N}_j} = [T_{\mathbb{N}_j}^L, T_{\mathbb{N}_j}^U], \tilde{I}_{\mathbb{N}_j} = [I_{\mathbb{N}_j}^L, I_{\mathbb{N}_j}^U], \tilde{F}_{\mathbb{N}_j} = [F_{\mathbb{N}_j}^L, F_{\mathbb{N}_j}^U]$ is an interval valued neutrosophic set in \check{V}^* simply denoted by $\mathbb{N}_j = \langle \tilde{T}_{\mathbb{N}_j}, \tilde{I}_{\mathbb{N}_j}, \tilde{F}_{\mathbb{N}_j}, T_{\mathbb{N}_j}, I_{\mathbb{N}_j}, F_{\mathbb{N}_j} \rangle$, $[0, 0] \leq \tilde{T}_{\mathbb{N}_j} + \tilde{I}_{\mathbb{N}_j} + \tilde{F}_{\mathbb{N}_j} \leq [3, 3]$ and $0 \leq T_{\mathbb{N}_j} + I_{\mathbb{N}_j} + F_{\mathbb{N}_j} \leq 3$.

Definition 2.5. [7] Let $C_j = \left\{ (\check{v}_i, \tilde{\mu}(\check{v}_i), \mu(\check{v}_i)) \mid \check{v}_i \in \check{V}^* \right\}$ be a cubic fuzzy set in \check{V}^* in which $\tilde{\mu}$ is interval fuzzy set in \check{V}^* , i.e., $\tilde{\mu} = [\mu^L, \mu^U]$ and μ is a fuzzy set in \check{V}^* .

2.1. Operations on NCSs

The algebraic addition and scalar multiplication on NCSs are discussed. Essential outcome of exponential multiplication that provides the basis for the concept of Dombi Hamy mean aggregation operators in neutrosophic cubic sets is based on these definitions.

Definition 2.6. The sum and product of the two neutrosophic cubic sets (NCSs), $\mathbb{N}_1 = \langle \tilde{T}_{\mathbb{N}_1}, \tilde{I}_{\mathbb{N}_1}, \tilde{F}_{\mathbb{N}_1}, T_{\mathbb{N}_1}, I_{\mathbb{N}_1}, F_{\mathbb{N}_1} \rangle$, where $\tilde{T}_{\mathbb{N}_1} = [T_{\mathbb{N}_1}^L, T_{\mathbb{N}_1}^U], \tilde{I}_{\mathbb{N}_1} = [I_{\mathbb{N}_1}^L, I_{\mathbb{N}_1}^U], \tilde{F}_{\mathbb{N}_1} = [F_{\mathbb{N}_1}^L, F_{\mathbb{N}_1}^U]$, and

$\mathbb{N}_2 = \langle \tilde{T}_{\mathbb{N}_2}, \tilde{I}_{\mathbb{N}_2}, \tilde{F}_{\mathbb{N}_2}, T_{\mathbb{N}_2}, I_{\mathbb{N}_2}, F_{\mathbb{N}_2} \rangle$, where $\tilde{T}_{\mathbb{N}_2} = [T_{\mathbb{N}_2}^L, T_{\mathbb{N}_2}^U]$, $\tilde{I}_{\mathbb{N}_2} = [I_{\mathbb{N}_2}^L, I_{\mathbb{N}_2}^U]$, $\tilde{F}_{\mathbb{N}_2} = [F_{\mathbb{N}_2}^L, F_{\mathbb{N}_2}^U]$ are defined as follows.

$$\begin{aligned} \tilde{\mathbb{N}}_1 \oplus \mathbb{N}_2 = & \left\{ \left[T_{\mathbb{N}_1}^L + T_{\mathbb{N}_2}^L - T_{\mathbb{N}_1}^L T_{\mathbb{N}_2}^L, T_{\mathbb{N}_1}^U + T_{\mathbb{N}_2}^U - T_{\mathbb{N}_1}^U T_{\mathbb{N}_2}^U \right], \right. \\ & \left[I_{\mathbb{N}_1}^L + I_{\mathbb{N}_2}^L - I_{\mathbb{N}_1}^L I_{\mathbb{N}_2}^L, I_{\mathbb{N}_1}^U + I_{\mathbb{N}_2}^U - I_{\mathbb{N}_1}^U I_{\mathbb{N}_2}^U \right], \\ & \left. \left[F_{\mathbb{N}_1}^L F_{\mathbb{N}_2}^L, F_{\mathbb{N}_1}^U F_{\mathbb{N}_2}^U \right]; \langle T_{\mathbb{N}_1} T_{\mathbb{N}_2}, I_{\mathbb{N}_1} I_{\mathbb{N}_2}, F_{\mathbb{N}_1} + F_{\mathbb{N}_2} - F_{\mathbb{N}_1} F_{\mathbb{N}_2} \rangle \right\} \end{aligned} \quad (1)$$

$$\begin{aligned} \mathbb{N}_1 \otimes \mathbb{N}_2 = & \left\{ \left[T_{\mathbb{N}_1}^L T_{\mathbb{N}_2}^L, T_{\mathbb{N}_1}^U T_{\mathbb{N}_2}^U \right], \left[I_{\mathbb{N}_1}^L I_{\mathbb{N}_2}^L, I_{\mathbb{N}_1}^U I_{\mathbb{N}_2}^U \right], \left[F_{\mathbb{N}_1}^L + F_{\mathbb{N}_2}^L - F_{\mathbb{N}_1}^L F_{\mathbb{N}_2}^L, F_{\mathbb{N}_1}^U + F_{\mathbb{N}_2}^U - F_{\mathbb{N}_1}^U F_{\mathbb{N}_2}^U \right]; \right. \\ & \left. \langle T_{\mathbb{N}_1} + T_{\mathbb{N}_2} - T_{\mathbb{N}_1} T_{\mathbb{N}_2}, I_{\mathbb{N}_1} + I_{\mathbb{N}_2} - I_{\mathbb{N}_1} I_{\mathbb{N}_2}, F_{\mathbb{N}_1} F_{\mathbb{N}_2} \rangle \right\} \end{aligned} \quad (2)$$

Definition 2.7. The scalar and exponential multiplication on neutrosophic cubic sets (NCSs), $\mathbb{N}_1 = \langle \tilde{T}_{\mathbb{N}_1}, \tilde{I}_{\mathbb{N}_1}, \tilde{F}_{\mathbb{N}_1}, T_{\mathbb{N}_1}, I_{\mathbb{N}_1}, F_{\mathbb{N}_1} \rangle$, where $\tilde{T}_{\mathbb{N}_1} = [T_{\mathbb{N}_1}^L, T_{\mathbb{N}_1}^U]$, $\tilde{I}_{\mathbb{N}_1} = [I_{\mathbb{N}_1}^L, I_{\mathbb{N}_1}^U]$, $\tilde{F}_{\mathbb{N}_1} = [F_{\mathbb{N}_1}^L, F_{\mathbb{N}_1}^U]$, and a scalar value ϱ are defined as respectively:

$$\begin{aligned} \varrho \mathbb{N}_1 = & \left\{ \left[1 - (1 - T_{\mathbb{N}_1}^L)^\varrho, 1 - (1 - T_{\mathbb{N}_1}^U)^\varrho \right], \left[1 - (1 - I_{\mathbb{N}_1}^L)^\varrho, 1 - (1 - I_{\mathbb{N}_1}^U)^\varrho \right], \right. \\ & \left. \left[(F_{\mathbb{N}_1}^L)^\varrho, (F_{\mathbb{N}_1}^U)^\varrho \right]; \langle (T_{\mathbb{N}_1})^\varrho, (I_{\mathbb{N}_1})^\varrho, 1 - (1 - F_{\mathbb{N}_1})^\varrho \rangle \right\} \end{aligned} \quad (3)$$

$$\begin{aligned} \mathbb{N}_1^\varrho = & \left\{ \left[(T_{\mathbb{N}_1}^L)^\varrho, (T_{\mathbb{N}_1}^U)^\varrho \right], \left[(I_{\mathbb{N}_1}^L)^\varrho, (I_{\mathbb{N}_1}^U)^\varrho \right], \left[1 - (1 - F_{\mathbb{N}_1}^L)^\varrho, 1 - (1 - F_{\mathbb{N}_1}^U)^\varrho \right]; \right. \\ & \left. \langle 1 - (1 - T_{\mathbb{N}_1})^\varrho, 1 - (1 - I_{\mathbb{N}_1})^\varrho, (F_{\mathbb{N}_1})^\varrho \rangle \right\} \end{aligned} \quad (4)$$

2.2. Similarity Measure of NCSs

Let \mathbb{N}_A and \mathbb{N}_B be two neutrosophic cubic sets in V^* . Then, the measure of similarity of \mathbb{N}_A and \mathbb{N}_B is defined by $\mathbb{N}_{SM} : \mathbb{N}_A^*(V) \times \mathbb{N}_B^*(V) \rightarrow [0, 1]$ which satisfies the following condition;

- (i). $0 \leq \mathbb{N}_{SM}(\mathbb{N}_A(\check{v}_i), \mathbb{N}_B(\check{v}_i)) \leq 1$
- (ii). $\mathbb{N}_{SM}(\mathbb{N}_A(\check{v}_i), \mathbb{N}_B(\check{v}_i)) = 1$ iff $\mathbb{N}_A(\check{v}_i) = \mathbb{N}_B(\check{v}_i)$
- (iii). $\mathbb{N}_{SM}(\mathbb{N}_A(\check{v}_i), \mathbb{N}_B(\check{v}_i)) = \mathbb{N}_{SM}(\mathbb{N}_B(\check{v}_i), \mathbb{N}_A(\check{v}_i))$
- (iv). If $\mathbb{N}_A(\check{v}_i) \subseteq \mathbb{N}_B(\check{v}_i) \subseteq \mathbb{N}_C(\check{v}_i)$, then $\mathbb{N}_{SM}(\mathbb{N}_A(\check{v}_i), \mathbb{N}_C(\check{v}_i)) \leq (\mathbb{N}_A(\check{v}_i), \mathbb{N}_B(\check{v}_i))$ and $\mathbb{N}_{SM}(\mathbb{N}_A(\check{v}_i), \mathbb{N}_C(\check{v}_i)) \leq (\mathbb{N}_B(\check{v}_i), \mathbb{N}_C(\check{v}_i)) \forall \mathbb{N}_A(\check{v}_i), \mathbb{N}_B(\check{v}_i), \mathbb{N}_C(\check{v}_i) \in NCSs^*(V)$

The similarity measure between two NCSs is expressed as follows:

$$\mathbb{N}_{SM}(\mathbb{N}_A, \mathbb{N}_B) = \frac{1}{n} \sum_{i=1}^n \left(1 - \frac{D_s(i)}{9} \right) \quad (5)$$

where

$$\begin{aligned} D_s(i) = & \left(\left| T_{\mathbb{N}_A}^L(\check{v}_i) - T_{\mathbb{N}_B}^L(\check{v}_i) \right| + \left| T_{\mathbb{N}_A}^U(\check{v}_i) - T_{\mathbb{N}_B}^U(\check{v}_i) \right| + \left| I_{\mathbb{N}_A}^L(\check{v}_i) - I_{\mathbb{N}_B}^L(\check{v}_i) \right| + \right. \\ & \left| I_{\mathbb{N}_A}^U(\check{v}_i) - I_{\mathbb{N}_B}^U(\check{v}_i) \right| + \left| F_{\mathbb{N}_A}^L(\check{v}_i) - F_{\mathbb{N}_B}^L(\check{v}_i) \right| + \left| F_{\mathbb{N}_A}^U(\check{v}_i) - F_{\mathbb{N}_B}^U(\check{v}_i) \right| + \left| T_{\mathbb{N}_A}(\check{v}_i) - T_{\mathbb{N}_B}(\check{v}_i) \right| \\ & \left. + \left| I_{\mathbb{N}_A}(\check{v}_i) - I_{\mathbb{N}_B}(\check{v}_i) \right| + \left| F_{\mathbb{N}_A}(\check{v}_i) - F_{\mathbb{N}_B}(\check{v}_i) \right| \right). \end{aligned}$$

The similarity measure follows the four conditions mentioned above.

Definition 2.8. Let \mathbb{N} be neutrosophic cubic fuzzy set in \tilde{V}^* , then the support of neutrosophic cubic fuzzy set \mathbb{N}^* is defined by

$$\mathbb{N}^* = \left\{ \left[T_{\mathbb{N}}^L(\check{v}), T_{\mathbb{N}}^U(\check{v}) \right] \supset [0, 0], \left[I_{\mathbb{N}}^L(\check{v}), I_{\mathbb{N}}^U(\check{v}) \right] \supset [0, 0], \left[F_{\mathbb{N}}^L(\check{v}), F_{\mathbb{N}}^U(\check{v}) \right] \subset [1, 1]; \right. \\ \left. \langle T_{\mathbb{N}}(\check{v}) > 0, I_{\mathbb{N}}(\check{v}) > 0, F_{\mathbb{N}}(\check{v}) < 1 \rangle | \check{v} \in \tilde{V}^* \right\}$$

Definition 2.9. Let \mathbb{N} be a non empty neutrosophic cubic fuzzy number given by $\mathbb{N} = \langle \tilde{T}_{\mathbb{N}}, \tilde{I}_{\mathbb{N}}, \tilde{F}_{\mathbb{N}}, T_{\mathbb{N}}, I_{\mathbb{N}}, F_{\mathbb{N}} \rangle$, where $\tilde{T}_{\mathbb{N}} = [T_{\mathbb{N}}^L, T_{\mathbb{N}}^U]$, $\tilde{I}_{\mathbb{N}} = [I_{\mathbb{N}}^L, I_{\mathbb{N}}^U]$, $\tilde{F}_{\mathbb{N}} = [F_{\mathbb{N}}^L, F_{\mathbb{N}}^U]$, then its functions for ranking, accuracy and certainty can be stated as follows:

$$s(\mathbb{N}) = \frac{\frac{[4 + T_{\mathbb{N}}^L(\check{v}) - I_{\mathbb{N}}^L(\check{v}) - F_{\mathbb{N}}^L(\check{v}) + T_{\mathbb{N}}^U(\check{v}) - I_{\mathbb{N}}^U(\check{v}) - F_{\mathbb{N}}^U(\check{v})]}{6}}{2} + \frac{[2 + T_{\mathbb{N}}(\check{v}) - I_{\mathbb{N}}(\check{v}) - F_{\mathbb{N}}(\check{v})]}{3}, \quad (6)$$

$$a(\mathbb{N}) = \frac{\left[\left(T_{\mathbb{N}}^L(\check{v}) - F_{\mathbb{N}}^L(\check{v}) + T_{\mathbb{N}}^U(\check{v}) - F_{\mathbb{N}}^U(\check{v}) \right) / 2 + T_{\mathbb{N}}(\check{v}) - F_{\mathbb{N}}(\check{v}) \right]}{2}, \quad (7)$$

$$c(\mathbb{N}) = \frac{\left[\left(T_{\mathbb{N}}^L(\check{v}) + T_{\mathbb{N}}^U(\check{v}) \right) / 2 + T_{\mathbb{N}}(\check{v}) \right]}{2}; \quad s(\mathbb{N}), a(\mathbb{N}), c(\mathbb{N}) \in [0, 1] \quad (8)$$

Definition 2.10. Let \mathbb{N}_1 and \mathbb{N}_2 be two non empty neutrosophic cubic values, where $S_{\mathbb{N}_1}$ and $S_{\mathbb{N}_2}$ are score values and $H_{\mathbb{N}_1}$ and $H_{\mathbb{N}_2}$ are accuracy functions of \mathbb{N}_1 and \mathbb{N}_2 respectively.

- (1) If $S_{\mathbb{N}_1} > S_{\mathbb{N}_2} \Rightarrow \mathbb{N}_1 > \mathbb{N}_2$
- (2) $S_{\mathbb{N}_1} = S_{\mathbb{N}_2}$ and $H_{\mathbb{N}_1} > H_{\mathbb{N}_2} \Rightarrow \mathbb{N}_1 > \mathbb{N}_2$; $H_{\mathbb{N}_1} = H_{\mathbb{N}_2} \Rightarrow \mathbb{N}_1 = \mathbb{N}_2$

3. Neutrosophic Cubic Dombi Hamy Mean Operation

Definition 3.1. The operator Hamy Mean (HM) is stated as follows:

$$HM^{(\mu)}(\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_k) = \frac{\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \left(\prod_{j=1}^{\mu} \mathbb{N}_{i_j} \right)^{1/\mu}}{C_k^{\mu}}$$

where μ is a parameter, $\mu = 1, 2, \dots, k$ and k_1, k_2, \dots, k_{μ} are μ integer values taken from the set $(1, 2, \dots, k)$ of k , C_k^{μ} is the binomial co-efficient, $C_k^{\mu} = \frac{k!}{\mu!(k-\mu)!}$

Definition 3.2. Dombi has formulated a generator for the development of Dombi T-norm and T-conorm that is shown as follows:

$$D(p, q) = \frac{1}{1 + \left(\left(\frac{1-p}{p} \right)^{\varrho} + \left(\frac{1-q}{q} \right)^{\varrho} \right)^{1/\varrho}}, \quad D^c(p, q) = 1 - \frac{1}{1 + \left(\left(\frac{p}{1-p} \right)^{\varrho} + \left(\frac{q}{1-q} \right)^{\varrho} \right)^{1/\varrho}}$$

where $\varrho > 0$, $(p, q) \in [0, 1]$.

Definition 3.3. For two neutrosophic cubic sets (NCs), $\mathbb{N}_1 = \langle \tilde{T}_1, \tilde{I}_1, \tilde{F}_1, T_1, I_1, F_1 \rangle$, where $\tilde{T}_1 = [T_1^L, T_1^U]$, $\tilde{I}_1 = [I_1^L, I_1^U]$, $\tilde{F}_1 = [F_1^L, F_1^U]$, and $\mathbb{N}_2 = \langle \tilde{T}_2, \tilde{I}_2, \tilde{F}_2, T_2, I_2, F_2 \rangle$, where $\tilde{T}_2 = [T_2^L, T_2^U]$, $\tilde{I}_2 = [I_2^L, I_2^U]$, $\tilde{F}_2 = [F_2^L, F_2^U]$. The basic Dombi Hamy operations are defined as follows:

$$\mathbb{N}_1 \oplus \mathbb{N}_2 = \left\{ \left[\begin{array}{l} 1 - \frac{1}{1 + \left[\left(\frac{T_1^L}{1-T_1^L} \right)^\varrho + \left(\frac{T_2^L}{1-T_2^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\left(\frac{T_1^U}{1-T_1^U} \right)^\varrho + \left(\frac{T_2^U}{1-T_2^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right], \right. \\ \left[\begin{array}{l} 1 - \frac{1}{1 + \left[\left(\frac{I_1^L}{1-I_1^L} \right)^\varrho + \left(\frac{I_2^L}{1-I_2^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\left(\frac{I_1^U}{1-I_1^U} \right)^\varrho + \left(\frac{I_2^U}{1-I_2^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right], \\ \left[\begin{array}{l} \frac{1}{1 + \left[\left(\frac{1-F_1^L}{F_1^L} \right)^\varrho + \left(\frac{1-F_2^L}{F_2^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\left(\frac{1-F_1^U}{F_1^U} \right)^\varrho + \left(\frac{1-F_2^U}{F_2^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \end{array} \right]; \\ \left. \left\langle \frac{1}{1 + \left[\left(\frac{1-T_1}{1-T_1} \right)^\varrho + \left(\frac{1-T_2}{1-T_2} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\left(\frac{1-I_1}{1-I_1} \right)^\varrho + \left(\frac{1-I_2}{1-I_2} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \right. \right. \\ \left. \left. 1 - \frac{1}{1 + \left[\left(\frac{F_1}{1-F_1} \right)^\varrho + \left(\frac{F_2}{1-F_2} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right\rangle \right\}$$

$$\mathbb{N}_1 \otimes \mathbb{N}_2 = \left\{ \left[\begin{array}{l} \frac{1}{1 + \left[\left(\frac{1-T_1^L}{T_1^L} \right)^\varrho + \left(\frac{1-T_2^L}{T_2^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\left(\frac{1-T_1^U}{T_1^U} \right)^\varrho + \left(\frac{1-T_2^U}{T_2^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right], \right. \\ \left[\begin{array}{l} \frac{1}{1 + \left[\left(\frac{1-I_1^L}{I_1^L} \right)^\varrho + \left(\frac{1-I_2^L}{I_2^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\left(\frac{1-I_1^U}{I_1^U} \right)^\varrho + \left(\frac{1-I_2^U}{I_2^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right], \\ \left[\begin{array}{l} 1 - \frac{1}{1 + \left[\left(\frac{F_1^L}{1-F_1^L} \right)^\varrho + \left(\frac{F_2^L}{1-F_2^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\left(\frac{F_1^U}{1-F_1^U} \right)^\varrho + \left(\frac{F_2^U}{1-F_2^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \end{array} \right]; \\ \left. \left\langle 1 - \frac{1}{1 + \left[\left(\frac{T_1}{1-T_1} \right)^\varrho + \left(\frac{T_2}{1-T_2} \right)^\varrho \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\left(\frac{I_1}{1-I_1} \right)^\varrho + \left(\frac{I_2}{1-I_2} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \right. \right. \\ \left. \left. \frac{1}{1 + \left[\left(\frac{1-F_1}{F_1} \right)^\varrho + \left(\frac{1-F_2}{F_2} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right\rangle \right\}$$

$$\begin{aligned}
\vartheta \mathbb{N}_1 = & \left\{ \left[1 - \frac{1}{1 + \left[\vartheta \left(\frac{T_1^L}{1-T_1^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\vartheta \left(\frac{T_1^U}{1-T_1^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right], \right. \\
& \left[1 - \frac{1}{1 + \left[\vartheta \left(\frac{I_1^L}{1-I_1^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\vartheta \left(\frac{I_1^U}{1-I_1^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right], \\
& \left[\frac{1}{1 + \left[\vartheta \left(\frac{1-F_1^L}{F_1^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\vartheta \left(\frac{1-F_1^U}{F_1^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right]; \\
& \left. \left\langle \frac{1}{1 + \left[\vartheta \left(\frac{1-T_1}{T_1} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\vartheta \left(\frac{1-I_1}{I_1} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\vartheta \left(\frac{F_1}{1-F_1} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right\rangle \right\} \\
[\mathbb{N}_1]^{\vartheta} = & \left\{ \left[\frac{1}{1 + \left[\vartheta \left(\frac{1-T_1^L}{T_1^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\vartheta \left(\frac{1-T_1^U}{T_1^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right], \right. \\
& \left[\frac{1}{1 + \left[\vartheta \left(\frac{1-I_1^L}{I_1^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\vartheta \left(\frac{1-I_1^U}{I_1^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right], \\
& \left[1 - \frac{1}{1 + \left[\vartheta \left(\frac{F_1^L}{1-F_1^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\vartheta \left(\frac{F_1^U}{1-F_1^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right]; \\
& \left. \left\langle 1 - \frac{1}{1 + \left[\vartheta \left(\frac{T_1}{1-T_1} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\vartheta \left(\frac{I_1}{1-I_1} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\vartheta \left(\frac{1-F_1}{F_1} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right\rangle \right\}
\end{aligned}$$

4. Dombi Hamy Mean Aggregation Operators to Neutrosophic Cubic Numbers

The NCFDHM Operator: The NCFDHM Operator is defined as follows, based on the Dombi and Hamy mean operations

Theorem 4.1. Let $\mathbb{N}_j = (\tilde{T}_j, \tilde{I}_j, \tilde{F}_j, T_j, I_j, F_j)$, where $\tilde{T}_j = [T_j^L, T_j^U]$, $\tilde{I}_j = [I_j^L, I_j^U]$, $\tilde{F}_j = [F_j^L, F_j^U]$ ($j = 1, 2, \dots, k$) be a non empty collection of NCFNs. The compressed value by the NCFDHM operators is also an NCFNs where

$$NCFDHM^{(\tilde{\nu})}(\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_k) = \bigoplus_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\left(\bigotimes_{j=1}^{\mu} \mathbb{N}_{ij} \right)^{1/\mu}}{C_k^{\mu}}$$

$$\begin{aligned}
&= \left\{ \left[1 - \frac{1}{1 + \left[\frac{\mu}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{1-T_{ij}^L}{T_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\frac{\mu}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{1-T_{ij}^U}{T_{ij}^U} \right)^\varrho} \right]^{\frac{1}{\varrho}}} \right], \right. \\
&\quad \left[1 - \frac{1}{1 + \left[\frac{\mu}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{1-I_{ij}^L}{I_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\frac{\mu}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{1-I_{ij}^U}{I_{ij}^U} \right)^\varrho} \right]^{\frac{1}{\varrho}}} \right], \\
&\quad \left[\frac{1}{1 + \left[\frac{\mu}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{F_{ij}^L}{1-F_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\frac{\mu}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{F_{ij}^U}{1-F_{ij}^U} \right)^\varrho} \right]^{\frac{1}{\varrho}}} \right]; \\
&\quad \left\langle \frac{1}{1 + \left[\frac{\mu}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{T_{ij}}{1-T_{ij}} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, \right. \\
&\quad \left. \frac{1}{1 + \left[\frac{\mu}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{I_{ij}}{1-I_{ij}} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, \right. \\
&\quad \left. 1 - \frac{1}{1 + \left[\frac{\mu}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{1-F_{ij}}{F_{ij}} \right)^\varrho} \right]^{\frac{1}{\varrho}}} \right\rangle \cdot \left. \right\}
\end{aligned}$$

Proof.

$$\text{Let } \bigotimes_{j=1}^{\mu} \mathbb{N}_{ij} = \left\{ \left[\frac{1}{1 + \left[\sum_{j=1}^{\mu} \left(\frac{1-T_{ij}^L}{T_{ij}^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\sum_{j=1}^{\mu} \left(\frac{1-T_{ij}^U}{T_{ij}^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right], \right. \\ \left[\frac{1}{1 + \left[\sum_{j=1}^{\mu} \left(\frac{1-I_{ij}^L}{I_{ij}^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\sum_{j=1}^{\mu} \left(\frac{1-I_{ij}^U}{I_{ij}^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right], \\ \left[1 - \frac{1}{1 + \left[\sum_{j=1}^{\mu} \left(\frac{F_{ij}^L}{1-F_{ij}^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\sum_{j=1}^{\mu} \left(\frac{F_{ij}^U}{1-F_{ij}^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right]; \\ \left. \left\langle 1 - \frac{1}{1 + \left[\sum_{j=1}^{\mu} \left(\frac{T_{ij}}{1-T_{ij}} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\sum_{j=1}^{\mu} \left(\frac{I_{ij}}{1-I_{ij}} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\sum_{j=1}^{\mu} \left(\frac{1-F_{ij}}{F_{ij}} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right\rangle \right\} \\ \text{Thus, } \left(\bigotimes_{j=1}^{\mu} \mathbb{N}_{ij} \right)^{1/\mu} = \left\{ \left[\frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} \left(\frac{1-T_{ij}^L}{T_{ij}^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} \left(\frac{1-T_{ij}^U}{T_{ij}^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right], \right. \\ \left[\frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} \left(\frac{1-I_{ij}^L}{I_{ij}^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} \left(\frac{1-I_{ij}^U}{I_{ij}^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right], \\ \left[1 - \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} \left(\frac{F_{ij}^L}{1-F_{ij}^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} \left(\frac{F_{ij}^U}{1-F_{ij}^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right]; \\ \left. \left\langle 1 - \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} \left(\frac{T_{ij}}{1-T_{ij}} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} \left(\frac{I_{ij}}{1-I_{ij}} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} \left(\frac{1-F_{ij}}{F_{ij}} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right\rangle \right\}$$

Thereafter, $\bigoplus_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \left(\bigotimes_{j=1}^{\mu} \mathbb{N}_{ij} \right)^{1/\mu}$

$$\begin{aligned}
 &= \left\{ \left[\begin{aligned} &1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} \left(\frac{1-T_{ij}^L}{T_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} \left(\frac{1-T_{ij}^U}{T_{ij}^U} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}} \end{aligned} \right] \right. \\
 &\quad \left[\begin{aligned} &1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} \left(\frac{1-I_{ij}^L}{I_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} \left(\frac{1-I_{ij}^U}{I_{ij}^U} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}} \end{aligned} \right] \\
 &\quad \left[\begin{aligned} &1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} \left(\frac{1-F_{ij}^L}{F_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} \left(\frac{1-F_{ij}^U}{F_{ij}^U} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}} \end{aligned} \right] \\
 &\quad \left. \left\langle \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} \left(\frac{1-T_{ij}}{T_{ij}} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} \left(\frac{1-I_{ij}}{I_{ij}} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, \right. \right. \\
 &\quad \left. \left. 1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} \left(\frac{1-F_{ij}}{F_{ij}} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}} \right\rangle \right\}.
 \end{aligned}$$

$$\begin{aligned}
\text{Therefore, } NCFDHM^{(\tilde{v})}(\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_k) &= \frac{\bigoplus_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \left(\bigotimes_{j=1}^{\mu} \mathbb{N}_{ij} \right)^{1/\mu}}{C_k^{\mu}} \\
&= \left\{ \left[1 - \frac{1}{1 + \left[\frac{\mu}{C_k^{\mu}} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{1 - T_{ij}^L}{T_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\frac{\mu}{C_k^{\mu}} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{1 - T_{ij}^U}{T_{ij}^U} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}} \right], \right. \\
&\quad \left[1 - \frac{1}{1 + \left[\frac{\mu}{C_k^{\mu}} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{1 - I_{ij}^L}{I_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\frac{\mu}{C_k^{\mu}} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{1 - I_{ij}^U}{I_{ij}^U} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}} \right], \\
&\quad \left[\frac{1}{1 + \left[\frac{\mu}{C_k^{\mu}} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{F_{ij}^L}{1 - F_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\frac{\mu}{C_k^{\mu}} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{F_{ij}^U}{1 - F_{ij}^U} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}} \right] \right\}; \\
&\quad \left\langle \frac{1}{1 + \left[\frac{\mu}{C_k^{\mu}} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{T_{ij}}{1 - T_{ij}} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, \right. \\
&\quad \left. \frac{1}{1 + \left[\frac{\mu}{C_k^{\mu}} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{I_{ij}}{1 - I_{ij}} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, \right. \\
&\quad \left. 1 - \frac{1}{1 + \left[\frac{\mu}{C_k^{\mu}} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{1}{\sum_{j=1}^{\mu} \left(\frac{1 - F_{ij}}{F_{ij}} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}} \right\rangle.
\end{aligned}$$

hence the proof. \square

Then we discuss some properties of NCFDHM Operator

1. Idempotency: If for all $\mathbb{N}_j = \langle \tilde{T}_j, \tilde{I}_j, \tilde{F}_j, T_j, I_j, F_j \rangle$, where $\tilde{T}_j = [T_j^L, T_j^U]$, $\tilde{I}_j = [I_j^L, I_j^U]$, $\tilde{F}_j = [F_j^L, F_j^U]$ ($j = 1, 2, \dots, k$) are equal, that is, $\mathbb{N}_j = \mathbb{N} \forall j$, then $NCFDHM^{(\tilde{v})}(\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_k) = \mathbb{N}$

2. Commutativity: Let $C_j = (\tilde{T}_j, \tilde{I}_j, \tilde{F}_j, T_j, I_j, F_j)$, where $\tilde{T}_j = [T_j^L, T_j^U]$, $\tilde{I}_j = [I_j^L, I_j^U]$, $\tilde{F}_j = [F_j^L, F_j^U]$ ($j = 1, 2, \dots, k$) is the collection of neutrosophic cubic numbers

$NCFDHM^{(\tilde{v})}(\tilde{\mathbb{N}}_1, \tilde{\mathbb{N}}_2, \dots, \tilde{\mathbb{N}}_k)$ be any permutation of $NCFDHM^{(\tilde{v})}(\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_k)$

$$NCFDHM^{(\tilde{v})}(\mathbb{N}_1), (\mathbb{N}_2), \dots, (\mathbb{N}_k) = NCFDHM^{(\tilde{v})}(\tilde{\mathbb{N}}_1), (\tilde{\mathbb{N}}_2), \dots, (\tilde{\mathbb{N}}_k)$$

3. Monotonicity: Let $\mathcal{C}_j = \langle \tilde{T}_j, \tilde{I}_j, \tilde{F}_j, T_j, I_j, F_j \rangle$, where $\tilde{T}_j = [T_j^L, T_j^U]$, $\tilde{I}_j = [I_j^L, I_j^U]$, $\tilde{F}_j = [F_j^L, F_j^U]$ ($j = 1, 2, \dots, k$) be the set of neutrosophic cubic numbers

If $S_{C_j}(\tilde{v}) \geq S_{\mathbb{N}_j}(\tilde{v})$ and $C_j(\tilde{v}) \geq \mathbb{N}_j(\tilde{v})$ then

$$NCFDHM^{(\tilde{v})}(\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_k) \leq NCFDHM^{(\tilde{v})}(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k)$$

4. Boundary: $\mathbb{N}_i^- \leq NCFDHM^{(\tilde{v})}(\mathbb{N}_1), (\mathbb{N}_2), \dots, (\mathbb{N}_n) \leq \mathbb{N}_i^+$, where

$$\begin{aligned} \mathbb{N}_i^- &= \{ \inf([T_i^-, T_i^+]), \sup([I_i^-, I_i^+]), \sup([F_i^-, F_i^+]); \min(T_i), \max(I_i), \max(F_i) \}, \\ \mathbb{N}_i^+ &= \{ \sup([T_i^-, T_i^+]), \inf([I_i^-, I_i^+]), \inf([F_i^-, F_i^+]); \max(T_i), \min(I_i), \min(F_i) \}. \end{aligned}$$

4.1. The Weighted NCFWDHM Aggregation Operator

Definition 4.2. Let $\mathbb{N}_j = \langle \tilde{T}_j, \tilde{I}_j, \tilde{F}_j, T_j, I_j, F_j \rangle$, where $\tilde{T}_j = [T_j^L, T_j^U]$, $\tilde{I}_j = [I_j^L, I_j^U]$, $\tilde{F}_j = [F_j^L, F_j^U]$ ($j = 1, 2, \dots, k$), be a set of NCFNs. The NCFWDHM Operator is

$$NCFWDHM_w^{(\tilde{v})}(\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_k) = \bigoplus_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\left(\bigotimes_{j=1}^{\mu} (\mathbb{N}_{i_j})^{w_{i_j}} \right)^{1/\mu}}{C_k^{\mu}}$$

Theorem 4.3. Let $\mathbb{N}_j = \langle \tilde{T}_j, \tilde{I}_j, \tilde{F}_j, T_j, I_j, F_j \rangle$, where $\tilde{T}_j = [T_j^L, T_j^U]$, $\tilde{I}_j = [I_j^L, I_j^U]$, $\tilde{F}_j = [F_j^L, F_j^U]$ ($j = 1, 2, \dots, k$) be a collection of non empty NCFNs. The compressed value by the NCFWDHM operators is also an NCFN where

$$NCFWDHM_w^{(\tilde{v})}(\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_k) = \bigoplus_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\left(\bigotimes_{j=1}^{\mu} (\mathbb{N}_{i_j})^{w_{i_j}} \right)^{1/\mu}}{C_k^{\mu}} =$$

$$\left\{ \left[\begin{array}{l} 1 - \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - T_{ij}^L}{T_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - T_{ij}^U}{T_{ij}^U} \right)^\varrho} \right]^{\frac{1}{\varrho}}} \end{array} \right], \right. \\
\left[\begin{array}{l} 1 - \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - I_{ij}^L}{I_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - I_{ij}^U}{I_{ij}^U} \right)^\varrho} \right]^{\frac{1}{\varrho}}} \end{array} \right], \\
\left[\begin{array}{l} \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{F_{ij}^L}{1 - F_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{F_{ij}^U}{1 - F_{ij}^U} \right)^\varrho} \right]^{\frac{1}{\varrho}}} \end{array} \right]; \\
\left\langle \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{T_{ij}^L}{1 - T_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, \right. \\
\left. \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{I_{ij}^L}{1 - I_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, \right. \\
\left. \frac{1}{1 - \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - F_{ij}^L}{F_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}} \right\rangle \cdot \left. \right\}$$

Proof.

$$\begin{aligned}
 \text{Let } \mathbb{N}_{ij}^{w_{ij}} = & \left\{ \left[\frac{1}{1 + \left[w_{ij} \left(\frac{1-T_{ij}^L}{T_{ij}^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[w_{ij} \left(\frac{1-T_{ij}^U}{T_{ij}^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right], \right. \\
 & \left[\frac{1}{1 + \left[w_{ij} \left(\frac{1-I_{ij}^L}{I_{ij}^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[w_{ij} \left(\frac{1-I_{ij}^U}{I_{ij}^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right], \\
 & \left[1 - \frac{1}{1 + \left[w_{ij} \left(\frac{F_{ij}^L}{1-F_{ij}^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[w_{ij} \left(\frac{F_{ij}^U}{1-F_{ij}^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right]; \\
 & \left. \left\langle 1 - \frac{1}{1 + \left[w_{ij} \left(\frac{T_{ij}}{1-T_{ij}} \right)^\varrho \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[w_{ij} \left(\frac{I_{ij}}{1-I_{ij}} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[w_{ij} \left(\frac{1-F_{ij}}{F_{ij}} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right\rangle \right\} \\
 \bigotimes_{j=1}^{\mu} (\mathbb{N}_{ij})^{w_{ij}} = & \left\{ \left[\frac{1}{1 + \left[\sum_{j=1}^{\mu} w_{ij} \left(\frac{1-T_{ij}^L}{T_{ij}^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\sum_{j=1}^{\mu} w_{ij} \left(\frac{1-T_{ij}^U}{T_{ij}^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right], \right. \\
 & \left[\frac{1}{1 + \left[\sum_{j=1}^{\mu} w_{ij} \left(\frac{1-I_{ij}^L}{I_{ij}^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\sum_{j=1}^{\mu} w_{ij} \left(\frac{1-I_{ij}^U}{I_{ij}^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right], \\
 & \left[1 - \frac{1}{1 + \left[\sum_{j=1}^{\mu} w_{ij} \left(\frac{F_{ij}^L}{1-F_{ij}^L} \right)^\varrho \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\sum_{j=1}^{\mu} w_{ij} \left(\frac{F_{ij}^U}{1-F_{ij}^U} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right]; \\
 & \left. \left\langle 1 - \frac{1}{1 + \left[\sum_{j=1}^{\mu} w_{ij} \left(\frac{T_{ij}}{1-T_{ij}} \right)^\varrho \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\sum_{j=1}^{\mu} w_{ij} \left(\frac{I_{ij}}{1-I_{ij}} \right)^\varrho \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\sum_{j=1}^{\mu} w_{ij} \left(\frac{1-F_{ij}}{F_{ij}} \right)^\varrho \right]^{\frac{1}{\varrho}}} \right\rangle \right\}
 \end{aligned}$$

Therefore,

$$\left(\bigotimes_{j=1}^{\mu} (\mathbb{N}_{ij})^{w_{ij}} \right)^{1/\mu} = \left\{ \left[\frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - T_{ij}^L}{T_{ij}^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - T_{ij}^U}{T_{ij}^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right], \right. \\ \left[\frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - I_{ij}^L}{I_{ij}^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - I_{ij}^U}{I_{ij}^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right], \\ \left[1 - \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} w_{ij} \left(\frac{F_{ij}^L}{1 - F_{ij}^L} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} w_{ij} \left(\frac{F_{ij}^U}{1 - F_{ij}^U} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right], \\ \left\langle 1 - \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} w_{ij} \left(\frac{T_{ij}}{1 - T_{ij}} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \right. \\ \left. \left. 1 - \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} w_{ij} \left(\frac{I_{ij}}{1 - I_{ij}} \right)^{\varrho} \right]^{\frac{1}{\varrho}}}, \right. \right. \\ \left. \left. \left. \frac{1}{1 + \left[\frac{1}{\mu} \sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - F_{ij}}{F_{ij}} \right)^{\varrho} \right]^{\frac{1}{\varrho}}} \right\rangle \right\} \right\};$$

Thereafter, $\bigoplus_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \left(\bigotimes_{j=1}^{\mu} (\mathbb{N}_{ij})^{w_{ij}} \right)^{1/\mu}$

$$= \left\{ \left[1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - T_{ij}^L}{T_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - T_{ij}^U}{T_{ij}^U} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}} \right], \right.$$

$$\left[\frac{1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - I_{ij}^L}{I_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}}{1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - I_{ij}^U}{I_{ij}^U} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}}, \frac{1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - F_{ij}^L}{1 - F_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}}{1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - F_{ij}^U}{1 - F_{ij}^U} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}} \right]^{\frac{1}{\varrho}}; \\
\left\langle \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{T_{ij}^L}{1 - T_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{I_{ij}^L}{1 - I_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}} \right\rangle \cdot \left. \frac{1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - F_{ij}^L}{F_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}}{1 - \frac{1}{1 + \left[\sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - F_{ij}^L}{F_{ij}^L} \right)^{\varrho}} \right]^{\frac{1}{\varrho}}}} \right\rangle \cdot \left. \right\}$$

Therefore,

$$NCFWDHM_w^{(\tilde{v})}(\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_k) = \frac{\bigoplus_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \left(\bigotimes_{j=1}^{\mu} (\mathbb{N}_{ij})^{w_{ij}} \right)^{1/\mu}}{C_k^{\mu}} =$$

$$\left\{ \left[\begin{aligned} &1 - \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - T_{ij}^L}{T_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^y w_{ij} \left(\frac{1 - T_{ij}^U}{T_{ij}^U} \right)^\varrho} \right]^{\frac{1}{\varrho}}} \right], \\ &1 - \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - I_{ij}^L}{I_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^y w_{ij} \left(\frac{1 - I_{ij}^U}{I_{ij}^U} \right)^\varrho} \right]^{\frac{1}{\varrho}}} \right], \\ &\left[\frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{F_{ij}^L}{1 - F_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^y w_{ij} \left(\frac{F_{ij}^U}{1 - F_{ij}^U} \right)^\varrho} \right]^{\frac{1}{\varrho}}} \right]; \\ &\left\langle \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{T_{ij}^L}{1 - T_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, \right. \\ &\quad \left. \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{I_{ij}^L}{1 - I_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}}, \right. \\ &\quad \left. 1 - \frac{1}{1 + \left[\frac{1}{C_k^\mu} \sum_{1 \leq \dots < k_{(\mu)} < \dots \leq k} \frac{\mu}{\sum_{j=1}^{\mu} w_{ij} \left(\frac{1 - F_{ij}^L}{F_{ij}^L} \right)^\varrho} \right]^{\frac{1}{\varrho}}} \right\rangle \right\}.
\end{aligned}$$

hence proved. \square

5. Algorithms and Illustration of the Proposed MCDM Method

5.1. Algorithm 1

Step 1. For an MCDM problem, a neutrosophic cubic decision matrix $A = (a_{ij})_{n \times k}$ is constructed

Step 2. Fix corresponding relative ideal point over attributes of neutrosophic cubic sets

$$\mathbb{N}_{C_i}^* = \left\{ \left[\max \left\{ \underline{T}_{\mathbb{N}^*}^L \right\}, \max \left\{ \underline{T}_{\mathbb{N}^*}^U \right\} \right], \left[\min \left\{ \underline{I}_{\mathbb{N}^*}^L \right\}, \min \left\{ \underline{I}_{\mathbb{N}^*}^U \right\} \right], \left[\min \left\{ \underline{F}_{\mathbb{N}^*}^L \right\}, \min \left\{ \underline{F}_{\mathbb{N}^*}^U \right\} \right] ; \right. \\ \left. \left\langle \max \left\{ \underline{T}_{\mathbb{N}^*} \right\}, \min \left\{ \underline{I}_{\mathbb{N}^*} \right\}, \min \left\{ \underline{F}_{\mathbb{N}^*} \right\} \right\rangle . \right\} \forall j = 1, 2, 3, \dots, k$$

Step 3. Calculate similarity measure between corresponding alternatives A_1, A_2, \dots, A_n and relative ideal point of neutrosophic cubic sets $\mathbb{N}_C^* = \mathbb{N}_{C_i}^* (j = 1, 2, 3, \dots, k)$ using equation.5

$$S(A_i, \mathbb{N}_C^*) \quad \forall i = 1, 2, 3, \dots, n$$

Step 4. Choose the best alternatives A_i according to similarity values of $S(A_i, \mathbb{N}_C^*)$, for all $i = 1, 2, \dots, n$

5.2. Algorithm 2

Step 1. For an MCDM problem, a neutrosophic cubic decision matrix $A = (a_{ij})_{n \times k}$ is constructed

Step 2. Compute neutrosophic cubic aggregated vaules for each alternative over attributes by $NCFWDHM_w^{(v)}(\mathbb{N}_1, \mathbb{N}_2, \dots, \mathbb{N}_k)$

Step 3. Utilize the score formula (Eq.6) to obtain the score values of the alternatives.

Step 4. Rank the alternatives A_i according to score values.

5.3. Illustration of the Models

In this section, an illustration has been chosen on the basis of finding the poor from among the target group based on education level, employment and income. We employ the proposed multi criteria decision making algorithms to the chosen problem. Here we have taken the households of target group of people as alternatives $A_i (i = 1, 2, 3, \dots, n)$ and we consider the attributes employment (C_1), education level (C_2) and income (C_3). The values are represented by neutrosophic cubic numbers which covers both interval and individual poverty information.

The decision matrix is given as follows:

$$DM = \left\{ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{cc} C_1 & C_2 & C_3 \\ \begin{bmatrix} [0.5, 0.6], & [0.1, 0.3], \\ [0.2, 0.4]; & \langle 0.6, 0.2, 0.3 \rangle \end{bmatrix} & \begin{bmatrix} [0.5, 0.6], & [0.1, 0.3], \\ [0.2, 0.4]; & \langle 0.6, 0.2, 0.3 \rangle \end{bmatrix} & \begin{bmatrix} [0.2, 0.4], & [0.7, 0.8], \\ [0.8, 0.9]; & \langle 0.3, 0.8, 0.9 \rangle \end{bmatrix} \\ \begin{bmatrix} [0.6, 0.8], & [0.1, 0.2], \\ [0.2, 0.3]; & \langle 0.7, 0.1, 0.2 \rangle \end{bmatrix} & \begin{bmatrix} [0.6, 0.7], & [0.1, 0.2], \\ [0.2, 0.3]; & \langle 0.6, 0.1, 0.2 \rangle \end{bmatrix} & \begin{bmatrix} [0.3, 0.4], & [0.6, 0.7], \\ [0.8, 0.9]; & \langle 0.3, 0.6, 0.9 \rangle \end{bmatrix} \\ \begin{bmatrix} [0.4, 0.6], & [0.2, 0.3], \\ [0.1, 0.3]; & \langle 0.6, 0.2, 0.2 \rangle \end{bmatrix} & \begin{bmatrix} [0.5, 0.6], & [0.2, 0.3], \\ [0.3, 0.4]; & \langle 0.6, 0.3, 0.4 \rangle \end{bmatrix} & \begin{bmatrix} [0.3, 0.5], & [0.7, 0.8], \\ [0.6, 0.7]; & \langle 0.4, 0.8, 0.7 \rangle \end{bmatrix} \\ \begin{bmatrix} [0.7, 0.8], & [0.1, 0.2], \\ [0.1, 0.2]; & \langle 0.8, 0.1, 0.2 \rangle \end{bmatrix} & \begin{bmatrix} [0.6, 0.7], & [0.1, 0.2], \\ [0.1, 0.3]; & \langle 0.7, 0.1, 0.2 \rangle \end{bmatrix} & \begin{bmatrix} [0.3, 0.4], & [0.6, 0.7], \\ [0.7, 0.8]; & \langle 0.3, 0.7, 0.8 \rangle \end{bmatrix} \end{array} \right\}$$

5.4. Algorithm 1

Step 2. Corresponding relative ideal point over attributes of neutrosophic cubic sets

$$\mathbb{N}_C^* : \langle \mathbb{N}_{C_1}^*, \mathbb{N}_{C_2}^*, \mathbb{N}_{C_3}^* \rangle$$

$$\mathbb{N}_{C_1}^* = \{[0.7, 0.8], [0.1, 0.2], [0.1, 0.2]; \langle 0.8, 0.1, 0.2 \rangle\}$$

$$\mathbb{N}_{C_2}^* = \{[0.6, 0.7], [0.1, 0.2], [0.1, 0.3]; \langle 0.7, 0.1, 0.2 \rangle\}$$

$$\mathbb{N}_{C_3}^* = \{[0.3, 0.5], [0.6, 0.7], [0.6, 0.7]; \langle 0.4, 0.6, 0.7 \rangle\}$$

Step 3. Similarity measure between alternatives A_1, A_2, \dots, A_m and relative ideal point of neutrosophic cubic sets \mathbb{N}_C^* .

$$S(A_1, \mathbb{N}_C^*) = 0.8778, \quad S(A_2, \mathbb{N}_C^*) = 0.9370, \quad S(A_3, \mathbb{N}_C^*) = 0.9000, \quad S(A_4, \mathbb{N}_C^*) = 0.9778$$

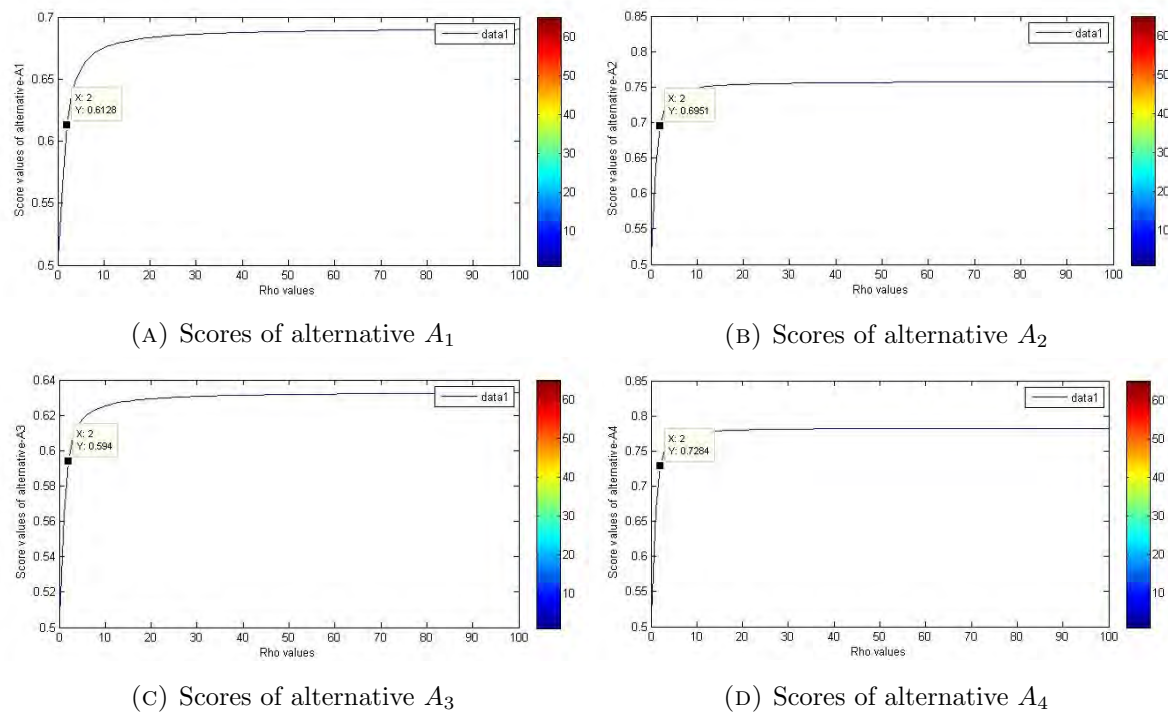
Step 4. The arrangement of alternatives according to similarity values of $S(A_i, \mathbb{N}_C^*)$, $i = 1, 2, \dots, n$. $A_4 > A_2 > A_3 > A_1$

5.5. Algorithm 2

Step 2. Let the weighted values of the attributes be $W = (0.32, 0.38, 0.3)$, respectively. Take $\mu = 2$, $\varrho = 2$ then using NCFWDHM operators on alternatives A_i ($i = 1, 2, 3, 4$) we get the aggregated values as shown in table.2

TABLE 2. Aggregated Values by NCFWDHM operator

Alternatives	Aggregated Values (NCFWDHM)
A_1	$[0.5239, 0.6594], [0.1955, 0.4830], [0.2025, 0.4028]; \langle 0.4133, 0.2025, 0.4838 \rangle$
A_2	$[0.6296, 0.7528], [0.1953, 0.3527], [0.2025, 0.3040]; \langle 0.4616, 0.1011, 0.3536 \rangle$
A_3	$[0.5249, 0.6925], [0.3527, 0.4830], [0.2323, 0.3423]; \langle 0.4218, 0.2649, 0.4966 \rangle$
A_4	$[0.6577, 0.7528], [0.1953, 0.3527], [0.1017, 0.2649]; \langle 0.5775, 0.1017, 0.3533 \rangle$

FIGURE 1. Scores of alternative A_i obtained by $NCFWDHM$ operator

Step 3. Score function of the alternatives are

$$S(A_1) = 0.6128, S(A_2) = 0.6951, S(A_3) = 0.5940, S(A_4) = 0.7284$$

Step 4. The alternatives are ranked based on their score values $A_4 > A_2 > A_1 > A_3$

5.6. Comparative Analysis

For comparison analysis, the proposed weighted dombi hamy neutrosophic cubic mean aggregation operator ($NCFWDHM$) is compared with an existing multi-criteria decision making method based on neutrosophic cubic aggregation operator ($WNCFGBM_w^{u,v}$) [9] and with the proposed decision making technique over similarity measure. The findings are shown in table.3

TABLE 3. Rank of the proposed methods

Proposed Methods	Ranking order
Algorithm 1 (Similarity Measure)	$A_4 > A_2 > A_3 > A_1$
Algorithm 2 ($NCFWDHM$)	$A_4 > A_2 > A_1 > A_3$
$WNCFGBM_w^{u,v}$ Operator [9]	$A_4 > A_2 > A_3 > A_1$

Also for a detailed comparison, we represent the score values of each alternative in Fig.1 by only changing the values of ϱ between 0 and 100 and we can find that the score values of each alternative are the same after certain values of ϱ . The values of the parameter μ have been chosen based on the total number of combinations of k data sets, that is, the total number of neutrosophic cubic numbers. When we have large data sets the values of μ will have more combinations to deal with. Therefore the WDHNCM operators are more applicable to large data sets for making decision. From table.3 we can see that the most optimistic results are A_4 and A_2 . The ranking order of the alternatives are almost the same and slight changes exist between alternatives A_1 and A_3 . The reason is that the proposed methods with the WDHNCM and $(WNCFGBM_w^{u,v})$ [9] aggregation operators have been applied for minimum data sets or, in other words, the given illustration exist for initial parameter, that is, for $\mu = 2$.

6. Conclusions

The advantage of neutrosophic cubic sets is presented in the article with a newly developed weighted dombi hamy neutrosophic cubic mean aggregation operators. Some of the basic neutrosophic cubic operations and properties are proved with respect to domi and hamy mean operations. Further the similarity measure and relative neutrosophic cubic ideal points are introduced and with the help of these ideas a new decision making method is developed. The comparative analysis has been made to prove the efficiency and validity of the proposed operators through numerical illustration. The important finding is that the proposed operators give more efficient results while having to deal with huge set of uncertain data or information due to their nature. Also the results are stable and more consistent with existing measures. The proposed aggregation operators can be extended to be used in existing decision making models like CODAS, AHP, MULTIMOORA, TOPSIS, VIKOR, etc. which would result in more new models. This study can further be extended to the field of artificial intelligence, medical and fault diagnosis with real time applications.

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Application of Similarity Measure on m-polar Interval-valued Neutrosophic Set in Decision Making in Sports

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Abstract. In real life, most of the problems occurred by wrong decision making, while in sports it is mandatory for every player, coach, and technique director to make a good and an ideal decision. In this paper, the concept of similarity measure is used in the neutrosophic environment for decision making in a football game for the selection of players. The data is collected in interval-valued, while the new concept m-polar is illustrated as previous records of m matches played by players. m-polar structures provide multiple data on the concerned problem, so as a result the best solution can be developed for the selection problem. An m-polar Interval-valued Neutrosophic Set (*mIVNS*) is derived for the targeted task of player selection problem. Then some operations, properties, and distance measures are introduced on m-polar Interval-valued Neutrosophic Set (*mIVNS*). Distance-base Similarity Measure is illustrated to each player with an ideal set in *mIVNS* structure. In the end, the Algorithm is given for ideal decision-making in sports for the selection of players.

Keywords: *mIVNS* Set; Operators on *mIVNS*; Properties; Distance and Similarity Measure; Decision-Making, Selection of Players

1. Introduction

Zadeh [1] has introduced a fuzzy set which describes membership degrees in $[0,1]$. They can be modeled to proceed towards including soft set theory [2,3] fuzzy soft set theory [4], probability theory, and also other mathematical tools. This theory has been used in many real-life applications to handle unpredictability. However, this theory doesn't deal with the hesitancy degree. To overcome this situation Atanassov [5] gave the idea of the intuitionistic fuzzy set as a generalized form of fuzzy theory, that handles incomplete data by considering both membership function values and non-membership function values. Intuitionistic Fuzzy

theory has been extended by many researchers in dealing with real-life problems. Intuitionistic fuzzy sets fail in grip patchy data because membership, hesitancy, and non-membership functions are dependent in this theory, while neutrosophic sets overcome with the solution where these functions are independent which exists in psychology theory and belief system. For several decades there are more concepts applied in soft set theory, different mathematicians relate their ideas and results on soft set theory in fuzzy set theory and other hybrid structures. To analyze the ability of ideal decision on positive effects and as well as on negative effects Zhang [6] gave bipolar concept on truth or false. It was further extended by Chen [7] to multipolar structure, where multiple data can be studied and evaluated. Akram [8-11] has applied to m-polar structure in decision-making problems and pattern recognition techniques.

Firstly, the concept of Neutrosophic Set (NS) introduced by Smarandache [12] is a tool that handles the problem with inconsistency and imprecise data with indeterminacy. Nowadays the NS has been extended in many hybrid structures. Single-valued NS is proposed in [12] and also Interval-valued Neutrosophic Set (IVNS) truth indeterminacy and falsity are determined by intervals that have huge information in real-life problems. Similarity measure technique is a well-known process to compare two sets. The Similarity measure is also used for the evaluation of pattern recognition of an object, set, and material. It is usually used between two independent objects on the basis of distance measure, while distance measure gives the numeric value of separation between two objects. A Similarity measures on fuzzy sets, soft sets, neutrosophic sets, etc. are done by several authors in their papers [13-26]. Aggregate operators, similarity measure, and a TOPSIS technique and their application in real life are introduced by [27-31] by Saeed et al. Application of fuzzy numbers in mobile selection in metros like Lahore is proposed by Saqlain [32,33]. TOPSIS technique of Multi-Criteria Decision Making (MCDM) can also be used for the prediction of games, and it's applied in FIFA 2018 by [34], prediction of games is a very complex topic and this game is also predicted by [35]. Liu et al. [36] introduced multi-valued neutrosophic numbers and utilized it with Bonferroni operator in multi-valued decision-making problems. Kamal et al. introduced a multi-polar neutrosophic soft structure with some operators and properties in [37]. Abdel-Basset et al. [38-42] proposed the solution to supply change problems, professional selection problem, time-cost tradeoff, and leveling problems in construction using a neutrosophic environment. Several authors [43-47] have done researches in m-polar structure with the fuzzy set, neutrosophic set, soft set topology in the past couple of years. From a scholastic point of view, operators on multi-valued neutrosophic soft sets need to be specified so that concepts can easily be applied in real-life applications. The concept of interval-valued neutrosophic sets was proposed where uncertain, vague, inconsistent, and incomplete data given in interval-valued. In this paper, we introduce m-polar Interval-valued neutrosophic sets that deal with multiple set of data that are used

in uncertain, vague, inconsistent, and incomplete data environment. An important issue is how we can represent, m-polar interval-valued neutrosophic set? It's operators and similarity measure? What should be the generalized form of interval-valued neutrosophic sets? What should be the application of this environment? To find the answers to all these questions, this study is done.

In this paper, the concept of Interval-valued neutrosophic set is extended to m-polar Interval-valued Neutrosophic Set (m-polar IVNS);

- (1) m-polar IVNS, its definition, and representation.
- (2) Aggregate operators of mIVNS and properties on operators of mIVNS.
- (3) Distance measure and Similarity measures of mIVNS.
- (4) m-polar IVNS Algorithm.
- (5) Application of the proposed environment

The paper is organized and structured in the following ways, also shown in Figure 1. In section 1, the introduction and literature review are presented. Section 2, consists of some basic definitions which will help read the rest of the article. In section 3, the definition, representation, and some operations like union, intersection, and complement, etc. of m-polar IVNS have been proposed. In section 4, properties concerned with operators have been studied. In section 5, distances on m-polar IVNS have been introduced and the similarity measure concept is revisited. In section 5, the application of the proposed environment with algorithm is presented. In section 6, the conclusion is presented.

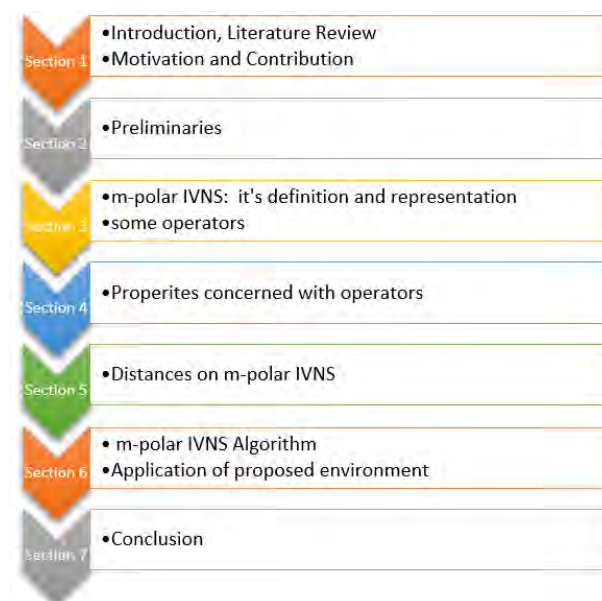


FIGURE 1. Pictorial view for the structure of the article

2. Preliminaries

This section studies some basic definitions related to this article.

2.1. Neutrosophic Set

Definition 2.1 [12] Let \ddot{U} be a space of points with a common element denoted by u . A neutrosophic set \ddot{A} over \ddot{U} is characterized by a truth-membership function $\phi_{\ddot{A}}$, an indeterminacy-membership function $\psi_{\ddot{A}}$ and a falsity-membership function $\eta_{\ddot{A}}$. The functions $\phi_{\ddot{A}}$, $\psi_{\ddot{A}}$, and $\eta_{\ddot{A}}$ are real standard belong to interval $[0, 1]$. The neutrosophic set can be represented as

$$\ddot{A} = \{u, (\phi_{\ddot{A}}(u), \psi_{\ddot{A}}(u), \eta_{\ddot{A}}(u)) \mid u \in \ddot{U}\}$$

where $0 \leq \phi_{\ddot{A}}(u) + \psi_{\ddot{A}}(u) + \eta_{\ddot{A}}(u) \leq 3$.

2.2. Interval-valued Neutrosophic set

Definition 2.2 [37] Let \ddot{U} be a space of objects with some element denoted by u . An interval-valued neutrosophic set \ddot{A} over \ddot{U} is characterized by interval-valued truth-membership function ${}^I\phi_{\ddot{A}}$, an interval-valued indeterminacy-membership function ${}^I\psi_{\ddot{A}}$ and an interval-valued falsity-membership function ${}^I\eta_{\ddot{A}}$, such that ${}^I\phi_{\ddot{A}}, {}^I\psi_{\ddot{A}}$ and ${}^I\eta_{\ddot{A}} \subseteq [0, 1]$. Thus, an interval-valued neutrosophic sets over \ddot{U} can be represented as

$$\ddot{A} = \{u, ({}^I\phi_{\ddot{A}}(u), {}^I\psi_{\ddot{A}}(u), {}^I\eta_{\ddot{A}}(u)) \mid u \in \ddot{U}\}$$

and

$$\begin{aligned} {}^I\phi_{\ddot{A}}(u) &= [\phi_{\ddot{A}}^-(u), \phi_{\ddot{A}}^+(u)] \\ {}^I\psi_{\ddot{A}}(u) &= [\psi_{\ddot{A}}^-(u), \psi_{\ddot{A}}^+(u)] \\ {}^I\eta_{\ddot{A}}(u) &= [\eta_{\ddot{A}}^-(u), \eta_{\ddot{A}}^+(u)] \end{aligned}$$

where $0 \leq \sup({}^I\phi_{\ddot{A}}(u)) + \sup({}^I\psi_{\ddot{A}}(u)) + \sup({}^I\eta_{\ddot{A}}(u)) \leq 3$

2.3. m-polar Neutrosophic Set

Definition 2.3 [37] An m-polar neutrosophic set is defined as

$$\ddot{A} = \{u, (\phi_{\ddot{A}}^1(u), \phi_{\ddot{A}}^2(u), \dots, \phi_{\ddot{A}}^m(u)), (\psi_{\ddot{A}}^1(u), \psi_{\ddot{A}}^2(u), \dots, \psi_{\ddot{A}}^m(u)), (\eta_{\ddot{A}}^1(u), \eta_{\ddot{A}}^2(u), \dots, \eta_{\ddot{A}}^m(u)) \mid u \in \ddot{U}\}$$

where $\phi_{\ddot{A}}^i : \ddot{U} \rightarrow [0, 1]$, $\psi_{\ddot{A}}^i : \ddot{U} \rightarrow [0, 1]$, and $\eta_{\ddot{A}}^i : \ddot{U} \rightarrow [0, 1]$; (for all $i = 1, 2, \dots, m$) denotes the degree of i -th truth-membership, i -th indeterminacy-membership, and i -th falsity-membership respectively for each element $u \in \ddot{U}$ to the set \ddot{A} and,

$$0 \leq \phi_{\ddot{A}}^i(u) + \psi_{\ddot{A}}^i(u) + \eta_{\ddot{A}}^i(u) \leq 3.$$

for all $i = 1, 2, \dots, m$

2.4. *m*-polar Interval-valued Neutrosophic Set

Definition 2.4 [37] Let \ddot{U} be a space of the objects and an *m*-polar IVNS \ddot{A} over universe \ddot{U} is defined as

$\ddot{A} = \{u, ({}^I\phi_{\ddot{A}}^1(u), {}^I\phi_{\ddot{A}}^2(u), \dots, {}^I\phi_{\ddot{A}}^m(u)), ({}^I\psi_{\ddot{A}}^1(u), {}^I\psi_{\ddot{A}}^2(u), \dots, {}^I\psi_{\ddot{A}}^m(u)), ({}^I\eta_{\ddot{A}}^1(u), {}^I\eta_{\ddot{A}}^2(u), \dots, {}^I\eta_{\ddot{A}}^m(u)) \mid u \in \ddot{U}\}$
where,

$$\begin{aligned} {}^I\phi_{\ddot{A}}^i(u) &= [\phi_{\ddot{A}}^{i-}(u), \phi_{\ddot{A}}^{i+}(u)], \\ {}^I\psi_{\ddot{A}}^i(u) &= [\psi_{\ddot{A}}^{i-}(u), \psi_{\ddot{A}}^{i+}(u)], \\ \text{and } {}^I\eta_{\ddot{A}}^i(u) &= [\eta_{\ddot{A}}^{i-}(u), \eta_{\ddot{A}}^{i+}(u)], \\ &\text{for all } i = 1, 2, \dots, m \end{aligned}$$

represents *i*-th interval-valued truth membership, an *i*-th interval-valued indeterminacy membership, and *i*-th interval-valued falsity membership respectively, and

$$\begin{aligned} 0 \leq \sup({}^I\phi_{\ddot{A}}^i(u)) + \sup({}^I\psi_{\ddot{A}}^i(u)) + \sup({}^I\eta_{\ddot{A}}^i(u)) &\leq 3. \\ &\text{for all } i = 1, 2, \dots, m \end{aligned}$$

Example 2.1

Let $\ddot{U} = \{u_1, u_2, u_3\}$ be a universal set and we define 3-polar IVNS \ddot{A} over universe \ddot{U} as,
 $\ddot{A} = \{u_1, (([0.2, 0.6], [0.3, 0.5], [0.6, 1]), ([0, 0.4], [0.2, 0.6], [0.4, 0.6]), ([0.5, 0.7], [0.8, 1], [0.6, 0.7])),$
 $u_2, (([0.3, 0.6], [0.3, 0.7], [0.1, 0.4]), ([0.5, 0.6], [0.8, 1], [0.5, 0.8]), ([0.3, 0.5], [0.6, 0.8], [0.2, 0.5])),$
 $u_3, (([0.4, 0.7], [0.5, 0.9], [0.6, 0.8]), ([0.7, 0.9], [0.6, 0.7], [0.5, 0.6]), ([0.2, 0.4], [0.3, 0.5], [0.4, 0.7]))\}$

3. Operations on *m*-polar interval valued neutrosophic sets

This section discusses some operators on these sets.

3.1. *m*-polar Interval-valued Neutrosophic Subset

Definition 3.1 Let \ddot{A} and \ddot{B} be two *m*-polar interval-valued neutrosophic sets over universal set \ddot{U} , then \ddot{A} is said to be a subset of \ddot{B} represented as $\ddot{A} \subseteq \ddot{B}$ if

$$\begin{aligned} \phi_{\ddot{A}}^{i-}(u) &\geq \phi_{\ddot{B}}^{i-}(u) \text{ and } \phi_{\ddot{A}}^{i+}(u) \leq \phi_{\ddot{B}}^{i+}(u); \\ \psi_{\ddot{A}}^{i-}(u) &\geq \psi_{\ddot{B}}^{i-}(u) \text{ and } \psi_{\ddot{A}}^{i+}(u) \leq \psi_{\ddot{B}}^{i+}(u); \\ \eta_{\ddot{A}}^{i-}(u) &\leq \eta_{\ddot{B}}^{i-}(u) \text{ and } \eta_{\ddot{A}}^{i+}(u) \geq \eta_{\ddot{B}}^{i+}(u); \end{aligned}$$

Example 3.1

Let $\ddot{U} = \{u_1, u_2, u_3\}$ be a universal set and we define two 3-polar IVNS \ddot{A} and \ddot{B} over universe \ddot{U} as,

$\ddot{A} = \{u_1, (([0.2, 0.6], [0.3, 0.5], [0.6, 1]), ([0, 0.4], [0.2, 0.6], [0.4, 0.6]), ([0.5, 0.7], [0.8, 1], [0.6, 0.7])),$
 $u_2, (([0.3, 0.6], [0.3, 0.7], [0.1, 0.4]), ([0.5, 0.6], [0.8, 1], [0.5, 0.8]), ([0.3, 0.5], [0.6, 0.8], [0.2, 0.5])),$
 $u_3, (([0.4, 0.7], [0.5, 0.9], [0.6, 0.8]), ([0.7, 0.9], [0.6, 0.7], [0.5, 0.6]), ([0.2, 0.4], [0.3, 0.5], [0.4, 0.7]))\}$
and

$\ddot{B} = \{u_1, ([0.1, 0.7], [0.2, 0.7], [0.3, 1]), ([0, 0.6], [0.2, 0.7], [0.2, 0.8]), ([0.6, 0.7], [0.9, 1], [0.6, 0.7])\},$
 $u_2, ([0.2, 0.7], [0.3, 0.8], [0.1, 0.5]), ([0.2, 0.7], [0.4, 1], [0.3, 0.8]), ([0.3, 0.5], [0.6, 0.7], [0.2, 0.4]),$
 $u_3, ([0.2, 0.7], [0.4, 1], [0.3, 0.8]), ([0.6, 0.9], [0.4, 0.7], [0.4, 0.8]), ([0.3, 0.4], [0.5, 0.5], [0.5, 0.6])\}$
 here \ddot{A} is a subset of \ddot{B} .

3.2. *m*-polar Interval-valued Neutrosophic Equal Set

Definition 3.2 Let \ddot{A} and \ddot{B} be two *m*-polar Interval-valued Neutrosophic Set over universal set \ddot{U} . Then two set \ddot{A} and \ddot{B} is said to be equal, represented as $\ddot{A} \doteq \ddot{B}$ if and only if

$$\ddot{A} \subseteq \ddot{B} \text{ and } \ddot{B} \subseteq \ddot{A}$$

3.3. *m*-polar Interval-valued Neutrosophic Null Set

Definition 3.3 An *m*-polar Interval-valued Neutrosophic Null Set $\ddot{\Phi}$ on universal set \ddot{U} is defined as

$$\ddot{\Phi} = \{u, ([1, 0], [1, 0], \dots, [1, 0]), ([1, 0], [1, 0], \dots, [1, 0]), ([0, 1], [0, 1], \dots, [0, 1])\}$$

3.4. *m*-polar Interval-valued Neutrosophic Absolute Set

Definition 3.4 An *m*-polar Interval-valued Neutrosophic Absolute Set \ddot{U} on universal set \ddot{U} is defined as

$$\ddot{U} = \{u, ([0, 1], [0, 1], \dots, [0, 1]), ([0, 1], [0, 1], \dots, [0, 1]), ([1, 0], [1, 0], \dots, [1, 0])\}$$

3.5. Union of *m*-polar Interval-valued Neutrosophic Set

Definition 3.5 Let \ddot{A} and \ddot{B} be two *m*-polar IVNS over a same universe \ddot{U} then the union of \ddot{A} and \ddot{B} defined as $\ddot{A} \cup \ddot{B} = \ddot{C}$ where $\ddot{C} = \{u, ({}^I\phi_{\ddot{C}}^i, {}^I\psi_{\ddot{C}}^i, {}^I\eta_{\ddot{C}}^i) \mid u \in \ddot{U}\}$ such that

$$\begin{aligned} {}^I\phi_{\ddot{C}}^i &= [\inf(\phi_{\ddot{A}}^{i-}, \phi_{\ddot{B}}^{i-}), \sup(\phi_{\ddot{A}}^{i+}, \phi_{\ddot{B}}^{i+})], \\ {}^I\psi_{\ddot{C}}^i &= [\inf(\psi_{\ddot{A}}^{i-}, \psi_{\ddot{B}}^{i-}), \sup(\psi_{\ddot{A}}^{i+}, \psi_{\ddot{B}}^{i+})], \\ {}^I\eta_{\ddot{C}}^i &= [\sup(\eta_{\ddot{A}}^{i-}, \eta_{\ddot{B}}^{i-}), \inf(\eta_{\ddot{A}}^{i+}, \eta_{\ddot{B}}^{i+})], \\ &\text{for all } i = 1, 2, \dots, m \end{aligned}$$

3.6. Intersection of *m*-polar Interval-valued Neutrosophic Set

Definition 3.6 Let \ddot{A} and \ddot{B} be two *m*-polar IVNS over a same universe \ddot{U} then the union of \ddot{A} and \ddot{B} defined as $\ddot{A} \cap \ddot{B} = \ddot{C}$ where $\ddot{C} = \{u, ({}^I\phi_{\ddot{C}}^i, {}^I\psi_{\ddot{C}}^i, {}^I\eta_{\ddot{C}}^i) \mid u \in \ddot{U}\}$ such that

$$\begin{aligned} {}^I\phi_{\ddot{C}}^i &= [\sup(\phi_{\ddot{A}}^{i-}, \phi_{\ddot{B}}^{i-}), \inf(\phi_{\ddot{A}}^{i+}, \phi_{\ddot{B}}^{i+})], \\ {}^I\psi_{\ddot{C}}^i &= [\sup(\psi_{\ddot{A}}^{i-}, \psi_{\ddot{B}}^{i-}), \inf(\psi_{\ddot{A}}^{i+}, \psi_{\ddot{B}}^{i+})], \\ {}^I\eta_{\ddot{C}}^i &= [\inf(\eta_{\ddot{A}}^{i-}, \eta_{\ddot{B}}^{i-}), \sup(\eta_{\ddot{A}}^{i+}, \eta_{\ddot{B}}^{i+})], \\ &\text{for all } i = 1, 2, \dots, m \end{aligned}$$

3.7. Complement of m -polar Interval-valued Neutrosophic Set

Definition 3.7 Let \ddot{U} be a universal set, and \ddot{A} be m -polar IVNS over universe \ddot{U} then the complement of \ddot{A} denoted by \ddot{A}^c and defined as

$$\ddot{A}^c = \{u, ({}^I\eta_{\ddot{A}}^i, [0 - 1] - {}^I\psi_{\ddot{A}}^i, {}^I\phi_{\ddot{A}}^i \mid u \in \ddot{U}\} \\ \text{for all } i = 1, 2, \dots, m$$

Example 3.2

Let $\ddot{U} = \{u_1, u_2, u_3\}$ be a universal set and we define two 3-polar IVNS \ddot{A} and \ddot{B} over universe \ddot{U} as,

$$\ddot{A} = \{u_1, (([0.2, 0.6], [0.3, 0.5], [0.6, 1]), ([0.2, 0.4], [0, 0.3], [0.3, 0.6]), ([0.4, 0.7], [0.3, 1], [0.2, 0.4])), \\ u_2, (([0.3, 0.6], [0.3, 0.7], [0.2, 0.5]), ([0.5, 0.6], [0.6, 1], [0.6, 0.9]), ([0.5, 0.7], [0.6, 0.9], [0.4, 0.9])), \\ u_3, (([0.4, 0.7], [0.5, 0.9], [0.6, 0.8]), ([0.7, 0.9], [0.7, 0.9], [0.1, 0.4]), ([0.3, 0.6], [0.4, 0.5], [0.2, 0.5]))\}$$

and

$$\ddot{B} = \{u_1, (([0.3, 0.5], [0.4, 0.7], [0, 0.6]), ([0.2, 0.4], [0, 0.1], [0.3, 0.6]), ([0, 0.4], [0.1, 0.4], [0.2, 0.3])), \\ u_2, (([0.3, 0.4], [0.2, 0.5], [0.5, 1]), ([0.6, 0.7], [0.7, 0.9], [0.4, 0.6]), ([0.5, 0.5], [0.7, 0.9], [0.8, 1])), \\ u_3, (([0.1, 0.4], [0.2, 0.7], [0.2, 0.6]), ([0.5, 0.8], [0.4, 0.7], [0.4, 0.6]), ([0.2, 0.4], [0.3, 0.6], [0.3, 0.5]))\}$$

then

$$\ddot{A}\ddot{B} = \{u_1, (([0.2, 0.6], [0.3, 0.7], [0, 1]), ([0.2, 0.4], [0, 0.3], [0.3, 0.6]), ([0.4, 0.4], [0.3, 0.4], [0.2, 0.3])), \\ u_2, (([0.3, 0.6], [0.2, 0.7], [0.2, 0.1]), ([0.5, 0.7], [0.6, 1], [0.4, 0.9]), ([0.5, 0.5], [0.7, 0.9], [0.8, 0.9])), \\ u_3, (([0.1, 0.7], [0.2, 0.9], [0.2, 0.8]), ([0.5, 0.9], [0.4, 0.9], [0.1, 0.6]), ([0.3, 0.4], [0.4, 0.5], [0.3, 0.5]))\} \\ \ddot{A}\tilde{\ddot{B}} = \{u_1, (([0.3, 0.5], [0.4, 0.5], [0.6, 0.6]), ([0.2, 0.4], [0, 0.1], [0.3, 0.6]), ([0, 0.7], [0.1, 1], [0.2, 0.4])), \\ u_2, (([0.3, 0.4], [0.3, 0.5], [0.5, 0.5]), ([0.6, 0.6], [0.7, 0.9], [0.6, 0.6]), ([0.5, 0.7], [0.6, 0.9], [0.4, 1])), \\ u_3, (([0.4, 0.4], [0.5, 0.7], [0.6, 0.6]), ([0.7, 0.8], [0.7, 0.7], [0.4, 0.4]), ([0.2, 0.6], [0.3, 0.6], [0.2, 0.5]))\} \\ \ddot{A}^c = \{u_1, (([0.4, 0.7], [0.3, 1], [0.2, 0.4]), ([0, 0.2] \cup [0.4, 1], [0.3, 1], [0, 0.3] \cup [0.6, 1]), ([0.2, 0.6], [0.3, 0.5], [0.6, 1])), \\ u_2, (([0.5, 0.7], [0.6, 0.9], [0.4, 0.9]), ([0, 0.5] \cup [0.6, 1], [0, 0.6], [0, 0.6] \cup [0.9, 1]), ([0.3, 0.6], [0.3, 0.7], [0.2, 0.5])), \\ u_3, (([0.3, 0.6], [0.4, 0.5], [0.2, 0.5]), ([0, 0.7] \cup [0.9, 1], [0, 0.7] \cup [0.9, 1], [0, 0.1] \cup [0.4, 1]), ([0.4, 0.7], [0.5, 0.9], [0.6, 0.8]))\}$$

4. Properties on m -polar IVNS set Operators

4.1. Idempotent Laws

- (i) $\ddot{A}\tilde{\ddot{A}} = \ddot{A}$
- (ii) $\ddot{A}\ddot{\ddot{A}} = \ddot{A}$

4.2. Identity Laws

- (iii) $\ddot{A}\ddot{\ddot{\Phi}} = \ddot{A} = \ddot{\Phi}\ddot{\ddot{A}}$
- (iv) $\ddot{A}\tilde{\ddot{U}} = \ddot{A} = \ddot{U}\tilde{\ddot{A}}$

4.3. *Domination Laws*

$$(v) \quad \check{A} \tilde{\cap} \check{\Phi} = \check{\Phi} = \check{\Phi} \tilde{\cap} \check{A}$$

$$(vi) \quad \check{A} \tilde{\cup} \tilde{U} = \tilde{U} = \tilde{U} \tilde{\cup} \check{A}$$

4.4. *Complement Laws*

$$(vii) \quad \check{\Phi}^c = \tilde{U}$$

$$(viii) \quad \tilde{U}^c = \check{\Phi}$$

4.5. *Double Complementation Law*

$$(ix) \quad (\check{A}^c)^c = \check{A}$$

4.6. *Commutative Laws*

$$(x) \quad \check{A} \tilde{\cup} \check{B} = \check{B} \tilde{\cup} \check{A}$$

$$(xi) \quad \check{A} \tilde{\cap} \check{B} = \check{B} \tilde{\cap} \check{A}$$

4.7. *Associative Laws*

$$(xii) \quad \check{A} \tilde{\cup} (\check{B} \tilde{\cup} \check{C}) = (\check{A} \tilde{\cup} \check{B}) \tilde{\cup} \check{C}$$

$$(xiii) \quad \check{A} \tilde{\cap} (\check{B} \tilde{\cap} \check{C}) = (\check{A} \tilde{\cap} \check{B}) \tilde{\cap} \check{C}$$

4.8. *Distributive Laws*

$$(xiv) \quad \check{A} \tilde{\cup} (\check{B} \tilde{\cap} \check{C}) = (\check{A} \tilde{\cup} \check{B}) \tilde{\cap} (\check{A} \tilde{\cup} \check{C})$$

$$(xv) \quad \check{A} \tilde{\cap} (\check{B} \tilde{\cup} \check{C}) = (\check{A} \tilde{\cap} \check{B}) \tilde{\cup} (\check{A} \tilde{\cap} \check{C})$$

4.9. *De Morgan's Laws*

$$(xvi) \quad (\check{A} \tilde{\cup} \check{B})^c = \check{A}^c \tilde{\cap} \check{B}^c$$

$$(xvii) \quad (\check{A} \tilde{\cap} \check{B})^c = \check{A}^c \tilde{\cup} \check{B}^c$$

The Proof of Commutative Laws, Associative Laws, Distributive Laws, and De Morgan's Laws are presented in this paper. They are the following:

Proof(x)

$$\check{A} \tilde{\cup} \check{B} = \{u, ([\inf(\phi_A^{i-}, \phi_B^{i-}), \sup(\phi_A^{i+}, \phi_B^{i+})], [\inf(\psi_A^{i-}, \psi_B^{i-}), \sup(\psi_A^{i+}, \psi_B^{i+})], [\sup(\eta_A^{i-}, \eta_B^{i-}), \inf(\eta_A^{i+}, \eta_B^{i+})]) \mid \text{for all } i = 1, 2, \dots, m\}$$

$$\check{A} \tilde{\cap} \check{B} = \{u, ([\inf(\phi_B^{i-}, \phi_A^{i-}), \sup(\phi_B^{i+}, \phi_A^{i+})], [\inf(\psi_B^{i-}, \psi_A^{i-}), \sup(\psi_B^{i+}, \psi_A^{i+})], [\sup(\eta_B^{i-}, \eta_A^{i-}), \inf(\eta_B^{i+}, \eta_A^{i+})]) \mid \text{for all } i = 1, 2, \dots, m\}$$

$$\check{A} \tilde{\cup} \check{B} = \check{B} \tilde{\cup} \check{A}$$

Proof(xii)

$$\begin{aligned}
\tilde{A}\tilde{U}(\tilde{B}\tilde{U}\tilde{C}) &= \{u, ([\inf(\phi_{\tilde{A}}^{i-}, \inf(\phi_{\tilde{B}}^{i-}, \phi_{\tilde{C}}^{i-})), \sup(\phi_{\tilde{A}}^{i+}, \sup(\phi_{\tilde{B}}^{i+}, \phi_{\tilde{C}}^{i+}))], \\
&\quad [\inf(\phi_{\tilde{A}}^{i-}, \inf(\psi_{\tilde{B}}^{i-}, \psi_{\tilde{C}}^{i-})), \sup(\phi_{\tilde{A}}^{i+}, \sup(\psi_{\tilde{B}}^{i+}, \psi_{\tilde{C}}^{i+}))], [\sup(\phi_{\tilde{A}}^{i-}, \sup(\eta_{\tilde{B}}^{i-}, \eta_{\tilde{C}}^{i-})), \inf(\phi_{\tilde{A}}^{i+}, \inf(\eta_{\tilde{B}}^{i+}, \eta_{\tilde{C}}^{i+}))]) \\
&\quad \text{for all } i = 1, 2, \dots, m \\
\tilde{A}\tilde{U}(\tilde{B}\tilde{U}\tilde{C}) &= \{u, ([\inf(\phi_{\tilde{A}}^{i-}, \phi_{\tilde{B}}^{i-}, \phi_{\tilde{C}}^{i-}), \sup(\phi_{\tilde{A}}^{i+}, \phi_{\tilde{B}}^{i+}, \phi_{\tilde{C}}^{i+})], \\
&\quad [\inf(\phi_{\tilde{A}}^{i-}, \psi_{\tilde{B}}^{i-}, \psi_{\tilde{C}}^{i-}), \sup(\phi_{\tilde{A}}^{i+}, \psi_{\tilde{B}}^{i+}, \psi_{\tilde{C}}^{i+})], [\sup(\phi_{\tilde{A}}^{i-}, \eta_{\tilde{B}}^{i-}, \eta_{\tilde{C}}^{i-}), \inf(\phi_{\tilde{A}}^{i+}, \eta_{\tilde{B}}^{i+}, \eta_{\tilde{C}}^{i+})]) \\
&\quad \text{for all } i = 1, 2, \dots, m \\
\tilde{A}\tilde{U}(\tilde{B}\tilde{U}\tilde{C}) &= \{u, ([\inf(\inf(\phi_{\tilde{A}}^{i-}, \phi_{\tilde{B}}^{i-}), \phi_{\tilde{C}}^{i-}), \sup(\sup(\phi_{\tilde{A}}^{i+}, \phi_{\tilde{B}}^{i+}), \phi_{\tilde{C}}^{i+})], \\
&\quad [\inf(\inf(\phi_{\tilde{A}}^{i-}, \psi_{\tilde{B}}^{i-}), \psi_{\tilde{C}}^{i-}), \sup(\sup(\phi_{\tilde{A}}^{i+}, \psi_{\tilde{B}}^{i+}), \psi_{\tilde{C}}^{i+})], [\sup(\sup(\phi_{\tilde{A}}^{i-}, \eta_{\tilde{B}}^{i-}), \eta_{\tilde{C}}^{i-}), \inf(\inf(\phi_{\tilde{A}}^{i+}, \eta_{\tilde{B}}^{i+}), \eta_{\tilde{C}}^{i+})]) \\
&\quad \text{for all } i = 1, 2, \dots, m \\
\tilde{A}\tilde{U}(\tilde{B}\tilde{U}\tilde{C}) &= (\tilde{A}\tilde{U}\tilde{B})\tilde{U}\tilde{C}
\end{aligned}$$

Proof(xiv)

$$\begin{aligned}
\tilde{A}\tilde{U}(\tilde{B}\tilde{U}\tilde{C}) &= \{u, ([\inf(\phi_{\tilde{A}}^{i-}, \sup(\phi_{\tilde{B}}^{i-}, \phi_{\tilde{C}}^{i-})), \sup(\phi_{\tilde{A}}^{i+}, \inf(\phi_{\tilde{B}}^{i+}, \phi_{\tilde{C}}^{i+}))], \\
&\quad [\inf(\phi_{\tilde{A}}^{i-}, \sup(\psi_{\tilde{B}}^{i-}, \psi_{\tilde{C}}^{i-})), \sup(\phi_{\tilde{A}}^{i+}, \inf(\psi_{\tilde{B}}^{i+}, \psi_{\tilde{C}}^{i+}))], [\sup(\phi_{\tilde{A}}^{i-}, \inf(\eta_{\tilde{B}}^{i-}, \eta_{\tilde{C}}^{i-})), \inf(\phi_{\tilde{A}}^{i+}, \sup(\eta_{\tilde{B}}^{i+}, \eta_{\tilde{C}}^{i+}))]) \\
&\quad \text{for all } i = 1, 2, \dots, m \\
\tilde{A}\tilde{U}(\tilde{B}\tilde{U}\tilde{C}) &= \{u, ([\sup(\inf(\phi_{\tilde{A}}^{i-}, \phi_{\tilde{B}}^{i-}), \inf(\phi_{\tilde{A}}^{i-}, \phi_{\tilde{C}}^{i-})), \inf(\sup(\phi_{\tilde{A}}^{i+}, \phi_{\tilde{B}}^{i+}, \sup(\phi_{\tilde{A}}^{i+}, \phi_{\tilde{C}}^{i+}))), \\
&\quad [\sup(\inf(\psi_{\tilde{A}}^{i-}, \psi_{\tilde{B}}^{i-}), \inf(\psi_{\tilde{A}}^{i-}, \psi_{\tilde{C}}^{i-})), \inf(\sup(\psi_{\tilde{A}}^{i+}, \psi_{\tilde{B}}^{i+}), \sup(\psi_{\tilde{A}}^{i+}, \psi_{\tilde{C}}^{i+}))], \\
&\quad [\inf(\sup(\eta_{\tilde{A}}^{i-}, \eta_{\tilde{B}}^{i-}), \sup(\eta_{\tilde{A}}^{i-}, \eta_{\tilde{C}}^{i-})), \sup(\inf(\eta_{\tilde{A}}^{i+}, \eta_{\tilde{B}}^{i+}), \inf(\eta_{\tilde{A}}^{i+}, \eta_{\tilde{C}}^{i+}))]) \\
&\quad \text{for all } i = 1, 2, \dots, m \\
\tilde{A}\tilde{U}(\tilde{B}\tilde{U}\tilde{C}) &= (\tilde{A}\tilde{U}\tilde{B})\tilde{U}(\tilde{A}\tilde{U}\tilde{C})
\end{aligned}$$

Proof(xvi)

$$\begin{aligned}
\tilde{A}^c\tilde{B}^c &= \{u, [\eta_{\tilde{A}}^{i-}, \eta_{\tilde{A}}^{i+}], [0, 1] - [\psi_{\tilde{A}}^{i-}, \psi_{\tilde{A}}^{i+}], [\phi_{\tilde{A}}^{i-}, \phi_{\tilde{A}}^{i+}]\} \tilde{\cap} \{u, [\eta_{\tilde{B}}^{i-}, \eta_{\tilde{B}}^{i+}], [0, 1] - [\psi_{\tilde{B}}^{i-}, \psi_{\tilde{B}}^{i+}], [\phi_{\tilde{B}}^{i-}, \phi_{\tilde{B}}^{i+}]\} \\
&\quad \text{for all } i = 1, 2, \dots, m \\
\tilde{A}^c\tilde{B}^c &= \{u, [\eta_{\tilde{A}}^{i-}, \eta_{\tilde{A}}^{i+}], [0, \psi_{\tilde{A}}^{i-}] \cup (\psi_{\tilde{A}}^{i+}, 1], [\phi_{\tilde{A}}^{i-}, \phi_{\tilde{A}}^{i+}]\} \tilde{\cap} \{u, [\eta_{\tilde{B}}^{i-}, \eta_{\tilde{B}}^{i+}], [0, \psi_{\tilde{B}}^{i-}] \cup (\psi_{\tilde{B}}^{i+}, 1], [\phi_{\tilde{B}}^{i-}, \phi_{\tilde{B}}^{i+}]\} \\
&\quad \text{for all } i = 1, 2, \dots, m \\
\tilde{A}^c\tilde{B}^c &= \{u, [\sup(\eta_{\tilde{A}}^{i-}, \eta_{\tilde{B}}^{i-}), \inf(\eta_{\tilde{A}}^{i+}, \eta_{\tilde{B}}^{i+})], [\sup(0, 0), \inf(\psi_{\tilde{A}}^{i-}, \psi_{\tilde{B}}^{i-})] \cup \\
&\quad (\sup(\psi_{\tilde{A}}^{i+}, \psi_{\tilde{B}}^{i+}), \inf(1, 1)], [\inf(\phi_{\tilde{A}}^{i-}, \phi_{\tilde{B}}^{i-}), \sup(\phi_{\tilde{A}}^{i+}, \phi_{\tilde{B}}^{i+})]\} \\
&\quad \text{for all } i = 1, 2, \dots, m \\
\tilde{A}^c\tilde{B}^c &= \{u, ([\sup(\eta_{\tilde{A}}^{i-}, \eta_{\tilde{B}}^{i-}), \inf(\eta_{\tilde{A}}^{i+}, \eta_{\tilde{B}}^{i+})], [0, \inf(\psi_{\tilde{A}}^{i-}, \psi_{\tilde{B}}^{i-})] \cup (\sup(\psi_{\tilde{A}}^{i+}, \psi_{\tilde{B}}^{i+}), 1], [\inf(\phi_{\tilde{A}}^{i-}, \phi_{\tilde{B}}^{i-}), \sup(\phi_{\tilde{A}}^{i+}, \phi_{\tilde{B}}^{i+})]) \\
&\quad \text{for all } i = 1, 2, \dots, m \\
\tilde{A}^c\tilde{B}^c &= \{u, ([\sup(\eta_{\tilde{A}}^{i-}, \eta_{\tilde{B}}^{i-}), \inf(\eta_{\tilde{A}}^{i+}, \eta_{\tilde{B}}^{i+})], [0, 1] - [\inf(\psi_{\tilde{A}}^{i-}, \psi_{\tilde{B}}^{i-}), \sup(\psi_{\tilde{A}}^{i+}, \psi_{\tilde{B}}^{i+})], [\inf(\phi_{\tilde{A}}^{i-}, \phi_{\tilde{B}}^{i-}), \sup(\phi_{\tilde{A}}^{i+}, \phi_{\tilde{B}}^{i+})]) \\
&\quad \text{for all } i = 1, 2, \dots, m \\
\tilde{A}^c\tilde{B}^c &= (\{u, ([\inf(\phi_{\tilde{A}}^{i-}, \phi_{\tilde{B}}^{i-}), \sup(\phi_{\tilde{A}}^{i+}, \phi_{\tilde{B}}^{i+})], [\inf(\psi_{\tilde{A}}^{i-}, \psi_{\tilde{B}}^{i-}), \sup(\psi_{\tilde{A}}^{i+}, \psi_{\tilde{B}}^{i+})], [\sup(\eta_{\tilde{A}}^{i-}, \eta_{\tilde{B}}^{i-}), \inf(\eta_{\tilde{A}}^{i+}, \eta_{\tilde{B}}^{i+})])\})^c \\
&\quad \text{for all } i = 1, 2, \dots, m \\
\tilde{A}^c\tilde{B}^c &= (\tilde{A}\tilde{U}\tilde{B})^c
\end{aligned}$$

Similarly, Other Laws can be proved.

5. Distance measure for m-polar Interval-valued Neutrosophic Sets

5.1. Distances

Let \tilde{A} and \tilde{B} be two m-polar Interval-valued Neutrosophic Sets corresponds to a universal set $\tilde{U} = \{u_1, u_2, \dots, u_n\}$ such that

$$\begin{aligned}\ddot{A} &= \{u_j, ([\phi_{\ddot{A}}^{i-}(u_j), \phi_{\ddot{A}}^{i+}(u_j)], [\psi_{\ddot{A}}^{i-}(u_j), \psi_{\ddot{A}}^{i+}(u_j)], [\eta_{\ddot{A}}^{i-}(u_j), \eta_{\ddot{A}}^{i+}(u_j)])\} \text{ and} \\ \ddot{B} &= \{u_j, ([\phi_{\ddot{B}}^{i-}(u_j), \phi_{\ddot{B}}^{i+}(u_j)], [\psi_{\ddot{B}}^{i-}(u_j), \psi_{\ddot{B}}^{i+}(u_j)], [\eta_{\ddot{B}}^{i-}(u_j), \eta_{\ddot{B}}^{i+}(u_j)])\} \\ &\text{for all } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n\end{aligned}$$

then distances between \ddot{A} and \ddot{B} is defined as;

(1) Hamming distance:

$$H|\ddot{A}, \ddot{B}| = \frac{1}{3m} \sum_{i=1}^m \sum_{j=1}^n \{|\bar{\phi}_{\ddot{A}}^i(u_j) - \bar{\phi}_{\ddot{B}}^i(u_j)| + |\bar{\psi}_{\ddot{A}}^i(u_j) - \bar{\psi}_{\ddot{B}}^i(u_j)| + |\bar{\eta}_{\ddot{A}}^i(u_j) - \bar{\eta}_{\ddot{B}}^i(u_j)|\} \quad (1)$$

(2) Normalized Hamming distance

$$NH|\ddot{A}, \ddot{B}| = \frac{1}{3mn} \sum_{i=1}^m \sum_{j=1}^n \{|\bar{\phi}_{\ddot{A}}^i(u_j) - \bar{\phi}_{\ddot{B}}^i(u_j)| + |\bar{\psi}_{\ddot{A}}^i(u_j) - \bar{\psi}_{\ddot{B}}^i(u_j)| + |\bar{\eta}_{\ddot{A}}^i(u_j) - \bar{\eta}_{\ddot{B}}^i(u_j)|\} \quad (2)$$

(3) Euclidean distance

$$E|\ddot{A}, \ddot{B}| = \left(\frac{1}{3m} \sum_{i=1}^m \sum_{j=1}^n \{(\bar{\phi}_{\ddot{A}}^i(u_j) - \bar{\phi}_{\ddot{B}}^i(u_j))^2 + (\bar{\psi}_{\ddot{A}}^i(u_j) - \bar{\psi}_{\ddot{B}}^i(u_j))^2 + (\bar{\eta}_{\ddot{A}}^i(u_j) - \bar{\eta}_{\ddot{B}}^i(u_j))^2\} \right)^{\frac{1}{2}} \quad (3)$$

(4) Normalized Euclidean distance

$$NE|\ddot{A}, \ddot{B}| = \frac{1}{3mn} \sum_{i=1}^m \sum_{j=1}^n \{(\bar{\phi}_{\ddot{A}}^i(u_j) - \bar{\phi}_{\ddot{B}}^i(u_j))^2 + (\bar{\psi}_{\ddot{A}}^i(u_j) - \bar{\psi}_{\ddot{B}}^i(u_j))^2 + (\bar{\eta}_{\ddot{A}}^i(u_j) - \bar{\eta}_{\ddot{B}}^i(u_j))^2\}^{\frac{1}{2}} \quad (4)$$

where

$$\begin{aligned}\bar{\phi}_{\ddot{A}}^i(u_j) &= \frac{\phi_{\ddot{A}}^{i-}(u_j) + \phi_{\ddot{A}}^{i+}(u_j)}{2}, & \bar{\phi}_{\ddot{B}}^i(u_j) &= \frac{\phi_{\ddot{B}}^{i-}(u_j) + \phi_{\ddot{B}}^{i+}(u_j)}{2}, \\ \bar{\psi}_{\ddot{A}}^i(u_j) &= \frac{\psi_{\ddot{A}}^{i-}(u_j) + \psi_{\ddot{A}}^{i+}(u_j)}{2}, & \bar{\psi}_{\ddot{B}}^i(u_j) &= \frac{\psi_{\ddot{B}}^{i-}(u_j) + \psi_{\ddot{B}}^{i+}(u_j)}{2}, \\ \bar{\eta}_{\ddot{A}}^i(u_j) &= \frac{\eta_{\ddot{A}}^{i-}(u_j) + \eta_{\ddot{A}}^{i+}(u_j)}{2}, & \bar{\eta}_{\ddot{B}}^i(u_j) &= \frac{\eta_{\ddot{B}}^{i-}(u_j) + \eta_{\ddot{B}}^{i+}(u_j)}{2},\end{aligned}$$

for all $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

Theorem 5.1 The distance between two mIVNS \ddot{A} and \ddot{B} satisfy the following inequalities;

- (1) $H|\ddot{A}, \ddot{B}| \leq n$,
- (2) $NH|\ddot{A}, \ddot{B}| \leq 1$,
- (3) $E|\ddot{A}, \ddot{B}| \leq \sqrt{n}$,
- (4) $NE|\ddot{A}, \ddot{B}| \leq 1$

Theorem 5.2 Distance between any two mIVNS \ddot{A} and \ddot{B} is a metric distance

Proof

- (i) $H|\ddot{A}, \ddot{B}| \geq 0$
- (ii) $H|\ddot{A}, \ddot{B}| = 0$

$$\Leftrightarrow \frac{1}{3m} \sum_{i=1}^m \sum_{j=1}^n \{|\bar{\phi}_{\ddot{A}}^i(u_j) - \bar{\phi}_{\ddot{B}}^i(u_j)| + |\bar{\psi}_{\ddot{A}}^i(u_j) - \bar{\psi}_{\ddot{B}}^i(u_j)| + |\bar{\eta}_{\ddot{A}}^i(u_j) - \bar{\eta}_{\ddot{B}}^i(u_j)|\} = 0$$

$$\Leftrightarrow |\bar{\phi}_{\check{A}}^i(u_j) - \bar{\phi}_{\check{B}}^i(u_j)| + |\bar{\psi}_{\check{A}}^i(u_j) - \bar{\psi}_{\check{B}}^i(u_j)| + |\bar{\eta}_{\check{A}}^i(u_j) - \bar{\eta}_{\check{B}}^i(u_j)| = 0$$

for all $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

$$\Leftrightarrow |\bar{\phi}_{\check{A}}^i(u_j) - \bar{\phi}_{\check{B}}^i(u_j)| = 0,$$

$$|\bar{\psi}_{\check{A}}^i(u_j) - \bar{\psi}_{\check{B}}^i(u_j)| = 0,$$

$$|\bar{\eta}_{\check{A}}^i(u_j) - \bar{\eta}_{\check{B}}^i(u_j)| = 0$$

for all $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

$$\Leftrightarrow \bar{\phi}_{\check{A}}^i(u_j) = \bar{\phi}_{\check{B}}^i(u_j),$$

$$\bar{\psi}_{\check{A}}^i(u_j) = \bar{\psi}_{\check{B}}^i(u_j),$$

$$\bar{\eta}_{\check{A}}^i(u_j) = \bar{\eta}_{\check{B}}^i(u_j)$$

for all $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

$$\Leftrightarrow \check{A} = \check{B}$$

$$(iii) H|\check{A}, \check{B}| = H|\check{B}, \check{A}|$$

(iv) For any three sets \check{A} , \check{B} , and \check{C}

$$|\bar{\phi}_{\check{A}}^i(u_j) - \bar{\phi}_{\check{B}}^i(u_j)| + |\bar{\psi}_{\check{A}}^i(u_j) - \bar{\psi}_{\check{B}}^i(u_j)| + |\bar{\eta}_{\check{A}}^i(u_j) - \bar{\eta}_{\check{B}}^i(u_j)|$$

for all $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

$$= |\bar{\phi}_{\check{A}}^i(u_j) - \bar{\phi}_{\check{C}}^i(u_j) + \bar{\phi}_{\check{C}}^i(u_j) - \bar{\phi}_{\check{B}}^i(u_j)|$$

$$+ |\bar{\psi}_{\check{A}}^i(u_j) - \bar{\psi}_{\check{C}}^i(u_j) + \bar{\psi}_{\check{C}}^i(u_j) - \bar{\psi}_{\check{B}}^i(u_j)|$$

$$+ |\bar{\eta}_{\check{A}}^i(u_j) - \bar{\eta}_{\check{C}}^i(u_j) + \bar{\eta}_{\check{C}}^i(u_j) - \bar{\eta}_{\check{B}}^i(u_j)|$$

for all $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

$$\leq |\bar{\phi}_{\check{A}}^i(u_j) - \bar{\phi}_{\check{C}}^i(u_j)| + |\bar{\phi}_{\check{C}}^i(u_j) - \bar{\phi}_{\check{B}}^i(u_j)|$$

$$+ |\bar{\psi}_{\check{A}}^i(u_j) - \bar{\psi}_{\check{C}}^i(u_j)| + |\bar{\psi}_{\check{C}}^i(u_j) - \bar{\psi}_{\check{B}}^i(u_j)|$$

$$+ |\bar{\eta}_{\check{A}}^i(u_j) - \bar{\eta}_{\check{C}}^i(u_j)| + |\bar{\eta}_{\check{C}}^i(u_j) - \bar{\eta}_{\check{B}}^i(u_j)|$$

for all $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

$$= |\bar{\phi}_{\check{A}}^i(u_j) - \bar{\phi}_{\check{C}}^i(u_j)| + |\bar{\psi}_{\check{A}}^i(u_j) - \bar{\psi}_{\check{C}}^i(u_j)| + |\bar{\eta}_{\check{A}}^i(u_j) - \bar{\eta}_{\check{C}}^i(u_j)|$$

$$+ |\bar{\phi}_{\check{C}}^i(u_j) - \bar{\phi}_{\check{B}}^i(u_j)| + |\bar{\psi}_{\check{C}}^i(u_j) - \bar{\psi}_{\check{B}}^i(u_j)| + |\bar{\eta}_{\check{C}}^i(u_j) - \bar{\eta}_{\check{B}}^i(u_j)|$$

for all $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

$$H|\check{A}, \check{B}| \leq H|\check{A}, \check{C}| + H|\check{C}, \check{B}|$$

5.2. Similarity Measure

Definition 5.2 [31] Similarity measures between two mIVNS is defined as

$$SM(\ddot{A}, \ddot{B}) = \frac{1}{1 + |\ddot{A}, \ddot{B}|} \quad (5)$$

where $|\ddot{A}, \ddot{B}|$ is any distance that are discussed above.

Lemma 5.3 Consider two mIVNS \ddot{A} and \ddot{B} corresponds to universal set \ddot{U} then we have the following properties

- $SM(\ddot{A}, \ddot{B}) = SM(\ddot{B}, \ddot{A})$
- $0 \leq SM(\ddot{A}, \ddot{B}) \leq 1$
- $SM(\ddot{A}, \ddot{B}) = 1$ iff $\ddot{A} = \ddot{B}$

5.3. Similarity

Definition 5.3 [31] Consider $N(U)$ be the set of all mIVNS corresponds to \ddot{U} . Suppose \ddot{A} and $\ddot{B} \in N(U)$. If $SM(\ddot{A}, \ddot{B}) \geq \check{\alpha}$, $\check{\alpha} \in [0, 1]$ then two \ddot{A} and \ddot{B} are said to be $\check{\alpha}$ similar and we denote the relation $\ddot{A} \cong_{\check{\alpha}} \ddot{B}$.

Lemma 5.4 Two mIVNS \ddot{A} and \ddot{B} are said to be significantly similar if

$$SM(\ddot{A}, \ddot{B}) > \frac{1}{2}$$

6. Case Study

Similarity Measure is well-known criteria for the solution of Decision Making Problems. The use of similarity measure can be very helpful in the selection of the best alternative. This criteria is considered the best tool for the comparison of two objects, set, and pattern. While it will better to compare some set with an ideal set to find the similarity with the idealness. This method will conclude that at which stage (from 0 to 1) an object can be ranked, which helps in ranking analysis and selection problem. From the ranking an object, a candidate can be selected more accurately.

In Sports, every game is important for a team, the selection of players for a team can affect the result of a game, so it is mandatory for the team, coach, and technique director to select good and excellent players for a game. Following is the algorithm of our proposed method, also shown in Figure 2.

6.1. Algorithm of Decision Making on mIVNS structure

- Construct set of attribute $\ddot{U} = \{u_1, u_2, \dots, u_n\}$ as n number of attribute of players.
- Construct t m-polar IVNS \ddot{A}_k , ($k = 1, 2, \dots, t$) corresponds to \ddot{U} based on previous m matches records of t Players

- Construct best ideal performance m-polar IVNS \ddot{B} corresponds to \ddot{U} , that is $\ddot{B} = \{u_j, ([1, 1], [1, 1], \dots, [1, 1]), ([0, 0], [0, 0], \dots, [0, 0]), ([0, 0], [0, 0], \dots, [0, 0])\}$ where $j = 1, 2, \dots, n$
- Calculate distance between each \ddot{A}_k and \ddot{B} using Euclidean distance formula $E|\ddot{A}, \ddot{B}| = (\frac{1}{3m} \sum_{i=1}^m \sum_{j=1}^n \{(\bar{\phi}_A^i(u_j) - \bar{\phi}_B^i(u_j))^2 + (\bar{\psi}_A^i(u_j) - \bar{\psi}_B^i(u_j))^2 + (\bar{\eta}_A^i(u_j) - \bar{\eta}_B^i(u_j))^2\})^{\frac{1}{2}}$
- Calculating the similarity measures between two m-polar IVNS by using the formula $SM(\ddot{A}, \ddot{B}) = \frac{1}{1+E|\ddot{A}, \ddot{B}|}$
- Arrange the similarity measure results in descending order as to rank players for next match.

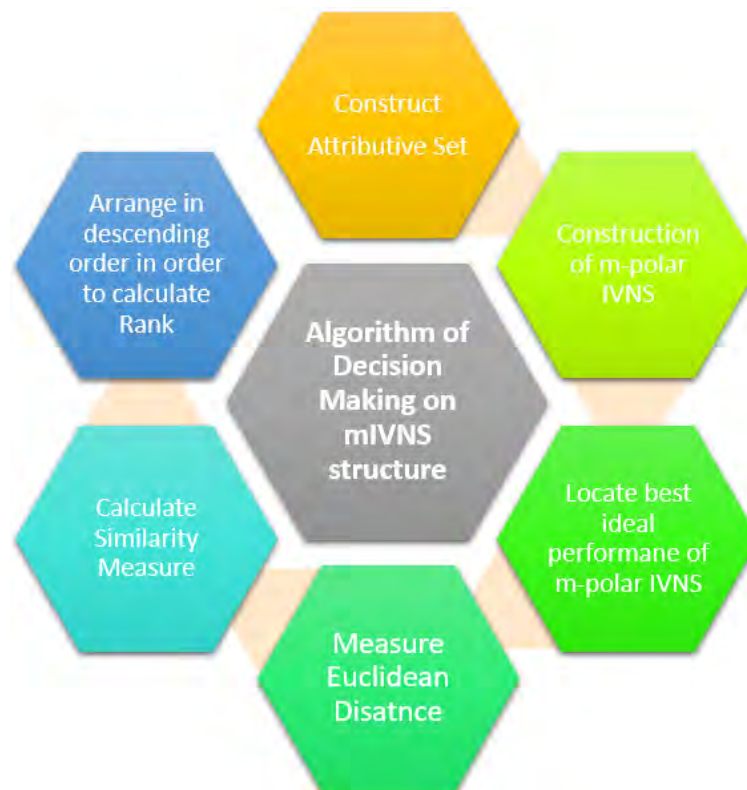


FIGURE 2. Pictorial view of Algorithm

6.2. Limitation of the Method

There are several limitations of the method that must be assured before implementing the similarity measure criteria.

- (1) Similarity measure can be found between two sets at a time to find comparison among themselves.
- (2) The two sets must be independent of each other and must be from the same structure.

6.3. Problem Formulation and Assumption

Here we illustrate a similarity measure for the selection of the best three forward players combination of Paris Saint-Germain Club (Football) as looking at previous matches records of forward players who played 90 minutes game. Since PSG is very strong in attacking and they have very good players in forward, mid and defending positions. But the PSG club is weak in scoring the good goals and misses good opportunities of scoring goals. In this paper, we took previous records of some matches of all forward players of PSG who played 90 minutes game for our decision making problem, we illustrate similarity measure based on Euclidean distance for the best 3 forward player combination.

6.4. Calculation

The data of previous matches are collected in distinct attributes (quality of and best ideal forward players), attribute set of players is defined as \ddot{U} such that,

$$\ddot{U} = \{u_1, u_2, u_3, u_4\}$$

where u_1 represents “shooting ability”, u_2 represents “first touch ability”, u_3 represents “speed”, and u_4 represents “switch with other forward players”,.

Although PSG has six forward players that are Edinson Cavani, Kylian Mbappe, Neymar Jr., Mauro Icardi, Pablo Sarabia, and Eric Maxim Chupo-Moting, but mostly three of them plays a match on the field.

Now we took data from the site of the last 3 matches played by six players mentioned above corresponds to attribute set \ddot{U} , then we construct six different 3-polar IVNS $\ddot{A}_1, \ddot{A}_2, \ddot{A}_3, \ddot{A}_4, \ddot{A}_5$, and \ddot{A}_6 , shown in Table 1 where these sets represents the last 3 matches performances of Edinson Cavani, Kylian Mbappe, Neymar Jr., Mauro Icardi, Pablo Sarabia, and Eric Maxim Chupo-Moting, respectively,

We take the best ideal performance of any forward player as mIVNS \ddot{B} , that is

$$\ddot{B} = \{u_1, (([1, 1][1, 1][1, 1]), ([0, 0][0, 0][0, 0]), ([0, 0][0, 0][0, 0])), u_2, (([1, 1][1, 1][1, 1]), ([0, 0][0, 0][0, 0]), ([0, 0][0, 0][0, 0])), \\ u_3, (([1, 1][1, 1][1, 1]), ([0, 0][0, 0][0, 0]), ([0, 0][0, 0][0, 0])), u_4, (([1, 1][1, 1][1, 1]), ([0, 0][0, 0][0, 0]), ([0, 0][0, 0][0, 0]))\}$$

Now We calculate the Euclidean distance between \ddot{A}_k , ($k = 1, 2, 3, 4, 5, 6$) and \ddot{B} , and then the Similarity measure is calculated as shown in Table 2.

Now we can easily rank the players as $\ddot{A}_3 \succ \ddot{A}_2 \succ \ddot{A}_4 \succ \ddot{A}_1 \succ \ddot{A}_5 \succ \ddot{A}_6$ as highest to lowest value from Table 2, so it shows that Neymar Jr., Kylian Mbappe, and Mauro Icardi are three key players for the very next match as they had a good performance in last 3 matches.

7. Discussion

Similarity Measure is a well-known tool for the evaluation of comparisons and finding the similarity between two objects, patterns, and sets. This tool is applicable in every structure

	u_1	u_2	u_3	u_4
\check{A}_1	$([0.71, 0.76][0.83, 0.90][0.80, 0.85]),$ $([0.23, 0.30][0.14, 0.22][0.19, 0.23]),$ $([0.30, 0.35][0.17, 0.20][0.30, 0.32])$	$([0.57, 0.63][0.65, 0.72][0.62, 0.67]),$ $([0.32, 0.38][0.40, 0.43][0.36, 0.40]),$ $([0.32, 0.39][0.42, 0.52][0.41, 0.49])$	$([0.74, 0.80][0.75, 0.80][0.70, 0.76]),$ $([0.23, 0.30][0.17, 0.22][0.25, 0.29]),$ $([0.27, 0.34][0.19, 0.28][0.22, 0.30])$	$([0.50, 0.57][0.52, 0.60][0.50, 0.55]),$ $([0.42, 0.48][0.43, 0.50][0.48, 0.55]),$ $([0.50, 0.53][0.49, 0.55][0.55, 0.60])$
\check{A}_2	$([0.90, 0.97][0.87, 0.92][0.86, 0.91]),$ $([0.10, 0.18][0.16, 0.22][0.20, 0.24]),$ $([0.20, 0.24][0.23, 0.30][0.18, 0.25])$	$([0.82, 0.87][0.84, 0.90][0.87, 0.93]),$ $([0.12, 0.20][0.20, 0.25][0.13, 0.20]),$ $([0.25, 0.32][0.18, 0.24][0.30, 0.33])$	$([0.60, 0.66][0.58, 0.64][0.53, 0.60]),$ $([0.33, 0.40][0.38, 0.45][0.34, 0.40]),$ $([0.38, 0.40][0.40, 0.50][0.43, 0.52])$	$([0.75, 0.80][0.70, 0.76][0.68, 0.75]),$ $([0.23, 0.30][0.30, 0.35][0.33, 0.39]),$ $([0.28, 0.34][0.30, 0.36][0.39, 0.46])$
\check{A}_3	$([0.94, 0.99][0.96, 1][0.95, 1]),$ $([0.13, 0.20][0.14, 0.22][0.18, 0.24]),$ $([0.20, 0.26][0.23, 0.27][0.19, 0.26])$	$([0.95, 1][0.96, 1][0.95, 0.98]),$ $([0.20, 0.22][0.15, 0.21][0.17, 0.23]),$ $([0.20, 0.26][0.21, 0.25][0.23, 0.29])$	$([0.65, 0.70][0.68, 0.76][0.74, 0.80]),$ $([0.39, 0.47][0.41, 0.46][0.38, 0.42]),$ $([0.40, 0.47][0.42, 0.50][0.46, 0.54])$	$([0.90, 0.95][0.96, 1][0.98, 1]),$ $([0.11, 0.15][0.14, 0.19][0.16, 0.24]),$ $([0.30, 0.33][0.25, 0.30][0.20, 0.30])$
\check{A}_4	$([0.78, 0.84][0.78, 0.84][0.83, 0.89]),$ $([0.20, 0.25][0.22, 0.29][0.14, 0.20]),$ $([0.34, 0.40][0.33, 0.35][0.23, 0.27])$	$([0.58, 0.64][0.60, 0.65][0.61, 0.67]),$ $([0.30, 0.36][0.28, 0.30][0.35, 0.40]),$ $([0.40, 0.45][0.39, 0.43][0.33, 0.37])$	$([0.65, 0.71][0.63, 0.68][0.60, 0.67]),$ $([0.50, 0.54][0.34, 0.40][0.40, 0.47]),$ $([0.29, 0.35][0.31, 0.39][0.35, 0.40])$	$([0.54, 0.60][0.54, 0.59][0.57, 0.60]),$ $([0.37, 0.40][0.39, 0.44][0.42, 0.46]),$ $([0.43, 0.45][0.40, 0.44][0.30, 0.33])$
\check{A}_5	$([0.74, 0.80][0.70, 0.76][0.70, 0.76]),$ $([0.20, 0.24][0.23, 0.28][0.30, 0.34]),$ $([0.23, 0.31][0.25, 0.31][0.30, 0.34])$	$([0.60, 0.68][0.58, 0.65][0.56, 0.62]),$ $([0.34, 0.39][0.38, 0.45][0.40, 0.48]),$ $([0.32, 0.35][0.45, 0.49][0.37, 0.42])$	$([0.50, 0.57][0.38, 0.45][0.37, 0.42]),$ $([0.37, 0.40][0.44, 0.51][0.38, 0.48]),$ $([0.41, 0.44][0.50, 0.54][0.59, 0.63])$	$([0.73, 0.82][0.75, 0.80][0.80, 0.85]),$ $([0.30, 0.32][0.17, 0.25][0.22, 0.26]),$ $([0.20, 0.26][0.20, 0.26][0.19, 0.22])$
\check{A}_6	$([0.64, 0.70][0.62, 0.70][0.69, 0.74]),$ $([0.31, 0.37][0.28, 0.33][0.22, 0.29]),$ $([0.21, 0.25][0.23, 0.25][0.24, 0.30])$	$([0.60, 0.69][0.59, 0.65][0.61, 0.69]),$ $([0.20, 0.30][0.34, 0.40][0.31, 0.40]),$ $([0.32, 0.38][0.30, 0.34][0.34, 0.37])$	$([0.48, 0.54][0.40, 0.48][0.40, 0.46]),$ $([0.51, 0.55][0.52, 0.58][0.48, 0.50]),$ $([0.42, 0.45][0.48, 0.53][0.50, 0.52])$	$([0.42, 0.50][0.55, 0.62][0.65, 0.70]),$ $([0.52, 0.57][0.43, 0.47][0.47, 0.53]),$ $([0.39, 0.44][0.38, 0.42][0.45, 0.50])$

TABLE 1. Represents six 3-IVNS $\check{A}_1, \check{A}_2, \check{A}_3, \check{A}_4, \check{A}_5$, and \check{A}_6

	Distance Measure	Similarity Measure (SM)	Rank
(\check{A}_1, \check{B})	0.2393	0.806	4
(\check{A}_2, \check{B})	0.1943	0.837	2
(\check{A}_3, \check{B})	0.1690	0.855	1
(\check{A}_4, \check{B})	0.2383	0.807	3
(\check{A}_5, \check{B})	0.2459	0.802	5
(\check{A}_6, \check{B})	0.2747	0.784	6

TABLE 2. Represents Distance Measures and Similarity Measures

of fuzzy, intuitionistic fuzzy, neutrosophic, and their hybrid structures while the first distance measure has to be derived for certain structures. It is mandatory that the two sets must be independent of each other and should be from the same structure family. In this paper, a new structure m-polar interval-valued neutrosophic set is derived and some operators are proposed on the derived structure. Furthermore, properties based on operators are discussed, and using distance-based similarity measure an application of player selection problems is dealt with the algorithm.

8. Conclusion

m-polar Interval-valued Neutrosophic set has wide application in decision making real-life problems. In this article, m-polar IVNS has been revisited, also some operators and properties on m-polar IVNS have been introduced. Furthermore, distance measure and similarity measure is introduced on m-polar IVNS, and evaluation of selecting of best three forward player

combination of PSG club for the very next match as watching their previous match records has been discussed using a similarity measure tool. The concept of m-polar is used as several number of previous m matches played by a player, so the m-polar concept can be used in many different fields of decision-making problems. In future, the this study can be extended and TOPSIS, VIKOR, etc can be applied to this new structure for decision making.

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Development of Hybrids of Hypersoft Set with Complex Fuzzy Set, Complex Intuitionistic Fuzzy set and Complex Neutrosophic Set

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Abstract. The complex fuzzy soft set and its generalized hybrids are such effective structures which not only minimize the impediments of all complex fuzzy-like structures for dealing uncertainties but also fulfill all the parametric requirements of soft sets. This feature makes it a completely new mathematical tool for solving problems dealing with uncertainties. Smarandache conceptualized hypersoft set as a generalization of soft set as it transforms the single attribute function into a multi-attribute function. This generalization demands an extension of complex fuzzy soft-like structures to hypersoft structure for more precise results. In this study, hybrids of hypersoft set with complex fuzzy set and its generalized structures i.e. complex intuitionistic fuzzy set and complex neutrosophic set, are developed along with illustrative examples to address the demand of literature. Moreover, some of their fundamentals i.e. subset, equal sets, null set, absolute set etc. and theoretic operations i.e. compliment, union, intersection etc. are discussed.

Keywords: Complex fuzzy sets (CF-Sets), soft set, hypersoft set and complex fuzzy hypersoft set.

1. Introduction

Zadeh's theory of fuzzy sets [1] is one of those theories which are considered as mathematical means to tackle many complicated problems involving various uncertainties in different fields of mathematical sciences. But these theories are unable to solve these problems successfully due to the inadequacy of the parametrization tool. This shortcoming is addressed by Molodtsov's soft set theory [2] which is free from all such Impediments and appeared as a new parameterized family of subsets of the universe of discourse. Classical complex analysis is useful in algebraic geometry, number theory, analytic combinatorics and many other branches of mathematical sciences. Ramot et al. [3, 4] introduced the concept of complex

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fuzzy set (CF-set) to tackle the problems of complex analysis under fuzzy environment. This novel concept used complex-valued state for the membership of its elements. Maji et al. [5] developed and conceptualized fuzzy soft set, a new hybrid of fuzzy set with soft set. They also discussed some of its fundamentals terminologies and operations like equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set, absolute soft set, AND, OR etc. in their work. Çağman et al. [6] extended this concept and discussed some other properties and operations. Nadia [7] developed a new hybrid of complex fuzzy set and soft set. Thirunavukarasu et al. [8] established aggregation properties of complex fuzzy soft set and discussed their applications. Atanassov [9] conceptualized intuitionistic fuzzy sets as generalization of fuzzy set. Alkouri et al. [10] extended this concept and developed complex intuitionistic fuzzy soft set and discussed some of its properties. Kumar et al. [11] further discussed its more properties and calculated its distance measures and entropies. Mumtaz et al. [12] extended neutrosophic set [13] to complex neutrosophic set and discussed its fundamentals, theoretic operations and applications. Broumi et al. [14] conceptualized complex neutrosophic soft set and discussed some of its fundamentals.

In 2018, Smarandache [15] introduced the concept of hypersoft set as a generalization of soft set. In 2020, Saeed et al. [16] extended the concept and discussed the fundamentals of hypersoft set such as hypersoft subset, complement, not hypersoft set, aggregation operators along with hypersoft set relation, sub relation, complement relation, function, matrices and operations on hypersoft matrices.

Having motivation from the work in [6], [8]- [16] and [21], novel hybrids of hypersoft set i.e. complex fuzzy hypersoft set, complex intuitionistic fuzzy hypersoft set and complex neutrosophic hypersoft set, are conceptualized along with their some fundamentals and theoretic operations. This is novel and more generalized work as compared to existing related literature for getting more precise results. Moreover, a comparative discussion is presented on particular cases of such hybrids.

The pattern of rest of the paper is: section 2 reviews the notions of soft sets, complex fuzzy set and relevant definitions used in the proposed work. Section 3, presents complex fuzzy hypersoft set and some of its fundamentals. Section 4, presents complex intuitionistic fuzzy hypersoft set and some of its fundamentals. Section 5, presents complex neutrosophic hypersoft set and some of its fundamentals and then concludes the paper.

2. Preliminaries

Here some existing fundamental concepts regarding fuzzy set, fuzzy soft set and fuzzy hypersoft set are presented along with their structures with complex fuzzy set from literature.

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Throughout the paper, \mathbb{U} , $P(\mathbb{U})$, $F(\mathbb{U})$, $C(\mathbb{U})$, $C_{Int}(\mathbb{U})$, $C_{Neu}(\mathbb{U})$, \coprod and \prod will present universe of discourse, power set of \mathbb{U} , collection of fuzzy sets, collection of complex fuzzy sets, collection of complex intuitionistic fuzzy sets, collection of complex neutrosophic sets, union and intersection respectively.

Definition 2.1. [1]

Suppose a universal set \mathbb{U} and a *fuzzy set* $X \subseteq \mathbb{U}$. The set X will be written as $X = \{(x, \alpha_X(x)) | x \in \mathbb{U}\}$ such that

$$\alpha_X : \mathbb{U} \rightarrow [0, 1]$$

where $\alpha_X(x)$ describes the membership percentage of $x \in X$.

Definition 2.2. [3]

A *complex fuzzy set* \mathbb{C}_f is of the form

$$\mathbb{C}_f = \{(\epsilon, \mu_{\mathbb{C}_f}(\epsilon)) : \epsilon \in \mathbb{U}\} = \{(\epsilon, r_{\mathbb{C}_f}(\epsilon)e^{i\omega_{\mathbb{C}_f}(\epsilon)}) : \epsilon \in \mathbb{U}\}.$$

where $\mu_{\mathbb{C}_f}(\epsilon)$ is a membership function of \mathbb{C}_f with $r_{\mathbb{C}_f}(\epsilon) \in [0, 1]$ and $\omega_{\mathbb{C}_f}(\epsilon) \in (0, 2\pi]$ as amplitude and phase terms respectively and $i = \sqrt{-1}$.

Zhang et al. [22] and Buckley [23]- [26] presented fuzzy complex number in different way. However, according to [3], [4], both amplitude and phase terms are captured by fuzzy sets.

Definition 2.3. [2]

A *soft set* \mathfrak{S} over \mathbb{U} , is defined as

$$\mathfrak{S} = \{(\epsilon, f_{\mathfrak{S}}(\epsilon)) : \epsilon \in E_1\}$$

where $f_{\mathfrak{S}} : E_1 \rightarrow P(\mathbb{U})$. and $E_1 \subseteq E$ (set of parameters).

Definition 2.4. [6]

A *fuzzy soft set* (FS-set) Γ_{E_1} on \mathbb{U} , is defined as

$$\Gamma_{E_1} = \{(\epsilon, \gamma_{E_1}(\epsilon)) : \epsilon \in E_1, \gamma_{E_1}(\epsilon) \in F(\mathbb{U})\}$$

where $\gamma_{E_1} : E_1 \rightarrow F(\mathbb{U})$ such that $\gamma_{E_1}(\epsilon) = \emptyset$ if $\epsilon \notin E_1$, and for all $\epsilon \in E_1$,

$$\gamma_{E_1}(\epsilon) = \left\{ \mu_{\gamma_{E_1}(\epsilon)}(v)/v : v \in \mathbb{U}, \mu_{\gamma_{E_1}(\epsilon)}(v) \in [0, 1] \right\}$$

is a fuzzy set over \mathbb{U} . Also γ_{E_1} is the approximate function of Γ_{E_1} and the value $\gamma_A(x)$ is a fuzzy set called ϵ -element of FS-set. Note that if $\gamma_{E_1}(\epsilon) = \emptyset$, then $(\epsilon, \gamma_{E_1}(\epsilon)) \notin \Gamma_{E_1}$.

Definition 2.5. [7]

A *complex fuzzy soft set* (CFS-set) χ_{E_1} over \mathbb{U} , is defined as

$$\chi_{E_1} = \{(\epsilon, \psi_{E_1}(\epsilon)) : \epsilon \in E_1, \psi_{E_1}(\epsilon) \in C(\mathbb{U})\}.$$

where $\psi_{E_1} : E_1 \rightarrow C(\mathbb{U})$ such that $\psi_{E_1}(\epsilon) = \emptyset$ if $\epsilon \notin E_1$ and it is complex fuzzy approximate function of CFS-set χ_{E_1} and its value $\psi_{E_1}(\epsilon)$ is called ϵ -member of CFS-set χ_{E_1} for all $\epsilon \in E_1$. Operations of CFS-sets and CF-sets were defined in [7] and [22] respectively.

Definition 2.6. [27] Let $A = \{(x; \mu_A(x)) : x \in \mathbb{U}\}$ and $B = \{(x; \mu_B(x)) : x \in \mathbb{U}\}$ be two complex fuzzy subsets of \mathbb{U} , with membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$, respectively. Then

- A complex fuzzy subset A is said to be a *homogeneous complex fuzzy set* if for all $x, y \in \mathbb{U}$, $r_A(x) \leq r_A(y)$ if and only if $\omega_A(x) \leq \omega_A(y)$
- A complex fuzzy subset A is said to be *homogeneous* with B, if for all $x, y \in \mathbb{U}$, $r_A(x) \leq r_B(y)$ if and only if $\omega_A(x) \leq \omega_B(y)$

Definition 2.7. [10] Let E be a set of attributes with $A \subseteq E$ and $\Psi(a)$ be a CIF-set over \mathbb{U} . Then, *complex intuitionistic fuzzy soft set* (CIFS-set) $\xi_A = (\Psi, A)$ over \mathbb{U} is defined as

$$\xi_A = \{(a, \Psi(a)) : a \in A, \Psi(a) \in C_{Int}(\mathbb{U})\}$$

where

$$\Psi : A \rightarrow C_{Int}(\mathbb{U}), \quad \Psi(a) = \emptyset \text{ if } a \notin A.$$

is a CIF approximate function of ξ_A and $\Psi(a) = \langle \Psi^T(a), \Psi^F(a) \rangle$.

$\Psi^T(a) = p_T e^{i\theta_T}$, and $\Psi^F(a) = p_F e^{i\theta_F}$ are complex-valued membership function, and complex-valued non-membership function of ξ_A respectively and their sum all are lying within the unit circle in the complex plane such that $p_T, p_F \in [0, 1]$ with $0 \leq p_T + p_F \leq 1$ (or $0 \leq |p_T + p_F| \leq 1$) and $\theta_T, \theta_F \in (0, 2\pi]$. The value $\Psi(a)$ is called a -member of CIFS-set $\forall a \in A$.

Definition 2.8. [14]

Let E be a set of attributes with $A \subseteq E$ and $\Psi(a)$ be a CN-set over \mathbb{U} . Then, *complex neutrosophic soft set* (CNS-set) $\xi_A = (\Psi, A)$ over \mathbb{U} is defined as

$$\xi_A = \{(a, \Psi(a)) : a \in A, \Psi(a) \in C_{Neu}(\mathbb{U})\}$$

where

$$\Psi : A \rightarrow C_{Neu}(\mathbb{U}), \quad \Psi(a) = \emptyset \text{ if } a \notin A.$$

is a CN approximate function of ξ_A and $\Psi(a) = \langle \Psi^T(a), \Psi^I(a), \Psi^F(a) \rangle$.

$\Psi^T(a) = p_T e^{i\theta_T}$, $\Psi^I(a) = p_I e^{i\theta_I}$ and $\Psi^F(a) = p_F e^{i\theta_F}$ are complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsity membership function of ξ_A respectively and their sum all are lying within the unit circle in the complex plane such that $p_T, p_I, p_F \in [0, 1]$ with $0 \leq p_T + p_I + p_F \leq 3$ (or $0 \leq |p_T + p_I + p_F| \leq 3$) and $\theta_T, \theta_I, \theta_F \in (0, 2\pi]$. The value $\Psi(a)$ is called a -member of CNS-set $\forall a \in A$.

For more study about neutrosophic sets see ([28]- [42]).

Definition 2.9. [15]

The pair (H, G) is called a *hypersoft set* over \mathbb{U} , where G is the cartesian product of n disjoint sets $H_1, H_2, H_3, \dots, H_n$ having attribute values of n distinct attributes $h_1, h_2, h_3, \dots, h_n$ respectively and $H : G \rightarrow P(\mathbb{U})$.

Definition 2.10. [15]

A hypersoft set over a fuzzy universe of discourse is called *fuzzy hypersoft set*.

For more definitions and operations of hypersoft set, see ([15]- [20]).

3. Complex Hypersoft set(CH-Set) and Complex Fuzzy Hypersoft Set(CFH-Set)

In this section, first we define complex hypersoft set then complex fuzzy hypersoft set is conceptualized with its some fundamentals.

Definition 3.1. Let \mathbb{C} be the set of complex numbers and $P(\mathbb{C})$ be the collection of all non-empty bounded subsets of the set of complex numbers. Let $A_1, A_2, A_3, \dots, A_n$ are disjoint sets having attribute values of n distinct attributes $a_1, a_2, a_3, \dots, a_n$ respectively for $n \geq 1$, $A = A_1 \times A_2 \times A_3 \times \dots \times A_n$ then a mapping $\psi : A \rightarrow P(\mathbb{C})$ is called a complex hypersoft set. It is denoted by (ψ, A) .

Example 3.2. Let $\mathbb{C} = \{2 + 3i, 1 + 2i, 3 + 5i, 4 + 2i, 3 + i\}$ be the set of complex numbers and $E = \{A_1, A_2, A_3\}$ with $A_1 = \{a_{11}, a_{12}\}$, $A_2 = \{a_{21}, a_{22}\}$ and $A_3 = \{a_{31}, a_{32}\}$ are disjoint set having attribute values then

$$A = \left\{ \begin{array}{l} (a_{11}, a_{21}, a_{31}), (a_{11}, a_{21}, a_{32}), (a_{11}, a_{22}, a_{31}), (a_{11}, a_{22}, a_{32}), \\ (a_{21}, a_{21}, a_{31}), (a_{21}, a_{21}, a_{32}), (a_{21}, a_{22}, a_{31}), (a_{21}, a_{22}, a_{32}) \end{array} \right\}$$

$A = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, then (ψ, A) can be considered as a complex hypersoft set where

$$(\psi, A) = \left\{ \begin{array}{l} (x_1, \{2 + 3i, 1 + 2i\}), (x_2, \{2 + 3i, 1 + 2i, 3 + 5i\}), (x_3, \{4 + 2i, 1 + 2i, 3 + 5i\}), \\ (x_4, \{2 + 3i, 4 + 2i, 3 + i\}), (x_5, \{3 + i, 1 + 2i\}), (x_6, \{3 + i, 2 + 3i, 3 + 5i\}), \\ (x_7, \{2 + 3i, 3 + i\}), (x_8, \{4 + 2i, 3 + 5i\}), \end{array} \right\}$$

Definition 3.3. Let $A_1, A_2, A_3, \dots, A_n$ are disjoint sets having attribute values of n distinct attributes $a_1, a_2, a_3, \dots, a_n$ respectively for $n \geq 1$, $G = A_1 \times A_2 \times A_3 \times \dots \times A_n$ and $\psi(\underline{\epsilon})$ be a CF-set over \mathbb{U} for all $\underline{\epsilon} = (d_1, d_2, d_3, \dots, d_n) \in G$ such that $d_1 \in A_1, d_2 \in A_2, d_3 \in A_3, \dots, d_n \in A_n$. Then, *complex fuzzy hypersoft set* (CFH-set) χ_G over \mathbb{U} is defined as

$$\chi_G = \{(\underline{\epsilon}, \psi(\underline{\epsilon})) : \underline{\epsilon} \in G, \psi(\underline{\epsilon}) \in C(\mathbb{U})\}$$

where

$$\psi : G \rightarrow C(\mathbb{U}), \quad \psi(\underline{\epsilon}) = \emptyset \text{ if } \underline{\epsilon} \notin G.$$

is a CF-approximate function of χ_G and its value $\psi(\underline{e})$ is called \underline{e} -member of CFH-set $\forall \underline{e} \in G$.

Example 3.4. Suppose a Department Promotion Committee (DPC) wants to observe(evaluate) the characteristics of some teachers by some defined indicators for departmental promotion. For this purpose, consider a set of teachers as a universe of discourse $\mathbb{U} = \{t_1, t_2, t_3, t_4\}$. The attributes of the teachers under consideration are the set $E = \{A_1, A_2, A_3\}$, where

A_1 = Total experience in years = $\{3, < 10\} = \{e_{11}, e_{12}\}$

A_2 = Total no. of publications = $\{10, 10 <\} = \{e_{21}, e_{22}\}$

A_3 = Performance Evaluation Report (PER) remarks = $\{eligible, not\ eligible\} = \{e_{31}, e_{32}\}$

and

$$G = A_1 \times A_2 \times A_3 = \left\{ \begin{array}{l} (e_{11}, e_{21}, e_{31}), (e_{11}, e_{21}, e_{32}), (e_{11}, e_{22}, e_{31}), \\ (e_{11}, e_{22}, e_{32}), (e_{12}, e_{21}, e_{31}), (e_{12}, e_{21}, e_{32}), \\ (e_{12}, e_{22}, e_{31}), (e_{12}, e_{22}, e_{32}) \end{array} \right\} = \{e_1, e_2, e_3, \dots, e_8\}$$

Complex fuzzy set $\psi_G(e_1), \psi_G(e_2), \dots, \psi_G(e_8)$ are defined as,

$$\psi_G(e_1) = \left\{ \frac{0.4e^{i0.5\pi}}{t_1}, \frac{0.8e^{i0.6\pi}}{t_2}, \frac{0.8e^{i0.8\pi}}{t_3}, \frac{1.0e^{i0.75\pi}}{t_4} \right\},$$

$$\psi_G(e_2) = \left\{ \frac{0.6e^{i0.7\pi}}{t_1}, \frac{0.9e^{i0.9\pi}}{t_2}, \frac{0.7e^{i0.9\pi}}{t_3}, \frac{0.75e^{i0.95\pi}}{t_4} \right\},$$

$$\psi_G(e_3) = \left\{ \frac{0.5e^{i0.6\pi}}{t_1}, \frac{0.8e^{i0.9\pi}}{t_2}, \frac{0.6e^{i0.9\pi}}{t_3}, \frac{0.65e^{i0.95\pi}}{t_4} \right\},$$

$$\psi_G(e_4) = \left\{ \frac{0.3e^{i0.7\pi}}{t_1}, \frac{0.7e^{i0.9\pi}}{t_2}, \frac{0.5e^{i0.9\pi}}{t_3}, \frac{0.75e^{i0.65\pi}}{t_4} \right\},$$

$$\psi_G(e_5) = \left\{ \frac{0.2e^{i0.5\pi}}{t_1}, \frac{0.3e^{i0.8\pi}}{t_2}, \frac{0.8e^{i0.7\pi}}{t_3}, \frac{0.45e^{i0.65\pi}}{t_4} \right\},$$

$$\psi_G(e_6) = \left\{ \frac{0.5e^{i0.9\pi}}{t_1}, \frac{0.3e^{i0.9\pi}}{t_2}, \frac{0.7e^{i0.8\pi}}{t_3}, \frac{0.85e^{i0.95\pi}}{t_4} \right\},$$

$$\psi_G(e_7) = \left\{ \frac{0.6e^{i0.9\pi}}{t_1}, \frac{0.9e^{i0.6\pi}}{t_2}, \frac{0.5e^{i0.6\pi}}{t_3}, \frac{0.85e^{i0.75\pi}}{t_4} \right\},$$

and

$$\psi_G(e_8) = \left\{ \frac{0.8e^{i0.9\pi}}{t_1}, \frac{0.8e^{i0.8\pi}}{t_2}, \frac{0.6e^{i0.8\pi}}{t_3}, \frac{0.65e^{i0.85\pi}}{t_4} \right\}$$

then CFH-set χ_G is written by,

$$\chi_G = \left\{ \begin{array}{l} (e_1, \frac{0.4e^{i0.5\pi}}{t_1}, \frac{0.8e^{i0.6\pi}}{t_2}, \frac{0.8e^{i0.8\pi}}{t_3}, \frac{1.0e^{i0.75\pi}}{t_4}), (e_2, \frac{0.6e^{i0.7\pi}}{t_1}, \frac{0.9e^{i0.9\pi}}{t_2}, \frac{0.7e^{i0.9\pi}}{t_3}, \frac{0.75e^{i0.95\pi}}{t_4}), \\ (e_3, \frac{0.5e^{i0.6\pi}}{t_1}, \frac{0.8e^{i0.9\pi}}{t_2}, \frac{0.6e^{i0.9\pi}}{t_3}, \frac{0.65e^{i0.95\pi}}{t_4}), (e_4, \frac{0.3e^{i0.7\pi}}{t_1}, \frac{0.7e^{i0.9\pi}}{t_2}, \frac{0.5e^{i0.9\pi}}{t_3}, \frac{0.75e^{i0.65\pi}}{t_4}), \\ (e_5, \frac{0.2e^{i0.5\pi}}{t_1}, \frac{0.3e^{i0.8\pi}}{t_2}, \frac{0.8e^{i0.7\pi}}{t_3}, \frac{0.45e^{i0.65\pi}}{t_4}), (e_6, \frac{0.5e^{i0.9\pi}}{t_1}, \frac{0.3e^{i0.9\pi}}{t_2}, \frac{0.7e^{i0.8\pi}}{t_3}, \frac{0.85e^{i0.95\pi}}{t_4}), \\ (e_7, \frac{0.6e^{i0.9\pi}}{t_1}, \frac{0.9e^{i0.6\pi}}{t_2}, \frac{0.5e^{i0.6\pi}}{t_3}, \frac{0.85e^{i0.75\pi}}{t_4}), (e_8, \frac{0.8e^{i0.9\pi}}{t_1}, \frac{0.8e^{i0.8\pi}}{t_2}, \frac{0.6e^{i0.8\pi}}{t_3}, \frac{0.65e^{i0.85\pi}}{t_4}) \end{array} \right\}$$

Definition 3.5. Let $\chi_{G_1} = (\psi_1, G_1)$ and $\chi_{G_2} = (\psi_2, G_2)$ be two CFH-sets over the same \mathbb{U} .

The set $\chi_{G_1} = (\psi_1, G_1)$ is said to be the *subset* of $\chi_{G_2} = (\psi_2, G_2)$, if

- i. $G_1 \subseteq G_2$

- ii. $\forall \underline{x} \in G_1, \psi_1(\underline{x}) \subseteq \psi_2(\underline{x})$ i.e. $r_{G_1}(\underline{x}) \leq r_{G_2}(\underline{x})$ and $\omega_{G_1}(\underline{x}) \leq \omega_{G_2}(\underline{x})$, where $r_{G_1}(\underline{x})$ and $\omega_{G_1}(\underline{x})$ are amplitude and phase terms of $\psi_1(\underline{x})$, whereas $r_{G_2}(\underline{x})$ and $\omega_{G_2}(\underline{x})$ are amplitude and phase terms of $\psi_2(\underline{x})$.

Definition 3.6. Two CFH-sets $\chi_{G_1} = (\psi_1, G_1)$ and $\chi_{G_2} = (\psi_2, G_2)$ over the same \mathbb{U} , are said to be *equal* if

- i. $(\psi_1, G_1) \subseteq (\psi_2, G_2)$
- ii. $(\psi_2, G_2) \subseteq (\psi_1, G_1)$.

Definition 3.7. Let (ψ, G) be a CFH-set over \mathbb{U} . Then

- i. (ψ, G) is called a *null CFH-set*, denoted by $(\psi, G)_\Phi$ if $r_G(\underline{x}) = 0$ and $\omega_G(\underline{x}) = 0\pi$ for all $\underline{x} \in G$.
- ii. (ψ, G) is called a *absolute CFH-set*, denoted by $(\psi, G)_\Delta$ if $r_G(\underline{x}) = 1$ and $\omega_G(\underline{x}) = 2\pi$ for all $\underline{x} \in G$.

Definition 3.8. Let (ψ_1, G_1) and (ψ_2, G_2) are two CFH-sets over the same universe \mathbb{U} . Then

- i. A CFH-set (ψ_1, G_1) is called a *homogeneous CFH-set*, denoted by $(\psi_1, G_1)_{Hom}$ if and only if $\psi_1(\underline{x})$ is a homogeneous CF-set for all $\underline{x} \in G_1$.
- ii. A CFH-set (ψ_1, G_1) is called a *completely homogeneous CFH-set*, denoted by $(\psi_1, G_1)_{CHom}$ if and only if $\psi_1(\underline{x})$ is a homogeneous with $\psi_1(\underline{y})$ for all $\underline{x}, \underline{y} \in G_1$.
- iii. A CFH-set (ψ_1, G_1) is said to be a completely homogeneous CFH-set with (ψ_2, G_2) if and only if $\psi_1(\underline{x})$ is a homogeneous with $\psi_2(\underline{x})$ for all $\underline{x} \in G_1 \coprod G_2$.

3.1. Set Theoretic Operations and Laws on CFH-Sets

Here some basic set theoretic operations (i.e. complement, union and intersection) and laws (commutative laws, associative laws etc.) are discussed on CFH-sets.

Definition 3.9. The *complement* of CFH-set (ψ, G) , denoted by $(\psi, G)^c$ is defined as

$$(\psi, G)^c = \{(\underline{x}, \psi^c(\underline{x})) : \underline{x} \in G, \psi^c(\underline{x}) \in C(\mathbb{U})\}$$

such that the amplitude and phase terms of the membership function $\psi^c(\underline{x})$ are given by $r_G^c(\underline{x}) = 1 - r_G(\underline{x})$ and $\omega_G^c(\underline{x}) = 2\pi - \omega_G(\underline{x})$ respectively.

Proposition 3.10. Let (ψ, G) be a CFH-set over \mathbb{U} . Then $((\psi, G)^c)^c = (\psi, G)$.

Proof. Since $\psi(\underline{x}) \in C(\mathbb{U})$, therefore (ψ, G) can be written in terms of its amplitude and phase terms as

$$(\psi, G) = \left\{ \left(\underline{x}, r_G(\underline{x})e^{i\omega_G(\underline{x})} \right) : \underline{x} \in G \right\} \quad (1)$$

Now

$$\begin{aligned}
 \psi^c(\underline{x}) &= \left\{ \left(\underline{x}, r_G^c(\underline{x}) e^{i\omega_G^c(\underline{x})} \right) : \underline{x} \in G \right\} \\
 \psi^c(\underline{x}) &= \left\{ \left(\underline{x}, (1 - r_G(\underline{x})) e^{i(2\pi - \omega_G(\underline{x}))} \right) : \underline{x} \in G \right\} \\
 ((\psi, G)^c)^c &= \left\{ \left(\underline{x}, (1 - r_G(\underline{x}))^c e^{i(2\pi - \omega_G(\underline{x})^c)} \right) : \underline{x} \in G \right\} \\
 ((\psi, G)^c)^c &= \left\{ \left(\underline{x}, (1 - (1 - r_G(\underline{x}))) e^{i(2\pi - (2\pi - \omega_G(\underline{x})))} \right) : \underline{x} \in G \right\} \\
 ((\psi, G)^c)^c &= \left\{ \left(\underline{x}, r_G(\underline{x}) e^{i\omega_G(\underline{x})} \right) : \underline{x} \in G \right\}
 \end{aligned} \tag{2}$$

from equations (1) and (2), we have $((\psi, G)^c)^c = (\psi, G)$. \square

Proposition 3.11. *Let (ψ, G) be a CFH-set over \mathbb{U} . Then*

- i. $((\psi, G)_\Phi)^c = (\psi, G)_\Delta$
- ii. $((\psi, G)_\Delta)^c = (\psi, G)_\Phi$

Definition 3.12. The *intersection* of two CFH-sets (ψ_1, G_1) and (ψ_2, G_2) over the same universe \mathbb{U} , denoted by $(\psi_1, G_1) \coprod (\psi_2, G_2)$, is the CFH-set (ψ_3, G_3) , where $G_3 = G_1 \coprod G_2$, and $\psi_3(\underline{x}) = \psi_1(\underline{x}) \coprod \psi_2(\underline{x})$ for all $\underline{x} \in G_3$.

Definition 3.13. The *difference* between two CFH-sets (ψ_1, G_1) and (ψ_2, G_2) is defined as

$$(\psi_1, G_1) \setminus (\psi_2, G_2) = (\psi_1, G_1) \coprod (\psi_2, G_2)^c$$

Definition 3.14. The *union* of two CFH-sets (ψ_1, G_1) and (ψ_2, G_2) over the same universe \mathbb{U} , denoted by $(\psi_1, G_1) \coprod (\psi_2, G_2)$, is the CFH-set (ψ_3, G_3) , where $G_3 = G_1 \coprod G_2$, and for all $\underline{x} \in G_3$,

$$\psi_3(\underline{x}) = \begin{cases} \psi_1(\underline{x}) & , \text{if } \underline{x} \in G_1 \setminus G_2 \\ \psi_2(\underline{x}) & , \text{if } \underline{x} \in G_2 \setminus G_1 \\ \psi_1(\underline{x}) \coprod \psi_2(\underline{x}) & , \text{if } \underline{x} \in G_1 \coprod G_2 \end{cases}$$

Proposition 3.15. *Let (ψ, G) be a CFH-set over \mathbb{U} . Then the following results hold true:*

- i. $(\psi, G) \coprod (\psi, G)_\Phi = (\psi, G)$
- ii. $(\psi, G) \coprod (\psi, G)_\Delta = (\psi, G)_\Delta$
- iii. $(\psi, G) \prod (\psi, G)_\Phi = (\psi, G)_\Phi$
- iv. $(\psi, G) \prod (\psi, G)_\Delta = (\psi, G)$
- v. $(\psi, G)_\Phi \coprod (\psi, G)_\Delta = (\psi, G)_\Delta$
- vi. $(\psi, G)_\Phi \prod (\psi, G)_\Delta = (\psi, G)_\Phi$

Proposition 3.16. *Let (ψ_1, G_1) , (ψ_2, G_2) and (ψ_3, G_3) are three CFH-sets over the same universe \mathbb{U} . Then the following commutative and associative laws hold true:*

- i. $(\psi_1, G_1) \prod (\psi_2, G_2) = (\psi_2, G_2) \prod (\psi_1, G_1)$
- ii. $(\psi_1, G_1) \coprod (\psi_2, G_2) = (\psi_2, G_2) \coprod (\psi_1, G_1)$
- iii. $(\psi_1, G_1) \prod ((\psi_2, G_2) \prod (\psi_3, G_3)) = ((\psi_1, G_1) \prod (\psi_2, G_2)) \prod (\psi_3, G_3)$
- iv. $(\psi_1, G_1) \coprod ((\psi_2, G_2) \coprod (\psi_3, G_3)) = ((\psi_1, G_1) \coprod (\psi_2, G_2)) \coprod (\psi_3, G_3)$

Proposition 3.17. Let (ψ_1, G_1) and (ψ_2, G_2) are two CFH-sets over the same universe \mathbb{U} . Then the following De Morgans laws hold true:

- i. $((\psi_1, G_1) \prod (\psi_2, G_2))^c = (\psi_1, G_1)^c \coprod (\psi_2, G_2)^c$
- ii. $((\psi_1, G_1) \coprod (\psi_2, G_2))^c = (\psi_1, G_1)^c \prod (\psi_2, G_2)^c$

4. Complex Intuitionistic Fuzzy Hypersoft Set (CIFH-Set)

In this section, fundamental theory of CIFH-set is developed.

Definition 4.1. Let $B_1, B_2, B_3, \dots, B_n$ are disjoint sets having attribute values of n distinct attributes $b_1, b_2, b_3, \dots, b_n$ respectively for $n \geq 1$, $B = B_1 \times B_2 \times B_3 \times \dots \times B_n$ and $\xi(\underline{\nu})$ be a CIF-set over \mathbb{U} for all $\underline{\nu} = (s_1, s_2, s_3, \dots, s_n) \in B$ such that $s_1 \in B_1, s_2 \in B_2, s_3 \in B_3, \dots, s_n \in B_n$. Then, complex intuitionistic fuzzy hypersoft set (CIFH-set) $\Gamma_B = (\xi, B)$ over \mathbb{U} is defined as

$$\Gamma_B = \{(\underline{\nu}, \xi(\underline{\nu})) : \underline{\nu} \in B, \xi(\underline{\nu}) \in C_{Int}(\mathbb{U})\}$$

where

$$\xi : B \rightarrow C_{Int}(\mathbb{U}), \quad \xi(\underline{\nu}) = \emptyset \text{ if } \underline{\nu} \notin B.$$

is a CIF approximate function of Γ_B and $\xi(\underline{\nu}) = \langle \xi^T(\underline{\nu}), \xi^F(\underline{\nu}) \rangle$.

$\xi^T(\underline{\nu}) = \alpha_T e^{i\beta_T}$ and $\xi^F(\underline{\nu}) = \alpha_F e^{i\beta_F}$ are complex-valued grade of membership and nonmembership of Γ_B respectively and their sum all are lying within the unit circle in the complex plane such that $\alpha_T, \alpha_F \in [0, 1]$ with $0 \leq \alpha_T + \alpha_F \leq 1$ and $\beta_T, \beta_F \in (0, 2\pi]$. The value $\xi(\underline{\nu})$ is called $\underline{\nu}$ -member of CIFH-set $\forall \underline{\nu} \in B$.

Example 4.2. Considering example 3.4 with $B = \{e_1, e_2, e_3, \dots, e_8\}$, CIF-sets $\xi_B(e_1), \xi_B(e_2), \dots, \xi_B(e_8)$ are defined as,

$$\begin{aligned} \xi_B(e_1) &= \left\{ \frac{\langle 0.6, 0.2 \rangle e^{i\langle 0.5, 0.3 \rangle \pi}}{t_1}, \frac{\langle 0.8, 0.1 \rangle e^{i\langle 0.5, 0.3 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.4 \rangle e^{i\langle 0.7, 0.2 \rangle \pi}}{t_3}, \frac{\langle 0.3, 0.1 \rangle e^{i\langle 0.65, 0.35 \rangle \pi}}{t_4} \right\}, \\ \xi_B(e_2) &= \left\{ \frac{\langle 0.5, 0.2 \rangle e^{i\langle 0.6, 0.3 \rangle \pi}}{t_1}, \frac{\langle 0.8, 0.01 \rangle e^{i\langle 0.8, 0.02 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.2 \rangle e^{i\langle 0.8, 0.03 \rangle \pi}}{t_3}, \frac{\langle 0.65, 0.25 \rangle e^{i\langle 0.85, 0.05 \rangle \pi}}{t_4} \right\}, \\ \xi_B(e_3) &= \left\{ \frac{\langle 0.4, 0.3 \rangle e^{i\langle 0.5, 0.1 \rangle \pi}}{t_1}, \frac{\langle 0.7, 0.02 \rangle e^{i\langle 0.8, 0.03 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.1 \rangle e^{i\langle 0.9, 0.01 \rangle \pi}}{t_3}, \frac{\langle 0.55, 0.25 \rangle e^{i\langle 0.85, 0.05 \rangle \pi}}{t_4} \right\}, \\ \xi_B(e_4) &= \left\{ \frac{\langle 0.3, 0.1 \rangle e^{i\langle 0.6, 0.1 \rangle \pi}}{t_1}, \frac{\langle 0.6, 0.01 \rangle e^{i\langle 0.8, 0.09 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.05 \rangle e^{i\langle 0.2, 0.01 \rangle \pi}}{t_3}, \frac{\langle 0.45, 0.25 \rangle e^{i\langle 0.55, 0.15 \rangle \pi}}{t_4} \right\}, \end{aligned}$$

$$\begin{aligned}\xi_B(e_5) &= \left\{ \frac{\langle 0.3, 0.2 \rangle e^{i\langle 0.4, 0.3 \rangle \pi}}{t_1}, \frac{\langle 0.7, 0.1 \rangle e^{i\langle 0.7, 0.08 \rangle \pi}}{t_2}, \frac{\langle 0.7, 0.01 \rangle e^{i\langle 0.6, 0.1 \rangle \pi}}{t_3}, \frac{\langle 0.55, 0.05 \rangle e^{i\langle 0.45, 0.05 \rangle \pi}}{t_4} \right\}, \\ \xi_B(e_6) &= \left\{ \frac{\langle 0.4, 0.01 \rangle e^{i\langle 0.5, 0.1 \rangle \pi}}{t_1}, \frac{\langle 0.4, 0.1 \rangle e^{i\langle 0.8, 0.1 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.070 \rangle e^{i\langle 0.7, 0.01 \rangle \pi}}{t_3}, \frac{\langle 0.65, 0.05 \rangle e^{i\langle 0.85, 0.15 \rangle \pi}}{t_4} \right\}, \\ \xi_B(e_7) &= \left\{ \frac{\langle 0.5, 0.09 \rangle e^{i\langle 0.8, 0.09 \rangle \pi}}{t_1}, \frac{\langle 0.4, 0.09 \rangle e^{i\langle 0.5, 0.06 \rangle \pi}}{t_2}, \frac{\langle 0.4, 0.05 \rangle e^{i\langle 0.5, 0.06 \rangle \pi}}{t_3}, \frac{\langle 0.75, 0.15 \rangle e^{i\langle 0.65, 0.25 \rangle \pi}}{t_4} \right\}, \\ \text{and} \\ \xi_B(e_8) &= \left\{ \frac{\langle 0.7, 0.08 \rangle e^{i\langle 0.1, 0.09 \rangle \pi}}{t_1}, \frac{\langle 0.5, 0.08 \rangle e^{i\langle 0.7, 0.02 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.06 \rangle e^{i\langle 0.8, 0.03 \rangle \pi}}{t_3}, \frac{\langle 0.4, 0.05 \rangle e^{i\langle 0.75, 0.15 \rangle \pi}}{t_4} \right\}\end{aligned}$$

then CIFH-set Γ_B is written by,

$$\Gamma_B = \left\{ \begin{aligned} &\left(e_1, \frac{\langle 0.6, 0.2 \rangle e^{i\langle 0.5, 0.3 \rangle \pi}}{t_1}, \frac{\langle 0.8, 0.1 \rangle e^{i\langle 0.5, 0.3 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.4 \rangle e^{i\langle 0.7, 0.2 \rangle \pi}}{t_3}, \frac{\langle 0.3, 0.1 \rangle e^{i\langle 0.65, 0.35 \rangle \pi}}{t_4} \right), \\ &\left(e_2, \frac{\langle 0.5, 0.2 \rangle e^{i\langle 0.6, 0.3 \rangle \pi}}{t_1}, \frac{\langle 0.8, 0.01 \rangle e^{i\langle 0.8, 0.02 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.2 \rangle e^{i\langle 0.8, 0.03 \rangle \pi}}{t_3}, \frac{\langle 0.65, 0.25 \rangle e^{i\langle 0.85, 0.05 \rangle \pi}}{t_4} \right), \\ &\left(e_3, \frac{\langle 0.4, 0.3 \rangle e^{i\langle 0.5, 0.1 \rangle \pi}}{t_1}, \frac{\langle 0.7, 0.02 \rangle e^{i\langle 0.8, 0.03 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.1 \rangle e^{i\langle 0.9, 0.01 \rangle \pi}}{t_3}, \frac{\langle 0.55, 0.25 \rangle e^{i\langle 0.85, 0.05 \rangle \pi}}{t_4} \right), \\ &\left(e_4, \frac{\langle 0.3, 0.1 \rangle e^{i\langle 0.6, 0.1 \rangle \pi}}{t_1}, \frac{\langle 0.6, 0.01 \rangle e^{i\langle 0.8, 0.09 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.05 \rangle e^{i\langle 0.2, 0.01 \rangle \pi}}{t_3}, \frac{\langle 0.45, 0.25 \rangle e^{i\langle 0.55, 0.15 \rangle \pi}}{t_4} \right), \\ &\left(e_5, \frac{\langle 0.3, 0.2 \rangle e^{i\langle 0.4, 0.3 \rangle \pi}}{t_1}, \frac{\langle 0.7, 0.1 \rangle e^{i\langle 0.7, 0.08 \rangle \pi}}{t_2}, \frac{\langle 0.7, 0.01 \rangle e^{i\langle 0.6, 0.1 \rangle \pi}}{t_3}, \frac{\langle 0.55, 0.05 \rangle e^{i\langle 0.45, 0.05 \rangle \pi}}{t_4} \right), \\ &\left(e_6, \frac{\langle 0.4, 0.01 \rangle e^{i\langle 0.5, 0.1 \rangle \pi}}{t_1}, \frac{\langle 0.4, 0.1 \rangle e^{i\langle 0.8, 0.1 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.070 \rangle e^{i\langle 0.7, 0.01 \rangle \pi}}{t_3}, \frac{\langle 0.65, 0.05 \rangle e^{i\langle 0.85, 0.15 \rangle \pi}}{t_4} \right), \\ &\left(e_7, \frac{\langle 0.5, 0.09 \rangle e^{i\langle 0.8, 0.09 \rangle \pi}}{t_1}, \frac{\langle 0.4, 0.09 \rangle e^{i\langle 0.5, 0.06 \rangle \pi}}{t_2}, \frac{\langle 0.4, 0.05 \rangle e^{i\langle 0.5, 0.06 \rangle \pi}}{t_3}, \frac{\langle 0.75, 0.15 \rangle e^{i\langle 0.65, 0.25 \rangle \pi}}{t_4} \right), \\ &\left(e_8, \frac{\langle 0.7, 0.08 \rangle e^{i\langle 0.1, 0.09 \rangle \pi}}{t_1}, \frac{\langle 0.5, 0.08 \rangle e^{i\langle 0.7, 0.02 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.06 \rangle e^{i\langle 0.8, 0.03 \rangle \pi}}{t_3}, \frac{\langle 0.4, 0.05 \rangle e^{i\langle 0.75, 0.15 \rangle \pi}}{t_4} \right) \end{aligned} \right\},$$

Definition 4.3. Let $\Gamma_{B_1} = (\xi_1, B_1)$ and $\Gamma_{B_2} = (\xi_2, B_2)$ be two CIFH-sets over the same \mathbb{U} .

The set $\Gamma_{B_1} = (\xi_1, B_1)$ is said to be the *subset* of $\Gamma_{B_2} = (\xi_2, B_2)$, if

- i. $B_1 \subseteq B_2$
- ii. $\forall p \in B_1, \xi_1(p) \subseteq \xi_2(p)$ implies $\xi_1^T(p) \subseteq \xi_2^T(p), \xi_1^F(p) \subseteq \xi_2^F(p)$ i.e.
 $\alpha_{TB_1}(p) \leq \alpha_{TB_2}(p), \alpha_{FB_1}(p) \leq \alpha_{FB_2}(p), \beta_{TB_1}(p) \leq \beta_{TB_2}(p)$ and $\beta_{FB_1}(p) \leq \beta_{FB_2}(p)$,

where

- $\alpha_{TB_1}(p)$ and $\beta_{TB_1}(p)$ are amplitude and phase terms of $\xi_1^T(p)$,
- $\alpha_{FB_1}(p)$ and $\beta_{FB_1}(p)$ are amplitude and phase terms of $\xi_1^F(p)$,
- $\alpha_{TB_2}(p)$ and $\beta_{TB_2}(p)$ are amplitude and phase terms of $\xi_2^T(p)$, and
- $\alpha_{FB_2}(p)$ and $\beta_{FB_2}(p)$ are amplitude and phase terms of $\xi_2^F(p)$.

Definition 4.4. Two CIFH-set $\Gamma_{B_1} = (\xi_1, B_1)$ and $\Gamma_{B_2} = (\xi_2, B_2)$ over the same \mathbb{U} , are said to be *equal* if

- i. $(\xi_1, B_1) \subseteq (\xi_2, B_2)$
- ii. $(\xi_2, B_2) \subseteq (\xi_1, B_1)$.

Definition 4.5. Let (ξ, B) be a CIFH-set over \mathbb{U} . Then

- i. (ξ, B) is called a *null CIFH-set*, denoted by $(\xi, B)_\Phi$ if $\alpha_{TB}(\underline{p}) = \alpha_{FB}(\underline{p}) = 0$ and $\beta_{TB}(\underline{p}) = \beta_{FB}(\underline{p}) = 0\pi$ for all $\underline{p} \in B$.
- ii. (ξ, B) is called a *absolute CIFH-set*, denoted by $(\xi, B)_\Delta$ if $\alpha_{TB}(\underline{p}) = \alpha_{FB}(\underline{p}) = 1$ and $\beta_{TB}(\underline{p}) = \beta_{FB}(\underline{p}) = 2\pi$ for all $\underline{p} \in B$.

Definition 4.6. Let (ξ_1, B_1) and (ξ_2, B_2) are two CIFH-sets over the same universe \mathbb{U} . Then

- i. A CIFH-set (ξ_1, B_1) is called a *homogeneous CIFH-set*, denoted by $(\xi_1, B_1)_{Hom}$ if and only if $\xi_1(\underline{p})$ is a homogeneous CIF-set for all $\underline{p} \in B_1$.
- ii. A CIFH-set (ξ_1, B_1) is called a *completely homogeneous CIFH-set*, denoted by $(\xi_1, B_1)_{CHom}$ if and only if $\xi_1(\underline{p})$ is a homogeneous with $\xi_1(\underline{q})$ for all $\underline{p}, \underline{q} \in B_1$.
- iii. A CIFH-set (ξ_1, B_1) is said to be a completely homogeneous CIFH-set with (ξ_2, B_2) if and only if $\xi_1(\underline{p})$ is a homogeneous with $\xi_2(\underline{p})$ for all $\underline{p} \in B_1 \coprod B_2$.

4.1. Set Theoretic Operations and Laws on CIFH-set

Here some basic set theoretic operations (i.e. complement, union and intersection) and laws (commutative laws, associative laws etc.) are discussed on CFH-set.

Definition 4.7. The *complement* of CIFH-set (ξ, B) , denoted by $(\xi, B)^c$ is defined as

$$(\xi, B)^c = \{(\underline{p}, (\xi(\underline{p}))^c) : \underline{p} \in B, (\xi(\underline{p}))^c \in C_{Int}(\mathbb{U})\}$$

such that the amplitude and phase terms of the membership function $(\xi(\underline{p}))^c$ are given by

$$(\alpha_{TB}(\underline{p}))^c = 1 - \alpha_{TB}(\underline{p})$$

$$(\alpha_{FB}(\underline{p}))^c = 1 - \alpha_{FB}(\underline{p})$$

and

$$(\beta_{TB}(\underline{p}))^c = 2\pi - \beta_{TB}(\underline{p}),$$

$$(\beta_{FB}(\underline{p}))^c = 2\pi - \beta_{FB}(\underline{p}) \text{ respectively.}$$

Proposition 4.8. Let (ξ, B) be a CIFH-set over \mathbb{U} . Then $((\xi, B)^c)^c = (\xi, B)$.

Proof. Since $\xi(\underline{p}) \in C_{Int}(\mathbb{U})$, therefore (ξ, B) can be written in terms of its amplitude and phase terms as

$$(\xi, B) = \left\{ \left(\underline{p}, \left(\alpha_{TB}(\underline{p})e^{i\beta_{TB}(\underline{p})}, \alpha_{FB}(\underline{p})e^{i\beta_{FB}(\underline{p})} \right) \right) : \underline{p} \in B \right\} \quad (3)$$

Now

$$\begin{aligned} (\xi, B)^c(\underline{p}) &= \left\{ \left(\underline{p}, \left((\alpha_{TB}(\underline{p}))^c e^{i(\beta_{TB}(\underline{p}))^c}, (\alpha_{FB}(\underline{p}))^c e^{i(\beta_{FB}(\underline{p}))^c} \right) \right) : \underline{p} \in B \right\} \\ (\xi, B)^c(\underline{p}) &= \left\{ \left(\underline{p}, \left((1 - \alpha_{TB}(\underline{p}))e^{i(2\pi - \beta_{TB}(\underline{p}))}, (1 - \alpha_{FB}(\underline{p}))e^{i(2\pi - \beta_{FB}(\underline{p}))} \right) \right) : \underline{p} \in B \right\} \\ ((\xi, B)^c)^c &= \left\{ \left(\underline{p}, \left((1 - \alpha_{TB}(\underline{p}))^c e^{i(2\pi - \beta_{TB}(\underline{p}))^c}, (1 - \alpha_{FB}(\underline{p}))^c e^{i(2\pi - \beta_{FB}(\underline{p}))^c} \right) \right) : \underline{p} \in B \right\} \\ ((\xi, B)^c)^c &= \left\{ \left(\underline{p}, \left((1 - (1 - \alpha_{TB}(\underline{p})))e^{i(2\pi - (2\pi - \beta_{TB}(\underline{p})))}, (1 - (1 - \alpha_{FB}(\underline{p})))e^{i(2\pi - (2\pi - \beta_{FB}(\underline{p})))} \right) \right) : \underline{p} \in B \right\} \\ ((\xi, B)^c)^c &= \left\{ \left(\underline{p}, \left(\alpha_{TB}(\underline{p})e^{i\beta_{TB}(\underline{p})}, \alpha_{FB}(\underline{p})e^{i\beta_{FB}(\underline{p})} \right) \right) : \underline{p} \in B \right\} \end{aligned} \quad (4)$$

from equations (3) and (4), we have $((\xi, B)^c)^c = (\xi, B)$. \square

Proposition 4.9. Let (ξ, B) be a CIFH-set over \mathbb{U} . Then

- i. $((\xi, B)_\Phi)^c = (\xi, B)_\Delta$
- ii. $((\xi, B)_\Delta)^c = (\xi, B)_\Phi$

Definition 4.10. The *intersection* of two CIFH-set (ξ_1, B_1) and (ξ_2, B_2) over the same universe \mathbb{U} , denoted by $(\xi_1, B_1) \prod (\xi_2, B_2)$, is the CIFH-set (ξ_3, B_3) , where $B_3 = B_1 \prod B_2$, and for all $\underline{p} \in B_3$,

$$\xi_3^T(\underline{p}) = \begin{cases} \alpha_{TB_1}(\underline{p})e^{i\beta_{TB_1}(\underline{p})} & , \text{if } \underline{p} \in B_1 \setminus B_2 \\ \alpha_{TB_2}(\underline{p})e^{i\beta_{TB_2}(\underline{p})} & , \text{if } \underline{p} \in B_2 \setminus B_1 \\ \min(\alpha_{TB_1}(\underline{p}), \alpha_{TB_2}(\underline{p}))e^{i\min(\beta_{TB_1}(\underline{p}), \beta_{TB_2}(\underline{p}))} & , \text{if } \underline{p} \in B_1 \prod B_2 \end{cases}$$

and

$$\xi_3^F(\underline{p}) = \begin{cases} \alpha_{FB_1}(\underline{p})e^{i\beta_{FB_1}(\underline{p})} & , \text{if } \underline{p} \in B_1 \setminus B_2 \\ \alpha_{FB_2}(\underline{p})e^{i\beta_{FB_2}(\underline{p})} & , \text{if } \underline{p} \in B_2 \setminus B_1 \\ \min(\alpha_{FB_1}(\underline{p}), \alpha_{FB_2}(\underline{p}))e^{i\min(\beta_{FB_1}(\underline{p}), \beta_{FB_2}(\underline{p}))} & , \text{if } \underline{p} \in B_1 \prod B_2 \end{cases}$$

Definition 4.11. The *difference* between two CFH-set (ξ_1, B_1) and (ξ_2, B_2) is defined as

$$(\xi_1, B_1) \setminus (\xi_2, B_2) = (\xi_1, B_1) \prod (\xi_2, B_2)^c$$

Definition 4.12. The *union* of two CFH-set (ξ_1, B_1) and (ξ_2, B_2) over the same universe \mathbb{U} , denoted by $(\xi_1, B_1) \coprod (\xi_2, B_2)$, is the CFH-set (ξ_3, B_3) , where $B_3 = B_1 \coprod B_2$, and for all $\underline{p} \in B_3$,

$$\xi_3^T(\underline{p}) = \begin{cases} \alpha_{TB_1}(\underline{p})e^{i\beta_{TB_1}(\underline{p})} & , \text{if } \underline{p} \in B_1 \setminus B_2 \\ \alpha_{TB_2}(\underline{p})e^{i\beta_{TB_2}(\underline{p})} & , \text{if } \underline{p} \in B_2 \setminus B_1 \\ \max(\alpha_{TB_1}(\underline{p}), \alpha_{TB_2}(\underline{p}))e^{i\max(\beta_{TB_1}(\underline{p}), \beta_{TB_2}(\underline{p}))} & , \text{if } \underline{p} \in B_1 \prod B_2 \end{cases}$$

and

$$\xi^F_3(\underline{p}) = \begin{cases} \alpha_{FB_1}(\underline{p})e^{i\beta_{FB_1}(\underline{p})} & , \text{if } \underline{p} \in B_1 \setminus B_2 \\ \alpha_{FB_2}(\underline{p})e^{i\beta_{FB_2}(\underline{p})} & , \text{if } \underline{p} \in B_2 \setminus B_1 \\ \max(\alpha_{FB_1}(\underline{p}), \alpha_{FB_2}(\underline{p}))e^{i\max(\beta_{FB_1}(\underline{p}), \beta_{FB_2}(\underline{p}))} & , \text{if } \underline{p} \in B_1 \cap B_2 \end{cases}$$

Proposition 4.13. Let (ξ, B) be a CIFH-set over \mathbb{U} . Then the following results hold true:

- i. $(\xi, B) \coprod (\xi, B)_\Phi = (\xi, B)$
- ii. $(\xi, B) \coprod (\xi, B)_\Delta = (\xi, B)_\Delta$
- iii. $(\xi, B) \prod (\xi, B)_\Phi = (\xi, B)_\Phi$
- iv. $(\xi, B) \prod (\xi, B)_\Delta = (\xi, B)$
- v. $(\xi, B)_\Phi \coprod (\xi, B)_\Delta = (\xi, B)_\Delta$
- vi. $(\xi, B)_\Phi \prod (\xi, B)_\Delta = (\xi, B)_\Phi$

Proposition 4.14. Let (ξ_1, B_1) , (ξ_2, B_2) and (ξ_3, B_3) are three CIFH-sets over the same universe \mathbb{U} . Then the following commutative and associative laws hold true:

- i. $(\xi_1, B_1) \prod (\xi_2, B_2) = (\xi_2, B_2) \prod (\xi_1, B_1)$
- ii. $(\xi_1, B_1) \coprod (\xi_2, B_2) = (\xi_2, B_2) \coprod (\xi_1, B_1)$
- iii. $(\xi_1, B_1) \prod ((\xi_2, B_2) \prod (\xi_3, B_3)) = ((\xi_1, B_1) \prod (\xi_2, B_2)) \prod (\xi_3, B_3)$
- iv. $(\xi_1, B_1) \coprod ((\xi_2, B_2) \coprod (\xi_3, B_3)) = ((\xi_1, B_1) \coprod (\xi_2, B_2)) \coprod (\xi_3, B_3)$

Proposition 4.15. Let (ξ_1, B_1) and (ξ_2, B_2) are two CIFH-sets over the same universe \mathbb{U} . Then the following De Morgan's laws hold true:

- i. $((\xi_1, B_1) \prod (\xi_2, B_2))^c = (\xi_1, B_1)^c \coprod (\xi_2, B_2)^c$
- ii. $((\xi_1, B_1) \coprod (\xi_2, B_2))^c = (\xi_1, B_1)^c \prod (\xi_2, B_2)^c$

5. Complex Neutrosophic Hypersoft Set (CNH-Set)

In this section, CNH-set and its some fundamentals are developed.

Definition 5.1. Let $N_1, N_2, N_3, \dots, N_n$ are disjoint sets having attribute values of n distinct attributes $n_1, n_2, n_3, \dots, n_n$ respectively for $n \geq 1$, $N = N_1 \times N_2 \times N_3 \times \dots \times N_n$ and $\zeta(\underline{\lambda})$ be a CN-set over \mathbb{U} for all $\underline{\lambda} = (a_1, a_2, a_3, \dots, a_n) \in N$ such that $a_1 \in N_1, a_2 \in N_2, a_3 \in N_3, \dots, a_n \in N_n$. Then, complex neutrosophic hypersoft set (CNH-set) $\Theta_N = (\zeta, N)$ over \mathbb{U} is defined as

$$\Theta_N = \{(\underline{\lambda}, \zeta(\underline{\lambda})) : \underline{\lambda} \in N, \zeta(\underline{\lambda}) \in C_{Neu}(\mathbb{U})\}$$

where

$$\zeta : N \rightarrow C_{Neu}(\mathbb{U}), \quad \zeta(\underline{\lambda}) = \emptyset \text{ if } \underline{\lambda} \notin N.$$

is a CN approximate function of Θ_N and $\zeta(\underline{\lambda}) = \langle \zeta^T(\underline{\lambda}), \zeta^I(\underline{\lambda}), \zeta^F(\underline{\lambda}) \rangle$.

$\zeta^T(\underline{\lambda}) = \delta_T e^{i\eta_T}$, $\zeta^I(\underline{\lambda}) = \delta_I e^{i\eta_I}$ and $\zeta^F(\underline{\lambda}) = \delta_F e^{i\eta_F}$ are complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsity

membership function of Θ_N respectively and their sum all are lying within the unit circle in the complex plane such that $\delta_T, \delta_I, \delta_F \in [0, 1]$ with $-0 \leq \delta_T + \delta_I + \delta_F \leq 3^+$ (or $0 \leq |\delta_T + \delta_I + \delta_F| \leq 3$) and $\eta_T, \eta_I, \eta_F \in (0, 2\pi]$. The value $\zeta(\underline{\lambda})$ is called $\underline{\lambda}$ -member of CNH-set $\forall \underline{\lambda} \in N$.

Example 5.2. Considering example 3.4 with $N = \{e_1, e_2, e_3, \dots, e_8\}$, CNF-sets $\zeta_N(e_1), \zeta_N(e_2), \dots, \zeta_N(e_8)$ are defined as,

$$\begin{aligned}\zeta_N(e_1) &= \left\{ \frac{\langle 0.6, 0.1, 0.2 \rangle e^{i\langle 0.5, 0.2, 0.3 \rangle \pi}}{t_1}, \frac{\langle 0.8, 0.3, 0.1 \rangle e^{i\langle 0.5, 0.4, 0.3 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.5, 0.4 \rangle e^{i\langle 0.7, 0.6, 0.2 \rangle \pi}}{t_3}, \frac{\langle 0.3, 0.7, 0.1 \rangle e^{i\langle 0.65, 0.55, 0.35 \rangle \pi}}{t_4} \right\}, \\ \zeta_N(e_2) &= \left\{ \frac{\langle 0.5, 0.2, 0.1 \rangle e^{i\langle 0.6, 0.3, 0.2 \rangle \pi}}{t_1}, \frac{\langle 0.8, 0.01, 0.2 \rangle e^{i\langle 0.8, 0.02, 0.3 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.2, 0.2 \rangle e^{i\langle 0.8, 0.03, 0.4 \rangle \pi}}{t_3}, \frac{\langle 0.65, 0.25, 0.5 \rangle e^{i\langle 0.85, 0.05, 0.5 \rangle \pi}}{t_4} \right\}, \\ \zeta_N(e_3) &= \left\{ \frac{\langle 0.4, 0.3, 0.3 \rangle e^{i\langle 0.5, 0.1, 0.8 \rangle \pi}}{t_1}, \frac{\langle 0.7, 0.02, 0.3 \rangle e^{i\langle 0.8, 0.03, 0.7 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.1, 0.9 \rangle e^{i\langle 0.9, 0.01, 0.7 \rangle \pi}}{t_3}, \frac{\langle 0.55, 0.25, 0.1 \rangle e^{i\langle 0.85, 0.05, 0.4 \rangle \pi}}{t_4} \right\}, \\ \zeta_N(e_4) &= \left\{ \frac{\langle 0.3, 0.1, 0.9 \rangle e^{i\langle 0.6, 0.1, 0.5 \rangle \pi}}{t_1}, \frac{\langle 0.6, 0.01, 0.4 \rangle e^{i\langle 0.8, 0.09, 0.5 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.05, 0.3 \rangle e^{i\langle 0.2, 0.01, 0.4 \rangle \pi}}{t_3}, \frac{\langle 0.45, 0.25, 0.01 \rangle e^{i\langle 0.55, 0.15, 0.3 \rangle \pi}}{t_4} \right\}, \\ \zeta_N(e_5) &= \left\{ \frac{\langle 0.3, 0.2, 0.1 \rangle e^{i\langle 0.4, 0.3, 0.4 \rangle \pi}}{t_1}, \frac{\langle 0.7, 0.1, 0.5 \rangle e^{i\langle 0.7, 0.08, 0.4 \rangle \pi}}{t_2}, \frac{\langle 0.7, 0.01, 0.4 \rangle e^{i\langle 0.6, 0.1, 0.5 \rangle \pi}}{t_3}, \frac{\langle 0.55, 0.05, 0.4 \rangle e^{i\langle 0.45, 0.05, 0.3 \rangle \pi}}{t_4} \right\}, \\ \zeta_N(e_6) &= \left\{ \frac{\langle 0.4, 0.01, 0.3 \rangle e^{i\langle 0.5, 0.1, 0.4 \rangle \pi}}{t_1}, \frac{\langle 0.4, 0.1, 0.3 \rangle e^{i\langle 0.8, 0.1, 0.3 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.070, 0.5 \rangle e^{i\langle 0.7, 0.01, 0.1 \rangle \pi}}{t_3}, \frac{\langle 0.65, 0.05, 0.3 \rangle e^{i\langle 0.85, 0.15, 0.4 \rangle \pi}}{t_4} \right\}, \\ \zeta_N(e_7) &= \left\{ \frac{\langle 0.5, 0.09, 0.3 \rangle e^{i\langle 0.8, 0.09, 0.5 \rangle \pi}}{t_1}, \frac{\langle 0.4, 0.09, 0.4 \rangle e^{i\langle 0.5, 0.06, 0.4 \rangle \pi}}{t_2}, \frac{\langle 0.4, 0.05, 0.1 \rangle e^{i\langle 0.5, 0.06, 0.5 \rangle \pi}}{t_3}, \frac{\langle 0.75, 0.15, 0.4 \rangle e^{i\langle 0.65, 0.25, 0.2 \rangle \pi}}{t_4} \right\},\end{aligned}$$

and

$$\zeta_N(e_8) = \left\{ \frac{\langle 0.7, 0.08, 0.3 \rangle e^{i\langle 0.1, 0.09, 0.01 \rangle \pi}}{t_1}, \frac{\langle 0.5, 0.08, 0.3 \rangle e^{i\langle 0.7, 0.02, 0.6 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.06, 0.5 \rangle e^{i\langle 0.8, 0.03, 0.3 \rangle \pi}}{t_3}, \frac{\langle 0.4, 0.05, 0.35 \rangle e^{i\langle 0.75, 0.15, 0.6 \rangle \pi}}{t_4} \right\}$$

then CNH-set Θ_N is written by,

$$\Theta_N = \left\{ \begin{aligned} &\left(e_1, \frac{\langle 0.6, 0.1, 0.2 \rangle e^{i\langle 0.5, 0.2, 0.3 \rangle \pi}}{t_1}, \frac{\langle 0.8, 0.3, 0.1 \rangle e^{i\langle 0.5, 0.4, 0.3 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.5, 0.4 \rangle e^{i\langle 0.7, 0.6, 0.2 \rangle \pi}}{t_3}, \frac{\langle 0.3, 0.7, 0.1 \rangle e^{i\langle 0.65, 0.55, 0.35 \rangle \pi}}{t_4} \right), \\ &\left(e_2, \frac{\langle 0.5, 0.2, 0.1 \rangle e^{i\langle 0.6, 0.3, 0.2 \rangle \pi}}{t_1}, \frac{\langle 0.8, 0.01, 0.2 \rangle e^{i\langle 0.8, 0.02, 0.3 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.2, 0.2 \rangle e^{i\langle 0.8, 0.03, 0.4 \rangle \pi}}{t_3}, \frac{\langle 0.65, 0.25, 0.5 \rangle e^{i\langle 0.85, 0.05, 0.5 \rangle \pi}}{t_4} \right), \\ &\left(e_3, \frac{\langle 0.4, 0.3, 0.3 \rangle e^{i\langle 0.5, 0.1, 0.8 \rangle \pi}}{t_1}, \frac{\langle 0.7, 0.02, 0.3 \rangle e^{i\langle 0.8, 0.03, 0.7 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.1, 0.9 \rangle e^{i\langle 0.9, 0.01, 0.7 \rangle \pi}}{t_3}, \frac{\langle 0.55, 0.25, 0.1 \rangle e^{i\langle 0.85, 0.05, 0.4 \rangle \pi}}{t_4} \right), \\ &\left(e_4, \frac{\langle 0.3, 0.1, 0.9 \rangle e^{i\langle 0.6, 0.1, 0.5 \rangle \pi}}{t_1}, \frac{\langle 0.6, 0.01, 0.4 \rangle e^{i\langle 0.8, 0.09, 0.5 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.05, 0.3 \rangle e^{i\langle 0.2, 0.01, 0.4 \rangle \pi}}{t_3}, \frac{\langle 0.45, 0.25, 0.01 \rangle e^{i\langle 0.55, 0.15, 0.3 \rangle \pi}}{t_4} \right), \\ &\left(e_5, \frac{\langle 0.3, 0.2, 0.1 \rangle e^{i\langle 0.4, 0.3, 0.4 \rangle \pi}}{t_1}, \frac{\langle 0.7, 0.1, 0.5 \rangle e^{i\langle 0.7, 0.08, 0.4 \rangle \pi}}{t_2}, \frac{\langle 0.7, 0.01, 0.4 \rangle e^{i\langle 0.6, 0.1, 0.5 \rangle \pi}}{t_3}, \frac{\langle 0.55, 0.05, 0.4 \rangle e^{i\langle 0.45, 0.05, 0.3 \rangle \pi}}{t_4} \right), \\ &\left(e_6, \frac{\langle 0.4, 0.01, 0.3 \rangle e^{i\langle 0.5, 0.1, 0.4 \rangle \pi}}{t_1}, \frac{\langle 0.4, 0.1, 0.3 \rangle e^{i\langle 0.8, 0.1, 0.3 \rangle \pi}}{t_2}, \frac{\langle 0.6, 0.070, 0.5 \rangle e^{i\langle 0.7, 0.01, 0.1 \rangle \pi}}{t_3}, \frac{\langle 0.65, 0.05, 0.3 \rangle e^{i\langle 0.85, 0.15, 0.4 \rangle \pi}}{t_4} \right), \\ &\left(e_7, \frac{\langle 0.5, 0.09, 0.3 \rangle e^{i\langle 0.8, 0.09, 0.5 \rangle \pi}}{t_1}, \frac{\langle 0.4, 0.09, 0.4 \rangle e^{i\langle 0.5, 0.06, 0.4 \rangle \pi}}{t_2}, \frac{\langle 0.4, 0.05, 0.1 \rangle e^{i\langle 0.5, 0.06, 0.5 \rangle \pi}}{t_3}, \frac{\langle 0.75, 0.15, 0.4 \rangle e^{i\langle 0.65, 0.25, 0.2 \rangle \pi}}{t_4} \right), \\ &\left(e_8, \frac{\langle 0.7, 0.08, 0.3 \rangle e^{i\langle 0.1, 0.09, 0.01 \rangle \pi}}{t_1}, \frac{\langle 0.5, 0.08, 0.3 \rangle e^{i\langle 0.7, 0.02, 0.6 \rangle \pi}}{t_2}, \frac{\langle 0.5, 0.06, 0.5 \rangle e^{i\langle 0.8, 0.03, 0.3 \rangle \pi}}{t_3}, \frac{\langle 0.4, 0.05, 0.35 \rangle e^{i\langle 0.75, 0.15, 0.6 \rangle \pi}}{t_4} \right) \end{aligned} \right\}$$

Definition 5.3. Let $\Theta_{N_1} = (\zeta_1, N_1)$ and $\Theta_{N_2} = (\zeta_2, N_2)$ be two CNH-sets over the same \mathbb{U} .

The set $\Theta_{N_1} = (\zeta_1, N_1)$ is said to be the *subset* of $\Theta_{N_2} = (\zeta_2, N_2)$, if

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- i. $N_1 \subseteq N_2$
- ii. $\forall \underline{u} \in N_1, \zeta_1(\underline{u}) \subseteq \zeta_2(\underline{u})$ implies $\zeta_1^T(\underline{u}) \subseteq \zeta_2^T(\underline{u}), \zeta_1^I(\underline{u}) \subseteq \zeta_2^I(\underline{u}), \zeta_1^F(\underline{u}) \subseteq \zeta_2^F(\underline{u})$ i.e.

$$\begin{aligned} \delta_{TN_1}(\underline{u}) &\leq \delta_{TN_2}(\underline{u}), \\ \delta_{IN_1}(\underline{u}) &\leq \delta_{IN_2}(\underline{u}), \\ \delta_{FN_1}(\underline{u}) &\leq \delta_{FN_2}(\underline{u}), \\ \eta_{TN_1}(\underline{u}) &\leq \eta_{TN_2}(\underline{u}), \\ \eta_{IN_1}(\underline{u}) &\leq \eta_{IN_2}(\underline{u}) \text{ and} \\ \eta_{FN_1}(\underline{u}) &\leq \eta_{FN_2}(\underline{u}), \end{aligned}$$
 where

$\delta_{TN_1}(\underline{u})$ and $\eta_{TN_1}(\underline{u})$ are amplitude and phase terms of $\zeta_1^T(\underline{u})$,
 $\delta_{IN_1}(\underline{u})$ and $\eta_{IN_1}(\underline{u})$ are amplitude and phase terms of $\zeta_1^I(\underline{u})$,
 $\delta_{FN_1}(\underline{u})$ and $\eta_{FN_1}(\underline{u})$ are amplitude and phase terms of $\zeta_1^F(\underline{u})$,
 $\delta_{TN_2}(\underline{u})$ and $\eta_{TN_2}(\underline{u})$ are amplitude and phase terms of $\zeta_2^T(\underline{u})$,
 $\delta_{IN_2}(\underline{u})$ and $\eta_{IN_2}(\underline{u})$ are amplitude and phase terms of $\zeta_2^I(\underline{u})$, and
 $\delta_{FN_2}(\underline{u})$ and $\eta_{FN_2}(\underline{u})$ are amplitude and phase terms of $\zeta_2^F(\underline{u})$.

Definition 5.4. Two CNH-set $\Theta_{N_1} = (\zeta_1, N_1)$ and $\Theta_{N_2} = (\zeta_2, N_2)$ over the same \mathbb{U} , are said to be *equal* if

- i. $(\zeta_1, N_1) \subseteq (\zeta_2, N_2)$
- ii. $(\zeta_2, N_2) \subseteq (\zeta_1, N_1)$.

Definition 5.5. Let (ζ, N) be a CNH-set over \mathbb{U} . Then

- i. (ζ, N) is called a *null CNH-set*, denoted by $(\zeta, N)_\Phi$ if $\delta_{TN}(\underline{u}) = \delta_{IN}(\underline{u}) = \delta_{FN}(\underline{u}) = 0$ and $\eta_{TN}(\underline{u}) = \eta_{IN}(\underline{u}) = \eta_{FN}(\underline{u}) = 0\pi$ for all $\underline{u} \in B$.
- ii. (ζ, N) is called a *absolute CNH-set*, denoted by $(\zeta, N)_\Delta$ if $\delta_{TN}(\underline{u}) = \delta_{IN}(\underline{u}) = \delta_{FN}(\underline{u}) = 1$ and $\eta_{TN}(\underline{u}) = \eta_{IN}(\underline{u}) = \eta_{FN}(\underline{u}) = 2\pi$ for all $\underline{u} \in B$.

Definition 5.6. Let (ζ_1, N_1) and (ζ_2, N_2) are two CNH-sets over the same universe \mathbb{U} . Then

- i. A CNH-set (ζ_1, N_1) is called a *homogeneous CNH-set*, denoted by $(\zeta_1, N_1)_{Hom}$ if and only if $\zeta_1(\underline{u})$ is a homogeneous CN-set for all $\underline{u} \in N_1$.
- ii. A CNH-set (ζ_1, N_1) is called a *completely homogeneous CNH-set*, denoted by $(\zeta_1, N_1)_{CHom}$ if and only if $\zeta_1(\underline{u})$ is a homogeneous with $\zeta_1(\underline{v})$ for all $\underline{u}, \underline{v} \in N_1$.
- iii. A CNH-set (ζ_1, N_1) is said to be a completely homogeneous CNH-set with (ζ_2, N_2) if and only if $\zeta_1(\underline{u})$ is a homogeneous with $\zeta_2(\underline{u})$ for all $\underline{u} \in N_1 \cap N_2$.

5.1. Set Theoretic Operations and Laws on CNH-set

Here some basic set theoretic operations (i.e. complement, union and intersection) and laws (commutative laws, associative laws etc.) are discussed on CNH-set.

Definition 5.7. The *complement* of CNH-set (ζ, N) , denoted by $(\zeta, N)^c$ is defined as

$$(\zeta, N)^c = \{(\underline{u}, (\zeta(\underline{u}))^c) : \underline{u} \in B, (\zeta(\underline{u}))^c \in C_{Neu}(\mathbb{U})\}$$

such that the amplitude and phase terms of the membership function $(\zeta(\underline{u}))^c$ are given by

$$(\delta_{TN}(\underline{u}))^c = \delta_{FN}(\underline{u}),$$

$$(\delta_{IN}(\underline{u}))^c = 1 - \delta_{IN}(\underline{u}),$$

$$(\delta_{FN}(\underline{u}))^c = \delta_{TN}(\underline{u}),$$

and

$$(\eta_{TN}(\underline{u}))^c = 2\pi - \eta_{TN}(\underline{u}),$$

$$(\eta_{IN}(\underline{u}))^c = 2\pi - \eta_{IN}(\underline{u}),$$

$$(\eta_{FN}(\underline{u}))^c = 2\pi - \eta_{FN}(\underline{u}) \text{ respectively.}$$

Proposition 5.8. Let (ζ, N) be a CNH-set over \mathbb{U} . Then $((\zeta, N)^c)^c = (\zeta, N)$.

Proof. Since $\zeta(\underline{u}) \in C_{Neu}(\mathbb{U})$, therefore (ζ, N) can be written in terms of its amplitude and phase terms as

$$(\zeta, N) = \left\{ \left(\underline{u}, \left\langle \delta_{TN}(\underline{u})e^{i\eta_{TN}(\underline{u})}, \delta_{IN}(\underline{u})e^{i\eta_{IN}(\underline{u})}, \delta_{FN}(\underline{u})e^{i\eta_{FN}(\underline{u})} \right\rangle \right) : \underline{u} \in N \right\} \quad (5)$$

Now

$$(\zeta, N)^c(\underline{u}) = \left\{ \left(\underline{u}, \left\langle (\delta_{TN}(\underline{u}))^c e^{i(\eta_{TN}(\underline{u}))^c}, (\delta_{IN}(\underline{u}))^c e^{i(\eta_{IN}(\underline{u}))^c}, (\delta_{FN}(\underline{u}))^c e^{i(\eta_{FN}(\underline{u}))^c} \right\rangle \right) : \underline{u} \in N \right\}$$

$$(\zeta, N)^c(\underline{u}) = \left\{ \left(\underline{u}, \left\langle (\delta_{FN}(\underline{u}))e^{i(2\pi-\eta_{TN}(\underline{u}))}, (1 - \delta_{IN}(\underline{u}))e^{i(2\pi-\eta_{IN}(\underline{u}))}, (\delta_{TN}(\underline{u}))e^{i(2\pi-\delta_{FN}(\underline{u}))} \right\rangle \right) : \underline{u} \in N \right\}$$

$$((\zeta, N)^c)^c = \left\{ \left(\underline{u}, \left\langle (\delta_{FN}(\underline{u}))^c e^{i(2\pi-\eta_{TN}(\underline{u}))^c}, (1 - \delta_{IN}(\underline{u}))^c e^{i(2\pi-\eta_{IN}(\underline{u}))^c}, (\delta_{TN}(\underline{u}))^c e^{i(2\pi-\eta_{TN}(\underline{u}))^c} \right\rangle \right) : \underline{u} \in N \right\}$$

$$((\zeta, N)^c)^c = \left\{ \left(\underline{u}, \left\langle \begin{array}{l} \delta_{TN}(\underline{u})e^{i(2\pi-(2\pi-\eta_{TN}(\underline{u})))}, \\ (1 - (1 - \delta_{IN}(\underline{u})))e^{i(2\pi-(2\pi-\eta_{IN}(\underline{u})))}, \\ \delta_{FN}(\underline{u})e^{i(2\pi-(2\pi-\eta_{FN}(\underline{u})))} \end{array} \right\rangle \right) : \underline{u} \in N \right\}$$

$$((\zeta, N)^c)^c = \left\{ \left(\underline{u}, \left\langle \delta_{TN}(\underline{u})e^{i\eta_{TN}(\underline{u})}, \delta_{IN}(\underline{u})e^{i\eta_{IN}(\underline{u})}, \delta_{FN}(\underline{u})e^{i\eta_{FN}(\underline{u})} \right\rangle \right) : \underline{u} \in N \right\} \quad (6)$$

from equations (5) and (6), we have $((\zeta, N)^c)^c = (\zeta, N)$. \square

Proposition 5.9. Let (ζ, N) be a CNH-set over \mathbb{U} . Then

$$\text{i. } ((\zeta, N)_{\Phi})^c = (\zeta, N)_{\Delta}$$

ii. $((\zeta, N)_\Delta)^c = (\zeta, N)_\Phi$

Definition 5.10. The *intersection* of two CNH-set (ζ_1, N_1) and (ζ_2, N_2) over the same universe \mathbb{U} , denoted by $(\zeta_1, N_1) \prod (\zeta_2, N_2)$, is the CNH-set (ζ_3, N_3) , where $N_3 = N_1 \prod N_2$, and for all $\underline{u} \in N_3$,

$$\zeta_3^T(\underline{u}) = \begin{cases} \delta_{TN_1}(\underline{u})e^{i\eta_{TN_1}(\underline{u})} & , if \underline{u} \in N_1 \setminus N_2 \\ \delta_{TN_2}(\underline{u})e^{i\eta_{TN_2}(\underline{u})} & , if \underline{u} \in N_2 \setminus N_1 \\ [\delta_{TN_1}(\underline{u}) \otimes \delta_{TN_2}(\underline{u})] \cdot e^{i[\eta_{TN_1}(\underline{u}) \otimes \eta_{TN_2}(\underline{u})]} & , if \underline{u} \in N_1 \prod N_2 \end{cases}$$

$$\zeta_3^I(\underline{u}) = \begin{cases} \delta_{IN_1}(\underline{u})e^{i\eta_{IN_1}(\underline{u})} & , if \underline{u} \in N_1 \setminus N_2 \\ \delta_{IN_2}(\underline{u})e^{i\eta_{IN_2}(\underline{u})} & , if \underline{u} \in N_2 \setminus N_1 \\ [\delta_{IN_1}(\underline{u}) \otimes \delta_{IN_2}(\underline{u})] \cdot e^{i[\eta_{IN_1}(\underline{u}) \otimes \eta_{IN_2}(\underline{u})]} & , if \underline{u} \in N_1 \prod N_2 \end{cases}$$

and

$$\zeta_3^F(\underline{u}) = \begin{cases} \delta_{FN_1}(\underline{u})e^{i\eta_{FN_1}(\underline{u})} & , if \underline{u} \in N_1 \setminus N_2 \\ \delta_{FN_2}(\underline{u})e^{i\eta_{FN_2}(\underline{u})} & , if \underline{u} \in N_2 \setminus N_1 \\ [\delta_{FN_1}(\underline{u}) \otimes \delta_{FN_2}(\underline{u})] \cdot e^{i[\eta_{FN_1}(\underline{u}) \otimes \eta_{FN_2}(\underline{u})]} & , if \underline{u} \in N_1 \prod N_2 \end{cases}$$

where \otimes denotes minimum operator.

Definition 5.11. The *difference* between two CNH-set (ζ_1, N_1) and (ζ_2, N_2) is defined as

$$(\zeta_1, N_1) \setminus (\zeta_2, N_2) = (\zeta_1, N_1) \prod (\zeta_2, N_2)^c$$

Definition 5.12. The *union* of two CNH-set (ζ_1, N_1) and (ζ_2, N_2) over the same universe \mathbb{U} , denoted by $(\zeta_1, N_1) \coprod (\zeta_2, N_2)$, is the CNH-set (ζ_3, N_3) , where $N_3 = N_1 \coprod N_2$, and for all $\underline{u} \in N_3$,

$$\zeta_3^T(\underline{u}) = \begin{cases} \delta_{TN_1}(\underline{u})e^{i\eta_{TN_1}(\underline{u})} & , if \underline{u} \in N_1 \setminus N_2 \\ \delta_{TN_2}(\underline{u})e^{i\eta_{TN_2}(\underline{u})} & , if \underline{u} \in N_2 \setminus N_1 \\ [\delta_{TN_1}(\underline{u}) \oplus \delta_{TN_2}(\underline{u})] \cdot e^{i[\eta_{TN_1}(\underline{u}) \oplus \eta_{TN_2}(\underline{u})]} & , if \underline{u} \in N_1 \prod N_2 \end{cases}$$

$$\zeta_3^I(\underline{u}) = \begin{cases} \delta_{IN_1}(\underline{u})e^{i\eta_{IN_1}(\underline{u})} & , if \underline{u} \in N_1 \setminus N_2 \\ \delta_{IN_2}(\underline{u})e^{i\eta_{IN_2}(\underline{u})} & , if \underline{u} \in N_2 \setminus N_1 \\ [\delta_{IN_1}(\underline{u}) \oplus \delta_{IN_2}(\underline{u})] \cdot e^{i[\eta_{IN_1}(\underline{u}) \oplus \eta_{IN_2}(\underline{u})]} & , if \underline{u} \in N_1 \prod N_2 \end{cases}$$

and

$$\zeta_3^F(\underline{u}) = \begin{cases} \delta_{FN_1}(\underline{u})e^{i\eta_{FN_1}(\underline{u})} & , if \underline{u} \in N_1 \setminus N_2 \\ \delta_{FN_2}(\underline{u})e^{i\eta_{FN_2}(\underline{u})} & , if \underline{u} \in N_2 \setminus N_1 \\ [\delta_{FN_1}(\underline{u}) \oplus \delta_{FN_2}(\underline{u})] \cdot e^{i[\eta_{FN_1}(\underline{u}) \oplus \eta_{FN_2}(\underline{u})]} & , if \underline{u} \in N_1 \prod N_2 \end{cases}$$

where \oplus denotes maximum operator.

Proposition 5.13. Let (ζ, N) be a CNH-set over \mathbb{U} . Then the following results hold true:

- i. $(\zeta, N) \coprod (\zeta, N)_\Phi = (\zeta, N)$
- ii. $(\zeta, N) \coprod (\zeta, N)_\Delta = (\zeta, N)_\Delta$

- iii. $(\zeta, N) \prod (\zeta, N)_\Phi = (\zeta, N)_\Phi$
- iv. $(\zeta, N) \prod (\zeta, N)_\Delta = (\zeta, N)$
- v. $(\zeta, N)_\Phi \coprod (\zeta, N)_\Delta = (\zeta, N)_\Delta$
- vi. $(\zeta, N)_\Phi \prod (\zeta, N)_\Delta = (\zeta, N)_\Phi$

Proposition 5.14. *Let (ζ_1, N_1) , (ζ_2, N_2) and (ζ_3, N_3) are three CNH-sets over the same universe \mathbb{U} . Then the following commutative and associative laws hold true:*

- i. $(\zeta_1, N_1) \prod (\zeta_2, N_2) = (\zeta_2, N_2) \prod (\zeta_1, N_1)$
- ii. $(\zeta_1, N_1) \coprod (\zeta_2, N_2) = (\zeta_2, N_2) \coprod (\zeta_1, N_1)$
- iii. $(\zeta_1, N_1) \prod ((\zeta_2, N_2) \prod (\zeta_3, N_3)) = ((\zeta_1, N_1) \prod (\zeta_2, N_2)) \prod (\zeta_3, N_3)$
- iv. $(\zeta_1, N_1) \coprod ((\zeta_2, N_2) \coprod (\zeta_3, N_3)) = ((\zeta_1, N_1) \coprod (\zeta_2, N_2)) \coprod (\zeta_3, N_3)$

Proposition 5.15. *Let (ζ_1, N_1) and (ζ_2, N_2) are two CNH-sets over the same universe \mathbb{U} . Then the following De Morgans laws hold true:*

- i. $((\zeta_1, N_1) \prod (\zeta_2, N_2))^c = (\zeta_1, N_1)^c \coprod (\zeta_2, N_2)^c$
- ii. $((\zeta_1, N_1) \coprod (\zeta_2, N_2))^c = (\zeta_1, N_1)^c \prod (\zeta_2, N_2)^c$

Discussion on particular cases of CFH-sets, CIFH-sets and CNH-sets

- If $\zeta(\lambda) = \langle \zeta^T(\lambda), \zeta^I(\lambda), \zeta^F(\lambda) \rangle$, $-0 \leq \delta_T + \delta_I + \delta_F \leq 3^+$ (or $0 \leq |\delta_T + \delta_I + \delta_F| \leq 3$) is replaced by $\zeta(\lambda) = \langle \zeta^T(\lambda), \zeta^F(\lambda) \rangle$, $0 \leq \delta_T + \delta_F \leq 1$ (or $0 \leq |\delta_T + \delta_F| \leq 1$) with omission of indeterminacy, then complex neutrosophic hypersoft set reduces to complex intuitionistic fuzzy hypersoft set.
- If $\zeta(\lambda) = \langle \zeta^T(\lambda), \zeta^I(\lambda), \zeta^F(\lambda) \rangle$ is replaced by $\zeta(\lambda) = \langle \zeta^T(\lambda) \rangle$ with omission of indeterminacy and falsity, then complex neutrosophic hypersoft set reduces to complex fuzzy hypersoft set.

This concludes that complex fuzzy hypersoft set and complex intuitionistic fuzzy hypersoft set are the particular cases of complex neutrosophic hypersoft set. Since Complex fuzzy hypersoft sets and complex intuitionistic fuzzy hypersoft sets cannot handle imprecise, indeterminate, inconsistent, and incomplete information of periodic nature so to overcome this hurdle, complex neutrosophic hypersoft set is conceptualized.

Conclusion

In this work, new hybrids of hypersoft set i.e. complex fuzzy hypersoft set, complex intuitionistic fuzzy hypersoft set and complex neutrosophic hypersoft set, are conceptualized with their some fundamentals and theoretic operations. Future study may include other hybrids

of hypersoft set with interval-valued complex fuzzy set etc., similarity and distance measures, aggregations operators and applications in multi-criteria decision making problems.

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Neutrosophic Φ -open sets and neutrosophic Φ -continuous functions

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Abstract: We introduce the notion of neutrosophic Φ -open set and neutrosophic Φ -continuous mapping via neutrosophic topological spaces and investigate several properties of it. By defining neutrosophic Φ -open set, neutrosophic Φ -continuous mapping, and neutrosophic Φ -open mapping, we prove some remarks, theorems on neutrosophic topological spaces.

Keywords: Neutrosophic set; Neutrosophic topology; Neutrosophic supra topology; Neutrosophic α -open set; Neutrosophic Φ -open set.

1. Introduction

Smarandache [53] defined the Neutrosophic Set (NS) in 1998 by extending fuzzy set [58], and intuitionistic fuzzy set [2] to deal with uncertain, inconsistent and indeterminate information. An NS Θ defined over the universe Ω , $\alpha = \alpha(\xi, \psi, \zeta) \in \Theta$ with ξ, ψ and ζ being the real standard or non-standard subsets of $]0, 1+[$. ξ, ψ and ζ are the degrees of true membership function, indeterminate membership function and falsity membership function respectively in the set Θ . Wang, Smarandache, Zhang, and Sunderraman [56] defined Interval NS (INS) as an instance and a subclass of NS by considering the subunitary interval $[0, 1]$. An INS τ defined on universe Ω , $\alpha = \alpha(\xi, \psi, \zeta) \in \tau$ with ξ, ψ and ζ being the subinterval of $[0, 1]$. Wang, Smarandache, Zhang, and Sunderraman [57] defined Single Valued NS (SVNS) as an instance of NS. In SVNS, the degrees of truth-membership function, indeterminacy-membership function and falsity-membership function lie in the interval $[0, 1]$. NS has drawn many researchers' much attention for theoretical as well as practical applications [3-18, 24, 26-34, 36-46, 54-55].

Salama and Alblowi [49] grounded the concept of Neutrosophic Topological Space (NTS). Salama and Alblowi [50] also studied the generalized NS and generalized NTS. Salama, Smarandache and Alblowi [51] presented a new concept on neutrosophic crisp topology. Iswaraya and Bageerathi [23] presented the neutrosophic semi-closed set and neutrosophic semi-open set. Arokianani, Dhavaseelan, Jafari, and Parimala [1] present the neutrosophic semi-open functions and established some relations between them. Rao and Srinivasa [48] presented neutrosophic pre-open

set and pre-closed set. Dhavaseelan, Parimala, Jafari, and Smarandache [20] presented the neutrosophic semi-supra open set and neutrosophic semi-supra continuous functions.

Dhavaseelan, Ganster, Jafari, and Parimala [21] presented the neutrosophic α -supra open set and neutrosophic α -supra continuous functions. Parimala, Karthika, Dhavaseelan, & Jafari [35] presented the neutrosophic supra pre-continuous functions, the neutrosophic supra pre-open maps, and the neutrosophic supra pre-closed maps in terms of neutrosophic supra pre-open sets and neutrosophic supra pre-closed sets. Dhavaseelan, and Jafari [21] studied Generalized Neutrosophic Closed Set (GNCS). Pushpalatha and Nandhini [47] defined the GNCS in NTSs. Ebenanjar, Immaculate, and Wilfred [22] studied neutrosophic b -open sets in NTSs. Maheswari, Sathyabama, and Chandrasekar [25] studied the neutrosophic generalized b -closed sets in NTSs. Das and Pramanik [17] presented the generalized neutrosophic b -open sets in NTSs.

Research gap: No study on neutrosophic Φ -open sets and neutrosophic Φ -continuous functions neutrosophic generalized b -open set has been reported in the recent literature.

Motivation: To fill the research gap, we introduce the neutrosophic Φ -open set.

In this paper, we develop the notion of neutrosophic Φ -open set and neutrosophic Φ -continuous mapping, neutrosophic Φ -open mapping, and neutrosophic Φ -closed mapping via NTSs.

The rest of the paper is designed as follows:

Section 2 recalls the definitions neutrosophic set, neutrosophic topological space, neutrosophic supra topological space, neutrosophic α -open sets, and neutrosophic α -closed sets. Section 3 introduces neutrosophic Φ -open set, neutrosophic Φ -continuous mapping, and neutrosophic Φ -open mapping and proofs of some remarks, and theorems on neutrosophic Φ -open sets and neutrosophic Φ -continuous mapping. Section 4 presents concluding remarks.

2. Preliminaries and some properties

In this section, we discuss some existing definitions and theorems which are already defined by many researchers.

Definition 2.1. Assume that W be a universal set. Then D , an NS [53] over W is denoted as follows:
 $D = \{(m, T_D(m), I_D(m), F_D(m)) : m \in W \text{ and } T_D(m), I_D(m), F_D(m) \in]0, 1^+[\}$ where T_D , I_D and F_D are the functions from D to $]0, 1^+]$ and for each $y \in W$, $-0 \leq T_D(m) + I_D(m) + F_D(m) \leq 3^+$.

Definition 2.2. Assume that $D = \{(m, T_D(m), I_D(m), F_D(m)) : m \in W\}$ and $K = \{(m, T_K(m), I_K(m), F_K(m)) : m \in W\}$ are any two NS over W , then $D \cup K$ and $D \cap K$ [53] are defined by

- i. $D \cup K = \{(m, T_D(m) \vee T_K(m), I_D(m) \wedge I_K(m), F_D(m) \wedge F_K(m)) : m \in W\};$
- ii. $D \cap K = \{(m, T_D(m) \wedge T_K(m), I_D(m) \vee I_K(m), F_D(m) \vee F_K(m)) : m \in W\}.$

Definition 2.3. Assume that $D = \{(m, T_D(m), I_D(m), F_D(m)) : m \in W\}$ is an NS over W . Then the complement [53] of D is defined by $D^c = \{(m, 1 - T_D(m), 1 - I_D(m), 1 - F_D(m)) : m \in W\}.$

Definition 2.4. Assume that $D = \{(m, T_D(m), I_D(m), F_D(m)): m \in W\}$ and $K = \{(m, T_K(m), I_K(m), F_K(m)): m \in W\}$ are any two NSs over W . Then D is contained in K [53] if and only if $T_D(m) \leq T_K(m)$, $I_D(m) \geq I_K(m)$, $F_D(m) \geq F_K(m)$, for all $m \in W$.

Now we may consider two NSs 0_N and 1_N over W as follows:

- 1) $0_N = \{(m, 0, 1, 1): m \in W\}$;
- 2) $1_N = \{(m, 1, 0, 0): m \in W\}$.

Clearly, $0_N \subseteq 1_N$.

Definition 2.5. Assume that W is a universe of discourse and τ is the collection of some NSs over W . Then the collection τ is said to be a Neutrosophic Topology (NT) [49] on W if the following axioms hold:

1. $0_N, 1_N \in \tau$
2. $C_1, C_2 \in \tau \Rightarrow C_1 \cap C_2 \in \tau$
3. $\cup C_i \in \tau$, for every $\{C_i: i \in \Delta\} \subseteq \tau$.

The pair (W, τ) is said to be an NTS. If $H \in \tau$, then H is called a Neutrosophic Open Set (NOS) and the complement of H i.e. H^c is called a Neutrosophic Closed Set (NCS).

Example 2.1. Assume that $W = \{s_1, s_2, s_3\}$ is a set with three NSs over W as follows:

$M_1 = \{(s_1, 0.9, 0.5, 0.7), (s_2, 0.7, 0.6, 0.8), (s_3, 0.7, 0.4, 0.7): s_1, s_2, s_3 \in W\}$;

$M_2 = \{(s_1, 1.0, 0.3, 0.4), (s_2, 0.9, 0.5, 0.5), (s_3, 1.0, 0.1, 0.5): s_1, s_2, s_3 \in W\}$;

$M_3 = \{(s_1, 0.9, 0.3, 0.5), (s_2, 0.8, 0.5, 0.8), (s_3, 0.9, 0.3, 0.5): s_1, s_2, s_3 \in W\}$;

Then (W, τ) is an NTS, where $\tau = \{0_N, 1_N, M_1, M_2, M_3\}$ is an NT on W .

Remark 2.1. The collection of all NOSs and NCSs in (W, τ) may be denoted as $NOS(W)$ and $NCS(W)$ respectively. The neutrosophic interior and neutrosophic closure [49] of a neutrosophic subset H of W is denoted by $N_{int}(H)$ and $N_{cl}(H)$ respectively and defined as follows:

$N_{int}(H) = \cup \{D: D \text{ is an NOS in } W \text{ and } D \subseteq H\}$,

$N_{cl}(H) = \cap \{L: L \text{ is an NCS in } W \text{ and } H \subseteq L\}$.

Clearly $N_{int}(H) \subseteq H \subseteq N_{cl}(H)$.

Definition 2.6. Assume that (W, τ) is an NTS and H be an NS over W . Then H is

- 1) Neutrosophic Pre-Open (NPO) set [48] iff $H \subseteq N_{int}N_{cl}(H)$;
- 2) Neutrosophic Semi-Open (NSO) set [23] iff $H \subseteq N_{cl}N_{int}(H)$;
- 3) Neutrosophic α -Open ($N\alpha$ -O) set [1] iff $H \subseteq N_{int}N_{cl}N_{int}(H)$.

Definition 2.7. Assume that W is a universal set and Ω be the collection of some NSs over W . Then Ω is said to be a Neutrosophic Supra Topology (NST) [19] on W if the following axioms hold:

- 1) $0_N, 1_N \in \Omega$
- 2) $\cup C_i \in \Omega$, for every $\{C_i: i \in \Delta\} \subseteq \Omega$.

The pair (W, Ω) is said to be a Neutrosophic Supra Topological Space (NSTS). If $H \in \Omega$, then H is called a Neutrosophic-Supra Open (N-SO) set and its complement H^c is called a Neutrosophic-Supra Closed (N-SC) set in (W, Ω) . The neutrosophic-supra interior and neutrosophic-supra closure of an NS H is denoted by $N_{int}^\Omega(H)$ and $N_{cl}^\Omega(H)$ respectively and are defined as follows:

$$N_{int}^\Omega(H) = \bigcup \{D : D \text{ is an N-SO set in } W \text{ and } D \subseteq H\},$$

$$N_{cl}^\Omega(H) = \bigcap \{L : L \text{ is an N-SC set in } W \text{ and } H \subseteq L\}.$$

Definition 2.8. Assume that (W, Ω) be an NSTS and H is an NS over W . Then H is

- 1) Neutrosophic-Pre Supra Open (N-PSO) set [35] iff $H \subseteq N_{int}^\Omega(N_{cl}^\Omega(H))$;
- 2) Neutrosophic-Semi Supra Open (N-SSO) set [20] if and only if $H \subseteq N_{cl}^\Omega(N_{int}^\Omega(H))$;
- 3) Neutrosophic- α -Supra Open (N- α SO) set [19] if and only if $H \subseteq N_{int}^\Omega(N_{cl}^\Omega(N_{int}^\Omega(H)))$.

The complement of N-PSO set, N-SSO set and N- α SO set are called Neutrosophic Pre Supra-Closed (N-PSC) set, Neutrosophic Semi Supra-Closed (N-SSC) set and Neutrosophic α -Supra-Closed (N- α SC) set respectively.

Theorem 2.1. Assume that (W, Ω) be an NSTS.

Then

- i. Every N-SO set is an N- α SO set.
- ii. Every N- α SO set is an N-PSO set (N-SSO set).

For proof, see Parimala, Karthika, Dhavaseelan, and Jafari (2018).

Theorem 2.2. Assume that (W, Ω) be an NSTS.

Then

- i. Union of two N- α SO sets is an N- α SO set.
- ii. Intersection of two N- α SO sets may not be an N- α SO set in general.

For proof, see [19].

Definition 2.9. Let (W, Ω) and (M, Π) be any two NSTSs. Then a function $\xi: (W, \Omega) \rightarrow (Y, M)$ is called a neutrosophic continuous function [52] if the inverse image of each NOS G in M is an NOS in W .

Definition 2.10. Let (W, Ω) and (M, Π) be any two NSTSs. Then a function $\xi: (W, \Omega) \rightarrow (Y, M)$ is called a neutrosophic supra continuous [19] if and only if the inverse image of each N-SO set G in M is an N-SO set in W .

Definition 2.11. A function $\xi: (W, \Omega) \rightarrow (M, \Pi)$, where (W, Ω) and (M, Π) are two NSTSs is said to be a neutrosophic α -supra [19] continuous iff $\xi^{-1}(G)$ is an N- α SO set in W whenever G is an N-SO set in M .

Theorem 2.3. Assume that ξ be a function from an NSTS (W, Ω) to another NSTS (M, Π) . Then the following statements [19] are equivalent:

- i. ξ is a neutrosophic α -supra continuous mapping.
- ii. $\xi^{-1}(G)$ is an N- α SC set in W whenever G is an N-SC set in M .

3. Neutrosophic Φ -open set and neutrosophic Φ -continuous mapping

Definition 3.1. Assume that (W, τ) is an NTS and H is an NS over W . Then H is called a Neutrosophic Φ -Open (N- Φ -O) set iff there exist an N α -O set K such that $K \subseteq H \subseteq N_{cl}(K)$, where $N_{cl}(K)$ denotes the neutrosophic closure of K with respect to the NT τ on W .

Theorem 3.1. In an NTS (W, τ) ,

- 1) Every NOS is a neutrosophic Φ -open set;
- 2) Every N α -O set is a neutrosophic Φ -open set.

Proof.

- 1) Assume that Q is an NOS in an NTS (W, τ) . Since every NOS is an N α -O set, so Q is an N α -O set in (W, τ) . Clearly $Q \subseteq Q \subseteq N_{cl}(Q)$. Therefore, Q is a neutrosophic Φ -open set. Hence every NOS in (W, τ) is a neutrosophic Φ -open set.
- 2) Assume that R is an N α -O set in an NTS (W, τ) . For any neutrosophic set R , $R \subseteq R \subseteq N_{cl}(R)$. Therefore, there exists an N α -O set R in (W, τ) such that $R \subseteq R \subseteq N_{cl}(R)$. Hence R is a neutrosophic Φ -open set. Thus, every N α -O set in (W, τ) is a neutrosophic Φ -open set.

Theorem 3.2. Assume that (W, τ) is an NTS and θ is a neutrosophic supra topology such that $\tau \subseteq \theta$. Then

- 1) Every neutrosophic Φ -open set in (W, τ) is a neutrosophic Φ -supra open set in (W, θ) ;
- 2) Every NOS in (W, τ) is a neutrosophic Φ -supra open set in (W, θ) .

Proof.

- 1) Assume that (W, τ) is an NTS and θ is an NST such that $\tau \subseteq \theta$. Assume that Q is an arbitrary neutrosophic Φ -open set in (W, τ) . Then there exists an N α -O set K such that $K \subseteq Q \subseteq N_{cl}(K)$, where $N_{cl}(K)$ denotes the neutrosophic closure of K with respect to the topology τ . Since $\tau \subseteq \theta$ and θ is an NST on W , so $N_{cl}(K) \subseteq N_{cl}^{\theta}(K)$, where $N_{cl}^{\theta}(K)$ denotes the neutrosophic supra-closure of K with respect to the NST θ . Therefore $K \subseteq Q \subseteq N_{cl}^{\theta}(K)$.

Hence Q is a neutrosophic Φ -supra open set in (W, θ) .

- 2) Assume that (W, τ) is an NTS and θ be an NST on W such that $\tau \subseteq \theta$.

Assume that Q be an arbitrary NOS in (W, τ) . From Theorem 3.1, it is clear that every NOS in an NTS (W, τ) is a neutrosophic Φ -open set. So, Q is a neutrosophic Φ -open set in (W, τ) .

From the first part of the theorem 3.2, it is clear that Q is a neutrosophic Φ -open set in (W, θ) .

Hence every NOS in an NTS (W, τ) is a neutrosophic Φ -supra open set in the NSTS (W, θ) .

Lemma 3.1. In an NTS (W, τ) , the union of two neutrosophic Φ -open sets is a neutrosophic Φ -open set.

Proof.

Assume that K and L are any two neutrosophic Φ -open sets in an NTS (W, τ) . Then there exist two $N\alpha$ -O sets Q_1 and Q_2 in (W, τ) such that $Q_1 \subseteq K \subseteq N_{cl}(Q_1)$, $Q_2 \subseteq L \subseteq N_{cl}(Q_2)$.

Now, $Q_1 \cup Q_2 \subseteq K \cup L \subseteq N_{cl}(Q_1) \cup N_{cl}(Q_2) = N_{cl}(Q_1 \cup Q_2)$ and $Q_1 \cup Q_2$ is an $N\alpha$ -O set in (W, τ) . Therefore $K \cup L$ is a neutrosophic Φ -open set in (W, τ) . Hence the union of two neutrosophic Φ -open sets in an NTS (W, τ) is a neutrosophic Φ -open set.

Theorem 3.3. Assume that (W, τ) is an NTS. Then

- 1) Union of an NOS and a neutrosophic Φ -open set is a neutrosophic Φ -open set.
- 2) Union of an $N\alpha$ -O set and a neutrosophic Φ -open set is a neutrosophic Φ -open set.

Proof. Let Q be an NOS and R be a neutrosophic Φ -open set in an NTS (W, τ) . From Theorem 3.1, Q is a neutrosophic Φ -open set. Again, from Lemma 3.1, it is clear that $Q \cup R$ is a neutrosophic Φ -open set in (W, τ) .

- 1) Assume that H is an $N\alpha$ -O set and G is a neutrosophic Φ -open set in an NTS (W, τ) . From Theorem 3.1, it is clear that H is a neutrosophic Φ -open set. Again, from Remark 3.1, it is clear that $H \cup G$ is a neutrosophic Φ -open set in (W, τ) .

Definition 3.2. Assume that (W, τ) and (M, δ) are two NTSs. Then a function $\xi: (W, \tau) \rightarrow (M, \delta)$ is called a neutrosophic Φ -continuous function iff the inverse image of every NOS G in M is a neutrosophic Φ -open set in W .

Definition 3.3. Assume that (W, τ) , and (M, δ) are two NTSs and θ is an NST on W such that $\tau \subseteq \theta$. Then a function $\xi: (W, \tau) \rightarrow (M, \delta)$ is called a neutrosophic Φ -supra continuous function iff the inverse image of every NOS G in M is a neutrosophic Φ -supra open set in W with respect to the NST θ on W .

Theorem 3.3. Every neutrosophic continuous function from an NTS (W, τ) to another NTS (M, δ) is a neutrosophic Φ -continuous function.

Proof. Assume that $\xi: (W, \tau) \rightarrow (M, \delta)$ is a neutrosophic continuous function and K be an arbitrary NOS in M . Then by hypothesis, $\xi^{-1}(K)$ is an NOS in W . Since each NOS is a neutrosophic Φ -open set, so $\xi^{-1}(K)$ is a neutrosophic Φ -open set in W . Therefore, for each NOS K in M , $\xi^{-1}(K)$ is a neutrosophic Φ -open set in W . Hence ξ is a neutrosophic Φ -continuous function. Therefore, every neutrosophic continuous function is a neutrosophic Φ -continuous function.

Theorem 3.4. Assume that (W, τ) and (M, δ) are two NTSs and $\tau \subseteq \theta$, where θ is an NST on W . Then every neutrosophic Φ -continuous function from (W, τ) to (M, δ) is a neutrosophic Φ -supra continuous function from (W, θ) to (M, δ) .

Proof. Assume that $\xi: (W, \tau) \rightarrow (M, \delta)$ is a neutrosophic Φ -continuous mapping. Let θ be an NST such that $\tau \subseteq \theta$. Let T be an NOS in M . Then by hypothesis $\xi^{-1}(T)$ is a neutrosophic Φ -open set in W . Since each neutrosophic Φ -open set (W, τ) is a neutrosophic Φ -supra open set in (W, θ) , so $\xi^{-1}(T)$ is a neutrosophic Φ -supra open set in (W, θ) . Therefore ξ is a neutrosophic Φ -supra continuous mapping from (W, θ) to (M, δ) .

Definition 3.4. Let (W, τ) and (M, δ) be two NTSs. A function $\xi: (W, \tau) \rightarrow (M, \delta)$ is called a neutrosophic Φ -open function if $\xi(Q)$ is a neutrosophic Φ -open set in M for each NOS Q in W .

Definition 3.5. Let (W, τ) and (M, δ) be two NTSs. A function $\xi: (W, \tau) \rightarrow (M, \delta)$ is called a neutrosophic Φ -closed function if $\xi(Q)$ is a neutrosophic Φ -closed set in M for each NCS Q in W .

Theorem 3.5. Assume that (W, τ) and (M, δ) are any two NTSs. Then $\xi: (W, \tau) \rightarrow (M, \delta)$ is a neutrosophic Φ -open function iff $\xi(N_{int}(K)) \subseteq N_{int}(\xi(K))$, for each neutrosophic subset K of W .

Proof. Let $\xi: (W, \tau) \rightarrow (M, \delta)$ be a neutrosophic Φ -open function and K be a neutrosophic subset of W . Clearly $N_{int}(K)$ is an NOS in W and $N_{int}(K) \subseteq K$. Since ξ is a neutrosophic Φ -open function, so $\xi(N_{int}(K))$ is a neutrosophic Φ -open set in M and $\xi(N_{int}(K)) \subseteq \xi(K)$. Since each NOS is a neutrosophic Φ -open set and $N_{int}(\xi(K))$ is the largest NOS contained in $\xi(K)$, so $N_{int}(\xi(K))$ is the largest neutrosophic Φ -open set contained in $\xi(K)$. Therefore $\xi(N_{int}(K)) \subseteq N_{int}(\xi(K)) \subseteq \xi(K)$ i.e. $\xi(N_{int}(K)) \subseteq N_{int}(\xi(K))$. Hence for each neutrosophic subset K of W , $\xi(N_{int}(K)) \subseteq N_{int}(\xi(K))$.

Conversely, let L be an NOS in (W, τ) . Therefore, $N_{int}(L) = L$. Now by hypothesis $\xi(N_{int}(L)) \subseteq N_{int}(\xi(L))$. This implies $\xi(L) \subseteq N_{int}(\xi(L))$. We know that $N_{int}(\xi(L)) \subseteq \xi(L)$. Therefore $\xi(L) = N_{int}(\xi(L))$. This means that $\xi(L)$ is an NOS in (M, δ) . Since each NOS is a neutrosophic Φ -open set, so $\xi(L)$ is a neutrosophic Φ -open set in (M, δ) . Hence for each NOS L in (W, τ) , $\xi(L)$ is a neutrosophic Φ -open set in (M, δ) . Therefore ξ is a neutrosophic Φ -open function.

Theorem 3.6. Assume that ξ is a bijective function from an NTS (W, τ) to another NTS (M, δ) . Then the following mathematical statements are equivalent:

- 1) ξ is a neutrosophic Φ -continuous function;
- 2) ξ is a neutrosophic Φ -closed function;
- 3) ξ is a neutrosophic Φ -open function.

Proof.

(1) \Rightarrow (2) Assume that $\xi: (W, \tau) \rightarrow (M, \delta)$ is a neutrosophic Φ -continuous function. Let Q be any arbitrary NCS in (W, τ) . Then Q^c is an NOS in (W, τ) . Since each NOS is a neutrosophic Φ -open set, so Q^c is a neutrosophic Φ -open set in (W, τ) . Since ξ is a bijective function, so $\xi(Q^c) = (\xi(Q))^c$ is an NOS in (M, δ) . Hence $\xi(Q)$ is an NCS in (M, δ) . Therefore, for each NCS Q in (W, τ) , $\xi(Q)$ is a neutrosophic Φ -closed set in (M, δ) . Hence ξ is a neutrosophic Φ -closed function.

(2) \Rightarrow (3) Assume that $\xi: (W, \tau) \rightarrow (M, \delta)$ be a neutrosophic Φ -closed function. Let L be any arbitrary NOS in (W, τ) . Then L^c is an NCS in (W, τ) . Since ξ is a neutrosophic Φ -closed function, so $\xi(L^c) = (\xi(L))^c$ is a neutrosophic Φ -closed set in (M, δ) . Then $\xi(L)$ is a neutrosophic Φ -open set in (M, δ) . Therefore, for each NOS L in (W, τ) , $\xi(L)$ is a neutrosophic Φ -open set in (M, δ) . Hence ξ is a neutrosophic Φ -open function.

(3) \Rightarrow (1) Assume that $\xi: (W, \tau) \rightarrow (M, \delta)$ is a neutrosophic Φ -open function. Let P be any arbitrary NOS in (M, δ) . Then P is a neutrosophic Φ -open set in (M, δ) . Since ξ is a bijective function, so $\xi^{-1}(P)$ is an NOS in (W, τ) . Again, since each NOS is a neutrosophic Φ -open set, so $\xi^{-1}(P)$ is a neutrosophic Φ -open set in (W, τ) . Therefore, for each NOS P in (M, δ) , $\xi^{-1}(P)$ is a neutrosophic Φ -open set in (W, τ) . Hence ξ is a neutrosophic Φ -continuous function.

4. Conclusion

In this study we have introduced neutrosophic Φ -open set, neutrosophic Φ -continuous mapping via NTSs and investigated their several properties. By defining neutrosophic Φ -open set, neutrosophic Φ -continuous mapping, we have proved some remarks, and theorems on NTSs. In the future, we hope that based on Φ -open set, neutrosophic Φ -continuous mapping via NTSs, many new investigations can be carried out.

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Blockchain Risk Evaluation on Enterprise Systems using an Intelligent MCDM based model

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Abstract: Blockchain technology (BT) has become popular in the firms in the present time, however, implementation of BT includes several risk factors from various points of view. Some of these risks can be serious for the processes of firms. These risks should be cautiously recognized and analyzed to reduce the negative impacts of them. Assessment of the risks can be recognized as a multi-criteria decision making (MCDM) problem. In this work, the risks that will occur when implementing BT are assessed by using MCDM methodology built on Single Valued Neutrosophic Sets (SVNSs), Analytic Hierarchy Process (AHP), and Decision Making and Trial Evaluation Laboratory (DEMATEL) methods. The main and sub-criteria risks are collected via a company in the smart village in Egypt and from previous research, hence, the hierarchical form of the problem is built. AHP is used to show the importance of risk factors and the relationships between risk factors obtained by using the DEMATEL method. The main goal of this study is to aid the firms mainly and the firm in Egypt especially to determine which risks are more serious and to which of them causing effect and are being affected. In this study 8 main criterion and 28 sub-criteria, risks are used. As result, the security risk is important in the main risks but energy costs and data leaks are important in sub risks.

Keywords: Blockchain technology (BT), Risks, SVNSs, AHP, DEMATEL

1. Introduction

Firms, industries, and businesses have a critical choice and decision in implementing new technology. The processes of the organization are affected by modern technology. For this reason, the implementation of new technology should be considered seriously. These days, technology can be found anywhere, 67% of adults use the internet based on a survey from 40 states. smartphones have also become common [1]. Technology has been profiled in several parts from the manufacture to service segment. It grows the well-being and life standard of people [2]. Technology choice depends on the competitiveness

and effectiveness of organizations [3]. Applying BT in firms has become more popular in the present time because of its importance. The transactions can be done by using a decentralized mechanism because BT is a distributed database. In BT, some blocks are related to each other and they cover many transactions. The transaction should be confirmed in terms of validness before adding to the system as a new block [4]. The chain of transactions can be represented as the blockchain. In Bitcoin, these transactions are public [5]. BT guarantees the transactions more secure for industries, businesses, organizations, and governments, hence the common use of BT will have a big influence on the firms in the future. The transaction data is reserved in various nodes in blockchain and it is known as a dispersed ledger. In the dispersed ledger, every user can enter the public ledger system. This can generate a stable environment and doesn't depend on third parties. The technology reduces system failure and other connected risks in the chain. BT can be a great area for keeping significant information. BT allows users to monitor prior transactions [6]. Implementing a new BT includes various risk factors from various parts. To apply BT at the maximum level, these risks should be assessed cautiously. In this research, these risks have been assessed in multi-criteria decision making (MCDM) and these are ranked by using Single Valued Neutrosophic Sets (SVNSs), Analytic Hierarchy Process (AHP), and Decision Making and Trial Evaluation Laboratory (DEMATEL). SVNSs are used to deal with uncertainties [7] and likely risk factors are hierarchical based on their importance by AHP [8] and the relationship between them with DEMATEL [9]. To get the best of information, ranking BT risks by using the MCDM technique has not been studied yet. This work will provide a decision to the firms to decide which of these risks are more serious and which of them should be reduced primarily. The remainder of the paper follows as section 2 provides a brief description of blockchain technologies. SVNSs are summarized in section 3. The proposed MCDM methodology based on SVNSs is presented in section 4. Section 5 shows the application for risk assessments of BT by using AHP and DEMATEL. The attained outcomes and future research directions have been discussed in section 6.

2. Blockchain Technologies

BT is considered as one of the most significant creations after the Internet [10]. BT and Internet technology are different in some significant parts. On the Internet, only the information and the copies of things are moved but the original information cannot. In BT, the value of the things is reserved in a time-stamped transaction in a common ledger in a safe way [11]. BT is an information technology [11] and is based on a dispersed ledger technology [6]. With this technology, there is no need to depend on a third party. In BT, when a transaction is done, it should be confirmed. The transaction is only accepted when the agreement is ensured. Then, the information about the transaction is kept on a new block and the new block is added after the other blocks on the chain [6]. Once the information is confirmed and added to the chain, it cannot be removed anymore [6, 10]. BT has become common with Bitcoin implementation [11, 12] and is used in various parts like the Internet of things, economics, and medicine, etc. [13]. Though BT suggests various chances for firms, it can only add value to the products if the processes are appropriate for BT implementation. For example, if there is a need for data transparency or immutability, BT will be beneficial, but if the transaction speed is important, BT will not be suitable [14].

3. The Proposed Model for Risk Evaluation of Blockchain Technologies

In this research, MCDM methodology based on SVNSs, AHP, and DEMATEL methods are suggested for risk assessment of BTs. Three key steps in methodology. The first step, factors of risk is recognized by conducting a literature review and specialist reviews. Then BT risk factors are determined and the

hierarchical structure of the problem is built. In the second step, the risk factors are assessed. For the second step to be achieved AHP method is used to attain main and sub-criteria weights and the DEMATEL method is used to show the importance of main and sub-criteria and the relationship between them. Finally, the risks are ranked according to the weights of the AHP method and showing the impact of the relationship between main and sub-criteria. The detailed framework of the proposed methodology is shown in figure 1.

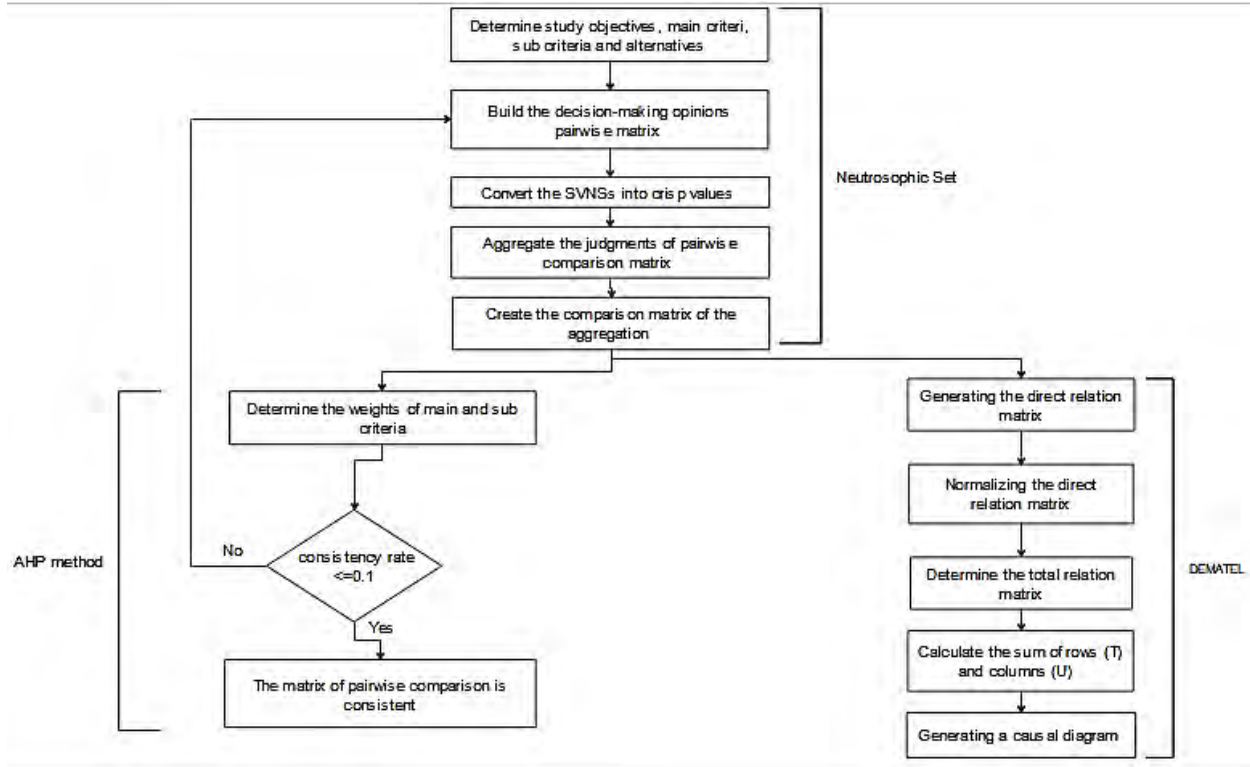


Fig 1. Steps of SVN, AHP, and DEMATEL methodology

3.1. Neutrosophic theory

The neutrosophic set can model the decision maker's perspectives in the neutrosophic single value scale [15] and apply aggregation to produce the final vision. Neutrosophic set multiplications and calculations are illustrated in [16]. The steps of the neutrosophic theory are illustrated in [17]:

Step 1. Build the decision-making opinions pairwise matrix according to SVN scale in table 1 using the mentioned form:

$$L^E = \begin{bmatrix} l_{11}^E & \cdots & l_{1y}^E \\ \vdots & \ddots & \vdots \\ l_{x1}^E & \cdots & l_{xy}^E \end{bmatrix} \quad (1)$$

Where E pointed to the number of decision-makers.

Table1. Single valued Neutrosophic scale

Linguistic term	SVNSs
Extremely evil	(0.00,1.00,1.00)
Very Highly evil	(0.10,0.90,0.90)
Very evil	(0.20,0.85,0.80)
Evil	(0.30,0.75,0.70)
Medium evil	(0.40,0.65,0.60)
Medium	(0.50,0.50,0.50)
Medium better	(0.60,0.35,0.40)
Better	(0.70,0.25,0.30)
Very better	(0.80,0.15,0.20)
Very Highly better	(0.90,0.10,0.10)
Extremely better	(1.00,0.00,0.00)

Step 2. Convert the SVNSs into crisp values by the use of the score function [18]:

$$V(I_{mn}^E) = \frac{2 + T_{mn}^E - I_{mn}^E - F_{mn}^E}{3} \quad (2)$$

$T_{mn}^E, I_{mn}^E, F_{mn}^E$ presents truth, indeterminacy, and falsity of the SVNSs.

Step 3. Aggregate the judgments of the pairwise comparison matrix as

$$l_{mn} = \frac{\sum_{E=1}^E l_{mn}}{E} \quad (3)$$

Step 4. Create the comparison matrix of the aggregation as following:

$$L = \begin{bmatrix} l_{11} & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{m1} & \cdots & l_{mn} \end{bmatrix} \quad (4)$$

3.2. The AHP method

The steps of the AHP method are shown in [17] as :

Step 1: Calculate the weights of the main criteria and sub-criteria.

Step 1.1: Calculate the normalization using the following equation.

$$w_m^x = \frac{w_m}{\sum_{m=1}^x w_m}; m = 1, 2, 3, \dots, x \quad (5)$$

Step 1.2: Calculate the row average.

$$w_m = \frac{\sum_{n=1}^y (l_{mn})}{y}; m = 1, 2, 3, \dots, x; n = 1, 2, 3, \dots, y; \quad (6)$$

Step 2: Check the consistency of matrix to ensure the consistency the pair-wise comparison matrix [17].

3.3. The DEMATEL method

The steps of the DEMATEL method are illustrated in [19].

Step 1: Generating the direct relation matrix

The matrix of direct relation $s \times s$ is obtained through step 4 in neutrosophic theory.

Step 2: Normalizing the direct relation matrix.

The normalized direct relation matrix uses the following equation:

$$B = \frac{1}{\max_{1 \leq x \leq s} \sum_{y=1}^s l_{xy}} \quad (7)$$

$$V = B \times L \quad (8)$$

Step 3: Determine the total relation matrix.

This step uses the Matlab software to obtain an identity matrix using the following equation:

$$O = V(I - V)^{-1} \quad (9)$$

Step 4: Calculate the sum of rows (T) and columns (U)**Step 5:** Generating a causal diagram

The causal diagram is attained by $(T + U)$ and $(T - U)$ is the outcome of the DEMATEL method.

4. Application

The case study for assessing risk factors of BT, in this section. A company in the smart village in Egypt needs to implement BT for its operations. But the managers recognize that some risks can happen during the implementation of operations, so they decided to assess these risks and calculate which of them have more important before the implementation. In the beginning, the factors of risks are collected by using previous work [10, 11, 13, 20-25] and decision-makers. As a consequence of this, 8 main criteria and 28 sub-criteria are calculated for risk assessment of BT as shown in Figure 2. Then three specialists assessed these main and sub-criteria by using AHP and DEMATEL method.

Evaluating Blockchain Technology Risks

Evaluating Blockchain Technology Risks	C1-Environmental/Cultural	S1: Negative image of BT
		S2: Uncertainty of customers
	C 2- legal and regulatory challenges	S3: Unclear Legal Jurisdictions
		S4: Regulatory barriers
		S5: Antitrust
	C3- Energy	S6: High consumption
		S7: Importing energy efficiency
		S8: Energy intensive cryptocurrency validation process
	C4- Adoption challenges	S9: System speed
		S10: User experience
		S11: Lack of knowledge
		S12: Technology usability
	C5- Organizational and strategic	S13: Need of skilled worker
		S14: Resistance to changing technology
		S15: Lack of equipment and tool
		S16: Lack of management support
	C6- Technical	S17: Lack of customer awareness
		S18: Access to technology
		S19: Limited transaction capacity
		S20: Scaling due to processing requirements
		S21: Untasted code
	C7- Financial	S22: Usage cost
		S23: Training cost
		S24: Energy cost
	C8- Security	S25: Cyberattacks
		S26: Privacy
		S27: Shared data among multiple peer
		S28: Data leaks

Fig 2. Evaluation risk factors (Criteria and sub-criteria)

4.1. Neutrosophic theory results

The neutrosophic set can model the decision maker's perspectives in neutrosophic single value scale as shown in table 1 and apply aggregation to produce the final vision. The steps of the neutrosophic theory are showed as follows:

Step 1: Build the decision-making opinions pairwise matrix according to SVNss scale using Eqs. (1).

Step 2: Convert the SVNss into crisp values by the use of the score function using Eqs. (2).

Step 3: Aggregate the judgments of the pairwise comparison matrix using Eqs. (3.)

Step 4: Create the comparison matrix of the aggregation as shown in table 2 using Eqs. (4).

4.2. The AHP results

Step 1: Compute the normalization matrix using Eq. (5) As shown in table 3.

Table 2. Crisp value of aggregated pairwise comparison matrix of criteria.

C _x	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆	C ₇	C ₈
C ₁	1	0.577333	0.494	0.577333	0.388667	0.316333	0.649333	0.605333
C ₂	1.748918	1	0.716	0.394	0.538333	0.538667	0.538667	0.427333
C ₃	2.206193	1.415172	1	0.538667	0.460667	0.499667	0.460667	0.394
C ₄	1.748918	3.232941	1.874459	1	0.461	0.505	0.394	0.610667
C ₅	2.714845	1.952573	2.513728	2.203655	1	0.605	0.649333	0.538333
C ₆	3.562138	1.874459	2.078114	2.260107	1.819944	1	0.538667	0.460667
C ₇	1.5478	2.751943	2.28177	3.464899	1.5478	1.874459	1	0.499667
C ₈	1.74183	3.157365	3.464899	1.673342	1.952573	2.28177	2.078114	1

Step 1.2: Determine the weights of criteria, local and global sub-criteria using Eq. (6) as shown in table 4. Figure 3 shows the weights of the main criteria.

Table 3. Normalization values of main criteria.

C _{yz}	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆	C ₇	C ₈
C ₁	0.06146	0.03617	0.034251	0.047666	0.047578	0.041509	0.102925	0.133451
C ₂	0.107489	0.06265	0.049643	0.03253	0.0659	0.070683	0.085384	0.094209
C ₃	0.135594	0.08866	0.069334	0.044474	0.056392	0.065565	0.07302	0.086861
C ₄	0.107489	0.202543	0.129963	0.082563	0.056433	0.066265	0.062453	0.134627
C ₅	0.166855	0.122328	0.174286	0.18194	0.122414	0.079387	0.102925	0.11868
C ₆	0.21893	0.117434	0.144084	0.186601	0.222787	0.131218	0.085384	0.101558
C ₇	0.095128	0.172408	0.158204	0.286072	0.189473	0.245963	0.158509	0.110156
C ₈	0.107054	0.197808	0.240235	0.138156	0.239023	0.29941	0.3294	0.220459

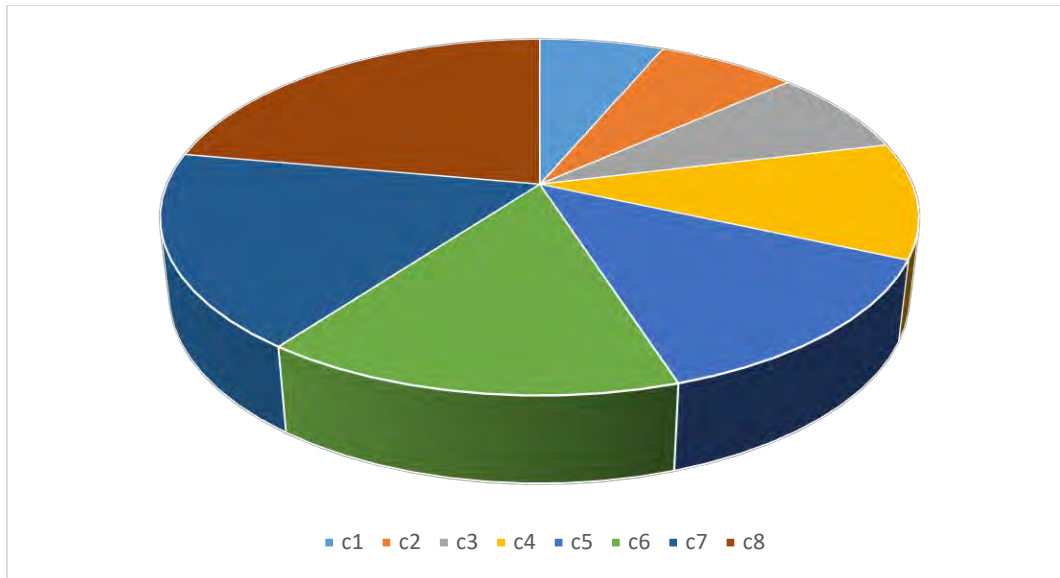


Fig 3. Weights of main criteria.

Step 2: The Consistency Ratio (CR) = 0.10. according to [17] such that $CR \leq 0.1$, therefore, the matrix of pairwise comparison is consistent. Table 5 displays the importance of local and global weights of main and sub-criteria based on AHP calculations. Hence C_8 (security) is the most important in the main criteria and C_1 (Environmental/Cultural) is the least important in the main criteria. For sub-criteria S_{24} (Energy cost) is the most important in sub-criteria and S_9 (System speed) is the least important in sub-criteria.

Table 4. Weights of main criteria, local and global sub-criteria.

Main Criteria	Sub criteria	Weights	Local weights	Global
C_1		0.063126		
	S1		0.32903	0.02077
	S2		0.67097	0.042356
C_2		0.071061		
	S3		0.283611	0.020154
	S4		0.315125	0.022393
	S5		0.401263	0.028514
C_3		0.077487		
	S6		0.236806	0.018349
	S7		0.305799	0.023695
	S8		0.457395	0.035442
C_4		0.105292		
	S9		0.137244	0.014451
	S10		0.204308	0.021512
	S11		0.349301	0.036779
	S12		0.309147	0.032551
C_5		0.133602		

	S13	0.159466	0.021305
	S14	0.292025	0.039015
	S15	0.247912	0.033122
	S16	0.300597	0.04016
C₆	0.150999		
	S17	0.122023	0.018425
	S18	0.200505	0.030276
	S19	0.198784	0.030016
	S20	0.226835	0.034252
	S21	0.251853	0.03803
C₇	0.176989		
	S22	0.211513	0.037435
	S23	0.286858	0.050771
	S24	0.501629	0.088783
C₈	0.221443		
	S25	0.14637	0.032413
	S26	0.25559	0.056599
	S27	0.238016	0.052707
	S28	0.360024	0.079725

4.3. The DEMATEL results

Step 1: Generating the direct relation matrix in table 5 of the main criteria and direct relation matrix for the sub-criteria of security criteria in table 5.

Step 2: Normalizing the direct relation matrix for the main criteria in table 6 using Eqs. (7, 8).

Step 3: Determine the total relation matrix using Eq. (9) In table 7.

Step 4: Calculate the sum of rows (T) and columns (U) in table 8 and rank according to the importance of the main criteria in table 8.

Step 5: Generating a causal diagram as shown in figure 4. It shows the security, financial, technical, and organizational is the most important main criteria. C₅ (Organizational), C₆ (Technical), C₇ (Financial), C₈ (Security) are causing effect while others are being affected

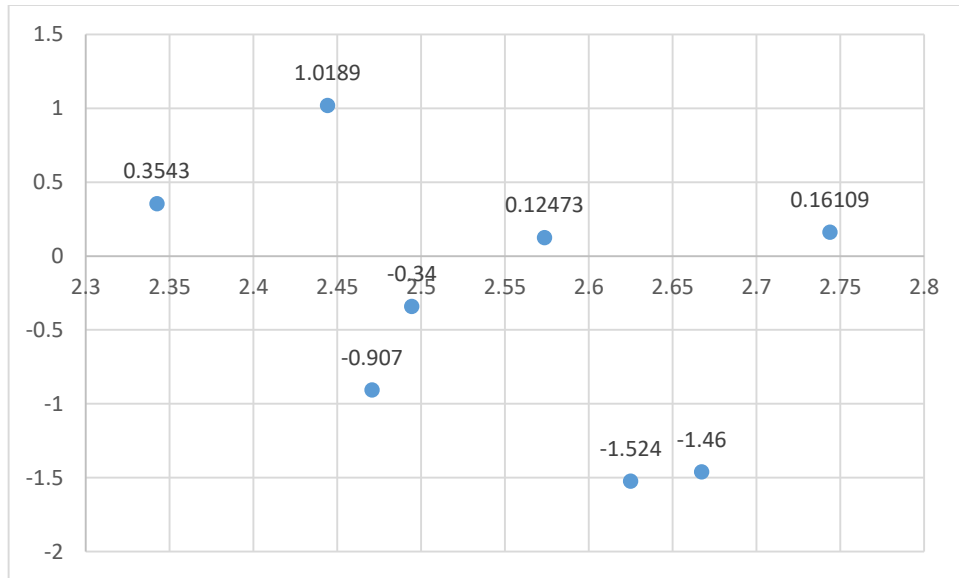


Fig 4. The causal diagram for the main criteria.

Table 5. The direct relation matrix for sub-criteria of security.

S_{yz}	S_{25}	S_{26}	S_{27}	S_{28}
S_{25}	1	0.394	0.671667	0.671667
S_{26}	4.051484	1	0.677333	0.571667
S_{27}	1.687315	1.540713	1	0.671667
S_{28}	1.687315	2.69554	1.687315	1

Table 6. Normalization of direct relation matrix of main criteria.

C_{yz}	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
C_1	0.0578	0.03337	0.028553	0.03337	0.022465	0.018284	0.037531	0.034988
C_2	0.101087	0.0578	0.041385	0.022773	0.031116	0.031135	0.031135	0.0247
C_3	0.127518	0.081797	0.0578	0.031135	0.026627	0.028881	0.026627	0.022773
C_4	0.101087	0.186864	0.108344	0.0578	0.026646	0.029189	0.022773	0.035297
C_5	0.156918	0.112859	0.145293	0.127371	0.0578	0.034969	0.037531	0.031116
C_6	0.205892	0.108344	0.120115	0.130634	0.105193	0.0578	0.031135	0.026627
C_7	0.089463	0.159062	0.131886	0.200271	0.089463	0.108344	0.0578	0.028881
C_8	0.100678	0.182496	0.200271	0.096719	0.112859	0.131886	0.120115	0.0578

Table 7. Total relation matrix of main criteria.

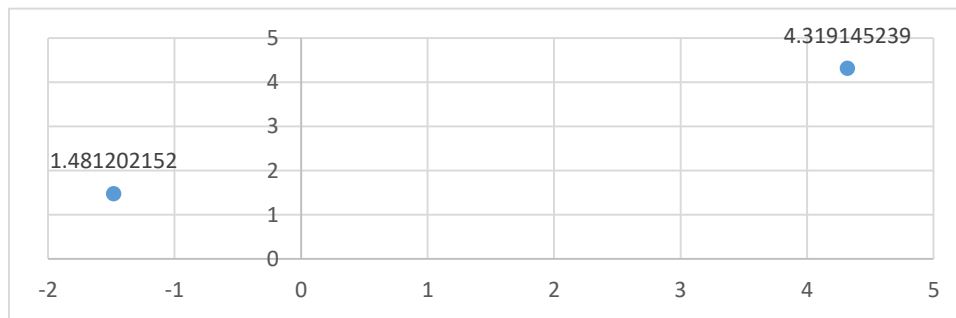
C_{yz}	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
C_1	0.121175	0.094524	0.081078	0.076915	0.05188	0.046394	0.062404	0.053891
C_2	0.176417	0.124154	0.098908	0.0713	0.064241	0.061554	0.059248	0.04676
C_3	0.211359	0.154533	0.119665	0.082342	0.062187	0.061692	0.057744	0.047614
C_4	0.220165	0.289444	0.193498	0.123948	0.074857	0.074542	0.065226	0.069124
C_5	0.307497	0.250131	0.257612	0.216011	0.11798	0.091106	0.090063	0.074551
C_6	0.376721	0.260514	0.247866	0.233394	0.173768	0.119465	0.089823	0.075681
C_7	0.295829	0.347913	0.287103	0.32493	0.173242	0.183664	0.124348	0.084594

C₈	0.348777	0.401178	0.387831	0.254874	0.219212	0.228671	0.20381	0.123685
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Table 7. The sum rows and columns of the main criteria.

C_{yz}	T	U	T+U	T – U	Ranking
C₁	0.588261	2.079	2.6673	-1.46	5
C₂	0.702581	1.9224	2.625	-1.524	1
C₃	0.797136	1.6736	2.4707	-0.907	7
C₄	1.110804	1.3837	2.4945	-0.34	4
C₅	1.40495	0.9374	2.3424	0.3543	8
C₆	1.577232	0.8671	2.4443	1.0189	6
C₇	1.821624	0.7527	2.5737	0.12473	2
C₈	2.168039	0.5759	2.7439	0.16109	3

Figure 5 shows S_2 (Uncertainty of customers) is causing effect while S_1 (Negative image of BT) is being affected in C_1 (Environmental/Cultural). Figure 6 shows S_5 (Antitrust) is causing effect while S_3 (Unclear Legal Jurisdictions) and S_4 (Regulatory barriers) are being affected in C_2 (legal and regulatory challenges). Figure 7 shows S_8 (Energy-intensive cryptocurrency validation process) is causing effect while S_6 (High consumption) and S_7 (Importing energy efficiency) are being affected in C_3 (Energy). Figure 8 shows S_{11} (Lack of knowledge) and S_{12} (Technology usability) are causing effect while S_9 (System speed) and S_{10} (User experience) are being affected in C_4 (Adoption challenges). Figure 9 shows S_{14} (Resistance to changing technology) and S_{16} (Lack of management support) are causing effect while S_{13} (Need of skilled worker) and S_{15} (Lack of equipment and tool) are being affected in C_5 (Organizational and strategic). Figure 10 shows S_{20} (Scaling due to processing requirements) are S_{21} (Untested code) are causing effect while S_{17} (Lack of customer awareness), S_{18} (Access to technology), and S_{19} (Limited transaction capacity) are being affected in C_6 (Technical). Figure 11 shows S_{24} (Energy cost) is causing effect while S_{22} (Usage cost) and S_{23} (Training cost) are being affected in C_7 (Financial). Figure 12 shows S_{27} (Shared data among multiple peers) and S_{28} (Data leaks) are causing effect while S_{25} (Cyberattacks) and S_{26} (Privacy) are being affected in C_8 (Security).

**Fig 5.** The causal diagram for C_1 (Environmental) sub-criteria.

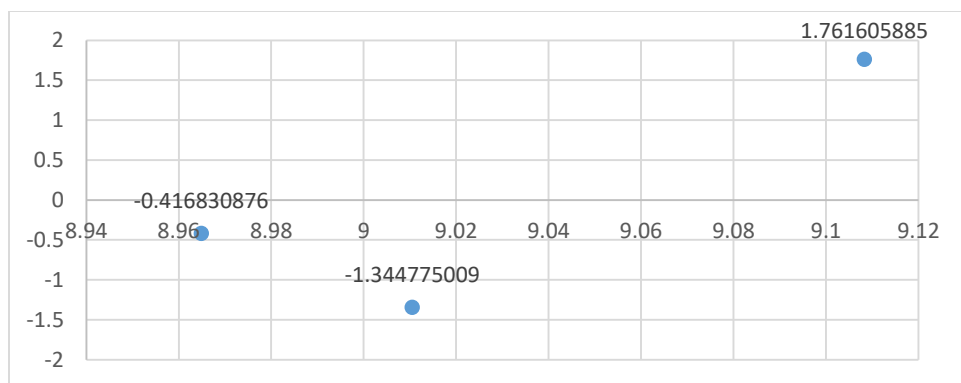


Fig 6. The causal diagram for C₂ (legal and regulatory challenges) sub-criteria.

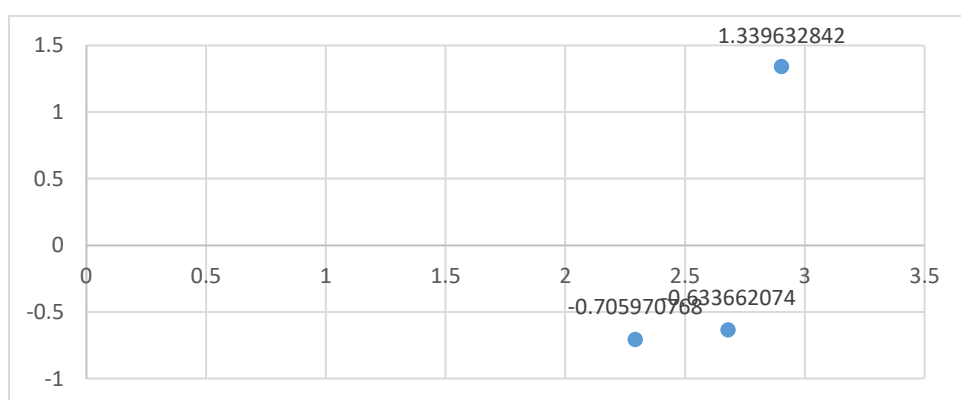


Fig 7. The causal diagram for C₃ (Energy) sub-criteria.

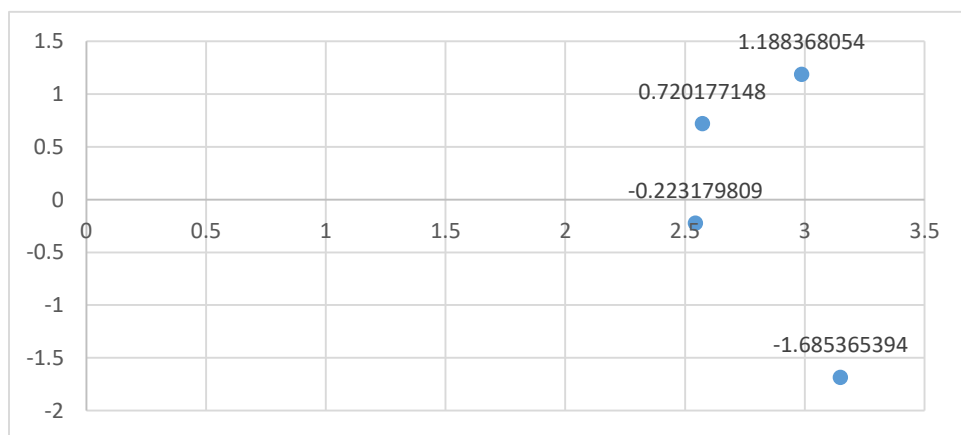


Fig 8. The causal diagram for C₄ (Adoption challenges) sub criteria.

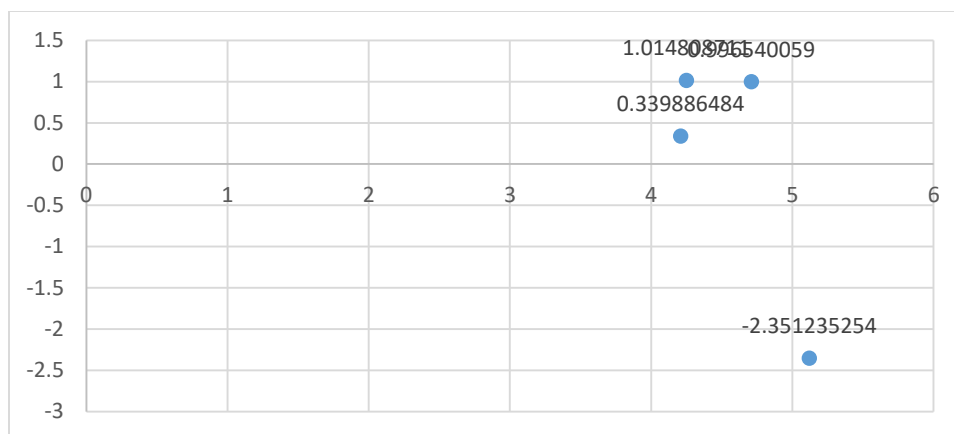


Fig 9. The causal diagram for C_5 (Organizational and strategic) sub-criteria.

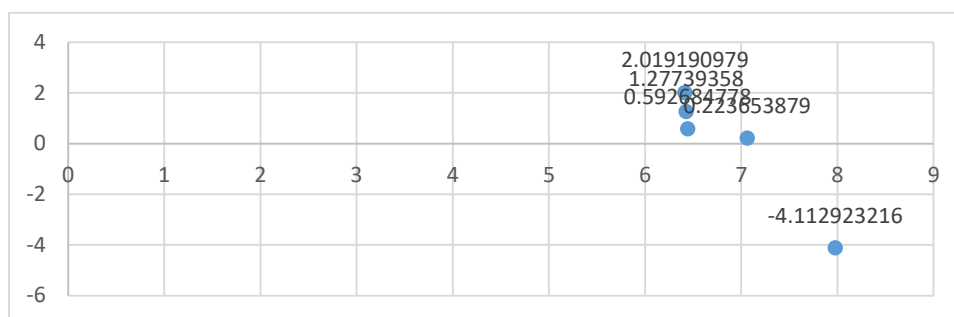


Fig 10. The causal diagram for C_6 (Technical) sub-criteria.

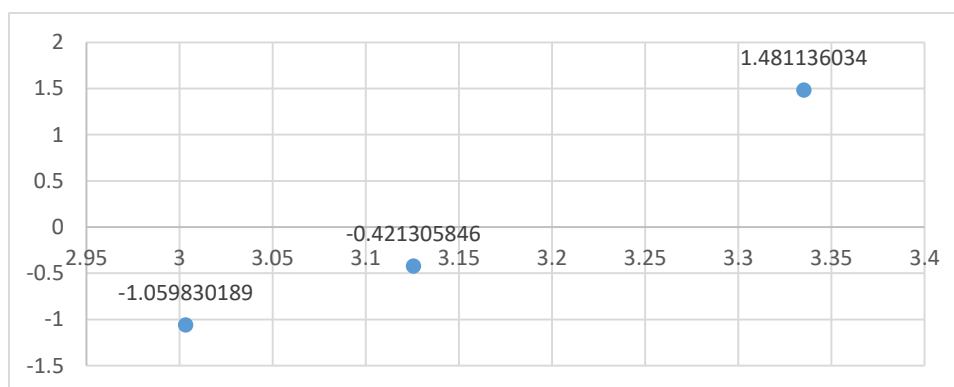


Fig 11. The causal diagram for C_7 (Financial) sub-criteria.

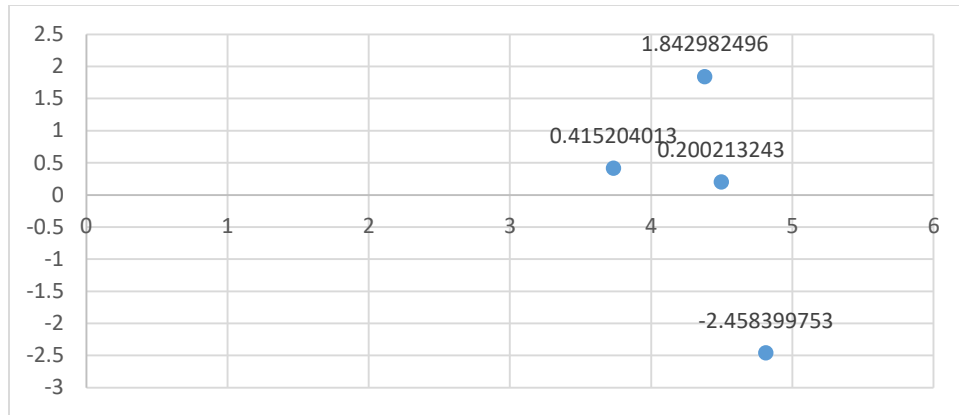


Fig 12. The causal diagram for Cs (Security) sub-criteria.

5. Conclusion and Future Works

BT is one of the most significant creations of the Internet. The usage of this system has become fairly common for firms. Though, implementing a new BT system in firms includes different risk factors. Consequently, firms need to address and analyze these risks. For this goal, the risks of BT in a firm are measured and ranked by using SVNSe, AHP, and DEMATEL method. In this ranking process, Energy, environmental/cultural, financial, security, organizational, technical, legal, and regulatory challenges and adoption challenges risks are taken into account. 28 sub-risks covered by these risks are assessed under these groups. As a result, security is considered as the most significant risk factor among the eight risks and energy cost, and data leaks are ranked as the first and second important sub-risks correspondingly. DEMATEL results show security, financial cost, technical and organizational are causing effect while others are being affected. So the administrators should give more importance to these types of risks. For future research, the scope of the problem can be extended and the solutions of minimizing the risks for BT can be added as alternatives and the problem can be solved by MCDM techniques.

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Neutrosophic Triplet m – Banach Spaces

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Abstract: Neutrosophic triplet theory has an important place in neutrosophic theory. Since the neutrosophic triplet set (Nts), which have the feature of having multiple unit elements, have different units than the classical unit, they have more features than the classical set. Also, Banach spaces are complete normed vector space defined by real and complex numbers that are studied historically in functional analysis. Thus, normed space and Banach space have an important place in functional analysis. In this article, neutrosophic triplet m - Banach spaces (NtmBs) are firstly obtained. Then, some definitions and examples are given for NtmBs. Based on these definitions, new theorems are given and proved. In addition, it is shown that NtmBs is different from neutrosophic triplet Banach space (NtBs). Furthermore, it is shown that relationship between NtmBs and NtBs. So, we added a new structure to functional analysis and neutrosophic triplet theory.

Keywords: neutrosophic triplet set, neutrosophic triplet normed space, neutrosophic triplet Banach space, neutrosophic triplet m - normed space, neutrosophic triplet m – Banach space

1 Introduction

Neutrosophic theory [1] has also supported the scientific world with more objective solutions by obtaining new solutions and methods in many fields in both application sciences and theoretical sciences. Neutrosophic theory was obtained by Smarandache in order to obtain more objective results by taking into account the effects of uncertainties encountered in science in 1998 [1]. A neutrosophic number is formulated by (T, I, F) . Where, T is truth function; I is indeterminacy function and F is falsity function and these functions's values are independently. Thus, neutrosophic theory is generalized of fuzzy theory [2] and intuitionistic fuzzy theory [3] and neutrosophic theory is more useful than fuzzy theory and intuitionistic fuzzy theory. Thus, many researchers studied neutrosophic theory for these reasons [4-6]. Recently, Olgun et al. studied neutrosophic logic on the decision tree

[7]; Şahin et al. obtained decision making application for single – valued neutrosophic set [8] and Uluçay et al. introduced decision making application for neutrosophic soft expert graphs [9]. Uluçay et al. [10] proposed an outranking approach for MCDM-problems with neutrosophic multi-sets. Uluçay and Şahin [11] defined the concepts of neutrosophic multi group. Bakbak et al. [12] developed some new operations. Uluçay et al. [13] introduced a new hybrid distance-based similarity measure for refined neutrosophic sets.

Neutrosophic triplet structures, which are a sub-branch of the neutrosophic theory, also aimed to carry the new advantages of neutrosophy to the algebraic structure. For this reason, many studies have been carried out on neutrosophic triplet structures. Thus, many structures in classical algebra were reconsidered in neutrosophic theory and new features emerged. Thus, a neutrosophic triplet structure has become available in fixed point theory Also, Smarandache and Ali studied neutrosophic triplet set (Nts) [14]. Since the neutrosophic triplet set (Nts), which have the feature of having multiple unit elements, have different units than the classical unit, they have more features than the classical set. A Nts k is formulated by $(k, \text{neut}(k), \text{anti}(k))$. Furthermore, the sets of have been studied by the scholars, on neutrosophic sets [15-20] neutrosophic triplet structures in neutrosophic triplet algebraic structures [21-26], some metric spaces on neutrosophic triplet [27-32]. Recently, Şahin et al. studied Nt m - metric space [33]; Zhang et al. obtained cyclic associative neutrosophic extended triplet groupoids [34]; Sahin et al. obtained Nt normed space [22], Shalla et al. introduced direct and semi-direct product of neutrosophic extended triplet group [35]; Şahin et al. Nt partial bipolar metric space [36]; Şahin et al. Nt partial g -metric space [37], Kandasamy et al. obtained Nts in neutrosophic rings [38], Shalla et al. introduced neutrosophic extended triplet group action [44].

Metric spaces, normed spaces and Banach spaces have an important place in classical mathematics. Metric spaces are widely used, especially in fixed point theory. Thus, Asadi, Karapınar and Salimi introduced m - metric spaces [39] in 2014. m - metric space is a generalized form of classical metric space and classical p - metric space. The m - metric spaces have an important role in fixed point theory. Recently, Souayah et al. obtained fixed point theorems for m – metric space [40]; Patle et al. studied mappings in m – metric space [41] and Pitchaimani et. al introduced $\$$ -contraction on m – metric space [42]. Also, Normed spaces and Banach spaces, which are special cases of normed spaces, have an important usage area, especially in the field of analysis.

In this article, we have defined Ntmns with a more specific structure than neutrosophic triplet m -metric spaces. We discussed the properties of this structure and proved the theorems related to this

structure. We also discussed the relationship between this structure and the Ntmms. In addition, with the help of some definitions in Ntmms, we obtained important definitions such as convergence and Cauchy sequence in Ntmns. Also, we have defined MtmBs. We compared these structures with previously obtained neutrosophic triplet structures. Thanks to this comparison, we have determined that the structures we have obtained have different and new features than others. Thus, we added a new structure to the neutrosophic triplet theory and prepared the ground for new structures that can be obtained. In Section 2, we give definitions and properties for neutrosophic structures [14], [36] and [37]. In Section 3, we define Ntmns and NtmBs and we give some properties for Ntmns and NtmBs. Furthermore, we obtain neutrosophic triplet m – metric space (NTmms) reduced by Ntmns. Also, we show that Ntns are different from the Ntmns due to triangular inequality. Then, we examine relationship between Ntmns and Ntns. In Section 4, we give conclusions.

2 Preliminaries

Definition 2.1 [14]: Let μ be a binary operation. An Nts (L, μ) is a set such that for $l \in L$,

- i) There is neutral of “ l ” such that $l \mu \text{neut}(l) = \text{neut}(l) \mu l = l$,
- ii) There is anti of “ l ” such that $l \mu \text{anti}(l) = \text{anti}(l) \mu l = \text{neut}(l)$.

Also, an Nt “ l ” is showed with $(l, \text{neut}(l), \text{anti}(l))$.

Furthermore, $\text{neut}(l)$ must different from classical unit element.

Definition 2.2: [43] Let (L, μ, π) be an Nts with two binary operations μ and π . Then (L, μ, π) is called Ntf if the following conditions are satisfied.

1. (L, μ) is a commutative Nt group with respect to μ .
2. (L, π) is an Nt group with respect to π .
3. $k\pi(l \mu m) = (k \pi l)\mu(k \pi m)$; $(l \mu m)\pi k = (l \pi k)\mu(m \pi k)$ for all $k, l, m \in L$.

Definition 2.3: [22] Let (L, μ_1, π_1) be an Ntf and let (V, μ_2, π_2) be an Nts with binary operations “ μ_2 ” and “ π_2 ”. Then (V, μ_2, π_2) is called an Ntvs if the following conditions are satisfied.

- i) $m \mu_2 n \in V$ and $m \#_2 s \in V$; $m, n \in V$ and $s \in L$;

- ii) $(m\mu_2n) \mu_2l = m\mu_2 (n\mu_2l); m, n, l \in V;$
 - iii) $m\mu_2n = n\mu_2m; m, n \in V;$
 - iv) $(m\mu_2n) \pi_2s = (m\pi_2s) \mu_2(n\pi_2s); s \in L \text{ and } m, n \in V;$
 - v) $(s\mu_1p) \pi_2m = (s\pi_2m) \mu_1(p\pi_2m); s, p \in L \text{ and } m \in V;$
 - vi) $(s\pi_1p) \pi_2m = s\pi_1(p\pi_2m); s, p \in L \text{ and } m \in V;$
 - vii) there exists at least an element $s \in L$ for each element m such that
- $$m\pi_2 \text{ neut}(s) = \text{neut}(s) \pi_2 m = m; m \in V.$$

Definition 2.4: [22] Let (V, μ_2, π_2) be an Ntvs on (L, μ_1, π_1) Ntf. If the function $\|\cdot\| : V \rightarrow \mathbb{R}^+ \cup \{0\}$ is satisfied the following properties, then the function $\|\cdot\|$ is an Ntn.

Where,

$f: L \times V \rightarrow \mathbb{R}^+ \cup \{0\}$ is a function such that

$f(k, l) = f(k, \text{anti}(l))$ and

if $l = \text{neut}(l)$, then $f(k, l) = 0; k \in L, l \in V$

a) $\|1\| \geq 0,$

b) If $l = \text{neut}(l)$, then $\|1\| = 0,$

c) $\|k \pi_2 l\| = f(k, l) \cdot \|1\|,$

d) $\|\text{anti}(l)\| = \|1\|,$

e) If there exists at least an element $m \in N$ for $k, l \in V$ pair such that

$\|k\mu_2 l\| \leq \|k\mu_2 l\mu_2 \text{neut}(m)\|$, then $\|k\mu_2 l\mu_2 \text{neut}(m)\| \leq \|k\| + \|1\|; k, l, m \in V.$

Also, $((V, \mu_2, \pi_2), \|\cdot\|)$ is called a Ntns.

Definition 2.5: [22] Let $((V, \mu_2, \pi_2), \|\cdot\|)$ be an Ntns. Let $\{a_n\}$ be a sequence in $((V, \mu_2, \pi_2), \|\cdot\|)$ and let m be an NTm reduced by $((V, \mu_2, \pi_2), \|\cdot\|)$. If each $\{a_n\}$ Nt Cauchy sequence in $((V, \mu_2, \pi_2), \|\cdot\|)$ is Nt convergent according to Ntmm, then $((V, \mu_2, \pi_2), \|\cdot\|)$ is called an Nt Banach space (NtBs).

Theorem 2.6:[22] Let (L, μ) be an NT group without zero divisors, with respect to μ . For $l \in L$;

- (i) $\text{neut}(\text{neut}(l)) = \text{neut}(l)$,
- (ii) $\text{anti}(\text{neut}(l)) = \text{neut}(l)$,
- (iii) $\text{anti}(\text{anti}(l)) = l$,
- (iv) $\text{neut}(\text{anti}(l)) = \text{neut}(l)$.

Definition 2.7:[33] Let L be a nonempty set and $m^d: L \times L \rightarrow \mathbb{R}^+ \cup \{0\}$ be a function. Then,

$$(i) m_{ln}^d = \min\{m(l, l), m(n, n)\} = m(l, l) \vee m(n, n); l, n \in L$$

$$(ii) M_{ln}^d = \max\{m(l, l), m(n, n)\} = m(l, l) \wedge m(n, n); l, n \in L$$

Definition 2.8: [33] Let (L, μ) be an Nts and $m: L \times L \rightarrow \mathbb{R}^+ \cup \{0\}$ be a function. If (L, μ) and m satisfy the following properties, then m is called an Nt m -metric space.

a) For all $l, n \in L$, $l \mu n \in L$,

b) If $m(l, l) = m(n, n) = m(l, n) = 0$, then $l = n$,

$$c) m_{ln}^d \leq m(l, n),$$

$$d) m(l, n) = m(l, n),$$

e) If there exists at least an element $s \in L$ for each pair $k, l \in L$ such that

$$m(k, l) \leq m(k, l \mu \text{neut}(s)), \text{ then } (m(k, l \mu \text{neut}(s))) - m_{kl} \leq (m(k, l) - m_{ks}) + (m(l, s) - m_{ls}).$$

Also, $((L, \mu), m)$ is called an Ntmms.

3 Neutrosophic Triplet m – Normed Space

In this section, we have defined Ntmns with a more specific structure than Ntmms. We discussed the properties of this structure and proved the theorems related to this structure. We also discussed the relationship between this structure and the Ntmms. In addition, with the help of some definitions in Ntmms, we obtained important definitions such as convergence and Cauchy sequence in Ntmns.

Definition 3.1: Let V be an Nts and let $m: V \times V \rightarrow \mathbb{R}^+ \cup \{0\}$ be a function. Then,

$$(i) m_{ln} = \min\{\|\text{neut}(l)\|_m, \|\text{neut}(n)\|_m\},$$

$$(ii) M_{ln} = \max\{\|\text{neut}(l)\|_m, \|\text{neut}(n)\|_m\}.$$

Definition 3.2: Let (V, μ_2, π_2) be an Ntv on the Ntf (L, μ_1, π_1) . Then function $\|\cdot\|_m : V \rightarrow \mathbb{R}^+ \cup \{0\}$ is an Nt m – norm (Ntmn) such that

$$a) 0 \leq \|\text{neut}(l)\|_m \leq \|l\|_m,$$

$$b) \text{ If } \|\mu_2 \text{neut}(n)\|_m = \|\text{neut}(l)\|_m = \|\text{neut}(n)\|_m = 0, \text{ then } l = n.$$

$$c) \|\beta \pi_2 l\|_m = f(\beta, l) \cdot \|l\|_m. \text{ Where, } f: L \times V \rightarrow \mathbb{R}^+ \cup \{0\}, f(\beta, l) = f(\beta, \text{anti}(l)) \text{ is a function.}$$

$$d) \|\text{anti}(l)\|_m = \|l\|_m,$$

(e) If there exists at least $l, p \in V$ for each $l, n \in V$ pair of elements such that

$$\|\mu_2 n\|_m \leq \|\mu_2 n \mu_2 \text{neut}(p)\|_m; \text{ then}$$

$$\|\mu_2 n \mu_2 \text{neut}(p)\|_m - m_{ln} \leq \|l\|_m + \|n\|_m - m_{lp} - m_{np}.$$

In this case, $((V, \mu_2, \pi_2), \|\cdot\|_m)$ is called Nt m – normed space (Ntmns).

Corollary 3.3: From Definition 3.2, we obtain that $m_{ln} \leq \|l\|_m$ and $m_{ln} \leq \|n\|_m$

since $0 \leq \|\text{neut}(l)\|_m \leq \|l\|_m$ and $0 \leq \|\text{neut}(n)\|_m \leq \|n\|_m$.

Corollary 3.4: From Definition 2.4 and Definition 3.2, an Ntmns is different from an Ntns since the triangle inequalities are different in these definitions.

Example 3.5: Let $P(M) = \{\emptyset, \{l\}, \{n\}, \{l, n\}\}$ be a set and μ be binary operation such that

$$K \mu N = \begin{cases} N \setminus K, & \text{if } s(K) < s(N) \\ K \setminus N, & \text{if } s(K) > s(N) \\ M, & \text{if } s(K) = s(N) \wedge K \neq N \\ K, & \text{if } K = N \end{cases}$$

Where, it is clear that $(P(M) \setminus \emptyset, \mu)$ be an Nts. Also,

$$\text{neut}(\{l\}) = \{l\}, \text{anti}(\{l\}) = \{l\}; \text{neut}(\{n\}) = \{n\}, \text{anti}(\{n\}) = \{n\}; \text{neut}(\{l, n\}) = \{l, n\}, \text{anti}(\{l, n\}) = \{l, n\}.$$

Then, from Definition 2.2, $(P(M) \setminus \emptyset, \mu, \cup)$ is an Ntf. Furthermore, from Definition 2.3, $(P(M) \setminus \emptyset, \mu, \cup)$ is an Ntns.

Now, we show that

$$\|K\|_m : P(M) \setminus \emptyset \rightarrow \mathbb{R}^+ \cup \{0\}, \|K\|_m = \begin{cases} 2^{s(K)}, & \text{if } K \neq \{l, n\} \\ 2^{s(K)-1}, & \text{if } K = \{l, n\} \end{cases} \text{ is an Ntmn such that}$$

$f: P(M) \setminus \emptyset \rightarrow P(M) \setminus \emptyset \rightarrow \mathbb{R}^+ \cup \{0\}$, $f(K, N) = \begin{cases} \frac{2^{s(K \cup N)-1}}{2^{s(N)}}, & \text{if } K \cup N = \{l, n\} \\ \frac{2^{s(K \cup N)}}{2^{s(N)}}, & \text{if } K \cup N \neq \{l, n\} \end{cases}$. Where $s(K)$ is the number of elements of the set K .

Since $\text{anti}(\{l\}) = \{l\}$, $\text{anti}(\{n\}) = \{n\}$, $\text{anti}(\{l, n\}) = \{l, n\}$; it is clear that $f(K, N) = f(K, \text{anti}(N))$ for $K, N \in P(M) \setminus \emptyset$.

a) Since $\text{neut}(\{l\}) = \{l\}$, $\text{neut}(\{n\}) = \{n\}$, $\text{neut}(\{l, n\}) = \{l, n\}$; it is clear that $0 \leq \|\text{neut}(l)\|_m \leq \|l\|_m$ for $K, N \in P(M) \setminus \emptyset$.

b) There are not $K, N \in P(M) \setminus \emptyset$ such that $\|K \mu \text{neut}(N)\|_m = \|\text{neut}(K)\|_m = \|\text{neut}(N)\|_m = 0$.

c) We assume that $K \cup N = \{l, n\}$. Thus, $\|K \cup N\|_m = 2^{s(K \cup N)-1} = \frac{2^{s(K \cup N)-1}}{2^{s(N)}} \cdot 2^{s(N)} = f(K, N) \cdot \|N\|_m$ for $K, N \in P(M) \setminus \emptyset$.

We assume that $K \cup N \neq \{l, n\}$. Thus, $\|K \cup N\|_m = 2^{s(K \cup N)} = \frac{2^{s(K \cup N)}}{2^{s(N)}} \cdot 2^{s(N)} = f(K, N) \cdot \|N\|_m$ for $K, N \in P(M) \setminus \emptyset$.

d) Since $\text{anti}(\{l\}) = \{n\}$, $\text{anti}(\{n\}) = \{n\}$, $\text{anti}(\{l, n\}) = \{l, n\}$; we obtain that $\|\text{anti}(l)\|_m = \|l\|_m$ for $K, N \in P(M) \setminus \emptyset$.

e) For $\{l\} \in P(M) \setminus \emptyset$, we obtain that $\|\{l\} \mu \{l\}\|_m \leq \|(\{l\} \mu \{l\}) \mu \text{neut}(\{n\})\|_m$. Thus,

$$\|\{l\} \mu \{l\} \mu \text{neut}(\{n\})\|_m \leq \|\{l\}\|_m + \|\{l\}\|_m + m_{\{l\}\{l\}} - m_{\{l\}\{n\}} - m_{\{l\}\{n\}}.$$

For $\{n\} \in P(M) \setminus \emptyset$, we obtain that $\|\{n\} \mu \{n\}\|_m \leq \|(\{n\} \mu \{n\}) \mu \text{neut}(\{l\})\|_m$. Thus,

$$\|\{n\} \mu \{n\} \mu \text{neut}(\{l\})\|_m \leq \|\{n\}\|_m + \|\{n\}\|_m + m_{\{n\}\{n\}} - m_{\{l\}\{n\}} - m_{\{l\}\{n\}}.$$

For $\{l, n\} \in P(M) \setminus \emptyset$, we obtain that $\|\{l, n\} \mu \{l, n\}\|_m \leq \|(\{l, n\} \mu \{l, n\}) \mu \text{neut}(\{n\})\|_m$. Thus,

$$\|\{l, n\} \mu \{l, n\} \mu \text{neut}(\{n\})\|_m \leq \|\{l, n\}\|_m + \|\{l, n\}\|_m + m_{\{l, n\}\{l, n\}} - m_{\{l, n\}\{n\}} - m_{\{l, n\}\{n\}}.$$

For $\{l\}, \{n\} \in P(M) \setminus \emptyset$, we obtain that $\|\{l\} \mu \{n\}\|_m \leq \|(\{l\} \mu \{n\}) \mu \text{neut}(\{l, n\})\|_m$. Thus,

$$\|\{l\} \mu \{n\} \mu \text{neut}(\{l, n\})\|_m \leq \|\{l\}\|_m + \|\{n\}\|_m + m_{\{l\}\{n\}} - m_{\{l, n\}\{n\}} - m_{\{l, n\}\{l\}}.$$

For $\{n\}, \{l\} \in P(M) \setminus \emptyset$, we obtain that $\|\{n\} \mu \{l\}\|_m \leq \|(\{n\} \mu \{l\}) \mu \text{neut}(\{l, n\})\|_m$. Thus,

$$\|\{n\} \mu \{l\} \mu \text{neut}(\{l, n\})\|_m \leq \|\{n\}\|_m + \|\{l\}\|_m + m_{\{l\}\{n\}} - m_{\{l, n\}\{n\}} - m_{\{l, n\}\{l\}}.$$

For $\{n\}, \{l, n\} \in P(M) \setminus \emptyset$, we obtain that $\|\{n\} \mu \{l, n\}\|_m \leq \|(\{n\} \mu \{l, n\}) \mu \text{neut}(\{l\})\|_m$. Thus,

$$\|\{n\} \mu \{l, n\} \mu \text{neut}(\{l\})\|_m \leq \|\{n\}\|_m + \|\{l, n\}\|_m + m_{\{n\}\{l, n\}} - m_{\{l, n\}\{x\}} - m_{\{n\}\{l\}}.$$

For $\{l, n\}, \{n\} \in P(M) \setminus \emptyset$, we obtain that $\|\{l, n\} \mu \{n\}\|_m \leq \|(\{l, n\} \mu \{n\}) \mu \text{neut}(\{l\})\|_m$. Thus,

$$\|\{l, n\} \mu \{n\} \mu \text{neut}(\{l\})\|_m \leq \|\{l, n\}\|_m + \|\{n\}\|_m + m_{\{l, n\}\{n\}} - m_{\{n\}\{l\}} - m_{\{l, n\}\{l\}}.$$

For $\{l\}, \{l, n\} \in P(M) \setminus \emptyset$, we obtain that $\|\{l\} \mu \{l, n\}\|_m \leq \|(\{l\} \mu \{l, n\} \mu \text{neut}(\{n\}))\|_m$. Thus,

$$\|\{l\} \mu \{l, n\} \mu \text{neut}(\{n\})\|_m \leq \|\{l\}\|_m + \|\{l, n\}\|_m + m_{\{l\}\{l, n\}} - m_{\{l, n\}\{n\}} - m_{\{n\}\{l\}}.$$

For $\{l, n\}, \{l\} \in P(M) \setminus \emptyset$, we obtain that $\|\{l, n\} \mu \{l\}\|_m \leq \|(\{l, n\} \mu \{l\} \mu \text{neut}(\{n\}))\|_m$. Thus,

$$\|\{l, n\} \mu \{n\} \mu \text{neut}(\{n\})\|_m \leq \|\{l, n\}\|_m + \|\{l\}\|_m + m_{\{l, n\}\{l\}} - m_{\{l, n\}\{n\}} - m_{\{l\}\{n\}}.$$

Hence, if there exists at least a $P \in P(M) \setminus \emptyset$ for each $K, N \in P(M) \setminus \emptyset$ pair of elements such that

$$\|K \mu N\|_m \leq \|K \mu N \mu \text{neut}(P)\|_m; \text{ then}$$

$$\|K \mu N \mu \text{neut}(P)\|_m - m_{KN} \leq \|K\|_m + \|N\|_m - m_{KP} - m_{PN}.$$

Therefore, $\|K\|_m$ is an Ntmns.

Theorem 3.6: Let $((V, \mu_2, \pi_2), \|\cdot\|_m)$ be an Ntmns. Then, the function

$$m: V \times V \rightarrow \mathbb{R}^+ \cup \{0\} \text{ defined by } m(l, n) = \|\{l \mu_2 \text{anti}(n)\}\|_m \text{ is an Ntmms.}$$

Proof: Let $l, n, p \in V$. From Definition 3.2,

a) Since (V, μ_2, π_2) is Ntvs, we obtain that $l \mu n \in V$, for all $l, n \in V$.

b) If $m(l, n) = \|\{l \mu_2 \text{anti}(n)\}\|_m = \|\{l \mu_2 \text{anti}(l)\}\|_m = \|\text{neut}(l)\|_m = \|\{n \mu_2 \text{anti}(n)\}\|_m = \|\text{neut}(n)\|_m = 0$, then $l = n$.

(c) We show that

$$m_{ln}^d = \min\{m(l, l), m(n, n)\} = \min\{\|\text{neut}(l)\|_m, \|\text{neut}(n)\|_m\} \leq m(l, n) = \|\{l \mu_2 \text{anti}(n)\}\|_m.$$

We assume that $\|\{l \mu_2 \text{anti}(n)\}\|_m \leq \|\{l \mu_2 \text{anti}(n) \mu_2 \text{neut}(p)\}\|_m$. From Definition 3.2,

$$m(l, n) = \|\{l \mu_2 \text{anti}(n)\}\|_m \leq \|l\|_m + \|\text{anti}(n)\|_m + m_{a\text{anti}(n)} - m_{lp} - m_{\text{anti}(n)p}. \text{ Also, since } \|\text{anti}(n)\|_m = \|n\|_m \text{ and Theorem 2.5; we obtain that}$$

$$m(l, n) = \|\{l \mu_2 \text{anti}(n)\}\|_m \leq \|l\|_m + \|n\|_m + m_{ln} - m_{lp} - m_{np}. \quad (1)$$

There are two cases:

1) Let $\|\text{neut}(l)\|_m \leq \|\text{neut}(n)\|_m$.

Then we show that $\|\text{neut}(l)\|_m \leq m(l, n) \leq \|l\|_m + \|n\|_m + m_{ln} - m_{lp} - m_{np}$.

We assume that

i) $\|\text{neut}(l)\|_m \leq \|\text{neut}(n)\|_m \leq \|\text{neut}(p)\|_m$. Thus, from (1),

$$\|\text{neut}(l)\|_m \leq m(l, n) \leq \|l\|_m + \|n\|_m + \|\text{neut}(l)\|_m - \|\text{neut}(l)\|_m - \|\text{neut}(n)\|_m.$$

Hence, $m_{ln}^d = \min(m(l, l), m(n, n)) = \min\{\|\text{neut}(l)\|_m, \|\text{neut}(n)\|_m\} = \|\text{neut}(l)\|_m \leq m(l, n)$.

ii) $\|\text{neut}(l)\|_m \leq \|\text{neut}(p)\|_m \leq \|\text{neut}(n)\|_m$. Thus, from (1)

$$\|\text{neut}(l)\|_m \leq m(l, n) \leq \|l\|_m + \|n\|_m + \|\text{neut}(l)\|_m - \|\text{neut}(l)\|_m - \|\text{neut}(p)\|_m. \quad \text{Hence,}$$

$m_{ln}^d = \min(m(l, l), m(n, n)) = \min\{\|\text{neut}(l)\|_m, \|\text{neut}(n)\|_m\} = \|\text{neut}(l)\|_m \leq m(l, n)$.

iii) Let $\|\text{neut}(p)\|_m \leq \|\text{neut}(l)\|_m \leq \|\text{neut}(n)\|_m$. Thus, from (1)

$$\|\text{neut}(l)\|_m \leq m(l, n) \leq \|l\|_m + \|n\|_m + \|\text{neut}(p)\|_m - \|\text{neut}(l)\|_m - \|\text{neut}(p)\|_m.$$

Therefore, $m_{ln}^d = \min(m(l, l), m(n, n)) = \min\{\|\text{neut}(l)\|_m, \|\text{neut}(n)\|_m\} = \|\text{neut}(l)\|_m \leq m(l, n)$.

2) Let $\|\text{neut}(n)\|_m \leq \|\text{neut}(l)\|_m$.

Then we show that $\|\text{neut}(n)\|_m \leq m(l, n) \leq \|l\|_m + \|n\|_m + m_{ln} - m_{lp} - m_{np}$.

We assume that

i) $\|\text{neut}(n)\|_m \leq \|\text{neut}(l)\|_m \leq \|\text{neut}(p)\|_m$. Thus, from (1),

$$\|\text{neut}(n)\|_m \leq m(l, n) \leq \|l\|_m + \|n\|_m + \|\text{neut}(n)\|_m - \|\text{neut}(n)\|_m - \|\text{neut}(l)\|_m.$$

Hence, $m_{ln}^d = \min(m(l, l), m(n, n)) = \min\{\|\text{neut}(l)\|_m, \|\text{neut}(n)\|_m\} = \|\text{neut}(n)\|_m \leq m(l, n)$.

ii) $\|\text{neut}(n)\|_m \leq \|\text{neut}(p)\|_m \leq \|\text{neut}(l)\|_m$. Thus, from (1)

$$\|\text{neut}(n)\|_m \leq m(l, n) \leq \|l\|_m + \|n\|_m + \|\text{neut}(n)\|_m - \|\text{neut}(n)\|_m - \|\text{neut}(p)\|_m.$$

Hence, $m_{ln}^d = \min(m(l, n), m(n, n)) = \min\{\|\text{neut}(l)\|_m, \|\text{neut}(n)\|_m\} = \|\text{neut}(n)\|_m \leq m(l, n)$.

iii) Let $\|\text{neut}(p)\|_m \leq \|\text{neut}(n)\|_m \leq \|\text{neut}(l)\|_m$. Thus, from (1)

$$\|\text{neut}(n)\|_m \leq m(l, n) \leq \|l\|_m + \|n\|_m + \|\text{neut}(n)\|_m - \|\text{neut}(p)\|_m - \|\text{neut}(p)\|_m.$$

Hence, $m_{ln}^d = \min(m(l, l), m(n, n)) = \min\{\|\text{neut}(l)\|_m, \|\text{neut}(n)\|_m\} = \|\text{neut}(n)\|_m \leq m(l, n)$.

(d) For any $n \in V$; suppose that $m(l, p) = \|l \mu_2 \text{anti}(p)\| \leq \|l \mu_2 \text{anti}(p) \mu_2 \text{neut}(n)\|$. Then

$$m(l, p) = \|l \mu_2 \text{anti}(p)\| \leq \|l \mu_2 \text{anti}(p) \mu_2 \text{neut}(n)\| = \|l \mu_2 \text{anti}(p) \mu_2 n \mu_2 \text{anti}(n)\|.$$

As V is an Nt commutative group, we obtain that

$$\begin{aligned}
\|l \mu_2 \text{anti}(p) \mu_2 n \mu_2 \text{anti}(n)\| &= \|(l \mu_2 \text{anti}(n)) \mu_2 (\text{anti}(p) \mu_2 n)\| \\
&\leq \|l \mu_2 \text{anti}(n)\| + \|\text{anti}(p) \mu_2 n\| + m_{lp} - m_{ln} - m_{np} \\
&= m(l, n) + m(n, p) + m_{lp} - m_{ln} - m_{np}
\end{aligned}$$

Thus; if $m(l, p) \leq m(l, p \mu_2 \text{neut}(n))$; then

$$(m(l, p \mu_2 \text{neut}(n)) - m_{lp}) \leq (m(l, n) - m_{ln}) + (m(n, p) - m_{np}).$$

Corollary 3.7: Every Ntmns is an Ntmms.

Definition 3.8: Let $((N, \mu_2, \pi_2), \|\cdot\|_m)$ be an Ntmns. Let $m: V \times V \rightarrow R$ be an Ntmms defined by

$$m(l, n) = \|l \mu_2 \text{anti}(n)\|_m.$$

Then, m is called the Ntmms reduced by $((V, \mu_2, \pi_2), \|\cdot\|_m)$.

Definition 3.9: Let $((V, \mu_2, \pi_2), \|\cdot\|_m)$ be a Ntmns. Let $\{l_n\}$ be a sequence in $((V, \mu_2, \pi_2), \|\cdot\|_m)$ and let m be an Ntmms reduced by $((V, \mu_2, \pi_2), \|\cdot\|_m)$. For all $\varepsilon > 0$ and $l \in V$ if there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$

$m(l, \{l_n\}) - m_{ll_n} = \|l \mu_2 \text{anti}(\{l_n\})\|_m - m_{ll_n} < \varepsilon$, then $\{l_n\}$ m -sequence is said to Nt converge to x . It is denoted by $\lim_{n \rightarrow \infty} l_n = l$ or $l_n \rightarrow l$.

Definition 3.10: Let $((V, \mu_2, \pi_2), \|\cdot\|_m)$ be an Ntmns. Let $\{l_n\}$ be a sequence in $((V, \mu_2, \pi_2), \|\cdot\|_m)$ and let m be an Ntmms reduced by $((V, \mu_2, \pi_2), \|\cdot\|_m)$. For all $\varepsilon > 0$ and $l \in V$ if there exists an $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$

$m(\{l_m\}, \{l_n\}) - m_{l_m l_n} = \|l_m \mu_2 \text{anti}(\{l_n\})\|_m - m_{l_m l_n} < \varepsilon$, then the sequence $\{l_n\}$ is called an Nt m - Cauchy sequence.

Definition 3.11: Let $((V, \mu_2, \pi_2), \|\cdot\|_m)$ be an Ntmns. Let $\{l_n\}$ be a sequence in $((V, \mu_2, \pi_2), \|\cdot\|_m)$ and let m be an Ntmms reduced by $((V, \mu_2, \pi_2), \|\cdot\|_m)$. If each $\{l_n\}$ Nt m - Cauchy sequence in $((V, \mu_2, \pi_2), \|\cdot\|_m)$ is Nt convergent according to Ntmms, then $((V, \mu_2, \pi_2), \|\cdot\|_m)$ is called an Nt m - Banach space.

Corollary 3.12: From Definition 2.5 and Definition 3.11, an NtBs is different from a NtmBs since the triangle inequalities are different in these definitions.

Theorem 3.13: Let $((V, \mu_2, \pi_2), \|\cdot\|)$ be an Ntns and $l = \text{neut}(l)$, for all $l \in V$. Then, $\|l\|_m = \|l\| + n$ is an Ntmn. Where, $n \in \mathbb{R}^+$.

Proof:

a) Since, $l = \text{neut}(l)$, it is clear that $0 \leq \|\text{neut}(l)\| + n \leq \|l\| + n$. Thus, $0 \leq \|\text{neut}(l)\|_m \leq \|l\|_m$

b) There are not $l, n \in V$ such that

$$\|l \mu_2 \text{neut}(n)\|_m = \|l \mu_2 \text{neut}(n)\| + n = \|\text{neut}(l)\|_m = \|\text{neut}(l)\| + n = \|\text{neut}(n)\|_m = \|\text{neut}(n)\| + n = 0.$$

c) Since $((V, \mu_2, \pi_2), \|\cdot\|)$ is an Ntns, we can define $\|k\pi_2 a\| = f(k, l) \cdot \|l\|$. Also, we assume that

$\|k\pi_2 l\|_m = f_m(k, l) \cdot \|l\|_m$ and $f_m(k, l) = (f(k, l) \cdot \|l\| + n) \setminus (\|l\| + n)$. Thus, we obtain that

$$\|k\pi_2 l\| + n = \|k\pi_2 l\|_m = f(k, a) \cdot \|l\| + n = f_m(k, l) \cdot \|l\|_m.$$

d) Since $((V, \mu_2, \pi_2), \|\cdot\|)$ is an Ntns, we obtain $\|\text{anti}(l)\| = \|l\|$.

Thus, $\|\text{anti}(l)\| + n = \|\text{anti}(l)\|_m = \|l\| + n = \|l\|_m$.

e) We assume that there exists $n \in V$ such that $\|l \mu_2 p\| \leq \|l \mu_2 p \mu_2 \text{neut}(n)\|$.
(2)

Also, we obtain $\|l \mu_2 p\| + n \leq \|l \mu_2 p \mu_2 \text{neut}(n)\| + n$.

Since $((V, \mu_2, \pi_2), \|\cdot\|)$ is an Ntns, from (2), $\|l \mu_2 p \mu_2 \text{neut}(n)\| \leq \|l\| + \|p\|$.
(3)

From (3), we obtain that $\|l \mu_2 p \mu_2 \text{neut}(n)\| + n \leq \|l\| + n + \|p\| + n$. Thus, we obtain that

$$\|l \mu_2 p \mu_2 \text{neut}(n)\|_m \leq \|l\|_m + \|p\|_m + n - n. \quad (4)$$

Furthermore, $m_{lp} = m_{ln} = m_{np} = n$, since, $l = \text{neut}(l)$ and $\|\text{neut}(l)\| = 0$. Thus, from (4),

$$\|l \mu_2 p \mu_2 \text{neut}(n)\|_m - m_{lp} \leq \|l\|_m + \|p\|_m - m_{ln} - m_{np}.$$

Corollary 3.14: If $l = \text{neut}(l)$, an Ntmns can be obtained from an Ntns.

Theorem 3.15: Let $((V, \mu_2, \pi_2), \|\cdot\|_m)$ be an Ntmns and $l = \text{neut}(l)$ for all $l \in V$. If the following condition is satisfied, then $\|\cdot\|_m$ an Ntn.

i) If $l = \text{neut}(l)$, then $\|l\|_m = 0$.

ii) If $l = \text{neut}(l)$, then $f(k, l) = 0$.

Proof:

Since $((V, \mu_2, \pi_2), \|\cdot\|_m)$ is an Ntmns, $f(k, l) = f(k, \text{anti}(l))$. Also, from condition ii, $f(k, l) = f(k, \text{anti}(l))$.

a) Since $((V, \mu_2, \pi_2), \|\cdot\|_m)$ is an Ntmns, it is clear that $\|l\|_m \geq 0$.

b) It is clear that from condition i.

c) Since $((V, \mu_2, \pi_2), \|\cdot\|_m)$ is an Ntmns, we obtain that $\|k\pi_2 l\|_m = f(k, l) \cdot \|l\|_m$.

d) Since $((V, \mu_2, \pi_2), \|\cdot\|_m)$ is an Ntmns, we obtain that $\|\text{anti}(l)\|_m = \|l\|_m$.

e) From condition i, If $l = \text{neut}(l)$, then $\|l\|_m = 0$. Since, $l = \text{neut}(l)$, we obtain that $\|\text{neut}(l)\|_m = 0$, for all $l \in V$. Thus, we obtain that $m_{ln} = 0$, for all $l, n \in V$.
(5)

Also, since $((V, \mu_2, \pi_2), \|\cdot\|_m)$ is an Ntmns, If there exists at least $n \in V$ for each $l, p \in V$ pair of elements such that

$$\|l \mu_2 p\|_m \leq \|l \mu_2 p \mu_2 \text{neut}(n)\|_m; \text{ then}$$

$$\|l \mu_2 p \mu_2 \text{neut}(n)\|_m - m_{lp} \leq \|l\|_m + \|p\|_m - m_{lp} - m_{np}.$$

Thus, from (5),

$$\|l \mu_2 p \mu_2 \text{neut}(n)\|_m \leq \|l\|_m + \|p\|_m.$$

Conclusion

Metric spaces, normed spaces and Banach spaces have an important place in classical mathematics. Metric spaces are widely used, especially in fixed point theory. For this purpose, m -metric space [39] has been defined and many studies have been carried out on fixed point theories with this definition. Normed spaces and Banach spaces, which are special cases of normed spaces, have an important usage area, especially in the field of analysis. Neutrosophic theory [1] has also supported the scientific world with more objective solutions by obtaining new solutions and methods

in many fields in both application sciences and theoretical sciences. Nt structures, which are a sub-branch of the neutrosophic theory, also aimed to carry the new advantages of neutrosophy to the algebraic structure. For this reason, many studies have been carried out on Nt structures. Thus, many structures in classical algebra were reconsidered in neutrosophic theory and new features emerged. In addition, m - metric space was considered in the Nt theory in 2020 and defined Ntmms [33]. Thus, a Nt structure has become available in fixed point theory. In this study, we have defined Ntmns with a more specific structure than Ntmms. We discussed the properties of this structure and proved the theorems related to this structure. We also discussed the relationship between this structure and the Ntmms. In addition, with the help of some definitions in Ntmms, we obtained important definitions such as convergence and Cauchy sequence in Ntmns. Also, we have defined MtmBs. We compared these structures with previously obtained Nt structures. Thanks to this comparison, we have determined that the structures we have obtained have different and new features than others. Thus, we added a new structure to the Nt theory and prepared the ground for new structures that can be obtained. In addition, by using Ntmns and NtmBs, researchers can obtain Nt m - inner product spaces and Nt m - Hilbert spaces. These structures can be the start of many new buildings.

Abbreviations

Nt: Neutrosophic triplet

Nts: Neutrosophic triplet set

Ntn: Neutrosophic triplet norm

Ntns: Neutrosophic triplet normed space

NtBs: Neutrosophic triplet Banach space

Ntmm: Neutrosophic triplet m - metric

NTmms: Neutrosophic triplet m - metric space

Ntf: Neutrosophic triplet field

Ntvs: Neutrosophic triplet vector space

Ntmn: Neutrosophic triplet m – norm

Ntmns: Neutrosophic triplet m – normed space

NtmBs: Neutrosophic triplet m –Banach space

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Neutrosophic Linear Diophantine Equations With Two Variables

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Abstract: This paper is devoted to study for the first time the neutrosophic linear Diophantine equations with two variables in the neutrosophic ring of integers $Z(I)$, and refined neutrosophic ring of integers $Z(I_1, I_2)$. This work introduces an algorithm to solve the linear Diophantine equation $AX + BY = C$ in $Z(I), Z(I_1, I_2)$.

Keywords: Neutrosophic ring, refined neutrosophic ring, neutrosophic linear Diophantine equation, refined neutrosophic linear Diophantine equation.

1. Introduction

Neutrosophy is a new kind of logic founded by F. Smarandache to deal with the indeterminacy in nature, mathematics and reality. It plays an interesting role in the progression of algebraic studies. Many neutrosophical algebraic structures were introduced and handled such as neutrosophic groups, neutrosophic rings, refined neutrosophic rings, and n-refined neutrosophic rings. See [1,2,3,4,5,6,8,10,11]. On the other hand neutrosophic sets were used to deal with health care [12], decision making [13], financial goals [14], computer science, and industry [15,16,17,18,20]. Recently, the interesting in neutrosophic number theory has increased. Relationships between neutrosophic rings and refined neutrosophic rings were studied in [1]. Also, some number theoretical concepts were presented in the neutrosophic ring of integers $Z(I)$ such as division, primes and factors [7]. The theory of neutrosophic numbers is concerning with properties of neutrosophic integers, by the same, refined neutrosophic number theory is dealing with the properties of refined neutrosophic

integers. One of the most important number theoretical concepts is the concept of linear Diophantine equations, these equations were solved in the case of classical integers [9]. Through this paper, we aim to find an algorithm to solve such equations in the case of neutrosophic integers and refined neutrosophic integers by using classical number theoretical methods, where a relationship between neutrosophic equations and classical equations is described.

This work continues the efforts of establishing neutrosophical number theory. It studies the concept of linear Diophantine equations with two variables with respect to neutrosophic integers and refined neutrosophic integers. We determine the sufficient condition for the solvability of these equations and introduce an algorithm which gives the solution in easy way.

2. Preliminaries

Theorem 2.1: [9]

Let $AX + BY = C$ be a linear Diophantine equation, where $A, B, C \in \mathbb{Z}$. Then it is solvable if and only if $\gcd(A, B) | C$. To check the solution's form of classical linear Diophantine equation based on Euclidean division theorem in the ring of integers \mathbb{Z} , see [9].

Definition 2.2:[6]

Let $(R, +, \times)$ be a ring, $R(I) = \{a + bI ; a, b \in R\}$ is called the neutrosophic ring where I is a neutrosophic element with condition $I^2 = I$.

If $R = \mathbb{Z}$, then $R(I)$ is called the neutrosophic ring of integers.

Definition 2.3:[4]

Let $(R, +, \times)$ be a ring, $(R(I_1, I_2), +, \times)$ is called a refined neutrosophic ring generated by R, I_1, I_2 .

If $R = \mathbb{Z}$, then $(R(I_1, I_2), +, \times)$ is called the refined neutrosophic ring of integers.

Definition 2.4: [5]

Let $(G, *)$ be a group. Then the neutrosophic group is generated by G and I under $*$ denoted by $N(G) = \langle G \cup I, * \rangle$.

I is called the indeterminate (neutrosophic element) with the property $I^2 = I$.

3. Main results

Definition 3.1:

Let $Z(I) = \{a + bI; a, b \in \mathbb{Z}\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation with two variables is defined as follows:

$$AX + BY = C; A, B, C \in Z(I).$$

Theorem 3.2:

Let $Z(I) = \{a + bI; a, b \in \mathbb{Z}\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation $AX + BY = C$ with two variables $X = x_1 + x_2I, Y = y_1 + y_2I$, where $A = a_1 + a_2I, B = b_1 + b_2I$ is equivalent to the following two classical Diophantine equations:

$$(1) \ a_1x_1 + b_1y_1 = c_1.$$

$$(2) \ (a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2.$$

Proof:

It is sufficient to show that $AX + BY = C$ implies (1) and (2).

$AX + BY = C$ is equivalent to:

$$(a_1 + a_2I)(x_1 + x_2I) + (b_1 + b_2I)(y_1 + y_2I) = c_1 + c_2I, \text{ by easy computing we find}$$

$$[a_1x_1 + b_1y_1] + [a_1x_2 + a_2x_1 + a_2x_2 + b_1y_2 + b_2y_1 + b_2y_2]I = c_1 + c_2I, \text{ hence}$$

$a_1x_1 + b_1y_1 = c_1$, and $a_1x_2 + a_2x_1 + a_2x_2 + b_1y_2 + b_2y_1 + b_2y_2 = c_2$. We can see that we get equation

(1). For equation (2) we take

$$a_1x_2 + a_2x_1 + a_2x_2 + b_1y_2 + b_2y_1 + b_2y_2 = c_2, \text{ by adding equation (1) to the two sides we obtain}$$

$$a_1x_1 + b_1y_1 + a_1x_2 + a_2x_1 + a_2x_2 + b_1y_2 + b_2y_1 + b_2y_2 = c_1 + c_2, \text{ which implies equation (2)}$$

$$(a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2.$$

The following theorem determines the criteria for the solvability of neutrosophic linear Diophantine equation.

Theorem 3.3:

Let $Z(I) = \{a + bI; a, b \in \mathbb{Z}\}$ be the neutrosophic ring of integers. The neutrosophic linear

Diophantine equation $AX + BY = C$ with two variables $X = x_1 + x_2I, Y = y_1 + y_2I$ and

$A = a_1 + a_2I, B = b_1 + b_2I$ is solvable if and only if $\gcd(a_1, b_1) | c_1, \gcd(a_1 + a_2, b_1 + b_2) | c_1 + c_2$.

Proof:

By Theorem 2.1, we can solve the neutrosophic linear Diophantine equation by solving (1) and (2).

Equation (1) is solvable if and only if $\gcd(a_1, b_1) | c_1$ according to Theorem 2.1.

Equation (2) is solvable if and only if $\gcd(a_1 + a_2, b_1 + b_2) | c_1 + c_2$ according to Theorem 2.1.

Thus our proof is complete.

Example 3.4:

(a) The neutrosophic Diophantine equation $(2 + 2I)X + (3 + 4I)Y = 5 + 5I$ is solvable, that is because

$$\gcd(2, 3) | 5, \text{ and } \gcd(4, 7) | 10.$$

(b) The neutrosophic Diophantine equation $(2 + 3I)X + (4 + 5I)Y = 5 + I$ is not solvable, since

$$\gcd(2, 4) = 2 \text{ does not divide } 5.$$

Now, we describe an algorithm to solve a neutrosophic linear Diophantine equation $AX + BY = C$.

Remark 3.5:

Let $Z(I) = \{a + bI; a, b \in \mathbb{Z}\}$ be the neutrosophic ring of integers. Consider a neutrosophic linear

Diophantine equation $AX + BY = C$ with two variables $X = x_1 + x_2I, Y = y_1 + y_2I$ and

$A = a_1 + a_2I, B = b_1 + b_2I$. To solve this equation follow these steps:

(a) Check the solvability of $AX + BY = C$ by Theorem 3.3.

(b) Solve $a_1x_1 + b_1y_1 = c_1$.

(c) Solve $(a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2$.

(d) Compute x_2, y_2 .

Example 3.6:

The neutrosophic Diophantine equation $(2 + 2I)X + (3 + 4I)Y = 5 + 5I$ is solvable according to Example 3.4 .

$2x_1 + 3y_1 = 5$ is a classical linear Diophantine equation. It has a solution $x_1 = 4, y_1 = -1$.

$(2 + 2)(x_1 + x_2) + (3 + 4)(y_1 + y_2) = 5 + 5$, i.e $4M + 7N = 10; M = x_1 + x_2, N = y_1 + y_2$. It is a

classical linear Diophantine equation with M, N as variables. It has a solution $M = -1, N = 2$.

$x_2 = M - x_1 = -5, y_2 = N - y_1 = 3$, thus the equation $(2 + 2I)X + (3 + 4I)Y = 5 + 5I$ has a solution

$X = 4 - 5I, Y = -1 + 3I$.

Definition 3.7:

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in \mathbb{Z}\}$ be the refined neutrosophic ring of integers. The refined neutrosophic linear Diophantine equation with two variables is defined as follows:

$$AX + BY = C; A, B, C \in Z(I_1, I_2).$$

Theorem 3.8:

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in \mathbb{Z}\}$ be the refined neutrosophic ring of integers,

$AX + BY = C; A, B, C \in Z(I_1, I_2)$ be a refined neutrosophic linear Diophantine equation, where

$$X = (x_0, x_1I_1, x_2I_2), Y = (y_0, y_1I_1, y_2I_2), A = (a_0, a_1I_1, a_2I_2),$$

$B = (b_0, b_1I_1, b_2I_2), C = (c_0, c_1I_1, c_2I_2)$. Then $AX + BY = C$ is equivalent to the following three

Diophantine equations:

$$(1) a_0x_0 + b_0y_0 = c_0.$$

$$(2) (a_0 + a_2)(x_0 + x_2) + (b_0 + b_2)(y_0 + y_2) = c_0 + c_2.$$

$$(3) (a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2.$$

Proof:

By replacing A, B, C, X, Y we find

$$AX = (a_0, a_1 I_1, a_2 I_2)(x_0, x_1 I_1, x_2 I_2) =$$

$$(a_0 x_0, [a_0 x_1 + a_1 x_0 + a_1 x_1 + a_1 x_2 + a_2 x_1] I_1, [a_0 x_2 + a_2 x_0 + a_2 x_2] I_2),$$

$$BY = (b_0, b_1 I_1, b_2 I_2)(y_0, y_1 I_1, y_2 I_2) =$$

$$(b_0 y_0, [b_0 y_1 + b_1 y_0 + b_1 y_1 + b_1 y_2 + b_2 y_1] I_1, [b_0 y_2 + b_2 y_0 + b_2 y_2] I_2), \text{ thus the equation}$$

$$AX + BY = C \text{ implies}$$

$$(*) a_0 x_0 + b_0 y_0 = c_0. \text{ (Equation (1)).}$$

$$(**) a_0 x_2 + a_2 x_0 + a_2 x_2 + b_0 y_2 + b_2 y_0 + b_2 y_2 = c_2.$$

$$(***) a_0 x_1 + a_1 x_0 + a_1 x_1 + a_1 x_2 + a_2 x_1 + b_0 y_1 + b_1 y_0 + b_1 y_1 + b_1 y_2 + b_2 y_1 = c_1.$$

$$\text{By adding } (*) \text{ to } (**) \text{ we get } (a_0 + a_2)(x_0 + x_2) + (b_0 + b_2)(y_0 + y_2) = c_0 + c_2. \text{ (Equation (2)).}$$

$$\text{By adding (2) to } (***) \text{ we get}$$

$$(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2. \text{ (Equation (3)).}$$

Theorem 3.9:

Let $Z(I_1, I_2) = \{(a, b I_1, c I_2); a, b, c \in Z\}$ be the refined neutrosophic ring of integers,

$AX + BY = C; A, B, C \in Z(I_1, I_2)$ be a refined neutrosophic linear Diophantine equation, where

$$X = (x_0, x_1 I_1, x_2 I_2), Y = (y_0, y_1 I_1, y_2 I_2), A = (a_0, a_1 I_1, a_2 I_2),$$

$$B = (b_0, b_1 I_1, b_2 I_2), C = (c_0, c_1 I_1, c_2 I_2). \text{ Then } AX + BY = C \text{ is solvable if and only if:}$$

$$(a) \gcd(a_0, b_0) | c_0.$$

$$(b) \gcd(a_0 + a_2, b_0 + b_2) | c_0 + c_2.$$

$$(c) \gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) | c_0 + c_1 + c_2.$$

The proof is similar to that of Theorem 3.3.

Example 3.10:

(a) Consider the refined neutrosophic linear Diophantine equation

$$(1, 2I_1, 3I_2).X + (3, 3I_1, 8I_2)Y = (2, 4I_1, I_2), \text{ we have}$$

$$\gcd(1, 3) = 1 | 2, \gcd(1 + 3, 3 + 8) = \gcd(4, 11) = 1 | (2 + 1 = 3),$$

$\gcd(1 + 2 + 3, 3 + 3 + 8) = \gcd(6, 14) = 2$ which does not divide $2 + 4 + 1 = 7$, thus it is not solvable.

(b) Consider the refined neutrosophic linear Diophantine equation

$$(1, 2I_1, 3I_2).X + (3, 3I_1, 8I_2)Y = (2, 4I_1, 2I_2), \text{ we have}$$

$$\gcd(1, 3) = 1 | 2, \gcd(1 + 3, 3 + 8) = \gcd(4, 11) = 1 | (2 + 2 = 4),$$

$$\gcd(1 + 2 + 3, 3 + 3 + 8) = \gcd(6, 14) = 2 | (2 + 4 + 2 = 8). \text{ Thus it is solvable.}$$

Remark 3.11:

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in \mathbb{Z}\}$ be the refined neutrosophic ring of integers,

$AX + BY = C; A, B, C \in Z(I_1, I_2)$ be a refined neutrosophic linear Diophantine equation, where

$$X = (x_0, x_1I_1, x_2I_2), Y = (y_0, y_1I_1, y_2I_2), A = (a_0, a_1I_1, a_2I_2),$$

$$B = (b_0, b_1I_1, b_2I_2), C = (c_0, c_1I_1, c_2I_2), \text{ we summarize the algorithm of solution as follows:}$$

(a) Check the solvability condition.

$$(b) \text{ Solve } a_0x_0 + b_0y_0 = c_0.$$

$$(c) \text{ Solve } (a_0 + a_2)(x_0 + x_2) + (b_0 + b_2)(y_0 + y_2) = c_0 + c_2.$$

$$(d) \text{ Compute } x_2, y_2.$$

(e) Solve $(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2$.

(f) Compute x_1, y_1 .

Example 3.12:

According to Example 3.10, we found that $(1, 2I_1, 3I_2).X + (3, 3I_1, 8I_2).Y = (2, 4I_1, 2I_2)$ is solvable.

We consider $x_0 + 3y_0 = 2$. It has a solution $x_0 = -1, y_0 = 1$. We take

$(1 + 3)(x_0 + x_2) + (3 + 8)(y_0 + y_2) = 2 + 2$, i.e. $4M + 11N = 4; M = x_0 + x_2$, and $N = y_0 + y_2$, it has

a solution $M = 1, N = 0$, thus $x_2 = M - x_0 = 2, y_2 = N - y_0 = -1$. The third equation is

$(1 + 2 + 3)(x_0 + x_1 + x_2) + (3 + 3 + 8)(y_0 + y_1 + y_2) = 2 + 4 + 2$, i.e.

$6S + 14T = 8; S = x_0 + x_1 + x_2, T = y_0 + y_1 + y_2$. It has a solution $S = -1, T = 1$, thus

$x_1 = S - x_0 - x_2 = -2, y_1 = T - y_0 - y_2 = 1$. The solution of

$(1, 2I_1, 3I_2).X + (3, 3I_1, 8I_2).Y = (2, 4I_1, 2I_2)$ is $X = (-1, -2I_1, 2I_2), Y = (1, I_1, -I_2)$.

5. Conclusion

In this paper, we have determined the criteria for the solvability of linear Diophantine equation in the neutrosophic ring of integers and refined neutrosophic rings of integers by finding the relationship between neutrosophic equations and classical equations. Also, we have presented an algorithm which gives a solution of these equations, and constructed some examples to clarify the validity of this work.

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On Deneutrosophication

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Abstract. Deneutrosophication is a process to evaluate real output from neutrosophic information. The paper presents on a novel deneutrosophication algorithm. The process is developed with similarity measure and probability density function (PDF). This similarity measure is newly defined to prepare a correct transformation from neutrosophic set (NS) to fuzzy set (FS). Then an approach to find PDF is formulated which relates with fuzzy set. Finally, the algorithm has been implemented in solving a critical path problem to find out the completion time of a certain project.

Keywords: Deneutrosophication; Similarity measure; Probability density function

1. Introduction

Modern technology and science can not be evolved without dealing with uncertainty. Before the era of fuzzy, problems of uncertainty would solved by theory of probability only. Fuzzy set (FS) [1] can easily cope up with uncertainty by its membership grade (Zadah, 1965). But the necessity of handling incomplete information addressed to the introduction of intuitionistic fuzzy set (IFS) [2] (Atanasov, 1985). IFS can not link indeterminacy. Neutrosophic set [3] (NS) can overcome all the limitations of FS, IFS due to its easy relationship between mathematical and formal language (Smarandache, 1998).

One of the most challenging issue to the neutrosophic researchers is to innovate better solution approach or method to reach to a better decision or conclusion from several pieces of neutrosophic informations. Similarity measure, a popular information measure method, is used in different researches to solve real life problems. Ye [4] in 2014 solved decision making (DM) problem by introducing Jaccard, Dice and cosine similarity measures for single valued neutrosophic sets (SVNSs). Further, the author (Ye, 2015) eliminated limitations of the cosine similarity measure in [5] and used in a medical diagnosis problem. In [6], using the

similarity function entropy measure was also developed (Aydogdu, 2015). Some more similarity functions [7] were introduced for recovery of some disadvantages of Jaccard, Dice and cosine similarity measures and applied to a minimum spanning tree problem (Mandal et al., 2016). It is also effectively used in medical diagnosis by using the euclidean distance based similarity measure [8] (Donghai Liu et al., 2018), hybrid distance based similarity measure for refined neutrosophic sets [9] (Vakkas Ulucay et al., 2019). Further similarity measure is applied in smart port development [10] (Jihong chen et al., 2019), selection of cricket players [11] (Muhammad et al., 2020), medical diagnosis as well as lecturer selection for universities [12] (Saeed et al., 2020).

In recent years different effective methods have been implemented to deal with several existing problems. Abdel-Basset et al. [13] developed a model on the basis of neutrosophic AHP which was successfully used in Egypt steel industry. In [14] Plithogenic aggregation operator was proposed to aggregate the opinions of decision makers and was used in best worst method in solving supply chain, warehouse location and plant evaluation problem (Abdel-Basset et al., 2020). In order to monitoring the spread of epidemic covid 19, a novel approach using best worst and Topsis method [15] is introduced (Abdel-Basset et al., 2020). Another technique health fog method was discussed in [16] to assist covid patient (Yasser et al., 2020). In [17] Carmen et al. analyzed emotional intelligence of some randomly selected university students. Further, contributions in neutrosophic researches are on medical diagnosis [18–21], smart product service systems [23], decision making in personnel selection [24], recommending in museum room [25], predicting tax time series [26], Sustainability of goat and sheep production systems [27], also in [28–32] etc.

The methods used in the above discussed literature correspond to real data, not the actual real output to solve problems. But sometimes it becomes necessary to evaluate the actual real output of the respective event from the neutrosophic information. Suppose a decision maker put his decision on some activity time of a certain project by a SVN $\langle 15, (0.3, 0.5, 0.6) \rangle$. Now a common thought arises upon us about the expected activity completion time. Deneutrosophication gives the answer, it is the process which can evaluate the crisp value from the neutrosophic data. Smarandache et al. [33] first discussed a deneutrosophication method in 2005 by using synthesization and center of gravity method. The synthesization process according to [33] corresponds a NS to different FSs resulting different crisp values. Using removal area method, a deneutrosophication technique [34] was studied on pentagonal neutrosophic number and utilized in MST problem (Chakraborty et al., 2019). A deneutrosophication equation computed in [35] gives the neat truth grade not the real output of the respective event (Azzah Awang et al., 2019). Deneutrosophicated value for a trapezoidal fuzzy number (TFN)

was found in [36] using the score function involving center of gravity of TFN (Said Broumi et al., 2019).

In this article first time similarity measure is used to have a meaningful relation between NS and FS. To establish this, a similarity function is posed by redefining the axiomatic definition. It also overcome the limitation of existing ones [5–7]. Then, an approach to relate FS with PDF is proposed. Finally, a deneutrosophication algorithm is established which is more generalized and robust in comparison with [33–36] as it can provide a single crisp value, deal with both continuous and discrete neutrosophic data and find the real output of the respective event. A comparative study to show its consistency and effectiveness has been delivered. At last a critical path method is solved as an application of the deneutrosophication approach.

2. Preliminaries

2.1. Neutrosophic set [3]

Let U be an universe of discourse, then the neutrosophic set A is defined as $A = \{\langle x : T_A(x), I_A(x), F_A(x) \rangle, x \in U\}$, where the functions $T, I, F: U \rightarrow]-0, 1+[$ define respectively the degree of membership (or Truth), the degree of indeterminacy and the degree of non-membership (or falsehood) of the element $x \in U$ to the set A with the condition $-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

2.2. Single Valued Neutrosophic Sets (SVNS) [38]

Let X be a space of points (objects) with generic elements in X denoted by x . A SVNS, A , in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity-membership function $F_A(x)$, for each point $x \in X$, $T_A(x), I_A(x), F_A(x) \in [0, 1]$. Therefore, a SVNS A can be written as $A_{SVNS} = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle, x \in X\}$.

2.3. Single valued triangular neutrosophic number [40]

A triangular neutrosophic number $A = \langle (a_1, b_1, c_1); w_a, u_a, y_a \rangle$ is a special neutrosophic set on the real number set R , whose truth-membership indeterminacy-membership and falsity-membership functions are defined as follows:

$$\begin{aligned}
T_A(x) &= \begin{cases} \frac{(x-a_1)w_a}{b_1-a_1} & , a_1 \leq x \leq b_1 \\ w_a & , x = b_1 \\ \frac{(c_1-x)w_a}{c_1-b_1} & , b_1 \leq x \leq c_1 \\ 0 & , otherwise. \end{cases} \\
I_A(x) &= \begin{cases} \frac{b_1-x+u_a(x-a_1)}{b_1-a_1} & , a_1 \leq x \leq b_1 \\ u_a & , x = b_1 \\ \frac{(x-b_1)+u_a(c_1-x)}{c_1-b_1} & , 0.5 \leq x \leq 0.7 \\ 1 & , otherwise. \end{cases} \\
F_A(x) &= \begin{cases} \frac{b_1-x+y_a(x-a_1)}{b_1-a_1} & , a_1 \leq x \leq b_1 \\ y_a & , x = b_1 \\ \frac{(x-b_1)+y_a(c_1-x)}{c_1-b_1} & , b_1 \leq x \leq c_1 \\ 1 & , otherwise. \end{cases}
\end{aligned}$$

2.4. Single valued trapezoidal neutrosophic number [40]

A single valued trapezoidal neutrosophic number $\langle A = (a_1, b_1, c_1, d_1); w_a, u_a, y_a \rangle$ is a special neutrosophic set on the real number set R , whose truth-membership, indeterminacy-membership, and a falsity-membership are given as follows:

$$\begin{aligned}
T_A(x) &= \begin{cases} \frac{(x-a_1)w_a}{b_1-a_1} & , a_1 \leq x \leq b_1 \\ w_a & , b_1 \leq x \leq c_1 \\ \frac{(d_1-x)w_a}{d_1-c_1} & , c_1 \leq x \leq d_1 \\ 0 & , otherwise. \end{cases} \\
I_A(x) &= \begin{cases} \frac{b_1-x+u_a(x-a_1)}{b_1-a_1} & , a_1 \leq x \leq b_1 \\ u_a & , b_1 \leq x \leq c_1 \\ \frac{(x-c_1)+u_a(d_1-x)}{d_1-c_1} & , c_1 \leq x \leq d_1 \\ 1 & , otherwise. \end{cases} \\
F_A(x) &= \begin{cases} \frac{b_1-x+y_a(x-a_1)}{b_1-a_1} & , a_1 \leq x \leq b_1 \\ y_a & , b_1 \leq x \leq c_1 \\ \frac{(x-c_1)+y_a(d_1-x)}{d_1-c_1} & , c_1 \leq x \leq d_1 \\ 1 & , otherwise. \end{cases}
\end{aligned}$$

2.5. similarity measure [5, 6]

Similarity measure s for $SVNS(X)$ is a real function on universe X such that $S : SVNS(X) \times SVNS(X) \rightarrow [0, 1]$ and satisfies the following properties: (i) $0 \leq s(A, B) \leq 1, \forall A, B \in SVNS(X)$,

(ii) $s(A, B) = s(B, A), \forall A, B \in SVNS(X)$,

(iii) $s(A, B) = 1$, if and only if $A = B, \forall A, B \in SVNS(X)$.

(iv) If $A \subset B \subset C$, $s(A, B) \geq s(A, C)$ and $s(B, C) \geq s(A, C) \forall A, B \in SVNS(X)$.

3. Limitations of existing similarity functions and redefinition

According to the definition of similarity measure function as defined in subsec 2.5, $s(A, B)$ should be 1 if and only if $A = B$. But there is no such necessary and sufficient condition for which $s(A, B)$ is zero.

$(1, 0, 0)$ and $(0, 0, 1)$ represent respectively the total affirmation and the total denial of the belongingness of an element to a given NS. Obviously, the similarity between $\{(1, 0, 0)\}$ and $\{(0, 0, 1)\}$ is zero. The existing formulee in [5–7], violate this argument. So, a new similarity measure is proposed by redefining the definition.

3.1. Proposed similarity measure

Along with the four properties in 2.5, the similarity measure is redefined with an extra fifth property:

(v) $s(A, B) = 0$, if $A = \{(1, 0, 0)\}$ and $B = \{(0, 0, 1)\}$,

which gives the sufficient condition for which the similarity between A and B will be zero.

Following the proposed function which satisfies all the property of similarity measure along with the fifth property.

$$s(A, B) = 1 - \frac{1}{2n} \sum_{i=1}^n [|T_A(x_i) - T_B(x_i)| + \max \{|I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\}] \quad (1)$$

Clearly, $s(A, B)$ satisfies all the required properties of similarity measure function as defined in 3.1.

3.2. Remark

Consider $A = \{(1, 0, 0)\}$ and $B = \{(0, 1, 1)\}$. Then, using equation 1, we get $s(A, B) = 0$. Thus, the condition (stated as property (v)) for which similarity is zero is sufficient but not necessary.

3.2.1. Example 1 [7]

Let $A = \{(x_1, (0.2, 0.5, 0.6)), (x_2, (0.2, 0.4, 0.4))\}$ and $B = \{(x_1, (0.2, 0.4, 0.4)), (x_2, (0.4, 0.2, 0.3))\}$ be two SVNNS.

TABLE 1. Similarity values ($s(A, B)$) in different methods

	Using [5]	Using [6]	using equ(3) in [7]	using equ(4) in [7]	By proposed formula
$s(A, B)$	0.977	0.867	0.815	0.975	0.85

3.2.2. Example 2 [7]

Consider $C = \{(x_1, (0.3, 0.2, 0.3)), (x_2, (0.5, 0.2, 0.3)), (x_3, (0.5, 0.3, 0.2))\}$ and $D = \{(x_1, (0.7, 0, 0.1)), (x_2, (0.6, 0.1, 0.2)), (x_3, (0.6, 0.1, 0.2))\}$.

TABLE 2. $s(C, D)$ using different methods

	Using [5]	Using [6]	using equ(3) in [7]	using equ(4) in [7]	By proposed formula
$s(C, D)$	0.9629	0.844	0.79	0.964	0.8167

From the above examples, it may be observed that the proposed formula gives consistent result.

Using the similarity measure function, we can convert neutrosophic fuzzy data to fuzzy data and hence PDF also which is discussed in the following section.

4. The proposed approach to formulate PDF from neutrosophic fuzzy data

We segregate the approach in two intermediate steps:

Step 1:

Synthesization is the process to convert a NS (here we consider SVNS) into a FS. The process evaluates the overall truth over the truth, indeterminacy and falsity membership function. In the neutrosophic set theory, $I_N = \{(1, 0, 0)\}$ can be considered as the reference NS, as it signifies the full membership of an element to a given set. Full membership in FS is indicated by the membership value 1. So, $(1, 0, 0)$ (in NS) is equivalent to the maximum membership 1 (in fuzzy) since both implies total belongingness to the respective set. The more the similarity between each $(T_A(x), I_A(x), F_A(x))$ and I_N is, the more the belongingness of the element to the set i.e., I_N stands for the reference NS. Thus, we get the following proposition to convert a NS into a FS:

Proposition:

Let $A_N = \{\langle x, (T_A(x), I_A(x), F_A(x)) \rangle, x \in X\}$ be a NS. Its equivalent fuzzy membership set is defined as $A_F = \{(x, \mu_A(x)) : x \in X\}$, where $\mu_A(x) = s((T_A(x), I_A(x), F_A(x)), (1, 0, 0))$. So, using equation 1,

$$\mu_A(x) = 1 - \frac{1}{2}[(1 - T_A(x)) + \max\{I_A(x), F_A(x)\}] \quad (2)$$

As the range of the similarity measure function is the unit interval $[0, 1]$, $\mu_A(x) \in [0, 1]$ $\forall x \in X$. Hence, the membership function of the derived fuzzy set belongs to $[0, 1]$ and thus it satisfies the property of membership function of a FS.

The larger is the similarity measure value between $(T_A(x), I_A(x), F_A(x))$ and $(1, 0, 0)$, the larger is the belongingness of x to the respective set, A , the more is the membership value.

Moreover, $(1, 0, 0)$, the neutrosophic number which denotes the full belongingness to a set, is transformed into the membership value 1 which confirms to be the full belongingness to the set. Again $(0, 0, 1)$, the neutrosophic number which denotes the zero belongingness to a set, is transformed into the membership value 0 which confirms to be the zero belongingness to the set. So, the results of the transformation are desirable and meaningful.

Step 2:

Formulate PDF from fuzzy membership function:

[Case 1.] The variable is discrete in (S) :

Let $S = \{x_1, x_2, \dots, x_n\}$ be a universe of discourse and $A = \{x_i, \mu_A(x_i) : x_i \in S\}$ be a fuzzy set with discrete membership function. Consider X to be the random variable corresponding to the event space S . The density function of the random variable X is defined as

$$P(X = x_i) = f(x_i) = \frac{\mu_A(x_i)}{\Delta} \quad (3)$$

where $\Delta = \sum_{-\infty}^{\infty} \mu_A(x_i)$. The proposed function $f(x_i)$ satisfies the required properties of a density function:

(1) $f(x_i) \geq 0$, as membership function $\mu_A(x_i) \geq 0$.

(2) $\sum_{-\infty}^{\infty} f(x_i) = \frac{\sum_{-\infty}^{\infty} \mu_A(x_i)}{\sum_{-\infty}^{\infty} \mu_A(x_i)} = 1$.

[Case 2.] The variable is continuous in (S) :

Let

$A = \{(x, \mu_A(x)) : x \in S\}$ be a fuzzy set. Consider X to be the random variable corresponding to the event space S . The density function of the random variable X is defined as

$$f(x) = \frac{\mu_A(x)}{\Delta} \quad (4)$$

where $\Delta = \int_{-\infty}^{\infty} \mu_A(x) dx$. Clearly $f(x)$ also satisfies the desired properties of a density function.

4.1. Example

Discrete case: Let $S = \{x_1, x_2, x_3\}$ be the universal set and the neutrosophic set is defined as

$$A_N = \{(x_1, (0.7, 0.5, 0.2)), (x_2, (0.7, 0.8, 0.9)), (x_3, (0.3, 0.8, 0.9))\}$$

Using equation (2), equivalent fuzzy set $A_F = \{(x_1, 0.6), (x_2, 0.4), (x_3, 0.2)\}$. So, the corresponding density function (from equation (3)) at x_i , ($i = 1, 2, 3$) are as follows:

$$f(x_1) = \frac{0.6}{1.2} = 0.5, f(x_2) = 0.33, f(x_3) = 0.17.$$

Continuous case: When the universal set S is a continuous, the degree of membership can be represented by a function which can take various shapes and forms like triangular membership function, trapezoidal membership function etc.

Consider the NS, A , defined on the interval $S = [0, 10]$ of real numbers by the truth, indeterminacy and falsity membership functions: $T_A(x) = \frac{1}{1+x}$, $I_A(x) = \frac{1}{1+x^2}$, $F_A(x) = \frac{1}{1+x^3}$.

Then its equivalent fuzzy membership function is

$$\mu_A(x) = \begin{cases} 1 - \frac{1}{2} \left[\left(1 - \frac{1}{1+x}\right) + \frac{1}{1+x^3} \right], & 0 \leq x \leq 1 \\ 1 - \frac{1}{2} \left[\left(1 - \frac{1}{1+x}\right) + \frac{1}{1+x^2} \right], & 1 \leq x \leq 10 \end{cases}$$

Also the corresponding density function

is $f_A(x) = \frac{\mu_A(x)}{\Delta}$, where $\Delta = \int_0^{10} \mu_A(x) dx = 5.4382$.

5. Proposed deneutrosophication method

Deneutrosophication is the process to convert neutrosophic data to crisp data corresponding universe of discourse. In this section, a deneutrosophication method is presented through the algorithm given below:

Step 1. Convert NS to FS using proposition given in section 4.

Step 2. Formulate PDF $f(x_i)$ or $f(x)$ of the random variable X according to discrete case or continuous case respectively from the fuzzy membership function using step 2 of section 4.

Step 3. Find the expectation of X . i.e., $E(X) = \sum_{x_i \in S} x_i f(x_i)$ (discrete case)
 $= \int_{x \in S} x f(x) dx$ (for continuous case).

Step 4. Deneutrosophicated value $= E(X)$.

5.1. Examples

5.1.1. Example 1

Let us consider the example (4.1). According to the proposed steps of the algorithm 5, the deneutrosophicated value is $\int_0^{10} x f(x) dx = 5.0816$.

5.1.2. Example 2

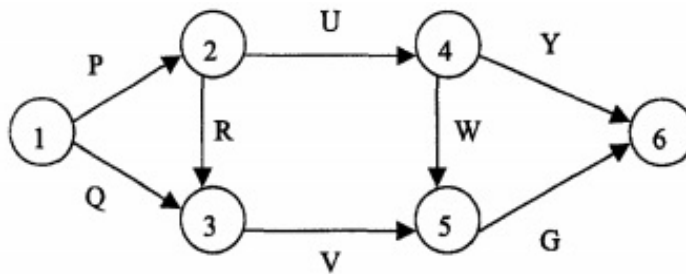


FIGURE 1. Capacitated network

We focus only on the arc 1 – 2 of the network (fig 1). Suppose flow through the arc is represented by a trapezoidal neutrosophic number P , where $P = ((0.3, 0.4, 0.5, 0.7); 0.5, 0.4, 0.3)$. To find the mean flow through the arc, deneutrosophication is useful.

Then using equation (2) the equivalent fuzzy membership function defined on the interval $[0.3, 0.7]$ is $\mu_P(x)$, where

$$\mu_P(x) = \begin{cases} 1 - \frac{1}{2} \left[1 - \frac{(x-0.3)0.5}{0.1} + \frac{0.4-x+0.3(x-0.3)}{0.1} \right], & 0.3 \leq x \leq 0.4 \\ 1 - \frac{1}{2} [1 - 0.5 + 0.4] & 0.4 \leq x \leq 0.5 \\ 1 - \frac{1}{2} \left[1 - \frac{(0.7-x)0.5}{0.2} + \frac{(x-0.5)+0.4(0.7-x)}{0.2} \right], & 0.5 \leq x \leq 0.7 \\ 0 & , \text{otherwise.} \end{cases}$$

the corresponding PDF $f_P(x) = \frac{\mu_P(x)}{\Delta}$, where $\Delta = \int_{0.3}^{0.7} \mu_P(x) dx = 0.1375$.

So, the deneutrosophicated value i.e., mean flow through the arc is $\int_{0.3}^{0.7} x f_P(x) dx = 0.48$.

5.1.3. Some more examples and comparative study

Following are some more examples shown in Table 3.

TABLE 3. Comparison study on deneutrosophicated values

<i>Different Single valued neutrosophic numbers</i>	<i>Deneutrosophi- cated value</i>			
	<i>using [33]</i>	<i>Using [33]</i>	<i>Using [33]</i>	<i>Using</i>
	<i>considering</i>	<i>considering</i>	<i>considering</i>	<i>proposed</i>
	<i>(a = 0.4, b = 0.3, c = 0.2, d = 0.1)</i>	<i>(a = 0.6, b = 0.2, c = 0.15, d = 0.05)</i>	<i>(a = 0.45, b = 0.4, c = 0.05, d = 0.1)</i>	<i>method</i>
<i>Triangular</i>				
$\langle (2, 4, 5); 0.3, 0.4, 0.5 \rangle$	3.5909	3.6042	3.6458	3.6667
$\langle (1, 2, 4); 0.7, 0.8, 0.9 \rangle$	2.4	2.38	2.35	2.33
<i>Trapezoidal</i>				
$\langle (0.3, 0.5, 0.8, 0.9); 0.2, 0.5, 0.8 \rangle$	0.6230	0.6220	0.5887	0.6424
$\langle (0.4, 0.6, 0.7, 0.8); 0.5, 0.4, 0.2 \rangle$	0.6169	0.6521	0.6222	0.6406
$\langle (1, 2, 5, 6); 0.8, 0.6, 0.4 \rangle$	3.5	3.5	3.5	3.5
<i>SVNN with discrete membership</i>				
$\langle 5, (0.6, 0.4, 0.3) \rangle$,				
$\langle 5.5, (0.8, 0.3, 0.2) \rangle, \langle 6, (0.5, 0.3, 0.4) \rangle$	5.51	5.516	5.507	5.519
$\langle 3, (0.7, 0.4, 0.3) \rangle$,				
$\langle 3.5, (0.6, 0.3, 0.3) \rangle, \langle 3.4, (0.7, 0.4, 0.4) \rangle$	3.292	3.291	3.29	3.3

5.1.4. Results discussion and comparison analysis

From the Table 3, it is shown that the transformed crisp value of each SVNN using proposed deneutrosophication method is almost similar to the different deneutrosophicated values using existing method [33] for different choices of a, b, c, d . So, the values evaluated by the proposed deneutrosophication are consistent. On the other hand, according to [33], the membership values of the FS obtained from the NS depend on the choice of the parameters a, b, c, d which

leads to the fact that for a given NS, there can be different membership values of the fuzzy set and so the different crisp values which is not desirable. Again, in the table, it is seen that the proposed technique is applicable on neutrosophic number with both continuous (Triangular, trapezoidal) and discrete membership grade. In this sense, the proposed deneutrosophication is more robust than the existing [34,36].

6. Numerical Example

6.1. Example 1

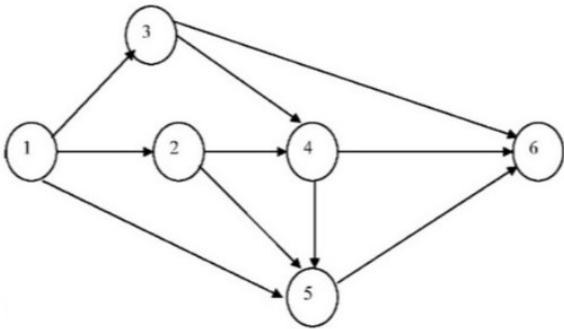


FIGURE 2. Project Network

Fig. 2 shows a construction project network. The duration of each activity are estimated by three estimators. To find the critical path and expected project completion time, a decision organization is assigned to evaluate the degrees of each estimated activity time. The estimated times and the corresponding degrees given by the organization in single valued neutrosophic forms are given in the table 4

TABLE 4. Estimated activity times

Activity	Time duration	and corresponding		degrees
	Expert I	Expert II	Expert III	
1 – 2	$\langle 8, (0.4, 0.3, 0.5) \rangle$	$\langle 8.5, (0.7, 0.3, 0.4) \rangle$	$\langle 9, (0.6, 0.4, 0.3) \rangle$	
1 – 3	$\langle 10, (0.4, 0.3, 0.2) \rangle$	$\langle 9.7, (0.4, 0.2, 0.3) \rangle$	$\langle 8.5, (0.2, 0.2, 0.5) \rangle$	
1 – 5	$\langle 13.5, (0.6, 0.3, 0.2) \rangle$	$\langle 13.33, (0.6, 0.1, 0.3) \rangle$	$\langle 14, (0.5, 0.3, 0.2) \rangle$	
2 – 4	$\langle 4, (0.3, 0.5, 0.4) \rangle$	$\langle 5.5, (0.5, 0.3, 0.2) \rangle$	$\langle 6, (0.5, 0.55, 0.3) \rangle$	
2 – 5	$\langle 5, (0.7, 0.3, 0.2) \rangle$	$\langle 4.4, (0.5, 0.3, 0.3) \rangle$	$\langle 4, (0.6, 0.3, 0.2) \rangle$	
3 – 4	$\langle 6, (0.73, 0.3, 0.3) \rangle$	$\langle 5.65, (0.8, 0.3, 0.2) \rangle$	$\langle 5, (0.85, 0.4, 0.3) \rangle$	
4 – 5	$\langle 4, (0.26, 0.37, 0.29) \rangle$	$\langle 4.45, (0.32, 0.21, 0.46) \rangle$	$\langle 4.8, (0.39, 0.34, 0.27) \rangle$	
3 – 6	$\langle 8, (0.69, 0.49, 0.20) \rangle$	$\langle 8.6, (0.65, 0.40, 0.3) \rangle$	$\langle 7.77, (0.75, 0.20, 0.30) \rangle$	
4 – 6	$\langle 7, (0.6, 0.4, 0.3) \rangle$	$\langle 6.5, (0.8, 0.3, 0.2) \rangle$	$\langle 5.8, (0.5, 0.3, 0.4) \rangle$	
5 – 6	$\langle 3.7, (0.6, 0.45, 0.33) \rangle$	$\langle 3.44, (0.55, 0.39, 0.4) \rangle$	$\langle 4, (0.7, 0.3, 0.2) \rangle$	

Let X be the random variable representing the time duration for the activity 1 – 2. Then X can be written as

$X = \{\langle 10, (0.4, 0.3, 0.5) \rangle, \langle 9.7, (0.7, 0.3, 0.4) \rangle, \langle 8.5, (0.6, 0.4, 0.3) \rangle\}$. So, to complete the activity 1 – 2, the expected time duration is the deneutrosophicated value = 8.544. (Using 5). All the expected time durations for activities are given in the table 5.

TABLE 5. Expected activity times

<i>Activity</i>	<i>deneutrosophicated values</i>
1 – 2	8.54
1 – 3	9.52
1 – 5	13.6
2 – 4	5.25
2 – 5	4.48
3 – 4	5.54
4 – 5	4.44
3 – 6	8.11
4 – 6	6.45
5 – 6	3.73

Now we reach to a project network with crisp duration time for each activity and apply forward pass method as well as backward pass method to find the critical path. Let E_i and L_j denote the earliest occurrence time and latest allowable time corresponding to the i^{th} event.

Forward pass method:

starting time $E_1 = 0$, $E_2 = 0 + t_{12} = 8.54$, $E_3 = 0 + t_{13} = 9.52$, $E_4 = \max\{E_2 + t_{2,4}, E_3 + t_{3,4}\} = 15.06$.

Similarly, $E_5 = 19.5$, $E_6 = \max\{E_3 + t_{3,6}, E_4 + t_{4,6}, E_5 + t_{5,6}\} = 23.23$.

Backward pass method:

$L_6 = E_6 = 23.23$, $L_5 = L_6 - t_{5,6} = 19.5$, $L_4 = \min\{L_6 - t_{4,6}, L_5 - t_{4,5}\} = 15.06$, similarly, $L_3 = 9.52$, $L_2 = 9.81$, $L_1 = 0$.

So, the critical path is 1 – 3 – 4 – 5 – 6 (as $E_1 = L_1$, $E_3 = L_3$, $E_4 = L_4$, $E_5 = L_5$, $E_6 = L_6$) and corresponding project completion time 23.23.

7. Conclusion and future scope

We discuss a deneutrosophication method for both the continuous and discrete cases, establishing a relation among neutrosophic fuzzy set, fuzzy set and probability density function. A well defined similarity measure function is formulated which meaningfully transform neutrosophic data to fuzzy data. The comparative study proves the consistency and effectiveness of the proposed algorithm. The method can be applied in any kinds of problem to deal with neutrosophic information. Various types of decision making methods to get the optimum solution is introduced in different researches but deneutrosophication not only measure the best among several informations but also can evaluate the crisp value of the universe of discourse.

In future our objective is to use this method on transformed neutrosophic covid 19 data of an area and comment on the infected and infective so that corrective measures can be taken to reduce the impact of the epidemic of the area.

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T-MBJ NEUTROSOPHIC SET UNDER M-SUBALGEBRA

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Abstract. In this paper, the idea of T-MBJ neutrosophic set is introduced in which MBJ-neutrosophic set is used to present this new set called T-MBJ neutrosophic set. furthermore the notion of T-MBJ neutrosophic M-subalgebra on G -algebra is also introduced and provide the conditions for T-MBJ neutrosophic M-subalgebra. The word M in the term M-subalgebra, represents the initial of author's first name Mohsin. We study the T-MBJ neutrosophic set through different characteristics and also prove some results for better understanding of newly define T-MBJ neutrosophic set.

Keywords: G -algebra; T-MBJ neutrosophic set; T-MBJ neutrosophic M-subalgebra.

1. Introduction

Smarandacha [3,4] extended the intuitionistic fuzzy set to neutrosophic set. Barbhuiya [22] wrote in detailed about t-intuitionistic fuzzy using the concept of subalgebra and different other characteristics. Takallo et al. [5] extensively explained the BMBJ-neutrosophic subalgebra. Senapati et al. [27] used cubic set and applied it to subalgebras, ideals and closed ideals of B -algebra. Imai and Iseki [7, 28] defined the BCK -algebra and BCI -algebra. Bandaru et al. [21] first time introduced the G -algebra. Zadeh [8,9] introduced unique sets called fuzzy set and interval-valued fuzzy set. Saeid [1] studied interval-valued fuzzy subalgebra of B -algebra. Khalid et al. [10] defined T-neutrosophic cubic set and explained this set with important results. Senapati et al. [25] studied L -fuzzy G -subalgebra of G -algebra. Lots of work on BG -algebras [2] have been done by the researchers [26]. Khalid et al. [11] investigated neutrosophic soft cubic subalgebra. Khalid et al. [12] studied translation and multiplication of intuitionistic fuzzy set through some theorems. Khalid et al. [13] first time done the magnification of translation of set MBJ-neutrosophic and proved the results to explain the magnification. Khalid et al. [14]

studied MBJ-neutrosophic T-ideal on B-algebra. Khalid et al. [15] wrote the extensively important results for multiplicative interpretation of neutrosophic cubic set. Takallo et al. [16] discussed the application of MBJ-neutrosophic set. Neggers et al. [6] studied the fundamental theorem of B -homomorphism for B -algebra. Biswas [23] investigated the membership function of interval valued fuzzy set. Ahn [24] proved different results for fuzzy subalgebra. Jun et al. [29] deeply studied neutrosophic cubic set with different characteristics. Basset et. al [17] done the detailed study on hybrid neutrosophic multiple criteria group decision making approach for project selection. Hemavati et. al [20] investigated the β -subalgebra using the interval valed fuzzy set. Surya et. al [18] worked on MBJ neutrsophic β -subalgebra. Basset et. al [19] worked on novel group decision making model based on neutrsophic set for the heart disease.

This paper is presented to define the T-MBJ neutrosophic set and provide the condition for T-MBJ neutrosophic set [T-MBJ NS] to be a T-MBJ neutrosophic M-subalgebra on G -algebra. We also investigate some properties and proved some results for T-MBJ neutrosophic M-subalgebra [T-MBJ NMSU].

2. Preliminaries

Here, some basic definitions are written that are helpful to present this paper.

Definition 2.1. A nonempty set Y with a constant 0 and a binary operation $*$ is said to be G -algebra [21] if it fulfills these axioms:

$$G1: t_1 * t_1 = 0$$

$$G2: t_1 * (t_1 * t_2) = t_2, \text{ for all } t_1, t_2 \in Y.$$

A G -algebra is denoted by $(Y, *, 0)$.

Definition 2.2. A nonempty set Y with a constant 0 and a binary operation $*$ is said to be B -algebra [6] if it fulfills these axioms:

$$B1: t_1 * t_1 = 0$$

$$B2: t_1 * 0 = t_1 \quad B2: (t_1 * t_2) * t_3 = t_1 * (t_3(0 * t_2)), \text{ for all } t_1, t_2, t_3 \in Y.$$

Definition 2.3. Let S be a subset of G -algebra is called a subalgebra [21] of Y if $t_1 * t_2 \in S$ $\forall t_1, t_2 \in S$.

Definition 2.4. Function $f | Y \rightarrow X$ of B -algebra is called homomorphic [6] if $f(t_1 * t_2) = f(t_1) * f(t_2) \forall t_1, t_2 \in Y$. If $f | Y \rightarrow X$ is a B -homomorphic, then $f(0) = 0$.

Definition 2.5. Let C be a fuzzy set [8] in Y is defined as $C = \{ \langle t_1, \vartheta_C(t_1) \rangle \mid t_1 \in Y \}$, where $\vartheta_C(t_1)$ is called the existence ship value of t_1 in C and $\vartheta_C(t_1) \in [0, 1]$.

For a fuzzy set's family $C_i = \{ \langle t_1, \vartheta_{C_i}(t_1) \rangle \mid t_1 \in Y \}$ in Y , where $i \in H$ and H is index set, Join (\vee) and meet (\wedge) are defined as follow:

$$\vee_{i \in H} C_i = (\vee_{i \in H} \vartheta_{C_i})(t_1) = \sup\{\vartheta_{C_i} \mid i \in H\},$$

and

$$\wedge_{i \in H} C_i = (\wedge_{i \in H} \vartheta_{C_i})(t_1) = \inf\{\vartheta_{C_i} \mid i \in H\}$$

respectively, $\forall t_1 \in Y$.

Definition 2.6. [23] Let two elements $D_1, D_2 \in D[0, 1]$. If $D_1 = [(t_1)_1^-, (t_1)_1^+]$ and $D_2 = [(t_1)_2^-, (t_1)_2^+]$, then $rmax(D_1, D_2) = [max((t_1)_1^-, (t_1)_2^-), max((t_1)_1^+, (t_1)_2^+)]$ which is denoted by $D_1 \vee^r D_2$ and $rmin(D_1, D_2) = [min((t_1)_1^-, (t_1)_2^-), min((t_1)_1^+, (t_1)_2^+)]$ which is denoted by $D_1 \wedge^r D_2$. Thus, if $D_i = [((t_1)_1)_i^-, ((t_1)_2)_i^+] \in D[0, 1]$ for $i = 1, 2, 3, \dots$, then they defined $rsup_i(D_i) = [sup_i(((t_1)_1)_i^-), sup_i(((t_1)_1)_i^+)]$, i.e., $\vee_i^r D_i = [\vee_i(((t_1)_1)_i^-), \vee_i(((t_1)_1)_i^+)]$. In the same way they defined $rinf_i(D_i) = [inf_i(((t_1)_1)_i^-), inf_i(((t_1)_1)_i^+)]$, i.e., $\wedge_i^r D_i = [\wedge_i(((t_1)_1)_i^-), \wedge_i(((t_1)_1)_i^+)]$. Now they called $D_1 \geq D_2 \iff (t_1)_1^- \geq (t_1)_2^-$ and $(t_1)_1^+ \geq (t_1)_2^+$. Similarly they defined the relations $D_1 \leq D_2$ and $D_1 = D_2$.

Ahn et al. [24] defined fuzzy subalgebra, which is defined below.

Definition 2.7. A nonempty set $C = \{ \langle t_1, \vartheta_C(t_1) \rangle \mid t_1 \in Y \}$ is called a fuzzy subalgebra [24] of Y if $\vartheta_C(t_1 * t_2) \geq \min\{\vartheta_C(t_1), \vartheta_C(t_2)\} \forall t_1, t_2 \in Y$.

Definition 2.8. For any $\mathcal{C}_i = (\rho_i, \lambda_i)$ [29] where $\rho_i = \{ \langle t_1; \rho_{iE}(t_1), \rho_{iI}(t_1), \rho_{iN}(t_1) \rangle \mid t_1 \in Y \}$, $\lambda_i = \{ \langle t_1; \lambda_{iE}(t_1), \lambda_{iI}(t_1), \lambda_{iN}(t_1) \rangle \mid t_1 \in Y \}$ for $i \in H$, P-union, P-intersection, R-union and R-intersection is defined respectively by **P-union** $\bigcup_{i \in H} \mathcal{C}_i = (\bigcup_{i \in H} \rho_i, \bigvee_{i \in H} \lambda_i)$, **P-intersection** $\bigcap_{i \in H} \mathcal{C}_i = (\bigcap_{i \in H} \rho_i, \bigwedge_{i \in H} \lambda_i)$, **R-union** $\bigcup_{i \in H} \mathcal{C}_i = (\bigcup_{i \in H} \rho_i, \bigwedge_{i \in H} \lambda_i)$, **R-intersection:** $\bigcap_{i \in H} \mathcal{C}_i = (\bigcap_{i \in H} \rho_i, \bigvee_{i \in H} \lambda_i)$, where

$$\begin{aligned} \bigcup_{i \in H} \rho_i &= \{ \langle t_1; (\bigcup_{i \in H} \rho_{iE})(t_1), (\bigcup_{i \in H} \rho_{iI})(t_1), (\bigcup_{i \in H} \rho_{iN})(t_1) \rangle \mid t_1 \in Y \}, \\ \bigvee_{i \in H} \lambda_i &= \{ \langle t_1; (\bigvee_{i \in H} \lambda_{iE})(t_1), (\bigvee_{i \in H} \lambda_{iI})(t_1), (\bigvee_{i \in H} \lambda_{iN})(t_1) \rangle \mid t_1 \in Y \}, \\ \bigcap_{i \in H} \rho_i &= \{ \langle t_1; (\bigcap_{i \in H} \rho_{iE})(t_1), (\bigcap_{i \in H} \rho_{iI})(t_1), (\bigcap_{i \in H} \rho_{iN})(t_1) \rangle \mid t_1 \in Y \}, \\ \bigwedge_{i \in H} \lambda_i &= \{ \langle t_1; (\bigwedge_{i \in H} \lambda_{iE})(t_1), (\bigwedge_{i \in H} \lambda_{iI})(t_1), (\bigwedge_{i \in H} \lambda_{iN})(t_1) \rangle \mid t_1 \in Y \}. \end{aligned}$$

Definition 2.9. Let $B = (\vartheta_B, \nu_B)$ be an IFS of BG-algebra Y and $t \in [0, 1]$, then the IFS B^t is said to be t -intuitionistic fuzzy subset [1] of Y w.r.t B and is defined as $B^t = \{ \langle t_1, \vartheta_{B^t}(t_1), \nu_{B^t}(t_1) \rangle \mid t_1 \in Y \} = \langle \vartheta_{B^t}, \nu_{B^t} \rangle$, where $\vartheta_{B^t}(t_1) = \min\{\vartheta_B(t_1), t\}$ and $\nu_{B^t}(t_1) = \max\{\nu_B(t_1), 1 - t\} \forall t_1 \in Y$.

Definition 2.10. Let $B^t = (\vartheta_{B^t}, \nu_{B^t})$ be a t -IFS of BG-algebra Y and $t \in [0, 1]$ then B^t is said to be t -IFSU [23] of Y if it fulfills these axioms.

- (i) $\vartheta_{B^t}(t_1 * t_2) \geq \min\{\vartheta_{B^t}(t_1), \vartheta_{B^t}(t_2)\}$ and
(ii) $\nu_{B^t}(t_1 * t_2) \leq \max\{\nu_{B^t}(t_1), \nu_{B^t}(t_2)\} \forall t_1, t_2 \in Y$.

Definition 2.11. A MBJ-neutrosophic set [16] in Y is a structure of the form $C = \{\langle M_C t_1, \hat{B}_C t_1, J_C t_1 \rangle \mid t_1 \in Y\}$, where M_C and J_C are fuzzy sets in Y and M_C is a truth membership function, J_C is a false membership function and \hat{B} is an indeterminate interval valued membership function.

3. T-MBJ Neutrosophic M-Subalgebras

Definition 3.1. A nonempty set of the form $\mathcal{C}^t = (M^t, \hat{B}^t, J^t)$ is called a T-MBJ neutrosophic set (**T-MBJ NS**) of Y , where $\mathcal{C}^t = \{\langle t_1, M^t(t_1), \hat{B}^t(t_1), J^t(t_1) \rangle \mid t_1 \in Y\} = \langle M^t, \hat{B}^t, J^t \rangle$ with two independent components is defined as $M^t(t_1) = \{\min(M, t)(t_1)\}$, $\hat{B}^t = \{rmin(\hat{B}, t')(t_1)\}$ and $J^t(t_1) = \{\max(J, 2 - t - t')(t_1)\} \forall t, t', 2 - t - t' \in [0, 1]$, where M^t is truth membership function, \hat{B} is an indeterminate interval valued membership function and J^t is a false membership function.

Definition 3.2. Let $\mathcal{C}^t = (M^t, \hat{B}, J^t)$ be a T-MBJ neutrosophic set. Then \mathcal{C}^t is T-MBJ NMSU under binary operation $*$, where $t_1, t_2, t, t', 2 - t - t', \aleph, \Re \in [0, 1]$ if it satisfies the following three conditions: N1:

$$\begin{aligned} \min(M((t_1 * \aleph) * (t_2 * \Re)), t) &= M^t((t_1 * \aleph) * (t_2 * \Re)) \succeq \min\{M^t(t_1 * \aleph), M^t(t_2 * \Re)\} \\ \min(\hat{B}((t_1 * \aleph) * (t_2 * \Re)), t') &= \hat{B}^t((t_1 * \aleph) * (t_2 * \Re)) \succeq rmin\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_2 * \Re)\} \\ \min(J((t_1 * \aleph) * (t_2 * \Re)), 2 - t - t') &= J^t((t_1 * \aleph) * (t_2 * \Re)) \preceq \max\{J^t(t_1 * \aleph), J^t(t_2 * \Re)\}. \end{aligned}$$

For our simplicity we replace the $2 - t - t'$ with \Im .

Example 3.3. Let $Y = \{0, t_1 * \aleph, t_2 * \Re\}$ be a G -algebra with the following Cayley table.

$*$	0	$t_1 * \aleph$	$t_2 * \Re$
0	0	$t_1 * \aleph$	$t_2 * \Re$
$t_1 * \aleph$	$t_1 * \aleph$	0	$t_2 * \Re$
$t_2 * \Re$	$t_2 * \Re$	$t_1 * \aleph$	0

A T-MBJ neutrosophic set $\mathcal{C}^t = (M^t, \hat{B}, J^t)$ of X is defined by

M^t_T	0	$t_1 * \aleph$	$t_2 * \Re$
	0.4	0.6	0.6
\hat{B}^t	0	$t_1 * \aleph$	$t_2 * \Re$
	[0.1, 0.2]	[0.3, 0.5]	[0.6, 0.9]
J^t	0	$t_1 * \aleph$	$t_2 * \Re$
	0.2	0.4	0.8

It is the routine work to check that set is T-MBJ neutrosophic M-subalgebra.

Proposition 3.4. Let $\mathcal{C}^t = \{\langle t_1, M^t(t_1), \hat{B}^t(t_1), J^t(t_1) \rangle\}$ be a T-MBJ NMSU of Y , then $\forall t_1 \in Y$, $M^t(0 * \aleph) \succeq M^t(t_1 * \aleph)$, $\hat{B}^t(0 * \aleph) \succeq \hat{B}^t(t_1 * \aleph)$ and $J^t(0 * \aleph) \preceq J^t(t_1 * \aleph)$. Thus, $M^t(0 * \aleph)$, $\hat{B}^t(0 * \aleph)$ and $J^t(0 * \aleph)$ are the upper bounds and lower bounds of $M^t(t_1 * \aleph)$, $\hat{B}^t(t_1 * \aleph)$ and $J^t(t_1 * \aleph)$ respectively.

Proof. $\forall t_1 \in Y$, we have $M^t((0 * \aleph)) = \min(M((0 * \aleph)), t) = \min(M((t_1 * \aleph) * (t_1 * \aleph)), t) \succeq \min\{\min(M((t_1 * \aleph)), t), \min(M(t_1 * \aleph), t)\} = \min(M(t_1 * \aleph), t) = M^t((t_1 * \aleph)) \Rightarrow M^t((0 * \aleph)) \succeq M^t((t_1 * \aleph))$, $\hat{B}^t(0 * \aleph) = r\min(\hat{B}(0 * \aleph), t') = r\min(\hat{B}((t_1 * \aleph) * (t_1 * \aleph)), t') \succeq r\min\{r\min(\hat{B}(t_1 * \aleph), t'), r\min(\hat{B}(t_1 * \aleph), t')\} = r\min(\hat{B}(t_1 * \aleph), t') = \hat{B}^t(t_1 * \aleph) \Rightarrow \hat{B}^t(0 * \aleph) \succeq \hat{B}^t(t_1 * \aleph)$ and $\max(J(0 * \aleph), \Im) = \max(J((t_1 * \aleph) * (t_1 * \aleph)), \Im) \succeq \max\{\max(J(t_1 * \aleph), \Im), \max(J(t_1 * \aleph), \Im)\} = \max(J(t_1 * \aleph), t) = J^t(t_1 * \aleph) \Rightarrow J^t(0 * \aleph) \preceq J^t(t_1 * \aleph)$. \square

Theorem 3.5. Let $\mathcal{C}^t = \{\langle (t_1), M^t(t_1), \hat{B}^t(t_1), J^t(t_1) \rangle\}$ be a T-MBJ NMSU of Y . If there exists a sequence $\{(t_1 * \aleph)_n\}$ of Y such that $\lim_{n \rightarrow \infty} M^t((t_1 * \aleph)_n) = 0$, $\lim_{n \rightarrow \infty} \hat{B}^t((t_1 * \aleph)_n) = [1, 1]$ and $\lim_{n \rightarrow \infty} J^t((t_1 * \aleph)_n) = 0$. Then $M^t(0) = 0$, $\hat{B}^t(0) = [1, 1]$ and $J^t(0) = 0$.

Proof. Using Proposition 3.4, $M^t(0 * \aleph) \succeq M^t(t_1 * \aleph) \forall t_1 \in Y$, then $M^t(0 * \aleph) \succeq M^t((t_1 * \aleph)_n)$ for $n \in \mathbf{Z}^+$. Consider, $0 \succeq M^t(0 * \aleph) \succeq \lim_{n \rightarrow \infty} M^t((t_1 * \aleph)_n) = 0$. Hence, $M^t(0 * \aleph) = 0$. Using Proposition 3.4, $\hat{B}^t(0 * \aleph) \succeq \hat{B}^t(t_1 * \aleph) \forall t_1 \in Y$, so therefore $\hat{B}^t(0 * \aleph) \succeq \hat{B}^t((t_1 * \aleph)_n)$ for $n \in \mathbf{Z}^+$. Consider, $[1, 1] \succeq \hat{B}^t(0 * \aleph) \succeq \lim_{n \rightarrow \infty} \hat{B}^t((t_1 * \aleph)_n) = [1, 1]$. Hence, $\hat{B}^t(0 * \aleph) = [1, 1]$. Again, using Proposition 3.4, $J^t(0 * \aleph) \preceq J^t(t_1 * \aleph) \forall t_1 \in Y$, so therefore $J^t(0 * \aleph) \preceq J^t((t_1 * \aleph)_n)$ for $n \in \mathbf{Z}^+$. Consider, $0 \preceq J^t(0 * \aleph) \preceq \lim_{n \rightarrow \infty} J^t((t_1 * \aleph)_n) = 0$. Hence, $J^t(0 * \aleph) = 0$. \square

Theorem 3.6. The R -intersection of any set of T-MBJ NMSU of Y is also a T-MBJ NMSU of Y .

Proof. Let $\mathcal{C}_i^t = \{\langle t_1, M_i^t, \hat{B}_i^t, J_i^t \rangle \mid t_1 \in Y\}$ where $i \in k$, be a set of T-MBJ NMSU of Y and $t_1, t_2 \in Y$ and $t, \aleph, \Re \in [0, 1]$. Then

$$\begin{aligned} (\vee(M_i^t)_i)((t_1 * \aleph) * (t_2 * \Re)) &= \vee(\min(M_i, t)((t_1 * \aleph) * (t_2 * \Re))) \\ &= \sup(\min(M_i, t)((t_1 * \aleph) * (t_2 * \Re))) \\ &\succeq \sup\{\min\{(\min(M_i, t)(t_1 * \aleph)), (\min(M_i, t)(t_2 * \Re))\}\} \\ &= \min\{\sup(\min(M_i, t)(t_1 * \aleph)), \sup(\min(M_i, t)(t_2 * \Re))\} \\ &= \min\{\sup(M_i^t)(t_1 * \aleph), \sup(M_i^t)(t_2 * \Re)\} \\ &= \min\{\vee(M_i^t)(t_1 * \aleph), \vee(M_i^t)(t_2 * \Re)\} \\ &\Rightarrow \vee(M_i^t)((t_1 * \aleph) * (t_2 * \Re)) \succeq \min\{\vee(M_i^t)(t_1 * \aleph), \vee(M_i^t)(t_2 * \Re)\} \end{aligned}$$

and

$$\begin{aligned}
 (\cap(\hat{B}^t)_i)((t_1 * \aleph) * (t_2 * \aleph)) &= \cap(\text{rmin}(\hat{B}_i, t')((t_1 * \aleph) * (t_2 * \aleph))) \\
 &= \text{rinf}(\text{rmin}(\hat{B}_i, t')((t_1 * \aleph) * (t_2 * \aleph))) \\
 &\succeq \text{rinf}\{\text{rmin}\{\text{rmin}(\hat{B}_i, t')(t_1 * \aleph), (\text{rmin}(\hat{B}_i, t')(t_1 * \aleph))\}\} \\
 &= \text{rmin}\{\text{rinf}(\text{rmin}(\hat{B}_i, t')(t_1 * \aleph)), \text{rinf}(\text{rmin}(\hat{B}_i, t')(t_1 * \aleph))\} \\
 &= \text{rmin}\{\text{rinf}(\hat{B}_i^t)(t_1 * \aleph), \text{rinf}(\hat{B}_i^t)(t_1 * \aleph)\} \\
 &= \text{rmin}\{\cap(\hat{B}_i^t)(t_1 * \aleph), \cap(\hat{B}_i^t)(t_1 * \aleph)\} \\
 \Rightarrow \cap(\hat{B}_i^t)((t_1 * \aleph) * (t_2 * \aleph)) &\succeq \text{rmin}\{\cap(\hat{B}_i^t)(t_1 * \aleph), \cap(\hat{B}_i^t)(t_1 * \aleph)\},
 \end{aligned}$$

and

$$\begin{aligned}
 (\vee(J^t)_i)((t_1 * \aleph) * (t_2 * \aleph)) &= \vee(\text{max}(J_i, \Im)((t_1 * \aleph) * (t_2 * \aleph))) \\
 &= \text{sup}(\text{max}(J_i, \Im)((t_1 * \aleph) * (t_2 * \aleph))) \\
 &\preceq \text{sup}\{\text{max}\{(\text{max}(J_i, \Im)(t_1 * \aleph)), (\text{max}(J_i, \Im)(t_1 * \aleph))\}\} \\
 &= \text{max}\{\text{sup}(\text{max}(J_i, \Im)(t_1 * \aleph)), \text{sup}(\text{max}(J_i, \Im)(t_1 * \aleph))\} \\
 &= \text{max}\{\text{sup}(J_i^t)(t_1 * \aleph), \text{sup}(J_i^t)(t_1 * \aleph)\} \\
 &= \text{max}\{\vee(J_i^t)(t_1 * \aleph), \vee(J_i^t)(t_1 * \aleph)\} \\
 \Rightarrow \vee(J_i^t)((t_1 * \aleph) * (t_2 * \aleph)) &\preceq \text{max}\{\vee(J_i^t)(t_1 * \aleph), \vee(J_i^t)(t_1 * \aleph)\},
 \end{aligned}$$

which show that R -intersection of \mathcal{C}_i^t is a T-MBJ NMSU of Y . \square

Theorem 3.7. Let $\mathcal{C}_i^t = \{\langle t_1, (M_i^t), (\hat{B}_i^t), (J_i^t) \rangle \mid t_1 \in Y\}$ be a set of T-MBJ NMSU of Y , where $i \in k$ and $t \in [0, 1]$. If $\inf\{\min\{(M_i^t)(t_1 * \aleph), (M_i^t)(t_1 * \aleph)\}\} = \min\{\inf(M_i^t)(t_1 * \aleph), \inf(M_i^t)(t_1 * \aleph)\}$ and $\inf\{\max\{(J_i^t)(t_1 * \aleph), (J_i^t)(t_1 * \aleph)\}\} = \max\{\inf(J_i^t)(t_1 * \aleph), \inf(J_i^t)(t_1 * \aleph)\} \forall t_1 \in Y$, then P -intersection of \mathcal{C}_i^t is also a T-MBJ NMSU of Y .

Proof. Suppose that $\mathcal{C}_i^t = \{\langle t_1, (M_i^t), (\hat{B}_i^t), (J_i^t) \rangle \mid t_1 \in Y\}$ where $i \in k$, is a family of sets of T-MBJ NMSU of Y such that $\inf\{\min\{(M_i^t)(t_1 * \aleph), (M_i^t)(t_1 * \aleph)\}\} = \min\{\inf(M_i^t)(t_1 * \aleph), \inf(M_i^t)(t_1 * \aleph)\}$ and $\inf\{\max\{(J_i^t)(t_1 * \aleph), (J_i^t)(t_1 * \aleph)\}\} = \max\{\inf(J_i^t)(t_1 * \aleph), \inf(J_i^t)(t_1 * \aleph)\}$

$\aleph\}\forall t_1, t_2 \in Y$ and $t \in [0, 1]$. Then

$$\begin{aligned}
 (\wedge(M_i^t))((t_1 * \aleph) * (t_2 * \aleph)) &= \wedge(\min(M_i, t)((t_1 * \aleph) * (t_2 * \aleph))) \\
 &= \inf(\min(M_i, t)((t_1 * \aleph) * (t_2 * \aleph))) \\
 &\succeq \inf\{\min\{(\min(M_i, t)(t_1 * \aleph)), (\min(M_i, t)(t_2 * \aleph))\}\} \\
 &= \min\{\inf(\min(M_i, t)(t_1 * \aleph)), \inf(\min(M_i, t)(t_2 * \aleph))\} \\
 &= \min\{\inf(M_i^t)(t_1 * \aleph), \inf(M_i^t)(t_2 * \aleph)\} \\
 &= \min\{\wedge(M_i^t)(t_1 * \aleph), \wedge(M_i^t)(t_2 * \aleph)\} \\
 \Rightarrow \wedge(M_i^t)((t_1 * \aleph) * (t_2 * \aleph)) &\succeq \min\{\wedge(M_i^t)(t_1 * \aleph), \wedge(M_i^t)(t_2 * \aleph)\}
 \end{aligned}$$

and

$$\begin{aligned}
 (\cap(\hat{B}_i^t))((t_1 * \aleph) * (t_2 * \aleph)) &= \cap(\text{rmin}(\hat{B}_i, t')((t_1 * \aleph) * (t_2 * \aleph))) \\
 &= \text{rinf}(\text{rmin}(\hat{B}_i, t')((t_1 * \aleph) * (t_2 * \aleph))) \\
 &\succeq \text{rinf}\{\text{rmin}\{(\text{rmin}(\hat{B}_i, t')(t_1 * \aleph)), (\text{rmin}(\hat{B}_i, t')(t_2 * \aleph))\}\} \\
 &= \text{rmin}\{\text{rinf}(\text{rmin}(\hat{B}_i, t')(t_1 * \aleph)), \text{rinf}(\text{rmin}(\hat{B}_i, t')(t_2 * \aleph))\} \\
 &= \text{rmin}\{\text{rinf}(\hat{B}_i^t)(t_1 * \aleph), \text{rinf}(\hat{B}_i^t)(t_2 * \aleph)\} \\
 &= \text{rmin}\{\cap(\hat{B}_i^t)(t_1 * \aleph), \cap(\hat{B}_i^t)(t_2 * \aleph)\} \\
 \Rightarrow \cap(\hat{B}_i^t)((t_1 * \aleph) * (t_2 * \aleph)) &\succeq \text{rmin}\{\cap(\hat{B}_i^t)(t_1 * \aleph), \cap(\hat{B}_i^t)(t_2 * \aleph)\},
 \end{aligned}$$

and

$$\begin{aligned}
 (\wedge(J_i^t))((t_1 * \aleph) * (t_2 * \aleph)) &= \wedge(\max(J_i, \Im)((t_1 * \aleph) * (t_2 * \aleph))) \\
 &= \inf(\max(J_i, \Im)((t_1 * \aleph) * (t_2 * \aleph))) \\
 &\preceq \inf\{\max\{(\max(J_i, \Im)(t_1 * \aleph)), (\max(J_i, \Im)(t_2 * \aleph))\}\} \\
 &= \max\{\inf(\max(J_i, \Im)(t_1 * \aleph)), \inf(\max(J_i, \Im)(t_2 * \aleph))\} \\
 &= \max\{\inf(J_i^t)(t_1 * \aleph), \inf(J_i^t)(t_2 * \aleph)\} \\
 &= \max\{\wedge(J_i^t)(t_1 * \aleph), \wedge(J_i^t)(t_2 * \aleph)\} \\
 \Rightarrow \wedge(J_i^t)((t_1 * \aleph) * (t_2 * \aleph)) &\preceq \max\{\wedge(J_i^t)(t_1 * \aleph), \wedge(J_i^t)(t_2 * \aleph)\},
 \end{aligned}$$

which show that P -intersection of \mathcal{C}_i^t is a T-MBJ NMSU of Y . \square

Theorem 3.8. Let $\mathcal{C}_i^t = \{\langle t_1, (M_i^t), (\hat{B}_i^t), (J_i^t) \rangle \mid t_1 \in Y\}$ where $i \in k$, be a family of sets of T-MBJ NMSU of Y . If $\sup\{\min\{(M_i^t)(t_1 * \aleph), (M_i^t)(t_2 * \aleph)\}\} = \min\{\sup(M_i^t)(t_1 * \aleph), \inf(M_i^t)(t_1 * \aleph)\}$ and $\text{rsup}\{\text{rmin}\{(\hat{B}_i^t)(t_1 * \aleph), (\hat{B}_i^t)(t_2 * \aleph)\}\} = \text{rmin}\{\text{rsup}(\hat{B}_i^t)(t_1 * \aleph), \text{rsup}(\hat{B}_i^t)(t_2 * \aleph)\}$

and $\sup\{\max\{(J_i^t)(t_1 * \aleph), (J_i^t)(t_1 * \aleph)\}\} = \max\{\sup(J_i^t)(t_1 * \aleph), \sup(J_i^t)(t_1 * \aleph)\} \forall t_1, t_2 \in Y$, then P -union of \mathcal{C}_i^t is also a T -MBJ NMSU of Y .

Proof. Let $\mathcal{C}_i^t = \{\langle t_1, (\hat{B}_i^t), (J_i^t) \rangle \mid t_1 \in Y\}$ where $i \in k$, be a family of sets of T -MBJ NMSU of Y such that $\sup\{\min\{(M_i^t)(t_1 * \aleph), (M_i^t)(t_1 * \aleph)\}\} = \min\{\sup(M_i^t)(t_1 * \aleph), \sup(M_i^t)(t_1 * \aleph)\}$ and $\sup\{\min\{(\hat{B}_i^t)(t_1 * \aleph), (\hat{B}_i^t)(t_1 * \aleph)\}\} = \min\{\sup(\hat{B}_i^t)(t_1 * \aleph), \sup(\hat{B}_i^t)(t_1 * \aleph)\}$ and $\sup\{\max\{(J_i^t)(t_1 * \aleph), (J_i^t)(t_1 * \aleph)\}\} = \max\{\sup(J_i^t)(t_1 * \aleph), \sup(J_i^t)(t_1 * \aleph)\} \forall t_1, t_2 \in Y$. Then for $t_1, t_2 \in Y$, and $t \in [0, 1]$.

$$\begin{aligned} (\vee(M^t)_i)((t_1 * \aleph) * (t_2 * \aleph)) &= \vee(\min(M_i, t)((t_1 * \aleph) * (t_2 * \aleph))) \\ &= \sup(\min(M_i, t)((t_1 * \aleph) * (t_2 * \aleph))) \\ &\succeq \sup\{\min\{(\min(M_i, t)(t_1 * \aleph)), (\min(M_i, t)(t_1 * \aleph))\}\} \\ &= \min\{\sup(\min(M_i, t)(t_1 * \aleph)), \sup(\min(M_i, t)(t_1 * \aleph))\} \\ &= \min\{\sup(M_i^t)(t_1 * \aleph), \sup(M_i^t)(t_1 * \aleph)\} \\ &= \min\{\vee(M_i^t)(t_1 * \aleph), \vee(M_i^t)(t_1 * \aleph)\} \\ \Rightarrow \vee(M_i^t)((t_1 * \aleph) * (t_2 * \aleph)) &\succeq \min\{\vee(M_i^t)(t_1 * \aleph), \vee(M_i^t)(t_1 * \aleph)\} \end{aligned}$$

and

$$\begin{aligned} (\cup(\hat{B}^t)_i)((t_1 * \aleph) * (t_2 * \aleph)) &= \cup(\min(\hat{B}_i, t')((t_1 * \aleph) * (t_2 * \aleph))) \\ &= \sup(\min(\hat{B}_i, t')((t_1 * \aleph) * (t_2 * \aleph))) \\ &\succeq \sup\{\min\{(\min(\hat{B}_i, t')(t_1 * \aleph)), (\min(\hat{B}_i, t')(t_1 * \aleph))\}\} \\ &= \min\{\sup(\min(\hat{B}_i, t')(t_1 * \aleph)), \sup(\min(\hat{B}_i, t')(t_1 * \aleph))\} \\ &= \min\{\sup(\hat{B}_i^t)(t_1 * \aleph), \sup(\hat{B}_i^t)(t_1 * \aleph)\} \\ &= \min\{\cup(\hat{B}_i^t)(t_1 * \aleph), \cup(\hat{B}_i^t)(t_1 * \aleph)\} \\ \Rightarrow \cup(\hat{B}_i^t)((t_1 * \aleph) * (t_2 * \aleph)) &\succeq \min\{\cup(\hat{B}_i^t)(t_1 * \aleph), \cup(\hat{B}_i^t)(t_1 * \aleph)\} \end{aligned}$$

and

$$\begin{aligned} (\vee(J^t)_i)((t_1 * \aleph) * (t_2 * \aleph)) &= \vee(\max(J_i, \Im)((t_1 * \aleph) * (t_2 * \aleph))) \\ &= \sup(\max(J_i, \Im)((t_1 * \aleph) * (t_2 * \aleph))) \\ &\preceq \sup\{\max\{(\max(J_i, \Im)(t_1 * \aleph)), (\max(J_i, \Im)(t_1 * \aleph))\}\} \\ &= \max\{\sup(\max(J_i, \Im)(t_1 * \aleph)), \sup(\max(J_i, \Im)(t_1 * \aleph))\} \\ &= \max\{\sup(J_i^t)(t_1 * \aleph), \sup(J_i^t)(t_1 * \aleph)\} \\ &= \max\{\vee(J_i^t)(t_1 * \aleph), \vee(J_i^t)(t_1 * \aleph)\} \\ \Rightarrow \vee(J_i^t)((t_1 * \aleph) * (t_2 * \aleph)) &\preceq \max\{\vee(J_i^t)(t_1 * \aleph), \vee(J_i^t)(t_1 * \aleph)\}, \end{aligned}$$

which show that P -union of \mathcal{C}_i^t is a T-MBJ NMSU of Y . \square

Theorem 3.9. Let $\mathcal{C}_i^t = \{ \langle t_1, (\hat{B}_i^t), (J_i^t) \rangle \mid t_1 \in Y \}$ where $i \in k$, be a family of sets of T-MBJ NMSU of Y . If $\inf\{\min\{(M_i^t)(t_1 * \aleph), (M_i^t)(t_1 * \mathfrak{R})\}\} = \min\{\inf(M_i^t)(t_1 * \aleph), \inf(M_i^t)(t_1 * \mathfrak{R})\}$ and $\inf\{\max\{(J_i^t)(t_1 * \aleph), (J_i^t)(t_1 * \mathfrak{R})\}\} = \max\{\inf(J_i^t)(t_1 * \aleph), \inf(J_i^t)(t_1 * \mathfrak{R})\}$ and $\sup\{\min\{(\hat{B}_i^t)(t_1 * \aleph), (\hat{B}_i^t)(t_1 * \mathfrak{R})\}\} = \min\{\sup(\hat{B}_i^t)(t_1 * \aleph), \sup(\hat{B}_i^t)(t_1 * \mathfrak{R})\} \forall t_1 \in Y$, and $t \in [0, 1]$ then R -union of \mathcal{C}_i^t is also a T-MBJ NMSU of Y .

Proof. Let $\mathcal{C}_i^t = \{ \langle t_1, (M_i^t), (\hat{B}_i^t), (J_i^t) \rangle \mid t_1 \in Y \}$ where $i \in k$, and $t \in [0, 1]$ be a family of sets of T-MBJ NMSU of Y such that $\inf\{\min\{(M_i^t)(t_1 * \aleph), (M_i^t)(t_1 * \mathfrak{R})\}\} = \min\{\inf(M_i^t)(t_1 * \aleph), \inf(M_i^t)(t_1 * \mathfrak{R})\}$ and $\inf\{\max\{(J_i^t)(t_1 * \aleph), (J_i^t)(t_1 * \mathfrak{R})\}\} = \max\{\inf(J_i^t)(t_1 * \aleph), \inf(J_i^t)(t_1 * \mathfrak{R})\}$ and $\sup\{\min\{(\hat{B}_i^t)(t_1 * \aleph), (\hat{B}_i^t)(t_1 * \mathfrak{R})\}\} = \min\{\sup(\hat{B}_i^t)(t_1 * \aleph), \sup(\hat{B}_i^t)(t_1 * \mathfrak{R})\} \forall t_1 \in Y$, and $t \in [0, 1]$. Then for $t_1, t_2 \in Y$ and $t \in [0, 1]$.

$$\begin{aligned} (\wedge(M_i^t))((t_1 * \aleph) * (t_2 * \mathfrak{R})) &= (\wedge(\min(M_i, t)((t_1 * \aleph) * (t_2 * \mathfrak{R}))) \\ &= \inf\{(\min(M_i, t)((t_1 * \aleph) * (t_2 * \mathfrak{R})))\} \\ &\succeq \inf\{\min\{\min(M_i, t)(t_1 * \aleph), \min(M_i, t)(t_1 * \mathfrak{R})\}\} \\ &= \min\{\inf(\min(M_i, t)(t_1 * \aleph)), \inf(\min(M_i, t)(t_1 * \mathfrak{R}))\} \\ &= \min\{\inf((M_i^t)(t_1 * \aleph)), \inf((M_i^t)(t_1 * \mathfrak{R}))\} \\ &= \min\{\wedge(M_i^t)(t_1 * \aleph), \wedge(M_i^t)(t_1 * \mathfrak{R})\} \\ \Rightarrow \wedge(M_i^t)((t_1 * \aleph) * (t_2 * \mathfrak{R})) &\succeq \min\{\wedge(M_i^t)(t_1 * \aleph), \wedge(M_i^t)(t_1 * \mathfrak{R})\}, \end{aligned}$$

and

$$\begin{aligned} (\cup(\hat{B}_i^t))((t_1 * \aleph) * (t_2 * \mathfrak{R})) &= (\cup(\min(\hat{B}_i, t')((t_1 * \aleph) * (t_2 * \mathfrak{R})))) \\ &= \sup\{\min(\hat{B}_i, t')((t_1 * \aleph) * (t_2 * \mathfrak{R}))\} \\ &\succeq \sup\{\min\{\min(\hat{B}_i, t')(t_1 * \aleph), \min(\hat{B}_i, t')(t_1 * \mathfrak{R})\}\} \\ &= \min\{\sup(\min(\hat{B}_i, t')(t_1 * \aleph)), \sup(\min(\hat{B}_i, t')(t_1 * \mathfrak{R}))\} \\ &= \min\{\sup((\hat{B}_i^t)(t_1 * \aleph)), \sup((\hat{B}_i^t)(t_1 * \mathfrak{R}))\} \\ &= \min\{\cup(\hat{B}_i^t)(t_1 * \aleph), \cup(\hat{B}_i^t)(t_1 * \mathfrak{R})\} \\ \Rightarrow \cup(\hat{B}_i^t)((t_1 * \aleph) * (t_2 * \mathfrak{R})) &\succeq \min\{\cup(\hat{B}_i^t)(t_1 * \aleph), \cup(\hat{B}_i^t)(t_1 * \mathfrak{R})\}, \end{aligned}$$

and

$$\begin{aligned}
 (\wedge(J_i^t))((t_1 * \aleph) * (t_2 * \aleph)) &= (\wedge(\max(J_i, \Im))((t_1 * \aleph) * (t_2 * \aleph))) \\
 &= \inf\{\max(J_i, \Im)((t_1 * \aleph) * (t_2 * \aleph))\} \\
 &\preceq \inf\{\max\{\max(J_i, \Im)(t_1 * \aleph), \max(J_i, \Im)(t_2 * \aleph)\}\} \\
 &= \max\{\inf(\max(J_i, \Im)(t_1 * \aleph)), \inf(\max(J_i, \Im)(t_2 * \aleph))\} \\
 &= \max\{\inf((J_i^t)(t_1 * \aleph)), \inf((J_i^t)(t_2 * \aleph))\} \\
 &= \max\{\wedge(J_i^t)(t_1 * \aleph), \wedge(J_i^t)(t_2 * \aleph)\} \\
 \Rightarrow \wedge(J_i^t)((t_1 * \aleph) * (t_2 * \aleph)) &\preceq \max\{\wedge(J_i^t)(t_1 * \aleph), \wedge(J_i^t)(t_2 * \aleph)\},
 \end{aligned}$$

which show that R -union of \mathcal{C}_i^t is a T-MBJ NMSU of Y . \square

Proposition 3.10. *If a T-MBJ neutrosophic set $\mathcal{C}^t = (M^t, \hat{B}^t, J^t)$ of Y is a TMBJ-netrosophic M -subalgebra, then $\forall t_1 \in Y$, $M^t(0 * t_1) \succeq M^t(t_1 * \aleph)$ and $\hat{B}^t(0 * t_1) \succeq \hat{B}^t(t_1 * \aleph)$ and $J^t(0 * t_1) \preceq J^t(t_1 * \aleph)$.*

Proof. $\forall t_1 \in Y$, $M^t(0 * t_1) = \min(M(0 * t_1), t) \succeq \min\{\min(M(0), t), \min(M(t_1 * \aleph), t)\} = \min\{\min(M(t_1 * t_1), t), \min(M(t_1 * \aleph), t)\} \succeq \min\{\min\{\min(M(t_1 * \aleph), t), \min(M(t_1 * \aleph), t)\}, \min(M(t_1 * \aleph), t)\} = \min(M(t_1 * \aleph), t) = M^t(t_1 * \aleph)$ and $\hat{B}^t(0 * t_1) = rmin(\hat{B}(0 * t_1), t') \succeq rmin\{rmin(\hat{B}(0), t'), rmin(\hat{B}(t_1 * \aleph), t')\} = rmin\{rmin(\hat{B}(t_1 * t_1), t'), rmin(\hat{B}(t_1 * \aleph), t')\} \succeq rmin\{rmin\{rmin(\hat{B}(t_1 * \aleph), t'), rmin(\hat{B}(t_1 * \aleph), t')\}, rmin(\hat{B}(t_1 * \aleph), t')\} = rmin(\hat{B}(t_1 * \aleph), t') = \hat{B}^t(t_1 * \aleph)$ and $J^t(0 * t_1) = \max(J(0 * t_1), \Im) \preceq \max\{\max(J(0), \Im), \max(J(t_1 * \aleph), \Im)\} = \max\{\max(J(t_1 * t_1), \Im), \max(J(t_1 * \aleph), \Im)\} \preceq \max\{\max\{\max(J(t_1 * \aleph), \Im), \max(J(t_1 * \aleph), \Im)\}, \max(J(t_1 * \aleph), \Im)\} = \max(J(t_1 * \aleph), \Im) = J^t(t_1 * \aleph)$ \square

Lemma 3.11. *If a T-MBJ neutrosophic set $\mathcal{C}^t = (M^t, \hat{B}^t, J^t)$ of Y is a T-MBJ neutrosophic M -subalgebra, then $\mathcal{C}^t((t_1 * \aleph) * (t_2 * \aleph)) = \mathcal{C}^t((t_1 * \aleph) * (0 * (0 * (t_2 * \aleph)))) \forall t_1, t_2 \in Y$.*

Proof. Let Y be a G -algebra and $t_1, t_2 \in Y$. Then we know by lemma that $t_2 * \aleph = 0 * (0 * (t_2 * \aleph))$. Hence, $M^t((t_1 * \aleph) * (t_2 * \aleph)) = M^t((t_1 * \aleph) * (0 * (0 * (t_2 * \aleph))))$ and $\hat{B}^t((t_1 * \aleph) * (t_2 * \aleph)) = \hat{B}^t((t_1 * \aleph) * (0 * (0 * (t_2 * \aleph))))$ and $J^t((t_1 * \aleph) * (t_2 * \aleph)) = J^t((t_1 * \aleph) * (0 * (0 * (t_2 * \aleph))))$. Therefore, $\mathcal{C}^t((t_1 * \aleph) * (t_2 * \aleph)) = \mathcal{C}^t((t_1 * \aleph) * (0 * (0 * (t_2 * \aleph))))$ \square

Proposition 3.12. *If T-MBJ neutrosophic set $\mathcal{C}^t = (M^t, \hat{B}^t, J^t)$ of Y is a T-MBJ NMSU, then $\forall t_1, t_2 \in Y$, $M^t((t_1 * \aleph) * (0 * (t_2 * \aleph))) \succeq \min\{M^t(t_1 * \aleph), M^t(t_2 * \aleph)\}$ and $\hat{B}^t((t_1 * \aleph) * (0 * (t_2 * \aleph))) \succeq rmin\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_2 * \aleph)\}$ and $J^t((t_1 * \aleph) * (0 * (t_2 * \aleph))) \preceq \max\{J^t(t_1 * \aleph), J^t(t_2 * \aleph)\}$.*

Proof. Let $t_1, t_2 \in Y$. Then we have $M^t((t_1 * \aleph) * (0 * (t_2 * \mathfrak{R}))) = \min(M((t_1 * \aleph) * (0 * (t_2 * \mathfrak{R}))), t) \succeq \min\{\min(M(t_1 * \aleph), t), \min(M(0 * (t_2 * \mathfrak{R})), t)\} \succeq \min\{\min(M(t_1 * \aleph), t), \min(M(t_2 * \mathfrak{R}), t)\} = \min\{M^t(t_1 * \aleph), M^t(t_2 * \mathfrak{R})\}$ and $\hat{B}^t((t_1 * \aleph) * (0 * (t_2 * \mathfrak{R}))) = rmin(\hat{B}((t_1 * \aleph) * (0 * (t_2 * \mathfrak{R}))), t') \succeq rmin\{rmin(\hat{B}(t_1 * \aleph), t'), rmin(\hat{B}(0 * (t_2 * \mathfrak{R})), t')\} \succeq rmin\{rmin(\hat{B}(t_1 * \aleph), t'), rmin(\hat{B}(t_2 * \mathfrak{R}), t')\} = rmin\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_2 * \mathfrak{R})\}$ and $J^t((t_1 * \aleph) * (0 * (t_2 * \mathfrak{R}))) = \max(J((t_1 * \aleph) * (0 * (t_2 * \mathfrak{R}))), \Im) \preceq \max\{\max(J(t_1 * \aleph), \Im), \max(J(0 * (t_2 * \mathfrak{R})), \Im)\} \preceq \max\{\max(J(t_1 * \aleph), t), \max(J(t_2 * \mathfrak{R}), \Im)\} = \max\{J^t(t_1 * \aleph), J^t(t_2 * \mathfrak{R})\}$ by Definition and Proposition. \square

Proposition 3.13. If T -MBJ neutrosophic set $\mathcal{C}^t = (M^t, \hat{B}^t, J^t)$ of Y fulfills the following statements, then \mathcal{C}^t refers to a T -MBJ NMSU of Y .

- (1) $M^t(0 * t_1) \succeq M^t(t_1 * \aleph)$ and $\hat{B}^t(0 * t_1) \succeq \hat{B}^t(t_1 * \aleph)$ and $J^t(0 * t_1) \preceq J^t(t_1 * \aleph) \forall t_1 \in Y$.
- (2) $M^t(t_1 * (0 * t_2)) \succeq \min\{M^t(t_1 * \aleph), M^t(t_1 * \mathfrak{R})\}$ and $\hat{B}^t(t_1 * (0 * t_2)) \succeq rmin\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_1 * \mathfrak{R})\}$ and $J^t(t_1 * (0 * t_2)) \preceq \max\{J^t(t_1 * \aleph), J^t(t_1 * \mathfrak{R})\} \forall t_1, t_2 \in Y$ and $t \in [0, 1]$.

Proof. Let T -MBJ neutrosophic set $\mathcal{C}^t = (M^t, \hat{B}^t, J^t)$ of Y fulfills the above statements (1 and 2). Then by Lemma 3.11, we have $M^t((t_1 * \aleph) * (t_2 * \mathfrak{R})) = \{\min(M((t_1 * \aleph) * (t_2 * \mathfrak{R})), t)\} = \{\min(M(t_1 * (0 * (0 * t_2))), t)\} \succeq \min\{\min(M(t_1 * \aleph), t), \min(M(0 * t_2), t)\} \succeq \min\{\min(M(t_1 * \aleph), t), \min(M(0 * t_2), t)\} = \min\{M^t(t_1 * \aleph), M^t(t_1 * \mathfrak{R})\}$ and $\hat{B}^t((t_1 * \aleph) * (t_2 * \mathfrak{R})) = \{rmin(\hat{B}((t_1 * \aleph) * (t_2 * \mathfrak{R}))), t'\} = \{rmin(\hat{B}(t_1 * (0 * (0 * t_2))), t')\} \succeq rmin\{rmin(\hat{B}(t_1 * \aleph), t), rmin(\hat{B}(0 * t_2), t')\} \succeq rmin\{rmin(\hat{B}(t_1 * \aleph), t), rmin(\hat{B}(0 * t_2), t')\} = rmin\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_1 * \mathfrak{R})\}$ and $J^t((t_1 * \aleph) * (t_2 * \mathfrak{R})) = \{\max(J((t_1 * \aleph) * (t_2 * \mathfrak{R}))), \Im\} = \{\max(J(t_1 * (0 * (0 * t_2))), \Im)\} \preceq \max\{\max(J(t_1 * \aleph), t), \max(J(0 * t_2), \Im)\} \preceq \max\{\max(J(t_1 * \aleph), \Im), \max(J(0 * t_2), \Im)\} = \max\{J^t(t_1 * \aleph), J^t(t_1 * \mathfrak{R})\} \forall t_1, t_2 \in Y$. Hence, \mathcal{C}^t is T -MBJ NMSU of Y . \square

Theorem 3.14. The T -MBJ neutrosophic set $\mathcal{C}^t = (M^t, \hat{B}^t, J^t)$ of Y is a T -MBJ NMSU of $Y \iff M^t$ and $\hat{B}^{t-}, \hat{B}^{t+}$ and J^t are fuzzy subalgebra of Y .

Proof. Suppose $M^t, \hat{B}^{t-}, \hat{B}^{t+}$ and J^t are fuzzy subalgebra of Y and $t_1, t_2 \in Y$ and $t, t', \Im \in [0, 1]$. Then $M^t((t_1 * \aleph) * (t_2 * \mathfrak{R})) = \{\min(M((t_1 * \aleph) * (t_2 * \mathfrak{R}))), t\} \succeq \min\{\min(M(t_1 * \aleph), t), \min(M(t_1 * \mathfrak{R}), t)\} = \min\{M^t(t_1 * \aleph), M^t(t_1 * \mathfrak{R})\}$ and $\hat{B}^{t-}((t_1 * \aleph) * (t_2 * \mathfrak{R})) = \{rmin(\hat{B}^-(t_1 * \aleph) * (t_2 * \mathfrak{R})), t'\} \succeq rmin\{rmin(\hat{B}^-(t_1 * \aleph), t'), rmin(\hat{B}^-(t_1 * \mathfrak{R}), t')\} = rmin\{\hat{B}^{t-}(t_1 * \aleph), \hat{B}^{t-}(t_1 * \mathfrak{R})\}$ and $\hat{B}^{t+}((t_1 * \aleph) * (t_2 * \mathfrak{R})) = \{rmin(\hat{B}^+(t_1 * \aleph) * (t_2 * \mathfrak{R})), t'\} \succeq rmin\{rmin(\hat{B}^+(t_1 * \aleph), t'), rmin(\hat{B}^+(t_1 * \mathfrak{R}), t')\} = rmin\{\hat{B}^{t+}(t_1 * \aleph), \hat{B}^{t+}(t_1 * \mathfrak{R})\}$ and $J^t((t_1 * \aleph) * (t_2 * \mathfrak{R})) = \{\max(J((t_1 * \aleph) * (t_2 * \mathfrak{R}))), \Im\} \preceq \max\{\max(J(t_1 * \aleph), \Im), \max(J(t_1 * \mathfrak{R}), \Im)\} = \max\{J^t(t_1 * \aleph), J^t(t_1 * \mathfrak{R})\}$. Now, $\hat{B}^t((t_1 * \aleph) * (t_2 * \mathfrak{R})) = [\hat{B}^{t-}((t_1 * \aleph) * (t_2 * \mathfrak{R})), \hat{B}^{t+}((t_1 * \aleph) * (t_2 * \mathfrak{R}))] = [rmin(\hat{B}^-(t_1 * \aleph) * (t_2 * \mathfrak{R})), t'), rmin(\hat{B}^+(t_1 * \aleph) * (t_2 * \mathfrak{R})), t')] \succeq$

$[rmin\{rmin(\hat{B}-(t_1 * \aleph), t'), rmin(\hat{B}-(t_1 * \aleph), t')\}, rmin\{rmin(\hat{B}+(t_1 * \aleph), t'), rmin(\hat{B}+(t_1 * \aleph), t')\}] = [rmin\{\hat{B}^{t-}(t_1 * \aleph), \hat{B}^{t-}(t_1 * \aleph)\}, rmin\{\hat{B}^{t+}(t_1 * \aleph), \hat{B}^{t+}(t_1 * \aleph)\}] \succeq rmin\{[\hat{B}^{t-}(t_1 * \aleph), \hat{B}^{t+}(t_1 * \aleph)], [\hat{B}^{t-}(t_1 * \aleph), \hat{B}^{t+}(t_1 * \aleph)]\} = rmin\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_1 * \aleph)\}$. Therefore, \mathcal{C}^t is T-MBJ NMSU of Y .

Conversely, assume that \mathcal{C}^t is a T-MBJ NMSU of Y . For any $t_1, t_2 \in Y$, $M^t((t_1 * \aleph) * (t_2 * \aleph)) = \{min(M((t_1 * \aleph) * (t_2 * \aleph)), t)\} \succeq min\{min(M(t_1 * \aleph), t), min(M(t_2 * \aleph), t)\} = min\{M^t(t_1 * \aleph), M^t(t_2 * \aleph)\}$ and $[\hat{B}^{t-}((t_1 * \aleph) * (t_2 * \aleph)), \hat{B}^{t+}((t_1 * \aleph) * (t_2 * \aleph))] = \hat{B}^t((t_1 * \aleph) * (t_2 * \aleph)) = \{rmin(\hat{B}((t_1 * \aleph) * (t_2 * \aleph)), t')\} \succeq rmin\{rmin(\hat{B}(t_1 * \aleph), t'), rmin(\hat{B}(t_2 * \aleph), t')\} = rmin\{[rmin(\hat{B}-(t_1 * \aleph), t'), rmin(\hat{B}+(t_1 * \aleph), t'), rmin(\hat{B}-(t_2 * \aleph), t'), rmin(\hat{B}+(t_2 * \aleph), t')]\} = [rmin\{rmin(\hat{B}-(t_1 * \aleph), t'), rmin(\hat{B}-(t_2 * \aleph), t')\}, rmin\{rmin(\hat{B}+(t_1 * \aleph), t'), rmin(\hat{B}+(t_2 * \aleph), t')\}] = [rmin\{\hat{B}^{t-}(t_1 * \aleph), \hat{B}^{t-}(t_2 * \aleph)\}, rmin\{\hat{B}^{t+}(t_1 * \aleph), \hat{B}^{t+}(t_2 * \aleph)\}]. Thus, $\hat{B}^{t-}((t_1 * \aleph) * (t_2 * \aleph)) \succeq min\{\hat{B}^{t-}(t_1 * \aleph), \hat{B}^{t-}(t_2 * \aleph)\}$, $\hat{B}^{t+}((t_1 * \aleph) * (t_2 * \aleph)) \succeq min\{\hat{B}^{t+}(t_1 * \aleph), \hat{B}^{t+}(t_2 * \aleph)\}$ and $J^t((t_1 * \aleph) * (t_2 * \aleph)) = \{max(J((t_1 * \aleph) * (t_2 * \aleph)), \Im)\} \preceq max\{max(J(t_1 * \aleph), \Im), max(J(t_2 * \aleph), \Im)\} = max\{J^t(t_1 * \aleph), J^t(t_2 * \aleph)\}$. Hence M^t and $\hat{B}^{t+}, \hat{B}^{t-}$ and J^t are fuzzy subalgebra of Y . $\square$$

Theorem 3.15. Let $\mathcal{C}^t = (M^t, \hat{B}^t, J^t)$ be a T-MBJ NMSU of Y . Then the sets I_{M^t} , $I_{\hat{B}^t}$ and I_{J^t} which are defined as $I_{M^t} = \{t_1 \in Y \mid M^t(t_1 * \aleph) = M^t(0)\}$, $I_{\hat{B}^t} = \{t_1 \in Y \mid \hat{B}^t(t_1 * \aleph) = \hat{B}^t(0)\}$ and $I_{J^t} = \{t_1 \in Y \mid J^t(t_1 * \aleph) = J^t(0)\}$ are T-MBJ neutrosophic M-subalgebra of Y .

Proof. Let $t_1, t_2 \in I_{M^t}$. Then $M^t(t_1 * \aleph) = M^t(0) = M^t(t_1 * \aleph)$ and so, $M^t((t_1 * \aleph) * (t_2 * \aleph)) = \{min(M((t_1 * \aleph) * (t_2 * \aleph)), t)\} \succeq min\{(min(M(t_1 * \aleph), t), (min(M(t_2 * \aleph), t))\} = M^t(0)$. By using Proposition 3.4, as we know that $M^t((t_1 * \aleph) * (t_2 * \aleph)) = M^t(0)$ or equivalently $(t_1 * \aleph) * (t_2 * \aleph) \in I_{M^t}$.

Now we let $t_1, t_2 \in I_{\hat{B}^t}$. Then $\hat{B}^t(t_1 * \aleph) = \hat{B}^t(0) = \hat{B}^t(t_1 * \aleph)$ and so, $\hat{B}^t((t_1 * \aleph) * (t_2 * \aleph)) = \{rmin(\hat{B}((t_1 * \aleph) * (t_2 * \aleph)), t')\} \succeq rmin\{(rmin(\hat{B}(t_1 * \aleph), t'), (rmin(\hat{B}(t_2 * \aleph), t'))\} = rmin\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_2 * \aleph)\} = \hat{B}^t(0)$. By using Proposition 3.4, as we know that $\hat{B}^t((t_1 * \aleph) * (t_2 * \aleph)) = \hat{B}^t(0)$ or equivalently $(t_1 * \aleph) * (t_2 * \aleph) \in I_{\hat{B}^t}$.

Again we let $t_1, t_2 \in I_{J^t}$. Then $J^t(t_1 * \aleph) = J^t(0) = J^t(t_1 * \aleph)$ and so, $J^t((t_1 * \aleph) * (t_2 * \aleph)) = \{max(J((t_1 * \aleph) * (t_2 * \aleph)), t)\} \preceq max\{(max(J(t_1 * \aleph), t), (max(J(t_2 * \aleph), t))\} = J^t(0)$. Again by using Proposition 3.4, as we know that $J^t((t_1 * \aleph) * (t_2 * \aleph)) = J^t(0)$ or equivalently $(t_1 * \aleph) * (t_2 * \aleph) \in I_{J^t}$. Hence the sets I_{M^t} , $I_{\hat{B}^t}$ and I_{J^t} are subalgebra of Y . \square

Definition 3.16. Let $\mathcal{C}^t = \{M^t, \hat{B}^t, J^t\}$ be a T-MBJ neutrosophic set of Y . For $[s_1, s_2] \in D[0, 1]$ and $t_1, t_2 \in [0, 1]$, the set $U(M^t \mid \hat{t}) = \{t_1 \in Y \mid M^t(t_1 * \aleph) \succeq \hat{t}\}$ is called upper t_1 -level of \mathcal{C}^t and the set $U(\hat{B}^t \mid [s_1, s_2]) = \{s_1, s_2 \in Y \mid \hat{B}^t(t_1 * \aleph) \succeq [s_1, s_2]\}$ is called upper $[s_1, s_2]$ -level

of \mathcal{C}^t and $L(J^t | \acute{t}) = \{t_1 \in Y \mid J^t(t_1 * \aleph) \preceq \acute{t}\}$ is called lower $(t_1 * \aleph)$ -level of \mathcal{C}^t .

Theorem 3.17. *If $\mathcal{C}^t = (M^t, \hat{B}^t, J^t)$ is T-MBJ NMSU of Y , then the upper \acute{t} -level, upper $[s_1, s_2]$ -level and lower \acute{t} -level of \mathcal{C}^t are subalgebra of Y .*

Proof. Let $t_1, t_2 \in U(M^t | \acute{t})$. Then $M^t(t_1 * \aleph) \succeq \acute{t}$ and $M^t(t_2 * \aleph) \succeq \acute{t}$. It follows that $M^t((t_1 * \aleph) * (t_2 * \aleph)) = \{\min(M((t_1 * \aleph) * (t_2 * \aleph)), t)\} \succeq \min\{\min(M(t_1 * \aleph), M(t_2 * \aleph)), t\} = \min\{\min(M(t_1 * \aleph), t), \min(M(t_2 * \aleph), t)\} = \min\{M^t(t_1 * \aleph), M^t(t_2 * \aleph)\} \succeq \acute{t} \Rightarrow (t_1 * \aleph) * (t_2 * \aleph) \in U(M^t | \acute{t})$. Hence $U(M^t | \acute{t})$ is a subalgebra of Y . Let $t_1, t_2 \in U(\hat{B}^t | [s_1, s_2])$. Then $\hat{B}^t(t_1 * \aleph) \succeq [s_1, s_2]$ and $\hat{B}^t(t_2 * \aleph) \succeq [s_1, s_2]$. It follows that $\hat{B}^t((t_1 * \aleph) * (t_2 * \aleph)) = \{rmin(\hat{B}((t_1 * \aleph) * (t_2 * \aleph)), t')\} \succeq rmin\{rmin(\hat{B}(t_1 * \aleph), \hat{B}(t_2 * \aleph)), t'\} = rmin\{rmin(\hat{B}(t_1 * \aleph), t'), rmin(\hat{B}(t_2 * \aleph), t')\} = rmin\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_2 * \aleph)\} \succeq [s_1, s_2] \Rightarrow (t_1 * \aleph) * (t_2 * \aleph) \in U(\hat{B}^t | [s_1, s_2])$. Hence, $U(\hat{B}^t | [s_1, s_2])$ is a subalgebra of Y . Let $t_1, t_2 \in L(J^t | \acute{t})$. Then $J^t(t_1 * \aleph) \preceq \acute{t}$ and $J^t(t_2 * \aleph) \preceq \acute{t}$. It follows that $J^t((t_1 * \aleph) * (t_2 * \aleph)) = \{\max(J((t_1 * \aleph) * (t_2 * \aleph)), \Im)\} \preceq \max\{\max(J(t_1 * \aleph), J(t_2 * \aleph)), \Im\} = \max\{\max(J(t_1 * \aleph), t), \max(J(t_2 * \aleph), t)\} = \max\{J^t(t_1 * \aleph), J^t(t_2 * \aleph)\} \preceq \acute{t} \Rightarrow (t_1 * \aleph) * (t_2 * \aleph) \in L(J^t | \acute{t})$. Hence $L(J^t | \acute{t})$ is a subalgebra of Y . \square

Theorem 3.18. *Any subalgebra of Y can be considered as upper \acute{t} -level, upper $[s_1, s_2]$ -level and lower \acute{t} -level of some T-MBJ NMSU of Y .*

Proof. Let \mathcal{D}^t be a T-MBJ NMSU of Y , and \mathcal{C}^t be a T-MBJ neutrosophic set on Y defined by

$$M^t = \begin{cases} [\nu] & \text{if } t_1 \in \mathcal{D}^t \\ 1, & \text{otherwise.} \end{cases} \quad \hat{B}^t = \begin{cases} [\mu_1, \mu_2] & \text{if } t_1 \in \mathcal{D}^t \\ [0, 0] & \text{otherwise.} \end{cases}, \quad J^t = \begin{cases} [\nu] & \text{if } t_1 \in \mathcal{D}^t \\ 0, & \text{otherwise.} \end{cases}$$

$\forall [\mu_1, \mu_2] \in D[0, 1]$ and $\nu \in [0, 1]$. Now we discuss the following cases.

Case 1. If $\forall t_1, t_2 \in \mathcal{D}^t$ then $M^t(t_1 * \aleph) = \nu$, $\hat{B}^t(t_1 * \aleph) = [\mu_1, \mu_2]$, $J^t(t_1 * \aleph) = \nu$ and $M^t(t_2 * \aleph) = \nu$, $\hat{B}^t(t_2 * \aleph) = [\mu_1, \mu_2]$, $J^t(t_2 * \aleph) = \nu$. Thus $M^t((t_1 * \aleph) * (t_2 * \aleph)) = \nu = \min\{\nu, \nu\} = \min\{M^t(t_1 * \aleph), M^t(t_2 * \aleph)\}$ and $\hat{B}^t((t_1 * \aleph) * (t_2 * \aleph)) = [\mu_1, \mu_2] = rmin\{[\mu_1, \mu_2], [\mu_1, \mu_2]\} = rmin\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_2 * \aleph)\}$ and $J^t((t_1 * \aleph) * (t_2 * \aleph)) = \nu = \max\{\nu, \nu\} = \max\{J^t(t_1 * \aleph), J^t(t_2 * \aleph)\}$.

Case 2. If $t_1 \in \mathcal{D}^t$ and $t_2 \notin \mathcal{D}^t$, then $M^t(t_1 * \aleph) = \nu$, $\hat{B}^t(t_1 * \aleph) = [\mu_1, \mu_2]$, $J^t(t_1 * \aleph) = \nu$ and $M^t(t_2 * \aleph) = 0$, $\hat{B}^t(t_2 * \aleph) = [0, 0]$, $J^t(t_2 * \aleph) = 1$. Thus $M^t((t_1 * \aleph) * (t_2 * \aleph)) \succeq 0 = \min\{\nu, 0\} = \min\{M^t(t_1 * \aleph), M^t(t_2 * \aleph)\}$, $\hat{B}^t((t_1 * \aleph) * (t_2 * \aleph)) \succeq [0, 0] = rmin\{[\mu_1, \mu_2], [0, 0]\} = rmin\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_2 * \aleph)\}$ and $J^t((t_1 * \aleph) * (t_2 * \aleph)) \preceq 1 = \max\{\nu, 1\} = \max\{J^t(t_1 * \aleph), J^t(t_2 * \aleph)\}$.

Case 3. If $t_1 \notin \mathcal{D}^t$ and $t_2 \in \mathcal{D}^t$, then $M^t(t_1 * \aleph) = 0$, $\hat{B}^t(t_1 * \aleph) = [0, 0]$, $J^t(t_1 * \aleph) = 1$ and $M^t(t_2 * \aleph) = \nu$, $\hat{B}^t(t_2 * \aleph) = [\mu_1, \mu_2]$, $J^t(t_2 * \aleph) = \nu$. Thus $M^t((t_1 * \aleph) * (t_2 * \aleph)) \succeq 0 =$

$\min\{0, v\} = \min\{M^t(t_1 * \aleph), M^t(t_2 * \aleph)\}$, $\hat{B}^t((t_1 * \aleph) * (t_2 * \aleph)) \succeq [0, 0] = \text{rmin}\{[0, 0], [\mu_1, \mu_2]\}$
 $= \text{rmin}\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_2 * \aleph)\}$ and $J^t((t_1 * \aleph) * (t_2 * \aleph)) \preceq 1 = \max\{1, \nu\} = \max\{J^t(t_1 * \aleph), J^t(t_2 * \aleph)\}$.

Case 4. If $t_1 \notin \aleph$ and $t_2 \notin \aleph$, then $M^t(t_1 * \aleph) = 0$, $\hat{B}^t(t_1 * \aleph) = [0, 0]$, $J^t(t_1 * \aleph) = 1$ and $M^t(t_2 * \aleph) = 0$, $\hat{B}^t(t_2 * \aleph) = [0, 0]$, $J^t(t_2 * \aleph) = 1$. Thus $M^t((t_1 * \aleph) * (t_2 * \aleph)) \succeq 1 = \min\{0, 0\} = \min\{M^t(t_1 * \aleph), M^t(t_2 * \aleph)\}$, $\hat{B}^t((t_1 * \aleph) * (t_2 * \aleph)) \succeq [0, 0] = \text{rmin}\{[0, 0], [0, 0]\} = \text{rmin}\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_2 * \aleph)\}$ and $J^t((t_1 * \aleph) * (t_2 * \aleph)) \preceq 1 = \max\{1, 1\} = \max\{J^t(t_1 * \aleph), J^t(t_2 * \aleph)\}$. Therefore, \mathcal{C}^t is a T-MBJ NMSU of Y . \square

Theorem 3.19. Let \mathcal{C}^t be a subset of Y and \mathcal{C}^t be a T-MBJ neutrosophic set on Y which is given in the proof of above Theorem. If \mathcal{C}^t is considered as lower level subalgebra and upper level subalgebra of some T-MBJ NMSU of Y , then \mathcal{C}^t is a T-MBJ neutrosophic cubic one of Y .

Proof. Let \mathcal{C}^t be a T-MBJ NMSU of Y , and $t_1, t_2 \in \mathcal{C}^t$. Then $M^t(t_1 * \aleph) = M^t(t_2 * \aleph) = \gamma$, $\hat{B}^t(t_1 * \aleph) = \hat{B}^t(t_2 * \aleph) = [\alpha_1, \alpha_2]$ and $J^t(t_1 * \aleph) = J^t(t_2 * \aleph) = \beta$. Thus $M^t((t_1 * \aleph) * (t_2 * \aleph)) = \{\min(M^t((t_1 * \aleph) * (t_2 * \aleph)), t)\} \succeq \min\{\min(M^t(t_1 * \aleph), t), \min(M^t(t_2 * \aleph), t)\} = \min\{M^t(t_1 * \aleph), M^t(t_2 * \aleph)\} = \min\{\gamma, \gamma\} = \gamma, \Rightarrow (t_1 * \aleph) * (t_2 * \aleph) \in \mathcal{C}^t$, $\hat{B}^t((t_1 * \aleph) * (t_2 * \aleph)) = \{\text{rmin}(\hat{B}^t((t_1 * \aleph) * (t_2 * \aleph)), t')\} \succeq \text{rmin}\{\text{rmin}(\hat{B}^t(t_1 * \aleph), t'), \text{rmin}(\hat{B}^t(t_2 * \aleph), t')\} = \text{rmin}\{\text{rmin}(\hat{B}^t(t_1 * \aleph), t'), \text{rmin}(\hat{B}^t(t_2 * \aleph), t')\} = \text{rmin}\{\hat{B}^t(t_1 * \aleph), \hat{B}^t(t_2 * \aleph)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$ and $J^t((t_1 * \aleph) * (t_2 * \aleph)) = \{\max(M^t((t_1 * \aleph) * (t_2 * \aleph)), \Im)\} \preceq \max\{\max(M^t(t_1 * \aleph), \Im), \max(M^t(t_2 * \aleph), \Im)\} = \max\{M^t(t_1 * \aleph), M^t(t_2 * \aleph)\} = \max\{\beta, \beta\} = \beta, \Rightarrow (t_1 * \aleph) * (t_2 * \aleph) \in \mathcal{C}^t$. Hence, proof is completed. \square

4. Conclusions

In this paper, T-MBJ neutrosophic set is defined and notion of T-MBJ neutrosophic M-subalgebra is also introduced by set of conditions on G-algebra. T-MBJ neutrosophic M-subalgebra of G-algebra has investigated by p-union, P-intersection, R-union, R-intersection and some results. For future work this study will be use to discuss the normal ideals, multiplication, translation and magnification of T-MBJ neutrosophic set.

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$N_{\delta^*g\alpha}$ -Continuous and Irresolute Functions in Neutrosophic Topological Spaces

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Abstract. In this paper, the notions of $N_{\delta^*g\alpha}$ -continuous and $N_{\delta^*g\alpha}$ -irresolute functions in neutrosophic topological spaces are given. Furthermore, we analyze their characterizations and investigate their properties.

Keywords: $N_{\delta^*g\alpha}$ -closed set; $N_{\delta^*g\alpha}$ -continuous; $N_{\delta^*g\alpha}$ -irresolute; $N_{\delta^*g\alpha}$ – homeomorphism; $N_{\delta^*g\alpha}$ -homeomorphism.

1. Introduction

The notion of fuzzy set (FS) and its logic are investigated and discussed by Zadeh [12]. Next, Chang [3] studied the conception of fuzzy topological space (FTS). After that, Atanassav [8] investigated the intuitionistic fuzzy set (IFS) in 1986. Neutrosophy has extend the grounds for a total family of new mathematical estimations. It is one of the non-classical sets, like fuzzy, nano, soft, permutation sets and so on, see ([17]-[39]). The neutrosophic set (NS) was presented by Smarandache[6]and expounded, (NS) is a popularization of (IFS) in intuitionistic fuzzy topological space ($IFTS$) by coker [4]. In 2012 [1], the conception of neutrosophic topological space (NTS) is presented. Further the fundamental sets like semi/pre/ α -open sets are presented in neutrosophic topological spaces ($NTSs$), see ([13]-[16]). The neutrosophic closed sets ($NCSs$) and neutrosophic continuous functions ($NCFs$) were presented by Salama et al.[2] in 2014. Arokiarani et al.[7] presented the neutrosophic α -closed set ($N\alpha CS$) in ($NTSs$). The concepts of δ -closure are auxiliary tools in standard topology in

the study of H-closed spaces. Damodharan et al. [9,10] present the idea of N_δ -closure and N_δ -Interior in $(NTSs)$. Further, N_δ -continuous and Neutrosophic almost continuous in $(NTSs)$ were presented and established some of their related attributes. Recently Damodharan and Vigneshwaran [11] presented the conception of $N_{\delta^*g\alpha}$ -closed sets in $(NTSs)$ and studied some of its characteristics. In 2020, some applications of (NS) are applied by Abdel-Basset and others, see ([40]) In this work, we presented the $N_{\delta^*g\alpha}$ -continuous functions and $N_{\delta^*g\alpha}$ -irresolute functions in $(NTSs)$. Furthermore, the conceptions of $N_{\delta^*g\alpha}$ -homeomorphism and $N_{\delta^*g\alpha}$ -homeomorphism are presented and investigate their characteristics.

2. Preliminaries

In this section, we mention some pertinent basic preliminaries about neutrosophic sets (NSs) and its operations.

2.1. Definition [1]

Assume S is a non-empty fixed set. A neutrosophic set (NS) P is an object having the form:

$P = \{\langle s, \mu_m(P(s)), \sigma_i(P(s)), \nu_{nm}(P(s)) \rangle \mid \forall s \in S\}$, where $\mu_m(P(s))$ represents the degree of membership, $\sigma_i(P(s))$ represents the degree of indeterminacy and $\nu_{nm}(P(s))$ represents the degree of nonmembership $\forall s \in S$ to P .

2.2. Remark [1]

A (NS) $P = \{\langle s, \mu_m(P(s)), \sigma_i(P(s)), \nu_{nm}(P(s)) \rangle \mid \forall s \in S\}$ can be identified to an ordered triple $\langle \mu_m(P(s)), \sigma_i(P(s)), \nu_{nm}(P(s)) \rangle$ in $]^{-0}, 1^+[$ on S .

2.3. Definition [1]

In (NTS) We have:

$$\begin{array}{ll}
 0_N \text{ may be defined as } \forall s \in S & 1_N \text{ may be defined as } \forall s \in S \\
 0_N = \langle s, 0, 0, 1 \rangle & 1_N = \langle s, 1, 0, 0 \rangle \\
 0_N = \langle s, 0, 1, 1 \rangle & 1_N = \langle s, 1, 0, 1 \rangle \\
 0_N = \langle s, 0, 1, 0 \rangle & 1_N = \langle s, 1, 1, 0 \rangle \\
 0_N = \langle s, 0, 0, 0 \rangle & 1_N = \langle s, 1, 1, 1 \rangle
 \end{array}$$

2.4. Definition [1]

Assume P is (NS) of the form:

$P = \{\langle s, \mu_m(P(s)), \sigma_i(P(s)), \nu_{nm}(P(s)) \rangle \mid \forall s \in S\}$, Then the complement of P [P^c] may be defined as

$$P^c = \{\langle s, \nu_{nm}(P(s)), \sigma_i(P(s)), \mu_m(P(s)) \rangle \mid \forall s \in S\}$$

2.5. Definition [1]

Assume P and Q are two (NSs) of the form,

$$P = \{\langle s, \mu_m(P(s)), \sigma_i(P(s)), \nu_{nm}(P(s)) \rangle \mid \forall s \in S\} \text{ and}$$

$$Q = \{\langle s, \mu_m(Q(s)), \sigma_i(Q(s)), \nu_{nm}(Q(s)) \rangle \mid \forall s \in S\}. \text{ Then,}$$

(1) Subsets $P \subseteq Q$ may be defined as follows

$$P \subseteq Q \Leftrightarrow \mu_m(P(s)) \leq \mu_m(Q(s)), \sigma_i(P(s)) \geq \sigma_i(Q(s)), \nu_{nm}(P(s)) \geq \nu_{nm}(Q(s))$$

(2) Subsets $P = Q \Leftrightarrow P \subseteq Q$ and $Q \subseteq P$

(3) Union of subsets $P \cup Q$ may be defined as follows

$$P \cup Q = \{s, \max\{\mu_m(P(s)), \mu_m(Q(s))\}, \min\{\sigma_i(P(s)), \sigma_i(Q(s))\}, \min\{\nu_{nm}(P(s)), \nu_{nm}(Q(s))\} \mid \forall s \in S\},$$

(4) Intersection of subsets $P \cap Q$ may be defined as follows

$$P \cap Q = \{s, \min\{\mu_m(P(s)), \mu_m(Q(s))\}, \max\{\sigma_i(P(s)), \sigma_i(Q(s))\}, \max\{\nu_{nm}(P(s)), \nu_{nm}(Q(s))\} \mid \forall s \in S\},$$

2.6. Proposition [9]

For any two (NSs) P and Q the following condition holds

$$\textbf{i): } (P \cap Q)^c = P^c \cup Q^c,$$

$$\textbf{ii): } (P \cup Q)^c = P^c \cap Q^c,$$

2.7. Definition [1]

A neutrosophic topology (NT) on a non-empty set S is a family τ of neutrosophic subsets in S satisfying the following axioms:

$$\textbf{i): } 0_N, 1_N \in \tau,$$

$$\textbf{ii): } G_1 \cap G_2 \in \tau \text{ for any } G_1, G_2 \in \tau,$$

$$\textbf{iii): } \cup G_i \in \tau \forall \{G_i : i \in J\} \subseteq \tau$$

Then the pair (S, τ) is named a neutrosophic topological space (NTS).

2.8. Definition [1]

Assume P is a (NS) in a (NTS) (S, τ) . Then

- i): $Nint(P) = \bigcup \{Q/Q \text{ is a neutrosophic open set (NOS) in } (S, \tau) \text{ and } Q \subseteq P\}$ is named the neutrosophic interior of P;
- ii): $Ncl(P) = \bigcap \{Q/Q \text{ is a neutrosophic closed set (NCS) in } (S, \tau) \text{ and } Q \supseteq P\}$ is named the neutrosophic closure of P;

2.9. Definition [7]

A subset A of (S, τ) is named

- i): neutrosophic semi-open set (NSOS) if $P \subseteq Ncl(Nint(P))$.
- ii): neutrosophic pre-open set (NPOS) if $P \subseteq Nint(Ncl(P))$.
- iii): neutrosophic semi-preopen set (NSPOS) if $P \subseteq Ncl(Nint(Ncl(P)))$.
- iv): neutrosophic α -open set ($N\alpha OS$) if $P \subseteq Nint(Ncl(Nint(P)))$.
- v): neutrosophic regular open set (NROS) if $P = Nint(Ncl(P))$.

The complement of a (NSOS) (resp. (NPOS), (NSPOS), ($N\alpha OS$), (NROS)) set is named (NSCS) (resp. (NPCS), (NSPCS), ($N\alpha CS$), (NRCS)).

2.10. Definition [9]

Assume $\alpha, \beta, \lambda \in [0, 1]$ and $\alpha + \beta + \lambda \leq 3$. A neutrosophic point $s_{(\alpha, \beta, \lambda)}$ of S is a neutrosophic point (NP) of S which is clarified by

$$s_{(\alpha, \beta, \lambda)}(y) = \begin{cases} (\alpha, \beta, \lambda) & \text{when } y = s, \\ (0, 0, 1) & \text{when } y \neq s. \end{cases}$$

Here, S is named the support of $s_{(\alpha, \beta, \lambda)}$ and α, β and λ , respectively. A (NP) $s_{(\alpha, \beta, \lambda)}$ is named belong to a (NS)

$P = \langle \mu_m(P(s)), \sigma_i(P(s)), \nu_{nm}(P(s)) \rangle$ in S , denoted by $s_{(\alpha, \beta, \lambda)} \in P$ if $\alpha \leq \mu_m(P(s)), \beta \geq \sigma_i(P(s))$ and $\lambda \geq \nu_{nm}(P(s))$. Clearly a (NP) can be represented by an ordered triple of (NP) as follows : $s_{(\alpha, \beta, \lambda)} = (s_\alpha, s_\beta, s_\lambda)$.

2.11. Definition [9]

Assume (S, τ) is a (NTS). Assume P is a (NS) and Assume $s_{(\alpha, \beta, \lambda)}$ is a (NP). $s_{(\alpha, \beta, \lambda)}$ is named neutrosophic quasi coincident with P [denoted by $s_{(\alpha, \beta, \lambda)} qP$] if $\alpha + \mu_m(P(s)) > 1; \beta + \sigma_i(P(s)) < 1$ and $\lambda + \nu_{nm}(P(s)) < 1$.

2.12. Definition [9]

Assume P and Q are two (NSs). P is named neutrosophic quasi coincident with Q [denoted by $P qQ$] if $\mu_m(P(s)) + \mu_m(Q(s)) > 1; \sigma_i(P(s)) + \sigma_i(Q(s)) < 1$ and $\nu_{nm}(P(s)) + \nu_{nm}(Q(s)) < 1$.

2.13. Definition [9]

Assume (S, τ) is an (NTS) . An (NP) $s_{(\alpha, \beta, \lambda)}$ is named an neutrosophic δ -cluster point of an (NS) P if AqP for each neutrosophic regular open q -neighborhood A of $s_{(\alpha, \beta, \lambda)}$. The set of all neutrosophic δ -cluster points of P is named the neutrosophic δ -closure of P and denoted by $Ncl_\delta(P)$. An (NS) P is named an N_δ -closed set (N_δ -CS) if $P = Ncl_\delta(P)$. The complement of an $(N_\delta$ -CS) is named an N_δ -open set (N_δ -OS).

3. $N_{\delta^*g\alpha}$ -continuous functions

Here, some new conceptions are given by the authors.

3.1. Definition

A map $T : (S, \tau) \rightarrow (Y, \sigma)$ is named a Neutrosophic delta star generalized alpha-continuous map (briefly $N_{\delta^*g\alpha}$ -CM) if $T^{-1}(K)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) for any (NCS) in (Y, σ) .

3.2. Theorem

Any $N_{\delta^*g\alpha}$ -CM is N_{gs} -CM. (resp $N_{\alpha g}$ -CM, N_{gsp} -CM, N_{gp} -CM). Also converse part is not true as shown through the following examples.

Proof. Assume K is a (NCS) in (Y, σ) . Since T is $N_{\delta^*g\alpha}$ -CM. $T^{-1}(K)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) . Since any $N_{\delta^*g\alpha}$ -CS is N_{gs} -CS (resp $N_{\alpha g}$ -CS, N_{gsp} -CS, N_{gp} -CS), therefore $T^{-1}(K)$ is N_{gs} -CS (resp $N_{\alpha g}$ -CS, N_{gsp} -CS, N_{gp} -CS) in (S, τ) . Hence T is N_{gs} -CM. (resp $N_{\alpha g}$ -CM, N_{gsp} -CM, N_{gp} -CM).

3.3. example

Assume $S = \{p, q, r\}$. Define the (NSs) D_1, D_2, D_3, D_4 and G_1, G_2, G_3, G_4 as follows:

$$D_1 = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right) \right\rangle$$

$$D_2 = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right) \right\rangle$$

$$D_3 = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right) \right\rangle$$

$$D_4 = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right) \right\rangle$$

$$\text{and } G_1 = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right) \right\rangle$$

$$G_2 = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right) \right\rangle$$

$$G_3 = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right) \right\rangle$$

$$G_4 = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right) \right\rangle$$

Then the families $\tau = \{0_N, 1_N, D_1, D_2, D_3, D_4\}$ and $\xi = \{0_N, 1_N, G_1, G_2, G_3, G_4\}$ are neutrosophic topologies (NTs) on S . Thus, (S, τ) and (S, ξ) are $(NTSs)$. Define $T : (S, \tau) \rightarrow (S, \xi)$ as $T(p) = p, T(q) = q, T(r) = r$. Then T is N_{gs} -CM but not $N_{\delta^*g\alpha}$ -CM. Hence in (S, τ) ,

$$N_{\delta^*g\alpha}\text{-CS is } \left\langle \left(\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.5} \right), \left(\frac{p}{0.5}, \frac{q}{0.5}, \frac{r}{0.5} \right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.3} \right) \right\rangle \text{ and}$$

N_{gs} -CS is $\langle (\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2}), (\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2}), (\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4}) \rangle$. Here $T^{-1}(G_3^c)$ is N_{gs} -CS but not $N_{\delta^*g\alpha}$ -CS.

3.4. example

Assume $S = \{p, q, r\}$. Define the (NSs) D_1, D_2, D_3, D_4 and H_1, H_2, H_3, H_4 as follows:

$$D_1 = \langle (\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}), (\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2}), (\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}) \rangle$$

$$D_2 = \langle (\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}), (\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}), (\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}) \rangle$$

$$D_3 = \langle (\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}), (\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2}), (\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}) \rangle$$

$$D_4 = \langle (\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}), (\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}), (\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}) \rangle$$

$$\text{and } H_1 = \langle (\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}), (\frac{p}{0.4}, \frac{q}{0.3}, \frac{r}{0.3}), (\frac{p}{0.3}, \frac{q}{0.4}, \frac{r}{0.4}) \rangle$$

$$H_2 = \langle (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.6}), (\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2}), (\frac{p}{0.3}, \frac{q}{0.5}, \frac{r}{0.5}) \rangle$$

$$H_3 = \langle (\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}), (\frac{p}{0.4}, \frac{q}{0.3}, \frac{r}{0.3}), (\frac{p}{0.3}, \frac{q}{0.5}, \frac{r}{0.5}) \rangle$$

$$H_4 = \langle (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.6}), (\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2}), (\frac{p}{0.3}, \frac{q}{0.4}, \frac{r}{0.4}) \rangle$$

Then the families $\tau = \{0_N, 1_N, D_1, D_2, D_3, D_4\}$ and $\xi = \{0_N, 1_N, H_1, H_2, H_3, H_4\}$ are (NTs) on S . Thus, (S, τ) and (S, ξ) are (NTSs). Define $T : (S, \tau) \rightarrow (S, \xi)$ as $T(p) = p, T(q) = q, T(r) = r$. Then T is $N_{\alpha g}$ -CM but not $N_{\delta^*g\alpha}$ -CM. Hence in (S, τ) ,

$$N_{\delta^*g\alpha}\text{-CS is } \langle (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.5}), (\frac{p}{0.5}, \frac{q}{0.5}, \frac{r}{0.5}), (\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.3}) \rangle \text{ and}$$

$$N_{\alpha g}\text{-CS is } \langle (\frac{p}{0.3}, \frac{q}{0.5}, \frac{r}{0.5}), (\frac{p}{0.4}, \frac{q}{0.3}, \frac{r}{0.3}), (\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}) \rangle. \text{ Here } T^{-1}(H_3^c) \text{ is } N_{\alpha g}\text{-CS but not } N_{\delta^*g\alpha}\text{-CS.}$$

3.5. example

Assume $Y = \{u, v, w\}$. Define the (NSs) F_1, F_2, F_3, F_4 and I_1, I_2, I_3, I_4 as follows:

$$F_1 = \langle (\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}), (\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4}), (\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}) \rangle$$

$$F_2 = \langle (\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}), (\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}), (\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.3}) \rangle$$

$$F_3 = \langle (\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}), (\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4}), (\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.3}) \rangle$$

$$F_4 = \langle (\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}), (\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}), (\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}) \rangle$$

$$\text{and } I_1 = \langle (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.6}), (\frac{p}{0.5}, \frac{q}{0.6}, \frac{r}{0.6}), (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.3}) \rangle$$

$$I_2 = \langle (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.3}), (\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}), (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.6}) \rangle$$

$$I_3 = \langle (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.6}), (\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}), (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.3}) \rangle$$

$$I_4 = \langle (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.3}), (\frac{p}{0.5}, \frac{q}{0.6}, \frac{r}{0.6}), (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.6}) \rangle$$

Then the families $\vartheta = \{0_N, 1_N, F_1, F_2, F_3, F_4\}$ and $\zeta = \{0_N, 1_N, I_1, I_2, I_3, I_4\}$ are (NTs) on Y . Thus, (Y, ϑ) and (Y, ζ) are (NTSs). Define $g : (Y, \vartheta) \rightarrow (Y, \zeta)$ as $g(u) = u, g(v) = v, g(w) = w$. Then g is N_{gp} -CM but not $N_{\delta^*g\alpha}$ -CM. Hence in (Y, ϑ) ,

$$N_{\delta^*g\alpha}\text{-CS is } \langle (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.5}), (\frac{p}{0.5}, \frac{q}{0.5}, \frac{r}{0.5}), (\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.3}) \rangle \text{ and}$$

$$N_{gp}\text{-CS is } \langle (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.3}), (\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}), (\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.6}) \rangle. \text{ Here } g^{-1}(I_3^c) \text{ is } N_{gp}\text{-CS but not } N_{\delta^*g\alpha}\text{-CS.}$$

3.6. example

Assume $Y = \{u, v, w\}$. Define the (NSs) F_1, F_2, F_3, F_4 and J_1, J_2, J_3, J_4 as follows:

$$\begin{aligned} F_1 &= \left\langle \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right) \right\rangle \\ F_2 &= \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.3} \right) \right\rangle \\ F_3 &= \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.3} \right) \right\rangle \\ F_4 &= \left\langle \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right) \right\rangle \\ \text{and } J_1 &= \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.5} \right), \left(\frac{p}{0.3}, \frac{q}{0.4}, \frac{r}{0.3} \right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.5} \right) \right\rangle \\ J_2 &= \left\langle \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.5} \right) \right\rangle \\ J_3 &= \left\langle \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.3}, \frac{q}{0.4}, \frac{r}{0.3} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.5} \right) \right\rangle \\ J_4 &= \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.5} \right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.5} \right) \right\rangle \end{aligned}$$

Then the families $\vartheta = \{0_N, 1_N, F_1, F_2, F_3, F_4\}$ and $\zeta = \{0_N, 1_N, J_1, J_2, J_3, J_4\}$ are (NTs) on Y .

Thus, (Y, ϑ) and (Y, φ) are (NTSs). Define $g : (Y, \vartheta) \rightarrow (Y, \varphi)$ as $g(u) = u, g(v) = w, g(w) = v$. Then g is N_{gsp} -C but not $N_{\delta^*g\alpha}$ -C. Hence in (Y, ϑ) ,

$N_{\delta^*g\alpha}$ -CS is $\left\langle \left(\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.5} \right), \left(\frac{p}{0.5}, \frac{q}{0.5}, \frac{r}{0.5} \right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.3} \right) \right\rangle$ and

N_{gsp} -CS is $\left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.5} \right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.3}, \frac{q}{0.4}, \frac{r}{0.3} \right) \right\rangle$. Here $g^{-1}(J_3^c)$ is N_{gsp} -CS but not $N_{\delta^*g\alpha}$ -CS.

3.7. Theorem

The composition of two $N_{\delta^*g\alpha}$ -CMs is also a $N_{\delta^*g\alpha}$ -CM. Proof. Assume $T : (S, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ are two $N_{\delta^*g\alpha}$ -CMs. Assume l is a NCS in (Z, η) . Since g is a $N_{\delta^*g\alpha}$ -CM, $g^{-1}(l)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . Since any $N_{\delta^*g\alpha}$ -CS is NCS, $g^{-1}(l)$ is NCSS in (Y, σ) . Since T is a $N_{\delta^*g\alpha}$ -CM, $T^{-1}(g^{-1}(l)) = goT(l)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) , therefore goT is also $N_{\delta^*g\alpha}$ -CM.

4. $N_{\delta^*g\alpha}$ -Irresolute functions

Here, some new conceptions are given by the authors.

4.1. Definition

A map $T : (S, \tau) \rightarrow (Y, \sigma)$ is named a Neutrosophic delta star generalized alpha-Irresolute map (briefly $N_{\delta^*g\alpha}$ -IMM) if $T^{-1}(K)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) for any $N_{\delta^*g\alpha}$ -CS in (Y, σ) .

4.2. Theorem

Assume $T : (S, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ are any two functions, then

- (i) $goT : (S, \tau) \rightarrow (Z, \eta)$ is $N_{\delta^*g\alpha}$ -CM if g is N -CM and T is $N_{\delta^*g\alpha}$ -CM.
- (ii) $goT : (S, \tau) \rightarrow (Z, \eta)$ is $N_{\delta^*g\alpha}$ -IM if both g and T are $N_{\delta^*g\alpha}$ -IM.
- (iii) $goT : (S, \tau) \rightarrow (Z, \eta)$ is $N_{\delta^*g\alpha}$ -CM if g is $N_{\delta^*g\alpha}$ -CM and T is $N_{\delta^*g\alpha}$ -IM.

Proof.

- (i) Assume K is a (NCS) in (Z, η) . Since g is N -CM, $g^{-1}(K)$ is NCS in (Y, σ) . Since T is $N_{\delta^*g\alpha}$ -CM, $T^{-1}(g^{-1}(K)) = (goT)^{-1}(K)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) , Therefore goT is $N_{\delta^*g\alpha}$ -CM.
- (ii) Assume K is a $N_{\delta^*g\alpha}$ -CS in (Z, η) . Since g is $N_{\delta^*g\alpha}$ -IM, $g^{-1}(K)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . Since T is $N_{\delta^*g\alpha}$ -IM, $T^{-1}(g^{-1}(K)) = (goT)^{-1}(K)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) , Therefore goT is $N_{\delta^*g\alpha}$ -IM.
- (iii) Assume K is a (NCS) in (Z, η) . Since g is $N_{\delta^*g\alpha}$ -CM, $g^{-1}(K)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . Since T is $N_{\delta^*g\alpha}$ -IM, $T^{-1}(g^{-1}(K)) = (goT)^{-1}(K)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) , Therefore goT is $N_{\delta^*g\alpha}$ -CM.

4.3. Theorem

Assume $T : (S, \tau) \longrightarrow (Y, \sigma)$ is $N_{\delta^*g\alpha}$ -CM (N_{gs} -CM, $N_{\alpha g}$ -CM, N_g -CM). If (S, τ) is an $N_{\alpha\delta T_{\frac{3}{4}}^{**g\alpha}}$ -space ($N_{\delta\alpha T_{\frac{1}{2}}^{**}}$ -space, $N_{\delta\alpha T_c^{**}}$ -space, $N_{\delta T_c^{**}}$ -space) then T is continuous.

Proof. Assume K is a (NCS) of (Y, σ) . Since T is $N_{\delta^*g\alpha}$ -CM (N_{gs} -CM, $N_{\alpha g}$ -CM, N_g -CM), then $T^{-1}(K)$ is $N_{\delta^*g\alpha}$ -CS (N_{gs} -CS, $N_{\alpha g}$ -CS, N_g -CS) in (S, τ) . Since (S, τ) is $N_{\alpha\delta T_{\frac{3}{4}}^{**g\alpha}}$ -space ($N_{\delta\alpha T_{\frac{1}{2}}^{**}}$ -space, $N_{\delta\alpha T_c^{**}}$ -space, $N_{\delta T_c^{**}}$ -space), then $T^{-1}(K)$ is N_{δ} -CS in (S, τ) . Any N_{δ} -CS is (NCS) in (S, τ) . Therefore T is continuous.

4.4. Theorem

Assume $T : (S, \tau) \longrightarrow (Y, \sigma)$ is a surjective, $N_{*g\alpha}$ -IM and N_{δ} -CM. Then $T(A)$ is $N_{\delta^*g\alpha}$ -CS of (Y, σ) for any $N_{\delta^*g\alpha}$ -CS A of (S, τ) .

Proof. Assume A is a $N_{\delta^*g\alpha}$ -CS of (S, τ) . Assume U is a $N_{*g\alpha}$ -OS of (Y, σ) . such that $T(A) \subseteq U$. Since T is surjective and $N_{*g\alpha}$ -IM, $T^{-1}(U)$ is $N_{*g\alpha}$ -OS in (S, τ) . Since $A \subseteq T^{-1}(U)$ and A is $N_{\delta^*g\alpha}$ -CS of (S, τ) , $Ncl_{\delta}(A) \subseteq T^{-1}(U)$. Then $T[Ncl_{\delta}(A)] \subseteq T[T^{-1}(U)] = U$, since T is N_{δ} -CS, $T[Ncl_{\delta}(A)] = Ncl_{\delta}[T[Ncl_{\delta}(A)]]$. This implies $Ncl_{\delta}[T(A)] \subseteq Ncl_{\delta}[T[Ncl_{\delta}(A)]] = T[Ncl_{\delta}(A)] \subseteq U$, Therefore $T(A)$ is a $N_{\delta^*g\alpha}$ -CS of (Y, σ) .

4.5. Theorem

Assume $T : (S, \tau) \longrightarrow (Y, \sigma)$ is a surjective, $N_{\delta^*g\alpha}$ -IM and N_{δ} -CM. If (S, τ) is an $N_{\alpha\delta T_{\frac{3}{4}}^{**g\alpha}}$ -space, then (Y, σ) is also an $N_{\alpha\delta T_{\frac{3}{4}}^{**g\alpha}}$ -space.

Proof. Assume A is a $N_{\delta^*g\alpha}$ -CS of (Y, σ) . Since T is $N_{\delta^*g\alpha}$ -IM, $T^{-1}(A)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) . Since (S, τ) is $N_{\alpha\delta T_{\frac{3}{4}}^{**g\alpha}}$ -space, $T^{-1}(A)$ is N_{δ} -CS of (S, τ) . Since T is N_{δ} -CM and surjective, $T[T^{-1}(A)] = A$ is N_{δ} -CS in (Y, σ) . Thus A is N_{δ} -CS in (Y, σ) , Therefore (Y, σ) is an $N_{\alpha\delta T_{\frac{3}{4}}^{**g\alpha}}$ -space.

5. $N_{\delta^*g\alpha}$ -Homeomorphism

Here, some new conceptions are given by the authors.

5.1. Definition

A map $T : (S, \tau) \longrightarrow (Y, \sigma)$ is named a neutrosophic delta star generalized alpha-homeomorphism (briefly $N_{\delta^*g\alpha}$ -H) if T is bijective, $N_{\delta^*g\alpha}$ -CM and $N_{\delta^*g\alpha}$ -OM.

5.2. Theorem

Any $N_{\delta^*g\alpha}$ -H is N_{gs} -H.

Proof. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be $N_{\delta^*g\alpha}$ -H then f is bijective, $N_{\delta^*g\alpha}$ -continuous and $N_{\delta^*g\alpha}$ -OM. Let V be N-CS in (Y, σ) , then $f^{-1}(V)$ is $N_{\delta^*g\alpha}$ -CS in (X, τ) . Since every $N_{\delta^*g\alpha}$ -CS is N_{gs} -CS, then $f^{-1}(V)$ is N_{gs} -CS in (X, τ) , Therefore f is N_{gs} -continuous. Let U be N-OS in (X, τ) , then $f(U)$ is $N_{\delta^*g\alpha}$ -OS in (Y, σ) . Since every $N_{\delta^*g\alpha}$ -OS is N_{gs} -OS, then $f(U)$ is N_{gs} -OS in (Y, σ) , Therefore f is N_{gs} -OM. Hence f is N_{gs} -H.

5.3. example

Assume $S = \{p, q, r\}$. Define the (NSs) D_1, D_2, D_3, D_4 and G_1, G_2, G_3, G_4 as follows:

$$D_1 = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right) \right\rangle$$

$$D_2 = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right) \right\rangle$$

$$D_3 = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right) \right\rangle$$

$$D_4 = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2} \right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right) \right\rangle$$

$$\text{and } G_1 = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right) \right\rangle$$

$$G_2 = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right) \right\rangle$$

$$G_3 = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right) \right\rangle$$

$$G_4 = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6} \right) \right\rangle$$

Then the families $\tau = \{0_N, 1_N, D_1, D_2, D_3, D_4\}$ and $\xi = \{0_N, 1_N, G_1, G_2, G_3, G_4\}$ are (NTs) on S. Thus, (S, τ) and (S, ξ) are (NTSs). Define $T : (S, \tau) \rightarrow (S, \xi)$ as $T(p) = p, T(q) = q, T(r) = r$. Then T is N_{gs} -H but not $N_{\delta^*g\alpha}$ -H. Hence in (S, τ) ,

$N_{\delta^*g\alpha}$ -CS is $\left\langle \left(\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.5} \right), \left(\frac{p}{0.5}, \frac{q}{0.5}, \frac{r}{0.5} \right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.3} \right) \right\rangle$ and

N_{gs} -CS is $\left\langle \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4} \right) \right\rangle$. Here $T^{-1}(G_3^c)$ is N_{gs} -CS but not $N_{\delta^*g\alpha}$ -CS.

$N_{\delta^*g\alpha}$ -OS is $\left\langle \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.3} \right), \left(\frac{p}{0.5}, \frac{q}{0.5}, \frac{r}{0.5} \right), \left(\frac{p}{0.4}, \frac{q}{0.5}, \frac{r}{0.5} \right) \right\rangle$ and

N_{gs} -OS is $\left\langle \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2} \right) \right\rangle$ is N_{gs} -OS but not $N_{\delta^*g\alpha}$ -OS.

5.4. Theorem

For any bijective map $T : (S, \tau) \longrightarrow (Y, \sigma)$ the following statement are equivalent.

- (i) $T^{-1} : (Y, \tau) \longrightarrow (S, \sigma)$ is $N_{\delta^*g\alpha}$ -CM.
- (ii) T is an $N_{\delta^*g\alpha}$ -OM.
- (iii) T is an $N_{\delta^*g\alpha}$ -CM.

Proof.

- (i) \Rightarrow (ii) Assume U is an (NOS) in (S, τ) , then $S-U$ is (NCS) in (S, τ) . Since T^{-1} is $N_{\delta^*g\alpha}$ -CM, then $(T^{-1})^{-1}(U)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . that is $T(S-U)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) , that is $Y-T(U)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . This implies that $T(U)$ is $N_{\delta^*g\alpha}$ -OS in (Y, σ) . Thus T is $N_{\delta^*g\alpha}$ -OM.
- (ii) \Rightarrow (iii) Assume F is an (NCS) in (S, τ) , then $S-F$ is N -OS in (S, τ) . Since T is $N_{\delta^*g\alpha}$ -OM, then $T(S-F)$ is $N_{\delta^*g\alpha}$ -OS in (Y, σ) . That is $Y-T(F)$ is $N_{\delta^*g\alpha}$ -OS in (Y, σ) . This implies that $T(F)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . hence T is $N_{\delta^*g\alpha}$ -CM.
- (iii) \Rightarrow (i) Assume K is an (NCS) in (S, τ) , Since T is $N_{\delta^*g\alpha}$ -CM, then $T(K)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . That is $[T^{-1}]^{-1}(K)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . hence T^{-1} is $N_{\delta^*g\alpha}$ -CM.

5.5. Theorem

Assume $T : (S, \tau) \longrightarrow (Y, \sigma)$ is bijective and $N_{\delta^*g\alpha}$ -CM. then the following statement are equivalent.

- (i) T is an $N_{\delta^*g\alpha}$ -OM.
- (ii) T is an $N_{\delta^*g\alpha}$ -H.
- (iii) T is an $N_{\delta^*g\alpha}$ -CM.

Proof.

- (i) \Rightarrow (ii) Assume T is an $N_{\delta^*g\alpha}$ -OM. Since T is bijective and $N_{\delta^*g\alpha}$ -CM, T is $N_{\delta^*g\alpha}$ -H.
- (ii) \Rightarrow (iii) Assume T is an $N_{\delta^*g\alpha}$ -H. Then T is $N_{\delta^*g\alpha}$ -OM. If F is (NCS) in S , then $T(S-F)$ is $N_{\delta^*g\alpha}$ -OS in (Y, σ) . That is $Y-T(F)$ is $N_{\delta^*g\alpha}$ -OS in (Y, σ) . This implies that $T(F)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . hence T is $N_{\delta^*g\alpha}$ -CM.
- (iii) \Rightarrow (i) Assume U is an (NOS) in (S, τ) , Then $S-U$ is (NCS) in (S, τ) . Since T is $N_{\delta^*g\alpha}$ -CS, then $T(S-U)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . That is $Y-T(U)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . Hence $T(U)$ is $N_{\delta^*g\alpha}$ -OS in (Y, σ) .

5.6. Theorem

The composition of two $N_{\delta^*g\alpha}$ -Hs is also a $N_{\delta^*g\alpha}$ -H.

Proof. Assume $T : (S, \tau) \longrightarrow (Y, \sigma)$ and $g : (Y, \sigma) \longrightarrow (Z, \eta)$ are two $N_{\delta^*g\alpha}$ -CM. Assume U is a (NCS) in (Z, η) . Since g is a $N_{\delta^*g\alpha}$ -CM, $g^{-1}(U)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . Since any $N_{\delta^*g\alpha}$ -CS is (NCS) , $g^{-1}(U)$ is (NCS) in (Y, σ) . Since T is a $N_{\delta^*g\alpha}$ -CM, $T^{-1}(g^{-1}(U)) = goT(U)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) , therefore goT is also $N_{\delta^*g\alpha}$ -CM.

Assume A is a (NCS) in (S, τ) then $S-A$ is a (NOS) in (S, τ) . Since T is $N_{\delta^*g\alpha}$ -H, then $T(S-A)$ is a $N_{\delta^*g\alpha}$ -OS in (Y, σ) , implies $T(A)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . Since any $N_{\delta^*g\alpha}$ -CS is (NCS) , then $T(A)$ is (NCS) in (Y, σ) , then $Y-T(A)$ is N -OS in (Y, σ) . Since g is $N_{\delta^*g\alpha}$ -H.

$g(Y - T(A))$ is $N_{\delta^*g\alpha}$ -OS in (Z, η) , implies $g(T(A)) = goT(A)$ is $N_{\delta^*g\alpha}$ -CS in (Z, η) therefore goT is $N_{\delta^*g\alpha}$ -CM and $N_{\delta^*g\alpha}$ -OM, implies goT is $N_{\delta^*g\alpha}$ -H.

5.7. Definition

A map $T : (S, \tau) \longrightarrow (Y, \sigma)$ is named $N_{\delta^*g\alpha}$ -c-H if T is bijective, T and T^{-1} are $N_{\delta^*g\alpha}$ -IM.

5.8. Theorem

The composition of two $N_{\delta^*g\alpha}$ -c-Hs is also a $N_{\delta^*g\alpha}$ -c-H.

Proof. Assume $T : (S, \tau) \longrightarrow (Y, \sigma)$ and $g : (Y, \sigma) \longrightarrow (Z, \eta)$ are two $N_{\delta^*g\alpha}$ -c-Hs. Assume U is a $N_{\delta^*g\alpha}$ -CS in (Z, η) . Since g is a $N_{\delta^*g\alpha}$ -IM, $g^{-1}(U)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . Since U is $N_{\delta^*g\alpha}$ -IM, $T^{-1}(g^{-1}(U))$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) . that is $(goT)^{-1}(T)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) , therefore $goT : (Y, \sigma) \longrightarrow (Z, \eta)$ is $N_{\delta^*g\alpha}$ -IM.

Assume G is a $N_{\delta^*g\alpha}$ -CS in (S, τ) , since T^{-1} is a $N_{\delta^*g\alpha}$ -IM, $(T^{-1})^{-1}(G)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) , that is $T(G)$ is $N_{\delta^*g\alpha}$ -CS in (Y, σ) . Since g^{-1} is $N_{\delta^*g\alpha}$ -IM, $(g^{-1})^{-1}(T(G))$ is $N_{\delta^*g\alpha}$ -CS in (Z, η) , that is $g(T(G))$ is $N_{\delta^*g\alpha}$ -CS in (Z, η) , therefore $(goT)(G)$ is $N_{\delta^*g\alpha}$ -CS in (Z, η) . This implies that $((goT)^{-1})^{-1}(G)$ is a $N_{\delta^*g\alpha}$ -CS in (Z, η) . This shows that $(goT)^{-1} : (Y, \sigma) \longrightarrow (Z, \eta)$ is $N_{\delta^*g\alpha}$ -IM. Hence (goT) is a $N_{\delta^*g\alpha}$ -c-H.

5.9. Theorem

Any $N_{\delta^*g\alpha}$ -H from a $N_{\alpha\delta T_{\frac{3}{4}}^{**}g\alpha}$ -space into another $N_{\alpha\delta T_{\frac{3}{4}}^{**}g\alpha}$ -space is a homeomorphism

Proof.

Assume $T : (S, \tau) \longrightarrow (Y, \sigma)$ is a $N_{\delta^*g\alpha}$ -H. Then T is bijective, $N_{\delta^*g\alpha}$ -OM and $N_{\delta^*g\alpha}$ -CM. Assume U is an (NOS) in (S, τ) . Since T is $N_{\delta^*g\alpha}$ -OM and since (Y, σ) is $N_{\alpha\delta T_{\frac{3}{4}}^{**}g\alpha}$ -space, $T(U)$ is (NOS) in (Y, σ) . This implies that T is N-open map. Assume K is a (NCS) in (Y, σ) , since T is $N_{\delta^*g\alpha}$ -CM and since (S, τ) is $N_{\alpha\delta T_{\frac{3}{4}}^{**}g\alpha}$ -space, $T^{-1}(K)$ is (NCS) in (S, τ) . Therefore T is continuous. Hence T is a homeomorphism.

5.10. Theorem

Assume (Y, σ) is $N_{\alpha\delta T_{\frac{3}{4}}^{**}g\alpha}$ -space. If $T : (S, \tau) \longrightarrow (Y, \sigma)$ and $g : (Y, \sigma) \longrightarrow (Z, \eta)$ are $N_{\delta^*g\alpha}$ -H then (goT) is $N_{\delta^*g\alpha}$ -H.

Proof. Assume $T : (S, \tau) \longrightarrow (Y, \sigma)$ and $g : (Y, \sigma) \longrightarrow (Z, \eta)$ are two $N_{\delta^*g\alpha}$ -H. Assume U is an (NOS) in (S, τ) . Since T is $N_{\delta^*g\alpha}$ -OM, $T(U)$ is $N_{\delta^*g\alpha}$ -OS in (Y, σ) . Since (Y, σ) is $N_{\alpha\delta T_{\frac{3}{4}}^{**}g\alpha}$ -space, $T(U)$ is N-OS in (Y, σ) . Also since g is $N_{\delta^*g\alpha}$ -OM, $g(T(U))$ is $N_{\delta^*g\alpha}$ -OS in (Z, η) . Hence goT is $N_{\delta^*g\alpha}$ -OM. Assume v is a (NCS) in (Z, η) . Since g is $N_{\delta^*g\alpha}$ -CM and since (Y, σ) is $N_{\alpha\delta T_{\frac{3}{4}}^{**}g\alpha}$ -space, $g^{-1}(V)$ is (NCS) in (Y, σ) . Since T is $N_{\delta^*g\alpha}$ -CM,

$T^{-1}(g^{-1}(V)) = (goT)^{-1}(V)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) . That is (goT) is $N_{\delta^*g\alpha}$ -continuous. Hence (goT) is $N_{\delta^*g\alpha}$ -H.

5.11. Theorem

Any $N_{\delta^*g\alpha}$ -H from $(N_{\alpha\delta T_{\frac{3}{4}}^{**}g\alpha}$ -S) into another $(N_{\alpha\delta T_{\frac{3}{4}}^{**}g\alpha}$ -S) is a $N_{\delta^*g\alpha}$ -c-H.

Proof. Assume $T : (S, \tau) \longrightarrow (Y, \sigma)$ is $N_{\delta^*g\alpha}$ -H. Assume U be $N_{\delta^*g\alpha}$ -CS in (Y, σ) . Then U is (NCS) in (Y, σ) . Since T is $N_{\delta^*g\alpha}$ -CM, $T^{-1}(U)$ is $N_{\delta^*g\alpha}$ -CS in (S, τ) . Then T is a $N_{\delta^*g\alpha}$ -IM. Let K be $N_{\delta^*g\alpha}$ -OS in (S, τ) . Then K is (NOS) in (S, τ) . Since T is $N_{\delta^*g\alpha}$ -OM, $T(K)$ is $N_{\delta^*g\alpha}$ -OS in (Y, σ) . That is $(T^{-1})^{-1}(K)$ is $N_{\delta^*g\alpha}$ -OS in (Y, σ) and hence T^{-1} is $N_{\delta^*g\alpha}$ -IM. Thus T is $N_{\delta^*g\alpha}$ -c-H.

6. Conclusion

The notions of $N_{\delta^*g\alpha}$ -continuous and $N_{\delta^*g\alpha}$ -irresolute functions in (NTS) are given in this work. Next, their characterizations and investigate their properties are analyzed. In future work, we will use the soft sets theory to investigate new classes of neutrosophic soft maps and then we can study these new classes of (NTS) in soft setting.

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Plithogenic sets and their application in decision making

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Abstract: There are various mathematical tools available to measure the level of accuracy such as Crisp, Fuzzy, Intuitionistic fuzzy sets and Neutrosophic set. Further, Plithogenic set is a generalization of these four sets. This paper aims to test whether Plithogenic aggregation operation is more effective than other sets in its accuracy, while decision making. In order to obtain a better accuracy, the Plithogenic aggregation operation such as Fuzzy set [FS], Intuitionistic fuzzy set [IFS] and Neutrosophic set [NS] used t -norm and t -conorm. An illustration is examined in this paper to prove the result of better accuracy using plithogenic aggregation operators in decision making.

Keywords: Plithogenic operators, t -norm, t -conorm, fuzzy union and intersection operators.

1. Introduction

In real life, there may be an uncertainty about any degree of membership in the variable assumption. In that situation, fuzzy sets and fuzzy logic formulated by Zadesh, 1965 (1) will become the proper mathematical tool to describe the conditions which are ambiguous. Fuzzy sets is an extension of Crisp set. That is why, fuzzy set theory has been developed for inexactness and vagueness. In mathematics, fuzzy set elements have degrees of membership functions. The membership of elements in a set is assessed by binary terms. According to a double fold condition an element either belongs or does not belong to the set in the interval $[0, 1]$. Fuzzy can be represented by a set of ordered pair $A = \{x, \mu_A(x)\}$. The contradiction of fuzzy set degree is 0. Fuzzy set is characterized by a single variable and its membership value is 1. t -norm of fuzzy constraints

the truth function of conjunction, whereas t-conorm of fuzzy constraints the truth functions of disjunction. The aggregative operators of fuzzy set are union (\vee_F), intersection (\wedge_F) and complement. However, in the computational aspects, membership is not enough. So we need an extension.

Intuitionistic fuzzy set [IFS], is an extension of fuzzy set and fuzzy logic, introduced by Atanassov and Baruah[9] to have a better accuracy level. Atanassov has given the definition for some operations on Intuitionistic fuzzy set and its properties [7]. IFS is based on only the membership and non-membership function. But it does not exist in the indeterminacy.

Then the next evaluation of Intuitionistic fuzzy set i.e Neutrosophic set have been developed by Smarandache [15]. It is a generalization of fuzzy sets and intuitionistic fuzzy set. It is a powerful tool to manipulate with some indeterminacy, inconsistency and incomplete information which are applied in day to day life. Neutrosophic sets deal with three components such as membership (M), indeterminacy (I) and non-membership (N) functions. It is a very helpful application to handle real life problems. But it is applicable only on three attribute values. In the stage of advanced research it is felt that the measurement of uncertainty of data needs to be handled with more attribute value so as to raise the accuracy level.

A precise level is one of the most important factors of decision making in day to day real life [30]. To, increase the preciseness, Smaranandache[21] introduced plithogenic. Plithogenic is a powerful tool which is generic of crisp set, fuzzy set, intuitionistic fuzzy set and neutrosophic set is collectively called as Plithogenic. Plithogenic is the base for all the plithogenic functions (plithogenic set, plithogenic probability, plithogenic statistics and plithgenic logic). These sets are characterized by a single appurtenance is plithogenic set. An element of Crisp set is characterized by one value (membership), fuzzy set is characterized by two values (membership, non-membership) and intuitionistic and Neutrosophic set is characterized by three values (membership, indeterminate, non- membership) but plithogenic sets are characterized by four or more values. Plithogenic is a set whose components are described by atleast one trait and each attribute may have numerous elements [30]. The linear combination of fuzzy operators are tnorm and tconorm are called as plithogenic aggregation operators (union and intersection). The aggregation operators of plithogenic set is related on contradiction degree to maximize the accuracy level.

2. Review of Literature

First study related to fuzzy combined operations based on conjugate pairs of t-norm and t-conorms [5]. In this paper M. J. Liberatore.et.al [11] analysed fuzzy logic as an alternative approach for modeling uncertainty in project schedule analysis. To solve a real-world problem by Fuzzy sets

of decision making on the basis of aggregation of experts opinions expressed in form of Z-valued t-norm and t-conorm operators [20]. Huchanget. al.[18] proposed some new hybrid hesitant fuzzy weighted aggregation operators, such as the hesitant fuzzy hybrid arithmetical averaging operator, the hesitant fuzzy hybrid arithmetical geometric operator and their properties satisfying the property named idempotency and used to multi-criteria single person decision making and multi-criteria group decision making respectively. Grabisch .et.al [8] represent an aggregation operator exhibits a set of mathematical properties, which depends on imposed axiomatic assumptions. MamoniDhar[16] introduced a new definition of cardinality of fuzzy sets on the basis of membership value.

Atanassov[4] introduced intuitionistic fuzzy set and its aggregative operations. Supriya.et.al[10] Kumar De, defined some operators (CON; DIL; NORM) with example, it is useful in intuitionistic fuzzy environment. Glad Deschrijver .et.al. [13] introduced the notion of intuitionistic fuzzy t-norm and t-conorm, and investigate under which conditions a similar theorem obtained. Monoranjan.et.al.[14] defined some new operations on intuitionistic neutrosophic set with examples for the implementation of the operations problems.

Neutrosophic set were introduced by FlorentinSmarandache [12], Neutrosophic set is an extension of intuitionistic fuzzy set. Hong-yu .et.al. [19] defined Interval Neutrosophic Sets and Their Application in Multicriteria Decision Making Problems.Wang.et.al [15] presented an instance of neutrosophic set called single valued neutrosophic set. Said Broumi.et.al [17] defined the distance between neutrosophic sets (NS) on the basis of the Hausdorff distance such a new distance called "extended Hausdorff distance for neutrosophic sets" or "neutrosophicHausdorff distance". Nagarajan.et.al[24] presented , Blockchain network has been used in terms of Bitcoin transaction and also the degree, total degree, minimum and maximum degree have been found using Blockchain single valued Neutrosophic graph. Nagarajan.et.al [25] proposed under triangular interval type-2 fuzzy and interval neutrosophic environments verified with the numerical example.Preethi.et.al.[23] verified the hyperstructure properties using the single valued neutrosophic set model through several hyperalgebraic structures .such as hyperrings and hyperideals. R.Jansi.et.al[26] defined the correlation measure of Pythagorean neutrosophic set with T and F are neutrosophic components and their properties . Abdel Nasser .et.al.[27] suggested Multi-Objective Optimization based on Ratio Analysis (MOORA) is the most suitable machine tools on the basis of Neutrosophic set. Smarandache, F [21,22] introduced Plithogenic, show that it is an extension of crisp set, fuzzy set, intuitionistic fuzzy set and neutrosophic set and it is applicable for many scientific experiments. Said Broumi.et.al. [28] proposed a new distance measure for the trapezoidal fuzzy neutrosophic number based on centroid with the graphical representation and its properties also proved. Nivetha et.al.[29] introduced the new concept of combined plithogenic hypersoft set and its application in multi attribute decision making

3. Preliminaries

In this section, preliminaries of the proposed concept are given

3.1. Fuzzy Set [1]

Fuzzy can be represented by an set ordered pair $A = \{x, \mu_A(x)\}$ where $\mu_A(x)$ is called the membership function such that $\mu_A(x) : X \rightarrow [0, 1]$.

3.1.1. Fuzzy set aggregative operators [1]

Let X be a non empty set in the unit interval $[0, 1]$. A fuzzy sets A and B are of the form.

$$A = \{x, \mu_A(x)/x \in X\} \text{ and } B = \{x, \mu_B(x)/x \in X\}$$

1. $A \vee_F B = \min\{\mu_A(x), \mu_B(x)\}$ (where (\vee_F) Union)
2. $A \wedge_F B = \max\{\mu_A(x), \mu_B(x)\}$ (where (\wedge_F) Intersection)
3. $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$ (Complement)

3.2. Intuitionistic fuzzy set [4, 7]

A Intuitionistic fuzzy set defined by ordered triplets $A = \{x, \mu_A(x), \rho_A(x)/x \in X\}$ $\mu_A(x) : X \rightarrow [0, 1]$ is the degree of membership function of $x \in X$, $\rho_A(x) : X \rightarrow [0, 1]$ is the degree non membership function of $x \in X$ where $\rho_A(x) = 1 - \mu_A(x)$ and $0 \leq \mu_A(x) + \rho_A(x) \leq 1$. $\gamma_A(x) = 1 - \mu_A(x) - \rho_A(x)$ is called of hesitancy. Its membership degree is 2.

3.2.2 Intuitionistic fuzzy set[IFS] aggregative operators:

Let X be a non empty set in the unit interval $[0, 1]$. A IFS A and B are of the form

$$A = \{x, \mu_A(x), \rho_A(x)/x \in X\} \text{ } B = \{x, \mu_B(x), \rho_B(x)/x \in X\}$$

1. $A \vee_{IF} B = \{x, \max(\mu_A(x), \mu_B(x)), \min(\rho_A(x), \rho_B(x))\}$ where (\vee_{IF}) union
2. $A \wedge_{IF} B = \{x, \min(\mu_A(x), \mu_B(x)), \max(\rho_A(x), \rho_B(x))\}$ where (\wedge_{IF}) intersection

4. Proposed Methodology

Plithogenic operator laws [23]:

Plithogenic aggregation operators are the linear combination of the fuzzy $t_{norm}(\wedge_F)$ and $t_{conorm}(\vee_N)$

Fuzzy operators:

Let P be a plithogenic set and w is an attribute value, $w \in W$, where W is a attribute value. The contradiction degree $c(w_d, w) = c_0 \in [0, 1]$ between dominant attribute element and the attribute element c_0 . The two expert, X and Y , each assigning single element fuzzy degree of attribute element w of x to the set P with respect to some given criteria:

$$d_X^F(W) = x \in [0, 1]$$

$$d_Y^F(W) = y \in [0, 1]$$

If \wedge_F be a fuzzy t-norm and \vee_F t - conorm respectively and the contradiction degree

$$c(w_d, w) = c_0 \in [0, 1]$$

Fuzzy set Union with Plithogenic :

$$x \vee_p y = (1 - c_0) [x \vee_F y] + c_0 [x \wedge_F y] \dots\dots\dots (a)$$

Proper Plithogenic intersection set means, $c(w_d, w) = c_0 \in [0, 0.5]$ then tconorm $(x, y) = x \vee_F y$ assigned more weight age than onto tnorm $(x, y) = x \wedge_F y$

Improper Plithogenic intersection set means, $c(w_d, w) = c_0 \in (0.5, 1]$ then tconorm $(x, y) = x \vee_F y$ assigned less weight age than onto tnorm $(x, y) = x \wedge_F y$

If $c(w_d, w) = c_0 \in 0.5$ then the same weight {0.5} is assigned onto the tconorm $(x, y) = x \vee_F y$ and tnorm $(x, y) = x \wedge_F y$

Fuzzy set Intersection with Plithogenic :

$$x \wedge_p y = (1 - c_0) [x \wedge_F y] + c_0 [x \vee_F y] \dots\dots\dots (b)$$

Proper Plithogenic intersection set means, $c(w_d, w) = c_0 \in [0, 0.5]$ then tnorm $(x, y) = x \wedge_F y$ assigned more weight age than onto tconorm $(x, y) = x \vee_F y$

Improper Plithogenic intersection set means, $c(w_d, w) = c_0 \in (0.5, 1]$ then tnorm $(x, y) = x \wedge_F y$ assigned less weight age than onto tconorm $(x, y) = x \vee_F y$

If $c(w_d, w) = c_0 \in 0.5$ then the same weight {0.5} is assigned onto the tnorm $(x, y) = x \wedge_F y$ and tconorm $(x, y) = x \vee_F y$

Intuitionistic Fuzzy set operators:

The intuitionistic fuzzy set degree of a single attribute value w of x to the Plithogenic set with some conditions –

$$d_X^{IF}(w) = (x_1, x_2) \in [0, 1]^2$$

$$d_Y^{IF}(w) = (y_1, y_2) \in [0, 1]^2$$

Intuitionistic Fuzzy set Union with Plithogenic :

$$(x_1, x_2) \vee_p (y_1, y_2) = (x_1 \vee_p y_1, x_2 \vee_p y_2) \dots\dots\dots (c)$$

Intuitionistic Fuzzy set intersection with Plithogenic :

$$(x_1, x_2) \wedge_p (y_1, y_2) = (x_1 \wedge_p y_1, x_2 \vee_p y_2) \dots\dots\dots (d)$$

Neutrosophic set operators:

The Neutrosophic set degree of a single attribute value w of x to the plithogenic set with some conditions –

$$d_X^N(w) = (x_1, x_2, x_3) \in [0, 1]^3$$

$$d_Y^N(w) = (y_1, y_2, y_3) \in [0, 1]^3$$

Neutrosophic set Union with Plithogenic :

$$(x_1, x_2, x_3) \vee_p (y_1, y_2, y_3) = (x_1 \vee_p y_1, 0.5 (x_2 \wedge_p y_2 + x_2 \vee_p y_2), x_3 \wedge_p y_3) \dots\dots\dots (e)$$

Neutrosophic set intersection with Plithogenic :

$$(x_1, x_2, x_3) \wedge_p (y_1, y_2, y_3) = (x_1 \wedge_p y_1, 0.5 (x_2 \vee_p y_2 + x_2 \wedge_p y_2), x_3 \vee_p y_3) \dots\dots\dots (f)$$

Theorems based on Plithogenic single attribute fuzzy set Unions and Intersections [21,22]:

Theorem 1 :

$c(w_d, w) = 0$, then

Result 1: If on w_d one applies the tnorm, on w one also applies the tnorm.

Result 2: If on w_d one applies the tconorm, on w one also applies the tconorm.

Theorem 2:

$c(w_d, w) = 1$, then

Result 1: If on w_d one applies the tnorm, on w one also applies the tconorm.

Result 2: If on w_d one applies the tconorm, on w one also applies the tnorm.

Theorem 3:

If $0 < c(w_d, w) < 1$, then on w one applies a linear combination of tnorm and tconorm.

Theorem 4:

Let x, y be fuzzy degrees of appurtenance of the attribute value with respect to Experts X and Y then,

$$x \wedge_p y + x \vee_p y = x \wedge_F y + x \vee_F y$$

Theorem 5:

Let x, y be fuzzy degrees of appurtenance of the attribute value with respect to Experts X and Y . If the degree of contradiction of x and y equal to 0.5 then

$$x \wedge_p y = x \vee_F y$$

5. Application:

In this phase to apply plithogenic operations, it takes four doctors and their reports so as obtain the accuracy of Plithogenic sets. There may be variation of information of the medical reports from different doctors. This may lead to uncertainties, hence an advanced operation is to be applied for higher accuracy. There is a possibility of various accuracy levels in each report. The proposed plithogenic operations that are given appurtenance will prove the increased level of accuracy.

Numerical Example of Plithogenic single valued set

Let \bar{U} be the whole set then $P \subseteq \bar{U}$ a plithogenic set.

For example, the Expert values between "Doctor" and "Report",

Doctor = {doctor1, doctor2, doctor3, doctor4} and

Report = {report1, report 2, report 3, report 4}

Then the objects elements are characterized by the Cartesian product

Doctor \times Report =

$$\left(\begin{array}{l} (\text{doctor1, report1}), (\text{doctor1, report2}), (\text{doctor1, report3}), (\text{doctor1, report4}), \\ (\text{doctor2, report1}), (\text{doctor2, report2}), (\text{doctor2, report3}), (\text{doctor2, report4}) \\ (\text{doctor3, report1}), (\text{doctor3, report2}), (\text{doctor3, report3}), (\text{doctor3, report4}) \\ (\text{doctor4, report1}), (\text{doctor4, report2}), (\text{doctor4, report3}), (\text{doctor4, report4}) \end{array} \right)$$

Let us consider the dominant value of attribute "Doctor" be "doctor1" and of attribute "Report" be "report 1".

The attribute value contradiction fuzzy degrees are:

$$c(\text{doctor1, doctor1}) = 0$$

$$c(\text{doctor1, doctor2}) = \frac{1}{4}$$

$$c(\text{doctor1, doctor3}) = \frac{2}{4}$$

$$c(\text{doctor1, doctor4}) = \frac{3}{4}$$

$$c(\text{report1, report 1}) = 0$$

$$c(\text{report 1, report2}) = \frac{1}{4}$$

$$c(\text{report 1, report3}) = \frac{2}{4}$$

$$c(\text{report 1, report 4}) = \frac{3}{4}$$

We have two plithogenic sets X and Y . Next, we consider the Fuzzy, Intuitionistic, or Neutrosophic degrees of attribute value to a plithogenic set with respect to some experts condition.

Single valued fuzzy set degrees appurtenance

Let $d_X(x, w_i)$ be the appurtenance degree of the attribute value w_i of the element x to the set X and $d_Y(x, w_i)$ be the appurtenance degree of the attribute value w_i of the element x to the set Y . Then w_i is a uni-attribute and its contradiction degree depends on uni-attribute w_d be $c(w_d, w_i) = c_i$.

Let us consider the fuzzy t-norm - $x \wedge_F y = x \cdot y$ (I)

The fuzzy t-conorm - $x \vee_F y = x + y - x \cdot y$ (II)

According to expert X:

$d_X : \{ \text{doctor1, doctor2, doctor3, doctor4; report1, report 2, report 3, report 4} \} \rightarrow [0,1]$

Contradiction degrees	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$		0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$
Attribute's Values	doctor 1	doctor 2	doctor 3	doctor 4		report 1	report 2	report 3	report 4
Fuzzy Degree	0.8	0.2	0.4	0.6		0.7	0.3	0.5	0.5

According to expert Y:

$d_Y : \{ \text{doctor1, doctor2, doctor3, doctor4; report1, report 2, report 3, report 4} \} \rightarrow [0,1]$

Contradiction degrees	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$		0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$
Attribute's Values	doctor 1	doctor 2	doctor 3	doctor 4		report 1	report 2	report 3	report 4
Fuzzy Degree	0.7	0.3	0.5	0.5		0.6	0.4	0.6	0.5

Single attribute value fuzzy set union with Plithogenic

Let us calculate all attribute value separately

$$d_X^F(x, \text{doctor 1}) \vee_p d_Y^F(x, \text{doctor 1}) = 0.8 \vee_p 0.7 \text{ (Contradiction degree is 0)}$$

Using the equation (a)

$$= (1 - 0) [0.8 \vee_F 0.7] + 0 [0.8 \wedge_F 0.7]$$

Using the equation (I) and (II)

$$= 0.8 + 0.7 - 0.56$$

$$= 0.94$$

$$d_X^F(x, \text{doctor 2}) \vee_p d_Y^F(x, \text{doctor 2}) = 0.2 \vee_p 0.3 \text{ (Contradiction degree is } \frac{1}{4})$$

$$= \left(1 - \frac{1}{4}\right) [0.2 \vee_F 0.3] + \frac{1}{4} [0.2 \wedge_F 0.3]$$

$$= \left(\frac{3}{4}\right) [0.2 + 0.3 - 0.2 \times 0.3] + \frac{1}{4} [0.2 \times 0.3]$$

$$= 0.35 \text{ (Using the equation (a) ,(I) and (II))}$$

Similarly, the calculation of the union of the experts results tabulated as follows.

Single valued fuzzy set intersection with Plithogenic

Let us calculate all attribute value separately

$$d_X^F(x, \text{doctor 1}) \wedge_F d_Y^F(x, \text{doctor 1}) = 0.8 \wedge_p 0.7 \text{ (Contradiction degree is 0)}$$

Using the equation (b)

$$= (1 - 0) [0.8 \wedge_F 0.7] + 0 [0.8 \vee_F 0.7] \text{ (Using the equation (I) and (II))}$$

$$= 0.56$$

$$d_X^F(x, \text{doctor 2}) \wedge_F d_Y^F(x, \text{doctor 2}) = 0.2 \wedge_p 0.3 \text{ (Contradiction degree is } \frac{1}{4})$$

$$= \left(1 - \frac{1}{4}\right) [0.2 \wedge_F 0.3] + \frac{1}{4} [0.2 \vee_F 0.3]$$

$$= 0.16 \text{ (Using the equation (b) ,(I) and (II))}$$

Contradiction degrees	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$		0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$
Attribute's Values	doctor 1	doctor 2	doctor 3	doctor 4		report 1	report 2	report 3	report 4

Fuzzy Degrees of Expert X	0.8	0.2	0.4	0.6		0.7	0.3	0.5	0.5
Fuzzy Degrees of Expert Y	0.7	0.3	0.5	0.5		0.6	0.4	0.6	0.5
Fuzzy Degrees of $X_X \vee_P X_Y$	0.94	0.35	0.45	0.43		0.88	0.47	0.55	0.38
Fuzzy Degrees of $X_X \wedge_P X_Y$	0.56	0.16	0.45	0.68		0.42	0.24	0.55	0.63

Similarly, the calculation of the intersection of the expert's results tabulated as follows

The above table calculation is the linear combination of tnorm and tconorm using the equations (a), (b), (I) and (II)

Single valued Intuitionistic fuzzy set degrees of appurtenance

According to expert X:

Contradiction degrees	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$		0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$
Attribute's Values	doctor 1	doctor 2	doctor 3	doctor 4		report 1	report 2	report 3	report 4
Intuitionistic Fuzzy Degree	(0.6,0.5)	(0.2,0.4)	(0.1,0.3)	(0.0,1)		(0.7,0.4)	(0.4,0.5)	(0.5,0.2)	(0.2,0.3)

According to expert Y :

Contradiction degrees	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$		0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$
Attribute's Values	doctor 1	doctor 2	doctor 3	doctor 4		report 1	report 2	report 3	report 4
Intuitionistic Fuzzy Degree	(0.8,0.7)	(0.4,0.5)	(0.3,0.6)	(0.1,0.4)		(0.6,0.5)	(0.5,0.3)	(0.3,0.3)	(0.3,0.1)

Single attribute value Intuitionistic Fuzzy set union with Plithogenic;

$$d_X^{IF}(x, \text{doctor1}) \vee_P d_Y^{IF}(x, \text{doctor1}) = (0.6, 0.5) \vee_P (0.8, 0.7) \text{ (contradiction degree is 0)}$$

Using the equation (c)

$$= (0.6 \vee_P 0.8, 0.5 \wedge_P 0.7)$$

Using the equation (a and b)

$$= ((1-0) [0.6 \vee_F 0.8] + 0[0.6 \wedge_F 0.8], (1-0) [0.5 \wedge_F 0.7] + 0[0.5 \vee_F 0.7])$$

Using the equation (I and II)

$$= (0.48, 0.85)$$

$$\begin{aligned}
 d_X^{IF}(x, doctor3) \vee_P d_Y^{IF}(x, doctor3) &= (0.1, 0.3) \vee_P (0.3, 0.6) \text{ (contradiction degree is } \frac{2}{4}) \\
 &= (0.1 \vee_P 0.3, 0.3 \wedge_P 0.6) \\
 &= ((1 - \frac{2}{4})[0.1 \vee_F 0.3] + \frac{2}{4}[0.1 \wedge_F 0.3], (1 - \frac{2}{4})[0.3 \wedge_F 0.6] + \frac{2}{4}[0.3 \vee_F 0.6]) \\
 &= (0.20, 0.45)
 \end{aligned}$$

Single attribute value Intuitionistic Fuzzy set intersection with Plithogenic;

$$d_X^{IF}(x, doctor1) \wedge_P d_Y^{IF}(x, doctor1) = (0.6, 0.5) \wedge_P (0.8, 0.7) \text{ (contradiction degree is 0)}$$

Using the equation (d)

$$= (0.6 \wedge_P 0.8, 0.5 \vee_P 0.7)$$

Using the equation (a and b)

$$= ((1-0)[0.6 \wedge_F 0.8] + 0[0.6 \vee_F 0.8], (1-0)[0.5 \vee_F 0.7] + 0[0.5 \wedge_F 0.7])$$

Using the equation (I and II)

$$= (0.92, 0.35)$$

$$d_X^{IF}(x, doctor3) \wedge_P d_Y^{IF}(x, doctor3) = (0.1, 0.3) \wedge_P (0.3, 0.6) \text{ (contradiction degree is } \frac{2}{4})$$

Using the equation (d)

$$= (0.1 \wedge_P 0.3, 0.3 \vee_P 0.6)$$

Using the equation (a and b)

$$= ((1 - \frac{2}{4})[0.1 \wedge_F 0.3] + \frac{2}{4}[0.1 \vee_F 0.3], (1 - \frac{2}{4})[0.3 \vee_F 0.6] + \frac{2}{4}[0.3 \wedge_F 0.6])$$

Using the equation (I and II)

$$= (0.20, 0.45)$$

Similarly, the calculation of the union and intersection of the expert's intuitionistic fuzzy results tabulated as follows.

Contradiction degrees	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$		0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$
Attribute's Values	doctor 1	doctor 2	doctor 3	doctor 4		report 1	report 2	report 3	report 4
Intuitionistic fuzzy degrees Expert X	(0.6,0.5)	(0.2,0.4)	(0.1,0.3)	(0,0.1)		(0.7,0.4)	(0.4,0.5)	(0.5,0.2)	(0.2,0.3)

Intuitionistic fuzzy degrees Expert Y	(0.8,0.7)	(0.4,0.5)	(0.3,0.6)	(0.1,0.4)		(0.6,0.5)	(0.5,0.3)	(0.3,0.3)	(0.3,0.1)
Intuitionistic Fuzzy Degrees of $X_X \vee_P X_Y$	0.92,0.35	0.41,0.33	0.20,0.45	0.03,0.36		0.88,0.2	0.58,0.28	0.4,0.25	0.16,0.29
Intuitionistic Fuzzy Degrees of $X_X \wedge_P X_Y$	0.48,0.85	0.19,0.58	0.2,0.45	0.08,0.15		0.42,0.7	0.33,0.53	0.4,0.25	0.35,0.12

The above table calculation is the linear combination of tnorm and tconorm using the equations (c), (d), (I) and (II)

Single valued Neutrosophic set degrees of appurtenance :

According to expert X:

Contradiction degrees	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$		0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$
Attribute's Values	doctor 1	doctor 2	doctor 3	doctor 4		report 1	report 2	report 3	report 4
Neutrosophic Degree	0.4,0.2, 0.6	0.2,0.4, 0.5	0.4,0.1, 0.5	0.5,0.2, 0.3		0.6,0.2, 0.5	0.4,0.1, 0.5	0.5,0.3, 0.4	0.3,0.1, 0.3

According to expert Y:

Contradiction degrees	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$		0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$
Attribute's Values	doctor 1	doctor 2	doctor 3	doctor 4		report 1	report 2	report 3	report 4
Neutrosophic Degree	0.6,0.1,0.3	0.5,0.2, 0.4	0.4,0.3, 0.3	0.7,0.1, 0.6		0.5,0.1, 0.3	0.4,0.2, 0.4	0.6,0.3, 0.5	0.4,0.1, 0.5

Single attribute value Neutrosophic set union with Plithogenic;

$$d_X^N(x, doctor1) \vee_P d_Y^N(x, doctor1) = (0.4, 0.2, 0.6) \vee_P (0.6, 0.1, 0.3) \text{ (contradiction degree is 0)}$$

Using the equation (e)

$$= (0.4 \vee_P 0.6, 0.5(0.2 \wedge_P 0.1 + 0.2 \vee_P 0.1), 0.6 \wedge_P 0.3)$$

Using the equation (a) and (b)

$$= ((1-0) [0.4 \vee_F 0.6] + 0[0.4 \wedge_F 0.6], 0.5(((1-0) [0.2 \wedge_F 0.1] + 0[0.2 \vee_F 0.1])(\\ ((1-0) [0.2 \vee_F 0.1] + 0[0.2 \wedge_F 0.1])), (1-0) [0.6 \wedge_F 0.3] + 0[0.6 \vee_F 0.3])$$

Using the equation (I and II)

$$= (0.76, 0.15, 0.18)$$

$$d_X^{IF}(x, \text{doctor3}) \vee_P d_Y^{IF}(x, \text{doctor3}) = (0.4, 0.1, 0.5) \vee_P (0.4, 0.3, 0.3) \text{ (contradiction degree is } \frac{2}{4})$$

$$= (0.4 \vee_P 0.4, 0.5(0.1 \wedge_P 0.3 + 0.1 \vee_P 0.3), 0.5 \wedge_P 0.3)$$

$$= ((1 - \frac{2}{4})[0.4 \vee_F 0.4] + \frac{2}{4}[0.4 \wedge_F 0.4], 0.5((1 - \frac{2}{4})[0.1 \wedge_F 0.3] + \frac{2}{4}[0.1 \vee_F 0.3]$$

$$+ (1 - \frac{2}{4})[0.1 \vee_F 0.3] + \frac{2}{4}[0.1 \wedge_F 0.3]), (1 - \frac{2}{4})[0.5 \wedge_F 0.3] + \frac{2}{4}[0.5 \vee_F 0.3])$$

$$= (0.40, 0.20, 40)$$

Single attribute value Neutrosophic set intersection with Plithogenic;

$$d_X^N(x, \text{doctor1}) \wedge_P d_Y^N(x, \text{doctor1}) = (0.4, 0.2, 0.6) \wedge_P (0.6, 0.1, 0.3) \text{ (contradiction degree is 0)}$$

Using the equation (f)

$$= (0.4 \wedge_P 0.6, 0.5 (0.2 \wedge_P 0.1 + 0.2 \vee_P 0.1), 0.6 \vee_P 0.3)$$

Using the equation (a) and (b)

$$= ((1-0) [0.4 \wedge_F 0.6] + 0 [0.4 \vee_F 0.6], 0.5(((1-0) [0.2 \vee_F 0.1] \\ + 0[0.2 \wedge_F 0.1]) ((1-0) [0.2 \vee_F 0.1] + 0 [0.2 \wedge_F 0.1])), (1-0) \\ [0.6 \vee_F 0.3] + 0[0.6 \wedge_F 0.3])$$

Using the equation (I and II)

$$= (0.24, 0.15, 0.72)$$

$$d_X^N(x, \text{doctor3}) \wedge_P d_Y^N(x, \text{doctor3}) = (0.4, 0.1, 0.5) \wedge_P (0.4, 0.3, 0.3) \text{ (contradiction degree is } \frac{2}{4})$$

Using the equation (f)

$$= (0.4 \wedge_P 0.4, 0.5 (0.1 \wedge_P 0.3 + 0.1 \vee_P 0.3), 0.5 \vee_P 0.3)$$

Using the equation (a) and (b)

$$= \left(\left(1 - \frac{2}{4} \right) [0.4 \wedge_P 0.4] + \frac{2}{4} [0.4 \vee_F 0.4], 0.5 \left(\left(1 - \frac{2}{4} \right) [0.1 \wedge_F 0.3] + \frac{2}{4} [0.1 \vee_F 0.3] \right. \right. \\ \left. \left. + \left(1 - \frac{2}{4} \right) [0.1 \vee_F 0.3] + \frac{2}{4} [0.1 \wedge_F 0.3] \right), \left(1 - \frac{2}{4} \right) [0.5 \vee_F 0.3] + \frac{2}{4} [0.5 \wedge_P 0.3] \right)$$

Using the equation (I and II)

$$= (0.40, 0.20, 40)$$

Similarly, the calculation of the union and intersection of the experts, intuitionistic fuzzy results tabulated as follows.

Contradiction degrees		0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$		0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$
Attribute's Values		doctor 1	doctor 2	doctor 3	doctor 4		report 1	report 2	report 3	report 4
Neutrosophic set	Expert X	0.4,0.2,0.6	0.2,0.4,0.5	0.4,0.1,0.5	0.5,0.2,0.3		0.6,0.2,0.5	0.4,0.1,0.5	0.5,0.3,0.4	0.3,0.1,0.3
	Expert Y	0.6,0.1,0.3	0.5,0.2,0.4	0.4,0.3,0.3	0.7,0.1,0.6		0.5,0.1,0.3	0.4,0.2,0.4	0.6,0.3,0.5	0.4,0.1,0.5
	Experts $X_X \vee_P X_Y$	0.76,0.15,0.18	0.48,0.3,0.33	0.4,0.2,0.4	0.45,0.15,0.59		0.8,0.15,0.15	0.52,0.15,0.33	0.55,0.3,0.45	0.24,0.1,0.53
	Experts $X_X \wedge_P X_Y$	0.24,0.15,0.72	0.23,0.3,0.58	0.4,0.2,0.4	0.73,0.15,0.32		0.3,0.15,0.65	0.28,0.15,0.58	0.55,0.3,0.45	0.47,0.1,0.28

The above table calculation is the linear combination of tnorm and tconorm using the equations (e), (f), (I) and (II)

6. Conclusion

The objective of this paper is to enhance the accuracy level of decision making, since the decision making level in the existing approaches Fuzzy, Intuitionistic fuzzy set and Neutrosophic set is less accurate. In this paper, it is an attempt to get more accurate value in the Plithogenic set

using aggregative operation t_{norm} and t_{conorm} . This method can be applied in Multiple Regression to get higher accurate level of evaluation. An example is given in this paper to find the level of accuracy for decision making using Plithogenic set with Fuzzy set, Intuitionistic fuzzy set and Neutrosophic set and it is proved practically how accurate the result is and its effectiveness. Hence from the above example, it is proved that Plithogenic set is a reliable and valuable tool for making decision.

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Neutrosophic Optimization of Industry 4.0 Production Inventory Model

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Abstract: In this age of information, the industrial sectors are embedding its functioning principles with the components of Industry 4.0. This article proposes a production inventory model discussing the paradigm shift towards smart production process involving many new cost parameters in addition to the conventional inventory costs. The proposed Industry 4.0 production inventory model is discoursed and compared in both deterministic and neutrosophic environments. The trapezoidal neutrosophic number representation of the parameters enhances the efficiency of the model in determining the optimal order time that minimizes the total costs. The model is highly comprehensive in nature and it is validated with a numerical example.

Keywords: Neutrosophic sets, Industry 4.0, production inventory model, optimization, decision making.

1. Introduction

Presently the industrial sectors are incorporating the techniques of digitalization to meet the requirements of the customer's demands at all ends. The production sectors practice new production methods to ease the process of production that comprises of several sequential steps and new cost parameters. The optimizing principle of manufacturing companies is costs minimization and profit maximization and the inventory models are utilized to make optimal decisions on order time and quantity. The Economic Production Quantity (EPQ) model proposed by Taft [1], a basic production inventory model to manage the levels of inventory by the production sectors. This model is the underlying model and it was developed and extended based on decision-making situations. The fundamental EPQ model was further modified with the integration of the cost parameters of shortages, trade discount, imperfect items, supply chain, deteriorating items, remanufacturing, waste disposal and so on. The production inventory models are extended to cater the requirements of the production sectors. Presently, the fourth industrial revolution is gaining significance amidst the developed and developing nations. Industry 4.0 will certainly bring a paradigm transition at all the levels of organization and control over the different stages of the product's life. The entire process of product production beginning from product conception, product design, product development,

initialization of product production, manufacturing of the product, product delivery and ending with product rework, recycle and disposal will get into the digitalized mode based on customer-centric approach.

The elements of Industry 4.0 are stepping into the production sectors of large, medium and small-sized and at all phases of production processes. Christian Decker et al [2] introduced a cost-benefit model for smart items in the supply chain which is an initial initiative in calculating the advantages of introducing smart items into the network of the supply chain. Andrew Kusiak [3] presented the benefits of smart manufacturing; its core components and the production pattern in future. Fei et al [4] developed IT-driven assistance arranged shrewd assembling with its structure and attributes. Sameer et al [5] introduced a basic audit on keen assembling and Industry 4.0 development models and the suggestions for the entrance of medium and little enterprises. Xiulong et al [6] developed CPS-based smart production system for Industry 4.0 based on the review of the existing literature on smart production systems. Pietro et al [7] built up a digital flexibly chain through the powerful stock and smart agreements. Marc Wins[8] introduced a wide depiction of the highlights of a smart stock administration framework. Souvik et al [9] investigated the savvy stock administration framework dependent on the web of things (IoT). Poti et al [10] introduced the prerequisite examination for shrewd flexibly chain the board for SMEs. Ghadge et al [11] tended to the effect of Industry 4.0 execution on flexibly chains; introduced the benefits and confinements of industry 4.0 in supply chain arrange alongside its cutting-edge headings; clarified the core Industry 4.0 innovations and their business applications and investigated the ramifications of Industry 4.0 with regards to operational and gainful proficiency. Iqra Asghar et al [12] presented a digitalized smart EPQ-based stock model for innovation subordinate items under stochastic failure and fix rate. The above examined stock models are deterministic in nature and the costs boundaries are traditional in nature and they don't mirror the real costs boundaries identified with industry 4.0 components.

In this paper, manufacturing inventory model incorporating a new range of smart costs is formulated, also in this industry 4.0 model, the cost parameters are characterized as neutrosophic sets. This is the novelty of this research work and as far as the literature is concerned, industry 4.0 neutrosophic production inventory models have not been discussed so far and related literature does not exist. Smarandache [13] introduced neutrosophic sets that deal with truth, indeterminacy and falsity membership functions. Neutrosophic sets are widely applied to handle the situations of indeterminacy and it has extensive applications in diverse fields. Sahidul et al [14] developed neutrosophic goal programming for choosing the optimal green supplier, Abdel Nasser [15] used an integrated neutrosophic approach for supplier selection, Lyzbeth [16] constructed neutrosophic decision-making model to determine the operational risks in financial management, Ranjan Kumar et al [17,18] developed neutrosophic multi-objective programming for finding the solution to shortest path problem, Vakkas et al [19] proposed MADM method with bipolar neutrosophic sets.

Abdel-Basst, Mohamed et al [20] has developed neutrosophic decision-making models for effective identification of COVID-19; constructed bipolar neutrosophic MCDM for professional selection [21]; formulated a model to solve supply chain problem using best-worst method [22] and to measure the

financial performance of the manufacturing industries [23]. Also, Abdel-Basset proposed presented a new framework for evaluating the innovativeness of the smart product – service systems [24]. As neutrosophic sets are highly viable, neutrosophic inventory models are formulated by many researchers. Chaitali Kar et al[25] developed inventory model with neutrosophic geometric programming approach. Mullai and Broumi[26] discussed neutrosophic inventory model without shortages, Mullai [27] developed neutrosophic model with price breaks. Mullai et al [28] constructed neutrosophic inventory model dealing with single-valued neutrosophic representation.

In all these neutrosophic inventory models, the cost parameters of the conventional inventory models are represented as neutrosophic sets or numbers, but these models did not discuss any new kind of cost parameters reflecting the transitions in the production processes. But the proposed model reflects the paradigm shift towards smart production process and incorporates new kinds of costs to cater the requirements of smart production inventory model. The industry 4.0 neutrosophic production inventory model with the inclusion of the respective costs to the core elements of smart production systems is highly essential as the existing production sectors are adapting to the environment of smart production set up, but to the best of our knowledge such models are still uncovered. This model primarily focuses on increase productivity and high quality of the product within low investment of finance. The composition of several components of industry 4.0 production inventory model result in diverse costs parameters such as smart ordering cost , internet connectivity initialization cost, holding costs ,smart product design cost, data management cost, customer data analysis cost, supplier data analysis cost, smart technology cost, production monitoring cost, reworking cost, smart training work personnel cost, smart tools purchase cost , smart disposal costs , smart environmental costs, holding cost. The term smart refers to the costs incurred with the integration of digital gadgets to the respective production departments.

The article is structured into the following sections: section 2 consists of the preliminary definitions of neutrosophic sets and its arithmetic operation; section 3 presents the industry 4.0 production inventory model; section 4 validates the proposed model with neutrosophic parameters; section 5 discusses the results and the last section concludes the paper.

2. Basics of Neutrosophic sets and operations

This section presents the fundamentals of neutrosophic sets, arithmetic operations and defuzzification

2.1 Neutrosophic set [13]

A neutrosophic set is characterized independently by a truth-membership function $\alpha(x)$, an indeterminacy-membership function $\beta(x)$, and a falsity-membership function $\gamma(x)$ and each of the function is defined from $X \rightarrow [0,1]$

2.2 Single valued Trapezoidal Neutrosophic Number

A single valued trapezoidal neutrosophic number $\tilde{A} = \langle (a, b, c, d): \rho_{\tilde{A}}, \sigma_{\tilde{A}}, \tau_{\tilde{A}} \rangle$ is a special neutrosophic set on the real number set R , whose truth –membership, indeterminacy-membership, and a falsity –membership is given as follows.

$$\mu_{\tilde{A}}(x) = \begin{cases} (x-a)\rho_{\tilde{A}}/(b-a) & (a \leq x < b) \\ \rho_{\tilde{A}} & (b \leq x \leq c) \\ (d-x)\rho_{\tilde{A}}/(d-c) & (c < x \leq d) \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_{\tilde{A}}(x) = \begin{cases} (b-x+\sigma_{\tilde{A}}(x-a))/(b-a) & (a \leq x < b) \\ \sigma_{\tilde{A}} & (b \leq x \leq c) \\ (x-c+\sigma_{\tilde{A}}(d-x))/(d-c) & (c < x \leq d) \\ 1 & \text{otherwise} \end{cases}$$

$$\varphi_{\tilde{A}}(x) = \begin{cases} (b-x+\tau_{\tilde{A}}(x-a))/(b-a) & (a \leq x < b) \\ \tau_{\tilde{A}} & (b \leq x \leq c) \\ (x-c+\tau_{\tilde{A}}(d-x))/(d-c) & (c < x \leq d) \\ 1 & \text{otherwise} \end{cases}$$

2.3. Operations on Single valued Trapezoidal Neutrosophic Numbers

Let $\tilde{A} = \langle (a_1, b_1, c_1, d_1); \rho_{\tilde{A}}, \sigma_{\tilde{A}}, \tau_{\tilde{A}} \rangle$ and $\tilde{B} = \langle (a_2, b_2, c_2, d_2); \rho_{\tilde{B}}, \sigma_{\tilde{B}}, \tau_{\tilde{B}} \rangle$ be two single valued trapezoidal neutrosophic numbers and $\mu \neq 0$, then

1. $\tilde{A} + \tilde{B} = \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2); \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle$
2. $\tilde{A} - \tilde{B} = \langle (a_1 - a_2, b_1 - b_2, c_1 - c_2, d_1 - d_2); \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle$
3. $\tilde{A} \tilde{B} = \begin{cases} \langle (a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2); \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 > 0, d_2 > 0) \\ \langle (a_1 d_2, b_1 c_2, c_1 b_2, d_1 a_2); \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 < 0, d_2 > 0) \\ \langle (d_1 d_2, c_1 c_2, b_1 b_2, a_1 a_2); \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 < 0, d_2 < 0) \end{cases}$
4. $\tilde{A}/\tilde{B} = \begin{cases} \langle (a_1/d_2, b_1/c_2, c_1/b_2, d_1/a_2); \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 > 0, d_2 > 0) \\ \langle (d_1/d_2, c_1/c_2, b_1/b_2, a_1/a_2); \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 < 0, d_2 > 0) \\ \langle (d_1/a_2, c_1/b_2, b_1/c_2, a_1/d_2); \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 < 0, d_2 < 0) \end{cases}$
5. $\mu \tilde{A} = \begin{cases} \langle (\mu a_1, \mu b_1, \mu c_1, \mu d_1); \rho_{\tilde{A}}, \sigma_{\tilde{A}}, \tau_{\tilde{A}} \rangle & (\mu > 0) \\ \langle (\mu d_1, \mu c_1, \mu b_1, \mu a_1); \rho_{\tilde{A}}, \sigma_{\tilde{A}}, \tau_{\tilde{A}} \rangle & (\mu < 0) \end{cases}$
6. $\tilde{A}^{-1} = \langle (1/d_1, 1/c_1, 1/b_1, 1/a_1); \rho_{\tilde{B}}, \sigma_{\tilde{B}}, \tau_{\tilde{B}} \rangle \quad (\tilde{A} \neq 0).$

2.4 Defuzzification of Neutrosophic set

A single valued trapezoidal neutrosophic numbers of the form $\tilde{A} = \langle (a, b, c, d); \rho, \sigma, \tau \rangle$ can be defuzzified by finding its respective score value $K(\tilde{A})$

$$K(\tilde{A}) = \frac{1}{16} [a + b + c + d] \times (2 + \mu_{\tilde{A}} - \pi_{\tilde{A}} - \varphi_{\tilde{A}}).$$

3. Model Development

3.1 Assumptions

Shortages are not allowed.

Demand is not deterministic in nature.

The products are not of deteriorating type.

Planning horizon is infinite.

3.2 Notations

The below notations are used throughout this paper.

P – Smart production rate per cycle

$D \rightarrow$ Uniform demand rate per cycle

General Costs

O_s – Smart Ordering cost

I_c – Internet Connectivity Initialization Cost

Costs for time period $0 \leq t \leq t_1$

PD_s – Smart Product design cost

DM - Data management Cost

CD – Customer Data Analysis cost

SD – Supplier Data Analysis cost

T_s - Smart Technology Cost

M – Production Monitoring cost

r - defective rate

R – Reworking Cost

TR_s – Smart training work personnel cost

TO_s – Smart tools purchase cost

Costs for time period $t_1 \leq t \leq T$

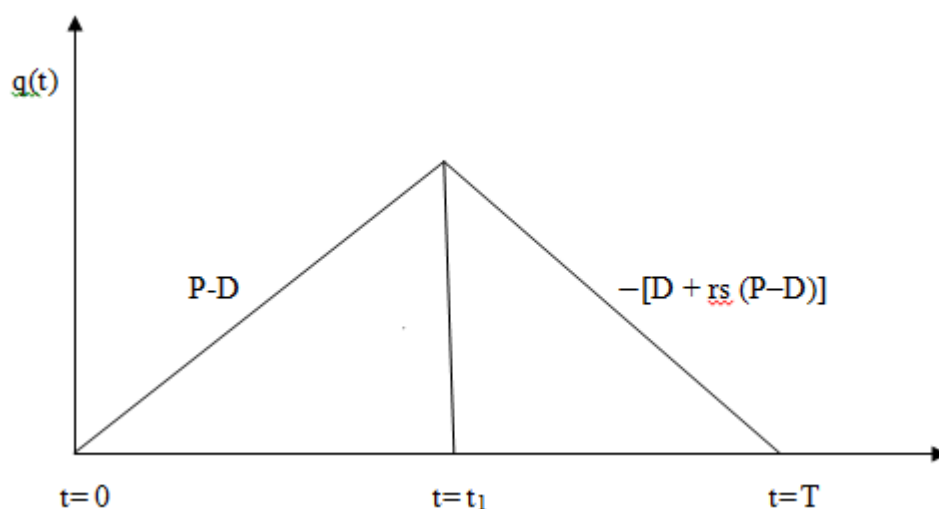
s – disposal rate

D_s – Smart disposal costs

E_s - Smart Environmental costs

Costs common for both the time periods

H - Holding costs



If $q(t)$ represents the inventory level at time $t \in [0, T]$, so the differential equation for the instantaneous inventory $q(t)$ at any time t over $[0, T]$ is

$$\frac{dq(t)}{dt} = P - D \quad 0 \leq t \leq t_1 \rightarrow (1)$$

$$= -[D + rs(P-D)] \quad t_1 \leq t \leq T \quad (2)$$

With initial condition $q(0) = 0$ and

Boundary condition $q(T) = 0$

$$\frac{dq(t)}{dt} = P - D$$

$$dq(t) = (P - D) dt$$

$$q(t) = (P - D) t + c$$

with initial condition $q(0) = 0$

$$q(0) = (P - D) 0 + c$$

$$0 = c$$

$$q(t) = (P - D) t \quad 0 \leq t \leq t_1 \rightarrow (3)$$

solving equation (3)

$$\frac{dq(t)}{dt} = -[D + rs(P-D)] \quad t_1 \leq t \leq T$$

$$dq(t) = -[D + rs(P-D)] dt$$

$$q(t) = -[D + rs(P-D)] t + c$$

With boundary condition $q(T) = 0$

$$q(T) = -[D + rs(P-D)] T + c$$

$$0 = -[D + rs(P-D)] T + c$$

$$c = [D + rs(P-D)] T$$

$$q(t) = -[D + rs(P-D)] t + [D + rs(P-D)] T \rightarrow (4)$$

using equation (3),(4), we get

$$I_{\max} = (P-D) t_1$$

$$I_{\max} = [D + rs(P-D)] (T - t_1)$$

$$t_1 = \frac{I_{\max}}{P-D}$$

$$T - t_1 = \frac{I_{\max}}{D + rs(P-D)}$$

We adding ,we get

$$t_1 + T - t_1 = I_{\max} \left[\frac{1}{(P-D)} + \frac{1}{D + rs(P-D)} \right]$$

$$T = I_{\max} \left[\frac{1}{(P-D)} + \frac{1}{D + rs(P-D)} \right]$$

$$T = I_{\max} \left[\frac{P + (P-D)rs}{P-D[D + rs(P-D)]} \right]$$

$$I_{\max} = \left[\frac{P-D[D+rs(P-D)]}{P+(P-D)rs} \right] T$$

$$\begin{aligned} \text{Smart product design cost} &= PD_s \int_0^{t_1} q(t) dt \\ &= PD_s \int_0^{t_1} (P-D) t dt \\ &= \frac{PD_s}{2} \left[P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} T \right)^2 \right] \end{aligned}$$

$$\begin{aligned} \text{Data management cost} &= DM \int_0^{t_1} q(t) dt \\ &= DM \int_0^{t_1} (P-D) t dt \\ &= \frac{DM}{2} \left[P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} T \right)^2 \right] \end{aligned}$$

$$\begin{aligned} \text{Customer data analysis cost} &= CD \int_0^{t_1} q(t) dt \\ &= CD \int_0^{t_1} (P-D) t dt \\ &= \frac{CD}{2} \left[P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} T \right)^2 \right] \end{aligned}$$

$$\begin{aligned} \text{Supplier data analysis cost} &= SD \int_0^{t_1} q(t) dt \\ &= SD \int_0^{t_1} (P-D) t dt \\ &= \frac{SD}{2} \left[P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} T \right)^2 \right] \end{aligned}$$

$$\begin{aligned} \text{Smart Technology cost} &= T_s \int_0^{t_1} q(t) dt \\ &= T_s \int_0^{t_1} (P-D) t dt \\ &= \frac{T_s}{2} \left[P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} T \right)^2 \right] \end{aligned}$$

$$\begin{aligned} \text{Production Monitoring cost} &= M \int_0^{t_1} q(t) dt \\ &= M \int_0^{t_1} (P-D) t dt \\ &= \frac{M}{2} \left[P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} T \right)^2 \right] \end{aligned}$$

$$\begin{aligned} \text{Reworking cost} &= R \int_0^{t_1} q(t) dt \\ &= R \int_0^{t_1} (P-D) t dt \\ &= \frac{R}{2} \left[P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} T \right)^2 \right] \end{aligned}$$

$$\begin{aligned} \text{Smart training work personal cost} &= TR_s \int_0^{t_1} q(t) dt \\ &= TR_s \int_0^{t_1} (P-D) t dt \\ &= \frac{TR_s}{2} \left[P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} T \right)^2 \right] \end{aligned}$$

$$\begin{aligned} \text{Smart tools purchase cost} &= TO_s \int_0^{t_1} q(t) dt \\ &= TO_s \int_0^{t_1} (P-D) t dt \\ &= \frac{TO_s}{2} \left[P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} T \right)^2 \right] \end{aligned}$$

$$\begin{aligned} \text{Smart disposal cost} &= D_s \int_{t_1}^T q(t) dt \\ &= D_s \int_{t_1}^T D + rs(P-D) t dt \end{aligned}$$

$$= \frac{O_s}{T} + \frac{I_c}{T} + \frac{C_1}{2} \left[\frac{P-D[D+rs(P-D)]}{P+(P-D)rs} \right] T + \frac{1}{2} [P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} \right)^2 T (PD_s + DM + CD + SD + T_s + M + R + TR_s + TO_s) +$$

$$[D + rs(P-D) \left(\frac{(P-D)}{P+(P-D)rs} \right)^2 T] (D_s + E_s)]$$

So the Classical EPQ model is

$$\text{Min TAC}(T) = \frac{O_s}{T} + \frac{I_c}{T} + \frac{C_1}{2} \left[\frac{P-D[D+rs(P-D)]}{P+(P-D)rs} \right] T + \frac{1}{2} [P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} \right)^2 T (PD_s + DM + CD + SD +$$

$$T_s + M + R + TR_s + TO_s) + [D + rs(P-D) \left(\frac{(P-D)}{P+(P-D)rs} \right)^2 T] (D_s + E_s)]$$

Such that $T > 0$

We can show that $\text{TAC}(T)$ will be minimum for

$$T^* = \sqrt{\frac{2(O_s + I_c)}{C_1 \left[\frac{P-D[D+rs(P-D)]}{P+(P-D)rs} \right] + [P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} \right)^2 (PD_s + DM + CD + SD + T_s + M + R + TR_s + TO_s) + [D + rs(P-D) \left(\frac{(P-D)}{P+(P-D)rs} \right)^2] (D_s + E_s)]}}$$

$$\text{TAC}^*(T^*) = \sqrt{2(O_s + I_c) + C_1 \left[\frac{P-D[D+rs(P-D)]}{P+(P-D)rs} \right] + [P-D \left(\frac{D+rs(P-D)}{P+(P-D)rs} \right)^2 (PD_s + DM + CD + SD + T_s + M + R + TR_s + TO_s) + [D + rs(P-D) \left(\frac{(P-D)}{P+(P-D)rs} \right)^2] (D_s + E_s)]}$$

4. Illustration

To validate the developed model, an inventory system with the below characteristics is taken into consideration

Smart production rate per cycle = Rs.500unit/per month , Uniform demand rate per cycle = Rs.250/month, Smart Ordering cost =Rs.310/run, Internet Connectivity Initialization Cost = Rs.370/year, Smart Product design cost = Rs.25/unit, Data management Cost = Rs.50/unit, Customer Data Analysis cost = Rs.45/unit, Supplier Data Analysis cost = Rs.25/unit, Smart Technology Cost = Rs.15/unit, Production Monitoring cost = Rs.45/unit, defective rate = Rs.1, Reworking Cost = Rs.22/run, Smart training work personnel cost = Rs.30/unit, Smart tools purchase cost= Rs.10/unit, disposal rate = Rs. 3/unit, Smart disposal costs = Rs. 5/unit, Smart Environmental costs = Rs.7/unit, Holding costs = Rs.1/unit/year. Find the time interval and find the total average cost.

The value of T^* and $\text{TAC}(T^*)$ is 0.179 and Rs.208.39 respectively

This model can be validated with the single valued neutrosophic trapezoidal fuzzy value representations as follows,

$$D = \langle (250, 350, 450, 550); 0.7, 0.2, 0.1 \rangle$$

$$O_s = \langle (350, 450, 550, 650); 0.9, 0.3, 0.1 \rangle$$

$$I_c = \langle (550, 650, 750, 850); 0.8, 0.3, 0.4 \rangle$$

$$PD_s = \langle (25, 35, 45, 55); 0.7, 0.3, 0.2 \rangle$$

$$DM = \langle (65, 75, 85, 95); 0.9, 0.3, 0.4 \rangle$$

$$CD = \langle (55, 65, 75, 85); 0.8, 0.1, 0.2 \rangle$$

$$SD = \langle (20, 30, 40, 50); 0.8, 0.3, 0.2 \rangle$$

$$T_s = \langle (15, 18, 22, 24); 0.7, 0.1, 0.2 \rangle$$

$$M = \langle (60, 70, 80, 90): 0.7, 0.2, 0.4 \rangle$$

$$r = \langle (1, 1.5, 2.5, 3): 0.9, 0.1, 0.2 \rangle$$

$$R = \langle (20, 25, 35, 40): 0.8, 0.2, 0.1 \rangle$$

$$TRs = \langle (35, 45, 55, 65): 0.7, 0.1, 0.3 \rangle$$

$$TOs = \langle (8, 12, 16, 20): 0.7, 0.1, 0.4 \rangle$$

$$S = \langle (3, 4, 6, 8): 0.8, 0.1, 0.4 \rangle$$

$$Ds = \langle (5, 7, 9, 11): 0.7, 0.2, 0.3 \rangle$$

$$Es = \langle (6, 9, 12, 15): 0.8, 0.2, 0.3 \rangle$$

$$C_1 = \langle (1, 1.5, 2.5, 3): 0.9, 0.3, 0.2 \rangle$$

The value of $T^* = 0.178$ and $TAC^*(T^*) = 210.29$

5. Discussion

A neutrosophic production inventory model incorporating the costs parameters of industry 4.0 is developed together with the presentation of its conceptual framework. Several key benefits of neutrosophic production inventory model have been emphasized in this paper, together with the additional cost parameters. Another point of discussion is the usage of the production inventory model to find the feasible time to place orders that confines the total expenses. The representation of these costs parameters as single valued trapezoidal neutrosophic number tackles the conditions of uncertainty.

The constructed manufacturing inventory model is validated with deterministic parameters and neutrosophic parameters. The optimal time that yields minimum costs is nearly equal in both the cases of deterministic and neutrosophic validation. The neutrosophic representation makes this model more comprehensive. In this paper shortages are not allowed, the products are not of deteriorating type, planning horizon is infinite. The developed model can be extended to neutrosophic production inventory model with shortages and deteriorating items. This model primarily focuses on increases productivity of high-quality products within low investment of finance. The discussion is summarized as follows, a novel neutrosophic production inventory model is developed with the cost parameters pertaining to the fourth industrial revolution. This proposed model will certainly assist the production sectors to incorporate new types of costs. A deeper investigation on the effects of our decision making is clearly an obligation for upcoming work.

6. Conclusion

The proposed industry 4.0 inventory model is a novel approach integrating the concept of smart production principles, and neutrosophic representations of cost parameters. This model is an underlying smart production model and this model can be further developed based on the needs of the production sectors. The proposed model is pragmatic in nature and it can be extended by including the concepts of customer acquisition and product propagation with additional cost parameters. These models will certainly unveil the new requirements of production scenario to meet the demands of the customers of this information age. The model constructed in this paper presents the present need of the production environment and it will certainly assist the decision makers to

optimize profit. The cost parameters of this model can be scaled to the requirements of small and medium sized enterprises which could be the future work.

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New Family of Neutrosophic Soft Sets

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Abstract : The goal of this paper is to study and discuss the neutrosophic soft set theory by introducing, new family of neutrosophic soft sets and because the concept of topological spaces is one of the most powerful concepts in system analysis, we introduced the concept of neutrosophic soft topological spaces depending on this the new family. Furthermore, we introduced new definitions, properties, concerning the neutrosophic soft closure, the neutrosophic soft interior, the neutrosophic soft exterior and the neutrosophic soft boundary in details.

Keywords : neutrosophic soft set theory, of neutrosophic soft topological spaces , new operations for neutrosophic soft sets. families of neutrosophic soft sets.

1. Introduction

D Molodtsov[1] introduced the notion of soft set in 1999 . In the same year F Smarandache firstly introduced the neutrosophic set theory [2]. Which is the generalization of the classical set conventional fuzzy set [3] and intuitionistic fuzzy set [4]. The soft set theory and the neutrosophic set theory have been applied to many different fields, (see for example [5-64]) .

In 2012 Maji[65] combined the concept of soft set and neutrosophic set together by introducing the current mathematical framework called neutrosophic soft set and later this concept has been modified by S.Broumi[66] . Faruk[67] redefined neutrosophic soft set, and their operations, also presented an application of neutrosophic soft set, in decision making . In 2017 Bera[68] introduced neutrosophic soft topological spaces using different subsets of the parameters set for each soft set . In 2019 and In 2020 Taha[69] and Evanzalin[70] introduced the neutrosophic soft topological spaces differently from the study[58] . More works on the concept, of neutrosophic soft set can be found in [71, 72, 73, 74, 75, 76, 77, 78, 79, 80,81,82] .

In this research , we studied and discussed the neutrosophic soft set theory by introducing, new family of neutrosophic soft sets, new operations for neutrosophic soft sets and we also we introduce the theory of neutrosophic soft topological spaces depending on this the new family.

The research is organized as follows: In section2, we first recall the necessary definitions needed in this work we then recall two families of neutrosophic soft sets with explaining the properties of each family. In section3, the neutrosophic soft set theory is studied and discussed by introducing, new family of neutrosophic soft sets [namely third family], new operations for neutrosophic soft sets, comparison between the new family and other families, new definitions and examples. In section4, the theory of neutrosophic soft topological spaces is investigated depending on the new family and also, new definitions, characterization, the neutrosophic soft closure, the neutrosophic soft interior, the neutrosophic soft exterior and the neutrosophic soft boundary are introduced in details .

2. Preliminaries

In this section, we will recall the necessary definitions needed in this work, we then recall two families of neutrosophic soft sets with explaining the properties of each family.

2.1. Definition [83]

If K is the initial universe then the neutrosophic set A is defined as follows:

$$A = \{ \langle k, T_A(k), I_A(k), V_A(k) \rangle, k \in K \}$$

where, the functions $T, I, V : K \rightarrow]-0, +1[$ and

$$-0 \leq T_A(k) + I_A(k) + V_A(k) \leq +3$$

For any two neutrosophic sets:

$$A = \{ \langle k, T_A(k), I_A(k), V_A(k) \rangle, k \in K \}$$

$$B = \{ \langle k, T_B(k), I_B(k), V_B(k) \rangle, k \in K \}$$

- ❖ $A \subseteq B \leftrightarrow T_A(k) \leq T_B(k), I_A(k) \leq I_B(k), V_A(k) \geq V_B(k)$, for all, $k \in K$.
- ❖ $A \cup B = \{ \langle k, T_A(k) \vee T_B(k), I_A(k) \vee I_B(k), V_A(k) \wedge V_B(k) \rangle, k \in K \}$.
- ❖ $A \cap B = \{ \langle k, T_A(k) \wedge T_B(k), I_A(k) \wedge I_B(k), V_A(k) \vee V_B(k) \rangle, k \in K \}$.
- ❖ $A = B \leftrightarrow A \subseteq B$ and $B \subseteq A$.
- ❖ The complement of A denoted by A^c is defined as:
 $(A)^c = \{ \langle k, 1 - T_A(k), 1 - I_A(k), 1 - V_A(k) \rangle, k \in K \}$.

2.2. Definition [1]

Let K be an initial universe set and E be a set of parameters. Consider a set $A \neq \emptyset, A \subseteq E$. A pair (F, A) is called a soft set (over K) if and only if F is a mapping from A into the set of all the subsets of K .

First family [65]

Let K be an initial universe set and E_K be a set of parameters. Consider a set $D \neq \emptyset, D \subseteq E_K$. A pair (F, D) is called a neutrosophic soft set (over K) if and only if F is a mapping from A into the set of all the neutrosophic sets over K .

Note that, we will denote simply by F_D of the pair (F, D) and the set of all the neutrosophic sets over K with respect to this family will be denoted by $N_1(K)$.

Let $F_{1D}, F_{2D} \in N_1(K)$. Then :

- 1) The union between them $(F_D \sqcup G_B)$ is defined by $H = F_D \sqcup G_B$ as follows :

$$T_{H(p)}(k) = \begin{cases} T_{F(p)}(k) & \text{if } p \in D \setminus B \\ T_{G(p)}(k) & \text{if } p \in B \setminus D \\ \max \{T_{F(p)}(k), T_{G(p)}(k)\} & \text{if } p \in D \cap B \end{cases}$$

$$I_{H(p)}(k) = \begin{cases} I_{F(p)}(k) & \text{if } p \in D \setminus B \\ I_{G(p)}(k) & \text{if } p \in B \setminus D \\ \frac{(I_{F(p)}(k) + I_{G(p)}(k))}{2} & \text{if } p \in D \cap B \end{cases}$$

$$V_{H(p)}(k) = \begin{cases} V_{F(p)}(k) & \text{if } p \in D \setminus B \\ V_{G(p)}(k) & \text{if } p \in B \setminus D \\ \min\{V_{F(p)}(k), V_{G(p)}(k)\} & \text{if } p \in D \cap B \end{cases}$$

2) The interstation between them($F_D \cap G_B$) is defined by $H = F_D \cap G_B$ as follows :

$$T_{H(p)}(k) = \min \{T_{F(p)}(k), T_{G(p)}(k)\}$$

$$I_{H(p)}(k) = \frac{(I_{F(p)}(k) + I_{G(p)}(k))}{2}$$

$$V_{H(p)}(k) = \max \{V_{F(p)}(k), V_{G(p)}(k)\}.$$

3) $F_D \subseteq G_B$ if and only if

1) $D \subseteq B$

2) $T_{F(p)}(k) \leq T_{G(p)}(k), I_{F(p)}(k) \leq I_{G(p)}(k), V_{F(p)}(k) \geq V_{G(p)}(k)$, for all $p \in D, k \in K$.

4) The complement of F_D is defined as:

$$(F_D)^c = \{(p, \{< k, T_{F(p)}(k), I_{F(p)}(k), V_{F(p)}(k) >, k \in K\}), p \in D\}.$$

2.3. Definition [70]

A neutrosophic soft set F_D over the universe K is called a null neutrosophic soft set and denoted by \emptyset_N if $T_{F(p)}(k) = 0, I_{F(p)}(k) = 0, V_{F(p)}(k) = 1$, for all $p \in D, k \in K$.

2.4. Definition [70]

A neutrosophic soft set F_D over the universe K is called an absolute is called an absolute neutrosophics soft set and denoted by K_N if $T_{F(p)}(k) = 1, I_{F(p)}(k) = 1, V_{F(p)}(k) = 0$, for all $p \in D, k \in K$.

Second family [66]

Let K be an initial universe set and E be a set of parameters, $P(Y)$ be the set of all the subsets of K and V be a neutrosophic set over E . Then a neutrosophic parameterized soft sets.

$$\Omega_V = \{(< p, T_V(p), I_V(p), W_V(p), f_V(p) >), p \in E\}$$

where, the functions $T_V, I_V, W_V : E \rightarrow [0,1]$ and $f_V : E \rightarrow P(K)$

and $f_V(p) = \emptyset$ if $T_V(p) = 0, I_V(p) = 1$ and $W_V(p) = 1$.

Here, the functions T_V, I_V, W_V are called membership function, indeterminacy function and non-membership function of parameterized soft set (for short, Np_soft set), respectively.

Let $\Omega_V, \bar{\Omega}_L \in Np_soft$ set.

Now : If $f_V(p) = K, T_V(p) = 0, I_V(p) = 0$ and $W_V(p) = 1, \forall p \in E$, then Ω_V is called a V_empty Np_soft set (for short Ω_{\emptyset_V}). If $V = \emptyset$, then the V_empty Np_soft set is called an empty Np_soft set (for short Ω_{\emptyset}). If $f_V(p) = K, T_V(p) = 1, I_V(p) = 0$ and $W_V(p) = 0, \forall p \in E$, then, Ω_V is called a $V_universal$ Np_soft set (for short $\Omega_{\bar{V}}$), if $V = E$, then the $V_universal$ Np_soft set is called an $V_universal$ Np_soft set (for short Ω_E).

$$\Omega_V \subseteq \bar{\Omega}_L \leftrightarrow T_V(p) \leq T_L(p), I_V(p) \geq I_L(p), W_V(p) \geq W_L(p), f_V(p) \subseteq f_L(p), p \in E.$$

$$\Omega_V \sqcup \bar{\Omega}_L = \{(< p, \max\{T_V(p), T_L(p)\}, \min\{I_V(p), I_L(p)\}, \min\{W_V(p), W_L(p)\}, f_V(p) \cup f_L(p) >), p \in E\}.$$

$$\Omega_V \cap \bar{\Omega}_L = \{(< p, \min\{T_V(p), T_L(p)\}, \max\{I_V(p), I_L(p)\}, \max\{W_V(p), W_L(p)\}, f_V(p) \cap f_L(p) >), p \in E\}.$$

The complement of Ω_V is defined as :

$$\Omega_V^c = \{(< p, W_V(p), I_V(p), L_V(p), f_{V^c}(p) >) p \in E\}, \text{ where, } f_{V^c}(p) = K - f_V(p).$$

3. Third family (New family)

In this section, we will study and discuss the neutrosophic soft set theory giving new definitions, example, new family of neutrosophic soft sets, new operations for neutrosophic soft sets and comparison between the new family and first family.

3.1. Definition

A neutrosophic soft set F_D on the universe K is denoted by the set of ordered pairs

$$F_D = \{(p, f_D(p)), p \in E_K, \}$$

It can be written as: $F_D = \{ (p, \{ < q^{(T_{f_D(p)}(q)), I_{f_D(p)}(q)), V_{f_D(p)}(q) > \}, p \in E_K \}$.

Where,

f_D is a mapping such that

$$f_D : \begin{cases} D \rightarrow P(K) \\ D^c \rightarrow \langle q^{(0,0,1)} \rangle \end{cases}, q \in K.$$

E_K is the set of all possible parameters under consideration with respect to K , $D \subseteq E_K$.

$P(K)$ is the set of all the neutrosophic sets over K .

From now on, the set of all the neutrosophic sets over K with respect to this family (Third family) will be denoted by $N_3(K)$.

3.2. Example

Let $K = \{q_1, q_2, q_3, q_4\}$ and $D \subseteq E_K = \{p_1, p_2, p_3, p_4\}$, such that $D = \{p_1, p_2\}$.

Suppose that :

$$f_{1D}(p_1) = \{ < q_1^{(0,6,0,3,0,7)}, q_2^{(0,5,0,4,0,5)}, q_3^{(0,7,0,4,0,3)}, q_4^{(0,8,0,4,0,3)} > \}.$$

$$f_{1D}(p_2) = \{ < q_1^{(0,7,0,3,0,5)}, q_2^{(0,6,0,7,0,3)}, q_3^{(0,7,0,3,0,5)}, q_4^{(0,6,0,3,0,6)} > \}.$$

$$f_{2D}(p_1) = \{ < q_1^{(0,6,0,4,0,5)}, q_2^{(0,6,0,5,0,4)}, q_3^{(0,7,0,4,0,5)}, q_4^{(0,7,0,5,0,6)} > \}.$$

$$f_{2D}(p_2) = \{ < q_1^{(0,7,0,6,0,6)}, q_2^{(0,8,0,4,0,5)}, q_3^{(0,7,0,4,0,6)}, q_4^{(0,6,0,3,0,5)} > \}.$$

Then, we can view the neutrosophic soft sets F_{1D}, F_{2D} as :

$$F_{1D} = \left\{ \begin{aligned} & (p_1, \{ < q_1^{(0,6,0,3,0,7)}, q_2^{(0,5,0,4,0,5)}, q_3^{(0,7,0,4,0,3)}, q_4^{(0,8,0,4,0,3)} > \}) \\ & (p_2, \{ < q_1^{(0,7,0,3,0,5)}, q_2^{(0,6,0,7,0,3)}, q_3^{(0,7,0,3,0,5)}, q_4^{(0,6,0,3,0,6)} > \}) \end{aligned} \right\}$$

$$F_{2D} = \left\{ \begin{aligned} & (p_1, \{ < q_1^{(0,6,0,4,0,5)}, q_2^{(0,6,0,5,0,4)}, q_3^{(0,7,0,4,0,5)}, q_4^{(0,7,0,5,0,6)} > \}) \\ & (p_2, \{ < q_1^{(0,7,0,6,0,6)}, q_2^{(0,8,0,4,0,5)}, q_3^{(0,7,0,4,0,6)}, q_4^{(0,6,0,3,0,5)} > \}) \end{aligned} \right\}$$

Note that, if $f_D(p) = \langle q^{(0,0,1)} \rangle$, for all $p \in E, q \in K$, the element $(p, f_D(p))$ is not appeared in neutrosophic soft set F_D .

3.3. Definition

The neutrosophic soft complement F_D^c of F_D is defined by the mapping $f_D^c(p) = f_D^c(p)$, where $f_D^c(p)$ is the complement of the set $f_D(p)$.

That is :

$$F_D^c = \{ \{p, \{ < q^{(1-T_{f_D(p)}(q))' (1-I_{f_D(p)}(q))' (1-V_{f_D(p)}(q)) > \}, p \in E_K \}.$$

3.4. Definition

Let $F_D \in N_3(K)$, if $f_D(p) = < q^{(0,0,1)} >, \forall p \in E_K, q \in K$, then F_D is called the null neutrosophic soft set and denoted by $\tilde{\emptyset}_D$.

3.5. Definition

Let $F_D \in N_3(K)$ if $f_D(p) = < q^{(1,1,0)} >, \forall p \in D, q \in K$, then F_D is called the absolute neutrosophic soft set and denoted by \tilde{K}_D .

3.6. Definition

Let $F_{1D} \in N_3(K)$, then F_{1D} is called a neutrosophic soft subset of F_{2D} and denoted by $F_{1D} \subseteq F_{2D}$ if $f_{1D}(p) \subseteq f_{2D}(p), \forall p \in E_K$.

3.7. Definition

Let $F_{1D}, F_{2D} \in N_3(K)$, then, the neutrosophic soft intersection $(F_{1D} \cap F_{2D})$ and the neutrosophic soft union $(F_{1D} \sqcup F_{2D})$ are defined by the mappings .

$$f_{1D}(p) \cap f_{2D}(p)$$

$$f_{1D}(p) \sqcup f_{2D}(p)$$

3.8. Example

Let us consider neutrosophic soft sets F_{1D}, F_{2D} in example .
Then ,

$$1) \quad F_{1D} \sqcup F_{2D} = \left\{ \left(p_1, \{ < q_1^{(0,6,0,4,0,5)}, q_2^{(0,6,0,5,0,4)}, q_3^{(0,7,0,4,0,3)}, q_4^{(0,8,0,5,0,3)} > \} \right), \left(p_2, \{ < q_1^{(0,7,0,6,0,5)}, q_2^{(0,8,0,7,0,3)}, q_3^{(0,7,0,4,0,5)}, q_4^{(0,6,0,3,0,5)} > \} \right) \right\}$$

$$2) \quad F_{1D} \cap F_{2D} = \left\{ \left(p_1, \{ < q_1^{(0,6,0,3,0,7)}, q_2^{(0,5,0,4,0,5)}, q_3^{(0,7,0,4,0,5)}, q_4^{(0,7,0,4,0,6)} > \} \right), \left(p_2, \{ < q_1^{(0,7,0,3,0,5)}, q_2^{(0,6,0,5,0,4)}, q_3^{(0,7,0,3,0,6)}, q_4^{(0,6,0,3,0,6)} > \} \right) \right\}$$

$$3) \quad (F_{2D})^c = \left\{ \left(p_1, \{ < q_1^{(0,4,0,6,0,5)}, q_2^{(0,4,0,5,0,6)}, q_3^{(0,3,0,6,0,5)}, q_4^{(0,3,0,5,0,4)} > \} \right), \left(p_2, \{ < q_1^{(0,3,0,4,0,4)}, q_2^{(0,2,0,6,0,5)}, q_3^{(0,3,0,6,0,4)}, q_4^{(0,4,0,7,0,5)} > \} \right), \left(p_3, \{ < q_1^{(1,1,1,0)}, q_2^{(1,1,1,0)}, q_3^{(1,1,1,0)}, q_4^{(1,1,1,0)} > \} \right), \left(p_4, \{ < q_1^{(1,1,1,0)}, q_2^{(1,1,1,0)}, q_3^{(1,1,1,0)}, q_4^{(1,1,1,0)} > \} \right) \right\}$$

3.9. Proposition

Let $F_{1D} \in N_3(K)$, then :

- $F_{1D} \sqcup F_{1D} = F_{1D}$.
- $F_{1D} \cap F_{1D} = F_{1D}$.
- $F_{1D} \sqcup \tilde{\emptyset}_D = F_{1D}$.
- $F_{1D} \sqsubseteq F_{1D}$.
- $\tilde{\emptyset}_D \sqsubseteq F_{1D}$.
- $F_{1D} \sqsubseteq \tilde{K}_D$.
- $F_{1D} \cap \tilde{\emptyset}_D = \tilde{\emptyset}_D$.
- $F_{1D} \sqcup \tilde{K}_D = \tilde{K}_D$.
- $F_{1D} \cap \tilde{K}_D = F_{1D}$.

Proof : The proof of the remark is direct from the definition

3.10. Proposition

Let $F_{1D} \in N_3(K)$, then :

- $(\tilde{\emptyset}_D)^c = \tilde{K}_D$
- $(\tilde{K}_D)^c = \tilde{\emptyset}_D$
- $((F_{1D})^c)^c = F_{1D}$

Proof : Straightforward .

3.11. Proposition

Let F_{1D}, F_{2D} and $F_{3D} \in N_3(K)$, then :

- $F_{1D} \sqcup F_{2D} = F_{2D} \sqcup F_{1D}$.
- $F_{1D} \cap F_{2D} = F_{2D} \cap F_{1D}$.
- $(F_{1D} \sqcup F_{2D})^c = (F_{1D})^c \cap (F_{2D})^c$.
- $(F_{1D} \cap F_{2D})^c = (F_{1D})^c \sqcup (F_{2D})^c$.
- $(F_{1D} \cap F_{2D}) \sqcup F_{3D} = (F_{1D} \sqcup F_{3D}) \cap (F_{2D} \sqcup F_{3D})$.
- $(F_{1D} \sqcup F_{2D}) \cap F_{3D} = (F_{1D} \cap F_{3D}) \sqcup (F_{2D} \cap F_{3D})$.

Proof : Straightforward .

Comparison

Next ,we will compare (new family) with the first family .

1- Dentition of neutrosophic soft sets	
First family	Third family(new family)
$F_D = \{(p, f_D(p)), p \in D, D \subseteq E_k\}$ Where, $F : D \rightarrow P(K)$	$F_D = \{(p, f_D(p)), p \in E_k, D \subseteq E_k\}$ Where, $F : \begin{cases} D \rightarrow P(K) \\ D^c \rightarrow \langle q^{(0,0,1)} \rangle, q \in K \end{cases}$

2- Intersection of neutrosophic soft sets	
First family	Third family(new family)
$F_{1A} \cap F_{2B} = F_{3(A \cap B)}$ $F_3: (A \cap B) \rightarrow P(K)$ $F_3(p) = F_1(p) \cap F_2(p)$	$F_{1D} \cap F_{2D} = F_{3D}$ $F_3: \begin{cases} D \rightarrow P(K) \\ D^c \rightarrow \langle q^{(0,0,1)} \rangle, q \in K \end{cases}$ $F_3(p) = F_1(p) \cap F_2(p)$

3- Union of neutrosophic soft sets	
First family	Third family(new family)
$F_{1A} \cup F_{2B} = F_{3(A \cup B)}$ $F_3: (A \cup B) \rightarrow P(K)$ $F_3(p) = \begin{cases} F_{1A} & \text{if } p \in A \setminus B \\ F_{2B} & \text{if } p \in B \setminus A \\ F_1(p) \cup F_2(p), & \text{if } p \in A \cap B \end{cases}$	$F_{1D} \cup F_{2D} = F_{3D}$ $F_3: \begin{cases} D \rightarrow P(K) \\ D^c \rightarrow \langle k^{(0,0,1)} \rangle, k \in K \end{cases}$ $F_3(p) = F_1(p) \cup F_2(p)$

4- Complement of neutrosophic soft sets	
First family	Third family(new family)
$(F_D)^c = F^c_{ D}$ <p>Where $F^c: D \rightarrow P(K)$ "</p> <p>Where, $\neg k$ is not k and $D = \{k: k \in D\}$</p>	$(F_D)^c = F^c_D$ <p>Where, $F^c: \begin{cases} D \rightarrow P(K) \\ D^c \rightarrow \langle q^{(0,0,1)} \rangle, q \in K \end{cases}$.</p>

4. Neutrosophic Soft Topology

In this section, we will investigate the theory of neutrosophic soft topological spaces depending on the new family $(N_3(K))$ and we present new definitions, characterization and properties concerning the neutrosophic soft closure, the neutrosophic soft interior, the neutrosophic soft exterior, the neutrosophic soft boundary .

4.1. Definition

Let K be the initial universe E_K be set of parameters , $D \subseteq E_K$ and $\mu \subseteq N_3(K)$, we say that μ is a neutrosophic soft topology on K , if it satisfies the following conditions :

- 1) $\tilde{\emptyset}_D, \tilde{K}_D \in \mu$.
- 2) $F_{1D} \cap F_{2D} \in \mu, \forall F_{1D}, F_{2D} \in \mu$.
- 3) $\cup \{F_{iD}, i \in I\} \in \mu, \forall F_{iD} \in \mu$.

The pair (K, μ) (or simply K) is a neutrosophic soft topological spaces or $((N_3 - \text{Top})$ for short) .

- ❖ The elements of μ are called a neutrosophic open the family sets .
- ❖ A neutrosophic soft F_{1D} is called a neutrosophic soft closed set, if its complement is a neutrosophic soft open set .

4.2. Proposition

Let (K, μ) be $(N_3 - \text{Top})$, then the family of neutrosophic soft closed sets $(C(\tilde{K}_D)$ for short) has the following properties :

- 1) $\tilde{\emptyset}_D, \tilde{K}_D \in C(\tilde{K}_D)$.
- 2) $F_{1D} \cap F_{2D} \in \mu, \forall F_{1D}, F_{2D} \in C(\tilde{K}_D)$.
- 3) $\cap \{F_{iD}, i \in I\} \in C(\tilde{K}_D), \forall F_{iD} \in C(\tilde{K}_D)$.

Proof : Straightforward.

4.3. Example

Let $K = \{q_1, q_2\}$, $D \subseteq E_K$, such that $D = \{p_1\}$ and $F_{1D}, F_{2D} \in N_3(K)$, such that :

$$F_{1D} = \left\{ (p_1, \{ \langle q_1^{(0,5',0,1',0,4)}, q_2^{(0,4',0,3',0,8)} \rangle \}) \right\}.$$

$$F_{2D} = \left\{ (p_1, \{ \langle q_1^{(0,5',0,1',0,3)}, q_2^{(0,5',0,3',0,6)} \rangle \}) \right\}.$$

Then , $\mu = \{ \tilde{\emptyset}_D, \tilde{K}_D, F_{1D}, F_{2D} \}$ is a neutrosophic soft topology on K and (K, μ) is a neutrosophic soft topological space .

4.4. Proposition

Let (K, μ_1) and (K, μ_2) be two neutrosophic soft topological spaces on K , then $(K, \mu_1 \cap \mu_2)$ is a neutrosophic soft topological spaces on K .

Poof:

Let (K, μ_1) and (K, μ_2) be two neutrosophic soft topological spaces on K . It can be seen clearly that $\tilde{\emptyset}_D, \tilde{K}_D \in \mu_1 \cap \mu_2$. If $F_{1D}, F_{2D} \in \mu_1 \cap \mu_2$, then $F_{1D}, F_{2D} \in \mu_1$ and $F_{1D}, F_{2D} \in \mu_2$. It is given that

$F_{1D} \cap F_{2D} \in \mu_1$, $F_{1D} \cap F_{2D} \in \mu_2$. Thus $F_{1D} \cap F_{2D} \in \mu_1 \cap \mu_2$. Let $\{F_{iD}, i \in I\} \in \mu_1 \cap \mu_2$, then $F_{iD} \in \mu_1, \forall i \in I$ and $F_{iD} \in \mu_2, \forall i \in I$. Then $F_{iD} \in \mu_1 \cap \mu_2, \forall i \in I$. So, we have $\cup \{F_{iD}, i \in I\} \in \mu_1 \cap \mu_2$

4.5. Remark

Let (K, μ_1) and (K, μ_2) be two neutrosophic soft topological spaces on K , then $(K, \mu_1 \cup \mu_2)$ may not be correct. It can be seen from the following example.

4.6. Example

Let $K = \{q_1, q_2\}$, $D \subseteq E_K$, such that $D = \{p_1\}$ and $F_{1D}, F_{2D} \in N_3(K)$,

such that :

$$F_{1D} = \left\{ (p_1, \{ < q_1^{(0,2,0,4,0,6)}, q_2^{(0,1,0,3,0,5)} > \}) \right\}.$$

$$F_{2D} = \left\{ (p_1, \{ < q_1^{(0,4,0,6,0,8)}, q_2^{(0,3,0,5,0,7)} > \}) \right\}.$$

Then, $\mu_1 = \{ \tilde{\emptyset}_D, \tilde{K}_D, F_{1D} \}$ and $\mu_2 = \{ \tilde{\emptyset}_D, \tilde{K}_D, F_{2D} \}$ are two neutrosophic soft topology on K . But

$\mu_1 \cup \mu_2 = \{ \tilde{\emptyset}_D, \tilde{K}_D, F_{1D}, F_{2D} \}$ is not neutrosophic soft topology on K .

4.7. Definition

Let $F_D \in N_3(K)$. The interior of F_D is union of all neutrosophic soft open sets contained in F_D , denoted by $\text{int}(F_D)$. That is

$$\text{int}(F_D) = \cup \{ F_{1D} : F_{1D} \text{ is neutrosophic soft open set, } F_{1D} \subseteq (F_D) \}.$$

4.8. Definition

Let $F_D \in N_3(K)$. The interior of F_D is intersection of all neutrosophic soft closed sets containing in F_D , denoted by $\text{cl}(F_D)$. That is

$$\text{cl}(F_D) = \cup \{ F_{1D} : F_{1D} \text{ is neutrosophic soft closed set, } F_{1D} \supseteq (F_D) \}.$$

4.9. Proposition

Let (K, μ) be $(N_3 - \text{Top})$, $F_D \in N_3(K)$. Then :

- 1) F_D is a neutrosophic soft open (closed) set if and only if $F_D = \text{int}(F_D)$ ($F_D = \text{cl}(F_D)$).
- 2) $\text{cl}((F_D)^c) = (\text{int}(F_D))^c$.
- 3) $\text{int}((F_D)^c) = (\text{cl}(F_D))^c$.

4.10. Proposition

Let $F_{1D}, F_{2D} \in N_3(K)$, Then :

1. $\text{int}(F_{1D}) \subseteq F_{1D}$.
2. $\text{int}(\text{int}(F_{1D})) = \text{int}(F_{1D})$.
3. $\text{int}(F_{1D}) \subseteq \text{int}(F_{2D})$, whenever $F_{1D} \subseteq F_{2D}$.
4. $\text{int}(F_{1D} \cap F_{2D}) = \text{int}(F_{1D}) \cap \text{int}(F_{2D})$.
5. $\text{int}(F_{1D} \cup F_{2D}) \supseteq \text{int}(F_{1D}) \cup \text{int}(F_{2D})$.
6. $F_{1D} \subseteq \text{cl}(F_{1D})$.
7. $\text{cl}(\text{cl}(F_{1D})) = \text{cl}(F_{1D})$.
8. $\text{cl}(F_{1D}) \subseteq \text{cl}(F_{2D})$, whenever $F_{1D} \subseteq F_{2D}$.

$$9. \quad \text{cl}(F_{1D} \cap F_{2D}) \subseteq \text{cl}(F_{1D}) \cap \text{cl}(F_{2D}) .$$

$$10. \quad \text{cl}(F_{1D} \cup F_{2D}) = \text{cl}(F_{1D}) \cup \text{cl}(F_{2D}) .$$

4.11. Remark

The converse of (property (1,3,6,9)) in above theorem is not true in general. It can be seen from the following examples.

4.12. Example

Let $K = \{q_1, q_2\}$, $D \subseteq E_K$, such that $D = \{p_1\}$ and $F_{1D}, F_{2D} \in N_3(K)$,
such that :

$$F_{1D} = \left\{ (p_1, \{ < q_1^{(0,5',0,5',0,5)}, q_2^{(0,4',0,4',0,4)} > \}) \right\} ,$$

$$F_{2D} = \left\{ (p_1, \{ < q_1^{(0,6',0,6',0,6)}, q_2^{(0,3',0,3',0,3)} > \}) \right\} ,$$

$$F_{3D} = \left\{ (p_1, \{ < q_1^{(0,5',0,5',0,6)}, q_2^{(0,3',0,3',0,4)} > \}) \right\} ,$$

$$F_{4D} = \left\{ (p_1, \{ < q_1^{(0,6',0,6',0,5)}, q_2^{(0,4',0,4',0,3)} > \}) \right\} ,$$

Then , $\mu = \{ \tilde{\emptyset}_D, \tilde{K}_D, F_{1D}, F_{2D}, F_{3D}, F_{4D} \}$ is a neutrosophic soft topology on K .

Note that :

- 1) $\text{int}(F_{1D} \cup F_{2D}) \not\subseteq \text{int}(F_{1D}) \cup \text{int}(F_{2D})$.
- 2) $F_{1D} \not\subseteq \text{int}(F_{1D})$

4.13. Example

Let $K = \{q_1, q_2\}$, $D \subseteq E_K$, such that $D = \{p_1\}$ and $F_{1D}, F_{2D} \in N_3(K)$,
such that :

$$F_{1D} = \left\{ (p_1, \{ < q_1^{(0,1',0,1',0,9)}, q_2^{(0,2',0,2',0,8)} > \}) \right\} ,$$

$$F_{2D} = \left\{ (p_1, \{ < q_1^{(0,9',0,9',0,1)}, q_2^{(0,8',0,8',0,2)} > \}) \right\} .$$

Then , $\mu = \{ \tilde{\emptyset}_D, \tilde{K}_D, F_{1D}, F_{2D} \}$ is a neutrosophic soft topology on K .

Note that :

- 1) $\text{cl}(F_{1D}) \cup \text{cl}(F_{2D}) \not\subseteq \text{cl}(F_{1D} \cup F_{2D})$.
- 2) $\text{cl}(F_{1D}) \not\subseteq F_{1D}$.

4.14. Definition

Let $F_D \in N_3(K)$. The neutrosophic soft exterior of F_D is denoted by $\text{ext}(F_D)$ and is defined as:
 $\text{ext}(F_D) = \text{int}((F_D)^c)$.

4.15. Definition

Let $F_D \in N_3(K)$. The neutrosophic soft boundary of F_D is denoted by $\text{br}(F_D)$ and is defined as:
 $\text{br}(F_D) = \text{cl}((F_D)^c) \cap (\text{cl}(F_D))$. It must be notion that $\text{br}(F_D) = \text{br}((F_D)^c)$.

4.16. Proposition

Let (K, μ) be $(N_3 - \text{Top})$, $F_D \in N_3(K)$. Then :

- 1) $\text{br}((F_D)^c) = \text{ext}(F_D) \sqcup \text{int}(F_D)$.
- 2) $\text{cl}(F_D) = \text{br}(F_D) \sqcup \text{int}(F_D)$.
- 3) $\text{br}(F_D) \cap \text{int}(F_D) = \tilde{\emptyset}_D$.
- 4) $\text{br}(\text{int}(F_D)) \subseteq \text{br}(F_D)$.

Proof : Straightforward .

4.17. Proposition

Let (K, μ) be $(N_3 - \text{Top})$, $F_D \in N_3(K)$. Then :

- 1) F_D is a neutrosophic soft open set $\leftrightarrow \text{br}(F_D) \cap (F_D) = \tilde{\emptyset}_D$.
- 2) F_D is a neutrosophic soft closed set $\leftrightarrow \text{br}(F_D) \subseteq (F_D)$.

Proof : Straightforward .

Conclusion

-The neutrosophic soft set theory is studied and discussed by introducing, new family of neutrosophic soft sets, new operations for neutrosophic soft set .

-The neutrosophic soft set theory is studied and discussed by introducing, new family of neutrosophic soft sets [namely third family], new operations for neutrosophic soft sets, comparison between the new family and other families, new definitions and examples.

- New definitions, characterization, the neutrosophic soft closure, the neutrosophic soft interior, the neutrosophic soft exterior and the neutrosophic soft boundary are introduced in details .

-We expect this research will promote the future study on theory of neutrosophic soft sets, the theory of neutrosophic soft topological spaces and many other general frameworks .

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Convex and Concave Hypersoft Sets with Some Properties

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Abstract. Convexity plays an imperative role in optimization, pattern classification and recognition, image processing and many other relating topics in different fields of mathematical sciences like operation research, numerical analysis etc. The concept of soft sets was first formulated by Molodtsov as a completely new mathematical tool for solving problems dealing with uncertainties. Smarandache conceptualized hypersoft set as a generalization of soft set (h_S, E) as it transforms the function h_S into a multi-attribute function h_{HS} . Deli introduced the concept of convexity cum concavity on soft sets to cover above topics under uncertain scenario. In this study, a theoretic and analytical approach is employed to develop a conceptual framework of convexity cum concavity on hypersoft set which is generalized and more effective concept to deal with optimization relating problems. Moreover, some generalized properties like δ -inclusion, intersection and union, are established. The novelty of this work is maintained with the help of illustrative examples and pictorial version first time in literature.

Keywords: Convex Soft Set; Concave Soft Set; hypersoft Set; convex hypersoft set; concave hypersoft set.

1. Introduction

The theories like theory of probability, theory of fuzzy sets, and the interval mathematics, are considered as mathematical means to tackle many Intricate problems involving various uncertainties in different fields of mathematical sciences. These theories have their own complexities which restrain them to solve these problems successfully. The reason for these hurdles is, possibly, the inadequacy of the parametrization tool. A mathematical tool is needed for dealing with uncertainties which should be free of all such Impediments. In 1999, Molodtsov [1] has the honor to introduce the such mathematical tool called soft sets in literature as a new

parameterized family of subsets of the universe of discourse. In 2003, Maji et al. [2] extended the concept and introduced some fundamental terminologies and operations like equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set, absolute soft set, AND, OR and also the operations of union and intersection. They verified De Morgan's laws and a number of other results too. In 2005, Pei et al. [3] discussed the relationship between soft sets and information systems. They showed the soft sets as a class of special information systems. In 2009, Ali et al. [4] pointed several assertions in previous work of Maji et al. and defined new notions such as the restricted intersection, the restricted union, the restricted difference and the extended intersection of two soft sets. In 2010, 2011, Babitha et al. [5,6] introduced the concepts of soft set relations as a sub soft set of the Cartesian product of the soft sets and also discussed many related concepts such as equivalent soft set relation, partition, composition and function. In 2011, Sezgin et al. [7], Ge et al. [8], Fuli [9] gave some modifications in the work of Maji et al. and also established some new results. Many researchers [10]- [19] developed certain hybrids with soft sets to get more generalized results for implementation in decision making and other related disciplines.

In 2013, Deli [20] defined soft convex and soft concave sets with some properties. In 2016, Majeed [21] investigated some more properties of convex soft sets. She developed the convex hull and the cone of a soft set with their generalized results. In 2018, Salih et al. [22] defined strictly soft convex and strictly soft concave sets and they discussed their properties.

In 2018, Smarandache [23] introduced the concept of hypersoft set and in 2020, M. Saeed et al. [24] extended the concept and discussed the fundamentals of hypersoft set such as hypersoft subset, complement, not hypersoft set, aggregation operators along with hypersoft set relation, sub relation, complement relation, function, matrices and operations on hypersoft matrices.

Convexity is an essential concept in optimization, recognition and classification of certain patterns, processing and decomposition of images, antismatroids, discrete event simulation, duality problems and many other related topics in operation research, mathematical economics, numerical analysis and other mathematical sciences. Deli provided a mathematical tool to tackle all such problems under soft set environment. Hypersoft set theory is more generalized than soft set theory so it's the need of the literature to carve out a conceptual framework for solving such kind of problems under more generalized version i.e. hypersoft set. Therefore, to meet this demand, an abstract and analytical approach is utilized to develop a basic framework of convexity and concavity on hypersoft sets along with some important results. Examples and pictorial version of convexity and concavity on hypersoft sets are presented first time in literature.

The rest of this article is structured as follows: Section 2 recalls some basic definitions and terms from literature to support main results. Section 3 discusses the main results i.e. convex

and concave hypersoft sets along with some generalized results. Section 4 concludes the paper and describes future directions. Throughout the paper, G , J^\bullet , Π and $P(\Pi)$, will play the role of R^n , unit interval, universal set and power set respectively.

2. Preliminaries

In this section, some fundamental terms regarding soft set, hypersoft set and their convexity-cum-concavity are presented.

Definition 2.1. [1](Soft Set)

Let Π be an initial universe set and let E be a set of parameters. A pair (h_S, E) is called a soft set over Π , where h_S is a mapping given by $h_S : E \rightarrow P(U)$. In other words, a soft set (h_S, E) over Π is a parameterized family of subsets of Π . For $\omega \in E$, $h_S(\omega)$ may be considered as the set of ω -elements or ω -approximate elements of the soft set (h_S, E) .

Definition 2.2. [2]

Let (f_S, A) and (g_S, B) be two soft sets over a common universe Π ,

- (1) we say that (f_S, A) is a **soft subset** of (g_S, B) denoted by $(f_S, A) \subseteq (g_S, B)$ if
 - i $A \subseteq B$, and
 - ii $\forall \omega \in A$, $f_S(\omega)$ and $g_S(\omega)$ are identical approximations.
- (2) the **union** of (f_S, A) and (g_S, B) , denoted by $(f_S, A) \cup (g_S, B)$, is a soft set (h_S, C) , where $C = A \cup B$ and $\omega \in C$,

$$h_S(\omega) = \begin{cases} f_S(\omega), & \omega \in A - B \\ g_S(\omega), & \omega \in B - A \\ f_S(\omega) \cup g_S(\omega), & \omega \in A \cap B \end{cases}$$

- (3) the **intersection** of (f_S, A) and (g_S, B) denoted by $(f_S, A) \cap (g_S, B)$, is a soft set (h_S, C) , where $C = A \cap B$ and $\omega \in C$, $h_S(\omega) = f_S(\omega)$ or $g_S(\omega)$ (as both are same set).

Definition 2.3. [2](Complement of Soft Set)

The complement of a soft set (h_S, A) , denoted by $(h_S, A)^c$, is defined as $(h_S, A)^c = (h_S^c, \neg A)$ where

$$h_S^c : \neg A \rightarrow P(\Pi)$$

is a mapping given by

$$h_S^c(\omega) = \Pi - h_S(\neg \omega) \forall \omega \in \neg A.$$

Definition 2.4. [23](Hypersoft Set)

Let Π be a universe of discourse, $P(\Pi)$ the power set of Π . Let $a_1, a_2, a_3, \dots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, A_3, \dots, A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, \dots, n\}$. Then the pair

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(h_{HS}, G) , where $G = A_1 \times A_2 \times A_3 \times \dots \times A_n$ and $h_{HS} : G \rightarrow P(\Pi)$ is called a hypersoft Set over Π .

Definition 2.5. [24](Union of Hypersoft Sets)

Let (Φ, G_1) and (Ψ, G_2) be two hypersoft sets over the same universal set Π , then their union $(\Phi, G_1) \cup (\Psi, G_2)$ is hypersoft set (h_{HS}, C) , where $C = G_1 \cup G_2$; $G_1 = A_1 \times A_2 \times A_3 \times \dots \times A_n$, $G_2 = B_1 \times B_2 \times B_3 \times \dots \times B_n$ and $\forall e \in C$ with

$$h_{HS}(e) = \begin{cases} \Phi(e), & e \in G_1 - G_2 \\ \Psi(e), & e \in G_2 - G_1 \\ \Phi(e) \cup \Psi(e), & e \in G_2 \cap G_1 \end{cases}$$

Definition 2.6. [24](Intersection of Hypersoft Sets)

Let (Φ, G_1) and (Ψ, G_2) be two hypersoft sets over the same universal set Π , then their intersection $(\Phi, G_1) \cap (\Psi, G_2)$ is hypersoft set (h_{HS}, C) , where $C = G_1 \cap G_2$; where $G_1 = A_1 \times A_2 \times A_3 \times \dots \times A_n$, $G_2 = B_1 \times B_2 \times B_3 \times \dots \times B_n$. and $\forall e \in C$ with $h_{HS}(e) = \Phi(e) \cap \Psi(e)$.

For more definition and results regarding hypersoft set, see [24–27].

Definition 2.7. [20](δ -inclusion)

The δ -inclusion of a soft set (h_S, Λ) (where $\delta \subseteq \Pi$) is defined by

$$(h_S, \Lambda)^\delta = \{\omega \in \Lambda : h_S(\omega) \supseteq \delta\}$$

Definition 2.8. [20](Convex Soft Set)

The soft set (h_S, Λ) on Λ is called a convex soft set if

$$h_S(\epsilon\omega + (1 - \epsilon)\mu) \supseteq h_S(\omega) \cap h_S(\mu)$$

for every $\omega, \mu \in \Lambda$ and $\epsilon \in J^\bullet$.

Definition 2.9. [20](Concave Soft Set)

The soft set (h_S, Λ) on Λ is called a concave soft set if

$$h_S(\epsilon\omega + (1 - \epsilon)\mu) \subseteq h_S(\omega) \cup h_S(\mu)$$

for every $\omega, \mu \in \Lambda$ and $\epsilon \in J^\bullet$.

For more about convex soft, see [20, 21].

3. Convex and Concave hypersoft sets

Here convex hypersoft sets and concave hypersoft sets are defined and some important results are proved.

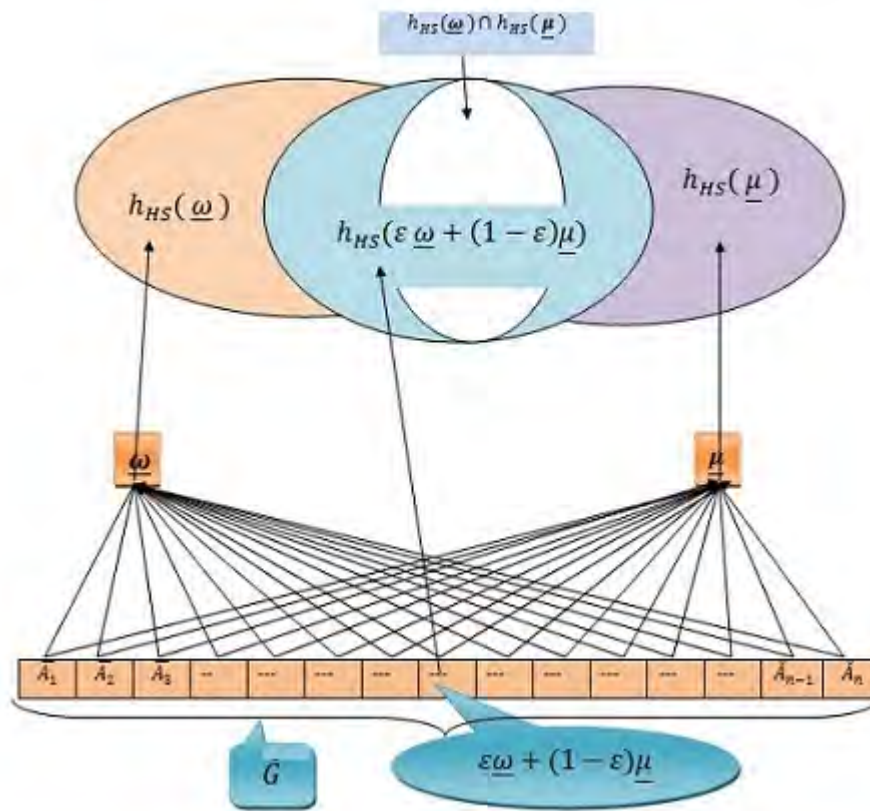


FIGURE 1. Convex hypersoft Set

Definition 3.1. δ -inclusion for hypersoft Set

The δ - inclusion of a hypersoft set (h_{HS}, G) (where $\delta \subseteq \Pi$) is defined by

$$(h_{HS}, G)^\delta = \{\underline{\omega} \in G : h_{HS}(\underline{\omega}) \supseteq \delta\}$$

Definition 3.2. Convex hypersoft Set

The hypersoft set (h_{HS}, G) is called a convex hypersoft set if

$$h_{HS}(\epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu}) \supseteq h_{HS}(\underline{\omega}) \cap h_{HS}(\underline{\mu})$$

for every $\underline{\omega}, \underline{\mu} \in G$ where, $G = A_1 \times A_2 \times A_3 \times \dots \times A_n$ with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, \dots, n\}$; $h_{HS} : G \rightarrow P(\Pi)$ and $\epsilon \in J^\bullet$.

Example 3.3. Suppose a university wants to observe(evaluate) the characteristics of its teachers by some defined indicators. For this purpose, consider a set of teachers as a universe of discourse $\Pi = \{t_1, t_2, t_3, \dots, t_{10}\}$. The attributes of the teachers under consideration are the set $\Lambda = \{A_1, A_2, A_3\}$, where

A_1 = Total experience in years

A_2 = Total no. of publications

A_3 = Student's evaluation against each teacher

such that the attributes values against these attributes respectively are the sets given as

$$A_1 = \{1year, 2years, 3years, 4years, 5years\}$$

$$A_2 = \{1, 2, 3, 4, 5\}$$

$$A_3 = \{Excellent(1), verygood(2), good(3), average(4), bad(5)\}$$

For simplicity, we write

$$A_1 = \{1, 2, 3, 4, 5\}$$

$$A_2 = \{1, 2, 3, 4, 5\}$$

$$A_3 = \{1, 2, 3, 4, 5\}$$

The hypersoft set (h_{HS}, G) is a function defined by the mapping $h_{HS} : G \rightarrow P(\Pi)$ where $G = A_1 \times A_2 \times A_3$.

Since the cartesian product of $A_1 \times A_2 \times A_3$ is a 3-tuple. we consider $\underline{\omega} = (2, 1, 3)$, then the function becomes $h_{HS}(\underline{\omega}) = h_{HS}(2, 1, 3) = \{t_1, t_5\}$. Also, consider $\underline{\mu} = (3, 2, 2)$, then the function becomes $h_{HS}(\underline{\mu}) = h_{HS}(3, 2, 2) = \{t_1, t_3, t_4\}$

Now

$$h_{HS}(\underline{\omega}) \cap h_{HS}(\underline{\mu}) = h_{HS}(\{2, 1, 3\}) \cap h_{HS}(\{3, 2, 2\}) = \{t_1, t_5\} \cap \{t_1, t_3, t_4\} = \{t_1\} \quad (1)$$

Let $\epsilon = 0.6 \in J^\bullet$, then, we have

$$\begin{aligned} \epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu} &= 0.6(2, 1, 3) + (10.6)(3, 2, 2) = 0.6(2, 1, 3) + 0.4(3, 2, 2) \\ &= (1.2, 0.6, 1.8) + (1.2, 0.8, 0.8) = (1.2 + 1.2, 0.6 + 0.8, 1.8 + 0.8) = (2.4, 1.4, 2.6) \end{aligned}$$

which is again a 3-tuple. By using the decimal round off property, we get $(2, 1, 3)$

$$h_{HS}(\epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu}) = h_{HS}(2, 1, 3) = \{t_1, t_5\} \quad (2)$$

it is vivid from equations (1) and (2), we have

$$h_{HS}(\epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu}) \supseteq h_{HS}(\underline{\omega}) \cap h_{HS}(\underline{\mu})$$

Theorem 3.4. $(f_{HS}, S) \cap (g_{HS}, T)$ is a convex hypersoft set when both (f_{HS}, S) and (g_{HS}, T) are convex hypersoft sets.

Proof. Suppose that $(f_{HS}, S) \cap (g_{HS}, T) = (h_{HS}, G)$ with $G = S \cap T$, for $\underline{\omega}_1, \underline{\omega}_2 \in G; \epsilon \in J^\bullet$, we have then

$$h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) = f_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \cap g_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2)$$

As (f_{HS}, S) and (g_{HS}, T) are convex hypersoft sets,

$$f_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \supseteq f_{HS}(\underline{\omega}_1) \cap f_{HS}(\underline{\omega}_2)$$

$$g_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \supseteq g_{HS}(\underline{\omega}_1) \cap g_{HS}(\underline{\omega}_2)$$

which implies

$$h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \supseteq (f_{HS}(\underline{\omega}_1) \cap f_{HS}(\underline{\omega}_2)) \cap (g_{HS}(\underline{\omega}_1) \cap g_{HS}(\underline{\omega}_2))$$

and thus

$$h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \supseteq h_{HS}(\underline{\omega}_1) \cap h_{HS}(\underline{\omega}_2)$$

□

Remark 3.5. If $\{(h_{HS}^i, G_i) : i \in \{1, 2, 3, \dots\}\}$ is any family of convex hypersoft sets, then the intersection $\bigcap_{i \in I} (h_{HS}^i, G_i)$ is a convex hypersoft set.

Remark 3.6. The union of any family $\{(h_{HS}^i, G_i) : i \in \{1, 2, 3, \dots\}\}$ of convex hypersoft sets is not necessarily a convex hypersoft set.

Theorem 3.7. (h_{HS}, G) is convex hypersoft set iff for every $\epsilon \in J^\bullet$ and $\delta \in P(\Pi)$, $(h_{HS}, G)^\delta$ is convex hypersoft set.

Proof. Suppose (h_{HS}, G) is convex hypersoft set. If $\underline{\omega}, \underline{\mu} \in G$ and $\delta \in P(\Pi)$, then $h_{HS}(\underline{\omega}) \supseteq \delta$ and $h_{HS}(\underline{\mu}) \supseteq \delta$, it implies that $h_{HS}(\underline{\omega}) \cap h_{HS}(\underline{\mu}) \supseteq \delta$.

So we have,

$$h_{HS}(\epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu}) \supseteq h_{HS}(\underline{\omega}) \cap h_{HS}(\underline{\mu}) \supseteq \delta$$

$$\Rightarrow h_{HS}(\epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu}) \supseteq \delta$$

thus $(h_{HS}, G)^\delta$ is convex hypersoft set.

Conversely suppose that $(h_{HS}, G)^\delta$ is convex hypersoft set for every $\epsilon \in J^\bullet$. For $\underline{\omega}, \underline{\mu} \in G$, $(h_{HS}, G)^\delta$ is convex hypersoft set with $\delta = h_{HS}(\underline{\omega}) \cap h_{HS}(\underline{\mu})$. Since $h_{HS}(\underline{\omega}) \supseteq \delta$ and $h_{HS}(\underline{\mu}) \supseteq \delta$, we have $\underline{\omega} \in (h_{HS}, G)^\delta$ and $\underline{\mu} \in (h_{HS}, G)^\delta$,
 $\Rightarrow \epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu} \in (h_{HS}, G)^\delta$.

Therefore,

$$h_{HS}(\epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu}) \supseteq \delta$$

So

$$h_{HS}(\epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu}) \supseteq h_{HS}(\underline{\omega}) \cap h_{HS}(\underline{\mu}),$$

Hence (h_{HS}, G) is convex hypersoft set. □

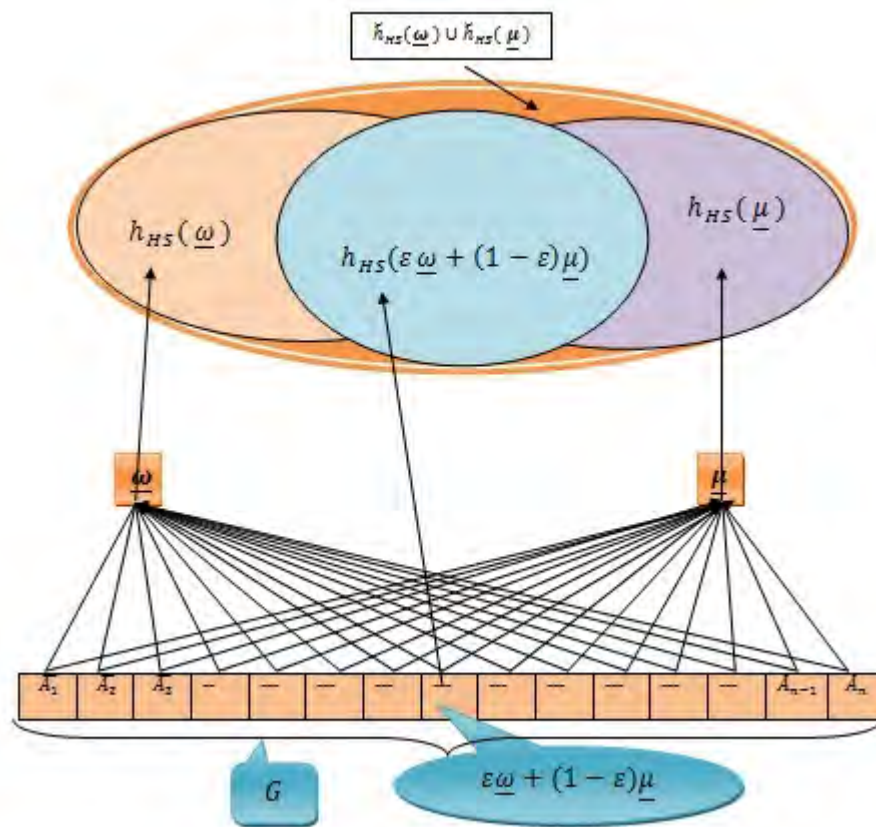


FIGURE 2. Concave hypersoft Set

Definition 3.8. Concave hypersoft Set

The hypersoft set (h_{HS}, G) on Λ is called a *concave hypersoft set* if

$$h_{HS}(\epsilon\omega + (1 - \epsilon)\mu) \subseteq h_{HS}(\omega) \cup h_{HS}(\mu)$$

for every $\omega = (A_1, A_2, A_3, \dots, A_n)$, $\mu = (B_1, B_2, B_3, \dots, B_n) \in G$ where, $G = A_1 \times A_2 \times A_3 \times \dots \times A_n$ with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, \dots, n\}$; $h_{HS} : G \rightarrow P(\Pi)$ and $\epsilon \in J^\bullet$.

Example 3.9. Considering given data in Example 3.3, we have

$$h_{HS}(\omega) \cup h_{HS}(\mu) = h_{HS}(\{2, 1, 3\}) \cup h_{HS}(\{3, 2, 2\}) = \{t_1, t_5\} \cup \{t_1, t_3, t_4\} = \{t_1, t_3, t_4, t_5\} \quad (3)$$

it is vivid from equations (2) and (3), we have

$$h_{HS}(\epsilon\omega + (1 - \epsilon)\mu) \subseteq h_{HS}(\omega) \cup h_{HS}(\mu)$$

Theorem 3.10. $(f_{HS}, S) \cup (g_{HS}, T)$ is a concave hypersoft set when both (f_{HS}, S) and (g_{HS}, T) are concave hypersoft sets.

Proof. Suppose that $(f_{HS}, S) \cup (g_{HS}, T) = (h_{HS}, G)$ with $G = S \cup T$, for $\underline{\omega}_1, \underline{\omega}_2 \in G; \epsilon \in J^\bullet$, we have then

$$h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) = f_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \cup g_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2)$$

As (f_{HS}, S) and (g_{HS}, T) are concave hypersoft sets,

$$f_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq f_{HS}(\underline{\omega}_1) \cup f_{HS}(\underline{\omega}_2)$$

$$g_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq g_{HS}(\underline{\omega}_1) \cup g_{HS}(\underline{\omega}_2)$$

which implies

$$h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq (f_{HS}(\underline{\omega}_1) \cup f_{HS}(\underline{\omega}_2)) \cup (g_{HS}(\underline{\omega}_1) \cup g_{HS}(\underline{\omega}_2))$$

and thus

$$h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq h_{HS}(\underline{\omega}_1) \cup h_{HS}(\underline{\omega}_2)$$

□

Remark 3.11. If $\{(h^i_{HS}, G_i) : i \in \{1, 2, 3, \dots\}\}$ is any family of concave hypersoft sets, then the union $\bigcup_{i \in I} (h^i_{HS}, G_i)$ is a concave hypersoft set.

Theorem 3.12. $(f_{HS}, S) \cap (g_{HS}, T)$ is a concave hypersoft set when both (f_{HS}, S) and (g_{HS}, T) are concave hypersoft sets.

Proof. Suppose that $(f_{HS}, S) \cap (g_{HS}, T) = (h_{HS}, G)$ with $G = S \cap T$, for $\underline{\omega}_1, \underline{\omega}_2 \in G; \epsilon \in J^\bullet$, we have then

$$h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) = f_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \cap g_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2)$$

As (f_{HS}, S) and (g_{HS}, T) are concave hypersoft sets,

$$f_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq f_{HS}(\underline{\omega}_1) \cup f_{HS}(\underline{\omega}_2)$$

$$g_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq g_{HS}(\underline{\omega}_1) \cup g_{HS}(\underline{\omega}_2)$$

which implies

$$h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq (f_{HS}(\underline{\omega}_1) \cup f_{HS}(\underline{\omega}_2)) \cap (g_{HS}(\underline{\omega}_1) \cup g_{HS}(\underline{\omega}_2))$$

and thus

$$h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq h_{HS}(\underline{\omega}_1) \cup h_{HS}(\underline{\omega}_2)$$

□

Remark 3.13. The intersection of any family $\{(h^i_{HS}, G_i) : i \in \{1, 2, 3, \dots\}\}$ of concave hypersoft sets is a concave hypersoft set.

Theorem 3.14. $(h_{HS}, G)^c$ is a convex hypersoft set when (h_{HS}, G) is a concave hypersoft set.

Proof. Suppose that for $\underline{\omega}_1, \underline{\omega}_2 \in G$, $\epsilon \in J^\bullet$ and (h_{HS}, G) be concave hypersoft set.

Since (h_{HS}, G) is concave hypersoft set,

$$h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq h_{HS}(\underline{\omega}_1) \cup h_{HS}(\underline{\omega}_2)$$

or

$$\Pi \setminus h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \supseteq \Pi \setminus \{h_{HS}(\underline{\omega}_1) \cup h_{HS}(\underline{\omega}_2)\}$$

If $h_{HS}(\underline{\omega}_1) \supset h_{HS}(\underline{\omega}_2)$ then $h_{HS}(\underline{\omega}_1) \cup h_{HS}(\underline{\omega}_2) = h_{HS}(\underline{\omega}_1)$ therefore,

$$\Pi \setminus h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \supseteq \Pi \setminus h_{HS}(\underline{\omega}_1). \quad (4)$$

If $h_{HS}(\underline{\omega}_1) \subset h_{HS}(\underline{\omega}_2)$ then $h_{HS}(\underline{\omega}_1) \cup h_{HS}(\underline{\omega}_2) = h_{HS}(\underline{\omega}_2)$ therefore,

$$\Pi \setminus h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \supseteq \Pi \setminus h_{HS}(\underline{\omega}_2). \quad (5)$$

From (4) and (5), we have

$$\Pi \setminus h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \supseteq (\Pi \setminus h_{HS}(\underline{\omega}_1)) \cap (\Pi \setminus h_{HS}(\underline{\omega}_2)).$$

So, $(h_{HS}, G)^c$ is a convex hypersoft set. \square

Theorem 3.15. $(h_{HS}, G)^c$ is a concave hypersoft set when (h_{HS}, G) is a convex hypersoft set.

Proof. Suppose that for $\underline{\omega}_1, \underline{\omega}_2 \in G$, $\epsilon \in J^\bullet$ and (h_{HS}, G) be convex hypersoft set.

since (h_{HS}, G) is convex hypersoft set,

$$h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \supseteq h_{HS}(\underline{\omega}_1) \cap h_{HS}(\underline{\omega}_2)$$

or

$$\Pi \setminus h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq \Pi \setminus \{h_{HS}(\underline{\omega}_1) \cap h_{HS}(\underline{\omega}_2)\}$$

If $h_{HS}(\underline{\omega}_1) \supset h_{HS}(\underline{\omega}_2)$ then $h_{HS}(\underline{\omega}_1) \cap h_{HS}(\underline{\omega}_2) = h_{HS}(\underline{\omega}_2)$ therefore,

$$\Pi \setminus h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq \Pi \setminus h_{HS}(\underline{\omega}_2). \quad (6)$$

If $h_{HS}(\underline{\omega}_1) \subset h_{HS}(\underline{\omega}_2)$ then $h_{HS}(\underline{\omega}_1) \cap h_{HS}(\underline{\omega}_2) = h_{HS}(\underline{\omega}_1)$ therefore,

$$\Pi \setminus h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \subseteq \Pi \setminus h_{HS}(\underline{\omega}_1). \quad (7)$$

From (6) and (7), we have

$$\Pi \setminus h_{HS}(\epsilon \underline{\omega}_1 + (1 - \epsilon) \underline{\omega}_2) \supseteq (\Pi \setminus h_{HS}(\underline{\omega}_1)) \cup (\Pi \setminus h_{HS}(\underline{\omega}_2)).$$

So $(h_{HS}, G)^c$ is a concave hypersoft set. \square

Theorem 3.16. (h_{HS}, G) is concave hypersoft set iff for every $\epsilon \in J^\bullet$ and $\delta \in P(\Pi)$, $(h_{HS}, G)^\delta$ is concave hypersoft set.

Proof. Suppose (h_{HS}, G) is concave hypersoft set. If $\underline{\omega}, \underline{\mu} \in G$ and $\delta \in P(\Pi)$, then $h_{HS}(\underline{\omega}) \supseteq \delta$ and $h_{HS}(\underline{\mu}) \supseteq \delta$, it implies that $h_{HS}(\underline{\omega}) \cup h_{HS}(\underline{\mu}) \supseteq \delta$.

So we have,

$$\delta \subseteq h_{HS}(\underline{\omega}) \cap h_{HS}(\underline{\mu}) \subseteq h_{HS}(\epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu}) \subseteq h_{HS}(\underline{\omega}) \cup h_{HS}(\underline{\mu})$$

$$\Rightarrow \delta \subseteq h_{HS}(\epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu})$$

thus $(h_{HS}, G)^\delta$ is concave hypersoft set.

Conversely suppose that $(h_{HS}, G)^\delta$ is concave hypersoft set for every $\epsilon \in J^\bullet$. For $\underline{\omega}, \underline{\mu} \in G$, $(h_{HS}, G)^\delta$ is concave hypersoft set with $\delta = h_{HS}(\underline{\omega}) \cup h_{HS}(\underline{\mu})$. Since $h_{HS}(\underline{\omega}) \subseteq \delta$ and $h_{HS}(\underline{\mu}) \subseteq \delta$, we have $\underline{\omega} \in (h_{HS}, G)^\delta$ and $\underline{\mu} \in (h_{HS}, G)^\delta$,

$$\Rightarrow \epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu} \in (h_{HS}, G)^\delta.$$

Therefore,

$$h_{HS}(\epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu}) \subseteq \delta$$

So

$$h_{HS}(\epsilon \underline{\omega} + (1 - \epsilon) \underline{\mu}) \subseteq h_{HS}(\underline{\omega}) \cup h_{HS}(\underline{\mu}),$$

Hence (h_{HS}, G) is concave hypersoft set. \square

4. Conclusion

In this study, convexity cum concavity on hypersoft sets, is conceptualized by adopting an abstract and analytical technique. This is novel addition in the literature and may enable the researchers to deal important applications of convexity under hypersoft environment with precise results. Moreover, some important results are established. Future work may include the introduction of strictly and strongly convexity cum concavity, convex hull, convex cone and many other types of convexity like (m, n) -convexity, ϕ -convexity, graded convexity, triangular convexity, concavoconvexity etc. on hypersoft set. It may also include the extension of this work by considering the modified versions of complement, intersection and union as discussed in [2]- [4].

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NeutroAlgebra of Neutrosophic Triplets using $\{Z_n, \times\}$

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Abstract. Smarandache in 2019 has generalized the algebraic structures to NeutroAlgebraic structures and AntiAlgebraic structures. In this paper, authors, for the first time, define the NeutroAlgebra of neutrosophic triplets group under usual $+$ and \times , built using $\{Z_n, \times\}$, n a composite number, $5 < n < \infty$, which are not partial algebras. As idempotents in Z_n alone are neutrals that contribute to neutrosophic triplets groups, we analyze them and build NeutroAlgebra of idempotents under usual $+$ and \times , which are not partial algebras. We prove in this paper the existence theorem for NeutroAlgebra of neutrosophic triplet groups. This proves the neutrals associated with neutrosophic triplet groups in $\{Z_n, \times\}$ under product is a NeutroAlgebra of triplets. We also prove the non-existence theorem of NeutroAlgebra for neutrosophic triplets in case of Z_n when $n = 2p, 3p$ and $4p$ (for some primes p). Several open problems are proposed. Further, the NeutroAlgebras of extended neutrosophic triplet groups have been obtained.

Keywords: neutrosophic triplets; neutrosophic extended triplets; neutrosophic triplet group; neutrosophic extended triplet group; NeutroAlgebra; partial algebra; NeutroAlgebra of neutrosophic triplets; NeutroAlgebra of neutrosophic extended triplets; AntiAlgebra

1. Introduction

The neutrosophic theory proposed by Smarandache in [1] has become a powerful tool in the study/analysis of real-world data as they are dominated by uncertainty, inconsistency, and indeterminacy. Neutrosophy deals with the neutralities and indeterminacies of real-world problems. The innovative concept of neutrosophic triplet groups was introduced by [2], which gives for any element a in $(G, *)$, the *anti*(a) and *neut*(a) satisfying conditions

$$a * \text{neut}(a) = \text{neut}(a) * a = a$$

and

$$a * anti(a) = anti(a) * a = neut(a)$$

where $neut(a)$ is not the identity element or the classical identity of the group. They call $(a, neut(a), anti(a))$ as the neutrosophic triplet group. These neutrosophic triplets built using Z_n are always symmetric about the neutral elements. For if $(a, neut(a), anti(a))$ is neutrosophic triplet then $(anti(a), neut(a), a)$ there by giving a perfect symmetry of a and $anti(a)$ about the $neut(a)$. The study of neutralities have been carried out by several researchers in neutrosophic algebraic structures like neutrosophic triplet rings, groups, neutrosophic quadruple vector spaces, neutrosophic semi idempotents, duplets and triplets in neutrosophic rings, neutrosophic triplet in biaglebras, neutrosophic triplet classical group and their applications, triplet loops, subgroups, cancellable semigroups and Abel-Grassman groupoids [2–24].

[13] has defined a classical group structure on these neutrosophic triplet groups and has obtained several interesting properties and given open conjectures. Smarandache [2] defined the Neutrosophic Extended Triplet, when the neutral element is allowed to be the classical unit element. Zhang et al has defined neutrosophic extended triplet group and have obtained several results in [25]. Later [26] have obtained some results on neutrosophic extended triplet groups with partial order defined on it. More results about neutrosophic triplet groups and neutrosophic extended triplet groups can be found in [25–32].

We in this paper study the very new notion of NeutroAlgebra introduced by [33]. Several interesting results are obtained in [12, 34–36], and they introduced Neutro BC Algebra and sub Neutro BI Algebra and so on. NeutroAlgebras and AntiAlgebras in the classical number systems were studied in [37].

Here we introduce NeutroAlgebra under the usual product and sum in case of idempotents in the semigroups $\{Z_n, \times\}$, n a composite number, $5 < n < \infty$. This study is very important for all the neutrosophic triplets in $\{Z_n, \times\}$, happen to be contributed only by the idempotents, which are the only neutrals in $\{Z_n, \times\}$. We obtain NeutroAlgebras under usual $+$ and \times in the case of neutrosophic triplet groups and neutrosophic extended triplet groups. It is pertinent to keep on record we define classical product on neutrosophic triplets, and they are classical groups under product of these triplets. This paper has six sections. Section one is introductory in nature, and basic concepts are recalled in section two. Section three obtains the existence and non-existence theorem on NeutroAlgebras under usual $+$ or \times using neutrosophic triplet groups. In section four, a similar study is carried out in the case of neutrosophic extended triplet groups. The fifth section provides a discussion on this topic, and the final section gives the conclusions based on our study and some open conjectures which will be taken for future research by the authors.

2. Basic Concepts

Here we recall some basic definitions which is important to make this paper a self contained one.

Definition 2.1. Let us assume that N is an empty set and with binary operation $*$ defined on it. N is called a neutrosophic triplet set (NTS) if for any $a \in N$, there exists a neutral of “a” (denoted by $\text{neut}(a)$), and an opposite of “a” (denoted by $\text{anti}(a)$) satisfying the following conditions:

$$\begin{aligned} a * \text{neut}(a) &= \text{neut}(a) * a = a \\ a * \text{anti}(a) &= \text{anti}(a) * a = \text{neut}(a). \end{aligned}$$

And, the neutrosophic triple is given by $(a, \text{neut}(a), \text{anti}(a))$.

In a neutrosophic triplet set $(N, *)$, $a \in N$, $\text{neut}(a)$ and $\text{anti}(a)$ may not be unique.

In the definition given in [2], the neutral element cannot be an unit element in the usual sense, and then this restriction is removed, using the concept of a neutrosophic extended triplet in [26].

The classical unit element can be regarded as a special neutral element. The notion of neutrosophic triplet groups and that of neutrosophic extended triplet groups are distinctly dealt with in this paper.

Definition 2.2. Let us assume that $(N, *)$ is a neutrosophic triplet set. Then, N is called a neutrosophic triplet group, if it satisfies:

- (1) Closure Law, i.e., $a * b \in N, \forall a, b \in N$;
- (2) Associativity, i.e., $(a * b) * c = a * (b * c), \forall a, b, c \in N$

A neutrosophic triplet group $(N, *)$ is said to be commutative, if $a * b = b * a, \forall a, b \in N$.

Let $\langle A \rangle$ be a concept (as in terms of attribute, idea, proposition, or theory). By the neutrosophication process, we split the non-empty space into three regions two opposite ones corresponding to $\langle A \rangle$ and $\langle \text{anti } A \rangle$, and one neutral (indeterminate) $\langle \text{neut } A \rangle$ (also denoted $\langle \text{neutro } A \rangle$) between the opposites, which may or may not be disjoint; depending on the application, but their union equals the whole space.

A NeutroAlgebra is an algebra that has at least one neutro operation or one neutro axiom (axiom that is true for some elements, indeterminate or false for the other elements) [33]. A partial algebra has at the minimum one partial operation, and all its axioms are classical. Through a theorem in [34], proved that NeutroAlgebra is a generalization of partial algebra, and also give illustrations of NeutroAlgebras that are not partial algebras. Boole has defined the Partial Algebra (based on Partial Function) as an algebra whole operation is partially well-defined, and partially undefined (this undefined goes under Indeterminacy with respect

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to NeutroAlgebra). Therefore, a Partial Algebra (Partial Function) has some elements for which the operation is undefined (not outer-defined). Similarly an AntiAlgebra is a nonempty set that is endowed with at least one anti-operation (or anti-function) or at least one anti-axiom.

3. NeutroAlgebras of neutrosophic triplets using $\{Z_n, \times\}$

Here for the first time authors build NeutroAlgebras using neutrosophic triplets group built using the modulo integers Z_n ; n a composite number. Neutrosophic triplet groups and extended neutrosophic triplet groups were studied by [25, 26]. First we define NeutroAlgebra using the non-trivial idempotents of Z_n , n a composite number. This study is mandatory as all the neutral elements of neutrosophic triplets build using Z_n are only the non-trivial idempotents of Z_n . Next we give the existence and non existence theorems in case of NeutroAlgebras for these neutrosophic triplet sets. We give some interesting properties about them. Further it is important to note unless several open conjectures about idempotents in Z_n given in [13], are solved or some progress is made in that direction it will not be possible to completely characterize NeutroAlgebras of the neutrosophic triplet groups or extended neutrosophic triplet groups. We will be using [13] to get NeutroAlgebras of idempotents and NeutroAlgebra of neutrosophic triplet sets. First we provide examples of NeutroAlgebra using subsets of the semigroup $\{Z_n, \times\}$ and then NeutroAlgebra of idempotents in $\{Z_n, \times\}$.

Example 3.1. Let $S = \{Z_{15}, \times\}$ be a semigroup under product modulo 15. Now consider the subset $A = \{5, 10, 14\} \in S$. The Cayley table for A is given in Table 1, where outer-defined elements are denoted by *od*.

TABLE 1. Cayley Table for A

\times	5	10	14
5	10	5	10
10	5	10	5
14	10	5	od

We see the table has outer-defined elements denoted by *od*. So A is a NeutroAlgebra which is not a partial algebra, since the operation 14×14 is outer-defined. $14 \times 14 \equiv 1 \pmod{15}$, but $1 \notin \{5, 10, 14\}$. Therefore Table 1 is only a NeutroAlgebra. Every subset of S need not be a NeutroAlgebra. For take $B = \{3, 6, 9, 12\}$ a subset in S . Consider the Cayley table for B is given in Table 2.

B is not a NeutroAlgebra as every term in the cell is defined and associativity axiom is totally true..

TABLE 2. Cayley Table for B

\times	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

Clearly B is a subsemigroup of S , in fact a group under \times modulo 15 with 6 as its multiplicative identity, so S is a Smarandache semigroup [10].

Consider $C = \{2, 7, 8\}$ a subset of S . The Cayley table for C is given in Table 3, this has every cell to be outer-defined.

TABLE 3. Cayley Table for C

\times	2	7	8
2	od	od	od
7	od	od	od
8	od	od	od

So C is not a NeutroAlgebra or a subsemigroup but an AntiAlgebra since the operation \times is totally outer-defined under \times modulo 15.

Thus we can categorically put forth the following facts.

Every classical algebraic structure A with binary operations defined on it is such that any proper subset B of A with inherited operation of A falls under the three categories;

- (1) B can be a proper substructure of a stronger structure of A with the inherited operations of A .
- (2) B can only be a NeutroAlgebra, which may be a Partial Algebra, when some operation is undefined, and all other operations are well-defined and all axioms are true.
- (3) B can be an AntiAlgebra when at least one operation is totally outer-defined. or at least one axiom is totally false.

Under these circumstances if one wants to get a NeutroAlgebra which is not a partial algebra for a proper subset of a classical algebraic structure one should exploit the special axioms satisfied by them, to this end we study the property of idempotents in the semigroup $\{Z_n, \times\}$.

We also in case of neutrosophic triplet group obtain a NeutroAlgebra which is not a partial algebra.

First we give examples of NeutroAlgebra which are not partial algebras using idempotents of the semigroup $S = \{Z_n, \times\}$.

Example 3.2. Let $S = \{Z_6, \times\}$ be the semigroup under product modulo 6. The nontrivial idempotents of S are $V = \{3, 4\}$. The Cayley table for V is given in Table 4,

TABLE 4. Cayley Table for V

\times	3	4
3	3	od
4	od	4

So V is a NeutroAlgebra under \times but not a partial algebra. For the same V define operation $+$ modulo 6, the Cayley table for V is given in Table 5 and V is AntiAlgebra and not a partial algebra either.

TABLE 5. Cayley Table for V

$+$	3	4
3	od	od
4	od	od

Suppose we take $W = \{0, 1, 3, 4\}$ the collection of trivial and non trivial idempotents of S , and if we take S as a whole set but study the idempotent axiom in W we see from Table 6.

TABLE 6. Cayley Table for W

\times	0	1	3	4
0	0	0	0	0
1	0	1	3	4
3	0	3	3	0
4	0	4	3	4

Suppose we find the Cayley table for W under $+$ we get the Cayley table given in the following Table 7.

W itself is a NeutroAlgebra under usual $+$ with several undefined terms. W under usual product is a subsemigroup of idempotents of S ; where as S under sum of idempotents is a NeutroAlgebra which is not a partial algebra under the axiom of the property of idempotency.

Now if we take for any subset of S the axiom of idempotent property we get NeutroAlgebras which are not partial algebras.

To this effect we provide an example.

TABLE 7. Cayley Table for W

+	0	1	3	4
0	0	1	3	4
1	1	od	4	od
3	3	4	0	1
4	4	od	1	od

Example 3.3. Let $S = \{Z_{42}, \times\}$ be the semigroup under product modulo 42. The trivial and non trivial idempotents of S are $B = \{0, 1, 7, 15, 21, 22, 28, 36\}$. We define $+$ modulo 42 on this set of idempotents keeping the resultant what we need is the axiom of idempotency. The Cayley table for B is given in Table 8.

TABLE 8. Cayley Table for B

+	0	1	7	15	21	22	28	36
0	0	1	7	15	21	22	28	36
1	1	od	od	od	22	od	od	od
7	7	od	od	22	28	od	od	1
15	15	od	22	od	36	od	1	od
21	21	22	28	36	0	1	7	15
22	22	od	od	od	1	od	od	od
28	28	od	od	1	7	od	od	22
36	36	od	1	od	15	od	22	od

Thus B is a NeutroAlgebra which is not a partial algebra under the axiom of idempotency. Thus we have a large class of NeutroAlgebras which are not partial algebras.

As the main theme of this paper is study of neutrosophic triplets using modulo integers $\{Z_n, \times\}$ and prove the existence theorem and non-existence theorem of NeutroAlgebra of neutrosophic triplet groups.

In view of all these we have the following existence theorem of NeutroAlgebra of neutrosophic triplets.

Theorem 3.4. Let $S = \{Z_n, \times\}$, n not a prime, $5 < n < \infty$. Let V be the collection of all non trivial idempotents that is all neutrals of S , where 0 and 1 are not in S . Then V under product is a NeutroAlgebra of triplets.

Proof. Let $W = \{w_1, w_2, \dots, w_t\}$ be the non trivial idempotents of S . It is proved in [13] that if W_i is the set of all neutrosophic triplets of a non trivial idempotent w_i in S which

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serves as the neutral for the collection W_i then $\{W_i, \times\}$ is a neutrosophic triplet classical group under usual product and i varies over all neutrals; $1 \leq i \leq t$. If V is the collection of all neutrosophic triplets (this V will include all W_i for different neutrals or non trivial idempotents in S), associated with $S = \{Z_n, \times\}$; then V is not closed under usual product [13] and there are many undefined elements under usual product so V is a NeutroAlgebra of neutrosophic triplets. Hence the claim. \square

In view of this we have the following partial non existence theorem of NeutroAlgebra of neutrosophic triplets under $+$ for Z_{np} where $n = 2, 3$ and 4 for some values of P provided in the Tables 9, 10 and 11. We have for Z_n , n a product of more than two primes can have NeutroAlgebra of neutrosophic triplets under $+$.

Theorem 3.5. *Let $S = \{Z_{np}, \times\}$; where $n = 2, 3$ and 4 , (p a specific prime and np is not a square of a prime, prime values refer Tables 9, 10, and 11) be a semigroup under product modulo np . If V denotes the collection of all idempotents associated with the non trivial idempotents of Z_{np} then $\{V, +\}$ is never a NeutroAlgebra of triplets for $n = 2, 3$ and 4 .*

Proof. Recall from [13] that there are two idempotents in all the three cases when $n = 2p$ or $3p$ or $4p$ given in Tables 9, 10 and 11. \square

TABLE 9. Idempotent table for Z_{2p}

S.no	Z_{2p}	p	$p+1$
1	Z_6	3	4
2	Z_{10}	5	6
3	Z_{14}	7	8
4	Z_{22}	11	12
5	Z_{26}	13	14
6	Z_{34}	17	18
7	Z_{38}	19	20
8	Z_{46}	23	24
9	Z_{58}	29	30

We see any sum of the idempotents is 1 and product is 0.

Here in Z_{3p} and Z_{4p} also sum of idempotents is 1 and that product is 0. Tables are provided for them [13]. In case of $2p$ the nontrivial idempotents are p and $p+1$, clearly under sum this is a set. Thus we have proved the non-existence of NeutroAlgebra of idempotents under ' $+$ '.

To this effect first provide an example.

TABLE 10. Idempotent table for Z_{3p}

S. No.	Z_{3p}	p	p + 1	2p	2p + 1
1	Z_{15}	-	6	10	-
2	Z_{21}	7	-	-	15
3	Z_{33}	-	12	22	-
4	Z_{39}	13	-	-	27
5	Z_{51}	-	18	34	-
7	Z_{57}	19	-	-	39
8	Z_{69}	-	24	46	-
9	Z_{159}	-	54	106	-

TABLE 11. Idempotent table for Z_{4p}

S. No.	Z_{4p}	p	p + 1	3p	3p + 1
1	Z_{12}	-	4	9	-
2	Z_{20}	5	-	-	16
3	Z_{28}	-	8	21	-
4	Z_{44}	-	12	33	-
5	Z_{52}	13	-	-	40
6	Z_{76}	-	20	57	-
7	Z_{212}	53	-	-	160
8	Z_{388}	97	-	-	292
9	Z_{332}	-	84	249	-

Example 3.6. Consider the semigroup $S = \{Z_{10}, \times\}$. The nontrivial idempotents of S which contribute to the neutrosophic triplet set are; $\{6, 5\}$ in Z_{10} . Consider the neutrosophic triplet set $V = \{(5, 5, 5), (6, 6, 6), (8, 6, 2), (2, 6, 8), (4, 6, 4)\}$. It is proved $V \setminus \{(5, 5, 5)\}$ is a neutrosophic triplet classical group under \times [13]. Now the Cayley table of V under usual product \times is given in Table 12.

TABLE 12. Cayley Table for V

\times	(5,5,5)	(6,6,6)	(8,6,2)	(2,6,8)	(4,6,4)
(5,5,5)	(5,5,5)	od	od	od	od
(6,6,6)	od	(6,6,6)	(8,6,2)	(2,6,8)	(4,6,4)
(8,6,2)	od	(8,6,2)	(4,6,4)	(6,6,6)	(2,6,8)
(2,6,8)	od	(2,6,8)	(6,6,6)	(4,6,4)	(8,6,2)
(4,6,4)	od	(4,6,4)	(2,6,8)	(8,6,2)	(6,6,6)

Clearly V is a NeutroAlgebra under usual product and not a partial algebra. Since we have not included the neutrals that is non trivial idempotents like 0 and 1 we have this to be only a NeutroAlgebra of triplets.

TABLE 13. Cayley Table for V

+	(5,5,5)	(6,6,6)	(8,6,2)	(2,6,8)	(4,6,4)
(5,5,5)	od	od	od	od	od
(6,6,6)	od	od	od	od	od
(8,6,2)	od	od	od	od	od
(2,6,8)	od	od	od	od	od
(4,6,4)	od	od	od	od	od

Thus the neutrosophic triplets collection yields only a set under addition where no pair of neutrosophic triplets gives under sum a neutrosophic triplet. Hence our claim no NeutroAlgebra neutrosophic triplets under addition. So V in Table 13 is an AntiAlgebra. Likewise the cases $3p$ and $4p$ from tables.

So if we include the non trivial idempotents 0 and 1 then we can get NeutroAlgebra of idempotents under $+$ which is carried out in the following section.

Example 3.7. Consider the semigroup $S = \{Z_{105}, \times\}$ under \times modulo 105. The non trivial idempotents are $V = \{15, 21, 36, 70, 85, 91\}$. Let M be the collection of all neutrosophic triplets using the idempotents in V . M contains elements say $\{(15, 15, 15), (21, 21, 21), (36, 36, 36), (30, 15, 60), (51, 36, 81)\}$, from the Cayley table of M under $+$ we see there are some undefined terms also given in Table 14.

TABLE 14. Cayley Table for M

+	(15,15,15)	(21,21,21)	(36,36,36)	(30,15,60)	(51,36,81)
(15,15,15)	od	(36,36,36)	od	od	od
(21,21,21)	(36,36,36)	od	od	(51,36,81)	od
(36,36,36)	od	od	od	od	od
(30,15,60)	od	(51,36,81)	od	od	od
(51,36,81)	od	od	od	od	od

Hence we have a NeutroAlgebra of neutrosophic triplets under $+$.

We propose some open problems in this regard in the final section of this paper.

Now we find ways to get NeutroAlgebra of neutrosophic triplets under $+$. The possibility is by using extended neutrosophic triplets group we can have for all Z_n , n any composite number

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NeutroAlgebra of neutrosophic triplets under $+$. Unless the conjectures proposed in [13] is solved complete characterization is not possible, only partial results and examples to that effect are possible.

In the following section we discuss NeutroAlgebra of extended neutrosophic triplet sets.

4. NeutroAlgebra of extended neutrosophic triplets using $\{Z_n, \times\}$

In this section we prove the existence of NeutroAlgebra of extended neutrosophic triplets using $\{Z_n, \times\}$, for more about extended neutrosophic triplets refer [2, 26] under both $+$ and \times . Throughout this section we assume the collection of idempotents contains both the trivial idempotents 1 and 0. It is thus mandatory the neutrosophic triplet set collection contains $(0, 0, 0)$ and $(1, 1, 1)$ apart from the neutrosophic triplets of the form $(a, 1, \text{anti } a = \text{inverse of } a)$, where a is in Z_n which has inverse in Z_n .

We first prove the collection of all trivial and non trivial idempotents in Z_n is a NeutroAlgebra under $+$ and also under \times .

Theorem 4.1. *Let $S = \{Z_n, \times\}$ be the semigroup under product modulo $n, 5 < n < \infty$. Let $V = \{\text{Collection of all idempotents in } Z_n \text{ including } 0 \text{ and } 1\}$.*

- (1) $V \setminus \{0, 1\}$ is a NeutroAlgebra of idempotents under \times modulo n .
- (2) V is a NeutroAlgebra of idempotents under $+$ mod n .

Proof. Consider $V \setminus \{0, 1\}$ for every x in $V \setminus \{1, 0\}$ is such that $x \times x = x$, so $V \setminus \{1, 0\}$ is a NeutroAlgebra under \times . Hence (1) is true.

Proof of (2): To show V is a NeutroAlgebra of idempotents under $+$. Since 0 is in V we have for every $x \in V$; $0 + x = x$ is in V , however we do not in general have the sum of two idempotents to be an idempotent. For instance $1 + 1 = 2$ is not an idempotent so $(V, +)$ has undefined elements, hence undefined. Thus (2) is proved. \square

We provide an example to this effect.

Example 4.2. Let $S = \{Z_{10}, n, \times\}$ be the semigroup under \times modulo 10. The trivial and non trivial idempotents are $V = \{0, 1, 5, 6\}$. It is easily verified V is a NeutroAlgebra under $+$, for $6 + 6 = 2$ modulo 10. However V is not a NeutroAlgebra under \times , but $V \setminus \{0, 1\}$ is a NeutroAlgebra under \times modulo 10. For $6 + 5 = 1$ modulo 10, so $V \setminus \{1, 0\}$ is a NeutroAlgebra.

Now the neutrosophic triplets of S associated with the idempotents V are $N = \{(0, 0, 0), (1, 1, 1), (5, 1, 5), (3, 1, 7), (7, 1, 3), (5, 5, 5), (6, 6, 6), (4, 6, 4), (2, 6, 8) \text{ and } (8, 6, 2)\}$. We see N under $+$ is a NeutroAlgebra, for $(1, 1, 1) + (7, 1, 3) = (8, 2, 4)$ is not in N . N is not a NeutroAlgebra under $+$. But $N \setminus \{(0, 0, 0), (1, 1, 1), (5, 1, 5), (3, 1, 7), (7, 1, 3)\} = W$ neutrosophic triplets formed by the non trivial idempotents 5 and 6 is a NeutroAlgebra as (5,

$5, 5) \times (2, 6, 8) = (0, 0, 0)$ which is not in W . Hence the claim. If $\{(0, 0, 0)\}$ is added, then the set V becomes a NeutroAlgebra under $+$.

TABLE 15. Cayley Table for V

$+$	$(0, 0, 0)$	$(5, 5, 5)$	$(6, 6, 6)$	$(8, 6, 2)$	$(2, 6, 8)$	$(4, 6, 4)$
$(0, 0, 0)$	$(0, 0, 0)$	$(5, 5, 5)$	$(6, 6, 6)$	$(8, 6, 2)$	$(2, 6, 8)$	$(4, 6, 4)$
$(5, 5, 5)$	$(5, 5, 5)$	od	od	od	od	od
$(6, 6, 6)$	$(6, 6, 6)$	od	od	od	od	od
$(8, 6, 2)$	$(8, 6, 2)$	od	od	od	od	od
$(2, 6, 8)$	$(2, 6, 8)$	od	od	od	od	od
$(4, 6, 4)$	$(4, 6, 4)$	od	od	od	od	od

Theorem 4.3. Let $S = \{Z_n, \times\}$ be a semigroup under \times modulo n , where n is not a prime and $5 < n < \infty$. Let $N = \{\text{collection of all extended neutrosophic triplet set including } (0, 0, 0), \text{ and all neutrosophic triplets associated with the trivial idempotent } 1\}$.

- (1) N is a NeutroAlgebra under $+$ of extended neutrosophic triplets set.
- (2) $N \setminus \{(0, 0, 0)\}$ is a NeutroAlgebra of extended neutrosophic triplet set under product modulo n .

Proof. Let N be the collection of all extended neutrosophic triplets including $(0, 0, 0)$ and $(1, 1, 1)$ and other triplets associated with the neutral 1.

Proof of (1): In the case extended triplet N we see sum of two idempotents need not be idempotent for $(1, 1, 1) + (1, 1, 1) = (2, 2, 2)$ is not in N , hence N is the NeutroAlgebra of extended neutrosophic triplets which is not a partial algebra as the axiom of neutrosophic triplets is not satisfied.

Proof of (2): Consider $N \setminus \{(0, 0, 0)\}$. Clearly in general the product of any two idempotents is not an idempotent in Z_n , and several triplets are undefined and do not in general satisfy the triplet relation [13]. Hence the claim. \square

5. Discussions

The study of NeutroAlgebra introduced by [33] is very new, here the authors built NeutroAlgebra using idempotents of $\{Z_n, \times\}$ a semigroup under \times modulo n for appropriate n which are not partial algebras. Likewise NeutroAlgebra built using neutrosophic triplets set and extended neutrosophic triplets set. Some open problems based on our study is proposed in the section on conclusions.

6. Conclusions

For the first time authors have NeutroAlgebra using idempotents of a semigroup $S = \{Z_n, \times\}$; n a composite number $5 < n < \infty$, neutrosophic triplets and extended neutrosophic triplets. We have obtained NeutroAlgebras of idempotents which are not partial algebras under the classical operation of $+$ and \times only using $S = \{Z_n, \times\}$, the semigroup under product for appropriate n . We have obtained both existence and non-existence theorem for NeutroAlgebras of idempotents in S . We suggest certain open problems for researchers as well as these problems will be taken by the authors for future study.

Problem 1: Does there exist a n (n a composite number) such that using $\{Z_n, \times\}$ there are no non trivial NeutroAlgebra of neutrosophic triplet set and NeutroAlgebra in extended neutrosophic triplet set?

Problem 2: Does there exist a n , n a composite number such that $\{Z_n, \times\}$ has its collection of trivial and non trivial idempotents denoted by N to be such that;

- $(N, +)$ is not NeutroAlgebra of idempotents ?
- (N, \times) is not a NeutroAlgebra of idempotents?

Problem 3: Prove in case of $\{Z_{3p}, \times\}$ and $\{Z_{4p}, \times\}$, the idempotents are only of the form mentioned in Tables 10 and 11 respectively.

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AH-Homomorphisms in Neutrosophic Rings and Refined Neutrosophic Rings

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Abstract: Algebraic relations between rings are determined by homomorphisms and isomorphisms. This paper introduces a new kind of algebraic functions between two neutrosophic rings to give more agility in the exploring of neutrosophic substructures properties, where it generalizes the concept of AHS-homomorphism in neutrosophic rings, and refined neutrosophic rings. Also, it determines the algebraic structure of neutrosophic AH-endomorphisms of the additive group of neutrosophic ring and refined neutrosophic ring.

Keywords: AH-homomorphism , AH-endomorphism , AH-ideal, AHS-ideal.

1. Introduction

Theory of neutrosophic algebra began with Smarandache and Kandasamy in [8]. They introduced interesting notions such as neutrosophic groups, neutrosophic rings, and neutrosophic loops. Recently, neutrosophic sets and their according concepts have many important applications in health care and Covid-19 identification, industry, optimization, and decision making algorithms [23,24,25,26,27].

More studies in neutrosophic algebra were carried out with a generalized view. The concepts of refined neutrosophic rings, and n-refined neutrosophic rings were defined and studied in [1,2,3,4,7]. Abobala and Smarandache presented AH-substructures in neutrosophic rings, refined neutrosophic rings, neutrosophic vector spaces, modules and n-refined neutrosophic rings. See

[1,2,9,12,13,17,18,19,20,22]. Concepts such as AH-ideal, AHS-ideal, AHS-homomorphism, and AHS-isomorphism were defined and handled. These structures in general consists of many similar algebraic objects which help us to build a bridge between classical and neutrosophical algebra. For

example, AH-ideal in a neutrosophic ring $R(I)$ is a set with form $P = Q + SI$ where Q, S are ideals in the classical ring R . If $Q=S$ we get an AHS-ideal. AHS-homomorphism is a well defined map

$f: R(I) \rightarrow T(I); f(a + bI) = f_R(a) + f_R(b)I$, where f_R is a homomorphism between R and T . It can be understood as a ring homomorphism with two equal parts, each one is a classical homomorphism.

AHS-homomorphisms play an important role in the study of AHS-ideals (subsets with two equal classical ideals parts) in neutrosophic rings and refined neutrosophic rings. Thus we need weaker conditions and a generalized view to study the properties of such ideals with two different parts. For this goal, we will discuss the concept of AH-homomorphisms which are considered as a natural generalization of AHS-homomorphisms to deal with more complex substructures in neutrosophic rings and refined neutrosophic rings.

In this article we define AH-homomorphism to be a well defined map with different parts which are not supposed to be equal homomorphisms (weaker condition). These homomorphisms lead to better comprehension of some complex neutrosophic structures, especially AH-ideals. We study AH-homomorphisms in neutrosophic rings, and refined neutrosophic rings. Also, we determine the algebraic structure of neutrosophic AH-endomorphisms of the additive group of neutrosophic ring and refined neutrosophic ring.

Motivation

In the literature, AHS-homomorphism was a tool to investigate AHS-ideals, kernels and factors properties. From this point, the motivation of our work is to present and study a stronger tool (AH-homomorphism) to deal with complex neutrosophic substructures, which allows us to explore more interesting properties of AH-ideals, and kernels in neutrosophic rings and refined neutrosophic rings.

Also, these kinds of algebraic functions have an interesting structure itself. We will prove that endomorphisms of this kind has a structure of neutrosophic/refined neutrosophic ring respectively.

2. Preliminaries

Definition 2.1:[8]

Let $(R, +, \times)$ be a ring, $R(I) = \{a + bI \mid a, b \in R\}$ is called the neutrosophic ring, where I is a neutrosophic element with condition $I^2 = I$.

Definition 2.2:[4]

Let $(R, +, \times)$ be a ring, $(R(I_1, I_2), +, \times)$ is called a refined neutrosophic ring generated by R, I_1, I_2 .

Definition 2.3: [2]

Let $(R(I_1, I_2), +, \cdot)$ be a refined neutrosophic ring and P_0, P_1, P_2 be ideals in the ring R then the set

$P = (P_0, P_1 I_1, P_2 I_2) = \{(a, bI_1, cI_2) : a \in P_0, b \in P_1, c \in P_2\}$ is called a refined neutrosophic AH-ideal.

If $P_0 = P_1 = P_2$ then P is called a refined neutrosophic AHS-ideal.

Definition 2.4: [12]

Let $(R, +, \times)$ be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define

$R_n(I) = \{a_0 + a_1 I + \cdots + a_n I_n \mid a_i \in R\}$ to be n -refined neutrosophic ring.

Addition and multiplication on $R_n(I)$ are defined as:

$$\sum_{i=0}^n x_i I_i + \sum_{i=0}^n y_i I_i = \sum_{i=0}^n (x_i + y_i) I_i, \sum_{i=0}^n x_i I_i \times \sum_{i=0}^n y_i I_i = \sum_{i,j=0}^n (x_i \times y_j) I_i I_j.$$

Where \times is the multiplication defined on the ring R .

Definition 2.5 :

Let $(R, +, \cdot), (T, +, \cdot)$ be two rings and $f_R: R \rightarrow T$ is a homomorphism:

The map $f: R(I_1, I_2) \rightarrow T(I_1, I_2); f(x, yI_1, zI_2) = (f_R(x), f_R(y)I_1, f_R(z)I_2)$ is called an

AHS-homomorphism.

It is easy to see that $\forall x, y \in R(I_1, I_2)$ then $f(x + y) = f(x) + f(y), f(xy) = f(x) \cdot f(y)$.

Definition 2.6: [2]

(a) Let $f: R(I_1, I_2) \rightarrow T(I_1, I_2)$ be an AHS-homomorphism we define AH-Kernel of f by :

$$AH - Ker f = \{ (a, bI_1, cI_2); a, b, c \in Ker f_R \} = (Ker f_R, Ker f_R I_1, Ker f_R I_2)$$

(b) let $S = (S_0, S_1 I_1, S_2 I_2)$ be a subset of $R(I_1, I_2)$, then :

$$f(S) = (f_R(S_0), f_R(S_1)I_1, f_R(S_2)I_2) = \{(f_R(a_0), f_R(a_1)I_1, f_R(a_2)I_2); a_i \in S_i\}.$$

Definition 2.7: [2]

Let $(R(I_1, I_2), +, \cdot)$ be a refined neutrosophic ring and $P = (P_0, P_1 I_1, P_2 I_2)$ be an AH-ideal:

(a) We say that P is a weak prime AH-ideal if $P_i; i \in \{0, 1, 2\}$ are prime ideals in R .

(b) We say that P is a weak maximal AH-ideal if $P_i; i \in \{0, 1, 2\}$ are maximal ideals in R .

(c) We say that P is a weak principal AH-ideal if $P_i; i \in \{0, 1, 2\}$ are principal ideals in R .

3. Main concepts and results

Definition 3.1:

Let R, T be two rings and $f, g, h: R \rightarrow T$ be three homomorphisms:

(a) The map $[f, g]: R(I) \rightarrow T(I); [f, g](a + bI) = f(a) + g(b)I$ is called an AH-homomorphism.

(b) The map $[f, g, h]: R(I_1, I_2) \rightarrow T(I_1, I_2); [f, g, h](a, bI_1, cI_2) = (f(a), g(b)I_1, h(c)I_2)$ is called a refined AH-homomorphism.

(c) If f, g, h are isomorphisms, then $[f, g], [f, g, h]$ are called AH-isomorphism and refined AH-isomorphism respectively.

$$(d) AH - Ker [f, g] = Ker (f) + (Ker (g))I.$$

$$(e) AH - ker [f, g, h] = (Ker(f), (Ker(g))I_1, (Ker(h))I_2).$$

Remark 3.2:

An AH-homomorphism is not supposed to be a neutrosophic homomorphism. See Example 3.11 in [1].

A refined AH-homomorphism is not supposed to be a refined neutrosophic homomorphism . See Example 3.3 in [2].

It is easy to see that if $f = g$, we get the concept of AHS-homomorphism in a neutrosophic ring $R(I)$.

Also, we find that if $f = g = h$, we get the concept of AHS-homomorphism in a refined

neutrosophic ring $R(I_1, I_2)$.

Theorem 3.3:

Let $R(I), T(I)$ be two neutrosophic rings and $[f, g]: R(I) \rightarrow T(I)$ be an AH-homomorphism:

$$(a) [f, g](R(I)) = f(R) + g(R)I.$$

$$(b) \forall x, y \in R(I), \text{ we have } [f, g](x + y) = [f, g](x) + [f, g](y) \text{ and } [f, g](x \cdot y) = [f, g](x) \cdot [f, g](y).$$

$$(c) \text{ If } P = Q + SI \text{ is an AH-ideal of } R(I), [f, g](P) \text{ is an AH-ideal of } T(I).$$

$$(d) AH - Ker [f, g] \text{ is an AH-ideal of } R(I).$$

Proof :

(a) It is simple.

(b) Let $x = a + bI, y = c + dI$. We have:

$$[f, g](x + y) = f(a + c) + g(b + d)I = f(a) + f(c) + [g(b) + g(d)]I = [f, g](x) + [f, g](y).$$

$$[f, g](x \cdot y) = f(a \cdot c) + g(a \cdot d + b \cdot c + b \cdot d)I =$$

$$f(a) \cdot f(c) + [g(a) \cdot g(d) + g(b) \cdot g(c) + g(b) \cdot g(d)]I = [f, g](x) \cdot [f, g](y).$$

(c) It is clear that $[f, g](P + QI) = f(P) + g(Q)I$, since $f(P), g(Q)$ are ideals in T , we get

$$f(P) + g(Q)I \text{ is an AH-ideal of } T(I).$$

(d) Since $\text{Ker}(f), \text{Ker}(g)$ are ideals of R we find that $\text{AH} - \text{Ker}[f, g] = \text{Ker } f + (\text{Ker } g)I$ is an AH-ideal of $R(I)$.

Theorem 3.4:

Let $R(I_1, I_2), T(I_1, I_2)$ be two refined neutrosophic rings and $[f, g, h]: R(I_1, I_2) \rightarrow T(I_1, I_2)$ be a refined AH-homomorphism, we have

$$(a) [f, g, h](R(I_1, I_2)) = (f(R), g(R)I_1, h(R)I_2).$$

$$(b) \forall x, y \in R(I_1, I_2), \text{ we have } [f, g, h](x + y) = [f, g, h](x) + [f, g, h](y) \text{ and}$$

$$[f, g, h](xy) = [f, g, h](x) \cdot [f, g, h](y).$$

(c) If $P = (Q, S I_1, M I_2)$ is an AH-ideal of $R(I_1, I_2)$, then $[f, g, h](P)$ is an AH-ideal of $T(I_1, I_2)$.

(d) $\text{AH} - \text{Ker}[f, g, h]$ is an AH-ideal of $R(I_1, I_2)$.

Proof :

The proof is similar to Theorem 3.3 .

Definition 3.5:

(a) Let $R(I)$ be a neutrosophic ring. The set of all AH-homomorphisms $[f, g]: R(I) \rightarrow R(I)$ is called AH-endomorphisms of $R(I)$. We denote it by $\text{AH} - \text{END}(R(I))$.

(b) Let $R(I_1, I_2)$ be a refined neutrosophic ring. The set of all refined AH-homomorphisms

$[f, g, h]: R(I_1, I_2) \rightarrow R(I_1, I_2)$ is called refined AH-endomorphisms of $R(I_1, I_2)$. We denote it by

$$\text{AH} - \text{END}(R(I_1, I_2)).$$

Theorem 3.6:

Let $R(I), T(I)$ be two neutrosophic rings and $[f, g]: R(I) \rightarrow T(I)$ is an AH-homomorphism and

$P = Q + SI$ is an AH-ideal of $R(I)$, where $\text{AH} - \text{Ker}[f, g] \leq P \neq R(I)$. We have

(a) P is a weak principal in $R(I)$ if and only if $[f, g](P)$ is weak principal in $T(I)$.

(b) P is a weak prime in $R(I)$ if and only if $[f, g](P)$ is weak prime in $T(I)$.

(c) P is a weak maximal in $R(I)$ if and only if $[f, g](P)$ is weak maximal in $T(I)$.

Proof :

(a) Since $[f, g](P) = f(Q) + g(S)I$, and $f(Q), g(S)$ are principal ideals in the ring T if and only if Q, S are principal in R , thus $[f, g](P)$ is weak principal in $T(I)$ if and only if P is weak principal in $R(I)$.

(b) Since $[f, g](P) = f(Q) + g(S)I$, and $f(Q), g(S)$ are prime ideals in the ring T if and only if Q, S are prime in R , thus $[f, g](P)$ is weak prime in $T(I)$ if and only if P is weak prime in $R(I)$.

(c) Since $[f, g](P) = f(Q) + g(S)I$, and $f(Q), g(S)$ are maximal ideals in the ring T if and only if Q, S are maximal in R , thus $[f, g](P)$ is weak maximal in $T(I)$ if and only if P is weak maximal in $R(I)$.

Theorem 3.7:

Let $R(I_1, I_2), T(I_1, I_2)$ be two refined neutrosophic rings and $[f, g, h]: R(I) \rightarrow T(I)$ is a refined

AH-homomorphism and $P = (Q, S, I_1, M, I_2)$ is a refined AH-ideal of $R(I)$, where

$AH - Ker [f, g, h] \leq P \neq R(I_1, I_2)$. Then :

(a) P is a weak principal in $R(I_1, I_2)$ if and only if $[f, g, h](P)$ is weak principal in $T(I_1, I_2)$.

(b) P is a weak prime in $R(I_1, I_2)$ if and only if $[f, g, h](P)$ is weak prime in $T(I_1, I_2)$.

(c) P is a weak maximal in $R(I_1, I_2)$ if and only if $[f, g, h](P)$ is weak maximal in $T(I_1, I_2)$.

Proof :

Since $[f, g, h](P) = (f(Q), g(S)I_1, h(M)I_2)$, we get the proof by similar argument to Theorem 3.6.

Example 3.8:

The following example clarifies the concept of AH-homomorphism between two neutrosophic rings.

Let $R = Z$ be the ring of integers, $T = Z_6$ be the ring of integers modulo 6, we have

(a) $f: R \rightarrow T; f(x) = x \bmod 6, g: R \rightarrow T; g(x) = 3x \bmod 6$ are two homomorphisms.

(b) $[f, g]: R(I) \rightarrow T(I); [f, g](x + yI) = f(x) + g(x)I = (x \bmod 6) + (3y \bmod 6)I$ is the corresponding AH-homomorphism.

(c) $[f, g](R(I)) = f(R) + g(R)I = Z_6 + \{0, 3\}I = \{0, 1, 2, 3, 4, 5, 3I, 1 + 3I, 2 + 3I, 3 + 3I, 4 + 3I, 5 + 3I\}$.

(d) $AH - Ker [f, g] = Ker(f) + Ker(g)I = 6Z + 2ZI = \{6x + 2yI; x, y \in Z\}$.

(e) We have $Q = \langle 3 \rangle, S = \langle 2 \rangle$ are two principal, maximal, and prime ideals in R ,

$$\text{Ker}(f) = 6\mathbb{Z} \leq Q \text{ and } \text{Ker}(g) = 2\mathbb{Z} \leq S,$$

$P = Q + SI$ is weak principal, maximal, and prime ideal in $R(I)$, $f(Q) = \{0, 3\}, g(S) = \{0\}$,

$[f, g](P) = f(Q) + g(S)I = \{0, 3\} + \{0\}I = \{0, 3\}$ which is a weak principal, maximal, and prime AH-ideal of $T(I)$.

Example 3.9:

This example clarifies the concept of AH-homomorphism between two refined neutrosophic rings.

Let $R = \mathbb{Z}$ be the ring of integers, $T = \mathbb{Z}_6$ be the ring of integers modulo 6, we have:

(a) $f: R \rightarrow T; f(x) = x \bmod 6, g: R \rightarrow T; g(x) = 3x \bmod 6, h: R \rightarrow T; h(x) = 4x \bmod 6$ are three homomorphisms.

(b) We have $Q = \langle 2 \rangle, S = \langle 3 \rangle$, are two principal, maximal, and prime ideals in R ,

$P = (Q, QI_1, SI_2) = \{(2x, 2yI_1, 3zI_2); x, y, z \in \mathbb{Z}\}$ is a weak principal, maximal, and prime AH-ideal of $R(I_1, I_2)$, $f(Q) = \{0, 2, 4\}, g(Q) = \{0\}, h(S) = \{0\}$.

(c) $[f, g, h](P) = (f(Q), g(Q)I_1, h(S)I_2) = \{(0, 0, 0), (2, 0, 0), (4, 0, 0)\}$ is a weak principal, maximal, and prime AH-ideal of $T(I_1, I_2)$.

(d) $AH - \text{Ker}[f, g, h] = (\text{Ker}(f), \text{Ker}(g)I_1, \text{Ker}(h)I_2) = (6\mathbb{Z}, 2\mathbb{Z}I_1, 3\mathbb{Z}I_2) = \{(6x, 2yI_1, 3zI_2); x, y, z \in \mathbb{Z}\}.$

4. Algebraic structure of some endomorphisms

Remark 4.1:

A famous result in classical ring theory ensures that ring endomorphisms do not have a structure of a ring, since the addition of two endomorphisms does not preserve multiplication, but if we consider the additive group of any ring R , then group endomorphisms together form a non-commutative ring in general.

To study algebraic structures of AH-endomorphisms we introduce the following definition:

Definition 4.2:

Let R be any ring, consider $R(I), R(I_1, I_2)$, the corresponding neutrosophic ring, and refined neutrosophic ring, respectively. We define:

(a) The set of all AH-homomorphisms on the additive group $(R(I), +)$ is denoted by

$$AAH - END(R(I)).$$

(b) The set of all AH-homomorphisms on the additive group $(R(I_1, I_2), +)$ is denoted by

$$RAAH - END(R(I_1, I_2)).$$

Now we define algebraic binary operations on these sets.

Definition 4.3:

(a) Let $R(I)$ be a neutrosophic ring. Addition and multiplication on $AAH-END(R(I))$ are defined as:

$$\text{Addition : } [f, g] + [h, t] = [f + h, g + t].$$

$$\text{Multiplication : } [f, g] \cdot [h, t] = [foh, fot + goh + got].$$

Where $f, g, h, t \in END(R, +)$.

(b) Let $R(I_1, I_2)$ be a refined neutrosophic ring. Addition and multiplication on

$RAAH-END(R(I_1, I_2))$ are defined as:

$$\text{Addition: } [f, g, m] + [h, t, n] = [f + h, g + t, m + n].$$

Multiplication:

$$[f, g, m] \cdot [h, t, n] = [foh, fot + got + goh + gon + mot, mon + fon + moh].$$

Where $f, g, h, m, n, t \in END(R, +)$.

Theorem 4.4:

Let $(R(I), +)$ be the additive abelian group of a neutrosophic ring $R(I)$. Then $AAH - END(R(I))$ is a ring.

Proof :

Suppose that $[f, g], [h, t], [k, s]$ are three arbitrary elements in $AAH-END(R(I))$, we have :

$(AAH - END(R(I)), +)$ is an abelian group clearly.

Multiplication is distributive with respect to addition since:

$$[f, g] \cdot ([h, t] + [k, s]) = [fo(h + s), fo(t + s) + go(h + k) + go(t + s)] =$$

$$[f, g] \cdot [h, t] + [f, g] \cdot [k, s].$$

Multiplication is associative:

$$[f, g] \cdot ([h, t] \cdot [k, s]) = [f, g] \cdot [hok, hos + tok + tos] =$$

$$[fohok, fo(hos + tok + tos) + gohok + go(hos + tok + tos)] = ([f, g], [h, t]), [k, s].$$

It is easy to check that multiplication and addition are well defined.

So, our proof is complete.

Theorem 4.5:

Let $(R(I_1, I_2), +)$ be the additive group of a refined neutrosophic ring. Then RAAH-END $R(I_1, I_2)$ is a ring.

Proof :

It is easy to see that $(\text{RAAH-END } R(I_1, I_2), +)$ is an abelian group.

Multiplication and addition are well defined clearly.

Multiplication is distributive with respect to addition, suppose that $[f, g, m], [h, t, n], [k, s, r]$ are

three arbitrary elements in RAAH-END $R(I_1, I_2)$, we have:

$$[f, g, m].([h, t, n] + [k, s, r]) =$$

$$[fo(h+k), fo(t+s) + go(t+s) + go(n+r) + mo(t+s) + go(h+k), fo(n+r) + mo(h+k) + mo(n+r)]$$

$$= [f, g, m].[h, t, n] + [f, g, m].[k, s, r].$$

. Multiplication is associative:

$$([f, g, m].[h, t, n]).[k, s, r] =$$

$$[foh, fot + got + goh + gon + mot, mon + fon + moh].[k, s, r], \text{ we put}$$

$$fot + got + goh + gon + mot = a, \quad mon + fon + moh = b, \text{ now we write:}$$

$$([f, g, m].[h, t, n]).[k, s, r] =$$

$$[foh, a, b].[k, s, r] = [fohok, aok + aos + aor + bos + fohos, fohor + bor + bok] =$$

$$[f, g, m].([h, t, n].[k, s, r]).$$

Thus we get the desired proof.

Theorem 4.6:

Let R be a ring. Then $\text{AAH-END}(R(I)) \cong \text{END}(R, +)(I)$,

$$\text{RAAH-END}(R(I_1, I_2)) \cong \text{END}(R, +)(I_1, I_2).$$

Where $END(R, +)(I)$ is the neutrosophic ring generated by $END(R, +)$ and I , $END(R, +)(I_1, I_2)$ is the refined neutrosophic ring generated by $END(R, +)$ and I_1, I_2 .

Proof:

We have $END(R, +)(I) = END(R, +) + END(R, +)I = \{f + gI; f, g \in END(R, +)\}$. Define the map

$f: AAH - END(R(I)) \rightarrow END(R, +)(I); f([g, h]) = g + hI$, f is a well defined map.

Let $x = [g, h], y = [m, n]$ be two arbitrary elements in $AAH - END(R(I))$,

$$f(x + y) = (g + m) + (h + n)I = (g + hI) + (m + nI) = f(x) + f(y).$$

$$f(x \cdot y) = f([gom, gon + hom + hon]) = (gom) + (gon + hom + hon)I = f(x) \cdot f(y).$$

It is easy to see that f is bijective, thus f is an isomorphism.

Also, we have $END(R, +)(I_1, I_2) = \{(f, gI_1, hI_2); f, g, h \in END(R, +)\}$. Define the map

$f: RAAH - END(R(I_1, I_2)) \rightarrow END(R, +)(I_1, I_2); f[g, h, k] = (g, hI_1, kI_2)$. By a similar argument we find

that f is an isomorphism.

Example 4.7:

This example clarifies operations on $AAH - END(R(I))$.

Let $R = \mathbb{Z}_6$ the ring of integers modulo 6,

$f: R \rightarrow R; f(x) = 3x, g: R \rightarrow R; g(x) = 4x, h: R \rightarrow R; h(x) = x$ are three endomorphisms on $(R, +)$.

$[f, g]: R(I) \rightarrow R(I); [f, g](x + yI) = 3x + 4yI, [h, g]: R(I) \rightarrow R(I); [h, g](x + yI) = x + 4yI$ are two AH-endomorphisms on $(R(I), +)$.

We can clarify addition by:

$$([f, g] + [h, g])(x + yI) = [f + h, g + g](x + yI) = (f + h)(x) + (2g)(y)I =$$

$$4x + 8yI = 4x + 2yI = [g, s](x + yI); s: R \rightarrow R; s(x) = 2x \text{ is an endomorphism on } (R, +).$$

Multiplication can be clarified as:

$$([f, g] \cdot [h, g])(x + yI) = foh(x) + (fog + goh + gog)(y)I = 3x + (12y + 4y + 16y)I =$$

$$3x + 2yI = [f, s](x + yI).$$

Example 4.8:

This example clarifies operations on RAAH-END $R(I_1, I_2)$.

Let $R = \mathbb{Z}_6$ the ring of integers modulo 6,

$f: R \rightarrow R; f(x) = 3x, g: R \rightarrow R; g(x) = 4x, h: R \rightarrow R; h(x) = x$ are three endomorphisms on $(R, +)$.

$[f, h, g]: R(I_1, I_2) \rightarrow R(I_1, I_2), [h, g, h]: R(I_1, I_2) \rightarrow R(I_1, I_2)$ are two AH-endomorphisms on $(R(I_1, I_2), +)$.

$$([f, h, g] + [h, g, h])(x, yI_1, zI_2) = ((f + h)(x), (h + g)(y)I_1, (g + h)(z)I_2) =$$

$$(4x, 5yI_1, 5zI_2) = [g, s, s](x, yI_1, zI_2); s: R \rightarrow R; s(x) = 5x \text{ is an endomorphism on } (R, +).$$

$$[f, h, g].[h, g, h](x, yI_1, zI_2) =$$

$$(f \circ h(x), (f \circ g + h \circ h + h \circ g + h \circ h + g \circ g)(y)I_1, (f \circ h + g \circ h + g \circ h)(z)I_2) =$$

$$(3x, (12y + y + 4y + y + 16y)I_1, (3z + 4z + 4z)I_2) = (3x, 4yI_1, 5zI_2) = [f, g, s](x, yI_1, zI_2).$$

Conclusion

In this work, we have generalized the concept of AHS-homomorphism by introducing the concept of AH-homomorphism in neutrosophic rings, and refined neutrosophic rings. These functions play an important role in determining the structures and properties of AH-ideal, that is because they generalize AHS-homomorphisms. Also, AH-homomorphisms between any neutrosophic/refined neutrosophic ring and itself have an interesting structure, since they make a neutrosophic ring/refined neutrosophic ring with a suitable defined operation. Through this work, we have characterized the algebraic structure of AH-endomorphisms of the additive group in the case of neutrosophic ring and case of refined neutrosophic ring.

Some important open questions arise according to this work, we can summarize it as follows:

Define AH-homomorphisms on an n-refined neutrosophic ring $R_n(I)$.

Is the set of all endomorphisms over the additive group of the n-refined neutrosophic ring

$R_n(I)$ form a ring?. If the answer is yes, then prove that it is isomorphic to the n-refined neutrosophic ring generated by $\text{END}(R, +)$.

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Solving Three Conjectures about Neutrosophic Quadruple Vector Spaces

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Abstract: The aim of this paper is to answer Smarandache conjectures about neutrosophic quadruple vector spaces. This paper depends on the concept of weak n-refined neutrosophic vector space to prove that an NQ vector space V defined over the field F is isomorphic to $F \times F \times F \times F$.

Keywords: n-Refined weak neutrosophic vector space , NQ vector space , vector space homomorphism.

1. Introduction

Neutrosophy as a new king of logic has an interesting effect into algebra. Algebraic studies began with Smarandache and Kandasamy in [5]. They introduced some interesting concepts such as neutrosophic group, neutrosophic ring, and neutrosophic semi group.

Many applications of neutrosophic set and related concepts into optimization, decision making and industry were proposed in [10,11,12,13,14].

Recently, more neutrosophical algebraic structures were defined and handled as a bridge between logical and algebraic concepts such as neutrosophic vector space, quadruple neutrosophic vector space, neutrosophic module, n-refined neutrosophic ring, and n-refined neutrosophic vector space. See [1,2,3,4,7,8,9].

Quadruple neutrosophic vector space is a logical space with some algebraic properties defined in [6], the most important question about this kind of spaces is the classification of them with respect to classical vector spaces.

In [6], Smarandache et al, proposed three open conjectures concerning the algebraic structure (classification structure) of neutrosophic quadruple vector spaces (NQ vector space). These conjectures can be described as follows:

Conjecture 1: Is the NQ vector space V defined over the field R isomorphic to $R \times R \times R \times R$?

Conjecture 2: Is the NQ vector space V defined over the field C isomorphic to $C \times C \times C \times C$?

Conjecture 3: Is the NQ vector space V defined over the field Z_p isomorphic to $Z_p \times Z_p \times Z_p \times Z_p$?

In this work we prove that the answer of previous conjectures is yes. We use the properties of weak n -refined neutrosophic vector space $V_n(I)$ to reach our goal.

These three conjectures are called Smarandache's conjectures.

Our motivation to write this paper is to give a classification theorem for n -refined weak neutrosophic vector spaces, as well as to answer three conjectures proposed by Smarandache about classifying quadruple neutrosophic vector spaces. This work shows for the first time the algebraic connection between n -refined neutrosophic weak vector spaces and quadruple neutrosophic vector spaces.

2. Preliminaries

Definition 2.1: [8]

Let $(R, +, \times)$ be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define

$R_n(I) = \{a_0 + a_1I + \dots + a_nI_n; a_i \in R\}$ to be n -refined neutrosophic ring.

Definition 2.2 :[4]

Let $(V, +, \cdot)$ be a vector space over the field K then $(V(I), +, \cdot)$ is called a weak neutrosophic vector space over the field K , and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field $K(I)$.

Definition 2.3: [9]

Let $(K, +, \cdot)$ be a field, we say that $K_n(I) = K + KI_1 + \dots + KI_n = \{a_0 + a_1I_1 + \dots + a_nI_n; a_i \in K\}$ is an n -refined neutrosophic field.

It is clear that $K_n(I)$ is an n -refined neutrosophic field, but not a field in the classical meaning.

Definition 2.4: [9]

Let $(V, +, \cdot)$ be a vector space over the field K . Then we say that

$V_n(I) = V + VI_1 + \dots + VI_n = \{x_0 + x_1I_1 + \dots + x_nI_n; x_i \in V\}$ is a weak n -refined neutrosophic vector

space over the field K . Elements of $V_n(I)$ are called n -refined neutrosophic vectors, elements of K are called scalars.

If we take scalars from the n -refined neutrosophic field $K_n(I)$, we say that $V_n(I)$ is a strong n -refined neutrosophic vector space over the n -refined neutrosophic field $K_n(I)$. Elements of $K_n(I)$ are called n -refined neutrosophic scalars.

Definition 2.5: [6]

The quadruple $(a, bT, cI, dF); a, b, c, d \in R \text{ or } C \text{ or } Z_p$, with T,I,F as in classical

Neutrosophic logic with a the known part and (bT, cI, dF) defined as the unknown part, denoted by $NQ = \{(a, bT, cI, dF) | a, b, c, d \in R \text{ or } C \text{ or } Z_n\}$ is called the Neutrosophic set of quadruple numbers.

Remark 2.6: [6]

$(NQ, +, \cdot)$ is a vector space, where (\cdot) is an external multiplication by a scalar from the same field that NQ is built over.

Open conjectures [6]

Conjecture 1: Is the NQ vector space V defined over the field R isomorphic to $R \times R \times R \times R$?

Conjecture 2: Is the NQ vector space V defined over the field C isomorphic to $C \times C \times C \times C$?

Conjecture 3: Is the NQ vector space V defined over the field Z_p isomorphic to $Z_p \times Z_p \times Z_p \times Z_p$?

3. Main concepts and results

Lemma 3.1:

Let V be a vector space over the field K , $V_n(I)$, be the corresponding weak n -refined neutrosophic vector space. Then

- (a) $V_{n-1}(I)$ is a homomorphic image of $V_n(I)$.
- (b) $V_n(I)/W \cong V_{n-1}(I)$, where W is a subspace of $V_n(I)$ with property $W \cong V$.
- (c) $V_m(I)$ is a homomorphic image of $V_n(I)$; $m \leq n$.

Proof:

- (a) We define $f: V_n(I) \rightarrow V_{n-1}(I); f(a_0 + a_1I_1 + \dots + a_nI_n) = a_0 + a_1I_1 + \dots + (a_{n-1} + a_n)I_{n-1}$.

It is easy to see that f is well defined. Let $x = \sum_{i=0}^n a_i I_i, y = \sum_{i=0}^n b_i I_i$ be two arbitrary elements in

$V_n(I)$, r be any element in the field K , we have:

$$f(x+y) = (a_0 + b_0) + (a_1 + b_1)I_1 + \dots + (b_{n-1} + a_{n-1} + b_n + a_n)I_{n-1} = f(x) + f(y).$$

$$f(r.x) = f(\sum_{i=0}^n r.a_i I_i) = r a_0 + r a_1 I_1 + \dots + (r a_{n-1} + r b_{n-1} + r a_n + r b_n) I_{n-1} = r.f(x).$$

Thus f is a homomorphism.

(b) $\text{Ker}(f) = \{x = \sum_{i=0}^n a_i I_i \in V_n(I); f(x) = 0\}$, this implies

$$a_i = 0 \text{ for all } 0 \leq i \leq n-2, a_{n-1} = -a_n, \text{ so } W = \text{Ker}(f) = \{a_n(I_n - I_{n-1}); a_n \in V\}.$$

Also, we have $V_n(I)/W \cong V_{n-1}(I)$. Now define $g: W \rightarrow V; g(a_n(I_n - I_{n-1})) = a_n$, g is a well defined

map and it is an isomorphism, thus $W \cong V$.

(c) According to (a), we get a series of vector space homomorphisms

$V_n(I) \xrightarrow{f_n} V_{n-1}(I) \xrightarrow{f_{n-1}} \dots \xrightarrow{f_{n-m}} V_m(I)$, $f_n \circ f_{n-1} \circ \dots \circ f_{n-m}$ is a homomorphism since it is a product of homomorphisms. Thus our proof is complete.

Example 3.2:

Let $V = R^2$ be a vector space over the field R , $R_2^2(I) = \{a + bI_1 + cI_2; a, b, c \in V\}$ be the

corresponding weak 2-refined neutrosophic vector space over R , $R_1^2(I) = \{a + bI_1; a, b \in V\}$ be the corresponding weak 1-refined neutrosophic vector space over R .

$f: R_2^2(I) \rightarrow R_1^2(I); f(a + bI_1 + cI_2) = a + (b+c)I_1$ is a homomorphism according to the previous theorem.

$$\text{Ker}(f) = \{c(I_2 - I_1); c \in V\} \cong V.$$

Theorem 3.3:

Let V be a vector space over the field K , $V(I) = V + VI$ be the corresponding weak neutrosophic vector space over the field K , $V_n(I)$ be the corresponding weak n -refined neutrosophic vector space over K .

Then:

$$(a) V \times V \cong V(I).$$

$$(b) V_n(I) \cong V \times V \times \dots \times V \text{ (n + 1 times)}.$$

Proof:

(a) Define $f: V \times V \rightarrow V(I); f(x, y) = x + yI; x, y \in V$, it is clear that f is well defined bijective map.

Let $(x, y), (z, t) \in V \times V, r \in K$, we have $(x, y) + (z, t) = (x + z, y + t), r.(x, y) = (r.x, r.y)$,

$$f[(x, y) + (z, t)] = (x + z) + (y + t)I = (x + yI) + (z + tI) = f(x, y) + f(z, t).$$

$f[r.(x, y)] = r.x + r.yI = r.(x + yI) = r.f(x)$. Hence f is a vector space isomorphism.

(b) Define $f: V_n(I) \rightarrow V \times V \times \dots \times V; f(x_0 + x_1I_1 + \dots + x_nI_n) = (x_0, x_1, \dots, x_n)$.

By similar argument we find that f is an isomorphism.

Result 3.4:

(a) Theorem 3.3 clarifies that the concept of weak neutrosophic vector space is a rediscovering of direct product of a vector space with itself, thus all results in [4] can be obtained easily according to this result.

(b) Theorem 3.3 clarifies that the concept of weak n-refined neutrosophic vector space is a rediscovering of direct product of a vector space with itself $n+1$ times, thus the question about defining basis of this kind of vector spaces in [9], can be answered easily.

Theorem 3.5:

Let $V = (a, bT, cI, dF)$ be a quadruple neutrosophic vector space defined over the field F , i.e

$a, b, c, d \in F$, then V is isomorphic to weak 3-refined neutrosophic vector space $F_3(I)$ over the field F .

Proof:

We define $f: V \rightarrow F_3(I); f[(a, bT, cI, dF)] = a + bI_1 + cI_2 + dI_3$.

f is well defined clearly. Let $m = (a, bT, cI, dF), n = (x, yT, zI, tF)$ be two arbitrary elements in V ,

where $a, b, c, d, x, y, z, t \in F$, we have:

$$m + n = (a + x, [b + y]T, [c + z]I, [d + t]F),$$

$$f(m + n) = (a + x) + (b + y)I_1 + (c + z)I_2 + (d + t)I_3 = f(m) + f(n).$$

Let s be an arbitrary element in the field F , then $s.m = (s.a, s.bT, s.cI, s.dF)$,

$$f(s.m) = s.a + (s.b)I_1 + (s.c)I_2 + (s.d)I_3 = s.f(m).$$

It is easy to see that f is a bijective map, thus we get the proof.

Solving conjectures

Theorem 3.6 :

Let $V = (a, bT, cI, dF)$ be a quadruple neutrosophic vector space defined over the field R , then V is isomorphic to $R \times R \times R \times R$.

Proof:

By Theorem 3.5 , we find that $V \cong R_2(I)$, by Theorem 3.3 , we find that $R_2(I) \cong R \times R \times R \times R$, thus $V \cong R \times R \times R \times R$.

Theorem 3.7:

Let $V = (a, bT, cI, dF)$ be a quadruple neutrosophic vector space defined over the field C , then V is isomorphic to $C \times C \times C \times C$.

Proof:

By Theorem 3.5 , we find that $V \cong C_2(I)$, by Theorem 3.3 , we find that $C_2(I) \cong C \times C \times C \times C$, thus $V \cong C \times C \times C \times C$.

Theorem 3.8:

Let $V = (a, bT, cI, dF)$ be a quadruple neutrosophic vector space defined over the field Z_p , then V is isomorphic to $Z_p \times Z_p \times Z_p \times Z_p$.

Proof:

By Theorem 3.5 , we find that $V \cong Z_{p_2}(I)$, by Theorem 3.3 , we find that $Z_{p_2}(I) \cong Z_p \times Z_p \times Z_p \times Z_p$, thus

$$V \cong Z_p \times Z_p \times Z_p \times Z_p.$$

5. Conclusion

In this paper we have answered Smarandache's conjectures in neutrosophic quadruple vector space (NQ vector space) by investigating the algebraic relations between n-refined neutrosophic vector spaces and quadruple neutrosophic vector spaces. Also, we have proved that every weak m-refined neutrosophic vector space is a homomorphic image of n-refined neutrosophic vector space, where $m \leq n$.

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A Two Stage Interval-valued Neutrosophic Soft Set Traffic Signal Control Model for Four Way Isolated signalized Intersections

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Abstract. One of the major problems of both developed and developing countries is traffic congestion in urban road transportation systems. Some of the adverse consequences of traffic congestion are loss of productive time, delay in transportation, increase in transportation cost, excess fuel consumption, safety of people, increase in air pollution level and disruption of day-to-day activities. Researches have shown that among others, traditional traffic control system is one of the main reasons for traffic congestion at traffic junctions. Most countries through out the world use pre-timed/ fixed cycle time traffic control systems. But these traffic control systems do not give an optimal signal time setting as they do not take into account the time dependent heavy traffic conditions at the junctions. They merely use a predetermined sequence or order for both signal phase change and time setting. Some times this also leads to more congestion at the junctions. As an improvement of fixed time traffic control method, fuzzy logic traffic control model was developed which takes into account the current traffic conditions at the junctions and works based on fuzzy logic principle under imprecise and uncertain conditions. But as a real life situation, in addition to uncertainty and impreciseness there is also indeterminacy in traffic signal control constraints which fuzzy logic can not handle. The aim of this research is to develop a new traffic signal control model that can solve the limitations of fixed time signal control and fuzzy logic signal control using a flexible approach based on interval-valued neutrosophic soft set and its decision making technique, specially developed for this purpose. We have developed an algorithm for controlling both phase change and green time extension / termination as warranted by the traffic conditions prevailing at any time. This algorithm takes into account the existing traffic conditions, its uncertainty and indeterminacy. The decision making technique developed allows both phase change and green time setting to be managed dynamically, depending on the current traffic intensity and queuing of vehicles at different lanes, as opposed to an order or a pre-determined sequence followed in existing traffic control models.

Keywords: Signal control; soft set; neutrosophic set; interval-valued neutrosophic set ;interval-valued neutrosophic soft set

1. INTRODUCTION

Most of the cities of developing countries suffer from traffic congestion in urban road transportation systems. Consequently this leads to loss of productive time, delay in transportation, increase in operating cost of transportation and excess fuel consumption. Increase in number of vehicles, insufficient roads, quality of roads and traditional traffic signal control systems are the major causes for traffic congestion in big cities[1].

Some of the adverse consequences of traffic congestion include increase in the level of pollution, safety of the people, disruption of day to day life /daily activities, health burden for the individuals as well as government and private organizations. In early days as well as at present, traffic is controlled by hand signs by traffic police or by signals and markings called the traditional traffic control systems. Researches have established that unless otherwise implemented properly the traditional traffic control system can contribute more to the congestion at intersections[2].

As a real world phenomenon traffic congestion involves uncertainty and impreciseness and this paved the way for the use of fuzzy logic controller in traffic control systems. Fuzzy logic is one of the most appropriate tool to handle imprecise characteristics accurately and scientifically. Fuzzy controller model makes use of expert's experience and knowledge in traffic control field to develop the linguistic protocol that produces input/output for the control system. Imprecise, inexact and linguistic traffic terms such as 'heavy traffic', 'moderate traffic', 'light traffic' and 'low traffic' can be manipulated using fuzzy logic controller to estimate signal timings and sequences. Currently most signalized intersections in almost all developing countries use fixed time traffic controllers or pre-timed traffic lights. The traffic lights change phase at a constant cycle time in fixed traffic light controller, without taking into account the peak period or highly varying traffic intensity with respect to time. Pre-timed traffic light also causes traffic congestion as it is incapable of detecting traffic intensity at the junction and allow the vehicles waiting in the lanes to cross the junction as per the urgency necessitated by the traffic conditions prevailing at that time.

The present day traffic signal controller models suffer from indeterminacy due to various factors like unawareness of the problem, inaccurate and imperfect data and poor forecasting in addition to uncertainty in the constraints. To overcome this we need to develop a new mathematical tool that can effectively address the problem of traffic congestion. The drawbacks of the present day traffic control models and their failure to address the problem of urban traffic congestion have motivated us to carry out this research. The main aim of this research is to develop a new traffic signal model that can overcome the limitations of present day traffic control models including fuzzy logic signal control.

Smarandache[41] introduced the concept of neutrosophic sets in 1998. Neutrosophic set (NS) is an extension of classical, fuzzy and intuitionistic fuzzy sets. A neutrosophic set is characterized by its components truth membership degree, indeterminacy membership degree and falsity membership degree, which are

considered to be independent of each other. These three membership degrees are more suitable to represent indeterminate and inconsistent information. Based on the pioneering work of Smarandache, Wang et al. [3] introduced the idea of interval-valued neutrosophic set (IVNS), an extension of neutrosophic set which is found to be more suitable for many real life applications like image processing, decision making, supply chain management, water resource management and medical diagnosis. Interval-valued neutrosophic set is more realistic, flexible and practical than neutrosophic set and is described by three intervals namely a membership interval, an indeterminacy interval and a non-membership interval. Thus IVNS provides a more reasonable and structured mathematical framework to cope with indeterminate and inconsistent information.

In a developed society, people are more concerned about their health. Thus, improvement of medical field application has been one of the greatest active study areas. Medical statistics show that heart disease is the main reason for morbidity and death in the world. The physician's job is difficult because of having too many factors to analyze in the diagnosis of heart disease. Besides, data and information gained by the physician for diagnosis are often partial and immersed. Recently, health care applications with the Internet of Things (IoT) have offered different dimensions and other online services. These applications have provided a new platform for millions of people to receive benefits from the regular health tips to live a healthy life. With this idea in their mind Abdel-Basset et al. [26, 27, 28, 29] proposed a neutrosophic multi criteria decision making (NMCDM) technique and applied this technique to medical diagnosis.

Many researchers have combined neutrosophic set with other mathematical structures to produce different hybrid structures and used them in suitable applications. These hybrid structures include neutrosophic soft set, intuitionistic soft set, generalized neutrosophic soft set and interval valued neutrosophic soft set. In this research we design a new model to facilitate smooth flow of traffic at isolated four way signalized intersections using a flexible approach based on interval-valued neutrosophic soft sets and its decision making technique developed for this purpose. We developed an algorithm that can dynamically control both of the main activities of traffic control namely, phase change and green time setting as necessitated by the traffic conditions prevailing at the time of consideration. Based on the decision making technique specifically developed for this purpose, this algorithm makes use of the existing traffic conditions and its uncertainty as well as indeterminacy to effectively control the phase change and green time setting to facilitate smooth flow of traffic at four way intersections.

The traffic signal control model, when implemented properly could eliminate the drawbacks of the present day traffic signal control models and make a complete solution to the problem of urban traffic congestion at four way signalized intersections.

To the best of our knowledge no research work seems to have been carried out so far on the performance of fixed time or pre-timed traffic signal controllers as well as on the use of fuzzy logic controller at four way traffic intersections in interval valued neutrosophic soft set environment.

2. REVIEW OF LITERATURE

Molodtsov [4], introduced the concept of soft sets for dealing with fuzzy, uncertain and ambiguously defined objects and outlined some problems that could be tackled using soft sets. The major advantage of soft set is its ability to use parametrization which was lacking with other tools used in the study of uncertainty.

Smarandache [5] defined the notion of neutrosophic set as a generalization of intuitionistic fuzzy set. He explained the difference between neutrosophic set and intuitionistic fuzzy set with illustrative examples. Wang et al. [6] defined the single valued neutrosophic set and proposed a framework for this set. They also defined the set theoretic operations and established some properties of this set.

Aggarwal et al. [7] discussed how neutrosophic logic can be utilized for modeling and control and proposed a block diagram for neutrosophic inference systems.

Maji [8] gave an application of neutrosophic set to a decision making problem on object recognition in imprecise environment. Maji [9] defined the notion of neutrosophic soft set and studied some properties of this set. Multi-criteria decision making using weighted correlation coefficient and weighted cosine similarity measure under single valued neutrosophic environment was studied by Ye [10].

Broumi and Smarandache [11] developed a new multi attribute decision making method based on neutrosophic trapezoidal linguistic weighted arithmetic and geometric averaging operators. They have demonstrated the new method with numerical example. Broumi et al. [12] have presented the definition of neutrosophic parametrized soft set and its operations. They have defined some aggregation operators to develop a soft decision making technique which is more efficient. Sahin [13] generalized the concept of neutrosophic soft set and gave an application of generalized neutrosophic soft set in decision making. Deli [14] further generalized the concepts of soft set, fuzzy soft set, interval valued fuzzy soft set, intuitionistic fuzzy soft set, interval valued intuitionistic fuzzy soft set and neutrosophic soft set by introducing interval valued neutrosophic soft set. He defined some operations on this set and developed a decision making method based on level soft sets and provided an illustrative example for the proposed method. Deli and Cogman [15] defined some operations on interval valued neutrosophic soft sets and developed a decision making method based on interval valued neutrosophic soft set and illustrated the method by example. Zhang et al. [16] defined some aggregation operators on interval valued neutrosophic set and developed a method for multi-criteria decision making using these operators. They have also illustrated the working of the method by example. Ye [17] explained the difficulty in applying the single valued and interval valued neutrosophic sets and introduced the concept of simplified neutrosophic set. He defined three vector similarity measures in vector space and applied them in a multi-criteria decision making problem with simplified neutrosophic information. Illustrative example is also provided. Ye [18] shown that simplified neutrosophic set is a sub class of neutrosophic set and defined some operational laws and aggregation operators such as weighted arithmetic and weighted geometric average operators. Based on these operators and cosine similarity measure, he developed a multicriteria decision making method

and using numerical example illustrated the ranking order of alternatives to select the ideal and best alternatives. Ye [19] defined the hamming and euclidean distances between interval valued neutrosophic sets and the similarity measures between them. He used the similarity measures to rank the alternatives and to determine the best one and demonstrated the method with example. Smarandache [20] introduced the concept of generalized interval neutrosophic soft set and discussed some operations on this set. He also presented an application of generalized interval neutrosophic soft set in decision making problem. Deli and Broumi [21] have redefined the notion of neutrosophic soft set and its operations. They defined neutrosophic soft matrix and their operators and explained the use of this matrix in storing a neutrosophic set in computer memory. They have also developed a decision making method based on neutrosophic soft matrix. Biswas et al. [22] have introduced the concept of single valued trapezoidal neutrosophic number. They defined the value and ambiguity indices of truth, indeterminacy and falsity membership functions of this number. A new ranking method of these numbers based on the two indices is developed and this ranking technique is applied to a multi-criteria decision making problem with a numerical illustration. Tian et al. [23] developed a method for multi-criteria decision making by combining simplified neutrosophic sets and normalized Bonferoni mean operators. They first introduced a comparison method for simplified neutrosophic linguistic numbers by employing linguistic scale function. Then they defined a normalized weighted Bonferoni mean operator and studied its properties. Based on the mean operator they developed a multi-criteria decision making method and illustrated with an example. Ye [24] introduced new exponential operational laws of interval valued neutrosophic set. He proposed an interval neutrosophic weighted exponential aggregation operator and its dual and developed a decision making method based on these operators. Yin-Xiang Ma et al. [25] developed a medical treatment selection method based on prioritized harmonic mean operators in an interval neutrosophic linguistic environment. They defined two aggregation operators based on harmonic mean and used them to develop an interval neutrosophic linguistic multi-criteria group decision making method and applied it to a practical treatment selection problem.

Abdel-Basset [26] proposed a novel framework based on computer supported diagnosis and IoT to detect and monitor heart failure infected patients, where the data are obtained from various other sources. The proposed healthcare system aims at obtaining better precision of diagnosis with ambiguous information and suggested neutrosophic multi criteria decision making (NMCDM) technique to aid patient and physician to know if patient is suffering from heart failure. The proposed model is validated by numerical examples on real case studies. The experimental results indicate that the proposed system provides a viable solution that can work at wide range and a new platform to millions of people getting benefit over decreasing of mortality and cost of clinical treatment related to heart failure.

Abdel-Basset et al. [27] suggested a novel framework based on computer propped diagnosis and IoT to detect and observe type-2 diabetes patients. The recommended healthcare system aims to obtain a better accuracy of diagnosis with mysterious data. The overall experimental results indicated the

validity and robustness of the proposed algorithms.

Abdel-Basset et al.[28] used a plithogenic multi criteria decision making (MCDM) strategy and VIKOR (Vlsekriterijumska Optimizacija I Kompromisno Resenje) technique to arrive at a methodological procedure to assess the infirmity serving under the framework of plithogenic theory ,which is more general than fuzzy,intuitionistic fuzzy and neutrosophic theory.In plithogenic theory ,the ambiguity ,incomplete information ,qualitative information,approximate evaluation ,imprecision and uncertainty are addressed with semantic expressions determined by plithogenic numbers and computing of contradiction degrees of attribute values.The theory is applied to 3 private and 2 general hospitals in Zagazig and found that the serving efficiency of private medical centers is superior than that of the general medical centers.

In some real world decision making environment ,similarity plays a vital role and unexpected outcomes from the decision making point of view.Based on this Abdel-Basset et al.[29] proposed the cosine similarity measures and weighted cosine similarity measures for bipolar and interval valued bipolar neutrosophic set and presented a two multi attribute decision making techniques based on the proposed measures .The feasibility of the proposed measures are verified using numerical examples and used for diagnosing bipolar disorder diseases. Thamaraiselvi and Santhil,[30] introduced a mathematical representation of a transportation problem in neutrosophic environment and proposed a new method for solving transportation problems with indeterminate and inconsistent information.Liang et al.[31] proposed a single valued trapezoidal neutrosophic preference relation as a strategy for tackling multicriteria decision making problems based on two aggregation operators, namely single valued trapezoidal weighted arithmetic and geometric average operators.Deli and Subas [32]have presented a methodology for solving multi-attribute decision making problems using single valued neutrosophic numbers.They have defined cut sets of these numbers and applied these cuts to single valued trapezoidal and triangular neutrosophic numbers and studied some properties.They developed a ranking method using the concepts of values and ambiguities and developed a multi-attribute decision making method using single valued trapezoidal neutrosophic numbers.Limin Su[33] developed a project procurement method selection under interval neutrosophic environment.He defined a similarity measure to handle this selection and applied it to develop a decision making model.A case study is presented to show the applicability of the proposed approach .The results are compared with three of the existing methods to show the superiority of the proposed method.Broumi et al.[34] proposed a new score function for interval valued neutrosophic numbers and used it to solve the shortest path problem.They proposed novel algorithms to find the neutrosophic shortest path by considering interval valued neutrosophic number,trapezoidal and triangular interval valued neutrosophic numbers for finding the length of the path in a network with illustrative examples.The effectiveness of the proposed algorithms are explained by a comparative analysis with the existing methods.Hongwa Qin and Xiagin Ma [35] proposed a new and complete system evaluation method based on interval valued fuzzy soft set.This method has four components viz data collection and processing,combination of data sets ,parameter reduction and decision making.They have

applied the method to three real life evaluation systems. Poyen Fb et al. [36] proposed a frame work for an intelligent traffic control system. They proposed a mechanism in which the time period of green light and red light are assigned on the basis of the density of the traffic present at that time. The density is calculated using proximity infrared sensors and the glowing time of green light is assigned with the help of micro controller. Ye [37] presented the concepts of neutrosophic linear equations, neutrosophic matrix and operations on such matrices. He proposed some methods to solve the system of neutrosophic linear equations. He has applied the method to solve the system of linear equations arising in indeterminate traffic flow. Broumi [38] has explained the importance of traffic management to ensure safe and peaceful travel for people. He defined some weighted aggregation operations on type 2 fuzzy sets and interval neutrosophic sets and constructed an improved score function for interval neutrosophic numbers. He proposed a method for traffic flow control based on this score function. Javed Alam and Pandey [40] proposed a two stage traffic light system for real-time traffic monitoring to dynamically manage both the phase and green time of traffic lights for an isolated signalized intersection with the objective of minimizing the average vehicle delay in different traffic flow rates. Software has been developed in MATLAB to simulate the situation of an isolated signalized intersection based on fuzzy logic. Simulation results verify the performance of the proposed two stage traffic light system using fuzzy logic. In this research we develop a new traffic signal control model. This model makes use of interval-valued neutrosophic soft set and its decision making technique developed specifically for this purpose. The model consists of two stages and can dynamically manage both phase change and green signal time extension for real time traffic monitoring.

3. PRELIMINARY CONCEPTS

In this section we present the necessary preliminary ideas and some basic results needed for the present research work. We start from the definition of a soft set.

3.1 Soft Set [4]

Let X be the universal set under consideration and E be a set of parameters which represents the attributes, properties or characteristics of objects in X . Let $P(X)$ denote the power set of X and $A \subseteq E$. A pair (F, A) is called a soft set over X where F is a mapping given by $F : A \rightarrow P(X)$. That is a soft set over X is a parametrized family of subsets of X .

For each $e \in A$, $F(e)$ may be considered as the set of e -approximate elements of X , called the value set of e . It is clear that soft set is not a set in the classical sense.

3.2 Complement of a soft set [4]

The complement of a soft set (F, A) denoted by $(F, A)^c$ is defined as $(F, A)^c = (F^c, \sim A)$, where $F^c : \sim A \rightarrow P(X)$ is the mapping given by $F^c(\alpha) = X - F(\sim \alpha)$, $\forall \alpha \in \sim A$. Obviously $(F^c)^c = F$ and $((F, A)^c)^c = (F, A)$.

3.3 Null soft set [4]

A soft set (F, A) over X is said to be a null soft set denoted by $\tilde{\Phi}$, if $\forall e \in A, F(e) = \tilde{\Phi}$ (null set).

3.4 Absolute soft set [4]

A soft set (F, A) over X is said to be an absolute soft set denoted by \tilde{A} , if $\forall e \in \tilde{A}, F(e) = X$.

3.5 Soft subset and soft super set [4]

Let (F, A) and (G, B) be two soft sets over a common universe X . We say that (F, A) is a soft subset of (G, B) if

- (i) $A \subseteq B$ and
- (ii) $\forall e \in A$, $F(e)$ and $G(e)$ are identical approximations.

In such a case we write $(F, A) \tilde{\subseteq} (G, B)$.

(F, A) is called a soft super set of (G, B) if (G, B) is a soft subset of (F, A) . In this case we write $(F, A) \tilde{\supseteq} (G, B)$.

3.6 Soft Equality [4]

Two soft sets (F, A) and (G, B) over a common universe X are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A) .

3.7 Single valued neutrosophic set [6]

Let X be the universal set. A neutrosophic set A in X is characterized by a truth membership function μ_A , an indeterminacy membership function ν_A and a falsity membership function ω_A , where $\mu_A, \nu_A, \omega_A : X \rightarrow [0, 1]$ are functions and $\forall x \in X, x \equiv x(\mu_x, \nu_x, \omega_x) \in A$ is a single valued neutrosophic element of A .

A single valued neutrosophic set A (SVNS in short) over a finite universe $X = \{x_1, x_2, \dots, x_n\}$ can be represented as $A = \sum_{i=1}^n \langle \mu_A(x_i), \nu_A(x_i), \omega_A(x_i) \rangle / x_i$.

The three membership functions form the fundamental concepts of neutrosophic set and they are independently and explicitly quantified subject to the following conditions.

$$0 \leq \mu_A(x), \nu_A(x), \omega_A(x) \leq 1 \text{ and} \\ 0 \leq \mu_A(x) + \nu_A(x) + \omega_A(x) \leq 3 \quad \forall x \in X.$$

3.8 Complement of a single valued neutrosophic set [6]

The complement of a SVNS A denoted by A^c is defined by $\mu_{A^c}(x) = \omega_A(x), \nu_{A^c}(x) = 1 - \nu_A(x)$ and $\omega_{A^c}(x) = \mu_A(x)$ for all $x \in X$.

3.9 Containment and equality of SVNS defined on the same universe X [6].

(i) A is contained in B denoted as $A \tilde{\subseteq} B$ if and only if $\mu_A(x) \leq \mu_B(x)$; $\nu_A(x) \leq \nu_B(x)$ and $\omega_A(x) \geq \omega_B(x) \forall x \in X$.

(ii) A and B are equal, that is., $A = B$ if and only if $A \tilde{\subseteq} B$ and $B \tilde{\subseteq} A$.

3.10 Union and Intersection of SVNS [6]

Let A and B be two SVNS defined on a common universe X . Then the union of A and B , written as $A \cup B = C$ is defined by

$$\mu_C(x) = \max(\mu_A(x), \mu_B(x)) \\ \nu_C(x) = \max(\nu_A(x), \nu_B(x)) \text{ and}$$

$$\omega_C(x) = \min(\omega_A(x), \omega_B(x)) \forall x \in X.$$

The intersection of A and B ,denoted by $A \cap B = C$ is defined by

$$\begin{aligned}\mu_C(x) &= \min(\mu_A(x), \mu_B(x)), \\ \nu_C(x) &= \min(\nu_A(x), \nu_B(x)) \text{ and} \\ \omega_C(x) &= \max(\omega_A(x), \omega_B(x)) \forall x \in X.\end{aligned}$$

3.11 Interval Valued Neutrosophic Set [39]

Definition: For an arbitrary sub interval set A of $[0, 1]$ we define $\underline{A} = \inf$ of A and $\bar{A} = \sup$ of A .

Let X be the universal set. An interval valued neutrosophic set A in X is characterized by a truth membership function μ_A , an indeterminacy membership function ν_A and a falsity membership function ω_A for each element $x \in X$ where

$$\mu_A(x) = [\underline{\mu}_A(x), \bar{\mu}_A(x)], \nu_A(x) = [\underline{\nu}_A(x), \bar{\nu}_A(x)], \omega_A(x) = [\underline{\omega}_A(x), \bar{\omega}_A(x)] \text{ are closed sub-intervals of } [0, 1].$$

Thus $A = \langle \mu_A(x), \nu_A(x), \omega_A(x) \rangle / x; x \in X$.

3.12 Empty and universal IVNS [39]

Let A be an IVNS over the universal set X .

(i) A is called an empty IVNS if $A = \bar{\Phi}_N = \langle [0, 0], [1, 1], [1, 1] \rangle / x; \forall x \in X$ (ii) A is called a universal IVNS if $A = \bar{E} = \langle [1, 1], [0, 0], [0, 0] \rangle / x; \forall x \in X$

3.13 Containment [39]

Let A and B be two IVNS over a common universe X . Then A is contained in B if and only if

$$\begin{aligned}\underline{\mu}_A(x) \leq \underline{\mu}_B(x); \bar{\mu}_A(x) \leq \bar{\mu}_B(x); \underline{\nu}_A(x) \leq \underline{\nu}_B(x); \bar{\nu}_A(x) \leq \bar{\nu}_B(x); \underline{\omega}_A(x) \geq \underline{\omega}_B(x); \bar{\omega}_A(x) \geq \bar{\omega}_B(x) \\ \forall x \in X.\end{aligned}$$

In this case we write $A \subseteq B$

3.14 Union and Intersection of IVNS [39]

let A and B be two IVNS defined over a common universe X . The union of A and B denoted by $A \tilde{\cup} B$ is defined as

$$\begin{aligned}A \tilde{\cup} B = \{ \langle [\max(\underline{\mu}_A(x), \underline{\mu}_B(x)), \max(\bar{\mu}_A(x), \bar{\mu}_B(x))], \\ [\max(\underline{\nu}_A(x), \underline{\nu}_B(x)), \max(\bar{\nu}_A(x), \bar{\nu}_B(x))], \\ [\min(\underline{\omega}_A(x), \underline{\omega}_B(x), \min(\bar{\omega}_A(x), \bar{\omega}_B(x))] \rangle / x; \forall x \in X \}\end{aligned}$$

Similarly the intersection of A and B denoted by $A \tilde{\cap} B$ is defined by

$$\begin{aligned}A \tilde{\cap} B = \{ \langle [\min(\underline{\mu}_A(x), \underline{\mu}_B(x)), \min(\bar{\mu}_A(x), \bar{\mu}_B(x))], \\ [\min(\underline{\nu}_A(x), \underline{\nu}_B(x)), \min(\bar{\nu}_A(x), \bar{\nu}_B(x))], \\ [\max(\underline{\omega}_A(x), \underline{\omega}_B(x), \max(\bar{\omega}_A(x), \bar{\omega}_B(x))] \rangle / x; \forall x \in X \}\end{aligned}$$

3.15 Complement of IVNS [39]

The complement A^c of an IVNS A is defined as $A^c = \{ \langle [\underline{\omega}_A(x), \bar{\omega}_A(x)], [1 - \bar{\nu}_A(x), 1 - \underline{\nu}_A(x)], [\underline{\mu}_A(x), \bar{\mu}_A(x)] \rangle / x; \forall x \in X \}$

It is clear that $A\tilde{\cup}B, A\tilde{\cap}B$ and A^c are all IVNS over X .

3.16 Neutrosophic Soft Set [20]

Let X be the universal set and E be the set of parameters, $A \subseteq E$. Let $P(X)$ denote the set of all neutrosophic subsets of X . A pair (F, A) is called a neutrosophic soft set over X , where F is a mapping given by $F : A \rightarrow P(X)$. For each $e \in A$, $F(e)$ is a neutrosophic subset of X .

3.17 Union and Intersection of neutrosophic Soft Set [20]

Let (F, A) and (G, B) be two neutrosophic soft sets over (X, E) . The union of (F, A) and (G, B) is defined as the neutrosophic soft set (H, C) where $C = A \cup B$ and $\forall e \in C$

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \cup G(e) & \text{if } e \in A \cup B \end{cases} \quad (1)$$

We write $(H, C) = (F, A) \tilde{\cup} (G, B)$.

The intersection of (F, A) and (G, B) denoted by (H, C) where $C = A \cup B$ and $\forall e \in C$

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases} \quad (2)$$

We write $(H, C) = (F, A) \tilde{\cap} (G, B)$.

Interval valued neutrosophic soft set is a hybrid structure combining interval-valued neutrosophic set and soft set. The first quantifies the indeterminacy and the second provides the parametrization tool. Interval valued neutrosophic soft set (IVNSS) is a better tool to express uncertainty than the other variants of fuzzy or soft sets. The mathematical definition of IVNSS is given below.

3.18 Interval Valued neutrosophic Soft Set [15]

Let X be the universal set and E be a set of parameters, $A \subseteq E$. Let $IVNS(X)$ denote the set of all interval-valued neutrosophic sets over X . A pair (G, A) is called an interval-valued neutrosophic soft set (IVNSS) over X where G is a mapping given by $G : A \rightarrow IVNS(X)$. For $e \in A$, $G(e)$ is an interval-valued neutrosophic set in X , called e -approximate value set. Thus for all $e \in A$, $G(e) = \langle x, \mu^e(x), v^e(x), \omega^e(x) \rangle \in IVNS(X)$.

3.19 Null and absolute IVNSS [15]

An IVNSS (G, A) of X is called a null or empty IVNSS denoted by $\tilde{\phi}_N$ if and only if $\forall e \in A$

$$\mu_e^G(x) = [0, 0], v_e^G(x) = [1, 1], \omega_e^G(x) = [1, 1] \forall x \in X$$

(G, A) is said to be an absolute IVNSS denoted by \tilde{A}_N if and only if

$$\mu_e^G(x) = [1, 1], v_e^G(x) = [0, 0], \omega_e^G(x) = [0, 0] \forall x \in X$$

For examples and illustration of related operations one can refer to the respective references.

4. THE PROPOSED TWO STAGE IVNSS TRAFFIC SIGNAL CONTROL MODEL

4.1 Introduction

It has been reported in the literature that fuzzy logic controller performs better when compared with pre-timed or fixed time controllers. However, in both the latter mentioned signal controllers the phase changes occur in a sequential order without any consideration on the current traffic conditions at the intersection. In this research it is proposed to implement a two stage traffic light control system for real time traffic monitoring to dynamically manage both phase change as well as extension of green signal time for an isolated signalized intersection with the objective of minimizing the average vehicle delay under varying traffic flow rates. Thus in the proposed model the phase change may not be in a sequential order among the roads at the intersection. IVNSS model has the ability to imitate the human intelligence for controlling traffic flow. It allows implementation of real life rules similar to the way human mind would work. This model is based on concepts graded to handle uncertainties and impreciseness to facilitate relatively smooth traffic flow. The graded concepts are more useful since real time situations in traffic control are very often difficult to describe precisely and are not deterministic.

IVNSS model allows linguistic expressions and inexact data to be manipulated in designing signal timings and phase change intervals. We consider an isolated traffic signal intersection with four approaches north, south, east and west, whose green time extension or termination at the traffic junction are tested during rush hour against the two parameters 'average number of vehicles entering the junction along the lanes with current green signal' (arrival rate) and 'the saturation flow rate' in the previous cycle. These input parameters are used to construct two neutrosophic sets on input variables, viz 'the quantity of traffic on the arrival side' and the quantity of traffic on the departure side' to estimate the output variable 'the extension time for the green signal' based on weight criteria. Thus based on the current traffic conditions the IVNSS model can estimate the output of the neutrosophic controller either to extend or terminate the ongoing green light signal. If there is no extension of the ongoing green time, there will be an immediate phase change of the traffic light allowing the flow of traffic from an alternate lane.

4.2 Design Criteria and Constraints

The design of IVNSS traffic signal controller depends on the experience and knowledge of experts in traffic control to formulate the linguistic protocol and to generate the input variables for the traffic signal control system. Vehicle detectors are installed on 'upstream line' and 'stop line'. The number of approaching vehicles for each approach during a given time interval can be estimated using the detectors.

The following assumptions are made for designing the IVNSS traffic control system:

- [1] The junction is an isolated four way intersection with traffic flowing from north, west, south and east directions.
- [2] When traffic moves from north and south, that from west and east stops, and vice-versa.

- [3] Right and left turns are permitted
- [4] The IVNSS controller observes the density of north and south traffic as one side and west and east traffic as another side.
- [5] The minimum and maximum green time in a cycle is specified.
- [6] If the north and south side is green, this would be the arrival side while the west and east would be considered as the queuing side, and vice-versa. Then the output variable would be the extension time needed for the green light on the arrival side based on some weight criteria.
- [7] In the proposed two stage traffic signal system, it is assumed that phase composition is pre-determined and the phase sequence as well as signal timings are changeable.

4.3 Structure of the Two Stage Traffic Signal Controller Model.

The general structure of an isolated four way intersection is illustrated in Figure 1. Each lane is equipped with two electromagnetic sensors. The first sensor is located behind each traffic light and the second is located behind the first at a distance S . The first sensor counts the number of cars passing the traffic light and the second counts the number of cars coming to the intersection. The number of cars waiting at the traffic light of each road is determined by the difference of the readings between the two sensors of that road. Each traffic light is a proximity sensor and can only sense the presence of a car in front of it waiting at the junction. The isolated intersection considered in Figure 1 is characterized by four phases as shown in Figure 2 with eight lanes. Each phase has two lanes. As discussed earlier the objective of phase design is to separate the conflicting movements at the intersection into various phases in such a way that there is no conflict in the movement during any phase. For example, when phase 1 is enabled, only the vehicles of lane EL_1 of the road direction east (E) and lane WL_1 of the road direction west (W) can go either straight or turn left, while all other lanes will have red light to stop movement of vehicles.

The flow rate and saturation flow rate of each phase is obtained as the maximum of the flow rates and saturation flow rates intended as the number of vehicles during the green/red light in their respective lanes as shown.

$$\text{Phase1} = \text{maximum (EL1, WL1)}$$

$$\text{Phase2} = \text{maximum (EL2, WL2)}$$

$$\text{Phase3} = \text{maximum (NL1, SL1)}$$

$$\text{Phase4} = \text{maximum (NL2, SL2)}$$

Table 1 summarizes the notation adopted in the two stage traffic control system.

4.4 Data Collection and Processing.

The necessary crisp data for the input parameters are obtained from vehicular detectors installed at traffic intersections. In order to derive useful information from the detector data, it is important to be selective and to use data that can give meaningful information about actual traffic flow. Factors which influence the data include the setting of signal group (red or green) and locating detection area. The

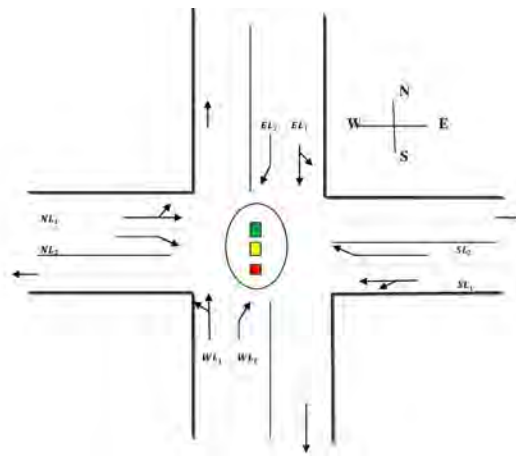


FIGURE 1. General Structure of the Intersection

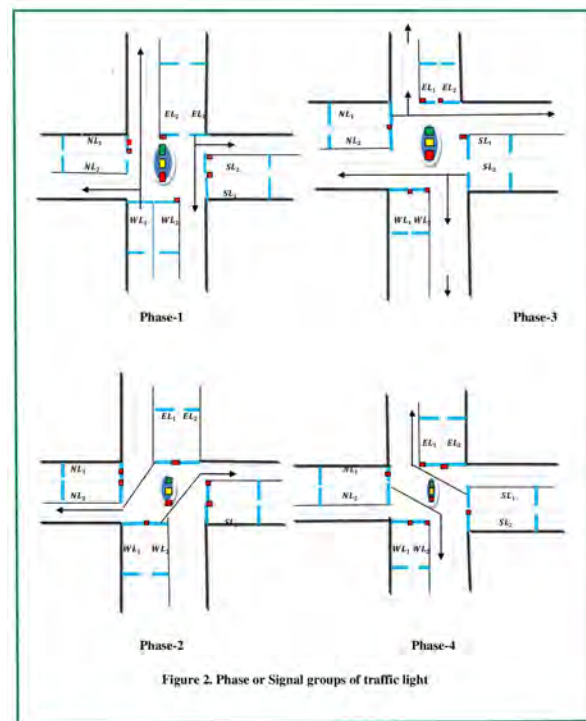


FIGURE 2. Phases/signal groups of Traffic light.

proposed IVNSS model uses the following two neutrosophic variables as input.

4.4.1 Saturation flow rate (vehicle/hour/green)

Saturation Flow Rate is the number of vehicles that would pass through the intersection when they approach signal which stays with green for an entire hour. Obviously, certain aspects of the traffic and the roadway will affect the saturation flow rate of the approach. If the approach has very narrow lanes, traffic will naturally provide longer gaps between vehicles, which will reduce our saturation flow rate. If there is large number of turning movements, or large number of trucks and busses, your saturation

Symbol	Lane movement	Allowed
EL_1	East Lane-1	Straight and left
EL_2	East Lane-2	right
WL_1	West Lane-1	Straight and left
WL_2	West Lane-2	right
NL_1	North Lane-1	Straight and left
NL_2	North Lane-2	right
SL_1	South Lane-1	Straight and left
SL_2	South Lane-2	right

TABLE 1. Notation of different phases/signal groups

flow rate will be reduced [64].

On another hand, the saturation flow rate (s) for a lane group is the maximum number of vehicles from that lane group that can pass through the intersection during one hour of continuous green under the prevailing traffic and roadway conditions.

4.4.2. Average arrival rate (veh/min/lane)

The detection area is located S meters down stream of the stop line of each approach, to record arrival rate. During the green period, the detector records the average arrival rate every minute and the average arrival rate for that period is used as an input. The output variable is the extension time of green signal time for the phase based on weight criteria.

4.5 Description of Two Stage IVNSS Traffic Signal Control Model (IVNSSTSC).

In the proposed IVNSS traffic signal control model there are two different modules, namely the phase selection decision module (PSDM) and the extension time calculation module (ETCM) for the selection of phase and the extension of green signal duration.

4.5.1 Phase Selection Decision Module (PSDM)

This stage performs the selection of the most appropriate phase /signal group for the next cycle with highest traffic urgency as the next phase to switch on using traffic data obtained from the previous cycle based on the input parameters. The proposed IVNSS model uses crisp traffic information from traffic sensors located at traffic junction for estimating the input parameters; saturation flow rate and average arrival rate collected during the previous cycle in order to decide on the number of seconds of green signal time required by each set of signal groups (phases) during the next cycle. The crisp data obtained has to be transformed into neutrosophic data using neutrosophication /normalization techniques.

The output parameter or decision criteria of the proposed IVNSSTSC is weight of the signal group. Weight is an indicator of the degree of need/urgency of the phase or signal group (SG) that requires green signal. For example, if the weight of SG_1 is 18 and that of SG_2 is 12, then SG_1 needs green time more urgently than SG_2 .

4.5.2 Extension Time Calculation Module (ETCM)

This stage calculates the green light time i.e., extension time of the signal group or phase which has highest urgency based on the weight value obtained in PSDM stage.

The weight values of SG_1, SG_2, SG_3 and SG_4 are used to calculate the total green time in a cycle. The weight values of each signal group and the total green time in a cycle are used to estimate the duration of green time that a signal group requires in the next cycle. Both total green time and green time of each signal group/phase are calculated as follows.

$$\text{Total Green Time (TGT)} = (\sum C_n - \text{Min}W) \times \left(\frac{\text{MaxGT} - \text{MinGT}}{\text{MaxW} - \text{MinW}} \right) + \text{MinGT}$$

The green time of each signal group is obtained by

$$\text{Green Time of } SG_n = \frac{C_n \times \text{TGT}}{\sum C_n}$$

where $\sum C_n$ is the total weight of signal groups, $\text{Min}W$ and $\text{Max}W$ are the minimum and maximum values of weights respectively. MinGT and MaxGT are the minimum and maximum values of green time in a cycle and n is the group index ($n = 1, 2, 3, 4$).

The proposed IVNSSTSC model consists of the following components namely, (i) data collection (ii) neutrosophication or normalization (iii) combination of data set (iv) parameter reduction and (v) decision making.

A flow chart of the proposed IVNSSTSC model is presented below.

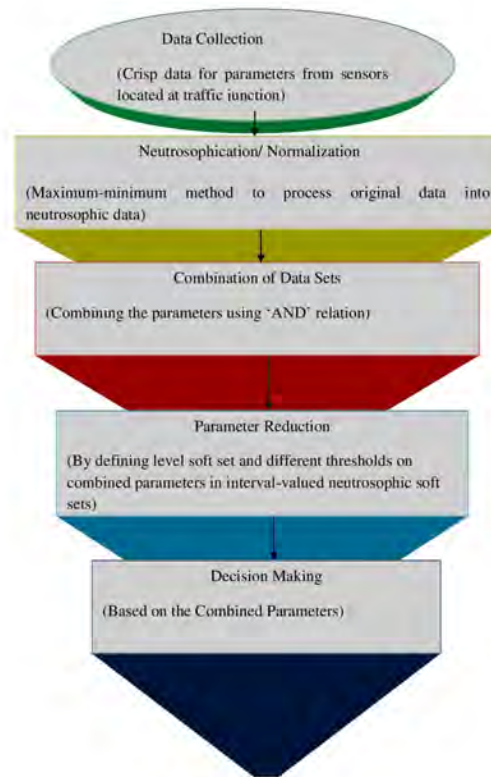


FIGURE 3. Structure of the Proposed Two Stage traffic Signal Control Model Using IVNSS

4.6 Neutrosophication /Normalization

In order to assess and evaluate a new system, we need to have original data. In the present case to analyze the working of the proposed IVNSSTSC system it is necessary to collect data from traffic sensors installed at traffic junction or traffic management directorate or using any other video graphic measures. But the data collected by this method will not be in the form of IVNSS. Hence it is essential to transform the original data into corresponding IVNSS.

Interval-valued data are used to describe truth, indeterminacy and falsity membership degrees by identifying their lower and upper limits in the unit interval $[0, 1]$. Consequently in this research the maximum-minimum neutrosophication/normalization method discussed by Hongwa Qin et al. [35] for IVFSS is extended to IVNSS. In the maximum-minimum normalization method there are more than one evaluators who give the scores of the objects aiming at one parameter.

The algorithm for the method is given below.

Maximum-Minimum Normalization Algorithm [35]

- (i) Input original data : the set of objects $U = \{x_1, x_2, \dots, x_n\}$; the set of parameters $E = \{e_1, e_2, \dots, e_m\}$
- (ii) For any $x_i \in U$ and for every $e_k \in E$, sort the data in ascending order.
- (iii) Find the maximum and minimum values of data for every $x_i \in U$ and $e_k \in E$.
- (iv) Transform the maximum and minimum evaluation scores into sub-intervals of $[0, 1]$ and normalize them as upper and lower membership degrees of the corresponding IVNSS.
- (v) Get the IVNSS for the evaluation system.

When the original data are numeric, the maximum and minimum evaluation scores are taken as the highest and lowest limits of such an evaluation respectively. Maximum and minimum evaluation scores can be transformed into sub-intervals of $[0, 1]$, which are considered as the normalized upper and lower membership degrees of membership in the corresponding IVNSS.

4.7 Defining Interval-Valued Neutrosophic Set for the Data

In this study we divide the input parameters into a finite number of states (levels) and construct the IVNSS based on this division. The parameter average arrival rate (e_1) is divided into five states/interval-valued neutrosophic set: very high, high, medium, low and very low. The saturation flow rate (e_2) is divided into four states/interval-valued neutrosophic set: very high, high, medium and low. The input data (traffic conditions) are first transformed into interval-valued neutrosophic data using the maximum-minimum normalization method for (e_1) and (e_2). This is carried out using the questionnaire from domain experts; their options could be a degree of "good performance" a degree of "indeterminacy" and a degree of "poor performance" with respect to each parameter. The interval valued neutrosophic soft set (F, A) describes the "average number of vehicles entering the junction" evaluated with respect to the levels of the parameter e_1 , where $A = \{a_1, a_2, a_3, a_4, a_5\}$ in which a_1 -stands for very high, a_2 -stands for high, a_3 -stands for medium, a_4 - stands for low and a_5 -stands for very low.

Similarly the interval-valued neutrosophic soft set (G, B) describes the 'the saturation flow rate' evaluated with respect to the levels of the parameter e_2 , where $B = \{q_1, q_2, q_3, q_4\}$ in which q_1 -represents very high, q_2 -represents high, q_3 -represents medium and q_4 -represents low.

Thus $X = \{SG_1, SG_2, SG_3, SG_4\}$ are the signal groups under consideration, $E = \{\text{average arrival rate } (e_1), \text{saturation flow rate } (e_2)\}$ is the parameter set, A is an IVNS describing the states of e_1 , along each signal group and B is an IVNS describing the states of e_2 along each signal group.

4.8. Combination of Data Set

In this section we define some operations on IVNSS and establish some properties of IVNSS.

4.8.1 Cartesian Product of Two IVNSS[12]

Let X be the universal set, E be a set of parameters and $A, B \subseteq E$. Let (F, A) and (G, B) be two IVNSS defined on X . Then the cartesian product of (F, A) and (G, B) denoted by $(H, A \times B)$ is defined as $H : A \times B \rightarrow IVNSS(X)$, where $H(a, b) = F(a) \cap G(b)$.

An interval valued neutrosophic relation from (F, A) to (G, B) is an interval-valued neutrosophic soft subset of $(H, A \times B)$.

4.8.2. AND and OR operations[12]

Let (F, A) and (G, B) be two IVNSS over the common universe X . Then (F, A) AND (G, B) denoted by $(F, A) \wedge (G, B)$ is an IVNSS defined as

$$(F, A) \wedge (G, B) = (H, A \times B) \text{ where } H(\alpha, \beta) = F(\alpha) \cap G(\beta) \quad \forall (\alpha, \beta) \in A \times B.$$

That is,

$$\begin{aligned} H(\alpha, \beta)(X) = & [\min(\underline{\mu}_{F(\alpha)}^{(x)}(x), \underline{\mu}_{G(\beta)}^{(x)}(x)), \min(\overline{\mu}_{F(\alpha)}^{(x)}(x), \overline{\mu}_{G(\beta)}^{(x)}(x))], \\ & [\min(\underline{\nu}_{F(\alpha)}^{(x)}(x), \underline{\nu}_{G(\beta)}^{(x)}(x)), \min(\overline{\nu}_{F(\alpha)}^{(x)}(x), \overline{\nu}_{G(\beta)}^{(x)}(x))], \\ & [\max(\underline{\omega}_{F(\alpha)}^{(x)}(x), \underline{\omega}_{G(\beta)}^{(x)}(x)), \max(\overline{\omega}_{F(\alpha)}^{(x)}(x), \overline{\omega}_{G(\beta)}^{(x)}(x))] > \quad \forall (\alpha, \beta) \in A \times B \text{ and } x \in X. \end{aligned}$$

Similarly, (F, A) OR (G, B) denoted by $(F, A) \vee (G, B)$ is an IVNSS defined as $(F, A) \vee (G, B) = (J, A \times B)$ where $J(\alpha, \beta) = F(\alpha) \cup G(\beta) \quad \forall (\alpha, \beta) \in A \times B$.

That is,

$$\begin{aligned} J(\alpha, \beta)(X) = & [\max(\underline{\mu}_{F(\alpha)}^{(x)}(x), \underline{\mu}_{G(\beta)}^{(x)}(x)), \max(\overline{\mu}_{F(\alpha)}^{(x)}(x), \overline{\mu}_{G(\beta)}^{(x)}(x))], \\ & [\max(\underline{\nu}_{F(\alpha)}^{(x)}(x), \underline{\nu}_{G(\beta)}^{(x)}(x)), \max(\overline{\nu}_{F(\alpha)}^{(x)}(x), \overline{\nu}_{G(\beta)}^{(x)}(x))], \\ & [\min(\underline{\omega}_{F(\alpha)}^{(x)}(x), \underline{\omega}_{G(\beta)}^{(x)}(x)), \min(\overline{\omega}_{F(\alpha)}^{(x)}(x), \overline{\omega}_{G(\beta)}^{(x)}(x))] > \quad \forall (\alpha, \beta) \in A \times B \text{ and } x \in X. \end{aligned}$$

4.9. Parameter Reduction and Decision making[15]

In this section we develop a parameter reduction algorithm of IVNSS for the smooth flow of traffic at isolated four way intersections. The algorithm produces a soft set from an IVNSS. For this we first define the concept of level sets for IVNSS. The notion of level set presents a flexible approach to IVNSS based decision making for facilitating the flow of traffic in an efficient way. The algorithm is designed to solve a decision making problem based on IVNSS using level soft sets. Level soft sets play a vital role in connecting IVNSS and crisp soft sets. Using level soft sets we do not deal with IVNSS directly, but work with crisp soft sets derived from them after choosing certain thresholds or decision strategies such

as top/bottom level decision rules or mid level decision rules. As per the algorithm, the choice value of the object in a level soft set is in fact the number of fair attributes which belong to that object on the premise that the degree of truth membership of x with respect to the parameter e is not less than the truth membership levels, the degree of indeterminacy membership of x with respect to the parameter e is not more than the indeterminacy levels and the degree of falsity membership of x with respect to the parameter e is not more than the falsity membership levels.

4.9.1 Relation From IVNSS[15]

Let $(G, A) \in IVNSS(X)$. Then a relation form of (G, A) is defined by

$R_{(G,A)} = \{(r_{(G,A)}(e, x)/(e, x); r_{(G,A)}(e, x) \in IVNS(X), e \in A, x \in U\}$ where $r_{(G,A)} : E \times X \rightarrow IVNS(X)$ and $r_{(G,A)}(e, x) = G_e(x)$ for all $x \in X, e \in A$.

That is $r_{(G,A)}(e, x) = G_e(x)$ is characterized by truth membership function μ , indeterminacy membership function v , and falsity membership function ω . For each point $e \in E$, and $x \in X$ and μ, v, ω are interval valued subsets of $[0, 1]$.

4.9.2. Example[15]

Let $X = \{x_1, x_2\}$ be the set of houses under consideration and $E = \{e_1, e_2, e_3, e_4, e_5\}$ be a set of parameters where e_1 = cheap, e_2 = beautiful, e_3 = in the green surroundings, e_4 = costly and e_5 = large respectively. Based on experts evaluation, an interval valued neutrosophic soft set (G, A) is defined and presented in tabular form as follows.

U	x_1	x_2
e_1	$[0.5, 0.8], [0.5, 0.9], [0.2, 0.5]$	$[0.4, 0.8], [0.2, 0.5], [0.5, 0.6]$
e_2	$[0.5, 0.8], [0.2, 0.8], [0.3, 0.7]$	$[0.1, 0.9], [0.6, 0.7], [0.2, 0.3]$
e_3	$[0.2, 0.7], [0.1, 0.5], [0.5, 0.8]$	$[0.5, 0.7], [0.1, 0.4], [0.6, 0.7]$
e_4	$[0.4, 0.5], [0.4, 0.9], [0.4, 0.9]$	$[0.3, 0.4], [0.6, 0.7], [0.1, 0.5]$
e_5	$[0.1, 0.7], [0.5, 0.6], [0.1, 0.5]$	$[0.6, 0.7], [0.2, 0.4], [0.3, 0.7]$

TABLE 2. The tabular representation of the IVNSS (G, A)

For this example $r_{(G,A)}(e, x) = G_e(x)$ is given below

$$G_{(e_1)}(x_1) = \langle [0.5, 0.8], [0.5, 0.9], [0.2, 0.5] \rangle$$

$$G_{(e_1)}(x_2) = \langle [0.4, 0.8], [0.2, 0.5], [0.5, 0.6] \rangle$$

$$G_{(e_2)}(x_1) = \langle [0.5, 0.8], [0.2, 0.8], [0.3, 0.7] \rangle$$

$$G_{(e_2)}(x_2) = \langle [0.1, 0.9], [0.6, 0.7], [0.2, 0.3] \rangle$$

$$G_{(e_3)}(x_1) = \langle [0.2, 0.7], [0.1, 0.5], [0.5, 0.8] \rangle$$

$$G_{(e_3)}(x_2) = \langle [0.5, 0.7], [0.1, 0.4], [0.6, 0.7] \rangle$$

$$G_{(e_4)}(x_1) = \langle [0.4, 0.5], [0.4, 0.9], [0.4, 0.9] \rangle$$

$$G_{(e_4)}(x_2) = \langle [0.3, 0.4], [0.6, 0.7], [0.1, 0.5] \rangle$$

$$G_{(e_5)}(x_1) = \langle [0.1, 0.7], [0.5, 0.6], [0.1, 0.5] \rangle$$

$$G_{(e_5)}(x_2) = \langle [0.6, 0.7], [0.2, 0.4], [0.3, 0.7] \rangle$$

The notion of level soft set and different thresholds on the parameters of interval valued intuitionistic fuzzy soft sets were introduced by Zhang et al. [16]. Irfan Deli [14] has extended these concepts to interval valued neutrosophic soft sets. In this work we extend the notion of level soft sets to IVNSS and different thresholds to group decision making by combining the input parameter sets.

4.9.3. Level Soft set of IVNSS [15]

Let $(G, A) \in IVNSS(X)$. For interval valued subsets $\alpha, \beta, \gamma \subseteq [0, 1]$, the $\langle \alpha, \beta, \gamma \rangle$ -level soft subset of (G, A) denoted by $((G, A); \langle \alpha, \beta, \gamma \rangle)$, is defined as

$((G, A); \langle \alpha, \beta, \gamma \rangle) = \{(e_i, (x_j \in X; \mu(x_{ij}) = 1)); e_i \in E\}$ where,

$$\mu(x_{ij}) = \begin{cases} 1 & \text{if } (\alpha, \beta, \gamma) \leq G_{e_i}(x_j) \\ 0 & \text{otherwise for all } x_j \in X \end{cases} \quad (3)$$

If $(\alpha, \beta, \gamma) \leq G_{e_i}(x_j)$, it means that the degree of truth membership of x with respect to the parameter e is not less than α , the degree of indeterminacy of x with respect to e is not more than β , and the degree of falsity of x with respect to e is not more than γ .

In practical application of IVNSS α, β, γ are the thresholds pre-established by the decision maker reflecting his requirements on truth, indeterminacy and falsity membership levels respectively.

4.9.4 Example

For the Example 4.9.2, the $([0.3, 0.4], [0.5, 0.7], [0.6, 0.8])$ level soft subset of (G, A) is

$$((G, A); \langle [0.3, 0.4], [0.5, 0.7], [0.6, 0.8] \rangle) = \{(e_1, u_2), (e_3, u_2), (e_5, u_1)\}$$

4.9.5 Average Threshold of an IVNSS [15]

Let $(G, A) \in IVNSS(X)$. For the given (G, A) , we define the average threshold denoted by $\langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Avg}$ as

$\langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Avg} : A \rightarrow IVNS(X)$ by

$$\langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Avg}(e_i) = \sum_{x \in X} G_{e_i}(x) / |x| \text{ for all } e_i \in A.$$

By average level decision rule we mean using the avg-threshold of the IVNSS in decision making procedure.

4.9.6 Example [15]

For the Example 4.9.2, the avg-threshold neutrosophic set is

$$\begin{aligned} \langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Avg}(e_1) &= \sum_{i=1}^2 e_{e_1}(x_i) / |x| = \langle [0.45, 0.8], [0.35, 0.7], [0.35, 0.55] \rangle \\ \langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Avg}(e_2) &= \sum_{i=1}^2 e_{e_2}(x_i) / |x| = \langle [0.3, 0.85], [0.4, 0.75], [0.25, 0.5] \rangle \\ \langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Avg}(e_3) &= \sum_{i=1}^2 e_{e_3}(x_i) / |x| = \langle [0.35, 0.7], [0.1, 0.45], [0.55, 0.75] \rangle \\ \langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Avg}(e_4) &= \sum_{i=1}^2 e_{e_4}(x_i) / |x| = \langle [0.35, 0.45], [0.5, 0.8], [0.25, 0.7] \rangle \\ \langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Avg}(e_5) &= \sum_{i=1}^2 e_{e_5}(x_i) / |x| = \langle [0.35, 0.7], [0.35, 0.5], [0.2, 0.6] \rangle \end{aligned}$$

Thus

$$\langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Avg} = \{ \langle [0.45, 0.8], [0.35, 0.7], [0.35, 0.55] \rangle / e_1, \dots \}$$

$$\begin{aligned} &\langle [0.3, 0.85], [0.4, 0.75], [0.25, 0.5] \rangle / e_2, \\ &\langle [0.35, 0.7], [0.1, 0.45], [0.55, 0.75] \rangle / e_3, \\ &\langle [0.35, 0.45], [0.5, 0.8], [0.25, 0.7] \rangle / e_4, \\ &\langle [0.35, 0.7], [0.35, 0.5], [0.2, 0.6] \rangle / e_5 \} \end{aligned}$$

For this example $((G, A); \langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Avg}) = \text{empty}$.

4.9.7 Max-min-min threshold of an IVNSS[15]

Let $(G, A) \in \text{IVNSS}(X)$. For the given (G, A) we define an interval-valued neutrosophic set $\langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Mmm} : A \rightarrow \text{IVNSS}(x)$ by

$$\begin{aligned} \langle (\alpha, \beta, \gamma) \rangle_{(G,A)}^{Mmm} = \{ & \langle [\max_{x \in X} \{ \underline{\mu}_{G(e_i)}^{(x)} \}, \max_{x \in X} \{ \overline{\mu}_{G(e_i)}^{(x)} \}], \\ & [\min_{x \in X} \{ \underline{\nu}_{G(e_i)}^{(x)} \}, \min_{x \in X} \{ \overline{\nu}_{G(e_i)}^{(x)} \}], \\ & [\min_{x \in X} \{ \underline{\omega}_{G(e_i)}^{(x)} \}, \min_{x \in X} \{ \overline{\omega}_{G(e_i)}^{(x)} \}] \rangle / e_i; e_i \in E \} \end{aligned}$$

$\langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Mmm}$ is called the maximum -minimum threshold of the IVNSS (G, A) . By Mmm-level decision rule we mean using the maximum-minimum-minimum threshold level soft set in IVNSS based decision making.

For this example $((G, A); \langle \alpha, \beta, \gamma \rangle_{(G,A)}^{Avg}) = \text{empty}$.

4.9.8 Min-min-min threshold of an IVNSS[15]

Let $(G, A) \in \text{IVNSS}(X)$. For the given (G, A) the min-min-min threshold denoted by $\langle \alpha, \beta, \gamma \rangle_{(G,A)}^{mmm} : A \rightarrow \text{IVNSS}(x)$ is an interval valued neutrosophic set defined as

$$\begin{aligned} \langle (\alpha, \beta, \gamma) \rangle_{(G,A)}^{mmm} = \{ & \langle [\min_{x \in X} \{ \underline{\mu}_{G(e_i)}^{(x)} \}, \min_{x \in X} \{ \overline{\mu}_{G(e_i)}^{(x)} \}], \\ & [\min_{x \in X} \{ \underline{\nu}_{G(e_i)}^{(x)} \}, \min_{x \in X} \{ \overline{\nu}_{G(e_i)}^{(x)} \}], \\ & [\min_{x \in X} \{ \underline{\omega}_{G(e_i)}^{(x)} \}, \min_{x \in X} \{ \overline{\omega}_{G(e_i)}^{(x)} \}] \rangle / e_i; e_i \in E \} \end{aligned}$$

By mmm-level decision rule we mean using the min-min-min threshold level soft set in IVNSS based decision making.

4.10 The Proposed Two Stage IVNSS Algorithm for Traffic Signal control

In this research the following algorithm is developed for controlling the flow of traffic using the proposed IVNSS decision making technique.

Steps of the Algorithm:

- (i) Input the data set $X = \{SG_1, SG_2, SG_3, SG_4\}$ and the control parameters $E = \{e_1, e_2\}$, where SG_1, SG_2, SG_3 and SG_4 are the four signal groups or phases under consideration, e_1 = Average arrival rate on lanes with current green light in veh/min/lane, e_2 = maximum queue length on lane with red light, which may receive green signal in the next phase, in veh/lane.
- (ii) Data Collection: Collect all the necessary crisp data for the input parameters from traffic sensors.
- (iii) Neutrosophication/Normalization: Transform the crisp data into related IVNSS data.
- (iv) Obtain the IVNSS (F, A) and (G, B) as explained in section 4.7.
- (v) Combination of data sets: Use (F, A) and (G, B) to find $(F, A) \text{ AND } (G, B)$. As there are five levels in A and four levels in B , we have $5 \times 4 = 20$ values of the form $e_{ij} = a_i \wedge q_j; i = 1, 2, 3, 4, 5 \text{ and } j = 1, 2, 3, 4$.

Let the resultant IVNSS of $(F, A) \text{ AND } (G, B) = (K, R)$.

(vi) Parameter reduction: Input a threshold interval valued neutrosophic set $\langle \alpha, \beta, \gamma \rangle_{(K,R)}^{avg}$ or $\langle \alpha, \beta, \gamma \rangle_{(K,R)}^{Mmm}$ or $\langle \alpha, \beta, \gamma \rangle_{(K,R)}^{mmm}$.

Using avg-level decision rule (or Mmm-level decision rule or mmm-level decision rule) for making decision based on the resultant IVNSS (K, R) .

(vii) Compute the avg-level soft set $((K, R); \langle \langle \alpha, \beta, \gamma \rangle_{(K,R)}^{avg} \rangle (or \langle \alpha, \beta, \gamma \rangle_{(K,R)}^{Mmm} or \langle \alpha, \beta, \gamma \rangle_{(K,R)}^{mmm}))$

(viii) Obtain the tabular form of the level soft set (K, R) calculated in step (vii).

(ix) Compute the choice values (weight) c_i of SG_i for every i in X .

(x) Select k such that $c_k = \max_{i \in X} C_i$. The optimal decision is to select signal group SG_k for the next phase.

(xi) Based on the choice value (weight) determine the extension time for each phase or signal group and extension time for total cycle length as explained in Section 4.5.

Note: If C_k attains maximum value for more than one signal group SG_k , then any one such group can be chosen arbitrarily. Alternatively, when more than one optimal k exists, we may go back to step (vi) and change the threshold (or decision rule) so that a unique optimal choice remains.

5. Verification of the Model

5.1 Traffic Control and Traffic Signal timing in Addis Ababa City.

Addis Ababa city road authority (AACRA) has installed traffic light signal controllers at many of the road junctions in the city to control and regulate traffic flow. At each of these junctions fixed time cycle traffic management system is employed to control traffic signals. These traffic control systems do not consider the congestion in lanes or need to extend or terminate green signal times due to congestion or no traffic. At times these signals are turned off and the phase change is controlled manually by traffic police officers who use predetermined sequential order to control traffic flow. Over 500 traffic police officers are deployed in the city every day. It is true that efficient and experienced traffic police officers can adjust the signal timings according to traffic intensity, especially during peak hours. On the other hand, the office that is responsible for traffic control in the city sets the length of each phase group in a cycle or the length of green light signal in a cycle to control traffic at the junction. Usually the maximum time given for green light in a signal group is 120 seconds with a maximum total time of 8 minutes for one full cycle. The minimum time given for green light in the city is 12 seconds. The maximum time is given for intersections that suffer from heavy traffic during the peak hours.

5.2 Site Selection and Description of the Verification Area

Addis Ababa, the capital city of Ethiopia has an estimated population of 5.5 millions and is located in the horn of Africa. The city is also the seat of African Union (AU), African Economic Commission (AEC) and more than 120 embassies of different countries. It is reported that about 80 percent of the vehicles in Ethiopia are found in Addis Ababa with an yearly growth rate of 5 percent. The total road length of the city is 1329.5 kilometers, out of which 29.7 percent is paved and 70.3 percent unpaved. According to

AACRA ,the city currently has 67 km of road with in 100 sq.km,which is minimal when compared to Nairobi ,Kenya where 155 km of road is found with in 100 sq.km.The city experiences traffic congestion at different intersections throughout the day ,the average traffic congestion intensity in Addis Ababa city expressed in vehicle minute or person minute is very high and the result shows that on the average about 38 vehicle days and 352 person days are wasted at each intersection leg or congestion spot per day.As per the information obtained from Addis Ababa city traffic management directorate office ,about 30 traffic intersections are identified out of which 14 traffic junctions are awarded installation of technologically advanced signal lights with traffic detectors.Unless otherwise implemented with proper care this could be a major cause for congestion.It has been identified that signal phase improvement is one of the most useful and cost effective method to reduce congestion.

For verifying the proposed IVNSS traffic signal control model and to compare the performance of existing fixed cycle time traffic signal controller with the proposed model ,the main traffic network in the inner city of Addis Ababa ,namely St. Stifanos traffic junction is selected.This junction consists of four roads :Bambis (West),Betemengist (south),Meskel square (East) and Dembel(north).The geometry of the intersection is shown in figure below.Existing delays and traffic volumes are measured on typical working days.The data collected in the field are used to estimate the optimal signal plans.The optimized delay and signal timings are compared with the existing values to evaluate the performance of the proposed IVNSS traffic control model.

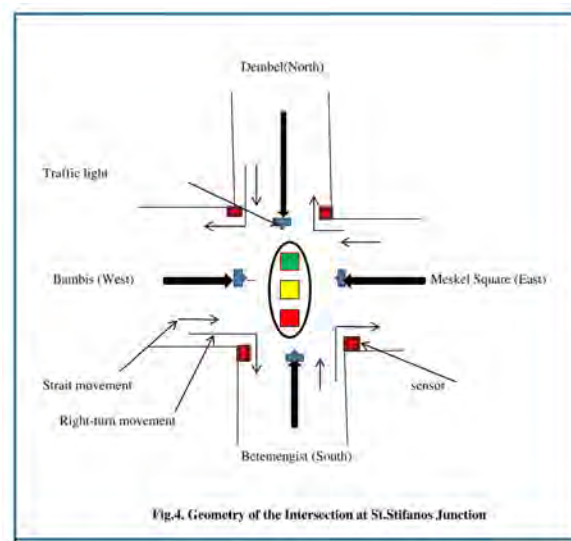


FIGURE 4. Geometry of the Intersection at St. Stifanos Junction

5.3 Verification of the Model using the Sample Data from the Selected Traffic Junction

The steps of the algorithm presented in Section 4.10 are applied to the sample data obtained from St. Stifanos junction .The decision making procedure of the Algorithm is verified with the data collected

as follows:

Step (i) $X = \{SG_1, SG_2, SG_3, SG_4\}$ where SG_1, SG_2, SG_3 and SG_4 are the four signal groups explained in Section 4.3. The parameter set is $E = \{e_1, e_2\}$ as defined in Section 4.4

Step(ii) A sample data obtained from vehicular detector at St. Stifanos junction is tabulated below.

X	e_1	e_2
SG_1	61	52
SG_2	54	79
SG_3	64	60
SG_4	59	62

TABLE 3. Sample Data obtained From Vehicular Detector at St. Stifanos Junction

Step(iii) The evaluation scores obtained from experts with respect to all roads and all levels of the parameter e_1 are given below.

For the signal group SG_1 with respect to a_1 obtained from five experts

(SG_1, a_1)	Truth membership degree (μ)	5.8	7.9	8.3	8.7	5.1
	Indeterminacy membership degree (v)	3.3	1.4	1.7	2.1	4.4
	falsity membership degree (ω)	4.1	2.4	2.6	2.1	3.7

Similarly the data collected from experts for other combinations are tabulated below.

(SG_1, a_2)	μ	9.1	6.3	7.1	6.9	8.1
	v	1.3	3.4	3.7	4.1	2.8
	ω	1.9	3.9	3.6	3.1	2.3
(SG_1, a_4)	μ	2.7	2.1	1.9	2.0	3.0
	v	2.3	1.4	1.5	2.0	1.0
	ω	6.1	6.9	6.5	7.1	5.5
(SG_2, a_1)	μ	2.8	4.9	1.3	4.5	2.1
	v	5.3	4.5	3.7	6.1	7.4
	ω	7.8	8.9	8.6	9.1	7.7
(SG_2, a_3)	μ	4.8	8.9	5.3	8.5	5.1
	v	7.3	8.4	5.7	6.1	5.4
	ω	3.1	2.5	2.1	1.4	3.7
(SG_2, a_5)	μ	4.5	3.9	8.5	3.7	4.6
	v	4.3	5.4	2.7	5.5	4.6
	ω	5.1	6.5	1.6	4.7	3.9
(SG_3, a_2)	μ	7.8	6.6	6.3	8.2	5.8
	v	5.7	6.5	3.7	5.1	8.4
	ω	2.1	3.9	4.6	2.0	4.2
(SG_3, a_4)	μ	5.5	2.7	5.4	3.2	6.2
	v	5.2	2.5	4.5	3.1	6.4
	ω	6.1	8.7	6.4	7.3	3.7
(SG_1, a_3)	μ	4.1	3.4	2.8	3.1	4.3
	v	4.3	5.4	2.9	4.1	4.0
	ω	4.1	5.9	4.6	5.1	3.9
(SG_1, a_5)	μ	2.8	1.9	2.3	1.7	3.1
	v	2.3	1.1	1.2	1.6	2.1
	ω	7.3	8.9	9.5	9.1	7.7
(SG_2, a_2)	μ	3.5	3.9	2.3	4.7	4.1
	v	5.3	6.4	5.5	3.8	4.7
	ω	4.5	5.8	6.6	7.1	4.5
(SG_2, a_4)	μ	8.8	7.5	6.5	8.5	5.5
	v	6.3	4.5	5.6	7.1	5.4
	ω	1.3	2.3	5.1	8.1	1.4
(SG_3, a_1)	μ	6.8	8.7	5.3	6.2	4.1
	v	8.3	3.5	4.7	7.1	6.4
	ω	6.1	1.9	6.6	2.8	4.7
(SG_3, a_3)	μ	5.8	4.7	9.3	7.2	4.1
	v	3.3	4.5	5.7	7.1	8.5
	ω	5.1	2.9	1.6	3.8	4.7
(SG_3, a_5)	μ	3.6	2.7	5.3	2.2	4.1
	v	4.3	2.5	4.6	5.1	7.4
	ω	7.3	8.8	5.4	8.0	4.7

(SG_4, a_1)	μ	4.8	6.2	5.3	6.2	4.3
	v	4.6	7.5	1.7	8.2	5.7
	ω	6.3	2.9	7.6	2.8	4.5
(SG_4, a_3)	μ	8.5	7.6	6.3	7.4	5.4
	v	2.1	7.2	8.2	5.6	2.2
	ω	1.1	3.5	4.7	5.6	1.5
(SG_4, a_5)	μ	5.5	4.9	8.3	3.4	5.3
	v	2.5	1.9	7.4	3.3	5.2
	ω	6.3	8.5	4.7	8.4	5.5

(SG_4, a_2)	μ	6.4	9.7	3.3	8.4	5.4
	v	8.1	3.2	4.3	7.6	6.2
	ω	3.1	1.6	8.6	2.1	4.5
(SG_4, a_4)	μ	6.1	5.7	8.3	2.7	5.8
	v	8.1	7.5	4.3	7.3	6.4
	ω	5.2	7.6	1.6	8.5	4.7

Step(iv) Using the maximum-minimum neutrosophication/normalization technique the maximum-minimum evaluation scores for all the above combinations are calculated and tabulated below.

(SG_1, a_1)	scores	Max.	Min.
	μ	8.7	5.1
	v	4.4	1.4
(SG_1, a_4)	scores	Max.	Min.
	μ	3.0	1.9
	v	2.3	1.0
(SG_2, a_2)	scores	Max.	Min.
	μ	4.7	2.3
	v	6.4	3.8
(SG_2, a_5)	scores	Max.	Min.
	μ	8.5	3.1
	v	5.5	2.7
(SG_3, a_3)	scores	Max.	Min.
	μ	9.3	4.1
	v	6.4	2.5
(SG_4, a_1)	scores	Max.	Min.
	μ	6.2	4.3
	v	8.2	1.7

(SG_1, a_2)	scores	Max.	Min.
	μ	9.1	6.3
	v	4.1	1.3
(SG_1, a_5)	scores	Max.	Min.
	μ	3.1	1.7
	v	2.3	1.1
(SG_2, a_3)	scores	Max.	Min.
	μ	8.9	4.8
	v	8.4	5.4
(SG_3, a_1)	scores	Max.	Min.
	μ	8.7	4.1
	v	8.3	3.5
(SG_3, a_4)	scores	Max.	Min.
	μ	6.2	2.7
	v	6.4	2.5
(SG_4, a_2)	scores	Max.	Min.
	μ	9.7	3.3
	v	8.1	3.2

(SG_1, a_3)	scores	Max.	Min.
	μ	4.3	2.8
	v	5.4	2.7
(SG_2, a_1)	scores	Max.	Min.
	μ	4.9	1.3
	v	7.4	3.7
(SG_2, a_4)	scores	Max.	Min.
	μ	8.8	5.5
	v	7.1	4.5
(SG_3, a_2)	scores	Max.	Min.
	μ	8.2	5.8
	v	8.4	3.7
(SG_3, a_5)	scores	Max.	Min.
	μ	5.3	2.2
	v	7.4	2.5
(SG_4, a_3)	scores	Max.	Min.
	μ	8.5	5.4
	v	8.2	2.1

(SG_4, a_4)

scores	Max.	Min.
μ	8.3	2.7
ν	8.1	4.3
ω	8.5	1.6

 (SG_4, a_5)

scores	Max.	Min.
μ	8.3	3.4
ν	7.4	1.9
ω	8.5	4.7

Similarly the data collected from experts for the parameter e_2 are presented below after maximum-minimum transformation.

 (SG_1, q_1)

scores	Max.	Min.
μ	8.3	3.5
ν	8.1	2.6
ω	6.8	2.2

 (SG_1, q_2)

scores	Max.	Min.
μ	8.6	4.4
ν	8.7	2.7
ω	8.6	1.5

 (SG_1, q_3)

scores	Max.	Min.
μ	9.3	5.7
ν	8.2	3.7
ω	7.9	1.7

 (SG_1, q_4)

scores	Max.	Min.
μ	8.7	3.6
ν	8.2	2.8
ω	9.1	1.4

 (SG_2, q_1)

scores	Max.	Min.
μ	9.2	5.4
ν	7.4	2.8
ω	5.5	1.9

 (SG_1, q_2)

scores	Max.	Min.
μ	9.1	5.7
ν	8.2	3.2
ω	7.1	2.6

 (SG_2, q_3)

scores	Max.	Min.
μ	7.3	3.4
ν	7.5	2.8
ω	7.5	4.2

 (SG_2, q_4)

scores	Max.	Min.
μ	5.7	1.5
ν	7.2	3.1
ω	8.1	4.5

 (SG_3, q_1)

scores	Max.	Min.
μ	8.6	4.4
ν	8.2	2.2
ω	7.6	3.5

 (SG_3, q_2)

scores	Max.	Min.
μ	8.3	5.4
ν	7.7	4.2
ω	7.6	1.6

 (SG_3, q_3)

scores	Max.	Min.
μ	7.6	3.6
ν	7.6	2.6
ω	7.7	2.5

 (SG_3, q_4)

scores	Max.	Min.
μ	7.6	2.6
ν	6.6	2.5
ω	8.4	4.2

 (SG_4, q_1)

scores	Max.	Min.
μ	7.4	2.5
ν	7.8	3.1
ω	7.3	4.5

 (SG_4, q_2)

scores	Max.	Min.
μ	8.6	3.7
ν	7.4	3.4
ω	6.4	2.5

 (SG_4, q_3)

scores	Max.	Min.
μ	8.3	3.4
ν	7.4	3.2
ω	5.6	2.5

 (SG_4, q_4)

scores	Max.	Min.
μ	8.7	2.6
ν	7.4	2.4
ω	7.5	1.7

Step (v) These maximum-minimum evaluation scores are converted into sub-intervals of $[0, 1]$ and the tabular representation of (F, A) and (G, B) are given below.

X	a_1	a_2	a_3	a_4	a_5
SG_1	[0.51, 0.87], [0.14, 0.44], [0.21, 0.41]	[0.63, 0.91], [0.13, 0.41], [0.19, 0.39]	[0.28, 0.43], [0.27, 0.54], [0.39, 0.59]	[0.19, 0.3], [0.1, 0.23], [0.55, 0.71]	[0.17, 0.31], [0.11, 0.23], [0.73, 0.95]
SG_2	[0.13, 0.49], [0.37, 0.74], [0.75, 0.91]	[0.23, 0.47], [0.38, 0.64], [0.45, 0.71]	[0.48, 0.89], [0.54, 0.84], [0.14, 0.37]	[0.55, 0.88], [0.45, 0.71], [0.13, 0.81]	[0.37, 0.85], [0.27, 0.55], [0.16, 0.65]
SG_3	[0.41, 0.87], [0.35, 0.83], [0.19, 0.66]	[0.58, 0.82], [0.37, 0.84], [0.2, 0.46]	[0.41, 0.93], [0.33, 0.85], [0.16, 0.51]	[0.27, 0.62], [0.25, 0.64], [0.37, 0.87]	[0.22, 0.53], [0.25, 0.74], [0.47, 0.88]
SG_4	[0.43, 0.62], [0.17, 0.82], [0.28, 0.76]	[0.33, 0.97], [0.32, 0.81], [0.16, 0.86]	[0.54, 0.85], [0.21, 0.82], [0.11, 0.56]	[0.27, 0.83], [0.43, 0.81], [0.16, 0.85]	[0.34, 0.83], [0.19, 0.74], [0.47, 0.85]

TABLE 4. Tabular Representation of (F, A)

Step(vi) Calculate (F, A) AND $(G, B) = (K, R)$ and present the tabular form of (K, R) .

X	q_1	q_2	q_3	q_4
SG_1	[0.35, 0.83], [0.26, 0.81], [0.22, 0.68]	[0.44, 0.86], [0.27, 0.87], [0.15, 0.86]	[0.57, 0.93], [0.37, 0.82], [0.17, 0.79]	[0.36, 0.87], [0.28, 0.82], [0.14, 0.91]
SG_2	[0.54, 0.92], [0.28, 0.78], [0.19, 0.55]	[0.57, 0.91], [0.32, 0.82], [0.26, 0.71]	[0.34, 0.73], [0.28, 0.75], [0.42, 0.75]	[0.15, 0.57], [0.31, 0.72], [0.45, 0.81]
SG_3	[0.44, 0.86], [0.22, 0.82], [0.35, 0.76]	[0.54, 0.83], [0.42, 0.77], [0.16, 0.76]	[0.36, 0.79], [0.26, 0.76], [0.25, 0.77]	[0.26, 0.76], [0.25, 0.66], [0.42, 0.84]
SG_4	[0.25, 0.74], [0.31, 0.78], [0.45, 0.73]	[0.37, 0.86], [0.34, 0.74], [0.25, 0.64]	[0.34, 0.83], [0.32, 0.74], [0.25, 0.56]	[0.26, 0.87], [0.24, 0.74], [0.17, 0.75]

TABLE 5. Tabular Representation of (G, B)

X	e_{11}	e_{12}	e_{13}	e_{14}
SG_1	[0.35, 0.83], [0.26, 0.80], [0.22, 0.68]	[0.44, 0.86], [0.27, 0.87], [0.21, 0.86]	[0.51, 0.87], [0.37, 0.82], [0.21, 0.79]	[0.36, 0.87], [0.28, 0.82], [0.21, 0.91]
SG_2	[0.13, 0.49], [0.37, 0.75], [0.75, 0.91]	[0.13, 0.49], [0.37, 0.82], [0.75, 0.91]	[0.13, 0.49], [0.37, 0.75], [0.75, 0.91]	[0.13, 0.49], [0.37, 0.74], [0.75, 0.91]
SG_3	[0.41, 0.86], [0.35, 0.83], [0.35, 0.76]	[0.41, 0.83], [0.42, 0.83], [0.19, 0.76]	[0.36, 0.79], [0.35, 0.83], [0.25, 0.77]	[0.26, 0.76], [0.35, 0.83], [0.42, 0.84]
SG_4	[0.25, 0.62], [0.31, 0.82], [0.45, 0.76]	[0.37, 0.62], [0.34, 0.82], [0.28, 0.76]	[0.34, 0.62], [0.32, 0.82], [0.28, 0.76]	[0.26, 0.62], [0.24, 0.82], [0.28, 0.76]
X	e_{21}	e_{22}	e_{23}	e_{24}
SG_1	[0.35, 0.83], [0.26, 0.80], [0.22, 0.68]	[0.44, 0.86], [0.27, 0.87], [0.19, 0.86]	[0.57, 0.91], [0.37, 0.82], [0.19, 0.79]	[0.36, 0.87], [0.28, 0.82], [0.19, 0.91]
SG_2	[0.23, 0.47], [0.38, 0.78], [0.45, 0.71]	[0.23, 0.47], [0.38, 0.82], [0.45, 0.71]	[0.23, 0.47], [0.38, 0.75], [0.45, 0.75]	[0.13, 0.49], [0.37, 0.74], [0.75, 0.91]
SG_3	[0.44, 0.82], [0.37, 0.84], [0.35, 0.76]	[0.54, 0.82], [0.42, 0.84], [0.20, 0.76]	[0.36, 0.79], [0.37, 0.84], [0.25, 0.77]	[0.26, 0.76], [0.35, 0.83], [0.42, 0.84]
SG_4	[0.25, 0.74], [0.32, 0.81], [0.45, 0.86]	[0.33, 0.86], [0.34, 0.81], [0.25, 0.86]	[0.33, 0.83], [0.32, 0.81], [0.25, 0.86]	[0.26, 0.62], [0.24, 0.82], [0.28, 0.76]
X	e_{31}	e_{32}	e_{33}	e_{34}
SG_1	[0.28, 0.43], [0.27, 0.81], [0.39, 0.68]	[0.28, 0.43], [0.27, 0.87], [0.39, 0.86]	[0.28, 0.43], [0.37, 0.82], [0.39, 0.79]	[0.28, 0.43], [0.28, 0.82], [0.39, 0.91]
SG_2	[0.48, 0.89], [0.54, 0.84], [0.19, 0.55]	[0.48, 0.89], [0.54, 0.84], [0.26, 0.71]	[0.34, 0.73], [0.54, 0.84], [0.42, 0.75]	[0.15, 0.57], [0.54, 0.84], [0.45, 0.81]
SG_3	[0.41, 0.86], [0.33, 0.85], [0.35, 0.76]	[0.41, 0.83], [0.42, 0.85], [0.16, 0.76]	[0.36, 0.79], [0.33, 0.85], [0.25, 0.77]	[0.26, 0.76], [0.33, 0.85], [0.42, 0.84]
SG_4	[0.25, 0.74], [0.31, 0.82], [0.45, 0.73]	[0.37, 0.85], [0.34, 0.82], [0.25, 0.64]	[0.34, 0.83], [0.32, 0.82], [0.25, 0.56]	[0.26, 0.85], [0.24, 0.82], [0.17, 0.75]
X	e_{41}	e_{42}	e_{43}	e_{44}
SG_1	[0.19, 0.3], [0.26, 0.81], [0.55, 0.71]	[0.19, 0.30], [0.27, 0.87], [0.55, 0.86]	[0.19, 0.30], [0.37, 0.82], [0.55, 0.79]	[0.19, 0.30], [0.28, 0.82], [0.55, 0.91]
SG_2	[0.54, 0.88], [0.45, 0.74], [0.19, 0.81]	[0.55, 0.88], [0.45, 0.82], [0.26, 0.81]	[0.34, 0.73], [0.45, 0.75], [0.42, 0.81]	[0.15, 0.57], [0.45, 0.72], [0.45, 0.81]
SG_3	[0.27, 0.62], [0.25, 0.82], [0.37, 0.87]	[0.27, 0.62], [0.42, 0.77], [0.37, 0.87]	[0.27, 0.62], [0.26, 0.76], [0.37, 0.87]	[0.26, 0.62], [0.25, 0.66], [0.42, 0.87]
SG_4	[0.25, 0.74], [0.43, 0.81], [0.45, 0.85]	[0.27, 0.83], [0.43, 0.81], [0.25, 0.85]	[0.27, 0.83], [0.43, 0.81], [0.25, 0.85]	[0.26, 0.83], [0.43, 0.81], [0.17, 0.85]
X	e_{51}	e_{52}	e_{53}	e_{54}
SG_1	[0.17, 0.31], [0.26, 0.81], [0.73, 0.95]	[0.17, 0.31], [0.27, 0.87], [0.73, 0.95]	[0.17, 0.31], [0.37, 0.82], [0.73, 0.95]	[0.17, 0.31], [0.28, 0.82], [0.73, 0.95]
SG_2	[0.37, 0.85], [0.28, 0.74], [0.19, 0.65]	[0.37, 0.85], [0.32, 0.82], [0.26, 0.71]	[0.34, 0.73], [0.28, 0.75], [0.42, 0.75]	[0.15, 0.57], [0.31, 0.72], [0.45, 0.81]
SG_3	[0.22, 0.53], [0.31, 0.78], [0.47, 0.88]	[0.22, 0.53], [0.42, 0.77], [0.47, 0.88]	[0.22, 0.53], [0.26, 0.76], [0.47, 0.88]	[0.22, 0.53], [0.25, 0.74], [0.47, 0.88]
SG_4	[0.25, 0.74], [0.31, 0.78], [0.47, 0.95]	[0.34, 0.83], [0.34, 0.74], [0.47, 0.95]	[0.34, 0.83], [0.32, 0.74], [0.47, 0.95]	[0.26, 0.83], [0.24, 0.74], [0.47, 0.85]

TABLE 6. Tabular Representation of $(K, R) = (F, A) \wedge (G, B)$

Step (vii) Evaluate the threshold interval-valued neutrosophic set $\langle \alpha, \beta, \gamma \rangle_{(K,R)}^{avg}$ and tabulate it.

$$\begin{aligned}
\langle \alpha, \beta, \gamma \rangle_{(K,R)}^{avg} = & \{ \langle [0.285, 0.7], [0.3225, 0.81] \rangle, [0.4425, 0.7775] \} / e_{11}, \\
& < [0.3375, 0.7], [0.35, 0.835], [0.3575, 0.8225] > / e_{12}, \\
& < [0.335, 0.6925], [0.3525, 0.805], [0.3725, 0.8325] > / e_{13}, \\
& < [0.2525, 0.685], [0.31, 0.8025], [0.415, 0.855] > / e_{14}, \\
& < [0.3175, 0.715], [0.3325, 0.81], [0.3675, 0.7525] > / e_{21}, \\
& < [0.385, 0.7525], [0.3525, 0.835], [0.2725, 0.7975] > / e_{22}, \\
& < [0.3225, 0.75], [0.36, 0.805], [0.285, 0.7925] > / e_{23}, \\
& < [0.2525, 0.695], [0.31, 0.8025], [0.41, 0.88] > / e_{24}, \\
& < [0.405, 0.73], [0.3625, 0.83], [0.345, 0.68] > / e_{31}, \\
& < [0.385, 0.75], [0.3925, 0.845], [0.265, 0.7425] > / e_{32}, \\
& < [0.33, 0.695], [0.39, 0.8325], [0.3275, 0.7175] > / e_{33}, \\
& < [0.2375, 0.6525], [0.3475, 0.8325], [0.3575, 0.8275] > / e_{34},
\end{aligned}$$

$$\begin{aligned}
&< [0.3125, 0.635], [0.3475, 0.795], [0.39, 0.81] > /e_{41}, \\
&< [0.32, 0.6575], [0.3925, 0.8175], [0.3575, 0.8475] > /e_{42}, \\
&< [0.2675, 0.62], [0.3775, 0.785], [0.345, 0.83] > /e_{43}, \\
&< [0.215, 0.58], [0.2525, 0.7525], [0.3975, 0.86] > /e_{44}, \\
&< [0.2525, 0.6075], [0.29, 0.7775], [0.465, 0.8575] > /e_{51}, \\
&< [0.275, 0.63], [0.3375, 0.8], [0.4825, 0.8725] > /e_{52}, \\
&< [0.2675, 0.6], [0.3075, 0.7675], [0.5225, 0.8825] > /e_{53}, \\
&< [0.2, 0.56], [0.27, 0.755], [0.53, 0.8725] > /e_{54}\}.
\end{aligned}$$

Step(viii) Compute avg-level soft set $((K, R); \langle \alpha, \beta, \gamma \rangle_{(K, R)}^{avg})$

$$\begin{aligned}
&((K, R); \langle \alpha, \beta, \gamma \rangle_{(K, R)}^{avg}) = \\
&\{(e_{11}, \{SG_1\}), (e_{13}, \{SG_3\}), (e_{33}, \{SG_4\}), (e_{34}, \{SG_4\}), (e_{11}, \{SG_1\}), (e_{53}, \{SG_2\}), \\
&(e_{54}, \{SG_3, SG_4\})\}
\end{aligned}$$

The tabular form of the level soft set $((K, R); \langle \alpha, \beta, \gamma \rangle_{(K, R)}^{avg})$

U	SG_1	SG_2	SG_3	SG_4
e_{11}	1	0	0	0
e_{12}	0	0	0	0
e_{13}	0	0	1	0
e_{14}	0	0	0	0
e_{21}	0	0	0	0
e_{22}	0	0	0	0
e_{23}	0	0	0	0
e_{24}	0	0	0	0
e_{31}	0	0	0	0
e_{32}	0	0	0	0
e_{33}	0	0	0	1
e_{34}	0	0	0	1
e_{41}	0	0	0	0
e_{42}	0	0	0	0
e_{43}	0	0	0	0
e_{44}	0	0	0	0
e_{51}	1	0	0	0
e_{52}	0	0	0	0
e_{53}	0	1	0	0
e_{54}	0	0	1	1

TABLE 7. The Tabular Representation of The Level Soft Set (K, R)

Step(ix) Compute the choice values (weights) c_i for SG_i for $i = 1, 2, 3, 4$.

$$c_1 = \sum_{SG_1} e_{ij} = 2, c_2 = \sum_{SG_2} e_{ij} = 1, c_3 = \sum_{SG_3} e_{ij} = 2, c_4 = \sum_{SG_4} e_{ij} = 3$$

Step (x) $c_4 = \max\{c_1, c_2, c_3, c_4\}$

Hence the optimal decision is to select SG_4 for extension of green signal time.

Step(xi) Based on the weights obtained in Step (ix) determine the extension time for each phase or signal group .

$$\begin{aligned} \text{Total Green Time (TGT)} &= (\sum C_n - \text{Min}W) * \left(\frac{\text{Max}ST - \text{Min}ST}{\text{Max}W - \text{Min}W} \right) + \text{Min}ST \\ &= (8 - 1) * \left(\frac{120 - 12}{3 - 1} \right) + 12 = 390 \text{seconds.} \end{aligned}$$

where

- $\sum C_n$ is the total weight of signal groups $= 2 + 1 + 2 + 3 = 8$
- $\text{Min}W = 1$ and $\text{Max}W = 3$ are the minimum and maximum values of weights respectively.
- $\text{Min}ST = 12$ seconds and $\text{Max}ST = 120$ seconds, are minimum and maximum value of green time in a cycle.
- n is the group index ($n = 1, 2, 3, 4$).

The green time of each signal group is

- Green Time of $SG_1 = \frac{C_1 * TGT}{\sum C_n} = \frac{2 * 390}{8} = 97.5$ seconds
- Green Time of $SG_2 = \frac{C_2 * TGT}{\sum C_n} = \frac{1 * 390}{8} = 48.75$ seconds
- Green Time of $SG_3 = \frac{C_3 * TGT}{\sum C_n} = \frac{2 * 390}{8} = 97.5$ seconds
- Green Time of $SG_4 = \frac{C_4 * TGT}{\sum C_n} = \frac{3 * 390}{8} = 146.25$ seconds.

6. A Comparative Analysis

In this section we make a comparative study of the proposed two stage traffic control model with the existing traffic control systems such as traditional traffic traffic control system ,fixed cycle or pre-timed traffic control models. Researches have established that traditional traffic control systems contribute to traffic congestion as they are one of the main reasons for congestion if not implemented properly.

The present day urban traffic control system uses fixed cycle time or pre-timed signal control. In this case the phase change occurs sequentially and the green times are fixed. The system neither takes into account the varying traffic intensity with respect to time nor the peak hour heavy traffic in to consideration. The present day traffic control systems also suffer from indeterminacy due to various factors like unawareness of the problem, inaccurate and imperfect data, poor forecasting and uncertainty in the constraints. Due to this delay or waiting time at traffic intersections increase and at time mounts even up to 5 or 6 full cycle times depending on the intensity of traffic. Some developed countries use fuzzy logic controllers to regulate traffic at intersections. Though fuzzy logic is capable of handling uncertainty and impreciseness in data ,it can not cope up with indeterminacy. This drawback can be overcome by the proposed two stage traffic control model as it takes into account indeterminacy and dynamically manages the traffic flow by extending/terminating green light timings and effecting phase

changes as necessitated by the traffic intensity at that time, not necessarily sequentially as followed in present day traffic control techniques.

7. CONCLUSION

In this research we have developed an algorithm for traffic signal control based on interval-valued neutrosophic soft set data. This algorithm can control both activities namely phase change and green time duration dynamically taking into consideration of the current traffic intensity and queue of vehicles estimated linguistically using experts opinions and converting this data into interval valued neutrosophic soft sets. Based on the decision making technique developed, the algorithm makes use of the existing traffic conditions together with its uncertainty and indeterminacy to control and facilitate smooth flow of traffic at four way signalized isolated intersections. The algorithm is verified with the sample data collected at St. Stefanos four way isolated junction in Addis Ababa city. As a future research direction we are developing a simulation technique to generate data and to test and validate the proposed traffic signal control model.

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Neutrosophic Vague Binary G – subalgebra of G - algebra

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Abstract: Nowadays, human society is using artificial intelligence in a large manner so as to upgrade the present existing applicational criteria's and tools. Logic is the underlying principle to these works. Algebra is inevitably inter-connected with logic. Hence its achievements to the scientific research outputs have to be addressed. For these reasons, nowadays, research on various algebraic structures are going on wide. Crisp set has also got developed in a parallel way in the forms as fuzzy, intuitionistic fuzzy, rough, vague, neutrosophic, plithogenic etc. Sets with one or more algebraic operations will form different new algebraic structures for giving assistance to these logics, which in turn acts to as, a support to artificial intelligence. BCH/BCI/BCK- are some algebras developed in the first phase of algebraic development output. After that, so many outputs got flashed out, individually and in combinations in no time. Q- algebra and QS –algebra are some of these and could be showed as such kind of productions. G - algebra is considered as an extension to QS – algebra. In this paper neutrosophic vague binary G – subalgebra of G – algebra is generated with example. Notions like, 0 – commutative G - subalgebra, minimal element, normal subset etc. are investigated. Conditions to define derivation and regular derivation for this novel concept are clearly presented with example. Constant of G – algebra can't be treated as the identity element, generally. In this paper, it is well explained with example. Cosets for neutrosophic vague binary G – subalgebra of G - algebra is developed with proper explanation. Homomorphism for this new concept has been also got commented. Its kernel, monomorphism and isomorphism are also have discussed with proper attention.

Keywords: neutrosophic vague binary G - subalgebra, neutrosophic vague binary G - normal set, neutrosophic vague binary G – normal subalgebra, neutrosophic vague binary G G - part, neutrosophic vague binary G - p radical , neutrosophic vague binary G - p semisimple, neutrosophic vague binary G - minimal element, 0- commutative neutrosophic vague binary G – subalgebra, neutrosophic vague binary G – Derivation, neutrosophic vague binary G – Regular Derivation, neutrosophic vague binary G - Coset, Kernel of neutrosophic vague binary G – Homomorphism.

Notations: NVBS – neutrosophic vague binary set, NVBSS – neutrosophic vague binary subset. In this paper NVB is used to indicate neutrosophic vague binary and NV is used to indicate neutrosophic vague and N is used to indicate neutrosophic.

1.Introduction

Without mathematics mobility in human-life even became an unthinkable process. But when get into the mathematical world, one faces with, versatile facets of maths, which again get take diversions. The thing is that, dry subject is less get commented on or even less get touched with!

Algebra can also be considered so. But the entry of artificial intelligence made things different. Human world can simply neither ignore nor reject 'robots and computers' from their presently existing life pattern, due to their high impact in changing life style. So the question is that, what is the importance of algebra to these new scenario? Is it really useful for this robotic framed world? Answer is, yes! Since artificial intelligence is the raw material to robotics and to all the other newly developing phenomenon's. Logic is a foundation to artificial intelligence. Here a rapport activity can be seen in the picture. For logical calculations, algebra is very important. So these mixed works of algebra and sets is needed for the future research works in higher level. Chaotic and turbulences in human life situations, made data mining more difficult. To handle these crisis, new kind of extensions to cantor set have also got arosed. Human – life is going to get controlled by chips in next step of evolution. So hereafter, have to think on, ' what algebra can do ? ' in these kind of cross- breed structures in a 'chip oriented human life'. In this point, some debates are necessary. Whether is it good or bad? If bad, how these bad impact can convert into good, by taming these research works? Definitely these robotic effect made human life much easier both in 'profit and labour' level. Some bad outputs are also there and have to think of removing such negatives! From our washing machines to rocket technology, one can found this logic and algebraic illuminations. So giving some applications to algebra is irrelevant in one sense. But can think of the other part, in a little bit humorously. Where algebra is 'not showed off, its face 'in this modern world ? Following will give an idea to the newly developed algebraic structures in the family of algebra.

In 1966, Yasuyuki Imai and Kiyoshi Iseki [13] introduced BCK/BCI – algebra based on two different ways. One approach is based on set theory and the other one is based on classical and non-classical propositional calculi. In 1983, followed by these works another wide class of algebra namely, BCH – algebra is introduced by Q. P. Hu and X. Li [12]. In 2001, Q- algebra is introduced by Joseph Neggers, Sun Shin Ahn, and Hee Sik Kim [14] as an extension to BCH/BCI/BCK – algebras. In 1999, Sun Shin Ahn and Hee Sik Kim [24] introduced QS – algebra and investigated some relations between QS – algebra and BCK/BCI – algebra. They also investigated G – part in QS – algebra. In 2012, Ravi Kumar Bandaru and Rafi. N [22] introduced G -algebra as a generalization of QS – algebra. Necessary and sufficient condition for a G – algebra to become a QS – algebra is presented. It has shown that every associative G – algebra is a group. Concepts like 0 - commutative, G – part and medial of a G – algebra have explained. In 1984, Mukherjee. N.P and Prabir Bhattacharya [21] introduced fuzzy cosets and fuzzy normal subgroups. They have proven several interesting elementary properties related to the context and finally presented fuzzy interpretation of Lagrange theorem. In 2002, Young Bae Jun, Eun Hwan Roh, Chinju, and Hee Sik Kim, Seoul [27] discussed on fuzzy B – algebra. They addressed fuzzy normal B – algebra and fuzzy normal set in B – algebra. In 2016, Ch. Mallika, N. Ramakrishna, G. Anandha Rao [2] studied Vague Cosets. They also presented notions like vague symmetric, vague invariant, vague normal and some related properties. In 2015, Chiranjibe Jana, Tapan Senapati, Monoranjan Bhowmik, Madhumangal Pal [3] applied the concept of intuitionistic fuzzy set to G – subalgebra. In 2015, Chiranjibe Jana and Tapan Senapati [4] introduced Cubic G - subalgebra with properties. They also defined homomorphism of cubic G – subalgebra and verified various theorems. In 2015, Tapan Senapati, Chiranjibe Jana, Monoranjan Bhowmik, Madhumangal Pal [25] introduced L-fuzzy- set G-subalgebra and gave a characterization of L - fuzzy G – subalgebra. They also discussed some related characterizations like group, homomorphism etc. In 2017, Deena Al-Kadi, and Rodyna Hosny [5] generated a G-algebra from a non-empty set. They obtained quotient G-algebra using a normal subalgebra and proved fundamental theorem on G-algebra homomorphism. They showed that every BP-algebra is a G-algebra and provided an additional condition for the existence of the converse part. They further contributed notions of left-right and right-left derivation of G – algebra. They proved that G-algebra satisfying associative law is a 2-group. In 2018, Apurba Das [1] explored homotopy G – algebraic structure on the cochain complex of hom-type algebras. In 2020, Wahiba Messirdi and Ahlam Fallatah [26] presented several results on derivations of G – algebra. In 2019, Muhammad Abdul Basit Khan, Junaid Alam Khan, Muhammad Ahsan Binyamin [20] introduced SAGBI bases in G – algebra. They

constructed an algorithm to compute SAGBI bases from a subset of polynomials contained in a subalgebra of a G – algebra. In 2019, Mohsin Khalid, Rakib Iqbal and Said Broumi [19] introduced Neutrosophic soft cubic G - subalgebras of G - algebras. Some basic operations like P - union, P – intersection, R -union, R – intersection are introduced. In 2020, Mohamed Abdel-Basset, Abdualah Gamal, Le Hoang Son and Florentin Smarandache [15] gave a case study in recruitment process based on bipolar neutrosophic set and bipolar neutrosophic number. In 2019, Mohamed Abdel-Basset, Rehab Mohamed, Abd El-Nasser H. Zaied and Florentin Smarandache [16] gave a hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics and conducted a study on thailand’s sugar industry. In 2020, Mohamed Abdel-Basst, Rehab Mohamed, Mohamed Elhoseny [17] suggested a novel framework to evaluate innovation value proposition for smart product–service systems. In 2020, Mohamed Abdel-Basst, Rehab Mohamed, Mohamed Elhoseny [18] developed a model for the effective COVID-19 identification in uncertainty environment using primary symptoms and CT scans.

In 1993, Gau. W. L and Buehrer. D. J [11] presented vague sets. In 2005, Florentin Smarandache [6] introduced neutrosophic set and its basic ideas. In 2005, Florentin Smarandache [7] illustrated the difference between neutrosophic set and intuitionistic fuzzy set with proper explanations and examples. In 2011, Florentin Smarandache [8] gave a geometric interpretation of neutrosophic set using a neutrosophic cube. In 2015, Shawkat Alkhazaleh [23], introduced a mixed form of neutrosophic and vague known as neutrosophic vague sets. In 2019, P. B. Remya and A. Francina Shalini [9] applied binary concept to neutrosophic vague set and developed neutrosophic vague binary set with its basic operations.

In 2020, P.B. Remya and A. Francina Shalini [10] developed BCK/BCI – algebra for neutrosophic vague binary sets. Authors proposed a new suggestion of ‘inclusion of new set’ in the structure in addition to the ‘underlying universal set’, for avoiding more confusions in theoretical calculations. In this paper, authors further modified that structure and proposed a new approach in the structure mentioned in [10], by presenting a single set in the structure instead of the above mentioned two sets. This will give a combined effect of those two sets discussed above. New structure, convey the same effect of the structure used in paper [10], with a single set outlook and by skipping the two set pattern from structure. So here authors tried to present a one more modified form to the structure discussed in paper [10]. This one more refined pattern can be used in the all existing algebraic structures of various sets like fuzzy, vague, neutrosophic, etc., and for their hybrid forms in future works. This new pattern will be helpful to get more clarity and stability in these works.

This paper focuses on the development of G – algebraic structure to neutrosophic vague binary set. Discussions on G – algebra need some more attention while comparing to other algebraic structures. Its axioms are very simple and can be handled in a very clear manner. Neutrosophic ideas and Neutrosophic Vague ideas in G – algebra deserve more attention due to its easily accessible practical applications. This paper concentrates on neutrosophic vague binary G – subalgebra and its theoretical implementations.

Follllwing are the newly introduced concepts in this paper.

- Neutrosophic Vague Binary G – subalgebra [Section 3]
 - ✓ Neutrosophic Vague Binary G – subalgebra [Definition 3.1]
- Different notions of Neutrosophic Vague Binary G – subalgebra [Section 4]
 - ✓ Neutrosophic Vague Binary G - G part [Definition 4.1 (i)]
 - ✓ Neutrosophic Vague Binary G - p radical [Definition 4.1 (ii)]

- ✓ Neutrosophic Vague Binary G – p semi simple [Definition 4.1 (iii)]
- ✓ Neutrosophic Vague Binary G - minimal element [Definition 4.1 (iv)]
- Neutrosophic Vague Binary G – normal subalgebra [Section 5]
 - ✓ Neutrosophic Vague Binary G – normal subalgebra [Definition 5.1]
 - ✓ Neutrosophic Vague Binary G – normal set [Definition 5.2]
- 0 – commutative Neutrosophic Vague Binary G – subalgebra [Section 6]
 - ✓ 0 – commutative Neutrosophic Vague Binary G – subalgebra [Definition 6.1]
- Derivations of Neutrosophic Vague Binary G – subalgebra [Section 7]
 - ✓ Neutrosophic Vague Binary G – derivation [Definition 7.1]
 - ✓ Neutrosophic Vague Binary G – regular derivation [Definition 7.4]
- Neutrosophic Vague Binary G – Coset [Section 8]
 - ✓ Neutrosophic Vague Binary G – Right Coset [Definition 8.1 (i)]
 - ✓ Neutrosophic Vague Binary G – Left Coset [Definition 8.1(ii)]
 - ✓ Neutrosophic Vague Binary G – Coset [Definition 8.3]
- Neutrosophic Vague Binary G – homomorphism [Section 9]
 - ✓ Neutrosophic Vague Binary G – homomorphism [Definition 9.1]
 - ✓ Neutrosophic Vague Binary G- Isomorphism [Definition 9.2]

2. Preliminaries

In this section some preliminaries are given.

Definition 2.1 [9] (Neutrosophic Vague Binary Set)

A neutrosophic vague binary set (NVBS in short) M_{NVB} over a common universe

$\{U_1 = \{x_j / 1 \leq j \leq n\}; U_2 = \{y_k / 1 \leq k \leq p\}\}$ is an object of the form

$$M_{NVB} = \left\{ \left(\frac{\hat{T}_{M_{NVB}}(x_j), \hat{I}_{M_{NVB}}(x_j), \hat{F}_{M_{NVB}}(x_j)}{x_j}; \forall x_j \in U_1 \right), \left(\frac{\hat{T}_{M_{NVB}}(y_k), \hat{I}_{M_{NVB}}(y_k), \hat{F}_{M_{NVB}}(y_k)}{y_k}; \forall y_k \in U_2 \right) \right\}$$

is defined as

$\hat{T}_{M_{NVB}}(x_j) = [T^-(x_j), T^+(x_j)]$, $\hat{I}_{M_{NVB}}(x_j) = [I^-(x_j), I^+(x_j)]$ and $\hat{F}_{M_{NVB}}(x_j) = [F^-(x_j), F^+(x_j)]$; $x_j \in U_1$ and $\hat{T}_{M_{NVB}}(y_k) = [T^-(y_k), T^+(y_k)]$, $\hat{I}_{M_{NVB}}(y_k) = [I^-(y_k), I^+(y_k)]$ and $\hat{F}_{M_{NVB}}(y_k) = [F^-(y_k), F^+(y_k)]$; $y_k \in U_2$ where

(1) $T^+(x_j) = 1 - F^-(x_j)$; $F^+(x_j) = 1 - T^-(x_j)$; $\forall x_j \in U_1$ and

$T^+(y_k) = 1 - F^-(y_k)$; $F^+(y_k) = 1 - T^-(y_k)$; $\forall y_k \in U_2$

(2) $-0 \leq T^-(x_j) + I^-(x_j) + F^-(x_j) \leq 2^+$; $-0 \leq T^-(y_k) + I^-(y_k) + F^-(y_k) \leq 2^+$

or

$$-0 \leq T^-(x_j) + I^-(x_j) + F^-(x_j) + T^-(y_k) + I^-(y_k) + F^-(y_k) \leq 4^+$$

and

$$-0 \leq T^+(x_j) + I^+(x_j) + F^+(x_j) \leq 2^+ ; \quad -0 \leq T^+(y_k) + I^+(y_k) + F^+(y_k) \leq 2^+$$

or

$$-0 \leq T^+(x_j) + I^+(x_j) + F^+(x_j) + T^+(y_k) + I^+(y_k) + F^+(y_k) \leq 4^+$$

(3) $T^-(x_j), I^-(x_j), F^-(x_j) : V(U_1) \rightarrow [0, 1]$ and $T^-(y_k), I^-(y_k), F^-(y_k) : V(U_2) \rightarrow [0, 1]$
 $T^+(x_j), I^+(x_j), F^+(x_j) : V(U_1) \rightarrow [0, 1]$ and $T^+(y_k), I^+(y_k), F^+(y_k) : V(U_2) \rightarrow [0, 1]$

Here $V(U_1), V(U_2)$ denotes power set of vague sets on U_1, U_2 respectively.

Definition 2.2 [14] (G – algebra)

A G- algebra is a non-empty set A with a constant 0 and a binary operation $*$ satisfying axioms:
 $(B_3) (x * x) = 0$ $(B_{12}) x * (x * y) = y$ for all $x, y \in A$;
 A G-algebra is denoted by $(A, *, 0)$

Proposition 2.3 [14]

Any G -algebra X satisfies the following axioms: for all $x, y, z \in X$,
 (i) $x * 0 = x$ (ii) $(x * (x * y)) * y = 0$ (iii) $0 * (0 * x) = x$ (iv) $x * y = 0$ implies $x = y$
 (v) $0 * x = 0 * y$ implies $x = y$

Definition 2.4 [14] (G - subalgebra)

A non-empty subset S of a G-algebra X , is called a G-subalgebra of X if $(x * y) \in S$, for all $x, y \in S$

Definition 2.5 [14] (0 – commutative G - algebra)

A G - algebra $(A, *, 0)$ is said to be 0 – commutative if: $x * (0 * y) = y * (0 * x)$, for any $x, y \in A$

Theorem 2.6 [14] (G -part, p -radical, p – semisimple)

Let A be a G – algebra. For any subset S of A , we define $G(S) = \{x \in S / 0 * x = x\}$. In particular, if $S = A$ then we say that $G(A)$ is the G – part of a G – algebra. For any G – algebra A , the set $B(A) = \{x \in A / 0 * x = x\}$ is called a p – radical of A . A G – algebra is said to be p -semisimple if $B(A) = \{0\}$. The following property is obvious: $G(A) \cap B(A) = \{0\}$

Proposition 2.7 [14]

Let $(U, *, 0)$ be a G - algebra. Then, the following conditions hold for any $x, y \in X$
 (1) $(x * (x * y)) * y = 0$ (2) $(x * y) = 0 \Rightarrow x = y$ (3) $(0 * x) = (0 * y) \Rightarrow x = y$

Definition 2.8 [17] (Fuzzy G – subalgebra)

Let $A = \{\langle x, \alpha_A(x) \rangle / x \in X\}$ be a fuzzy set in X , where X is a G-subalgebra. Then the set A is a fuzzy G - subalgebra over the binary operator $*$ if it satisfies the condition $\alpha_A(x * y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$ for all $x, y \in X$

Definition 2.9 [3] (Intuitionistic Fuzzy G – subalgebra)

An IFS $A = (\alpha_A, \beta_A)$ in X is called an intuitionistic fuzzy G - subalgebra of X if for all $x, y \in X$ it satisfies:

GS1 $\alpha_A(x * y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$; GS2 $\beta_A(x * y) \leq \max\{\beta_A(x), \beta_A(y)\}$

Definition 2.10 [18] (Normal Subalgebra of a G - algebra)

Let N be a non – empty subset of a G – algebra X . We say that N is a normal subset of X if for all x, y, z and t in X such that $(x * y) \in N$ and $(z * t) \in N$, we have $((x * z) * (y * t)) \in N$

Definition 2.11 [19] (Fuzzy normal subset of a B - algebra)

Let $(X, *, 0)$ be a B – algebra and let a fuzzy set μ in X is said to be fuzzy normal if it satisfies the inequality $\mu((x * a) * (y * b)) \geq \min \{\mu(x * y), \mu(a * b)\}$ for all $a, b, x, y \in X$.

Definition 2.12 [19] (Fuzzy normal subalgebra of B – algebra)

A fuzzy set μ in a B – algebra X is called a fuzzy normal B - algebra if it is a fuzzy B – algebra which is fuzzy normal

Definition 2.13 [5] (Derivation of G – algebra)

Let X be a G – algebra and d is a self-map on X . We say that,

d is (l, r)- derivation of X if $d(x * y) = (d(x) * y) \wedge (x * d(y))$

d is (r, l)- derivation of X if $d(x * y) = (x * d(y)) \wedge (d(x) * y)$. If d is both (l, r)- derivation and (r, l)- derivation of X then we say that d is a derivation of X . (l, r) indicates left-right and (r, l) indicates right-left

Remark 2.14 [5]

In a G – algebra, $(x \wedge y) = x$

Definition 2.15 [5] (Modified definition of G -derivation)

Let X be a G – algebra and d a self – map on X . We say that d is a derivation of X if,

d is (l, r)- derivation of X and (r, l)- derivation of X .

That is, for all $x, y \in X$: $d(x * y) = d(x) * y$ and $d(x * y) = x * d(y)$, respectively.

Definition 2.16 [2] (Vague Coset)

Let A be a vague group of a group (G, \cdot) . For any $a \in G$.

(i) A vague left coset of A is denoted by aA and defined by $V_{aA}(x) = V_A(a^{-1}x)$.

i.e., $t_{aA}(x) = t_A(a^{-1}x)$ and $f_{aA}(x) = f_A(a^{-1}x)$

(ii) A vague right coset of A is denoted by Aa and defined by $V_{Aa}(x) = V_A(xa^{-1})$.

i.e., $t_{Aa}(x) = t_A(xa^{-1})$ and $f_{Aa}(x) = f_A(xa^{-1})$

Definition 2.17 [5] (Homomorphism, Epimorphism, Endomorphism of G - algebra)

Let X and Y be G-algebras. A mapping $\varphi: X \rightarrow Y$ is called a homomorphism if $\varphi(x * y) = \varphi(x) * \varphi(y)$, $\forall x, y \in X$. The homomorphism φ is said to be a monomorphism (resp., an epimorphism) if it is injective (resp., surjective). If the map φ is both injective and surjective then X and Y are said to be isomorphic, written $X \cong Y$. For any homomorphism $\varphi: X \rightarrow Y$, the set $\{x \in X / \varphi(x) = 0_Y\}$ is called the kernel of φ and denoted by $\text{Ker } \varphi$

3. Neutrosophic vague binary G - subalgebra

In this section neutrosophic vague binary G - subalgebra is developed with its properties and with some theorems.

Definition 3.1 (Neutrosophic vague binary G - subalgebra)

Let \mathbf{M}_{NVB} be a neutrosophic vague binary set (in short, NVBS) with two universes \mathbf{U}_1 and \mathbf{U}_2 .

A neutrosophic vague binary G - subalgebra is a structure $\mathfrak{G}_{\mathbf{M}_{NVB}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{NVB}}}, *, \mathbf{0})$ which satisfies, the following $\mathfrak{G}_{\mathbf{M}_{NVB}}$ inequality:

$\mathfrak{G}_{\mathbf{M}_{NVB}}$ inequality :

$$NVB_{\mathbf{M}_{NVB}}(x * y) \geq r \min \{NVB_{\mathbf{M}_{NVB}}(x), NVB_{\mathbf{M}_{NVB}}(y)\} ; \forall x, y \in \mathbf{U}$$

That is, $\forall x, y \in \mathbf{U}$

$$\begin{aligned} \hat{T}_{\mathbf{M}_{NVB}}(x * y) &\geq \min \{\hat{T}_{\mathbf{M}_{NVB}}(x), \hat{T}_{\mathbf{M}_{NVB}}(y)\} ; \hat{I}_{\mathbf{M}_{NVB}}(x * y) \leq \max \{\hat{I}_{\mathbf{M}_{NVB}}(x), \hat{I}_{\mathbf{M}_{NVB}}(y)\} ; \hat{F}_{\mathbf{M}_{NVB}}(x * y) \leq \max \{\hat{F}_{\mathbf{M}_{NVB}}(x), \hat{F}_{\mathbf{M}_{NVB}}(y)\} \\ &\left[* \text{ and } \mathbf{0} \text{ are as in } \mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{NVB}}} \text{ \& } \hat{T} = [T^-, T^+]; \hat{I} = [I^-, I^+]; \hat{F} = [F^-, F^+] \right]. \end{aligned}$$

Here,

- M_{NVB} is a neutrosophic vague binary set with two universes U_1 and U_2
- $U^{\mathfrak{M}_{NVB}} = (U = \{U_1 \cup U_2\}, *, 0)$ is a G – algebraic structure with a binary operation $*$ & a constant 0 , which satisfies following axioms : $\forall x, y \in U$, (i) $(x * x) = 0$; (ii) $x * (x * y) = y$

Remark 3.2

(i) Neutrosophic vague binary G – subalgebra is written in short as NVB G - subalgebra.
(ii) In NVB G – subalgebra universal set U is taken as “union” of elements of U_1 and U_2 .
(iii) Before applying \mathfrak{M}_{NVB} condition, neutrosophic vague binary union [Here, (max, min, min)] have to take for common elements of U_1 and U_2 . Combined neutrosophic vague binary membership grades will draw and implement combined effect to neutrosophic vague binary values of U_1 and U_2 . This will fulfill the binary effect in neutrosophic vague sets in the practical point of view.

Example 3.3

Let $U_1 = \{0, a, b\}$, $U_2 = \{0, b, c\}$ be two universes. Combined universe $U = \{U_1 \cup U_2\} = \{0, a, b, c\}$. Binary operation $*$ is defined as given by the Cayley table given below:

*	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Clearly, $(U, *, 0)$ is a G –algebra. Consider a NVBS formed based on U_1 & U_2 .

$$M_{NVB} = \left\{ \left(\frac{[0.9, 0.9][0.2, 0.8][0.1, 0.1]}{0}, \frac{[0.8, 0.9][0.3, 0.7][0.1, 0.2]}{a}, \frac{[0.6, 0.9][0.4, 0.6][0.1, 0.4]}{b} \right), \left(\frac{[0.6, 0.9][0.3, 0.6][0.1, 0.4]}{0}, \frac{[0.8, 0.8][0.2, 0.7][0.2, 0.2]}{b}, \frac{[0.8, 0.9][0.3, 0.7][0.1, 0.2]}{c} \right) \right\}$$

Combined neutrosophic vague binary membership grade is given by

$$M_{NVB}(s) = \begin{cases} [0.9, 0.9][0.2, 0.6][0.1, 0.1] & ; \text{ if } s = 0 \\ [0.8, 0.9][0.3, 0.7][0.1, 0.2] & ; \text{ if } s = a \\ [0.8, 0.9][0.2, 0.6][0.1, 0.2] & ; \text{ if } s = b \\ [0.8, 0.9][0.3, 0.7][0.1, 0.2] & ; \text{ if } s = c \end{cases}$$

Calculations shows that M_{NVB} is a NVB G – subalgebra.

Remark 3.4

In a NVB G – algebra, construction of the underlying G – algebraic structure, using a binary operation $*$ deserves prime importance. Instead of $*$ different symbols like $+$, $-$, \times , $+$ etc can be applied. Binary operation can be formed in different ways. Construction of G – algebra using the following points always defines a G - algebra. In the Binary Operation,

(i) If “first operand = second operand” then the output will be zero.

[Using definition of G – algebra, $(x * x) = 0 \Rightarrow$ principal diagonal elements should occupy with constant 0 , in the Cayley table of a G – algebra].

(ii) If “first operand \neq second operand” with “first operand $\neq 0$ & second operand = 0 ”, then output will be first operand

(iii) If “first operand \neq second operand” with “first operand $\neq 0$ & the second operand $\neq 0$ ”,

then output will be second operand

Following Cayley Table will make idea clear. $U = \{0, a_1 \neq 0, a_2 \neq 0, \dots, a_n \neq 0\}$.

From above, numbers in the square brackets indicates specific points used to frame the output.

*	0	$a_1 \neq 0$	$a_2 \neq 0$	$\dots \neq 0$	$\dots \neq 0$	$a_n \neq 0$
0	0 [i]	a_1 [iii]	a_2 [iii]	----	----	a_n [iii]
$a_1 \neq 0$	a_1 [ii]	0 [i]	a_2 [iii]	----	----	a_n [iii]
$a_2 \neq 0$	a_2 [ii]	a_1 [iii]	0 [i]	----	----	a_n [iii]
$\dots \neq 0$	---[ii]	--- [iii]	--- [iii]	----	----	--- [iii]
$\dots \neq 0$	---[ii]	--- [iii]	--- [iii]	----	----	a_n [iii]
$a_n \neq 0$	a_n [ii]	a_1 [iii]	a_2 [iii]	----	----	0 [i]

Remark 3.5

A neutrosophic vague G - subalgebra is a structure $\mathfrak{G}_{M_{NV}} = (U^{\mathfrak{G}_{M_{NV}}}, *, 0)$ which satisfies,

$NV_{M_{NV}}(x * y) \geq r \min \{NV_{M_{NV}}(x), NV_{M_{NV}}(y)\}$ [known as, $\mathfrak{G}_{M_{NV}}$ condition]. That is,

$$\begin{aligned} \hat{T}_{M_{NV}}(x * y) &\geq \min \{\hat{T}_{M_{NV}}(x), \hat{T}_{M_{NV}}(y)\}; \hat{I}_{M_{NV}}(x * y) \leq \max \{\hat{I}_{M_{NV}}(x), \hat{I}_{M_{NV}}(y)\}; \hat{F}_{M_{NV}}(x * y) \leq \max \{\hat{F}_{M_{NV}}(x), \hat{F}_{M_{NV}}(y)\} \\ [* \text{ and } 0 \text{ are as in } U^{\mathfrak{G}_{M_{NV}}} \text{ \& } \hat{T} = [T^-, T^+]; \hat{I} = [I^-, I^+]; \hat{F} = [F^-, F^+]]. \end{aligned}$$

Here,

- M_{NV} is a neutrosophic vague set with a single universe U
- $U^{\mathfrak{G}_{M_{NV}}} = (U, *, 0)$ is a G - algebraic structure with a binary operation $*$ & a constant 0, which satisfies following axioms: $\forall x, y \in U, (i) (x * x) = 0$; (ii) $x * (x * y) = y$

Remark 3.6

It is straight forward to check that, intersection of neutrosophic vague binary G - subalgebras produce a neutrosophic vague binary G - subalgebra itself. But union may not be!

4. Different notions of Neutrosophic Vague Binary G - subalgebra

In this section following notions to a NVB - G subalgebra are discussed.

- ✓ G - part of a Neutrosophic Vague Binary G - subalgebra
- ✓ G - p radical of a Neutrosophic Vague Binary G - subalgebra
- ✓ G - p semi simple of a Neutrosophic Vague Binary G - subalgebra
- ✓ G - minimal element of a Neutrosophic Vague Binary G - subalgebra

Definition 4.1

Let M_{NVB} be a NVB G -subalgebra with structure $\mathfrak{G}_{M_{NVB}} = (U^{\mathfrak{G}_{M_{NVB}}}, *, 0)$

i. G - part of a Neutrosophic Vague Binary G - subalgebra

Let S_{NVB} be any NVBSS of M_{NVB} . Define, $G(S_{NVB}) = \{x \in S_{NVB} / NVB_{S_{NVB}}(0 * x) = NVB_{S_{NVB}}(x)\}$.

In particular, if $S_{NVB} = M_{NVB}$ then $G(M_{NVB})$ is called the *neutrosophic vague binary G - G part* (in short, NVB G - G part) of the NVB G - subalgebra.

ii. p - radical of a Neutrosophic Vague Binary G - subalgebra

$B(\mathbf{M}_{\text{NVB}}) = \{x \in \mathbf{U} / \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0 * x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0)\}$ is called, a *neutrosophic vague binary G – p radical* (in short, *NVB G – p radical*) of the NVB G -subalgebra \mathbf{M}_{NVB}

iii. *p- semi simple of a Neutrosophic Vague Binary G – subalgebra*

\mathbf{M}_{NVB} is called *neutrosophic vague binary G - p semi simple* (in short, *NVB G – p semi simple*), if $B(\mathbf{M}_{\text{NVB}}) = \{x \in \mathbf{U} / \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0 * x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0)\} = \{0\}$

iv. *Minimal Element for Neutrosophic Vague Binary G – subalgebra*

Any element $x \in \mathbf{U}$ is *neutrosophic vague binary G – minimal element* (in short, *NVB G – minimal element*), if $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x * y) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0) \Rightarrow \text{NVB}_{\mathbf{M}_{\text{NVB}}}(y) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$

Remark 4.2

It is clear that, $G(\mathbf{M}_{\text{NVB}}) \cap B(\mathbf{M}_{\text{NVB}}) = \{0\}$

Theorem 4.3

Let \mathbf{M}_{NVB} be a NVB G - subalgebra with structure $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}}, *, 0)$. Then, $x \in G(\mathbf{M}_{\text{NVB}})$ if and only if $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(0 * x) \in G(\mathbf{M}_{\text{NVB}})$

Proof

$x \in G(\mathbf{M}_{\text{NVB}}) \Rightarrow \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0 * x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$
 $\Rightarrow \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0 * x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0 * (0 * x))$, [from proposition 3.8]
 $\Rightarrow (0 * x) \in G(\mathbf{M}_{\text{NVB}})$, [By definition 6.1].
 Conversely, if $(0 * x) \in G(\mathbf{M}_{\text{NVB}})$, then $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(0 * (0 * x)) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0 * x)$
 $\Rightarrow \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0 * x) \Rightarrow x \in G(\mathbf{M}_{\text{NVB}})$

Theorem 4.4

Let \mathbf{M}_{NVB} be a NVB G - subalgebra with structure $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}}, *, 0)$.

(i) \mathbf{M}_{NVB} is NVB G - p semi simple. (ii) Every element in \mathbf{U} is a NVB G - minimal element.

Proof

(i) From definition 6.1(iii), $B(\mathbf{M}_{\text{NVB}}) = \{x \in \mathbf{U} / \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0 * x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0)\}$
 $\Rightarrow B(\mathbf{M}_{\text{NVB}}) = \{x \in \mathbf{U} / \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)\} = \{0\}$
 (ii) Assume (ii). Let x be an arbitrary element in \mathbf{U} such that $y \leq x$ for some $y \in \mathbf{U}$
 $\Rightarrow (x * y) = 0 \Rightarrow \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x * y) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0) \Rightarrow \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(y)$

Theorem 4.5

If $G(\mathbf{M}_{\text{NVB}}) = \mathfrak{G}_{\mathbf{M}_{\text{NVB}}}$, then \mathbf{M}_{NVB} is NVB G - p semi simple.

That is, if a NVB G – subalgebra coincides with its G – part then it is NVB G – p semi simple

Proof

Let \mathbf{M}_{NVB} be a NVB G - subalgebra with structure $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}}, *, 0)$.

From definition 6.1(ii), $G(\mathbf{M}_{\text{NVB}}) = \{x \in \mathbf{M}_{\text{NVB}} / \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0 * x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)\}$

If $G(\mathbf{M}_{\text{NVB}}) = \mathfrak{G}_{\mathbf{M}_{\text{NVB}}}$ then $B(\mathbf{M}_{\text{NVB}}) = \{0\} \Rightarrow \mathbf{M}_{\text{NVB}}$ is NVB G - p semi simple

Remark 4.6

(1) In any NVB G – subalgebra: $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x) \leq \text{NVB}_{\mathbf{M}_{\text{NVB}}}(y) \Leftrightarrow \text{NVB}_{\mathbf{M}_{\text{NVB}}}(y * x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(0)$
 (2) Denote $\text{NVB}_{\mathbf{M}_{\text{NVB}}}[y * (y * x)]$ by $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x \wedge y)$ for all $x, y \in \mathbf{U}$. From definition 3.1 (i) (2), $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x \wedge y)$

Theorem 4.7

Let \mathbf{M}_{NVB} be a NVB G - subalgebra with structure $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}}, *, \mathbf{0})$.

Then for any $x, y, z \in \mathbf{U}$,

- (1) For $x \neq y$, $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x \wedge y) \neq \text{NVB}_{\mathbf{M}_{\text{NVB}}}(y \wedge x)$
- (2) $\text{NVB}_{\mathbf{M}_{\text{NVB}}}[x \wedge (y \wedge z)] = \text{NVB}_{\mathbf{M}_{\text{NVB}}}[(x \wedge y) \wedge z]$
- (3) $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x \wedge \mathbf{0}) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$ and $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0} \wedge x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0})$
- (4) For $x \neq \mathbf{0}$, $\text{NVB}_{\mathbf{M}_{\text{NVB}}}[x \wedge (y \wedge z)] \neq \text{NVB}_{\mathbf{M}_{\text{NVB}}}[(x \wedge y) * (x \wedge z)]$

Proof

- (1) For a NVB G - subalgebra, $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x \wedge y) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$ & $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(y \wedge x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(y)$
 \therefore If $x \neq y$, then $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x \wedge y) \neq \text{NVB}_{\mathbf{M}_{\text{NVB}}}(y \wedge x)$
- (2) $\text{NVB}_{\mathbf{M}_{\text{NVB}}}[x \wedge (y \wedge z)] = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x \wedge y) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$ &
 $\text{NVB}_{\mathbf{M}_{\text{NVB}}}[(x \wedge y) \wedge z] = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x \wedge z) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$. $\therefore \text{NVB}_{\mathbf{M}_{\text{NVB}}}[x \wedge (y \wedge z)] = \text{NVB}_{\mathbf{M}_{\text{NVB}}}[(x \wedge y) \wedge z]$
- (3) For a NVB G - subalgebra, $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x \wedge \mathbf{0}) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}[\mathbf{0} * (\mathbf{0} * x)] = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$ &
 $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0} \wedge x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}[x * (x * \mathbf{0})] = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0})$
- (4) $\text{NVB}_{\mathbf{M}_{\text{NVB}}}[x \wedge (y \wedge z)] = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x \wedge y) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$, for a NVB G - subalgebra
 $\text{NVB}_{\mathbf{M}_{\text{NVB}}}[(x \wedge y) * (x \wedge z)] = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x * x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0})$, for a NVB G - subalgebra
 \therefore it is clear that, for a NVB G - subalgebra,
 $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x) \neq \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0}) \Rightarrow \text{NVB}_{\mathbf{M}_{\text{NVB}}}[x \wedge (y \wedge z)] \neq \text{NVB}_{\mathbf{M}_{\text{NVB}}}[(x \wedge y) * (x \wedge z)]$

Theorem 4.8

Every NVB G - subalgebra satisfies the inequality, $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0}) \geq \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$; $\forall x \in \mathbf{U}$

Proof

$\text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0}) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x * x) \geq r \min \{\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x), \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)\} = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$
 $\therefore \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0}) \geq \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$; $\forall x \in \mathbf{U}$

Theorem 4.9

Let $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}}, *, \mathbf{0})$ be a NVB G - subalgebra. Then the following conditions hold :

- (i) $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x * \mathbf{0}) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$, $\forall x \in \mathbf{U}$ (ii) $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0} * (\mathbf{0} * x)) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$, $\forall x \in \mathbf{U}$

Proof

Let $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}}, *, \mathbf{0})$ be a NVB G - subalgebra and $x, y \in \mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}}$. Then,

- (i) $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x * \mathbf{0}) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x * (x * x))$
 [Using first condition in the G - algebraic structure of NVB G - subalgebra]
 $= \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x)$ [Using second condition in the G - algebraic structure of NVB G - subalgebra]
- (ii) (ii) Since $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}$ is a NVB G - subalgebra, $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x * (x * y)) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(y)$
 [Using second condition in the G - algebraic structure of NVB G - subalgebra] Put
 $x = \mathbf{0}$ and $y = x$ in the above then (ii) follows

Theorem 4.10

Let $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}}, *, \mathbf{0})$ be a NVB G - subalgebra. Then following conditions hold: $\forall x, y \in \mathbf{U}$,

- (i) $\text{NVB}_{\mathbf{M}_{\text{NVB}}}((x * (x * y)) * y) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0})$
- (ii) $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(x * y) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0}) \Rightarrow \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(y)$
- (iii) $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0} * x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{0} * y) \Rightarrow \text{NVB}_{\mathbf{M}_{\text{NVB}}}(x) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}(y)$

Proof

- (i) $\text{NVB}_{\text{MNVB}}((x * (x * y)) * y) = \text{NVB}_{\text{MNVB}}((y * (y * y)) * y)$ by putting $x = y$
 $= \text{NVB}_{\text{MNVB}}((y * 0) * y) = \text{NVB}_{\text{MNVB}}(y * y) = \text{NVB}_{\text{MNVB}}(0)$
- (ii) Assume $\text{NVB}_{\text{MNVB}}(x * y) = \text{NVB}_{\text{MNVB}}(0)$
 $\therefore \text{NVB}_{\text{MNVB}}(x) = \text{NVB}_{\text{MNVB}}(x * 0) = \text{NVB}_{\text{MNVB}}(x * (x * y))$, [by assumption]
 $= \text{NVB}_{\text{MNVB}}(y)$
- (iii) Assume $\text{NVB}_{\text{MNVB}}(0 * x) = \text{NVB}_{\text{MNVB}}(0 * y)$
 $\therefore \text{NVB}_{\text{MNVB}}(x) = \text{NVB}_{\text{MNVB}}(0 * (0 * x)) = \text{NVB}_{\text{MNVB}}(0 * (0 * y))$, [by assumption]
 $= \text{NVB}_{\text{MNVB}}(y)$

Theorem 4.11

Let $\mathfrak{G}_{\text{MNVB}} = (\mathbf{U}^{\mathfrak{G}_{\text{MNVB}}}, *, 0)$ be a NVB G – subalgebra. Then,

$$\text{NVB}_{\text{MNVB}}(a * x) = \text{NVB}_{\text{MNVB}}(a * y) \Rightarrow \text{NVB}_{\text{MNVB}}(x) = \text{NVB}_{\text{MNVB}}(y), \text{ for any } a, x, y \in \mathbf{U}$$

Proof

If $\mathfrak{G}_{\text{MNVB}} = (\mathbf{U}^{\mathfrak{G}_{\text{MNVB}}}, *, 0)$ be a NVB G – subalgebra satisfying,

$\text{NVB}_{\text{MNVB}}(a * x) = \text{NVB}_{\text{MNVB}}(a * y)$, for any $a, x, y \in \mathbf{U}$. Then,

$$\text{NVB}_{\text{MNVB}}(x) = \text{NVB}_{\text{MNVB}}(a * (a * x)) = \text{NVB}_{\text{MNVB}}(a * (a * y)) = \text{NVB}_{\text{MNVB}}(y)$$

Theorem 4.12

Let $\mathfrak{G}_{\text{MNVB}} = (\mathbf{U}^{\mathfrak{G}_{\text{MNVB}}}, *, 0)$ be a NVB G – subalgebra. Then the following are equivalent:

- (1) $\text{NVB}_{\text{MNVB}}((x * y) * (x * z)) = \text{NVB}_{\text{MNVB}}(z * y); \forall x, y, z \in \mathbf{U}$
- (2) $\text{NVB}_{\text{MNVB}}((x * z) * (y * z)) = \text{NVB}_{\text{MNVB}}(x * y); \forall x, y, z \in \mathbf{U}$

Proof

(i) \Rightarrow (ii)

Assume (i). i.e., $\text{NVB}_{\text{MNVB}}((x * y) * (x * z)) = \text{NVB}_{\text{MNVB}}(z * y); \forall x, y, z \in \mathbf{U}$

$$\therefore \text{NVB}_{\text{MNVB}}((x * z) * (x * y)) = \text{NVB}_{\text{MNVB}}(y * z)$$

Consider, $\text{NVB}_{\text{MNVB}}((x * z) * (y * z))$

$$= \text{NVB}_{\text{MNVB}}((x * z) * ((x * z) * (x * y)))$$

$$= \text{NVB}_{\text{MNVB}}(x * y), \text{ since } \text{NVB}_{\text{MNVB}}(x * (x * y)) = \text{NVB}_{\text{MNVB}}(y)$$

(ii) \Rightarrow (i)

Assume (ii). i.e., $\text{NVB}_{\text{MNVB}}((x * z) * (y * z)) = \text{NVB}_{\text{MNVB}}(x * y)$

$$\therefore \text{NVB}_{\text{MNVB}}((x * y) * (z * y)) = \text{NVB}_{\text{MNVB}}(x * z)$$

Consider, $\text{NVB}_{\text{MNVB}}((x * y) * (x * z))$

$$= \text{NVB}_{\text{MNVB}}((x * y) * ((x * y) * (z * y)))$$

$$= \text{NVB}_{\text{MNVB}}(z * y), \text{ since } \text{NVB}_{\text{MNVB}}(x * (x * y)) = \text{NVB}_{\text{MNVB}}(y)$$

5. Neutrosophic Vague Binary G – normal subalgebra

In this section neutrosophic vague binary G – normal subalgebra is introduced

Definition 5.1 (Neutrosophic vague binary G – normal subalgebra)

Let \mathbf{M}_{NVB} be a neutrosophic vague binary set (in short, NVBS) with two universes \mathbf{U}_1 and \mathbf{U}_2 .

Neutrosophic vague binary G – normal subalgebra is a structure $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}^{\mathcal{N}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}^{\mathcal{N}}}, *, 0)$

which satisfies, the following 2 conditions known as $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}^{\mathcal{N}}$ inequalities:

$\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}^{\mathcal{N}}$ inequality (1) :

$$\text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}) \geq r \min \{ \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x}), \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{y}) \} ; \forall \mathbf{x}, \mathbf{y} \in \mathbf{U}$$

That is, $\forall \mathbf{x}, \mathbf{y} \in \mathbf{U}$

$$\hat{\mathbf{T}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}) \geq \min \{ \hat{\mathbf{T}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x}), \hat{\mathbf{T}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{y}) \}; \hat{\mathbf{I}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}) \leq \max \{ \hat{\mathbf{I}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x}), \hat{\mathbf{I}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{y}) \}; \hat{\mathbf{F}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}) \leq \max \{ \hat{\mathbf{F}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x}), \hat{\mathbf{F}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{y}) \}$$

$\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}^{\mathcal{N}}$ inequality (2) :

$$\text{NVB}_{\mathbf{M}_{\text{NVB}}}((\mathbf{x} * \mathbf{a}) * (\mathbf{y} * \mathbf{b})) \geq r \min \{ \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}), \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{a} * \mathbf{b}) \} ; \forall \mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y} \in \mathbf{U}$$

That is, $\forall \mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y} \in \mathbf{U}$. $\hat{\mathbf{T}}_{\mathbf{M}_{\text{NVB}}}((\mathbf{x} * \mathbf{a}) * (\mathbf{y} * \mathbf{b})) \geq \min \{ \hat{\mathbf{T}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}), \hat{\mathbf{T}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{a} * \mathbf{b}) \} ; \hat{\mathbf{I}}_{\mathbf{M}_{\text{NVB}}}((\mathbf{x} * \mathbf{a}) * (\mathbf{y} * \mathbf{b})) \leq$

$$\max \{ \hat{\mathbf{I}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}), \hat{\mathbf{I}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{a} * \mathbf{b}) \}. \hat{\mathbf{F}}_{\mathbf{M}_{\text{NVB}}}((\mathbf{x} * \mathbf{a}) * (\mathbf{y} * \mathbf{b})) \leq \max \{ \hat{\mathbf{F}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}), \hat{\mathbf{F}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{a} * \mathbf{b}) \}$$

$$\left[* \text{ and } 0 \text{ are as in } \mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}^{\mathcal{N}}} \quad \& \quad \hat{\mathbf{T}} = [\mathbf{T}^-, \mathbf{T}^+]; \hat{\mathbf{I}} = [\mathbf{I}^-, \mathbf{I}^+]; \hat{\mathbf{F}} = [\mathbf{F}^-, \mathbf{F}^+] \right]$$

Here,

- \mathbf{M}_{NVB} is a neutrosophic vague binary set with two universes \mathbf{U}_1 and \mathbf{U}_2
- $\mathbf{U}^{\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}^{\mathcal{N}}} = (\mathbf{U} = \{\mathbf{U}_1 \cup \mathbf{U}_2\}, *, 0)$ is a G - algebraic structure with a binary operation $*$ & a constant 0, which satisfies following axioms : $\forall \mathbf{x}, \mathbf{y} \in \mathbf{U}$, (i) $(\mathbf{x} * \mathbf{x}) = 0$; (ii) $\mathbf{x} * (\mathbf{x} * \mathbf{y}) = \mathbf{y}$

Definition 5.2 (Neutrosophic vague binary G – normal set)

Let \mathbf{M}_{NVB} be a NVBS with two universes $\mathbf{U}_1, \mathbf{U}_2$. Take $\mathbf{U} = \{\mathbf{U}_1 \cup \mathbf{U}_2\}$.

A NVBS \mathbf{M}_{NVB} in \mathbf{U} is said to be NVB G – normal set if it satisfies the inequality

$$\{ \text{NVB}_{\mathbf{M}_{\text{NVB}}}((\mathbf{x} * \mathbf{a}) * (\mathbf{y} * \mathbf{b})) \geq r \min \{ \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}), \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{a} * \mathbf{b}) \} \mid \forall \mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} \in \mathbf{U} \}$$

That is,

$$\left\{ \begin{array}{l} \hat{\mathbf{T}}_{\mathbf{M}_{\text{NVB}}}((\mathbf{x} * \mathbf{a}) * (\mathbf{y} * \mathbf{b})) \geq \min \{ \hat{\mathbf{T}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}), \hat{\mathbf{T}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{a} * \mathbf{b}) \} \\ \hat{\mathbf{I}}_{\mathbf{M}_{\text{NVB}}}((\mathbf{x} * \mathbf{a}) * (\mathbf{y} * \mathbf{b})) \leq \max \{ \hat{\mathbf{I}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}), \hat{\mathbf{I}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{a} * \mathbf{b}) \} \\ \hat{\mathbf{F}}_{\mathbf{M}_{\text{NVB}}}((\mathbf{x} * \mathbf{a}) * (\mathbf{y} * \mathbf{b})) \leq \max \{ \hat{\mathbf{F}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}), \hat{\mathbf{F}}_{\mathbf{M}_{\text{NVB}}}(\mathbf{a} * \mathbf{b}) \} \end{array} \right\} \forall \mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} \in \mathbf{U}$$

Remark 5.3

(i) Neutrosophic vague binary G - normal subalgebra is written in short as **NVB G – normal subalgebra**. It is denoted by $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}^{\mathcal{N}}$.

(ii) In other words, a **NVBS** \mathbf{M}_{NVB} in a G – algebra \mathbf{U} is called a **NVB G – normal subalgebra** if it is a **NVB G – subalgebra** which is **NVB G – normal set**.

Theorem 5.4

Every **NVB G – normal set** \mathbf{M}_{NVB} in \mathbf{U} is a **NVB G – subalgebra** of \mathbf{U} .

That is, every **NVB G – normal set** \mathbf{M}_{NVB} is a $\mathfrak{G}_{\mathbf{M}_{\text{NVB}}}^{\mathcal{N}}$.

Proof

Let \mathbf{M}_{NVB} be a **NVB G – normal set** in $\mathbf{U} \Rightarrow \text{NVB}_{\mathbf{M}_{\text{NVB}}}((\mathbf{x} * \mathbf{a}) * (\mathbf{y} * \mathbf{b})) \geq r \min \{ \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}), \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{a} * \mathbf{b}) \}$

Consider, $\text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * \mathbf{y}) = \text{NVB}_{\mathbf{M}_{\text{NVB}}}((\mathbf{x} * \mathbf{y}) * (0 * 0)) \geq r \min \{ \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{x} * 0), \text{NVB}_{\mathbf{M}_{\text{NVB}}}(\mathbf{y} * 0) \}$

$= r \min \{NVB_{M_{NVB}}(x), NVB_{M_{NVB}}(y)\} \Rightarrow NVB_{M_{NVB}}(x * y) \geq r \min \{NVB_{M_{NVB}}(x), NVB_{M_{NVB}}(y)\} ; \forall x, y \in U \Rightarrow M_{NVB}$ is a NVB G – subalgebra

Remark 5.5

Converse of theorem 5.4 is not true.

That is, a NVB G – subalgebra M_{NVB} in U is not a NVB G – normal set, generally.

Proof

Consider example 4.3, in which M_{NVB} is a NVB G – subalgebra. In this example, $NVB_{M_{NVB}}((a * a) * (b * a)) \not\geq r \min \{NVB_{M_{NVB}}(a * b), NVB_{M_{NVB}}(a * a)\} \Rightarrow M_{NVB}$ is not a NVB G – normal set.

Theorem 5.6

If a neutrosophic vague binary set M_{NVB} in U is a NVB G – normal subalgebra, then $NVB_{M_{NVB}}(x * y) = NVB_{M_{NVB}}(y * x) ; \forall x, y \in U$

Proof

Let $x, y \in U$. Then, $NVB_{M_{NVB}}(x * y) = NVB_{M_{NVB}}((x * y) * 0)$ [From theorem 4.8] $= NVB_{M_{NVB}}((x * y) * (x * x)) \geq r \min \{NVB_{M_{NVB}}(x * x), NVB_{M_{NVB}}(y * x)\} = r \min \{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(y * x)\} = NVB_{M_{NVB}}(y * x)$ [From theorem 4.7]. $\therefore NVB_{M_{NVB}}(x * y) \geq NVB_{M_{NVB}}(y * x)$. Similarly, $NVB_{M_{NVB}}(y * x) \geq NVB_{M_{NVB}}(x * y) \Rightarrow NVB_{M_{NVB}}(x * y) = NVB_{M_{NVB}}(y * x)$

6. 0 – commutative neutrosophic vague binary G – subalgebra

In this section 0 – commutative neutrosophic vague binary G – subalgebra with its properties are introduced

Definition 6.1 (0 - commutative of a NVB G - subalgebra)

Let M_{NVB} be a neutrosophic vague binary set (in short, NVBS) with two universes U_1 and U_2 . 0 - commutative neutrosophic vague binary G - subalgebra is a structure $\mathfrak{G}_{M_{NVB}}^0 = (U^{\mathfrak{G}_{M_{NVB}}^0}, *, 0)$ which satisfies the following $\mathfrak{G}_{M_{NVB}}^0$ inequality:

$\mathfrak{G}_{M_{NVB}}^0$ inequality :

$NVB_{M_{NVB}}(x * y) \geq r \min \{NVB_{M_{NVB}}(x), NVB_{M_{NVB}}(y)\} ; \forall x, y \in U$

i.e., $\hat{T}_{M_{NVB}}(x * y) \geq \min \{\hat{T}_{M_{NVB}}(x), \hat{T}_{M_{NVB}}(y)\} ; \hat{I}_{M_{NVB}}(x * y) \leq \max \{\hat{I}_{M_{NVB}}(x), \hat{I}_{M_{NVB}}(y)\} ; \hat{F}_{M_{NVB}}(x * y) \leq \max \{\hat{F}_{M_{NVB}}(x), \hat{F}_{M_{NVB}}(y)\}$

$\left[* \text{ and } 0 \text{ are as in } U^{\mathfrak{G}_{M_{NVB}}^0} \quad \& \quad \hat{T} = [T^-, T^+]; \hat{I} = [I^-, I^+]; \hat{F} = [F^-, F^+] \right]$

Here,

- M_{NVB} is a neutrosophic vague binary set with two universes U_1 and U_2
- $U^{\mathfrak{G}_{M_{NVB}}^0} = (U = \{U_1 \cup U_2\}, *, 0)$ is a 0 – commutative G - algebraic structure with a binary operation $*$ & a constant 0, which satisfies the following axioms :
 $\forall x, y \in U$, (i) $(x * x) = 0$;(ii) $x * (x * y) = y$; (iii) $x * (0 * y) = y * (0 * x)$

0 - commutative neutrosophic vague binary G - subalgebra is denoted by $\mathfrak{G}_{M_{NVB}}^0$

Example 6.2

Consider example 4.3. with a different binary operation defined as given in following Cayley table:

*	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

$\mathcal{U}_{\mathcal{M}_{\text{NVB}}}^0$ will become 0 - commutative $\mathcal{G}_{\mathcal{M}_{\text{NVB}}}^0$ only if $\mathcal{G}_{\mathcal{M}_{\text{NVB}}}^0$ inequality got satisfied. Verifications showed that \mathcal{M}_{NVB} given in example 3.3 is not only a $\mathcal{G}_{\mathcal{M}_{\text{NVB}}}$ but it is clearly a $\mathcal{G}_{\mathcal{M}_{\text{NVB}}}^0$ too!

Theorem 6.3

Let $\mathcal{G}_{\mathcal{M}_{\text{NVB}}} = (\mathcal{U}_{\mathcal{M}_{\text{NVB}}}, *, 0)$ be a $\mathcal{G}_{\mathcal{M}_{\text{NVB}}}^0$.

Then, $\text{NVB}_{\mathcal{M}_{\text{NVB}}}((0 * x) * (0 * y)) = \text{NVB}_{\mathcal{M}_{\text{NVB}}}(y * x)$ for any $x, y \in \mathcal{U}$

Proof

Let $x, y \in \mathcal{U}$ where $\mathcal{U} \in \mathcal{U}_{\mathcal{M}_{\text{NVB}}}$. Then, $((0 * x) * (0 * y)) = (y * (0 * (0 * x))) = (y * x)$

$\Rightarrow x, y \in \mathcal{U}$ where $\mathcal{U} \in \mathcal{U}_{\mathcal{M}_{\text{NVB}}}^0 \Rightarrow \mathcal{U}_{\mathcal{M}_{\text{NVB}}}$ becomes $\mathcal{U}_{\mathcal{M}_{\text{NVB}}}^0 \Rightarrow \mathcal{G}_{\mathcal{M}_{\text{NVB}}}$ becomes $\mathcal{G}_{\mathcal{M}_{\text{NVB}}}^0$

7. Derivations of Neutrosophic Vague Binary G – subalgebra

In this section following points are developed

- neutrosophic vague binary **G** – derivation
- neutrosophic vague binary **G** – regular derivation

Definition 7.1 (**G** – derivation of neutrosophic vague binary **G** – subalgebra)

Let \mathcal{M}_{NVB} be a neutrosophic vague binary set (in short, **NVBS**) with two universes \mathcal{U}_1 and \mathcal{U}_2 .

Also let considered \mathcal{M}_{NVB} is a **NVB G** - subalgebra with structure $\mathcal{G}_{\mathcal{M}_{\text{NVB}}} = (\mathcal{U}_{\mathcal{M}_{\text{NVB}}}, *, 0)$ and with a self – map $d : \mathcal{U} \rightarrow \mathcal{U}$ on \mathcal{M}_{NVB} with $\mathcal{U} = \{\mathcal{U}_1 \cup \mathcal{U}_2\}$. Then,

(i) d is (l, r) neutrosophic vague binary **G** - derivation of \mathcal{M}_{NVB} if,

$$\text{NVB}_{\mathcal{M}_{\text{NVB}}}[d(x * y)] = \text{NVB}_{\mathcal{M}_{\text{NVB}}}[(d(x) * y) \wedge (x * d(y))]$$

(ii) d is (r, l) neutrosophic vague binary **G** - derivation of \mathcal{M}_{NVB} if,

$$\text{NVB}_{\mathcal{M}_{\text{NVB}}}[d(x * y)] = \text{NVB}_{\mathcal{M}_{\text{NVB}}}[(x * d(y)) \wedge (d(x) * y)]$$

d is a neutrosophic vague binary **G** – derivation (in short, **NVB G** - derivation) of \mathcal{M}_{NVB} only if

d is both (l, r) neutrosophic vague binary **G** – derivation [in short, (l, r) **NVB G** -derivation]

& (r, l) neutrosophic vague binary **G** – derivation [in short, (r, l) **NVB G** -derivation] of \mathcal{M}_{NVB} .

In this definition, (l, r) indicates left-right and (r, l) indicates right-left

Remark 7.2

For a **NVB G** - subalgebra, $\text{NVB}_{\mathcal{M}_{\text{NVB}}}(x \wedge y) = \text{NVB}_{\mathcal{M}_{\text{NVB}}}(x)$

- (i) \therefore To check, \mathbf{d} is (l, r) $\mathbf{NVB} \mathbf{G}$ - derivation of $\mathbf{M}_{\mathbf{NVB}}$, it is enough to check that,
 $\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{y})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x}) * \mathbf{y})$; [Using definition 2.13 & by remark 2.14]
(ii) \therefore To check, $\mathbf{d}_{\mathfrak{M}_{\mathbf{NVB}}}$ is (r, l) $\mathbf{NVB} \mathbf{G}$ - derivation of $\mathbf{M}_{\mathbf{NVB}}$, it is enough to check that,
 $\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{y})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{d}(\mathbf{y}))$ [Using definition 2.13 & by remark 2.14]
 \therefore Definition 7.1, can be re – written as, definition 7.3

Definition 7.3

Let $\mathfrak{M}_{\mathbf{NVB}}$ be a $\mathbf{NVB} \mathbf{G}$ – subalgebra and \mathbf{d} be a self – map on \mathbf{U} .

\mathbf{d} is a neutrosophic vague binary \mathbf{G} - derivation of \mathbf{U} if

- (i) \mathbf{d} is (l, r) – neutrosophic vague binary \mathbf{G} - derivation of \mathbf{U}

i.e., $\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{y})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x}) * \mathbf{y})$; for all $\mathbf{x}, \mathbf{y} \in \mathbf{U}$ & it is denoted by $\mathbf{d}_{(l,r)}^{\mathfrak{M}_{\mathbf{NVB}}}$

- (ii) \mathbf{d} is a (r, l) – neutrosophic vague binary \mathbf{G} - derivation of \mathbf{U} .

i.e., $\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{y})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{d}(\mathbf{y}))$; for all $\mathbf{x}, \mathbf{y} \in \mathbf{U}$ & it is denoted by $\mathbf{d}_{(r,l)}^{\mathfrak{M}_{\mathbf{NVB}}}$.

\mathbf{d} is a $\mathbf{NVB} \mathbf{G}$ – derivation on a $\mathfrak{M}_{\mathbf{NVB}}$ if \mathbf{d} is both $\mathbf{d}_{(l,r)}^{\mathfrak{M}_{\mathbf{NVB}}}$ and $\mathbf{d}_{(r,l)}^{\mathfrak{M}_{\mathbf{NVB}}}$. It is denoted by $\mathbf{d}^{\mathfrak{M}_{\mathbf{NVB}}}$.

Definition 7.4 (Regular derivation of a Neutrosophic Vague Binary \mathbf{G} – subalgebra)

A derivation $\mathbf{d}^{\mathfrak{M}_{\mathbf{NVB}}}$ of a $\mathbf{NVB} \mathbf{G}$ – subalgebra is said to be regular if,

$\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{0})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{0})$. It is denoted by $\mathbf{d}_r^{\mathfrak{M}_{\mathbf{NVB}}}$.

Example 7.5

From example 3.3, $\mathbf{M}_{\mathbf{NVB}}$ is a $\mathfrak{M}_{\mathbf{NVB}}$.

Case (i) Define a self – map, $\mathbf{d} : \mathbf{U} = \{0, \mathbf{a}, \mathbf{b}\} \rightarrow \mathbf{U} = \{0, \mathbf{a}, \mathbf{b}\}$ by $\mathbf{d}(\mathbf{s}) = \begin{cases} 0 & \text{if } \mathbf{s} = 0 \\ \mathbf{a} & \text{if } \mathbf{s} = \mathbf{a} \\ \mathbf{b} & \text{if } \mathbf{s} = \mathbf{b} \end{cases}$

Here the given self – map is an identity map.

From calculations, \mathbf{d} is a $\mathbf{d}_{(l,r)}^{\mathfrak{M}_{\mathbf{NVB}}}$ & $\mathbf{d}_{(r,l)}^{\mathfrak{M}_{\mathbf{NVB}}} \Rightarrow \mathbf{d}$ is a $\mathbf{d}^{\mathfrak{M}_{\mathbf{NVB}}}$

Case (ii) Define a self-map, $\mathbf{d} : \mathbf{U} = \{0, \mathbf{a}, \mathbf{b}\} \rightarrow \mathbf{U} = \{0, \mathbf{a}, \mathbf{b}\}$ by $\mathbf{d}(\mathbf{s}) = \begin{cases} \mathbf{a} & \text{if } \mathbf{s} = 0 \\ 0 & \text{if } \mathbf{s} = \mathbf{a} \\ \mathbf{b} & \text{if } \mathbf{s} = \mathbf{b} \end{cases}$

\mathbf{d} is not a $\mathbf{NVB} \mathbf{G}$ – derivation on $\mathbf{M}_{\mathbf{NVB}}$. One violation is attached below.

$$\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{b} * \mathbf{a})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{a})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{0}) = [0.9, 0.9][0.1, 0.1][0.1, 0.1]$$

$$\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}[\mathbf{d}(\mathbf{b}) * \mathbf{a}] = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{b} * \mathbf{a}) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{a}) = [0.7, 0.9][0.3, 0.4][0.1, 0.3]$$

$$\mathbf{d}_{(l,r)}^{\mathfrak{M}_{\mathbf{NVB}}}(\mathbf{b} * \mathbf{a}) \text{ does not exist, since } \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{b} * \mathbf{a})) \neq \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{b}) * \mathbf{a})$$

$$\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}[\mathbf{b} * \mathbf{d}(\mathbf{a})] = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{b} * \mathbf{0}) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{b}) = [0.2, 0.6][0.1, 0.2][0.4, 0.8]$$

$$\mathbf{d}_{(r,l)}^{\mathfrak{M}_{\mathbf{NVB}}}(\mathbf{b} * \mathbf{a}) \text{ does not exist, since } \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{b} * \mathbf{a})) \neq \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{b} * \mathbf{d}(\mathbf{a}))$$

$\Rightarrow \mathbf{d}$ is not a $\mathbf{d}^{\mathfrak{M}_{\mathbf{NVB}}}$.

Theorem 7.6

In a $\mathfrak{M}_{\mathbf{NVB}}$, the identity map \mathbf{d} on \mathbf{U} is a $\mathbf{d}^{\mathfrak{M}_{\mathbf{NVB}}}$. Converse not true in general. But if $\mathbf{d}^{\mathfrak{M}_{\mathbf{NVB}}}$ is a $\mathbf{d}_r^{\mathfrak{M}_{\mathbf{NVB}}}$ then converse hold good. That is, if $\mathbf{d}^{\mathfrak{M}_{\mathbf{NVB}}}$ is a $\mathbf{d}_r^{\mathfrak{M}_{\mathbf{NVB}}}$ then \mathbf{d} is the identity map on \mathbf{U}

Proof

- (i) Let $\mathbf{x}, \mathbf{y} \in \mathbf{U}$ & also let \mathbf{d} is an identity map on \mathbf{U}

Case (i) : $\mathbf{x} = \mathbf{y}$; $\mathbf{y} \neq 0$

$$\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{y})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{x})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{0})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{0}).$$

$$\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x}) * \mathbf{y}) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x}) * \mathbf{x}) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{x}) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{0}).$$

$$\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{d}(\mathbf{y})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{d}(\mathbf{x})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{x}) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{0}).$$

$$\therefore \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{y})) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{d}(\mathbf{x}) * \mathbf{y}) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{d}(\mathbf{y}))$$

Case (ii) : $x \neq y$; $y \neq 0$

Either $NVB_{M_{NVB}}(d(x * y)) = NVB_{M_{NVB}}(d(x)) = NVB_{M_{NVB}}(x)$ or $NVB_{M_{NVB}}(d(x * y)) = NVB_{M_{NVB}}(d(y)) = NVB_{M_{NVB}}(y)$
 $\Rightarrow d(x * y) = d(x)$ or $d(x * y) = d(y) \Rightarrow$ either $(x * y) = x$ or $(x * y) = y$, since d is identity map
 \Rightarrow either $y = 0$ or $y \neq 0$.

Consider $y \neq 0$, i.e., $d(x * y) = d(y)$, i.e., $(x * y) = y$. $NVB_{M_{NVB}}(d(x * y)) = NVB_{M_{NVB}}(d(y)) = NVB_{M_{NVB}}(y)$. $NVB_{M_{NVB}}(d(x) * y) = NVB_{M_{NVB}}(x * y) = NVB_{M_{NVB}}(y)$. $NVB_{M_{NVB}}(x * d(y)) = NVB_{M_{NVB}}(x * y) = NVB_{M_{NVB}}(y)$. $\therefore NVB_{M_{NVB}}(d(x * y)) = NVB_{M_{NVB}}(d(x) * y) = NVB_{M_{NVB}}(x * d(y))$

Case (iii) : $x \neq y$; $y = 0$

Either $NVB_{M_{NVB}}(d(x * y)) = NVB_{M_{NVB}}(d(x)) = NVB_{M_{NVB}}(x)$ or $NVB_{M_{NVB}}(d(x * y)) = NVB_{M_{NVB}}(d(y)) = NVB_{M_{NVB}}(y)$
 $\Rightarrow d(x * y) = d(x)$ or $d(x * y) = d(y) \Rightarrow$ either $(x * y) = x$ or $(x * y) = y$, since d is identity map
 \Rightarrow either $y = 0$ or $y \neq 0$.

Consider $y = 0$, i.e., $d(x * y) = d(x)$, i.e., $(x * y) = x$. $NVB_{M_{NVB}}(d(x * y)) = NVB_{M_{NVB}}(d(x)) = NVB_{M_{NVB}}(x)$. $NVB_{M_{NVB}}(d(x) * y) = NVB_{M_{NVB}}(x * y) = NVB_{M_{NVB}}(x)$. $NVB_{M_{NVB}}(x * d(y)) = NVB_{M_{NVB}}(x * y) = NVB_{M_{NVB}}(x)$. $\therefore NVB_{M_{NVB}}(d(x * y)) = NVB_{M_{NVB}}(d(x) * y) = NVB_{M_{NVB}}(x * d(y))$
 $\therefore d$ is both $d_{(l,r)}^{M_{NVB}}$ & $d_{(r,l)}^{M_{NVB}}$. Hence d is a $d^{M_{NVB}}$

Converse

$d^{M_{NVB}}$ is a $d_r^{M_{NVB}} \Rightarrow NVB_{M_{NVB}}(d(0)) = NVB_{M_{NVB}}(0) \Rightarrow NVB_{M_{NVB}}(d(x * x)) = NVB_{M_{NVB}}(0)$
 $\Rightarrow NVB_{M_{NVB}}(d(x) * x) = NVB_{M_{NVB}}(0) \Rightarrow d(x) = x$ [By proposition 3.11 (ii)]
 $\Rightarrow d$ is the identity map on U

Remark 7.7

Let M_{NVB} be a NVB G - subalgebra with structure $\mathfrak{G}_{M_{NVB}} = (U^{M_{NVB}}, *, 0)$. A NVB G - derivation on M_{NVB} is a mapping $d: U \rightarrow U$ such that $NVB_{M_{NVB}}(d(x * y)) = NVB_{M_{NVB}}(d(x) * y) = NVB_{M_{NVB}}(x * d(y))$, $\forall x, y \in U$. Set of all neutrosophic vague binary G -derivations on M_{NVB} is denoted as $\Gamma_d^{M_{NVB}}$

8. Neutrosophic vague binary G – Coset

General properties that are true for abstract algebra and G – algebra may not be true in the case of neutrosophic G – subalgebra/neutrosophic vague G – subalgebra/ neutrosophic vague binary G – subalgebra. In this section coset for neutrosophic vague binary G – subalgebra is developed. Neutrosophic vague binary G – Coset is considered as a shifted (or translated) neutrosophic vague binary G – subalgebra. Existence of identity element and inverse element can't be assured in every neutrosophic vague binary G – subalgebra. In generalization process, this will become a crisis. As a result, generalization is confined to a particular area. It will lead to the formation of different concepts like Lagrange neutrosophic vague binary G – subalgebra etc.

Definition 8.1 (Neutrosophic Vague Binary G – Right Coset & Neutrosophic Vague Binary G – Left Coset)

Let M_{NVB} be a neutrosophic vague binary set (in short, NVBS) with two universes U_1 and U_2 . and also let the considered M_{NVB} is a NVB G – subalgebra of a G – algebra with algebraic structure $\mathfrak{G}_{M_{NVB}} = (U^{M_{NVB}}, *, 0)$ where $U^{M_{NVB}} = (U, *, 0_{M_{NVB}})$. Also, $\hat{T} = [T^-, T^+]$; $\hat{I} = [I^-, I^+]$; $\hat{F} = [F^-, F^+]$ and $U = \{U_1 \cup U_2\}$

Case (i) (Neutrosophic Vague Binary G – Right Coset)

Let $a \in U_1$ and $b \in U_2$ be fixed elements. Then define, for every $c \in U_1$ and for every $d \in U_2$ a neutrosophic vague binary G – right coset of M_{NVB} is denoted by $M_{NVB}(a, b)$ and defined by,

$$(M_{NVB}(a, b))(c, d) = NVB_{M_{NVB}(a, b)}(c, d) = \{ \langle NVB_{M_{NVB}}(c * (a)^{-1}) \mid \forall c \in U_1 \rangle \langle NVB_{M_{NVB}}(d * (b)^{-1}) \mid \forall d \in U_2 \rangle \}$$

$$\text{i.e., } \{ \langle (\hat{T}_{M_{NVB}(a)}(c), \hat{I}_{M_{NVB}(a)}(c), \hat{F}_{M_{NVB}(a)}(c) \mid \forall c \in U_1) \rangle \langle (\hat{T}_{M_{NVB}(b)}(d), \hat{I}_{M_{NVB}(b)}(d), \hat{F}_{M_{NVB}(b)}(d) \mid \forall d \in U_2) \rangle \}$$

$$= \{ \langle (\hat{T}_{M_{NVB}}(c * (a)^{-1}), \hat{I}_{M_{NVB}}(c * (a)^{-1}), \hat{F}_{M_{NVB}}(c * (a)^{-1}) \mid \forall c \in U_1) \rangle \langle (\hat{T}_{M_{NVB}}(d * (b)^{-1}), \hat{I}_{M_{NVB}}(d * (b)^{-1}), \hat{F}_{M_{NVB}}(d * (b)^{-1}) \mid \forall d \in U_2) \rangle \}$$

Then $M_{NVB}(a, b)$ is called a neutrosophic vague binary G -Right Coset (in short NVB G – Right Coset) determined by M_{NVB} and (a, b) .

Case (ii) (Neutrosophic vague binary G – Left Coset)

Let $a \in U_1$ and $b \in U_2$ be fixed elements. Then define, for every $c \in U_1$ and for every $d \in U_2$ a neutrosophic vague binary G – right coset of M_{NVB} is denoted by $(a, b) M_{NVB}$ and defined by,

$$((a, b) M_{NVB})(c, d) = NVB_{(a, b) M_{NVB}}(c, d) = \{ \langle NVB_{M_{NVB}}((a)^{-1} * c) \mid \forall c \in U_1 \rangle \langle NVB_{M_{NVB}}((b)^{-1} * d) \mid \forall d \in U_2 \rangle \}$$

$$= \{ \langle (\hat{T}_{(a) M_{NVB}}(c), \hat{I}_{(a) M_{NVB}}(c), \hat{F}_{(a) M_{NVB}}(c) \mid \forall c \in U_1) \rangle \langle (\hat{T}_{(b) M_{NVB}}(d), \hat{I}_{(b) M_{NVB}}(d), \hat{F}_{(b) M_{NVB}}(d) \mid \forall d \in U_2) \rangle \}$$

$$= \{ \langle (\hat{T}_{M_{NVB}}((a)^{-1} * c), \hat{I}_{M_{NVB}}((a)^{-1} * c), \hat{F}_{M_{NVB}}((a)^{-1} * c) \mid \forall c \in U_1) \rangle \langle (\hat{T}_{M_{NVB}}((b)^{-1} * d), \hat{I}_{M_{NVB}}((b)^{-1} * d), \hat{F}_{M_{NVB}}((b)^{-1} * d) \mid \forall d \in U_2) \rangle \}$$

Then $(a, b) M_{NVB}$ is called a neutrosophic vague binary left coset (in short NVB G – left coset) determined by M_{NVB} and (a, b) .

Remark 8.2

NVB G – right coset is a NVBS. Similarly, a NVB G – left coset is a NVBS.

Definition 8.3 (Neutrosophic Vague Binary G – Coset)

Let the neutrosophic vague binary set M_{NVB} be a neutrosophic vague binary G – subalgebra of a G – algebra. If M_{NVB} is both neutrosophic vague binary G – right coset and neutrosophic vague binary G – left coset then M_{NVB} is called as a Neutrosophic Vague Binary G – Coset

Example 8.4

Let $U_1 = \{0, u_1, u_3\}$ and $U_2 = \{0, u_2, u_4, u_5\}$ be two universes.

$$\text{Let } M_{NVB} = \left\{ \begin{array}{c} \langle \frac{[0.7, 0.8][0.3, 0.4][0.2, 0.3]}{0}, \frac{[0.2, 0.7][0.5, 0.7][0.3, 0.8]}{u_1}, \frac{[0.6, 0.7][0.1, 0.4][0.3, 0.4]}{u_3} \rangle \\ \langle \frac{[0.2, 0.9][0.1, 0.7][0.1, 0.8]}{0}, \frac{[0.3, 0.5][0.6, 0.7][0.5, 0.7]}{u_2}, \frac{[0.2, 0.8][0.4, 0.7][0.2, 0.8]}{u_4}, \frac{[0.6, 0.9][0.3, 0.7][0.1, 0.4]}{u_5} \rangle \end{array} \right\} \text{ be a NVBS.}$$

Here, combined universe $U = \{0, u_1, u_2, u_3, u_4, u_5\}$ & combined NVB membership grades are,

$$NVB_{M_{NVB}}(s) = \begin{cases} [0.7, 0.9][0.1, 0.4][0.1, 0.3] & ; s = 0 \\ [0.2, 0.7][0.5, 0.7][0.3, 0.8] & ; s = u_1 \\ [0.3, 0.5][0.6, 0.7][0.5, 0.7] & ; s = u_2 \\ [0.6, 0.7][0.1, 0.4][0.3, 0.4] & ; s = u_3 \\ [0.2, 0.8][0.4, 0.7][0.2, 0.8] & ; s = u_4 \\ [0.6, 0.9][0.3, 0.7][0.1, 0.4] & ; s = u_5 \end{cases}$$

Corresponding Cayley table is :

*	0	u_1	u_2	u_3	u_4	u_5
0	0	u_1	u_2	u_3	u_4	u_5
u_1	u_1	0	u_2	u_3	u_4	u_5
u_2	u_2	u_1	0	u_3	u_4	u_5
u_3	u_3	u_1	u_2	0	u_4	u_5

u_4	u_4	u_1	u_2	u_3	0	u_5
u_5	u_5	u_1	u_2	u_3	u_4	0

Obviously, M_{NVB} is a NVB G – subalgebra

In every G – algebra 0 may not be the identity element. But in the present case it is clear that 0 acts as an identity element. Hence inverses got as :

$$(0)^{-1} = 0 ; (u_1)^{-1} = u_1 ; (u_2)^{-1} = u_2 ; (u_3)^{-1} = u_3 ; (u_4)^{-1} = u_4 ; (u_5)^{-1} = u_5$$

To construct a NVB G -right coset:

Let $u_a = u_1 \in U_1$ and $\forall u_c \in U_1 = \{0, u_1, u_3\}$

$$\begin{aligned} NVB_{M_{NVB} u_1}(0) &= NVB_{M_{NVB}}(0 * (u_1^{-1})) = NVB_{M_{NVB}}(0 * u_1) = NVB_{M_{NVB}}(u_1) = [0.2, 0.7][0.5, 0.7][0.3, 0.8] \\ NVB_{M_{NVB} u_1}(u_1) &= NVB_{M_{NVB}}(u_1 * (u_1^{-1})) = NVB_{M_{NVB}}(u_1 * u_1) = NVB_{M_{NVB}}(0) = [0.7, 0.9][0.1, 0.4][0.1, 0.3] \\ NVB_{M_{NVB} u_1}(u_3) &= NVB_{M_{NVB}}(u_3 * (u_1^{-1})) = NVB_{M_{NVB}}(u_3 * u_1) = NVB_{M_{NVB}}(u_1) = [0.2, 0.7][0.5, 0.7][0.3, 0.8] \end{aligned}$$

&

Let $u_b = u_2 \in U_2$ and $\forall u_d \in U_2 = \{0, u_2, u_4, u_5\}$

$$\begin{aligned} NVB_{M_{NVB} u_2}(0) &= NVB_{M_{NVB}}(0 * (u_2^{-1})) = NVB_{M_{NVB}}(0 * u_2) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7] \\ NVB_{M_{NVB} u_2}(u_2) &= NVB_{M_{NVB}}(u_2 * (u_2^{-1})) = NVB_{M_{NVB}}(u_2 * u_2) = NVB_{M_{NVB}}(0) = [0.7, 0.9][0.1, 0.4][0.1, 0.3] \\ NVB_{M_{NVB} u_2}(u_4) &= NVB_{M_{NVB}}(u_4 * (u_2^{-1})) = NVB_{M_{NVB}}(u_4 * u_2) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7] \\ NVB_{M_{NVB} u_2}(u_5) &= NVB_{M_{NVB}}(u_5 * (u_2^{-1})) = NVB_{M_{NVB}}(u_5 * u_2) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7] \end{aligned}$$

$$M_{NVB} u_1 : u_2 =$$

$$\left\{ \frac{[0.2, 0.7][0.5, 0.7][0.3, 0.8]}{0}, \frac{[0.7, 0.9][0.1, 0.4][0.1, 0.3]}{u_1}, \frac{[0.2, 0.7][0.5, 0.7][0.3, 0.8]}{u_3}, \frac{[0.3, 0.5][0.6, 0.7][0.5, 0.7]}{0}, \frac{[0.7, 0.9][0.1, 0.4][0.1, 0.3]}{u_2}, \frac{[0.3, 0.5][0.6, 0.7][0.5, 0.7]}{u_4}, \frac{[0.3, 0.5][0.6, 0.7][0.5, 0.7]}{u_5} \right\}$$

To construct a NVB G - left coset:

Let $a = u_1 \in U_1$ and $\forall c \in U_1 = \{0, u_1, u_3\}$

$$\begin{aligned} NVB_{u_1 M_{NVB}}(0) &= NVB_{M_{NVB}}((u_1^{-1}) * 0) = NVB_{M_{NVB}}(u_1 * 0) = NVB_{M_{NVB}}(u_1) = [0.2, 0.7][0.5, 0.7][0.3, 0.8] \\ NVB_{u_1 M_{NVB}}(u_1) &= NVB_{M_{NVB}}(u_1 * (u_1^{-1})) = NVB_{M_{NVB}}(u_1 * u_1) = NVB_{M_{NVB}}(0) = [0.7, 0.9][0.1, 0.4][0.1, 0.3] \\ NVB_{u_1 M_{NVB}}(u_3) &= NVB_{M_{NVB}}((u_1^{-1}) * u_3) = NVB_{M_{NVB}}(u_1 * u_3) = NVB_{M_{NVB}}(u_3) = [0.6, 0.7][0.1, 0.4][0.3, 0.4] \end{aligned}$$

Let $u_b = u_2 \in U_2$ and $\forall u_d \in U_2 = \{0, u_2, u_4, u_5\}$

$$\begin{aligned} NVB_{u_2 M_{NVB}}(0) &= NVB_{M_{NVB}}((u_2^{-1}) * 0) = NVB_{M_{NVB}}(u_2 * 0) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7] \\ NVB_{u_2 M_{NVB}}(u_2) &= NVB_{M_{NVB}}((u_2^{-1}) * u_2) = NVB_{M_{NVB}}(u_2 * u_2) = NVB_{M_{NVB}}(0) = [0.7, 0.9][0.1, 0.4][0.1, 0.3] \\ NVB_{u_2 M_{NVB}}(u_4) &= NVB_{M_{NVB}}((u_2^{-1}) * u_4) = NVB_{M_{NVB}}(u_2 * u_4) = NVB_{M_{NVB}}(u_4) = [0.2, 0.8][0.4, 0.7][0.2, 0.8] \\ NVB_{u_2 M_{NVB}}(u_5) &= NVB_{M_{NVB}}((u_2^{-1}) * u_5) = NVB_{M_{NVB}}(u_2 * u_5) = NVB_{M_{NVB}}(u_5) = [0.6, 0.9][0.3, 0.7][0.1, 0.4] \end{aligned}$$

$$u_1 : u_2 M_{NVB} =$$

$$\left\{ \frac{[0.2, 0.7][0.5, 0.7][0.3, 0.8]}{0}, \frac{[0.3, 0.5][0.6, 0.7][0.5, 0.7]}{u_1}, \frac{[0.6, 0.7][0.1, 0.4][0.3, 0.4]}{u_3}, \frac{[0.3, 0.5][0.6, 0.7][0.5, 0.7]}{0}, \frac{[0.7, 0.9][0.1, 0.4][0.1, 0.3]}{u_2}, \frac{[0.2, 0.8][0.4, 0.7][0.2, 0.8]}{u_4}, \frac{[0.6, 0.9][0.3, 0.7][0.1, 0.4]}{u_5} \right\}$$

Remark 8.5

(i) In example 8.4, $M_{NVB} u_1 : u_2 \neq u_1 : u_2 M_{NVB}$

(ii) Constant 0 is not an identity element in G – algebra. For example, let $X = \{0, u_1, u_2, u_3, u_4, u_5\}$.

$(X, *, 0)$ is a G – algebra, with binary operation $*$ is defined by the following Cayley table:

*	0	u_1	u_2	u_3	u_4	u_5
0	0	u_2	u_1	u_3	u_4	u_5
u_1	u_1	0	u_3	u_2	u_5	u_4
u_2	u_2	u_4	0	u_5	u_1	u_3
u_3	u_3	u_5	u_4	0	u_2	u_1
u_4	u_4	u_3	u_5	u_1	0	u_2
u_5	u_5	u_1	u_2	u_4	u_3	0

It is clear that X is a G – algebra without an identity element. And hence inverse does not exist. So neutrosophic vague binary G – cosets cannot construct in this case. This construction is possible, only for those cases where identity element exists in the basic G – algebraic structure.

(ii) If the basic G – algebraic structure is formed using the following rules, then definitely there exist identity element and hence can construct a NVB G – right coset & NVB G – left coset.

Rules in Cayley table:

- (i) Principal diagonal elements = 0
- (ii) Column occupied with constant 0 is a copy of column of operands
- (iii) Fill each of the remaining columns (except principal diagonal entries) with the element given in the column head (i.e., elements from row of operands)

Definition 8.6 (Neutrosophic Vague G – Right Coset & Neutrosophic Vague G - Left Coset)

Let M_{NV} be a neutrosophic vague set (in short, NV Set) with a single universe U and also let M_{NV} be a neutrosophic vague G – subalgebra (in short, NV G – subalgebra) of a G – algebra. Algebraic structure of M_{NV} is given by $\mathfrak{G}_{M_{NV}} = (U^{\mathfrak{G}_{M_{NV}}}, *, 0)$ where $U^{\mathfrak{G}_{M_{NV}}} = (U, *, 0)$. Also $\hat{T} = [T^-, T^+]$; $\hat{I} = [I^-, I^+]$; $\hat{F} = [F^-, F^+]$

Case (i) (Neutrosophic Vague G – Right coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic vague G – right coset of M_{NV} which is denoted by $M_{NV} a$ and defined by,

$$(M_{NV}(a))(c) = NV_{M_{NV}}(a)(c) = \{NV_{M_{NV}}(c * (a)^{-1}) / \forall c \in U\}$$

$$\text{i.e.} \left\{ \left(\hat{T}_{M_{NV} a}(c), \hat{I}_{M_{NV} a}(c), \hat{F}_{M_{NV} a}(c) \right) / \forall c \in U \right\} = \left\{ \left(\hat{T}_{M_{NV}}(c * (a)^{-1}), \hat{I}_{M_{NV}}(c * (a)^{-1}), \hat{F}_{M_{NV}}(c * (a)^{-1}) \right) / \forall c \in U \right\}$$

Then $M_{NV} a$ is called a neutrosophic vague G -right coset (in short NV G – right coset) determined by M_{NV} and a .

Case (ii) (Neutrosophic Vague G – Left Coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic vague G – right coset of M_{NV} is denoted by $a M_{NV}$ and defined by,

$$((a) M_{NV})(c) = NV_{a M_{NV}}(c) = \{NV_{M_{NV}}((a)^{-1} * c) / \forall c \in U\}$$

$$= \left\{ \left(\hat{T}_{a M_{NV}}(c), \hat{I}_{a M_{NV}}(c), \hat{F}_{a M_{NV}}(c) \right) / \forall c \in U \right\}$$

$$= \left\{ \left(\hat{T}_{M_{NV}}((a)^{-1} * c), \hat{I}_{M_{NV}}((a)^{-1} * c), \hat{F}_{M_{NV}}((a)^{-1} * c) \right) / \forall c \in U \right\}$$

Then a M_{NV} is called a neutrosophic vague left coset (in short NV G – left coset) determined by M_{NV} and a .

Definition 8.7 (Neutrosophic Vague G – Coset)

Let the neutrosophic vague set M_{NV} be a neutrosophic vague G – subalgebra of a G – algebra. If M_{NV} is both neutrosophic vague G – Right Coset and neutrosophic vague G – Left Coset then M_{NV} is called as a Neutrosophic Vague G – Coset

Definition 8.8 (Neutrosophic G – Right Coset & Neutrosophic G – Left Coset)

Let M_N be a neutrosophic set (in short, N set) with single universe U and also let M_N be a neutrosophic G – subalgebra (in short, N G – subalgebra) of a G – algebra. Algebraic structure of M_N is given by $\mathfrak{G}_{M_N} = (U^{\mathfrak{G}_{M_N}}, *, 0)$ where $U^{\mathfrak{G}_{M_N}} = (U, *, 0)$.

Case (i) (Neutrosophic G – Right Coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic G – right coset of M_N which is denoted by $M_N a$ and defined by,

$$\begin{aligned} (M_N(a))(c) &= N_{M_N(a)}(c) = \{N_{M_N}(c * (a)^{-1}) / \forall c \in U\} \\ \text{i.e., } \{ (T_{M_N a}(c), I_{M_N a}(c), F_{M_N a}(c)) / \forall c \in U \} \\ &= \{ (T_{M_N}(c * (a)^{-1}), I_{M_N}(c * (a)^{-1}), F_{M_N}(c * (a)^{-1})) / \forall c \in U \} \end{aligned}$$

Then $M_N a$ is called a neutrosophic G -right coset (in short N G – right coset) determined by M_N and a .

Case (ii) (Neutrosophic G – Left Coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic G– right coset of M_N is denoted by aM_N and defined by,

$$\begin{aligned} ((a)M_N)(c) &= N_{aM_N}(c) = \{N_{M_N}((a)^{-1} * c) / \forall c \in U\} \\ &= \{ (T_{aM_N}(c), I_{aM_N}(c), F_{aM_N}(c)) / \forall c \in U \} \\ &= \{ (T_{M_N}((a)^{-1} * c), I_{M_N}((a)^{-1} * c), F_{M_N}((a)^{-1} * c)) / \forall c \in U \} \end{aligned}$$

Then aM_N is called a neutrosophic left coset (in short N G – left coset) determined by M_N and a .

Definition 8.9 (Neutrosophic G – Coset)

Let the neutrosophic set M_N be a neutrosophic G – subalgebra of a G – algebra. If M_N is both neutrosophic G – Right Coset and neutrosophic G – Left Coset then M_N is called as a Neutrosophic G – Coset

9. Neutrosophic Vague Binary G – homomorphism

In this section homomorphism on **NVB G** – subalgebra is presented with some related theorems.

Definition 9.1

Let $\mathfrak{G}_{M_{NVB}} = (U^{\mathfrak{G}_{M_{NVB}}}, *, 0_{M_{NVB}})$ and $\mathfrak{G}_{P_{NVB}} = (U^{\mathfrak{G}_{P_{NVB}}}, *, 0_{P_{NVB}})$ be two NVB G – subalgebras based on common universe $\{U_1, U_2\}$.

A mapping $\Psi^G : \mathfrak{G}_{M_{NVB}} = (U^{\mathfrak{G}_{M_{NVB}}}, *, 0_{M_{NVB}}) \rightarrow \mathfrak{G}_{P_{NVB}} = (U^{\mathfrak{G}_{P_{NVB}}}, *, 0_{P_{NVB}})$ is called a neutrosophic vague binary G - homomorphism if, $\Psi^G(u_x * u_y) = \Psi^G(u_x) *' \Psi^G(u_y), \forall u_x, u_y \in U$.

Remark 9.2

- (i) The NVB G - homomorphism Ψ^G is said to be a neutrosophic vague binary G - monomorphism (resp., a neutrosophic vague binary G - epimorphism) if it is injective (resp., surjective).
- (ii) If the map Ψ^G is both injective and surjective then $\mathfrak{G}_{M_{NVB}}$ and $\mathfrak{G}_{P_{NVB}}$ are said to be isomorphic,

written $\mathfrak{G}_{M_{NVB}} \cong \mathfrak{G}_{P_{NVB}}$. For any NVB G - homomorphism $\Psi^G : \mathfrak{G}_{M_{NVB}} \rightarrow \mathfrak{G}_{P_{NVB}}$, the set $\{x \in U / \Psi^G(x) = 0_{P_{NVB}}\}$ is called the kernel of Ψ^G and denoted by $\text{Ker } \Psi^G$

Theorem 9.3

Let $\Psi^G : \mathfrak{G}_{M_{NVB}} = (U^{\mathfrak{G}_{M_{NVB}}}, *, 0_{M_{NVB}}) \rightarrow \mathfrak{G}_{P_{NVB}} = (U^{\mathfrak{G}_{P_{NVB}}}, *, 0_{P_{NVB}})$ be a neutrosophic vague binary G - homomorphism of NVB G - subalgebras, then :

- (i) $\Psi^G(0_{M_{NVB}}) = 0_{P_{NVB}}$
- (ii) $\text{Ker } \Psi^G$ is a normal neutrosophic vague binary G - subalgebra of U
- (iii) $\text{Im } \Psi^G = \{y \in \mathfrak{G}_{P_{NVB}} / y = \Psi^G(x) \text{ for some } x \in \mathfrak{G}_{M_{NVB}}\}$ is a NVB G - subalgebra

Proof

$$(i) \Psi^G(0_{M_{NVB}}) = \Psi^G(0_{M_{NVB}} * 0_{M_{NVB}}) = \Psi^G(0_{M_{NVB}}) *' \Psi^G(0_{M_{NVB}}) = 0_{P_{NVB}} *' 0_{P_{NVB}} = 0_{P_{NVB}}$$

$$(ii) 0_{M_{NVB}} \in \text{Ker } \Psi^G \Rightarrow \text{Ker } \Psi^G \neq \emptyset$$

$$\text{Let } (x * y), (a * b) \in \text{Ker } \Psi^G \Rightarrow \Psi(x * y) = 0_{P_{NVB}} = \Psi(a * b)$$

$$\Rightarrow \Psi^G(x) *' \Psi^G(y) = 0_{P_{NVB}} = \Psi^G(a) *' \Psi^G(b) \Rightarrow \Psi^G(x) = \Psi^G(y) \ \& \ \Psi^G(a) = \Psi^G(b)$$

[By proposition 3.9(ii)]

$$\Rightarrow \Psi^G((x * a) * (y * b)) = \Psi^G(x * a) *' \Psi^G(y * b)$$

$$= (\Psi^G(x) *' \Psi^G(a)) * (\Psi^G(y) *' \Psi^G(b)) = (\Psi^G(x) *' \Psi^G(a)) * (\Psi^G(x) *' \Psi^G(a)) = 0_{P_{NVB}}$$

[From definition of NVB G - subalgebra]

$$\Rightarrow \Psi^G(x * a) *' \Psi^G(x * a) = 0_{P_{NVB}} \text{ [since } \Psi^G \text{ is a NVB } G \text{ - homomorphism]}$$

$$\Rightarrow \Psi^G[(x * a) * (x * a)] = 0_{P_{NVB}} \text{ [since } \Psi^G \text{ is a NVB } G \text{ - homomorphism]}$$

$$\Rightarrow (x * a) * (y * b) \in \text{Ker } \Psi^G \Rightarrow \text{Ker } \Psi^G \text{ is a NVB } G \text{ - subalgebra of } U$$

$$(iii) \text{ Let } y, z \in \mathfrak{G}_{P_{NVB}} \Rightarrow y = \Psi^G(a) \ \& \ z = \Psi^G(b) \text{ for some } a, b \in \mathfrak{G}_{M_{NVB}}$$

$$\Psi^G(a) *' \Psi^G(b) = \Psi^G(a * b) \geq r \min \{\Psi^G(a), \Psi^G(b)\}. \text{ Hence the proof.}$$

Theorem 9.4

A neutrosophic vague binary G - homomorphism $\chi^G : \mathfrak{G}_{T_{NVB}} = (U^{\mathfrak{G}_{T_{NVB}}}, *, 0_{T_{NVB}}) \rightarrow \mathfrak{G}_{L_{NVB}} = (U^{\mathfrak{G}_{L_{NVB}}}, *, 0_{L_{NVB}})$ is a neutrosophic vague binary G - monomorphism $\Leftrightarrow \text{ker}(\chi^G) = \{0\}$

Proof

$$\text{Let } x \in \text{Ker } (\chi^G) \Rightarrow \chi^G(x) = 0_{L_{NVB}} = \chi^G(0_{T_{NVB}}).$$

χ^G is a neutrosophic vague binary G - monomorphism, then it is clearly got that, $\text{Ker } (\chi^G) = \{0\}$.

Conversely, let $\text{ker } (\chi^G) = \{0\}$ and also let $\chi^G(x) = \chi^G(y), \forall x, y \in U$

$$\Rightarrow \chi^G(x) *' \chi^G(y) = 0_{L_{NVB}} \Rightarrow \chi^G(x * y) = 0_{L_{NVB}}, \text{ since } \chi^G \text{ is a NVB } G \text{ - homomorphism}$$

$$\Rightarrow (x * y) \in \text{Ker } (\chi^G) = \{0_{T_{NVB}}\} \Rightarrow (x * y) = 0_{T_{NVB}}. \text{ Hence, } x = y \Rightarrow \chi^G \text{ is a neutrosophic vague binary } G \text{ - monomorphism.}$$

10. Conclusion

In this paper, NVB G - subalgebraic structure is developed with its properties for NVBS's. Some basic ideas as NVB G - normal set of a G - algebra, NVB G - normal subalgebra and 0 - commutative NVB G - subalgebra are illustrated with examples and basic properties. Notions like G - part, p radical and p semi simple are defined in NVB G - subalgebra with characterizations. NVB G - minimal element, Derivations of NVB G - subalgebra, Regular Derivation of a NVB G - subalgebra, NVB G - homomorphism are also explained. Formation of Cosets is a basic idea in any algebraic structure. Coset for neutrosophic vague binary G - subalgebra is also got developed. Based on this, present work can be extended to NVB vague binary G - groups, NVB G - rings, NVB G - product, NVB G - factor group, Lagrange NVB G - subalgebra etc. As a future scope neutrosophic vague binary models can be tried to use in hazard detection, especially in switching circuits. Another application can be given in geographical area. Development of a neutrosophic vague binary spatial

algebra could be more helpful in this area than the already existing crisp spatial algebraic concepts. Since the already existing pattern got failed to provide an accurate output when collected data becomes vague.

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Decision Making Methods with Linguistic Neutrosophic Information: A Review

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Abstract: Linguistic neutrosophic information and its extension have been long recognized as a useful tool in decision-making problems in many areas. This paper briefly describes the development process of linguistic neutrosophic information expressions, and gives in-depth studies on seven different concepts and tools. At the same time, a brief evaluation and summary of the decision-making methods of its various measures and aggregation operators are also made. A comparative analysis of different linguistic neutrosophic sets is made with examples to illustrate the effectiveness and practicability of decision making methods based on multiple aggregation operators and measures. Finally, according to the analysis of the current situation of linguistic neutrosophic information, the related trends of its future development are discussed.

Keywords: linguistic; neutrosophic; decision making, aggregation operator, measures

1. Introduction

In a complex decision-making problem where humans are accustomed to use language to express their idea, decision makers may use linguistic variables (LVs) to qualitatively evaluate attributes. With this regard, Zadeh [1] first proposed the use of LVs to describe preference information and applied it to fuzzy reasoning, and attracted the attention of scholars at home and abroad. Since then, several studies have been carried out to solve problems in different application area [2-6]. However, previous studies [2-6] have reported that merely incomplete information can effectually expressed, while uncertain and conflicting information, are not. To fill the shortcomings mentioned above, Smarandache proposed the neutrosophic sets [7-8] and neutrosophic numbers (NNs) [7-9]. Since the concept of the neutrosophic set was established, some scholars focused on the combination of neutrosophic set and linguistic set to come up with their new concepts.

Fang and Ye [10] first introduced a linguistic neutrosophic number (LNN) concept. LNN has three-part the truth linguistic probability, indeterminacy linguistic probability, and falsity linguistic probability and can express three kinds of linguistic information in this situation. And they also provided score and accuracy functions and some aggregation operators of LNNs. Fan et al. [11] presented an LNN normalized weighted Bonferroni mean operator and an LNN normalized weighted geometric Bonferroni mean operator and applied them to deal with decision-making(DM) problems in LNN environment. Shi and Ye [12] proposed two cosine measures based on the distance and cosine of the included angle between two vectors of LNNs for describing indeterminate linguistic information. Meanwhile, Shi and Ye [13] presented three correlation coefficients of LNNs and showed how they can apply on multiple attribute group decision-making (MAGDM) problems.

On the basis of combining LNNs and NLNs, Cui et al. [14] defined a linguistic neutrosophic uncertain number (LNUN) and the score and accuracy function of LNUNs and then developed related aggregation operators to tackle MAGDM problems. Cui and Ye [15] further introduced a hesitant linguistic neutrosophic number (HLNN) and put forward a MADM method based on similarity measures for DM problems in HLNN sets. On the other hand, Ye et al. [16] proposed a Q-linguistic neutrosophic variable set (Q-LNVs), which extended linguistic neutrosophic evaluation to two-dimensional universal sets (TDUSs). Then, Fan et al. [17] presented a linguistic neutrosophic multiset (LNM) and two Heronian mean operators to handle the multiplicity information under LNM environment. Besides, Ye [18] originally put forward the concept of a linguistic cubic variable (LCV), which consists both uncertain and certain LV synchronously, then he developed some operators to aggregate linguistic cubic information. Next, Lu and Ye [19] integrated Dombi operators with LCVs to better handle DM problems of linguistic cubic sets. Further, Ye and Cui [20] proposed a linguistic neutrosophic hesitant variable (LCHV), and applied aggregation operators to figure out DM problems with interval and hesitant linguistic information. Then, Lu and Ye [21] presented cosine similarity measures of LCHVs which is characterized by the least common multiple number extension method, and its applications in decision-making with LCHV information. Also, Ye and Cui [22] put forward single-valued linguistic neutrosophic interval linguistic numbers (SVLN-ILN) and correlative aggregation operators together with its decision-making approach. Meanwhile, Ye [23] first proposed a new linguistic neutrosophic notion, named linguistic neutrosophic cubic numbers (LNCNs), which is made up of an inconclusive linguistic neutrosophic number and an LNN. Fan and Ye [24] extended the Heronian mean operator to LNCNs and adopt this idea to solve decision-making problems.

The main purpose of this paper is to carry out research on the decision-making methods under the linguistic neutrosophic environment. Firstly, it will be possible to describe some concepts of NLN, LNS, LNUN, HLNS, Q-LNS, LCS, and LNCS. Secondly, insight will be gained into the decision-making methods of using various measures and aggregation operators. Lastly, it gives conclusions and future study of this paper. These findings have significant implications for solving decision making problems in various field.

2. Linguistic Neutrosophic Information Expressions

2.1. Neutrosophic Linguistic numbers

Smarandache [7-8] originally presented the conception of a neutrosophic number that can express incomplete, indeterminate, inconsonant information, represented by $B=t+vI$, where t stands for the determinate part and vI for the indeterminate part, and $t, v \in R$ (all real numbers), $I \in [\inf I, \sup I]$ (indeterminacy). To better express uncertainty on linguistic information, Smarandache [25] introduced NNs into the LV and proposed a neutrosophic linguistic number (NLN) concept and described by l_{t+vI} where $t+vI$ is NN.

It can be known that on the above method only a single neutrosophic linguistic number is used to evaluate the linguistic information. However, in a complicated DM environment, decision makers may enforce to give several linguistic term values from a linguistic term set (LTS) due to their hesitancy. It means that a single linguistic term value is not sufficient to express the results of the assessment. Hence, it is clearly that the existing NLN method [26] is not suitable for such case. In order to deal with this situation, Ye began to see hesitant neutrosophic linguistic numbers as key components in linguistic decision-making field. As a result, Ye [27] proposed hesitant neutrosophic linguistic numbers (HNLNs) that consist of a series of NLNs, standing for the decision makers' different proposals respectively. Hence, HNLNs can easily be applied to hesitant decision-making problems involving the NLNs consist of partial determinacy and partial uncertainty.

Definition 2.1.1. [27] Let $X = \{x_1, x_2, \dots, x_n\}$ be a universe of discourse and $L = \{l_0, l_1, \dots, l_{2t}\}$ be a finite and fully ordered set of discrete linguistic terms. An HNLN set H_l on X described mathematically as the following form:

$$H_l = \left\{ \left\langle x_j, h_l(x_j) \right\rangle \mid x_j \in X \right\}, \quad (1)$$

where $h_l(x_j)$ is the set of x_j NLNs for $x_j \in X$ and L , and x_j is the number of NLNs with $j=1, 2, \dots, n$. Therefore, $h_l(x_j)$ can be denoted by $h_l(x_j) = \left\{ l_{a_j^k + b_j^k} \mid l_{a_j^k + b_j^k} \in L, k=1, 2, \dots, s_j \right\}$ for $x_j \in X$ and $j=1, 2, \dots, n$.

2.2. Linguistic neutrosophic sets

The existing NLN can provide useful tools to deal with incomplete, indeterminate, and inconsistent linguistic information. However, it cannot use for DM problems with information expressed with their truth, indeterminacy and false functions. An LNN proposed by Fang and Ye [10] can better address the drawback shown above since it is characterized by the truth, indeterminacy, and falsity LVs respectively rather than exact values. In fact, LNNs can also be considered as a new LV added to LIFN to indicate the degree of indeterminacy and the incomplete and inconsistent linguistic information. LNNs are a useful tool in depicting the indeterminate and inconsistent decision-making information by using three linguistic variables.

Definition 2.2.1. [10] Let $L = \{l_0, l_1, \dots, l_{2t}\}$ is a finitely linguistic term set. If $g = \langle l_T, l_I, l_F \rangle$ is defined as $l_T, l_I, l_F \in L$ and $T, I, F \in [0, 2t]$, where l_T, l_I and l_F use linguistic terms to show the truth, indeterminacy, and falsity degree, severally, then g is called an LNN.

2.3. Linguistic neutrosophic uncertain numbers/sets

Motivated by NLNs and LNNs, Cui et al. [14] defined a new notion of an LNUN constructed respectively by three uncertain linguistic variables representing linguistic truth, indeterminacy and falsity. In general, the LNUN is the expansion of LNN and NLN with partial linguistic certain and partial linguistic uncertain evaluations. It turns out that LNUNs can describe the different complex linguistic neutrosophic decision-making information under an LNUN environment.

Definition 2.3.1. [14] Assume that $L = \{l_0, l_1, \dots, l_{2t}\}$ is a finite and fully ordered set of linguistic term set. An LNUN in L is constructed as $h = \left\langle l_{T_a+T_bI}, l_{U_a+U_bI}, l_{F_a+F_bI} \right\rangle$ with three uncertain linguistic variables $l_{T_a+T_bI}$, $l_{U_a+U_bI}$, and $l_{F_a+F_bI}$ representing the truth, uncertainty, and falsity NLNs independently, where $T_a+T_{bl}, U_a+U_{bl}, F_a+F_{bl} \in [0, 2t]$ and $I \in [\inf I, \sup I]$.

2.4. Hesitant linguistic neutrosophic sets

It is obvious that much DM information in the real world is fuzzy rather than precise, in which decision-makers may be entangled in a certain decision. However, LNN cannot express the hesitation of decision-makers in the evaluation of linguistic alternatives. A HLNN introduced by Cui and Ye [15] can express much more information given by decision-makers since it is composed of several LNNs related to an objective thing. Essentially, HLNNs are combined form of HFSs and LNNs, which can simultaneously express both the hesitancy information and LNN information of decision-makers.

Definition 2.4.1. [15] Set $X = \{x_1, x_2, \dots, x_n\}$ as a universe of discourse and a finite linguistic term set $L = \{l_0, l_1, \dots, l_{2t}\}$, and then an HLNN set N_l on X can be given by

$$N_l = \left\{ \left\langle x_j, E_l(x_j) \right\rangle \mid x_j \in X, j = 1, 2, \dots, n \right\}, \quad (2)$$

Where $E_l(x_j)$ is a set of x_j LNNs for $x_j \in X$ and L , expressed by an HLNN $E_l(x_j) = \left\{ \left\langle l_{T_j^k}, l_{U_j^k}, l_{F_j^k} \right\rangle \mid l_{T_j^k} \in L, l_{U_j^k} \in L, l_{F_j^k} \in L, k = 1, 2, \dots, x_j \right\}$ for $x_j \in X$.

2.5. Q-linguistic neutrosophic set

A majority of linguistic concepts only process indeterminate, uncertain and incompatible data of the subject being evaluated in one-dimensional universal sets. This prompted researchers to amplify them to have the ability to depict linguistic arguments in TDUSs. Then, Ye et al. [16] first proposed a Q-LNVS to explain linguistic neutrosophic claims in DM problems of TDUSs. Therefore, Q-LNVS was primarily used to define its linguistic values of truth, indeterminacy and falsity corresponding to TDUSs, respectively.

Definition 2.5.1. [16] Assume that $X = \{x_1, x_2, \dots, x_n\}$ and $Q = \{q_1, q_2, \dots, q_m\}$ are two-dimensional universal sets and a finite linguistic term set $L = \{l_0, l_1, \dots, l_{2t}\}$, and then a Q-LNVS P on X and Q can be denoted by

$$P = \left\{ \left\langle \left\langle x_i, q_j \right\rangle, l_t(x_i, q_j), l_u(x_i, q_j), l_v(x_i, q_j) \mid x_i \in X, q_j \in Q \right\rangle \mid l_t(x_i, q_j), l_u(x_i, q_j), l_v(x_i, q_j) \in L, i = 1, 2, \dots, n; j = 1, 2, \dots, m \right\}, \quad (3)$$

where $l_t(x_i, q_j)$, $l_u(x_i, q_j)$, $l_v(x_i, q_j)$ denoted the truth, indeterminacy, and falsity LVs, independently, in TDUSs for $t, u, v \in [0, 2t]$.

Then, the basic element $\langle (x_i, q_j), l_t(x_i, q_j), l_u(x_i, q_j), l_v(x_i, q_j) \rangle$ in L is simply expressed as $l_{ij} = \left\langle (x_i, q_j), s_{t_{ij}}, s_{u_{ij}}, s_{v_{ij}} \right\rangle$, which is known as a Q-linguistic neutrosophic element (Q-LNE).

Later on, based on the linguistic multiplicity evaluation in some real situations, Fan et al. [17] developed an LNM, which is extended from neutrosophic multiset. An LNM can use pure linguistic value to express and process the multiplicity information and can represent the truth, indeterminacy, and falsity through three values, severally.

Definition 2.5.2. [17] Set a universe $X = \{x_1, x_2, \dots, x_n\}$ and $L = \{l_0, l_1, \dots, l_{2t}\}$ be an LTS, and $Z = \{1, 2, 3, \dots, \infty\}$, then LNM R represented with the following mathematical expression.

$$R = \left\{ x, \left[\left(f_{R1}, \left\langle l_{\mu_{R1}}(x), l_{\nu_{R1}}(x), l_{\xi_{R1}}(x) \right\rangle \right), \left(f_{R2}, \left\langle l_{\mu_{R2}}(x), l_{\nu_{R2}}(x), l_{\xi_{R2}}(x) \right\rangle \right), \dots, \left(f_{Ry}, \left\langle l_{\mu_{Ry}}(x), l_{\nu_{Ry}}(x), l_{\xi_{Ry}}(x) \right\rangle \right) \right] \mid x \in X \right\}, \quad (4)$$

where $l_{\mu_{Rt}}(x)$, $l_{\nu_{Rt}}(x)$, $l_{\xi_{Rt}}(x) \in L$, μ_{Rt} , ν_{Rt} , $\xi_{Rt} \in [0, 2t]$. An LNM consists of the truth degree membership function $l_{\mu_{Rt}}(x)$, the indeterminacy degree membership function $l_{\nu_{Rt}}(x)$, and the falsity degree membership function $l_{\xi_{Rt}}(x)$. Among them, $l_{\mu_{R1}}(x)$, $l_{\mu_{R2}}(x)$, ..., $l_{\mu_{Ry}}(x) \in [0, 1]$, $l_{\nu_{R1}}(x)$, $l_{\nu_{R2}}(x)$, ..., $l_{\nu_{Ry}}(x) \in [0, 1]$ and $l_{\xi_{R1}}(x)$, $l_{\xi_{R2}}(x)$, ..., $l_{\xi_{Ry}}(x) \in [0, 1]$, that is,

$$0 \leq l_{\mu_{Rt}}(x) + l_{\nu_{Rt}}(x) + l_{\xi_{Rt}}(x) \leq 3 \quad (t = 1, 2, \dots, y), \quad y \in Z, \quad f_{R1}, f_{R2}, \dots, f_{Ry} \in Z, \quad \text{and} \\ f_{R1} + f_{R2} + \dots + f_{Ry} \geq 2.$$

The above expression for an LNM R can be simplified to the following form:

$$R = \left\{ x, \left(f_{Rt}, \left(l_{\mu_{Rt}}(x), l_{\nu_{Rt}}(x), l_{\xi_{Rt}}(x) \right) \right) \mid x \in X \right\}, \quad (5) \\ \text{for } t = 1, 2, \dots, y.$$

2.6. Linguistic cubic sets

In reality, some real decision-making problems may contain mixed evaluation information of uncertain and certain linguistic arguments simultaneously. To handle this, Ye [18] proposed an LCV by merging LVs and cubic set together and can be applied on can have consists of an uncertain LV and a specific LV.

Definition 2.6.1. [18] Let $L = \{l_0, l_1, \dots, l_{2t}\}$ is a finite LTS. An LCV V in L is denoted using $V = (\tilde{L}, L_c)$,

where $\tilde{L} = [L_a, L_b]$ is an uncertain LV and L_c is an LV for $b \geq a$ and $L_a, L_b, L_c \in L$. If $a \leq b \leq c$, $V = ([L_a, L_b], L_c)$ is an internal LCV. If $c \notin (a, b)$, $V = ([L_a, L_b], L_c)$ is an external LCV.

However, due to uncertainty and hesitation on the part of decision-makers on the subject of evaluation, in some decision-making problems, information on decision-making is made up of an interval of LV and a hesitant linguistic set. To deal with such a situation, based on the concepts of an LCV and hesitant fuzzy sets, Ye and Cui [20] proposed an LCHV. The proposed LCHV reasonably express the combined information from uncertain and hesitant linguistic arguments and efficiently tackled LCHV problems.

Definition 2.6.2. [20] Set a linguistic variable term set as $L = \{l_j \mid j \in [0, 2t]\}$. An LCHV z in L is built by z

$= (\tilde{l}_u, \tilde{l}_h)$, where $\tilde{l}_u = [l_a, l_b]$ for $b \geq a$ and $l_a, l_b \in L$ is an interval linguistic variable and $\tilde{l}_h = \{l_{\lambda_k} \mid l_{\lambda_k} \in L, k = 1, 2, \dots, j\}$ is a set of j possible LVs (i.e., a hesitant LV is listing in an increasing order.)

Furthermore, Ye and Cui [22] presented the idea of an SVLN-ILN, composed entirely of its uncertain / interval linguistic number and its single valued neutrosophic linguistic number. In the case of a DM problem, the SVLN-ILN represents both the linguistic judgment of the decision-maker and the affirmative linguistic judgment of the evaluated object.

Definition 2.6.3 [22] Let a linguistic variable set be $L = \{l_0, l_1, \dots, l_{2t}\}$. A SVLN-ILN W in L is denoted by $W = \langle [l_a, l_b]; l_T, l_I, l_F \rangle$, where $[l_a, l_b]$ is the interval linguistic number part of W and l_a and l_b are linguistic lower and upper limits of l_j for $l_a \leq l_j \leq l_b$ and $l_j \in L$, and then $\langle l_T, l_I, l_F \rangle$ is the SVLNN part of W . Here, the truth linguistic function $T_W(l_j)$ of W can be constructed by

$$T_W(l_j) = \begin{cases} l_T, & l_a \leq l_j \leq l_b \\ l_0, & \text{otherwise} \end{cases}, \quad (6)$$

The indeterminacy linguistic function $I_W(l_j)$ of W can be constructed by

$$I_W(l_j) = \begin{cases} l_I, & l_a \leq l_j \leq l_b \\ l_z, & \text{otherwise} \end{cases}, \quad (7)$$

The falsity linguistic function $F_W(l_j)$ of W can be constructed by

$$F_W(l_j) = \begin{cases} l_F, l_a \leq l_j \leq l_b \\ l_z, \text{ otherwise} \end{cases}, \quad (8)$$

where $l_0 \leq l_T \leq l_z, l_0 \leq l_I \leq l_z$ and $l_0 \leq l_F \leq l_z$.

2.7. Linguistic neutrosophic cubic sets

A new notion of linguistic neutrosophic cubic set, as presented by Ye, extending the concept of cubic sets to linguistic neutrosophic sets, called linguistic neutrosophic cubic sets. A proposed LNCN contains an uncertain LNN and a single-valued LNN at the same time as the linguistic variables of truth, indeterminacy and falsity [23]. In LNCN, the uncertain LNN expresses the truth, indeterminacy, and falsity values of uncertain LVs, and the single-valued LNN is composed of the truth, indeterminacy, and falsity LVs, which are used to describe their mixed information.

Definition 2.7.1. [23] Let an LTS be $L = \{l_j \mid j \in [0, 2t]\}$. An LNCN O in L is defined as $O = (u, c)$, where $u = \langle [l_{Ta}, l_{Tb}], [l_{Ia}, l_{Ib}], [l_{Fa}, l_{Fb}] \rangle$ is an uncertain LNN with the truth linguistic variables $[l_{Ta}, l_{Tb}]$, indeterminacy linguistic variables $[l_{Ia}, l_{Ib}]$, and falsity uncertain linguistic variables $[l_{Fa}, l_{Fb}]$, for $l_{Ta}, l_{Tb}, l_{Ia}, l_{Ib}, l_{Fa}, l_{Fb} \in L$ and $Ta \leq Tb, Ia \leq Ib, Fa \leq Fb$; $c = \langle l_T, l_I, l_F \rangle$ is consisted of an LNN with l_T, l_I and l_F each on behalf of the truth, indeterminacy, and falsity LVs, respectively, where $l_T, l_I, l_F \in L$.

3. Decision making methods regarding various measures and aggregation operators

Because of the inherent vagueness of human thinking and the complexity of the objective world, a clear description of decision information is the most crucial part in the real evaluation processes. Hence, to better describe the decision information, the forms of decision information need to be continuously expanded and enriched according to the specific situation. In the process of dealing with information that is incomplete, uncertain and inconsistent, the introduction of linguistic neutrosophic sets play an important role. Smarandache [25] firstly defined NLNs in symbolic neutrosophic theory. Later, to address the problems of neutrosophic linguistic number decision-making, Ye [26] further suggested basic operations and two weighted NLN aggregation operators, namely, the NLN weighted arithmetic average (NLNWAA) operator and the NLN weighted geometric average (NLNWGA) operator. Next, they have been widely used to make alternative manufacturing decisions in flexible manufacturing systems. Then, Ye [27] put forward the concept of HNLNs and the excepted value together with their similarity measure. HNLNs were further developed to use under hesitant and indeterminate linguistic environment. Apart from that, the application is illustrated by taking the problem of manufacturing scheme selection as an example.

To express the truth, falsity, and indeterminacy linguistic information respectively, linguistic neutrosophic numbers containing three independently linguistic variables were presented. After that, some aggregation operators of LNNs, such as the LNN-weighted arithmetic averaging (LNNWAA) and the LNN-weighted geometric averaging (LNNWGA) operators [10], the LNN normalized weighted Bonferroni mean (LNNNWBM) and LNN normalized weighted geometric Bonferroni mean (LNNNWGBM) operators [11], a cosine similarity measure of LNNs [12] and correlation coefficients of LNNs [13] were proposed to tackle decision-making problems in linguistic neutrosophic sets. LNNWAA and LNNWGA operators, as two basic aggregation operators, are often used to select investment alternatives under LNN information. Bonferroni mean (BM), which is an effective aggregation operator that not only considers the importance weights of attributes, but also reflects the interrelationship between attribute values [28] and it is extended to fuzzy sets [29-34] and neutrosophic theory [35-36] to apply Bonferroni mean operators for DM. Motivated by the idea of LNN and Bonferroni mean (BM) operators, Fan proposed the LNNNWBM operator and the LNNNWGBM operator. At the same time, he took different parameter values of p and q to analyze

their impact on the decision results. Meanwhile, similarity measures have aroused widespread concerns, which is a vital tool in decision-making process [37-41]. The cosine measures between LNNs were proposed based on distance and the included cosine of the angle between LNNs in vector space that can sort the alternatives and choose the most ideal one(s) [12]. The similarity measure methods have a good application prospect in ideal investment alternatives under linguistic decision-making environments. Further, correlation coefficient is also an available tool for making decisions in complex problems [42-46]. Shi extended correlation coefficients to LNNs and put forward three new correlation coefficients between a substitution and the ideal substitution of LNNs and introduced an example of the investment substitution selection problem.

Also, LNUNs with corresponding weighted aggregation operators were put forward to depict three uncertain linguistic variables for decision-making in the uncertain linguistic environment [14]. Some weighted operators, such as a LNUNWAA operator and a LNUNWGA operator, are raised to aggregate LNUN information and exploited to demonstrate the effectiveness of an investment company decisions.

In fact, the degree of similarity or difference between the research objects plays a decisive role in the DM results. The similarity measures of HLNNs were put forward under the hesitant neutrosophic environment. Cui and Ye [15] presented similarity measures and generalized distance of HLNNs. Then he utilized similarity measures of HLNNs regarding least common multiple cardinality to satisfy the demand of hesitant decision-making and showed their application in investment alternatives.

In vector space, in particular, the Jaccard, Dice, and cosine similarity measures are usually used in diversified fields [41, 47-50]. Applying the Jaccard, Dice, and cosine similarity measures thus improve the decision-making process and produce better results. In this way, a Q-LNVS, which can depict linguistic neutrosophic arguments to two-dimensional universal sets, was presented [16]. And the vector similarity measures that contain Jaccard, Dice, and cosine measures were used for settling linguistic neutrosophic decision-making problems regarding TDUSs. Thereafter, the LNM and its two Heronian mean operators were raised to handle multiplicity information under linguistic neutrosophic multiplicity number environment [17].

On the basis of LCVs, a LCVWAA operator and a LCVWGA operator are presented to aggregate linguistic cubic information [18]. Next, Lu and Ye [19] extended the Dombi operators to LCV, which contain variable operational parameters and more flexible representation of decision information and developed a LCVDWAA operator and a LCVDWGA operator to aggregate linguistic cubic information. These two methods are well applied in the optimal selective problems. Hereafter, a target expansion method of LCHVs using least common multiple/cardinality, and the WAA and WGA operators of LCHVs to reasonably aggregate LCHV information, were proposed [20]. Next, the similarity measures were developed to measure the degree of similarity between LCHVs and an example of engineering selection was used to solve practical problems [21]. Utilizing the mixed information of interval linguistic number and single-valued LNN, SVLN-ILNs and corresponding weighted aggregation operators were given to provide a comprehensively description of interval linguistic parameters and confident linguistic parameters [22].

Meanwhile, by the combined form of uncertain linguistic and certain linguistic neutrosophic numbers, LNCNs and related aggregation operators, like two weighted aggregation and Heronian mean operators were introduced to work out linguistic decision-making problems [23-24]. The DM method based on a LNCNWAA operator and a LNCNWGA operator was constructed in machinal design schemes problems. All of the above methods assume that the set variables are independent of each other. However, because of the complexity of the real world, most of the information variables are related to each other. This correlation will directly affect the decision results. To overcome the shortcomings, Fan combined the Heronian mean operator with the LNCN to develop a MADM method of mechanical design schemes using the LNCNGWHM operator or LNCNTPWHM operator under LNCN setting.

These various measures and aggregation operators are further shown in Table 1.

Table 1. Regarding Various Measures and Aggregation Operators.

Authors	Sets	Tools and approaches
Jun Ye	NLNs	NLNWAA and NLNWGA operators to aggregate NLN information
Jun Ye	NLNs	the expected value and the similarity measure of HNLNs
Zebo Fang; Jun Ye	LNS	LNNWAA and LNNWGA operators to aggregate LNN information
Changxing Fan et al.	LNS	extend Bonferroni mean to LNN, and propose LNNNWBM and LNNNWGBM operators
Lilian Shi; Jun Ye	LNS	extend cosine similarity measures to LNNs
Lilian Shi; Jun Ye	LNS	put forward three new correlation coefficients of LNNs
Wenhua Cui et al.	LNUNs	LNUNWAA and LNUNWGA operators to aggregate LNUN information
Wenhua Cui, Jun Ye	HLNS	present similarity measures of HLNNs based on LCMC
Jun Ye et al.	Q-LNS	put forward vector similarity measures of Q-LNVSS
Changxing Fan et al.	Q-LNS	extend Heronian mean to LNM, and propose LNMNGWHM and LNMNIGWHM operators
Jun Ye	LCS	LCVWAA and LCVWGA operators to aggregate linguistic cubic information
Jun Ye; Wenhua Cui	LCS	WAA and WGA operators of LCHVs
Jun Ye; Wenhua Cui	LCS	SVLN-ILNWAA and SVLN-ILNWGA operators to aggregate SVLN-ILN information
Jun Ye	LNCS	expand neutrosophic cubic sets to linguistic neutrosophic arguments, and propose LNCNWAA and LNCNWGA operators
Changxing Fan; Jun Ye	LNCS	extend Heronian mean to LNCN, and propose LNCNGWHM and LNCNTPWHM operators
Lilian Shi; Jun Ye	LNS	extend cosine similarity measures to LNNs
Lilian Shi; Jun Ye	LNS	put forward three new correlation coefficients of LNNs

Obviously, the main advantage of NLN is that it can express and process ubiquitous imprecise, incomplete, and indeterminate linguistic information under a linguistic DM environment, which is more suitable for practical scientific and engineering applications. However, in the event of complex DM problems due to the hesitation and uncertainty of the cognition of decision-makers, this method cannot accurately reflect the actual meaning of the decision makers. They may not put their evaluation of a certain attribute with a single NLN. In such a case, the hesitation and uncertain evaluation are expressed by a series of NLNs known as HNLN which is an effective method in a hesitant linguistic environment. Through a comparative analysis of the two MADM methods proposed under the HNLN setting and the present MADM methods proposed in the NLN environment, it is found that the best choice is the same. But it can also be known that their ranking order is slightly different. This is because the MADM method in the HNLN and NLN environments differs in the information expression and algorithm, which explains that there may be differences in the sort order under HNLN and NLN environments.

In fact, LNNs can express uncertain and inconsistent linguistic information corresponding to human's vague thinking on intricate problems, particularly the qualitative evaluation of some attributes, which solve the problem of uncertain and inconsistent linguistic information. After comparison, it is found that the two sorting orders and the ideal choice based on the LNNWAA and LNNWGA operators are the same, which is consistent with the result in the literature [51]. The LNNNWBM and LNNNWGBM operators take into account the influence of the parameters p and q on the decision results. By diverse values of the parameters p and q , we can know that the arrangement order of the study is the same. Therefore, these two parameters have little effect on this decision problem [11]. The ranking results of this example are consistent, but in contrast, the LNNNWBM operator and LNNNWGBM operator consider the correlation between attributes for MAGDM, making the information aggregation more objective and reliable. The cosine similarity measures of LNNs are simpler than the LNNWGA operator and LNNWAA operator. In addition,

the correlation coefficients of LNN are compared with LNNWGA and LNNWAA operators, and it can be seen from the literature [13] that the sort order based on these three new correlation coefficients is consistent with the results proposed in the literature [10]. What counts is that the correlation coefficients of LNNs are relatively simple and can even further avoid some unreasonable phenomena existing in LNNWGA and LNNWAA operators.

Similarity measures of HLNNs based on the LCMC extension method can reflect the indecisiveness of decision-makers under a HLNN environment. The similarity measures not just process the HLNN, but also the LNN as LNN is just a special case of the HLNN without decision-makers hesitation. LNUNWAA and LNUNWGA operators are two types of LNUN information aggregation operators, in which the indeterminacy range of I will lead to different order of the schemes. Therefore, with the MAGDM method based on LNUN information, decision makers can pick disparate indeterminacy ranges according to their own preferences or actual needs, making the actual decision-making problem more flexible. It is worth noting that if the indeterminacy I is not considered (i.e., $I = 0$), LNN is just a special case of LNUN.

Compared with the LNNs decision-making method [10], LNCNs contain more information which can simultaneously express uncertain LNNs and certain LNNs under linguistic DM environment. The aggregation of linguistic neutrosophic cubic information can performed by the LNCNWAA operator and LNCNWGA operator. Therefore, decision-makers have two choices of LNCNs weighted set operators to settle the linguistic neutrosophic cubic decision problem depend on their own preferences and actual needs. The MADM method based on a LNCNGWHM operator and a LNCNTPWHM operator combine the LNCN with Heronian mean operator which can reflect the interaction between attributes. The literature [24] analyzed the possibility that the various parameters p, q, r may could affect decision results differently. Therefore, sort the operation results by adjusting the values of the three parameters. The results show that the parameters in the LNCNGWHM or LNCNTPWHM operator have little effect on the decision of this example. Compared with the results of LNCNWAA and LNCNWGA [23], their sort order is the same. However, LNCNGWHM and LNCNTPWHM operators reflect the interactions between attributes, and take into account different p, q , and r values, making the outcome more convincing and comprehensive than those of LNCNWAA and LNCNWGA.

A linguistic neutrosophic MADM method based on Q-LNVs includes the Jaccard, Dice, and cosine similarity measures. Then LNV is a particular case of Q-LNVS for a general set.

The LNMNGWHM and LNMNIGWHM operators represent and deal with the problem of multiplicity, and can obtain more complicated results by considering the interrelationship between attributes, which make the results more realistic. The ranking results are analyzed by different values of d and f that show no matter how these two values are taken, the sort orders are consistent, so d and f have tiny effect on the ranking results of the study. Compared with the proposed operators in the literature [10], it is found that their results are coincident, but the operators of LNM have the advantage of expressing and handling the multiplicity problems. Therefore, this method can make the decision result more reliable and has certain practicability in practical application.

4. Conclusions

Linguistic neutrosophic information has been extended to various types and these extensions have been used in many areas of decision making. This review paper mainly focused on the overview of the development process of linguistic neutrosophic information expressions from seven aspects (NLNs, LNS, LNUNs, HLNS, Q-LNS, LCS, LNCS), and makes in-depth research on its application in decision-making. Analysis shows that they can be combined with commonly used mathematical tools, such as aggregation operators, measures, etc. These methods are being employed increasingly for the evaluation of alternatives and comparative analysis in different decision problems. Despite their advantage of getting a better result, the currently proposed linguistic neutrosophic information hasn't widely used outside MADM problems.

As a result, in the future study, we will further combine with other fuzzy theories (such as rough sets, etc.) to develop new linguistic sets and expand its application to other domains, such as fault diagnosis, medical diagnosis, picture analysis, and pattern recognition.

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