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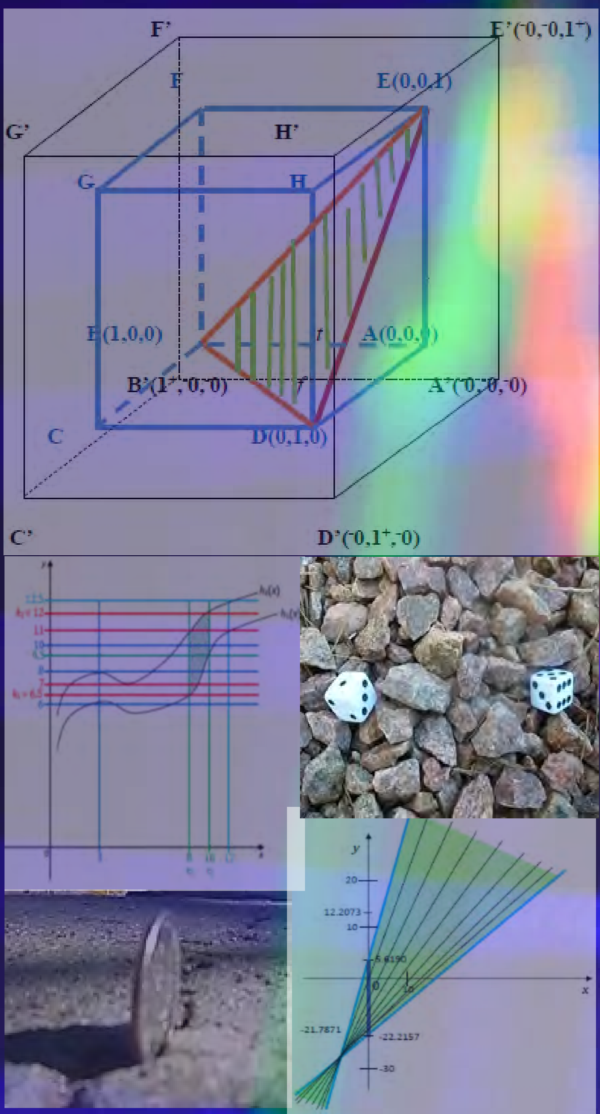

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset
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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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March 20, 2019

Prof. Florentin Smarandache
Univ New Mexico, Gallup Campus

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Eight Kinds of Graphs of BCK-algebras Based on Ideal and Dual Ideal

Atena Tahmasbpour Meikola [1,*]

[1] Department of mathematics. Islamic azad university, central Tehran branch, Iran;

* Correspondence: atena.tahmasbpour63@gmail.com.

Abstract: In this paper, at first we introduce the concepts of ideal-annihilator, dual ideal- annihilator, right- ideal- annihilator, left- ideal- annihilator, right- dual ideal- annihilator, left- dual ideal- annihilator. Then by using of these concepts, we constructed six new types of graphs in a bounded BCK-algebra $(X,*,0)$ based on ideal and dual ideal which are denoted by $\Phi_I(X), \Phi_{I^N}(X), \Delta_I(X), \Sigma_I(X)$, $\Delta_{I^N}(X)$, and $\Sigma_{I^N}(X)$, respectively. Then basic properties of graph theory such as connectivity, regularity, and planarity on the structure of these graphs are investigated. Finally, by utilizing of binary operations $\wedge$ and $\vee$, we construct graphs $\Upsilon_I(X)$ and $\Upsilon_{I^N}(X)$, respectively, some their interesting properties are presented.

Keywords: BCK- algebra; Diameter; Chromatic number; Euler graph.

1. Introduction

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. Associating a graph to an algebraic structure is a research subject in this area and has attracted considerable attention. In fact, the research in this subject aims at exposing the relationship between algebra and graph theory and at advancing the application of one to the other. The story goes back to a paper of Beck [4] in 1998, where he introduced the idea of a zero-divisor graph of a commutative ring R with identity. He defined $\Gamma(R)$ to be the graph whose vertices are elements of R and in which two vertices x and y are adjacent if and only if $xy = 0$. Recently, Halas and Jukl in [7] introduced the zero divisor graphs of posets. The study of the zero-divisor graphs of posets was then continued by Xue and Liu in [23], Maimani in [12]. More recently, a different method of associating a zero-divisor graph to a poset P was proposed by Lu and Wu in [11]. In this paper, we deal with zero-divisor graphs of BCI/BCK-algebras based on ideal and dual ideal. Imai and Iseki [8] in 1966 introduced the notion of BCK- algebra. In the same year, Iseki [9] introduced BCI-algebra as a super class of the class of BCK- algebras. Jun and Lee [10] defined the concept of associated graph of BCK- algebra and verified some properties of this graph. Zahiri and Borzooei [24] associated a new graph to a BCI-algebra X which is denoted by $G(X)$, this definition is based on branches of X, Tahmasbpour in [16, 17] studied chordality of graph defined by Zahiri and Borzooei and introduced four types of graphs of BCK- algebras which are constructed by equivalence classes determined by ideal I and dual ideal $I^{\vee}$. Also, Tahmasbpour in [18, 21] introduced two new graphs of lattice implication algebras based on LI-ideal. Further, Tahmasbpour in [19, 20] introduced two new graphs of BCK-algebras based on fuzzy ideal μ_I and fuzzy dual ideal $\mu_{I^{\vee}}$, two new graphs of lattice implication algebras based on fuzzy filter μ_F and fuzzy LI- ideal μ_A. Futhermore, Tahmasbpour in [22] introduced twelve kinds of graphs of lattice implication algebras based on filter and LI- ideal. This paper is divided into six parts.

In Section 2, we recall some concepts of graph theory such as connected graph, planar graph, outerplanar graph, Eulerian graph, and chromatic number, among others.

Section 3, is an introduction to a general theory of BCK- algebras. We will first give the notions of BCI/BCK- algebras, and investigate their elementary and fundamental properties , and then deal with a number of basic concepts, such as ideal, and dual ideal, among others.

In Section 4, inspired by ideas from Behzadi et al. [5], we study the graphs of BCK-algebras which are constructed from ideal-annihilator and dual ideal-annihilator, denoted by $\Phi_I(X), \Phi_{I^{\vee}}(X)$, respectively.

In Sction 5, inspired by ideas from Behzadi et al. [5], we study the graphs of BCK- algebras which are constructed from right- ideal- annihilator, left- ideal- annihilator, right- dual ideal- annihilator, left- dual ideal- annihilator, denoted by $\Delta_I(X), \Sigma_I(X), \Delta_{I^{\vee}}(X)$, $\Sigma_{I^{\vee}}(X)$, respectively.

In Section 6, inspired by ideas from Alizadeh et al. [3], we introduce the associated graphs $\Upsilon_I(X)$ and $\Upsilon_{I^{\vee}}(X)$ which are constructed from binary operations $\wedge$ and $\vee$, respectively.

2. Preliminaries of graph theory

In this section, for convenience of the reader, we recall some definitions and notations concerning graphs and posets for later use.

Definition 2.1. ([3, 6]) For a graph G, we denote the set of vertices of G as $V(G)$ and the set of edges as $E(G)$. A graph G is said to be complete if every two distinct vertices are joined by exactly one edge. The greatest induced complete subgraph denotes a clique. If graph G contains a clique with n elements, and every clique has at most n elements, we say that the clique number of G is n and write $\omega(G) = n$. Also, a graph G is said to be connected if there is a path between any given pairs of vertices, otherwise the graph is disconnected. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y and if there is no such path we define $d(x, y) \coloneqq \infty$. The diameter of G is $diam(G) \coloneqq \sup\{d(x, y); x, y \in V(G)\}$. Also, the girth of a graph G, is denoted by $gr(G)$, is the length of the shortest cycle in G if G has a cycle; otherwise, we get $gr(G) \coloneqq \infty$. The neighborhood of a vertex x is the set $N_G(x) = \{y \in V(G); xy \in E(G)\}$. Graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph G is called regular of degree k when every vertex has precisely k neighbors. A cubic graph is a graph in which all vertices have degree three. In other words, a cubic graph is a 3- regular graph. Moreover, for distinct vertices x and y, we use the notation $x - y$ to show that is x connected to y. Let $P = (V, \leq)$ be a poset. If $x \leq y$ but $x \neq y$, then we write $x < y$. If x and y are in V, then y covers x in P if $x < y$ and there is no $z \in V$, with $x < z < y$. Two sets $\{x \in P; x\ covers\ 0\}$ and $\{x \in P; 1\ covers x\}$, denoted by $Atom(P)$ and $Coatom(P)$, respectively. Let $L \subseteq P$, we say L is a chain if for all $x,y \in L, x \leq y$ or $y \leq x$. Chain L is maximal if for all chain $L', L \subseteq L'$ implies that $L = L'$.

Definition 2.2. ([4]) If K is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color, we say that the chromatic number of G is K and write $\chi(G) = K$. Moreover, we have $\chi(G) \geq \omega(G)$.

Definition 2.3. ([6]) A closed walk in a graph G containing all the edges of G is called an Euler line in G. A graph containing an Euler line is called an Euler graph. We know that a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an Euler graph is connected. Euler's theorem says that the connected graph G is Eulerian if and only if all vertices of G are of even degree.

Definition 2.4. ([2]) A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. Graph G is planar if it can be drawn in a plane without the edges having to cross. Proving that a graph is planar amounts to redrawing the edges in such a way that no edges will cross. One may need to move the vertices around and the edges may have to be drawn in a very indirect fashion. Kuratowski's theorem says that a finite graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. The clique number of any planar graph is less than or equal to four.

Definition 2.5. ([15]) Let G be a plane graph. A face is a region bounded by edges. An undirected graph is an outerplanar graph if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

Definition 2.6. ([14]) The number g is called the genus of the surface if it is homeomorphic to a sphere with g handles or equivalently holes. Also, the genus g of a graph G is the smallest genus of all surfaces in such a way that the graph G can be drawn on it without any edge-crossing. The graphs of genus zero are precisely the planar graphs since the genus of a plane is zero. The graphs that can be drawn on a torus without edge- crossing are called toroidal. They have a genus of one since the genus of a torus is one. The notation $\gamma(G)$ stands for the genus of a graph G.

Theorem 2.7. ([1]) For the positive integers m and n, we have:

(i) $\gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil$ if $n \geq 3$,

(ii) $\gamma(K_{m,n}) = \left\lceil \frac{1}{4}(m-2)(n-2) \right\rceil$ if $m, n \geq 2$.

3. Introduction of BCI/BCK- algebras

In this section, we submit some concepts related to BCI/BCK-algebra, which are necessary for our discussion.

Definition 3.1. ([13]) A BCI- algebra $(X,*,0)$ is an algebra of type $(2,0)$ satisfying in the following conditions:

$(BCI\ 1)$ $((x*y)*(x*z))*(z*y)=0,$

$(BCI\ 2)$ $x*0=0,$

$(BCI\ 3)$ $x*y=0, y*x=0$ imply $y=x$.

If X satisfies in the following identity:

$$(\forall x\in X)\ (0*x=0),$$

Therefore X is called a BCK-algebra. Any BCI/BCK- algebra X satisfies in the following conditions:

$(i)(x*(x*y))*y=0,$

$(ii)x*x=0,$

$(iii)(x*y)*z=(x*z)*y,$

$(iv)x\leq y$ implies $x*z\leq y*z$ and $z*y\leq z*x$, for any $z\in X$.

Moreover, the relation $\leq$ was defined by $x\leq y\leftrightarrow x*y=0$, for any $x,y\in X$, which is a partial order on X. $(X,*,0)$ is said to be commutative if it satisfies for all $x,y\in X,$

$$x*(x*y)=y*(y*x)$$

Definition 3.2. ([13]) A subset I is called an ideal of X if it satisfies the following conditions:

$(i)\ 0\in I,$

$(ii)\ (\forall x,y\in X), (x*y\in I, y\in I\rightarrow x\in I).$

An ideal P of X is prime if $x*(x*y)\in P$ implies $x\in P$ or $y\in P$.

Note: A BCK- algebra X is said to be bounded if there exists $e \in X$ in such a way that $x \leq e$ for any $x \in X$, and the element e is said to be the unit of X. In a bounded BCK-algebra, we denote $e * x$ by $N(X)$.

Definition 3.3. ([13]) A nonempty subset $I^{\vee}$ of a bounded BCK-algebra X is said to be a dual ideal of X if

$(i)\, 1 \in I^{\vee}$.

(ii) $N(Nx * Ny) \in I^{\vee}$ and $y \in I^{\vee}$ imply $x \in I^{\vee}$, for any $x, y \in X$.

A dual ideal $P^{\vee}$ of X is prime if $N(Nx * (Nx * Ny)) \in P^{\vee}$ implies $x \in P^{\vee}$ or $y \in P^{\vee}$.

Theorem 3.4. ([13]) Let X be a bounded BCK-algebra with the greatest element 1. Then, the following statements hold for any $x, y \in X$:

$(i)\, N1 = 0$ and $N0 = 1$.

$(ii)\, Nx * Ny \leq y * x$.

$(iii)\, y \leq x$ implies $Nx \leq Ny$.

Theorem 3.7. ([13]) Let X be a bounded BCK-algebra. Then X is commutative if and only if $x \wedge y = x * (x * y), x \vee y = N(Nx \wedge Ny)$.

4. Graphs of BCK-algebras based on ideal and dual ideal by the concepts of ideal- annihilator, dual ideal- annihilator

Definition 4.1. Let A be a nonempty subset of X, I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the set of all zero-divisors of A by I and $I^{\vee}$ are defined as follows:

(i) $Ann_I A = \{x \in X; x * a \in I \text{ or } a * x \in I, \forall a \in A\}$.

(ii) $Ann_{I^\vee} A = \{x \in X; N(Nx * Na) \in I^\vee \text{ or } N(Na * Nx) \in I^\vee, \forall a \in A\}$.

Proposition 4.2. Let A and B be nonempty subsets of X, I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) $I \cup \{1\} \subseteq Ann_I A$, $I^\vee \cup \{0\} \subseteq Ann_{I^\vee} A$.

(ii) If $A \subseteq B$, then $Ann_I B \subseteq Ann_I A$ and $Ann_{I^\vee} B \subseteq Ann_{I^\vee} A$.

(iii) If $0 \in A$, then $Ann_I A = Ann_I(A - \{0\})$ and $Ann_{I^\vee} A = Ann_{I^\vee}(A - \{0\})$.

(iv) If $1 \in A$, then $Ann_I A = Ann_I(A - \{1\})$ and $Ann_{I^\vee} A = Ann_{I^\vee}(A - \{1\})$.

(v) $Ann_I I = X$ and $Ann_{I^\vee} I^\vee = X$.

(vi) If $I = \{0\}, I^\vee = \{1\}$, then we have

$Ann_I A = \{y; y \text{ is comparable to any element in } A\}$,

$Ann_{I^\vee} A = \{y; y \text{ is comparable to any element in } A\}$.

Proof. (i) Let $x \in I$, then by Definition 3.1 (iii), we have $x * a \in I, \forall a \in A$. Also, $x * 1 = 0, \forall x \in X$, So $I \cup \{1\} \subseteq Ann_I A$. Similarly, we can prove $I^\vee \cup \{0\} \subseteq Ann_{I^\vee} A$.

(ii) Suppose that $x \in Ann_I B$, then $x * b \in I$ or $b * x \in I$, $\forall b \in B$, but $A \subseteq B$, therefore $x * b \in I$ or $b * x \in I, \forall b \in A$. i.e $x \in Ann_I A$, hence $Ann_I B \subseteq Ann_I A$.

(iii) According to Definition 4.1 (i), we have $Ann_I A = \cap_{a \in A} Ann_I a$. Also, $Ann_I\{0\} = X$. Then, $Ann_I A = Ann_I(A - \{0\})$. Similarly, we can prove $Ann_{I^\vee} A = Ann_{I^\vee}(A - \{0\})$.

(iv) According to Definition 4.1 (i), we have $Ann_I A = \cap_{a \in A} Ann_I a$. Also, $Ann_I\{1\} = X$. Then, $Ann_I A = Ann_I(A - \{1\})$. Similarly, we can prove $Ann_{I^\vee} A = Ann_{I^\vee}(A - \{1\})$.

(v) Let $x \in X$, we know by Definition 3.1 (iii), $a * x \in I, \forall a \in I$, then $x \in Ann_I I$, hence $Ann_I I = X$. Similarly we can prove $Ann_{I^\vee} I^\vee = X$.

(vi) The proof is easy.

Definition 4.3. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, we have:

(i) $\Phi_I(X)$ is a simple graph, with vertex set X and two distinct vertices x and y being adjacent if and only if $Ann_I\{x,y\} = I \cup \{1\}$.

(ii) $\Phi_{I^\vee}(X)$ is a simple graph, with vertex set X and two distinct vertices x and y being adjacent if and only if $Ann_{I^\vee}\{x,y\} = I^\vee \cup \{0\}$.

Example 4.4. Let $X = \{0, a, b, c, 1\}$ and the operation $*$ be defined by the following table:

*	0	a	b	c	1
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	b	0
c	c	c	c	0	0
1	1	1	1	1	0

TABLE 1

Therefore, $(X, *, 0)$ is a bounded BCK-algebra. Also, we have

$E\left(\Phi_{\{0\}}(X)\right) = E\left(\Phi_{\{1\}}(X)\right) = \{ab, bc, ac\}$.

Theorem 4.5. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then the following statements hold:

(i) $N_G(\{0\}) = N_G(\{1\}) = \emptyset$, where $G = \Phi_I(X)$.

(ii) $N_G(\{0\}) = N_G(\{1\}) = \emptyset$, where $G = \Phi_{I^\vee}(X)$.

Proof. (i) We know $Ann_I\{0\} = X$ and $Ann_I\{1\} = X$, for all $x \in X, x \neq 0, 1,$ we have, $I \cup \{x, 1\} \subseteq Ann_I\{x\}$. Then $I \cup \{x, 1\} \subseteq Ann_I\{0, x\}$ and $I \cup \{x, 1\} \subseteq Ann_I\{x, 1\}$, for all $x \in X, x \neq 0, 1$. So, by Definition 4.3 (i) of graph $\Phi_I(X)$, for all $x \in X, x \neq 0, 1, x$ is connected to elements $0, 1$ if and only if $x \in I$, if $x \in I$, then by proposition 4.2 (v), $Ann_I\{x\} = X$. So, $0, 1$ are not connected to x, for all $x \in X$.

(ii) We know $Ann_{I^\vee}\{0\} = X$ and $Ann_{I^\vee}\{1\} = X$, for all $x \in X, x \neq 0, 1$, we have, $I^\vee \cup \{0, x\} \subseteq Ann_{I^\vee}\{x\}$. Then $I^\vee \cup \{0, x\} \subseteq Ann_{I^\vee}\{0, x\}$ and $I^\vee \cup \{0, x\} \subseteq Ann_{I^\vee}\{x, 1\}$, for all $x \in X, x \neq 0, 1$. So, by Definition 4.3 (ii) of graph $\Phi_{I^\vee}(X)$, for all $x \in X, x \neq 0, 1, x$ is connected to elements $0, 1$ if and only if $x \in I^\vee$, then by Proposition 4.2 (v), $Ann_{I^\vee}\{x\} = X$. So, $0, 1$ are not connected to x, for all $x \in X$.

Theorem 4.6. Let $X = \{0, 1\} \cup Atom(X), I = \{0\}$ and $I^\vee = \{1\}$ be an ideal, a dual ideal of X, respectively. Then, $E(\Phi_I(X)) = E(\Phi_{I^\vee}(X)) = \{xy; x, y \in Atom(X)\}$.

Proof. We know $Ann_{\{0\}}\{0\} = X$ and $Ann_{\{0\}}\{1\} = X$, by proposition 4.2 (vi), since $X = Atom(X) \cup \{0, 1\}$, we have, for all $x \in Atom(X)$, $Ann_{\{0\}}\{x\} = \{0, x, 1\}$. On the other hand we know $Ann_{\{0\}}\{x, y\} = Ann_{\{0\}}\{x\} \cap Ann_{\{0\}}\{y\}$. Then by Definition 4.3$(i)$ of graph $\Phi_{\{0\}}(X)$, x and y are adjacent if and only if $x, y \in Atom(X)$. Similarly, we have $Ann_{\{1\}}\{0\} = X$ and $Ann_{\{1\}}\{1\} = X$, for all $x \in Atom(X)$, $Ann_{\{1\}}\{x\} = \{0, x, 1\}$. Then by Definition 4.3$(ii)$ of graph $\Phi_{\{1\}}(X)$, x and y are adjacent if and only if $x, y \in Atom(X)$.

Theorem 4.7. Let $X = \{0, 1\} \cup Atom(X)$. Then, the following statements hold:

(i) $\omega\left(\Phi_{\{0\}}(X)\right) = |Atom(X)|$.

(ii) $\omega\left(\Phi_{\{1\}}(X)\right) = |Atom(X)|$.

Proof. (i) Straightforward by Theorem 4.6(i).

(ii) Straightforward by Theorem 4.6(ii).

Theorem 4.8. Let $I = \{0\}$ and $I^\vee = \{1\}$ be an ideal, a dual ideal of X, respectively. Then the following statements hold:

$(i)\, N_G(\{x\}) = \{y; y \text{ is not comparable to } x\}$, where $G = \Phi_I(X), x \neq 0, 1$.

$(ii)\, N_G(\{x\}) = \{y; y \text{ is not comparable to } x\}$, where $G = \Phi_{I^\vee}(X), x \neq 0, 1$.

Proof. (i) We have, for all $x \in X, x \neq 0, 1$, $Ann_{\{0\}}\{x\} = \{y; y \text{ is comparable to } x\}$. On the other hand we know $Ann_{\{0\}}\{x, y\} = Ann_{\{0\}}\{x\} \cap Ann_{\{0\}}\{y\}$. Then by Definition 4.3 (i) of graph $\Phi_{\{0\}}(X)$, x and y are adjacent if and only if x and y are not comparable to each other.

(ii) We have, for all $x \in X, x \neq 0, 1$, $Ann_{\{1\}}\{x\} = \{y; y \text{ is comparable to } x\}$. On the other hand we know $Ann_{\{1\}}\{x, y\} = Ann_{\{1\}}\{x\} \cap Ann_{\{1\}}\{y\}$. Then by Definition 4.3 (ii) of graph $\Phi_{\{1\}}(X)$, x and y are adjacent if and only if x and y are not comparable to each other.

Theorem 4.9. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then the following statements hold:

$(i)\, \alpha(\Phi_I(X)) \geq |I|$.

$(ii)\, \alpha(\Phi_{I^\vee}(X)) \geq |I^\vee|$.

Proof. (i) We suppose that $x, y \in I$. Then by Proposition 4.2 (v), we have, $Ann_I\{x\} = X$, $Ann_I\{y\} = X$. Therefore, by Definition 4.3 (i) of graph $\Phi_I(X)$, $xy \notin E(\Phi_I(x))$. Therefore, by Definition 2.1 of independent set, we have $\alpha(\Phi_I(X)) \geq |I|$.

(ii) We suppose that $x, y \in I^\vee$. Then by Proposition 4.2 (v), we have, $Ann_{I^\vee}\{x\} = X$, $Ann_{I^\vee}\{y\} = X$. Therefore, by Definition 4.3 (ii) of graph $\Phi_{I^\vee}(X)$, $xy \notin E(\Phi_{I^\vee}(X))$. Therefore, by Definition 2.1 of independent set, we have $\alpha(\Phi_{I^\vee}(X)) \geq |I^\vee|$.

Theorem 4.10. Let $|X| > 2$ and I be a prime ideal, $I^\vee$ be a prime dual ideal of X. Then the following statements hold:

$(i)\ \Phi_I(X)$ is an empty graph.

$(i)\ \Phi_{I^{\vee}}(X)$ is an empty graph.

Proof. (i) We suppose, on the contrary, that $\Phi_I(X)$ is not an empty graph. Then there exist $x, y \in X$, such that $xy \in E(\Phi_I(X))$. So, by Definition 4.3 (i) of graph $\Phi_I(X)$, we have, $Ann_I\{x, y\} = I \cup \{1\}$. On the other hand, since $|X - I| > 1$, we can choose $z \in X, z \notin I, z \neq 1$. Since I is a prime ideal, then $z * x \in I$ or $* z \in I$, and $z * y \in I$ or $y * z \in I$, hence $z \in Ann_I\{x, y\}$ that is contradiction, complete proof.

(ii) We suppose, on the contrary, that $\Phi_{I^{\vee}}(X)$ is not an empty graph. Then there exist $x, y \in X$, such that $xy \in E(\Phi_{I^{\vee}}(X))$. So, by Definition 4.3 (ii) of graph $\Phi_{I^{\vee}}(X)$, we have, $Ann_{I^{\vee}}\{x, y\} = I^{\vee} \cup \{0\}$. On the other hand, since $|X - I^{\vee}| > 1$, we can choose $z \in X, z \notin I^{\vee}, z \neq 0$. Since $I^{\vee}$ is a prime dual ideal, then $N(Nz * Nx) \in I^{\vee}$ or $N(Nx * Nz) \in I^{\vee}$ and $N(Nz * Ny) \in I^{\vee}$ or $N(Ny * Nz) \in I^{\vee}$, hence $z \in Ann_{I^{\vee}}\{x, y\}$ that is contradiction, complete proof.

5. Graphs of BCK- algebras based on ideal and dual ideal by the concepts of right- ideal- annihilator, left- ideal- annihilator, right- dual ideal- annihilator, and left- dual ideal- annihilator.

Definition 5.1. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Denote $Ann_I^R\{x\} = \{y \in X; x * y \in I\}$, $Ann_I^L\{x\} = \{y \in X; y * x \in I\}$, $Ann_{I^{\vee}}^R\{x\} = \{y \in X; N(Nx * Ny) \in I^{\vee}\}$, $Ann_{I^{\vee}}^L\{x\} = \{y \in X; N(Ny * Nx) \in I^{\vee}\}$

, which are called right- ideal- annihilator, left- ideal- annihilator, right- dual ideal- annihilator, left- dual ideal- annihilator, respectively.

Definition 5.2. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, we have:

$(i)\ \Delta_I(X)$ is a simple graph, with vertex set X and two distinct vertices x and y being adjacent if and only if $Ann_I^R\{x\} \subseteq Ann_I^R\{y\}$ or $Ann_I^R\{y\} \subseteq Ann_I^R\{x\}$, there is an edge between x and y in the graph $\Sigma_I(X)$ if and only if $Ann_I^L\{x\} \subseteq Ann_I^L\{y\}$ or $Ann_I^L\{y\} \subseteq Ann_I^L\{x\}$.

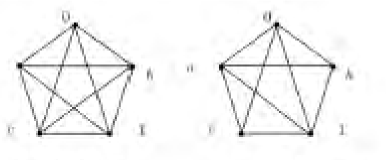
(ii) $\Delta_{I^{\vee}}(X)$ is a simple graph, with vertex set X and two distinct vertices x and y being adjacent if and only if $Ann^{R}_{I^{\vee}}\{x\} \subseteq Ann^{R}_{I^{\vee}}\{y\}$ or $Ann^{R}_{I^{\vee}}\{y\} \subseteq Ann^{R}_{I^{\vee}}\{x\}$, there is an edge between x and y in the graph $\Sigma_{I^{\vee}}(X)$ if and only if $Ann^{L}_{I^{\vee}}\{x\} \subseteq Ann^{L}_{I^{\vee}}\{y\}$ or $Ann^{L}_{I^{\vee}}\{y\} \subseteq Ann^{L}_{I^{\vee}}\{x\}$.

Example 5.3. Let $X = \{0, a, b, c, 1\}$ and the operation $*$ is given by the following table:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	a	0
c	c	c	c	0	0
1	1	c	c	a	0

TABLE 2

Therefore, $(X, *, 0)$ is a bounded BCK- algebra, $I^{\vee} = \{c, 1\}$ is a dual ideal of X. Also, in the Figure 2, we can see the graphs $\Delta_{\{0\}}(X), \Sigma_{\{0\}}(X), \Delta_{I^{\vee}}(X)$, and $\Sigma_{I^{\vee}}(X)$.



$\Delta_{I^{\vee}}(X), \Sigma_{I^{\vee}}(X)$ $\Delta_{\{0\}}(X), \Sigma_{\{0\}}(X)$

Proposition 5.4. Let I and $I^{\vee}$ be an ideal , a dual ideal of X, respectively. Then, the following statements hold:

$(i)\,\omega(\Delta_I(X)) \geq max\{|A|; A \text{ is a chain in } X\}$.

$(ii)\,\omega(\Sigma_I(X)) \geq max\{|A|; A \text{ is a chain in } X\}$.

$(iii)\,\omega(\Delta_{I^{\vee}}(X)) \geq max\{|A|; A \text{ is a chain in } X\}$.

$(iv)\,\omega(\Sigma_{I^{\vee}}(X)) \geq max\{|A|; A \text{ is a chain in } X\}$.

Proof. (i) According to Definition 3.1 (iv), if $x \leq y$ then, $x * z \leq y * z$. On the other hand now we let $x \leq y, z \in Ann^{R}_{I}\{y\}$. Then, by Definition 5.1, $y * z \in I$. So, by Definition 3.2 of ideal, $x * z \in I$. So, $z \in Ann^{R}_{I}\{x\}$. Then, $Ann^{R}_{I}\{y\} \subseteq Ann^{R}_{I}\{x\}, xy \in E(\Delta_I(X))$, complete proof.

(ii) According to Definition 3.1 (iv), if $x \leq y$ then, $z * y \leq z * x$. On the other hand now we let $x \leq y, z \in Ann_I^L\{x\}$. Then, by Definition 5.1, $z * x \in I$. So, by Definition 3.2 of ideal, $z * y \in I$. So, $z \in Ann_I^L\{y\}$. Then, $Ann_I^L\{x\} \subseteq Ann_I^L\{y\}, xy \in E\big(\Sigma_I(X)\big)$, complete proof.

(iii) According to Definition 3.1 (iv), Theorem 3.4 (iii), if $x \leq y$ then $N(Nx * Nz) \leq N(Ny * Nz)$. On the other hand now we let $x \leq y, z \in Ann_{I^\vee}^R\{x\}$ then, by Definition 5.1 $N(Nx * Nz) \in I^\vee$. So, by Definition 3.3 of dual ideal, $N(Ny * Nz) \in I^\vee$. So, $z \in Ann_{I^\vee}^R\{y\}$ then, $Ann_{I^\vee}^R\{x\} \subseteq Ann_{I^\vee}^R\{y\}, xy \in E\big(\Delta_{I^\vee}(X)\big)$, complete proof.

(iv) According to Definition 3.1 (iv), Theorem 3.4 (iii), if $x \leq y$ then $N(Nz * Ny) \leq N(Nz * Nx)$. On the other hand now we let $x \leq y, z \in Ann_{I^\vee}^L\{y\}$ then, by Definition 5.1 $N(Nz * Ny) \in I^\vee$. So, by **Definition 3.3** of dual ideal, $N(Nz * Nx) \in I^\vee$. So, $z \in Ann_{I^\vee}^L\{x\}$ then, $Ann_{I^\vee}^L\{y\} \subseteq Ann_{I^\vee}^L\{x\}, xy \in E\big(\Sigma_{I^\vee}(X)\big)$, complete proof.

Theorem 5.5. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

$(i)\, \Delta_I(X)$ is connected, $diam\big(\Delta_I(X)\big) \leq 2, gr\big(\Delta_I(X)\big) = 3.$

$(ii)\, \Sigma_I(X)$ is connected, $diam\big(\Sigma_I(X)\big) \leq 2,\ gr\big(\Sigma_I(X)\big) = 3.$

$(iii)\, \Delta_{I^\vee}(X)$ is connected, $diam\big(\Delta_{I^\vee}(X)\big) \leq 2, gr\big(\Delta_{I^\vee}(X)\big) = 3.$

$(iv)\, \Sigma_{I^\vee}(X)$ is connected, $diam\big(\Sigma_{I^\vee}(X)\big) \leq 2, gr\big(\Sigma_{I^\vee}(X)\big) = 3.$

Proof. (i) For all $x \in X, 0 \leq x \leq 1$, then by Proposition 5.4 (i), $0, 1$ are connected to any element in X. So, $\Delta_I(X)$ is connected, $diam\big(\Delta_I(X)\big) \leq 2, gr\big(\Delta_I(X)\big) = 3.$

(ii) For all $x \in X, 0 \leq x \leq 1$, then by Proposition 5.4 (ii), $0, 1$ are connected to any element in X. So, $\Sigma_I(X)$ is connected, $diam(\Sigma_I(X)) \leq 2, gr(\Sigma_I(X)) = 3$.

(iii) For all $x \in X, 0 \leq x \leq 1$, then by Proposition 5.4 (iii), $0, 1$ are connected to any element in X. So, $\Delta_{I^\vee}(X)$ is connected, $diam(\Delta_{I^\vee}(X)) \leq 2, gr(\Delta_{I^\vee}(X)) = 3$.

(iv) For all $x \in X, 0 \leq x \leq 1$, then by Proposition 5.4 (iv), $0, 1$ are connected to any element in X. So, $\Sigma_{I^\vee}(X)$ is connected, $diam(\Sigma_{I^\vee}(X)) \leq 2, gr(\Sigma_{I^\vee}(X)) = 3$.

Theorem 5.6. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) $\Delta_I(X)$ is regular if and only if it is complete.

(ii) $\Sigma_I(X)$ is regular if and only if it is complete.

(iii) $\Delta_{I^\vee}(X)$ is regular if and only if it is complete.

(iv) $\Sigma_{I^\vee}(X)$ is regular if and only if it is complete.

Proof. (i) Suppose that $\Delta_I(X)$ is regular. By Theorem 5.5(i), $\deg(0) = |X| - 1$. Since $\Delta_I(X)$ is regular, for all $x \in X, \deg(x) = |X| - 1$. Hence, $\Delta_I(X)$ is complete. Conversely, a complete graph is regular.

(ii) Suppose that $\Sigma_I(X)$ is regular. By Theorem 5.5(ii), $\deg(0) = |X| - 1$. Since $\Sigma_I(X)$ is regular, for all $x \in X, \deg(x) = |X| - 1$. Hence, $\Sigma_I(X)$ is complete. Conversely, a complete graph is regular.

(iii) Suppose that $\Delta_{I^\vee}(X)$ is regular. By Theorem 5.5(iii), $\deg(0) = |X| - 1$. Since $\Delta_{I^\vee}(X)$ is regular, for all $x \in X, \deg(x) = |X| - 1$. Hence, $\Delta_{I^\vee}(X)$ is complete. Conversely, a complete graph is regular.

(iv) Suppose that $\Sigma_{I^\vee}(X)$ is regular. By Theorem 5.5(iv), $\deg(0) = |X| - 1$. Since $\Sigma_{I^\vee}(X)$ is regular, for all $x \in X, \deg(x) = |X| - 1$. Hence, $\Sigma_{I^\vee}(X)$ is complete. Conversely, a complete graph is regular.

Proposition 5.7. Let X be a chain, I and $I^\vee$ be an ideal, a dual of X, respectively. Then, the following statements hold:

$(i)\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are planar graphs if and only if $|X| \leq 4$.

$(ii)\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are outerplanar graphs if and only if $|X| \leq 3$.

$(iii)\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are toroidal graphs if and only if $|X| \leq 7$.

Proof. (i) According to Proposition 5.4, $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are complete graphs, respectively, if $|X| \geq 5$, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have a subgraph isomorphic to K_5, respectively, then by Kuratowski's theorem $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not planar, respectively. Conversely, we knowK_5 has five vertices, hence if $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not planar, respectively, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have at least five vertices, respectively, which is contrary to $|X| \leq 4$.

(ii) According to Proposition 5.4, $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are complete graphs, respectively, if $|X| \geq 4$, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have a subgraph isomorphic to K_4, respectively, then by Definition 2.5 $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not outerplanar, respectively. Conversely, we knowK_4 has four vertices, hence if $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not outerplanar, respectively, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have at least four vertices, respectively, which is contrary to $|X| \leq 3$.

(iii) According to Proposition 5.4, $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are complete graphs, respectively, if $|X| \geq 8$, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have a subgraph isomorphic to K_8, respectively, then by Theorem 2.7 $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not toroidal, respectively. Conversely, we know K_8 has eight vertices, hence if $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ are not toroidal, respectively, then $\Delta_I(X), \Sigma_I(X), \Delta_{I^\vee}(X)$, and $\Sigma_{I^\vee}(X)$ have at least eight vertices, respectively, which is contrary to $|X| \leq 7$.

6. Graphs of BCK- algebras based on ideal and dual ideal by the binary operations ∧ and ∨.

From now on, X is a bounded commutative BCK-algebra.

Definition 6.1. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, we have:

$(i)\, \Upsilon_I(X)$ is a simple graph, with vertex set X and two distinct vertices x and y are adjacent if and only if $x \wedge y \in I$.

$(ii)\, \Upsilon_{I^\vee}(X)$ is a simple graph, with vertex set X and two distinct vertices x and y are adjacent if and only if $x \vee y \in I^\vee$.

Example 6.2. Let $X = \{0, a, b, c, d, 1\}$ and the operation $*$ be defined by the table:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	a	a	0	0
b	b	b	0	0	0	0
c	c	c	b	0	b	0
d	d	b	a	a	0	0
1	1	c	d	a	b	0

TABLE 3

Therefore, $(X, *, 0)$ is a bounded commutative BCK-algebra. It is easy to verify that $I = \{0, a\}$ is an ideal of X. Also, we let $I^\vee = \{1\}$ be a dual ideal of X, then in the Figure 3 , we can see the graphs $\Upsilon_I(X)$ and $\Upsilon_{I^\vee}(X)$.

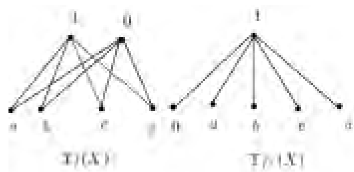


FIGURE 3

Lemma 6.3. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) $\deg(x) = |X| - 1$, in the graph $\Upsilon_I(X)$, where $x \in I$.

(ii) $\deg(x) = |X| - 1$, in the graph $\Upsilon_{I^{\vee}}(X)$, where $x \in I^{\vee}$.

Proof. (i) Let $x \in I, y$ be an arbitrary element in X, then $y * (y * x) \in I$. Since $y * (y * x) \leq x, I$ is an ideal of X. So, $xy \in E(\Upsilon_I(X))$, complete proof.

(ii) Let $x \in I^{\vee}, y$ be an arbitrary element in X, then $N(Ny * (Ny * Nx)) \in I^{\vee}$. Since $N(Ny * (Ny * Nx)) \geq x, I^{\vee}$ is a dual ideal of X. So, $xy \in E(\Upsilon_{I^{\vee}}(X))$, complete proof.

Theorem 6.4. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) $\Upsilon_I(X)$ is regular if and only if it is complete.

(ii) $\Upsilon_{I^{\vee}}(X)$ is regular if and only if it is complete.

Proof. (i) Let $\Upsilon_I(X)$ be a regular graph. By Lemma 6.3 (i), we have $\deg(0) = |X| - 1$. Now, since $\Upsilon_I(X)$ is regular, then for any $x \in X, \deg(x) = |X| - 1$. This means that $\Upsilon_I(X)$ is a complete graph. Conversely, a complete graph is regular.

(ii) Let $\Upsilon_{I^{\vee}}(X)$ be a regular graph. By Lemma 6.3 (ii), we have $\deg(1) = |X| - 1$. Now, since $\Upsilon_{I^{\vee}}(X)$ is regular, then for any $x \in X, \deg(x) = |X| - 1$. This means that $\Upsilon_{I^{\vee}}(X)$ is a complete graph. Conversely, a complete graph is regular.

Proposition 6.5. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) $\omega(\Upsilon_I(X)) \geq |I|$.

$(ii)\ \omega(\Upsilon_{I^{\vee}}(X)) \geq |I^{\vee}|$.

Proof. (i) Straightforward by Lemma 6.3 (i).

(ii) Straightforward by Lemma 6.3 (ii).

Theorem 6.6. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) $\Upsilon_I(X)$ is connected, $diam(\Upsilon_I(X)) \leq 2$.

(ii) $\Upsilon_{I^{\vee}}(X)$ is connected, $diam(\Upsilon_{I^{\vee}}(X)) \leq 2$.

Proof. (i) Straightforward by Lemma 6.3 (i).

(ii) Straightforward by Lemma 6.3 (ii).

Theorem 6.7. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

$(i)\ gr(\Upsilon_I(X)) = 3$.

$(ii)\ gr(\Upsilon_{I^{\vee}}(X)) = 3$.

Proof. (i) Let $a \neq 0$ be an element in I, x be an arbitrary element in X, then $0 - a - x - 0$ is a cycle of length 3 in $\Upsilon_I(X)$, complete proof.

(ii) Let $a \neq 1$ be an element in $I^{\vee}, x$ be an arbitrary element in X, then $1 - a - x - 1$ is a cycle of length 3 in $\Upsilon_{I^{\vee}}(X)$, complete proof.

Proposition 6.8. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) If $\Upsilon_I(X)$ is planar, then $|I| \leq 4$.

(ii) If $\Upsilon_I(X)$ is outerplanar, then $|I| \leq 3$.

(iii) If $\Upsilon_I(X)$ is toroidal, then $|I| \leq 7$.

(iv) If $\Upsilon_I(X)$ is planar, then $|I^{\vee}| \leq 4$.

(v) If $Y_I(X)$ is outerplanar, then $|I^{\vee}| \leq 3$.

(vi) If $Y_I(X)$ is toroidal, then $|I^{\vee}| \leq 7$.

Proof. (i) According to Lemma 6.3 (i), $Y_I(X)$ is a complete graph on I, if $|I| \geq 5$ then $Y_I(X)$ has a subgraph isomorphic to K_5 which by Kuratowski's theorem, $Y_I(X)$ is not planar.

(ii) According to Lemma 6.3 (i), $Y_I(X)$ is a complete graph on I, if $|I| \geq 4$ then $Y_I(X)$ has a subgraph isomorphic to K_4 which by Definition 2.5, $Y_I(X)$ is not outerplanar.

(iii) According to Lemma 6.3 (i), $Y_I(X)$ is a complete graph on I, if $|I| \geq 7$ then $Y_I(X)$ has a subgraph isomorphic to K_8 which by Theorem 2.7, $Y_I(X)$ is not toroidal.

(iv) According to Lemma 6.3 (ii), $Y_{I^{\vee}}(X)$ is a complete graph on $I^{\vee}$, if $|I^{\vee}| \geq 5$ then $Y_{I^{\vee}}(X)$ has a subgraph isomorphic to K_5 which by Kuratowski's theorem, $Y_{I^{\vee}}(X)$ is not planar.

(v) According to Lemma 6.3 (ii), $Y_{I^{\vee}}(X)$ is a complete graph on $I^{\vee}$, if $|I^{\vee}| \geq 4$ then $Y_{I^{\vee}}(X)$ has a subgraph isomorphic to K_4 which by Definition 2.5, $Y_{I^{\vee}}(X)$ is not outerplanar.

(vi) According to Lemma 6.3 (ii), $Y_{I^{\vee}}(X)$ is a complete graph on $I^{\vee}$, if $|I^{\vee}| \geq 7$ then $Y_{I^{\vee}}(X)$ has a subgraph isomorphic to K_8 which by Theorem 2.7, $Y_{I^{\vee}}(X)$ is not toroidal.

Theorem 6.9. Let I and $I^{\vee}$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) If $Y_I(X)$ is an Euler graph then $|X|$ is odd.

(ii) If $Y_{I^{\vee}}(X)$ is an Euler graph then $|X|$ is odd.

Proof. (i) According to Lemma 6.3 (i), for all $x \in I$, $\deg(x) = |X| - 1$. Now, if $Y_I(X)$ is an Euler graph then degree of every vertex in I is even. So, $|X|$ is odd, complete proof.

(ii) According to Lemma 6.3 (ii), for all $x \in I$, $\deg(x) = |X| - 1$. Now, if $\Upsilon_{I^\vee}(X)$ is an Euler graph then degree of every vertex in $I^\vee$ is even. So, $|X|$ is odd, complete proof.

Theorem 6.10. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) If $I = \cap_{1 \le i \le n} P_i$ and, for each $1 \le j \le n, I \neq \cap_{1 \le i \le n, i \neq j} P_i$, where P_i are prime ideals of X. Then $\omega(\Upsilon_I(X)) = n = \chi(\Upsilon_I(X))$.

(ii) If $I^\vee = \cap_{1 \le i \le n} P_i^\vee$ and, for each $1 \le j \le n, I^\vee \neq \cap_{1 \le i \le n, i \neq j} P_i^\vee$, where $P_i^\vee$ are prime dual ideals of X. Then $\omega(\Upsilon_{I^\vee}(X)) = n = \chi(\Upsilon_{I^\vee}(X))$.

Proof. (i) For each j with $1 \le j \le n$, consider an element x_j in $(\cap_{1 \le i \le n, i \neq j} P_i) - P_j$. We have $A = \{x_1, \dots, x_n\}$ is a clique in $\Upsilon_I(X)$. Hence $\omega(\Upsilon_I(X)) \ge n$. Now, we prove that $\chi(\Upsilon_I(X)) \le n$. Define a coloring f by putting $f(x) = \min\{i; x \notin P_i\}$. Let $f(x) = k$, x and y be adjacent vertices. So, $x \notin P_k$ and $x \wedge y \in I$. Since P_k is prime, $y \in P_k$, and so $f(y) \neq k$. Now, since $\omega(\Upsilon_I(X)) \le \chi(\Upsilon_I(X))$, the result hold.

(ii) For each j with $1 \le j \le n$, consider an element x_j in $(\cap_{1 \le i \le n, i \neq j} P_i^\vee) - P_j^\vee$. We have $A = \{x_1, \dots, x_n\}$ is a clique in $\Upsilon_{I^\vee}(X)$. Hence $\omega(\Upsilon_{I^\vee}(X)) \ge n$. Now, we prove that $\chi(\Upsilon_{I^\vee}(X)) \le n$. Define a coloring f by putting $f(x) = \min\{i; x \notin P_i^\vee\}$. Let $f(x) = k$, x and y be adjacent vertices. So, $x \notin P_k^\vee$ and $x \vee y \in I^\vee$. Since $P_k^\vee$ is prime, $y \in P_k^\vee$, and so $f(y) \neq k$. Now, since $\omega(\Upsilon_{I^\vee}(X)) \le \chi(\Upsilon_{I^\vee}(X))$, the result hold.

Theorem 6.11. Let I and $I^\vee$ be an ideal, a dual ideal of X, respectively. Then, the following statements hold:

(i) If $I = \cap_{j \in J} P_j$, where P_j are prime ideals of X, J is an infinite set and, for each $i \in J$, $I \neq \cap_{j \neq i} P_j$. Then $\omega(\Upsilon_I(X)) = \infty = \chi(\Upsilon_I(X))$.

(ii) If $I = \cap_{j \in J} P_j^{\vee}$ where $P_j^{\vee}$ are prime dual ideals of X, J is an infinite set and, for each $i \in J$, $I^{\vee} \neq \cap_{j \neq i} P_j^{\vee}$. Then $\omega(\Upsilon_{I^{\vee}}(X)) = \infty = \chi(\Upsilon_{I^{\vee}}(X))$.

Proof. (i) For each $i \in J$, there exists $x_i \in (\cap_{j \neq i} P_j - P_i)$. Now, one can easily see that the set of x_i forms an infinite clique in $\Upsilon_I(X)$. Since $\omega(\Upsilon_I(X)) \leq \chi(\Upsilon_I(X))$, the assertion holds.

(ii) For each $i \in J$, there exists $x_i \in (\cap_{j \neq i} P_j^{\vee} - P_i^{\vee})$. Now, one can easily see that the set of x_i forms an infinite clique in $\Upsilon_{I^{\vee}}(X)$. Since $\omega(\Upsilon_{I^{\vee}}(X)) \leq \chi(\Upsilon_{I^{\vee}}(X))$, the assertion holds.

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Neutrosophic Soft Structures

Arif Mehmood[1,*], Wadood Ullah[2], Said Broumi[3], Muhammad Imran Khan[4], Humera Qureshi[5], Muhammad Ibrar Abbas[6], Humaira Kalsoom[7] and Fawad Nadeem[8]

[1] Department of Mathematics and Statistics, Riphah International University, Sector I-14, Islamabad, Pakistan
mehdaniyal@gmail.com

[2, 6, 8] Department of Mathematics, University of Science and Technology, Bannu, Khyber Pakhtunkhwa, Pakistan.
[2] wadoodkhan2131@gmail.com, [6]mibrarabbas86@gmail.con, [8]fawadnadeem2@gmail.com,

[3] Laboratory of Information Processing, Faculty of Science, University Hassan II, Casablanca, Morocco
broumisaid78@gmail.com

[4, 5] College of Statistical and Actuarial Science, Punjab University, Pakistan
[4] imranktk1745@yahoo.com, [5]humiimran1745@yahoo.com

[7] School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China. humaira87@zju.edu.cn

* Correspondence: mehdaniyal@gmail.com

Abstract: In paper, neutrosophic soft points with the concept of one point greater than the other and their properties, generalized neutrosophic soft open set known as soft b-open set, neutrosophic soft separation axioms theoretically with support of suitable examples with respect to soft points, neutrosophic soft b_0-space engagement with generalized neutrosophic soft closed set, neutrosophic soft b_2-space engagement with generalized neutrosophic soft open set are addressed. In continuation, neutrosophic soft b_0-space behave as neutrosophic soft b_2-space with the plantation of some extra condition on soft b_0-space, neutrosophic soft b_3-space and related theorems, neutrosophic soft b_4-space, monotonous behavior of neutrosophic soft function with connection of different neutrosophic soft separation axioms, monotonous behavior of neutrosophic soft function with connection of different neutrosophic soft close sets are reflected. Secondly, long touched has been given to neutrosophic soft countability connection with bases and sub-bases, neutrosophic soft product spaces and its engagement through different generalized neutrosophic soft open set and close sets, neutrosophic soft coordinate spaces and its engagement through different generalized neutrosophic soft open set and close sets, Finally, neutrosophic soft countability and its relationship with Bolzano Weirstrass Property through engagement of compactness, neutrosophic soft strongly spaces and its related theorems, neutrosophic soft sequences and its relation with neutrosophic soft compactness, neutrosophic soft Lindelof space and related theorems are supposed to address.

Keywords Neutrosophic soft set (NSS),neutrosophic soft point, neutrosophic soft b-open set and neutrosophic b-separation axioms.

1. Introduction

Cagman et al. [1] defined the concept of soft topology on a soft set, and presented its related properties. The authors also discussed the foundations of the theory of soft topological spaces. Shabir and Naz [2] addressed soft topological spaces with a fixed set of parameters over an initial universe. The notions of soft open sets, soft closed sets, soft closure, soft interior points, soft point neighborhood, soft separation axioms and their basic characteristics are addressed. The authors reflected that a soft topological space gives birth to a family of crisp topological spaces that are parameterized. The authors scanned a soft topological space's subspaces, and explored open and soft closed sets of characterization w. r. t. soft open set. Finally, the authors tackled in depth the notion T_i spaces, soft normal space and soft regular spaces.

Bayramov and Gunduz [3] investigated some basic notions of STS by using soft point concept. Later on the authors addressed T_i-soft spaces and the ties between them. Finally, the authors defined soft compactness and leaked out some of its important characteristics.

Khattak et al [4] introduced the concept of softα-open soft β-open, soft α-separations axioms and soft β-separation axioms in soft single point topology. The authors have addressed soft (α, β) separation axioms with regard to ordinary points and soft points in soft topological spaces.

Zadeh [5] exposed the concept of fuzzy set. The author described that a fuzzy set is a class of objects with a continuum of grades of membership. The authors furthers defined the set through a membership feature, assigning membership grade to each group candidate. The notions of inclusion, union, intersection, complement, relation, convexity, etc. have been applied to such sets, different properties of these notions have been developed in the sense of fuzzy sets. In particular, it has been proven that a soft separation axiom theorem for convex fuzzy set ignored the prerequisites of mutually exclusive fuzzy sets.

Atanassov [6] developed the 'intuitionistic fuzzy set' (IFS) concept, which is an extension of the 'fuzzy set' definition. The authors explored different properties including operations and set-over relationships. Bayramov and Gunduz [7] introduced some important features of intuitive fuzzy soft topological spaces and established the intuitive soft closure and interior of an intuitive soft set.
In addition, their research also addressed intuitionistic fuzzy continuous mapping and structural characteristics. Deli and Broumi [8] defined for the first a relation on neutrosophic soft sets.
The new concept allows two neutrosophical soft sets to be composed. It is conceived to extract useful information by combing neutrosophical soft sets. Eventually a decision making approach is based on neutrosophic soft sets.

In a new approach, Bera and Mahapatra [9] introduced the concept of $cartesian$ product and the neutrosophic soft sets in a new approach. Some properties of this principle were discussed and checked with relevant examples from real life. Smarandache [10] for the first time initiated the concept of neutrosophic set which is generalization of the intuitionistic fuzzy set (IFS), and intuitionistic set (NS). Some related examples are presented. Peculiarities between NS and IFS are underlined.

Maji [11] broadened the Smarandache analysis. The author used the idea of soft set neutrosophic set and incorporated neutrosophic soft set. On neutrosophic soft set those meaning and related operations were addressed.

Bera and Mahapatra [12] developed topology formulation on a neutrosophic soft set (NSS). This study studies the notion of neutrosophic soft interior, neutrosophic soft closure, neutrosophic soft neighborhood, neutrosophic soft boundary, normal NSS and their basic properties.

Topology and topology for subspaces on the NSS are described with appropriate examples. It also developed some related properties. In addition to this, the concept of separation axioms on neutrosophic soft topological space was introduced along with investigation of several structural features.

Khattak et al. [13] for the first time leaked out the idea of neutrosophic soft b-open set, neutrosophic soft b-closed sets and their properties. Also the idea of neutrosophic soft b-neighborhood and neutrosophic soft b-separation axioms in neutrosophic soft topological structures are reflected. Later on the important results are discussed related to these newly defined concepts $with\ respect\ to\ soft\ points$. The concept of neutrosophic soft b-separation axioms of neutrosophic soft topological spaces is diffused in different results with respect to soft points. Furthermore, properties of neutrosophic soft bT_i -space (i = 0, 1, 2, 3, 4) and some associations between them are discussed.

C.G. Aras et al. [14] leaked out $some\ basic$ notions of $neutrosophic\ soft\ sets$ and redefined some n$eutr$osophic soft point concept. Later on the authors addressed some neutrosophic soft T_i-space and the relationships among them.

T. Y. Ozturk et al. [15] re-defined some operations on neutrosophic soft sets differently as defined by others authors. The authors supported and defended their approach through interesting examples. The authors further beautifully addressed different results with this new approach.

M Al-Tahan, B Davvaz [16] discussed a relationship between SVNS and neutrosophic ℵ-structures and study it. Moreover, the authors apply results to algebraic structures (hyper structures) and prove that the results on neutrosophic ℵ-substructure (sub hyper structure) of a given algebraic structure (hyper structure) can be deduced from single valued neutrosophic algebraic structure (hyper structure) and vice versa.

Adeleke et al. [17] studied refined neutrosophic rings, Substructures of refined neutrosophic rings and their elementary properties and it is shown that every refined neutrosophic ring is a ring.

Adeleke et al. [18] studied refined neutrosophic ideals and refined neutrosophic homomorphism along their elementary properties. Madeleine et al. [19] provided a connection between neutrosophic ℵ–structures and subtraction algebras. In this regard, the authors introduced the concept of neutrosophic ℵ–ideals in subtraction algebra. Moreover, the authors studied its properties and find out a necessary and sufficient condition for a neutrosophic ℵ–structure to be a neutrosophic ℵ–ideal. M. Parimala et al. [20] introduced the notion of neutrosophic $\alpha\omega$-closed sets and study some of the properties of neutrosophic $\alpha\omega$-closed sets. Further, the authors investigated neutrosophic $\alpha\omega$-continuity, neutrosophic $\alpha\omega$-irresoluteness, neutrosophic $\alpha\omega$ connectedness and neutrosophic contra $\alpha\omega$ continuity along with examples. Abdel-Basset et al. [21] proposed a powerful framework based on neutrosophic sets to aid with patients with cancer. Abdel-Basset et al. [22] developed a novel intelligent medical decision support model based on soft computing and IOT as the use of neutrosophical sets decision-making. Abdel-Basset et al. [23] concentrated on the evaluation of supply chain sustainability based on the two critical dimensions. The authors further added that the first is the importance of evolution metrics based on economic, environmental and social aspect, and the second is the degree of difficulty of information gathering. The authors guaranteed that the aim of this paper increase the accuracy of the evacuation. Abdel-Basset et al. [24] suggested that this article proposed a hybrid combination between analytical hierarchical process (AHP) as an MCDM method and neutrosophic theory to successfully detect and handle the uncertainty and inconsistency challenges.

A. Mehmood et al. [26] introduced generalized neutrosophic separation axioms in neutrosophic soft topological spaces. A. Mehmood et al. [27] discussed soft α-connectedness, soft α-dis-connectedness and soft α-compact spaces in bi-polar soft topological spaces with respect to ordinary points. For better understanding the authors provided suitable examples.

2. Preliminaries

In this section we now state certain useful definitions, theorems, and several existing results for neutrosophic soft sets that we require in the next sections.

Definition2.1 [13] $NSS\ \ on\ a\ father\ set\ \langle\mathcal{X}\rangle$is $characterized\ as$:

$$\mathcal{A}^{\text{neutrosophic}} = \begin{cases} \left(x, \mathbb{T}_{\mathcal{A}^{\text{ntophic}}}(x), \mathbb{I}_{\mathcal{A}^{\text{ntophic}}}(x), \mathbb{F}_{\mathcal{A}^{\text{ntophic}}}(x) : x \in \langle\mathcal{X}\rangle\right) \\ \mathbb{T}: \langle\mathcal{X}\rangle \longrightarrow]0^-, 1^+[\\ , \mathbb{I}: \langle\mathcal{X}\rangle \longrightarrow]0^-, 1^+[, \\ \mathbb{F}: \langle\mathcal{X}\rangle \longrightarrow]0^-, 1^+[\\ \text{, so that's it} \\ 0^- \preccurlyeq \{\ \mathbb{T} + \mathbb{I} + \mathbb{F}\ \} \preccurlyeq 3^+. \end{cases}$$

Definition 2.2[10] let $\langle\mathcal{X}\rangle$ be a father set, $ꟼ^{parameter}$be a set of all conditions, and $\mathcal{L}(\langle\mathcal{X}\rangle)$ denote the efficiency set of$\langle\mathcal{X}\rangle$. A pair $(\mathfrak{f}, ꟼ^{parameter})$is referred to as a soft set over$\langle\mathcal{X}\rangle$, where$\mathfrak{f}$is a map given by$\mathfrak{f}: ꟼ^{parameter} \rightarrow \mathcal{L}(\langle\mathcal{X}\rangle)$. For n$\in ꟼ^{parameter}$, $\mathfrak{f}(n)$ may be viewed as the set of soft$set\ elements$ $(\mathfrak{f}, ꟼ^{parameter})$, or as a set of n –estimated the soft set components, i.e. $(\mathfrak{f}, ꟼ^{parameter}) = \{(n, \mathfrak{f}(n)\ : n \in ꟼ^{parameter}\ , \mathfrak{f}: ꟼ^{parameter} \rightarrow \mathcal{L}(\langle\mathcal{X}\rangle)\}$.

Definition 2.3 [7] Let $\langle\mathcal{X}\rangle$ be a father set, $ꟼ^{parameter}$ be a set of all conditions, and $\mathcal{L}(\langle\mathcal{X}\rangle)$ denote the efficiency

Set of$\langle\mathcal{X}\rangle$. A pair $(\mathfrak{f}, ꟼ^{parameter})$is referred to as a soft set over$\langle\mathcal{X}\rangle$, where $\mathfrak{f}$is a map given by: $ꟼ^{parameter} \rightarrow \mathcal{L}(\langle\mathcal{X}\rangle)$. $Then\ a$ (NS)set$(\tilde{\mathfrak{f}}, ꟼ^{parameter})$over $\langle\mathcal{X}\rangle$is a set defined by a set of valued functions signifying a mapping $ꟼ^{parameter} \rightarrow$

$\mathcal{L}(\langle\mathcal{X}\rangle)$,is referred to as the approximate (NS) function$(\tilde{\mathfrak{f}}, ꟼ^{parameter})$.In other words, the (NS) is a group of con ditions of certain elements of the set $\mathcal{L}(\langle\mathcal{X}\rangle)$so it can be written as a set of ordered pairs:

$(\tilde{\mathfrak{f}}, ꟼ^{parameter}) = \{\left(\left(n, [\![x, \mathbb{T}_{\tilde{\mathfrak{f}}(x)^{(x)}}, \mathbb{I}_{\tilde{\mathfrak{f}}(x)^{(x)}}, \mathbb{F}_{\tilde{\mathfrak{f}}(x)^{(x)}} : x \in \langle\mathcal{X}\rangle]\!]\right) : n \in ꟼ^{parameter}\right\}$,Obviously, $\mathbb{T}_{\tilde{\mathfrak{f}}(x)^{(x)}}, \mathbb{I}_{\tilde{\mathfrak{f}}(x)^{(x)}}, \mathbb{F}_{\tilde{\mathfrak{f}}(x)^{(x)}} \in$ $[0,1]$are $membership\ of\ truth$, $membership\ of\ indeterminacy$ and membership of falsehood$\tilde{\mathfrak{f}}(n)$.Since the supremum of each$\mathbb{T}, \mathbb{I}, \mathbb{F}$ is 1, the inequality that $0^- \preccurlyeq \mathbb{T}_{\tilde{\mathfrak{f}}(x)^{(x)}} + \mathbb{I}_{\tilde{\mathfrak{f}}(x)^{(x)}} + \mathbb{F}_{\tilde{\mathfrak{f}}(x)^{(x)}} \preccurlyeq 3^+$ is obvious.

Definition 2.4 [5] let$\left(\tilde{\mathfrak{f}}, \text{Ⅎ}^{parameter}\right)$ be a(NSS) over the father set$\langle\mathcal{X}\rangle$. The complement of $\left(\tilde{\mathfrak{f}}, \text{Ⅎ}^{parameter}\right)$ is signified$\left(\tilde{\mathfrak{f}}, \text{Ⅎ}^{parameter}\right)^{c}$ and is defined as follows:

$$\left(\tilde{\mathfrak{f}}, \text{Ⅎ}^{parameter}\right)^{c} = \{\left(\left(n, \left[\!\left[x, \mathbb{T}_{\tilde{\mathfrak{f}}(x)^{(x)}}, 1 - \mathbb{I}_{\tilde{\mathfrak{f}}(x)^{(x)}}, \mathbb{F}_{\tilde{\mathfrak{f}}(x)^{(x)}} : x \in \langle\mathcal{X}\rangle\right]\!\right]\right) : n \in \text{Ⅎ}^{parameter}\right\}$$

It's clear that

$$((\tilde{\mathfrak{f}}, \text{Ⅎ}^{parameter})^{c})^{c} = (\tilde{\mathfrak{f}}, \text{Ⅎ}^{parameter}).$$

Definition 2.5 [9] Let $(\tilde{\mathfrak{f}}, n)$ and $(\tilde{\rho}, n)$ two (NSS) over a father$\langle\mathcal{X}\rangle$. $(\tilde{\mathfrak{f}}, n)$is supposed to be $NSSS$ of $(\tilde{\rho}, n)$ if $\mathbb{T}_{\tilde{\mathfrak{f}}(x)^{(x)}} \preccurlyeq \mathbb{T}_{\tilde{\rho}(x)^{(x)}}$,$\mathbb{I}_{\tilde{\mathfrak{f}}(x)^{(x)}} \preccurlyeq \mathbb{I}_{\tilde{\rho}(x)^{(x)}}$,$\mathbb{F}_{\tilde{\mathfrak{f}}(x)^{(x)}} \succcurlyeq \mathbb{F}_{\tilde{\rho}(x)^{(x)}}$,$\forall\, n \in \text{Ⅎ}^{parameter}$&$\forall\; x \in \langle\mathcal{X}\rangle$. It is signifies as $(\tilde{\mathfrak{f}}, n) \sqsubseteq (\tilde{\rho}, n)$. $(\tilde{\mathfrak{f}}, n)$is is said to be (NS) equal to $(\tilde{\rho}, n)$ if $(\tilde{\mathfrak{f}}, n)$ is $(NSSS)$ of $(\tilde{\rho}, n)$ and $(\tilde{\rho}, n)$ is $NSSS$ of if $(\tilde{\mathfrak{f}}, n)$. *It is symbolizedas*$(\tilde{\mathfrak{f}}, n) = (\tilde{\rho}, n)$.

3. Neutrosophic Soft Points and Their Characteristics

Definition 3.1 Let$(\tilde{\mathfrak{f}}_{1}, n)$&$(\tilde{\mathfrak{f}}_{2}, n)$ be two $(NSSS)$ over a father$set\langle\mathcal{X}\rangle s.t.$ $(\tilde{\mathfrak{f}}_{1}, n) \neq (\tilde{\mathfrak{f}}_{2}, n)$. Then their union is signifies as $(\tilde{\mathfrak{f}}_{1}, n)\tilde{\sqcup}(\tilde{\mathfrak{f}}_{2}, n) = (\tilde{\mathfrak{f}}_{3}, n)$&*is defined as*$\left(\tilde{\mathfrak{f}}_{3}, n\right) = \left\{\begin{matrix}\left(n, x, \mathbb{T}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}}, \mathbb{I}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}}, \mathbb{F}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}} : x\right) \\ : n \in \text{Ⅎ}^{parameter}\end{matrix}\right.$

$$Where,\begin{cases}\mathbb{T}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}} = max\left[\mathbb{T}_{\tilde{\mathfrak{f}}_{1}(x)^{(x)}}, \mathbb{T}_{\tilde{\mathfrak{f}}_{2}(x)^{(x)}},\right. \\ \mathbb{I}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}} = max\left[\mathbb{I}_{\tilde{\mathfrak{f}}_{1}(x)^{(x)}}, \mathbb{I}_{\tilde{\mathfrak{f}}_{2}(x)^{(x)}},\right. \\ \mathbb{F}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}} = min\left[\mathbb{F}_{\tilde{\mathfrak{f}}_{1}(x)^{(x)}}, \mathbb{F}_{\tilde{\mathfrak{f}}_{2}(x)^{(x)}}\right.\end{cases}$$

Definition 3.2 Let$(\tilde{\mathfrak{f}}_{1}, n)$&$(\tilde{\mathfrak{f}}_{2}, n)$ be two $(NSSS)$ over the father $set\langle\mathcal{X}\rangle s.t.$ $(\tilde{\mathfrak{f}}_{1}, n) \neq (\tilde{\mathfrak{f}}_{2}, n)$. Then their *intersection is signifies as* $(\tilde{\mathfrak{f}}_{1}, n) \tilde{\sqcap} (\tilde{\mathfrak{f}}_{2}, n) = (\tilde{\mathfrak{f}}_{3}, n)$&is defined as follows$\left(\tilde{\mathfrak{f}}_{3}, n\right) = \{\left(\left(n, \left[\!\left[x, \mathbb{T}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}}, \mathbb{I}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}}, \mathbb{F}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}} : \langle\mathcal{X}\rangle\right]\!\right]\right) : n\text{Ⅎ}^{parameter}\right\}$*where* $\mathbb{T}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}} = min\left[\mathbb{T}_{\tilde{\mathfrak{f}}_{1}(x)^{(x)}}, \mathbb{T}_{\tilde{\mathfrak{f}}_{2}(x)^{(x)}}, \mathbb{I}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}} = min\left[\mathbb{I}_{\tilde{\mathfrak{f}}_{1}(x)^{(x)}}, \mathbb{I}_{\tilde{\mathfrak{f}}_{2}(x)^{(x)}}, \mathbb{F}_{\tilde{\mathfrak{f}}_{3}(x)^{(x)}} = max\left[\mathbb{F}_{\tilde{\mathfrak{f}}_{1}(x)^{(x)}}, \mathbb{F}_{\tilde{\mathfrak{f}}_{2}(x)^{(x)}}\right.\right.\right.$

Definition 3.3 $NSSet\left(\tilde{\mathfrak{f}}, n\right)$ be a (NSS) over the father set $\langle\mathcal{X}\rangle$ is said to be a null neutrosophic soft set If$\mathbb{T}_{\tilde{\mathfrak{f}}(x)^{(x)}} = 0, \mathbb{I}_{\tilde{\mathfrak{f}}(x)^{(x)}} = 0, \mathbb{F}_{\tilde{\mathfrak{f}}(x)^{(x)}} = 1;, \forall\; e \in n$ &$\forall\; x \in \langle\mathcal{X}\rangle$. it is signifies as $0_{(\langle\mathcal{X}\rangle, n)}$.

Definition3.4 NSS $\left(\tilde{\mathfrak{f}}, n\right)$ *over the father set*$\langle\mathcal{X}\rangle$It is said to be an absolute $neutrosophical$ softness i f$\mathbb{T}_{\tilde{\mathfrak{f}}(x)^{(x)}} = 1, \mathbb{I}_{\tilde{\mathfrak{f}}(x)^{(x)}} = 1, \mathbb{F}_{\tilde{\mathfrak{f}}(x)^{(x)}} = 0;, \curlyvee\; e \in n\; \&\; \curlyvee\; x \in \langle\mathcal{X}\rangle$.

It is signifies as $1_{(\langle\mathcal{X}\rangle, n)}$.clearly, $0_{(\langle\mathcal{X}\rangle, n)}{}^{c} = 1_{(\langle\mathcal{X}\rangle, n)}$&$1_{(\langle\mathcal{X}\rangle, n)}{}^{c} = 0_{(\langle\mathcal{X}\rangle, n)}$.

Definition 3.5 Let NSS$(\langle\widetilde{\mathcal{X}}\rangle, ꟻ^{parameter})$be the family of all $NS\ soft\ sets$ over the father set $\langle\widetilde{\mathcal{X}}\rangle$and $\tau \subset NSS(\langle\widetilde{\mathcal{X}}\rangle, ꟻ^{parameter})$.then$\tau$ is said to be a$NS\ soft\ topology$ on $\langle\widetilde{\mathcal{X}}\rangle$if:

(1).$0_{(\langle\mathcal{X}\rangle, n)}, 1_{(\langle\mathcal{X}\rangle, n)} \in \tau$,

(2).The union of any number of $NS\ soft\ sets$ in τbelongs to τ,

(3).The intersection of a finite number of $(NS)\ soft\ sets$ in in τbelongs to τ.Then $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ is said to be a $(NSTS)$ over $\langle\widetilde{\mathcal{X}}\rangle$. Each member of τ is said to be a $NS\ soft$ open set.

Definition 3.6 Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a $NSTS$ over $\langle\widetilde{\mathcal{X}}\rangle$&$(\tilde{\mathfrak{f}}, ꟻ^{parameter})$ be a NSset over $\langle\widetilde{\mathcal{X}}\rangle$.Then $(\tilde{\mathfrak{f}}, ꟻ^{parameter})$ is supposed to be a NS closed set iff its complement is a NS open set.

Definition 3.7 Let$NS be\ the\ family\ of\ all\ NS\ over\ father\ set\langle\widetilde{\mathcal{X}}\rangle$ and $x \in \langle\widetilde{\mathcal{X}}\rangle$the$NS\ x_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})}$ is supposed to be a N point, for 0 <𝔞,𝔟, 𝔠 ≤ 1 and is defined as follows:$x_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})}(y) = \begin{cases}(\mathfrak{a},\mathfrak{b},\mathfrak{c})\ provided\ y = x\\ (0,0,1) provided\ y \neq x\end{cases}$. It is obvious that every (NS) is actually the union of its N points

Example 3.8 Suppose that $\langle\widetilde{\mathcal{X}}\rangle = \{x_1, x_2\}$ then N set $A = \{\langle x_1, 0.1,0.3,0.5\rangle, \langle x_2, 0.5,0.4,0.7\rangle\}$is the union of N points$x_{1_{(0.1,0.3,0.5)}}$&$x_{2_{(0.5,0.4,0.7)}}$. Now we define the concept of NSpoints for NSsets.

Definition 3.9 Let $NSS(\widetilde{\langle\mathcal{X}\rangle})$be the family of all N soft sets over the father set $\langle\widetilde{\mathcal{X}}\rangle$. Then $NSS(x_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})})^e$is called a NS point, for every $x \in \langle\widetilde{\mathcal{X}}\rangle, 0 \prec \{\mathfrak{a},\mathfrak{b},\mathfrak{c}\} \preccurlyeq 1, e \propto ꟻ^{parameter}$, and is defined as follows:$x^e_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})_{e`}(y)} = \begin{cases}(\mathfrak{a},\mathfrak{b},\mathfrak{c})\text{provided } e` = e \wedge y = x\\ (\mathfrak{a},\mathfrak{b},\mathfrak{c})\text{provided } e` \neq e \wedge y \neq x\end{cases}$

Definition 3.10 Suppose that the father set $\langle\widetilde{\mathcal{X}}\rangle$is assumed to be

$\langle\widetilde{\mathcal{X}}\rangle = \{x_1, x_2\}$& the set of conditions by$ꟻ^{parameter} = \{e_1, e_2\}$. Let us consider $NSS(\tilde{\mathfrak{f}}, ꟻ^{parameter})$ over the father set $\langle\widetilde{\mathcal{X}}\rangle$ as follows:$(\tilde{\mathfrak{f}}, ꟻ^{parameter}) = \begin{cases}e_1 = \{\langle x_1, 0.3,0.7,0.6\rangle, \langle x_2, 0.4,0.3,0.8\rangle\}\\ e_2 = \{\langle x_1, 0.4,0.6,0.8\rangle, \langle x_2, 0.3,0.7,0.2\rangle\}\end{cases}$. It is clear that $(\tilde{\mathfrak{f}}, ꟻ^{parameter})$is the union of its NS points $\begin{cases}x^{e_1}{}_{1(0.3,0.7,0.6)},\\ x^{e_2}{}_{1(0.4,0.6,0.8)},\\ x^{e_1}{}_{2(0.4,0.3,0.8)}\\ x^{e_2}{}_{2(0.3,0.7,0.2)}.\end{cases}$

Where,$\begin{cases}x^{e_1}{}_{1(0.3,0.7,0.6)} = \left[\begin{cases}e_1 = \{\langle x_1, 0.3,0.7,0.6\rangle, \langle x_2, 0,0,1\rangle\}\\ e_2 = \{\langle x_1, 0,0,1\rangle, \langle x_2, 0,0,1\rangle\}\end{cases}\right]\\ x^{e_2}{}_{1(0.4,0.6,0.8)} = \left[\begin{cases}e_1 = \{\langle x_1, 0,0,1\rangle, \langle x_2, 0,0,1\rangle\}\\ e_2 = \{\langle x_1, 0.4,0.6,0.8\rangle, \langle x_2, 0,0,1\rangle\}\end{cases}\right]\\ x^{e_1}{}_{2(0.3,0.7,0.6)} = \left[\begin{cases}e_1 = \{\langle x_1, 0,0,1\rangle, \langle x_2, 0.4,0.3,0.8\rangle\}\\ e_2 = \{\langle x_1, 0,0,1\rangle, \langle x_2, 0,0,1\rangle\}\end{cases}\right]\\ x^{e_2}{}_{2(0.3,0.7,0.2)} = \left[\begin{cases}e_1 = \{\langle x_1, 0,0,1\rangle, \langle x_2, 0,0,1\rangle\}\\ e_2 = \{\langle x_1, 0,0,1\rangle, \langle 0.3,0.7,0.2\rangle\}\end{cases}\right]\end{cases}$

Definition 3.11 Let$(\tilde{\mathfrak{f}}, ⅎ^{parameter})$be a NSS over the father $set\langle\widetilde{\mathcal{X}}\rangle$.we say that $x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \in (\tilde{\mathfrak{f}}, ⅎ^{parameter})$ read as belonging to the NSS $(\tilde{\mathfrak{f}}, ⅎ^{parameter})$ whenever $\mathfrak{a} \preccurlyeq \mathbb{T}_{\tilde{\mathfrak{f}}(x)}(x)$,$\mathfrak{b} \preccurlyeq \mathbb{I}_{\tilde{\mathfrak{f}}(x)}(x)$,$\mathfrak{c} \succcurlyeq \mathbb{F}_{\tilde{\mathfrak{f}}(x)}(x)$.

Definition 3.12 Let $x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})}$ and $x^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})}$ be two NSpoints. For the NS points Over father set$\langle\widetilde{\mathcal{X}}\rangle$, we say that the NSpoints are distinct points$x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \sqcap x^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})} = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ⅎ^{parameter})}$. It is clear that $x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})}$ and $x^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})}$ are distinct NS points if and only if $x \succ y$ or $x \prec y$ or $e^{/} \succ e$ or $e^{/} \prec e$.

4. Neutrosophic Soft b-Separation Axioms

In this phase we define generalized neutrosophic soft separation axioms.

Definition 4.1 Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ⅎ^{\text{parameter}})$ be a NSTS over $\langle\widetilde{\mathcal{X}}\rangle$ & $(\tilde{\mathfrak{f}}, ⅎ^{\text{parameter}})$ be a neutrosophic soft set over $\langle\widetilde{\mathcal{X}}\rangle$.Then $(\tilde{\mathfrak{f}}, ⅎ^{\text{parameter}})$ is supposed to be a NS b-open if $(\tilde{\mathfrak{f}}, ⅎ^{\text{parameter}}) \subseteq \text{cl}\left(int\left((\tilde{\mathfrak{f}}, ⅎ^{\text{parameter}})\right)\right) \cup \text{int}\left(cl\left((\tilde{\mathfrak{f}}, ⅎ^{\text{parameter}})\right)\right)$ and NS b-close if $(\tilde{\mathfrak{f}}, ⅎ^{\text{parameter}}) \supseteq \text{int}\left(cl\left((\tilde{\mathfrak{f}}, ⅎ^{\text{parameter}})\right)\right) \cap \text{cl}\left(in\left((\tilde{\mathfrak{f}}, ⅎ^{\text{parameter}})\right)\right)$.

Definition 4.2 Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ⅎ^{parameter})$ be a $NSTS$ over$\langle\widetilde{\mathcal{X}}\rangle$, and $x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \succ y^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})}$ or $x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \prec y^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})}$ areNSPoints. If there exist NSb open sets $(\tilde{\mathfrak{f}}, ⅎ^{parameter})$&$(\tilde{\mathcal{g}}, ⅎ^{parameter})$ such that

$x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \in (\tilde{\mathfrak{f}}, ⅎ^{parameter}), x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \sqcap (\tilde{\mathcal{g}}, ⅎ^{parameter}) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ⅎ^{parameter})}$ or $y^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})} \in (\tilde{\mathcal{g}}, ⅎ^{parameter})$,

$y^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})} \sqcap (\tilde{\mathfrak{f}}, ⅎ^{parameter}) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ⅎ^{parameter})}$,Then $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ⅎ^{parameter})$ is called a NSb_0

Definition 4.3 Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ⅎ^{parameter})$ be a $NSTS$ over$\langle\widetilde{\mathcal{X}}\rangle$, and $x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \succ y^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})}$ or$x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \prec y^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})}$ are NS points. If there exist NSb open sets$(\tilde{\mathfrak{f}}, ⅎ^{parameter})$&$(\tilde{\mathcal{g}}, ⅎ^{parameter})$:

$x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \in (\tilde{\mathfrak{f}}, ⅎ^{parameter}), x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \sqcap (\tilde{\mathcal{g}}, ⅎ^{parameter}) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ⅎ^{parameter})} or y^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})} \in (\tilde{\mathcal{g}}, ⅎ^{parameter})$,

$y^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})} \sqcap (\tilde{\mathfrak{f}}, ⅎ^{parameter}) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ⅎ^{parameter})}$,Then $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ⅎ^{parameter})$ is called a NSb_1.

Definition 4.4 Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ⅎ^{parameter})$ be a $NSTS$over$\langle\widetilde{\mathcal{X}}\rangle$, and $x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \succ y^{e^{/}}{}_{(\mathfrak{a}^{/},\mathfrak{b}^{/},\mathfrak{c}^{/})}$

$x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \prec y^{e/}{}_{(\mathfrak{a}/,\mathfrak{b}/,\mathfrak{c}/)}$ *are NS*points. if $\exists$ *NSb* open sets $(\tilde{\mathfrak{f}}, \mathfrak{U}^{parameter})\&(\tilde{\mathcal{G}}, \mathfrak{U}^{parameter})s.t.$

$x^{e}{}_{(\mathfrak{a},\mathfrak{b},\mathfrak{c})} \in (\tilde{\mathfrak{f}}, \mathfrak{U}^{parameter})$ & $y^{e/}{}_{(\mathfrak{a}/,\mathfrak{b}/,\mathfrak{c}/)} \in (\tilde{\mathcal{G}}, \mathfrak{U}^{parameter})$ & $(\tilde{\mathfrak{f}}, \mathfrak{U}^{parameter}) \sqcap (\tilde{\mathcal{G}}, \mathfrak{U}^{parameter}) =$

$0_{(\langle\widetilde{\mathcal{X}}\rangle,\mathfrak{U}^{parameter})}$,Then $(\langle\widetilde{\mathcal{X}}\rangle, \tau, \mathfrak{U}^{parameter})$ is called a NSb_2 space.

Example 4.5 Suppose that the father set $\langle\widetilde{\mathcal{X}}\rangle$is *assumed to be*

$\langle\widetilde{\mathcal{X}}\rangle = \{x_1, x_2\}$ & the set of conditions by $\mathfrak{U}^{parameter} = \{e_1, e_2\}$. Let us consider NS set $(\tilde{\mathfrak{f}}, \mathfrak{U}^{parameter})$*over the father* set $\langle\widetilde{\mathcal{X}}\rangle$ & $x^{e_1}{}_{1(0.1,0.4,0.7)}$, $x^{e_2}{}_{1(0.2,0.5,0.6)}$, $x^{e_1}{}_{2(0.3,0.3,0.5)}$ & $x^{e_1}{}_{2(0.4,0.4,0.4)}$ be NS points. Then the family $\tau = \{0_{(\langle\widetilde{\mathcal{X}}\rangle,\mathfrak{U}^{parameter})}, 1_{(\langle\widetilde{\mathcal{X}}\rangle,\mathfrak{U}^{parameter})}, (\widetilde{\mathfrak{f}_1}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}),$ $,(\widetilde{\mathfrak{f}_3}, \mathfrak{U}^{parameter}), ,(\widetilde{\mathfrak{f}_4}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_5}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_6}, \mathfrak{U}^{parameter}), ,(\widetilde{\mathfrak{f}_7}, \mathfrak{U}^{parameter}), ,(\widetilde{\mathfrak{f}_8}, \mathfrak{U}^{parameter})\}$, , where $(\widetilde{\mathfrak{f}_1}, \mathfrak{U}^{parameter}) = x^{e_1}{}_{1(0.1,0.4,0.7)}, (\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}) = x^{e_2}{}_{1(0.2,0.5,0.6)}, (\widetilde{\mathfrak{f}_3}, \mathfrak{U}^{parameter}) =$ $x^{e_1}{}_{2(0.3,0.3,0.5)}, (\widetilde{\mathfrak{f}_4}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_1}, \mathfrak{U}^{parameter}) \cup (\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_5}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_1}, \mathfrak{U}^{parameter}) \cup$ $(\widetilde{\mathfrak{f}_3}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_6}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}) \cup (\widetilde{\mathfrak{f}_3}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_7}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_1}, \mathfrak{U}^{parameter}) \cup$ $(\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}) \cup (\widetilde{\mathfrak{f}_3}, \mathfrak{U}^{parameter}),$

$(\widetilde{\mathfrak{f}_8}, \mathfrak{U}^{parameter}) = \{x^{e_1}{}_{1(0.1,0.4,0.7)}, x^{e_2}{}_{1(0.2,0.5,0.6)}, x^{e_1}{}_{2(0.3,0.3,0.5)}, x^{e_2}{}_{2(0.4,0.4,0.4)}$ is a $NSTS$over the father set $\langle\widetilde{\mathcal{X}}\rangle$. Thus $(\langle\widetilde{\mathcal{X}}\rangle, \tau, \mathfrak{U}^{parameter})$ be a$NSTS$over the father set $\langle\widetilde{\mathcal{X}}\rangle$. Also $(\langle\widetilde{\mathcal{X}}\rangle, \tau, \mathfrak{U}^{parameter})$ is NSb_0 structure but it is not NSb_1because for NSpoints $x^{e_1}{}_{1(0.1,0.4,0.7)}, x^{e_2}{}_{2(0.4,0.4,0.4)}, (\langle\widetilde{\mathcal{X}}\rangle, \tau, \mathfrak{U}^{parameter})$ not NSb_1.

Example 4.6 Suppose that the father set $\langle\widetilde{\mathcal{X}}\rangle$is assumed to be

$\langle\widetilde{\mathcal{X}}\rangle = \{x_1, x_2\}$& the set of *conditions* by$\mathfrak{U}^{parameter} = \{e_1, e_2\}$. Let us consider NSset $(\tilde{\mathfrak{f}}, \mathfrak{U}^{parameter})$ over the father set $\langle\widetilde{\mathcal{X}}\rangle$&$x^{e_1}{}_{1(0.1,0.4,0.7)}, x^{e_2}{}_{1(0.2,0.5,0.6)}, x^{e_1}{}_{2(0.3,0.3,0.5)}$&$x^{e_1}{}_{2(0.4,0.4,0.4)}$ be NSpoints. Then the family $\tau =$ $\{0_{(\langle\widetilde{\mathcal{X}}\rangle,\mathfrak{U}^{parameter})}, 1_{(\langle\widetilde{\mathcal{X}}\rangle,\mathfrak{U}^{parameter})}, (\widetilde{\mathfrak{f}_1}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}) (\widetilde{\mathfrak{f}_3}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_4}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_5}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_6}, \mathfrak{U}^{parameter})$ $,(\widetilde{\mathfrak{f}_7}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_8}, \mathfrak{U}^{parameter}), \ldots\ldots\ldots (\widetilde{\mathfrak{f}_{15}}, \mathfrak{U}^{parameter})\},, where (\widetilde{\mathfrak{f}_1}, \mathfrak{U}^{parameter}) x^{e_1}{}_{1(0.1,0.4,0.7)}, (\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}) =$ $x^{e_2}{}_{1(0.2,0.5,0.6)}, (\widetilde{\mathfrak{f}_3}, \mathfrak{U}^{parameter}) = x^{e_1}{}_{2(0.3,0.3,0.5)}, (\widetilde{\mathfrak{f}_4}, \mathfrak{U}^{parameter}) = x^{e_2}{}_{2(0.4,0.4,0.4)} (\widetilde{\mathfrak{f}_5}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_1}, \mathfrak{U}^{parameter}) \cup$ $(\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_6}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_1}, \mathfrak{U}^{parameter}) \cup (\widetilde{\mathfrak{f}_3}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_7}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}) \cup$ $(\widetilde{\mathfrak{f}_4}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_8}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}) \cup (\widetilde{\mathfrak{f}_3}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_9}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}) \cup$ $(\widetilde{\mathfrak{f}_4}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_{10}}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_3}, \mathfrak{U}^{parameter}) \cup (\widetilde{\mathfrak{f}_4}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_{11}}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_1}, \mathfrak{U}^{parameter}) \cup$ $(\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}) \cup (\widetilde{\mathfrak{f}_3}, \mathfrak{U}^{parameter}), (\widetilde{\mathfrak{f}_{12}}, \mathfrak{U}^{parameter}) = (\widetilde{\mathfrak{f}_1}, \mathfrak{U}^{parameter}) \cup (\widetilde{\mathfrak{f}_2}, \mathfrak{U}^{parameter}) \cup$

$\left(\widetilde{\mathfrak{f}_4}, ꟻ^{parameter}\right), \left(\widetilde{\mathfrak{f}_{13}}, ꟻ^{parameter}\right) = \left(\widetilde{\mathfrak{f}_2}, ꟻ^{parameter}\right) \cup \left(\widetilde{\mathfrak{f}_3}, ꟻ^{parameter}\right) \cup \left(\widetilde{\mathfrak{f}_4}, ꟻ^{parameter}\right), \left(\widetilde{\mathfrak{f}_{14}}, ꟻ^{parameter}\right) =$

$\left(\widetilde{\mathfrak{f}_1}, ꟻ^{parameter}\right) \cup \left(\widetilde{\mathfrak{f}_3}, ꟻ^{parameter}\right) \cup \left(\widetilde{\mathfrak{f}_4}, ꟻ^{parameter}\right) \cup \left(\widetilde{\mathfrak{f}_{15}}, ꟻ^{parameter}\right) =$

$\left\{x^{e_1}{}_{1(0.1,0.4,0.7)}, x^{e_2}{}_{1(0.2,0.5,0.6)}, x^{e_1}{}_{2(0.3,0.3,0.5)}, x^{e_2}{}_{2(0.4,0.4,0.4)}\right\}$is a $NSTS$over the father set $\langle\widetilde{\mathcal{X}}\rangle$. Thus $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a$(NSTS)$ over the father set $\langle\widetilde{\mathcal{X}}\rangle$. Also $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ is NSb_1 structure.$x^{e_1}{}_{1(0.1,0.4,0.7)}, x^{e_2}{}_{2(0.4,0.4,0.4)}$, $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ not NSb_2.

Theorem 4.7 Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a$(NSTS)$ over the father $\widetilde{set\langle\mathcal{X}\rangle}$. Then $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a NSb_1 structure iffeach NSpointis a NS b-closed set.

Proof. Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a $NSTS$over the father $\widetilde{set\langle\mathcal{X}\rangle}$. $(x^e{}_{(a,b,c)}, ꟻ^{parameter})$be an arbitrary NSpoint.We establish $\left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right)$ $is\ a\ N\ soft\ b$-open set. Let$\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) \in \left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right)$. Then either $\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) \succ \left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right) or \left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) \prec \left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) or$

$$\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) \succ\succ \left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right) or \left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) \prec\prec \left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right).$$

$This\ means\ that$ $\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) \& \left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right)$ are mutually exclusive $NSpoints$.

Thus $x \succ y$ or $x \prec y$ or $e/ \succ e$ or $e/ \prec e$.or $x \succ\succ y$ or $x \prec\prec y$ or $e/ \succ\succ e$ or $e/ \prec\prec e$. Since$(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a NSb_1 structure, $\exists$ a NSb-open set $(\widetilde{\mathcal{G}}, ꟻ^{parameter})$ so that $\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) \in (\widetilde{\mathcal{G}}, ꟻ^{parameter})$&$\left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right) \sqcap (\widetilde{\mathcal{G}}, ꟻ^{parameter}) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ꟻ^{parameter})}$. Since,$\left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right) \sqcap (\widetilde{\mathcal{G}}, ꟻ^{parameter}) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ꟻ^{parameter})}$. So $\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) \in (\widetilde{\mathcal{G}}, ꟻ^{parameter}) \subset \left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right)$. Thus $\left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right)$is a NSb-open set, i.e.,$\left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right)$is a NSb-closed set. Suppose that each NSpoint $\left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right)$is a NSb-closed set.Then $\left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right)^c$is a NSb-open set. Let $\left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right) \sqcap \left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ꟻ^{parameter})}$. $Thus$ $\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) \in$ $\left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right)^c$&$\left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right) \sqcap \left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right)^c = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ꟻ^{parameter})}$. So $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a NSb-b_1 space.

Theorem 4.8 Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a $(NSTS)$over the father $\widetilde{set\langle\mathcal{X}\rangle}$. Then $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ is NS-$b_2$space iff for distinct NS points $\left(x^e{}_{(a,b,c)}, ꟻ^{parameter}\right)$&$\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)$, there exists a NSb-open set $\left(\tilde{\mathfrak{f}}, ꟻ^{parameter}\right)$ containing $\exists$ but not $\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) s.t. \left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) \notin \overline{\left(\tilde{\mathfrak{f}}, ꟻ^{parameter}\right)}$.

Proof Let$(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \succ (y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter})$ be two NSpoints in $NSbT_2$ space. Then ∃ disjoint NSopen sets $(\tilde{f}, Ⅎ^{parameter})$&$(\tilde{g}, Ⅎ^{parameter})$ $s.t.$ $(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \in (\tilde{f}, Ⅎ^{parameter})$& $(y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter}) \in (\tilde{g}, Ⅎ^{parameter})$. *Since* $(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \sqcap (y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter}) = 0_{(\langle\widetilde{X}\rangle, Ⅎ^{parameter})}$&

$(\tilde{f}, Ⅎ^{parameter}) \sqcap (\tilde{g}, Ⅎ^{parameter}) = 0_{(\langle\widetilde{X}\rangle, Ⅎ^{parameter})}$. $(y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter}) \notin (\tilde{f}, Ⅎ^{parameter}) \Longrightarrow$

$(y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter}) \notin \overline{(\tilde{f}, Ⅎ^{parameter})}$. Next suppose that,$(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \succ (y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter})$, ∃a *NSb open set* $(\tilde{f}, Ⅎ^{parameter})$*containing* $(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter})$ *but not* $(y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter})$ $s.t.$ $(y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter}) \notin \overline{(\tilde{f}, Ⅎ^{parameter})}^{c}$ that is $(\tilde{f}, Ⅎ^{parameter})$&$\overline{(\tilde{f}, Ⅎ^{parameter})}^{c}$ are *mutually exclusiveNSb*open sets supposing $(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter})$& $(y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter})$in turn.

Theorem 4.9 Let $(\langle\widetilde{X}\rangle, \tau, Ⅎ^{parameter})$ be a $NSTS$ over the father $set\widetilde{\langle X\rangle}$. Then $(\langle\widetilde{X}\rangle, \tau, Ⅎ^{parameter})$ isNS b-T_1space if every NS point $(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \in (\tilde{f}, Ⅎ^{parameter}) \in (\langle\widetilde{X}\rangle, \tau, Ⅎ^{parameter})$. If there exists a NSb open set $(\tilde{g}, Ⅎ^{parameter})$ $s.t.$ $(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \in (\tilde{g}, Ⅎ^{parameter}) \subset \overline{(\tilde{g}, Ⅎ^{parameter})} \subset (\tilde{f}, Ⅎ^{parameter})$,Then $(\langle\widetilde{X}\rangle, \tau, Ⅎ^{parameter})$ a NSb_2 space.

Proof. Suppose$(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \sqcap (y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter}) = 0_{(\langle\widetilde{X}\rangle, Ⅎ^{parameter})}$. Since $(\langle\widetilde{X}\rangle, \tau, Ⅎ^{parameter})$ is NSb_1 space. $(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter})$& $(y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter})$ *areNS* b *close sets* in $(\langle\widetilde{X}\rangle, \tau, Ⅎ^{parameter})$ .Then $(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \in ((y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter}))^{c} \in (\langle\widetilde{X}\rangle, \tau, Ⅎ^{parameter})$. Thus ∃a NS b open set $(\tilde{g}, Ⅎ^{parameter}) \in (\langle\widetilde{X}\rangle, \tau, Ⅎ^{parameter})$s. t. $(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \in (\tilde{g}, Ⅎ^{parameter}) \subset \overline{(\tilde{g}, Ⅎ^{parameter})} \subset ((y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter}))^{c}$. *So we have* $(y^{e/}{}_{(a/,b/,c/)}, Ⅎ^{parameter}) \in (\tilde{g}, Ⅎ^{parameter})$&

$(\tilde{g}, Ⅎ^{parameter}) \sqcap ((\tilde{g}, Ⅎ^{parameter}))^{c} = 0_{(\langle\widetilde{X}\rangle, Ⅎ^{parameter})}$, *i. e.* $(\langle\widetilde{X}\rangle, \tau, Ⅎ^{parameter})$ is a NS soft $NSb_2 space$.

5. Characterization of other NS b-Separation Axioms

Definition 5.1. Let $(\langle\widetilde{X}\rangle, \tau, Ⅎ^{parameter})$ be a $NSTS$ over the father set $\widetilde{\langle X\rangle}$. $(\tilde{f}, Ⅎ^{parameter})$ be a NSbclosed set and $(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \sqcap (\tilde{f}, Ⅎ^{parameter}) = 0_{(\langle\widetilde{X}\rangle, Ⅎ^{parameter})}$. If ∃ NS b-open sets $(\widetilde{g_1}, Ⅎ^{parameter})$&$(\widetilde{g_2}, Ⅎ^{parameter})$ $s.t.$ $(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \in (\widetilde{g_1}, Ⅎ^{parameter})$, $(\tilde{f}, Ⅎ^{parameter}) \subset (\widetilde{g_2}, Ⅎ^{parameter})$&$(x^{e}{}_{(a,b,c)}, Ⅎ^{parameter}) \sqcap (\widetilde{g_1}, Ⅎ^{parameter}) = 0_{(\langle\widetilde{X}\rangle, Ⅎ^{parameter})}$, then $(\langle\widetilde{X}\rangle, \tau, Ⅎ^{parameter})$ is called a NS b-regular space. $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ is said to be NSb_3space, if is both a NSregular and NSb_1space.

Theorem 5.2. Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a $NSTSover$ the father set $\widetilde{\langle\mathcal{X}\rangle}$. $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ is soft b-T_3 $spaceiff$ for every $(x^e{}_{(a,b,c)}, ꟻ^{parameter}) \in (\tilde{f}, ꟻ^{parameter}), t.e, (\widetilde{\mathcal{G}}, ꟻ^{parameter}) \in (\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ $s.t. (x^e{}_{(a,b,c)}, ꟻ^{parameter}) \in (\widetilde{\mathcal{G}}, ꟻ^{parameter}) \subset \overline{(\widetilde{\mathcal{G}}, ꟻ^{parameter})} \subset (\tilde{f}, ꟻ^{parameter})$.

Proof. Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ is NSb_3space $\&(x^e{}_{(a,b,c)}, ꟻ^{parameter}) \in (\tilde{f}, ꟻ^{parameter}) \in (\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$. Since $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ is $NSNST_3\ space$ for $the\ NS\ point(x^e{}_{(a,b,c)}, ꟻ^{parameter})\&$ b-closed set $(\tilde{f}, ꟻ^{parameter})^c, \exists(\widetilde{\mathcal{G}_1}, ꟻ^{parameter})\&(\widetilde{\mathcal{G}_2}, ꟻ^{parameter})$ s.t. $(x^e{}_{(a,b,c)}, ꟻ^{parameter}) \in (\widetilde{\mathcal{G}_1}, ꟻ^{parameter}), (\tilde{f}, ꟻ^{parameter})^c \subset (\widetilde{\mathcal{G}_2}, ꟻ^{parameter})\&(\widetilde{\mathcal{G}_1}, ꟻ^{parameter}) \cap (\widetilde{\mathcal{G}_2}, ꟻ^{parameter}) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ꟻ^{parameter})}$. $Then\ we\ have(x^e{}_{(a,b,c)}, ꟻ^{parameter}) \in (\widetilde{\mathcal{G}_1}, ꟻ^{parameter}) \subset (\widetilde{\mathcal{G}_2}, ꟻ^{parameter})^c \subset (\tilde{f}, ꟻ^{parameter})$. Since $(\widetilde{\mathcal{G}_2}, ꟻ^{parameter})^c NSb$closed set.$\overline{(\widetilde{\mathcal{G}_1}, ꟻ^{parameter})} \subset (\widetilde{\mathcal{G}_2}, ꟻ^{parameter})^c$.

$Conversely$, let $(x^e{}_{(a,b,c)}, ꟻ^{parameter}) \cap (\widetilde{\mathcal{H}}, ꟻ^{parameter}) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ꟻ^{parameter})}\&(\widetilde{\mathcal{H}}, ꟻ^{parameter})$ be a NSb closed set. $(x^e{}_{(a,b,c)}, ꟻ^{parameter}) \propto (\widetilde{\mathcal{H}}, ꟻ^{parameter})^c$ & from the $condition\ of\ the\ theorem$, wehave $(x^e{}_{(a,b,c)}, ꟻ^{parameter}) \in (\widetilde{\mathcal{G}}, ꟻ^{parameter}) \subset \overline{(\widetilde{\mathcal{G}}, ꟻ^{parameter})} \subset (\widetilde{\mathcal{H}}, ꟻ^{parameter})^c$ $.Thus\ (x^e{}_{(a,b,c)}, ꟻ^{parameter}) \in (\widetilde{\mathcal{G}}, ꟻ^{parameter}), (\widetilde{\mathcal{H}}, ꟻ^{parameter}) \subset \overline{(\widetilde{\mathcal{G}}, ꟻ^{parameter})}^c \&(\widetilde{\mathcal{G}}, ꟻ^{parameter}) \cap \overline{(\widetilde{\mathcal{G}}, ꟻ^{parameter})}^c = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ꟻ^{parameter})}$. So $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ is NSb_3space.

Definition 5.3. Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a $NSTSover$ the father set $\widetilde{\langle\mathcal{X}\rangle}$. This space is a NSnormal space, if for every pair of disjoint NSb closed sets $(\tilde{f}_1, ꟻ^{parameter})\&(\tilde{f}_2, ꟻ^{parameter}), \exists$ disjoint NSb open sets $(\widetilde{\mathcal{G}}_1, ꟻ^{parameter})\&(\widetilde{\mathcal{G}}_2, ꟻ^{parameter}) s.t. (\tilde{f}_1, ꟻ^{parameter}) \subset (\widetilde{\mathcal{G}}_1, ꟻ^{parameter})\&(\tilde{f}_2, ꟻ^{parameter}) \subset (\widetilde{\mathcal{G}}_2, ꟻ^{parameter})$.

$(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ is said to be a NSb_4spaceif it is both a NSnormal and NSb_1space.

Theorem 5.4. Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a $NSTSover$ the father set $\widetilde{\langle\mathcal{X}\rangle}$. This space is a NS bT_4 space ⟺, for each NS b closed set $(\tilde{f}, ꟻ^{parameter})$and NSbopen set $(\widetilde{\mathcal{G}}, ꟻ^{parameter})$ with $(\tilde{f}, ꟻ^{parameter}) \subset (\widetilde{\mathcal{G}}, ꟻ^{parameter})$, ∃a NSbopen set $(\widetilde{\mathcal{D}}, ꟻ^{parameter}) s.t. (\tilde{f}, ꟻ^{parameter}) \subset (\widetilde{\mathcal{D}}, ꟻ^{parameter}) \subset \overline{(\widetilde{\mathcal{D}}, ꟻ^{parameter})} \subset (\widetilde{\mathcal{G}}, ꟻ^{parameter})$.

Proof. Let $(\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a NS b_4 over the father set $\widetilde{\langle\mathcal{X}\rangle}$. & let $(\tilde{f}, ꟻ^{parameter}) \subset (\widetilde{\mathcal{G}}, ꟻ^{parameter})$.Then$(\widetilde{\mathcal{G}}, ꟻ^{parameter})^c$ is is a NSb closed set and $(\tilde{f}, ꟻ^{parameter}) \cap (\widetilde{\mathcal{G}}, ꟻ^{parameter}) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ꟻ^{parameter})}$. Since$\langle\widetilde{\mathcal{X}}\rangle, \tau, ꟻ^{parameter})$ be a $NS\ b_4\ space, \exists NS$ b-open sets $(\widetilde{\mathcal{D}}_1, ꟻ^{parameter})\&(\widetilde{\mathcal{D}}_2, ꟻ^{parameter}) s.t. (\tilde{f}, ꟻ^{parameter}) \subset (\widetilde{\mathcal{D}}_1, ꟻ^{parameter}), (\widetilde{\mathcal{G}}, ꟻ^{parameter})^c \subset (\widetilde{\mathcal{D}}_2, ꟻ^{parameter})\&(\widetilde{\mathcal{D}}_1, ꟻ^{parameter}) \cap (\widetilde{\mathcal{D}}_2, ꟻ^{parameter}) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ꟻ^{parameter})}$. Thus $(\tilde{f}, ꟻ^{parameter}) \subset (\widetilde{\mathcal{D}}_1, ꟻ^{parameter}) \subset (\widetilde{\mathcal{D}}_2, ꟻ^{parameter})^c \subset (\widetilde{\mathcal{G}}, ꟻ^{parameter}), (\widetilde{\mathcal{D}}_2, ꟻ^{parameter})^c$is a NS b closed set and $\overline{(\widetilde{\mathcal{D}}_1, ꟻ^{parameter})} \subset (\widetilde{\mathcal{D}}_2, ꟻ^{parameter})^c$. So $(\tilde{f}, ꟻ^{parameter}) \subset (\widetilde{\mathcal{D}}_1, ꟻ^{parameter}) \subset \overline{(\widetilde{\mathcal{D}}_1, ꟻ^{parameter})} \subset (\widetilde{\mathcal{G}}, ꟻ^{parameter})$.

Conversely, let $(\tilde{f}_1, ɏ^{parameter})$&$(\tilde{f}_2, ɏ^{parameter})$ be two disjoint NSb closed sets. Then $(\tilde{f}_1, ɏ^{parameter}) \subset (\tilde{f}_2, ɏ^{parameter})^c$.From the condition of theorem, there exists a NS b open set$(\widetilde{\mathcal{D}}, ɏ^{parameter})$ $s.t.$ $(\tilde{f}_1, ɏ^{parameter}) \subset (\widetilde{\mathcal{D}}, ɏ^{parameter}) \subset \overline{(\widetilde{\mathcal{D}}_1, ɏ^{parameter})} \subset (\tilde{f}_2, ɏ^{parameter})^c$. Thus $(\widetilde{\mathcal{D}}, ɏ^{parameter})$&$(\widetilde{\mathcal{D}}, ɏ^{parameter})^c$ are NSb open sets and $(\tilde{f}_1, ɏ^{parameter}) \subset (\tilde{f}_2, ɏ^{parameter})^c$, $(\tilde{f}_2, ɏ^{parameter}) \subset \overline{(\widetilde{\mathcal{D}}, ɏ^{parameter})}^c$&$(\widetilde{\mathcal{D}}, ɏ^{parameter})$&$\overline{(\widetilde{\mathcal{D}}, ɏ^{parameter})}^c = 0_{(\langle\widetilde{\mathcal{X}}\rangle, ɏ^{parameter})}$.

$(\langle\widetilde{\mathcal{X}}\rangle, \tau, ɏ^{parameter})$ be a NS b_4 $space$

6. Monotonous behavior of NS b-Separation Axioms

Theorem 6.1. Let $\langle\mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ be $NSST$ such that it is NSb Hausdorff space and $\langle Y^{crip}, \mathfrak{F}, \partial\rangle$ be any$NSST$. Let$\langle f, \partial\rangle$:$\langle\mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \longrightarrow \langle Y^{crip}, \mathfrak{F}, \partial\rangle$ be a soft fuction such that it is soft monotone and continuous. Then $\langle Y^{crip}, \mathfrak{F}, \partial\rangle$ is also of characteristics of NSb Hausdorfness.

Proof: Suppose $(x^e{}_{(a,b,c)}, ɏ^{parameter})_1, (x^e{}_{(a,b,c)}, ɏ^{parameter})_2 \in \mathcal{X}^{crip}$ such that either $(x^e{}_{(a,b,c)}, ɏ^{parameter})_1 \succ (x^e{}_{(a,b,c)}, ɏ^{parameter})_2 or (x^e{}_{(a,b,c)}, ɏ^{parameter})_1 \prec (x^e{}_{(a,b,c)}, ɏ^{parameter})_2$. Since $\langle f, \partial\rangle$ is soft monotone. Let us suppose the monotonically increasing case. So, $(x^e{}_{(a,b,c)}, ɏ^{parameter})_1 \succ f_{(x^e{}_{(a,b,c)}, ɏ^{parameter})_2} or (x^e{}_{(a,b,c)}, ɏ^{parameter})_1 \prec f_{(x^e{}_{(a,b,c)}, ɏ^{parameter})_2}$ implies that $f_{(x^e{}_{(a,b,c)}, ɏ^{parameter})_1} \succ f(x_2\ ) or f_{(x^e{}_{(a,b,c)}, ɏ^{parameter})_1} \prec f_{(x^e{}_{(a,b,c)}, ɏ^{parameter})_2}$ respectively. Suppose $(y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter})_1, (y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter})_2 \in Y^{crip}$ such that $(y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter})_1 \succ (y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter})_2$ or $(y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter})_1 \prec (y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter})_2$ So, $(y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter})_1 \succ (y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter})_2$ or $(y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter})_1 \prec (y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter})_2$ respectively such that $(y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter}) = f_{(x^e{}_{(a,b,c)}, ɏ^{parameter})_1}, (y^{e/}{}_{(a/,b/,c/)}, ɏ^{parameter})_2 = f_{(x^e{}_{(a,b,c)}, ɏ^{parameter})_2}$.Since, $\langle\mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ is NSb Hausdorff space so there exists mutually disjoint NS b-open sets $\langle k_1, \partial\rangle$ and $\langle k_2, \partial\rangle \in \langle\mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \Longrightarrow$ $f(\langle k_1, \partial\rangle)$ and $f(\langle k_2, \partial\rangle) \in \langle Y^{crip}, \mathfrak{F}, \partial\rangle$. We claim that$f(\langle k_1, \partial\rangle) \tilde{\cap} f(\langle k_2, \partial\rangle) = 0_{(\langle\widetilde{\mathcal{X}}\rangle, \widetilde{\partial^{parameter}})}$. Otherwise $f(\langle k_1, \partial\rangle) \tilde{\cap} f(\langle k_2, \partial\rangle) \neq 0_{(\langle\widetilde{\mathcal{X}}\rangle, \widetilde{\partial^{parameter}})}$. Suppose $\exists\ (t^{e//}{}_{(a//,b//,c//)}, ɏ^{parameter})_1 \in f(\langle k_1, \partial\rangle) \tilde{\cap} f(\langle k_2, \partial\rangle) \Longrightarrow$

$(t^{e//}{}_{(a//,b//,c//)}, ɏ^{parameter})_1 \in f(\langle k_1, \partial\rangle)$&$(t^{e//}{}_{(a//,b//,c//)}, ɏ^{parameter})_1 \in f(\langle k_2, \partial\rangle)$,

$(t^{e//}{}_{(a//,b//,c//)}, ɏ^{parameter})_1 \in f(\langle k_1, \partial\rangle), f\ is\ soft\ one-one$ $\Longrightarrow \exists\ (t^{e//}{}_{(a//,b//,c//)}, ɏ^{parameter})_2 \in \langle k_1, \partial\rangle$s.t.

$\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_1=\not{f}\left(\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_2\right),\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_1\in\not{f}(\langle\mathscr{k}_2,\partial\rangle)\Longrightarrow$

$\exists\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_3\in\langle\mathscr{k}_2,\partial\rangle$ s.t. $\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_1=\not{f}\left(\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_3\right)\Longrightarrow$

$\not{f}\left(\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_2\right)=\not{f}\left(\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_3\right) Since, \not{f}\ is\ soft\ one-one\Longrightarrow$

$\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_2=\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_3\Longrightarrow\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_2\in$

$\not{f}(\langle\mathscr{k}_1,\partial\rangle),\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_2\in\not{f}(\langle\mathscr{k}_2,\partial\rangle)\Longrightarrow\left(t^{e^{//}}{}_{(a^{//},b^{//},c^{//})},\daleth^{parameter}\right)_2\in\not{f}(\langle\mathscr{k}_1,\partial\rangle)\,\widetilde{\cap}\,\not{f}(\langle\mathscr{k}_2,\partial\rangle)$. This $is\ contradiction\ because\ \langle\mathscr{k}_1,\partial\rangle\,\widetilde{\cap}\,\langle\mathscr{k}_2,\partial\rangle=0_{(\langle\widetilde{\mathcal{X}}\rangle,\daleth^{parameter})}.\ so,\not{f}(\langle\mathscr{k}_1,\partial\rangle)\,\widetilde{\cap}\,\not{f}(\langle\mathscr{k}_2,\partial\rangle)=$ $0_{(\langle\widetilde{\mathcal{X}}\rangle,\daleth^{parameter})}$. Finally,

$$\begin{cases}\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1\succ\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2\ or\\ \left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1\prec\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2\Longrightarrow\\ \left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1\neq\not{f}\left(\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2\right)\quad \not{f}\left(\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1\right)\succ\\ or\ \not{f}\left(\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1\right)\prec\not{f}\left(\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2\right)\\ \Longrightarrow\not{f}\left(\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1\right)\neq\not{f}\left(\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2\right)\end{cases}$$

Given a pair of points

$\left(\mathscr{y}^{e^{/}}{}_{(a^{/},b^{/},c^{/})},\daleth^{parameter}\right)_1,\left(\mathscr{y}^{e^{/}}{}_{(a^{/},b^{/},c^{/})},\daleth^{parameter}\right)_2\in Y^{crip}\ni\left(\mathscr{y}^{e^{/}}{}_{(a^{/},b^{/},c^{/})},\daleth^{parameter}\right)_1\neq$

$\left(\mathscr{y}^{e^{/}}{}_{(a^{/},b^{/},c^{/})},\daleth^{parameter}\right)_2$ We are able to find out mutually exclusive NSb open sets $\not{f}(\langle\mathscr{k}_1,\partial\rangle),\not{f}(\langle\mathscr{k}_2,\partial\rangle)\in$ $\langle Y^{crip},\mathfrak{F},\partial\rangle$ s.t. $\left(\mathscr{y}^{e^{/}}{}_{(a^{/},b^{/},c^{/})},\daleth^{parameter}\right)_1\in\not{f}(\langle\mathscr{k}_1,\partial\rangle),\left(\mathscr{y}^{e^{/}}{}_{(a^{/},b^{/},c^{/})},\daleth^{parameter}\right)_2\in\not{f}(\langle\mathscr{k}_2,\partial\rangle)$. this proves that $\langle Y^{crip},\mathfrak{F},\partial\rangle$ is NSb Husdorff space.

Theorem 6.2. Let$\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ be $NSST$ and $\langle Y^{crip},\mathfrak{F},\partial\rangle$ be an-other $NSST$ which satisfies one more condition of NSb Hausdorffness.Let $\langle\not{f},\partial\rangle:\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle\longrightarrow\langle Y^{crip},\mathfrak{F},\partial\rangle$ be a soft fuction $s.t.$ it is soft monotone and continuous. Then $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ is also of characteristics of NSb Hausdorfness.

Proof: Suppose $\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1,\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2\in\mathcal{X}^{crip}$ such that either $\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1\succ$ $\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2\ or\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2 Let\ us\ suppose\ the\ NS\ monotonically$ $increasing\ case. So,\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1\succ\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2\ or\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1\prec$ $\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2\ implies\ that\ \not{f}(\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1)\succ$ $\not{f}(\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2)or\not{f}(\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_1)\prec$ $\not{f}(\left(x^e{}_{(a,b,c)},\daleth^{parameter}\right)_2)\ respectively. Suppose\ \left(\mathscr{y}^{e^{/}}{}_{(a^{/},b^{/},c^{/})},\daleth^{parameter}\right)_1,\left(\mathscr{y}^{e^{/}}{}_{(a^{/},b^{/},c^{/})},\daleth^{parameter}\right)_2\in Y^{crip}$ $such\ that\ \left(\mathscr{y}^{e^{/}}{}_{(a^{/},b^{/},c^{/})},\daleth^{parameter}\right)_1\succ\left(\mathscr{y}^{e^{/}}{}_{(a^{/},b^{/},c^{/})},\daleth^{parameter}\right)_2\ or\left(\mathscr{y}^{e^{/}}{}_{(a^{/},b^{/},c^{/})},\daleth^{parameter}\right)_1\prec$

$\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_2$. So, $\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_1 \succ \left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_2$ or $\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_1 \prec$ $\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_2$ $respectively\ such\ that$ $\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_1 =$ $f(\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)_1), \left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_2 = f(\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)_2) s.t. \left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)_1 =$ $f^{-1}(\mathcal{y}_1)\ and\ \left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)_2 =$ $f^{-1}(\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_2). since \left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_1, \left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_2 \in Y^{crip}$ but $\langle Y^{crip}, \mathfrak{T}, \partial\rangle$ is NSb Hausdorff space. So according to definition $\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_1 \succ$ $\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_2$ or $\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_1 \prec \left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_2$. There definitely exists NS b-open sets $\langle \mathcal{k}_1, \partial\rangle$ and $\langle \mathcal{k}_2, \partial\rangle \in \langle Y^{crip}, \mathfrak{T}, \partial\rangle$ such that $\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_1 \in \langle \mathcal{k}_1, \partial\rangle$ and $\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_2 \in \langle \mathcal{k}_2, \partial\rangle$ and these two NS b-open sets are guaranteedly mutually exclusive because the possibility of one rules out the possibility of other. Since $f^{-1}(\langle \mathcal{k}_1, \partial\rangle)$ and $f^{-1}(\langle \mathcal{k}_2, \partial\rangle)$ are NS open in $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle. Now, f^{-1}(\langle \mathcal{k}_1, \partial\rangle) \tilde{\cap} f^{-1}(\langle \mathcal{k}_1, \partial\rangle) = f^{-1}(\langle \mathcal{k}_1, \partial\rangle \tilde{\cap} \langle \mathcal{k}_2, \partial\rangle) = f^{-1}(\tilde{\emptyset}) =$ $0_{(\widetilde{(\langle\mathcal{X}\rangle, \partial parameter)})}\ and\ \left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_1 \in \langle \mathcal{k}_1, \partial\rangle \Rightarrow f^{-1}(\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_1) \in f^{-1}(\langle \mathcal{k}_1, \partial\rangle) \Rightarrow$ $\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)_1 \in (\langle \mathcal{k}_1, \partial\rangle), \left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_2 \in \langle \mathcal{k}_2, \partial\rangle \Rightarrow f^{-1}(\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)_2) \in$ $f^{-1}(\langle \mathcal{k}_2, \partial\rangle)$ implies that $\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)_2 \in (\langle \mathcal{k}_2, \partial\rangle)$. We see that it has been shown for every pair of distinct points of $\mathcal{X}^{crip}$, there suppose disjoint NS b-open sets namely, $f^{-1}(\langle \mathcal{k}_1, \partial\rangle)$ and $f^{-1}(\langle \mathcal{k}_2, \partial\rangle)$ belong to $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ such that $\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)_1 \in f^{-1}(\langle \mathcal{k}_1, \partial\rangle)$ and $\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)_2 \in f^{-1}(\langle \mathcal{k}_2, \partial\rangle)$. Accordingly, $NSST$ is $NS\ b$ Hausdorff space.

Theorem 6.3. Let$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ be $NSST$ and $\langle Y^{crip}, \mathfrak{F}, \partial\rangle$ be an-other $NSST$.Let$\langle f, \partial\rangle$:$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \longrightarrow \langle Y^{crip}, \mathfrak{F}, \partial\rangle$ be a soft mapping such that it is continuous mapping. Let $\langle Y^{crip}, \mathfrak{T}, \partial\rangle$ is NSb Hausdorff space Then it is guaranteed that $\left\{\left(\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right), \left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)\right) : f\left(\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)\right) = f\left(\left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)\right)\right\}$ is a NSb closed sub-set of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \times \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$.

Proof: Given that$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ be $NSST$ and $\langle Y^{crip}, \mathfrak{F}, \partial\rangle$ be an-other $NSST$.Let$\langle f, \partial\rangle$:$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \longrightarrow \langle Y^{crip}, \mathfrak{F}, \partial\rangle$ be a soft mapping such that it is continuous mapping. $\langle Y^{crip}, \mathfrak{T}, \partial\rangle$ is NSb Hausdorff space Then we $will\ prove\ that$ $\left\{\left(\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right), \left(\mathcal{y}^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)\right) : f\left(\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)\right) =$

$\not{f}\left(\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right)\}$ is a NSb closed sub-set of $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle \times \langle Y^{crip},\mathfrak{T},\partial\rangle$. Equavilintly, we will prove that $\left\{\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right),\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right):\not{f}\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\right)=\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right\}^{c}$ is NS b-open sub-set of $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle \times \langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle$. Let $\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right),\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right)\in$

$\left\{\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right),\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right) with \left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\succ\right.$

$\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right):\not{f}\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\right)\succ$

$\left.\not{f}\left(\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right)\right\}^{c}$ *or* $\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right),\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right)\in$

$\left\{\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right),\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right) with \left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\prec\right.$

$\left.\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right):\not{f}\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\right)\prec\not{f}\left(\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right)\right\}^{c}$ *.Then,*

$\not{f}\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\right)\succ\not{f}\left(\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right) or \prec\not{f}\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\right)\prec$

$\not{f}\left(\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right)$ accordingly. Since, $\langle Y^{crip},\mathfrak{F},\partial\rangle$ is NSb Hausdorff space. Definitely, $\not{f}\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\right),\not{f}\left(\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right)$ are points of $\langle Y^{crip},\mathfrak{F},\partial\rangle$, there exists NS b-open sets $\langle\mathcal{G},\partial\rangle,\langle \hslash,\partial\rangle\in\langle Y^{crip},\mathfrak{F},\partial\rangle$ such that $\not{f}\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\right)\in\langle\mathcal{G},\partial\rangle\&\not{f}\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\right)\in\langle \hslash,\partial\rangle$ provided $\langle\mathcal{G},\partial\rangle\,\tilde{\cap}\,\langle \hslash,\partial\rangle=0_{\widetilde{(\langle\mathcal{X}\rangle,\text{ⅎ}^{parameter})}_{Y}}$. Since, $\langle\not{f},\partial\rangle$ is soft continuous, $\not{f}^{-1}(\langle\mathcal{G},\partial\rangle\&\not{f}^{-1}(\langle \hslash,\partial\rangle$ are NS b-open sets in $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ supposing $\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)$ and $\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)$ respectively and so $\not{f}^{-1}(\langle\mathcal{G},\partial\rangle\times\not{f}^{-1}(\langle\mathcal{H},\partial\rangle$ is basic NS b-open set in $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle\times\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ containing $\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right),\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right)$. Since $\langle\mathcal{G},\partial\rangle\,\tilde{\cap}\,\langle \hslash,\partial\rangle=0_{\widetilde{(\langle\mathcal{X}\rangle,\text{ⅎ}^{parameter})}_{Y}}$, it is clear by the definition of $\left\{\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right),\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right):\not{f}\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\right)=\not{f}\left(\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right)\right\}$ that

$\{\not{f}^{-1}(\langle\mathcal{G},\partial\rangle\&\not{f}^{-1}(\langle \hslash,\partial\rangle\}\,\tilde{\cap}\left\{\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right),\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right):\not{f}(x)=\right.$

$\left.\not{f}\left(\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right)\right\}=0_{(\widetilde{\langle\mathcal{X}\rangle},\text{ⅎ}^{parameter})}$, *that is* $\not{f}^{-1}(\langle\mathcal{G},\partial\rangle\times\not{f}^{-1}(\langle \hslash,\partial\rangle\Subset$

$\left\{\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right),\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right):\not{f}\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\right)=\not{f}\left(\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right)\right\}^{c}$.

Hence, $\left\{\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right),\left(y^{e'}{}_{(a',b',c')},\text{ⅎ}^{parameter}\right)\right):\not{f}\left(\left(x^{e}{}_{(a,b,c)},\text{ⅎ}^{parameter}\right)\right)=\right.$

$\mathcal{f}\left(\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)\Big\}^{c}$ *implies that* $\Big\{\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right),\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right):\mathcal{f}\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\right)=$

$\mathcal{f}\left(\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)\Big\}$is NS b-closed.

7. Mixed NS b-Separation Axioms

Theorem 7.1. Let$\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ be $(NSSTS)$ and $\langle Y^{crip},\mathfrak{F},\partial\rangle$ be an-other$(NSSTS)$. Let$\langle \mathcal{f},\partial\rangle:\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle\longrightarrow\langle Y^{crip},\mathfrak{F},\partial\rangle$ be NSb open mapping such that it is onto. If the soft set $\Big\{\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right),\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right):\mathcal{f}\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\right)=\mathcal{f}\left(\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)\Big\}$ is NS b-closed in $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle\times\langle Y^{crip},\mathfrak{F},\partial\rangle$, then $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ will behave as NSb Hausdorff space.

Proof: Suppose $\mathcal{f}\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\right),\mathcal{f}(y)$ be two points of Y^{crip} such that either $\mathcal{f}\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\right)\succ\mathcal{f}\left(\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)\,or\,\prec\mathcal{f}\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\right)\prec$

$\mathcal{f}\left(\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right).Then\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right),\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)\notin$

$\Big\{\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right),\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)with\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\succ$

$\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right):\mathcal{f}\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\right)\succ$

$\mathcal{f}\left(\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)\Big\}\,or\,\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right),\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)\notin$

$\Big\{\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right),\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)with\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\prec$

$\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right):\mathcal{f}\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\right)\prec\mathcal{f}\left(\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)\Big\}$, that is

$\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right),\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)\in$

$\Big\{\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right),\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)with\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\succ$

$\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right):\mathcal{f}\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\right)\succ$

$\mathcal{f}\left(\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)\Big\}^{c}\,or\,\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right),\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)\in$

$\Big\{\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right),\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)\right)with\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\prec$

$\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right):\mathcal{f}\left(\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)\right)\prec$

$f\left(\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)\Big\}^{c}. Since, \left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right),\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)\in$

$\Big\{\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter},y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right) with \left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\succ$

$\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right): f\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\right)\succ$

$f\left(\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)\Big\}^{c}$ or $\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right),\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)\in$

$\Big\{\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right),\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right) with \left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\prec$

$\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right): f\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\right)\prec f\left(\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)\Big\}^{c}$ is soft in $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle\times$

$\langle Y^{crip},\mathfrak{F},\partial\rangle, then\ \exists NS\ b-open\ sets$

$\langle \mathcal{G},\partial\rangle\ and\ \langle k,\partial\rangle\ in\ \langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle s.t. \left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right),\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\in$

$\langle \mathcal{G},\partial\rangle\times\langle k,\partial\rangle \Subset \Big\{\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right),\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right) with \left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\succ$

$\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right): f\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\right)\succ$

$f\left(\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)\Big\}^{c}$ or $\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right),\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)\in\langle \mathcal{G},\partial\rangle\times\langle k,\partial\rangle\Subset$

$\Big\{\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right),\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right) with \left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\prec$

$\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right): f\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\right)\prec f\left(\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)\Big\}^{c}$. Then, since f is NS b-open, $f(\langle \mathcal{G},\partial\rangle)$ and $f(\langle k,\partial\rangle)$ are NS b-open sets in $\langle Y^{crip},\mathfrak{F},\partial\rangle$ containing $f\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\right)$ and $f\left(\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)$ respectively,and $f(\langle \mathcal{G},\partial\rangle)\ \widetilde{\cap}\ f(\langle k,\partial\rangle)=0_{(\widetilde{\langle \mathcal{X}\rangle,\text{Ⅎ}^{parameter}})}$ otherwise $f(\langle \mathcal{G},\partial\rangle)\times$

$f(\langle k,\partial\rangle)\ \widetilde{\cap}\ \Big\{\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right),\right) with \left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\succ$

$\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right): f\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\right)\succ$

$f(y)\Big\} or \left(\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right),\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)\right)\notin$

$\Big\{\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right),\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right) with \left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\prec$

$\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right): f\left(\left(x^{e}_{(a,b,c)},\text{Ⅎ}^{parameter}\right)\right)\prec f\left(\left(y^{e'}_{(a',b',c')},\text{Ⅎ}^{parameter}\right)\right)\Big\}=0_{(\widetilde{\langle \mathcal{X}\rangle,\text{Ⅎ}^{parameter}})}$. It follows that $\langle Y^{crip},\mathfrak{F},\partial\rangle$ is NSb Hausdorff space.

Theorem 7.2. Let$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a NS second countable space then it is guaranteed that every family of non-empty dis-joint NS b-open subsets of a NS second countable space $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is NSb countable.

Proof: Given that $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a NS second countable space.

Then, $\exists$ a NS countable base$\mathfrak{B} = \langle \mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3, \mathcal{B}^4, \ldots\ldots \mathcal{B}^n : n \in \mathbb{N} \rangle for\ \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Let $\langle \mathcal{C}, \partial \rangle$be a family of non-vacuous mutually exclusive NS b-open sub-sets of$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Then, for each $\langle f, \partial \rangle$ of in $\langle \mathcal{C}, \partial \rangle \exists$ a soft $\mathcal{B}^n \in \mathfrak{B}$ in such a way that$\mathcal{B}^n \Subset \langle f, \partial \rangle$. Let us attach with$\langle f, \partial \rangle$, the smallest positive interger n such that$\mathcal{B}^n \Subset \langle f, \partial \rangle$. Since the candidates of $\langle \mathcal{C}, \partial \rangle$ are mutully exclusive because of this behavior distinct candidates will be associated with distinct positive integers. Now, if we put the elements of $\langle \mathcal{C}, \partial \rangle$ in order so that the order is increasing relative to the positive integers associated with them, we obtain a sequence of of candidates of $\langle \mathcal{C}, \partial \rangle$. This verifies that $\langle \mathcal{C}, \partial \rangle$ is NS countable.

Theorem 7.3. Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a NS second countable space and let $\langle f, \partial \rangle$ be NS uncountable suset of$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Then, at least one point of $\langle f, \partial \rangle$ is a soft limit point of$\langle f, \partial \rangle$.

Proof: Let $\mathfrak{B} = \langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \ldots\ldots \mathcal{B}_n : n \in \mathbb{N} \rangle\ for\ \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$.Let, if possible, no point of $\langle f, \partial \rangle$ be a soft $limitpoint\ of\ \langle f, \partial \rangle. Then, for\ each\ \left(x^e{}_{(a,b,c)}, d^{parameter}\right) \in \langle f, \partial \rangle \exists NSb - open\ set$

$\langle \rho, \partial \rangle_{\left(x^e{}_{(a,b,c)}, d^{parameter}\right)} such\ that \left(x^e{}_{(a,b,c)}, d^{parameter}\right) \in \langle \rho, \partial \rangle_{\left(x^e{}_{(a,b,c)}, d^{parameter}\right)}$ and

$\langle \rho, \partial \rangle_{\left(x^e{}_{(a,b,c)}, d^{parameter}\right)} \tilde{\cap} \langle f, \partial \rangle = \left\{\left(x^e{}_{(a,b,c)}, d^{parameter}\right)\right\}$. Since $\mathfrak{B}$ is soft base $\exists \mathcal{B}_{n\left(x^e{}_{(a,b,c)}, d^{parameter}\right)} \in \mathfrak{B}$ such that $\left(x^e{}_{(a,b,c)}, d^{parameter}\right) \in \mathcal{B}_{n\left(x^e{}_{(a,b,c)}, d^{parameter}\right)} \Subset \langle \rho, \partial \rangle_{\left(x^e{}_{(a,b,c)}, d^{parameter}\right)}. Therefore, \mathcal{B}_{n\left(x^e{}_{(a,b,c)}, d^{parameter}\right)} \tilde{\cap} \langle f, \partial \rangle \Subset \langle \rho, \partial \rangle_{\left(x^e{}_{(a,b,c)}, d^{parameter}\right)} \tilde{\cap} \langle f, \partial \rangle = \left\{\left(x^e{}_{(a,b,c)}, d^{parameter}\right)\right\}. More -$

$over, if \left(x^e{}_{(a,b,c)}, d^{parameter}\right)_1 and \left(x^e{}_{(a,b,c)}, d^{parameter}\right)_2$ be any two NS points such that $\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_1 \neq \left(x^e{}_{(a,b,c)}, d^{parameter}\right)_2$ which means either $\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_1 \succ \left(x^e{}_{(a,b,c)}, d^{parameter}\right)_2$ or $\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_1 \prec \left(x^e{}_{(a,b,c)}, d^{parameter}\right)_2$ then $\exists \mathcal{B}_{n\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_1}$ and $\mathcal{B}_{n\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_2}$ in $\mathfrak{B}$ such that $\mathcal{B}_{n\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_1} \tilde{\cap} \langle f, \partial \rangle = \left\{\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_1\right\}$ and $\mathcal{B}_{n\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_2} \tilde{\cap} \langle f, \partial \rangle = \left\{\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_2\right\}$.

Now, $\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_1 \neq \left(x^e{}_{(a,b,c)}, d^{parameter}\right)_2$ which guarantees that $\left\{\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_1\right\} \neq \left\{\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_2\right\}$ which $\Longrightarrow$

$\mathcal{B}_{n\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_1} \tilde{\cap} \langle f, \partial \rangle \neq \mathcal{B}_{n\left(x^e{}_{(a,b,c)}, d^{parameter}\right)_2} \tilde{\cap} \langle f, \partial \rangle which\ imlplies\ \mathcal{B}_{\mathrm{n}\left(x^e{}_{(a,b,c)}, d^{\mathrm{parameter}}\right)_1} \neq \mathcal{B}_{\mathrm{n}\left(x^e{}_{(a,b,c)}, d^{\mathrm{parameter}}\right)_2}$. Thus, $\exists$ a one to one soft correspondence of $\langle f, \partial \rangle$ on to $\left\{\mathcal{B}_{\mathrm{n}\left(x^e{}_{(a,b,c)}, d^{\mathrm{parameter}}\right)} : \left(x^{\mathrm{e}}{}_{(a,b,c)}, d^{\mathrm{parameter}}\right) \in \langle f, \partial \rangle\right\}$. Now, $\langle f, \partial \rangle$ being NS uncountable,it follows that $\left\{\mathcal{B}_{\mathrm{n}\left(x^e{}_{(a,b,c)}, d^{\mathrm{parameter}}\right)} : \left(x^{\mathrm{e}}{}_{(a,b,c)}, d^{\mathrm{parameter}}\right) \in \langle f, \partial \rangle\right\}$ is NS uncountable.But,this is purely a contradiction, since $\left\{\mathcal{B}_{\mathrm{n}\left(x^e{}_{(a,b,c)}, d^{\mathrm{parameter}}\right)} : \left(x^{\mathrm{e}}{}_{(a,b,c)}, d^{\mathrm{parameter}}\right) \in \langle f, \partial \rangle\right\}$ benig a NS sub-family of the NS countable collection $\mathfrak{B}$. This contradiction is taking birth that on point of $\langle \mathscr{f}, \partial \rangle$ is a soft limit point of $\langle \mathscr{f}, \partial \rangle$,so at least one point of $\langle \mathscr{f}, \partial \rangle$is a soft limit point of $\langle \mathscr{f}, \partial \rangle$.

Theorem 7.4. Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle NSSTS$ such that is is NS countably compact then this space has the characteristics of Bolzano Weirstrass Property(BWP).

Proof: Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a NS countably compact space and suppose, if possible,that it negate the Bolzano Weierstrass Property(BWP). Then there must exists an infinite NS set $\langle \mathscr{f}, \partial \rangle$supposing no soft limit point. Further suppose $\langle \rho, \partial \rangle$ be soft countably infinite soft sub-set $\langle \mathscr{f}, \partial \rangle$ that is$\langle \rho, \partial \rangle \Subset \langle \mathscr{f}, \partial \rangle$. Then this guarantees $\langle \rho, \partial \rangle$ has no soft limit poit. It follows that $\langle \rho, \partial \rangle$ is NSb closed set. Also for each $\left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\mathrm{Ɥ}^{parameter}}\right)_n \in \langle \rho, \partial \rangle$, $\left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\mathrm{Ɥ}^{parameter}}\right)_n$is not a soft limit point of $\langle \rho, \partial \rangle$. Hence there exists NS b-open set $\langle \mathcal{G}_n, \partial \rangle$,such that $\left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\mathrm{Ɥ}^{parameter}}\right)_n \in \langle \mathcal{G}_n, \partial \rangle$ and $\langle \mathcal{G}_n, \partial \rangle \tilde{\cap} \langle \rho, \partial \rangle = \left\{\left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\mathrm{Ɥ}^{parameter}}\right)_n\right\}$. The the collection $\{\langle \mathcal{G}_n, \partial \rangle : n \in N\} \tilde{\cap} \langle \rho, \partial \rangle^c$ is countable NS b-open cover of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$.this soft cover has no finite sub-cover. For this we exhaust a single $\langle \mathcal{G}_n, \partial \rangle$, it would not be a soft cover of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ since then $\langle \left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\mathrm{Ɥ}^{parameter}}\right)_n \rangle$ would be covered. Result in $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$is not NS countably compact. But this contradicts the given. Hence, we are compelled to accept $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ must have Bolzano Weierstrass Property.

Theorem 7.5. Let$\left(\mathcal{X}^{crisp^1}, \mathfrak{F}, \mathcal{A}\right)$ and $\left(\mathcal{X}^{crisp^2}, \boldsymbol{\tau}, \mathcal{A}\right)$ be two $NSSTS$ and suppose $\langle \mathrm{f}, \partial \rangle$ be a NS continuous function such that $\langle \mathrm{f}, \partial \rangle : \left(\mathcal{X}^{crisp^1}, \mathfrak{F}, \mathcal{A}\right) \longrightarrow \left(\mathcal{X}^{crisp^2}, \boldsymbol{\tau}, \mathcal{A}\right)$ is NS continuous function and let $\langle \mathcal{L}, \partial \rangle \Subset \left(\mathcal{X}^{crisp^1}, \mathfrak{F}, \mathcal{A}\right)$ supposes the $B.V.P.$ then safely $\mathrm{f}(\langle \mathcal{L}, \partial \rangle)$has the $B.V.P.$

Proof: Suppose $\langle \mathcal{L}, \partial \rangle$ be an infinite NS sub-set of $\langle \mathrm{f}, \partial \rangle$, so that $\langle \mathcal{L}, \partial \rangle$ contains an enumerable NS set $\langle \left(x^{e}{}_{(\mathrm{a,b,c})}, \mathrm{Ɥ}^{parameter}\right)_n : n \in N \rangle$ then there exists enumerable NS set $\langle \left(\mathcal{Y}^{e'}{}_{(\mathrm{a}',\mathrm{b}',\mathrm{c}')}, \mathrm{Ɥ}^{parameter}\right)_n : n \in N \rangle \Subset \langle \mathcal{L}, \partial \rangle$s.t. $\mathrm{f}\left(\left(\mathcal{Y}^{e'}{}_{(\mathrm{a}',\mathrm{b}',\mathrm{c}')}, \mathrm{Ɥ}^{parameter}\right)_n\right) = \left(x^{e}{}_{(\mathrm{a,b,c})}, \mathrm{Ɥ}^{parameter}\right)_n$. $\langle \mathcal{L}, \partial \rangle$ *has* $B.V.P. \Rightarrow$every infinite soft subset of $\langle \mathcal{L}, \partial \rangle$ supposes soft accumulation poit belonging to $\langle \mathcal{L}, \partial \rangle. \Rightarrow \langle \left(\mathcal{Y}^{e'}{}_{(\mathrm{a}',\mathrm{b}',\mathrm{c}')}, \mathrm{Ɥ}^{parameter}\right)_n : n \in N \rangle$ has soft neutrosophic limit poit,say, $\left(\mathcal{Y}^{e'}{}_{(\mathrm{a}',\mathrm{b}',\mathrm{c}')}, \mathrm{Ɥ}^{parameter}\right)_0 \in \langle \mathcal{L}, \partial \rangle. \Rightarrow$ the limit of soft sequence $\langle \left(\mathcal{Y}^{e'}{}_{(\mathrm{a}',\mathrm{b}',\mathrm{c}')}, \mathrm{Ɥ}^{parameter}\right)_n : n \in N \rangle$ is $\left(\mathcal{Y}^{e'}{}_{(\mathrm{a}',\mathrm{b}',\mathrm{c}')}, \mathrm{Ɥ}^{parameter}\right)_0 \in \langle \mathcal{L}, \partial \rangle \Rightarrow \left(\mathcal{Y}^{e'}{}_{(\mathrm{a}',\mathrm{b}',\mathrm{c}')}, \mathrm{Ɥ}^{parameter}\right)_n \longrightarrow \left(\mathcal{Y}^{e'}{}_{(\mathrm{a}',\mathrm{b}',\mathrm{c}')}, \mathrm{Ɥ}^{parameter}\right)_0 \in \langle \mathcal{L}, \partial \rangle$. f is soft continuous $\Rightarrow$ it is soft continuous. Furthermore $\left(\mathcal{Y}^{e'}{}_{(\mathrm{a}',\mathrm{b}',\mathrm{c}')}, \mathrm{Ɥ}^{parameter}\right)_n \longrightarrow \left(\mathcal{Y}^{e'}{}_{(\mathrm{a}',\mathrm{b}',\mathrm{c}')}, \mathrm{Ɥ}^{parameter}\right)_0 \in \langle \mathcal{L}, \partial \rangle \Rightarrow \mathrm{f}(\left(\left(\mathcal{Y}^{e'}{}_{(\mathrm{a}',\mathrm{b}',\mathrm{c}')}, \mathrm{Ɥ}^{parameter}\right)_n\right)) \longrightarrow$

$\mathfrak{f}(\left(\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_0\right)) \in \mathfrak{f}(\langle \mathcal{L}, \partial\rangle) \Rightarrow \left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)_n \to \mathfrak{f}(\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_0) \in$ $\mathfrak{f}(\langle \mathcal{L}, \partial\rangle) \Rightarrow$ limit of a soft sequence $\langle\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)_n : n \in N\rangle$ is $\mathfrak{f}(\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_0) \in$ $\mathfrak{f}(\langle \mathcal{L}, \partial\rangle) \Rightarrow$limit of a soft sequence $\langle\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)_n : n \in N\rangle$ *is* $\mathfrak{f}(\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_0) \in \langle \mathfrak{f}, \partial\rangle(\langle \mathcal{L}, \partial\rangle)$ .Finally we have shown that there exists infinite soft subset$\langle\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)_n : n \in N\rangle$ of $\mathfrak{f}(\langle \mathcal{L}, \partial\rangle)$ containing a limit point $\mathfrak{f}\left(\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_0\right) \in \mathfrak{f}(\langle \mathcal{L}, \partial\rangle)$. This guarantees that $\mathfrak{f}(\langle \mathcal{L}, \partial\rangle)$ has $B.V.P$.

Theorem 7.6. Let$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ be a $NSSTS$ so that $(i)\langle \mathcal{F}, \partial\rangle$ is NSb compact soft sub-set of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ and $\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)$ be a crisp point in $\mathcal{X}^{crip}$ such that $\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)$ can be strongly separated from every point $\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)$in $\langle \mathcal{F}, \partial\rangle$,then it is guaranteed that $\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)$ and $\langle \mathcal{F}, \partial\rangle$ can also be soft strongly separated in $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$. (ii) if $\langle \wp, \partial\rangle$ and $\langle \mathcal{F}, \partial\rangle$ are two NSb compact soft sub-sets of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ such that every point $\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)$ in $\langle \wp, \partial\rangle$ can be strongly separated from every point $\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)$in $\langle \mathcal{F}, \partial\rangle$, then it is guaranteed that $\langle \wp, \partial\rangle$ and $\langle \mathcal{F}, \partial\rangle$ ca n be strongly separated in $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$.

Proof i) Let $\langle \mho, \partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)}\left(\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)\right)$ and $\langle \mathcal{L}, \partial\rangle_{\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)}\left(\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)\right)$ separate strongly the point$x$ from a point$\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right) \in \langle \mathcal{F}, \partial\rangle$. As $\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)$ runs\rush over $\langle \mathcal{F}, \partial\rangle$, the corresponding NS sets $\langle \mathcal{L}, \partial\rangle_{\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)}\left(\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)\right)$ form NS b-open covering of$\langle \mathcal{F}, \partial\rangle$, for which there exists a finite soft sub-covering, $\langle \mathcal{L}, \partial\rangle_{\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)}\left(\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_1\right), \ldots,$ $\langle \mathcal{L}, \partial\rangle_{\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)}\left(\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_n\right)$, *say, since* $\langle \mathcal{F}, \partial\rangle$ *is NSb compact. Let*

$$\langle \mho, \partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_3}\left(\langle \mho, \partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_1}, \langle \mho, \partial\rangle_{y_n}\right)$$
$$\langle \mho, \partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_1}, \ldots \langle \mho, \partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_n} \langle \mho, \partial\rangle_{y_4}\left(\langle \mho, \partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_1}, \ldots \langle \mho, \partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_n}\right),$$
$$\langle \mho, \partial\rangle_{y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}{}_5}\left(\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)\right)$$

be the corresponding NSb open sets supposing the point $\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)$.

$$Let\ \langle \mho, \partial\rangle_{\langle \mathcal{F}, \partial\rangle}\left(\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)\right) = \langle \mho, \partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{Ⅎ}^{parameter}\right)_1}\left(\left(x^e{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)\right) \widetilde{\cap}, \ldots \ldots \ldots$$

$$.\widetilde{\cap}\ \langle \mho,\partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)},\text{parameter}\right)_1}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)\right) and \langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)}(\langle \mathcal{F},\partial\rangle)$$

$$= \langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)}\left(\left(y^{e/}{}_{(a/,b/,c/)},\text{parameter}\right)_1\right)\widetilde{\cup}$$

$$\langle V,\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)}\left(\left(y^{e/}{}_{(a/,b/,c/)},\text{parameter}\right)_2\right)\widetilde{\cup}\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)\left(\left(y^{e/}{}_{(a/,b/,c/)},\text{parameter}\right)_3\right)\widetilde{\cup}\ldots$$

$\widetilde{\cup}\ \langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)}\left(\left(y^{e/}{}_{(a/,b/,c/)},\text{parameter}\right)_n\right). Then \left(x^{e}{}_{(a,b,c)},\text{parameter}\right) \in$

$\langle \mho,\partial\rangle_{\langle \mathcal{F},\partial\rangle}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)\right) and \langle \mathcal{F},\partial\rangle \Subset \langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)}(\langle \mathcal{F},\partial\rangle)$. Also, since $\langle \mho,\partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)},\text{parameter}\right)_i}\left(\left(y^{e/}{}_{(a/,b/,c/)},\text{parameter}\right)\right)\widetilde{\cap}\ \langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)}\left(\left(y^{e/}{}_{(a/,b/,c/)},\text{parameter}\right)_i\right) = 0_{\left(\langle\widetilde{\mathcal{X}}\rangle,\text{parameter}\right)}$, $and\ \ \langle \mho,\partial\rangle_{\langle \mathcal{F},\partial\rangle}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)\right) \Subset \langle \mho,\partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)},\text{parameter}\right)_i}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)\right), for\ i = 1,2,3,4,\ldots,n, it\ follows\ that\ \langle \mho,\partial\rangle_{\langle \mathcal{F},\partial\rangle}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)\right)\widetilde{\cap}\ \langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)}(\langle \mathcal{F},\partial\rangle) = 0_{\left(\langle\widetilde{\mathcal{X}}\rangle,\text{parameter}\right)}$.

Thus $\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)$ and $\langle \mathcal{F},\partial\rangle$ are separated strongly by the pair of disjoint NSb open sets $\langle \mho,\partial\rangle_{\langle \mathcal{F},\partial\rangle}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)\right)$ and $\langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)}(\langle \mathcal{F},\partial\rangle)$ in $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle$.

(***ii***) Suppose $\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)$ runs over $\langle \wp,\partial\rangle$,then corresponding soft sets $\langle \mho,\partial\rangle_{\langle \mathcal{F},\partial\rangle}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)\right)$ generate soft covering of $\langle \wp,\partial\rangle$, for which there exists a finite soft sub-covering

$$\left\{\begin{array}{l}\langle \mho,\partial\rangle_{\langle \mathcal{F},\partial\rangle}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_1\right),\\ \langle \mho,\partial\rangle_{\langle \mathcal{F},\partial\rangle}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_2\right),\\ \ldots,\langle \mho,\partial\rangle_{\langle \mathcal{F},\partial\rangle}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_m\right)\end{array}\right\}, say, for \langle \wp,\partial\rangle (since\ \langle \wp,\partial\rangle\ is\ soft\ NSb\ compact). Let$$

$\langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_1}(\langle \mathcal{F},\partial\rangle), \langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_2}(\langle \mathcal{F},\partial\rangle),\ldots$

$\langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_m}(\langle \mathcal{F},\partial\rangle)$ be the corresponding NSb open sets containing $\langle \mathcal{F},\partial\rangle$. Then $\langle \mho,\partial\rangle(\langle \wp,\partial\rangle) =$

$$\left\{\begin{array}{l}\langle \mho,\partial\rangle_{\langle \mathcal{F},\partial\rangle}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_1\right)\widetilde{\cup}\\ \langle \mho,\partial\rangle_{\langle \mathcal{F},\partial\rangle}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_2\right)\widetilde{\cup},\ldots\\ ,\widetilde{\cup}\ \langle \mho,\partial\rangle_{\langle \mathcal{F},\partial\rangle}\left(\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_m\right)\end{array}\right\}, \text{and } \langle \mathcal{L},\partial\rangle\left((\langle \mathcal{F},\partial\rangle)\right) = \langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_1}(\langle \mathcal{F},\partial\rangle)$$

$$\widetilde{\cap}\ \langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_2}(\langle \mathcal{F},\partial\rangle)\ \widetilde{\cap},\ldots$$

$\widetilde{\cap}\ \langle \mathcal{L},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},\text{parameter}\right)_m}(\langle \mathcal{F},\partial\rangle)$are two disjoint NS open sets ,(as in(i),which separate $\langle \wp,\partial\rangle$ and $\langle \mathcal{F},\partial\rangle$ strongly.

Theorem 7.7. Let$\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ be a $NSSTS$ and $\langle \mathfrak{f}^{crip},\mathfrak{H},\partial\rangle$ be NS sub-space of $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle$. The necessary and sufficient condition for$\langle \mathfrak{f}^{crip},\partial\rangle$ to be NS compact relative to $\langle \mathfrak{f}^{crip},\mathfrak{T},\partial\rangle$ is that $\langle \mathfrak{f}^{crip},\partial\rangle$ is NS compact relative to $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle$.

Proof: First we prove that $\langle \mathfrak{f}^{crip},\partial\rangle$ relative to $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle$. Let $\{\langle \mathfrak{k},\partial\rangle_i : i \in I\}$ that is $\{\langle \mathfrak{k},\partial\rangle_1,\langle \mathfrak{k},\partial\rangle_2,\langle \mathfrak{k},\partial\rangle_3,\langle \mathfrak{k},\partial\rangle_4,\ldots\}$ be $\langle \mathcal{X}^{crip},\mathfrak{T},\partial\rangle - NSb$ open cover of $\langle \mathfrak{f}^{crip},\partial\rangle$,then$\langle \mathfrak{f}^{crip},\partial\rangle \Subset \widetilde{\cup}_i\ \langle \mathfrak{k},\partial\rangle_i$. $\langle \mathfrak{k},\partial\rangle_i \in$

$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \Rightarrow \exists\langle g, \partial\rangle_i \in \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle s.t. \langle \mathfrak{k}, \partial\rangle_i = \langle g, \partial\rangle_i \tilde{\cap} \langle \mathfrak{f}^{crip}, \partial\rangle \Subset \langle g, \partial\rangle_i \Rightarrow \exists\langle g, \partial\rangle_i \in$ $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle s.t. \langle \mathfrak{k}, \partial\rangle_i \Subset \langle g, \partial\rangle_i \Rightarrow \tilde{\cup}_\iota \langle \mathfrak{k}, \partial\rangle_i \Subset \tilde{\cup}_\iota$ but $\langle \mathfrak{f}^{crip}, \partial\rangle \Subset \langle \mathfrak{k}, \partial\rangle_i$. So that $\langle \mathfrak{f}^{crip}, \partial\rangle \Subset \tilde{\cup}_\iota \langle \mathfrak{k}, \partial\rangle_i$. This guarantees that $\{\langle g, \partial\rangle_i : i \in I\}$is a $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle - NSb$ open cover of $\langle \mathfrak{f}^{crip}, \partial\rangle$ which is known to be NSb compact relative $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ and hence the soft cover $\{\langle g, \partial\rangle_i : i(x^{e}{}_{(a,b,c)}, \text{Ꟊ}^{parameter}) I\}$ must be freezable to a finite soft sub cover, say,

$\{\langle g, \partial\rangle_{ir} : r = 1,2,3,4, \dots, n\}$,Then $\langle \mathfrak{f}^{crip}, \partial\rangle \Subset \cup_{r=1}^{n} \widetilde{\langle G, \partial\rangle}_{\iota r} \Rightarrow$

$$\langle \mathfrak{f}^{crip}, \partial\rangle \tilde{\cap} \langle \mathfrak{f}^{crip}, \partial\rangle \Subset \langle \mathfrak{f}^{crip}, \partial\rangle \tilde{\cap} \left[\bigcup_{r=1}^{n} \widetilde{\langle G, \partial\rangle_{\iota r}}\right]$$

$= \widetilde{\cup_{r=1}^{n}}(\langle \mathfrak{f}^{crip}, \partial\rangle \tilde{\cap} \langle g, \partial\rangle_{ir} = \cup_{r=1}^{n} \widetilde{\langle \mathfrak{k}, \partial\rangle}_{\iota r}\ or \langle \mathfrak{f}^{crip}, \partial\rangle \Subset \cup_{r=1}^{n} \widetilde{\langle \mathfrak{k}, \partial\rangle}_{\iota r} \Rightarrow \{\langle \mathfrak{k}, \partial\rangle_{ir} : 1 \le r \le n\}$ is a $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle - NS$ open cover of $\langle \mathfrak{f}^{crip}, \partial\rangle$. Steping from an arbitrary $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ −open cover of $\langle \mathfrak{f}^{crip}, \partial\rangle$, we are able to show that the NS cover is freezable to a finite soft subcover$\{\langle \mathfrak{k}, \partial\rangle_{ir} : 1 \le r \le n\}$ of $\langle \mathfrak{f}^{crip}, \partial\rangle$, meaning there by $\langle \mathfrak{f}^{crip}, \partial\rangle$ is $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle - NSbcompact$. The condition is sufficient: Suppose $\langle \mathfrak{f}^{crip}, \mathfrak{H}, \partial\rangle$ be soft sub-space of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ and also $\langle \mathfrak{f}^{crip}, \partial\rangle$ is $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle - NSb\ compact$. We have to prove that $\langle \mathfrak{f}^{crip}, \partial\rangle$ is $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle - NScompact$. Let $\{\langle \mathfrak{k}, \partial\rangle_1, \langle \mathfrak{k}, \partial\rangle_2, \langle \mathfrak{k}, \partial\rangle_3, \langle \mathfrak{k}, \partial\rangle_4, \dots\}$ be soft $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle - NS$ b-open cover of $\langle \mathfrak{f}^{crip}, \partial\rangle$, so that $\langle \mathfrak{f}^{crip}, \partial\rangle \Subset \tilde{\cup}_\iota \langle g, \partial\rangle_i$ from which $\langle \mathfrak{f}^{crip}, \partial\rangle \tilde{\cap} \langle \mathfrak{f}^{crip}, \partial\rangle \Subset \langle \mathfrak{f}^{crip}, \partial\rangle \tilde{\cap} (\tilde{\cup}_\iota \langle g, \partial\rangle_i) or, \langle \mathfrak{f}^{crip}, \partial\rangle \Subset \tilde{\cup}_\iota (\langle \mathfrak{f}^{crip}, \partial\rangle \tilde{\cap} \langle g, \partial\rangle_i)$. On taking $\langle \mathfrak{k}, \partial\rangle_i = \langle g, \partial\rangle_i \tilde{\cap} \langle \mathfrak{f}^{crip}, \partial\rangle$, we get $\langle \mathfrak{f}^{crip}, \partial\rangle \Subset \tilde{\cup} \langle \mathfrak{k}, \partial\rangle_i, \langle g, \partial\rangle_i \in \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \Rightarrow \langle \mathfrak{k}, \partial\rangle_i =$ $\langle g, \partial\rangle_i \tilde{\cap} \langle \mathfrak{f}^{crip}, \partial\rangle \in \langle \mathfrak{f}^{crip}, \mathfrak{H}, \partial\rangle \dots (1)$.Now from (1) it is clear that $\{\langle \mathfrak{k}, \partial\rangle_1, \langle \mathfrak{k}, \partial\rangle_2, \langle \mathfrak{k}, \partial\rangle_3, \langle \mathfrak{k}, \partial\rangle_4, \dots\}$ is $\langle \mathfrak{f}^{crip}, \mathfrak{H}, \partial\rangle -$ $NSopen$ soft cover of $\langle \mathfrak{f}^{crip}, \partial\rangle$ which is known to be $\langle \mathfrak{f}^{crip}, \mathfrak{H}, \partial\rangle - NS\ b\ compact$ hence this sof cover must be reducible to a finite soft sub-cover.say, $\{\langle \mathfrak{k}, \partial\rangle_{ir} : 1 \le r \le n\}$. $This \Rightarrow \langle \mathfrak{f}^{crip}, \partial\rangle \Subset \cup_{r=1}^{n} \widetilde{\langle \mathfrak{k}, \partial\rangle}_{\iota r} =$ $\cup_{r=1}^{n}((\langle g, \partial\rangle_{ir}) \tilde{\cap} \langle \mathfrak{f}^{crip}, \partial\rangle \in \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle), or$

$$\langle \mathfrak{f}^{crip}, \partial\rangle \Subset (\bigcup_{r=1}^{n}((\langle g, \partial\rangle_{ir}) \tilde{\cap} \langle \mathfrak{f}^{crip}, \partial\rangle) \Subset \bigcup_{r=1}^{n} \widetilde{\langle g, \partial\rangle_{\iota r}}, or$$

$\langle \mathfrak{f}^{crip}, \partial\rangle \cup_{r=1}^{n} \langle g, \partial\rangle_{ir}$. This proves that $\{\langle g, \partial\rangle_{ir} : 1 \le r \le n\}$is a finite soft sub-cover ot the soft cover$\langle g, \partial\rangle_i$. Starting from an arbitrary $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle - NS$ b-open soft cover of $\langle \mathfrak{f}^{crip}, \partial\rangle$,we are able to show that this soft neutrosophic b- open cover is freezable to a finite soft sub-cover, showing there by $\langle \mathfrak{f}^{crip}, \partial\rangle$ is $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle -$ $Nbcompact$.

Theorem 7.8. Let$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle NSSTS$ and let $\langle \widetilde{(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})}_n\rangle$ be a NS sequence in $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ such that it converges to a point $(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_0$ then the soft set $\langle \mathcal{g}, \partial\rangle$consisting of the points $(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_{n_0}$ and $(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_n (n = 1,2,3, \dots \dots)$is soft NSb compact.

Proof: Given$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle NSSTS$ and let $\langle \widetilde{(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})}_n\rangle$ be a NS sequence in $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ such that it converges to a point $(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_{n_0}$ that is $\widetilde{(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})}_n \to (x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_{n_0} \in \mathcal{X}^{crip}$. Let

$$\langle \mathcal{g}, \partial\rangle = \langle \begin{matrix} \widetilde{(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})}_1, \widetilde{(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})}_2, \widetilde{(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})}_3, \\ \widetilde{(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})}_4, \widetilde{(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})}_5, \widetilde{(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})}_7, \dots \end{matrix} \rangle$$

.Let $\{\langle \mathfrak{S}, \partial\rangle_\alpha : \alpha \in \Delta\}$ be NS b-open covering of $\langle \mathcal{g}, \partial\rangle$ so that $\langle \mathcal{g}, \partial\rangle \Subset \tilde{\cup} \{\langle \mathfrak{S}, \partial\rangle_\alpha : \alpha \in \Delta\}$, $(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_{n_0} \in \langle \mathcal{g}, \partial\rangle \Rightarrow \exists \alpha_0 \in \Delta\, s.t. (x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_{n_0} \in \langle \mathfrak{S}, \partial\rangle_{\alpha_0}$. According to the definition of soft convergence,$(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_{n_0} \in \langle \mathfrak{S}, \partial\rangle_{\alpha_0} \in \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \Rightarrow \exists n_0 \in N\, s.t. n \geq n_0$ and $(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_n \in \langle \mathfrak{S}, \partial\rangle_{\alpha_0}$ .Evidently, $\langle \mathfrak{S}, \partial\rangle_{\alpha_0}$ contains the points $(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_{n_0}$,

$$(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_{n_{0+1}}, (x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_{n_{0+2}},$$

$$(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_{n_{0+3}}, (x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter}), \dots.$$

$(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_{n_{0+n}}, \dots..$ Look carefully at the points and train them in a way as, $(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_1, (x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_2,$

$$(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_3, (x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_4, \dots.$$

$(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter}), \dots \dots$.generating a finite soft set. Let $1 \leq n_{0-1}$. Then $(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_i \in \langle \mathcal{g}, \partial\rangle$. For this i, $(x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_i \in \langle \mathcal{g}, \partial\rangle$. Hence $\exists\, \alpha_i \in \Delta\, s.t. (x^e{}_{(a,b,c)}, \text{ꟻ}^{parameter})_i \in \langle \mathfrak{S}, \partial\rangle_{\alpha_i}$. Evidently $\langle \mathcal{g}, \partial\rangle \Subset \widetilde{\cup_{r=0}^{n_{0-1}}} \langle \mathfrak{S}, \partial\rangle_{\alpha_i}$.This shows that $\{\langle \mathfrak{S}, \partial\rangle_{\alpha_i} : 0 \leq n_{0-1}\}$is NS b-open cover of $\langle \mathcal{g}, \partial\rangle$. Thus an arbitrary soft neutrosophic open cover $\{\langle \mathfrak{S}, \partial\rangle_\alpha : \alpha \in \Delta\}$ of $\langle \mathcal{g}, \partial\rangle$ is reducible to a finite NS sub-cover $\{\langle \mathfrak{S}, \partial\rangle_{\alpha i} : i = 0,1,2,3, \dots n_{0-1}\}$,it follows that $\langle \mathcal{g}, \partial\rangle$ is soft NSb compact.

Theorem 7.9. Let$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ be a $NSSTS$ such that it is soft countably compact.Then every NSb closed sub-set of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ is NSb countably compact

Proof: Let$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle NSSTS$ such that it is NSb countably compact and suppose $\langle \mathcal{f}, \partial\rangle$ be a NSb closed sub-set of$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$. Let $\complement = \{\langle \mathcal{G}_1, \partial\rangle, \langle \mathcal{G}_2, \partial\rangle, \langle \mathcal{G}_3, \partial\rangle, \langle \mathcal{G}_4, \partial\rangle, \langle \mathcal{G}_5, \partial\rangle, \langle \mathcal{G}_6, \partial\rangle, \dots \dots . \langle \mathcal{G}_n, \partial\rangle \dots . \}$ That is $\{\langle \mathcal{G}_n, \partial\rangle : n \in N\}$ be any countably NS b-open covering of $\langle \mathcal{f}, \partial\rangle$. Then, $\langle \mathcal{f}, \partial\rangle \Subset \tilde{\cup} \langle \mathcal{G}_n, \partial\rangle$ .This qualifying us to write $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle = \langle \mathcal{f}, \partial\rangle \tilde{\cup} \langle \mathcal{f}, \partial\rangle^c \Subset \tilde{\cup} (\langle \mathcal{G}_n, \partial\rangle) \tilde{\cup} \langle \mathcal{f}, \partial\rangle^c$. This guarantee that the collection $\{\langle \mathcal{G}_n, \partial\rangle : n \in N\}$ is a NS countable b-open covering of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. But $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ being soft countably NSb compact and $\langle \mathscr{f}, \partial \rangle^c$ obviously absorbing no piece of $\langle \mathscr{f}, \partial \rangle$. It follows that there exists finite soft number of indices $n_1, , n_1, n_1, n_1, \dots . n_k$ such that $\langle \mathscr{f}, \partial \rangle \Subset \cup_{i=1}^{k} \langle \mathcal{G}_{ni}, \partial \rangle$. This shows that $\{\langle \mathcal{G}_{ni}, \partial \rangle : i = 1,2,3,4, \dots, k\}$ is a finite soft neutrosophic sub-covering of C.

Theorem 7.10. If $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle NSSTS$ such that it has the characteristics of soft neutrosophic sequentially compactness. Then$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is safely NSbcountably compact.

Proof: Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ $NSSTS$ and let $\langle \rho, \partial \rangle$ be finite soft sub-set of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Let $\left\langle \begin{matrix} \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_1, \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_2, \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_3, \\ \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_4, \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_5, \\ \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_6, \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_7, \dots \end{matrix} \right\rangle$ be a soft sequence of soft points of $\langle \rho, \partial \rangle$. Then,$\langle \rho, \partial \rangle$ being finite, at least one of the elements In $\langle \rho, \partial \rangle$say $\left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_0$ must be duplicated an in- $finite\ number\ of\ times\ in\ the\ NS\ sequence.$

Hence, $\left\langle \begin{matrix} \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_0, \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_0, \\ \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_0, \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_0, \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_0, \\ \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_0, \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_0, \dots \end{matrix} \right\rangle$ is soft sub-sequence of $\left\langle \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_n \right\rangle$ such that it is soft constant sequence and repeatedly constructed by single soft number $\left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_0$ and we know that a soft constant sequence converges on its self. So it converges to $\left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_0$ which belongs to $\langle \rho, \partial \rangle$. Hence, $\langle \rho, \partial \rangle$ is soft sequentially NSb compact.

Theorem 7.11. Let$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ $NSSTS$ and $\langle \mathcal{Y}^{crip}, \mathfrak{T}, \partial \rangle$ be another $NSSTS$. Let $\langle \mathscr{f}, \partial \rangle$ be a soft continuous mapping of a soft neutrosophic sequentially compact NSb space $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ into $\langle \mathcal{Y}^{crip}, \mathfrak{T}, \partial \rangle$. Then, $\langle \mathscr{f}, \partial \rangle(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle)$ is NSb sequentially compact.

Proof: Given$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ $NSSTS$ and $\langle \mathcal{Y}^{crip}, \mathfrak{T}, \partial \rangle$ be another $NSSTS$. Let $\langle \mathscr{f}, \partial \rangle$ be a soft continuous mapping of a NSb sequentially compact space $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$into $\langle \mathcal{Y}^{crip}, \mathfrak{T}, \partial \rangle$.Then we have to prove $\langle \mathscr{f}, \partial \rangle(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle) NSb$ sequentially. For this we proceed as.Let $\left\langle \begin{matrix} \left(y^{e/}{}_{(a/,b/,c/)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_1, \left(y^{e/}{}_{(a/,b/,c/)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_2, \\ , \left(y^{e/}{}_{(a/,b/,c/)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_5, \left(y^{e/}{}_{(a/,b/,c/)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_6, \\ \left(y^{e/}{}_{(a/,b/,c/)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_7, \dots \left(y^{e/}{}_{(a/,b/,c/)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_n, \dots . \end{matrix} \right\rangle$ be a soft sequence of NS points in $\langle \mathscr{f}, \partial \rangle(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle)$, $Then\ for\ each\ n \in N \exists$ $\left\langle \begin{matrix} \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_1, \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_2, \\ \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_4, \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_5, \\ \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_7, \dots \left(x^e{}_{(a,b,c)}, \widetilde{\text{Ⴙ}^{parameter}}\right)_n, \dots . \in \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle \end{matrix} \right\rangle$ such that

$$\langle \not{f}, \partial \rangle \left(\langle \begin{matrix} (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_1, (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_2, \\ (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_3, \\ (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_7, \dots (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_n, \dots . \end{matrix} \rangle \right) =$$

$$\langle \begin{matrix} (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_1, (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_2, \\ (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_3, (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_4, \\ (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_6, (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_7, \\ \dots (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_n, \dots . \end{matrix} \rangle \textit{Thus we obtain a soft sequence}$$

$$\langle \begin{matrix} (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_1, (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_2, \\ (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_3, (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_4, \\ (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}}), (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_6, \\ (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_7, \dots (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_n, \dots . \end{matrix} \rangle$$ in $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. But $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ being soft sequentially NS compact, there is a NS sub-sequence $\langle (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_{n_\iota} \rangle$ of

$\langle (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_n \rangle$ *such that* $\langle (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_{n_\iota} \rangle \longrightarrow (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}}) \in \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. *So, by NS contiuity of* $\langle \not{f}, \partial \rangle$, $\langle (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_{n_\iota} \rangle \longrightarrow (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}}) \longrightarrow \langle \not{f}, \partial \rangle (\langle (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_{n_\iota} \rangle) \longrightarrow$

$\langle \not{f}, \partial \rangle (\langle (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_n \rangle) \in \langle \not{f}, \partial \rangle (\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle)$. Thus, $\langle \not{f}, \partial \rangle (\langle (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_{n_\iota} \rangle)$ is a soft sub-sequence

of $$\langle \begin{matrix} (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_1, (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_2, \\ (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}}), (x^{e}{}_{(a,b,c)}, \widetilde{Ɖ^{parameter}})_4, \\ (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_5, (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_6, \\ (y^{e/}{}_{(a/,b/,c/)} \widetilde{Ɖ^{parameter}})_7, \dots (y^{e/}{}_{(a/,b/,c/)}, \widetilde{Ɖ^{parameter}})_n, \dots . \end{matrix} \rangle$$ converges to $(\langle \not{f}, \partial \rangle)(\tilde{\varkappa})$ in $\langle \not{f}, \partial \rangle (\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle)$.

Hence, $\langle \not{f}, \partial \rangle (\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle)$ is NSb sequentially compact.

Theorem 7.12. Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be $NSSTS$ such that is NS sequentially comapct then it is guaranteed that it must be NSb countably compact.

Proof: Suppose $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is NS sequentially compact. If $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is finite, then nothing to prove in this case because it it is then automatically NSb countably compact. Suppose $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be in-finite. We prove the contrapositive of the statement given in the theorem. Let $\{\langle \mathcal{G}_i, \partial \rangle : i \in N\}$ that is $\{\langle \mathcal{G}_1, \partial \rangle, \langle \mathcal{G}_2, \partial \rangle, \langle \mathcal{G}_3, \partial \rangle, \langle \mathcal{G}_4, \partial \rangle, \dots : i \in N\}$ be a NS b-open cover of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ which has no finite soft sub-cover. Now, we generate a soft sequence $\langle \begin{matrix} \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{1}, \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{2}, \\ \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{3}, \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{4}, \\ \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{5}, \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{6}, \\ \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{7}, \dots \end{matrix} \rangle$ which may be soft monotone. That is soft monotonically non-increasing or soft monotonically strictly increasing or soft monotonically non-decreasing or soft monotonically strictly decreasing. Whatever the case may be we proceed as follows. Let n_1 be the smallest positive integer such that $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_1}, \partial\rangle \neq 0_{(\widetilde{\langle \mathcal{X}\rangle}, Ꟗ^{parameter})}$. Choose $\left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{1} \in \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_1}, \partial\rangle$ .Now let n_2 be the least positive integer greater than n_1 such that $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_2}, \partial\rangle \neq 0_{(\widetilde{\langle \mathcal{X}\rangle}, Ꟗ^{parameter})}$. Choose $\left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{2} \in \left(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_2}, \partial\rangle\right) \backslash`` \left(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_1}, \partial\rangle\right)$. It is important to be noted that such a point $\left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{2}$ always exists, for otherwise $\langle \mathcal{G}_{n_1}, \partial\rangle$will be a soft cover of$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$. Choose $\left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{3} \in \left(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_3}, \partial\rangle\right) \backslash \left(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_2}, \partial\rangle\right)$. It is important to be noted that such a point $\left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{3}$ always exists, for otherwise $\mathcal{G}_{n_2}$ will be a soft cover of$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$. Choose $\left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{4} \in \left(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \mathcal{G}_{n_4}\right) \backslash \left(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_3}, \partial\rangle\right)$. Choose $\left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{5} \in \left(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_5}, \partial\rangle\right) \backslash \left(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_4}, \partial\rangle\right)$. .... Choose $\left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{n} \in \left(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_n}, \partial\rangle\right) \backslash \left(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_{n-1}}, \partial\rangle\right)$. we get the soft sequence $\left(\begin{matrix} \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{1}, \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{2}, \\ \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{3}, \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{4}, \\ \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{5}, \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{6}, \dots\dots, \\ \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{n}, \dots \end{matrix}\right)$ having the characteristics that, for each

$i \in N. \left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{\iota} \in \left(\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \langle \mathcal{G}_{n_i}, \partial\rangle\right),$

$\left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{\iota}$ *does not belong to* $\tilde{\cup}\left\{\begin{matrix} \begin{pmatrix} \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle \tilde{\cap} \\ (\langle \mathcal{G}_{k}, , \partial\rangle \end{pmatrix}: \\ k = 1,2,3, \dots\dots, n-1 \end{matrix}\right\}$, $m \geq n_i - 1$. It can be seen that $\left(\left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right)_{n}\right)$ supposes no soft convergent sub-sequence in $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$. For let $\left(x^{e}{}_{(a,b,c)}, \widetilde{Ꟗ^{parameter}}\right) \in \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$. Then there exists $\langle \mathcal{G}_{m_0}, , \partial\rangle \in V\{\langle \mathcal{G}_{n}, , \partial\rangle : n \in N\ \}$ such that $\left(x^{e}{}_{(a,b,c)}, \widetilde{\text{Ⅎ}^{parameter}}\right) \in \langle \mathcal{G}_{m_0}, , \partial \rangle$. Since $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle \tilde{\cap} \langle \mathcal{G}_{m_0}, , \partial \rangle \neq 0_{(\langle \widetilde{\mathcal{X}} \rangle, \text{Ⅎ}^{parameter})}$, there exists $k_0 \in N$ such that $\langle \mathcal{G}_{n_{k_0}}, , \partial \rangle = \langle \mathcal{G}_{m_0}, , \partial \rangle$. But by the choice of of the soft sequence

$$\left(\begin{array}{c} \left(x^{e}{}_{(a,b,c)}, \widetilde{\text{Ⅎ}^{parameter}}\right)_1, \left(x^{e}{}_{(a,b,c)}, \widetilde{\text{Ⅎ}^{parameter}}\right)_2, \\ \left(x^{e}{}_{(a,b,c)}, \widetilde{\text{Ⅎ}^{parameter}}\right)_3, \left(x^{e}{}_{(a,b,c)}, \widetilde{\text{Ⅎ}^{parameter}}\right)_4, \\ \left(x^{e}{}_{(a,b,c)}, \widetilde{\text{Ⅎ}^{parameter}}\right)_5, \left(x^{e}{}_{(a,b,c)}, \widetilde{\text{Ⅎ}^{parameter}}\right)_6, \ldots\ldots, \\ \left(x^{e}{}_{(a,b,c)}, \widetilde{\text{Ⅎ}^{parameter}}\right)_n, \ldots \end{array}\right)$$

we have $i > k_0$ this implies that $\left(x^{e}{}_{(a,b,c)}, \widetilde{\text{Ⅎ}^{parameter}}\right)$ *does not* $\langle \mathcal{G}_{m_0}, , \partial \rangle$. Since $\langle \mathcal{G}_{m_0}, , \partial \rangle$ is NS b-open set containing $\left(x^{e}{}_{(a,b,c)}, \widetilde{\text{Ⅎ}^{parameter}}\right)$, Since $\left(x^{e}{}_{(a,b,c)}, \widetilde{\text{Ⅎ}^{parameter}}\right)$ was arbitrary, $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is not NS sequentially compact,which is planely contradiction. Hence $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is NSb countably compact.

Theorem 7.13. Every NS co-finite $NSTS$ $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is NSb separable.

Proof: case1: If $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is NS countable, then clearly $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is a NSb countable dense soft sub-set of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ and therefore, in this case, $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is NSb separable.

Case2: suppose $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is NS uncountable. Then, there exists an infinite NSb countable soft sub-set $\langle \not{f}, \partial \rangle$ of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Now, $\overline{\langle \not{f}, \partial \rangle}$ is the smallest NSb closed superset of $\langle \not{f}, \partial \rangle$ and in the soft co-finite space $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$, the only NSb closed sub-sets of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ are $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ and finite soft sets.Results in, $\overline{\langle \not{f}, \partial \rangle} = \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Thus, $\langle \not{f}, \partial \rangle$ is a NSb countable dense soft sub-set of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. This signifies that $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is soft separable. Hence, every NSb co-finite $NSSTS$.

Theorem 7.14. If $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is $NSSTS$ such that it is NS second countable then it has the characteristics NSseparability.

Proof: Suppose $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be NSsecond countable space.

Let $\mathfrak{W} = \langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \ldots\ldots \mathcal{B}_n : n \in \mathbb{N} \rangle$ be a NSb countable base for $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Choose $\left(x^{e}{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)_n \in \mathcal{B}_n$ for each n . Then, the set $\langle \not{f}, \partial \rangle = \left\{ n: \left(x^{e}{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)_n \in \mathcal{B}_n \right\}$ is NS countable. Only remaining to prove that $\langle \not{f}, \partial \rangle$ is soft dense in $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$.suppose $\left(x^{e}{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right) \in \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ and let $\langle \mathcal{G}, \partial \rangle_{(x^{e}{}_{(a,b,c)}, \text{Ⅎ}^{parameter})}$ be NS b open set absorbing $\left(x^{e}{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)$. Then, $\mathfrak{W}$ being a NSbase, there exists a NS b open set $\mathcal{B}_{n0}$ in $\mathfrak{W}$ such that $\left(x^{e}{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right) \in \mathcal{B}_{n0} \Subset \langle \mathcal{G}, \partial \rangle_{(x^{e}{}_{(a,b,c)}, \text{Ⅎ}^{parameter})}$. But, by our choice of $\langle \not{f}, \partial \rangle$, the soft set $\mathcal{B}_{n0}$ contains a point $\left(x^{e}{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)_{n0}$ of $\langle \not{f}, \partial \rangle$ that is every NSb open set containing $\left(x^{e}{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)$ contain at least one point of $\langle \not{f}, \partial \rangle$. So, $\left(x^{e}{}_{(a,b,c)}, \text{Ⅎ}^{parameter}\right)$ is soft adherent poit of

$\langle \not{f}, \partial \rangle$. Thus, every point of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is soft adherent point of $\langle \not{f}, \partial \rangle$.that is $\overline{\langle \not{f}, \partial \rangle} = \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. It follows, therefore,that $\langle \not{f}, \partial \rangle$ is soft countable dense soft sub-set of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Hence,$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is NSb seperable.

Theorem 7.15. Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a second soft neutrosophic countable NS space is NS Lindelof space.

Proof: Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a second NS countable topological space and let $\mathfrak{W} = \langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \ldots\ldots \mathcal{B}_n : n \in \mathbb{N} \rangle$ soft base for $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Let $C = \{\langle \mathcal{G}_i, \partial \rangle : i \in N\}$ that is $\{\langle \mathcal{G}_1, \partial \rangle, \langle \mathcal{G}_2, \partial \rangle, \langle \mathcal{G}_3, \partial \rangle, \langle \mathcal{G}_4, \partial \rangle, \ldots : i \in N\}$ be any NSb open cover of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Then for each $\left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right) \in \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ there exists a NSb open set

$\langle \mathcal{G}, \partial \rangle_{\alpha_{\left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right)}}$ and,$\mathfrak{W}$ being a NSbbases, corresponding to each such NSb open set there exists a NSb open set $\langle \mathcal{G}, \partial \rangle_{n_{\left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right)}}$ in $\mathfrak{W}$ such that $\left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right) \in \mathcal{B}_{n_{\left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right)}} \Subset \langle \mathcal{G}, \partial \rangle_{n_{\left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right)}}$. Therefore, $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle = \tilde{\cup} \left\{ \mathcal{B}_{n_{\left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right)}} : \widetilde{\left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right)} \in \mathcal{X} \right\} \Subset \tilde{\cup} \left\{ \langle \mathcal{G}, \partial \rangle_{n_{\left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right)}} : \left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right) \in \mathcal{X} \right\}$. Now, $\left\{ \mathcal{B}_{n_{\left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right)}} : \left(x^{e}{}_{(\mathrm{a,b,c})}, \text{Ⅎ}^{parameter}\right) \in \mathcal{X} \right\}$ being s soft sub-family of Ccovering$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Thus, every NSbopen covering of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is reducible to a soft sub-covering Hence, $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is a NSb Lindelof space

Theorem 7.16. Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a NS Lindelof space, then this space need not always be second NScountable.

Proof: Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a NS co-finite topological space provided $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is NS un-coutable.Now, Let $C = \{\langle \mathcal{G}_i, \partial \rangle : i \in N\}$ that is $\{\langle \mathcal{G}_1, \partial \rangle, \langle \mathcal{G}_2, \partial \rangle, \langle \mathcal{G}_3, \partial \rangle, \langle \mathcal{G}_4, \partial \rangle, \ldots : i \in N\}$ be any NSb open cover of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. $\langle \mathcal{G}, \partial \rangle_{\alpha_0}$ be an arbitrary member of $\{\langle \mathcal{G}_1, \partial \rangle, \langle \mathcal{G}_2, \partial \rangle, \langle \mathcal{G}_3, \partial \rangle, \langle \mathcal{G}_4, \partial \rangle, \ldots : i \in N\}$. Then, $(\langle \mathcal{G}, \partial \rangle_{\alpha_0})^{c}$ is finite.Let,

$$(\langle \mathcal{G}, \partial \rangle_{\alpha_0})^{c} = \begin{pmatrix} \left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\text{Ⅎ}^{parameter}}\right)_1, \left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\text{Ⅎ}^{parameter}}\right)_2, \\ \left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\text{Ⅎ}^{parameter}}\right)_3, \left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\text{Ⅎ}^{parameter}}\right)_4, \\ \left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\text{Ⅎ}^{parameter}}\right)_5, \left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\text{Ⅎ}^{parameter}}\right)_6, \\ \ldots\ldots, \left(x^{e}{}_{(\mathrm{a,b,c})}, \widetilde{\text{Ⅎ}^{parameter}}\right)_n, \ldots \end{pmatrix}$$

. Now, $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle = \langle \mathcal{G}, \partial \rangle_{\alpha_0} \tilde{\cup} (\langle \mathcal{G}, \partial \rangle_{\alpha_0})^{c}$, Where, $(\langle \mathcal{G}, \partial \rangle_{\alpha_0})^{c}$ absorbs $(n)\, points$ of $(\langle \mathcal{G}, \partial \rangle_{\alpha_0})^{c}$ are NSb covered by at the most $(n)\, sets\ in\, \{\langle \mathcal{G}_i, \partial \rangle : i \in N\}$ that is $\{\langle \mathcal{G}_1, \partial \rangle, \langle \mathcal{G}_2, \partial \rangle, \langle \mathcal{G}_3, \partial \rangle, \langle \mathcal{G}_4, \partial \rangle, \ldots : i \in N\}$ and so $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is covered by the most $(n+1)$ sets in

$\{\langle\mathcal{G}_i,\partial\rangle: i \in N\}$ that is $\{\langle\mathcal{G}_1,\partial\rangle,\langle\mathcal{G}_2,\partial\rangle,\langle\mathcal{G}_3,\partial\rangle,\langle\mathcal{G}_4,\partial\rangle,\dots : i \in N\}$.Thus, $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ is NSb compact space and therefore, a NS Lindelof space. Now, if possible,let there be a NS countable base $\mathfrak{W}$ for $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$. let $\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right) \in \mathcal{X}$. Then, $\tilde{\cap}\left\{\langle\mathcal{G},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)} \in \langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle \; and \; \left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right) \propto \langle\mathcal{G},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)} = \left\{\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)\right\}\right\}$, for, if $\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) \neq \left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)$, then either $\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) > \left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)$ or $\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right) < \left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)$,then $\mathcal{X}\backslash \left\{\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)\right\}$ is cleary NSb open sets containing $\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)$ but not $\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)$ and therefore any $\left(y^{e/}{}_{(a/,b/,c/)}, ꟻ^{parameter}\right)$, different from $\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)$. More-over, $\mathfrak{W} = \langle\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\dots\dots\mathcal{B}_n : n \in \mathbb{N}\rangle$ being soft base,for each $\langle\mathcal{G},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)}\; \exists\mathcal{B}_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)} \in \mathfrak{W}$ such that $\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right) \in \mathcal{B}_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)} \Subset \langle\mathcal{G},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)}$. Obviously, $\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right) \in \tilde{\cap}\left\{\mathcal{B}_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)} : \mathcal{B}_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)}\mathfrak{W}, \left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right) \in \mathcal{B}_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)} \Subset \langle\mathcal{G},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)}\right\} \Subset \tilde{\cap}\left\{\langle\mathcal{G},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)} : \langle\mathcal{G},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)} \in \langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle, \left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right) c\langle\mathcal{G},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)}\right\} = \left\{\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)\right\}$ or $\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right) \in \tilde{\cup}\left\{\left(\mathcal{B}_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)}\right)^{c} : \mathcal{B}_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)} \in \mathfrak{W}, \left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right) \in \mathcal{B}_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)} \Subset \langle\mathcal{G},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)}\right\} = \mathcal{X}\backslash\left\{\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)\right\}$ .But, this is false, since $\tilde{\cup}\left\{\left(\mathcal{B}_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)}\right)^{c} : \mathcal{B}_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)}\mathfrak{W}, \left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right) \in \mathcal{B}_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)} \Subset \langle\mathcal{G},\partial\rangle_{\left(x^{e}{}_{(a,b,c)},ꟻ^{parameter}\right)}\right\}$ being a NS countable union of finite soft sets is NS countable while $\mathcal{X}^{crip}\backslash \left\{\left(x^{e}{}_{(a,b,c)}, ꟻ^{parameter}\right)\right\}$ is soft un-countable. So, there does not exist a NS countable soft base for $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$, that is, $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ is not NS countable. Thus, $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ is a NS Lindelof space space which is not second soft count-able.

Theorem 7.17. Let $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ be a NS Lindelof space and $\langle Y^{crip},\mathfrak{T}_Y,\partial\rangle$ be soft sub-space of $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$, then guaranteedly, $\langle Y^{crip},\mathfrak{T}_Y,\partial\rangle$ is NS Lindelof space.

Proof: Given $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a NSLindel space and $\langle Y^{crip}, \mathfrak{T}_Y, \partial \rangle$ be soft sub-space of$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Let $C = \{\langle \mathcal{H}, \partial \rangle_\alpha\}$ be any $\langle Y^{crip}, \mathfrak{T}_Y, \partial \rangle NSb$ open coveringY^{crip}. Then, $Y^{crip} = \tilde{\cup} \langle \mathcal{H}, \partial \rangle_\alpha$. Also,$\langle \mathcal{H}, \partial \rangle_\alpha = Y \tilde{\cap} \langle \mathcal{G}, \partial \rangle_\alpha$ where $\langle \mathcal{G}, \partial \rangle_\alpha \in \langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Therefor $Y = \widetilde{\cup_\alpha} (Y \tilde{\cap} \langle \mathcal{G}, \partial \rangle_\alpha) \Subset \widetilde{\cup_\alpha} \langle \mathcal{G}, \partial \rangle_\alpha$. So $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle = Y \widetilde{\cup Y^c} \Subset (\langle \mathcal{G}, \partial \rangle_\alpha) \widetilde{\cup Y^C}$. Thus, $\mathbb{C}^* = \{\langle \mathcal{G}, \partial \rangle_\alpha, Y^C\}$ is soft b open covering of the NS Lindel space of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$.Since, Y^C covers no part of$Y$, so there exists a NScountable nuzmber of $\langle \mathcal{G}, \partial \rangle_{\alpha i}$,s in $\mathbb{C}^*$ such that $Y \Subset \tilde{\cup} \{\langle \mathcal{G}, \partial \rangle_{\alpha i} : i \in \Lambda \Subset N\}$ or $Y = \tilde{\cup} \{Y \tilde{\cap} \langle \mathcal{G}, \partial \rangle_{\alpha i} : i \in \Lambda \Subset N\}$. Therefore, $Y = \tilde{\cup} \{\langle \mathcal{H}, \partial \rangle_{\alpha i} : i \in \Lambda \Subset N\}$. This shows that $\mathbb{C}^*$ is reducible to a NScountable subcovering.Hence,$\langle Y^{crip}, \mathfrak{T}_Y, \partial \rangle$ is also a NSb Lindelof space.

Theorem 7.18. Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a NSb_1 space and $\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right), \left(y^{e'}{}_{(a',b',c')}, Ⅎ^{parameter}\right) \in \mathcal{X}^{crip}$ such that $\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right) > \left(y^{e'}{}_{(a',b',c')}, Ⅎ^{parameter}\right)$ or $\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right) < \left(y^{e'}{}_{(a',b',c')}, Ⅎ^{parameter}\right)$. If $\mathfrak{W}_{\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right)}$ is a NSb local base at $\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right)$, then there exists at least one member of $\mathfrak{W}_{\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right)}$ which does not supposes $\left(y^{e'}{}_{(a',b',c')}, Ⅎ^{parameter}\right)$.

Proof: Since $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a NSb_1 space and $\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right) > \left(y^{e'}{}_{(a',b',c')}, Ⅎ^{parameter}\right)$ or $\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right) < \left(y^{e'}{}_{(a',b',c')}, Ⅎ^{parameter}\right)$, $\exists$ NSb open sets $\langle \mathcal{G}, \partial \rangle$ and $\langle \mathcal{H}, \partial \rangle$ such that $\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right) \in \langle \mathcal{G}, \partial \rangle$ but $\left(y^{e'}{}_{(a',b',c')}, Ⅎ^{parameter}\right) \notin \langle \mathcal{G}, \partial \rangle$ and $\left(y^{e'}{}_{(a',b',c')}, Ⅎ^{parameter}\right) \in \langle \mathcal{H}, \partial \rangle$ but $\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right) \notin \langle \mathcal{H}, \partial \rangle$. Since, $\mathfrak{W}_{\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right)}$ is NS local base at $\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right)$ *there exists* $\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right) \in B \Subset \langle \mathcal{G}, \partial \rangle$. *Since* $\left(y^{e'}{}_{(a',b',c')}, Ⅎ^{parameter}\right) \notin \langle \mathcal{G}, \partial \rangle$*and* $B \Subset \langle \mathcal{G}, \partial \rangle$,so$\left(y^{e'}{}_{(a',b',c')}, Ⅎ^{parameter}\right) \notin B$.Thus, $B \in \mathfrak{W}_{\left(x^e{}_{(a,b,c)}, Ⅎ^{parameter}\right)}$ such that $\left(y^{e'}{}_{(a',b',c')}, Ⅎ^{parameter}\right) \notin B$.

Theorem 7.19. Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ be a $NSSTS$ such that it is $NS$of b_1 space in which every in-finite soft subset has a soft limit point.Then, $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ is definitely NSbcompact.

Proof: Let C NS open covering of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Then $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ being NS Lindelof space, C is reducible to a NS countable sub-covering,say $C^* = \{\langle \mathcal{G}_n, \partial \rangle : n \in \Lambda \Subset N\}$ that is $\{\langle \mathcal{G}_1, \partial \rangle, \langle \mathcal{G}_2, \partial \rangle, \langle \mathcal{G}_3, \partial \rangle, \langle \mathcal{G}_4, \partial \rangle, \dots : n \in \Lambda \Subset N\}$. If possible, let C^* is not reducible to a finite soft subcovering.Then,for any positive integer k,the soft $(\cup_{i=1}^{k} \langle \mathcal{G}_n, \partial \rangle)^c$ is NSb open proper subset of

$\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ and therefore,itscomplement, $F_k = (\cup_{i=1}^{k}\langle \mathcal{G}_n, \partial \rangle)^c$ is non-empty NS closed subset of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$. Now,taking $k = 1,2,3, \dots \dots$ we obtain a nested soft sequence $\langle F_k \rangle$ of soft neutrosophic closed subsets of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ such that $(F_1, \partial) \Supset (F_2, \partial) \Supset (F_3, \partial) \Supset (F_4, \partial) \Supset (F_5, \partial) \Supset (F_6, \partial) \Supset, \dots \Supset (F_n, \partial) \Supset \dots$. Let $A = \left\{ \left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right)_k : \left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right)_k \in F_k \right\}$.then, the soft set A is obviously an infinite soft set. So, by the given hypothesis, A has a soft limit point, suppose $\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right)$. But $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ being NSb_1 space, so every NSb open set containing $\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right)$ must therefore contain an infinite number of points of A. Result in $\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right)$ is a soft limit point of each F_k that is $(F_1, \partial), (F_1, \partial)$, $(F_2, \partial), (F_3, \partial)$, $(F_4, \partial), (F_5, \partial)$,… But each of $(F_1, \partial), (F_1, \partial)$, $(F_2, \partial), (F_3, \partial)$, $(F_4, \partial), (F_5, \partial)$,… is soft b-closed,$\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right) \in (F_1, \partial)$, $\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right) \in (F_2, \partial)$, $\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right) \in (F_3, \partial)$, $\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right) \in (F_4, \partial)$, $\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right) \in (F_5, \partial)$, $\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right) \in (F_6, \partial), \dots$ .This contradicts the fact that C^* os a soft covering of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ and hence C is reducible to a finite soft sub covering.

Theorem 7.20. Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$be NSregular Lindelop space then it is safely NSbnormal.

Proof: $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle NSb$regular Lindelop space and let $\langle \Psi_1, \partial \rangle \langle \Psi_2, \partial \rangle$ be two NSbclosed sub-sets of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$ such that these are mutually excusive. Then,everyNSb closed sub-space of a NSLindelop space is soft Lindelop space. It is then guaranteed that $\langle \Psi_1, \partial \rangle \langle \Psi_2, \partial \rangle$ are NS Lindelop spaces. Now, by the NSb regularity of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$, correspondingto NSb closed set $\langle \Psi_1, \partial \rangle$ and every $\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right) \in \langle \Psi_2, \partial \rangle \ \exists$ soft NSb open set $\langle \mathcal{G}, \partial \rangle_{\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right)}$ such that $\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right) \in \langle \mathcal{G}, \partial \rangle_{x} \Subset \overline{\langle \mathcal{G}, \partial \rangle_{\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right)}} \Subset (\langle \Psi_1, \partial \rangle)^c$. More-over, the soft family $\left\{ \langle \mathcal{G}, \partial \rangle_{\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right)} : \left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right) \in \langle \Psi_2, \partial \rangle \right\}$ is clrearly NSb open covering of the Lindelop space NS set $\langle \Psi_2, \partial \rangle$. So,it must supposes NS countable sub-covering $\left\{ \langle \mathcal{G}, \partial \rangle_{\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right)_i} : i \in N \right\}$. Again, by the$NSb$regularity of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$,corresponding to the NSb closed set $\langle \Psi_2, \partial \rangle$ and every $\left(x^{e}{}_{(a,b,c)}, \text{ꟻ}^{parameter} \right) \in \langle \Psi_1, \partial \rangle \exists \ NSb$ open set $\langle \Psi_3, \partial \rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{ꟻ}^{parameter} \right)}$ such that $\left(y^{e/}{}_{(a/,b/,c/)}, \text{ꟻ}^{parameter} \right) \in \langle \Psi_3, \partial \rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{ꟻ}^{parameter} \right)} \Subset \overline{\langle \Psi_3, \partial \rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{ꟻ}^{parameter} \right)}} \Subset (\langle \Psi_2, \partial \rangle)^c$.

Clearly, $\left\{ \langle \Psi_3, \partial \rangle_{\left(y^{e/}{}_{(a/,b/,c/)}, \text{ꟻ}^{parameter} \right)} : \left(y^{e/}{}_{(a/,b/,c/)}, \text{ꟻ}^{parameter} \right) \in \langle \Psi_1, \partial \rangle \right\}$ is NS open covering oft the

Lindelop space $\langle\Psi_1,\partial\rangle$ and therefore,it is frezzable to a NSb countable sub-covering $\langle\Psi_3,\partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)_i}: i \in N$. Let $\langle\mathcal{M},\partial\rangle_n = \langle\Psi_1,\partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)_n} - \widetilde{\cup_{\iota\in n}}\left\{\overline{\langle\Psi_3,\partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)}}: i \le n\right\} = \langle\Psi_1,\partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)_n} \cap [\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle - \widetilde{\cup}\left\{\overline{\langle\Psi_3,\partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)}}: i \le n\right\}]$ and $\langle\omega,\partial\rangle_n = \langle\mathcal{G},\partial\rangle_{xn} - \widetilde{\cup_{\iota\in n}}\left\{\overline{\langle\mathcal{G},\partial\rangle_{(x^e{}_{(a,b,c)},ꟻ^{parameter})_\iota}}: i \le n\right\} = \langle\mathcal{G},\partial\rangle_{(x^e{}_{(a,b,c)},ꟻ^{parameter})_n} \widetilde{\cap} [\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle - \widetilde{\cup}\left\{\overline{\langle\mathcal{G},\partial\rangle_{x\iota}}: i \le n\right\}]$. Then, $\langle\mathcal{M},\partial\rangle_n$ and $\langle\omega,\partial\rangle_n$ are clearly NSb open sets and therefore,so are the sets $\langle\mathbb{G},\partial\rangle = \widetilde{\cup}\{\langle\mathcal{M},\partial\rangle_n : n \in N\}$ and $\langle\mathbb{H},\partial\rangle = \langle\omega,\partial\rangle_n : n \in N$ .Now, $\langle\Psi_1,\partial\rangle \Subset \widetilde{\cup}\langle\Psi_3,\partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)_i}: i \in N$ and $\overline{\langle\Psi_3,\partial\rangle_{\left(y^{e/}{}_{(a/,b/,c/)},ꟻ^{parameter}\right)_\iota}} \widetilde{\cap} \langle\Psi_1,\partial\rangle = (x^e{}_{(a,b,c)},ꟻ^{parameter})$. So,it follows that $\{\langle\mathcal{M},\partial\rangle_n : n \in N\}$ is NSb open covering of $\langle\Psi_1,\partial\rangle$. Therefore, $\langle\Psi_1,\partial\rangle \Subset \widetilde{\cup}\{\langle\mathcal{M},\partial\rangle_n : n \in N\} = \langle\mathbb{G},\partial\rangle$,Similarly, $\langle\Psi_2,\partial\rangle \Subset \langle\mathbb{H},\partial\rangle$. Also, $\langle\mathcal{M},\partial\rangle_n \widetilde{\cap} \langle\omega,\partial\rangle_n = \widetilde{\emptyset}$ for each n that is $\langle\mathcal{M},\partial\rangle_1 \widetilde{\cap} \langle\omega,\partial\rangle_1 = \widetilde{\emptyset}$, $\langle\mathcal{M},\partial\rangle_2 \widetilde{\cap} \langle\omega,\partial\rangle_2 = \widetilde{\emptyset}$, $\langle\mathcal{M},\partial\rangle_3 \widetilde{\cap} \langle\omega,\partial\rangle_3 = \widetilde{\emptyset}$, $\langle\mathcal{M},\partial\rangle_4 \widetilde{\cap} \langle\omega,\partial\rangle_4 = \widetilde{\emptyset}$, $\langle\mathcal{M},\partial\rangle_5 \widetilde{\cap} \langle\omega,\partial\rangle_5 = \widetilde{\emptyset}$, $\langle\mathcal{M},\partial\rangle_6 \widetilde{\cap} \langle\omega,\partial\rangle_6 = \widetilde{\emptyset}$………………… . This guarantees that $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ is soft neutrosophic normal.

Theorem 7.21. Let $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle NSSTS$ and Suppose $\langle f,\partial\rangle, \langle g,\partial\rangle$ be two NS continuous function on a NS topological space $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$ in to a $NSSTS$ $\langle Y^{crip},\mathfrak{F},\partial\rangle$ which is NSb Hausdorf.Then, soft set $\{(x^e{}_{(a,b,c)},ꟻ^{parameter}) \in \mathcal{X}^{crip} : (f)((x^e{}_{(a,b,c)},ꟻ^{parameter})) = (g)((x^e{}_{(a,b,c)},ꟻ^{parameter}))\}$ is NSb closed of $\langle\mathcal{X}^{crip},\mathfrak{T},\partial\rangle$.

Proof: Let If $\{(x^e{}_{(a,b,c)},ꟻ^{parameter}) \in \mathcal{X}^{crip} : (f)((x^e{}_{(a,b,c)},ꟻ^{parameter})) = (g)((x^e{}_{(a,b,c)},ꟻ^{parameter}))\}$ is a NS set of function.If $\{(x^e{}_{(a,b,c)},ꟻ^{parameter}) \in \mathcal{X}^{crip} : (f)((x^e{}_{(a,b,c)},ꟻ^{parameter})) = (g)(\varkappa)\}^c = \widetilde{\emptyset}$, it is clearly NSb open and therefore, $\{(x^e{}_{(a,b,c)},ꟻ^{parameter}) \in \mathcal{X}^{crip} : (f)((x^e{}_{(a,b,c)},ꟻ^{parameter})) = (g)((x^e{}_{(a,b,c)},ꟻ^{parameter}))\}$ is NSb closed, that is nothing is proved in this case. Let us consider the case when $\{(x^e{}_{(a,b,c)},ꟻ^{parameter}) \in \mathcal{X}^{crip} : (f)((x^e{}_{(a,b,c)},ꟻ^{parameter})) = (g)((x^e{}_{(a,b,c)},ꟻ^{parameter}))\}^c \neq (x^e{}_{(a,b,c)},ꟻ^{parameter})$. And let $\rho \in \left\{(x^e{}_{(a,b,c)},ꟻ^{parameter}) \in \mathcal{X}^{crip} : (f)\left((x^e{}_{(a,b,c)},ꟻ^{parameter})\right) = (g)\left((x^e{}_{(a,b,c)},ꟻ^{parameter})\right)\right\}^c$. Then ρ *does not belong* $\{(x^e{}_{(a,b,c)},ꟻ^{parameter}) \in \mathcal{X}^{crip} : (f)((x^e{}_{(a,b,c)},ꟻ^{parameter})) = (g)(\varkappa)\}$ .Result in $(f)(\rho) \neq (g)(\rho)$. Now, $\langle Y^{crip},\mathfrak{F},\partial\rangle$ being

NSbHausdorff space so there exists NSbopen sets $\langle g, \partial\rangle$and $\langle \mathfrak{H}, \partial\rangle$ of $(f)(\rho)$ and $(g)(\rho)$ respectively such that $\langle g, \partial\rangle$and $\langle \mathfrak{H}, \partial\rangle$ such that these NS sets such that the possibility of one rules out the possibility of other. By soft continuity of $\langle f, \partial\rangle, \langle g, \partial\rangle, \langle f, \partial\rangle^{-1}$ as well as $\langle g, \partial\rangle^{-1}$is NSbopen nhd of ρ and therefore,so is $\langle f, \partial\rangle^{-1} \tilde{\cap} \langle g, \partial\rangle^{-1}$ is contained in $\{(x^e{}_{(a,b,c)}, Ⴏ^{parameter}) \in \mathcal{X}^{crip}: (f)((x^e{}_{(a,b,c)}, Ⴏ^{parameter})) = (g)((x^e{}_{(a,b,c)}, Ⴏ^{parameter}))\}$, for, $(x^e{}_{(a,b,c)}, Ⴏ^{parameter}) \in (\langle f, \partial\rangle^{-1} \tilde{\cap} \langle g, \partial\rangle^{-1}) \Rightarrow (f)((x^e{}_{(a,b,c)}, Ⴏ^{parameter})) \in \langle g, \partial\rangle$ and $(g)((f)((x^e{}_{(a,b,c)}, Ⴏ^{parameter})) \neq (g)((x^e{}_{(a,b,c)}, Ⴏ^{parameter}))$ because $\langle g, \partial\rangle$ and $\langle \mathfrak{H}, \partial\rangle$ are mutually exclusive. This implies that $x\ does\ not\ belong\ to\ \{(x^e{}_{(a,b,c)}, Ⴏ^{parameter}) \in \mathcal{X}^{crip}: (f)((x^e{}_{(a,b,c)}, Ⴏ^{parameter})) = (g)((x^e{}_{(a,b,c)}, Ⴏ^{parameter}))\}$. $Therefore \left\{\begin{matrix} \rho \in (f)^{-1}(\langle g, \partial\rangle) \tilde{\cap} (g)^{-1}(\langle g, \partial\rangle) \\ \Subset \\ \left\{\begin{matrix} (x^e{}_{(a,b,c)}, Ⴏ^{parameter}) \in \mathcal{X}^{crip} \\ : (f)\left((x^e{}_{(a,b,c)}, Ⴏ^{parameter})\right) = \\ (g)\left((x^e{}_{(a,b,c)}, Ⴏ^{parameter})\right) \end{matrix}\right\}^c \end{matrix}\right.$ $This\ shows\ that$ $\left\{\begin{matrix} (x^e{}_{(a,b,c)}, Ⴏ^{parameter}) \in \mathcal{X}^{crip}: \\ (f)\left((x^e{}_{(a,b,c)}, Ⴏ^{parameter})\right) = \\ (g)\left((x^e{}_{(a,b,c)}, Ⴏ^{parameter})\right) \end{matrix}\right\}^c$ is nhd of each of its points. So, $\left\{\begin{matrix} (x^e{}_{(a,b,c)}, Ⴏ^{parameter}) \in \mathcal{X}^{crip}: \\ (f)\left((x^e{}_{(a,b,c)}, Ⴏ^{parameter})\right) = \\ (g)\left((x^e{}_{(a,b,c)}, Ⴏ^{parameter})\right) \end{matrix}\right\}^c$ NSb open and hence $\{(x^e{}_{(a,b,c)}, Ⴏ^{parameter}) \in \mathcal{X}^{crip}: (f)((x^e{}_{(a,b,c)}, Ⴏ^{parameter})) = (g)((x^e{}_{(a,b,c)}, Ⴏ^{parameter}))\}$ is NSb closed.

Theorem 7.22. Let $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle NSSTS$ such that it is NSb Hausdorff space and let (f) be soft continuous function of $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial\rangle$ into itself. Then, the NS set of fixed points under (f) is a NSb closed set.

Proof: Let $\delta = \{(f)\left((x^e{}_{(a,b,c)}, Ⴏ^{parameter})\right) = (x^e{}_{(a,b,c)}, Ⴏ^{parameter})\}$. If $\delta^c = \tilde{\emptyset}$, Then is NSb open and therefore $\{(f)\left((x^e{}_{(a,b,c)}, Ⴏ^{parameter})\right) = (x^e{}_{(a,b,c)}, Ⴏ^{parameter})\}NSb$ closed. So, let $\{(f)\left((x^e{}_{(a,b,c)}, Ⴏ^{parameter})\right) = (x^e{}_{(a,b,c)}, Ⴏ^{parameter})\}^c \neq \tilde{\emptyset}$ and let $(y^{e/}{}_{(a/,b/,c/)}, Ⴏ^{parameter}) \in \{(f)\left((x^e{}_{(a,b,c)}, Ⴏ^{parameter})\right) = (x^e{}_{(a,b,c)}, Ⴏ^{parameter})\}^c$ .Then, $(y^{e/}{}_{(a/,b/,c/)}, Ⴏ^{parameter})\ does\ not\ belong\ to\ \{(f)\left((x^e{}_{(a,b,c)}, Ⴏ^{parameter})\right) = (x^e{}_{(a,b,c)}, Ⴏ^{parameter})\}$ and therefore $(f)\left((y^{e/}{}_{(a/,b/,c/)}, Ⴏ^{parameter})\right) \neq (y^{e/}{}_{(a/,b/,c/)}, Ⴏ^{parameter})$. Now, $(y^{e/}{}_{(a/,b/,c/)}, Ⴏ^{parameter})$ and

$(\not{f})\left(\left(\mathcal{Y}^{e/}{}_{(\mathrm{a}/,\mathrm{b}/,\mathrm{c}/)}, \mathrm{Ⅎ}^{parameter}\right)\right)$ being two distinct points of the NSb Hasdorff space $\langle \mathcal{X}^{crip}, \mathfrak{T}, \partial \rangle$, so there exists NSb open sets $\langle \mathcal{g}, \partial \rangle$ and $\langle \mathfrak{H}, \partial \rangle$ such that $\left(\mathcal{Y}^{e/}{}_{(\mathrm{a}/,\mathrm{b}/,\mathrm{c}/)}, \mathrm{Ⅎ}^{parameter}\right) \in \langle \mathcal{g}, \partial \rangle, (\not{f})\left(\left(\mathcal{Y}^{e/}{}_{(\mathrm{a}/,\mathrm{b}/,\mathrm{c}/)}, \mathrm{Ⅎ}^{parameter}\right)\right) \in \langle \mathfrak{H}, \partial \rangle$ and $\langle \mathcal{g}, \partial \rangle, \langle \mathfrak{H}, \partial \rangle$ are disjoint. Also, by the NS continuity of $(\not{f})$, $(\not{f})^{-1}(\langle H, \partial \rangle$ is NS b open set containing $\mathcal{Y}$. We prtend that $\langle \mathcal{g}, \partial \rangle \tilde{\cap} (\not{f})^{-1}(\langle \mathfrak{H}, \partial \rangle) \Subset \{(\not{f})\left(\left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\right) = \left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\}^{c}$. Snice, $\mu \in \langle \mathcal{g}, \partial \rangle \tilde{\cap} (\not{f})^{-1}(\langle \mathfrak{H}, \partial \rangle) \Rightarrow \mu \in \langle \mathcal{g}, \partial \rangle \& \mu \in (\not{f})^{-1} \Rightarrow \mu \in \langle \mathcal{g}, \partial \rangle \& (\not{f})(\mu) \in \langle \mathfrak{H}, \partial \rangle \Rightarrow \mu \neq (\not{f})(\mu). As \langle \mathcal{g}, \partial \rangle \tilde{\cap} \langle \mathfrak{H}, \partial \rangle = \tilde{\emptyset}. \Rightarrow$

$\mu\ does\ not\ belong\ to\ \{(\not{f})\left(\left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\right) = \left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\} \Rightarrow \mu \in$

$\{(\not{f})\left(\left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\right) = \left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\}^{c}. Therefore, \left(\mathcal{Y}^{e/}{}_{(\mathrm{a}/,\mathrm{b}/,\mathrm{c}/)}, \mathrm{Ⅎ}^{parameter}\right) \in$

$\langle \mathcal{g}, \partial \rangle \tilde{\cap} (\not{f})^{-1}(\langle \mathfrak{H}, \partial \rangle) \Subset \{(\not{f})\left(\left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\right) = \left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\}^{c}$. Thus, $\{(\not{f})\left(\left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\right) = \left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\}^{c}$ is the NS nhd of each of its points. So, $\{(\not{f})\left(\left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\right) = \left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\}^{c}$ is NSb open and hence $\{(\not{f})\left(\left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\right) = \left(x^{e}{}_{(\mathrm{a},\mathrm{b},\mathrm{c})}, \mathrm{Ⅎ}^{parameter}\right)\}$ is NS b-closed.

8. Conclusion

In this paper, neutrosophic soft points with one point greater than the other and their properties, generalized neutrosophic soft open set known as b-open set, neutrosophic soft separation axioms theoretically and with support of suitable examples with respect to soft points, neutrosophic soft b_0-space engagement with generalized neutrosophic soft closed set, neutrosophic soft b_2 -space engagement with generalized neutrosophic soft open set are addressed. In continuation, neutrosophic soft b_0-space behave as neutrosophic soft b_2-space with the plantation of some extra condition on soft b_0-space, neutrosophic soft b_3-space and related theorems, neutrosophic soft b_4-space, monotonous behavior of neutrosophic soft function with connection of different neutrosophic soft separation axioms, monotonous behavior of neutrosophic soft function with connection of different neutrosophic soft close sets are reflected. Secondly, long touched has been given to neutrosophic soft countability connection with bases and sub-bases, neutrosophic soft product spaces and its engagement through different generalized neutrosophic soft open set and close sets, neutrosophic soft coordinate spaces and its engagement through different generalized neutrosophic soft open set and close sets, Finally, neutrosophic soft countability and its relationship with Bolzano Weirstrass Property through engagement of compactness, neutrosophic soft strongly spaces and its related theorems, neutrosophic soft sequences and its relation with neutrosophic soft compactness, neutrosophic soft Lindelof space and related theorems are supposed to address.

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Conflicts of Interest

The authors declare that they have no conflict of interest.

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Neutrosophic Boolean Rings

T.Chalapathi [1,*] and L. Madhavi [2]

1 Department of Mathematics, Sree Vidyanikethan Engg. College, Tirupati-517 102, Andhra Pradesh, India.
chalapathi.tekuri@gmail.com
2 Department of Applied Mathematics, Yogi Vemana University, Kadapa-516 003, Andhra Pradesh, India.
lmadhaviyvu@gmail.com
* Correspondence: chalapathi.tekuri@gmail.com; Tel.: (+91 9542865332)

Abstract: In this paper, we are going to define Neutrosophic Boolean rings and study their algebraic structure. A finite Boolean ring R satisfies the identity $a^2 = a$ for all $a \in R$, which implies the identity $a^n = a$ for each positive integer $n \geq 1$. With this as motivation, we consider a Neutrosophic Boolean ring $N(R,I)$ which fulfils the identity $(a+bI)^2 = a+bI$ for all $a+bI \in N(R,I)$ and describes several Neutrosophic rings which are Neutrosophic Boolean rings with various algebraic personalities. First, we show a necessary and sufficient condition for a Neutrosophic ring of a classical ring to be a Neutrosophic Boolean ring. Further, we achieve a couple of properties of Neutrosophic Boolean rings satisfied by utilizing the Neutrosophic self-additive inverse elements and Neutrosophic compliments.

Keywords: Boolean ring; Neutrosophic Boolean ring; Neutrosophic self-additive inverse elements; Neutrosophic compliments; Self and Mutual additive inverses.

1. Introduction

Essentially, a component a of a ring is idempotent if $a^2 = a$. A Boolean ring is a ring with unity wherein each component is idempotent. In any case, a ring with unity is by definition a ring with a recognized component 1 that goes about as a multiplicative identity and that is particular from the added substance character 0. The impact of the last stipulation is to prohibit from thought the insignificant ring comprising of 0 alone. The expression with unit is in some cases excluded from the meaning of a Boolean ring; in that facilitate our current idea is known as a Boolean ring with unity. Every Boolean ring contains 0 and 1; the simplest Boolean ring contains nothing else. To be sure, the ring of numbers modulo 2 is a Boolean ring. This specific Boolean ring will be signified all through by a similar image as the ordinary integer 2. However, it is exceptionally helpful. It is accordance with von Neumann's meaning of ordinal number, with sound general standards of notational economy, and in logical expressions such as two-honored with idiomatic linguistic usage. A non-trivial and common case of a Boolean ring is the set 2^X arrangement of all functions from an arbitrary non-empty set X into 2. The components of 2^X will be called 2-valued functions on X. The recognized components and operations in 2^X are defined point wise. This means that 0 and 1 in 2^X are the functions defined, for each x in X, by $0(x)=0$ and $1(x)=1$, and, if f and g are 2-valued functions on X, then the functions $f+g$ and fg are defined by $(f+g)=f(x)+g(x)$ and $(fg)=f(x)g(x)$. These equations make sense; their right sides refer to elements of 2. The assumption $X\neq\phi$ needed to guarantee that 0 and 1 are distinct.

Next the usefulness of finite non-trivial Boolean rings has become increasingly apparent in the modern computer system theory, modern design theory, algebraic coding theory, algebraic cryptography, and electric circuit design theory. In particular, the electric circuit design of computer chips can be expressed in terms of finite Boolean rings with two components 0 and 1 as major elements. In this paper, we will consider Neutrosophic Boolean rings with three components 0, 1 and I of its significant components, and the results of these Neutrosophic Boolean rings can easily be generalized to the design of modern systems and the construction of integrated modern computer circuitry with indeterminate I.

As often occurs, the primary Neutrosophic theory research in pure and applied mathematics became indispensable in a large variety of applications in engineering and applied sciences. But Neutrosophic Boolean logic, Neutrosophic Boolean rings, and Neutrosophic Boolean algebra have become essential in the modern design of the large scale integrated circuitry found on today's modern computer chips. Additionally, sociologists and philosophical theorists have used Neutrosophic Boolean logic and their corresponding algebras to model social hierarchies; biologists, genetic engineers, and neurologists have used them to describe Neutrosophic biosystems with indeterminate I, see [1-8].

Throughout this paper, let all classical rings and Neutrosophic rings are considered to be finite and commutative structures with unity 1 and indeterminate I. Also, the present paper deals with Neutrosophic Boolean rings with generalized algebraic properties and define corresponding Neutrosophic units and Neutrosophic compliments. Further, we consider the cardinality of the finite Neutrosophic Boolean ring $N(R,I)$ which is defined by $|N(R,I)|>2$.

Furthermore, almost all our classical notions and their corresponding results are standard and follow those from [9-10]. The other non-classical ring concepts and their terminology will be explained in detail. Let R be a finite ring. The non-empty set

$$N(R,I)=\langle R,I\rangle=\{a+bI:a,b\in R,I^2=I\}$$

is called Neutrosophic ring generated by R and I under the operations of R, where I is called Neutrosophic unit with specific properties

(i). $I\neq 0,1$,

(ii). $I^2=I$,

(iii). $I+I=2I$

(iv). I^{-1} does not exist.

(v) $aI=0$ if and only if $a=0$, and

(vi) $aI=bI$ if and only if $a=b$.

If R is a commutative ring with unity 1 then $N(R,I)$ is also a Neutrosophic commutative ring with unity 1. An element u in R is a unit (multiplicative inverse element) if there exists u^{-1} in R such that $u^{-1}u=1=uu^{-1}$. The set of units of R is denoted by $R^{\times}$. But the set of Neutrosophic units denoted by $R^{\times}I$ and defined as $R^{\times}I=\{uI:u\in R^{\times}\}$, see{14}. But, the Neutrosophic group units denoted by $N(R,I)$ and defined as $(a+bI)^2=a+bI$ $=R^{\times}\cup R^{\times}I$ where $R^{\times}\cap R^{\times}I=\phi$. For further details about finite Neutrosophic rings, the reader should refer [11-17].

In this paper, we shall adopt the definition of a modern abstract mathematical structure known as the Boolean ring introduced by famous mathematician George Boole (1815 – 1864). This ring became an essential tool for the analysis and design modern digital systems, electronic computers, dial telephones, switching systems and many kinds of electronic devices and Fuzzy systems. First, we consider some definitions and results related to finite Boolean rings. An element a of a ring R is called idempotent if $a^2 = a$. In the integral domain, the only idempotent are 0 and 1. But, there exist many rings, which contain idempotent elements of different from 0 and 1. A ring with unity is called the Boolean ring if every element of R is an idempotent element. A finite Boolean ring R is a field if and only if R is isomorphic to Z_2, where Z_2 is the ring of integers modulo 2. Also, every nontrivial Boolean ring is commutative and its characteristic is 2. These results tend to particularly easy. Most of the results in this section can be found in [18].

2. Properties of Neutrosophic Boolean Rings

In this section, we are going to define Neutrosophic Boolean rings and study their properties with different illustrations and examples.

Definition.2.1 A Neutrosophic ring $N(R,I)$ is called **Neutrosophic Boolean ring** if $(a+bI)^2 = a+bI$ for all Neutrosophic elements $a+bI$ in $N(R,I)$.

Example.2.2 The Neutrosophic Boolean ring $N(Z_2,I) = \{0,1,I,1+I\}$ is a Neutrosophic Boolean ring of integers modulo 2 because $0^2 = 0$, $1^2 = 1$, $I^2 = I$, $(1+I)^2 = 1+I$.

Now we begin a necessary and sufficient condition for Neutrosophic Boolean rings.

Theorem. 2.3 The ring R is Boolean if and only if the Neutrosophic ring $N(R,I)$ is Neutrosophic Boolean ring.

Proof: Suppose R is a finite ring with unity 1. Then by the concepts of Boolean rings, we have R is a Boolean ring if and only if $a^2 = a$, $b^2 = b$, $ab = ba$, $2ab = 0$ for every $a,b \in R$. It is clear that for any Neutrosophic element $a+bI$ in the Neutrosophic ring $N(R,I)$,

$$(a+bI)^2 = (a+bI)(a+bI) = a^2 + 2abI + b^2I = a + 0I + bI \ a + bI.$$

Hence, $N(R,I)$ is Neutrosophic Boolean ring. The converse part is trivial.

Corollary.2.4 For any finite Neutrosophic Boolean ring, the following identities hold good.

(1) $(a+bI)^n = a+bI$

(2) $(aI+bI)^n = (a+b)I$ where n **is a** positive integer .

Proof: Follows from the identities $I^2 = I$, $a^n = a$, $b^n = b$,and $(a+b)^n = a+b$ for every positive integer n.

Theorem (Cauchy's theorem).2.5 Every finite abelian group has an element of prime order.

We recall that the notion $|N(R,I)|$ for the cardinality of a Neutrosophic ring $N(R,I)$. If $N(R,I)$ is a finite nontrivial Neutrosophic ring then we denote the subset of its non zero elements by $N(R,I)^*$ and the subset of Neutrosophic units by $R^{\times}I$. Note that $|R| \neq |N(R,I)| \neq \{0\}$. For finite

Neutrosophic Boolean rings, we have the following two preparatory results. Recall from[14], let $R^{\times} = \{u \in R : \exists v \in R, uv = 1 = vu\}$ be the set of group units of a ring R . Then

$N(R^{\times}, I) = R^{\times} \cup R^{\times}I$ and

$N(R,I)^{\times} = \{a+bI : \exists c+dI \in N(R,I), (a+bI)(c+dI) = 1\}$

be the set of Neutrosophic group units and Neutrosophic ring units of the Neutrosophic ring $N(R,I)$, respectively. For instance,

$Z_4 = \{0,1,2,3\}$,

$N(Z_4, I) = \{0,1,2,3,I,2I,3I,1+I,2+I,3+I,1+2I,2+2I,3+2I,1+3I,2+3I,3+3I\}$,

$N(Z_4^{\times}, I) = \{1,3,I,3I\}$, and

$N(Z_4, I)^{\times} = \{1+2I, 3+2I\}$.

Theorem.2.6 For some positive integer k , the total number of elements in the finite Neutrosophic Boolean ring $N(R,I)$ is 2^{2k} .

Proof: Suppose $|N(R,I)| = n$. We shall show that $n = 2^{2k}$ for some positive integer k . Assume that $n \neq 2^{2k}$ then n has a prime factor p other than 2 . Since $N(R,I)$ is an additive group with respect to Neutrosophic addition (+), $(a+bI)+(c+dI) = (a+c)+(b+dI)$ for all Neutrosophic elements $a+bI$ and $c+dI$ in $N(R,I)$. By the Cauchy's Theorem [14] for finite abelian groups, the group $N(R,I)$ contains an element $a+bI \neq 0$ with order prime p . Therefore,

$p(a+bI) = 0 \Rightarrow (2m+1)(a+bI) = 0$ where $p = 2m+1, m > 1$

$\Rightarrow 2m(a+bI)+(a+bI) = 0$, since the characteristic of $N(R,I)$ is 2

$\Rightarrow a+bI = 0$,

which is a contradiction to the fact that $a+bI \neq 0$. This completes the proof..

Theorem.2.7 If $N(R,I)$ is a Neutrosophic Boolean ring with unity 1 then $N(R^{\times}, I) = \{1, I\}$

Proof: Since $I^2 = I$. It is evident that I is the Neutrosophic unit of the Neutrosophic Boolean ring $N(R,I)$. Therefore, $N(R^{\times}, I) = \langle R^{\times}, I\rangle = R^{\times} \cup R^{\times}I$ where $R^{\times}$ and $R^{\times}I$ are disjoint. Suppose now that $u \in R^{\times}$. Then, now multiplying the expression $u^2 = u$ by u^{-1} , we obtain $u = 1$. Thus $R^{\times}$ contains the unique element 1 if and only if R is a nontrivial Boolean ring. This implies that $R^{\times}I = \{I\}$. Hence, $N(R^{\times}, I) = R^{\times} \cup R^{\times}I = \{1\} \cup \{I\} = \{1, I\}$.

Theorem. 2. 8 For any finite non trivial Boolean ring R , we have $N(R,I)^{\times}$ is empty.

Proof. Suppose that R is a finite Boolean ring. Then its definition satisfies the identity $a^2 = a$ for every a in R . Now we shall show that $N(R,I)^{\times}$ is empty. If possible assume that $N(R,I)^{\times}$ is non empty, then there is a Neutrosophic element $a+bI$ in $N(R,I)^{\times}$ such that $a \neq 0$, $b \neq 0$ and $(a+bI)^2 = 1$. This implies that

$a^2 + (b^2 + 2ab)I = 1 + 0I \Rightarrow a^2 = 1, b^2 + 2ab = 0$

$\Rightarrow a = 1, b(b+2a) = 0$

$\Rightarrow a = 1, b = 0$, or, $a = 1, b = -2$, which is a contradiction to the fact that R is a finite non trivial Boolean ring and $N(R,I)$ is its corresponding Neutrosophic Boolean ring. So our assumption is not true, and hence $N(R,I)^{\times}$ is empty.

Now we can immediately prove that a special relationship between divisor of zero and simple Neutrosophic field.

Theorem. 2.9 If a Neutrosophic Boolean ring $N(R,I)$ contains no divisor of zero, then it is either $\{0\}$, or, is isomorphic to Neutrosophic field $N(Z_2,I)$.

Proof: Suppose $N(R,I)$ is a Neutrosophic Boolean ring. Then for any two Neutrosophic elements $a+bI$ and $c+dI$ in $N(R,I)$ we have the following relation

$$(a+bI)(c+dI)[(a+bI)+(c+dI)] = (a+bI)^2(c+dI)+(a+bI)(c+dI)^2$$
$$= (a+bI)(c+dI)+(a+bI)(c+dI)$$
$$= 2(a+bI)(c+dI) = 0,$$

Since $N(R,I)$ is a Neutrosophic Boolean ring and its characteristic is 2. This implies that

$$(a+bI)(c+dI)[(a+bI)+(c+dI)] = 0$$

Therefore, either $(a+bI)(c+dI)=0$, or, $(a+bI)+(c+dI)=0$. Hence, either $N(R,I)$ has a divisor of zero, or, $(a+bI)+(c+dI)=0$ for any two Neutrosophic elements $a+bI$ and $c+dI$ in $N(R,I)$**.** In later case, that is, $(a+bI)+(c+dI)=0$ implies that $(a+bI) = -(c+dI) = c+dI$, it follows that $a=c$ and $b=d$, and R can have only one non zero element, that is, R is isomorphic to the field Z_2, and thus $N(R,I)$ isomorphic to Neutrosophic field $N(Z_2,I)$.

For general Neutrosophic ring $N(R,I)$, the following theorem is obvious when the characteristic of $N(R,I)$ is 2, and after we shall show that a Neutrosophic ring is Neutrosophic Boolean ring when it satisfies the identity $(a+bI)^3 = a+bI$.

Theorem. 2.10 Let $N(R,I)$ be a Neutrosophic commutative ring with unity and its characteristic is 2. Then the following identities are held good in $N(R,I)$.

(1) $(1+(a+bI))^2 = 1+(a+bI)^2$

(2) $(1+(a+bI))^4 = 1+(a+bI)^4$

(3) $(I+(a+bI))^2 = I+(a+bI)^2$.

Theorem.2.11 Let $N(R,I)$ be a Neutrosophic commutative ring with unity and it satisfies the identity $(a+bI)^3 = a+bI$ for all $a+bI$ in $N(R,I)$. Then $N(R,I)$ is a Neutrosophic Boolean ring.

Proof: Since $|N(R,I)| \geq 4$ for any ring R with $|R|>1$. Then clearly $1, I \in N(R,I)$ and the identity $(a+bI)^3 = a+bI$ for all $a+bI \in N(R,I)$ implies that the characteristic of $N(R,I)$ is 2. For this reason, the following are true in $N(R,I)$.

$$a+bI = (a+bI+1)+1 \text{ and } a+bI = (a+bI+I)+I.$$

Hence, by the Theorem [2.10] and by the identity $(a+bI)^3 = a+bI$, the following is holds good.

$$1+(a+bI) = (1+(a+bI))^3 = (1+(a+bI))(1+(a+bI))^2$$
$$= (1+(a+bI))(1+(a+bI)^2)$$
$$= 1+(a+bI)+(a+bI)^2+(a+bI)^3$$
$$= 1+(a+bI)+(a+bI)^2+(a+bI)$$
$$= 1+(a+bI)^2+2(a+bI)$$
$$= 1+(a+bI)^2+0 = 1+(a+bI)^2.$$

Now using the additive left cancellation law of Neutrosophic rings, we obtain the identity $(a+bI)^2 = a+bI$ for all $a+bI \in N(R,I)$, and hence $N(R,I)$ is a Neutrosophic Boolean ring.

Remark.2.12 The Theorem [2.11] shows that, if Neutrosophic ring with identity $(a+bI)^3 = a+bI$ is Neutrosophic Boolean ring. From this identity, we observe that the characteristic of $N(R,I)$ is 2, which is essential. Otherwise, it is evidence that the Neutrosophic ring $N(Z_6,I)$ satisfies the identity $(a+bI)^3 = a+bI$ but it is not a Neutrosophic Boolean ring because of the characteristic $N(Z_6,I)$ is 6.

Next, the following table [2.13] illustrates the main differences between Boolean rings (classical rings) and Neutrosophic Boolean rings. Consider R and $N(R,I)$ be a finite Boolean ring and its corresponding Neutrosophic Boolean ring, respectively.

Boolean rings	Neutrosophic Boolean rings
(i). $\|R\| = 2^k$.	(i). $\|N(R,I)\| = 2^{2k}$.
(ii). R contains two logical components 0 and 1.	(ii). $N(R,I)$ contains three logical components 0, 1 and I.
(iii). $R^{\times} = \{1\}$.	(iii). $N(R^{\times},I) = \{1,I\}$.
(iv). If R is a field then R isomorphic to Z_2.	(iv). If $N(R,I)$ is a Neutrosophic field then $N(R,I)$ is isomorphic to $N(Z_2,I)$.
(v). $1 \le \|R\| \le 2^k$.	(v). $4 \le \|N(R,I)\| \le 2^{2k}$.

Table. 2.13 Differences between Boolean rings and Neutrosophic Boolean rings.

3. Neutrosophic Complements

In this section, we have mainly obtained some properties satisfied by the Neutrosophic complements of Neutrosophic Boolean rings with unity. Note that the element a is called compliment of b in the ring R if $a+b=1$. The set of all compliments of R is denoted by $Comp(R)$, that is, $Comp(R) = \{(a,b) : a+b=1\}$. Also, the two distinct elements x and y of R are called mutual additive inverses of R if $x+y=0$, and the set of all mutual additive inverses of R is denoted by $M(R)$ and $M(R) = \{(x,y) : x+y=0\}$. In particular, the set $S(R) = \{(x,y) : x+x=0\}$ is called the set of all self additive inverses of R. For more information about self and mutual additive inverses of R, reader refer [15]. Now begin the definition of compliments in Neutrosophic ring.

Definition.3.1 Let $N(R,I)$ be a Neutrosophic ring with unity 1 An element $a+bI$ is called **Neutrosophic compliment** of $c+dI$ in $N(R,I)$ if $(a+bI)+(c+dI)=1$. The set of all these Neutrosophic complement pairs in $N(R,I)$ is denoted by $Comp(N(R,I))$ and defined as

$$Comp(N(R,I)) = \{(a+bI, c+dI) : (a+bI)+(c+dI)=1\}.$$

Note that if $2(a+bI)=1$ then $a+bI$ is called **Neutrosophic self-complement** and the set of all Neutrosophic self-compliments of $N(R,I)$ is denoted by $SComp(N(R,I))$. For example, the pair $(I, 1+I)$ is a Neutrosophic complement pair in $N(Z_2,I)$ because $I+(1+I)=1$.

Theorem. 3.2 Let R be a finite ring with unity 1. Then the pair $(a+bI, c+dI)$ is a Neutrosophic complement pair in $N(R,I)$ if and only if (a,c) is a complimenting pair and (b,d) mutual additive inverse pair in R.

Proof: Suppose $a+bI$ and $c+dI$ be two elements in $N(R,I)$. By the definition of Neutrosophic complement pair, the pair $(a+bI, c+dI)$ is a Neutrosophic complement pair in $N(R,I)$ if and only if $(a+bI)+(c+dI)=1$ if and only if $(a+c)+(b+d)I = 1+0I$ if and only if $a+c=1$ and $b+d=0$.

Corollary. 3.3 If $n > 1$ be a positive integer then the total number of Neutrosophic Complement pairs in $N(Z_n, I)$ is $n/2$ if n is even and is $(n-1)/2$ if n is odd.

Proof: It is obvious from the well-known formula that

$$|Comp(N(Z_n, I))| = n/2 \text{ if } n \text{ is even and } (n-1)/2 \text{ if } n \text{ is odd.}$$

Example.3.4 Since the ring of integers modulo 4 is $Z_4 = \{1,2,3,4\}$. The set complement and Neutrosophic complement pairs of the ring Z_4 is

$$Comp(Z_4) = \{(0,1),(2,3)\} \text{ and } Comp(N(Z_4, I)) = \{(2I, 1+2I),(2+2I, 3+2I)\},$$

respectively.

Theorem.3.5 The following conditions on the Neutrosophic ring $N(R, I)$ with unity 1 are equivalent.

(i). $N(R, I)$ is a Neutrosophic Boolean ring.

(ii). The complement of the Neutrosophic idempotent element is Neutrosophic idempotent

(iii). The Neutrosophic complements are Neutrosophic zero divisors.

Proof: (i)⇒(ii). First suppose $N(R, I)$ is a Neutrosophic Boolean ring with unity 1. Let $c+dI$ be the Neutrosophic complement of Neutrosophic idempotent $a+bI$ in $N(R, I)$. Then

$$(c+dI)^2 = (c+dI)(c+dI) = (1-(a+bI))(1-(a+bI))$$
$$= 1-(a+bI)-(a+bI)+(a+bI)^2 = 1-(a+bI) = c+dI \text{ .}$$

This proves (ii).

(ii) ⇒ (iii). From (ii) we have $(a+bI)+(c+dI)=1$.

Therefore,

$$(a+bI)(c+dI) = (a+bI)(1-(a+bI)) = (a+bI)-(a+bI)^2 = (a+bI)-(a+bI) = 0 \text{ .}$$

This completes (iii).

(iii) ⇒ (i). From (iii) we have

$$(a+bI)+(c+dI)=1 \Rightarrow (a+bI)(c+dI)=0$$

It is clear that the Neutrosophic elements $a+bI$ and $c+dI$ are both Neutrosophic Complements to each other, and this forces that the identity $(1-(a+bI))^2 = 1-(a+bI)$. Hence $N(R, I)$ is a Neutrosophic Boolean ring.

4. Conclusions

In this paper, we address our self a twofold aim: first to review the theory of classical Boolean rings, as we understand it recent, and to construct certain Neutrosophic Boolean rings. Next, we have introduced Neutrosophic complement elements and mainly obtained some properties satisfied by the Neutrosophic complement elements of Neutrosophic Boolean rings. This study understands the new structure basis in Neutrosophic hypothesis which builds up another idea for the comparison of classical Boolean ring and Neutrosophic Boolean ring structures dependent on the use of the indeterminacy idea and the structural information.

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Neutrosophic Weakly Generalized open and Closed Sets

R. Suresh [1,*] and S. Palaniammal [2]

[1] Department of Science and Humanities, Sri Krishna College of Engineering and Technology, Coimbatore, Tamil Nadu, India.
E-mail: rsuresh6186@gmail.com

[2] Principal, Sri Krishna Adithya College of Arts and Science, Coimbatore, Tamil Nadu, India.
E-mail: splvlb@yahoo.com

* Correspondence: rsuresh6186@gmail.com;

Abstract: Smarandache presented and built up the new idea of Neutrosophic concepts from the Neutrosophic sets. A.A. Salama presented Neutrosophic topological spaces by utilizing the Neutrosophic sets. Point of this paper is we present and concentrate the ideas Neutrosophic Weakly Generalized Closed Set in Neutrosophic topological spaces and its Properties are talked about subtleties

Keywords: Neutrosophic Generalized closed sets, Neutrosophic Weakly Generalized Closed Sets, Neutrosophic Weakly Generalized open Sets, Neutrosophic topological spaces

1. Introduction

Smarandache's neutrosophic framework have wide scope of constant applications for the fields of Electrical & Electronic, Artificial Intelligence, Mechanics, Computer Science ,Information Systems, Applied Mathematics , basic leadership. Prescription and Management Science and so forth,.In 1965 ,Zadeh proposed Fuzzy set(FS), and Atanassov [1] proposed intuitionistic Fuzzy set (IFS) in 1983 .Topology is an old style subject, as a speculation topological spaces many sort of topological spaces presented over the year. Smarandache [5] characterized the Neutrosophic set on three segment Neutrosophic sets(T Truth, I-Indeterminacy, F-Falsehood). Neutrosophic topological spaces(NS-T-S) presented by Salama [10] et al., R.Dhavaseelan [3], SaiedJafari are introduced Neutrosophic generalized closed sets Point of this paper is we present and concentrate the ideas Neutrosophic Weakly Generalized Closed Set in Neutrosophic topological spaces and its Properties are talked about subtleties

2. Preliminaries

In this part, we review required essential definition and results of Neutrosophic sets

Definition 2.1 [5] Let N_X^* be a non-empty fixed set. A Neutrosophic set R_1^* is a object having the form

$R_1^* = \{< r, \mu_{R_1^*}(r), \sigma_{R_1^*}(r), \gamma_{R_1^*}(r) >: r \in N_X^*\}$,

$\mu_{R_1^*}(r)$-represents the degree of membership function

$\sigma_{R_1^*}(r)$-represents degree indeterminacy function and then

$\gamma_{R_1^*}(r)$-represents the degree of non-membership function

Definition 2.2 [5] Neutrosophic set $R_1^* = \{< r, \mu_{R_1^*}(r), \sigma_{R_1^*}(r), \gamma_{R_1^*}(r) >: r \in N_X^*\}$, on N_X^* and $\forall r \in N_X^*$ then complement of R_1^* is $R_1^{*C} = \{< r, \gamma_{R_1^*}((r)), 1 - \sigma_{R_1^*}(r), \mu_{R_1^*}(r) >: r \in N_X^*\}$

Definition 2.3 [5] Let R_1^* and R_2^* are two Neutrosophic sets, $\forall r \in N_X^*$

$R_1^* = \{< r, \mu_{R_1^*}(r), \sigma_{R_1^*}(r), \gamma_{R_1^*}(r) >: r \in N_X^*\}$,$R_2^* = \{< r, \mu_{R_2^*}(r), \sigma_{R_2^*}(r), \gamma_{R_2^*}(r) >: r \in N_X^*\}$

Then $R_1^* \subseteq R_2^* \Leftrightarrow \mu_{R_1^*}(r) \leq \mu_{R_2^*}(r), \sigma_{R_1^*}(r) \leq \sigma_{R_2^*}(r) \& \gamma_{R_1^*}(r) \geq \gamma_{R_2^*}(r)\}$

Definition 2.4 [5] Let N_X^* be a non-empty set, and Let $R_1^*$and R_2^* be two Neutrosophic sets are

$R_1^* = \{< r, \mu_{R_1^*}(r), \sigma_{R_1^*}(r), \gamma_{R_1^*}(r) >: r \in N_X^*\}$, $R_2^* = \{< r, \mu_{R_2^*}(r), \sigma_{R_2^*}(r), \gamma_{R_2^*}(r) >: r \in N_X^*\}$Then

1. $R_1^* \cap R_2^* = \{< r, \mu_{R_1^*}(r) \cap \mu_{R_2^*}(r), \sigma_{R_1^*}(r) \cap \sigma_{R_2^*}(r), \gamma_{R_1^*}(r) \cup \gamma_{R_2^*}(r) >: r \in N_X^*\}$

2. $R_1^* \cup R_2^* = \{< r, \mu_{R_1^*}(r) \cup \mu_{R_2^*}(r), \sigma_{R_1^*}(r) \cup \sigma_{R_2^*}(r), \gamma_{R_1^*}(r) \cap \gamma_{R_2^*}(r) >: r \in N_X^*\}$

Definition 2.5 [11] Let N_X^* be non-empty set and NS_τ be the collection of Neutrosophic subsets of N_X^*satisfying, the accompanying properties:
1.$0_N, 1_N \in NS_\tau$
2. $N_{T_1} \cap N_{T_2} \in NS_\tau$ for any $N_{T_1}, N_{T_2} \in NS_\tau$
3. $\cup N_{T_i} \in NS_\tau$ for every $\{N_{T_i}: i \in j\} \subseteq NS_\tau$
Then the space (N_X^*, NS_τ), is called a Neutrosophic topological spaces (NS-T-S).
The component of of NS_τ are called NS-OS (Neutrosophic open set)
and its complement is NS-CS(Neutrosophic closed set)

Example 2.6. Let N_X^* ={r} and $\forall r \in N_X^*$

$R_1^* = \langle r, \frac{5}{10}, \frac{5}{10}, \frac{4}{10}\rangle$, $R_2^* = \langle r, \frac{4}{10}, \frac{6}{10}, \frac{8}{10}\rangle$

$R_3^* = \langle r, \frac{5}{10}, \frac{6}{10}, \frac{4}{10}\rangle$,$R_4^* = \langle r, \frac{4}{10}, \frac{5}{10}, \frac{8}{10}\rangle$

Then the collection $NS_\tau = \{0_N, R_1^*, R_2^*, R_3^*, R_4^* 1_N\}$ is called a NS-T-S on X.

Definition 2.7 Let (N_X^*, NS_τ), be a NS-T-S and $R_1^* = \{< r, \mu_{R_1^*}(r), \sigma_{R_1^*}(r), \gamma_{R_1^*}(r) >: r \in N_X^*\}$ be a Neutrosophic set in N_X^*. Then R_1^* is said to be

1. Neutrosophic α-closed set [2] (NS- αCS in short) NS-cl(NS-in(NS-cl(R_1^*)))⊆ R_1^*,
2. Neutrosophic pre-closed set [14] (NS-PCS in short) NS-cl(NS-in(R_1^*))⊆ R_1^*,
3. Neutrosophic regular closed set [5] (NS-RCS in short) NS-cl(NS-in(R_1^*)) = R_1^*,
4. Neutrosophic semi closed set [7] (NS-SCS in short) NS-in(NS-cl(R_1^*))⊆ R_1^*,
5. Neutrosophic generalized closed set [3] (NS-GCS in short) NS-cl(R_1^* ⊆H whenever R_1^* ⊆H and H is a NS-OS,
6. Neutrosophic generalized pre closed set [9] (NS-GPCS in short) NS-pcl(R_1^*) ⊆ H whenever R_1^* ⊆ H and H is a NS-OS,

7. Neutrosophic α generalized closed set [8] (NS- αGCS in short) NS-α-cl(R_1^*)⊆H whenever R_1^* ⊆H And H is aNS-OS,
8. Neutrosophic generalized semi closed set [13](NS-GSCS in short) NS-Scl(R_1^*)⊆H whenever R_1^* ⊆H and H is aNS-OS.

Definition 2.8. (N_X^*, NS_τ), be a NS-T-S and $R_1^* = \{< r, \mu_{R_1^*}(r), \sigma_{R_1^*}(r), \gamma_{R_1^*}(r) >: r \in N_X^*\}$ N_X^*.Then

Neutrosophic closure of R_1^* is

NS-Cl(R_1^*)= ∩ {H:H is a NS-CS in N_X^* and R_1^* ⊆H}

Neutrosophic interior of R_1^* is

NS-Int(R_1^*)= ∪{M:M is a NS-OS in N_X^* and M⊆ R_1^*}.

Definition 2.9. Let (N_X^*, NS_τ), be a NS-T-S and $R_1^* = \{< r, \mu_{R_1^*}(r), \sigma_{R_1^*}(x), \gamma_{R_1^*}(r) >: r \in N_X^*\}$

NS-Sint(R_1^*)= ∪{ G/G is a NS-SOS in N_X^* and G⊆ R_1^*},
NS-Scl(R_1^*)= ∩{ K/K is a NS-SCS in N_X^* and R_1^* ⊆K }.
NS-αint(R_1^*)= ∪{ G/G is a NS-αOS in N_X^* and G⊆ R_1^*},
NS-αcl(R_1^*)= ∩{ K/K is a NS-αCS in N_X^* and R_1^* ⊆K }.

3. Neutrosophic Weakly Generalized Closed Set

In this section we introduce Neutrosophic weakly generalized closed set and have studied some of its properties.

Definition 3.1 An (NS)S R_1^* in an (NS)TS (N_X^*, NS_τ), is said to be aNeutrosophic weakly generalized closed set ((NS-WG)CS) NS-cl(NS-in(R_1^*))⊆U whenever R_1^* ⊆U, U is (NS)OS in N_X^*.
The family of all (NS-WG)CSs of an (NS)TS (N_X^*, NS_τ), is denoted by (NS-WG)CS(N_X^*).

Example 3.2: Let N_X^* = {r_1^*, r_2^*} and let NS_τ = {0~, N_T, 1~} be a(NS)T on N_X^*

Where $N_T = < r, \left(\frac{1}{10}, \frac{5}{10}, \frac{2}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{6}{10}\right) >$.

Then the (NS)S $R_1^* = < r, \left(\frac{1}{10}, \frac{5}{10}, \frac{1}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{6}{10}\right) >$ is a(NS-WG)CS in N_X^*.

Theorem 3.3: Every (NS)CS is a(NS-WG)CS but not conversely.

Proof: Let R_1^* be a(NS)CS in (N_X^*, NS_τ),.Let U be a Neutrosophic open set such that R_1^* ⊆ U. Since R_1^* is Neutrosophic closed, NS-cl(R_1^*) = R_1^* and hence NS-cl(R_1^*) ⊆U. But NS-cl(NS-in(R_1^*)) ⊆NS-cl(A)⊆U. Therefore NS-cl(NS-in(R_1^*))⊆U. Hence R_1^* is a(NS-WG)CS in N_X^*.

Example 3.4: Let N_X^* = {r_1^*, r_2^*} and let NS_τ = {0~, N_T, 1~} be a(NS) N_T on N_X^*,

where $N_T =< r, \left(\frac{3}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right) >$.

Then the (NS)S $R_1^* =< r, \left(\frac{1}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{6}{10}\right) >$ is a(NS-WG)CS in N_X^*

but not an (NS)CS in N_X^* since NS-cl(R_1^*) =T^C≠ R_1^*

Theorem 3.5: Every (NS) CS is a(NS-WG)CS but not conversely.

Proof: Let R_1^* be a(NS) CS in N_X^* and let R_1^* ⊆U and U is a(NS)OS in (N_X^*, NS_τ),. By hypothesis, NS-cl(NS-in(NS-cl(R_1^*))) ⊆ R_1^*. Therefore NS-cl(NS-in((R_1^*))⊆NS-cl(NS-in(NS-cl(R_1^*)))⊆ R_1^* ⊆U. Therefore NS-cl(NS-in((R_1^*))⊆U. Hence R_1^* is a(NS-WG)CS in N_X^*.

Example 3.6: Let N_X^* = {r_1^*, r_2^*} and
let NS_τ = {0~, N_T, 1~} be a(NS)Ton N_X^* where

N_T =$< r, \left(\frac{4}{10},\frac{5}{10},\frac{4}{10}\right), \left(\frac{2}{10},\frac{5}{10},\frac{6}{10}\right) >$.

Then the (NS)S R_1^* = $< r, \left(\frac{3}{10},\frac{5}{10},\frac{5}{10}\right), \left(\frac{1}{10},\frac{5}{10},\frac{7}{10}\right) >$ is a(NS-WG)CS

but not an (NS) CS in N_X^*

since R_1^* ⊆ N_T but NS-cl(NS-in(NS-cl(R_1^*)))=$< r, \left(\frac{4}{10},\frac{5}{10},\frac{4}{10}\right), \left(\frac{6}{10},\frac{5}{10},\frac{2}{10}\right) > \not\subseteq R_1^*$.

Theorem 3.7: Every (NS)GCS is a(NS-WG)CS but not conversely.

Proof: Let R_1^* be a(NS)GCS in N_X^* and let R_1^* ⊆U and U is a(NS)OS in (N_X^*, NS_τ). Since NS-cl(R_1^*) ⊆U, NS-cl(NS-in(R_1^*)) ⊆ NS-cl(R_1^*).That is NS-cl(NS-in(R_1^*))⊆NS-cl(R_1^*)⊆U. Therefore NS-cl(NS-in(R_1^*))⊆U.Hence R_1^* is a(NS-WG)CS in N_X^*.

Example 3.8: Let N_X^* = {r_1^*, r_2^*} and
let NS_τ = {0~, N_T, 1~} be a(NS) N_T on N_X^*

where N_T =$< r, \left(\frac{2}{10},\frac{5}{10},\frac{6}{10}\right), \left(\frac{3}{10},\frac{5}{10},\frac{5}{10}\right) >$.

Then (NS)S R_1^* = $< r, \left(\frac{1}{10},\frac{5}{10},\frac{7}{10}\right), \left(\frac{3}{10},\frac{5}{10},\frac{5}{10}\right) >$is a(NS-WG)CS

but not an (NS)GCS in N_X^*

since R_1^* ⊆ N_T but NS-cl(R_1^*) = $< r, \left(\frac{6}{10},\frac{5}{10},\frac{2}{10}\right), \left(\frac{5}{10},\frac{5}{10},\frac{3}{10}\right) >\not\subseteq N_T$.

Theorem 3.9: Every (NS)RCS is a(NS-WG)CS but not conversely.

Proof: Let R_1^* be a(NS)RCS in N_X^* and let R_1^* ⊆U and U is a(NS)OS in (N_X^*, NS_τ),.Since R_1^* is (NS)RCS, NS-cl(NS-in(R_1^*)) = R_1^* ⊆ U. This implies NS-cl(NS-in(R_1^*))⊆U. Hence R_1^* is a(NS-WG)CS in N_X^*.

Example 3.10:

Let N_X^* = {r_1^*, r_2^*} and
let NS_τ = {0~, N_T, 1~} be a(NS) Ton N_X^*, where

N_T = $< r, \left(\frac{5}{10},\frac{5}{10},\frac{3}{10}\right), \left(\frac{7}{10},\frac{5}{10},\frac{1}{10}\right) >$.

The (NS)S R_1^* = $< r, \left(\frac{1}{10},\frac{5}{10},\frac{6}{10}\right), \left(\frac{1}{10},\frac{5}{10},\frac{7}{10}\right) >$is a(NS-WG)CS

but not an (NS)RCS in N_X^* since NS-cl(NS-in(R_1^*)) ≠ 0~ R_1^*.

Theorem 3.11: Every (NS)PCS is a(NS-WG)CS but not conversely.

Proof: Let R_1^* be a(NS)PCS in N_X^* and let R_1^* ⊆U and U is a(NS)OS in (N_X^*, NS_τ). By Definition, NS-cl(NS-in(R_1^*))⊆ R_1^* and R_1^* ⊆U. Therefore NS-cl(NS-in(R_1^*)⊆U. Hence R_1^* is a(NS-WG)CS in N_X^*.

Example 3.12:

Let N_X^* = {r_1^*, r_2^*} and
let NS_τ = {0~, N_T, 1~} be a(NS)T on N_X^*,

N_T = $< r, \left(\frac{4}{10},\frac{5}{10},\frac{4}{10}\right), \left(\frac{2}{10},\frac{5}{10},\frac{6}{10}\right) >$.

Then the (NS)S R_1^* =$< r, \left(\frac{8}{10}, \frac{5}{10}, \frac{0}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{5}{10}\right) >$ is a(NS-WG)CS

but not an (NS)PCS in N_X^*

since NS-cl(NS-in(R_1^*)) =$T^C \nsubseteq R_1^*$.

Theorem 3.13: Every (NS) GCS is a(NS-WG)CS .

Proof: Let R_1^* be a(NS) GCS in N_X^* and let $R_1^* \subseteq$ U and U is a (NS)OS in (N_X^*, NS_τ), By Definition, $R_1^* \subseteq$ NS-cl(NS-in(NS-cl(R_1^*)))⊆U. This implies NS-cl(NS-in(NS-cl(R_1^*)))⊆U and NS-cl(NS-in(R_1^*)) ⊆NS-cl(NS-in(NS-cl(R_1^*)))⊆U. Therefore NS-cl(NS-in(R_1^*))⊆U. Hence R_1^* is a(NS-WG)CS in N_X^*.

Example 3.14:

Let N_X^* = {r_1^*, r_2^*} and

let NS_τ = {0~, N_T, 1~} be a(NS)T on N_X^*,

where N_T = $< r, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right) >$.

Then the (NS)S R_1^* = $< r, \left(\frac{2}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right) >$ is a(NS-WG)CS

but not an (NS) GCS in N_X^* since NS-αcl(R_1^*) = 1~ $\nsubseteq N_T$.

Proposition 3.15: (NS)SCS and (NS-WG)CS are independent to each other which can be seen from the following example.

Example 3.16: Let N_X^* = {r_1^*, r_2^*} and

let NS_τ = {0~, N_T, 1~} be a(NS)T on N_X^*

N_T = $< r, \left(\frac{3}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{6}{10}\right) >$.

Then (NS)S R_1^* = N_T is a(NS)SCS

but not an (NS-WG)CS in N_X^* since $R_1^* \subseteq N_T$

but NS-cl(NS-in(R_1^*)) = $< r, \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{2}{10}\right) > \nsubseteq N_T$.

Example 3.17:

Let N_X^* = {r_1^*, r_2^*} and

let NS_τ = {0~, N_T, 1~} be a(NS)T on N_X^*,

N_T =$< r, \left(\frac{8}{10}, \frac{5}{10}, \frac{0}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{1}{10}\right) >$.

Then the (NS)S R_1^* =$< r, \left(\frac{6}{10}, \frac{5}{10}, \frac{2}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right) >$is a(NS-WG)CS

but not an (NS)SCS in N_X^* since NS-in(NS-cl(R_1^*))= 1~ $\nsubseteq R_1^*$.

Proposition 3.18: (NS)GSCS and (NS-WG)CS are independent to each other.

Example 3.19: Let N_X^* = {r_1^*, r_2^*} and

let NS_τ = {0~, N_T, 1~} be a(NS)T on N_X^*,

where N_T = $< r, \left(\frac{2}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right) >$.

Then the (NS)S R_1^* = N_T is a(NS-WG)CS

but not an (NS)GPCS in N_X^* since $R_1^* \subseteq N_T$

but NS-cl(NS-in(R_1^*)= $< r, \left(\frac{6}{10}, \frac{5}{10}, \frac{2}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right) > \nsubseteq R_1^*$.

Example 3.20: Let N_X^* = {r_1^*, r_2^*} and
let NS_τ = {0~, N_T, 1~} be a(NS)T on N_X^*, where

$$N_T = < r,\ \left(\frac{7}{10},\frac{5}{10},\frac{1}{10}\right), \left(\frac{8}{10},\frac{5}{10},\frac{0}{10}\right) > .$$

Then the (NS)S $R_1^* = < r,\ \left(\frac{6}{10},\frac{5}{10},\frac{2}{10}\right), \left(\frac{7}{10},\frac{5}{10},\frac{1}{10}\right) >$ is a(NS-WG)CS

but not an (NS)GSCS in N_X^*
since NS-scl(R_1^*)) = 1~ $\nsubseteq$ N_T.

Remark 3.21:
The union of any two (NS-WG)CSs need not be a(NS-WG)CS in general as seen from the following example.

Example 3.22:
Let N_X^* = {r_1^*, r_2^*} be a(NS)TS and

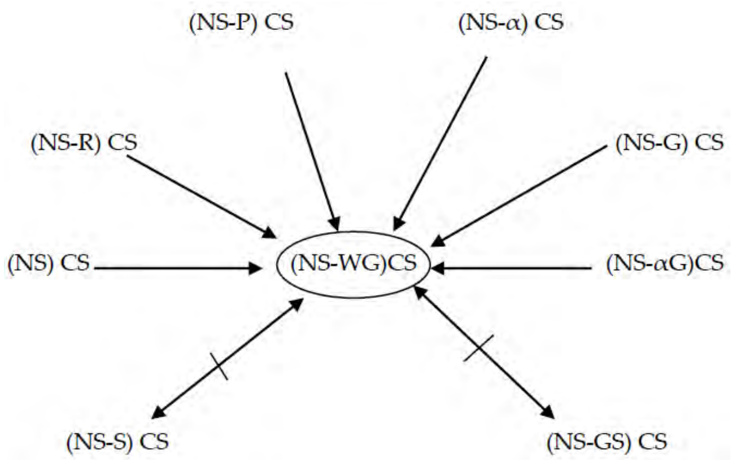
let $N_T = < r,\ \left(\frac{5}{10},\frac{5}{10},\frac{3}{10}\right), \left(\frac{7}{10},\frac{5}{10},\frac{1}{10}\right) >$.

Then NS_τ = { 0~,N_T, 1~ }is a(NS)T on N_X^* and the (NS)Ss

$$R_1^* = < r,\ \left(\frac{0}{10},\frac{5}{10},\frac{8}{10}\right), \left(\frac{7}{10},\frac{5}{10},\frac{1}{10}\right) >,$$

$$R_2^* = < r,\ \left(\frac{5}{10},\frac{5}{10},\frac{3}{10}\right), \left(\frac{6}{10},\frac{5}{10},\frac{2}{10}\right) >$$

are (NS-WG)CSs but $R_1^* \cup R_2^*$ is not an (NS-WG)CS in N_X^*.
The following implications are true:
Fig.1



4. NEUTROSOPHIC WEAKLY GENERALIZED OPEN SET

In this section we introduce Neutrosophic weakly generalized open set and have studied some of its properties.

Definition 4.1: An (NS)S R_1^* is said to be a Neutrosophic weakly generalized open set ((NS-WG)OS in short) in (N_X^*, NS_τ), (NS) the complement $(R_1^*)^C$ is a(NS-WG)CS in N_X^*.
The family of all (NS-WG)OS of an (NS)TS (N_X^*, NS_τ), is denoted by (NS-WG)O(N_X^*).

Example 4.2: Let N_X^* = {r_1^*, r_2^*} and
let NS_τ = {0~, N_T, 1~} be a(NS)T on N_X^*,

where $N_T = < r, \left(\frac{6}{10}, \frac{5}{10}, \frac{1}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{4}{10}\right) >$.

Then the (NS)S $R_1^* =< r, \left(\frac{7}{10}, \frac{5}{10}, \frac{1}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{1}{10}\right) >$ is a(NS-WG)OS in N_X^*.

Theorem 4.3: For any (NS)TS (N_X^*, NS_τ), we have the following:

(i)Every (NS)OS is a(NS-WG)OS.

(ii)Every (NS)SOS is a(NS-WG)OS.

(iii)Every (NS)αOS is a(NS-WG)OS.

(iv)Every (NS)GOS is a(NS-WG)OS. But the converses are not true in general.

Proof: Straight forward.

The converse of the above statement need not be true in general which can be seen from the following examples.

Example 4.4: Let $N_X^* = \{r_1^*, r_2^*\}$ and

$N_T = < r, \left(\frac{6}{10}, \frac{5}{10}, \frac{2}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{3}{10}\right) >$.

Then $NS_\tau = \{0\sim, N_T, 1\sim\}$ is a(NS)T on N_X^*. The (NS)S

$R_1^* =< r, \left(\frac{7}{10}, \frac{5}{10}, \frac{0}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{1}{10}\right) >$ is a(NS-WG)OS in (N_X^*, NS_τ),

but not an (NS)OS in N_X^*.

$N_T = < r, \left(\frac{0}{10}, \frac{5}{10}, \frac{8}{10}\right), \left(\frac{1}{10}, \frac{5}{10}, \frac{6}{10}\right) >$.

Then the (NS)S$R_1^* = < r, \left(\frac{2}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{5}{10}\right) >$is a(NS-WG)OS

but not an (NS)SOS in N_X^*.

Example 4.6: Let $N_X^* = \{r_1^*, r_2^*\}$ and let $NS_\tau = \{0\sim, N_T, 1\sim\}$ be a(NS)T on N_X^*, where $N_T = < r, (0.5, 0.7), (0.5, 0.3) >$.

Then the (NS)S $R_1^* = < r, \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{1}{10}\right) >$ is a(NS-WG)OS

but not an (NS)αOS in N_X^*.

Example 4.7: Let $N_X^* = \{r_1^*, r_2^*\}$ and

$N_T = < r, \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right) >$.

Then $NS_\tau = \{0\sim, N_T, 1\sim\}$ is a(NS)T on N_X^*.

The (NS)S $R_1^* =< r, \left(\frac{6}{10}, \frac{5}{10}, \frac{2}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right) >$ is a(NS-WG)OS

but not an (NS)POS in N_X^*.

Remark 4.8:

The intersection of any two (NS-WG)OSs need not be a(NS-WG)OS in general.

Example 4.9: Let $N_X^* = \{r_1^*, r_2^*\}$ be a(NS)TS and

let $N_T =< r, \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{1}{10}\right) >$.

Then $NS_\tau = \{0\sim, N_T, 1\sim\}$ is a(NS)T on N_X^*.

The (NS)Ss R_1^*= $< r,\ \left(\frac{8}{10},\frac{5}{10},\frac{0}{10}\right),\left(\frac{1}{10},\frac{5}{10},\frac{7}{10}\right) >$ and

$R_2^* = < r,\ \left(\frac{3}{10},\frac{5}{10},\frac{5}{10}\right),\left(\frac{2}{10},\frac{5}{10},\frac{6}{10}\right)$ >are (NS-WG)OS's but $R_1^* \cap R_2^*$ is not an (NS-WG)OS in N_X^*.

Theorem 4.10:

An (NS)S R_1^* of an (NS)TS (N_X^*, NS_τ), is a(NS-WG)OS (NS) and only (NS) F⊆NS-in(NS-cl(R_1^*)) whenever F is a(NS)CS and F⊆ R_1^*.

Proof:

Necessity:

Suppose R_1^* is a(NS-WG)OS in N_X^*. Let F be a(NS)CS and F ⊆ R_1^*. Then F^C is a(NS)OS in N_X^* such that $(R_1^*)^C$ F^C. Since $(R_1^*)^C$ is a(NS-WG)CS,NS-cl(NS-in($(R_1^*)^C$)) ⊆ F^C .Hence (NS-in(NS-cl(R_1^*)))^C ⊆ F^C.This implies F ⊆ NS-in(NS-cl(R_1^*)).

Sufficiency:

Let R_1^* be a(NS)S of N_X^* and let F ⊆ NS-in(NS-cl(R_1^*)) whenever F is a(NS)CS and F⊆ R_1^*. Then $(R_1^*)^C$ ⊆ F^C and F^C is a(NS)OS. By hypothesis, (NS-in(NS-cl(R_1^*)))^C ⊆ F^C. Hence NS-cl(NS-in($(R_1^*)^C$)) ⊆ F^C .

Hence R_1^* is a(NS-WG)OS of N_X^*.

5. APPLICATIONS

In this section, we introduce Neutrosophic$wT^{\frac{1}{2}}$ space and wgqT space, which utilize Neutrosophic weakly generalized closed set and its characterizations are proved.

Definition 5.1:

An (NS)TS (N_X^*, NS_τ), is called an Neutrosophic $wT^{\frac{1}{2}}$ ($(NS)w\,T^{\frac{1}{2}}$in short) space (NS) every (NS-WG)CS in N_X^* is a(NS)CS in N_X^*.

Definition 5.2:

An (NS)TS (N_X^*, NS_τ), is called an Neutrosophic wgqT ((NS-WG)qT in short) space (NS) every (NS-WG)CS in N_X^* is a(NS)PCS in N_X^*.

Theorem 5.3: Every (NS) $wT^{\frac{1}{2}}$ space is a(NS-WG)qT space. But reversal isn't true in general.

Proof: Let N_X^* be a$(NS)wT^{\frac{1}{2}}$space and let R_1^* be a(NS-WG)CS in N_X^*.By hypothesis R_1^* is a(NS)CS in N_X^*. Since every (NS)CS is a(NS)PCS, R_1^* is a(NS)PCS in N_X^*. Hence N_X^* is a(NS)wgqT space.But reversal isn't true in general.

Example 5.4: Let N_X^*= {r_1^*, r_2^*} and

let NS_τ = {0~, N_T, 1~}

$N_T = < r,\ \left(\frac{9}{10},\frac{5}{10},\frac{1}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{1}{10}\right) >$. Then (N_X^*, NS_τ), is a(NS-WG)qT space.

But it is not an $(NS)wT^{\frac{1}{2}}$ space

since the (NS)S R_1^* =$< r,\left(\frac{2}{10},\frac{5}{10},\frac{8}{10}\right),\left(\frac{3}{10},\frac{5}{10},\frac{7}{10}\right) >$ is (NS-WG)CS but not (NS)CS in N_X^*.

Theorem 5.5: Let $(N_X^*,\ NS_\tau)$, be a(NS)TS and N_X^* is a$(NS)wT^{\frac{1}{2}}$space then

(i) Any union of (NS-WG)CS is a(NS-WG)CS.

(ii) Any intersection of (NS-WG)OS is a(NS-WG)OS.

Proof:

(i): Let {Ai}i∈J be a collection of (NS-WG)CS in an $(NS)wT^{\frac{1}{2}}$space $(N_X^*,\ NS_\tau)$. Therefore every (NS-WG)CS is a(NS)CS. But the union of (NS)CS is a(NS)CS. Hence the Union of (NS-WG)CS is a(NS-WG)CS in N_X^*.

(ii): It tends to be demonstrated by taking complement at(i).

Theorem 5.6:

An (NS)TS N_X^* is a(NS-WG)qT space (NS) and only (NS) (NS-WG)OS(N_X^*) = (NS)POS(N_X^*).

Proof:

Necessity:

Let R_1^* be a(NS-WG)OS in N_X^*. Then $(R_1^*)^C$ is a(NS-WG)CS in N_X^*. By hypothesis $(R_1^*)^C$ is a(NS)PCS in N_X^*. Therefore R_1^* is a(NS)POS in N_X^*. Hence (NS-WG)OS(N_X^*) = (NS)POS(N_X^*).

Sufficiency:

Let R_1^* be a(NS-WG)CS in N_X^*. Then $(R_1^*)^C$ is a(NS-WG)OS in N_X^*. By hypothesis $(R_1^*)^C$ is a(NS)POS in N_X^*. Therefore R_1^* is a(NS)PCS in N_X^*. Hence N_X^* is a(NS-WG)qT space.

Theorem 5.7: An (NS)TS N_X^* is a$(NS)wT^{\frac{1}{2}}$space (NS) and only (NS) (NS-WG)OS (N_X^*) = (NS)OS(N_X^*).

Proof: Necessity:

Let R_1^* be a(NS-WG)OS in N_X^*. Then $(R_1^*)^C$ is a(NS-WG)CS in N_X^*. By hypothesis $(R_1^*)^C$ is a(NS)CS in N_X^*. Therefore R_1^* is a(NS)OS in N_X^*. Hence (NS-WG)OS(N_X^*) = (NS)OS(N_X^*).

Sufficiency:

Let R_1^* be a(NS-WG)CS in N_X^*. Then $(R_1^*)^C$ is a(NS-WG)OS in N_X^*. By hypothesis $(R_1^*)^C$ is a(NS)OS in N_X^*. Therefore R_1^* is a(NS)CS in N_X^*. Hence N_X^* is a $(NS)wT^{\frac{1}{2}}$ space.

6. CONCLUSION

In this paper we have presented another class of Neutrosophic closed set to be specific (NS)WG closed set and have examined the connection between Neutrosophic weakly generalized closed set and other existing Neutrosophic closed sets. Likewise we have explored a portion of the properties of Neutrosophic weakly generalized closed set. As an utilization of Neutrosophic weakly generalized closed set we have presented two new spaces specifically $(NS)wT^{\frac{1}{2}}$ space and Neutrosophicwg qT ((NS-WG)qT in short) space and concentrated a portion of their properties.

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Concentric Plithogenic Hypergraph based on Plithogenic Hypersoft sets – A Novel Outlook

Nivetha Martin [1,*] and Florentin Smarandache [2]

[1]Department of Mathematics, Arul Anandar College (Autonomous), Karumathur, Madurai, Tamil Nadu, India; nivetha.martin710@gmail.com
[2] Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA,smarand@unm.edu

* Correspondence: nivetha.martin710@gmail.com

Abstract: Plithogenic Hypersoft sets (PHS) introduced by Smarandache are the extensions of soft sets and hypersoft sets and it was further protracted to plithogenic fuzzy whole Hypersoft set to make it more applicable to multi attribute decision making environment. The fuzzy matrix representation of the plithogenic hypersoft sets lighted the spark of concentric plithogenic hypergraph. This research work lays a platform for presenting the concept of concentric plithogenic hypergraph, a graphical representation of plithogenic hypersoft sets. This paper comprises of the definition, classification of concentric plithogenic hypergraphs, extended hypersoft sets, extended concentric plithogenic hypergraphs and it throws light on its application. Concentric Plithogenic hypergraphs will certainly open the new frontiers of hypergraphs and this will undoubtedly bridge hypersoft sets and hypergraphs.

Keywords: Plithogenic sets; Hypergraph; Plithogenic Hypersoft sets; Concentric Plithogenic Hypergraphs

1. Introduction

The structure of a graph comprises of vertices and edges, has a range of applications in diverse fields. In general an edge in a graph represents the relation between two vertices. Berge [1] extended this basic idea and introduced hypergraph as the generalization of graph. In a hypergraph hyperedge links one or more vertices and they are mainly used to explore configuration of the systems by clustering and segmentation, but to handle the uncertain and imprecise environment; Kaufmann [2] introduced fuzzy hypergraphs. The concept of fuzzy set was introduced by Lofti.A. Zadeh [3]. The fuzzy hypergraph introduced by Kaufmann was later generalized by Hyung and Keon [4] to overcome the limitations of inappropriate representation of fuzzy partition by redefining fuzzy hypergraph and developing many expedient concepts which finds extensive applications in system analysis, circuit clustering and pattern recognition. In a fuzzy hypergraph the hyper edges are fuzzy sets of vertices. Mordeson and Nair [5] have made significant contributions to fuzzy graphs and fuzzy hypergraphs. Parvathi et al [6] extended of intuitionistic fuzzy graphs to intuitionistic fuzzy hypergraph in which (α, β)-cut hypergraph represent intuitionistic fuzzy partition. In an intuitionistic fuzzy hypergraph the hyperedge sets are intuitionistic fuzzy sets of vertices consisting of both membership and non-membership values as Atanssov [7] introduced in Intuitionistic set. Akram and Dudek [8] discussed the properties and applications of intuitionistic fuzzy hypergraph. Akram et al [9] introduced neutrosophic hypergraphs and single valued neutrosophic hypergraphs. Neutrosophic sets introduced by Smarandache [10] deals with truth function, indeterminacy function and falsity function, based on conceptualization of neutrosophic sets, Akram.et al [11] investigated the properties of line graph of neutrosophic hypergraph, dual neutrosophic hypergraph, tempered neutrosophic hypergraph and transversal neutrosophic hypergraph with illustrations. Neutrosophic theory has extensive applications in the domain of decision- making. Abdel-Baset et al [12] introduced a novel neutrosophic approach to assess the green supply chain management practices and the recent research in multi criteria decision- making uses the neutrosophic representations. On other hand Vasantha Kanthasamy et al [13]discussed Plithogenic graph, special type of graphs based on fuzzy intuitionistic and single valued neutrosophic graphs. The characteristics of plithogenic intuitionistic fuzzy graph, plithogenic neutrosophic graphs and plithogenic complex graphs are also examined, but notion of plithogenic hypergraph was not discoursed.

The plithogenic sets introduced by Smarandache [14] deals with attributes and it is extension of crisp, fuzzy, intuitionistic and neutrosophic sets. Plithogenic sets are widely used in multi attribute decision -making systems as it plays a vital role in deriving optimal solutions to the decision-making problems. Abdel-Basset et al [15,16] has framed a novel plithogenic TOPSIS-CRITIC model for sustainable supply chain risk management and formulated a hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. These proposed models are highly advantageous, compatible to make decisions as it handles multi attributes or multi criteria environment. The selection process of alternatives based on different attributes containing several attribute values becomes easier in plithogenic representation. Furthermore plithogenic hypersoft introduced by Smarandache [17] also has significant contribution in multi attribute decision-making methods. Molodtsov [18] introduced soft sets and Smarandache generalized to hypersoft set by modifying single attribute function to multi attribute function. The plithogenic hypersoft set, generalization of crisp, fuzzy , intuitionistic and neutrosophic soft sets. Shazia Rana et al [19] extended plithogenic fuzzy hypersoft set to plithogenic fuzzy whole hypersoft set ;developed Frequency Matrix Multi Attributes Decision making scheme to rank the alternatives and proposed a new ranking approach based on frequency matrix in their research work. Plithogenic hypersoft sets are finding new avenue in decision- making and in ranking process.

Nivetha and Pradeepa [20,21] initiated integration of hypergraphs and fuzzy hypergraphs with Fuzzy Cognitive Maps (FCM). Kosko [22] introduced FCMs, directed graphs consisting of nodes and edges which represent the casual factors and its relationship. FCM assumes simple weights such as -1 if factors have negative impact over another, 0 if no impact and 1 for positive impact. Weighted FCM assumes values from [-1,1]. The approach of FCM is analogues to the reasoning and decision -making of human and it facilitates conception of intricate social systems. Peláezand Bowles [23],Miao and Liu [24], Papageorgiou et al [25]have proposed various algorithms and methods to handle various forms of FCM. The nature of weights classifies FCMs as intuitionistic and neutrosophic FCM. One of the most difficult aspects in handling FCM is consideration of large number of study factors. Confinement of the number of inputs is essential to make optimal decision-making and this has to take place step wise. It is helpful to limit the study factors for analyzing their inter impacts, to make so, FCMs with hypergraphic and fuzzy hypergraphic approaches facilitated to formulate student’s low academic performance model and assessment model of blended method of teaching. Nivetha and Pradeepa [26] also introduced concentric fuzzy hypergraphs for inclusive decision -making and this kind of hypergraph deals with hyper envelopes instead of hyper edges; examined the properties of concentric fuzzy hypergraph and justified with suitable illustrations and applications. Concentric fuzzy hypergraphs was further extended to concentric neutrosophic hypergraphs to explore the factors causing autoimmune diseases using Fuzzy Cognitive Maps (FCM).

The proposed integrated models of FCM with hypergraphs, fuzzy hypergraphs and concentric fuzzy hypergraphs focus only on the factors based on single criteria. Suppose if the factors are dependent on multi criteria then the above integrated models do not meet the need. This is limitation of the above described integrated models. Concentric plithogenic hypergraphs integrated with FCMs helps to overcome such shortcomings. As the concept of plithogenic hypergraph was not disclosed so far, this research work extends concentric hypergraphic approach of FCM to concentric plithogenic hypergraph with plithogenic hypersoft representation to make optimal decisions by ranking the study factors based on multi attribute. Such representations will he highly pragmatic and it will certainly ease the decision-making process. The frequency matrices ranks the factors represented as plithogenic hypersoft sets and the core factors considered for determining the inter relationship and inter impacts using FCM method. This approach will definitely yield optimal results with simplified computations.

The objectives of this research work are to introduce notion of concentric plithogenic hypergraph; define concentric plithogenic hypergraph based on concentric fuzzy hypergraph and plithogenic graphs; classify concentric plithogenic hypergraph based on degree of appurtenance; widens concentric plithogenic hypergraph to extended concentric plithogenic hypergraph; proposes FCM decision making model integrated with extended concentric plithogenic hypergraphic approach. But this research work concentrates on proposing concentric plithogenic hypergraphs in a more distinct way. With the brief introduction of the research work in section 1, the rest of the paper is organized with construction of concentric plithogenic hypergraph in section 2, extension and classification of concentric plithogenic hypergraphs in section 3, applications of the proposed approach in section 4, discussion of the results in section 5 and the concluding remarks in the section 6.

2. Construction of Concentric Plithogenic Hypergraphs

The concept of concentric fuzzy hypergraphs evolved at times of integrating hypergraphs with Fuzzy Cognitive Maps after integration of hypergraphic and fuzzy hypergraphic approaches. Concentric plithogenic hypergraph is an integration concentric hypergraph with plithogenity. To make it more comprehensive, the following definitions are put forward.

2.1 Hypergraph

A hypergraph H is an ordered pair H = (X,E), where

(i) $X = \{x_1, x_2, .. x_n\}$ is a finite set of vertices.

(ii) $E = \{E_1, E_2, .. E_n\}$ is a family of subsets of X and each E_j is a hyper edge.

(iii) $E_j \neq \varphi$, j = 1,2,..3 and $\cup_j E_j = X$

2.2 Concentric Fuzzy Hypergraph

A concentric fuzzy hypergraph $\mathcal{C}_H$ is defined as follows

$\mathcal{C}_H = (X, \mathcal{E})$

X- finite set of vertex set

$\mathcal{E}$- Concentric fuzzy hyper envelope – family of fuzzy sets of X

$\mathcal{E}_j = \{(x_i, \mu_j(x_i)) / \mu_j(x_i) > 0 \text{ and } \forall x_i \in X\}$ j = 1,2,…m

Supp ($\mathcal{E}$) = X = Supp ($\mathcal{E}_j$) $\forall$ j = 1,2,…m

To illustrate concentric fuzzy hypergraph,

let $\mathcal{C}_H = (X, \mathcal{E})$, where

$X = \{x_1, x_2, x_3, x_4\}$

$\mathcal{E}_1 = \{(x_1,0.4), (x_2,0.6), (x_3,0.5), (x_4,0.3)\}$

$\mathcal{E}_2$= {(x1,0.3), (x2,0.8), (x3,0.4),(x4,0.5)}

$\mathcal{E}_3$={(x1,0.6), (x2,0.6), (x3,0.7),(x4,0.4)}

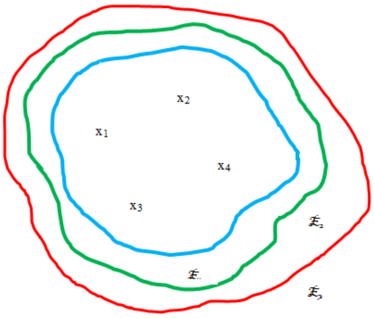

Fig.2.1. Concentric Fuzzy Hypergraph

2.3 Concentric Plithogenic Hypergraph

A hypergraph with non- empty, disjoint hyperedges and plithogenic envelopes is called as concentric plithogenic hypergraph P_{C_H}.

A concentric Plithogenic hypergraph P_{C_H} is defined as follows

P_{C_H} = (X, E, $\mathcal{P}_{\mathcal{E}}$)

- X- Finite vertex set
- E – Hyperedge set
- $\mathcal{P}_{\mathcal{E}}$ - Plithogenic envelope
- $\mathcal{P}_{\mathcal{E}j} = \{(x_{ij}, \mu(x_{ij})) / \mu(x_{ij}) > 0, x_{ij} \in E_i\}$
- $E_i \cap E_k = \emptyset$ and $\cup E_i = X$

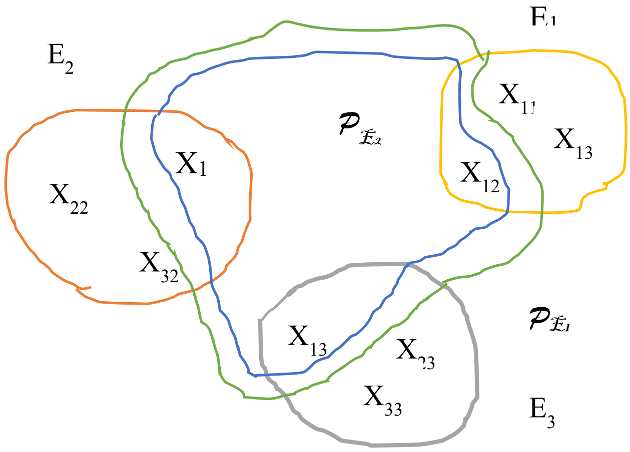

Fig.2.2. Concentric Plithogenic Hypergraph

2.4 Integration of Plithogenic Hypersoft sets and Concentric Plithogenic Hypergraph

Plithogenic hypersoft sets are extensively used in decision making situations. Let us consider the plithogenic hypersoft set presented below.

Let U = { M_1,M_2,M_3,M_4, M_5, M_6,M_7,M_8,M_9, M_{10}} be the university of discourse and set T = {M_1,M_3,M_6} ⊂ U. The attribute system is represented as follows $\mathcal{A}$ = { (A_1)**Maintenance Cost** {Maximum in the initial years of utility(A_1^1), Maximum in the latter years of utility(A_1^2), Moderate throughout (A_1^3)}, (A_2)**Reliability** {High with additional expenditure(A_2^1), Moderate with no extra expense(A_2^2), Moderate with high expense(A_2^3)}, (A_3)**Flexibility** {Single task oriented(A_3^1), Multi task oriented(A_3^2), Dual task oriented(A_3^3) }, (A_4)**Durability** {Very high in the beginning years of service(A_4^1), High in the latter years of service(A_4^2),Moderate (A_4^3) }, (A_5)**Profitability** {Moderate in the initial years(A_5^1), Maximum in the latter years(A_5^2), Moderate throughout the years (A_5^3)}}.

G: $A_1^1 \times A_2^2 \times A_3^2 \times A_4^1 \times A_5^2 \longrightarrow (\mathcal{U})$.

G(A_1^1,A_2^2 , A_3^2 , A_4^1,A_5^2) =

{M_1(0.9,0.875,0.8,0.75,0.5),M_3(0.67,0.5,0.4,0.8,0.7),M_6(0.8,0.7,0.6,0.7,0.5)}

In the below graphical representations the attributes A_1 to A_5 are the hyperedges consisting of the vertices (x_{ij}) A_i^j , i = 1,2,3,4,5 and j = 1,2,3. M_1, M_3 and M_6 are the plithogenic envelopes

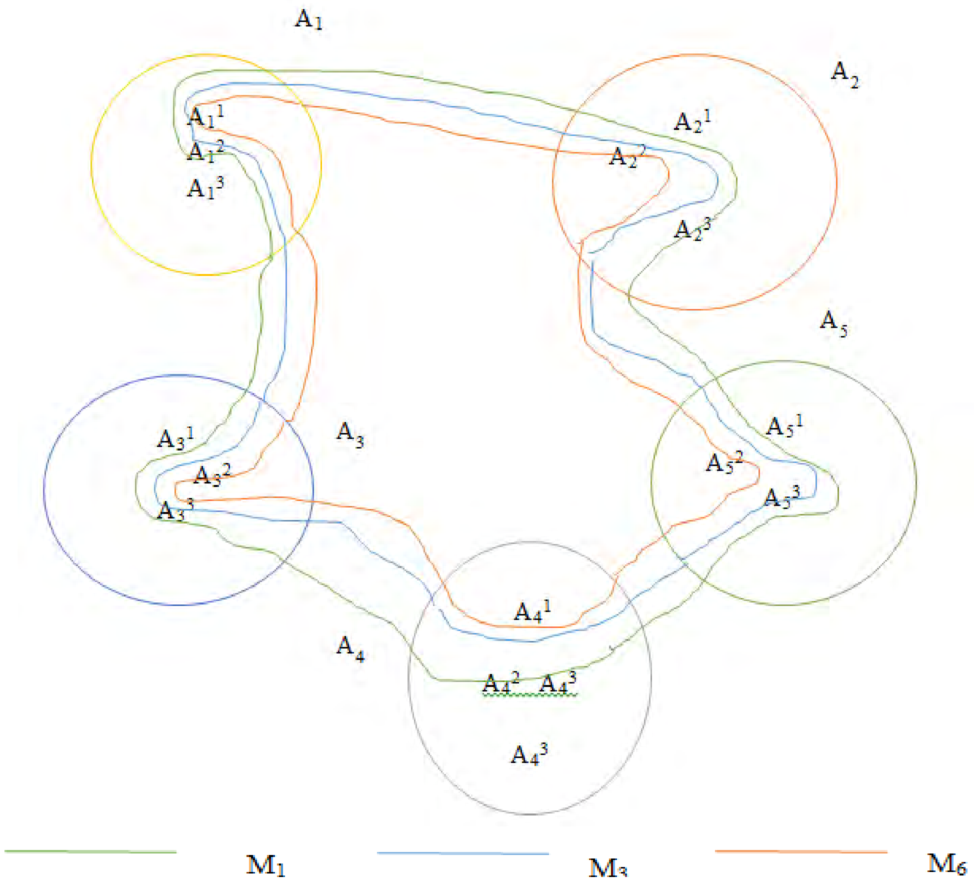

Fig 2.3 Graphical Representation of Plithogenic Hypersoft set by Concentric Plithogenic Hypergraph

To illustrate Concentric Plithogenic Hypergraph P_{C_H}based on Plithogenic Hypersoft sets, Let X= { $A_1^1, A_1^2, A_1^3, A_2^1, A_2^2, A_2^3, A_3^1, A_3^2, A_3^3, A_4^1, A_4^2, A_4^3, A_5^1, A_5^2, A_5^3$}

E = {A_1, A_2,A_3,A_4, A_5}

$\mathcal{P}_{E_1}$ = {(A_1^1,0.9),(A_2^2,0.875),(A_3^2,0.8), (A_4^1,0.75),(A_5^2,0.5)}

$\mathcal{P}_{E_2}$ = {(A_1^1,0.67),(A_2^2,0.5),(A_3^2,0.4), (A_4^1,0.8),(A_5^2,0.7)}

$\mathcal{P}_{E_3}$ = {(A_1^1,0.8),(A_2^2,0.7),(A_3^2,0.6), (A_4^1,0.7),(A_5^2,0.5)}

The formulation of the notion of concentric plithogenic hypergraph based on plithogenic hypersoft will be more rational. Also the plithogenic hypergraphs and concentric plithogenic hypergraphs can be defined based on plithogenic graphs as in the below fig.2.4 and fig.2.5

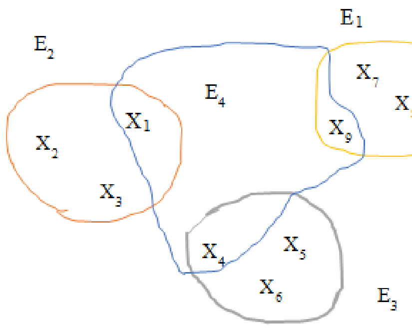

Fig.2.4 Plithogenic hypergraph

To illustrate Plithogenic Hypergraph P_{H^*}, based on Plithogenic graphs,

Let X= {x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9}

E_1= {x_7(0.2, 0.3, 0.4), x_8 (0.8, 0.6, 0.3), x_9 (0.4, 0.2, 0.6)}

E_2= {x_1(0.1, 0.5, 0.4), x_2 (0.5, 0.6, 0.8), x_3 (0.3, 0.2, 0.7)}

E_3= {x_4(0.9, 0.3, 0.5), x_5 (0.2, 0.4, 0.3), x_6(0.6, 0.2, 0.1)}

E_4= {x_1(0.2, 0.9, 0.7), x_4 (0.3, 0.7, 0.9), x_9 (0.1, 0.8, 0.6)}

X is the vertex set and E_i, i = 1,2,3 are the plithogenic hyperedges of plithogenic hypergraph.

A hypergraph with hyperedges possessing plithogenic weight representations is defined as plithogenic hypergraph otherwise Plithogenic hypergraphs can be defined as hypergraphs with plithogenic hyperedges. A hypergraph with plithogenic hyper envelopes is defined as concentric plithogenic hypergraphs.

Fig.2.5 Concentric Plithogenic hypergraph

To illustrate Concentric Plithogenic Hypergraph P_{CH^*}, based on Plithogenic graphs

Let $X = \{x_1,x_2,x_3\}$

$\mathcal{PE}_1 = \{(x_1, (0.4,0.2,0.1), (x_2,(0.2,0.3,0.4)), (x_3,(0.4,0.8,0.5))\}$

$\mathcal{PE}_2 = \{(x_1, (0.8,0.5,0.6), (x_2,(0.4,0.3,0.2)), (x_3,(0.7,0.6,0.1))\}$

$\mathcal{PE}_3 = \{(x_1, (0.7,0.6,0.3), (x_2,(0.1,0.5,0.6)), (x_3,(0.4,0.5,0.8))\}$

X is the vertex set and $\mathcal{PE}_i$, i = 1,2,3 are the plithogenic hyper envelopes of concentric plithogenic hypergraph.

Plithogenic hypergraphs and concentric plithogenic hypergraphs based on plithogenic graphs are distinct from the concentric plithogenic hypergraphs defined based on plithogenic hypersoft sets. The former definition doesn't take the condition of $E_i \cap E_k = \emptyset$. The graphs are called as plithogenic based on their dimension of membership values. In comparison of both kinds of representation, the latter is more feasible in nature as it incorporates the degree of appurtenance and the holistic meaning of plithogeny is reflected.

3. Classification and Extension of Concentric Plithogenic Hypergraphs

The concentric plithogenic hypergraphs is classified into crisp, fuzzy, intuitionistic and neutrosophic based on the values of the degree of appurtenance.

3.1 Crisp Concentric Plithogenic Hypergraphs

Let $X = \{A_1^1, A_1^2, A_1^3, A_2^1, A_2^2, A_2^3, A_3^1, A_3^2, A_3^3, A_4^1, A_4^2, A_4^3, A_5^1, A_5^2, A_5^3\}$

$E = \{A_1, A_2, A_3, A_4, A_5\}$

$\mathcal{P}_{\mathcal{E}_1} = \{(A_1^1,1),(A_2^2,1),(A_3^2,1),(A_4^1,1),(A_5^2,1)\}$

$\mathcal{P}_{\mathcal{E}_2} = \{(A_1^1,1),(A_2^2,1),(A_3^2,1),(A_4^1,1),(A_5^2,1)\}$

$\mathcal{P}_{\mathcal{E}_3} = \{(A_1^1,1),(A_2^2,1),(A_3^2,1),(A_4^1,1),(A_5^2,1)\}$

3.2 Fuzzy Concentric Plithogenic Hypergraphs

Let $X = \{A_1^1, A_1^2, A_1^3, A_2^1, A_2^2, A_2^3, A_3^1, A_3^2, A_3^3, A_4^1, A_4^2, A_4^3, A_5^1, A_5^2, A_5^3\}$

$E = \{A_1, A_2, A_3, A_4, A_5\}$

$\mathcal{P}_{\mathcal{E}_1} = \{(A_1^1,0.9),(A_2^2,0.875),(A_3^2,0.8),(A_4^1,0.75),(A_5^2,0.5)\}$

$\mathcal{P}_{\mathcal{E}_2} = \{(A_1^1,0.67),(A_2^2,0.5),(A_3^2,0.4),(A_4^1,0.8),(A_5^2,0.7)\}$

$\mathcal{P}_{\mathcal{E}_3} = \{(A_1^1,0.8),(A_2^2,0.7),(A_3^2,0.6),(A_4^1,0.7),(A_5^2,0.5)\}$

3.3 Intuitionistic Concentric Plithogenic Hypergraphs

Let $X = \{A_1^1, A_1^2, A_1^3, A_2^1, A_2^2, A_2^3, A_3^1, A_3^2, A_3^3, A_4^1, A_4^2, A_4^3, A_5^1, A_5^2, A_5^3\}$

$E = \{A_1, A_2, A_3, A_4, A_5\}$

$\mathcal{P}_{\mathcal{E}_1} = \{(A_1^1,(0.9,0.1)),(A_2^2,(0.5,0.2)),(A_3^2,(0.8,0.2)),(A_4^1,(0.75,0.5)),(A_5^2,(0.5,0.2))\}$

$\mathcal{P}_{\mathcal{E}_2} = \{(A_1^1,(0.6,0.7),(A_2^2,(0.7,0.5),(A_3^2,(0.9,0.4),(A_4^1,(0.7,0.1),(A_5^2,(0.7,0.2)\}$

$\mathcal{P}_{\mathcal{E}_3} = \{(A_1^1,(0.6,0.4),(A_2^2,(0.2,0.58),(A_3^2,(0.19,0.54),(A_4^1,(0.7,0.2),(A_5^2,(0.8,0.2)\}$

3.4 Neutrosophic Concentric Plithogenic Hypergraphs

Let $X = \{A_1^1, A_1^2, A_1^3, A_2^1, A_2^2, A_2^3, A_3^1, A_3^2, A_3^3, A_4^1, A_4^2, A_4^3, A_5^1, A_5^2, A_5^3\}$

$E = \{A_1, A_2, A_3, A_4, A_5\}$

$\mathcal{P}_{\mathcal{E}_1} = \{(A_1^1,(0.9,0.1,0.2),(A_2^2,(0.8,0.5,0.2),(A_3^2,(0.2,0.2,0.8),(A_4^1,(0.1,0.3,0.75),(A_5^2,(0.4,0.2,0.5)\}$

$\mathcal{P}_{\mathcal{E}_2} = \{(A_1^1,(0.7,0.1,0.2),(A_2^2,(0.6,0.2,0.1),(A_3^2,(0.3,0.2,0.5),(A_4^1,(0.4,0.3,0.5),(A_5^2,(0.6,0.2,0.3)\}$

$\mathcal{P}_{\mathcal{E}_3} = \{(A_1^1,(0.7,0.1,0.2),(A_2^2,(0.6,0.4,0.2),(A_3^2,(0.6,0.2,0.3),(A_4^1,(0.5,0.2,0.7),(A_5^2,(0.6,0.1,0.4)\}$

3.5 Extended Plithogenic Hypersoft sets

Extended plithogenic hypersoft sets comprises of the degree of appurtenance of the elements to the corresponding attributes along with multi expert's opinion. These sets comprise of the opinion of several experts. The representation of such kind of set is presented as follows

Let us consider a situation that the below values are given by two experts for the same example discussed under plithogenic hypersoft sets.

G ($A_1^1, A_2^2, A_3^2, A_4^1, A_5^2$) given by the first expert =

{M_1(0.9,0.875,0.8,0.75,0.5),M_3(0.67,0.5,0.4,0.8,0.7),M_6(0.8,0.7,0.6,0.7,0.5)}

G ($A_1^1, A_2^2, A_3^2, A_4^1, A_5^2$) given by the second expert =

{M_1(0.6,0.875,0.8,0.5,0.5),M_3(0.7,0.6,0.3,0.9,0.7),M_6(0.8,0.7,0.7,0.7,0.6)}

The aggregate representation will be

$\mathcal{G}$ ($A_1^1, A_2^2, A_3^2, A_4^1, A_5^2$) = {$M_1${(0.9,0.875,0.8,0.75,0.5), (0.6,0.875,0.8,0.5,0.5)}, M_3 {(0.67,0.5,0.4,0.8,0.7), (0.7,0.6,0.3,0.9,0.7)}, M_6 {(0.8,0.7,0.6,0.7,0.5), (0.8,0.7,0.7,0.7,0.6)}}

The extended concentric plithogenic hypergraphs are based on the extended plithogenic hypersoft sets. The graphical representation is depicted in fig 3.1

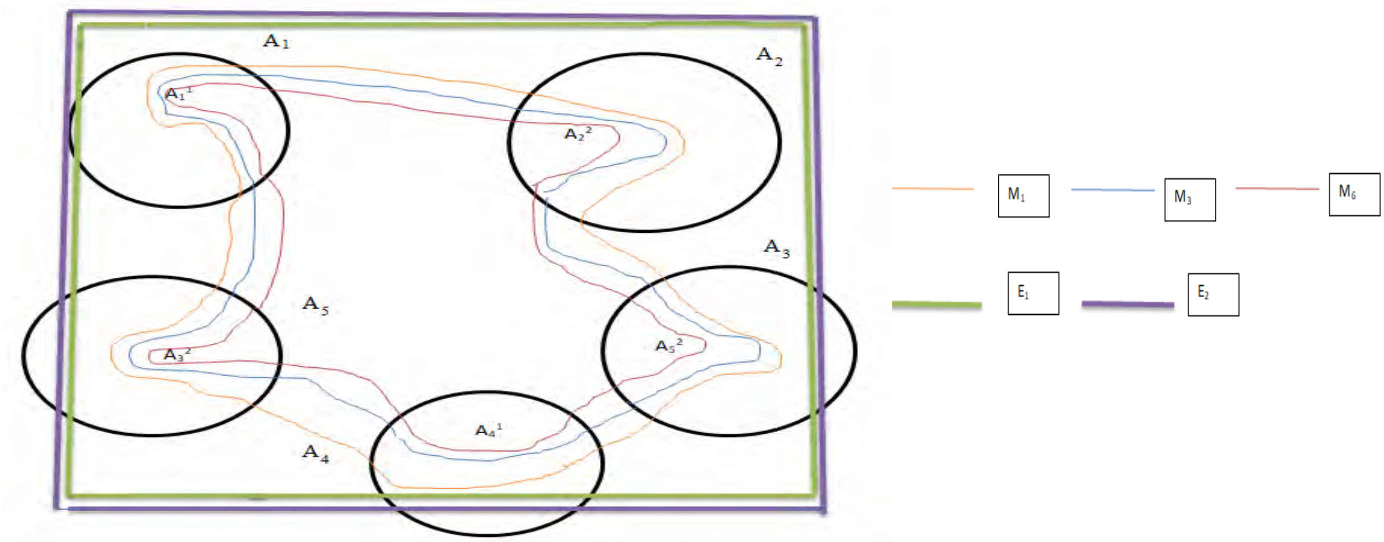

Fig.3.1. Graphical Representation of Extended Concentric Plithogenic Hypergraph

3.6 Extended Concentric Plithogenic Hypergraph

A hypergraph with non- empty, disjoint hyperedges and extended plithogenic envelopes is called as extended concentric plithogenic hypergraph $E_{P_{C_H}}$

An extended concentric Plithogenic hypergraph P_{C_H} is defined as follows

$E_{P_{C_H}}$ = (X, E, $E_{\mathcal{P}\mathcal{E}}$)

- X- finite vertex set
- E- Hyperedge set
- $E\mathcal{P}_{\mathcal{E}}$ =Extended Plithogenic hyper envelope
- $E\mathcal{P}_{\mathcal{E}j} = \{(x_{ij}, \mu_{El}(x_{ij})) / \mu(x_{ij}) > 0, x_{ij} \in E_i \}$

- $E_i \cap E_k = \emptyset$ and $UE_i = X$

4. Fuzzy Cognitive Maps integrated with Extended Concentric Plithogenic Hypergraphic Approach.

Fuzzy Cognitive Maps (FCM) is a decision making tools that are predominantly used in finding the cause and effect relationship. Basically FCM is a directed graph consisting of nodes and edges representing the factors of study and its relationship respectively. The adjacency or the connection matrix is the representation of the relationship between the nodes. Let us consider a decision making problem, which comprises of several factors, then the connection matrix will be of higher order which will make the computational process complicated. In order to handle such situations the core factors can be determined by using the approach of extended concentric plithogenic hypergraph.

Let us consider a decision making situation to find inter relational impacts of the factors contributing towards the sales promotion of a manufacturing firm. The promotion of sales generally depends on major attributes of a company such as Customers, Pricing stratagem and marketing strategies. Different companies follow various other aspects to foster their sales promotion, but the above three attributes play predominant roles.

If customer (A_1), pricing stratagem (A_2) and marketing strategies (A_3) are considered as the attribute sets then A_1 = {**Potential**, Impulsive, Novel, Loyal}, A_2 = {Competition- Based, Skimming, **Penetration**, Dynamic}, A_3 = {**Social**, Service, Green, Holistic, Direct} are the attribute values. A company has decided to launch a new product in the market with the focus towards potential customers by following penetration pricing strategy through social marketing, the ultimate target of the company to increase the sales and attain the target within the stipulated time, so it has called the experts to present the aspects the company has to concentrate in deep. The expert's perception is presented as factors.

F_1 The touch of innovation in the entire lifecycle of the product

F_2 Extensive design of the product

F_3 Customer centric approach

F_4 Product improvisation suiting the contemporary needs

F_5 Widening of the distribution channels

F_6 Implementation of Price breaks

F_7 Enriching the portals of communication

F_8 Employment of self- assessment tools

F_9 Placement of suitable personnel

F_{10} Counter actions to the competitors

The extended concentric plithogenic hyper envelopes with linguistic representations in accordance to the expert's opinion are presented below in Table 4.1

Table 4.1. Representation of Expert's opinion

Factors	$E\mathcal{P}_{\mathcal{E}_1}$	$E\mathcal{P}_{\mathcal{E}_2}$
F_1	(H,M,M)	(M,H,VH)
F_2	(VH,H,H)	(H,H,M)
F_3	(VH,H,VH)	(H,H,H)
F_4	(H,VH,H)	(VH,VH,M)
F_5	(VH,H,VH)	(VH,VH,VH)
F_6	(H,H,L)	(H,VH,M)
F_7	(H,H,VH)	(H,H,VH)
F_8	(H,H,H)	(H,H,VH)
F_9	(VH,VH,H)	(VH,VH,VH)
F_{10}	(H,H,H)	(H,H,H)

The linguistic representations of the experts are quantified using hexagonal fuzzy numbers based on the below values in Table 4.2.

Table 4.2. Hexagonal Quantification of values

Very Low (VL)	(0,0.05,0.1,0.15,0.2,0.25)	0.125
Low (L)	(0.15,0.2,0.25,0.3,0.35,0.4)	0.275
Moderate (M)	(0.3,0.35,0.4,0.45,0.5,0.55)	0.425
High (H)	(0.45,0.5,0.55,0.6,0.65,0.7)	0.575
Very High (VH)	(0.65,0.7,0.75,0.8,0.9,1)	0.8

The quantified representations of the experts are presented in Table 4.3

Table 4.3. Modified representation of Expert's opinion

Factors	$E\mathcal{P}_{\mathcal{E}_1}$	$E\mathcal{P}_{\mathcal{E}_2}$
F_1	(0.575,0.425,0.425)	(0.425,0.575,0.8)
F_2	(0.8,0.575,0.575)	(0.575,0.575,0.425)
F_3	(0.8,0.575,0.8)	(0.575,0.575,0.575)
F_4	(0.575,0.8,0.575)	(0.8,0.8,0.425)
F_5	(0.8,0.575,0.8)	(0.8,0.8,0.8)
F_6	(0.575,0.575,0.275)	(0.575,0.8,0.425)
F_7	(0.575,0.575,0.8)	(0.575,0.575,0.8)
F_8	(0.575,0.575,0.575)	(0.575,0.575,0.8)
F_9	(0.8,0.8,0.575)	(0.8,0.8,0.8)
F_{10}	(0.575,0.575,0.575)	(0.575,0.575,0.575)

The combined values of each factor are presented in below Table 4.4.

Table 4.4 Combined values of the factors

Factors	EP_E
F_1	(1,1,1.225)
F_2	(1.375,1.15,1)
F_3	(1.375,1.15,1.375)
F_4	(1.375,1.6,1)
F_5	(1.6,1.375,1.6)
F_6	(1.15,1.375,0.7)
F_7	(1.15,1.15,1.6)
F_8	(1.15,1.15,1.375)
F_9	(1.6,1.6,1.375)
F_{10}	(1.15,1.15,1.15)

These factors are to be ranked and the above values corresponding to each factor are represented in matrix form

	A_1^1	A_2^3	A_3^1
F_1	1	1	1.225
F_2	1.375	1.15	1
F_3	1.375	1.15	1.375
F_4	1.375	1.6	1
F_5	1.6	1.375	1.6
F_6	1.15	1.375	0.7
F_7	1.15	1.15	1.6
F_8	1.15	1.15	1.375
F_9	1.6	1.6	1.375
F_{10}	1.15	1.15	1.15

By using the procedure of ranking as discussed by Shazia Rana et.al [19] the factors are ranked

The frequency matrix F representing the ranking of the factors is

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}
F_1	0	0	0	0	0	0	1	0	2	0
F_2	0	0	0	0	1	0	2	0	0	0
F_3	0	0	1	1	1	0	0	0	0	0
F_4	1	0	1	0	0	0	1	0	0	0
F_5	3	0	0	0	0	0	0	0	0	0
F_6	0	0	0	0	1	0	0	0	1	1
F_7	1	0	1	1	0	0	0	0	0	0
F_8	0	0	1	0	1	1	0	0	0	0
F_9	3	0	0	0	0	0	0	0	0	0
F_{10}	0	0	1	0	0	0	0	1	0	1

Based on the percentage measure of authenticity of ranking of the factors, the following factors F_3, F_4, F_5, F_8, F_9 are considered for analyzing their inter relation using fuzzy cognitive maps (FCM). These factors are taken as C_1, C_2, C_3, C_4,C_5 and by using the procedure of finding the cause and effect relationship, the limit points are presented in Table 4.5.

The core factors considered for the study are

C_1 Customer centric approach

C_2 Product improvisation suiting the contemporary needs

C_3 Widening of the distribution channels

C_4 Employment of self- assessment tools

C_5 Placement of suitable personnel

The connection matrix M representing the degree of association between the core factors and the graphical representation in Fig 4.1 is presented as follows

	C_1	C_2	C_3	C_4	C_5
C_1	0	1	1	0	1
C_2	1	0	0	0	0
C_3	1	0	0	0	0
C_4	1	1	1	0	1
C_5	1	1	1	1	0

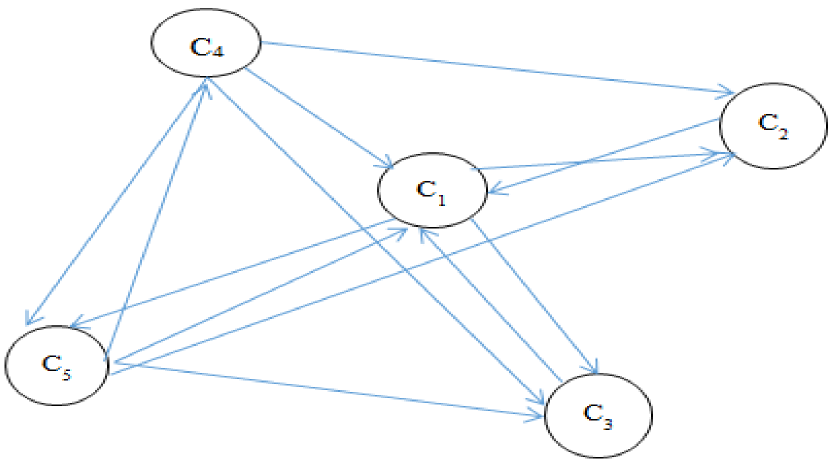

Fig 4.1 Graphical Representation of the association of core factors

Table 4.5. Limit points of the Core Factors

Core Factors in On Position	Limit Points
(10000)	(11111)
(01000)	(11111)
(00100)	(11111)
(00010)	(11111)
(00001)	(11111)

5. Discussion

The integration of FCM with extended concentric plithogenic hypergraphic approach is an innovative effort in minimizing the number of factors considered for studying interrelationship. In the decision making problem as discussed in section 4, the factors which are to be concentrated deeply for sales promotion are considered based on the fulfilment of the three attributes. The consideration of the factors are very specific, but if the company wishes to find the association and impact between the factors using FCM, then the proposed ten factors are to be considered and the computation of the limit points using higher order matrix will be tedious and time consuming, but if the core factors are only considered, the process becomes compatible and the intervention of extended concentric plithogenic hypergraphic approach makes it highly objective and reliable.

6. Conclusion

This research article presents the evolution of the concept of concentric plithogenic hypergraphs based on plithogenic hypersoft sets and plithogenic graphs. This work also introduces extended hypersoft sets and extended concentric plithogenic hypergraphs. The integration of extended concentric plithogenic hypergraphs with fuzzy cognitive maps using linguistic expert's representation is presented with a decision making problem and such integrations are highly advantageous in dealing with several multi attribute factors of study using FCM approach. The linguistic representation of expert's opinion is a new initiative made in this research work and the proposed model of making decisions can be extended using refined plithogenic sets in representations.

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Systems of Neutrosophic Linear Equations

S. A. Edalatpanah [1,*]

[1] Department of Research Center, Ayandegan Institute of Higher Education, Tonekabon, Iran;

* Correspondence: saedalatpanah@aihe.ac.ir; Tel.: (+98)11 5431 04 29

Abstract: In the present paper, for first time, a System of Neutrosophic Linear Equations (SNLE) is investigated based on the embedding approach. To this end, the (α,β,γ)-cut is used for transformation of SNLE into a crisp linear system. Furthermore, the existence of a neutrosophic solution to n x n linear system is proved in details and a computational procedure for solving the SNLE is designed. Finally, numerical experiments are presented to show the reliability and efficiency of the method.

Keywords: Neutrosophic set; Neutrosophic number; Neutrosophic linear equation; Neutrosophic linear system; Embedding method.

1. Introduction

A system of linear equations can be defined as:

$$Ax=b, \tag{1}$$

Various equations in the field of scientific modeling that describe the realistic issues like engineering problems and natural phenomena such as differential equations, computational fluid, circuit simulation, cryptography, quantum and structural mechanics, MRI reconstructions, vibroacoustics, linear and non-linear optimization, portfolios, economic modeling, astrophysics, Google page rank, image processing, nano-technology, natural language processing, deep learning, etc., must be solved mathematically. These issues can regularly be diminished to solving of linear systems. There are a huge amount of models to solve this problem, for more details, see [1-15] and the references therein.

Nevertheless, if the assessment of the coefficients of systems is uncertain and imprecise and just some ambiguous understanding regarding the real values of the parameters is accessible, it might be advantageous to characterize them with special numbers related to soft computing. Fuzzy set was introduced by Zadeh [16, 17], as a suitable instrument to express uncertainty in real life situation. After the introduction of fuzzy set, numerous scholars deliberate on this topic (information of some studies can be observed in [18-23]).

Numerous researchers also suggested several strategies to solve linear systems under fuzzy situation. Fuzzy linear systems emerged at least until 1980 [24]; however Friedman et al. [25] launched a particular model to solve a fuzzy linear system where, the matrix coefficient is crisp and the right-hand hand vector is a fuzzy number. Their model later modified by some other scholars; see [26-46].

However, when there is not clarity in information then the measure of non-membership is not the complement of the measure of membership. In these cases, individual measure of membership and non-membership are needed. Keeping this type of situation in consideration, intuitionistic fuzzy set (IFS) was established by Atanassov [47]. Nevertheless, in different branches of sciences and engineering, it was found that two mentioned components are not sufficient to represent some special types of information. In such cases, a component namely 'neutrality' is needed to represent the information completely. Thus, to remove the limitation of IFS and to handle with more possible types of uncertainty in practical situation, Smarandache [48-51] initiated neutrosophic set (NS) as an extension of the classical and all types of fuzzy sets.

This concept divided into two category of the neutrosophic numbers (NNs) and the neutrosophic sets (NSs). The neutrosophic number (NN) introduce a concept of indeterminacy, denoted by $A = m + nI$ ($m, n \in R$), consists of its determinate part m and its indeterminate part nI. In the worst scenario, A can be unknown, *i.e.*, $A = nI$. However, when there is no indeterminacy related to A, in the best scenario, there is only its determinate part *i.e.*, $A = m$ [50, 51]. But, the neutrosophic sets (NSs) represented by a truth-membership degree, an indeterminacy-membership degree, a falsity-membership degree and have some subclasses such as interval neutrosophic set [52-54] , bipolar neutrosophic set [55-57], single-valued neutrosophic set [58-66], multi-valued neutrosophic set [67-98], and neutrosophic linguistic set [69-70] and applied to solve various problems; see [71-78]. It is worth mentioning that NSs and NNs are two different branches in neutrosophic theory and indicate different forms and concepts of information.

Like any other framework, system of linear equations has also been the topic of evolution. One of the important developments in this field related to situations that coefficients are defined under conditions of uncertainty and indeterminacy. In fact, one of the expectations of classic linear systems is their crispness of data. However, in circumstances where uncertainty and indeterminacy is an inevitable feature of a real life environment, the assumption of crispness of data seems questionable. Also, there is a lot of ambiguity, indeterminacy, and uncertainty in these problems. The system of linear equations under neutrosophic environment are more useful than crisp and other fuzzy linear systems because user in his/her formulation of the problem is not forced to make a delicate formulation. The use of system of Neutrosophic linear equations (SNLE) is recommended to avert unrealistic modeling. Though there are numerous methodologies to solve various issues under NSs and also some models presented to solve linear systems with NNs [79-80], but to the best of our knowledge, the SNLE has not been discussed sets until now. Therefore, the contributions of this study are as follow:

(i) We present for first time, the system of Neutrosophic Linear Equations (SNLE) problem.
(ii) Based on the (α,β,γ)-cut, we design a strategy for solving SNLE with the single valued neutrosophic numbers (SVNNs).
(iii) Some theorems about SNLE are investigated and the conditions of a strong neutrosophic solution to n x n system of linear equations is proved in details.

This study prearranged as follows: some fundamental information, notions and operations on SVNNs are announced in Section 2. In Section 3, we introduce the SNLE and propose a general model to solve it. To show the efficiency and reliability of the method, numerical tests are provided in Section 4. Lastly, conclusions are offered in Section 5.

2. Some Basic Definitions and Arithmetic Operations

Here, we have deliberated some fundamental definitions regarding the neutrosophic sets and single-valued neutrosophic numbers.

Definition 1 [48-49]. A neutrosophic set A in objects X is described by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$ where, $T_A(x)$: $X \to]0^-, 1^+[$, $I_A(x)$: $X \to]0^-, 1^+[$, and $F_A(x)$: $X \to]0^-, 1^+[$, and

$$0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+.$$

Definition 2 [58].When three membership functions of neutrosophic set A be singleton subsets in the real standard [0, 1],we have a single-valued neutrosophic set (SVNS) *A* that is denoted by

$$A=\{(x,T_A(x),I_A(x),F_A(x))\,|\,x\in X\}.$$

Definition 3 [59]. A single valued triangular neutrosophic number (SVTrN-number) is denoted by $A^{\aleph}=<(a,b,c),(\mu,\nu,\omega)>$ whose its three membership functions are given as follows:

$$T_{A^{\aleph}}(x)=\begin{cases}\frac{(x-a)}{(b-a)}\mu & a\le x<b,\\ \mu & x=b,\\ \frac{(c-x)}{(c-b)}\mu & b\le x<c,\\ 0 & otherwise.\end{cases}\quad I_{A^{\aleph}}(x)=\begin{cases}\frac{(b-x)}{(b-a)}\nu & a\le x<b,\\ \nu & x=b\\ \frac{(x-c)}{(c-b)}\nu & b\le x<c,\\ 1 & otherwise.\end{cases}\quad F_{A^{\aleph}}(x)=\begin{cases}\frac{(b-x)}{(b-a)}\omega & a\le x<b,\\ \omega & x=b\\ \frac{(x-c)}{(c-b)}\omega & b\le x<c,\\ 1 & otherwise.\end{cases}$$

Definition 3 [59]. Let $A_1^{\aleph}=<(a_1,b_1,c_1),(\mu_1,\nu_1,\omega_1)>$ and $A_2^{\aleph}=<(a_2,b_2,c_2),(\mu_2,\nu_2,\omega_2)>$ be two SVTrN-numbers. Then the arithmetic relations are defined as:

$$(i)A_1^{\aleph}\oplus A_2^{\aleph}=<(a_1+a_2,b_1+b_2,c_1+c_2),(\mu_1\wedge\mu_2,\nu_1\vee\nu_2,\omega_1\vee\omega_2)> \tag{2}$$

$$(ii)\lambda A_1^{\aleph}=\begin{cases}<(\lambda a_1,\lambda b_1,\lambda c_1),(\mu_1,\nu_1,\omega_1)>, & if \quad \lambda>0\\ <(\lambda c_1,\lambda b_1,\lambda a_1),(\mu_1,\nu_1,\omega_1)>, & if \quad \lambda<0\end{cases} \tag{3}$$

Definition 4 [59]. The (α,β,γ)-cut Neutrosophic set F is denoted by $F_{(\alpha,\beta,\gamma)}$, where α, β, γ ∈ [0,1] and are fixed numbers such that α + β + γ ≤ 3 is defined as by F(α,β,γ)= {<T_A(x), I_A(x), F_A(x)> : x ∈ X, T_A(x) ≥ α , I_A(x) ≤ β, F_A(x) ≤ γ }.

Also, If $A^{\aleph}=<(a,b,c),(\mu,\nu,\omega)>$ then (α,β,γ)-cut is given by:

$$A^{\aleph}_{(\alpha,\beta,\gamma)}=\left\langle\begin{matrix}[(a+\alpha(b-a))\mu,(c-\alpha(c-b))\mu],\\ [(b-\beta(b-a))\nu,(b+\beta(c-b))\nu],\\ [(b-\gamma(b-a))\omega,(b+\gamma(c-b))\omega]\end{matrix}\right\rangle \tag{4}$$

3. System of Neutrosophic Linear Equations (SNLE)

Consider the $n\times n$ linear system with the following equations:

$$\begin{cases}a_{11}x_1^{\aleph}+a_{12}x_2^{\aleph}+.............+a_{1n}x_n^{\aleph}=b_1^{\aleph},\\ a_{21}x_1^{\aleph}+a_{22}x_2^{\aleph}+.............+a_{2n}x_n^{\aleph}=b_2^{\aleph},\\ .\\ .\\ .\\ .\\ .\\ a_{n1}x_1^{\aleph}+a_{n2}x_2^{\aleph}+.............+a_{nn}x_n^{\aleph}=b_n^{\aleph}.\end{cases} \tag{5}$$

The matrix form of the Eq.(5) is as follows:

$$Ax^{\aleph} = b^{\aleph}, \qquad (6)$$

where, the coefficient matrix $A = (a_{ij})$ is a crisp $n \times n$ matrix and $b_i^{\aleph}, i = 1,2,....,n$ is a neutrosophic number. The Eq.(6) is called a system of neutrosophic linear equations (SNLE).

Let the solution of the SNLE of Eq.(6) be $x^{\aleph}$ and its (α,β,γ)-cut be $x^{\aleph}_{(\alpha,\beta,\gamma)} = ([\underline{x}^T(\alpha), \overline{x}^T(\alpha)],$ $[\underline{x}^I(\beta), \overline{x}^I(\beta)], [\underline{x}^F(\gamma), \overline{x}^F(\gamma)])$ .If the (α,β,γ)-cut of $b^{\aleph}$ be $x^{\aleph}_{(\alpha,\beta,\gamma)} = ([\underline{b}^T(\alpha), \overline{b}^T(\alpha)],$ $[\underline{b}^I(\beta), \overline{b}^I(\beta)], [\underline{b}^F(\gamma), \overline{b}^F(\gamma)])$, then The SNLE of (6) can be written as:

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij} x_j^T(\alpha)} = \sum_{j=1}^{n} \underline{a_{ij} x_j^T(\alpha)} = \underline{b_i^T(\alpha)}, \\ \overline{\sum_{j=1}^{n} a_{ij} x_j^T(\alpha)} = \sum_{j=1}^{n} \overline{a_{ij} x_j^T(\alpha)} = \overline{b_i^T(\alpha)}, \\ \underline{\sum_{j=1}^{n} a_{ij} x_j^I(\beta)} = \sum_{j=1}^{n} \underline{a_{ij} x_j^I(\beta)} = \underline{b_i^I(\beta)}, \\ \overline{\sum_{j=1}^{n} a_{ij} x_j^I(\beta)} = \sum_{j=1}^{n} \overline{a_{ij} x_j^I(\beta)} = \overline{b_i^I(\beta)}, \\ \underline{\sum_{j=1}^{n} a_{ij} x_j^F(\gamma)} = \sum_{j=1}^{n} \underline{a_{ij} x_j^F(\gamma)} = \underline{b_i^F(\gamma)}, \\ \overline{\sum_{j=1}^{n} a_{ij} x_j^F(\gamma)} = \sum_{j=1}^{n} \overline{a_{ij} x_j^F(\gamma)} = \overline{b_i^F(\gamma)}. \end{cases} \qquad (7)$$

If we define $x_i^{\aleph} = (\underline{x}_1^T,....,\underline{x}_n^T, \overline{x}_1^T,....,\overline{x}_n^T, \underline{x}_1^I,....,\underline{x}_n^I, \overline{x}_1^I,....,\overline{x}_n^I, \underline{x}_1^F,....,\underline{x}_n^F, \overline{x}_1^F,....,\overline{x}_n^F)^T$ and $b_i^{\aleph} = (\underline{b}_1^T,....,\underline{b}_n^T, \overline{b}_1^T,....,\overline{b}_n^T, \underline{b}_1^I,....,\underline{b}_n^I, \overline{b}_1^I,....,\overline{b}_n^I, \underline{b}_1^F,....,\underline{b}_n^F, \overline{b}_1^F,....,\overline{b}_n^F)^T$, then following Friedman et al., (1998) we must solve an 6n×6n crisp linear system as:

$$HX = B \qquad (8)$$

Where,

$$H = \begin{bmatrix} D_{2n\times 2n} & [0]_{2n\times 2n} & [0]_{2n\times 2n} \\ [0]_{2n\times 2n} & D_{2n\times 2n} & [0]_{2n\times 2n} \\ [0]_{2n\times 2n} & [0]_{2n\times 2n} & D_{2n\times 2n} \end{bmatrix}, B = \begin{bmatrix} B^T \\ B^I \\ B^F \end{bmatrix}. \qquad (9)$$

Also $D = (d_{ij})$,and obtain as follows:

$$\begin{cases} a_{ij} \geq 0 \to d_{ij} = a_{ij}, & d_{i+n,j+n} = a_{ij}, \\ a_{ij} < 0 \to d_{i,j+n} = -a_{ij}, & d_{i+n,j} = -a_{ij} \end{cases} \quad (10)$$

and any d_{ij} which is not determined by (10) is zero. Also:

$$D = \begin{bmatrix} D_1 & -D_2 \\ -D_2 & D_1 \end{bmatrix}, B^T = \begin{bmatrix} \underline{b}^T \\ \overline{b}^T \end{bmatrix}, B^I = \begin{bmatrix} \underline{b}^I \\ \overline{b}^I \end{bmatrix}, B^F = \begin{bmatrix} \underline{b}^F \\ \overline{b}^F \end{bmatrix}.$$

Where, $D_1, D_2 \geq 0, D = D_1 - D_2$.

Since H is a block diagonal matrix, to reduce the computational complexity, we need only to solve the following 2n×2n crisp linear systems:

$$Dx^i = B^i, \quad i = T, I, F. \quad (11)$$

Worthy mentioning that the matrix D may be singular even if A is nonsingular; see the following example:

Example 1. The matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ of the SNLE is nonsingular, while $D = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is singular.

In other sense, a SNLE represented by a nonsingular matrix *A* may be have no solution or an infinite number of solutions. Next, following the Friedman et al., (1998), we study some theorems regarding the properties of D.

Theorem 1. D is nonsingular iff $A = D_1 + D_2$ and $D_1 - D_2$ are nonsingular.

Theorem 2. If D^{-1} exists it must have the same structure as D, *i.e,*

$$D^{-1} = \begin{pmatrix} E & F \\ F & E \end{pmatrix}$$

Definition 5. Let $x_i^{\aleph} = (\underline{x}_1^T, \ldots, \underline{x}_n^T, \overline{x}_1^T, \ldots, \overline{x}_n^T, \underline{x}_1^I, \ldots, \underline{x}_n^I, \overline{x}_1^I, \ldots, \overline{x}_n^I, \underline{x}_1^F, \ldots, \underline{x}_n^F, \overline{x}_1^F, \ldots, \overline{x}_n^F)^T$ be the unique solution of Eq.(5). If $\forall k \in \{1, 2, \ldots, n\}$: $\underline{x}_k^T \leq \overline{x}_k^T, \underline{x}_k^I \leq \overline{x}_k^I$ and $\underline{x}_k^F \leq \overline{x}_k^F$, then the solution $x_i^{\aleph}$ is called a strong neutrosophic solution. Otherwise, it is a weak neutrosophic solution.

Theorem 3. Assume that $D = \begin{pmatrix} D_1 & D_2 \\ D_2 & D_1 \end{pmatrix}$ be a nonsingular matrix. Then Eq.(5) has a strong solution if and only if:

$$(D_1 - D_2)^{-1}(\underline{b}^i - \overline{b}^i) \leq 0, \quad i = T, I, F. \quad (12)$$

Proof. From the system (11) we obtain:

$$\begin{pmatrix} D_1 & D_2 \\ D_2 & D_1 \end{pmatrix}\begin{pmatrix} \underline{x}^i \\ \overline{x}^i \end{pmatrix}=\begin{pmatrix} \underline{b}^i \\ \overline{b}^i \end{pmatrix}, \qquad i=T,I,F$$

Hence,

$$D_1\underline{x}^i - D_2\overline{x}^i = \underline{b}^i \tag{13}$$

$$-D_2\underline{x}^i + D_1\overline{x}^i = \overline{b}^i \tag{14}$$

From (13) and (14) we have:

$$\begin{cases} (D_1+D_2)\underline{x}^i - (D_1+D_2)\overline{x}^i = \underline{b}^i - \overline{b}^i, \\ (D_1+D_2)(\underline{x}-\overline{x}) = \underline{b}^i - \overline{b}^i. \end{cases}$$

From Theorem 1, $D_1 - D_2$ is nonsingular. So,

$$(\underline{x}^i - \overline{x}^i) = (D_1 - D_2)^{-1}(\underline{b}^i - \overline{b}^i) \tag{15}$$

By the Definition 5, $\underline{x}^i - \overline{x}^i \le 0$ if Eq. (5) has a strong solution. Henceforth (12) holds. Conversely, if (12) holds, by Eq.(15), we have $\underline{x}^i - \overline{x}^i \le 0$.□

From the theorems 1 and 3, we conclude this result:

Theorem 4. The SNLE has a strong solution if and only if the following conditions hold:

1. The matrices $A = D_1 + D_2$ and $D_1 - D_2$ are both nonsingular.
2. $(D_1 - D_2)^{-1}(\underline{b}^i - \overline{b}^i) \le 0.$

4. Numerical Example

Here, we provide an experiment to demonstrate the consequences gained in former sections.

Example 2. Consider the following SNLE:

$$\begin{cases} x_1^{\aleph} - x_2^{\aleph} = <(0,1,2);(0.9,0.4,0.2)>, \\ x_1^{\aleph} + 3x_2^{\aleph} = <(4,5,7);(0.8,0.3,0.3)>. \end{cases} \tag{16}$$

The extended 4 × 4 matrix is

$$D=\begin{bmatrix}1&0&0&-1\\1&3&0&0\\0&-1&1&0\\0&0&1&3\end{bmatrix}.$$

Since the matrices $A=D_1+D_2$ and $D_1-D_2=\begin{bmatrix}1&1\\1&3\end{bmatrix}$ are both nonsingular, then by Theorem1, it is easy to see that the matrix D is nonsingular. Therefor, D^{-1} exists and based on Theorem 2, it must have the same structure as D. If we obtain this inverse, we can see that the Theorem 2 is true:

$$D^{-1}=\begin{bmatrix}\frac{9}{8}&\frac{-1}{8}&\frac{-3}{8}&\frac{3}{8}\\\frac{-3}{8}&\frac{3}{8}&\frac{1}{8}&\frac{-1}{8}\\\frac{-3}{8}&\frac{3}{8}&\frac{9}{8}&\frac{-1}{8}\\\frac{1}{8}&\frac{-1}{8}&\frac{-3}{8}&\frac{3}{8}\end{bmatrix}.$$

Now, we obtain the (α,β,γ)-cut of the right hand side vector. By Definition 4, we get:

$$b_{1\,(\alpha,\beta,\gamma)}^{\aleph}=<[0.9(\alpha),0.9(2-\alpha)],[0.4(1-\beta),0.4(1+\beta)],[0.2(1-\gamma),0.2(1+\gamma)]>,$$
$$b_{2\,(\alpha,\beta,\gamma)}^{\aleph}=<[0.8(4+\alpha),0.8(8-2\alpha)],[0.3(5-\beta),0.3(5+2\beta)],[0.3(5-\gamma),0.3(5+2\gamma)]>.$$

Also, $(D_1+D_2)^{-1}=\begin{bmatrix}\frac{3}{2}&\frac{-1}{2}\\\frac{-1}{2}&\frac{1}{2}\end{bmatrix}.$

So:

$$(D_1+D_2)^{-1}(\underline{b}^T-\overline{b}^T)=\begin{bmatrix}\frac{3}{2}&\frac{-1}{2}\\\frac{-1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{9}{5}(\alpha-1)\\\frac{12}{5}(\alpha-1)\end{bmatrix}=\begin{bmatrix}\frac{3}{2}(\alpha-1)\\\frac{3}{10}(\alpha-1)\end{bmatrix}\leq\begin{bmatrix}0\\0\end{bmatrix},$$

$$(D_1+D_2)^{-1}(\underline{b}^I-\overline{b}^I)=\begin{bmatrix}\frac{3}{2}&\frac{-1}{2}\\\frac{-1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}-0.8\beta\\-0.9\beta\end{bmatrix}=\begin{bmatrix}\frac{-3}{4}\beta\\\frac{-1}{20}\beta\end{bmatrix}\leq\begin{bmatrix}0\\0\end{bmatrix},$$

$$(D_1+D_2)^{-1}(\underline{b}^F-\overline{b}^F)=\begin{bmatrix}\frac{3}{2}&\frac{-1}{2}\\\frac{-1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}-0.4\gamma\\-0.9\gamma\end{bmatrix}=\begin{bmatrix}\frac{-3}{20}\gamma\\\frac{-1}{4}\gamma\end{bmatrix}\leq\begin{bmatrix}0\\0\end{bmatrix},$$

Therefore, by theorems 3 and 4, The SNLE (16) should has a strong solution. To obtain this solution, form Eq.(11) we have:

$$x^T = \begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \\ \overline{x}_1^T \\ \overline{x}_2^T \end{bmatrix} = D^{-1}B^T = \left[\frac{1}{40}(26\alpha+41),\quad \frac{1}{40}(2\alpha+29),\quad \frac{1}{40}(-34\alpha+101),\quad \frac{1}{40}(-10\alpha-41)\right]^t,$$

$$x^I = \begin{bmatrix} \underline{x}_1^I \\ \underline{x}_2^I \\ \overline{x}_1^I \\ \overline{x}_2^I \end{bmatrix} = D^{-1}B^I = \left[\frac{-27}{80}(\beta-2),\quad \frac{1}{80}(\beta+22),\quad \frac{3}{80}(11\beta+18),\quad \frac{1}{80}(5\beta-22)\right]^t,$$

$$x^F = \begin{bmatrix} \underline{x}_1^F \\ \underline{x}_2^F \\ \overline{x}_1^F \\ \overline{x}_2^F \end{bmatrix} = D^{-1}B^F = \left[\frac{-3}{80}(\gamma-14),\quad \frac{-1}{80}(7\gamma+26),\quad \frac{3}{80}(3\gamma+14),\quad \frac{13}{80}(\gamma+2)\right]^t.$$

$$x_{1\,(\alpha,\beta,\gamma)}^{\aleph} = <[\frac{1}{40}(26\alpha+41),\frac{1}{40}(-34\alpha+101)],[\frac{-27}{80}(\beta-2),\frac{3}{80}(11\beta+18)],[\frac{-3}{80}(\gamma-14),\frac{3}{80}(3\gamma+14)]>,$$

$$x_{2\,(\alpha,\beta,\gamma)}^{\aleph} = <[\frac{1}{40}(2\alpha+29),\frac{1}{40}(-10\alpha-41)],[\frac{1}{80}(\beta+22),\frac{1}{80}(5\beta-22)],[\frac{-1}{80}(7\gamma+26),\frac{13}{80}(\gamma+2)]>.$$

For different values of $0 \le \alpha,\beta,\gamma \le 1$,the graphical interpretation of the above results is shown in figures 1and 2.

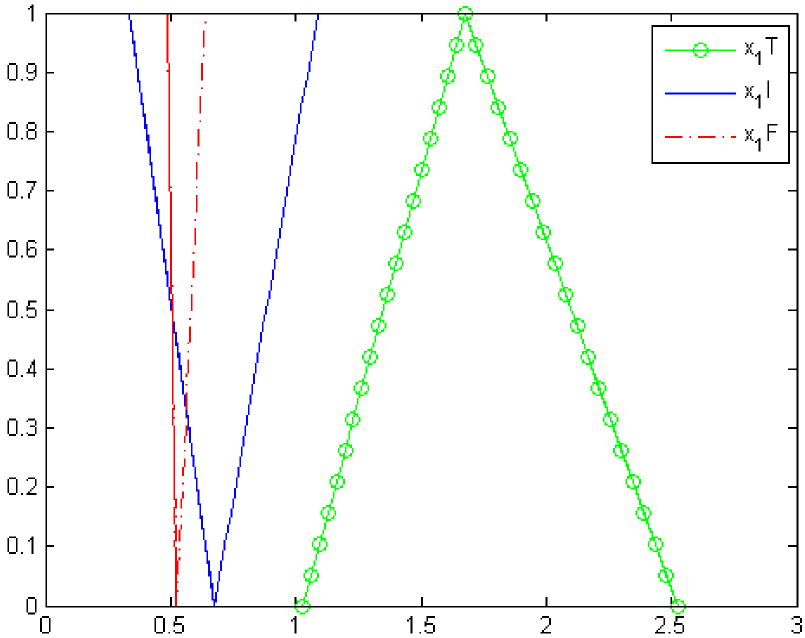

Figure 1. The value of $x_1^{\aleph}$.

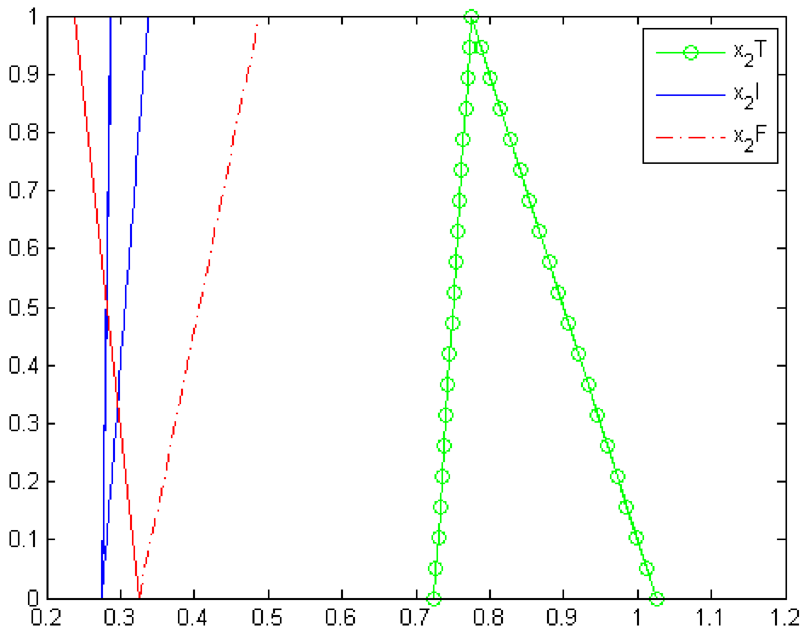

Figure 2. The value of $\tilde{x}_2^{\aleph}$.

5. Conclusions

In this study, we present for first time, the system of Neutrosophic Linear Equations (SNLE) and establish a general model to solve it. Some theorems about SNLE are investigated and the conditions of a strong neutrosophic solution to n x n system of linear equations is proved in details. Finally, from numerical and theoretical studies it can be concluded that the model is efficient and convenient.

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The Neutrosophic Time Series-Study Its Models (Linear-Logarithmic) and test the Coefficients Significance of Its linear model

Rafif Alhabib[1] and A. A. Salama[2]
[1] Department of Mathematical Statistics, Faculty of Science, Albaath University, Homs, Syria; rafif.alhabib85@gmail.com
[2] Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said, Egypt; drsalama44@gmail.com

Abstract: In this paper, we present the Neutrosophic time series by studying the classical time series within the framework of the Neutrosophic logic. (Logic established by the American philosopher and mathematician Florentin Smarandache presented it as a generalization of fuzzy logic, especially intuitionistic fuzzy logic). As an extension of this, A.A. Salama presented the theory of Neutrosophic crisp sets as a generalization of crisp sets theory. This study enables us to deal with all the time series values whether it is specified or not specified, we present the linear model for the Neutrosophic time series, and we test the significant of its coefficient based on Student's distribution. We present an example in which we pave the Neutrosophic time series according to the linear model, test the significant of its coefficient, and show how to deal with the unspecified values of the time series. Then we present the logarithmic model of the Neutrosophic time series. We conclude that the existence of indeterminacy in the matter we cannot ignored because it actually affects the estimated values of the time series and thus affects the prediction of the future of the series.

Keywords: Time Series, Neutrosophic logic, Neutrosophic Time Series, the linear model of the Neutrosophic time series, the significant of coefficients to the Neutrosophic linear model. The logarithmic model of the Neutrosophic time series.

1. Introduction

Neutrosophic is a new view of Modeling , designed to effectively deal underlying doubts in the real world, as it came to replace binary logic that recognized right and wrong by introducing a third neutral case which could be interpreted as non-specific or uncertain. Founded by Florentin Smarandache, the American philosopher and mathematician, he presented it in 1999 as a generalization of fuzzy logic and as an extension of the theory of fuzzy sets presented by Lotfi A. Zadeh (1965) [6]. As an extension of this, A. A. Salama presented the theory of Neutrosophic crisp sets as a generalization of classical sets theory [5], [7], [12] and developed, inserted and formulated new concepts in the fields of mathematics, statistics, computer science and classical information systems through Neutrosophic [1-3] , [4], [5], [12]. Neutrosophic has grown significantly in recent years through its application in measurement, sets and graphs and in many scientific and practical fields [14-29].

Smarandache defined the Neutrosophic logic as a new, non-classical logic that studies the origin, nature, and field of Indeterminacy, as well as the interaction of all the different spectra that a person imagines in an Issue, so that he takes into account every idea with against it (its opposite) with the indeterminacy [11], [9].

In this paper, we highlight the application of the Neutrosophic logic to the concept of time series, so we know the concept of the Neutrosophic time series, which opens the way for dealing with time series that take precisely unspecified values such as taking range of values instead of one value (as in the classic) . We also provide the linear model for this Neutrosophic time series with an example showing how to deal with non-specified values in time series, and we test the significance of the coefficient of the model that we obtained using the Student's distribution. Finally, we present the logarithmic model of the Neutrosophic time series.

2. The Neutrosophic Time Series:

A time series is a set of data arranged in chronological order, the data of this series are associate to each other in the general case, and this correlation gives us reliable future forecasts. We also define it as a set of consecutive values (observations) that describe the evolution of a phenomenon over time. We say about this time series that it is Neutrosophic, if some or all of its values (its observations) are not explicitly specific, such as being a range of values instead of one value.

3. The Discussion:

The aim of studying time series is to monitor the changes that accompany the phenomenon during a specific period. In addition, description, analysis and classification of the phenomenon. As well as studying the reasons, that led to these changes in the phenomenon, and trying to evaluate it by accurate scientific methods. In addition, predicting what will happen to the series in the future. this is done based on the previous history of the series, by relying on statistical and mathematical laws that describe the phenomenon in the past well , and have the ability to Evaluate and estimate their future values with the least amount of errors possible. Proceeding from the importance of this goal, we must look at the data in a more comprehensive and accurate manner than they are, considering that it is the basis in predicting the future of the time series, and based on this we present the concept of the Neutrosophic time series.

The Neutrosophic time series is similar to the classic time series in terms of their types according to the unit of time that the phenomenon is measured. Where there are decennium Neutrosophic series whose values are taken every ten years such as population census (from a Neutrosophic point of view). Annual Neutrosophic time series that record their values each year such as estimating wheat production for a particular country. Quarterly Neutrosophic series such as the production of some seasonal crops. Monthly Neutrosophic series such as monthly factor production of medications. In addition, daily Neutrosophic series that record their values on a daily such as temperature, humidity, and wind speed.

Many conditions and attributes must be achieved by the values in order for us to call them a Neutrosophic time series:

1. The unit of time measurement, which is that all elements of the series have the same units of measurement (day - month - year ...).

2. The one place, all elements of the series must be measured in the same place and it is not permissible to take part of the values in one city and the rest in another city.

3. The unit of measurement for the series elements, where all the elements of the series must be measured in the same unit (m, kg ...).

4. The number of series values must be finite, and it may contain some indeterminate values.

4. The Linear model of a Neutrosophic time series:

The general form of the linear model of a Neutrosophic time series is:

$$\hat{Y}_t = a_N + b_N\ t$$

Where:

Y_t :The real values of the time series.
$\hat{Y}_t$: The estimated values.
a_N : Constant coefficient.
b_N : Regression coefficient.
T : Time.

We compute a_N , b_N by the least squares method [10], which is:

$$\hat{b}_N = \frac{\sum(t-\bar{t})(Y_t - \overline{Y_t})}{\sum(t-\bar{t})^2}$$

4.1 Example:

We have the following Neutrosophic data representing the production of a machine in a factory (by piece). We want to pave this data linearly:

T	Y_t	$(t-\bar{t})$	$(t-\bar{t})^2$	$(Y_t - \overline{Y_t})$	$(t-\bar{t})(Y_t - \overline{Y_t})$
1	[25,27]	-5.2	02.02	[-37.9 , -37.5]	[168.75 , 170.55]
0	02	-3.5	10.02	[-34.5 , -32.9]	[115.15 , 120.75]
0	52	-2.5	5.02	[-19.5 , -17.9]	[44.75 , 48.75]
5	[50,55]	-1.5	0.02	[-12.9 , -9.5]	[14.25 , 19.35]
2	[44,46]	-0.5	2.02	[-18.9 , -18.5]	[9.25 , 9.45]
5	52	0.5	2.02	[-4.5 , -2.9]	[-2.25 , -1.45]
7	02	1.5	0.02	[15.5 , 17.1]	[23.25 , 25.65]
0	[90,95]	2.5	5.02	[27.1 , 30.5]	[67.75 , 76.25]
9	122	3.5	10.02	[35.5 , 37.1]	[124.25 , 129.25]
12	[105,107]	5.2	02.02	[42.1 , 42.5]	[189.45 , 191.25]
the sum					[754.6 , 789.8]

Calculation of the following: $\bar{t} = \frac{55}{10} = 5.5$

$\bar{y} = \frac{\sum Y_t}{n} = \frac{[25,27]+[30,30]+[45,45]+[50,55]+[44,46]+[60,60]+[80,80]+[90,95]+[100,100]+[105,107]}{10} = \frac{[629,645]}{10} =$

[62.9 , 64.5]

Subsequently $: \overline{Y_t} = [\,62.9\ ,64.5]$

$\hat{b}_N = \frac{\sum(t-\bar{t})(Y_t - \overline{Y_t})}{\sum(t-\bar{t})^2} = \frac{[754.6\ ,789.8]}{82.5} = [\,9.147\,,\ \ 9.573]$

$\hat{a}_N = \bar{y} - b\bar{t} = [62.9\,,64.5] - [9.147\,,9.573](5.5) \quad = [62.9\,,64.5] - [\,50.31\,,52.65] = [11.85\,,12.59]$

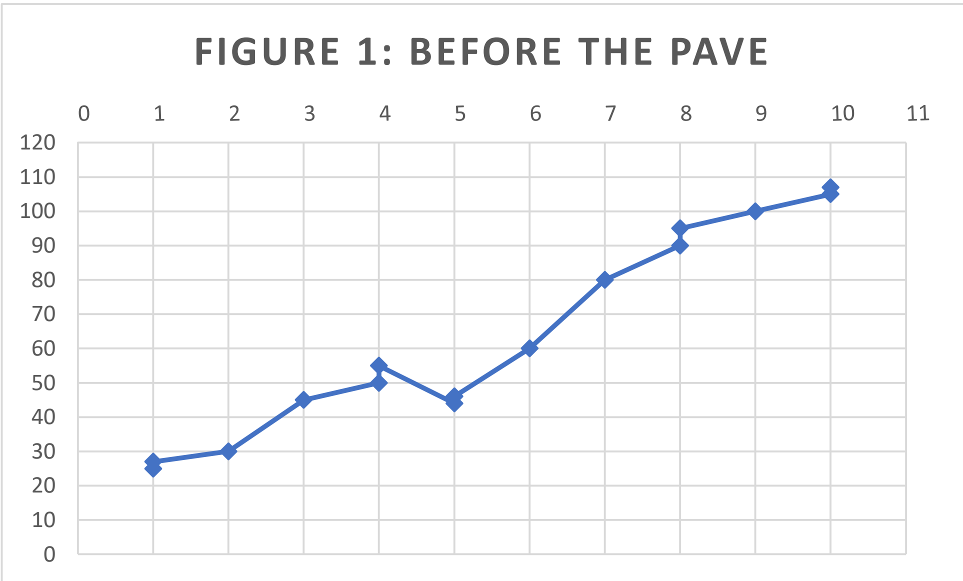
$\hat{a}_N = [\ 11.85\ ,\ 12.59]$

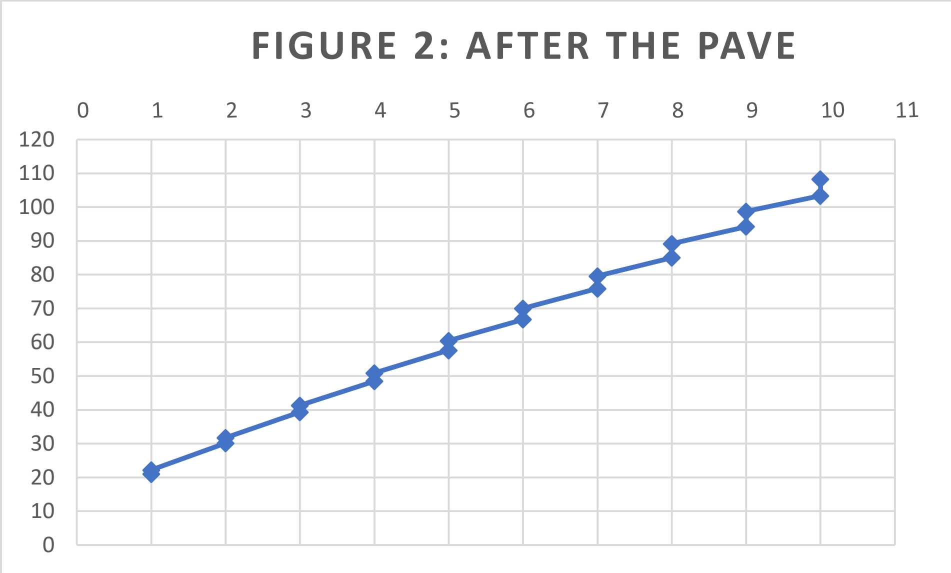
Subsequently $: \hat{Y}_t = a_N + b_N\, t = [\mathbf{11.85}\,,\mathbf{12.59}] + [\mathbf{9.147}\,,\mathbf{9.573}]\ \boldsymbol{t}$

Now we calculate the estimated values for $\hat{Y}_t$:

T	$\hat{Y}_t$
1	[21 , 22.16]
2	[30.15 , 31.73]
3	[39.3 , 41.3]
4	[48.45 , 50.87]
5	[57.6 , 60.44]
6	[66.75 , 70.01]
7	[75.9 , 79.58]
8	[85.05 , 89.15]
9	[94.2 , 98.72]
10	[103.35 , 108.29]

4.1.1 The graph:





We note that:

1. From figures (1) and (2) it is clear that the model represents the data well, and this appears from the approximation of the graphic line of the estimated values from the graphic line of the real values.

2. We did not merge the diagram of series before and after the pave into a single diagram, so that the Neutrosophic values appear clearly, and no confusion occurs.
3. We note in both cases (before and after the pave of the series) that the Neutrosophic values of the series did not affect its general direction neither by increasing nor decreasing (only the time series moves slightly within the range), without affecting the properties of the series and the characteristics of its general direction. It gives us an accurate view of the series and thus more accurate and objective prediction of the future of the series.

5. Coefficient's Significance test of the Neutrosophic linear model:

We say about the regression coefficient (the coefficient of the model) that it is significant if its value is effective and its effect cannot be ignored. In addition, we say that the regression coefficient is not significant if its value can be neglected and its effect neglected.

We perform the test on the coefficient 'b' as in the classic method, but here we deal with Neutrosophic data. (The constant coefficient 'a' mostly has no statistical significance).

The model coefficient test is subjected to a student test with (n-2) degrees of freedom, and his Neutrosophic hypothesis is: [10]

$$NH_0 = \ b_N = 0$$
$$NH_1 = \ b_N \neq 0$$

The test index is: $tt = \frac{b_N}{s_b}$

Where:

$$S_b = \frac{e_r}{\sqrt{\sum(t-\bar{t})^2}}$$

$$e_r = \sqrt{\frac{\sum(Y_t-\widehat{Y_t})^2}{n-2}}$$

In which:

b_N :The coefficient of the Neutrosophic model we want to test its value.

S_b : the error of Neutrosophic model Coefficient.

$\widehat{Y_t}$: Estimated values in the Neutrosophic model.

Y_t : the Neutrosophic real values of the series.

Coefficient significance test 'b_N' in the previous example:

We had:

$$\hat{Y}_t = a_N + b_N\, t = [\mathbf{11.85}\,, \mathbf{12.59}] + [\mathbf{9.147}\,, \mathbf{9.573}]\ \boldsymbol{t}$$

Let us calculate the test index value:

$$tt = \frac{b_N}{S_b}$$

To do this, we calculate first:

$$\boldsymbol{e_r} = \sqrt{\frac{\sum(Y_t-\widehat{Y_t})^2}{n-2}}$$

Y_t	$\widehat{Y_t}$	$(Y_t - \widehat{Y_t})^2$
[25,27]	[21 , 22.16]	[16 , 23.4]
02	[30.15 , 31.73]	[0.02 , 2.99]
52	[39.3 , 41.3]	[13.7 , 32.5]
[50,55]	[48.45 , 50.87]	[2.4 , 17.1]
[44,46]	[57.6 , 60.44]	[184.96 , 208.5]
52	[66.75 , 70.01]	[45.6 , 100.2]
02	[75.9 , 79.58]	[0.18 , 16.8]
[90,95]	[85.05 , 89.15]	[24.5 , 34.2]
122	[94.2 , 98.72]	[1.64 , 33.6]
[105,107]	[103.35 , 108.29]	[1.67 , 2.7]
The sum		**[290.7 , 472]**

$$e_r = \sqrt{\frac{\sum(Y_t - \hat{Y}_t)^2}{n-2}} = \sqrt{\frac{[290.7\ ,472]}{8}} = \sqrt{[36.3\ ,59]} = [6.02\ ,7.68]$$

$$S_b = \frac{e_r}{\sqrt{\sum(t-\bar{t})^2}} = \frac{[6.02\ ,7.68]}{62.25} = [0.097\ ,0.123]$$

$$tt = \frac{b_N}{S_b} = \frac{[9.147\ ,9.573]}{[0.097\ ,0.123]} = [77.8\ ,94.3]$$

Let us discuss the significance of ' b_N', by comparing the test index ' tt ' with the tabular value, at the significance level of $\alpha = 0.05$ and the degree of freedom (n-2 = 8), then:

$$tt\left(\alpha\ ,\mathrm{n}-2\right) = \mathrm{tt}\ (0.05\ ,8) = 1.8595$$

We note that:

$$tt = [77.8\ ,94.3]\ >\ tt\left(\alpha\ ,\mathrm{n}-2\right) = 1.8595$$

Thus, we reject the primary hypothesis and take the alternative hypothesis. That is, the coefficient of the Neutrosophic linear model is significant. In addition, the model can be used to pave the series and prediction.

6. The logarithmic model of a Neutrosophic time series:

The general form of the logarithmic model of the Neutrosophic time series is:

$$\hat{Y}_t = a_N + b_N\ lnt$$

Where:

Y_t :The real values of the time series.
$\hat{Y}_t$: The estimated values.
a_N : Constant coefficient.
b_N : Regression coefficient.
T : Time.

We transform this model into a linear model by the following assumption: $T = lnt$

Then, the form becomes linear as follows:

$$\hat{Y}_t = a_N + b_N\ T$$

We compute a_N , b_N by the least squares method [10], which is:

(We used the second formula) $\hat{b}_N = \frac{(\sum Y.T - \sum Y \sum T)/n}{(\sum T^2 - (\sum T)^2)/n}$

$$\hat{a}_N = \bar{y} - b\bar{T}$$

We want pave the Neutrosophic time series "in the previous example" using the logarithmic model:

t	$T = lnt$	Y_t	$Y_t \ .T$	T^2
1	2	[25,27]	2	2
0	2.59	02	02.7	2.50
0	1.12	52	59.2	1.01
5	1.09	[50,55]	[69.5 , 76.45]	1.90
2	1.51	[44,46]	[70.84 , 74.06]	0.29
5	1.79	52	127.5	0.01
7	1.92	02	125	0.79
0	0.20	[90,95]	[187.2 , 197.6]	5.00
9	0.02	122	002	5.00
12	0.02	[105,107]	[241.5 , 246.1]	2.02
22	**52.51**	**[629 , 645]**	**[1122.64 , 1147.81]**	**56.72**

$$\hat{b}_N = \frac{(\sum Y.T - \sum Y \sum T)/n}{(\sum T^2 - (\sum T)^2)/n} = \frac{\left([1122.64 , 1147.81] - [629 , 645](15.10)\right)/10}{\left(27.65 - (15.10)^2\right)/10} = [41.8 \ \ , 42.9]$$

$$\hat{a}_N = \bar{y} - b\bar{T} = [\,62.9 \ \ , 64.5] - [41.8 \ \ , 42.9](1.51) = [-0.3 \, , -0.22]$$

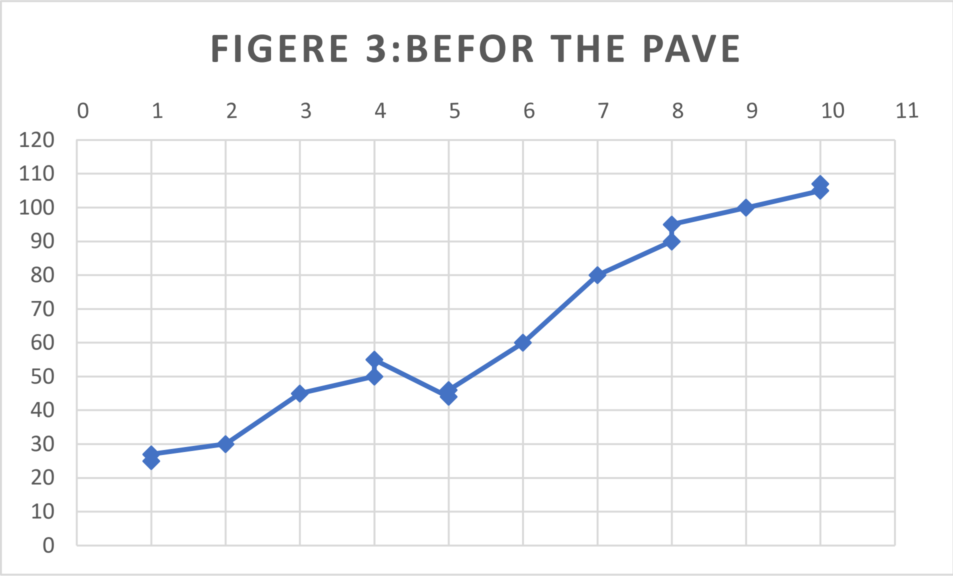
Subsequently:

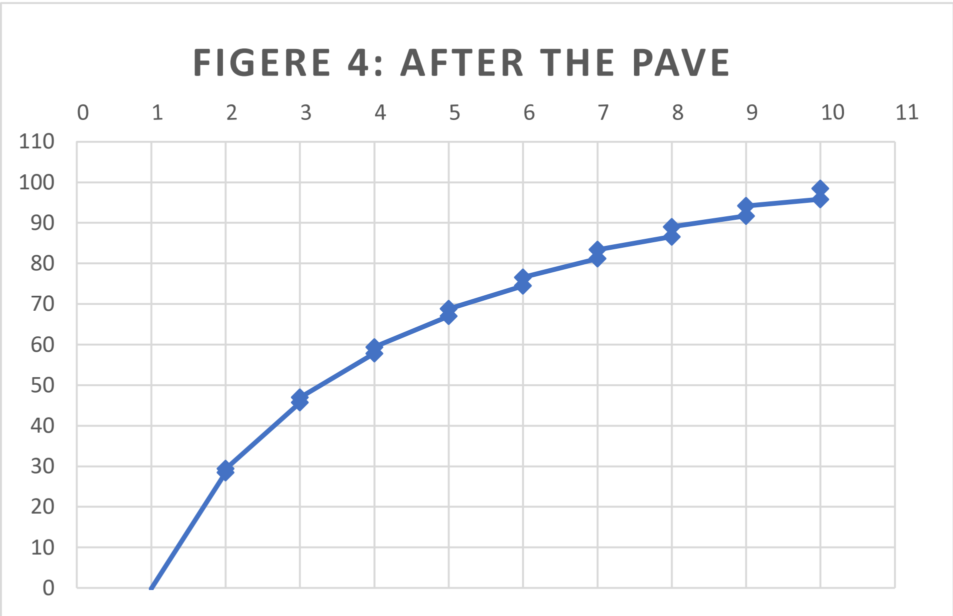
$$\boldsymbol{\hat{Y}_t = a_N + b_N \ lnt = \ [-0.3 \, , -0.22] + [41.8 \ \ , 42.9] \ln t}$$

Now we calculate the estimated values for $\hat{Y}_t$:

$T = lnt$	t	$\hat{Y}_t$
2	1	$[-0.3 \, , -0.22]$
2.59	2	[28.5 , 29.4]
1.12	3	[45.7 , 46.97]
1.09	4	[57.8 , 59.4]
1.51	5	[66.998 , 68.8]
1.79	6	[74.5 , 76.57]
1.92	7	[81.2 , 83.4]
0.20	8	[86.6 , 89.01]
0.02	9	[91.7 , 94.2]
0.02	10	[95.8 , 98.5]

6.1 The graph:





We notice from figures (3) and (4) it is clear that the model represents the data well, and this appears from the approximation of the graphic line of the estimated values from the graphic line of the real values.

7. Conclusions:

We conclude that dealing with Neutrosophic time series provides us with a comprehensive and general study of the studied phenomenon, in which we do not neglect any data only because it is not specific. Working within the framework of classical logic is not sufficient now. Where we found that the indeterminacy actually affects the estimated values, and therefore the unspecified values we cannot be ignored and removed from the study framework in order to obtain the most accurate results as possible, thus drawing the future of the series and predicting it better and more accurately. In the near future, we look forward to applying the Neutrosophic logic to other models of time series, and to studying the seasonal, periodic and random components of the Neutrosophic time series.

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University of New Mexico

Neutrosophic Triplet Partial g - Metric Spaces

Memet Şahin[1], Abdullah Kargın[2,*] and Murat Yücel[3]

[1]Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey. mesahin@gantep.edu.tr
[2,*] Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey. abdullahkargin27@gmail.com
[3]Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey. muratycl012@gmail.com
*Correspondence: abdullahkargin27@gmail.com; Tel.:+9005542706621

Abstract: In this chapter, neutrosophic triplet partial g - metric spaces are obtained. Then, some definitions and examples are given for neutrosophic triplet partial g - metric space. Based on these definitions, new theorems are given and proved. In addition, it is shown that neutrosophic triplet partial g - metric spaces are different from the classical g - metric spaces, neutrosophic triplet metric spaces. Thus, we add a new structure in neutrosophic triplet theory. Also, thanks to neutrosophic triplet partial g – metric space, researchers can obtain new fixed point theorems for neutrosophic triplet theory.

Keywords: g - metric space, neutrosophic triplet set, neutrosophic triplet metric space, neutrosophic triplet g - metric space, neutrosophic triplet partial g - metric space

1 Introduction

There are many uncertainties in daily life. The logic of classical mathematics is often insufficient to explain these uncertainties. Because it is not always possible to call a situation or event absolutely right or wrong. For example, we cannot always call the weather cold or hot. It can be hot for some, cold for some and cool for others. Similar situations in which we remain indecisive may appear in the professional proficiency assessment. It is often difficult to determine whether a work done or a product produced is always definite good or definite bad. Such a situation reduces the reliability of evaluating professional proficiencies. In order to cope with these uncertainties, Smarandache (1998) defined the concept of neutrosophic logic and neutrosophic set. In the concept of neutrosophic logic and neutrosophic sets, there is T degree of membership, I degree of undeterminacy and F degree of non-membership. These degrees are defined independently of each other. A neutrosophic value is shown by (T, I, F). In other words, a condition is handled according to both its accuracy and its inaccuracy and its uncertainty. Therefore, neutrosophic logic and neutrosophic set help us to explain many uncertainties in our lives. In addition, many researchers have made studies on this theory [2-27, 50 - 56]. Recently, Baset et al. studied TOPSIS-CRITIC model for sustainable supply chain risk management [51]; Baset et al. obtained resource levelling problem in construction projects under neutrosophic environment [52].

In fact, in the concept of fuzzy logic and fuzzy sets [28] there is only a degree of membership. In addition, the concept of intuitionistic fuzzy logic and intuitionistic fuzzy set [29] includes membership degree, degree of undeterminacy and degree of non-membership. But these degrees are defined dependently of each other. Therefore, neutrosophic set is a generalized state of fuzzy and intuitionistic fuzzy set.

Also, Smarandache and Ali obtained neutrosophic triplet set (NTS) and neutrosophic triplet groups (NTG) [30]. For every element “x” in NTS A, there exist a neutral of “x” and an opposite of “x”. Also, neutral of “x” must different from the classical neutral element. Therefore, the NTS is different from the classical set. Furthermore, a neutrosophic triplet (NT) “x” is showed by <x, neut(x), anti(x)>. Also, many researchers have introduced NT structures [31-44]. Recently, Şahin, Kargın, Yücel and Özkartepe obtain neutrosophic triplet g – metric spaces [45].

Furthermore, Mustafa and Sims introduced g - metric spaces [46] in 2006. g - metric space is generalized form of metric space. The g - metric spaces have an important role in fixed point theory. Recently, researchers studied g - metric space [46-48]. Also, Salimi and Vetro introduced partial g – metric spaces [49].

In this chapter, we introduce neutrosophic triplet partial g - metric space (NTpgMS). In Section 2, we give definitions and properties for partial g - metric space (pgMS) [49], neutrosophic triplet sets (NTS) [30], neutrosophic triplet metric spaces (NTMS) [32] and neutrosophic triplet g – metric space (NTgMS) [45]. In Section 3, we define NTpgMS and we give some properties for NTpgMS. Also, we show that NTpgMSs are different from the pgMSs, NTMSs and NTgMSs, because the triangle inequality in the NTgMS, NTMS and pgMS differ from the triangle inequality in the NTpgMS. Then, we examine relationship between NTpgMS and NTgMS. In Section 4, we give conclusions.

2 Preliminaries

Definition 2.1: [30] Let # be a binary operation. A NTS (X, #) is a set such that for $x \in$ X,

i) There exists neutral of "x" such that x*neut(x) = neut(x)* x = x.

ii) There exists anti of "x" such that x*anti(x) = anti(x)* x = neut(x).

Also, a neutrosophic triplet "x" is denoted by (x, neut(x), anti(x)).

Definition 2.2: [32] Let (N,*) be a NTS and d_N:NxN→ $\mathbb{R}^+ \cup \{0\}$ be a function. If d_N:NxN→ $\mathbb{R}^+ \cup \{0\}$ and (N, *) satisfies the following conditions, then d_N is called NTM.

a) x*y ∈ N;

b) d_N(x, y) ≥ 0;

c) If x = y, then d_N(x, y) = 0;

d) d_N(x, y) = d_N(y, x);

e) If there exits at least a y ∈ N for each x, z ∈ N such that d_N(x, z) ≤ d_N(x, z*neut(y)), then

d_N(x, z*neut(y)) ≤ d_N(x, y) + d_N(y, z).

In this case, ((N,*), d_N) is called a NTMS.

Definition 2.3: [36] Let (N,*) be a NTS. If d_p:NxN→ $\mathbb{R}^+ \cup \{0\}$ function satisfies the following conditions, then d_p is a NTpM. For all x, y, z ∈ N,

a) x*y ∈ N,

b) d_p (x, y) ≥ d_p (x, x) ≥ 0,

c) If d_p (x, y) = d_p (x, x) = d_p (y, y) = 0, then there exists at least one pair of elements x, y ∈ N such that

$x = y$,

d) d_p (x, y) = d_p (y, x),

e) If for each pair of x, z ∈N, there exists at least one y ∈N such that d_p (x, z) ≤ d_p (x, z*neut(y)), then d_p (x, z*neut(y)) ≤ d_p (x, y) + d_p (y, z) - d_p (y, y).

In this case, ((N,*), d_p) is called a NTpMS.

Definition 2.4: [45] Let $(X,*)$ be a NTS. If the following conditions hold, then $g: X \times X \times X \to R^+ \cup \{0\}$ is an NTgM.

a) $\forall x, y \in X$; $x * y \in X$,

b) If $x = y = z$, then $g(x, y, z) = 0$,

c) If $x \neq y$, then $g(x, y, z) > 0$,

d) If $z \neq y$, then $g(x, x, y) \leq g(x, y, z)$,

e) $g(x, y, z) = g(x, z, y) = g(y, x, z) = g(y, z, x) = g(z, x, y) = g(z, y, x)$, for every $x, y, z \in X$,

f) If there exists at least an $a \in X$ for each $x, y, z \in X$ such that

$g(x, y, z) \leq g(x * neut(a), y * neut(a), z * neut(a))$, then

$g(x * neut(a), y * neut(a), z * neut(a)) \leq G(x, a, a) + G(a, y, z)$.

In this case, (*X*, *), g) is called NTgMS.

Definition 2.5:[45] Let (X, *), g) be a NTgMS and $\{x_n\}$ be a sequence in this space. A point x ∈ X is said to be limit of the sequence $\{x_n\}$, if $\lim_{n,m \to \infty} g(x, x_n, x_m) = 0$ and $\{x_n\}$ is called NT g – convergent to x.

Definition 2.6:[45] Let (X, *), g) be a NTgMS and $\{x_n\}$ be a sequence in this space. $\{x_n\}$ is called NT g – Cauchy sequence if $\lim_{n,m,l \to \infty} g(x_n, x_m, x_l) = 0$.

Definition 2.7:[45] Let (X, *), g) be a NTgMS. If every $\{x_n\}$ NT g - Cauchy sequence is NT g - convergent, then (X, *), g) is called NT complete NTgMS.

Definition 2.8:[49] Let X be a neutrosophic triplet set. If the following conditions hold, then $g: X \times X \times X \to R^+ \cup \{0\}$ is *a pgM. For all a, x, y, z ∈X;*

a) If $x = y = z$, then $g(x, y, z) = g(x, x, x) = g(y, y, y) = g(z, z, z)$,

b) $g(x, x, x) + g(y, y, y) + g(z, z, z) \leq 3\, g(x, y, z)$,

c)If x ≠y, then $\frac{1}{3}\, g(x, x, x) + \frac{2}{3}\, g(x, x, x) < g(x, y, y)$,

d) If y ≠ z, then $g(x, x, y) - \frac{1}{3}\, g(x, x, x) \leq g(x, y, z) - \frac{1}{3}\, g(x, x, x)$,

e) $g(x, y, z) = g(x, z, y) = g(y, x, z) = g(y, z, x) = g(z, x, y) = g(z, y, x)$,

f) g(x, y, z) ≤ g(x, a, a) + g(a, y, z) – g(a, a, a).

Definition 2.9: [49] Let (X, g) be a pgMS and $\{x_n\}$ be a sequence in this space. A point x ∈ X is said to be limit of the sequence $\{x_n\}$, if $\lim_{n,m \to \infty} g(x, x_n, x_m) = g(x, x, x)$ and $\{x_n\}$ is called NT p– g – convergent to x.

3 Neutrosophic Triplet Partial g - Metric Space

Definition 3.1: Let (A,*) be a NTS. If the function $\mathrm{d_{NG}}: \mathrm{A} \times \mathrm{A} \times \mathrm{A} \to \mathrm{R}^+ \cup \{0\}$ satisfies the below conditions, then $\mathrm{p_{NG}}$ is called a NTpgMS. For $\forall \mathrm{x, y, z} \in \mathrm{A}$;

a) $\mathrm{x} * \mathrm{y} \in \mathrm{A}$,

b) $0 \leq \mathrm{p_{NG}}(\mathrm{x, x, x}) \leq \mathrm{p_{NG}}(\mathrm{x, y, z})$,

c) If $\mathrm{p_{NG}}(\mathrm{x, x, x}) = \mathrm{p_{NG}}(\mathrm{y, y, y}) = \mathrm{p_{NG}}(\mathrm{z, z, z}) = \mathrm{p_{NG}}(\mathrm{x, y, z}) = 0$, then x = y = z,

d) If $z \neq y$, then $p_{NG}(x,x,y) \leq p_{NG}(x,y,z)$,

e) $p_{NG}(x,y,z) = p_{NG}(x,z,y) = p_{NG}(y,x,z) = p_{NG}(y,z,x) = p_{NG}(z,x,y) = p_{NG}(z,y,x)$

f) If there exists at least an $a \in X$ for each $x,y,z \in X$ such that

$p_{NG}(x,y,z) \leq p_{NG}(x * neut(a), y * neut(a), z * neut(a))$, then

$p_{NG}(x * neut(a), y * neut(a), z * neut(a)) \leq p_{NG}(x,a,a) + p_{NG}(a,y,z) - p_{NG}(a,a,a)$.

In this case, $((A,*), p_{NG})$ is called NTpgMS.

Example 3.2: Let $X = \{0,3,4,6,9\}$ be a set. We show that $(X, *)$ is a NTS on $\mathbb{Z}_{12}$. Also, we obtain that $neut(0) = 0$, $anti(0) = 0$; $neut(3) = 9$, $anti(3) = 6$; $neut(4) = 4$, $anti(4) = 4$; $neut(6) = 6$, $anti(6) = 6$; $neut(9) = 9$, $anti(9) = 9$.

Thus, $(X,.)$ is a NTS and NTs are (0,0,0), (3,6,9), (4,4,4), (6,6,6) and (9,9,9).

Now, we define the function $p_{NG}: X \times X \times X \to R^+ \cup \{0\}$ such that

$p_{NG}(x,y,z) = 1 + |4^x - 4^y| + |4^x - 4^z| + |4^y - 4^z|$.

We show that p_{NG} is a NTpgM.

a) From Table 1, it is clear that $\forall x,y \in X$; $x * y \in X$.

*	0	3	4	6	9
0	0	0	0	0	0
3	0	9	0	6	3
4	0	0	4	0	0
6	0	6	0	0	6
9	0	3	0	6	9

Table 1: "*" binary operator under $\mathbb{Z}_{12}$

b) It is clear that $0 \leq p_{NG}(x,x,x) = 1 \leq p_{NG}(x,y,z)$.

c) $p_{NG}(x,y,z) = 1 + |4^x - 4^y| + |4^x - 4^z| + |4^y - 4^z| \geq 0$.

d) If $y \neq z$, it is clear that

$p_{NG}(x,x,y) = 1 + |4^x - 4^x| + |4^x - 4^y| + |4^x - 4^y| \leq p_{NG}(x,y,z) = 1 + |4^x - 4^y| + |4^x - 4^z| + |4^y - 4^z|$.

e) By absolute value function, it is clear that

$p_{NG}(x,y,z) = p_{NG}(x,z,y) = p_{NG}(y,x,z) = p_{NG}(y,z,x) = p_{NG}(z,x,y) = p_{NG}(z,y,x)$, for every $x,y,z \in X$.

f)

For $x = 0$, $y = 6$, $z = 3$, $a = 3$, $neut(a) = 6$;

since $p_{NG}(0,6,3) \leq p_{NG}(0 * 6, 6 * 6, 3 * 6) = p_{NG}(0,6,6)$, we obtain that

$p_{NG}(0,6,6) \leq p_{NG}(0,3,3) + p_{NG}(3,6,6) - p_{NG}(3,3,3)$.

For $x = 0$, $y = 3$, $z = 9$, $a = 3$, $neut(a) = 6$;

since $p_{NG}(0,3,9) \leq p_{NG}(0 * 6, 3 * 6, 9 * 6) = p_{NG}(0,6,6)$, we obtain that

$p_{NG}(0,6,6) \leq p_{NG}(0,3,3) + p_{NG}(3,6,6) - p_{NG}(3,3,3)$.

For $x = 0$, $y = 9$, $z = 3$, $a = 6$, $neut(a) = 6$;

since $p_{NG}(0,9,3) \leq p_{NG}(0*6,9*6,3*6) = p_{NG}(0,6,6)$, we obtain that

$p_{NG}(0,6,6) \leq p_{NG}(0,6,6) + p_{NG}(6,6,6) - p_{NG}(6,6,6)$.

For $x = 0$, $y = 6$, $z = 9$, $a = 3$, $neut\ (a) = 6$;

since $p_{NG}(0,6,9) \leq p_{NG}(0*6,6*6,9*6) = p_{NG}(0,6,6)$, we obtain that

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For $x = 0$, $y = 9$, $z = 6$, $a = 3$, $neut\ (a) = 6$;

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For $x = 3$, $y = 6$, $z = 9$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(3,6,9) \leq p_{NG}(3*9,6*9,9*9) = p_{NG}(3,6,9)$, we obtain that

$p_{NG}(9,6,9) \leq p_{NG}(3,6,9) + p_{NG}(9,6,9) - p_{NG}(9,9,9)$.

For $x = 3$, $y = 9$, $z = 6$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(3,9,6) \leq p_{NG}(3*9,9*9,6*9) = p_{NG}(3,9,6)$, we obtain that

$p_{NG}(3,9,6) \leq p_{NG}(3,9,9) + p_{NG}(9,9,6) - p_{NG}(9,9,9)$.

For $x = 3$, $y = 0$, $z = 6$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(3,0,6) \leq p_{NG}(3*9,0*9,6*9) = p_{NG}(3,0,6)$, we obtain that

$p_{NG}(3,0,6) \leq p_{NG}(3,9,9) + p_{NG}(9,0,6) - p_{NG}(9,9,9)$.

For $x = 3$, $y = 6$, $z = 0$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(3,6,0) \leq p_{NG}(3*9,6*9,0*9) = p_{NG}(3,6,0)$, we obtain that

$p_{NG}(3,6,0) \leq p_{NG}(3,9,9) + p_{NG}(9,6,0) - p_{NG}(9,9,9)$.

For $x = 3$, $y = 9$, $z = 0$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(3,9,0) \leq p_{NG}(3*9,9*9,0*9) = p_{NG}(3,9,0)$, we obtain that

$p_{NG}(3,9,0) \leq p_{NG}(3,9,9) + p_{NG}(9,9,0) - p_{NG}(9,9,9)$.

For $x = 3$, $y = 0$, $z = 9$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(3,0,9) \leq p_{NG}(3*9,0*9,9*9) = p_{NG}(3,0,9)$, we obtain that

$p_{NG}(3,0,9) \leq p_{NG}(3,9,9) + p_{NG}(9,0,9) - p_{NG}(9,9,9)$.

For $x = 6$, $y = 0$, $z = 3$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(6,0,3) \leq p_{NG}(6*9,0*9,3*9) = p_{NG}(6,0,3)$, we obtain that

$p_{NG}(6,0,3) \leq p_{NG}(6,9,9) + p_{NG}(9,0,3) - h_{p_{NG}g}(9,9,9)$.

For $x = 6$, $y = 3$, $z = 0$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(6,3,0) \leq p_{NG}(6*9,3*9,0*9) = p_{NG}(6,3,0)$, we obtain that

$p_{NG}(6,3,0) \leq p_{NG}(6,9,9) + p_{NG}(9,3,0) - p_{NG}(3,3,3)$.

For $x = 6$, $y = 3$, $z = 9$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(6,3,9) \leq p_{NG}(6*9,3*9,9*9) = p_{NG}(6,3,9)$, we obtain that

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For $x = 6$, $y = 9$, $z = 3$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(6,9,3) \le p_{NG}(6*9,9*9,3*9) = p_{NG}(6,9,3)$, we obtain that
$p_{NG}(6,9,3) \le p_{NG}(6,9,9) + p_{NG}(9,9,3) - p_{NG}(9,9,9)$.
For $x = 6, y = 0, z = 9, a = 9, neut(a) = 9$;
since $p_{NG}(6,0,9) \le p_{NG}(6*9,0*9,9*9) = p_{NG}(6,0,9)$, we obtain that
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For $x = 6, y = 9, z = 0, a = 9, neut(a) = 9$;
since $p_{NG}(6,9,0) \le p_{NG}(6*9,9*9,0*9) = p_{NG}(6,9,0)$, we obtain that
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For $x = 9, y = 0, z = 3, a = 9, neut(a) = 9$;
since $p_{NG}(9,0,3) \le p_{NG}(9*9,0*9,3*9) = p_{NG}(9,0,3)$, we obtain that
$p_{NG}(9,0,3) \le p_{NG}(9,9,9) + p_{NG}(9,0,3) - p_{NG}(9,9,9)$.
For $x = 9, y = 3, z = 0, a = 9, neut(a) = 9$;
since $p_{NG}(9,3,0) \le p_{NG}(9*9,3*9,0*9) = p_{NG}(9,3,0)$, we obtain that
$p_{NG}(9,3,0) \le p_{NG}(9,9,9) + p_{NG}(9,3,0) - p_{NG}(9,9,9)$.
For $x = 9, y = 3, z = 6, a = 9, neut(a) = 9$;
since $p_{NG}(9,3,6) \le p_{NG}(9*9,3*9,6*9) = p_{NG}(9,3,6)$, we obtain that
$p_{NG}(9,3,6) \le p_{NG}(9,9,9) + p_{NG}(9,3,6) - p_{NG}(9,9,9)$.
For $x = 9, y = 6, z = 3, a = 9, neut(a) = 9$;
since $p_{NG}(9,6,3) \le p_{NG}(9*9,6*9,3*9) = p_{NG}(9,6,3)$, we obtain that
$p_{NG}(9,6,3) \le p_{NG}(9,9,9) + p_{NG}(9,6,3) - p_{NG}(9,9,9)$.
For $x = 9, y = 0, z = 6, a = 9, neut(a) = 9$;
since $p_{NG}(9,0,6) \le p_{NG}(9*9,0*9,6*9) = p_{NG}(9,0,6)$, we obtain that
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For $x = 9, y = 6, z = 0, a = 9, neut(a) = 9$;
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For $x = 0, y = 0, z = 3, a = 6, neut(a) = 6$;
since $p_{NG}(0,0,3) \le p_{NG}(0*6,0*6,3*6) = p_{NG}(0,0,6)$, we obtain that
$p_{NG}(0,0,6) \le p_{NG}(0,6,6) + p_{NG}(6,0,6) - p_{NG}(6,6,6)$.
For $x = 0, y = 3, z = 0, a = 6, neut(a) = 6$;
since $p_{NG}(0,3,0) \le p_{NG}(0*6,3*6,0*6) = p_{NG}(0,6,0)$, we obtain that
$p_{NG}(0,6,0) \le p_{NG}(0,6,6) + p_{NG}(6,6,0) - p_{NG}(6,6,6)$.
For $x = 3, y = 0, z = 0, a = 6, neut(a) = 6$;
since $p_{NG}(3,0,0) \le p_{NG}(3*6,0*6,0*6) = p_{NG}(6,0,0)$, we obtain that
$p_{NG}(6,0,0) \le p_{NG}(6,6,6) + p_{NG}(6,0,0) - p_{NG}(6,6,6)$.
For $x = 0, y = 0, z = 6, a = 9, neut(a) = 9$;

since $\mathrm{p_{NG}}(0,0,6) \leq \mathrm{p_{NG}}(0*9,0*9,6*9) = \mathrm{p_{NG}}(0,0,6)$, we obtain that

$\mathrm{p_{NG}}(0,0,6) \leq \mathrm{p_{NG}}(0,9,9) + \mathrm{p_{NG}}(9,0,6) - \mathrm{p_{NG}}(9,9,9)$.

For $x = 0, y = 6, z = 0, a = 9, neut\ (a) = 9$;

since $\mathrm{p_{NG}}(0,6,0) \leq \mathrm{p_{NG}}(0*9,6*9,0*9) = \mathrm{p_{NG}}(0,6,0)$, we obtain that

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For $x = 6, y = 0, z = 0, a = 9, neut\ (a) = 9$;

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For $x = 0, y = 0, z = 9, a = 9, neut\ (a) = 9$;

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For $x = 0, y = 9, z = 0, a = 9, neut\ (a) = 9$;

since $\mathrm{p_{NG}}(0,9,0) \leq \mathrm{p_{NG}}(0*9,9*9,0*9) = \mathrm{p_{NG}}(0,9,0)$, we obtain that

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For $x = 9, y = 0, z = 0, a = 9, neut\ (a) = 9$;

since $\mathrm{p_{NG}}(9,0,0) \leq \mathrm{p_{NG}}(9*9,0*9,0*9) = \mathrm{p_{NG}}(9,0,0)$, we obtain that

$\mathrm{p_{NG}}(9,0,0) \leq \mathrm{p_{NG}}(9,9,9) + \mathrm{p_{NG}}(9,0,0) - \mathrm{p_{NG}}(9,9,9)$.

For $x = 3, y = 3, z = 0, a = 6, neut\ (a) = 6$;

since $\mathrm{p_{NG}}(3,3,0) \leq \mathrm{p_{NG}}(3*6,3*6,0*6) = \mathrm{p_{NG}}(6,6,0)$, we obtain that

$\mathrm{p_{NG}}(6,6,0) \leq \mathrm{p_{NG}}(6,6,6) + \mathrm{p_{NG}}(6,6,0) - \mathrm{p_{NG}}(6,6,6)$.

For $x = 3, y = 0, z = 3, a = 6, neut\ (a) = 6$;

since $\mathrm{p_{NG}}(3,0,3) \leq \mathrm{p_{NG}}(3*6,0*6,3*6) = \mathrm{p_{NG}}(6,0,6)$, we obtain that

$\mathrm{p_{NG}}(6,0,6) \leq \mathrm{p_{NG}}(6,6,6) + \mathrm{p_{NG}}(6,0,6) - \mathrm{p_{NG}}(6,6,6)$.

For $x = 0, y = 3, z = 3, a = 6, neut\ (a) = 6$;

since $\mathrm{p_{NG}}(0,3,3) \leq \mathrm{p_{NG}}(0*6,3*6,3*6) = \mathrm{p_{NG}}(0,6,6)$, we obtain that

$\mathrm{p_{NG}}(0,6,6) \leq \mathrm{p_{NG}}(0,6,6) + \mathrm{p_{NG}}(6,6,6) - \mathrm{p_{NG}}(6,6,6)$.

For $x = 3, y = 3, z = 6, a = 9, neut\ (a) = 9$;

since $\mathrm{p_{NG}}(3,3,6) \leq \mathrm{p_{NG}}(3*9,3*9,6*9) = \mathrm{p_{NG}}(3,3,6)$, we obtain that

$\mathrm{p_{NG}}(3,3,6) \leq \mathrm{p_{NG}}(3,9,9) + \mathrm{p_{NG}}(9,3,6) - \mathrm{p_{NG}}(9,9,9)$.

For $x = 3, y = 6, z = 3, a = 9, neut\ (a) = 9$;

since $\mathrm{p_{NG}}(3,6,3) \leq \mathrm{p_{NG}}(3*9,6*9,3*9) = \mathrm{p_{NG}}(3,6,3)$, we obtain that

$\mathrm{p_{NG}}(3,6,3) \leq \mathrm{p_{NG}}(3,9,9) + \mathrm{p_{NG}}(9,6,3) - \mathrm{p_{NG}}(9,9,9)$.

For $x = 6, y = 3, z = 3, a = 9, neut\ (a) = 9$;

since $\mathrm{p_{NG}}(6,3,3) \leq \mathrm{p_{NG}}(6*9,3*9,3*9) = \mathrm{p_{NG}}(6,3,3)$, we obtain that

$\mathrm{p_{NG}}(6,3,3) \leq \mathrm{p_{NG}}(6,9,9) + \mathrm{p_{NG}}(9,3,3) - \mathrm{p_{NG}}(9,9,9)$.

For $x = 3, y = 3, z = 9, a = 9, neut\ (a) = 9$;

since $p_{NG}(3,3,9) \leq p_{NG}(3*9, 3*9, 9*9) = p_{NG}(3,3,9)$, we obtain that
$p_{NG}(3,3,9) \leq p_{NG}(3,9,9) + p_{NG}(9,3,9) - p_{NG}(9,9,9)$.
For $x = 3$, $y = 9$, $z = 3$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(3,9,3) \leq p_{NG}(3*9, 9*9, 3*9) = p_{NG}(3,9,3)$, we obtain that
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since $p_{NG}(9,3,3) \leq p_{NG}(9*9, 3*9, 3*9) = p_{NG}(9,3,3)$, we obtain that
$p_{NG}(9,3,3) \leq p_{NG}(9,9,9) + p_{NG}(9,3,3) - p_{NG}(9,9,9)$.
For $x = 6$, $y = 6$, $z = 0$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(6,6,0) \leq p_{NG}(6*9, 6*9, 0*9) = p_{NG}(6,6,0)$, we obtain that
$p_{NG}(6,6,0) \leq p_{NG}(6,9,9) + p_{NG}(9,6,0) - p_{NG}(9,9,9)$.
For $x = 6$, $y = 0$, $z = 6$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(6,0,6) \leq p_{NG}(6*9, 0*9, 6*9) = p_{NG}(6,0,6)$, we obtain that
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For $x = 0$, $y = 6$, $z = 6$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(0,6,6) \leq p_{NG}(0*9, 6*9, 6*9) = p_{NG}(0,6,6)$, we obtain that
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For $x = 6$, $y = 6$, $z = 3$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(6,6,3) \leq p_{NG}(6*9, 6*9, 3*9) = p_{NG}(6,6,3)$, we obtain that
$p_{NG}(6,6,3) \leq p_{NG}(6,9,9) + p_{NG}(9,6,3) - p_{NG}(9,9,9)$.
For $x = 6$, $y = 3$, $z = 6$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(6,3,6) \leq p_{NG}(6*9, 3*9, 6*9) = p_{NG}(6,3,6)$, we obtain that
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For $x = 3$, $y = 6$, $z = 6$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(3,6,6) \leq p_{NG}(3*9, 6*9, 6*9) = p_{NG}(3,6,6)$, we obtain that
$p_{NG}(3,6,6) \leq p_{NG}(9,9,9) + p_{NG}(9,6,6) - p_{NG}(9,9,9)$.
For $x = 6$, $y = 6$, $z = 3$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(6,6,3) \leq p_{NG}(6*9, 6*9, 3*9) = p_{NG}(6,6,3)$, we obtain that
$p_{NG}(6,6,3) \leq p_{NG}(6,9,9) + p_{NG}(9,6,3) - p_{NG}(9,9,9)$.
For $x = 6$, $y = 6$, $z = 9$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(6,6,9) \leq p_{NG}(6*9, 6*9, 9*9) = p_{NG}(6,6,9)$, we obtain that
$p_{NG}(6,6,9) \leq p_{NG}(6,9,9) + p_{NG}(9,6,9) - p_{NG}(9,9,9)$.
For $x = 6$, $y = 9$, $z = 6$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(6,9,6) \leq p_{NG}(6*9, 9*9, 6*9) = p_{NG}(6,9,6)$, we obtain that
$p_{NG}(6,9,6) \leq p_{NG}(6,9,9) + p_{NG}(9,9,6) - p_{NG}(9,9,9)$.
For $x = 3$, $y = 6$, $z = 6$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(3, 6, 6) \le p_{NG}(3 * 9, 6 * 9, 6 * 9) = p_{NG}(3, 6, 6)$, we obtain that
$p_{NG}(3, 6, 6) \le p_{NG}(3, 9, 9) + p_{NG}(9, 6, 6) - p_{NG}(9, 9, 9)$.
For $x = 9 , y = 9 , z = 0 , a = 9 , neut\ (a) = 9$;
since $p_{NG}(9, 9, 0) \le p_{NG}(9 * 9, 9 * 9, 0 * 9) = p_{NG}(9, 9, 0)$, we obtain that
$p_{NG}(9, 9, 0) \le p_{NG}(9, 9, 9) + p_{NG}(9, 9, 0) - p_{NG}(9, 9, 9)$.
For $x = 9 , y = 0 , z = 9 , a = 9 , neut\ (a) = 9$;
since $p_{NG}(9, 0, 9) \le p_{NG}(9 * 9, 0 * 9, 9 * 9) = p_{NG}(9, 0, 9)$, we obtain that
$p_{NG}(9, 0, 9) \le p_{NG}(9, 9, 9) + p_{NG}(9, 0, 9) - p_{NG}(9, 9, 9)$.
For $x = 0 , y = 9 , z = 9 , a = 9 , neut\ (a) = 9$;
since $p_{NG}(0, 9, 9) \le p_{NG}(0 * 9, 9 * 9, 9 * 9) = p_{NG}(0, 9, 9)$, we obtain that
$p_{NG}(0, 9, 9) \le p_{NG}(0, 9, 9) + p_{NG}(9, 9, 9) - p_{NG}(9, 9, 9)$.
For $x = 9 , y = 9 , z = 3 , a = 9 , neut\ (a) = 9$;
since $p_{NG}(9, 9, 3) \le p_{NG}(9 * 9, 9 * 9, 3 * 9) = p_{NG}(9, 9, 3)$, we obtain that
$p_{NG}(9, 9, 3) \le p_{NG}(9, 9, 9) + p_{NG}(9, 9, 3) - p_{NG}(9, 9, 9)$.
For $x = 9 , y = 3 , z = 9 , a = 9 , neut\ (a) = 9$;
since $p_{NG}(9, 3, 9) \le p_{NG}(9 * 9, 3 * 9, 9 * 9) = p_{NG}(9, 3, 9)$, we obtain that
$p_{NG}(9, 3, 9) \le p_{NG}(9, 9, 9) + p_{NG}(9, 3, 9) - p_{NG}(9, 9, 9)$.
For $x = 3 , y = 9 , z = 9 , a = 9 , neut\ (a) = 9$;
since $p_{NG}(3, 9, 9) \le p_{NG}(3 * 9, 9 * 9, 9 * 9) = p_{NG}(3, 9, 9)$, we obtain that
$p_{NG}(3, 9, 9) \le p_{NG}(3, 9, 9) + p_{NG}(9, 9, 9) - p_{NG}(9, 9, 9)$.
For $x = 9 , y = 9 , z = 6 , a = 9 , neut\ (a) = 9$;
since $p_{NG}(9, 9, 6) \le p_{NG}(9 * 9, 9 * 9, 6 * 9) = p_{NG}(9, 9, 6)$, we obtain that
$p_{NG}(9, 9, 6) \le p_{NG}(9, 9, 9) + p_{NG}(9, 9, 6) - p_{NG}(9, 9, 9)$.
For $x = 9 , y = 6 , z = 9 , a = 9 , neut\ (a) = 9$;
since $p_{NG}(9, 6, 9) \le p_{NG}(9 * 9, 6 * 9, 9 * 9) = p_{NG}(9, 6, 9)$, we obtain that
$p_{NG}(9, 6, 9) \le p_{NG}(9, 9, 9) + p_{NG}(9, 6, 9) - p_{NG}(9, 9, 9)$.
For $x = 6 , y = 9 , z = 9 , a = 9 , neut\ (a) = 9$;
since $p_{NG}(6, 9, 9) \le p_{NG}(6 * 9, 9 * 9, 9 * 9) = p_{NG}(6, 9, 9)$, we obtain that
$p_{NG}(6, 9, 9) \le p_{NG}(6, 9, 9) + p_{NG}(9, 9, 9) - p_{NG}(9, 9, 9)$.
For $x = 0 , y = 0 , z = 0 , a = 9 , neut\ (a) = 9$;
since $p_{NG}(0, 0, 0) \le p_{NG}(0 * 9, 0 * 9, 0 * 9) = p_{NG}(0, 0, 0)$, we obtain that
$p_{NG}(0, 0, 0) \le p_{NG}(0, 9, 9) + p_{NG}(9, 0, 0) - p_{NG}(9, 9, 9)$.
For $x = 3 , y = 3 , z = 3 , a = 6 , neut\ (a) = 6$;
since $p_{NG}(3, 3, 3) \le p_{NG}(3 * 6, 3 * 6, 3 * 6) = p_{NG}(6, 6, 6)$, we obtain that
$p_{NG}(6, 6, 6) \le p_{NG}(6, 6, 6) + p_{NG}(6, 6, 6) - p_{NG}(6, 6, 6)$.
For $x = 6 , y = 6 , z = 6 , a = 9 , neut\ (a) = 9$;

since $p_{NG}(6,6,6) \leq p_{NG}(6*9,6*9,6*9) = p_{NG}(6,6,6)$, we obtain that
$p_{NG}(6,6,6) \leq p_{NG}(6,9,9) + p_{NG}(9,6,6) - p_{NG}(9,9,9)$.
For $x = 9$, $y = 9$, $z = 9$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(9,9,9) \leq p_{NG}(9*9,9*9,9*9) = p_{NG}(9,9,9)$, we obtain that
$p_{NG}(9,9,9) \leq p_{NG}(9,9,9) + p_{NG}(9,9,9) - p_{NG}(9,9,9)$.
For $x = 4$, $y = 0$, $z = 0$, $a = 4$, $neut\ (a) = 4$;
since $p_{NG}(4,0,0) \leq p_{NG}(4*4,0*4,0*4) = p_{NG}(4,0,0)$, we obtain that
$p_{NG}(4,0,0) \leq p_{NG}(4,4,4) + p_{NG}(4,0,0) - p_{NG}(4,4,4)$.
For $x = 4$, $y = 0$, $z = 3$, $a = 6$, $neut\ (a) = 6$;
since $p_{NG}(4,0,3) \leq p_{NG}h_{dg}(4*6,0*6,3*6) = p_{NG}(0,0,6)$, we obtain that
$p_{NG}(4,0,3) \leq p_{NG}(4,6,6) + p_{NG}(6,0,3) - p_{NG}(6,6,6)$.
For $x = 4$, $y = 3$, $z = 0$, $a = 6$, $neut\ (a) = 6$;
since $p_{NG}(4,3,0) \leq p_{NG}(4*6,3*6,0*6) = p_{NG}(0,6,0)$, we obtain that
$p_{NG}(0,6,0) \leq p_{NG}(0,6,6) + p_{NG}(6,6,0) - p_{NG}(6,6,6)$.
For $x = 3$, $y = 0$, $z = 4$, $a = 6$, $neut\ (a) = 6$;
since $p_{NG}(3,0,4) \leq p_{NG}(3*6,0*6,4*6) = p_{NG}(6,0,0)$, we obtain that
$p_{NG}(6,0,0) \leq p_{NG}(6,6,6) + p_{NG}(6,0,0) - p_{NG}(6,6,6)$.
For $x = 3$, $y = 4$, $z = 0$, $a = 6$, $neut\ (a) = 6$;
since $p_{NG}(3,4,0) \leq p_{NG}(3*6,4*6,0*6) = p_{NG}(6,0,0)$, we obtain that
$p_{NG}(6,0,0) \leq p_{NG}(6,6,6) + p_{NG}(6,0,0) - p_{NG}(6,6,6)$.
For $x = 4$, $y = 0$, $z = 4$, $a = 4$, $neut\ (a) = 4$;
since $p_{NG}(4,0,4) \leq p_{NG}(4*4,0*4,4*4) = p_{NG}(4,0,4)$, we obtain that
$p_{NG}(4,0,4) \leq p_{NG}(4,4,4) + p_{NG}(4,0,4) - p_{NG}(4,4,4)$.
For $x = 4$, $y = 4$, $z = 0$, $a = 4$, $neut\ (a) = 4$;
since $p_{NG}(4,4,0) \leq p_{NG}(4*4,4*4,0*4) = p_{NG}(4,4,0)$, we obtain that
$p_{NG}(4,4,0) \leq p_{NG}(4,4,4) + p_{NG}(4,4,0) - p_{NG}(4,4,4)$.
For $x = 0$, $y = 4$, $z = 4$, $a = 4$, $neut\ (a) = 4$;
since $p_{NG}(0,4,4) \leq p_{NG}(0*4,4*4,4*4) = p_{NG}(0,4,4)$, we obtain that
$p_{NG}(0,4,4) \leq p_{NG}(0,4,4) + p_{NG}(4,4,4) - p_{NG}(4,4,4)$.
For $x = 4$, $y = 0$, $z = 6$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(4,0,6) \leq p_{NG}(4*9,0*9,6*9) = p_{NG}(0,0,6)$, we obtain that
$p_{NG}(0,0,6) \leq p_{NG}(0,9,9) + p_{NG}(9,0,6) - p_{NG}(9,9,9)$.
For $x = 4$, $y = 6$, $z = 0$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(4,6,0) \leq p_{NG}(4*9,6*9,0*9) = p_{NG}(0,6,0)$, we obtain that
$p_{NG}(0,6,0) \leq p_{NG}(0,9,9) + p_{NG}(9,6,0) - p_{NG}(9,9,9)$.
For $x = 6$, $y = 4$, $z = 0$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(6,4,0) \le p_{NG}(6*9,4*9,0*9) = p_{NG}(6,0,0)$, we obtain that
$p_{NG}(6,0,0) \le p_{NG}(6,9,9) + p_{NG}(9,0,0) - p_{NG}(9,9,9)$.
For $x = 6$, $y = 0$, $z = 4$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(6,0,4) \le p_{NG}(6*9,0*9,4*9) = p_{NG}(6,0,0)$, we obtain that
$p_{NG}(6,0,0) \le p_{NG}(6,9,9) + p_{NG}(9,0,0) - p_{NG}(9,9,9)$.
For $x = 4$, $y = 0$, $z = 9$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(4,0,9) \le p_{NG}(4*9,0*9,9*9) = p_{NG}(0,0,9)$, we obtain that
$p_{NG}(0,0,9) \le p_{NG}(0,9,9) + p_{NG}(9,0,9) - p_{NG}(9,9,9)$.
For $x = 4$, $y = 9$, $z = 0$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(4,9,0) \le p_{NG}(4*9,9*9,0*9) = p_{NG}(0,9,0)$, we obtain that
$p_{NG}(0,9,0) \le p_{NG}(0,9,9) + p_{NG}(9,9,0) - p_{NG}(9,9,9)$.
For $x = 9$, $y = 0$, $z = 4$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(9,0,4) \le p_{NG}(9*9,0*9,4*9) = p_{NG}(9,0,0)$, we obtain that
$p_{NG}(9,0,0) \le p_{NG}(9,9,9) + p_{NG}(9,0,0) - p_{NG}(9,9,9)$.
For $x = 9$, $y = 4$, $z = 0$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(9,4,0) \le p_{NG}(9*9,4*9,0*9) = p_{NG}(9,0,0)$, we obtain that
$p_{NG}(9,0,0) \le p_{NG}(9,9,9) + p_{NG}(9,0,0) - p_{NG}(9,9,9)$.
For $x = 4$, $y = 3$, $z = 3$, $a = 3$, $neut\ (a) =6$;
since $p_{NG}(4,3,3) \le p_{NG}(4*6,3*6,3*6) = p_{NG}(0,6,6)$, we obtain that
$p_{NG}(0,6,6) \le p_{NG}(0,6,6) + p_{NG}(6,6,6) - p_{NG}(6,6,6)$.
For $x = 4$, $y = 3$, $z = 4$, $a = 3$, $neut\ (a) =6$;
since $p_{NG}(4,3,4) \le p_{NG}(4*6,3*6,4*6) = p_{NG}(0,6,0)$, we obtain that
$p_{NG}(0,6,0) \le p_{NG}(0,6,6) + p_{NG}(6,6,0) - p_{NG}(6,6,6)$.
For $x = 4$, $y = 4$, $z = 3$, $a = 3$, $neut\ (a) =6$;
since $p_{NG}(4,4,3) \le p_{NG}(4*6,4*6,3*6) = p_{NG}(0,0,6)$, we obtain that
$p_{NG}(0,0,6) \le p_{NG}(0,6,6) + p_{NG}(6,0,6) - p_{NG}(6,6,6)$.
For $x = 3$, $y = 4$, $z = 4$, $a = 3$, $neut\ (a) =6$;
since $p_{NG}(3,4,4) \le p_{NG}(3*6,4*6,4*6) = p_{NG}(6,0,0)$, we obtain that
$p_{NG}(6,0,0) \le p_{NG}(6,6,6) + p_{NG}(6,0,0) - p_{NG}(6,6,6)$.
For $x = 4$, $y = 3$, $z = 6$, $a = 9$, $neut\ (a) =9$;
since $p_{NG}(4,3,6) \le p_{NG}(4*9,3*9,6*9) = p_{NG}(0,3,6)$, we obtain that
$p_{NG}(0,3,6) \le p_{NG}(0,9,9) + p_{NG}(9,3,6) - p_{NG}(9,9,9)$.
For $x = 4$, $y = 6$, $z = 3$, $a = 9$, $neut\ (a) =9$;
since $p_{NG}(4,6,3) \le p_{NG}(4*9,6*9,3*9) = p_{NG}(0,6,3)$, we obtain that
$p_{NG}(0,6,3) \le p_{NG}(0,9,9) + p_{NG}(9,6,3) - p_{NG}(9,9,9)$.
For $x = 6$, $y = 4$, $z = 3$, $a = 9$, $neut\ (a) =9$;

since $p_{NG}(6,4,3) \le p_{NG}(6*9,4*9,3*9) = p_{NG}(6,0,3)$, we obtain that
$p_{NG}(6,0,3) \le p_{NG}(6,9,9) + p_{NG}(9,0,3) - p_{NG}(9,9,9)$.
For $x = 6$, $y = 3$, $z = 4$, $a = 9$, $neut\ (a)$ =9;
since $p_{NG}(6,3,4) \le p_{NG}(6*9,3*9,4*9) = p_{NG}(6,3,0)$, we obtain that
$p_{NG}(6,3,0) \le p_{NG}(6,9,9) + p_{NG}(9,3,0) - p_{NG}(9,9,9)$.
For $x = 3$, $y = 6$, $z = 4$, $a = 9$, $neut\ (a)$ =9;
since $p_{NG}(3,6,4) \le p_{NG}(3*9,6*9,4*9) = p_{NG}(3,6,0)$, we obtain that
$p_{NG}(3,6,0) \le p_{NG}(3,9,9) + p_{NG}(9,6,0) - p_{NG}(9,9,9)$.
For $x = 3$, $y = 4$, $z = 6$, $a = 9$, $neut\ (a)$ =9;
since $p_{NG}(3,4,6) \le p_{NG}(3*9,4*9,6*9) = p_{NG}(3,0,6)$, we obtain that
$p_{NG}(3,0,6) \le p_{NG}(3,9,9) + p_{NG}(9,0,6) - p_{NG}(9,9,9)$.
For $x = 4$, $y = 3$, $z = 9$, $a = 6$, $neut\ (a) = 6$;
since $p_{NG}(4,3,9) \le p_{NG}(4*6,3*6,9*6) = p_{NG}(0,6,6)$, we obtain that
$p_{NG}(0,6,6) \le p_{NG}(0,6,6) + p_{NG}(6,6,6) - p_{NG}(6,6,6)$.
For $x = 4$, $y = 9$, $z = 3$, $a = 6$, $neut\ (a) = 6$;
since $p_{NG}(4,9,3) \le p_{NG}(4*6,9*6,3*6) = p_{NG}(0,6,6)$, we obtain that
$p_{NG}(0,6,6) \le p_{NG}(0,6,6) + p_{NG}(6,6,6) - p_{NG}(6,6,6)$.
For $x = 9$, $y = 4$, $z = 3$, $a = 6$, $neut\ (a) = 6$;
since $p_{NG}(9,4,3) \le p_{NG}(9*6,4*6,3*6) = p_{NG}(6,0,6)$, we obtain that
$p_{NG}(6,0,6) \le p_{NG}(6,6,6) + p_{NG}(6,0,6) - p_{NG}(6,6,6)$.
For $x = 9$, $y = 3$, $z = 4$, $a = 6$, $neut\ (a) = 6$;
since $p_{NG}(9,3,4) \le p_{NG}(9*6,3*6,4*6) = p_{NG}(6,6,0)$, we obtain that
$p_{NG}(6,6,0) \le p_{NG}(6,6,6) + p_{NG}(6,6,0) - p_{NG}(6,6,6)$.
For $x = 3$, $y = 9$, $z = 4$, $a = 6$, $neut\ (a) = 6$;
since $p_{NG}(3,9,4) \le p_{NG}(3*6,9*6,4*6) = p_{NG}(6,6,0)$, we obtain that
$p_{NG}(6,6,0) \le p_{NG}(6,6,6) + p_{NG}(6,6,0) - p_{NG}(6,6,6)$.
For $x = 3$, $y = 4$, $z = 9$, $a = 6$, $neut\ (a) = 6$;
since $p_{NG}(3,4,9) \le p_{NG}(3*6,4*6,9*6) = p_{NG}(6,0,6)$, we obtain that
$p_{NG}(6,0,6) \le p_{NG}(6,6,6) + p_{NG}(6,0,6) - p_{NG}(6,6,6)$.
For $x = 4$, $y = 4$, $z = 4$, $a = 4$, $neut\ (a) = 4$;
since $p_{NG}(4,4,4) \le p_{NG}(4*4,4*4,4*4) = p_{NG}(4,4,4)$, we obtain that
$p_{NG}(4,4,4) \le p_{NG}(4,4,4) + p_{NG}(4,4,4) - p_{NG}(4,4,4)$.
For $x = 4$, $y = 4$, $z = 6$, $a = 9$, $neut\ (a) = 9$;
since $p_{NG}(4,4,6) \le p_{NG}(4*9,4*9,6*9) = p_{NG}(0,0,6)$, we obtain that
$p_{NG}(0,0,6) \le p_{NG}(0,9,9) + p_{NG}(9,0,6) - p_{NG}(9,9,9)$.
For $x = 4$, $y = 6$, $z = 4$, $a = 9$, $neut\ (a) = 9$;

since $p_{NG}(4,6,4) \leq p_{NG}(4*9,6*9,4*9) = p_{NG}(0,6,0)$, we obtain that
$p_{NG}(0,6,0) \leq p_{NG}(0,9,9) + p_{NG}(9,6,0) - p_{NG}(9,9,9)$.
For $x = 6, y = 4, z = 4, a = 9, neut\,(a) = 9$;
since $p_{NG}(6,4,4) \leq p_{NG}(6*9,4*9,4*9) = p_{NG}(6,0,0)$, we obtain that
$p_{NG}(6,0,0) \leq p_{NG}(0,9,9) + p_{NG}(9,0,0) - p_{NG}(9,9,9)$.
For $x = 4, y = 4, z = 9, a = 9, neut\,(a) = 9$;
since $p_{NG}(4,4,9) \leq p_{NG}(4*9,4*9,9*9) = p_{NG}(0,0,9)$, we obtain that
$p_{NG}(0,0,9) \leq p_{NG}(0,9,9) + p_{NG}(9,0,9) - p_{NG}(9,9,9)$.
For $x = 4, y = 9, z = 4, a = 9, neut\,(a) = 9$;
since $p_{NG}(4,9,4) \leq p_{NG}(4*9,9*9,4*9) = p_{NG}(0,9,0)$, we obtain that
$p_{NG}(0,9,0) \leq p_{NG}(0,9,9) + p_{NG}(9,9,0) - p_{NG}(9,9,9)$.
For $x = 9, y = 4, z = 4, a = 9, neut\,(a) = 9$;
since $p_{NG}(9,4,4) \leq p_{NG}(9*9,4*9,4*9) = p_{NG}(9,0,0)$, we obtain that
$p_{NG}(9,0,0) \leq p_{NG}(9,9,9) + p_{NG}(9,0,0) - p_{NG}(9,9,9)$.
For $x = 4, y = 6, z = 6, a = 9, neut\,(a) = 9$;
since $p_{NG}(4,6,6) \leq p_{NG}(4*9,6*9,6*9) = p_{NG}(0,6,6)$, we obtain that
$p_{NG}(0,6,6) \leq p_{NG}(0,9,9) + h_{p_{NG}}(9,6,6) - p_{NG}(9,9,9)$.
For $x = 6, y = 4, z = 6, a = 9, neut\,(a) = 9$;
since $p_{NG}(6,4,6) \leq p_{NG}(6*9,4*9,6*9) = p_{NG}(6,0,6)$, we obtain that
$p_{NG}(6,0,6) \leq p_{NG}(6,9,9) + p_{NG}(9,0,6) - p_{NG}(9,9,9)$.
For $x = 6, y = 6, z = 4, a = 9, neut\,(a) = 9$;
since $p_{NG}(6,6,4) \leq p_{NG}(6*9,6*9,4*9) = p_{NG}(6,6,0)$, we obtain that
$p_{NG}(6,6,0) \leq p_{NG}(6,9,9) + p_{NG}(9,6,0) - p_{NG}(9,9,9)$.
For $x = 4, y = 6, z = 9, a = 9, neut\,(a) = 9$;
since $p_{NG}(4,6,9) \leq p_{NG}(4*9,6*9,9*9) = p_{NG}(0,6,9)$, we obtain that
$p_{NG}(0,6,9) \leq p_{NG}(0,9,9) + p_{NG}(9,6,9) - p_{NG}(9,9,9)$.
For $x = 4, y = 9, z = 6, a = 9, neut\,(a) = 9$;
since $p_{NG}(4,9,6) \leq p_{NG}(4*9,9*9,6*9) = p_{NG}(0,9,0)$, we obtain that
$p_{NG}(0,9,0) \leq p_{NG}(0,9,9) + p_{NG}(9,9,0) - p_{NG}(9,9,9)$.
For $x = 6, y = 9, z = 4, a = 9, neut\,(a) = 9$;
since $p_{NG}(6,9,4) \leq p_{NG}(6*9,9*9,4*9) = p_{NG}(6,9,0)$, we obtain that
$p_{NG}(6,9,0) \leq p_{NG}(6,9,9) + p_{NG}(9,9,0) - p_{NG}(9,9,9)$.
For $x = 6, y = 4, z = 9, a = 9, neut\,(a) = 9$;
since $p_{NG}(6,4,9) \leq p_{NG}(6*9,4*9,9*9) = p_{NG}(6,0,9)$, we obtain that
$p_{NG}(6,0,9) \leq p_{NG}(6,9,9) + p_{NG}(9,0,9) - p_{NG}(9,9,9)$.
Therefore, p_{NG} is a NTpgM.

Corollary 3.3:

1) The NTpgMS differs from the pgMS. Because, there is not a * binary operation in pgMS. Also, triangle inequalities are different in this spaces.

2) The NTpgMS differs from the NTMS due to triangle inequalities.

3) The NTpgMS differs from the NTgMS. Because the triangle inequality in the NTgMS differs from the triangle inequality in the NTpgMS. Also, in a NTpgMS, it can be that $p_{NG}(x, x) \neq 0$.

Theorem 3.4: Let $((X,*), p_{NG})$ be a NTpgMS and $d_p: X \times X \to R^+ \cup \{0\}$ be a function such that

$d_p(x,y) = p_{NG}(x,y,y) + p_{NG}(x,x,y)$. Then, d_p is a NTpM.

Proof:

i) Since $((X,*), p_{NG})$ is a NTpgMS, it is clear that for $\forall x, y \in X$; $x * y \in X$.

ii) Since p_{NG} is a NTpgMS, $0 \leq d_p(x,x) \leq d_p(x,y)$ implies that

$0 \leq p_{NG}(x,x,x) + p_{NG}(x,x,x) \leq p_{NG}(x,y,y) + p_{NG}(x,x,y)$.

iii) Since p_{NG} is a NTpgMS, if $d_p(x,x) = d_p(y,y) = d_p(x,y) = 0$, then we obtain x = y.

iv) Since p_{NG} is a NTpgMS, we obtain

$d_p(x,y) = p_{NG}(x,y,y) + p_{NG}(x,x,y) = p_{NG}(y,x,x) + p_{NG}(y,y,x) = d_p(y,x)$.

v) We assume that there exists at least an element $a \in X$ for each x, y and z such that $p_{NG}(x,y,z) \leq p_{NG}(x * neut(a), y * neut(a), z * neut(a))$. Thus, if we assume a = x, It is clear that

$d_p(x,y) \leq d_p(x, y*neut(a))$.

Also, since $((X,*), p_{NG})$is a NTpgMS, it is obvious that

$p_{NG}(x,y,z) \leq p_{NG}(x * neut(a), y * neut(a), z * neut(a)) \leq p_{NG}(x,a,a) + p_{NG}(a,y,z) - p_{NG}(a,a,a)$.

Hence, we obtain

$p_{NG}(x,y,y) + p_{NG}(x,x,y) \leq$

$p_{NG}(x,a,a) + p_{NG}(x,x,a) + p_{NG}(x,x,a) + p_{NG}(a,y,y) + p_{NG}(a,a,y) - p_{NG}(a,a,a) - p_{NG}(a,a,a)$.

Thus, we have $d_p(x,y) \leq d_p(x,a) + d_p(a,y) - d_p(a,a)$.

Theorem 3.5: Let $((X, *), p_{NG})$ be a NTpgMS. If for all $x \in X$, $p_{NG}(x,x,x) = 0$, then $((X, *), p_{NG})$ is a NTgMS.

Proof: We suppose that $(X,*)$ is a NTS and $((X, *), p_{NG})$ is a NTpgMS.

i) Since $((X, *), p_{NG})$ is a NTpgMS; then for all $x, y \in X$; $x*y \in X$.

ii) Since $p_{NG}(x,x,x) = 0$, it is clear that $0 \leq p_{NG}(x,x,x) = 0 \leq p_{NG}(x,y,z)$.

iii) Since $((X, *), p_{NG})$ is a NTpgMS, it is clear that if $x \neq y$, then $p_{NG}(x,y,z) > 0$.

iv) Since $((X, *), p_{NG})$ is a NTpgMS, it is clear that if $y \neq z$, then $p_{NG}(x,x,y) \leq p_{NG}(x,y,z)$.

v) Since $((X, *), p_{NG})$ is a NTpgMS, it is clear that

$p_{NG}(x,y,z) = p_{NG}(x,z,y) = p_{NG}(y,x,z) = p_{NG}(y,z,x) = p_{NG}(z,x,y) = p_{NG}(z,y,x)$.

vi) We assume that there exists at least an element $a \in X$ for each x, y, z such that

$p_{NG}(x,y,z) = p_{NG}(x * neut(a), y * neut(a), z * neut(a))$, then

$p_{NG}(x * neut(a), y * neut(a), z * neut(a)) \leq p_{NG}(x,a,a) + p_{NG}(a,y,z) - p_{NG}(a,a,a)$.

Since $p_{NG}(x, x, x) = 0$, we obtain that

$p_{NG}(x * neut(a), y * neut(a), z * neut(a)) \le p_{NG}(x, a, a) + p_{NG}(a, y, z)$.

Thus, ((X, *), p_{NG}) is a NTpgMS.

Theorem 3.6: Let $((X,*), d_{NG})$ be a NTgM. Then, the function $p_{NG}(x, y, z) = d_{NG}(x, y, z) + k$, $k \in R^{+}$ is an NTpgMS.

Proof:

i) Since d_{NG} is a NTgMS, for all x, y ∈ X; x*y ∈ X.

ii) Since d_{NG} is a NTgMS, we obtain $d_{NG}(x, x, x,) \le d_{NG}(x, y, z)$. Thus, it is clear that

$p_{NG}(x, x, x) = d_{NG}(x, x, x) + k \le p_{NG}(x, y, z) = d_{NG}(x, y, z) + k$.

iii) $p_{NG}(x, y, z) = d_{NG}(x, y, z) + k > 0$.

iv) Since d_{NG} is a NTgMS, if y ≠ z, then $d_{NG}(x, x, y) \le d_{NG}(x, y, z)$. Thus, it is clear that

$p_{NG}(x, x, y) = d_{NG}(x, x, y) + k \le p_{NG}(x, y, z) = d_{NG}(x, y, z) + k$

v) Since d_{NG} is a NTgMS, we obtain

$d_{NG}(x, y, z) = d_{NG}(x, z, y) = d_{NG}(y, x, z) = d_{NG}(y, z, x) = d_{NG}(z, x, y) = d_{NG}(z, y, x)$. Thus, it is clear that $d_{NG}(x, y, z) + k = d_{NG}(x, z, y) + k = d_{NG}(y, x, z) + k = d_{NG}(y, z, x) + k = d_{NG}(z, x, y) + k = d_{NG}(z, y, x) + k$.

Therefore,

$p_{NG}(x, y, z) = p_{NG}(x, z, y) = p_{NG}(y, x, z) = p_{NG}(y, z, x) = p_{NG}(z, x, y) = p_{NG}(z, y, x)$.

vi) We assume that there exists at least an $a \in X$ for each $x, y, z \in X$ such that

$d_{NG}(x, y, z) \le d_{NG}(x * neut(a), y * neut(a), z * neut(a))$. Thus, we obtain

$p_{NG}(x, y, z) = d_{NG}(x, y, z) + k \le$

$p_{NG}(x * neut(a), y * neut(a), z * neut(a)) = d_{NG}(x * neut(a), y * neut(a), z * neut(a)) + k$. (1)

Also, since d_{NG} is a NTgMS, we obtain

$d_{NG}(x * neut(a), y * neut(a), z * neut(a)) \le d_{NG}(x, a, a) + d_{NG}(a, y, z)$. Therefore, we obtain

$d_{NG}(x * neut(a), y * neut(a), z * neut(a)) + k \le d_{NG}(x, a, a) + k + d_{NG}(a, y, z) + k - k$. Thus,

$p_{NG}(x * neut(a), y * neut(a), z * neut(a)) \le p_{NG}(x, a, a) + p_{NG}(a, y, z) - k =$

$p_{NG}(x * neut(a), y * neut(a), z * neut(a)) \le p_{NG}(x, a, a) + p_{NG}(a, y, z) - p_{NG}(a, a, a)$. (2)

From (1) and (2), this condition is hold.

In this case, $((X,*), p_{NG})$ is called a NTpgMS.

Corollary 3.7: A NTpgMS can be obtained from a NTgMS.

Definition 3.8: Let ((X, *), p_{NG}) be a NTpgMS and $\{x_n\}$ be a sequence in this space. A point x ∈ X is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m \to \infty} p_{NG}(x, x_n, x_m) - p_{NG}(x, x, x) = 0$ and $\{x_n\}$ is called a NT pg – convergent to x.

Definition 3.9: Let ((X, *), p_{NG}) be a NTpgMS and $\{x_n\}$ be a sequence in this space. $\{x_n\}$ is called a NT pg – Cauchy sequence if there exists at least a x ∈X such that $\lim_{n,m,l \to \infty} p_{NG}(x_n, x_m, x_l) - p_{NG}(x, x, x) = 0$.

Definition 3.10: Let ((X, *), p_{NG}) be a NTpgMS. If every $\{x_n\}$ NT pg - Cauchy sequence is a NT pg - convergent, then ((X, *), p_{NG}) is called a NT complete NTpgMS.

Conclusion

In this study we first obtained NTpgMS. We show that NTpgMS is different from pgMS, NTgMS and NTMS. Also, we show that a NTpgMS will provide the properties of a NTgMS under which conditions are met. Thus, we added a new structure to neutrosophic triple structures. Also, thanks to neutrosophic triplet partial g – metric space, researchers can obtain new fixed point theorems for neutrosophic triplet theory and neutrosophic triplet partial g - normed space, neutrosophic triplet partial g – inner product space.

Abbreviations

gM: g - metric

gMS: g - metric space

NT: Neutrosophic triplet

NTS: Neutrosophic triplet set

NTM: Neutrosophic triplet metric

NTMS: Neutrosophic triplet metric space

NTpM: Neutrosophic triplet partial metric

NTpMS: Neutrosophic triplet partial metric space

NTgM: Neutrosophic triplet g - metric

NTgMS: Neutrosophic triplet g - metric space

NTpgM: Neutrosophic triplet partial g - metric

NTpgMS: Neutrosophic triplet partial g - metric space

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A Note on Neutrosophic Bitopological Spaces

Dimacha Dwibrang Mwchahary[1] and Bhimraj Basumatary [2,*]
[1]Department of Mathematics, Kokrajhar Govt. College, Kokrajhar, BTAD, India Email: ddmwchahary@kgc.edu.in
[2] Department of Mathematical Sciences, Bodoland University, Kokrajhar, BTAD, India, Email: brbasumatary14@gmail.com, brbasumatary@bodolanduniversity.ac.in
* Correspondence: brbasumatary14@gmail.com, brbasumatary@bodolanduniversity.ac.in

Abstract: In this paper, we have introduced the idea on neutrosophic bitopological space and studied its properties with examples. We have defined several definitions of neutrosophic interior, closure and boundary also we have studied all of its properties.
Keywords: Neutrosophic Closed set; Neutrosophic Open set; (τ_i, τ_j) - N-Interior; (τ_i, τ_j) - N-Closure ; (τ_i, τ_j)- N- Boundary; Neutrosophic Bitopological Space.

1. Introduction

In 1995 neutrosophic set has been proposed by F. Smarandache [3, 4] as a new branch of philosophy dealing with ancient roots, origin, nature and scope of neutralities as well as their interactions with different ideational spectra. The term "neutron-sophy" means knowledge of neutral thoughts with natural represents the main distinction between fuzzy set and intuitionistic fuzzy set.

In 1965, L. A. Zadeh defined the concept of membership function and discovered the fuzzy set [1]. With the help of fuzzy set [1] Zadeh explained the idea of uncertainty. In 1989, K. T. Atanassov [2] generalized the concepts of fuzzy set and introduced the degree of non-membership as an independent component and proposed the intuitionistic fuzzy set.

After the introduction of fuzzy sets, several researches were conducted on the generalizations of the notions of fuzzy set. After the generalization of fuzzy sets, many researchers have applied generalization of fuzzy set theory in many branches of science and technology. Chang [5] introduced fuzzy topology. Coker (1997) defined the notion of intuitionistic fuzzy topological spaces. In 1963, J.C. Kely [12] defined the study of Bitopological spaces. A. Kandil et al.[13] discussed on fuzzy bitopological spaces. Lee et al. [14] discussed on some properties of Intuitionistic Fuzzy Bitopological Spaces. Now a day many researchers have studied topology on neutrosophic sets, such as Lupianez [7–10] and Salama [11]. Abdel-Baset et al. [17] discussed on Hybride plitogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. Recently Abdel-Baset et al. [18] studied on Novel plithogenic TOPSIS-CRITIC model for sustainable supply chain risk management.

In this paper, we introduce the concept of Netrosophic Bitopological Spaces. Next, we introduce the concepts of neutrosophic interior set, neutrosophic closure set and neutrosophic boundary set. Also, we have discussed some propositions related to neutrosophic interior set, neutrosophic closure set and neutrosophic boundary set.

2. Basic operations

Definition 2.1 [20] A neutrosophic set A on the universe of discourse X is defined as

$A = \{< x, \mu_{A,} \ \sigma_A, \ \gamma_A >: x \epsilon X\}$

Where $\mu_{A,} \ \sigma_A, \ \gamma_A : X \longrightarrow]0^-, 1^+[$and $0^- \leq \mu_A + \sigma_A + \gamma_A \leq 3^+$, μ_A represents degrees of membership function, $\sigma_A$is the degree of indeterminacy and γ_A is the degree of non-membership function.

Let $= \{< x, \mu_{A,} \ \sigma_A, \ \gamma_A >: x \epsilon X \}$ and $B = \{< x, \mu_{B,} \ \sigma_B, \ \gamma_B >: x \epsilon X \}$ be two neutrosophic sets on X. Then

i. Neutrosophic subset: A≤B if $\mu_A \leq \mu_B$, $\sigma_A \geq \sigma_B$ and $\gamma_A \geq \gamma_B$, That is A is neutrosophic subset of B
ii. Neutrosophic equality: If A ≤B and A ≥B then A=B
iii. Neutrosophic intersection : A∧B = $\{< x, \mu_A \wedge \mu_A, \ \sigma_A \vee \sigma_A, \ \gamma_A \vee \gamma_B >: x \epsilon X\}$
iv. Neutrosophic union: A∨B = $\{< x, \mu_A \vee \mu_A, \ \sigma_A \wedge \sigma_A, \ \gamma_A \wedge \gamma_B >: x \epsilon X\}$
v. Neutrosophic complement: $A^C = \{< x, \ \gamma_A, \ 1 - \sigma_A, \ \mu_{A,} \ >: x \epsilon X\}$
vi. Neutrosophic universal set: $1_X = \{< x, \ 1, \ 0, \ 0 >: x \epsilon X \}$
vii. Neutrosophic empty set: $0_X = \{< x, \ 0, \ 1, \ 1 >: x \epsilon X \}$

Theorem 2.1 [20] Let A and B be two neutrosophic sets on X then

i. A∨A= A and A∧A=A
ii. A∨B= B∨A and A∧B = B ∧ A
iii. A ∨ 0_X= A and A ∨ 1_X=1_X
iv. A∧ 0_X= 0_X and A∧ 1=A
v. A ∨ (B∨C)= (A∨B) ∨C and A ∧ (B∧C)= (A∧B) ∧C
vi. $(A^C)^C$=A

Theorem 2.2 [20] Let A and B be two neutrosophic sets on X then De Morgan's law is valid.

i. $[\vee_{i \in I} A_{i \in I}]^C = \wedge_{i \in I} {A_i}^C$
ii. $[\wedge_{i \in I} A_i]^C = \vee_{i \in I} {A_i}^C$

Definition 2.2 [7] Neutrosophic topological spaces

Let τ be a collection of all neutrosophic subsets on X. Then τ is called a neutrosophic topology in X if the following conditions hold

i. 0_X and 1_X is belong to τ.
ii. Union of any number of neutrosophic sets in τ is again belong to τ.
iii. Intersection of any two neutrosophic set in τ is belong to τ.

Then the pair (X, τ) is called neutrosophic topology on X.

Definition 2.3 [7, 8, 9]

Let (X, τ) be a neutrosophic topological space over X and A is neutrosophic subset on X. Then, the neutrosophic interior of A is the union of all neutrosophic open subsets of A. Clearly neutrosophic interior of A is the biggest neutrosophic open set over X which containing A.

Definition 2.4 [7, 8, 9]

Let (X, τ) be a neutrosophic topological space over X and A is neutrosophic subset on X. Then, the neutrosophic closure of A is the intersection of all neutrosophic closed super sets of A. Clearly neutrosophic closure of A is the smallest neutrosophic closed set over X which contains A.

3. Main Results

Definition 3.1

A system (X, τ_i, τ_j) consisting of a set X with two neutrosophic topologies τ_i and τ_j on X is called Neutrosophic Bitopological space. Throughout in this paper the indices i, j take the value $\in \{1, 2\}$ and $i \neq j$.

Example 3.1

Let X= {a,b} and $A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.4, 0.4, 0.4 >\}$, $B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.3, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$, $D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X,\ 1_X,\ A,\ B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X,\ 1_X,\ C,\ D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutrosophic bitopological space

Definition 3.2

Let (X, τ_i, τ_j) be a neutroscopic bitopological space. Then for a set $A = \{< x, \mu_{ij},\ \sigma_{ij},\ \gamma_{ij} >: x \epsilon X\}$, neutrosophic (τ_i, τ_j)- N-interior of A is the union of all (τ_i, τ_j)-N-open sets of X contained in A and is defined as follows

(τ_i, τ_j)-N-Int(A) = $\{< x, \vee_{\tau_i} \vee_{\tau_j} \mu_{ij},\ \wedge_{\tau_i} \wedge_{\tau_j} \sigma_{ij},\ \wedge_{\tau_i} \wedge_{\tau_j} \gamma_{ij} >: x \epsilon X\}$

Note : Here $\mu_{ij,}$ represents degrees of membership function, σ_{ij} is the degree of indeterminacy and γ_{ij} is the degree of non-membership function of a neutrosophic set and i is interrelated with neutrosophic topology τ_i and j is interrelated with neutrosophic topologie τ_j when we discussed on (τ_i, τ_j)-N-Int(A).

Example 3.2

Let X={a, b} and $A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.4, 0.4, 0.4 >\}$, $B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.3, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$, $D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X,\ 1_X,\ A,\ B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X,\ 1_X,\ C,\ D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutrosophic bitopological space

Let $Q = \{< a, 0.6, 0.4, 0.4 >, < b, 0.3, 0.3, 0.4 >\}$

τ_2-N-Int (Q) = 0_X and τ_1-N-Int (0_X) =0_X

Hence (τ_1, τ_2)-N-Int(Q)= 0_X

Theorem 3.1

Let (X, τ_i, τ_j) be neutrosophic bitopological space then

i. (τ_i, τ_j)-N-Int(0_X) = 0_X, (τ_i, τ_j)-N-Int(1_X) = 1_X

ii. (τ_i, τ_j)-N-Int(A)≤ A.

iii. A is neutroscopic open set if and only if A=(τ_i, τ_j)-N-Int(A)

iv. (τ_i, τ_j)-N-Int [(τ_i, τ_j)-N-Int(A)] = A

v. A≤B implies (τ_i, τ_j)-N-Int(A)≤(τ_i, τ_j)-N-Int(B)

vi. (τ_i, τ_j)-N-Int(A)∨ (τ_i, τ_j)-N-Int(B) ≤ (τ_i, τ_j)-N-Int(A∨B)

vii. (τ_i, τ_j)-N-Int(A ∧ B) = (τ_i, τ_j)-N-Int(A) ∧ (τ_i, τ_j)-N-Int(B).

Proof of the theorems are straightforward.

Remark 3.1: (τ_i, τ_j)-N-Int(A)≠(τ_j, τ_i)-N-Int(A) when i≠ j. For this we cite an example.

Example 3.3

Let X={a, b} and $A = \{< a, 0.5, 0.6, 0.7 >, < b, 0.4, 0.5, 0.6 >\}$, $B = \{< a, 0.6, 0.6, 0.7 >, < b, 0.6, 0.4, 0.5 >\}$, $C = \{< a, 0.6, 0.6, 0.7 >, < b, 0.3, 0.2, 0.3 >\}$, $D = \{< a, 0.7, 0.6, 0.7 >, < b, 0.7, 0.2, 0.3 >\}$. Then $\tau_1 = \{0_X,\ 1_X,\ A,\ B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X,\ 1_X,\ C, D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutrosophic bitopological space.

Let $P = \{< a, 0.8, 0.4, 0.5 >, < b, 0.7, 0.1, 0.2 >\}$

Then τ_2-N-Int(P) = D and (τ_1, τ_2)-N-Int(P) = B.

Now τ_1-N-Int(P) = B and (τ_2, τ_1)-N-Int(P) = C.

Hence the result that is (τ_1, τ_2)-N-Int(A)≠((τ_2, τ_1)-N-Int(A).

Definition 3.3

Let (X, τ_i, τ_j) be a neutrosophic bitopological space. Then for a set $A = \{< x, \mu_{ij},\ \sigma_{ij},\ \gamma_{ij} >: x \epsilon X\}$, neutrosophic (τ_i, τ_j)- N-closure of A is the intersection of all (τ_i, τ_j)-N-closed sets of X contained in A and is defined as follows

$$(\tau_i, \tau_j)\text{-N-Cl(A)} = \{< x,\ \wedge_{\tau_i} \wedge_{\tau_j} \mu_{ij}\ ,\ \vee_{\tau_i} \vee_{\tau_j} \sigma_{ij},\ \ \vee_{\tau_i} \vee_{\tau_j} \gamma_{ij} >: x \epsilon X\}$$

Example 3.4

Let X={a, b} and $A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.4, 0.4, 0.4 >\}$, $B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.3, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$, $D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X,\ 1_X,\ A,\ B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X,\ 1_X,\ C, D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutroscopic bitopological space

Let $P = \{< a, 0.6, 0.5, 0.4 >, < b, 0.4, 0.3, 0.2 >\}$

$P^C = \{< a, 0.4, 0.5, 0.6 >, < b, 0.2, 0.7, 0.4 >\}$.

Now τ_2-N-Cl(P)= 1_X and τ_1-N-Cl(1_X) =1_X

Hence (τ_1, τ_2)-N-Cl(P)= 1_X.

Theorem 3.2 Let (X, τ_i, τ_j) be neutrosophic bitopological space then

i. (τ_i, τ_j)-N-Cl(0_X) = 0_X, (τ_i, τ_j)-N-Cl(1_X) = 1_X

ii. A ≤(τ_i, τ_j)-N-Cl(A).

iii. A is neutrosophic closed set if and only if A=(τ_i, τ_j)-N-Cl(A)

iv. (τ_i, τ_j)-N-Cl [(τ_i, τ_j)-N-Cl(A)] = A

v. A≤B implies (τ_i, τ_j)-N-Cl(A) ≤(τ_i, τ_j)-N-Cl(B).

vi. (τ_i, τ_j)-N-Cl(A∨B) = (τ_i, τ_j)-N-Cl(A)∨(τ_i, τ_j)-N-Cl(B)

vii. (τ_i, τ_j)-N-Cl(A∧B)≤(τ_i, τ_j)-N-Cl(A) ∧ (τ_i, τ_j)-N-Cl(B).

Prove of the theorems are straightforward.

Remark 3.2 (τ_i, τ_j)-N-Cl(A)≠(τ_j, τ_i)-N-Cl(A) when i≠ j. For this we cite an example.

Example 3.5

Let X={a, b} and $A=\{<a,0.5,0.5,0.5>,<b,0.4,0.4,0.4>\}$,
$B=\{<a,0.4,0.6,0.6>,<b,0.2,0.8,0.4>\}$, $C=\{<a,0.6,0.6,0.6>,<b,0.2,0.2,0.2>\}$,
$D=\{<a,0.7,0.7,0.7>,<b,0.4,0.4,0.4>\}$. Then $\tau_1=\{0_X,\ 1_X,\ A,\ B, A\wedge B, A\vee B\}$ and $\tau_2=\{0_X,\ 1_X,\ C,\ D, C\wedge D, C\vee D\}$ then (X, τ_1,τ_2) is neutroscopic bitopological space
Let $P=\{<a,0.6,0.5,0.7>,<b,0.3,0.7,0.4>\}$
τ_2-N-Cl(P)= D^C and τ_1-N-Cl(D^C)=1_X and (τ_1,τ_2)-N-Cl(P)= 1_X.
Now, τ_1-N-Cl(P)= B^C and τ_2-N-Cl(B^C)=D^C and (τ_1,τ_2)-N-Cl(P)= D^C.
Hence (τ_i,τ_j)-N-Cl(A) ≠ (τ_j,τ_i)-N-Cl(A).

Theorem 3.3
Let (X, τ_i,τ_j) be neutrosophic bitopological space then
i. (τ_i,τ_j)-N-Int(A^C) =[(τ_i,τ_j)-N-Cl(A)]^C
ii. (τ_i,τ_j)-N-Cl(A^C) =[(τ_i,τ_j)-N-Int(A)]^C.
iii. (τ_i,τ_j)-N-Int(A) =[(τ_i,τ_j)-N-Cl(A^C)]^C
iv. (τ_i,τ_j)-N-Cl(A) =[(τ_i,τ_j)-N-Int(A^C)]^C

Proof of (i)
Let $A=\{<x,\mu_{ij},\ \sigma_{ij},\ \gamma_{ij}>: x\epsilon X\}$.
Then $A^C=\{<x,\ \gamma_{ij},\ 1-\sigma_{ij},\ \mu_{ij}\ >: x\epsilon X\}$.
Now (τ_i,τ_j)-N-Int(A^C) = $\{<x,\ \vee_{\tau_i}\vee_{\tau_j}\gamma_{ij},\ \wedge_{\tau_i}\wedge_{\tau_j}(1-\sigma_{ij}),\ \wedge_{\tau_i}\wedge_{\tau_j}\mu_{ij}\ >: x\epsilon X\}$
$=\{<x,\ \vee_{\tau_i}\vee_{\tau_j}\gamma_{ij},\ 1-\vee_{\tau_i}\vee_{\tau_j}\sigma_{ij},\ \wedge_{\tau_i}\wedge_{\tau_j}\mu_{ij}\ >: x\epsilon X\}$
(τ_i,τ_j)-N-Cl(A)= $\{<x,\ \wedge_{\tau_i}\wedge_{\tau_j}\mu_{ij}\ ,\ \vee_{\tau_i}\vee_{\tau_j}\sigma_{ij},\ \ \vee_{\tau_i}\vee_{\tau_j}\gamma_{ij}>: x\epsilon X\}$
[(τ_i,τ_j)-N-Cl(A)]^C=$\{<x,\ \ \vee_{\tau_i}\vee_{\tau_j}\gamma_{ij},\ 1-\vee_{\tau_i}\vee_{\tau_j}\sigma_{ij},\ \wedge_{\tau_i}\wedge_{\tau_j}\mu_{ij}\ >: x\epsilon X\}$
Hence (τ_i,τ_j)-N-Int(A^C) =[(τ_i,τ_j)-N-Cl(A)]^C.

Example 3.6
From the **Example 3.4,** we have
τ_2-N-Int(P^C) = 0_X , (τ_1,τ_2)-N-Int(P^C)= 0_X.
τ_2-N-Cl(P) = 1_X, (τ_1,τ_2)-N-Cl(P)= 1_X and [(τ_1,τ_2)-N-Cl(P)]^C= 0_X.
Hence (τ_i,τ_j)-N-Int(A^C) =[(τ_i,τ_j)-N-Cl(A)]^C.
Proof of (ii) is straight forward
Proof of (iii)
Let $A=\{<x,\mu_{ij},\ \sigma_{ij},\ \gamma_{ij}>: x\epsilon X\}$.
Then $A^C=\{<x,\ \gamma_{ij},\ 1-\sigma_{ij},\ \mu_{ij}\ >: x\epsilon X\}$ and

(τ_i,τ_j)-N-Int(A) = $\{<x,\vee_{\tau_i}\vee_{\tau_j}\mu_{ij},\ \wedge_{\tau_i}\wedge_{\tau_j}\sigma_{ij},\ \wedge_{\tau_i}\wedge_{\tau_j}\gamma_{ij}\ >: x\epsilon X\}$

Now

$$(\tau_i,\tau_j)-\mathrm{N}-\mathrm{Cl}(A^C)=\{<x,\ \wedge_{\tau_i}\wedge_{\tau_j}\gamma_{ij},\qquad 1-\wedge_{\tau_i}\wedge_{\tau_j}\sigma_{ij},\ \ \vee_{\tau_i}\vee_{\tau_j}\mu_{ij}\ \ >: x\epsilon X\}$$

So,

$$[(\tau_i,\tau_j)-\mathrm{N}-\mathrm{Cl}(A^C)]^C=\{<x,\vee_{\tau_i}\vee_{\tau_j}\mu_{ij},\ \wedge_{\tau_i}\wedge_{\tau_j}\sigma_{ij},\ \wedge_{\tau_i}\wedge_{\tau_j}\gamma_{ij}\ >: x\epsilon X\}$$

Hence (τ_i,τ_j)-N-Int(A) =[(τ_i,τ_j)-N-Cl(A^C)]^C

Example 3.7 Let X={a, b} and $A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.4, 0.4, 0.4 >\}$, $B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.3, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$, $D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X, 1_X, A, B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X, 1_X, C, D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutrosophic bitopological space

Let $P = \{< a, 0.6, 0.5, 0.4 >, < b, 0.2, 0.3, 0.2 >\}$ and Let $P^C = \{< a, 0.4, 0.5, 0.6 >, < b, 0.2, 0.7, 0.2 >\}$

τ_2-N-Int(P) = 0_X , (τ_1, τ_2)-N-Int(P)= 0_X.

τ_2-N-Cl(P^C) = 1_X, (τ_1, τ_2)-N-Cl(P^C)= 1_X and [(τ_1, τ_2)-N-Cl(P^C)]^C= 0_X.

Hence (τ_i, τ_j)-N-Int(A) =[(τ_i, τ_j)-N-Cl(A^C)]^C.

Proof of (iv) is straight forward

Definition 3.4

Let A be a neutrosophic set in (X, τ_i, τ_j), then (τ_i, τ_j)-N-neutrosophic boundary of A is defined as (τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Cl(A) ∧(τ_i, τ_j)-N-Cl(A^C).

Proposition 3.1

Let A be neutrosophic set in (X, τ_i, τ_j). Then (τ_i, τ_j)-N-Bd(A) ∨A ≤(τ_i, , τ_j)-N-Cl(A).

Proof : We have from the definition (τ_i, τ_j)-N-Bd(A)≤(τ_i, , τ_j)-N-Cl(A) and A≤(τ_i, , τ_j)-N-Cl(A) and hence (τ_i, τ_j)-N-Bd(A)∨A≤ (τ_i, τ_j)-N-Cl(A).

Remark 3.3: The converse part of the proposition is not true. For this we cite an example.

Example 3.8

Let X={a, b} and $A = \{< a, 0.8, 0.7, 0.8 >, < b, 0.5, 0.4, 0.5 >\}$, $B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.3, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$,
$D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X, 1_X, A, B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X, 1_X, C, D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutrosophic bitopological space

Let $P = \{< a, 0.7, 0.4, 0.7 >, < b, 0.4, 0.4, 0.3 >\}$

P^C= $\{< a, 0.7, 0.6, 0.7 >, < b, 0.3, 0.6, 0.4 >\}$.

Now τ_2-N-Cl(P)= 1_X and (τ_i, τ_j)-N-Cl(P) = 1_X

τ_2-N-Cl(P^C)= (C∧D)^C and (τ_i, τ_j)-N-Cl((C∧D)^C) = (A∧B)^C

Now (τ_1, τ_2)-N-Bd(P)= (A∧B)^C and

(τ_1, τ_2)-N-Bd(P) ∨P =$\{< a, 0.8, 0.3, 0.6 >, < b, 0.5, 0.6, 0.3 >\}$

Hence (τ_i, τ_j)-N-Bd(A) ∨ A≠(τ_i, τ_j)-N-Cl(A).

Propositions 3.2

Let A and B be neutrosophic sets in (X, τ_i, τ_j). Then

i. (τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Bd(A^C).
ii. If A be (τ_i, τ_j)-N- neutrosophic closed set then (τ_i, τ_j)-N-Bd(A) ≤A
iii. If A be (τ_i, τ_j)-N- neutrosophic open set then (τ_i, τ_j)-N-Bd(A) ≤ A^C

Proof of (i)

(τ_i, τ_j)-N-Bd(A) =(τ_i, τ_j)-N-cl(A) ∧ (τ_i, τ_j)-N-Cl(A^C)

$$=\{< x, \wedge_{\tau_i}\wedge_{\tau_j}\mu_{ij}\ , \vee_{\tau_i}\vee_{\tau_j}\sigma_{ij}, \vee_{\tau_i}\vee_{\tau_j}\gamma_{ij} >: x\epsilon X\}\wedge\{< x, \wedge_{\tau_i}\wedge_{\tau_j}\gamma_{ij}, 1 - \wedge_{\tau_i}\wedge_{\tau_j}\sigma_{ij}, \vee_{\tau_i}\vee_{\tau_j}\mu_{ij} >: x\epsilon X\}$$

Also (τ_i, , τ_j)-N-Bd(A^C) = (τ_i, τ_j)-N-Cl(A^C) ∧ (τ_i, τ_j)-N-Cl(A)

$=\{< x, \wedge_{\tau_i}\wedge_{\tau_j}\gamma_{ij}, 1 - \wedge_{\tau_i}\wedge_{\tau_j}\sigma_{ij}, \vee_{\tau_i}\vee_{\tau_j}\mu_{ij} >: x\epsilon X\}\wedge$
$\{< x, \wedge_{\tau_i}\wedge_{\tau_j}\mu_{ij} , \vee_{\tau_i}\vee_{\tau_j}\sigma_{ij}, \vee_{\tau_i}\vee_{\tau_j}\gamma_{ij} >: x\epsilon X\}$

Hence (τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Bd(A^C).

Proof of (ii)

Let A be (τ_i, τ_j)-N- neutrosophic closed set then (τ_i, τ_j)-N-Cl(A) = A

Now (τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Cl(A) ∧ (τ_i, τ_j)-N-Cl(A^C) ≤ (τ_i, τ_j)-N-Cl(A) = A

Hence (τ_i, τ_j)-N-Bd(A) ≤ A.

Converse part is not true.

Remark 3.4: The converse part of the proposition is not true. For this we cite an example.

Example 3.9

Let X={a, b} and $A = \{< a, 0.8, 0.7, 0.8 >, < b, 0.5, 0.4, 0.5 >\}$, $B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$, $D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X, 1_X, A, B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X, 1_X, C, D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutroscopic bitopological space

Let $S = \{< a, 0.9, 0.3, 0.2 >, < b, 0.6, 0.2, 0.3 >\}$

$S^C= \{< a, 0.2, 0.7, 0.9 >, < b, 0.3, 0.8, 0.6 >\}$.

Now τ_2-N-Cl(S)= 1_X and (τ_i, τ_j)-N-Cl(1_X) = 1_X

τ_2-N-Cl(S^C)= $(C\wedge D)^C$ and (τ_i, τ_j)-N-Cl($((C\wedge D)^C)$) = $(A\wedge B)^C$

Now (τ_1, τ_2)-N-Bd(S)= $(A\wedge B)^C$ ≤ S.

But S is not a (τ_i, τ_j)-N-closed set.

Hence the converse part is not true.

Proof of (iii) is straight forward.

Proposition 3.3

Let A be neutrosophic set in (X, τ_i, τ_j), then

$[(\tau_i, \tau_j) - \mathrm{N} - \mathrm{Bd(A)}]^C$ = (τ_i, τ_j)-N-Int(A) ∨(τ_i, τ_j)-N-Int(A^C)

Proof:

From the definition we have (τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Cl(A) ∧ (τ_i, τ_j)-N-Cl(A^C)

[(τ_i, τ_j)-N-Bd(A)]^C= [(τ_i, τ_j)-N-Cl(A)]^C ∨ [(τ_i, τ_j)-N-Cl(A^C)]^C

= (τ_i, τ_j)-N-Int(A) ∨ [(τ_i, τ_j)-N-Int(A^C).

Example 3.10 Let X={a, b} and $A = \{< a, 0.8, 0.7, 0.8 >, < b, 0.5, 0.4, 0.5 >\}$, $B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$, $D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X, 1_X, A, B, A \wedge B, A \vee B\}$ and $\tau_2 = \{ 0_X, 1_X, C, D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutrosophic bitopological space

Let $P = \{< a, 0.9, 0.3, 0.2 >, < b, 0.6, 0.2, 0.3 >\}$

$P^C= \{< a, 0.2, 0.7, 0.9 >, < b, 0.3, 0.8, 0.6 >\}$.

Now τ_2-N-Cl(P)= 1_X and (τ_1, τ_2)-N-Cl(1_X) = 1_X

τ_2-N-Cl(P^C)= $(C\wedge D)^C$ and (τ_1, τ_2)-N-Cl($(C\wedge D)^C$) = $(A\wedge B)^C$

So, (τ_1, τ_2)-N-Bd(P)= $(A\wedge B)^C$ and $[(\tau_1, \tau_2)$-N-Bd(P)$]^C$= A∧B.

Now τ_2-N-Int(P)= C∨D, (τ_1, τ_2)-N-Int(P)= A∧B

τ_2-N-Int(P^C)= ϕ, (τ_1, τ_2)-N-Int(P)= ϕ and (τ_1, τ_2)-N-Int(A)∨[(τ_1, τ_2)-N-Int(A^C)= A∧B.

Thus $[(\tau_1, \tau_2) - \mathrm{N} - \mathrm{Bd}(A)]^C =$ (τ_1, τ_2)-N-Int(A) ∨(τ_1, τ_2)-N-Int(A^C) .

Proposition 3.4

Let A be neutrosophic set in (X, τ_i, τ_j), then

(τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Cl(A) - (τ_i, τ_j)-N-Int(A)

Proof: From the definition of (τ_i, τ_j)-N-Bd(A) we have

(τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Cl(A)∧(τ_i, τ_j)-N-Cl(A^C)

= (τ_i, τ_j)-N-Cl(A) – [(τ_i, τ_j)-N-Cl(A^C)$]^C$

= (τ_i, τ_j)-N-Cl(A) –(τ_i, τ_j)-N-Int(A).

Example 3.11

From the **Example 3.10** we have

τ_2-N-Int(P)= C∨D, (τ_1, τ_2)-N-Int(P)= A∧B and 1_X - A∧B = $(A\wedge B)^C$.

Hence (τ_1, τ_2)-N-Bd(A) = (τ_1, τ_2)-N-Cl(A) - (τ_1, τ_2)-N-Int(A).

Proposition 3.5

Let A be neutrosophic set in (X, τ_i, τ_j), then

(τ_i, τ_j)-N-Bd(Int(A)) ≤ (τ_i, τ_j)-N-Bd(A).

Proof :

(τ_i, τ_j)-N-Bd(Int(A))= (τ_i, τ_j)-N-Cl(Int A)∧(τ_i, τ_j)-N-Cl(Int A^C)

= (τ_i, τ_j)-N-Cl(Int A) – [(τ_i, τ_j)-N-Cl(Int A^C)$]^C$

= (τ_i, τ_j)-N-Cl(Int A) – (τ_i, τ_j)-N-Int(A)

≤(τ_i, τ_j)-N-Cl(A) – (τ_i, τ_j)-N-Int(A)

= (τ_i, τ_j)-N-Bd(A).

Remark 3.5: The converse of the proposition is not true. For this we cite an example.

Example 3.12

From Example 3.10, we have

τ_2-N-Int(P^C)= 0_X, (τ_1, τ_2)-N-Int(P^C)= 0_X

(τ_1, τ_2)-N-Bd(Int(A))= (τ_1, τ_2)-N-Cl(0_X) ∧(τ_1, τ_2)-N-Cl(Int 1_X)

= 0_X

Also Now τ_2-N-Cl(P)= 1_X and (τ_1, τ_2)-N-Cl(1_X) = 1_X

τ_2-N-Cl(P^C)= $(C\wedge D)^C$ and (τ_1, τ_2)-N-Cl($(C\wedge D)^C$) = $(A\wedge B)^C$

Now (τ_1, τ_2)-N-Bd(P)= $(A\wedge B)^C$.

Hence (τ_1, τ_2)-N-Bd(Int(A))≤(τ_1, τ_2)-N-Bd(A) but (τ_1, τ_2)-N-Bd(Int(A)) ≠(τ_1, τ_2)-N-Bd(A).

Proposition 3.6

Let A be neutrosophic set in (X, τ_i, τ_j), then
(τ_i, τ_j)-N-Bd(Cl(A)) ≤(τ_i, τ_j)-N-Bd(A).
Proof : Straightforward.

Remark 3.6: The converse of the proposition is not true. For this we cite an example.
Example 3.13
From Example 3.10, we have
(τ_1, τ_2)-N-Bd(Cl(P)) = (τ_i, τ_j)-N-Cl(1_X)∧(τ_i, τ_j)-N-Cl(0_X)
= 0_X
Also Now τ_2-N-Cl(P)= 1_X and (τ_1, τ_2)-N-Cl(1_X) = 1_X
τ_2-N-Cl(P^C)= $(C∧D)^C$ and (τ_1, τ_2)-N-Cl(($(C∧D)^C$) = $(A∧B)^C$
Now (τ_1, τ_2)-N-Bd(P)= $(A∧B)^C$.
Hence (τ_1, τ_2)-N-Bd(Cl(A)) ≤(τ_1, τ_2)-N-Bd(A) but (τ_1, τ_2)-N-Bd(Int(A)) ≠(τ_1, τ_2)-N-Bd(A).
Proposition 3.7
Let A be neutrosophic set in (X, τ_i, τ_j), then

(τ_i, τ_j)-N-Int (A) =A - (τ_i, τ_j)-N-Bd(A)

Proof: Straightforward.
Proposition 3.8
Let A and B be neutrosophic set in (X, τ_i, τ_j). Then
(τ_i, τ_j)-N-Bd(A∨B)≤(τ_i, τ_j)-N-Bd(A) ∨ (τ_i, τ_j)-N-Bd(B)
Proof: Straightforward.
Remark 3.7: The converse of the proposition is not true
Example 3.14
From Example 3.10, we have
Let $Q = \{< a, 0.8, 0.8, 0.8 >, < b, 0.5, 0.5, 0.5 >\} Q^C = \{< a, 0.8, 0.2, 0.8 >, < b, 0.5, 0.5, 0.5 >\}$
$P \vee Q = \{< a, 0.9, 0.3, 0.2 >, < b, 0.6, 0.2, 0.3 >\}$
Now τ_2-N-Cl(Q)= 1_X and (τ_i, τ_j)-N-Cl(Q) = 1_X
τ_2-N-Cl(Q^C)= $1_X$and (τ_i, τ_j)-N-Cl((Q^C) = 1_X
So, (τ_1, τ_2)-N-Bd(Q)= 1_X
Now τ_2-N-Cl($P \vee Q$)= 1_X and (τ_i, τ_j)-N-Cl($P \vee Q$) = 1_X
τ_2-N-Cl($[P \vee Q]^C$)= $(C∧D)^C$ and (τ_i, τ_j)-N-Cl($[P \vee Q]^C$) = $(A∧B)^C$
So, (τ_1, τ_2)-N-Bd($P \vee Q$)= $(A∧B)^C$
Now (τ_i, τ_j)-N-Bd(P∨Q) =$(A∧B)^C$ and (τ_i, τ_j)-N-Bd(P) ∨ (τ_i, τ_j)-N-Bd(Q)= 1_X
Hence (τ_i, τ_j)-N-Bd(P∨Q) ≠(τ_i, τ_j)-N-Bd(P) ∨ (τ_i, τ_j)-N-Bd(Q).

Proposition 3.9
Let A and B be neutrosophic set in (X, τ_i, τ_j). Then

(τ_i, τ_j)-N-Bd(A∧B) ≤ (τ_i, τ_j)-N-Bd(A) ∨ (τ_i, τ_j)-N-Bd(B)

Proof: Straightforward.

Conclusion: In this work we have redefined the definition of Bitopological space with the help of netrosophic set. Then we have investigated the properties of interior, closure and boundary of neutrosophic bitopological spaces. Hope our work will help in further study of neutrosophic generalized closed sets in neutrosophiv bitopological space. This may lead a new beginning for further research on the study of generalized closed sets in neutrosophic bitopological space associated with digraph and directed graphs. This may also lead to the new properties of separation axioms on neutrosophic bitopological space.

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Neutrosophic Fuzzy Soft BCK-submodules

R. S. Alghamdi [1],* and N. O. Alshehri [2]

[1] Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.
[2] Department of Mathematics, Faculty of Science, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia; noal-shehri@uj.edu.sa

* Correspondence: rsaalghamdi@uj.edu.com

Abstract: The target of this study is to apply the notion of neutrosophic soft sets to the theory of BCK-modules by introducing the notion of neutrosophic fuzzy soft BCK-submodules and deriving their basic properties. Also, (α, β, γ)-soft top of neutrosophic fuzzy soft sets in BCK-modules is presented. The concept of Cartesian product of neutrosophic fuzzy soft BCK-submodules is defined and some results are investigated. Finally, an application of neutrosophic fuzzy soft sets in decision making is investigated and an example demonstrating the successfully application of this method is provided.

Keywords: BCK-algebras; BCK-modules; soft sets, fuzzy soft sets, neutrosophic sets; neutrosophic soft sets; neutrosophic fuzzy soft BCK-submodules.

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1. Introduction

A soft set theory as a new mathematical tool for dealing with uncertainties was proposed by Molodtsov in 1999 [21]. He pointed out several directions for the applications of soft sets. In 2002, Maji et al. [19] described the application of soft set theory to a decision-making problem. They [18] also studied several operations on the theory of soft sets. Few years later, Chen et al. [11] presented a new definition of soft set parametrization reduction and compared this definition to the related concept of attributes reduction in rough set theory. At present, works on the soft set theory are progressing rapidly. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with vagueness is the theory of fuzzy sets developed by Zadeh [34]. Since then it has become a vigorous area of research in different domains such as engineering, medical science, social science, physics, statistics, graph theory, artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, expert systems, decision making and automata theory.

Neutrosophic set theory was introduced by F. Smarandache in 1998 [28]. It is considered as a generalization of the fuzzy set. For the first time V.Kandasamy and F. Smarandache [14] introduced the concept of algebraic structures which has caused a pattern shift in the study of algebraic structures. Maji [17] had combined the neutrosophic sets with soft sets and introduced a new mathematical model neutrosophic soft set. The neutrosophic sets aims to model vagueness and ambiguity in complex system. In recent years, it is applied by many researchers in various fields such as group of decision making [3, 22], Project scheduling [1,2] and image processing [26, 33] etc.

In 1994, the notion of BCK-modules was introduced by H. Abujable, M. Aslam and A. Thaheem as an action of BCK-algebras on abelian group [4]. BCK-modules theory then was developed by Z. perveen, M. Aslam and A. Thaheem [25]. Bakhshi [8] presented the concept of fuzzy BCK-submodules and investigated their properties. Recently, H. Bashir and Z. Zahid applied the theory of soft sets on BCK-modules in [16].

In this paper, the concept of neutrosophic fuzzy soft BCK-submodules of BCK-algebra will be introduced and some related properties will be established. Also, (α,β,γ)-soft top of neutrosophic fuzzy soft sets in BCK-modules will be presented. We will define the concept of Cartesian product of neutrosophic fuzzy soft BCK-submodules and investigate some results. Finally, an application of neutrosophic fuzzy soft sets in decision making is going to be investigated and an example demonstrating the successfully application of this method will be given.

This paper is classified as follows. Section 2 gives a brief introduction of neutrosophic fuzzy set, neutrosophic fuzzy soft set, BCK-algebra and BCK-submodule. The notion of neutrosophic fuzzy soft BCK-submodules and some related results are introduced in section 3. The concept of Cartesian product of neutrosophic fuzzy soft BCK-submodules and some properties are obtained in section 4. Section 5 investigates the application of neutrosophic fuzzy soft set in group decision making problems. Finally, in section 6 conclusion is given.

2. Preliminaries

In this section, some preliminaries from the soft set theory, neutrosophic soft sets, BCK-algebras and BCK-modules are induced.

Definition 2.1.[21] Let U be an initial universe and E be a set of parameters. Let $P(U)$ denote the power set of U and let A be a nonempty subset of E. A pair $F_A = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \rightarrow P(U)$ is a set-valued mapping, called the approximate function of the soft set (F, A). It is easy to represent a soft set (F, A) by a set of ordered pairs as follows:

$$(F, A) = \{(x, F(x)) : x \in A\}$$

Neutrosophic set is a generalization of the fuzzy set especially of intuitionistic fuzzy set. The intuitionistic fuzzy set has the degree of non-membership as introduced by K. Atanassov [7]. Smarandache in 1998 [28] has introduced the degree of indeterminacy as an independent component and defined the neutrosophic set on three components: truth, indeterminacy and falsity.

Definition 2.2.[28] A neutrosophic set A on the universe of discourse U is defined as $A = \{(x, T_A(x), I_A(x), F_A(x)), x \in U\}$ where $T_A : U \rightarrow]-0, 1+[$ is a truth membership function, $I_A : U \rightarrow]-0, 1+[$ is an indeterminate membership function, and $F_A : U \rightarrow]-0, 1+[$ is a false membership function and $0- \leq T_A(x) + I_A(x) + F_A(x) \leq 3+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or nonstandard subsets of $]-0, 1+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]-0, 1+[$. Hence, we consider the neutrosophic set which takes the value from the subset of $[0, 1]$.

Definition 2.3.[17] Let U be an initial universe set and E be a set of parameters. Consider $A \subset E$. Let $P(U)$ denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the neutrosophic soft set (NSS) over U, where F is a mapping given by $F : A \rightarrow P(U)$.

Definition 2.4.[17] Let (F, A) and (G, B) be two neutrosophic soft sets over the common universe U. (F, A) is said to be neutrosophic soft subset of (G, B) if $A \subset B$, and $T_{F(e)}(x) \leq T_{G(e)}(x), I_{F(e)}(x) \leq I_{G(e)}(x), F_{F(e)}(x) \geq F_{G(e)}(x), \forall e \in A, x \in U$. We denote it by $(F, A) \subseteq (G, B)$.

Definition 2.5. [17] The complement of a neutrosophic soft set (F, A) denoted by $(F, A)^c$ and is defined as $(F, A)^c = (F^c, \sim A)$, where $F^c : \sim A \rightarrow P(U)$ is a mapping given by

$$F^c(e) = (T_{F^c(e)} = F_{F(e)}, I_{F^c(e)} = I_{F(e)}, F_{F^c(e)} = T_{F(e)}) \quad \text{for all } e \in \sim A$$

Definition 2.6.[12,13] An algebra $(X,*,0)$ of type $(2,0)$ is called BCK-algebra if it is satisfying the following axioms:

(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$,

(BCK-2) $(x * (x * y)) * y = 0$,

(BCK-3) $x * x = 0$,

(BCK-4) $0 * x = 0$,

(BCK-5) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$, *for all* $x, y, z \in X$.

A partial ordering " $\leq$ " is defined on X by $x \leq y \Leftrightarrow x * y = 0$. A BCK-algebra X is said to be bounded if there is an element $1 \in X$ such that $x \leq 1$, for all $x \in X$, commutative if it satisfies the identity $x \wedge y = y \wedge x$, where $x \wedge y = y * (y * x)$, for all $x, y \in X$ and implicative if $x * (y * x) = x$, for all $x, y \in X$.

Definition 2.7.[4] Let X be a BCK-algebra. Then by a left X-module (abbreviated X-module), we mean an abelian group M with an operation $X \times M \to M$ with $(x, m) \mapsto xm$ satisfies the following axioms for all $x, y \in X$ and $m, n \in M$:

(i) $(x \wedge y)m = x(ym)$,

(ii) $x(m + n) = xm + xn$,

(iii) $0m = 0$.

A subgroup N of a BCK-module M is called submodule of M if N is also a BCK-module.

Definition 2.8.[8] Let X be a BCK-algebra. A subset N of a BCK-module M is a BCK-submodule of M if and only if $n_1 - n_2 \in N$ and $xn \in N$ for all $n, n_1, n_2 \in N$ and $x \in X$.

Moreover, if X is bounded and M satisfies $1m = m$, for all $m \in M$, then M is said to be unitary. A mapping $\mu : X \to [0,1]$ is called a fuzzy set in a BCK-algebra X. For any fuzzy set μ in X and any $t \in [0,1]$, we define set $U(\mu; t) = \mu_t = \{x \in X | \mu(x) \geq t\}$, which is called an upper t-level cut of μ.

Definition 2.9.[8] A fuzzy subset μ of M is said to be a fuzzy BCK-submodule if for all $m, m_1, m_2 \in M$ and $x \in X$, the following axioms hold:

(FBCKM1) $\mu(m_1 + m_2) \geq \min\{\mu(m_1), \mu(m_2)\}$,

(FBCKM2) $\mu(-m) = \mu(m)$,

(FBCKM3) $\mu(xm) \geq \mu(m)$.

Definition 2.10.[16] A soft set (F, A) over a BCK-module M is said to be a soft BCK-submodule over M if for all $\varepsilon \in A$, $F(\varepsilon)$ is a BCK-submodule of M.

3. Neutrosophic fuzzy soft BCK-submodules

In this section, we introduce the notion of neutrosophic fuzzy soft BCK-submodules and some related results.

Definition 3.1. A neutrosophic fuzzy soft set (F, A) over a BCK-module M in a BCK-algebra X is said to be a neutrosophic fuzzy soft BCK-submodule over M if for all $m, m_1, m_2 \in M$, $x \in X$ and $\varepsilon \in A$ the following axioms hold:

(NFSS1) $T_{F(\varepsilon)}(m_1 + m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,

$I_{F(\varepsilon)}(m_1 + m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,

$F_{F(\varepsilon)}(m_1 + m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$,

(NFSS2) $T_{F(\varepsilon)}(-m) = T_{F(\varepsilon)}(m)$, $I_{F(\varepsilon)}(-m) = I_{F(\varepsilon)}(m)$, $F_{F(\varepsilon)}(-m) = F_{F(\varepsilon)}(m)$,

(NFSS3) $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m)$, $I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m)$, $F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$.

Example 3.2. Let $X = \{0, a, b, c, d, 1\}$ be a set with a binary operation $*$ defined in Table 1, then $(X, *, 0)$ forms a bounded commutative, non-implicative BCK-algebra (see [20]). $(M, +)$ forms a commutative group defined in Table 2 where $M = \{0, a, c, d\}$ be a subset of X. Consequently, M forms an X-module (see [15]).

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	0	a	0	0
b	b	a	0	b	a	0
c	c	c	c	0	0	0
d	d	c	c	a	0	0
1	1	d	c	b	a	0

Table 1

$+$	0	a	c	d
0	0	a	c	d
a	a	0	d	c
c	c	d	0	a
d	d	c	a	0

Table 2

$\wedge$	0	a	c	d
0	0	0	0	0
a	0	a	0	a
b	0	a	0	a
c	0	0	c	c
d	0	a	c	d
1	0	a	c	d

Table 3

Let $A = \{0, a\}$. Define a neutrosophic fuzzy soft set (F, A) over M as shown in Table 4.

(F, A)	0	a	c	d
$T_{F(0)}$	0.9	0.7	0.8	0.7
$I_{F(0)}$	0.8	0.5	0.6	0.5
$F_{F(0)}$	0.1	0.1	0.1	0.1
$T_{F(a)}$	0.5	0.2	0.3	0.2
$I_{F(a)}$	0.3	0.1	0.3	0.1
$F_{F(a)}$	0.1	0.5	0.4	0.5

Table 4

Consequently, a routine calculations shows that (F, A) forms a neutrosophic fuzzy soft BCK-submodule over M. Note that, Table 3 explains the action of X on M under the operation $xm = x \wedge m$ for all $x \in X$ and $m \in M$.

For the sake of simplicity, we shall use the symbol $NFSS(M)$ for the set of all neutrosophic fuzzy soft BCK-submodules over M.

Theorem 3.3. Let X be a BCK-algebra then a neutrosophic fuzzy soft set $(F, A) \in NFSS(M)$ if and only if

(i) $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m), I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m), F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$,

(ii) $T_{F(\varepsilon)}(m_1 - m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,

$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,

$F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$.

for all $m, m_1, m_2 \in M, x \in X$ and $\varepsilon \in A$.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M then by the Definition (3.1)

condition (i) holds.
(ii) $T_{F(\varepsilon)}(m_1 - m_2) = T_{F(\varepsilon)}(m_1 + (-m_2)) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(-m_2)\} = \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,
Similarly for

$$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$$

and

$$F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Conversely suppose (F, A) satisfies the conditions (i), (ii). Then we have by (i)
$T_{F(\varepsilon)}(-m) = T_{F(\varepsilon)}((-1)m) \geq T_{F(\varepsilon)}(m)$, and $T_{F(\varepsilon)}(m) = T_{F(\varepsilon)}((-1)(-1)m) \geq T_{F(\varepsilon)}(-m)$.
Thus, $T_{F(\varepsilon)}(m) = T_{F(\varepsilon)}(-m)$. Similarly for $I_{F(\varepsilon)}(-m) = I_{F(\varepsilon)}(m)$ *and* $F_{F(\varepsilon)}(-m) = F_{F(\varepsilon)}(m)$.
(ii)$T_{F(\varepsilon)}(m_1 + m_2) = T_{F(\varepsilon)}(m_1 - (-m_2)) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(-m_2)\} = \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,
Similarly for

$$I_{F(\varepsilon)}(m_1 + m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$$

and

$$F_{F(\varepsilon)}(m_1 + m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Hence (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

Theorem 3.4. A neutrosophic fuzzy soft set (F, A) belongs to $NFSS(M)$ in a BCK-algebra X if and only if for all $m, m_1, m_2 \in M, x, y \in X$ *and* $\varepsilon \in A$ the following statements hold:

(i) $T_{F(\varepsilon)}(0) \geq T_{F(\varepsilon)}(m)$, $I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(m)$, $F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(m)$,
(ii) $T_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,
$I_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,
$F_{F(\varepsilon)}(xm_1 - ym_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$.

Proof. Let (F, A) be a $NFSS(M)$, by Theorem (3.3) and since $0m = 0$ for all $m \in M$, we have
(i) $T_{F(\varepsilon)}(0) = T_{F(\varepsilon)}(0m) \geq T_{F(\varepsilon)}(m)$.

The same way for $I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(m)$.

(ii) $T_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{T_{F(\varepsilon)}(xm_1), T_{F(\varepsilon)}(ym_2)\} \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$.

Similarly for

$$I_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}.$$

and

$$F_{F(\varepsilon)}(xm_1 - ym_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Conversely suppose (F, A) satisfies (i), (ii), then

$$T_{F(\varepsilon)}(xm) = T_{F(\varepsilon)}(x(m-0)) = T_{F(\varepsilon)}(xm - x0) \geq \min\{T_{F(\varepsilon)}(m), T_{F(\varepsilon)}(0)\} = T_{F(\varepsilon)}(m).$$

Similarly for $I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$.
Also,

$$T_{F(\varepsilon)}(m_1 - m_2) = T_{F(\varepsilon)}(1m_1 - 1m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}.$$

Similarly for

$$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$$

and

$$F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$$

Hence by Theorem (3.3), (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

Definition 3.5. Let (F, A) be a neutrosophic fuzzy soft set over M. Then (α, β, γ)-soft top of (F, A) is a soft set given by $(H, C_{(\alpha,\beta,\gamma)}(A)) = ((T)_{\alpha}, (I)_{\beta}, (F)^{\gamma})$ where

$$H\ (a) = \{\, m \in M : T_{F(\varepsilon)}(m) \geq \alpha, I_{F(\varepsilon)}\ (m) \geq \beta,\ F_{F(\varepsilon)}(m) \leq \gamma\}$$

for all $\varepsilon \in A, a \in C_{(\alpha,\beta,\gamma)}(A)$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma \leq 3$.

Proposition 3.6. A soft set over BCK-module M is a neutrosophic fuzzy soft BCK-submodule over M if and only if the (α, β, γ)-soft top is either empty or soft BCK-submodule over M for all $\alpha, \beta\ , \gamma \in [0, 1]$ with $\alpha + \beta + \gamma \leq 3$.

Proof. Let (F, A) be a $NFSS(M)$, $(H, C_{(\alpha,\beta,\gamma)}\ (A))$ is non-empty (α, β, γ)-soft top of (F, A) and X is a BCK-algebra. Let $m, n \in H\ (a)$ then by Definition (3.5) we have

$$T_{F(\varepsilon)}(m) \geq \alpha, T_{F(\varepsilon)}(n) \geq \alpha \Rightarrow \min\{T_{F(\varepsilon)}\ (m), T_{F(\varepsilon)}\ (n)\} \geq \alpha,$$
$$I_{F(\varepsilon)}(m) \geq \beta,\ I_{F(\varepsilon)}(n) \geq \beta \Rightarrow \min\{I_{F(\varepsilon)}(m), I_{F(\varepsilon)}(n)\} \geq \beta,$$
$$F_{F(\varepsilon)}(m) \leq \gamma, F_{F(\varepsilon)}(n) \leq \gamma \Rightarrow \max\{F_{F(\varepsilon)}(m), F_{F(\varepsilon)}(n)\} \leq \gamma.$$

By Theorem (3.3), we have

$$T_{F(\varepsilon)}(m - n) \geq \min\{T_{F(\varepsilon)}(m), T_{F(\varepsilon)}(n)\} \geq \alpha,$$
$$I_{F(\varepsilon)}(m - n) \geq \min\{I_{F(\varepsilon)}(m), I_{F(\varepsilon)}(n)\} \geq \beta,$$
$$F_{F(\varepsilon)}(m - n) \leq \max\{F_{F(\varepsilon)}(m), F_{F(\varepsilon)}(n)\} \leq \gamma.$$

Hence $m - n \in H\ (a)$.
Now let $m \in H\ (a), x \in X$. Then

$$T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m) \geq \alpha,$$
$$I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m) \geq \beta\ ,$$
$$F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m) \leq \gamma.$$

Hence $xm \in H\ (a)$. Therfore $H\ (a)$ is a BCK-submodule of M and $(H, C_{(\alpha,\beta,\gamma)}\ (A))$ is a soft BCK-submodule over M.
Conversely, let $(H, C_{(\alpha,\beta,\gamma)}\ (A))$ is a soft BCK-submodule over M for all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma \leq 3$. Let$\alpha = \min\{T_{F(\varepsilon)}(m), T_{F(\varepsilon)}(n)\}, \beta = \min\{I_{F(\varepsilon)}(m), I_{F(\varepsilon)}(n)\}$ and $\gamma = \max\{F_{F(\varepsilon)}(m), F_{F(\varepsilon)}(n)\}$ for $m, n \in M$. Then $m, n \in H\ (a)$. Since $H(a)$is a BCK-submodule of M, therefore $m - n \in H\ (a)$ which mean

$$T_{F(\varepsilon)}\ (m - n) \geq \alpha = \min\{T_{F(\varepsilon)}\ (m), T_{F(\varepsilon)}\ (n)\}\ ,$$
$$I_{F(\varepsilon)}\ (m - n) \geq \beta = \min\{I_{F(\varepsilon)}\ (m), I_{F(\varepsilon)}\ (n)\}\ ,$$
$$F_{F(\varepsilon)}(m - n) \leq \gamma = \max\{F_{F(\varepsilon)}\ (m), F_{F(\varepsilon)}\ (n)\}\ .$$

Now let $\alpha = T_{F(\varepsilon)}(m)$, $\beta = I_{F(\varepsilon)}(m)$ and $\gamma = F_{F(\varepsilon)}(m)$ then $m \in H\ (a)$. Since $H\ (a)$ is a BCK-submodule of M then $xm \in H\ (a)$ for all $x \in X$ i.e.
$T_{F(\varepsilon)}(xm) \geq \alpha = T_{F(\varepsilon)}(m)$, $I_{F(\varepsilon)}(xm) \geq \beta = I_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(xm) \leq \gamma = F_{F(\varepsilon)}(m)$. By Theorem (3.3) we have, (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

Definition 3.7. Let (F, A) be a neutrosophic fuzzy soft set over M, then $(\tilde{F}\ , A_0^1\ )$ is called soft support of (F, A) if it satisfies

$$\tilde{F}\ (\delta) = \{\, m \in M : T_{F(\varepsilon)}\ (m) > 0, I_{F(\varepsilon)}\ (m) > 0, F_{F(\varepsilon)}\ (m) < 1\}$$

for all $\delta \in A_0^1$ and $m \in M$.

Theorem 3.8. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M, then $(\tilde{F}, A_0^1)$ is a soft BCK-submodule over M.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M in a BCK-algebra X and let $m_1, m_2 \in \tilde{F}(a)$, $a \in A_0^1$ then
$T_{F(\varepsilon)}(m_1 - m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\} > 0$, $I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\} > 0$, and $F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\} < 1$. So, $m_1 - m_2 \in \tilde{F}(a)$.
Now let $m \in \tilde{F}(a), x \in X$, then we have $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m) > 0$, $I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m) > 0$, and $F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m) < 1$. So, $xm \in \tilde{F}(a)$. Hence $\tilde{F}(a)$ is a BCK-submodule of M. Therefore $(\tilde{F}, A_0^1)$ a soft BCK-submodule over M.

Proposition 3.9 If a neutrosophic fuzzy soft set over M is a neutrosophic fuzzy soft BCK-submodule over M, then the complement of a neutrosophic fuzzy soft set is also neutrosophic fuzzy soft BCK-submodule over M.

Proof. The proof follow from the Theorem (3.3) and Definition (2.5).

Corollary 3.10 Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M if and only if $(F, A)^c$ is a neutrosophic fuzzy soft BCK-submodule over M.

4. Cartesian Product of Neutrosophic Fuzzy Soft BCK-submodules

In this section, we defined the concept of Cartesian product of neutrosophic fuzzy soft BCK-submodules and obtained some properties on it.

Definition 4.1. Let (F, A) and (G, B) be two neutrosophic fuzzy soft BCK-submodules over M. Then the Cartesian product $(F, A) \times (G, B) = (H, C)$ where $C = A \times B$ and $H(\varepsilon, \delta) = F(\varepsilon) \times G(\delta)$ for all $(\varepsilon, \delta) \in A \times B$ defined as $H(\varepsilon, \delta) = (T_{F\times G}(m, n), I_{F\times G}(m, n), F_{F\times G}(m, n))$ where

$$T_{H(\varepsilon,\delta)}(m, n) = T_{F\times G}(m, n) = \min\{T_{F(\varepsilon)}(m), T_{G(\delta)}(n)\},$$
$$I_{H(\varepsilon,\delta)}(m, n) = I_{F\times G}(m, n) = \min\{I_{F(\varepsilon)}(m), I_{G(\delta)}(n)\},$$
$$F_{H(\varepsilon,\delta)}(m, n) = F_{F\times G}(m, n) = \max\{F_{F(\varepsilon)}(m), F_{G(\delta)}(n)\}.$$

For all $m, n \in M$ and $T_H, I_H, F_H: M \times M \to [0,1]$.

Theorem 4.2. Let (F, A) and (G, B) be two neutrosophic fuzzy soft BCK-submodules over M. Then $(F, A) \times (G, B)$ is a neutrosophic fuzzy soft BCK-submodule over $M \times M$.

Proof. Let X be a BCK-algebra and (F, A), (G, B) be a neutrosophic fuzzy soft BCK-submodules over M. Let $m \in M$, then by Definition (4.1) and Theorem (3.4)

$$T_{F\times G}(0,0) = \min\{T_{F(\varepsilon)}(0), T_{G(\delta)}(0)\} \geq \min\{T_{F(\varepsilon)}(m), T_{G(\delta)}(m)\} = T_{F\times G}(m, m),$$

The same for $I_{F\times G}(0,0) \geq I_{F\times G}(m, m)$ and $F_{F\times G}(0,0) \leq F_{F\times G}(m, m)$ for all $(\varepsilon, \delta) \in A \times B$.
Also, for any $(m_1, n_1), (m_2, n_2) \in M \times M$ and $x, y \in X$ we have

$$\begin{aligned} T_{F\times G}(xm_1 - ym_2, xn_1 - yn_2) &= \min\{T_{F(\varepsilon)}(xm_1 - ym_2), T_{G(\delta)}(xn_1 - yn_2)\} \\ &\geq \min\{\min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}, \min\{T_{G(\delta)}(n_1), T_{G(\delta)}(n_2)\}\} \\ &= \min\{\min\{T_{F(\varepsilon)}(m_1), T_{G(\delta)}(n_1)\}, \min\{T_{F(\varepsilon)}(m_2), T_{G(\delta)}(n_2)\}\} \\ &= \min\{T_{F\times G}(m_1, n_1), T_{F\times G}(m_2, n_2)\}. \end{aligned}$$

Similarly for

$$I_{F\times G}(xm_1 - ym_2, xn_1 - yn_2) \geq \min\{I_{F\times G}(m_1, n_1), I_{F\times G}(m_2, n_2)\}$$

and

$$F_{F\times G}(xm_1 - ym_2, xn_1 - yn_2) \leq \max\{F_{F\times G}(m_1, n_1), F_{F\times G}(m_2, n_2)\}$$

Hence $(F,A)\times(G,B)$ is a neutrosophic fuzzy soft BCK-submodule over $M\times M$. The converse of Theorem 4.2 is not true in general as seen in the following example.

Example 4.3. Let $X=\{0,a,b,c\}$ with a binary operation $*$ defined in Table 5, and then $(X,*,0)$ forms a bounded implicative BCK-algebra (see [20]). Let $M=\{0,a\}$ be a subset of X with a binary operation $+$ defined by Table 6. Then M is a commutative group. Define scalar multiplication $(X,M)\to M$ by $xm=x\wedge m$ for all $x\in X$ and $m\in M$ that is given in Table 7. Consequently, M forms an X-module (see [15]).

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Table 5

$+$	0	a
0	0	a
a	a	0

Table 6

$\wedge$	0	a
0	0	0
a	0	a
b	0	0
c	0	a

Table 7

Let $A=B=M$. Then $C=A\times B=\{(0,0),(0,a),(a,0),(a,a)\}$. Define a neutrosophic fuzzy soft set (H,C) on $M\times M$ as shown in Table 8.

(H,C)	(0,0)	$(0,a)$	$(a,0)$	(a,a)
$T_{H(0,0)}$	0.3	0.3	0.2	0.2
$I_{H(0,0)}$	0.7	0.5	0.6	0.5
$F_{H(0,0)}$	0.1	0.5	0.4	0.5
$T_{H(0,a)}$	0.1	0.1	0.1	0.1
$I_{H(0,a)}$	0.1	0.1	0.1	0.1
$F_{H(0,a)}$	0.5	0.6	0.5	0.6
$T_{H(a,0)}$	0.2	0.2	0.2	0.2
$I_{H(a,0)}$	0.1	0.1	0.1	0.1
$F_{H(a,0)}$	0.4	0.5	0.5	0.5
$T_{H(a,a)}$	0.1	0.1	0.1	0.1
$I_{H(a,a)}$	0.1	0.1	0.1	0.1
$F_{H(a,a)}$	0.5	0.6	0.5	0.6

Table 8

Then $(H,C)=(F,A)\times(G,B)$ is a neutrosophic fuzzy soft BCK-submodule over $M\times M$. But if we consider the neutrosophic fuzzy soft sets (F,A) and (G,B) defined as in Table 9 and Table 10.

(F,A)	0	a
$T_{F(0)}$	0.3	0.2
$I_{F(0)}$	0.8	0.6
$F_{F(0)}$	0.1	0.4
$T_{F(a)}$	0.2	0.2
$I_{F(a)}$	0.1	0.1
$F_{F(a)}$	0.4	0.5

Table9

(G,B)	0	a
$T_{G(0)}$	0.3	0.4
$I_{G(0)}$	0.7	0.5
$F_{G(0)}$	0.1	0.5
$T_{G(a)}$	0.1	0.1
$I_{G(a)}$	0.1	0.1
$F_{G(a)}$	0.5	0.6

Table 10

we can observe that (G,B) is not a neutrosophic fuzzy soft BCK-submodule over M since

$$T_{G(0)}\,(0a)=T_{G(0)}\,(0\wedge a)=T_{G(0)}\,(0)=0.3\ngeq T_{G(0)}\,(a)=0.4$$

Proposition.4.4. Let (F,A) and (G,B) be two neutrosophic fuzzy soft BCK-submodules over M. Then the following equalities are satisfied for the (α,β,γ)-soft top:

$$(T_{F\times G})_\alpha = (T_{F(\varepsilon)})_\alpha \times (T_{G(\delta)})_\alpha, (I_{F\times G})_\beta = (I_{F(\varepsilon)})_\beta \times (I_{G(\delta)})_\beta \text{ and } (F_{F\times G})^\gamma = (F_{F(\varepsilon)})^\gamma \times (F_{G(\delta)})^\gamma$$

For all $(\varepsilon, \delta) \in A \times B$.

Proof: Let $(x, y) \in (T_{F\times G})_\alpha$ be arbitrary. So

$$\begin{aligned} T_{F\times G}(x,y) \geq \alpha &\Leftrightarrow \min\{T_{F(\varepsilon)}(x), T_{G(\delta)}(y)\} \geq \alpha \\ &\Leftrightarrow T_{F(\varepsilon)}(x) \geq \alpha, T_{G(\delta)}(y) \geq \alpha \\ &\Leftrightarrow (x,y) \in (T_{F(\varepsilon)})_\alpha \times (T_{G(\delta)})_\alpha. \end{aligned}$$

$(I_{F\times G})_\beta = (I_{F(\varepsilon)})_\beta \times (I_{G(\delta)})_\beta$ is proved in similar way. Now let$(x, y) \in (F_{F\times G})^\gamma$. Then

$$\begin{aligned} F_{F\times G}(x,y) \leq \gamma &\Leftrightarrow \max\{F_{F(\varepsilon)}(x), F_{G(\delta)}(y)\} \leq \gamma \\ &\Leftrightarrow F_{F(\varepsilon)}(x) \leq \gamma, F_{G(\delta)}(y) \leq \gamma \\ &\Leftrightarrow (x,y) \in (F_{F(\varepsilon)})^\gamma \times (F_{G(\delta)})^\gamma. \end{aligned}$$

Hence the equalities $(T_{F\times G})_\alpha = (T_{F(\varepsilon)})_\alpha \times (T_{G(\delta)})_\alpha, (I_{F\times G})_\beta = (I_{F(\varepsilon)})_\beta \times (I_{G(\delta)})_\beta$ and $(F_{F\times G})^\gamma = (F_{F(\varepsilon)})^\gamma \times (F_{G(\delta)})^\gamma$ are satisfied for all $(\varepsilon, \delta) \in A \times B$.

5. The Neutrosophic Fuzzy Soft Set Application in a Decision-Making Problems

In this section we have investigated the application of neutrosophic fuzzy soft set in group decision making problems. Let $U = \{u_1, u_2, \dots, u_n\}$ be a universal set consisting set of alternatives. Let $E = \{e_1, e_2, \dots, e_m\}$ be a set of criteria. We can represent a group decision making problem using the neutrosophic fuzzy soft approach in the following way.

Let (F, A) denotes the corresponding neutrosophic fuzzy soft set in which $F(e_j)$ represents the neutrosophic fuzzy set for the alternative u_i corresponding to the criteria e_j.

Definition.5.1.[17] Let $A = \langle T_A, I_A, F_A\rangle$ be a neutrosophic fuzzy number, and then the score function $S(A)$ is defined as follows

$$S(A) = (T_A + 1 - I_A + 1 - F_A)/3$$

For two neutrosophic fuzzy numbers A and B, if $S(A) > S(B)$ then $A > B$.

Algorithm

Step 1: Input the neutrosophic soft set (F, A).

Step 2: Compute the score function $S(A)$ of a neutrosophic fuzzy number $A = \langle T_A, I_A, F_A\rangle$, based on the truth-membership degree, indeterminacy-membership degree and falsity membership degree by $S(A) = (T_A + 1 - I_A + 1 - F_A)/3$ and the induced fuzzy soft set $\Delta_{\tilde{F}} = (\tilde{F}, A)$.

Step 3: Calculate the average of $\tilde{F}(e_j)$ for each u_i and let it be denoted as a_i , this is the decision table.

Step 4: Select the optimal alternative u_i if $a_i = \max_k(a_k)$.

Step 5: If there are more than one u_i's then any one of u_i may be chosen.

Remark 5.2:

In the case of multicriteria decision making problems, sometimes every criteria e_j associated with the value $w_j \in [0,1]$ called its weight, which used to represent the different importance of the concerned criteria. In this case there is a small change in the above algorithm. In step 3 instead of average we take weighted average

$$\frac{\sum_{j=1}^{m} \tilde{F}(e_j)w_j}{m}$$

and follows the next steps.

We adopt the following example to illustrate the idea of algorithm given above.

Example 5.3. Suppose that someone wants to invest his money in a stock exchange company. Let $U = \{u_1, u_2, u_3, u_4\}$ the set of alternative companies. Then the four alternatives are evaluated over the set of criteria $E = \{e_1, e_2, e_3, e_4\}$ where e_1=Earnings Per Share, e_2=Dividend, e_3=Book Value and e_4=Price/Earning Ratio. The Evaluation values of the four alternatives on the basis of the above four criteria using the form of neutrosophic fuzzy soft set. The problem is the selection of best company which satisfies the criteria.

Step 1: Neutrosophic fuzzy soft set (F, A) can describe in Table 11.

F	e_1	e_2	e_3	e_4
u_1	⟨0.4,0.2,0.5⟩	⟨0.5,0.3,0.3⟩	⟨0.2,0.7,0.5⟩	⟨0.4,0.6,0.5⟩
u_2	⟨0.3,0.6,0.1⟩	⟨0.2,0.6,0.1⟩	⟨0.4,0.2,0.5⟩	⟨0.2,0.7,0.5⟩
u_3	⟨0.3,0.5,0.2⟩	⟨0.4,0.5,0.2⟩	⟨0.9,0.5,0.7⟩	⟨0.3,0.7,0.6⟩
u_4	⟨0.6,0.7,0.5⟩	⟨0.8,0.4,0.6⟩	⟨0.6,0.3,0.6⟩	⟨0.8,0.3,0.2⟩

Table 11

Step 2: calculate the score of each neutrosophic fuzzy number and obtain the induced fuzzy soft set
$\Delta_{\tilde{F}} = (\tilde{F}, A)$, which is shown in Table 12.

$\tilde{F}$	e_1	e_2	e_3	e_4
u_1	0.57	0.63	0.33	0.43
u_2	0.53	0.5	0.57	0.33
u_3	0.53	0.57	0.57	0.33
u_4	0.47	0.6	0.57	0.77

Table 12

Step 3: Calculate the average of $\tilde{F}(e_j)$ and the decision table for each company u_i obtained in Table 13.

a_i	values
a_1	0.49
a_2	0.4825
a_3	0.5
a_4	0.6025

Table 13

Step 4: Rank all the alternative companies according to the average values a_i $(i = 1,2,3,4)$ as:

$$u_4 \succ u_3 \succ u_1 \succ u_2$$

and thus u_4 is the most desirable alternative.

6. Conclusion

In this paper, we introduced the concept of neutrosophic fuzzy soft BCK-submodules of BCK-algebra and established some related properties. Also, (α, β, γ)-soft top of neutrosophic fuzzy soft sets in BCK-modules was presented. We defined the concept of Cartesian product of neutrosophic fuzzy soft BCK-submodules and investigated some results. Then, we presented an application method for the neutrosophic fuzzy soft set theory in decision making problem. Finally, we provided an example demonstrating the successfully application of this method. The study of neutrosophic fuzzy soft set and their properties have a considerable significance in the sense of applications as well as in understanding the fundamentals of uncertainty. In the future, we shall further develop more algorithms for neutrosophic fuzzy soft set and apply them to solve practical applications in areas such as group decision making, image processing, fusion images and so on.

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Subtraction operational aggregation operators of simplified neutrosophic numbers and their multi-attribute decision making approach

Shigui Du [1,2], Rui Yong [2] and Jun Ye [1,2,*]

[1] Institute of Rock Mechanics, Ningbo University, Ningbo 315211, Zhejiang, P.R. China; dushigui@nbu.edu.cn
[2] School of Civil Engineering, Shaoxing University; Shaoxing 312000, Zhejiang, P.R. China; yongrui@usx.edu.cn

* Correspondence: yejun1@nbu.edu.cn; Tel.: 86 575 88327323

Abstract: The simplified form of a neutrosophic set (NS) was introduced as the simplified NS (S-NS) containing an interval-valued NS (IV-NS) and a single-valued NS (SV-NS) when its truth, indeterminacy and falsity membership degrees are constrained in the real standard interval [0, 1] for the convenience of actual applications. Then, Ye presented subtraction operations of simplified neutrosophic numbers (S-NNs), containing the subtraction operations of interval-valued neutrosophic numbers (IV-NNs) and single-valued neutrosophic numbers (SV-NNs) in S-NN setting. However, the subtraction operations of S-NSs lack actual applications in current research. Since simplified neutrosophic aggregation operators are one of critical mathematical tools in decision making (DM) applications, they have been not investigated so far. Regarding the subtraction operations of S-NNs (SV-NNs and IV-NNs), this work proposes an IV-NN subtraction operational weighted arithmetic averaging (IV-NNSOWAA) operator and a SV-NN subtraction operational weighted arithmetic averaging (SV-NNSOWAA) operator as a necessary complement to existing aggregation operators of S-NNs to aggregate S-NNs (SV-NNs and IV-NNs). Then, a DM approach is developed by means of the SV-NNSOWAA and IV-NNSOWAA operators. Finally, an illustrative example is presented to indicate the applicability and effectiveness of the developed approach.

Keywords: Decision making; Simplified neutrosophic set; Subtraction operation; Subtraction operational aggregation operator

1. Introduction

Neutrosophic set (NS) introduced by Smarandache [1] can depict indeterminate and inconsistent information, which is characterized independently by its truth, falsity and indeterminacy membership degrees in the real standard or non-standard interval $]^-0, 1^+[$. As a simplified form of NS, however, a simplified NS (S-NS) can be introduced when the truth, falsity and indeterminacy membership degrees are constrained in the real standard [0, 1] for the convenience of actual applications. Thus, Ye [2] introduced the S-NS composed of an interval-valued NS (IV-NS) [3] and a single-valued NS (SV-NS) [4], as the subclass of NS. S-NSs can depict the inconsistent and indeterminate information which exists in actual situations, while (interval-valued) intuitionistic fuzzy sets (IFSs) cannot do it. Therefore, S-NSs have received more and more attention in various fields. So far S-NSs (SV-NSs and IV-NSs) have been utilized in image processing [5], medical diagnosis [6], clustering analysis [7, 8], fault diagnosis [9-11], decision making (DM) [12-23] and so on.

Then, division and subtraction operations of (interval-valued) IFSs were presented in existing literature [26-28], and then the subtraction operational aggregation operators (SOAOs) of IFSs were developed for DM problems of clay-brick selection [29]. After that, division and subtraction operations of S-NSs [30], containing the division and subtraction operations of IV-NNs and SV-NNs, were proposed as the operational generalization of (interval-valued) IFSs. However, the subtraction operations of S-NSs lack actual applications in current research. Since simplified neutrosophic number (S-NN) aggregation operators are one of critical mathematical tools in DM applications, the SOAOs of S-NNs have been not investigated so far. Since S-NSs are the extension of (interval-valued) IFSs, the SOAOs of (interval-valued) IFSs can be also generalized to S-NSs to form the SOAOs of S-NSs as a necessary complement to existing aggregation operators of S-NNs. Hence, this paper presents an IV-NN subtraction operational weighted arithmetic averaging (IV-NNSOWAA) operator and a SV-NN subtraction operational weighted arithmetic averaging (SV-NNSOWAA) operator based on the S-NN (IV-NN and SV-NN) aggregation operators [18-20] and establishes their multi-attribute DM approach in S-NN setting.

For this study, the remainder of this paper is formed as the structure. Section 2 introduces some basic notion of S-NSs and operations of S-NNs (IV-NNs and SV-NNs). Section 3 proposes the SV-NNSOWAA and IV-NNSOWAA operators of S-NNs. A multi-attribute DM approach is developed by using the SV-NNSPOWAA or IV-NNSOWAA operator in Section 4. In Section 5, an illustrative example is provided to indicate the applicability and effectiveness of the developed approach. Some conclusions and future work are contained in Section 6.

2. Some basic notion of S-NSs and operations of S-NNs

By the truth, falsity and indeterminacy membership degrees constrained in the real standard interval [0, 1] for the convenience of actual applications, Ye [2] introduced the S-NS notion as a subclass of NS.

Definition 1 [2]. A S-NS *N* in a universal of discourse *U* is characterized by a truth-membership function $TM_N(u)$, a falsity-membership function $FM_N(u)$, and an indeterminacy-membership function $IM_N(u)$, where the values of the three functions $TM_N(u)$, $IM_N(u)$ and $FM_N(u)$ are three real single/interval values in the real standard interval [0, 1], such that $TM_N(u), IM_N(u), FM_N(u) \in [0, 1]$ and $0 \le TM_N(u) + IM_N(u) + FM_N(u) \le 3$ for SV-NS and then $TM_N(u), IM_N(u), FM_N(u) \subseteq [0, 1]$ and $0 \le \sup TM_N(u) + \sup IM_N(u) + \sup FM_N(u) \le 3$ for IV-NS. Thus, a S-NS *N* is denoted as the following mathematical symbol:

$$N = \{\langle u, TM_N(u), IM_N(u), FM_N(u)\rangle \mid u \in U\}.$$

For the simplified representation, the element $\langle u, TM_N(u), IM_N(u), FM_N(u)\rangle$ in the S-NS *N* is simply denoted as the S-NN $a = <TM_a, IM_a, FM_a>$, including IV-NN and SV-NN.

Suppose that two S-NNs are $a = \langle TM_a, IM_a, FM_a\rangle$ and $b = \langle TM_b, IM_b, FM_b\rangle$, then there are the following relations [2]:

(i) $a^c = \langle FM_a, 1 - IM_a, TM_a\rangle$ for the complement of the SV-NN *a* and $a^c = \langle [\inf FM_a, \sup FM_a], [1 - \sup IM_a, 1 - \inf IM_a], [\inf TM_a, \sup TM_a]\rangle$ for the complement of the IV-NN *a*;

(ii) $a \subseteq b$ if and only if $TM_a \le TM_b$, $IM_a \ge IM_b$, and $FM_a \ge FM_b$ for the SV-NN *a* and $\inf TM_a \le \inf TM_b$, $\inf IM_a \ge \inf IM_b$, $\inf FM_a \ge \inf FM_b$ $\sup TM_a \le \sup TM_b$, $\sup IM_a \ge \sup IM_b$, and $\sup FM_a \ge \sup FM_b$ for the IV-NN *a*;

(iii) $a = b$ if and only if $a \subseteq b$ and $b \subseteq a$.

For two S-NNs $a = \langle TM_a, IM_a, FM_a \rangle$ and $b = \langle TM_b, IM_b, FM_b \rangle$, their operational laws are introduced as follows [18, 20]:

(i) $a + b = \langle TM_a + TM_b - TM_a TM_b, IM_a IM_b, FM_a FM_b \rangle$ for SV-NNs and $a + b = \left\langle \begin{array}{l} [\inf TM_a + \inf TM_b - \inf TM_a \inf TM_b, \sup TM_a + \sup TM_b - \sup TM_a \sup TM_b], \\ [\inf IM_a \inf IM_b, \sup IM_a \sup IM_b], [\inf FM_a \inf FM_b, \sup FM_a \sup FM_b] \end{array} \right\rangle$ for IV-NNs;

(ii) $a \times b = \langle TM_a TM_b, IM_a + IM_b - IM_a IM_b, FM_a + FM_b - FM_a FM_b \rangle$ for SV-NNs and $a \times b = \left\langle \begin{array}{l} [\inf TM_a \inf TM_b, \sup TM_a \sup TM_b], \\ [\inf IM_a + \inf IM_b - \inf IM_a \inf IM_b, \sup IM_a + \sup IM_b - \sup IM_a \sup IM_b], \\ [\inf FM_a + \inf FM_b - \inf FM_a \inf FM_b, \sup FM_a + \sup FM_b - \sup FM_a \sup FM_b] \end{array} \right\rangle$ for IV-NNs;

(iii) $\rho a = \langle 1 - (1 - TM_a)^\rho, IM_a^\rho, FM_a^\rho \rangle$ for SV-NN and ρ > 0 and $\rho a = \left\langle \begin{array}{l} [1 - (1 - \inf TM_a)^\rho, 1 - (1 - \sup TM_a)^\rho], \\ [(\inf IM_a)^\rho, (\sup IM_a)^\rho], [(\inf FM_a)^\rho, (\sup FM_a)^\rho] \end{array} \right\rangle$ for IV-NN and ρ > 0;

(iv) $a^\rho = \langle TM_a^\rho, 1 - (1 - IM_a)^\rho, 1 - (1 - FM_a)^\rho \rangle$ for SV-NN and ρ > 0 and $a^\rho = \left\langle \begin{array}{l} [(\inf TM_a)^\rho, (\sup TM_a)^\rho], [1 - (1 - \inf IM_a)^\rho, 1 - (1 - \sup IM_a)^\rho], \\ [1 - (1 - \inf FM_a)^\rho, 1 - (1 - \sup FM_a)^\rho] \end{array} \right\rangle$ for IV-NN and ρ > 0.

For any S-NN $a = \langle TM_a, IM_a, FM_a \rangle$, its score functions can be introduced as follows [18]:

$$S(a) = (2 + TM_a - IM_a - FM_a) / 3, \quad S(a) \in [0,1] \text{ for SV-NN,} \tag{1}$$

$$S(a) = (4 + \inf TM_a - \inf IM_a - \inf FM_a + \sup TM_a - \sup IM_a - \sup FM_a) / 6, \quad S(a) \in [0,1] \text{ for IV-NN.} \tag{2}$$

Set $a_j = \langle TM_{a_j}, IM_{a_j}, FM_{a_j} \rangle$ (*j* = 1, 2, …, *n*) as a group of S-NNs. Then we can introduce the following SV-NN weighted arithmetic averaging (SV-NNWAA) and IV-NN weighted arithmetic averaging (IV-NNWAA) operators [18, 20]:

$$SV - NNWAA(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} w_j a_j = \left\langle 1 - \prod_{j=1}^{n} (1 - TM_{a_j})^{w_j}, \prod_{j=1}^{n} (IM_{a_j})^{w_j}, \prod_{j=1}^{n} (FM_{a_j})^{w_j} \right\rangle, \tag{3}$$

$$IV - NNWAA(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} w_j a_j = \left\langle \begin{array}{l} \left[1 - \prod_{j=1}^{n} (1 - \inf TM_{a_j})^{w_j}, 1 - \prod_{j=1}^{n} (1 - \sup TM_{a_j})^{w_j} \right], \\ \left[\prod_{j=1}^{n} (\inf IM_{a_j})^{w_j}, \prod_{j=1}^{n} (\sup IM_{a_j})^{w_j} \right], \left[\prod_{j=1}^{n} (\inf FM_{a_j})^{w_j}, \prod_{j=1}^{n} (\sup FM_{a_j})^{w_j} \right] \end{array} \right\rangle, \tag{4}$$

where w_j (*j* = 1, 2, …, *n*) is the weight of a_j (*j* = 1, 2, …, *n*) for $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

3. SOAOs of S-NNs (SV-NNs and IV-NNs)

In this section, we present SOAOs based on the subtraction operation of S-NNs (SV-NNs and IV-NNs).

Definition 2 [30]. Set $a = \langle TM_a, IM_a, FM_a \rangle$ and $b = \langle TM_b, IM_b, FM_b \rangle$ as two S-NNs (SV-NNs and IV-NNs), then the subtraction operations of the S-NNs *a* and *b* are defined below:

$$a-b=\left\langle \frac{TM_a-TM_b}{1-TM_b},\frac{IM_a}{IM_b},\frac{FM_a}{FM_b}\right\rangle \text{ for SV-NNs,} \quad (5)$$

$$a-b=\left\langle \begin{array}{l}\left[\frac{\inf TM_a-\inf TM_b}{1-\inf TM_b},\frac{\sup TM_a-\sup TM_b}{1-\sup TM_b}\right],\\ \left[\frac{\inf IM_a}{\inf IM_b},\frac{\sup IM_a}{\sup IM_b}\right],\left[\frac{\inf FM_a}{\inf FM_b},\frac{\sup FM_a}{\sup FM_b}\right]\end{array}\right\rangle \text{ for IV-NNs,} \quad (6)$$

which is valid under the conditions $a \geq b$, $TM_b \neq 1$, $IM_b \neq 0$, and $FM_b \neq 0$ for the SV-NNs a and b, and then $a \supseteq b$, $TM_b \neq [1, 1]$, $IM_b \neq [0, 0]$, and $FM_b \neq [0, 0]$ for the IV-NNs a and b.

Corresponding to the operational laws of S-NNs, we give the following theorem.

Theorem 1. Set $a = <TM_a, IM_a, FM_a>$ and $b = <TM_b, IM_b, FM_b>$ as two S-NNs and $\rho > 0$. Then, there are the following subtraction operational laws:

$$\rho(a-b)=\left\langle 1-\left(1-\frac{TM_a-TM_b}{1-TM_b}\right)^{\rho},\left(\frac{IM_a}{IM_b}\right)^{\rho},\left(\frac{FM_a}{FM_b}\right)^{\rho}\right\rangle, \textit{if } a\geq b, TM_b\neq 1, IM_b, FM_b\neq 0 \text{ for SV-NNs,} \quad (7)$$

$$\rho(a-b)=\left\langle \begin{array}{l}\left[1-\left(1-\frac{\inf TM_a-\inf TM_b}{1-\inf TM_b}\right)^{\rho},1-\left(1-\frac{\sup TM_a-\sup TM_b}{1-\sup TM_b}\right)^{\rho}\right],\\ \left[\left(\frac{\inf IM_a}{\inf IM_b}\right)^{\rho},\left(\frac{\sup IM_a}{\sup IM_b}\right)^{\rho}\right],\left[\left(\frac{\inf FM_a}{\inf FM_b}\right)^{\rho},\left(\frac{\sup FM_a}{\sup FM_b}\right)^{\rho}\right]\end{array}\right\rangle, \textit{if } a\supseteq b, TM_b\neq[1,1], IM_b, FM_b\neq[0,0]$$

for IV-NNs, (8)

$$(a-b)^{\lambda}=\left\langle \left(\frac{TM_a-TM_b}{1-TM_b}\right)^{\lambda},1-\left(1-\frac{IM_a}{IM_b}\right)^{\lambda},1-\left(1-\frac{FM_a}{FM_b}\right)^{\lambda}\right\rangle, \textit{if } a\geq b, TM_b\neq 1, IM_b, FM_b\neq 0 \text{ for}$$

SV-NNs, (9)

$$(a-b)^{\rho}=\left\langle \begin{array}{l}\left[\left(\frac{\inf TM_a-\inf TM_b}{1-\inf TM_b}\right)^{\rho},\left(\frac{\sup TM_a-\sup TM_b}{1-\sup TM_b}\right)^{\rho}\right],\\ \left[1-\left(1-\frac{\inf IM_a}{\inf IM_b}\right)^{\rho},1-\left(1-\frac{\sup IM_a}{\sup IM_b}\right)^{\rho}\right],\\ \left[1-\left(1-\frac{\inf FM_a}{\inf FM_b}\right)^{\rho},1-\left(1-\frac{\sup FM_a}{\sup FM_b}\right)^{\rho}\right]\end{array}\right\rangle, \textit{if } a\supseteq b, TM_b\neq[1,1], IM_b, FM_b\neq[0,0]$$

for IV-NNs. (10)

Obviously, Eqs. (7)-(10) are true according to the operational laws of S-NNs.

Definition 3. Set $a_j=\left\langle TM_{a_j},IM_{a_j},FM_{a_j}\right\rangle$ and $b_j=\left\langle TM_{b_j},IM_{b_j},FM_{b_j}\right\rangle$ (j = 1, 2, …, n) as two groups of S-NNs and $c_j=a_j-b_j=\left\langle TM_{c_j},IM_{c_j},FM_{c_j}\right\rangle$ (j = 1, 2, …, n) as a group of subtraction operations between a_j and b_j. Based on Eqs. (3), (4) and (7)-(10), we can present the SV-NNSOWAA and IV-NNSOWAA operators:

$$\begin{array}{l}SV-NNSOWAA(c_1,c_2,...,c_n)=\sum_{j=1}^{n}w_jc_j=\sum_{j=1}^{n}w_j(a_j-b_j)\\ =\left\langle 1-\prod_{j=1}^{n}(1-TM_{c_j})^{w_j},\prod_{j=1}^{n}(IM_{c_j})^{w_j},\prod_{j=1}^{n}(FM_{c_j})^{w_j}\right\rangle\end{array} \text{ for SV-NNs,} \quad (11)$$

$$IV-NNSOWAA(c_1,c_2,...,c_n)=\sum_{j=1}^{n}w_jc_j=\sum_{j=1}^{n}w_j(a_j-b_j)$$
$$=\left\langle \begin{array}{l}\left[1-\prod_{j=1}^{n}(1-\inf TM_{c_j})^{w_j},1-\prod_{j=1}^{n}(1-\sup TM_{c_j})^{w_j}\right],\\ \left[\prod_{j=1}^{n}(\inf IM_{c_j})^{w_j},\prod_{j=1}^{n}(\sup IM_{c_j})^{w_j}\right],\left[\prod_{j=1}^{n}(\inf FM_{c_j})^{w_j},\prod_{j=1}^{n}(\sup FM_{c_j})^{w_j}\right]\end{array}\right\rangle \text{ for IV-NNs, (12)}$$

where w_j (j = 1, 2, …, n) is the weight of $c_j = a_j - b_j$ (j = 1, 2, …, n) for $w_j \in [0, 1]$ and $\sum_{j=1}^{n}w_j=1$, and the three elements in c_j for the SV-NNs a and b contain the following forms:

$$TM_{c_j}=\begin{cases}\dfrac{TM_{a_j}-TM_{b_j}}{1-TM_{b_j}}\in[0,1],\ if\ TM_{a_j}\ge TM_{b_j}\ and\ TM_{b_j}\ne 1\\ 0,\ otherwise\end{cases}, \quad (13)$$

$$IM_{c_j}=\begin{cases}\dfrac{IM_{a_j}}{IM_{b_j}}\in[0,1],\ if\ IM_{a_j}\le IM_{b_j}\ and\ IM_{b_j}\ne 0\\ 1,\ otherwise\end{cases}, \quad (14)$$

$$FM_{c_j}=\begin{cases}\dfrac{FM_{a_j}}{FM_{b_j}}\in[0,1],\ if\ FM_{a_j}\le FM_{b_j}\ and\ FM_{b_j}\ne 0\\ 1,\ otherwise\end{cases}, \quad (15)$$

or the three elements in c_j for the IV-NNs a and b contain the following forms:

$$TM_{c_j}=\begin{cases}\left[\dfrac{\inf TM_{a_j}-\inf TM_{b_j}}{1-\inf TM_{b_j}},\dfrac{\sup TM_{a_j}-\sup TM_{b_j}}{1-\sup TM_{b_j}}\right]\subseteq[0,1],\ if\ TM_{a_j}\supseteq TM_{b_j}\ and\ TM_{b_j}\ne[1,1]\\ [0,0],\ otherwise\end{cases}, \quad (16)$$

$$IM_{c_j}=\begin{cases}\left[\dfrac{\inf IM_{a_j}}{\inf IM_{b_j}},\dfrac{\sup IM_{a_j}}{\sup IM_{b_j}}\right]\subseteq[0,1],\ if\ IM_{a_j}\subseteq IM_{b_j}\ and\ IM_{b_j}\ne[0,0]\\ [1,1],\ otherwise\end{cases}, \quad (17)$$

$$FM_{c_j}=\begin{cases}\left[\dfrac{\inf FM_{a_j}}{\inf FM_{b_j}},\dfrac{\sup FM_{a_j}}{\sup FM_{b_j}}\right]\subseteq[0,1],\ if\ FM_{a_j}\subseteq FM_{b_j}\ and\ FM_{b_j}\ne[0,0]\\ [1,1],\ otherwise\end{cases}. \quad (18)$$

4. Multi-attribute DM approach corresponding to the SV-NNSOWAA and IV-NNSOWAA operators

Regarding the SV-NNSOWAA and IV-NNSOWAA operators, we can establish a multi-attribute DM approach to deal with the DM problem with S-NNs (SV-NNs and IV-NNs).

As for a multi-attribute DM problem in S-NN setting, suppose that $P = \{P_1, P_2, …, P_m\}$ is a set of alternatives and $R = \{r_1, r_2, …, r_n\}$ is a set of attributes. Then the suitability assessment of an alternative P_i (i = 1, 2, …, m) over an attribute r_j (j = 1, 2, …, n) is expressed by a S-NN $a_{ij}=\langle TM_{a_{ij}},IM_{a_{ij}},FM_{a_{ij}}\rangle$ (i = 1, 2, …, m; j = 1, 2, …, n), where $TM_{a_{ij}}$ indicates the degree that the alternative P_i is satisfactory to the attribute r_j, $IM_{a_{ij}}$ indicates the indeterminate degree that the alternative P_i is satisfactory and/or unsatisfactory to the attribute r_j, and $FM_{a_{ij}}$ indicates the degree that the alternative P_i is unsatisfactory to the attribute r_j. Thus, all the assessment values of S-NNs can be structured as a S-NN decision matrix $D = (a_{ij})_{m\times n}$. Then, the weight of each attribute r_j (j = 1, 2, …, n) is w_j (j = 1, 2, …, n) for $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Regarding the DM problem with S-NNs (SV-NNs or IV-NNs), the decision steps is indicated as follows:

Step 1. From the S-NN decision matrix $D = (a_{ij})_{m\times n}$, the j-th S-NN positive ideal solution can be determined by the SV-NN $a_j^+ = \left\langle TM_j^+, IM_j^+, FM_j^+ \right\rangle = \left\langle \max_i(TM_{ij}), \min_i(IM_{ij}), \min_i(FM_{ij}) \right\rangle$ or the IV-NN $a_j^+ = \left\langle TM_j^+, IM_j^+, FM_j^+ \right\rangle = \left\langle \begin{array}{l} \left[\max_i(\inf TM_{ij}), \max_i(\sup TM_{ij})\right], \\ \left[\min_i(\inf IM_{ij}), \min_i(\sup IM_{ij})\right], \\ \left[\min_i(\inf FM_{ij}), \min_i(\sup FM_{ij})\right] \end{array} \right\rangle$ (j = 1, 2, …, n; i = 1, 2, …, m), while the j-th S-NN negative ideal solution can be determined by the SV-NN $a_j^- = \left\langle TM_j^-, IM_j^-, FM_j^- \right\rangle = \left\langle \min_i(TM_{ij}), \max_i(IM_{ij}), \max_i(FM_{ij}) \right\rangle$ or the IV-NN $a_j^- = \left\langle TM_j^-, IM_j^-, FM_j^- \right\rangle = \left\langle \begin{array}{l} \left[\min_i(\inf TM_{ij}), \min_i(\sup TM_{ij})\right], \\ \left[\max_i(\inf IM_{ij}), \max_i(\sup IM_{ij})\right], \\ \left[\max_i(\inf FM_{ij}), \max_i(\sup FM_{ij})\right] \end{array} \right\rangle$ (j = 1, 2, …, n; i = 1, 2, …, m).

Step 2. Two collective values d_i^+ and d_i^- (i = 1, 2, …, m) for each alternative P_i (i = 1, 2, …, m) can be obtained by the SV-NNSOWAA and IV-NNSOWAA operators:

$$d_i^+ = SV-NNSOWAA(c_{i1}^+, c_{i2}^+, ..., c_{in}^+) = \sum_{j=1}^{n} w_j c_{ij}^+ = \sum_{j=1}^{n} w_j (a_j^+ - a_{ij}) = \left\langle 1-\prod_{j=1}^{n}(1-TM_{c_{ij}^+})^{w_j}, \prod_{j=1}^{n}(IM_{c_{ij}^+})^{w_j}, \prod_{j=1}^{n}(FM_{c_{ij}^+})^{w_j} \right\rangle$$ for the SV-NN a_{ij}, (19)

$$d_i^+ = IV-NNSOWAA(c_{i1}^+, c_{i2}^+, ..., c_{in}^+) = \sum_{j=1}^{n} w_j c_{ij}^+ = \sum_{j=1}^{n} w_j (a_j^+ - a_{ij}) = \left\langle \begin{array}{l} \left[1-\prod_{j=1}^{n}(1-\inf TM_{c_{ij}^+})^{w_j}, 1-\prod_{j=1}^{n}(1-\sup TM_{c_{ij}^+})^{w_j}\right], \\ \left[\prod_{j=1}^{n}(\inf IM_{c_{ij}^+})^{w_j}, \prod_{j=1}^{n}(\sup IM_{c_{ij}^+})^{w_j}\right], \left[\prod_{j=1}^{n}(\inf FM_{c_{ij}^+})^{w_j}, \prod_{j=1}^{n}(\sup FM_{c_{ij}^+})^{w_j}\right] \end{array} \right\rangle$$ for the IV-NN a_{ij}, (20)

$$d_i^- = SV-NNSOWAA(c_{i1}^-, c_{i2}^-, ..., c_{in}^-) = \sum_{j=1}^{n} w_j c_{ij}^- = \sum_{j=1}^{n} w_j (a_{ij} - a_j^-) = \left\langle 1-\prod_{j=1}^{n}(1-TM_{c_{ij}^-})^{w_j}, \prod_{j=1}^{n}(IM_{c_{ij}^-})^{w_j}, \prod_{j=1}^{n}(FM_{c_{ij}^-})^{w_j} \right\rangle$$ for the SV-NN a_{ij}, (21)

$$d_i^- = IV-NNSOWAA(c_{i1}^-, c_{i2}^-, ..., c_{in}^-) = \sum_{j=1}^{n} w_j c_{ij}^- = \sum_{j=1}^{n} w_j (a_{ij} - a_j^-)$$
$$= \left\langle \begin{array}{l} \left[1-\prod_{j=1}^{n}(1-\inf TM_{c_{ij}^-})^{w_j}, 1-\prod_{j=1}^{n}(1-\sup TM_{c_{ij}^-})^{w_j}\right], \\ \left[\prod_{j=1}^{n}(\inf IM_{c_{ij}^-})^{w_j}, \prod_{j=1}^{n}(\sup IM_{c_{ij}^-})^{w_j}\right], \left[\prod_{j=1}^{n}(\inf FM_{c_{ij}^-})^{w_j}, \prod_{j=1}^{n}(\sup FM_{c_{ij}^-})^{w_j}\right] \end{array} \right\rangle$$

for the IV-NN a_{ij}, (22)

where w_j (j = 1, 2, …, n) is the attribute weight for $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and the components in the SV-NNs c_{ij}^+ and c_{ij}^- contain the following forms:

$$TM_{c_{ij}^+} = \begin{cases} \dfrac{TM_{a_j^+} - TM_{a_{ij}}}{1 - TM_{a_{ij}}}, if\ TM_{a_j^+} \geq TM_{a_{ij}}\ and\ TM_{a_{ij}} \neq 1 \\ 0, otherwise \end{cases},$$

$$IM_{c_{ij}^+} = \begin{cases} \dfrac{IM_{a_j^+}}{IM_{a_{ij}}}, if\ IM_{a_j^+} \leq IM_{a_{ij}}\ and\ IM_{a_{ij}} \neq 0 \\ 1, otherwise \end{cases},$$

$$FM_{c_{ij}^+} = \begin{cases} \dfrac{FM_{a_j^+}}{FM_{a_{ij}}}, if\ FM_{a_j^+} \leq FM_{a_{ij}}\ and\ FM_{a_{ij}} \neq 0 \\ 1, otherwise \end{cases},$$

$$TM_{c_{ij}^-} = \begin{cases} \dfrac{TM_{a_{ij}} - TM_{a_j^-}}{1 - TM_{a_j^-}}, if\ TM_{a_j^-} \leq TM_{a_{ij}}\ and\ TM_{a_j^-} \neq 1 \\ 0, otherwise \end{cases},$$

$$IM_{c_{ij}^-} = \begin{cases} \dfrac{IM_{a_{ij}}}{IM_{a_j^-}}, if\ IM_{a_j^-} \geq IM_{a_{ij}}\ and\ IM_{a_j^-} \neq 0 \\ 1, otherwise \end{cases},$$

$$FM_{c_{ij}^-} = \begin{cases} \dfrac{FM_{a_{ij}}}{FM_{a_j^-}}, if\ FM_{a_j^-} \geq FM_{a_{ij}}\ and\ FM_{a_j^-} \neq 0 \\ 1, otherwise \end{cases},$$

and the components in the IV-NNs c_{ij}^+ and c_{ij}^- contain the following forms:

$$TM_{c_{ij}^+} = \begin{cases} \left[\dfrac{\inf TM_{a_j^+} - \inf TM_{a_{ij}}}{1 - \inf TM_{a_{ij}}}, \dfrac{\sup TM_{a_j^+} - \sup TM_{a_{ij}}}{1 - \sup TM_{a_{ij}}}\right], if\ TM_{a_j^+} \supseteq TM_{a_{ij}}\ and\ TM_{a_{ij}} \neq [1,1] \\ [0,0], otherwise \end{cases},$$

$$IM_{c_{ij}^+} = \begin{cases} \left[\dfrac{\inf IM_{a_j^+}}{\inf IM_{a_{ij}}}, \dfrac{\sup IM_{a_j^+}}{\sup IM_{a_{ij}}}\right], if\ IM_{a_j^+} \subseteq IM_{a_{ij}}\ and\ IM_{a_{ij}} \neq [0,0] \\ [1,1], otherwise \end{cases},$$

$$FM_{c_{ij}^{+}}=\begin{cases}\left[\dfrac{\inf FM_{a_j^{+}}}{\inf FM_{a_{ij}}},\dfrac{\sup FM_{a_j^{+}}}{\sup FM_{a_{ij}}}\right], if\ FM_{a_j^{+}}\subseteq FM_{a_{ij}}\ and\ FM_{a_{ij}}\neq[0,0]\\ [1,1], otherwise\end{cases},$$

$$TM_{c_{ij}^{-}}=\begin{cases}\left[\dfrac{\inf TM_{a_{ij}}-\inf TM_{a_j^{-}}}{1-\inf TM_{a_j^{-}}},\dfrac{\sup TM_{a_{ij}}-\sup TM_{a_j^{-}}}{1-\sup TM_{a_j^{-}}}\right], if\ TM_{a_j^{-}}\subseteq TM_{a_{ij}}\ and\ TM_{a_j^{-}}\neq[1,1]\\ [0,0], otherwise\end{cases},$$

$$IM_{c_{ij}^{-}}=\begin{cases}\left[\dfrac{\inf IM_{a_{ij}}}{\inf IM_{a_j^{-}}},\dfrac{\sup IM_{a_{ij}}}{\sup IM_{a_j^{-}}}\right], if\ IM_{a_j^{-}}\supseteq IM_{a_{ij}}\ and\ IM_{a_j^{-}}\neq[0,0]\\ [1,1], otherwise\end{cases},$$

$$FM_{c_{ij}^{-}}=\begin{cases}\left[\dfrac{\inf FM_{a_{ij}}}{\inf FM_{a_j^{-}}},\dfrac{\sup FM_{a_{ij}}}{\sup FM_{a_j^{-}}}\right], if\ FM_{a_j^{-}}\supseteq FM_{a_{ij}}\ and\ FM_{a_j^{-}}\neq[0,0]\\ [1,1], otherwise\end{cases}.$$

Step 3. By Eq. (1) and (2), we calculate the score values of $S(d_i^+)$ and $S(d_i^-)$ (i = 1, 2, …, m).

Step 4. The relative closeness degree of each alternative with respect to the S-NN ideal solution (i = 1, 2, …, m) is calculated by

$$C_i=\frac{S(d_i^-)}{S(d_i^-)+S(d_i^+)}\quad \text{for } C_i \in [0, 1]. \tag{23}$$

Clearly, the larger value of C_i reveals that an alternative is closer to the ideal solution and farther from the negative ideal solution simultaneously. Therefore, all the alternatives can be ranked in the descending order according to the values of C_i (i = 1, 2, …, m). The alternative with the largest value is chosen as the best one.

Step 5. End.

5. Illustrative example

For convenient comparison, we consider the multi-attribute DM problem adapted from [12]. Some investment company needs to invest a sum of money into the best company. Then, the panel indicates four possible alternatives as their set $P = \{P_1, P_2, P_3, P_4\}$, where P_1, P_2, P_3, and P_4 are denoted as a car company, a food company, a computer company, and an arms company, respectively. To select the best company, they must be assessed by the three attributes: r_1 (risk), r_2 (growth) and r_3 (environmental impact), while their weight vector is specified by w = (0.35, 0.25, 0.4). The four alternatives are assessed over the three attributes by the suitable assessments, then their assessment values are represented by the form of S-NNs (SV-NNs and IV-NNs) and constructed as the SV-NN decision matrix:

$$D=(a_{ij})_{m\times n}=\begin{bmatrix}\langle 0.4,0.2,0.3\rangle & \langle 0.4,0.2,0.3\rangle & \langle 0.2,0.2,0.5\rangle\\ \langle 0.6,0.1,0.2\rangle & \langle 0.6,0.1,0.2\rangle & \langle 0.5,0.2,0.2\rangle\\ \langle 0.3,0.2,0.3\rangle & \langle 0.5,0.2,0.3\rangle & \langle 0.5,0.3,0.2\rangle\\ \langle 0.7,0.1,0.1\rangle & \langle 0.6,0.1,0.2\rangle & \langle 0.4,0.3,0.2\rangle\end{bmatrix},$$

and the IV-NN decision matrix:

$$D=(a_{ij})_{m\times n}$$
$$=\begin{bmatrix} \langle[0.4,0.5],[0.2,0.3],[0.3,0.4]\rangle & \langle[0.4,0.5],[0.2,0.3],[0.3,0.4]\rangle & \langle[0.2,0.3],[0.2,0.3],[0.5,0.6]\rangle \\ \langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle & \langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle & \langle[0.5,0.6],[0.2,0.3],[0.2,0.3]\rangle \\ \langle[0.3,0.4],[0.2,0.3],[0.3,0.4]\rangle & \langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle & \langle[0.5,0.6],[0.3,0.4],[0.2,0.3]\rangle \\ \langle[0.7,0.8],[0.1,0.2],[0.1,0.2]\rangle & \langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle & \langle[0.4,0.5],[0.3,0.4],[0.2,0.3]\rangle \end{bmatrix}.$$

On the one hand, the proposed DM approach can be applied in the DM problem with SV-NNs and depicted by the following decision steps:

Step 1. By $a_j^+=\langle TM_j^+,IM_j^+,FM_j^+\rangle=\left\langle \max_i(TM_{ij}),\min_i(IM_{ij}),\min_i(FM_{ij})\right\rangle$ (i = 1, 2, 3, 4; j = 1, 2, 3) for the SV-NN decision matrix $D=(a_{ij})_{m\times n}$, we can determine the SV-NN positive ideal solution (ideal alternative):

$$P^+=\{a_1^+,a_2^+,a_3^+\}=\{<0.7,0.1,0.1>,<0.6,0.1,0.2>,<0.5,0.2,0.2>\},$$

then by $a_j^-=\langle TM_j^-,IM_j^-,FM_j^-\rangle=\left\langle \min_i(TM_{ij}),\max_i(IM_{ij}),\max_i(FM_{ij})\right\rangle$ (j = 1, 2, …, n), we can determine the SV-NN negative ideal solution (non-ideal alternative):

$$P^-=\{a_1^-,a_2^-,a_3^-\}=\{<0.3,0.2,0.3>,<0.4,0.2,0.3>,<0.2,0.3,0.5>\}.$$

Step 2. By using Eqs. (19) and (21), we can obtain the two aggregated values d_i^+ and d_i^- (i = 1, 2, …, m) for each alternative P_i (i = 1, 2, …, m):

d_1^+ = <0.4126, 0.6598, 0.4264>, d_2^+ = <0.0958, 1.0000, 0.7846>, d_3^+ = <0.2970, 0.5610, 0.6152>, and d_4^+ = <0.0703, 0.8503, 1.0000>;

d_1^- = <0.0525, 0.8503, 1.0000>, d_2^- = <0.3844, 0.5610, 0.5435>, d_3^- = <0.2083, 1.0000, 0.6931>, and d_4^- = <0.4013, 0.6598, 0.4264>.

Step 3. By applying Eq. (1), we calculate the score values of $S(d_i^+)$ and $S(d_i^-)$ (i = 1, 2, 3, 4):

$S(d_1^+)$ = 0.4421, $S(d_2^+)$ = 0.1037, $S(d_3^+)$ = 0.3736, and $S(d_4^+)$ = 0.0733;

$S(d_1^-)$ = 0.0674, $S(d_2^-)$ = 0.4267, $S(d_3^-)$ = 0.1717, and $S(d_4^-)$ = 0.4384.

Step 4. By using Eq. (23), we calculate the relative closeness degrees of each alternative with respect to the SV-NN ideal solution:

C_1 = 0.1323, C_2 = 0.8044, C_3 = 0.3149, and C_4 = 0.8567.

Since the ranking order of the relative closeness degrees is $C_4 > C_2 > C_3 > C_1$, the ranking order of the four alternatives is $P_4 \succ P_2 \succ P_3 \succ P_1$. Hence, the best alternative is P_4.

On the other hand, the proposed DM approach can be also applied in the DM problem with IV-NNs and depicted by the following decision steps:

Step 1. By $a_j^+=\langle TM_j^+,IM_j^+,FM_j^+\rangle=\left\langle \begin{matrix}\left[\max_i(\inf TM_{ij}),\max_i(\sup TM_{ij})\right],\\ \left[\min_i(\inf IM_{ij}),\min_i(\sup IM_{ij})\right],\\ \left[\min_i(\inf FM_{ij}),\min_i(\sup FM_{ij})\right]\end{matrix}\right\rangle$ (i = 1, 2, 3, 4; j = 1, 2, 3) for the IV-NN decision matrix $D=(a_{ij})_{m\times n}$, we can determine the IV-NN positive ideal solution (ideal alternative):

$$P^+=\{a_1^+,a_2^+,a_3^+\}=\left\langle \begin{matrix}<[0.7,0.8],[0.1,0.2],[0.1,0.2]>,\\ <[0.6,0.7],[0.1,0.2],[0.2,0.3]>,\\ <[0.5,0.6],[0.2,0.3],[0.2,0.3]>\end{matrix}\right\rangle.$$

By $a_j^- = \langle TM_j^-, IM_j^-, FM_j^- \rangle = \left\langle \begin{matrix} \left[\min_i(\inf TM_{ij}), \min_i(\sup TM_{ij})\right], \\ \left[\max_i(\inf IM_{ij}), \max_i(\sup IM_{ij})\right], \\ \left[\max_i(\inf FM_{ij}), \max_i(\sup FM_{ij})\right] \end{matrix} \right\rangle$ (j = 1, 2, …, n), we can determine the IV-NN negative ideal solution (non-ideal alternative):

$$P^- = \{a_1^-, a_2^-, a_3^-\} = \left\langle \begin{matrix} <[0.3, 0.4], [0.2, 0.3], [0.3, 0.4]>, \\ <[0.4, 0.5], [0.2, 0.3], [0.3, 0.4]>, \\ <[0.2, 0.3], [0.3, 0.4], [0.5, 0.6]> \end{matrix} \right\rangle.$$

Step 2. By using Eqs. (20) and (22), we can obtain the two aggregated values d_i^+ and d_i^- (i = 1, 2, …, m) for each alternative P_i (i = 1, 2, …, m):

d_1^+ = <[0.4126, 0.4894], [0.6598, 0.7841], [0.4264, 0.5533]>, d_2^+ = <[0.0958, 0.1323], [1.0000, 1.0000], [0.7846, 0.8677]>, d_3^+ = <[0.2970, 0.3665], [0.5610, 0.6988], [0.6152, 0.7301]>, and d_4^+ = <[0.0703, 0.0854], [0.8503, 0.8913], [1.0000, 1.0000]>;

d_1^- = <[0.0525, 0.0618], [0.8503, 0.8913], [1.0000, 1.0000]>, d_2^- = <[0.3844, 0.4480], [0.5610, 0.6988], [0.5435, 0.6377]>, d_3^- = <[0.2083, 0.2439], [1.0000, 1.0000], [0.6931, 0.7579]>, and d_4^- = <[0.4013, 0.4763], [0.6598, 0.7841], [0.4264, 0.5533]>.

Step 3. By applying Eq. (2), we calculate the score values of $S(d_i^+)$ and $S(d_i^-)$ (i = 1, 2, 3, 4):

$S(d_1^+)$ = 0.4131, $S(d_2^+)$ = 0.0960, $S(d_3^+)$ = 0.3431, and $S(d_4^+)$ = 0.0690;

$S(d_1^-)$ = 0.0621, $S(d_2^-)$ = 0.3986, $S(d_3^-)$ = 0.1669, and $S(d_4^-)$ = 0.4090.

Step 4. By using Eq. (23), we calculate the relative closeness degrees of each alternative with respect to the SV-NN ideal solution:

C_1 = 0.1307, C_2 = 0.8059, C_3 = 0.3272, and C_4 = 0.8556.

Since the ranking order of the relative closeness degrees is $C_4 > C_2 > C_3 > C_1$, the ranking order of the four alternatives is $P_4 \succ P_2 \succ P_3 \succ P_1$. Hence, the best alternative is P_4.

By the comparison of the above decision results with the decision results obtained in [12], both the ranking order of the four alternatives and the best one above are the same as in [12].

Table 1. Decision results of multi-attribute DM approaches regarding various weighted aggregation operators of SV-NNs and IV-NNs

Aggregation operator	Score value	Relative closeness degree	Ranking order
SV-NNWSOAA	$S(d_1^+)$ = 0.4421, $S(d_2^+)$ = 0.1037, $S(d_3^+)$ = 0.3736, $S(d_4^+)$ = 0.0733; $S(d_1^-)$ = 0.0674, $S(d_2^-)$ = 0.4267, $S(d_3^-)$ = 0.1717, $S(d_4^-)$ = 0.4384	C_1 = 0.1323, C_2 = 0.8044, C_3 = 0.3149, C_4 = 0.8567	$P_4 \succ P_2 \succ P_3 \succ P_1$
IV-NNWSOAA	$S(d_1^+)$ = 0.4131, $S(d_2^+)$ = 0.0960, $S(d_3^+)$ = 0.3431, $S(d_4^+)$ = 0.0690; $S(d_1^-)$ = 0.0621, $S(d_2^-)$ = 0.3986, $S(d_3^-)$ = 0.1669, $S(d_4^-)$ = 0.4090	C_1 = 0.1307, C_2 = 0.8059, C_3 = 0.3272, C_4 = 0.8556	$P_4 \succ P_2 \succ P_3 \succ P_1$
SV-NNWAA [18, 20]	$S(d_1)$ = 0.5611, $S(d_2)$ = 0.6891, $S(d_3)$ = 0.6194, $S(d_4)$ = 0.6901	/	$P_4 \succ P_2 \succ P_3 \succ P_1$
IV-NNWAA [18, 20]	$S(d_1)$ = 0.5407, $S(d_2)$ = 0.6696, $S(d_3)$ = 0.5993, $S(d_4)$ = 0.6712	/	$P_4 \succ P_2 \succ P_3 \succ P_1$

To demonstrate the effectiveness and rationality of the proposed DM approach in this paper, we compare it with existing DM approaches based on the SV-NNWAA and IV-NNWAA operators [18, 20]. By directly using the SV-NNWAA operator of Eq. (3) and IV-NNWAA operator of Eq. (4) and the score function of Eqs. (1) and (2), we can obtain all the aggregated values of $d_i = SV - NNWAA(a_{i1}, a_{i2}, a_{i3}, a_{i4})$ and $d_i = IV - NNWAA(a_{i1}, a_{i2}, a_{i3}, a_{i4})$, and then the score values of $S(d_i)$ and decision results for each alternative P_i (i = 1, 2, …, m) are tabulated in Table 1.

In Table 1, all the ranking orders of the four alternatives given by the multi-attribute DM approaches based on the SV-NNWSOAA, IV-NNWSOAA, SV-NNWAA, and IV-NNWAA operators are identical, and then the best choices indicate the same alternative P_4, which show the effectiveness of the proposed approach. Clearly, the DM results obtained by the SV-NNWSOAA and IV-NNWSOAA operators reveals stronger identification than the DM results obtained by existing SV-NNWAA and IV-NNWAA operators [18, 19] because the values of the relative closeness degrees show bigger difference than the score values in existing approaches [18, 19]. Therefore, the DM method proposed in this paper is reasonable and provides an effective DM way for decision-makers.

6. Conclusion

Regarding existing subtraction operations of S-NNs (SV-NNs and IV-NNs), this paper firstly presented the SV-NNSOWAA and IV-NNSOWAA operators for S-NNs as a necessary complement to existing aggregation operators of S-NNs. Next, we developed a multi-attribute DM approach based on the SV-NNWSOAA and IV-NNSOWAA operators for the first time. Finally, an illustrative example was presented to demonstrate the applicability and effectiveness of the developed approach. However, the main advantage of the proposed DM approach is that the DM results in this study reveals stronger identification than the DM results of existing DM approaches. In the future work, the developed approach will be further extended to other fields, such as image processing and clustering analysis.

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Neutrosophic Quadruple Algebraic Codes over Z_2 and their Properties

Vasantha Kandasamy[1], Ilanthenral Kandasamy [1,*] and Florentin Smarandache[2]

[1] School of Computer Science and Engineering, VIT, Vellore 632014, India; E-mail: vasantha.wb@vit.ac.in

[2] Department of Mathematics, University of New Mexico, 705 Gurley Avenue, Gallup, NM 87301, USA; E-mail: smarand@unm.edu

*Correspondence: ilanthenral.k@vit.ac.in;

Abstract. In this paper we for the first time develop, define and describe a new class of algebraic codes using Neutrosophic Quadruples which uses the notion of known value, and three unknown triplets (T, I, F) where T is the truth value, I is the indeterminate and F is the false value. Using this Neutrosophic Quadruples several researchers have built groups, NQ-semigroups, NQ-vector spaces and NQ-linear algebras. However, so far NQ algebraic codes have not been developed or defined. These NQ-codes have some peculiar properties like the number of message symbols are always fixed as 4-tuples, that is why we call them as Neutrosophic Quadruple codes. Here only the check symbols can vary according to the wishes of the researchers. Further we find conditions for two NQ-Algebraic codewords to be orthogonal. In this paper we study these NQ codes only over the field Z_2. However, it can be carried out as a matter of routine in case of any field Z_p of characteristics p.

Keywords: Neutrosophic Quadruples; NQ-vector spaces; NQ-groups; Neutrosophic Quadruple Algebraic codes (NQ-algebraic codes); Dual NQ-algebraic codes; orthogonal NQ- algebraic codes; NQ generator matrix; parity check matrix; self dual NQ algebraic codes

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1. Introduction

Neutrosophic Quadruples (NQ) was introduced by Smarandache [1] in 2015, it assigns a value to known part in addition to the truth, indeterminate and false values, it happens to be very interesting and innovative. NQ numbers was first introduced by [1] and algebraic operations like addition, subtraction and multiplication were defined. Neutrosophic Quadruple algebraic structures where studied in [2]. Smarandache and et al introduced Neutrosophic triplet groups, modal logic Hedge algebras in [3, 4]. Zhang and et al in [5–7] defined and described Neutrosophic duplet semigroup and triplet loops and strong AG(1, 1) loops. In [8–12], various structures like Neutrosophic triplet and neutrosophic rings application to mathematical modelling, classical group of neutrosophic triplets on $\{Z_{2p}, \times\}$ and neutrosophic duplets in neutrosophic rings were developed and analyzed.

Algebraic structures of neutrosophic duplets and triplets like quasi neutrosophic triplet loops, AG-groupoids, extended triplet groups and NT-subgroups were studied in [7,13,16,17]. Various types of refined neutrosophic sets were introduced, developed and applied to real world problems by [18–24]. In 2015, [18] has obtained several algebraic structures on refined Neutrosophic sets. Neutrosophy has found immense applications in [25–28]. Neutrosophic algebraic structures in general were studied in [29–32]. The algebraic structure of Neutrosophic Quadruples, such as groups, monoids, ideals, BCI-algebras, BCI-positive implicative ideals, hyper structures and BCK/BCI algebras have been developed recently and studied in [34–39]. In 2016 [33] have developed some algebraic structures using Neutrosophic Quadruples $(NQ, +)$ groups and $(NQ, .)$ monoids and scalar multiplication on Neutrosophic Quadruples. [41] have recently developed the notion of NQ vector spaces over R(reals) (or Complex numbers C or Z_p the field of characteristic p, p a prime). They have also defined NQ dual vector subspaces and proved all these NQ-vectors though are distinctly different, yet they are of dimension 4.

The main aim of this paper is to introduce Neutrosophic Quadruple (NQ) algebraic codes over Z_2. (However it can be extended for any Z_p, p a prime). Any NQ codeword is an ordered quadruple with four message symbols which can be a real or complex value, truth value, indeterminate or complex value and the check symbols are combinations of these four elements. We have built a new class of NQ algebraic codes which can measure the four aspects of any code word.

The proposed work is important for Neutrosophic codes have been studied Neutrosophic codes have been studied by [42] but it has the limitations for it could involve only the indeterminacy present and not all the four factors which are present in Neutrosophic Quadruple codes. Hence when the codes are endowed with all the four features it would give in general a better result of detecting the problems while transmission takes place.

It is to be recalled any classical code gives us only the approximately received code word. However the degrees of truth or false or indeterminacy present in the correctness of the received code word is never studied. So our approach would not only be novel and innovative but give a better result when used in real channels.

The main objective of this study is to assess the quality of the received codeword for the received code word may be partially indeterminate or partially false or all the four, we can by this method assess the presence of these factors and accordingly go for re-transmission or rejection.

Hexi codes were defined in [43,44] which uses 16 symbols, 0 to 9 and A to F. Likewise these NQ codes uses the symbols 0, 1, T, I and F.

This paper is organized into six sections. Section one is introductory in nature. Basic concepts needed to make this paper a self-contained one is given in section two. Neutrosophic Quadruple algebraic codes (NQ-codes) are introduced and some interesting properties about them are given in section three. Section four defines the new notion of special orthogonal NQ codes using the inner product of two NQ codewords. The uses of NQ codes and comparison with classical linear algebraic codes are carried out in section five. The final section gives the conclusions based on our study.

2. Basic Concepts

In this section we first give the basic properties about the NQ algebraic structures needed for this study. Secondly we give some fundamental properties associated with algebraic codes in general. For NQ algebraic structures refer [29,33].

Definition 2.1. A Neutrosophic quadruple number is of the form (x, yT, zI, wF) where T, I, F are the usual truth value, indeterminate value and the false value respectively and $x, y, z, w \in Z_p$ (or R or C). The set NQ is defined by $NQ = \{(x, yT, zI, wF)|x, y, z, w \in R$ (or Z_p or C); p a prime$\}$ is defined as the Neutrosophic set of quadruple numbers.

A Neutrosophic quadruple number (x, yT, zI, wF) represents any entity or concept which may be a number an idea etc., x is called the known part and (yT, zI, wF) is called the unknown part. Addition, subtraction and scalar multiplication are defined in [33] in the following way. Let $x = (x_1, x_2T, x_3I, x_4F)$ and $y = (y_1, y_2T, y_3I, y_4F) \in NQ$.

$x + y = (x_1 + y_1, (x_2 + y_2)T, (x_3 + y_3)I, (x_4 + y_4)F)$

$x - y = (x_1 - y_1, (x_2 - y_2)T, (x_3 - y_3)I, (x_4 - y_4)F)$

For any $a \in R$ (or C or Z_p) and $x = (x_1, x_2T, x_3I, x_4F)$ where $a \in R$ (or C or Z_P) will be known as scalars and $x \in NQ$ the scalar product of a with x in defined by

$$a.x = a(x_1, x_2T, x_3I, x_4F)$$
$$= (ax_1, ax_2T, ax_3I, ax_4F).$$

If $a = 0$ then $a.x = (0, 0, 0, 0)$. $(0, 0, 0, 0)$ is the additive identity in $(NQ, +)$. For every $x \in NQ$ there exists a unique element $-x = (-x_1, -x_2T, -x_3I, -x_4F)$, in NQ such that $x + (-x) = (0, 0, 0, 0)$. x is called the additive inverse of $-x$ and vice versa.

Finally for $a, b \in C$ (or R or Z_p) and $x, y, \in NQ$ we have $(a + b).x = a.x + b.x$ and $(a \times b).x = a \times (b.x); a(x + y) = a.x + a.y$.

These properties are essential for us to build NQ-algebraic codes.

We use the following results; proofs of which can be had form [33].

Theorem 2.2. *$(NQ, +)$ is an abelian group.*

[33] defines product of any pair of elements $x, y \in NQ$ as follows. Let $x = (x_1, x_2T, x_3I, x_4F)$ and $y = (y_1, y_2T, y_3I, y_4F) \in NQ$.

$$x.y = (x_1, x_2T, x_3I, x_4F).(y_1, y_2T, y_3I, y_4F)$$

$$(x_1y_1, (x_1y_2 + x_2y_1 + x_2y_2)T,$$

$$(x_1y_3 + x_2y_3 + x_3y_1 + x_3y_2 + x_3y_3)I,$$

$$(x_1y_4 + x_2y_4 + x_3y_4 + x_4y_4 + x_4y_1 + x_4y_2 + x_4y_3)F).$$

Theorem 2.3. *$(NQ, .)$ is a commutative monoid.*

Now we just recall some of the properties associated with basic algebraic codes.

Through out this paper Z_2 will denote the finite field of characteristic two. V a finite dimensional vector space over $F = Z_2$ [40].

We call a n-tuple to be $C = C(n, k)$ codeword if C has k message symbols and $n - k$ check symbols. For $c = (c_1, c_2, \dots, c_k, c_{k+1}, \dots c_n)$ where $(c_1, c_2, \dots, c_k) \in V$ (dimension of V over Z_2) and $c_{k+1}, \dots, c_n$ are check symbols calculated using the $(c_1, c_2, \dots, c_k) \in V$. To basically generate the code words we use the concept of generator matrix denoted by G and G is a $k \times n$ matrix with entries from Z_2 and to evaluate the correctness of the received codeword we use the parity check matrix H, which is a $n - k \times n$ matrix with entries from Z_2. We in this paper use only the standard form of the generator matrix and parity check matrix for any $C(n, k)$ code of length n with k message symbols. The standard form of the generator matrix G for an $C(n, k)$ code is as follows:

$$G = (I_k, -A^T)$$

where I_k is a $k \times k$ identity matrix and $-A^T$ is a $k \times n - k$ matrix with entries from Z_2. Here the standard form of the parity check matrix $H = (A, I_{n-k})$ where A is a $n - k \times k$ matrix with entries fromZ_2 and I_{n-k} is the $n - k \times n - k$ identity matrix. We have $GH^T = (0)$. In this paper, we use both the generator matrix and the parity check matrix of a NQ code to be only in the standard form.

3. Definition of NQ algebraic codes and their properties

In this section we proceed on to define the new class of algebraic codes called Neutrosophic Quadruple algebraic codes (NQ-algebraic codes) using the NQ vector spaces over the finite field Z_2. We have defined NQ vector spaces over Z_2 in [41].

$$NQ = \{(a, bT, cI, dF) | a, b, c, d \in Z_2\}$$

under + is an abliean group.

Now we proceed on to define $\times$ on NQ. Let

$$x = x_1 + x_2T + x_3I + x_4F$$

and

$$y = y_1 + y_2T + y_3I + y_4F$$

where $x_i, y_i \in R$ or C or Z_p (p a prime) and T, I and F satisfy the following table for product $\times$.

×	T	I	F	0
T	T	0	0	0
I	0	I	0	0
F	0	0	F	0
0	0	0	0	0

So the set $\{T, I, F, 0\}$ under product is an idempotent semigroup. now we find

$$x \times y = (x_1 + x_2T + x_3I + x_4F) \times (y_1 + y_2T + y_3I + y_4F)$$

$$= x_1y_1 + (y_1x_2 + x_1y_2 + x_2y_2)T + (x_3y_1 + y_3x_1 + x_3y_3)I + (x_1y_4 + y_1x_4 + x_4y_4)F \in NQ$$

$\{NQ, \times\}$ is a semigroup which is commutative.

In this section we introduce the new notion of algebraic codes using the set NQ which is a group under '+'

$$NQ = \{(0\ 0\ 0\ 0),\ (1\ 0\ 0\ 0),\ (0\ T\ 0\ 0),\ (0\ 0\ I\ 0),\ (0\ 0\ 0\ F),\ (1\ T\ 0\ 0),\ (1\ 0\ I\ 0),\ (1\ 0\ 0\ F),$$
$$(0\ T\ I\ 0),\ (0\ T\ 0\ F),\ (0\ 0\ I\ F),\ (1\ T\ I\ 0),\ (1\ T\ 0\ F),\ (1\ 0\ I\ F),\ (0\ T\ I\ F),\ (1\ T\ I\ F)\ \};$$

$\{NQ, +\}$ is a NQ vector space over $Z_2 = \{0, 1\}$. NQ coding comprises of transforming a block of message symbols in NQ into a NQ code word $a_1a_2a_3a_4x_5x_6 \ldots x_n$, where $a_1a_2a_3a_4 \in NQ$ that is $a_1a_2a_3a_4 = (a_1a_2a_3a_4) \in NQ$ is a quadruple and $x_5, x_6, \ldots, x_n$ belongs to the set $T = \{a + bT + cI + dF/a, b, c, d$ takes its values from $Z_2 = \{0, 1\}\}$. The first four terms $a_1a_2a_3a_4$ symbols are always the message symbols taken from NQ and the remaining $n-4$ are the check symbols or the control symbols which are from T.

In this paper NQ codewords will be written as $a_1a_2a_3a_4x_5x_6x_7 \ldots x_n$, where $(a_1a_2a_3a_4) \in NQ$ and $x_i \in T, 4 < i \leq n$. The check symbols can be obtained from the NQ message symbols in such a way that the NQ code words $a = (a_1a_2a_3a_4)$ satisfy the system of linear equations $Ha^T = (0)$, where H is the $n-4 \times n$ parity check matrix in the standard form with elements from Z_2. Throughout this paper we assume $H = (A, I_{n-4})$, with A, a $n-4\times 4$ matrix and I_{n-4} the $n-4\times n-4$ identity matrix with entries from Z_2.

The matrix $G = (I_{4\times 4}, -A^T)$ is called the canonical generator matrix of the linear $(n, 4)$ NQ code with parity check matrix $H = (A, I_{n-4})$.

We use only standard form of the generator matrix and parity check matrix to generate the NQ-codewords for general matrix of appropriate order will not serve the purpose which is a limitation in this case.

We provide some examples of a HQ linear algebraic code.

Example 3.1. Let $C(7, 4)$ be a NQ code of length 7. G be the NQ generator matrix of the $(7, 4)NQ$ code.

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

G takes the entries from Z_2, over which the NQ vector space is defined and the message symbols are from NQ. Consider the set of NQ message symbols, $P = \{(0\ 0\ 0\ 0), (0\ T\ 0\ 0),$ (0 0 I 0), (0 0 0 F), (0 0 I F), (0 T I F), (1 0 I F), (0 T I 0), (0 T 0 F)$\} \subseteq NQ$. We now give the NQ code words of

$C(7, 4) = \{$ (0 0 0 0 0 0 0), (0 T 0 0 0 T 0), (0 0 0 F 0 0 F), (0 0 I 0 I 0 0), (0 0 I F I 0 F), (0 T I F I T F), (1 0 I F 1 + I I F) (0 T I 0 I T 0), (0 T 0 F 0 T F) $\}$ which are associated with $P \subseteq$ NQ. The NQ parity check matrix associated with this generator matrix G is as follows;

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

It is easily verified $Hx^t = (0)$; for all NQ code words $x \in C(7, 4)$. Suppose one receives a NQ code word $y = (0\ I\ 0\ T\ I\ 0\ 0)$; how to find out if the received NQ code word y is a correct one or not. For this we find out Hy^t, if $Hy^t = (0)$, then y is a correct code word; if $Hy^t \neq (0)$, then some error has occurred during transmission. Clearly $Hy^t \neq (0)$. Thus y is not a correct NQ code word.

How to correct it? These NQ code behave differently as these codewords, which is a $1 \times n$ row matrix does not take the values from Z_2, but from NQ and T; message symbols from NQ and check symbols from T. Hence, we cannot use the classical method of coset leader method for error correction, however we use the parity check matrix for error detection.

We have to adopt a special method to find the corrected version of the received NQ code word which has error.

Here we describe the procedure for error correction which is carried out in three steps; Suppose y is the received NQ code word;

(1) We first find Hy^t, if Hy^t is zero no error; on the other hand if Hy^t is not zero there is error so we go to step two for correction.

(2) Now consider the NQ received code word with error. We observe and correct only the first four component in the y that is we correct the message symbols; if the first component is 1 or 0 then it is accepted as the correct component in y; if on the other hand the first component is T (or I or F) and if 1 has occurred in the rest of any three components then replace T (or I or F) by one if 1 has not occurred in the 2nd or 3rd or 4th component replace the first component by 0.

Now observe the second component if it is T accept, if not T but 0 or 1 or I or F, then replace by zero if T has not occurred in the first or third or fourth place. If T has occurred in any of the 3 other components replace it by T. Next observe the third component if it is I accept else replace by I if I has occurred as first or second or fourth component. If in none of the first four places I has occurred, then fill the third place by zero. Now observe the fourth component if it is F accept it, if not replace by 0 if in none of the other places F has occurred or by F if F has occurred in first or second or third place, now the message word is in NQ by this procedure. If the corrected NQ code word z of y is such that $Hz^t = (0)$ then accept it if not we go for the next step. We check only for the correctness of the message symbols.

(3) For check symbols we use the table of codewords or check matrix H and find the check symbols.

Table of NQ codewords related to $P \subset NQ$ given in example 2.

Table 1. Table of NQ codewords related to P

Sno	Message symbols in P	NQ Codeword
1	(0 0 0 0)	(0 0 0 0 0 0 0)
2	(0 T 0 0)	(0 T 0 0 0 T 0)
3	(0 0 I 0)	(0 0 I 0 I 0 0)
4	(0 0 0 F)	(0 0 0 F 0 0 F)
5	(0 0 I F)	(0 0 I F I 0 F)
6	(0 T I F)	(0 T I F I T F)
7	(0 T I 0)	(0 T I 0 I T 0)
8	(0 T 0 F)	(0 T 0 F 0 T F)
9	(1 0 I F)	(1 0 I F 1+I 1 F)

We provide one example of the codeword given in Example 2.1. Let $y = (I\ 1\ F\ 0\ 1{+}I\ 1\ F)$, we see Hy^t is not zero, so we have found the error hence we proceed to next step. We see first component cannot be I so replace I by 1 for 1 has occurred as second component. As second component cannot be one we see in none of the four components T has occurred so we replace 1 by zero. In the second place. Third component is F which is incorrect so we replace it by I as I has occurred in the first place. We observe the fourth component it can be 0 or F; 0 only in case F has not occurred in the first three places but F has occurred as the third component so we replace the zero of the fourth component by F. So the corrected message symbol is (1 0 I F). In step three we check from the table of codes the check symbols and the check symbols matches with the check symbols of the corrected message symbols so we take this as the corrected version of corrected code word as (1 0 I F I+I 1 F).

We give the definition of the procedure.

Definition 3.2. Let C(n, 4) be a NQ code of length n defined over Z_2. The message symbols are always from the set NQ; whatever be n there are only 16 codewords only check symbols increase and not the message symbol length, for it is always four. If $y = (A_1\ A_2\ A_3\ A_4\ a_5\ a_6\ a_7 \ldots a_n)$ is a received NQ codeword and it has some error, then we define the rearrangement technique of error correction in the message symbols $A_1\ A_2\ A_3\ A_4$ only, where if $A_1\ A_2\ A_3\ A_4$ is to be in NQ then A_1 can only values 1 or 0, A_2 can take values 0 or T; A_3 can take values 0 or I and A_4 can take values 0 or F. If this is taken care of the message symbol will be correct and will be in NQ.

If not the following rearrangement process is carried out;

Observe if A_1 is different from 0 or 1 then see values in the 2nd, third and the fourth components if 1 has occurred in any one of them replace the first component by 1, if 1 has not occurred in any one of the four components fill the first component by zero. Now go for the second component A_2 if A_2 is T then it is correct ;if not and 1 or 0 or I or F has occurred and T has occurred in any one of the other three places replace the second component by T; if T has not occurred as any one of the four components replace the second component by 0. Inspect the third component if it is I then it is correct, if not I and if T or 0 or 1 or F has occurred and I has occurred in any of the four components replace the third component by I, if I has failed to occur in any of the four places replace the third component by zero. Now for the fourth component if it is F it is correct, if not and if F has occurred in any one of the other three components replace it by F, if not by zero. After this arrangement certainly the message symbols will be in NQ.

This method of getting the correct code word is defined as the rearrangement technique.

4. **Orthogonal NQ codes and special orthogonal NQ codes**

In this section we define the notion of orthogonality of two HQ code words and the special orthogonal HQ code words and suggest some open problems in this direction in the last section of this paper. Now we define first inner product on the NQ code words of the NQ algebraic code $C(n, 4)$ defined over Z_2.

Definition 4.1. Let $C(n, 4)$ be a NQ code of length n defined over Z_2. Let $x = (A_1\ A_2\ A_3\ A_4\ a_5\ a_6\ a_7\ \ldots\ a_n)$ and $y = (B_1\ B_2\ B_3\ B_4\ b_5\ b_6\ b_7 \ldots b_n)$ be any two NQ code words from $C(n, 4)$, where $A_i, B_i \in NQ$, $i = 1, 2, 3, 4$ and $a_j, b_j \in T$; $j = 5, 6, \ldots n$. We define the dot product of x and y as follows:

$$x.y = A_1 \times B_1 + A_2 \times B_2 + A_3 \times B_3 + A_4 \times B_4 + a_5 \times b_5 + \ldots + a_n \times b_n$$

.

If $x.y = 0$ then we say the two NQ codes words are orthogonal or dual with each other.

Example 4.2. Let $C(6, 4)$ be a NQ code of length 4 defined over Z_2; with associated generated matrix G in the standard form with entries from Z_2 given in the following:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The $C(6, 4)$ NQ code words generated by G is as follows; $C(6, 4) = \{$(0 0 0 0 0 0), (1 0 0 0 1 0), (0 T 0 0 0 T), (0 0 I 0 I I), (0 0 0 F F 0), (1 T 0 0 1 T), (1 0 I 0 1+I I), (1 0 0 F F + 1 0), (0 T I 0 I T + I), (0 T 0 F F T), (0 0 I F I+F I), (1 T I 0 1 +I T + I), (1 T 0 F 1+ F T), (1 0 I F 1 +I +F I),(0 T I F I + F I +T), (1 T I F 1 + I + F I + T)$\}$.

We see (0 0 0 0 0 0) is orthogonal with every other NQ code word in the NQ code (6, 4). Consider the NQ code word (1 0 0 0 1 0) in C (6, 4), NQ code words orthogonal to (1 0 0 0 1 0) are $\{$(1 0 0 0 1 0), (0 0 0 0 0 0), (0 T 0 0 0 T), (1 T 0 0 1 T) $\}$. The NQ codes orthogonal to (0 T 0 0 0 T) are given by

$\{$(0 0 0 0 0 0), (0 T 0 0 0 T), (1 0 0 0 1 0), (0 0 I 0 I I), (0 0 0 F F 0), (1 T 0 0 1 T), (1 0 0 F 1 + F 0), (0 T I 0 1 + I T + I), (1 0 I 0 1 +I I), (0 T 0 F F T), (0 0 I F I + F I), (1 I T 0 1 +I T + I), (1 T 0 F 1 + F T), (1 0 I F 1 + I + F I), (0 T I F I + F T + I), (1 T I F 1 + T +F T + I) $\} = C(6, 4)$.

Thus every element in $C(6, 4)$ is orthogonal with (0 T 0 0 0 T). However (1 0 0 0 0 1) is not orthogonal with every element in $C(6, 4)$. We call all those NQ codes words which are orthogonal to every code word in $C(6, 4)$ including it as the special orthogonal NQ code. A NQ code word which is orthogonal to itself is defined as the self orthogonal NQ code word.

We define them in the following;

Definition 4.3. Let C(n, 4) be a NQ code of length n. We say a NQ code word is self orthogonal if x . x = 0 for x in C(n, 4). A NQ code word x in C(n, 4) is defined as a special orthogonal NQ code word if x is self orthogonal and x is orthogonal with every NQ code word in C(n, 4). (0 0 0 ...0) is a trivial special NQ code word.

We give yet another example of a NQ code which has NQ special orthogonal code word.

Example 4.4. Let C(7, 4) be a NQ code word of length 7. Let G be the associated generator matrix of the NQ code C.

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

It is easily verified that only the NQ code word (0 0 I 0 0 0 I) in C is the special orthogonal NQ code word. We have yet another extreme case where every NQ code word in that NQ code is a special orthogonal NQ code word.

We give examples of them.

Example 4.5. Let C(8, 4) be a NQ code generated by the following generator matrix G

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

It is easily verified every NQ code word in $C(8, 4)$ is a special orthogonal NQ code word. We call such NQ codes as special self orthogonal NQ code or self orthogonal NQ code.

Definition 4.6. Let $C = C(n, 4)$ be a NQ code word defined over Z_2. We define C to be a NQ special self orthogonal code if every NQ code word in C is a special orthogonal NQ code word of C.

5. Uses of NQ codes and comparison of NQ codes with classical linear algebraic codes

NQ codes are best suited for data transmission where one does not require security. They are also very useful in data storage for one can easily retrieve the data even if the data is corrupted. The disadvantage of these NQ codes is that they always have a fixed number of message symbols namely four. They are not compatible in channels were one needs security. The only flexibility is one can have any number of check symbols. NQ codes are entirely different from the classical linear algebraic code ; for these code words take the message symbols from NQ and the check symbols from T where as the later take their values from Z_2 (or Z_p).

Classical linear algebraic codes takes its code words from Z_p, p a prime or more commonly from Z_2; and are defined over Z_p or Z_2; but in case of NQ codes the code words take their values from NQ for message symbols and from T for their check symbols which is a big difference as we can only use the standard form of the generator matrix and the parity check matrix, in this case also both the matrices take their values from Z_2 (or Z_p) only. The similarity is both the codes take the entries of the matrices from the finite field over which they are defined. All NQ codes are only of a fixed form that is they can have only 4 message symbols from NQ, but the classical codes can have any value from 1 to m , m ¡ n, which is a major difference between the two class of codes. Both NQ codes and the classical linear code use parity matrix to detect the error in the received code word, that is error detection procedure for both of them is the same. For error correction we have to adopt a special technique of rearrangement of the message symbols once an error is detected in the received NQ code word, as the coset leader method of error correction cannot be carried out as the NQ code words do not belong to the field over which the NQ code words are defined.

6. **Conclusions**

In this paper for the first time we have defined the new class of codes called NQ codes which are distinctly different from the classical algebraic linear codes. All these NQ codes can have only fixed number of message symbols viz four. NQ codes are of the form $C(n, 4)$, n can vary from 5 to any finite integer. We have defined orthogonality of these NQ codes. This has lead us to define NQ special orthogonal code word and NQ special orthogonal codes. We suggest the following problems:

(1) Prove or disprove all NQ codes have a non trivial code word which is orthogonal to all codes in C (n, 4).

(2) Characterize all NQ codes C (n, 4) which are NQ special orthogonal codes.

For future research we would be defining super NQ structures and NQ codes over Z_p, p an odd prime. Also application of these codes can be done in case of Hexi codes [43] in McEliece Public Key crypto-systems [44] and in coding applications like T-Direct codes [45] and multi covering radius with rank metric [46].

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Multi-Polar Neutrosophic Soft Sets with Application in Medical Diagnosis and Decision-Making

Muhammad Saeed[1], Muhammad Saqlain[1*], Asad Mehmood[2], Khushbakht Naseer[3] and Sonia Yaqoob[4]

[1]Department of Mathematics, University of Management and Technology, Lahore, Pakistan; muhammad.saeed@umt.edu.pk

[1*]Department of Mathematics, Lahore Garrison University, Lahore, Pakistan; msaqlain@lgu.edu.pk

[2]Department of Mathematics, University of Management and Technology, Lahore, Pakistan; a.asadkhan.khi@gmail.com

[3]Department of Mathematics, University of Management and Technology, Lahore, Pakistan; khushbakht433@gmail.com

[4]Department of Mathematics, University of Management and Technology, Lahore, Pakistan; soniayaqoob1122@gmail.com

Abstract. A Similarity measure for Neutrosophic function performs a fundamental role in tackling the problems that include blurred and hazed information but is not able to handle the fuzziness and vagueness of the problems which have numerous information. The objective of this research paper is to generalize neutrosophic soft set to the multi-polar neutrosophic soft set (mNS set), aggregation operators and their properties on mNS sets. It also discusses the distance-based similarity measures that rely on between two mNS sets. It explains with the help of examples that the intended similarity measures of mNS sets are applicable in the field of medical diagnosis and decision-making problem for selection of lecturer in universities. Eventually, this proposed method is concluded as an algorithm in the application.

Keywords: mNS Set; Operators on mNS; Properties; Distance and Similarity Measure; Medical Diagnosis; Decision-Making

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1. Introduction

Zadeh [1] introduced fuzzy sets as an additional classical conception of set. The theory of fuzzy set can be widely utilized in those domains where the information is deficient or incomplete like in bio-informatics fuzzy set logics, the members in a set are allowed to have a moderate assessment of membership, this is explained by the help of membership function admired in real unit interim [0,1]. Fuzzy sets have been derived from crisp sets, because the different aspects of functions of crisp sets are extraordinary occasions of degree functions of fuzzy sets if the later contains the values of fuzzy relations like 0 or 1 used in different areas like clustering (Bezdek [16]), linguistics (Cock [17]), and decision-making (Deli [18]) are significant theories of L-relations when the unit interval L is [0,1]. Further than intuitionistic fuzzy set was being proposed by Atanassov [2] which was an extension of Zadeh's conviction and it was in itself adjunct the classical conviction of a set. The Intuitionistic fuzzy set was only able to grasp insufficient particulars and facts, not unspecified particulars and facts. Neutrosophy was proposed by Florentin [19]. It is a limb of philosophy that scrutinizes the origination, essence, and range of neutralities along with interconnection with various ideational spectra. Neutrosophic set is a powerful general authorized substructure which deducts the theory of fuzzy sets and intuitionistic fuzzy set. The Soft set supposition is induction of fuzzy set supposition that was being presented by Molodtsov [3] to trade with unpredictability in a parametric model. He designated strains and troubles that were in mathematical representations and tackle the complications by proposing a soft set theory. Maji [4,23] expanded the soft set scheme into a fuzzy soft set and neutrosophic soft set theory. Feng [20] further explored decision-making that was supported by fuzzy soft sets.

Bipolar fuzzy sets and its connections were further introduced by Zhang [5], then Chen [6] proposed the notion of multi-polar fuzzy sets that was an abstraction of bipolar fuzzy sets. It was being manifested that bipolar fuzzy sets were isomorphic mathematical conceptions. Further than Wang and Liu [22] proposed decision-making on the multi-polar neutrosophic numbers. Deli [24] studied the neutrosophic soft multi-set theory with the multi distinct universe.

Chen [9,10] researched the notion of similarity measure of a vague set which was unsuccessful to retain in some scenarios. To conquer this problem Hong and Kim [21] proposed some altered calculations. Majumdar and Samanta [11,12,13] proposed uncertainty measures of soft sets and fuzzy sets. Kharal [14] proposed similarity measures of soft sets on set-theoretic functions. Li and Cheng [7] suggested the vision of new SMs allying in intuitionistic fuzzy sets. Deli [18,24] introduced decision-making methods on interval-valued neutrosophic soft set and neutrosophic soft multi-set theory. Huang [25] proposed TOPSIS for group decision-making problems. Furthermore, Anisseh [27] extend the TOPSIS model for group decision-making problem under multiple criteria.

Smarandache [28] generalized the soft set to hypersoft set by converting the function into a multi-decision function. Aggregate operators, similarity measure and a TOPSIS technique is introduced by [29-34] in his work. Application of fuzzy numbers in mobile selection in metros like Lahore is proposed by [35]. TOPSIS technique of MCDM can also be used for the prediction of games, and it's applied in FIFA 2018 by [36], prediction of games is a very complex topic and this game is also predicted by [37]. Abdel-Basset [38-41] has published a set of articles on Medical disease diagnosis based on neutrosophic environment.

1.1. *Motivation*

A huge number of articles is published in neutrosophic field, and as well as this theory is applied and extended in different branches such as Decision Making. The extension of neutrosophic environment to m-polar Nuetrosophic Soft set is totally new. Here, a set of questions arises that how mNS set can be represented? what is the purpose of m-polar structure? and how m-polar structures can be utilized in medical diagnosis and in decision making problems? From this point of view, mNS structure is good choice to get better results to the problems in decision making.

1.2. *Paper Presentation*

This article visualizes new concept m-polar Neutrosophic Soft Set as an extension of Neutrosophic Soft Set.

- Basic operators such as union, intersection on mNS set are introduced
- Properties related to operators on mNS structure
- Distance measure of different types are introduced between any two mNS sets
- A case study of two applications are concluded with an algorithm in the field of medical diagnosis and decision-making problem.

2. **Preliminaries**

This section studies some basic definitions related to this article.

2.1. *Neutrosophic Set*

Definition 2.1 [19] Let Z be a universal set. A neutrosophic set X is defined as :

$$X = \{z, (T_X(z), I_X(z), F_X(z)) : z \in Z\},$$

where,

$$T_X(z), I_X(x), F_X(z) \in [0, 1]$$
$$0 \leq T_X(z) + I_X(z) + F_X(z) \leq 3$$
$$\text{for all } z \in Z$$

2.2. *Multi-Polar Neutrosophic Set*

Definition 2.2 [22] An mN set on a universal set Z is a mapping

$$X = ((s_1 \circ T_X(z), s_2 \circ T_X(z), \cdots, s_m \circ T_X(z)), (s_1 \circ I_X(z), s_2 \circ I_X(z), \cdots, s_m \circ I_X(z)),$$
$$(s_1 \circ F_X(z), s_2 \circ F_X(z), \cdots, s_m \circ F_X(z))) : Z \to ([0, 1]^m, [0, 1]^m, [0, 1]^m)$$

where $i-th$ mapping is defined as

$$s_i \circ T_X : [0,1]^m \to [0,1]$$
$$s_i \circ I_X : [0,1]^m \to [0,1]$$
$$s_i \circ F_X : [0,1]^m \to [0,1]$$

and

$$0 \le s_i \circ T_X(z) + s_i \circ I_X + s_i \circ F_X \le 3$$
$$\text{for all } i = 1, 2, \cdots, m \text{ and } z \in Z$$

2.3. *Soft Set*

Definition 2.3 [3] Let Z be a universal set and E be the set of attributes of elements in Z. Take X to be a subset of E then a function F defined as

$$F : X \to P(Z),$$

then a pair (F, X) is called a soft set over Z such as

$$(F, X) = \{e, F(e) : e \in X,\ F(e) \in P(Z)\}$$

2.4. *Not Set*

Definition 2.4 [23] The Not Set of set of parameters $E = \{e_1, e_2, \cdots, e_q\}$, denoted by $\neg E$ is defined as

$$\neg E = \{\neg e_1, \neg e_2, \cdots, \neg e_q\}$$

where $\neg e_j$ means not e_j for all $j = 1, 2, \cdots, q$.

2.5. *Neutrosophic Soft Sets*

Definition 2.5 [23] A Neutrosophic Soft set (ω, X) over a universal set Z is a mapping from X to $P(Z)$ and defined as

$$(\omega, X) = \Omega_X = \{(e, (z, T_X(e)(z), I_X(e)(z), F_X(e)(z)) : z \in Z,\ e \in E)\}$$

where, $P(Z)$ denotes collection of all neutrosophic subsets of Z. Each of $T_X(e), I_X(e)$ and $F_X(e)$ is a mapping from Z to interval $[0, 1]$ and

$$0 \le T_X(e)(z) + I_X(e)(z) + F_X(e)(z) \le 3$$
$$\text{for all } e \in E \text{ and } z \in Z$$

2.6. *Multi-Polar Nuetrosophic Soft set*

Definition 2.6 [24] Let Z be a universal set, E be a set of attributes and $X \subseteq E$. Define $\omega : X \to m\mathrm{N}^Z$ where $m\mathrm{N}^Z$ is the collection of all mN subsets of set Z. Then (ω, X) is called an mNS set over Z as follows

$$\Omega_X = (\omega, X) = \{e, \omega_X(e) : e \in E, \omega_X(e) \in mN^Z\}$$

and $\omega_X(e)$ is an mN set denoted by,

$$\omega_X(e) = \{z, s_i \circ T_X(e)(z), s_i \circ I_X(e)(z), s_i \circ F_X(e)(z) : z \in Z\}$$

and

$$0 \leq s_i \circ T_X(e)(z), s_i \circ I_X(e)(z), s_i \circ F_X(e)(z) \leq 3$$
$$\text{for all } i = 1, 2, \cdots, m;\ e \in E \text{ and } z \in Z$$

3. Operations on mN Soft Sets

This section discusses some operators on mNS sets.

3.1. *Multi-Polar Neutrosophic Soft subset*

Definition 3.1 Let Z be a universal set, X and Y are subsets of a set of attributes E. A set Ω_X is an mNS subset of Ψ_Y denoted by $\Omega_X \breve{\subseteq} \Psi_Y$ if

(i) $X \subseteq Y$
(ii) $\omega_X(e) \subseteq \psi_Y(e)$ i.e.

$$s_i \circ T_X(e)(z) \leq s_i \circ T_Y(e)(z),\ s_i \circ I_X(e)(z) \leq s_i \circ I_Y(e)(z) \text{ and } s_i \circ F_X(e)(z) \geq s_i \circ F_Y(e)(z)$$
$$\text{for all } i = 1, 2, \cdots, m;\ e \in X \text{ and } z \in Z$$

Example 3.1 Let $Z = \{z_1, z_2\}$ be a universal set and $E = \{e_1, e_2, e_3,\}$ be a set of attributes. $X = \{e_1, e_2\}, Y = \{e_1, e_2\} \subseteq E$.Let Ω_X and Ψ_Y be two 3−NS set defined as:

$$\Omega_X = \{e_1, (z_1, (0.7, 0.5, 0.5), (0.2, 0.2, 0.3), (0.3, 0.5, 0.3)), (z_2, (0.3, 0.4, 0.5), (0.3, 0.2, 0.4), (0.3, 0.5, 0.8)),$$
$$e_2, (z_1, (0.3, 0.5, 0.6), (0.1, 0.4, 0.2), (0.7, 0.5, 0.3)), (z_2, (0.7, 0.2, 0.7), (0.3, 0.4, 0.4), (0.2, 0.7, 0.4))\}$$

$$\Psi_Y = \{e_1, (z_1, (0.8, 0.8, 0.7), (0.4, 0.5, 0.4), (0.3, 0.3, 0.2)), (z_2, (0.4, 0.7, 0.4), (0.6, 0.4, 0.5), (0.3, 0.3, 0.6)),$$
$$e_2, (z_1, (0.7, 0.8, 0.8), (0.3, 0.8, 0.2), (0.3, 0.4, 0.3)), (z_2, (0.8, 0.5, 0.8), (0.5, 0.8, 0.4), (0.2, 0.5, 0.3))\}$$

this implies $\Omega_X \breve{\subseteq} \Psi_Y$

3.2. *Equal Multi-Polar Neutrosophic Soft set*

Definition 3.2 Let Ω_X and Ψ_Y be two mNS set over a universal set Z, where X and Y are subsets of sets of attributes E. Then two mNS sets Ω_X and Ψ_Y are said to be equal denoted as $\Omega_X \breve{=} \Psi_Y$ if and only if $\Omega_X \breve{\subseteq} \Psi_Y$ and $\Psi_Y \breve{\subseteq} \Omega_X$

3.3. *Relative Null Multi-Polar Neutrosophic Soft set*

Definition 3.3 An mNS set over the universal set Z is said to be relative empty or relative null mNS set concering the set of attributes $X \subseteq E$, denoted by $\breve{\Phi}_X$ if

$$s_i \circ T_X(e)(z) = 0$$
$$s_i \circ I_X(e)(z) = 0$$
$$s_i \circ F_X(e)(z) = 1$$
for all $i = 1, 2, \cdots, m$; $e \in X$ and $z \in Z$

that is

$$\breve{\Phi}_X = \{e, (z, ((0, 0, \cdots, 0), (0, 0, \cdots, 0), (1, 1, \cdots, 1))) : z \in Z, e \in X\}$$

3.4. *Relative Whole Multi-Polar Neutrosophic Soft set*

Definition 3.4 An mNS set over the universal set Z is said to be relative whole mNS set concering the set of attributes $X \subseteq E$, denoted by $\breve{Z}_X$ if

$$s_i \circ T_X(e)(z) = 1$$
$$s_i \circ I_X(e)(z) = 1$$
$$s_i \circ F_X(e)(z) = 0$$
for all $i = 1, 2, \cdots, m$; $e \in X$ and $z \in Z$

that is

$$\breve{Z}_X = \{e, (z, ((1, 1, \cdots, 1), (1, 1, \cdots, 1), (0, 0, \cdots, 0))) : z \in Z, e \in X\}$$

3.5. *Absolute Multi-Polar Neutrosophic Soft set*

Definition 3.5 An mNS set over the universal set Z is said to be an absolute mNS set concering the set of attributes E, denoted by $\breve{Z}_E$ if

$$s_i \circ T_X(e)(z) = 1$$
$$s_i \circ I_X(e)(z) = 1$$
$$s_i \circ F_X(e)(z) = 0$$
for all $i = 1, 2, \cdots, m$; $e \in E$ and $z \in Z$

that is

$$\breve{Z}_E = \{e, (z, ((1, 1, \cdots, 1), (1, 1, \cdots, 1), (0, 0, \cdots, 0))) : z \in Z, e \in E\}$$

Example 3.2 Let $Z = \{z_1, z_2\}$ be universal set and $E = \{e_1, e_2, e_3\}$ is set of attributes, if $X = \{e_1, e_2\} \subseteq E$, A $3-$NS set Ω_X such that

$$\Omega_X = \{e_1, (z_1, (0, 0, 0), (0, 0, 0), (1, 1, 1)), (z_2, (0, 0, 0), (0, 0, 0), (1, 1, 1)),$$
$$e_2, (z_1, (0, 0, 0), (0, 0, 0), (1, 1, 1)), (z_2, (0, 0, 0), (0, 0, 0), (1, 1, 1))\} = \breve{\Phi}_X$$

then Ω_X is a relative null $3-$NS set $\breve{\Phi}_X$.

if $Y = \{e_1, e_3\} \subseteq E$, A $3-$NS set Ψ_Y such that

$$\Psi_Y = \{e_1, (z_1, (1, 1, 1), (1, 1, 1), (0, 0, 0)), (z_2, (1, 1, 1), (1, 1, 1), (0, 0, 0)),$$
$$e_3, (z_1, (1, 1, 1), (1, 1, 1), (0, 0, 0)), (z_2, (1, 1, 1), (1, 1, 1), (0, 0, 0))\} = \breve{Z}_Y$$

then Ψ_Y is a relative whole $3-$NS set $\breve{Z}_Y$.

if $W = E = \{e_1, e_2, e_3\}$, A $3-$NS set Λ_W such that

$$\Lambda_W = \{e_1, (z_1, (1,1,1), (1,1,1), (0,0,0)), (z_2, (1,1,1), (1,1,1), (0,0,0)),$$
$$e_2, (z_1, (1,1,1), (1,1,1), (0,0,0)), (z_2, (1,1,1), (1,1,1), (0,0,0)),$$
$$e_3, (z_1, (1,1,1), (1,1,1), (0,0,0)), (z_2, (1,1,1), (1,1,1), (0,0,0))\} = \breve{Z}_E$$

then Λ_W is an absolute $3-$NS set $\breve{Z}_E$.

Proposition 3.1 Let Z be a universal set, E a set of attributes, $X, Y, W \subseteq E$, If Ω_X, Ψ_Y and Λ_W are mNS sets over Z, then

(i) $\Omega_X \breve{\subseteq} \breve{Z}_X$

(ii) $\breve{\Phi}_X \breve{\subseteq} \Omega_X$

(iii) $\Omega_X \breve{\subseteq} \Omega_X$

(iv) $\Omega_X \breve{\subseteq} \Psi_Y$ and $\Psi_Y \breve{\subseteq} \Lambda_W$, then $\Omega_X \breve{\subseteq} \Lambda_W$

(v) $\Omega_X \breve{=} \Psi_Y$ and $\Psi_Y \breve{=} \Lambda_W$, then $\Omega_X \breve{=} \Lambda_W$

3.6. *Complement of mN Soft Set*

Definition 3.6 The complement of an mNS set Ω_X over a universal set Z with respect to the set of attributes $X \subseteq E$, denoted by $\Omega_X^c = (\omega^c, X)$ where $\omega^c : \neg X \to m\text{NS}^Z$ is a mapping given as

$$\omega^c(\neg e) = \{z, ((s_i \circ T_X^c(\neg e)(z) = s_i \circ F_X(e)(z)), (s_i \circ I_X^c(\neg e)(z) = (1, 1, \cdots, 1) - s_i \circ I_X(e)(z)),$$
$$(s_i \circ F_X^c(\neg e)(z) = s_i \circ T_X(e)(z)))\}$$
$$\text{for all } i = 1, 2, \cdots, m, \ \neg e \in \neg X \text{ and } z \in Z$$

3.7. *Relative Complement of mN Soft Set*

Definition 3.7 The relative complement of an mNS set Ω_X over a universal set Z with respect to the set of attributes $X \subseteq E$, denoted by $\Omega_X^r = (\omega^r, X)$ where $\omega^r : X \to m\text{NS}^Z$ is a mapping given as

$$\omega^r(e) = \{z, ((s_i \circ T_X^r(e)(z) = s_i \circ F_X(e)(z)), (s_i \circ I_X^r(e)(z) = (1, 1, \cdots, 1) - s_i \circ I_X(e)(z)),$$
$$(s_i \circ F_X^r(e)(z) = s_i \circ T_X(e)(z)))\}$$
$$\text{for all } i = 1, 2, \cdots, m; \ e \in X \text{ and } z \in Z$$

Example 3.3 Let $Z = \{z_1, z_2\}$ be universal set and $E = \{e_1, e_2, e_3\}$ is set of attributes, if $X = \{e_1, e_2\} \subseteq E$, A $3-$NS set Ω_X such that

$$\Omega_X = \{e_1, (z_1, (0.4, 0.4, 0.6), (0.2, 0.4, 0.5), (0.5, 0.3, 0.7)), (z_2, (0.5, 0.7, 0.5), (0.6, 0.5, 0.3), (0.5, 0.7, 0.3)),$$
$$e_2, (z_1, (0.7, 0.4, 0.6), (0.8, 0.6, 0.4), (0.2, 0.6, 0.7)), (z_2, (0.5, 0.7, 0.6), (0.8, 0.4, 0.7), (0.9, 0.3, 0.6))\}$$

Then,

$$\Omega_X^c = \{\neg e_1, (z_1, (0.5, 0.3, 0.7), (0.8, 0.6, 0.4), (0.4, 0.4, 0.6)), (z_2, (0.5, 0.7, 0.3), (0.4, 0.5, 0.7), (0.5, 0.7, 0.5)),$$
$$\neg e_2, (z_1, (0.2, 0.6, 0.7), (0.2, 0.4, 0.6), (0.7, 0.4, 0.6)), (z_2, (0.9, 0.3, 0.6), (0.2, 0.6, 0.3), (0.5, 0.7, 0.6))\}$$

$$\Omega^r_X = \{e_1, (z_1, (0.5, 0.3, 0.7), (0.8, 0.6, 0.4), (0.4, 0.4, 0.6)), (z_2, (0.5, 0.7, 0.3), (0.4, 0.5, 0.7), (0.5, 0.7, 0.5)),$$
$$e_2, (z_1, (0.2, 0.6, 0.7), (0.2, 0.4, 0.6), (0.7, 0.4, 0.6)), (z_2, (0.9, 0.3, 0.6), (0.2, 0.6, 0.3), (0.5, 0.7, 0.6))\}$$

Proposition 3.2 Let Ω_X be an mNS set over a universal set Z. Then
(i) $(\Omega^c_X)^c = \Omega_X$
(ii) $(\Omega^r_X)^r = \Omega_X$
(iii) $\breve{Z}^c_X = \breve{\Phi}_X = \breve{Z}^r_X$
(iv) $\breve{\Phi}^c_X = \breve{Z}_X = \breve{\Phi}^r_X$

3.8. *Union of Two mN Soft Sets*

Definition 3.8 Let Z be a universal set and X and Y are subsets of the set of attributes E. The set Ω_X and Ψ_Y are two mNS sets. Let $W = X \cup Y$, then the union of Ω_X and Ψ_Y is an mNS set denoted by $\Omega_X \breve{\cup} \Psi_Y$ and defined as for all $e \in W$

$$\Omega_X \breve{\cup} \Psi_Y = \begin{cases} \omega_X(e), & e \in X \setminus Y; \\ \psi_Y(e), & e \in Y \setminus X; \\ \omega_X(e) \cup \psi_Y(e), & e \in X \cap Y. \end{cases}$$

where,

$$\omega_X(e) \cup \psi_Y(e) = (max(s_i \circ\ T_X(e)(z), s_i \circ\ T_Y(e)(z)), max(s_i \circ\ I_X(e)(z), s_i \circ\ I_Y(e)(z)),$$
$$min(s_i \circ\ F_X(e)(z), s_i \circ\ F_Y(e)(z)))$$
$$\text{for all } i = 1, 2, \cdots, m;\ e \in W \text{ and } z \in Z$$

3.9. *Intersection of Two mN Soft Sets*

Definition 3.9 Let Z be a universal set and X and Y are subsets of the set of attributes E. The set Ω_X and Ψ_Y are two mNS sets. Let $W = X \cap Y$, then the intersection of Ω_X and Ψ_Y is an mNS set denoted by $\Omega_X \breve{\cap} \Psi_Y$ and defined as for all $e \in W$

$$\Omega_X \breve{\cap} \Psi_Y = \omega_X(e) \cap \psi_Y(e)$$

where,

$$\omega_X(e) \cap \psi_Y(e) = (min(s_i \circ\ T_X(e)(z), s_i \circ\ T_Y(e)(z)), min(s_i \circ\ I_X(e)(z), s_i \circ\ I_Y(e)(z)),$$
$$max(s_i \circ\ F_X(e)(z), s_i \circ\ F_Y(e)(z)))$$
$$\text{for all } i = 1, 2, \cdots, m;\ e \in W \text{ and } z \in Z$$

3.10. *Restricted Union of Two mN Soft Sets*

Definition 3.10 Let Z be a universal set and X and Y are subsets of the set of attributes E. The set Ω_X and Ψ_Y are two mNS sets. Let $W = X \cap Y$, then the Restricted union of Ω_X and Ψ_Y is an mNS set denoted by $\Omega_X \cup_R \Psi_Y$ and defined as for all $e \in W$

$$\Omega_X \cup_R \Psi_Y = \omega_X(e) \cup \psi_Y(e)$$

3.11. *Extended Intersection of Two mN Soft Sets*

Definition 3.11 Let Z be a universal set and X and Y are subsets of the set of attributes E. The set Ω_X and Ψ_Y are two mNS sets. Let $W = X \cup Y$, then the Extended intersection of Ω_X and Ψ_Y is an mNS set denoted by $\Omega_X \cap_\varepsilon \Psi_Y$ and defined as for all $e \in W$

$$\Omega_X \cap_\varepsilon \Psi_Y = \begin{cases} \omega_X(e), & e \in X \setminus Y; \\ \psi_Y(e), & e \in Y \setminus X; \\ \omega_X(e) \cap \psi_Y(e), & e \in X \cap Y. \end{cases}$$

3.12. *OR-operator of Two mN Soft Sets*

Definition 3.12 Let Z be a universal set and X and Y are subsets of the set of attributes E. The set Ω_X and Ψ_Y are two mNS sets, then the OR-operator of Ω_X and Ψ_Y is an mNS set denoted by $\Omega_X \vee \Psi_Y$ and defined as $\Omega_X \vee \Psi_Y = \Lambda_{X\times Y}$ where

$$\lambda_{X\times Y}(x,y) = \omega_X(x) \cup \psi_Y(y)$$
$$\text{for all } (x,y) \in X \times Y$$

3.13. *AND-operator of Two mN Soft Sets*

Definition 3.13 Let Z be a universal set and X and Y are subsets of the set of attributes E. The set Ω_X and Ψ_Y are two mNS sets, then the AND-operator of Ω_X and Ψ_Y is an mNS set denoted by $\Omega_X \wedge \Psi_Y$ and defined as $\Omega_X \wedge \Psi_Y = \Lambda_{X\times Y}$ where

$$\lambda_{X\times Y}(x,y) = \omega_X(x) \cap \psi_Y(y)$$
$$\text{for all } (x,y) \in X \times Y$$

Example 3.4 Let $Z = \{z_1, z_2\}$ be a universal set and $E = \{e_1, e_2, e_3,\}$ be a set of attributes. Let $X = \{e_1, e_2\}, Y = \{e_2, e_3\} \subseteq E$. Let Ω_X and Ψ_Y be two $3-$N soft set defined as.

$$\begin{aligned}\Omega_X = \{&e_1,(z_1,(0.7,0.5,0.5),(0.4,0.5,0.4),(0.3,0.5,0.3)),(z_2,(0.3,0.4,0.5),(0.6,0.4,0.5),(0.3,0.5,0.8)),\\ &e_2,(z_1,(0.3,0.5,0.6),(0.3,0.8,0.2),(0.7,0.5,0.3)),(z_2,(0.7,0.2,0.7),(0.5,0.8,0.4),(0.2,0.7,0.4))\}\end{aligned}$$

$$\begin{aligned}\Psi_Y = \{&e_2,(z_1,(0.3,0.5,0.7),(0.2,0.6,0.3),(0.3,0.4,0.6)),(z_2,(0.4,0.6,0.7),(0.3,0.5,0.7),(0.4,0.7,0.3)),\\ &e_3,(z_1,(0.4,0.7,0.4),(0.7,0.5,0.3),(0.8,0.5,0.7)),(z_2,(0.2,0.7,0.4),(0.3,0.8,0.9),(0.2,0.1,0.6))\}\end{aligned}$$

Then,

$$\begin{aligned}\Omega_X \breve{\cup} \Psi_Y = \{&e_1,(z_1,(0.7,0.5,0.5),(0.4,0.5,0.4),(0.3,0.5,0.3)),(z_2,(0.3,0.4,0.5),(0.6,0.4,0.5),(0.3,0.5,0.8)),\\ &e_2,(z_1,(0.3,0.5,0.7),(0.3,0.8,0.3),(0.3,0.4,0.3)),(z_2,(0.7,0.6,0.7),(0.5,0.8,0.7),(0.2,0.7,0.3)),\\ &e_3,(z_1,(0.4,0.7,0.4),(0.7,0.5,0.3),(0.8,0.5,0.7)),(z_2,(0.2,0.7,0.4),(0.3,0.8,0.9),(0.2,0.1,0.6))\}\end{aligned}$$

$$\Omega_X \breve{\cap} \Psi_Y = \{e_2,(z_1,(0.3,0.5,0.6),(0.2,0.6,0.2),(0.7,0.5,0.6)),(z_2,(0.4,0.2,0.7),(0.3,0.5,0.4),(0.4,0.7,0.4))\}$$

$$\Omega_X \cup_R \Psi_Y = \{e_2,(z_1,(0.3,0.5,0.7),(0.3,0.8,0.3),(0.3,0.4,0.3)),(z_2,(0.7,0.6,0.7),(0.5,0.8,0.7),(0.2,0.7,0.3))\}$$

$$\begin{aligned}\Omega_X \cap_\varepsilon \Psi_Y = \{&e_1, (z_1, (0.7, 0.5, 0.5), (0.4, 0.5, 0.4), (0.3, 0.5, 0.3)), (z_2, (0.3, 0.4, 0.5), (0.6, 0.4, 0.5), (0.3, 0.5, 0.8)),\\ &e_2, (z_1, (0.3, 0.5, 0.6), (0.2, 0.6, 0.2), (0.7, 0.5, 0.6)), (z_2, (0.4, 0.2, 0.7), (0.3, 0.5, 0.4), (0.4, 0.7, 0.4)),\\ &e_3, (z_1, (0.4, 0.7, 0.4), (0.7, 0.5, 0.3), (0.8, 0.5, 0.7)), (z_2, (0.2, 0.7, 0.4), (0.3, 0.8, 0.9), (0.2, 0.1, 0.6))\}\end{aligned}$$

$$\begin{aligned}\Omega_X \vee \Psi_Y = \{&(e_1, e_2), (z_1, (0.7, 0.5, 0.7), (0.4, 0.6, 0.4), (0.3, 0.4, 0.3)), (z_2, (0.4, 0.6, 0.7), (0.6, 0.5, 0.7), (0.3, 0.5, 0.3)),\\ &(e_1, e_3), (z_1, (0.7, 0.7, 0.5), (0.7, 0.5, 0.4), (0.3, 0.5, 0.3)), (z_2, (0.3, 0.7, 0.5), (0.6, 0.8, 0.9), (0.2, 0.1, 0.6)),\\ &(e_2, e_2), (z_1, (0.3, 0.5, 0.7), (0.3, 0.8, 0.3), (0.3, 0.4, 0.3)), (z_2, (0.7, 0.6, 0.7), (0.5, 0.8, 0.7), (0.2, 0.7, 0.3)),\\ &(e_2, e_3), (z_1, (0.4, 0.7, 0.6), (0.7, 0.8, 0.3), (0.7, 0.5, 0.3)), (z_2, (0.7, 0.7, 0.7), (0.5, 0.8, 0.9), (0.2, 0.1, 0.4))\}\end{aligned}$$

$$\begin{aligned}\Omega_X \wedge \Psi_Y = \{&(e_1, e_2), (z_1, (0.3, 0.5, 0.5), (0.2, 0.5, 0.3), (0.3, 0.5, 0.6)), (z_2, (0.3, 0.4, 0.5), (0.3, 0.4, 0.5), (0.4, 0.7, 0.8)),\\ &(e_1, e_3), (z_1, (0.4, 0.5, 0.4), (0.4, 0.5, 0.3), (0.8, 0.5, 0.7)), (z_2, (0.2, 0.4, 0.4), (0.3, 0.4, 0.5), (0.3, 0.5, 0.8)),\\ &(e_2, e_2), (z_1, (0.3, 0.5, 0.6), (0.2, 0.6, 0.2), (0.7, 0.5, 0.6)), (z_2, (0.4, 0.2, 0.7), (0.3, 0.5, 0.4), (0.4, 0.7, 0.4)),\\ &(e_2, e_3), (z_1, (0.3, 0.5, 0.4), (0.3, 0.5, 0.2), (0.8, 0.5, 0.7)), (z_2, (0.2, 0.2, 0.4), (0.3, 0.8, 0.4), (0.4, 0.7, 0.6))\}\end{aligned}$$

4. Properties of mNS Set Operators

In this section, we define some properties of mNS set operators that satisfied among mNS sets. We also give proof of some of them, while others can also be proved. Let Ω_X, Ψ_Y and Λ_W be three mNS sets over universal set Z with respect to parameter set E where X, Y and W are subsets of E. The approximation functions of Ω_X, Ψ_Y and Λ_W are defined as

$$\omega_X(e) = \{(z, s_i \circ T_X(e)(z), s_i \circ I_X(e)(z), s_i \circ F_X(e)(z)) : z \in Z,\ e \in X\}$$
$$\psi_Y(e) = \{(z, s_i \circ T_Y(e)(z), s_i \circ I_Y(e)(z), s_i \circ F_Y(e)(z)) : z \in Z,\ e \in Y\}$$
$$\lambda_W(e) = \{(z, s_i \circ T_W(e)(z), s_i \circ I_W(e)(z), s_i \circ F_W(e)(z)) : z \in Z,\ e \in W\}$$
$$\text{for all } i = 1, 2, \cdots, m$$

4.1. *Idempotent properties*

(i) $\Omega_X \breve{\cup} \Omega_X = \Omega_X = \Omega_X \cup_R \Omega_X$
(ii) $\Omega_X \breve{\cap} \Omega_X = \Omega_X = \Omega_X \cap_\varepsilon \Omega_X$

4.2. *Identity Properties*

(i) $\Omega_X \breve{\cup} \breve{\Phi}_X = \Omega_X = \Omega_X \cup_R \breve{\Phi}_X$
(ii) $\Omega_X \breve{\cap} \breve{Z}_X = \Omega_X = \Omega_X \cap_\varepsilon \breve{Z}_X$

4.3. *Domination Properties*

(i) $\Omega_X \breve{\cup} \breve{Z}_X = \breve{Z}_X = \Omega_X \cup_R \breve{Z}_X$
(ii) $\Omega_X \breve{\cap} \breve{\Phi}_X = \breve{\Phi}_X = \Omega_X \cap_\varepsilon \breve{\Phi}_X$

4.4. *Complementation Properties*

(i) $\breve{Z}^c_X = \breve{\Phi}_X = \breve{Z}^r_X$
(ii) $\breve{\Phi}^c_X = \breve{Z}_X = \breve{\Phi}^r_X$

4.5. *Double Complementation Property*

(i) $(\Omega_X^c)^c = \Omega_X = (\Omega_X^r)^r$

4.6. *Absorption Properties*

(i) $\Omega_X \breve{\cup} (\Omega_X \breve{\cap} \Psi_Y) = \Omega_X$

(ii) $\Omega_X \breve{\cap} (\Omega_X \breve{\cup} \Psi_Y) = \Omega_X$

(iii) $\Omega_X \cup_R (\Omega_X \cap_\varepsilon \Psi_Y) = \Omega_X$

(iv) $\Omega_X \cap_\varepsilon (\Omega_X \cup_R \Psi_Y) = \Omega_X$

Remark 4.1

(i) Union $\breve{\cup}$ and extended intersection $\cap_\varepsilon$ do not absorb over each other among mNS sets

(ii) Restricted Union $\cup_R$ and intersection $\breve{\cap}$ do not absorb over each other among mNS sets

4.7. *Commutative Properties*

(i) $\Omega_X \breve{\cup} \Psi_Y = \Psi_Y \breve{\cup} \Omega_X$

(ii) $\Omega_X \cup_R \Psi_Y = \Psi_Y \cup_R \Omega_X$

(iii) $\Omega_X \breve{\cap} \Psi_Y = \Psi_Y \breve{\cap} \Omega_X$

(iv) $\Omega_X \cap_\varepsilon \Psi_Y = \Psi_Y \cap_\varepsilon \Omega_X$

Remark 4.2

(i) OR-operator $\vee$ and AND-operator $\wedge$ do not commute among mNS sets

4.8. *Associative Properties*

(i) $\Omega_X \breve{\cup} (\Psi_Y \breve{\cup} \Lambda_W) = (\Omega_X \breve{\cup} \Psi_Y) \breve{\cup} \Lambda_W$

(ii) $\Omega_X \breve{\cap} (\Psi_Y \breve{\cap} \Lambda_W) = (\Omega_X \breve{\cap} \Psi_Y) \breve{\cap} \Lambda_W$

(iii) $\Omega_X \cup_R (\Psi_Y \cup_R \Lambda_W) = (\Omega_X \cup_R \Psi_Y) \cup_R \Lambda_W$

(iv) $\Omega_X \cap_\varepsilon (\Psi_Y \cap_\varepsilon \Lambda_W) = (\Omega_X \cap_\varepsilon \Psi_Y) \cap_\varepsilon \Lambda_W$

(v) $\Omega_X \vee (\Psi_Y \vee \Lambda_W) = (\Omega_X \vee \Psi_Y) \vee \Lambda_W$

(vi) $\Omega_X \wedge (\Psi_Y \wedge \Lambda_W) = (\Omega_X \wedge \Psi_Y) \wedge \Lambda_W$

Proof(i)

$$\Rightarrow \omega_X(e) \cup (\psi_Y(e) \cup \lambda_Y(e)) = max\{s_i \circ T_X(e)(z), max(s_i \circ T_Y(e)(z), s_i \circ T_W(e)(z))\},$$
$$max\{s_i \circ I_X(e)(z), max(s_i \circ I_Y(e)(z), s_i \circ I_W(e)(z))\},$$
$$min\{s_i \circ F_X(e)(z), min(s_i \circ F_Y(e)(z), s_i \circ F_W(e)(z))\}$$
$$\textit{for all } i = 1, 2, \cdots, m; \ e \in X \cup (Y \cup Z) = (X \cup Y) \cup Z \ \textit{and} \ z \in Z$$

$$\Rightarrow \omega_X(e) \cup (\psi_Y(e) \cup \lambda_Y(e)) = max(s_i \circ T_X(e)(z), s_i \circ T_Y(e)(z), s_i \circ T_W(e)(z)),$$
$$max(s_i \circ I_X(e)(z), s_i \circ I_Y(e)(z), s_i \circ I_W(e)(z)), min(s_i \circ F_X(e)(z), s_i \circ F_Y(e)(z), s_i \circ F_W(e)(z))$$
$$\textit{for all } i = 1, 2, \cdots, m; \ e \in X \cup (Y \cup Z) = (X \cup Y) \cup Z \ \textit{and} \ z \in Z$$

$$\Rightarrow \omega_X(e) \cup (\psi_Y(e) \cup \lambda_Y(e)) = max\{max(s_i \circ T_X(e)(z), s_i \circ T_Y(e)(z)), s_i \circ T_W(e)(z)\},$$
$$max\{max(s_i \circ I_X(e)(z), s_i \circ I_Y(e)(z)), s_i \circ I_W(e)(z)\},$$
$$min\{min(s_i \circ F_X(e)(z), s_i \circ F_Y(e)(z)), s_i \circ F_W(e)(z)\}$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cup (Y \cup Z) = (X \cup Y) \cup Z \textit{ and } z \in Z$$

$$\Rightarrow \omega_X(e) \cup (\psi_Y(e) \cup \lambda_Y(e)) = (\omega_X(e) \cup \psi_Y(e)) \cup \lambda_Y(e)$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cup (Y \cup Z) = (X \cup Y) \cup Z \textit{ and } z \in Z$$

$$\Rightarrow \Omega_X \breve{\cup} (\Psi_Y \breve{\cup} \Lambda_W) = (\Omega_X \breve{\cup} \Psi_Y) \breve{\cup} \Lambda_W$$

Proof(ii)

$$\Rightarrow \omega_X(e) \cap (\psi_Y(e) \cap \lambda_Y(e)) = min\{s_i \circ T_X(e)(z), min(s_i \circ T_Y(e)(z), s_i \circ T_W(e)(z))\},$$
$$min\{s_i \circ I_X(e)(z), min(s_i \circ I_Y(e)(z), s_i \circ I_W(e)(z))\},$$
$$max\{s_i \circ F_X(e)(z), max(s_i \circ F_Y(e)(z), s_i \circ F_W(e)(z))\}$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap (Y \cap Z) = (X \cap Y) \cap Z \textit{ and } z \in Z$$

$$\Rightarrow \omega_X(e) \cap (\psi_Y(e) \cap \lambda_Y(e)) = min(s_i \circ T_X(e)(z), s_i \circ T_Y(e)(z), s_i \circ T_W(e)(z)),$$
$$min(s_i \circ I_X(e)(z), s_i \circ I_Y(e)(z), s_i \circ I_W(e)(z)), max(s_i \circ F_X(e)(z), s_i \circ F_Y(e)(z), s_i \circ F_W(e)(z))$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap (Y \cap Z) = (X \cap Y) \cap Z \textit{ and } z \in Z$$

$$\Rightarrow \omega_X(e) \cap (\psi_Y(e) \cap \lambda_Y(e)) = min\{min(s_i \circ T_X(e)(z), s_i \circ T_Y(e)(z)), s_i \circ T_W(e)(z)\},$$
$$min\{min(s_i \circ I_X(e)(z), s_i \circ I_Y(e)(z)), s_i \circ I_W(e)(z)\},$$
$$max\{max(s_i \circ F_X(e)(z), s_i \circ F_Y(e)(z)), s_i \circ F_W(e)(z)\}$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap (Y \cap Z) = (X \cap Y) \cap Z \textit{ and } z \in Z$$

$$\Rightarrow \omega_X(e) \cap (\psi_Y(e) \cap \lambda_Y(e)) = (\omega_X(e) \cup \psi_Y(e)) \cup \lambda_Y(e)$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap (Y \cap Z) = (X \cap Y) \cap Z \textit{ and } z \in Z$$

$$\Rightarrow \Omega_X \breve{\cap} (\Psi_Y \breve{\cap} \Lambda_W) = (\Omega_X \breve{\cap} \Psi_Y) \breve{\cap} \Lambda_W$$

Similarly, others associative properties also satisfy equality.

4.9. *Distributive Properties*

(i) $\Omega_X \breve{\cup} (\Psi_Y \breve{\cap} \Lambda_W) = (\Omega_X \breve{\cup} \Psi_Y) \breve{\cap} (\Omega_X \breve{\cup} \Lambda_W)$
(ii) $\Omega_X \breve{\cap} (\Psi_Y \breve{\cup} \Lambda_W) = (\Omega_X \breve{\cap} \Psi_Y) \breve{\cup} (\Omega_X \breve{\cap} \Lambda_W)$
(iii) $\Omega_X \cup_R (\Psi_Y \cap_\varepsilon \Lambda_W) = (\Omega_X \cup_R \Psi_Y) \cap_\varepsilon (\Omega_X \cup_R \Lambda_W)$
(iv) $\Omega_X \cap_\varepsilon (\Psi_Y \cup_R \Lambda_W) = (\Omega_X \cap_\varepsilon \Psi_Y) \cup_R (\Omega_X \cap_\varepsilon \Lambda_W)$
(v) $\Omega_X \cup_R (\Psi_Y \breve{\cap} \Lambda_W) = (\Omega_X \cup_R \Psi_Y) \breve{\cap} (\Omega_X \cup_R \Lambda_W)$
(vi) $\Omega_X \breve{\cap} (\Psi_Y \cup_R \Lambda_W) = (\Omega_X \breve{\cap} \Psi_Y) \cup_R (\Omega_X \breve{\cap} \Lambda_W)$

Proof(i)

$$\Rightarrow \omega_X(e) \cup (\psi_Y(e) \cap \lambda_W(e)) = max\{s_i \circ T_X(e)(z), min(s_i \circ T_Y(e)(z), s_i \circ T_W(e)(z))\},$$
$$max\{s_i \circ I_X(e)(z), min(s_i \circ I_Y(e)(z), s_i \circ I_W(e)(z))\},$$
$$min\{s_i \circ T_X(e)(z), max(s_i \circ T_Y(e)(z), s_i \circ T_W(e)(z))\}$$

$$for\ all\ i=1,2,\cdots,m;\ e\in X\cup(Y\cap W)=(X\cup Y)\cap(X\cup W)\ and\ z\in Z$$

$$\Rightarrow \omega_X(e)\cup(\psi_Y(e)\cap\lambda_W(e)=$$
$$min\{max(s_i\circ T_X(e)(z),s_i\circ T_Y(e)(z)),max(s_i\circ T_X(e)(z),s_i\circ T_W(e)(z))\},$$
$$min\{max(s_i\circ I_X(e)(z),s_i\circ I_Y(e)(z)),max(s_i\circ I_X(e)(z),s_i\circ I_W(e)(z))\},$$
$$max\{min(s_i\circ F_X(e)(z),s_i\circ F_Y(e)(z)),min(s_i\circ F_X(e)(z),s_i\circ F_W(e)(z))\}$$
$$for\ all\ i=1,2,\cdots,m;\ e\in X\cup(Y\cap W)=(X\cup Y)\cap(X\cup W)\ and\ z\in Z$$

$$\Rightarrow \omega_X(e)\cup(\psi_Y(e)\cap\lambda_W(e))=(\omega_X(e)\cup\psi_Y(e))\cap(\omega_X(e)\cup\lambda_W(e))$$
$$for\ all\ i=1,2,\cdots,m;\ e\in X\cup(Y\cap W)=(X\cup Y)\cap(X\cup W)\ and\ z\in Z$$

$$\Rightarrow \Omega_X\breve{\cup}(\Psi_Y\breve{\cap}\Lambda_W)=(\Omega_X\breve{\cup}\Psi_Y)\breve{\cap}(\Omega_X\breve{\cup}\Lambda_W)$$

Proof(ii)

$$\Rightarrow \omega_X(e)\cap(\psi_Y(e)\cup\lambda_W(e))=min\{s_i\circ T_X(e)(z),max(s_i\circ T_Y(e)(z),s_i\circ T_W(e)(z))\},$$
$$min\{s_i\circ I_X(e)(z),max(s_i\circ I_Y(e)(z),s_i\circ I_W(e)(z))\},$$
$$max\{s_i\circ T_X(e)(z),min(s_i\circ T_Y(e)(z),s_i\circ T_W(e)(z))\}$$
$$for\ all\ i=1,2,\cdots,m;\ e\in X\cap(Y\cup W)=(X\cap Y)\cup(X\cap W)\ and\ z\in Z$$

$$\Rightarrow \omega_X(e)\cap(\psi_Y(e)\cup\lambda_W(e)=$$
$$max\{min(s_i\circ T_X(e)(z),s_i\circ T_Y(e)(z)),min(s_i\circ T_X(e)(z),s_i\circ T_W(e)(z))\},$$
$$max\{min(s_i\circ I_X(e)(z),s_i\circ I_Y(e)(z)),min(s_i\circ I_X(e)(z),s_i\circ I_W(e)(z))\},$$
$$min\{max(s_i\circ F_X(e)(z),s_i\circ F_Y(e)(z)),max(s_i\circ F_X(e)(z),s_i\circ F_W(e)(z))\}$$
$$for\ all\ i=1,2,\cdots,m;\ e\in X\cap(Y\cup W)=(X\cap Y)\cup(X\cap W)\ and\ z\in Z$$

$$\Rightarrow \omega_X(e)\cap(\psi_Y(e)\cup\lambda_W(e))=(\omega_X(e)\cap\psi_Y(e))\cup(\omega_X(e)\cap\lambda_W(e))$$
$$for\ all\ i=1,2,\cdots,m;\ e\in X\cap(Y\cup W)=(X\cap Y)\cup(X\cap W)\ and\ z\in Z$$

$$\Rightarrow \Omega_X\breve{\cap}(\Psi_Y\breve{\cup}\Lambda_W)=(\Omega_X\breve{\cap}\Psi_Y)\breve{\cup}(\Omega_X\breve{\cap}\Lambda_W)$$

Similarly, others distributive properties also satisfy equality.

Remark 4.3

(i) Union $\breve{\cup}$ and extended intersection $\cap_\varepsilon$ do not distribute over each other among mNS sets

(ii) OR-operator $\vee$ and AND-operator $\wedge$ do not distribute over each other among mNS sets

(iii) Restricted union $\cup_R$ distribute over union $\breve{\cup}$ but converse does not hold true

(iv) Intersection $\breve{\cap}$ distribute over extended intersection $\cap_\varepsilon$ but converse does not hold true

Counter-Example 4.1

Let $\Omega_X=\{e_2,(z_1,(0.5,0.6),(0.3,0.2),(0.9,0.7)),(z_2,(0.3,0.6),(0.4,0.7),(0.5,0.8))\}$;
$\Psi_Y=\{e_1,(z_1(0.3,0.2),(0.4,0.6),(0.1,0.8)),(z_2,(0.6,0.4),(0.6,0.8),(0.8,0.8))\}$ and
$\Lambda_W=\{e_2,(z_1,(0.7,0.2),(0.3,0.5),(0.2,0.1)),(z_2,(0.6,0.5),(0.3,0.6),(0.5,0.4))\}$ be three $2-$NS sets over the universal set $Z=\{z_1,z_2\}$ with respect to set of attributes $E=\{e_1,e_2\}$, then

$$\Omega_X \cap_\varepsilon (\Psi_Y \breve{\cup} \Lambda_W) = \{e_1, (z_1(0.3, 0.2), (0.4, 0.6), (0.1, 0.8)), (z_2, (0.6, 0.4), (0.6, 0.8), (0.8, 0.8)),$$
$$e_2, (z_1, (0.5, 0.2), (0.3, 0.2), (0.9, 0.7)), (z_2, (0.3, 0.5), (0.3, 0.6), (0.5, 0.8))\}$$

and

$$(\Omega_X \cap_\varepsilon \Psi_Y) \breve{\cup} (\Omega_X \cap_\varepsilon \Lambda_W) = \{e_1, (z_1(0.3, 0.2), (0.4, 0.6), (0.1, 0.8)), (z_2, (0.6, 0.4), (0.6, 0.8), (0.8, 0.8)),$$
$$e_2, (z_1, (0.5, 0.6), (0.3, 0.2), (0.9, 0.7)), (z_2, (0.3, 0.6), (0.4, 0.7), (0.5, 0.8))\}$$

Hence, $\Omega_X \cap_\varepsilon (\Psi_Y \breve{\cup} \Lambda_W) \neq (\Omega_X \cap_\varepsilon \Psi_Y) \breve{\cup} (\Omega_X \cap_\varepsilon \Lambda_W)$

4.10. *De Morgan's Properties*

(i) $(\Omega_X \cup_R \Psi_Y)^r = \Omega_X^r \breve{\cap} \Psi_Y^r$
(ii) $(\Omega_X \breve{\cap} \Psi_Y)^r = \Omega_X^r \cup_R \Psi_Y^r$
(iii) $(\Omega_X \wedge \Psi_Y)^r = \Omega_X^r \vee (\Psi_Y^r$
(iv) $(\Omega_X \vee \Psi_Y)^r = \Omega_X^r \wedge \Psi_Y^r$
(v) $(\Omega_X \breve{\cup} \Psi_Y)^r = \Omega_X^r \cap_\varepsilon \Psi_Y^r$
(vi) $(\Omega_X \cap_\varepsilon \Psi_Y)^r = \Omega_X^r \breve{\cup} \Psi_Y^r$
(vii) $(\Omega_X \cup_R \Psi_Y)^c = \Omega_X^c \breve{\cap} \Psi_Y^c$
(viii) $(\Omega_X \breve{\cap} \Psi_Y)^c = \Omega_X^c \cup_R \Psi_Y^c$
(ix) $(\Omega_X \wedge \Psi_Y)^c = \Omega_X^c \vee \Psi_Y^c$
(x) $(\Omega_X \vee \Psi_Y)^c = \Omega_X^c \wedge \Psi_Y^c$
(xi) $(\Omega_X \breve{\cup} \Psi_Y)^c = \Omega_X^c \cap_\varepsilon \Psi_Y^c$
(xii) $(\Omega_X \cap_\varepsilon \Psi_Y)^c = \Omega_X^c \breve{\cup} \Psi_Y^c$

Proof(i)

$$\Rightarrow (\omega_X(e) \cup \psi_Y(e))^r = [max(s_i \circ T_X(e)(z), s_i \circ T_Y(e)(z)),$$
$$max(s_i \circ I_X(e)(z), s_i \circ I_Y(e)(z)), min(s_i \circ F_X(e)(z), s_i \circ F_Y(e)(z))]^r$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\omega_X(e) \cup \psi_Y(e))^r = min(s_i \circ F_X(e)(z), s_i \circ F_Y(e)(z)),$$
$$(1, 1, \cdots, 1) - max(s_i \circ I_X(e)(z), s_i \circ I_Y(e)(z)), max(s_i \circ T_X(e)(z), s_i \circ T_Y(e)(z))$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\omega_X(e) \cup \psi_Y(e))^r = min(s_i \circ F_X(e)(z), s_i \circ F_Y(e)(z)),$$
$$min\{(1, 1, \cdots, 1) - s_i \circ I_X(e)(z), (1, 1, \cdots, 1) - s_i \circ I_Y(e)(z))\}, max(s_i \circ T_X(e)(z), s_i \circ T_Y(e)(z))$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\omega_X(e) \cup \psi_Y(e))^r = [s_i \circ F_X(e)(z), (1, 1, \cdots, 1) - s_i \circ I_X(e)(z), s_i \circ T_X(e)(z)]$$
$$\cap [s_i \circ F_Y(e)(z)), (1, 1, \cdots, 1) - s_i \circ I_Y(e)(z)), s_i \circ T_Y(e)(z))]$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\omega_X(e) \cup \psi_Y(e))^r = [s_i \circ T_X(e)(z), s_i \circ I_X(e)(z), s_i \circ F_X(e)(z)]^r$$
$$\cap [s_i \circ T_Y(e)(z)), s_i \circ I_Y(e)(z)), s_i \circ F_Y(e)(z))]^r$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\omega_X(e) \cup \psi_Y(e))^r = \omega^r_X(e) \cap \psi^r_Y(e)$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\Omega_X \cup_R \Psi_Y)^r = (\Omega_X)^r \breve{\cap} (\Psi_Y)^r$$

Proof(ii)

$$\Rightarrow (\omega_X(e) \cap \psi_Y(e))^r = [min(s_i \circ T_X(e)(z), s_i \circ T_Y(e)(z)),$$
$$min(s_i \circ I_X(e)(z), s_i \circ I_Y(e)(z)), max(s_i \circ F_X(e)(z), s_i \circ F_Y(e)(z))]^r$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\omega_X(e) \cap \psi_Y(e))^r = max(s_i \circ F_X(e)(z), s_i \circ F_Y(e)(z)),$$
$$(1, 1, \cdots, 1) - min(s_i \circ I_X(e)(z), s_i \circ I_Y(e)(z)), min(s_i \circ T_X(e)(z), s_i \circ T_Y(e)(z))$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\omega_X(e) \cap \psi_Y(e))^r = max(s_i \circ F_X(e)(z), s_i \circ F_Y(e)(z)),$$
$$max\{(1, 1, \cdots, 1) - s_i \circ I_X(e)(z), (1, 1, \cdots, 1) - s_i \circ I_Y(e)(z))\}, min(s_i \circ T_X(e)(z), s_i \circ T_Y(e)(z))$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\omega_X(e) \cap \psi_Y(e))^r = [s_i \circ F_X(e)(z), (1, 1, \cdots, 1) - s_i \circ I_X(e)(z), s_i \circ T_X(e)(z)]$$
$$\cup [s_i \circ F_Y(e)(z)), (1, 1, \cdots, 1) - s_i \circ I_Y(e)(z)), s_i \circ T_Y(e)(z))]$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\omega_X(e) \cap \psi_Y(e))^r = [s_i \circ T_X(e)(z), s_i \circ I_X(e)(z), s_i \circ F_X(e)(z)]^r$$
$$\cup [s_i \circ T_Y(e)(z)), s_i \circ I_Y(e)(z)), s_i \circ F_Y(e)(z))]^r$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\omega_X(e) \cap \psi_Y(e))^r = \omega^r_X(e) \cup \psi^r_Y(e)$$
$$\textit{for all } i = 1, 2, \cdots, m;\ e \in X \cap Y \textit{ and } z \in Z$$

$$\Rightarrow (\Omega_X \breve{\cap} \Psi_Y)^r = (\Omega_X)^r \cup_R (\Psi_Y)^r$$

Similarly, all the other De Morgan's properties can be proved in the same way.

4.11. *Exclusion and Contradiction Properties*

The Exclusion and Contradiction Properties among mNS set do not hold, we show it by a counter-example

(i) $\Omega_X \breve{\cup} \Omega^r_X \neq \breve{Z}_X \neq \Omega_X \cup_R \Omega^r_X$
(ii) $\Omega_X \breve{\cup} \Omega^c_X \neq \breve{Z}_X \neq \Omega_X \cup_R \Omega^c_X$
(iii) $\Omega_X \breve{\cap} \Omega^r_X \neq \breve{\Phi}_X \neq \Omega_X \cap_\varepsilon \Omega^r_X$
(iv) $\Omega_X \breve{\cap} \Omega^c_X \neq \breve{\Phi}_X \neq \Omega_X \cap_\varepsilon \Omega^c_X$

Counter-Example 4.2

Let $\Omega_X = \{e_1, (z_1, (0.5, 0.6), (0.3, 0.2), (0.9, 0.3)), (z_2, (0.3, 0.6), (0.4, 0.7), (0.5, 0.8))\}$ be a $2-$NS set over universal set $Z = \{z_1, z_2\}$ with respect to the set of attributes $X \subseteq E$, relative complement of $2-$NS set Ω_X will be

$$\Omega^r_X = \{e_1, (z_1, (0.9, 0.3), (0.7, 0.8), (0.5, 0.6)), (z_2, (0.5, 0.8), (0.6, 0.3), (0.3, 0.6))\}, \text{ then}$$
$$\Omega_X \breve{\cup} \Omega^r_X = \{e_1, (z_1, (0.9, 0.6), (0.7, 0.8), (0.5, 0.3)), (z_2, (0.5, 0.8), (0.6, 0.7), (0.3, 0.6))\} \neq \breve{Z}_X$$
and
$$\Omega_X \breve{\cap} \Omega^r_X = \{e_1, (z_1, (0.5, 0.3), (0.3, 0.2), (0.9, 0.6)), (z_2, (0.3, 0.6), (0.4, 0.3), (0.5, 0.8))\} \neq \breve{\Phi}_X$$

Similarly, others can also be proved by counter-example.

5. **Distances and Similarity Measure**

In this section we define distances and similarity measure formulas for mNS set as follows:

5.1. *Distances*

Definition 4.1 Let $Z = \{z_1, z_2, ..., z_n\}$ be a universal set, $E = \{e_1, e_2, ..., e_q\}$ be a set of attributes and $X, Y \in E$. Let Ω_X, Ψ_Y are two mNS sets over Z with their mN approximate mapping

$$\omega_X(e_j) = \{(z, s_i \circ T_X(e_j)(z_k), s_i \circ I_X(e_j)(z_k), s_i \circ F_X(e_j)(z_k))\}$$
$$\psi_Y(e_j) = \{(z, s_i \circ T_Y(e_j)(z_k), s_i \circ I_Y(e_j)(z_k), s_i \circ F_Y(e_j)(z_k))\}$$
$$\text{for all } i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, q \text{ and } k = 1, 2, \cdots, n$$

respectively, then the distance measure between Ω_X and Ψ_Y is defined as

(1) Hamming distance:

$$d_H(\Omega_X, \Psi_Y) = \frac{1}{3mq}\{\sum_{i=1}^{m}\sum_{j=1}^{q}\sum_{k=1}^{n}(|s_i \circ\ T_X(e_j)(z_k) - s_i \circ\ T_Y(e_j)(z_k)| + |s_i \circ\ I_X(e_j)(z_k) - s_i \circ\ I_Y(e_j)(z_k)| + |s_i \circ\ F_X(e_j)(z_k) - s_i \circ\ F_Y(e_j)(z_k)|)\} \quad (1)$$

(2) Normalized Hamming distance:

$$d_{NH}(\Omega_X, \Psi_Y) = \frac{1}{3mqn}\{\sum_{i=1}^{m}\sum_{j=1}^{q}\sum_{k=1}^{n}(|s_i \circ\ T_X(e_j)(z_k) - s_i \circ\ T_Y(e_j)(z_k)| + |s_i \circ\ I_X(e_j)(z_k) - s_i \circ\ I_Y(e_j)(z_k)| + |s_i \circ\ F_X(e_j)(z_k) - s_i \circ\ F_Y(e_j)(z_k)|)\} \quad (2)$$

(3) Euclidean distance:

$$d_E(\Omega_X, \Psi_Y) = \{\frac{1}{3mq}\sum_{i=1}^{m}\sum_{j=1}^{q}\sum_{k=1}^{n}((s_i \circ\ T_X(e_j)(z_k) - s_i \circ\ T_Y(e_j)(z_k))^2 + (s_i \circ\ I_X(e_j)(z_k) - s_i \circ\ I_Y(e_j)(z_k))^2 + (s_i \circ\ F_X(e_j)(z_k) - s_i \circ\ F_Y(e_j)(z_k))^2)\}^{\frac{1}{2}} \quad (3)$$

(4) Normalized Euclidean distance:

$$d_{NE}(\Omega_X,\Psi_Y)=\{\frac{1}{3mqn}\sum_{i=1}^{m}\sum_{j=1}^{q}\sum_{k=1}^{n}((s_i\circ\ T_X(e_j)(z_k)-s_i\circ\ T_Y(e_j)(z_k))^2$$
$$+(s_i\circ\ I_X(e_j)(z_k)-s_i\circ\ I_Y(e_j)(z_k))^2$$
$$+(s_i\circ\ F_X(e_j)(z_k)-s_i\circ\ F_Y(e_j)(z_k))^2)\}^{\frac{1}{2}} \quad (4)$$

Theorem 5.1 The distance measures between Ω_X and Ψ_Y satisfy the following inequality

(1) $d_H(\Omega_X,\Psi_Y)\leq n$

(2) $d_{NH}(\Omega_X,\Psi_Y)\leq 1$

(3) $d_E(\Omega_X,\Psi_Y)\leq\sqrt{n}$

(4) $d_{NE}(\Omega_X,\Psi_Y)\leq 1$

Theorem 5.2

The distance mappings d_H, d_{NH}, d_E and d_{NE} are defined from $m\mathrm{N}^Z\rightarrow R^+$ are metric

Proof

Let $\Omega_X=(\omega,X),\Psi_Y=(\psi,Y)$ and $\Lambda_W=(\lambda,W)$ be three mNS sets over Z, then

(1) $d_H(\Omega_X,\Psi_Y)\geq 0$

(2) Suppose $d_H(\Omega_X,\Psi_Y)=0$

$$\Longleftrightarrow\frac{1}{3mqn}\{\sum_{i=1}^{m}\sum_{j=1}^{q}\sum_{k=1}^{n}(|s_i\circ\ T_X(e_j)(z_k)-p_i\circ\ T_Y(e_j)(z_k)|$$
$$+|s_i\circ\ I_X(e_j)(z_k)-p_i\circ\ I_Y(e_j)(z_k)|$$
$$+|s_i\circ\ F_X(e_j)(z_k)-p_i\circ\ F_Y(e_j)(z_k)|)\}=0$$
$$\text{for all } i=1,2,\cdots,m;\ j=1,2,\cdots,q \text{ and } k=1,2,\cdots,n$$

$$\Longleftrightarrow|s_i\circ\ T_X(e_j)(z_k)-s_i\circ\ T_Y(e_j)(z_k)|$$
$$+|s_i\circ\ I_X(e_j)(z_k)-s_i\circ\ I_Y(e_j)(z_k)|$$
$$+|s_i\circ\ F_X(e_j)(z_k)-s_i\circ\ F_Y(e_j)(z_k)|=0$$
$$\text{for all } i=1,2,\cdots,m;\ j=1,2,\cdots,q \text{ and } k=1,2,\cdots,n$$

$$\Longleftrightarrow|s_i\circ\ T_X(e_j)(z_k)-s_i\circ\ T_Y(e_j)(z_k)|=0$$
$$+|s_i\circ\ I_X(e_j)(z_k)-s_i\circ\ I_Y(e_j)(z_k)|=0$$
$$+|s_i\circ\ F_X(e_j)(z_k)-s_i\circ\ F_Y(e_j)(z_k)|=0$$
$$\text{for all } i=1,2,\cdots,m;\ j=1,2,\cdots,q \text{ and } k=1,2,\cdots,n$$

$$\iff s_i \circ\ T_X(e_j)(z_k) = s_i \circ\ T_Y(e_j)(z_k),$$
$$s_i \circ\ I_X(e_j)(z_k) = s_i \circ\ I_Y(e_j)(z_k),$$
$$s_i \circ\ F_X(e_j)(z_k) = s_i \circ\ F_Y(e_j)(z_k)$$
$$\text{for all } i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, q \text{ and } k = 1, 2, \cdots, n$$

$$\iff \Omega_X = \Psi_Y$$

(3) $d_H(\Omega_X, \Psi_Y) = d_H(\Psi_Y, \Omega_X)$

(4) For any three mNS sets Ω_X, Ψ_Y and Λ_W

$$|s_i \circ\ T_X(e_j)(z_k) - s_i \circ\ T_Y(e_j)(z_k)|$$
$$+|s_i \circ\ I_X(e_j)(z_k) - s_i \circ\ I_Y(e_j)(z_k)|$$
$$+|s_i \circ\ F_X(e_j)(z_k) - s_i \circ\ F_Y(e_j)(z_k)|$$
$$\text{for all } i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, q \text{ and } k = 1, 2, \cdots, n$$

$$= |s_i \circ\ T_X(e_j)(z_k) - s_i \circ\ T_W(e_j)(z_k) + s_i \circ\ T_W(e_j)(z_k) - s_i \circ\ T_Y(e_j)(z_k)|$$
$$+|s_i \circ\ I_X(e_j)(z_k) - s_i \circ\ I_W(e_j)(z_k) + s_i \circ\ I_W M(e_j)(z_k) - s_i \circ\ I_Y(e_j)(z_k)|$$
$$+|s_i \circ\ F_X(e_j)(z_k) - s_i \circ\ F_W(e_j)(z_k) + s_i \circ\ F_W(e_j)(z_k) - s_i \circ\ F_Y(e_j)(z_k)|$$
$$\text{for all } i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, q \text{ and } k = 1, 2, \cdots, n$$

$$\leq |s_i \circ\ T_X(e_j)(z_k) - s_i \circ\ T_W(e_j)(z_k)| + |s_i \circ\ T_W(e_j)(z_k) - s_i \circ\ T_Y(e_j)(z_k)|$$
$$+|s_i \circ\ I_X(e_j)(z_k) - s_i \circ\ I_W(e_j)(z_k)| + |s_i \circ\ I_W(e_j)(z_k) - s_i \circ\ I_Y(e_j)(z_k)|$$
$$+|s_i \circ\ F_X(e_j)(z_k) - s_i \circ\ F_W(e_j)(z_k)| + |s_i \circ\ F_W(e_j)(z_k) - s_i \circ\ F_Y(e_j)(z_k)|$$
$$\text{for all } i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, q \text{ and } k = 1, 2, \cdots, n$$

$$= \{|s_i \circ\ T_X(e_j)(z_k) - s_i \circ\ T_W(e_j)(z_k)| + |s_i \circ\ I_X(e_j)(z_k) - s_i \circ\ I_W(e_j)(z_k)|$$
$$+|s_i \circ\ F_X(e_j)(z_k) - s_i \circ\ F_W(e_j)(z_k)|\} + \{|s_i \circ\ T_W(e_j)(z_k) - s_i \circ\ T_Y(e_j)(z_k)|$$
$$+|s_i \circ\ I_W(e_j)(z_k) - s_i \circ\ I_Y(e_j)(z_k)| + |s_i \circ\ F_W(e_j)(z_k) - s_i \circ\ F_Y(e_j)(z_k)|\}$$
$$\text{for all } i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, q \text{ and } k = 1, 2, \cdots, n$$

Thus,

$$d_H(\Omega_X, \Psi_Y) \leq d_H(\Omega_X, \Lambda_W) + d_H(\Lambda_W, \Psi_Y)$$

5.2. *Similarity Measure*

Definition 5.2 [16] The SM of Ω_X and Ψ_Y is defined as

$$S(\Omega_X, \Psi_Y) = \frac{1}{1 + d(\Omega_X, \Psi_Y)} \tag{5}$$

where $d(\Omega_X, \Psi_Y)$ is any of the above distance.

5.3. *Similarity of two mN Soft Set*

Definition 5.3 [16] The two mN soft sets Ω_X and Ψ_Y are γ similar if and only if

$$S(\Omega_X, \Psi_Y) \geq \gamma, \text{ i.e.,}$$

$$\Omega_X \approx^{\gamma} \Psi_Y \Leftrightarrow S(\Omega_X, \Psi_Y) \geq \gamma, \gamma \in (0,1) \tag{6}$$

Ω_X and Ψ_Y are significantly similar if $S(\Omega_X, \Psi_Y) \geq 0.5$

Theorem 5.3

The SM of Ω_X and Ψ_Y over Z satisfies the following.

(1) $0 \leq S(\Omega_X, \Psi_Y) \leq 1$

(2) $S(\Omega_X, \Psi_Y) = S(\Psi_Y, \Omega_X)$

(3) $S(\Omega_X, \Psi_Y) = 1 \Leftrightarrow \Omega_X = \Psi_Y$

6. **Application of SM for mN Soft Set**

In this section, we utilize similarity measure for mNS set in two different real-life applications like as in medical diagnosis and decision-making for selection of a lecturer for university.

6.1. *Case Study I*

We use the notion of Similarity Measure to analyze whether the patient has dengue fever or not. An algorithm is given as follows

6.1.1. *Algorithm*

Step 1: Construct set of parameters $E = \{e_1, e_2, \cdots, e_q\}$ as all symptoms of a disease.

Step 2: Construct an mN soft set Ω_X of disease by a medical expert.

Step 3: Construct an mN soft set Ψ_Y by the medical report of the patient.

Step 4: Compute the distance between Ω_X and Ψ_Y by using the distance formula

$$\begin{aligned} d_H(\Omega_X, \Psi_Y) = \frac{1}{3mq}\{\sum_{i=1}^{m}\sum_{j=1}^{q}\sum_{k=1}^{n}(&|s_i \circ\ T_X(e_j)(z_k) - s_i \circ\ T_Y(e_j)(z_k)| \\ &+|s_i \circ\ I_X(e_j)(z_k) - s_i \circ\ I_Y(e_j)(z_k)| \\ &+|s_i \circ\ F_X(e_j)(z_k) - s_i \circ\ F_Y(e_j)(z_k)|)\} \end{aligned}$$

Step 5: Calculate similarity measure between Ω_X and Ψ_Y using formula

$$S(\Omega_X, \Psi_Y) = \frac{1}{1 + d(\Omega_X, \Psi_Y)}$$

Step 6: Analyze the result using similarity.

6.1.2. *Problem Formulation and Assumptions*

The proposed algorithm can be utilized in medical diagnosis problems, here we are giving one numerical example of solution for such medical diagnosis problem in the light of mathematics. This proposed algorithm can be applied for any medical disease diagnosis problems. We consider dengue fever disease as an medical diagnosis problem, wether a considered patient has dengue fever or not, since many of the symptoms of dengue fever are matched with other diseases such as malaria. For specification of disease we applied similarity measure on mNS structure to get insured and accurate results. The m-polar structure gives us data of m medical experts evaluation for particular disease.

6.1.3. *Application of Algorithm*

Now we consider a universal set $Z = \{z_1 = \text{dengue fever}, z_2 = \text{not dengue fever}\}$
We consider set of parameters $E = \{e_1 = \text{High Fever}, e_2 = \text{Bleeding}, e_3 = \text{Severe Pain}\}$ as some of the symptoms of dengue fever disease, where these parameters can be described as,
The patient may have "High Fever" may also suffering from irritability and headache
"Bleeding" from gums or under the skin or from nose
"Severe Pain" in joints or in muscles
Let $X, Y \subseteq E$. Then we construct an $3-$NS set Ω_X with the help of 3 medical expert (doctor) as follows:

Ω_X	z_1	z_2
e_1	(0.69,0.52,0.61),(0.37,0.44,0.23),(0.46,0.37,0.29)	(0.54,0.63,0.55),(0.48,0.44,0.26),(0.63,0.47,0.59)
e_2	(0.43,0.66,0.62),(0.48,0.45,0.53),(0.47,0.52,0.36)	(0.17,0.23,0.29),(0.37,0.41,0.47),(0.53,0.59,0.61)
e_3	(0.34,0.47,0.27),(0.46,0.48,0.37),(0.75,0.58,0.69)	(0.58,0.53,0.55),(0.37,0.35,0.32),(0.65,0.63,0.59)

Table 1: $3-$NS set by 3 medical expert (doctor)

Then we construct a $3-$NS set Ψ_Y by a medical report of the patient as follows:

Ψ_Y	z_1	z_2
e_1	(0.63,0.57,0.54),(0.47,0.46,0.32),(0.62,0.75,0.67)	(0.45,0.71,0.50),(0.50,0.43,0.26),(0.61,0.50,0.47)
e_2	(0.47,0.59,0.69),(0.53,0.50,0.60),(0.43,0.58,0.32)	(0.15,0.25,0.25),(0.32,0.40,0.43),(0.53,0.60,0.60)
e_3	(0.27,0.38,0.24),(0.58,0.37,0.47),(0.65,0.69,0.70)	(0.47,0.46,0.64),(0.44,0.40,0.30),(0.61,0.60,0.68)

Table 2: $3-$NS set by a medical report of a patient

Computing distances between Ω_X and Ψ_Y and the results are

$$d_H(\Omega_X, \Psi_Y) = 0.1381$$
$$d_{NH}(\Omega_X, \Psi_Y) = 0.0690$$
$$d_E(\Omega_X, \Psi_Y) = 0.0195$$
$$d_{NE}(\Omega_X, \Psi_Y) = 0.0097$$

Using Eculidean distance to calculate similarity measure of Ω_X and Ψ_Y and result is as follows

$$S(\Omega_X, \Psi_Y) = 0.98 \geq 0.5$$

Since $S(\Omega_X, \Psi_Y)$ is greater than 0.5, i.e. the similarity measure of two $3-$NS sets is significantly similar, this implies a patient is suffering from dengue fever.

6.2. *Case Study II*

Here, we generate an example of selecting lecturer for university after seeing candidate's interview reports. An algorithm is given as follows;

6.2.1. *Algorithm*

Step 1: Construct a set of attribute of selection purpose as $E = \{e_1, e_2, \cdots, e_q\}$
Step 2: Construct an mNS set Ω_X as the requirements of a firm concluded by decision-making team.
Step 3: Construct t mNS sets Ψ_Y^h by the help of evaluation of different alternatives given by decision-making team, where $h = 1, 2, \cdots, t$
Step 4: Compute the distance between Ω_X and Ψ_Y^h by using the distance formula

$$d_E(\Omega_X, \Psi_Y) = \{\frac{1}{3mq}\sum_{i=1}^{m}\sum_{j=1}^{q}\sum_{k=1}^{n}((s_i \circ\ T_X(e_j)(z_k) - s_i \circ\ T_Y(e_j)(z_k))^2 + (s_i \circ\ I_X(e_j)(z_k) - s_i \circ\ I_Y(e_j)(z_k))^2 + (s_i \circ\ F_X(e_j)(z_k) - s_i \circ\ F_Y(e_j)(z_k))^2)\}^{\frac{1}{2}}$$

Step 5: Calculate the similarity measure between Ω_X and Ψ_Y^h using formula

$$S(\Omega_X, \Psi_Y^h) = \frac{1}{1 + d(\Omega_X, \Psi_Y^h)}$$

Step 6: Analyze the result using similarity that which alternative is more suitable to be select as a lecturer.

6.2.2. *Problem Formulation and Assumption*

A Similarity measure of mNS sets can help in decision-making problem of selection of best alternative corresponds to attribute of selection purpose. It can be applicable in problems of group decision making where a group of people gives their own evaluation to all alternatives corresponds to attribute and wants an alternative to be selected which fulfills or near to the evaluation that was given individually by them. We consider one example of such type of problems that is department of mathematics wants to hire a new lecturer for university. The new lecturer must well aware of both Pure and Applied Mathematics. They made a decision-making team of three-person for the selection of a lecturer. All member of team first evaluate the requirements of a department for selecting the purpose of a new lecturer individually, then they took interviews and demo classes of four applicants who are willing to be a lecturer in that university and made reports of every applicant for decision-making process.

6.2.3. *Application of Algorithm*

Consider a universal set $Z = \{z_1 = \text{Pure Math}, z_2 = \text{Applied Math}\}$ and set of attributes for the selection purpose as $E = \{e_1 = \text{Teaching Techniques}, e_2 = \text{Research Work}, e_3 = \text{Expertise}\}$. Let $X = Y \subseteq E$, then we construct a $3-$NS set Ω_X as requirements of a department.

Ω_X	z_1	z_2
e_1	(0.82,0.55,0.63),(0.55,0.46,0.28),(0.43,0.38,0.60)	(0.50,0.62,0.52),(0.93,0.57,0.80),(0.66,0.48,0.52)
e_2	(0.43,0.68,0.86),(0.47,0.67,0.56),(0.42,0.51,0.33)	(0.77,0.54,0.82),(0.75,0.54,0.72),(0.53,0.54,0.69)
e_3	(0.73,0.48,0.53),(0.87,0.43,0.77),(0.76,0.53,0.62)	(0.64,0.48,0.59),(0.32,0.58,0.22),(0.94,0.64,0.62)

Table 3: $3-$NS set of requirement of a department

Now we will construct four $3-$NS sets $\Psi_Y^1, \Psi_Y^2, \Psi_Y^3$ and Ψ_Y^4 of different applicants A_1 A_2 A_3 and A_4 respectively with the help of reports made by decision-making team.

Ψ_Y^1	z_1	z_2
e_1	(0.13,0.15,0.22),(0.89,0.78,0.83),(0.77,0.82,0.91)	(0.79,0.84,0.93),(0.36,0.18,0.26),(0.21,0.24,0.16)
e_2	(0.07,0.23,0.32),(0.12,0.18,0.20),(0.74,0.79,0.88)	(0.23,0.13,0.22),(0.31,0.25,0.43),(0.19,0.22,0.27)
e_3	(0.23,0.12,0.17),(0.25,0.16,0.22),(0.14,0.16,0.18)	(0.10,0.13,0.11),(0.91,0.84,0.69),(0.31,0.30,0.28)

Table 4: $3-$NS set a report of Applicant 1 (A_1)

Ψ_Y^2	z_1	z_2
e_1	(0.16,0.20,0.27),(0.83,0.87,0.89),(0.70,0.75,0.86)	(0.88,0.81,0.90),(0.40,0.20,0.26),(0.22,0.27,0.17)
e_2	(0.13,0.21,0.24),(0.18,0.20,0.20),(0.70,0.84,0.90)	(0.15,0.16,0.25),(0.32,0.33,0.43),(0.22,0.25,0.30)
e_3	(0.20,0.16,0.27),(0.29,0.17,0.26),(0.14,0.15,0.12)	(0.16,0.17,0.14),(0.85,0.84,0.70),(0.30,0.30,0.30)

Table 5: $3-$NS set a report of Applicant 2 (A_2)

Ψ_Y^3	z_1	z_2
e_1	(0.76,0.59,0.56),(0.47,0.52,0.33),(0.52,0.45,0.67)	(0.45,0.70,0.56),(0.90,0.50,0.85),(0.61,0.42,0.47)
e_2	(0.47,0.59,0.89),(0.53,0.60,0.60),(0.46,0.58,0.30)	(0.70,0.59,0.76),(0.70,0.50,0.68),(0.56,0.60,0.62)
e_3	(0.77,0.40,0.48),(0.83,0.37,0.74),(0.66,0.62,0.70)	(0.72,0.45,0.64),(0.39,0.50,0.31),(0.85,0.60,0.68)

Table 6: $3-$NS set a report of Applicant 3 (A_3)

Ψ_Y^4	z_1	z_2
e_1	(0.40,0.32,0.33),(0.30,0.29,0.40),(0.24,0.67,0.83)	(0.28,0.45,0.73),(0.64,0.86,0.49),(0.48,0.70,0.80)
e_2	(0.63,0.40,0.59),(0.69,0.44,0.30),(0.70,0.70,0.60)	(0.55,0.31,0.60),(0.52,0.76,0.58),(0.72,0.79,0.90)
e_3	(0.49,0.28,0.34),(0.58,0.63,0.50),(0.60,0.69,0.79)	(0.47,0.65,0.80),(0.56,0.37,0.47),(0.61,0.76,0.80)

Table 7: $3-$NS set a report of Applicant 4 (A_4)

The Euclidean distance between Ω_X and Ψ_Y^h is calculated as

$$d_E(\Omega_X, \Psi_Y^1) = 0.3798$$
$$d_E(\Omega_X, \Psi_Y^2) = 0.3610$$

$$d_E(\Omega_X, \Psi_Y^3) = 0.0076$$
$$d_E(\Omega_X, \Psi_Y^4) = 0.1087$$

The Similarity Measure of Ω_X and Ψ_Y^h is calculated as

$$S(\Omega_X, \Psi_Y^1) = 0.72$$
$$S(\Omega_X, \Psi_Y^2) = 0.73$$
$$S(\Omega_X, \Psi_Y^3) = 0.99$$
$$S(\Omega_X, \Psi_Y^4) = 0.90$$

Since, similarity measure of $S(\Omega_X, \Psi_Y^3) = 0.99$ is greater than 0.5 and greater than all others, the two $3-$NS sets are significantly similar, which shows applicant 3 (A_3) is more suitable and he also fulfills requirements of the university.

7. Conclusion

The existing multi-polar information is not completely defined by using the existing methods. now multi-polar neutrosophic models explain the things in a better way to solve the undetermined data having multi-polar information and having the vast applications in different fields. similarity measures based on distance play an important role to solve the problems that have indeterminacy. In this paper, we defined some basic operations and their properties on mN soft sets. Moreover, we have defined the distance-based similarity measure of multi-polar neutrosophic soft sets. We have used the concept of distance-based similarity measures in medical diagnosis and decision-making of the selection of lecturers for university with along algorithms. Moreover, we defined some basic operations and their properties on mN soft sets. In the future, Group MCDM problems can be solved using different methods of MCDM (TOPSIS, VIKOR, etc).

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Introduction to Plithogenic Hypersoft Subgroup

Sudipta Gayen[1], Florentin Smarandache[2], Sripati Jha[1], Manoranjan Kumar Singh[3], Said Broumi[4] and Ranjan Kumar[5,*]

[1] National Institute of Technology Jamshedpur, India; sudi23dipta@gmail.com

[2] University of New Mexico, USA

[3] Magadh University, Bodh Gaya, India

[4] Faculty of Science Ben MSik, University Hassan II, Morocco

[5] Jain Deemed to be University, Jayanagar, Bengaluru, India; ranjank.nit52@gmail.com

[*] Correspondence: ranjank.nit52@gmail.com

Abstract. In this article, some essential aspects of plithogenic hypersoft algebraic structures have been analyzed. Here the notions of plithogenic hypersoft subgroups i.e. plithogenic fuzzy hypersoft subgroup, plithogenic intuitionistic fuzzy hypersoft subgroup, plithogenic neutrosophic hypersoft subgroup have been introduced and studied. For doing that we have redefined the notions of plithogenic crisp hypersoft set, plithogenic fuzzy hypersoft set, plithogenic intuitionistic fuzzy hypersoft set, and plithogenic neutrosophic hypersoft set and also given their graphical illustrations. Furthermore, by introducing function in different plithogenic hypersoft environments, some homomorphic properties of plithogenic hypersoft subgroups have been analyzed.

Keywords: Hypersoft set; Plithogenic set; Plithogenic hypersoft set; Plithogenic hypersoft subgroup

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A LIST OF ABBREVIATIONS

US signifies universal set.

CS signifies crisp set.

FS signifies fuzzy set.

IFS signifies intuitionistic fuzzy set.

NS signifies neutrosophic set.

PS signifies plithogenic set.

SS signifies soft set.

HS signifies hypersoft set.

CHS signifies crisp hypersoft set.

FHS signifies fuzzy hypersoft set.

IFHS signifies intuitionistic fuzzy hypersoft set.

NHS signifies neutrosophic hypersoft set.

PHS signifies plithogenic hypersoft set.

PCHS signifies plithogenic crisp hypersoft set.

PFHS signifies plithogenic fuzzy hypersoft set.

PIFHS signifies plithogenic intuitionistic fuzzy hypersoft set.

PNHS signifies plithogenic neutrosophic hypersoft set.

CG signifies crisp group.

FSG signifies fuzzy subgroup.

IFSG signifies intuitionistic fuzzy subgroup.

NSG signifies neutrosophic subgroup.

DAF signifies degree of appurtenance function.

DCF signifies degree of contradiction function.

PSG signifies plithogenic subgroup.

PCHSG signifies plithogenic crsip hypersoft subgroup.

PFHSG signifies plithogenic fuzzy hypersoft subgroup.

PIFHSG signifies plithogenic intuitionistic fuzzy hypersoft subgroup.

PNHSG signifies plithogenic neutrosophic hypersoft subgroup.

DMP signifies decision making problem.

$\boldsymbol{\rho(U)}$ signifies power set of U.

1. Introduction

FS [1] theory was first initiated by Zadeh to handle uncertain real-life situations more precisely than CSs. Gradually, some other set theories like IFS [2], NS [3], Pythagorean FS [4], PS [5], etc., have emerged. These sets are able to handle ambiguous situations more appropriately than FSs. NS theory was introduced by Smarandache which was generalizations of IFS and FS. He has also introduced neutrosophic probability, measure [6,7] , psychology [8], pre-calculus and calculus [9], etc. Presently, NS theory is vastly used in various pure as well as applied fields. For instance, in medical diagnosis [10, 11], shortest path problem [12–20], DMP [21–26], transportation problem [27, 28], forecasting [29], mobile edge computing [30], abstract algebra [31], pattern recognition problem [32], image segmentation [33], internet of things [34], etc. Another set theory of profound importance is PS theory which is extensively used in handling various uncertain situations. This set theory is more general than CS, FS, IFS, and NS theory. Gradually, plithogenic probability and statistics [35], plithogenic logic [35], etc., have evolved which are generalizations of crisp probability, statistics, and logic. Smarandache has also introduced the notions of plithogenic number, plithogenic measure function, bipolar PS, tripolar PS, multipolar PS, complex PS, refined PS, etc. Presently, PS theory is extensively used in numerous research domains.

The notion of SS [36] theory is another fundamental set theory. Presently, SS theory has become one of the most popular branches in mathematics for its huge areas of applications in various research fields. For instance, nowadays in DMP [37], abstract algebra [38–40], etc., it is widely used. Again, there exist concepts like vague sets [41,42], rough set [43], hard set [44], etc., which are well known for their vast applications in various domains. Gradually, based on SS theory the notions of fuzzy SS [45], intuitionistic SS [46], neutrosophic SS [47] theory, etc., have been introduced by various researchers. In fuzzy abstract algebra, the notions of FSG [48], IFSG [49], NSG [31], etc., have been developed and studied by different mathematicians. SS theory has opened some new windows of opportunities for researchers working not only in applied fields but also in pure fields. As a result, the notions of soft FSG [39], soft IFSG [50], soft NSG [51], etc., were introduced. Later on, Smarandache has proposed the concept of HS [52] theory which is a generalization of SS theory. Also, he has extended and introduced the concept of HS in the plithogenic environment and generalized that further. As a result, a new branch has emerged which can be a fruitful research field for its promising potentials. The following Table 1 contains some significant contributions in SS and PS theory by numerous researchers.

Table 1. Significance and influences of PS & SS theory in various fields.

Author & references	**Year**	**Contributions in various fields**
Majhi et al. [53]	2002	Applied SS theory in a DMP.
Feng et al. [54]	2010	Described an adjustable approach to fuzzy SS based DMP with some examples.
aman [55]	2011	Defined fuzzy soft aggregation operator which allows the construction of more efficient DMP.
Broumi et al. [56]	2014	Defined neutrosophic parameterized SS and neutrosophic parameterized aggregation operator and applied it in DMP.
Broumi et al. [57]	2014	Defined interval-valued neutrosophic parameterized SS a reduction method for it.
Deli et al. [58]	2014	Introduced neutrosophic soft multi-set theory and studied some of its properties.
Deli & Naim [59]	2015	Introduced intuitionistic fuzzy parameterized SS and studied some of its properties.
Smarandache [60]	2018	Introduced physical PS.
Smarandache [61]	2018	Studied aggregation plithogenic operators in physical fields.

continued . . .

Author & references	Year	Contributions in various fields
Gayen et al. [62]	2019	Introduced the notions of plithogenic subgroups and studied some of their homomorphic properties.
Abdel-Basset et al. [63]	2019	Described a novel model for evaluation of hospital medical care systems based on PSs.
Abdel-Basset et al. [64]	2019	Described a novel plithogenic TOPSIS-CRITIC model for sustainable supply chain risk management.
Abdel-Basset et al. [65]	2019	Proposed a hybrid plithogenic decision-making approach with quality function deployment.

This Chapter has been systematized as the following: In Section 2, literature reviews of FS, IFS, NS, FSG, IFSG, NSG, PS, PHS, etc., are mentioned. In Section 3, the concepts of PCHS, PFHS, PIFHS, and PNHS have been redefined in a different way and their graphical illustrations have been given. Also, the notions of PFHSG, PIFHSG, and PNHSG have been introduced and further the effects of homomorphism on those notions are studied. Finally, in Section 4, the conclusion is given mentioning some scopes of future researches.

2. Literature Survey

In this segment, some important notions like, FS, IFS, NS, FSG, IFSG, NSG, etc., have been discussed. We have also mentioned PS, SS, HS and some aspects of PHS. These notions will play vital roles in developing the concepts of PHSGs.

Definition 2.1. [1] Let U be a CS. A function $\sigma : U \to [0,1]$ is called a FS.

Definition 2.2. [2] Let U be a CS. An IFS γ of U is written as $\gamma = \{(m, t_\gamma(m), f_\gamma(m)) : m \in U\}$, where $t_\gamma(m)$ and $f_\gamma(m)$ are two FSs of U, which are called the degree of membership and non-membership of any $m \in U$. Here $\forall m \in U$, $t_\gamma(m)$ and $f_\gamma(m)$ satisfy the inequality $0 \leq t_\gamma(m) + f_\gamma(m) \leq 1$.

Definition 2.3. [3] Let U be a CS. A NS η of U is denoted as $\eta = \{(m, t_\eta(m), i_\eta(m), f_\eta(m)) : m \in U\}$, where $t_\eta(m), i_\eta(m), f_\eta(m) : U \to]^-0, 1^+[$ are the corresponding degree of truth, indeterminacy, and falsity of any $m \in U$. Here $\forall m \in U$ $t_\eta(m)$, $i_\eta(m)$ and $f_\eta(m)$ satisfy the inequality $^-0 \leq t_\eta(m) + i_\eta(m) + f_\eta(m) \leq 3^+$.

2.1. *Fuzzy, Intuitionistic fuzzy & Neutrosophic subgroup*

Definition 2.4. [48] A FS of a CG U is called as a FSG iff $\forall m, u \in U$, the conditions mentioned below are satisfied:

(i) $\alpha(mu) \geq min\{\alpha(m), \alpha(u)\}$

(ii) $\alpha(m^{-1}) \geq \alpha(m)$.

Definition 2.5. [49] An IFS $\gamma = \{(m, t_\gamma(m), f_\gamma(m)) : m \in U\}$ of a CG U is called an IFSG iff $\forall m, u \in U$,

(i) $t_\gamma(mu^{-1}) \geq min\{t_\gamma(m), t_\gamma(u)\}$
(ii) $f_\gamma(mu^{-1}) \leq \max\{f_\gamma(m), f_\gamma(u)\}$.

The set of all the IFSG of U will be denoted as IFSG(U).

Definition 2.6. [31] Let U be a CG and δ be a NS of U. δ is called a NSG of U iff the conditions mentioned below are satisfied:

(i) $\delta(mu) \geq \min\{\delta(m), \delta(u)\}$, i.e. $t_\delta(mu) \geq \min\{t_\delta(m), t_\delta(u)\}$, $i_\delta(mu) \geq \min\{i_\delta(m), i_\delta(u)\}$ and $f_\delta(mu) \leq \max\{f_\delta(m), f_\delta(u)\}$
(ii) $\delta(m^{-1}) \geq \delta(m)$ i.e. $t_\delta(m^{-1}) \geq t_\delta(u)$, $i_\delta(m^{-1}) \geq i_\delta(u)$ and $f_\delta(m^{-1}) \leq f_\delta(u)$.

Theorem 2.1. *[66] Let g be a homomorphism of a CG U_1 into another CG U_2. Then preimage of an IFSG γ of U_2 i.e. $g^{-1}(\gamma)$ is an IFSG of U_1.*

Theorem 2.2. *[66] Let g be a surjective homomorphism of a CG U_1 to another CG U_2. Then the image of an IFSG γ of U_1 i.e. $g(\gamma)$ is an IFSG of U_2.*

Theorem 2.3. *[31] The homomorphic image of any NSG is a NSG.*

Theorem 2.4. *[31] The homomorphic preimage of any NSG is a NSG.*

Some more references in the domains of FSG, IFSG, NSG, etc., which can be helpful to various other researchers are [67–71].

2.2. *Plithogenic set & Plithogenic hypersoft set*

Definition 2.7. [5] Let U be a US and $P \subseteq U$. A PS is denoted as $P_s = (P, \psi, V_\psi, a, c)$, where ψ be an attribute, V_ψ is the respective range of attributes values, $a : P \times V_\psi \to [0,1]^s$ is the DAF and $c : V_\psi \times V_\psi \to [0,1]^t$ is the corresponding DCF. Here $s, t \in \{1, 2, 3\}$.

In Definition 2.7, for $s = 1$ and $t = 1$ a will become a FDAF and c will become a FDCF. In general, we consider only FDAF and FDCF. Also, $\forall (u_i, u_j) \in V_\psi \times V_\psi$, c satisfies $c(u_i, u_i) = 0$ and $c(u_i, u_j) = c(u_j, u_i)$.

Definition 2.8. [36] Let U be a US, V_A be a set of attribute values. Then the ordered pair (Γ, U) is called a SS over U, where $\Gamma : V_A \to \rho(U)$.

Definition 2.9. [52] Let U be a US. Let $r_1, r_2, ..., r_n$ be n attributes and corresponding attribute value sets are respectively $D_1, D_2, ..., D_n$ (where $D_i \cap D_j = \phi$, for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$). Let $V_\psi = D_1 \times D_2 \times \cdots \times D_n$. Then the ordered pair (Γ, V_ψ) is called a HS of U, where $\Gamma : V_\psi \to \rho(U)$.

Definition 2.10. [72] A US U_C is termed as a crisp US if $\forall u \in U_C$, u fully belongs to U_C i.e. membership of u is 1.

Definition 2.11. [72] A US U_F is termed as a fuzzy US if $\forall u \in U_F$, u partially belongs to U_F i.e. membership of u belonging to $[0, 1]$.

Definition 2.12. [72] A US U_{IF} is termed as an intuitionistic fuzzy US if $\forall u \in U_{IF}$, u partially belongs to U_{IF} and also partially does not belong to U_{IF} i.e. membership of u belonging to $[0, 1] \times [0, 1]$.

Definition 2.13. [72] A US U_N is termed as an neutrosophic US if $\forall u \in U_N$, u has truth belongingness, indeterminacy belongingness, and falsity belongingness to U_N i.e. membership of u belonging to $[0, 1] \times [0, 1] \times [0, 1]$.

Definition 2.14. [72] A US U_P over an attribute value set ψ is termed as a plithogenic US if $\forall u \in U_P$, u belongs to U_P with some degree on the basis of each attribute value. This degree can be crisp, fuzzy, intuitionistic fuzzy, or neutrosophic.

Definition 2.15. [52] Let U_C be a crisp US and $\psi = \{r_1, r_2, ..., r_n\}$ be a set of n attributes with attribute value sets respectively as $D_1, D_2, ..., D_n$ (where $D_i \cap D_j = \phi$ for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$). Also, let $V_\psi = D_1 \times D_2 \times \cdots \times D_n$. Then (Γ, V_ψ), where $\Gamma : V_\psi \to \rho(U_C)$ is termed as a CHS over U_C.

Definition 2.16. [52] Let U_F be a fuzzy US and $\psi = \{r_1, r_2, ..., r_n\}$ be a set of n attributes with attribute value sets respectively as $D_1, D_2, ..., D_n$ (where $D_i \cap D_j = \phi$ for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$). Also, let $V_\psi = D_1 \times D_2 \times \cdots \times D_n$. Then (Γ, V_ψ), where $\Gamma : V_\psi \to \rho(U_F)$ is called a FHS over U_F.

Definition 2.17. [52] Let U_{IF} be an intuitionistic fuzzy US and $\psi = \{r_1, r_2, ..., r_n\}$ be a set of n attributes with attribute value sets respectively as $D_1, D_2, ..., D_n$ (where $D_i \cap D_j = \phi$ for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$). Also, let $V_\psi = D_1 \times D_2 \times \cdots \times D_n$. Then (Γ, V_ψ), where $\Gamma : V_\psi \to \rho(U_{IF})$ is called an IFHS over U_{IF}.

Definition 2.18. [52] Let U_N be a neutrosophic US and $\psi = \{r_1, r_2, ..., r_n\}$ be a set of n attributes with attribute value sets respectively as $D_1, D_2, ..., D_n$ (where $D_i \cap D_j = \phi$ for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$). Also, let $V_\psi = D_1 \times D_2 \times \cdots \times D_n$. Then (Γ, V_ψ), where $\Gamma : V_\psi \to \rho(U_N)$ is called a NHS over U_N.

Definition 2.19. [52] Let U_P be a plithogenic US and $\psi = \{r_1, r_2, ..., r_n\}$ be a set of n attributes with attribute value sets respectively as $D_1, D_2, ..., D_n$ (where $D_i \cap D_j = \phi$ for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$). Also, let $V_\psi = D_1 \times D_2 \times \cdots \times D_n$. Then (Γ, V_ψ), where $\Gamma : V_\psi \to \rho(U_P)$ is called a PHS over U_P.

Further, depending on someones preferences PHS can be categorized as PCHS, PFHS, PIFHS, and PNHS. In [52], Smarandache has wonderfully introduced and illustrated these categories with proper examples.

In the next section, we have mentioned an equivalent statement of Definition 2.19 and described its categories in a different way. Also, we have given some graphical representations of PCHS, PFHS, PIFHS, and PNHS. Again, we have introduced functions in the environments of PFHS, PIFHS, and PNHS. Furthermore, we have introduced the notions of PFHSG, PIFHSG, and PNHSG and studied their homomorphic characteristics.

3. Proposed Notions

As an equivalent statement to Definition 2.19, we can conclude that $\forall M \in$ range(Γ) and $\forall i \in \{1, 2, ..., n\}$, $\exists a_i : M \times D_i \to [0, 1]^s$ $(s = 1, 2$ or $3)$ such that $\forall (m, d) \in M \times D_i,\ a_i(m, d)$ represent the DAFs of m to the set M on the basis of the attribute value d. Then the pair (Γ, V_ψ) is called a PHS.

So, based on someones requirement one may choose $s = 1, 2$ or 3 and further, depending on these choices PHS can be categorized as PFHS, PIFHS, and PNHS. Also, by defining DAF as $a_i : M \times D_i \to \{0, 1\}$, the notion of PCHS can be introduced. The followings are those aforementioned notions:

Let $\psi = \{r_1, r_2, ..., r_n\}$ be a set of n attributes and corresponding attribute value sets are respectively $D_1, D_2, ..., D_n$ (where $D_i \cap D_j = \phi$, for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$). Let $V_\psi = D_1 \times D_2 \times \cdots \times D_n$ and (Γ, V_ψ) be a HS over U, where $\Gamma : V_\psi \to \rho(U)$.

Definition 3.1. The pair (Γ, V_ψ) is called a PCHS if $\forall M \in$ range(Γ) and $\forall i \in \{1, 2, ..., n\}$ $\exists a_{C_i} : M \times D_i \to \{0, 1\}$ such that $\forall (m, d) \in M \times D_i,\ a_{C_i}(m, d) = 1$.

A set of all the PCHSs over a set U will be denoted as PCHS(U).

Example 3.2. Let a balloon seller has a set $U = \{b_1, b_2, ..., b_{20}\}$ of a total of 20 balloons some which are of different size, color, and cost. Also, let for the aforementioned attributes corresponding attribute value sets are $D_1 =$ {small, medium, large}, $D_2 =$ {red, orange, blue} and $D_3 =$ {small, medium, large}. Let a person is willing to buy some balloons having the attributes as big, red and expensive. Lets assume (Γ, V_ψ) be a HS over U, where $\Gamma : V_\psi \to \rho(U)$ and $V_\psi = D_1 \times D_2 \times D_3$. Also, let Γ(big, red, expensive) $= \{b_3, b_{10}, b_{12}\}$.

Then corresponding PCHS will be $\Gamma(\text{big, red, expensive}) = \{b_3(1,1,1), b_{10}(1,1,1), b_{12}(1,1,1)\}$. Its graphical representation is shown in Figure 1.

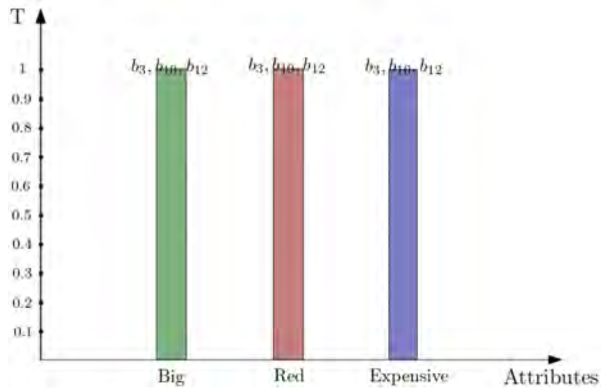

FIGURE 1. PCHS according to Example 3.2

Definition 3.3. The pair (Γ, V_ψ) is called a PFHS if $\forall M \in \text{range}(\Gamma)$ and $\forall i \in \{1,2,...,n\}$, $\exists a_{F_i} : M \times D_i \to [0,1]$ such that $\forall (m,d) \in M \times D_i,\ a_{F_i}(m,d) \in [0,1]$.

A set of all the PFHSs over a set U will be denoted as PFHS(U).

Example 3.4. In Example 3.2 let corresponding PFHS is $\Gamma(\text{big, red, expensive}) = \{b_3(0.75, 0.3, 0.8), b_{10}(0.45, 0.57, 0.2), b_{12}(0.15, 0.57, 0.95)\}$. Its graphical representation is shown in Figure 2.

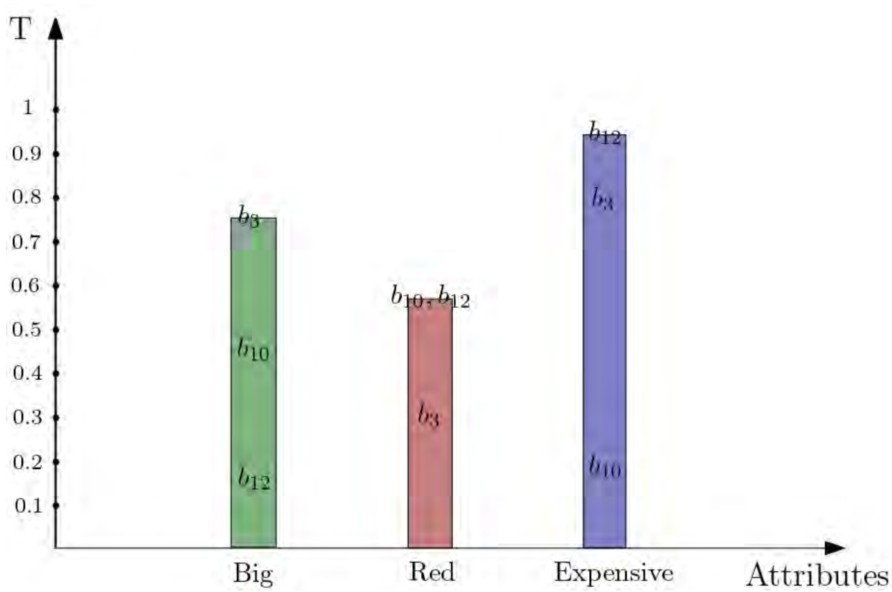

FIGURE 2. PFHS according to Example 3.4

Definition 3.5. The pair (Γ, V_ψ) is called a PIFHS if $\forall M \in \text{range}(\Gamma)$ and $\forall i \in \{1, 2, ..., n\}$, $\exists a_{IF_i} : M \times D_i \rightarrow [0,1] \times [0,1]$ such that $\forall (m,d) \in M \times D_i, a_{F_i}(m,d) \in [0,1] \times [0,1]$.

A set of all the PIFHSs over a set U will be denoted as PIFHS(U).

Example 3.6. In Example 3.2 let corresponding PIFHS is

$$\Gamma(\text{big, red, expensive}) = \begin{Bmatrix} b_3(0.87, 0.52, 0.66), b_{10}(0.6, 0.52, 0.2), b_{12}(0.33, 0.2, 0.83) \\ b_3(0.3, 0.4, 0.72), b_{10}(0.5, 0.19, 0.98), b_{12}(1, 0.72, 0.3) \end{Bmatrix}.$$

Its graphical representation is shown in Figure 3.

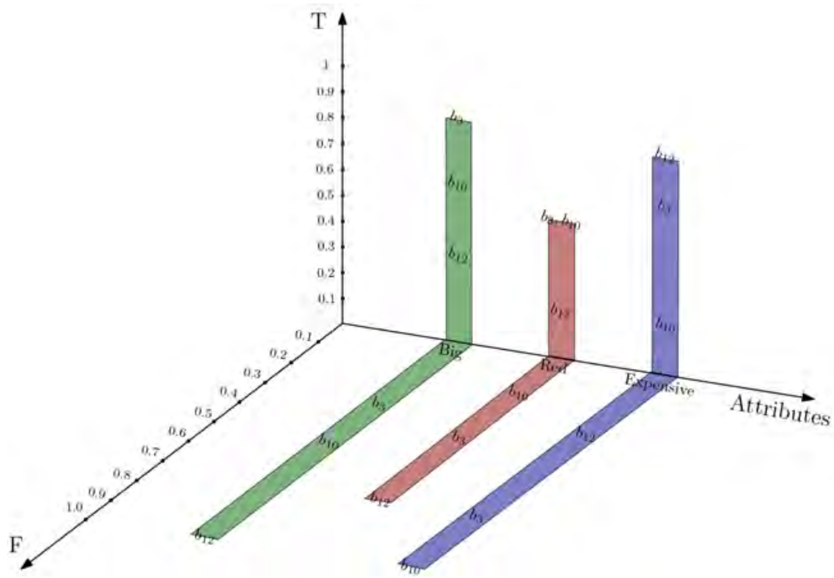

FIGURE 3. PIFHS according to Example 3.6

Definition 3.7. The pair (Γ, V_ψ) is called a PNHS if $\forall M \in \text{range}(\Gamma)$ and $\forall i \in \{1, 2, ..., n\}$, $\exists a_{N_i} : M \times D_i \rightarrow [0,1] \times [0,1] \times [0,1]$ such that $\forall (m,d) \in M \times D_i,\ a_{N_i}(m,d) \in [0,1] \times [0,1] \times [0,1]$.

A set of all the PNHSs over a set U will be denoted as PNHS(U).

Example 3.8. In Example 3.2 let corresponding PNHS is

$$\Gamma(\text{big, red, expensive}) = \begin{Bmatrix} b_3(0.87, 1, 0.66), b_{10}(0.61, 0.25, 0.2), b_{12}(0.32, 0.7, 0.83) \\ b_3(0.15, 0.72, 0.47), b_{10}(0.77, 0.4, 0.48), b_{12}(0.37, 0.18, 0.2) \\ b_3(0.76, 0.17, 0.29), b_{10}(0.5, 0.71, 0.98), b_{12}(1, 0.35, 0.67) \end{Bmatrix}.$$

Its graphical representation is shown in Figure 4.

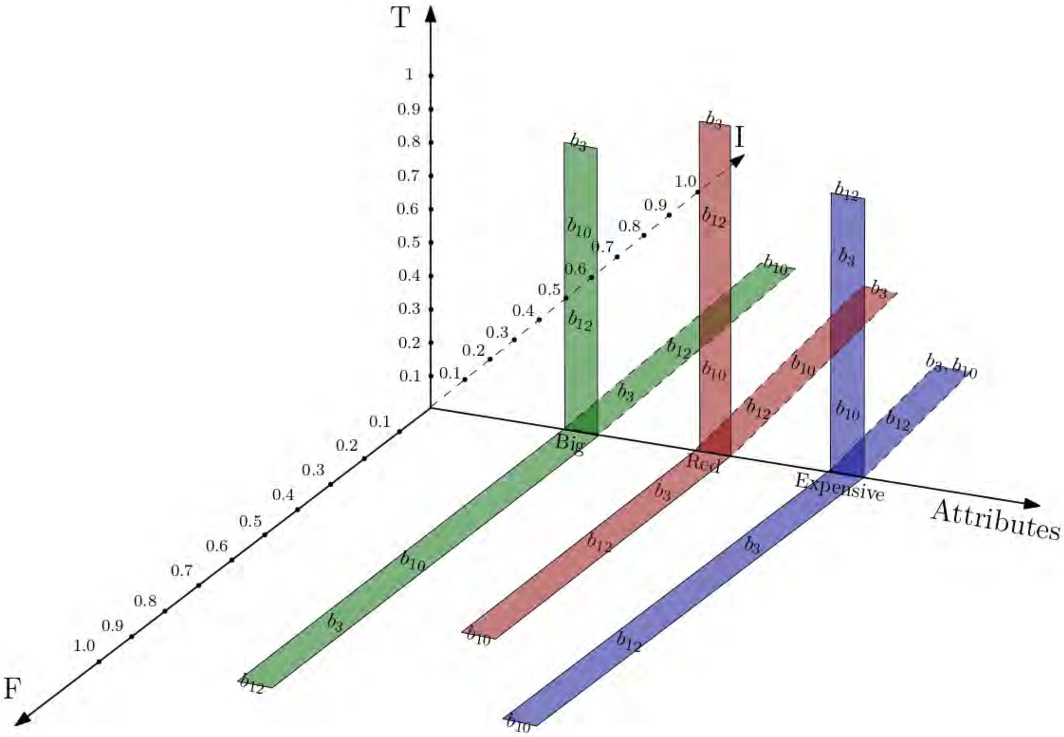

FIGURE 4. PNHS according to Example 3.8

3.1. *Images & Preimages of PFHS, PIFHS & PNHS under a function*

Let U_1 and U_2 be two CSs and $\forall i, j \in \{1, 2, ..., n\}$, D_i and P_j are attribute value sets consisting of some attribute values. Again, let $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are some functions. Then the followings can be defined:

Definition 3.9. Let $(\Gamma_1, V_\psi^1) \in \text{PFHS}(U_1)$ and $(\Gamma_2, V_\psi^2) \in \text{PFHS}(U_2)$, where $V_\psi^1 = D_1 \times D_2 \times \cdots \times D_n$ and $V_\psi^2 = P_1 \times P_2 \times \cdots \times P_n$. Also, let $\forall M \in \text{range}(\Gamma_1)$, $a_{F_i} : M \times D_i \to [0, 1]$ are the corresponding FDAFs. Again, let $\forall N \in \text{range}(\Gamma_2)$, $b_{F_j} : N \times P_j \to [0, 1]$ are the corresponding FDAFs. Then the images of (Γ_1, V_ψ^1) under the functions $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are PFHS over U_2 and they are denoted as $g_{ij}(\Gamma_1, V_\psi^1)$, where the corresponding FDAFs are defined as:

$$g_{ij}(a_{F_i})(n, p) = \begin{cases} \max a_{F_i}(m, d) \text{ if } (m, d) \in g_{ij}^{-1}(n, p) \\ 0 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

The preimages of (Γ_2, V_ψ^2) under the functions $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are PFHSs over U_1, which are denoted as $g_{ij}^{-1}(\Gamma_2, V_\psi^2)$ and the corresponding FDAFs are defined as $g_{ij}^{-1}(b_{F_j})(m, d) = b_{F_j}(g_{ij}(m, d))$.

Definition 3.10. Let $(\Gamma_1, V_\psi^1) \in \text{PIFHS}(U_1)$ and $(\Gamma_2, V_\psi^2) \in \text{PIFHS}(U_2)$, where $V_\psi^1 = D_1 \times D_2 \times \cdots \times D_n$ and $V_\psi^2 = P_1 \times P_2 \times \cdots \times P_n$. Also, let $\forall M \in \text{range}(\Gamma_1)$, a_{IF_i} :

$M \times D_i \to [0,1] \times [0,1]$ with $a_{IF_i}(m,d) = \{((m,d), a^T_{IF_i}(m,d), a^F_{IF_i}(m,d)) : (m,d) \in M \times D_i\}$ are the corresponding IFDAFs. Again, let $\forall N \in \text{range}(\Gamma_2)$, $b_{IF_j} : N \times P_j \to [0,1] \times [0,1]$ with $b_{IF_j}(n,p) = \{((n,p), b^T_{IF_j}(n,p), b^F_{IF_j}(n,p)) : (n,p) \in N \times P_j\}$ are the corresponding IFDAFs. Then the images of (Γ_1, V^1_ψ) under the functions $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are PIFHS over U_2, which are denoted as $g_{ij}(\Gamma_1, V^1_\psi)$ and the corresponding IFDAFs are defined as: $g_{ij}(a_{IF_i})(n,p) = (g_{ij}(a^T_{IF_i})(n,p), g_{ij}(a^F_{IF_i})(n,p))$, where

$$g_{ij}(a^T_{IF_i})(n,p) = \begin{cases} \max a^T_{IF_i}(m,d) & \text{if } (m,d) \in g^{-1}_{ij}(n,p) \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_{ij}(a^F_{IF_i})(n,p) = \begin{cases} \min a^F_{IF_i}(m,d) & \text{if } (m,d) \in g^{-1}_{ij}(n,p) \\ 1 & \text{otherwise} \end{cases}$$

The preimages of (Γ_2, V^2_ψ) under the functions $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are PIFHSs over U_1, which are denoted as $g^{-1}_{ij}(\Gamma_2, V^2_\psi)$ and the corresponding IFDAFs are defined as
$g^{-1}_{ij}(b_{IF_j})(m,d) = (g^{-1}_{ij}(b^T_{IF_j})(m,d), g^{-1}_{ij}(b^F_{IF_j})(m,d))$, where $g^{-1}_{ij}(b^T_{IF_j})(m,d) = b^T_{IF_j}(g_{ij}(m,d))$ and $g^{-1}_{ij}(b^F_{IF_j})(m,d) = b^F_{IF_j}(g_{ij}(m,d))$

Definition 3.11. Let $(\Gamma_1, V^1_\psi) \in \text{PNHS}(U_1)$ and $(\Gamma_2, V^2_\psi) \in \text{PNHS}(U_2)$, where $V^1_\psi = D_1 \times D_2 \times \cdots \times D_n$ and $V^2_\psi = P_1 \times P_2 \times \cdots \times P_n$. Also, let $\forall M \in \text{range}(\Gamma_1)$, $a_{N_i} : M \times D_i \to [0,1] \times [0,1] \times [0,1]$ with $a_{N_i}(m,d) = \{((m,d), a^T_{N_i}(m,d), a^I_{N_i}(m,d), a^F_{N_i}(m,d)) : (m,d) \in M \times D_i\}$ are the corresponding NDAFs. Again, let $\forall N \in \text{range}(\Gamma_2)$, $b_{N_j} : N \times P_j \to [0,1] \times [0,1] \times [0,1]$ with $b_{N_j}(n,p) = \{((n,p), b^T_{N_i}(n,p), b^I_{N_i}(n,p), b^F_{N_i}(n,p)) : (n,p) \in N \times P_i\}$ are the corresponding NDAFs. Then the images of (Γ_1, V^1_ψ) under the functions $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are PNHS over U_2, which are denoted as $g_{ij}(\Gamma_1, V^1_\psi)$ and the corresponding NDAFs are defined as:
$g_{ij}(a_{N_i})(n,p) = (g_{ij}(a^T_{N_i})(n,p), g_{ij}(a^I_{N_i})(n,p), g_{ij}(a^F_{N_i})(n,p))$, where

$$g_{ij}(a^T_{N_i})(n,p) = \begin{cases} \max a^T_{N_i}(m,d) & \text{if } (m,d) \in g^{-1}_{ij}(n,p) \\ 0 & \text{otherwise} \end{cases},$$

$$g_{ij}(a^I_{N_i})(n,p) = \begin{cases} \max a^I_{N_i}(m,d) & \text{if} (m,d) \in g^{-1}_{ij}(n,p) \\ 0 & \text{otherwise} \end{cases},$$

and

$$g_{ij}(a^F_{N_i})(n,p) = \begin{cases} \min a^F_{N_i}(m,d) & \text{if } (m,d) \in g^{-1}_{ij}(n,p) \\ 1 & \text{otherwise} \end{cases}$$

The preimages of (Γ_2, V_ψ^2) under the functions $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are PNHS over U_1, which are denoted as $g_{ij}^{-1}(\Gamma_2, V_\psi^2)$ and the corresponding NDAFs are defined as
$g_{ij}^{-1}(b_{N_j})(m,d) = (g_{ij}^{-1}(b_{N_j}^T)(m,d), g_{ij}^{-1}(b_{N_j}^I)(m,d), g_{ij}^{-1}(b_{N_j}^F)(m,d))$, where
$g_{ij}^{-1}(b_{N_j}^T)(m,d) = b_{N_j}^T(g_{ij}(m,d))$, $g_{ij}^{-1}(b_{N_j}^I)(m,d) = b_{N_j}^I(g_{ij}(m,d))$ and
$g_{ij}^{-1}(b_{N_j}^F)(m,d) = b_{N_j}^F(g_{ij}(m,d))$.

In the next segment, we have defined plithogenic hypersoft subgroups in fuzzy, intuitionistic fuzzy, and neutrosophic environments. We have also, analyzed their homomorphic properties.

3.2. *Plithogenic Hypersoft Subgroup*

3.2.1. *Plithogenic Fuzzy Hypersoft Subgroup*

Definition 3.12. Let the pair (Γ, V_ψ) be a PFHS of a CG U, where $V_\psi = D_1 \times D_2 \times \cdots \times D_n$ and $\forall i \in \{1,2,...,n\}$, D_i are CGs. Then (Γ, V_ψ) is called a PFHSG of U if and only if $\forall M \in$ range(Γ), $\forall (m_1,d),(m_2,d') \in M \times D_i$ and $\forall a_{F_i} : M \times D_i \to [0,1]$, the conditions mentioned below are satisfied:

(i) $a_{F_i}((m_1,d)\cdot(m_2,d')) \geq \min\{a_{F_i}(m_1,d), a_{F_i}(m_2,d')\}$ and
(ii) $a_{F_i}(m_1,d)^{-1} \geq a_{F_i}(m_1,d)$.

A set of all PFHSG of a CG U is denoted as PFHSG(U).

Example 3.13. Let $U = \{e, m, u, mu\}$ be the Kleins 4-group and $\psi = \{r_1, r_2\}$ is a set of two attributes and corresponding attribute value sets are respectively, $D_1 = \{1, i, -1, -i\}$ and $D_2 = \{1, w, w^2\}$, which are two cyclic groups. Let $V_\psi = D_1 \times D_2$ and (Γ, V_ψ) be a HS over U, where $\Gamma : V_\psi \to \rho(U)$ such that the range of Γ i.e. R$(\Gamma) = \{\{e,m\},\{e,u\},\{e,mu\}\}$. Let for $M = \{e,m\}$, $a_{F_1} : M \times D_1 \to [0,1]$ is defined in Table 2 and $a_{F_2} : M \times D_2 \to [0,1]$ is defined in Table 3 respectively.

TABLE 2. Membership values of a_{F_1}

a_{F_1}	1	i	-1	$-i$
e	0.4	0.2	0.4	0.2
m	0.2	0.2	0.2	0.2

TABLE 3. Membership values of a_{F_2}

a_{F_2}	1	w	w^2
e	0.8	0.5	0.5
m	0.6	0.5	0.5

Let for $M = \{e,u\}$, $a_{F_1} : M \times D_1 \to [0,1]$ is defined in Table 4 and $a_{F_2} : M \times D_2 \to [0,1]$ is defined in Table 5 respectively.

TABLE 4. Membership values of a_{F_1}

a_{F_1}	1	i	-1	$-i$
e	0.8	0.2	0.7	0.2
u	0.5	0.2	0.5	0.2

TABLE 5. Membership values of a_{F_2}

a_{F_2}	1	w	w^2
e	0.7	0.4	0.4
u	0.3	0.3	0.3

Let for $M = \{e, mu\}$, $a_{F_1} : M \times D_1 \to [0,1]$ is defined in Table 6 and $a_{F_2} : M \times D_2 \to [0,1]$ is defined in Table 7 respectively.

TABLE 6. Membership values of a_{F_2}

a_{F_1}	1	i	-1	$-i$
e	0.9	0.2	0.4	0.2
mu	0.7	0.2	0.7	0.2

TABLE 7. Membership values of a_{F_1}

a_{F_2}	1	w	w^2
e	1	0.3	0.3
mu	0.2	0.2	0.2

Here, for any $M \in \text{range}(\Gamma)$ and $\forall i \in \{1,2\}$, a_{F_i} satisfy Definition 3.12. Hence, $(\Gamma, V_\psi) \in$ PFHSG(U).

Proposition 3.1. *Let U be a CG and $(\Gamma, V_\psi) \in$ PFHSG(U), where $V_\psi = D_1 \times D_2 \times \cdots \times D_n$ and $\forall i \in \{1,2,...,n\}$, D_i are CGs. Then for any $M \in$ range(Γ), $\forall (m,d) \in M \times D_i$ and $\forall a_{F_i} : M \times D_i \to [0,1]$, the followings are satisfied:*

(i) $a_{F_i}(e, d_e^i) \geq a_{F_i}(m,d)$, *where e and d_e^i are the neutral elements of U and D_i.*

(ii) $a_{F_i}(m,d)^{-1} = a_{F_i}(m,d)$

Proof. (i) Let e and d_e^i be the neutral elements of U and D_i. Then $\forall (m,d) \in M \times D_i$,

$$\begin{aligned} a_{F_i}(e, d_e^i) &= a_{F_i}((m,d) \cdot (m,d)^{-1}), \\ &\geq \min\{a_{F_i}(m,d), a_{F_i}(m,d)^{-1}\} \text{ (by Definition 3.12)} \\ &\geq \min\{a_{F_i}(m,d), a_{F_i}(m,d)\} \text{ (by Definition 3.12)} \\ &\geq a_{F_i}(m,d) \end{aligned}$$

(ii) Let U be a group and $(\Gamma, V_\psi) \in$ PFHSG(U). Then by Definition 3.12,

$$a_{F_i}(m,d)^{-1} \geq a_{F_i}(m,d) \tag{3.1}$$

Again,

$$\begin{aligned} a_{F_i}(m,d) &= a_{F_i}((m,d)^{-1})^{-1} \\ &\geq a_{F_i}(m,d)^{-1} \end{aligned} \tag{3.2}$$

Hence, from Equation 3.1 and Equation 3.2, $a_{F_i}(m,d)^{-1} = a_{F_i}(m,d)$. □

Proposition 3.2. *Let the pair* (Γ, V_ψ) *be a PFHS of a CG* U*, where* $V_\psi = D_1 \times D_2 \times \cdots \times D_n$ *and* $\forall i \in \{1, 2, ..., n\}$*,* D_i *are CGs. Then* (Γ, V_ψ) *is called a PFHSG of* U *if and only if* $\forall M \in$ *range*(Γ)*,* $\forall (m_1, d), (m_2, d') \in M \times D_i$ *and* $\forall a_{F_i} : M \times D_i \to [0, 1]$*,* $a_{F_i}((m_1, d) \cdot (m_2, d')^{-1}) \geq \min\{a_{F_i}(m_1, d), a_{F_i}(m_2, d')\}$.

Proof. Let U be a CG and $(\Gamma, V_\psi) \in$ PFHSG(U). Then by Definition 3.12 and Proposition 3.1

$$\begin{aligned} a_{F_i}((m_1, d) \cdot (m_2, d')^{-1}) &\geq \min\{a_{F_i}(m_1, d), a_{F_i}(m_2, d')^{-1}\} \\ &= \min\{a_{F_i}(m_1, d), a_{F_i}(m_2, d')\} \end{aligned}$$

Conversely, let $a_{F_i}((m_1, d) \cdot (m_2, d')^{-1}) \geq \min\{a_{F_i}(m_1, d), a_{F_i}(m_2, d')\}$. Also, let e and d_e^i be the neutral elements of U and D_i. Then,

$$\begin{aligned} a_{F_i}(m, d)^{-1} &= a_{F_i}((e, d_e^i) \cdot (m, d)^{-1}) \\ &\geq \min\{a_{F_i}(e, d_e^i), a_{F_i}(m, d)\} \\ &= \min\{a_{F_i}((m, d) \cdot (m, d)^{-1}), a_{F_i}(m, d)\} \\ &\geq \min\{a_{F_i}(m, d), a_{F_i}(m, d), a_{F_i}(m, d)\} \\ &= a_{F_i}(m, d) \end{aligned} \tag{3.3}$$

Now,

$$\begin{aligned} a_{F_i}((m_1, d) \cdot (m_2, d')) &= a_{F_i}((m_1, d) \cdot ((m_2, d')^{-1})^{-1}) \\ &\geq \min\{a_{F_i}(m_1, d), a_{F_i}(m_2, d')^{-1}\} \\ &= \min\{a_{F_i}(m_1, d), a_{F_i}(m_2, d')\} \text{ (by Equation 3.3)} \end{aligned} \tag{3.4}$$

Hence, by Equation 3.3 and Equation 3.4, $(\Gamma, V_\psi) \in$ PFHSG(U). □

Proposition 3.3. *Intersection of two PFHSGs is also a PFHSG.*

Theorem 3.4. *The homomorphic image of a PFHSG is a PFHSG.*

Proof. Let U_1 and U_2 be two CGs and $\forall i, j \in \{1, 2, ..., n\}$, D_i and P_j are attribute value sets consisting of some attribute values and let $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are homomorphisms. Also, let $(\Gamma_1, V_\psi^1) \in$ PFHSG(U_1), where $V_\psi^1 = D_1 \times D_2 \times \cdots \times D_n$. Again, let $\forall M \in$ range(Γ_1), $a_{F_i} : M \times D_i \to [0, 1]$ are the corresponding FDAFs.
Assuming $(n_1, p_1), (n_2, p_2) \in U_2 \times P_j$, if $g_{ij}^{-1}(n_1, p_1) = \phi$ and $g_{ij}^{-1}(n_2, p_2) = \phi$, then $g_{ij}(\Gamma_1, V_\psi^1) \in$ PFHSG(U_2).
Lets assume that $\exists (m_1, d_1), (m_2, d_2) \in U_1 \times D_i$ such that $g_{ij}(m_1, d_1) = (n_1, p_1)$ and

$g_{ij}(m_2, d_2) = (n_2, p_2)$. Then

$$\begin{aligned} g_{ij}(a_{F_i})(n_1, p_1) \cdot (n_2, p_2)^{-1} &= \max_{(n_1,p_1)\cdot(n_2,p_2)^{-1}=g_{ij}(m,d)} a_{F_i}(m, d) \\ &\geq a_{F_i}(m_1, d_1) \cdot (m_2, d_2)^{-1} \\ &\geq \min\{a_{F_i}(m_1, d_1), a_{F_i}(m_2, d_2)\} \text{ (as } (\Gamma_1, V_\psi^1) \in \text{PFHSG}(U_1)) \\ &\geq \min\{\max_{(n_1,p_1)=g_{ij}(m_1,d_1)} a_{F_i}(m_1, d_1), \max_{(n_2,p_2)=g_{ij}(m_2,d_2)} a_{F_i}(m_2, d_2)\} \\ &\geq \min\{g_{ij}(a_{F_i})(n_1, p_1), g_{ij}(a_{F_i})(n_2, p_2)\} \end{aligned}$$

Hence, $g_{ij}(\Gamma_1, V_\psi^1) \in \text{PFHSG}(U_2)$. □

Theorem 3.5. *The homomorphic preimage of a PFHSG is a PFHSG.*

Proof. Let U_1 and U_2 be two CGs and $\forall i, j \in \{1, 2, ..., n\}$, D_i and P_j are attribute value sets consisting of some attribute values and let $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are homomorphisms. Also, let $(\Gamma_2, V_\psi^2) \in \text{PFHSG}(U_2)$, $V_\psi^2 = P_1 \times P_2 \times \cdots \times P_n$. Again, $\forall N \in \text{range}(\Gamma_2)$, $b_{F_j} : N \times P_j \to [0, 1]$ are the corresponding FDAFs. Lets assume $(m_1, d_1), (m_2, d_2) \in U_1 \times D_i$. As g_{ij} is a homomorphism the followings can be concluded:

$$\begin{aligned} & g_{ij}^{-1}(b_{F_i})(m_1, d_1) \cdot (m_2, d_2)^{-1} \\ &= b_{F_i}(g_{ij}((m_1, d_1) \cdot (m_2, d_2)^{-1})) \\ &= b_{F_i}(g_{ij}(m_1, d_1) \cdot g_{ij}(m_2, d_2)^{-1}) \text{ (As } g_{ij} \text{ is a homomorphism)} \\ &\geq \min\{b_{F_i}(g_{ij}(m_1, d_1)), b_{F_i}(g_{ij}(m_2, d_2))\} \text{ (As } (\Gamma_2, V_\psi^2) \in \text{PFHSG}(U_2)) \\ &\geq \min\{g_{ij}^{-1}(b_{F_i})(m_1, d_1), g_{ij}^{-1}(b_{F_i})(m_2, d_2)\} \end{aligned}$$

Then $g_{ij}^{-1}(\Gamma_2, V_\psi^2) \in \text{PFHSG}(U_1)$. □

3.2.2. *Plithogenic Intuitionistic Fuzzy Hypersoft Subgroup*

Definition 3.14. Let the pair (Γ, V_ψ) be a PIFHS of a CG U, where $V_\psi = D_1 \times D_2 \times \cdots \times D_n$ and $\forall i \in \{1, 2, ..., n\}$, D_i are CGs. Then (Γ, V_ψ) is called a PIFHSG of U if and only if $\forall M \in \text{range}(\Gamma)$, $\forall (m_1, d), (m_2, d') \in M \times D_i$ and $\forall a_{IF_i} : M \times D_i \to [0, 1] \times [0, 1]$ with $a_{IF_i}(m, d) = \{((m, d), a^T_{IF_i}(m, d), a^F_{IF_i}(m, d)) : (m, d) \in M \times D_i\}$, the subsequent conditions are fulfilled:

(i) $a^T_{IF_i}((m_1, d) \cdot (m_2, d')) \geq \min\{a^T_{IF_i}(m_1, d), a^T_{IF_i}(m_2, d')\}$
(ii) $a^T_{IF_i}(m_1, d)^{-1} \geq a^T_{IF_i}(m_1, d)$
(iii) $a^F_{IF_i}((m_1, d) \cdot (m_2, d')) \leq \max\{a^F_{IF_i}(m_1, d), a^F_{IF_i}(m_2, d')\}$
(iv) $a^F_{IF_i}(m_1, d)^{-1} \leq a^F_{IF_i}(m_1, d)$

A set of all PIFHSG of a CG U is denoted as PIFHSG(U).

Example 3.15. Let $U = S_3$ be a CG and $\psi = \{r_1, r_2\}$ is a set of two attributes and corresponding attribute value sets are respectively, $D_1 = A_3$ and $D_2 = S_2$, which are respectively an alternating group of order 3 and a symmetric group of order 2. Let $V_\psi = D_1 \times D_2$ and (Γ, V_ψ) be a HS over U, where $\Gamma : V_\psi \to \rho(U)$ such that the range of Γ i.e. $\mathrm{R}(\Gamma) = \{\{(1),(13)\},\{(1),(23)\}\}$. Let for $M = \{(1),(13)\}$, $a_{IF_1} : M \times D_1 \to [0,1] \times [0,1]$ is defined in Table 8–9 and $a_{IF_2} : M \times D_2 \to [0,1] \times [0,1]$ is defined in Table 10–11 respectively.

TABLE 8. Membership values of a_{IF_1}

$a^T_{IF_1}$	(1)	(123)	(132)
(1)	0.4	0.5	0.5
(13)	0.2	0.2	0.2

TABLE 9. Non-membership values of a_{IF_1}

$a^F_{IF_1}$	(1)	(123)	(132)
(1)	0.4	0.7	0.7
(13)	0.8	0.8	0.8

TABLE 10. Membership values of a_{IF_2}

$a^T_{IF_2}$	(1)	(12)
(1)	0.8	0.4
(13)	0.3	0.3

TABLE 11. Non-membership values of a_{IF_2}

$a^F_{IF_2}$	(1)	(12)
(1)	0.4	0.8
(13)	0.9	0.9

Let for $M = \{(1),(23)\}$ $a_{IF_1} : M \times D_1 \to [0,1] \times [0,1]$ is defined in Table 12–13 and $a_{IF_2} : M \times D_2 \to [0,1] \times [0,1]$ is defined in Table 14–15 respectively.

TABLE 12. Membership values of a_{IF_1}

$a^T_{IF_1}$	(1)	(123)	(132)
(1)	0.6	0.4	0.4
(23)	0.5	0.4	0.4

TABLE 13. Non-membership values of a_{IF_1}

$a^F_{IF_1}$	(1)	(123)	(132)
(1)	0.4	0.7	0.7
(23)	0.6	0.7	0.7

TABLE 14. Membership values of a_{IF_2}

$a^T_{IF_2}$	(1)	(12)
(1)	0.7	0.6
(23)	0.7	0.6

TABLE 15. Non-membership values of a_{IF_2}

$a^F_{IF_2}$	(1)	(12)
(1)	0.5	0.9
(23)	0.8	0.9

Here, for any $M \in \mathrm{range}(\Gamma)$ and $\forall i \in \{1,2\}$, a_{IF_i} satisfy Definition 3.14. Hence, $(\Gamma, V_\psi) \in \mathrm{PIFHSG}(U)$.

Proposition 3.6. *Let U be a CG and $(\Gamma, V_\psi) \in PIFHSG(U)$, where $V_\psi = D_1 \times D_2 \times \cdots \times D_n$ and $\forall i \in \{1, 2, ..., n\}$, D_i are CGs. Then for any $M \in range(\Gamma)$ and $\forall (m, d_i) \in M \times D_i$ and $\forall a_{IF_i} : M \times D_i \to [0, 1] \times [0, 1]$ with $a_{IF_i}(m, d) = \{((m, d), a^T_{IF_i}(m, d), a^F_{IF_i}(m, d)) : (m, d) \in M \times D_i\}$, the subsequent conditions are satisfied:*

(i) $a^T_{IF_i}(e, d^i_e) \geq a^T_{IF_i}(m, d)$, *where e and d^i_e are the neutral elements of U and D_i.*
(ii) $a^T_{IF_i}(m, d)^{-1} = a^T_{IF_i}(m, d)$
(iii) $a^F_{IF_i}(e, d^i_e) \leq a^F_{IF_i}(m, d)$, *where e and d^i_e are the neutral elements of U and D_i.*
(iv) $a^F_{IF_i}(m, d)^{-1} = a^F_{IF_i}(m, d)$

Proof. Here, (i) and (ii) can be easily proved using Proposition 3.1.
(iii) Let e and d^i_e be the neutral elements of U and D_i. Then $\forall (m, d) \in M \times D_i$,

$$\begin{aligned} a^F_{IF_i}(e, d^i_e) &= a^F_{IF_i}((m, d) \cdot (m, d)^{-1}) \\ &\leq \max\{a^F_{IF_i}(m, d), a^F_{IF_i}(m, d)^{-1}\} \text{ (by Definition 3.14)} \\ &\leq \max\{a^F_{IF_i}(m, d), a^F_{IF_i}(m, d)\} \text{ (by Definition 3.14)} \\ &\leq a^F_{IF_i}(m, d) \end{aligned}$$

(iv) Let U be a CG and $(\Gamma, V_\psi) \in$ PFHSG(U). Then by Definition 3.14,

$$a^F_{IF_i}(m, d)^{-1} \leq a^F_{IF_i}(m, d) \tag{3.5}$$

Again,

$$\begin{aligned} a^F_{IF_i}(m, d) &= a^F_{IF_i}((m, d)^{-1})^{-1} \\ &\leq a^F_{IF_i}(m, d)^{-1} \end{aligned} \tag{3.6}$$

Hence, by Equation 3.5 and Equation 3.6, $a^F_{IF_i}(m, d)^{-1} = a^F_{IF_i}(m, d)$. □

Proposition 3.7. *Let the pair (Γ, V_ψ) be a PIFHS of a CG U, where $V_\psi = D_1 \times D_2 \times \cdots \times D_n$ and $\forall i \in \{1, 2, ..., n\}$, D_i are CGs. Then (Γ, V_ψ) is called a PIFHSG of U if and only if $\forall M \in range(\Gamma)$, $\forall (m_1, d), (m_2, d') \in M \times D_i$ and $\forall a_{IF_i} : M \times D_i \to [0, 1] \times [0, 1]$ with $a_{IF_i}(m, d) = \{((m, d), a^T_{IF_i}(m, d), a^F_{IF_i}(m, d)) : (m, d) \in M \times D_i\}$, the subsequent conditions are fulfilled:*

(i) $a^T_{IF_i}((m_1, d) \cdot (m_2, d')^{-1}) \geq \min\{a^T_{IF_i}(m_1, d), a^T_{IF_i}(m_2, d')\}$ *and*
(ii) $a^F_{IF_i}((m_1, d) \cdot (m_2, d')^{-1}) \leq \max\{a^F_{IF_i}(m_1, d), a^F_{IF_i}(m_2, d')\}$

Proof. Here, (i) can be proved using Proposition 3.2.
(ii) Let U be a CG and $(\Gamma, V_\psi) \in$ PFHSG(U). Then by Definition 3.14 and Proposition 3.6

$$\begin{aligned} a^F_{IF_i}((m_1, d) \cdot (m_2, d')^{-1}) &\leq \max\{a^F_{IF_i}(m_1, d), a^F_{IF_i}(m_2, d')^{-1}\} \\ &\leq \max\{a^F_{IF_i}(m_1, d), a^F_{IF_i}(m_2, d')\} \end{aligned}$$

Conversely, let $a^F_{IF_i}((m_1,d)\cdot(m_2,d')^{-1}) \le \max\{a^F_{IF_i}(m_1,d), a^F_{IF_i}(m_2,d')\}$. Also, let e and d^i_e be the neutral elements of U and D_i. Then

$$\begin{aligned}
a^F_{IF_i}(m,d)^{-1} &= a^F_{IF_i}((e,d^i_e)\cdot(m,d)^{-1}) \\
&\le \max\{a^F_{IF_i}(e,d^i_e), a^F_{IF_i}(m,d)\} \\
&\le \max\{a^F_{IF_i}((m,d)\cdot(m,d)^{-1}), a^F_{IF_i}(m,d)\} \\
&\le \max\{a^F_{IF_i}(m,d), a^F_{IF_i}(m,d), a^F_{IF_i}(m,d)\} \\
&= a^F_{IF_i}(m,d) \qquad (3.7)
\end{aligned}$$

Now,

$$\begin{aligned}
a^F_{IF_i}((m_1,d)\cdot(m_2,d')) &= a^F_{IF_i}((m_1,d)\cdot((m_2,d')^{-1})^{-1}) \\
&\le \max\{a^F_{IF_i}(m_1,d), a^F_{IF_i}(m_2,d')^{-1}\} \\
&= \max\{a^F_{IF_i}(m_1,d), a^F_{IF_i}(m_2,d')\} \text{ (by Equation 3.7)} \qquad (3.8)
\end{aligned}$$

Hence, by Equation 3.7 and Equation 3.8, $(\Gamma, V_\psi) \in$ PFHSG(U). □

Proposition 3.8. *Intersection of two PIFHSGs is also a PIFHSG.*

Theorem 3.9. *The homomorphic image of a PIFHSG is a PIFHSG.*

Proof. Let U_1 and U_2 be two CGs and $\forall i,j \in \{1,2,...,n\}$, D_i and P_j are attribute value sets consisting of some attribute values and let $g_{ij}: U_1 \times D_i \to U_2 \times P_j$ are homomorphisms. Also, let $(\Gamma_1, V^1_\psi) \in$ PIFHSG(U_1), where $V^1_\psi = D_1 \times D_2 \times \cdots \times D_n$. Again, let $\forall M \in$ range(Γ_1) and $a_{IF_i}: M \times D_i \to [0,1]\times[0,1]$ with $a_{IF_i}(m,d) = \{((m,d), a^T_{IF_i}(m,d), a^F_{IF_i}(m,d)) : (m,d) \in M \times D_i\}$ are the corresponding IFDAFs. Assuming $(n_1,p_1),(n_2,p_2) \in U_2 \times P_j$, if $g^{-1}_{ij}(n_1,p_1) = \phi$ and $g^{-1}_{ij}(n_2,p_2) = \phi$, then $g_{ij}(\Gamma_1, V^1_\psi) \in$ PIFHSG(U_2). Lets assume that $\exists (m_1,d_1),(m_2,d_2) \in U_1 \times D_i$ such that $g_{ij}(m_1,d_1) = (n_1,p_1)$ and $g_{ij}(m_2,d_2) = (n_2,p_2)$. Then by Theorem 3.4

$$g_{ij}(a^T_{IF_i})(n_1,p_1)\cdot(n_2,p_2)^{-1} \ge \min\{g_{ij}(a^T_{IF_i})(n_1,p_1), g_{ij}(a^T_{IF_i})(n_2,p_2)\}.$$

Again,

$$\begin{aligned}
g_{ij}(a^F_{IF_i})(n_1,p_1)\cdot(n_2,p_2)^{-1} &= \min_{(n_1,p_1)\cdot(n_2,p_2)^{-1}=g_{ij}(m,d)} a^F_{IF_i}(m,d) \\
&\le a^F_{IF_i}(m_1,d_1)\cdot(m_2,d_2)^{-1} \\
&\le \max\{a^F_{IF_i}(m_1,d_1), a^F_{IF_i}(m_2,d_2)\} \text{ (as } (\Gamma_1, V^1_\psi) \in PIFHSG(U_1)) \\
&\le \max\{\min_{(n_1,p_1)=g_{ij}(m_1,d_1)} a^F_{IF_i}(m_1,d_1), \min_{(n_2,p_2)=g_{ij}(m_2,d_2)} a^F_{IF_i}(m_2,d_2)\} \\
&\le \max\{g_{ij}(a^F_{IF_i})(n_1,p_1), g_{ij}(a^F_{IF_i})(n_2,p_2)\} \text{ (by Definition 3.10)}
\end{aligned}$$

Hence, $g_{ij}(\Gamma_1, V_\psi^1) \in$ PIFHSG(U_2). □

Theorem 3.10. *The homomorphic preimage of a PIFHSG is a PIFHSG.*

Proof. Let U_1 and U_2 be two CGs and $\forall i, j \in \{1, 2, ..., n\}$, D_i and P_j are attribute value sets consisting of some attribute values and let $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are homomorphisms. Also, let $(\Gamma_2, V_\psi^2) \in$ PIFHS(U_2), where $V_\psi^2 = P_1 \times P_2 \times \cdots \times P_n$. Again, let $\forall N \in$ range(Γ_2), $b_{IF_j} : N \times P_j \to [0,1] \times [0,1]$ with $b_{IF_j}(n,p) = \{((n,p), b^T_{IF_j}(n,p), b^F_{IF_j}(n,p)) : (n,p) \in N \times P_j\}$ are the corresponding IFDAFs. Lets assume$(m_1, d_1), (m_2, d_2) \in U_1 \times D_i$. Since, g_{ij} is a homomorphism, by Theorem 3.5

$$g_{ij}^{-1}(b^T_{IF_j})(m_1, d_1) \cdot (m_2, d_2)^{-1} \geq \min\{g_{ij}^{-1}(b^T_{IF_j})(m_1, d_1), g_{ij}^{-1}(b^T_{IF_j})(m_2, d_2)\}.$$

Again,

$$\begin{aligned} g_{ij}^{-1}(b^F_{IF_j})(m_1, d_1) \cdot (m_2, d_2)^{-1} &= b^F_{IF_j}(g_{ij}((m_1, d_1) \cdot (m_2, d_2)^{-1})) \\ &= b^F_{IF_j}(g_{ij}(m_1, d_1) \cdot g_{ij}(m_2, d_2)^{-1}) \text{ (As } g_{ij} \text{ is a homomorphism)} \\ &\leq \max\{b^F_{IF_j}(g_{ij}(m_1, d_1)), b^F_{IF_j}(g_{ij}(m_2, d_2))\} \\ &\qquad \text{(As } (\Gamma_2, V_\psi^2) \in PIFHSG(U_2)) \\ &\leq \max\{g_{ij}^{-1}(b^F_{IF_j})(m_1, d_1), g_{ij}^{-1}(b^F_{IF_j})(m_2, d_2)\} \end{aligned}$$

Hence, $g_{ij}^{-1}(\Gamma_2, V_\psi^2) \in$ PIFHSG(U_1). □

3.2.3. *Plithogenic Neutrosophic Hypersoft Subgroup*

Definition 3.16. content.Let the pair (Γ, V_ψ) be a PNHS of a CG U, where $V_\psi = D_1 \times D_2 \times \cdots \times D_n$ and $\forall i \in \{1, 2, ..., n\}$, D_i are CGs. Then (Γ, V_ψ) is called a PNHSG of U if and only if $\forall M \in$ range(Γ), $\forall (m_1, d), (m_2, d') \in M \times D_i$ and $\forall a_{N_i} : M \times D_i \to [0,1] \times [0,1] \times [0,1]$, with $a_{N_i}(m, d) = \{((m, d), a^T_{N_i}(m, d),\ a^I_{N_i}(m, d), a^F_{N_i}(m, d)) : (m, d) \in M \times D_i\}$, the subsequent conditions are fulfilled:

(i) $a^T_{N_i}((m_1, d) \cdot (m_2, d')^{-1}) \geq \min\{a^T_{N_i}(m_1, d), a^T_{N_i}(m_2, d')\}$
(ii) $a^T_{N_i}(m_1, d)^{-1} \geq a^T_{N_i}(m_1, d)$
(iii) $a^I_{N_i}((m_1, d) \cdot (m_2, d')^{-1}) \geq \min\{a^I_{N_i}(m_1, d), a^I_{N_i}(m_2, d')\}$
(iv) $a^I_{N_i}(m_1, d)^{-1} \geq a^I_{N_i}(m_1, d)$
(v) $a^F_{N_i}((m_1, d) \cdot (m_2, d')^{-1}) \leq \max\{a^F_{N_i}(m_1, d), a^F_{N_i}(m_2, d')\}$
(vi) $a^F_{N_i}(m_1, d)^{-1} \leq a^F_{N_i}(m_1, d)$

A set of all PNHSG of a CG U is denoted as PNHSG(U).

Example 3.17. Let $D_6 = \{e, m, u, mu, um, mum\}$ be a dihedral group of order 6 and $\psi = \{r_1, r_2\}$ is a set of two attributes and corresponding attribute value sets are respectively, $D_1 = \{1, w, w^2\}$ and $D_2 = A_3$, which are respectively a cyclic group of order 3 and an alternating group of order 3. Let $V_\psi = D_1 \times D_2$ and (Γ, V_ψ) be a HS over U, where $\Gamma : V_\psi \to \rho(U)$ such that the range of Γ i.e. $\mathrm{R}(\Gamma) = \{\{e, mu, um\}, \{e, mum\}\}$.
Let for $M = \{e, mu\}$, $a_{N_1} : M \times D_1 \to [0,1] \times [0,1] \times [0,1]$ is defined in Table 16–18 and $a_{N_2} : M \times D_2 \to [0,1] \times [0,1] \times [0,1]$ is defined in Table 19–21 respectively.

TABLE 16. Truth values of a_{N_1}

$a^T_{N_1}$	1	w	w^2
e	0.7	0.5	0.5
mu	0.3	0.3	0.3
um	0.3	0.3	0.3

TABLE 17. Indeterminacy values of a_{N_1}

$a^I_{N_1}$	1	w	w^2
e	0.8	0.4	0.4
mu	0.5	0.4	0.4
um	0.5	0.4	0.4

TABLE 18. Falsity values of a_{N_1}

$a^F_{N_1}$	1	w	w^2
e	0.3	0.5	0.5
mu	0.7	0.7	0.7
um	0.7	0.7	0.7

TABLE 19. Truth values of a_{N_2}

$a^T_{N_2}$	(1)	(123)	(132)
e	0.7	0.2	0.2
mu	0.1	0.1	0.1
um	0.1	0.1	0.1

TABLE 20. Indeterminacy values of a_{N_2}

$a^I_{N_2}$	(1)	(123)	(132)
e	0.8	0.5	0.5
mu	0.8	0.5	0.5
um	0.8	0.5	0.5

TABLE 21. Falsity values of a_{N_2}

$a^F_{N_2}$	(1)	(123)	(132)
e	0.3	0.8	0.8
mu	0.9	0.9	0.9
um	0.9	0.9	0.9

Let for $M = \{e, mum\}$, $a_{N_1} : M \times D_1 \to [0,1] \times [0,1] \times [0,1]$ is defined in Table 22–24 and $a_{N_2} : M \times D_2 \to [0,1] \times [0,1] \times [0,1]$ is defined in Table 25–27 respectively.

TABLE 22. Truth values of a_{N_1}

$a^T_{N_1}$	1	w	w^2
e	0.8	0.4	0.4
mum	0.2	0.2	0.2

TABLE 23. Indeterminacy values of a_{N_1}

$a^I_{N_1}$	1	w	w^2
e	0.8	0.6	0.6
mum	0.7	0.6	0.6

TABLE 24. Falsity values of a_{N_1}

$a^F_{N_1}$	1	w	w^2
e	0.2	0.6	0.6
mum	0.8	0.8	0.8

TABLE 25. Truth values of a_{N_2}

$a^T_{N_2}$	(1)	(123)	(132)
e	0.9	0.8	0.8
mum	0.9	0.8	0.8

TABLE 26. Indeterminacy values of a_{N_2}

$a^I_{N_2}$	(1)	(123)	(132)
e	0.5	0.2	0.2
mum	0.1	0.1	0.1

TABLE 27. Falsity values of a_{N_2}

$a^F_{N_2}$	(1)	(123)	(132)
e	0.1	0.2	0.2
mum	0.1	0.2	0.2

Here, for any $M \in$ range(Γ) and $\forall i \in \{1,2\}$, a_{N_i} satisfy Definition 3.16. Hence, $(\Gamma, V_\psi) \in$ PNHSG(U).

Proposition 3.11. *Let the pair (Γ, V_ψ) be a PNHS of a CG U, where $V_\psi = D_1 \times D_2 \times \cdots \times D_n$ and $\forall i \in \{1,2,...,n\}$, D_i are CGs. Then (Γ, V_ψ) is called a PNHSG of U if and only if $\forall M \in$ range(Γ), $\forall (m_1,d),(m_2,d') \in M \times D_i$ and $\forall a_{N_i} : M \times D_i \to [0,1]\times[0,1]\times[0,1]$, with $a_{N_i}(m,d) = \{((m,d), a^T_{N_i}(m,d), a^I_{N_i}(m,d), a^F_{N_i}(m,d)) : (m,d) \in M \times D_i\}$. Then the subsequent conditions are satisfied:*

(i) $a^T_{N_i}(e,d) \geq a^T_{N_i}(m,d)$, *where e is the neutral element of U.*
(ii) $a^T_{N_i}(m,d)^{-1} = a^T_{N_i}(m,d)$
(iii) $a^I_{N_i}(e,d) \geq a^I_{N_i}(m,d)$, *where e is the neutral element of U.*
(iv) $a^I_{N_i}(m,d)^{-1} = a^I_{N_i}(m,d)$
(v) $a^F_{N_i}(e,d) \leq a^F_{N_i}(m,d)$, *where e is the neutral element of U.*
(vi) $a^F_{N_i}(m,d)^{-1} = a^F_{N_i}(m,d)$

Proof. This can be proved using Proposition 3.1 and Proposition 3.6. □

Proposition 3.12. *Let the pair (Γ, V_ψ) be a PNHS of a CG U, where $V_\psi = D_1 \times D_2 \times \cdots \times D_n$ and $\forall i \in \{1,2,...,n\}$, D_i are CGs. Then (Γ, V_ψ) is called a PNHSG of U if and only if $\forall M \in$ range(Γ), $\forall (m_1,d),(m_2,d') \in M \times D_i$ and $\forall a_{N_i} : M \times D_i \to [0,1]\times[0,1]\times[0,1]$, with $a_{N_i}(m,d) = \{((m,d), a^T_{N_i}(m,d), a^I_{N_i}(m,d), a^F_{N_i}(m,d)) : (m,d) \in M \times D_i\}$. Then the subsequent conditions are fulfilled:*

(i) $a^T_{N_i}((m_1,d)\cdot(m_2,d')^{-1}) \geq \min\{a^T_{N_i}(m_1,d), a^T_{N_i}(m_2,d')\}$
(ii) $a^I_{N_i}((m_1,d)\cdot(m_2,d')^{-1}) \geq \min\{a^I_{N_i}(m_1,d), a^I_{N_i}(m_2,d')\}$
(iii) $a^F_{N_i}((m_1,d)\cdot(m_2,d')^{-1}) \leq \max\{a^F_{N_i}(m_1,d), a^F_{N_i}(m_2,d')\}$

Proof. This can be proved using Proposition 3.2 and Proposition 3.7. □

Proposition 3.13. *Intersection of two PNHSGs is also a PNHSG.*

Theorem 3.14. *The homomorphic image of a PNHSG is a PNHSG.*

Proof. Let U_1 and U_2 be two CGs and $\forall i, j \in \{1, 2, ..., n\}$, D_i and P_j are attribute value sets consisting of some attribute values and let $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are homomorphisms. Also, let $(\Gamma_1, V_\psi^1) \in$ PNHSG(U_1), where $V_\psi^1 = D_1 \times D_2 \times \cdots \times D_n$. Again, let $\forall M \in$ range(Γ_1), $a_{N_i} : M \times D_i \to [0,1] \times [0,1] \times [0,1]$ with $a_{N_i}(m,d) = \{((m,d), a^T_{N_i}(m,d), a^I_{N_i}(m,d), a^F_{N_i}(m,d)) : (m,d) \in M \times D_i\}$ are the corresponding NDAFs.
Assuming $(n_1,p_1),(n_2,p_2) \in U_2 \times P_j$, if $g_{ij}^{-1}(n_1,p_1) = \phi$ and $g_{ij}^{-1}(n_2,p_2) = \phi$ then $g_{ij}(\Gamma_1, V_\psi^1) \in$ NHSG(U_2). Lets assume that $\exists (m_1,d_1),(m_2,d_2) \in U_1 \times D_i$ such that $g_{ij}(m_1,d_1) = (n_1,p_1)$ and $g_{ij}(m_2,d_2) = (n_2,p_2)$. Then by Theorem 3.4 and Theorem 3.9, we can prove the followings:

$$g_{ij}(a^T_{N_i})(n_1,p_1) \cdot (n_2,p_2)^{-1} \geq \min\{g_{ij}(a^T_{N_i})(n_1,p_1), g_{ij}(a^T_{N_i})(n_2,p_2)\},$$

$$g_{ij}(a^I_{N_i})(n_1,p_1) \cdot (n_2,p_2)^{-1} \geq \min\{g_{ij}(a^I_{N_i})(n_1,p_1), g_{ij}(a^I_{N_i})(n_2,p_2)\},$$

and

$$g_{ij}(a^F_{N_i})(n_1,p_1) \cdot (n_2,p_2)^{-1} \leq \max\{g_{ij}(a^F_{N_i})(n_1,p_1), g_{ij}(a^F_{N_i})(n_2,p_2)\}.$$

Hence, $g_{ij}(\Gamma_1, V_\psi^1) \in$ PNHSG(U_2). □

Theorem 3.15. *The homomorphic preimage of a PNHSG is a PNHSG.*

Proof. Let U_1 and U_2 be two CGs and $\forall i, j \in \{1, 2, ..., n\} D_i$ and P_j are attribute value sets consisting of some attribute values and let $g_{ij} : U_1 \times D_i \to U_2 \times P_j$ are homomorphisms. Also, let $(\Gamma_2, V_\psi^2) \in$ PNHSG(U_2), where $V_\psi^2 = P_1 \times P_2 \times \cdots \times P_n$. Again, let $\forall N \in$ range(Γ_2),b_{N_j} : $N \times P_j \to [0,1] \times [0,1] \times [0,1]$ with $b_{N_j}(n,p) = \{((n,p), b^T_{N_j}(n,p), b^I_{N_j}(n,p), b^F_{N_j}(n,p)) : (n,p) \in N \times P_j\}$ are the corresponding IFDAFs. Lets assume $(m_1,d_1),(m_2,d_2) \in U_1 \times D_i$. Since g_{ij} is a homomorphism by Theorem 3.5 and Theorem 3.10, the followings can be proved:

$$g_{ij}^{-1}(b^T_{N_j})(m_1,d_1) \cdot (m_2,d_2)^{-1} \geq \min\{g_{ij}^{-1}(b^T_{N_j})(m_1,d_1), g_{ij}^{-1}(b^T_{N_j})(m_2,d_2)\},$$

$$g_{ij}^{-1}(b^I_{N_j})(m_1,d_1) \cdot (m_2,d_2)^{-1} \geq \min\{g_{ij}^{-1}(b^I_{N_j})(m_1,d_1), g_{ij}^{-1}(b^I_{N_j})(m_2,d_2)\},$$

and

$$g_{ij}^{-1}(b^F_{N_j})(m_1,d_1) \cdot (m_2,d_2)^{-1} \leq \max\{g_{ij}^{-1}(b^F_{N_j})(m_1,d_1), g_{ij}^{-1}(b^F_{N_j})(m_2,d_2)\}.$$

Hence, $g_{ij}^{-1}(\Gamma_2, V_\psi^2) \in$ PNHSG(U_1). □

4. Conclusions

Hypersoft set theory is more general than soft set theory and it has a huge area of applications. That is why we have adopted and implemented it in plithogenic environment so that we can introduce various algebraic structures. Because of this, the notions of plithogenic hypersoft subgroups have become general than fuzzy, intuitionistic fuzzy, neutrosophic subgroups, and plithogenic subgroups. Again, we have introduced functions in different plithogenic hypersoft environments. Hence, homomorphism can be introduced and its effects on these newly defined plithogenic hypersoft subgroups can be studied. In the future, to extend this study one may introduce general T-norm and T-conorm and further generalize plithogenic hypersoft subgroups. Also, one may extend these notions by introducing different normal versions of plithogenic hypersoft subgroups and by studying the effects of homomorphism on them.

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Neutrosophic Cubic Hamacher Aggregation Operators and Their Applications in Decision Making

Hüseyin Kamacı

Department of Mathematics, Faculty of Science and Arts, Yozgat Bozok University, 66100 Yozgat, Turkey; huseyin.kamaci@hotmail.com, huseyin.kamaci@bozok.edu.tr

Correspondence: huseyin.kamaci@hotmail.com

Abstract. In this paper, firstly, novel approaches of score function and accuracy function are introduced to achieve more practical and convincing comparison results of two neutrosophic cubic values. Furthermore, the neutrosophic cubic Hamacher weighted averaging operator and the neutrosophic cubic Hamacher weighted geometric operator are developed to aggregate neutrosophic cubic values. Some desirable properties of these operators such as idempotency, monotonicity and boundedness are discussed. To deal with the multi-criteria decision making problems in which attribute values take the form of the neutrosophic cubic elements, the decision making algorithms based on some Hamacher aggregation operators, which are extensions of the algebraic aggregation operators and Einstein aggregation operators, are constructed. Finally, the illustrative examples and comparisons are given to verify the proposed algorithms and to demonstrate their practicality and effectiveness.

Keywords: Neutrosophic set; Neutrosophic cubic set; Score function; Accuracy function; Hamacher operations; Decision making

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1. Introduction

In real life, there are many problems with inconsistent, indeterminate and incomplete information which cannot be described by crisp numbers. Under these circumstances, Zadeh [34] proposed the fuzzy set, which is an effective method to deal with such problems. To express uncertainty, Sambuc [26] extended the fuzzy set and initiated the interval valued fuzzy theory. In [33], the researchers discussed the multipolar types of fuzzy sets. In 2012, Jun combined the idea of fuzzy sets and interval valued fuzzy sets to form cubic sets. Some researchers used the cubic sets in different directions to have more applications [23, 24]. In some situations, hesitancy may exist when ones determine the membership degree of an object. Torra [29] improved the hesitant fuzzy set to depict this hesitant information. Moreover, Smarandache [27] introduced the neutrosophic set to reflect the truth, indeterminate and false information simultaneously. In addition, Wang et al. pointed out that the neutrosophic set is difficult to truly apply to practical problems in real world scenarios. To overcome this flaw, they proposed single valued neutrosophic sets [32]. In addition, they put forward that in many real life problems, the degrees of truth, indeterminacy and falsity of a certain statement may be adaptly preferred by interval forms, instead of real numbers [31]. Moreover, many papers were published on the neutrosophic set's case studies [1, 2, 19, 20, 30], their some extensions [3–5, 12, 16], and combining with other theories, like graph theory [11, 18], soft set theory [8, 15, 28], rough set theory [6].

By combining the single valued neutrosophic set and interval neutrosophic set, Jun et al. [14], and Ali et al. [7] introduced the notion of neutrosophic cubic set. These sets enable us to choose both interval values and single values for the membership, indeterminacy and non-membership. This characteristic of neutrosophic cubic sets enables us to deal with ambiguous and uncertain data more efficiently. In addition, the application of sundry extensions of neutrosophic cubic sets studied by researchers in a variety of fields, like decision-making, supplier selection and similarity measure [9, 21, 22, 35].

The aggregation operators are an indispensable part of decision making in neutrosophic cubic environments. In 2019, Khan et al. [17] developed the neutrosophic cubic Einstein weighted geometric operator, and also defined the score and accuracy functions to reveal the superiority among the neutrosophic cubic numbers. It is known that Einstein t-norm and Einstein t-conorm are special forms of Hamacher t-norm and Hamacher t-conorm respectively, that is, Hamacher t-norm and Hamacher t-conorm are the more general version. This paper aims to introduce the neutrosophic cubic Hamacher weighted averaging operator and neutrosophic cubic Hamacher weighted geometric operator, which generalize the aggregation operators proposed by Khan et al. [17]. Furthermore, it proposes new score function and accuracy function, which provide more efficient outputs than Khan et al.'s functions. By using these emerging operators and functions, the phenomenal algorithms are elaborated to solve multi-criteria decision making problems. The contributions of this study can be summarized as follows. The models are proposed to compare neutrosophic cubic numbers, and the operators which are more efficient than some existing netrosophic cubic aggregation operators are developed. In addition to these, it is instilled that these concepts can be used to handle the problems with neutrosophic cubic information.

This paper is arranged as follows. Section 2 presents some fundamental concepts of fuzzy set, neutrosophic set, interval neutrosophic set, cubic set and neutrosophic cubic set. Section 3 presents comparison strategy of two neutrosophic cubic elements. Section 4 is devoted to improve the Hamacher operations of neutrosophic cubic elements. Section 5 introduces neutrosophic cubic Hamacher weighted aggregation operators and their basic properties. Section 6 is devoted to proposing the neutrosophic cubic decision making algorithms with possible applications and analyzing the ranking order with different reducing factors. Section 7 is the conclusion and the future scope of research.

2. Preliminaries

In this part, we briefly remind the definitions of fuzzy set, neutrosophic set, interval neutrosophic set, cubic set, neutrosophic cubic set and neutrosophic cubic element.

Definition 2.1. ([34]) Let $\mathcal{O}$ be a universal set. Then, a fuzzy set (FS) Ψ in $\mathcal{O}$ is defined by

$$\Psi = \{\mu_\Psi(o)/o : o \in \mathcal{O}\}$$

where $\mu_\Psi : \mathcal{O} \to [0,1]$ is said to be the membership function and $\mu_\Upsilon(o)$ denotes the degrees of membership of $o \in \mathcal{O}$ to the set Ψ.

Definition 2.2. ([26]) Let $\mathcal{O}$ be a universal set and $D[0,1]$ be the set of all closed subintervals of the interval [0,1]. Then, an interval-valued fuzzy set (IFS) $\widetilde{\Psi}$ in $\mathcal{O}$ is characterized by

$$\widetilde{\Psi} = \{\widetilde{\mu}_{\widetilde{\Psi}}(o)/o : o \in \mathcal{O}\}$$

where $\widetilde{\mu}_{\widetilde{\Psi}} = [\widetilde{\mu}^L_{\widetilde{\Psi}}, \widetilde{\mu}^U_{\widetilde{\Psi}}] : \mathcal{O} \to D[0,1]$ is said to be the membership function, and $\widetilde{\mu}^L_{\widetilde{\Psi}}(o)$ and $\widetilde{\mu}^U_{\widetilde{\Psi}}(o)$ (where $\widetilde{\mu}^L_{\widetilde{\Psi}}(o) \leq \widetilde{\mu}^U_{\widetilde{\Psi}}(o)$) denote the lower degree and upper degree of membership of $o \in \mathcal{O}$ to the set $\widetilde{\Psi}$, respectively.

Definition 2.3. ([27]) Let $\mathcal{O}$ be a universal set. Then, a nuetrosophic set (NS) Υ in $\mathcal{O}$ is described in the following form

$$\Upsilon = \{(\mu_\Upsilon, \iota_\Upsilon, \eta_\Upsilon)/o : o \in \mathcal{O}\}$$

where $\mu_\Upsilon, \iota_\Upsilon, \eta_\Upsilon : \mathcal{O} \to]0^-, 1^+[$ are said to be the functions of membership, indeterminacy and non-membership, respectively. Also, $\mu_\Upsilon(o)$, $\iota_\Upsilon(o)$ and $\eta_\Upsilon(o)$ denote the degrees of membership, indeterminacy and non-membership of $o \in \mathcal{O}$ to the set Υ respectively.

Definition 2.4. ([32]) Let $\mathcal{O}$ be a universal set. Then, a single nuetrosophic set Γ in $\mathcal{O}$ is described in the following form

$$\Upsilon = \{(\mu_\Upsilon, \iota_\Upsilon, \eta_\Upsilon)/o : o \in \mathcal{O}\}$$

where $\mu_\Upsilon, \iota_\Upsilon, \eta_\Upsilon : \mathcal{O} \to [0,1]$ are called the functions of membership, indeterminacy and non-membership, respectively. Also, $\mu_\Upsilon(o)$, $\iota_\Upsilon(o)$ and $\eta_\Upsilon(o)$ denote the degrees of membership, indeterminacy and non-membership of $o \in \mathcal{O}$ to the set Υ respectively.

Remark. Throughout the paper, Υ means the single valued neutrosophic set.

Definition 2.5. ([31]) Let $\mathcal{O}$ be a universal set and $D[0,1]$ be the set of all closed subintervals of the interval [0,1]. Then, an interval neutrosophic set (INS) Υ in $\mathcal{O}$ is characterized by

$$\widetilde{\Upsilon} = \{(\widetilde{\mu}_{\widetilde{\Upsilon}}, \widetilde{\iota}_{\widetilde{\Upsilon}}, \widetilde{\eta}_{\widetilde{\Upsilon}})/o : o \in \mathcal{O}\}$$

where $\widetilde{\mu}_{\widetilde{\Upsilon}}, \widetilde{\iota}_{\widetilde{\Upsilon}}, \widetilde{\eta}_{\widetilde{\Upsilon}} : \mathcal{O} \to D[0,1]$ are termed to be the functions of membership, indeterminacy and non-membership, respectively. Also, $\widetilde{\mu}^L_{\widetilde{\Upsilon}}(o)$, $\widetilde{\mu}^U_{\widetilde{\Upsilon}}(o)$ denote the lower and upper degrees of membership, $\widetilde{\iota}^L_{\widetilde{\Upsilon}}(o)$, $\widetilde{\iota}^U_{\widetilde{\Upsilon}}(o)$ denote the lower and upper degrees of indeterminacy and $\widetilde{\eta}^L_{\widetilde{\Upsilon}}(o)$, $\widetilde{\eta}^U_{\widetilde{\Upsilon}}(o)$ denote the lower and upper degrees of non-membership, respectively.

Definition 2.6. ([13]) Let $\mathcal{O}$ be a universal set. Then, a cubic set (CS) Δ in $\mathcal{O}$ is a structure in the following form

$$\Delta = \{(\widetilde{\Psi}(o), \Psi(o))/o : o \in \mathcal{O}\}$$

where $\widetilde{\Psi}$ is an IFS in $\mathcal{O}$ and Ψ is an FS in $\mathcal{O}$

Definition 2.7. ([7,14]) Let $\mathcal{O}$ be a universal set. Then, a neutrosophic cubic set (NCS) Λ in $\mathcal{O}$ is a structure in the following form

$$\Lambda = \{(\widetilde{\Upsilon}(o), \Upsilon(o))/o : o \in \mathcal{O}\}$$

where $\widetilde{\Upsilon}$ is an INS in $\mathcal{O}$ and Υ is an NS in $\mathcal{O}$

Simply, the structure of neutrosophic cubic set can be considered as follows

$$\widetilde{\Upsilon} = \{((\widetilde{\mu}_{\widetilde{\Upsilon}}(o), \widetilde{\iota}_{\widetilde{\Upsilon}}(o), \widetilde{\eta}_{\widetilde{\Upsilon}}(o)), (\mu_{\Upsilon}(o), \iota_{\Upsilon}(o), \eta_{\Upsilon}(o)))/o : o \in \mathcal{O}\}$$

Furthermore, $((\widetilde{\mu}_{\widetilde{\Upsilon}}(o), \widetilde{\iota}_{\widetilde{\Upsilon}}(o), \widetilde{\eta}_{\widetilde{\Upsilon}}(o)), (\mu_{\Upsilon}(o), \iota_{\Upsilon}(o), \eta_{\Upsilon}(o)))$, which is an element in Λ, is called a neutrsophic cubic element (NCE). For simplicity, an NCE is denoted by $\upsilon_k = (\widetilde{\mu}_k, \widetilde{\iota}_k, \widetilde{\eta}_k, \mu_k, \iota_k, \eta_k)$.

Example 2.8. Suppose that $\mathcal{O} = \{o_1, o_2, o_3, o_4\}$ be a universal set. Then,

(i): a fuzzy set Ψ in $\mathcal{O}$ can be exemplified as follows.

$$\Psi = \{0.3/o_1, 0.7/o_2, 1/o_3, 0.1/o_4\}.$$

(ii): an interval-valued fuzzy set $\widetilde{\Psi}$ in $\mathcal{O}$ can be illustrated as follows.

$$\widetilde{\Psi} = \{[0.3, 0.4]/o_1, [0.4, 0.7]/o_2, [0, 1]/o_3, [0.1, 0.1]/o_4\}$$

(iii): As a sample of a neutrosophic set Υ in $\mathcal{O}$, the following set can be given.

$$\Upsilon = \{(0.3, 0.7, 0.2)/o_1, (0.1, 0.1, 0.1)/o_2, (1, 0.7, 0.3)/o_3, (0, 0, 0.9)/o_4\}$$

(iv): an interval neutrosophic set $\widetilde{\Upsilon}$ in $\mathcal{O}$ can be shown in the following form.

$$\widetilde{\Upsilon} = \left\{ \begin{array}{l} ([0.2, 0.6], [0.4, 0.4], [0.1, 0.8])/o_1, ([0.5, 1], [0.3, 0.4], [0.6, 0.7])/o_2, \\ ([0, 0], [0.1, 0.8], [0.2, 0.4])/o_3, ([0.1, 0.4], [0.3, 0.5], [0.2, 0.2])/o_4 \end{array} \right\}.$$

(v): a cubic set Δ in $\mathcal{O}$ is can be exemplified as follows.

$$\Delta = \{([0.2, 0.6], 0.5)/o_1, ([0.1, 0.5], 0.2)/o_2, ([0.5, 0.7], 1)/o_3, ([0.1, 1], 0.4)/o_4\}.$$

(vi): a neutrosophic cubic set Λ in $\mathcal{O}$ is an object having the following form

$$\Lambda = \left\{ \begin{array}{l} (([0.1, 0.4], [0.1, 0.4], [0.3, 0.6]), (0.5, 0.3, 0.8))/o_1, \\ (([0.8, 0.9], [0.1, 0.7], [0.2, 0.7]), (0.6, 1, 0.7))/o_2, \\ (([0.3, 1], [0, 0.5], [0.4, 0.6]), (0, 0.3, 0.7))/o_3, \\ (([0.4, 0.9], [0.2, 0.2], [0.6, 0.8]), (0.1, 0.1, 0.1))/o_4 \end{array} \right\}.$$

3. **Score and Accuracy Functions of Neutrosophic Cubic Element**

We can develop the score and accuracy functions to compare two NCEs. For comparison of two NCEs, firstly, we use the score functions, sometimes the score values of two NCEs can be equal although they have different components of membership, indeterminacy and non-membership functions. In such cases, it is aimed to achieve a ranking priority between the NCEs using the accuracy function.

Definition 3.1. Let $\upsilon_k = (\widetilde{\mu}_k, \widetilde{\iota}_k, \widetilde{\eta}_k, \mu_k, \iota_k, \eta_k)$ be an NCE, where $\widetilde{\mu}_k = [\widetilde{\mu}_k^L, \widetilde{\mu}_k^U]$, $\widetilde{\iota}_k = [\widetilde{\iota}_k^L, \widetilde{\iota}_k^U]$ and $\widetilde{\eta}_k = [\widetilde{\eta}_k^L, \widetilde{\eta}_k^U]$. Then, the score function f_{scr} is defined by

$$f_{scr} = \frac{\frac{1}{2}\big(6 + (\widetilde{\mu}_k^L + \widetilde{\mu}_k^U) - 2(\widetilde{\iota}_k^L + \widetilde{\iota}_k^U) - (\widetilde{\eta}_k^L + \widetilde{\eta}_k^U)\big) + \big(3 + \mu_k - 2\iota_k - \eta_k\big)}{8}. \tag{1}$$

Proposition 3.2. *The score function of any NCE lies between 0 to 1, i.e., $f_{scr}(\upsilon_k) \in [0, 1]$ for any υ_k.*

Proof. Consider $\upsilon_k = (\widetilde{\mu}_k, \widetilde{\iota}_k, \widetilde{\eta}_k, \mu_k, \iota_k, \eta_k)$. By using the definitions of INS and NS, we have all $\widetilde{\mu}_k^L$, $\widetilde{\mu}_k^U$, $\widetilde{\iota}_k^L$, $\widetilde{\iota}_k^U$, $\widetilde{\eta}_k^L$,$\widetilde{\eta}_k^U$, μ_k, ι_k, $\eta_k \in [0, 1]$.
Then, it is easily seen that

$$0 \leq \widetilde{\mu}_k^L \leq 1, \quad 0 \leq \widetilde{\mu}_k^U \leq 1 \Rightarrow 0 \leq \widetilde{\mu}_k^L + \widetilde{\mu}_k^U \leq 2, \tag{2}$$

$$0 \leq \widetilde{\iota}_k^L \leq 1, \quad 0 \leq \widetilde{\iota}_k^U \leq 1 \Rightarrow 0 \leq \widetilde{\iota}_k^L + \widetilde{\iota}_k^U \leq 2 \Rightarrow -4 \leq -2(\widetilde{\iota}_k^L + \widetilde{\iota}_k^U) \leq 0, \tag{3}$$

and

$$0 \leq \widetilde{\eta}_k^L \leq 1, \quad 0 \leq \widetilde{\eta}_k^U \leq 1 \Rightarrow 0 \leq \widetilde{\eta}_k^L + \widetilde{\eta}_k^U \leq 2 \Rightarrow -2 \leq -\widetilde{\eta}_k^L - \widetilde{\eta}_k^U \leq 0. \tag{4}$$

By adding Eqs. (2), (3) and (4), we obtain

$$\begin{aligned} &-6 \leq (\widetilde{\mu}_k^L + \widetilde{\mu}_k^U) - 2(\widetilde{\iota}_k^L + \widetilde{\iota}_k^U) - (\widetilde{\eta}_k^L + \widetilde{\eta}_k^U) \leq 2 \\ &\Rightarrow 0 \leq \frac{1}{2}\big(6 + (\widetilde{\mu}_k^L + \widetilde{\mu}_k^U) - 2(\widetilde{\iota}_k^L + \widetilde{\iota}_k^U) - (\widetilde{\eta}_k^L + \widetilde{\eta}_k^U) \leq 4 \end{aligned} \tag{5}$$

In addition, we obtain

$$\begin{aligned} 0 \leq \mu_k \leq 1, \quad -2 \leq -2\iota_k \leq 0, \quad -1 \leq -\eta_k \leq 0 \Rightarrow -3 \leq \mu_k - 2\iota_k - \eta_k \leq 1 \\ \Rightarrow 0 \leq 3 + \mu_k - 2\iota_k - \eta_k \leq 4. \end{aligned} \tag{6}$$

By adding Eqs. (5) and (6) and then dividing by 8, we have

$$0 \leq \frac{\frac{1}{2}\big(6 + (\widetilde{\mu}_k^L + \widetilde{\mu}_k^U) - 2(\widetilde{\iota}_k^L + \widetilde{\iota}_k^U) - (\widetilde{\eta}_k^L + \widetilde{\eta}_k^U)\big) + \big(3 + \mu_k - 2\iota_k - \eta_k\big)}{8} \leq 1. \tag{7}$$

This result completes the proof. □

Definition 3.3. Let $\upsilon_k = (\widetilde{\mu}_k, \widetilde{\iota}_k, \widetilde{\eta}_k, \mu_k, \iota_k, \eta_k)$ be an NCE, where $\widetilde{\mu}_k = [\widetilde{\mu}_k^L, \widetilde{\mu}_k^U]$, $\widetilde{\iota}_k = [\widetilde{\iota}_k^L, \widetilde{\iota}_k^U]$ and $\widetilde{\eta}_k = [\widetilde{\eta}_k^L, \widetilde{\eta}_k^U]$. Then, the accuracy function f_{acr} is defined by

$$f_{acr} = \frac{\frac{1}{2}\left(\widetilde{\mu}_k^L + \widetilde{\mu}_k^U + \widetilde{\iota}_k^L + \widetilde{\iota}_k^U + \widetilde{\eta}_k^L + \widetilde{\eta}_k^U\right) + \mu_k + \iota_k + \eta_k}{6}. \tag{8}$$

Proposition 3.4. *The accuracy function of any NCE lies between 0 to 1, i.e., $f_{acr}(\upsilon_k) \in [0,1]$ for any υ_k.*

Proof. Consider $\upsilon_k = (\widetilde{\mu}_k, \widetilde{\iota}_k, \widetilde{\eta}_k, \mu_k, \iota_k, \eta_k)$. Since $\widetilde{\mu}_k^L$, $\widetilde{\mu}_k^U$, $\widetilde{\iota}_k^L$, $\widetilde{\iota}_k^U$, $\widetilde{\eta}_k^L$,$\widetilde{\eta}_k^U$, μ_k, ι_k, $\eta_k \in [0,1]$ from the definitions of INS and NS, it is obvious that

$$0 \leq \frac{1}{2}\left(\widetilde{\mu}_k^L + \widetilde{\mu}_k^U + \widetilde{\iota}_k^L + \widetilde{\iota}_k^U + \widetilde{\eta}_k^L + \widetilde{\eta}_k^U\right) + \mu_k + \iota_k + \eta_k \leq 6. \tag{9}$$

Dividing by 6, we have

$$0 \leq \frac{\frac{1}{2}\left(\widetilde{\mu}_k^L + \widetilde{\mu}_k^U + \widetilde{\iota}_k^L + \widetilde{\iota}_k^U + \widetilde{\eta}_k^L + \widetilde{\eta}_k^U\right) + \mu_k + \iota_k + \eta_k}{6} \leq 1. \tag{10}$$

Thus, the proof is complete. □

The following definition is proposed to compare two NCEs, thereby ensuring the order priority between the NCEs.

Definition 3.5. Let υ_1 and υ_2 be two NCEs. The comparison method for any two NCEs υ_1 and υ_2 is defined as follows:

(1) If $f_{scr}(\upsilon_1) < f_{scr}(\upsilon_1)$ then $\upsilon_1 \prec \upsilon_2$
(2) $f_{scr}(\upsilon_1) > f_{scr}(\upsilon_1)$ then $\upsilon_1 \succ \upsilon_2$
(3) $f_{scr}(\upsilon_1) = f_{scr}(\upsilon_1)$ then
- when $f_{acr}(\upsilon_1) < f_{acr}(\upsilon_1)$, $\upsilon_1 \prec \upsilon_2$
- when $f_{acr}(\upsilon_1) > f_{acr}(\upsilon_1)$, $\upsilon_1 \succ \upsilon_2$
- when $f_{acr}(\upsilon_1) = f_{acr}(\upsilon_1)$, $\upsilon_1 = \upsilon_2$

Example 3.6. We consider any two NCEs as $\upsilon_1 = ([0.4, 0.6], [0.3, 0.4], [0.4, 0.5], 0.8, 0.6, 0.5)$ and $\upsilon_2 = ([0.5, 0.7], [0.2, 0.5], [0.5, 0.6], 0.5, 0.6, 0.2)$. Then, it is obtain $f_{scr}(\upsilon_1) = f_{scr}(\upsilon_2) = 0.5968$. If we compare this two NCEs by using the accuracy functions, then we have $\upsilon_1 \succ \upsilon_2$ since $f_{acr}(\upsilon_1) = 0.5333 > 0.4666 = f_{acr}(\upsilon_2)$.

4. **Hamacher Operations of Neutrosophic Cubic Elements**

The concepts of t-norm and t-conorm, which are useful notions in fuzzy set theory and neutrosophic set theory, are proposed by Roychowdhury and Wang [25]. In 1978, Hamacher [10] defined Hamacher sum ($\oplus_\hbar$) and Hamacher product ($\otimes_\hbar$), which are samples of t-conorm and t-norm, respectively. Hamacher t-norm and Hamacher t-conorm are given as follows.
For all $\hat{a}, \hat{b} \in [0,1]$,

$$\hat{a} \oplus_{\hbar} \hat{b} = \frac{\hat{a}+\hat{b}-\hat{a}\hat{b}-(1-\xi)\hat{a}\hat{b}}{1-(1-\xi)\hat{a}\hat{b}},$$
$$\hat{a} \otimes_{\hbar} \hat{b} = \frac{\hat{a}\hat{b}}{\xi+(1-\xi)(\hat{a}+\hat{b}-\hat{a}\hat{b})}$$

where $\xi > 0$.

Especially, if it is taken $\xi = 1$, then Hamacher t-norm and Hamacher t-conorm will reduce to the form

$$\hat{a} \oplus_{\hbar} \hat{b} = \hat{a} + \hat{b} - \hat{a}\hat{b},$$
$$\hat{a} \otimes_{\hbar} \hat{b} = \hat{a}\hat{b}$$

which represent algebraic t-norm and t-conorm, respectively.

If it is taken $\xi = 2$, then Hamacher t-norm and Hamacher t-conorm will conclude to the form

$$\hat{a} \oplus_{\hbar} \hat{b} = \frac{\hat{a}+\hat{b}}{1-\hat{a}\hat{b}},$$
$$\hat{a} \otimes_{\hbar} \hat{b} = \frac{\hat{a}\hat{b}}{1+(1-\hat{a})(1-\hat{b})}$$

which are called Einstein t-norm and Einstein t-conorm, respectively.

By using the Hamacher t-norm and Hamacher t-conorm, we can create the Hamacher sum and Hamacher product of two NCEs.

Definition 4.1. Let $\upsilon_1 = (\widetilde{\mu}_1, \widetilde{\iota}_1, \widetilde{\eta}_1, \mu_1, \iota_1, \eta_1)$ and $\upsilon_2 = (\widetilde{\mu}_2, \widetilde{\iota}_2, \widetilde{\eta}_2, \mu_2, \iota_2, \eta_2)$ be two CNEs and $\xi > 0$, then the operational rules based on the Hamacher t-norm and Hamacher t-conorm are established as follows:

(a):

$$\upsilon_1 \oplus_{\hbar} \upsilon_2 = \begin{pmatrix} \left[\frac{\widetilde{\mu}_1^L+\widetilde{\mu}_2^L-\widetilde{\mu}_1^L\widetilde{\mu}_2^L-(1-\xi)\widetilde{\mu}_1^L\widetilde{\mu}_2^L}{1-(1-\xi)\widetilde{\mu}_1^L\widetilde{\mu}_2^L}, \frac{\widetilde{\mu}_1^U+\widetilde{\mu}_2^U-\widetilde{\mu}_1^U\widetilde{\mu}_2^U-(1-\xi)\widetilde{\mu}_1^U\widetilde{\mu}_2^U}{1-(1-\xi)\widetilde{\mu}_1^U\widetilde{\mu}_2^U}\right], \\ \left[\frac{\widetilde{\iota}_1^L\widetilde{\iota}_2^L}{\xi+(1-\xi)(\widetilde{\iota}_1^L+\widetilde{\iota}_2^L-\widetilde{\iota}_1^L\widetilde{\iota}_2^L)}, \frac{\widetilde{\iota}_1^U\widetilde{\iota}_2^U}{\xi+(1-\xi)(\widetilde{\iota}_1^U+\widetilde{\iota}_2^U-\widetilde{\iota}_1^U\widetilde{\iota}_2^U)}\right], \\ \left[\frac{\widetilde{\eta}_1^L\widetilde{\eta}_2^L}{\xi+(1-\xi)(\widetilde{\eta}_1^L+\widetilde{\eta}_2^L-\widetilde{\eta}_1^L\widetilde{\eta}_2^L)}, \frac{\widetilde{\eta}_1^U\widetilde{\eta}_2^U}{\xi+(1-\xi)(\widetilde{\eta}_1^U+\widetilde{\eta}_2^U-\widetilde{\eta}_1^U\widetilde{\eta}_2^U)}\right], \\ \frac{\mu_1\mu_2}{\xi+(1-\xi)(\mu_1+\mu_2-\mu_1\mu_2)}, \frac{\iota_1+\iota_2-\iota_1\iota_2-(1-\xi)\iota_1\iota_2}{1-(1-\xi)\iota_1\iota_2}, \frac{\eta_1+\eta_2-\eta_1\eta_2-(1-\xi)\eta_1\eta_2}{1-(1-\xi)\eta_1\eta_2} \end{pmatrix}. \tag{11}$$

(b):

$$\upsilon_1 \otimes_{\hbar} \upsilon_2 = \begin{pmatrix} \left[\frac{\widetilde{\mu}_1^L\widetilde{\mu}_2^L}{\xi+(1-\xi)(\widetilde{\mu}_1^L+\widetilde{\mu}_2^L-\widetilde{\mu}_1^L\widetilde{\mu}_2^L)}, \frac{\widetilde{\mu}_1^U\widetilde{\mu}_2^U}{\xi+(1-\xi)(\widetilde{\mu}_1^U+\widetilde{\mu}_2^U-\widetilde{\mu}_1^U\widetilde{\mu}_2^U)}\right], \\ \left[\frac{\widetilde{\iota}_1^L+\widetilde{\iota}_2^L-\widetilde{\iota}_1^L\widetilde{\iota}_2^L-(1-\xi)\widetilde{\iota}_1^L\widetilde{\iota}_2^L}{1-(1-\xi)\widetilde{\iota}_1^L\widetilde{\iota}_2^L}, \frac{\widetilde{\iota}_1^U+\widetilde{\iota}_2^U-\widetilde{\iota}_1^U\widetilde{\iota}_2^U-(1-\xi)\widetilde{\iota}_1^U\widetilde{\iota}_2^U}{1-(1-\xi)\widetilde{\iota}_1^U\widetilde{\iota}_2^U}\right], \\ \left[\frac{\widetilde{\eta}_1^L+\widetilde{\eta}_2^L-\widetilde{\eta}_1^L\widetilde{\eta}_2^L-(1-\xi)\widetilde{\eta}_1^L\widetilde{\eta}_2^L}{1-(1-\xi)\widetilde{\eta}_1^L\widetilde{\eta}_2^L}, \frac{\widetilde{\eta}_1^U+\widetilde{\eta}_2^U-\widetilde{\eta}_1^U\widetilde{\eta}_2^U-(1-\xi)\widetilde{\eta}_1^U\widetilde{\eta}_2^U}{1-(1-\xi)\widetilde{\eta}_1^U\widetilde{\eta}_2^U}\right], \\ \frac{\mu 1+\mu_2-\mu_1\mu_2-(1-\xi)\mu_1\mu_2}{1-(1-\xi)\mu_1\mu_2}, \frac{\iota_1\iota_2}{\xi+(1-\xi)(\iota_1+\iota_2-\iota_1\iota_2)}, \frac{\eta_1\eta_2}{\xi+(1-\xi)(\eta_1+\eta_2-\eta_1\eta_2)} \end{pmatrix}. \tag{12}$$

(c):

$$q\upsilon_1 = \begin{pmatrix} \left[\frac{(1+(\xi-1)\widetilde{\mu}_1^L)^q-(1-\widetilde{\mu}_1^L)^q}{(1+(\xi-1)\widetilde{\mu}_1^L)^q+(\xi-1)(1-\widetilde{\mu}_1^L)^q}, \frac{(1+(\xi-1)\widetilde{\mu}_1^U)^q-(1-\widetilde{\mu}_1^U)^q}{(1+(\xi-1)\widetilde{\mu}_1^U)^q+(\xi-1)(1-\widetilde{\mu}_1^U)^q}\right], \\ \left[\frac{\xi(\widetilde{\iota}_1^L)^q}{(1+(\xi-1)(1-\widetilde{\iota}_1^L))^q+(\xi-1)(\widetilde{\iota}_1^L)^q}, \frac{\xi(\widetilde{\iota}_1^U)^q}{(1+(\xi-1)(1-\widetilde{\iota}_1^U))^q+(\xi-1)(\widetilde{\iota}_1^U)^q},\right], \\ \left[\frac{\xi(\widetilde{\eta}_1^L)^q}{(1+(\xi-1)(1-\widetilde{\eta}_1^L))^q+(\xi-1)(\widetilde{\eta}_1^L)^q}, \frac{\xi(\widetilde{\eta}_1^U)^q}{(1+(\xi-1)(1-\widetilde{\eta}_1^U))^q+(\xi-1)(\widetilde{\eta}_1^U)^q},\right], \\ \frac{\xi\mu_1^q}{(1+(\xi-1)(1-\mu_1))^q+(\xi-1)\mu_1^q}, \frac{(1+(\xi-1)\iota_1)^q-(1-\iota_1)^q}{(1+(\xi-1)\iota_1)^q+(\xi-1)(1-\iota_1)^q}, \frac{(1+(\xi-1)\eta_1)^q-(1-\eta_1)^q}{(1+(\xi-1)\eta_1)^q+(\xi-1)(1-\eta_1)^q} \end{pmatrix} \tag{13}$$

where $q > 0$.

(d):

$$\upsilon_1^q = \left(\begin{array}{c} \big[\frac{\xi(\widetilde{\mu}_1^L)^q}{(1+(\xi-1)(1-\widetilde{\mu}_1^L))^q+(\xi-1)(\widetilde{\mu}_1^L)^q}, \frac{\xi(\widetilde{\mu}_1^U)^q}{(1+(\xi-1)(1-\widetilde{\mu}_1^U))^q+(\xi-1)(\widetilde{\mu}_1^U)^q},\big], \\ \big[\frac{(1+(\xi-1)\widetilde{\iota}_1^L)^q-(1-\widetilde{\iota}_1^L)^q}{(1+(\xi-1)\widetilde{\iota}_1^L)^q+(\xi-1)(1-\widetilde{\iota}_1^L)^q}, \frac{(1+(\xi-1)\widetilde{\iota}_1^U)^q-(1-\widetilde{\iota}_1^U)^q}{(1+(\xi-1)\widetilde{\iota}_1^U)^q+(\xi-1)(1-\widetilde{\iota}_1^U)^q}\big], \\ \big[\frac{(1+(\xi-1)\widetilde{\eta}_1^L)^q-(1-\widetilde{\eta}_1^L)^q}{(1+(\xi-1)\widetilde{\eta}_1^L)^q+(\xi-1)(1-\widetilde{\eta}_1^L)^q}, \frac{(1+(\xi-1)\widetilde{\eta}_1^U)^q-(1-\widetilde{\eta}_1^U)^q}{(1+(\xi-1)\widetilde{\eta}_1^U)^q+(\xi-1)(1-\widetilde{\eta}_1^U)^q}\big], \\ \frac{(1+(\xi-1)\mu_1)^q-(1-\mu_1)^q}{(1+(\xi-1)\mu_1)^q+(\xi-1)(1-\mu_1)^q}, \frac{\xi\iota_1^q}{(1+(\xi-1)(1-\iota_1))^q+(\xi-1)\iota_1^q}, \frac{\xi\eta_1^q}{(1+(\xi-1)(1-\eta_1))^q+(\xi-1)\eta_1^q} \end{array} \right) \tag{14}$$

where $q > 0$.

Example 4.2. Assume that two NCEs are $\upsilon_1 = ([0.4, 1], [0.7, 0.8], [0, 0.2], 0.5, 0, 0.7)$ and $\upsilon_1 = ([0.2, 0.4], [0.5, 0.6], [0.5, 0.6], 0.1, 1, 0.4)$ and $q = 2$. Then, for $\xi = 3$

$\upsilon_1 \oplus_\hbar \upsilon_2 = ([0.8095, 1], [0.2692, 0.4137], [0, 0.0731], 0.0263, 1, 0.8846)$,

$\upsilon_1 \otimes_\hbar \upsilon_2 = ([0.0408, 0.4], [0.9117, 0.9591], [0.5, 0.7419], 0.5909, 0, 0.2058)$,

$q\upsilon_2 = ([0.4074, 0.7272], [0.1666, 0.2727], [0.1666, 0.2727], 0.0038, 1, 0.7272)$,

$\upsilon_1^q = ([0.1237, 1], [0.9545, 0.9824], [0, 0.4074], 0.8333, 0, 0.4152)$.

Proposition 4.3. *Let υ_1 and υ_2 be two NCEs and $q, q' > 0$.*

(1): $\upsilon_1 \oplus_\hbar \upsilon_2 = \upsilon_2 \oplus_\hbar \upsilon_1$.

(2): $\upsilon_1 \otimes_\hbar \upsilon_2 = \upsilon_2 \otimes_\hbar \upsilon_1$.

(3): $q(\upsilon_1 \oplus_\hbar \upsilon_2) = q\upsilon_1 \oplus_\hbar q\upsilon_2$.

(4): $q\upsilon_1 \oplus_\hbar q'\upsilon_1) = (q + q')\upsilon_1$.

(5): $(\upsilon_1 \otimes_\hbar \upsilon_2)^q = \upsilon_1^q \otimes_\hbar \upsilon_2^q$.

(6): $\upsilon_1^q \otimes_\hbar \upsilon_1^{q'} = \upsilon_1^{q+q'}$.

Proof. They are easily seen from the formulas in Definition 4.1, hence omitted. □

5. Neutrosophic Cubic Hamacher Weighted Aggregation Operators

In this section, we will introduce the neutrodophic cubic Hamacher weighted averaging operator and neutrosophic cubic Hamacher weighted geometric operator.

Definition 5.1. Let υ_k $(k = 1, 2, ..., r)$ be a collection of the CNEs. Then, neutrosophic cubic Hamacher weighted averaging (NCHWA) operator is defined as the mapping $NCHWA_\varpi : \mathcal{N}^r \to \mathcal{N}$ such that

$$NCHWA_\varpi(\upsilon_1, \upsilon_2, ..., \upsilon_r) = \bigoplus_{k=1}^{r}{}_\hbar(\varpi_k \upsilon_k) \tag{15}$$

where $\mathcal{N}$ is the set of all NCEs and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_r)^T$ is weight vector of $(\upsilon_1, \upsilon_2, ..., \upsilon_r)$ such that $\varpi_k \in [0, 1]$ and $\sum_{k=1}^{r} \varpi_k = 1$.

Theorem 5.2. *The aggregation value of NCEs by using the NCHWA operator is still an NCE, and even*

$$NCHWA_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_r)=\bigoplus_{\hbar\, k=1}^{r}(\varpi_k\upsilon_k)=$$
$$\left(\begin{array}{l}\left[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\mu}_k^L)^{\varpi_k}},\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\mu}_k^U)^{\varpi_k}}\right],\\ \left[\frac{\xi\prod_{k=1}^{r}(\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\iota}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\iota}_k^L)^{\varpi_k}},\frac{\xi\prod_{k=1}^{r}(\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\iota}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\iota}_k^U)^{\varpi_k}}\right],\\ \left[\frac{\xi\prod_{k=1}^{r}(\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\eta}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\eta}_k^L)^{\varpi_k}},\frac{\xi\prod_{k=1}^{r}(\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\eta}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\eta}_k^U)^{\varpi_k}}\right],\\ \frac{\xi\prod_{k=1}^{r}(\mu_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\mu_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\mu_k)^{\varpi_k}},\frac{\prod_{k=1}^{r}(1+(\xi-1)\iota_k)^{\varpi_k}-\prod_{k=1}^{r}(1-\iota_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\iota_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\iota_k)^{\varpi_k}},\frac{\prod_{k=1}^{r}(1+(\xi-1)\eta_k)^{\varpi_k}-\prod_{k=1}^{r}(1-\eta_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\eta_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\eta_k)^{\varpi_k}}\end{array}\right) \quad (16)$$

Proof. This can be proved by mathematical induction.
When $r=1$, for the left side of Eq. (16), $NCHWA_{\varpi}(\upsilon_1)=\varpi_1\upsilon_1=\upsilon_1$ and for the right side of Eq. (16) we have

$$\left(\begin{array}{l}\left[\frac{1+(\xi-1)\widetilde{\mu}_1^L-(1-\widetilde{\mu}_1^L)}{1+(\xi-1)\widetilde{\mu}_1^L+(\xi-1)(1-\widetilde{\mu}_1^L)},\frac{1+(\xi-1)\widetilde{\mu}_1^U-(1-\widetilde{\mu}_1^U)}{1+(\xi-1)\widetilde{\mu}_1^U+(\xi-1)(1-\widetilde{\mu}_1^U)}\right],\\ \left[\frac{\xi\widetilde{\iota}_1^L}{1+(\xi-1)(1-\widetilde{\iota}_1^L)+(\xi-1)\widetilde{\iota}_1^L},\frac{\xi\widetilde{\iota}_1^U}{1+(\xi-1)(1-\widetilde{\iota}_1^U)+(\xi-1)\widetilde{\iota}_1^U}\right],\\ \left[\frac{\xi\widetilde{\eta}_1^L}{1+(\xi-1)(1-\widetilde{\eta}_1^L)+(\xi-1)\widetilde{\eta}_1^L},\frac{\xi\widetilde{\eta}_1^U}{1+(\xi-1)(1-\widetilde{\eta}_1^U)+(\xi-1)\widetilde{\eta}_1^U}\right],\\ \frac{\xi\mu_1}{1+(\xi-1)(1-\mu_1)+(\xi-1)\mu_1},\frac{1+(\xi-1)\iota_1-(1-\iota_1)}{1+(\xi-1)\iota_1+(\xi-1)(1-\iota_1)},\frac{1+(\xi-1)\eta_1-(1-\eta_1)}{1+(\xi-1)\eta_1+(\xi-1)(1-\eta_1)}\end{array}\right).$$

Suppose that Eq. (16) holds for $r=t$, i.e., we have

$$NCHWA_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_t)=\bigoplus_{\hbar\, k=1}^{t}(\varpi_k\upsilon_k)=$$
$$\left(\begin{array}{l}\left[\frac{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}-\prod_{k=1}^{t}(1-\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\widetilde{\mu}_k^L)^{\varpi_k}},\frac{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}-\prod_{k=1}^{t}(1-\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\widetilde{\mu}_k^U)^{\varpi_k}}\right],\\ \left[\frac{\xi\prod_{k=1}^{t}(\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\iota}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\iota}_k^L)^{\varpi_k}},\frac{\xi\prod_{k=1}^{t}(\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\iota}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\iota}_k^U)^{\varpi_k}}\right],\\ \left[\frac{\xi\prod_{k=1}^{t}(\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\eta}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\eta}_k^L)^{\varpi_k}},\frac{\xi\prod_{k=1}^{t}(\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\eta}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\eta}_k^U)^{\varpi_k}}\right],\\ \frac{\xi\prod_{k=1}^{t}(\mu_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\mu_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\mu_k)^{\varpi_k}},\frac{\prod_{k=1}^{t}(1+(\xi-1)\iota_k)^{\varpi_k}-\prod_{k=1}^{t}(1-\iota_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\iota_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\iota_k)^{\varpi_k}},\frac{\prod_{k=1}^{t}(1+(\xi-1)\eta_k)^{\varpi_k}-\prod_{k=1}^{t}(1-\eta_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\eta_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\eta_k)^{\varpi_k}}\end{array}\right).$$

When $r = t + 1$,

$NCHWA_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_{t+1}) = NCHWA_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_t) \oplus_{\hbar} (\varpi_{t+1}\upsilon_{t+1}) =$

$$\begin{pmatrix} \left[\frac{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}-\prod_{k=1}^{t}(1-\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\widetilde{\mu}_k^L)^{\varpi_k}}, \frac{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}-\prod_{k=1}^{t}(1-\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\widetilde{\mu}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^{t}(\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\iota}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\iota}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{t}(\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\iota}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\iota}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^{t}(\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\eta}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\eta}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{t}(\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\eta}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\eta}_k^U)^{\varpi_k}}\right], \\ \frac{\xi\prod_{k=1}^{t}(\mu_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\mu_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\mu_k)^{\varpi_k}}, \frac{\prod_{k=1}^{t}(1+(\xi-1)\iota_k)^{\varpi_k}-\prod_{k=1}^{t}(1-\iota_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\iota_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\iota_k)^{\varpi_k}}, \frac{\prod_{k=1}^{t}(1+(\xi-1)\eta_k)^{\varpi_k}-\prod_{k=1}^{t}(1-\eta_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\eta_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\eta_k)^{\varpi_k}} \end{pmatrix}$$

$$\oplus_{\hbar} \begin{pmatrix} \left[\frac{1+(\xi-1)\widetilde{\mu}_{t+1}^L-(1-\widetilde{\mu}_{t+1}^L)}{1+(\xi-1)\widetilde{\mu}_{t+1}^L+(\xi-1)(1-\widetilde{\mu}_{t+1}^L)}, \frac{1+(\xi-1)\widetilde{\mu}_{t+1}^U-(1-\widetilde{\mu}_{t+1}^U)}{1+(\xi-1)\widetilde{\mu}_{t+1}^U+(\xi-1)(1-\widetilde{\mu}_{t+1}^U)}\right], \\ \left[\frac{\xi\widetilde{\iota}_1^L}{1+(\xi-1)(1-\widetilde{\iota}_{t+1}^L)+(\xi-1)\widetilde{\iota}_{t+1}^L}, \frac{\xi\widetilde{\iota}_{t+1}^U}{1+(\xi-1)(1-\widetilde{\iota}_{t+1}^U)+(\xi-1)\widetilde{\iota}_{t+1}^U}\right], \\ \left[\frac{\xi\widetilde{\eta}_{t+1}^L}{1+(\xi-1)(1-\widetilde{\eta}_{t+1}^L)+(\xi-1)\widetilde{\eta}_{t+1}^L}, \frac{\xi\widetilde{\eta}_{t+1}^U}{1+(\xi-1)(1-\widetilde{\eta}_{t+1}^U)+(\xi-1)\widetilde{\eta}_{t+1}^U}\right], \\ \frac{\xi\mu_{t+1}}{1+(\xi-1)(1-\mu_{t+1})+(\xi-1)\mu_{t+1}}, \frac{1+(\xi-1)\iota_{t+1}-(1-\iota_{t+1})}{1+(\xi-1)\iota_{t+1}+(\xi-1)(1-\iota_{t+1})}, \frac{1+(\xi-1)\eta_{t+1}-(1-\eta_{t+1})}{1+(\xi-1)\eta_{t+1}+(\xi-1)(1-\eta_{t+1})} \end{pmatrix}.$$

$$= \begin{pmatrix} \left[\frac{\prod_{k=1}^{t+1}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}-\prod_{k=1}^{t+1}(1-\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(1-\widetilde{\mu}_k^L)^{\varpi_k}}, \frac{\prod_{k=1}^{t+1}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}-\prod_{k=1}^{t+1}(1-\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(1-\widetilde{\mu}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^{t+1}(\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)(1-\widetilde{\iota}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(\widetilde{\iota}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{t+1}(\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)(1-\widetilde{\iota}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(\widetilde{\iota}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^{t+1}(\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)(1-\widetilde{\eta}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(\widetilde{\eta}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{t+1}(\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)(1-\widetilde{\eta}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(\widetilde{\eta}_k^U)^{\varpi_k}}\right], \\ \frac{\xi\prod_{k=1}^{t+1}(\mu_k)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)(1-\mu_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(\mu_k)^{\varpi_k}}, \frac{\prod_{k=1}^{t+1}(1+(\xi-1)\iota_k)^{\varpi_k}-\prod_{k=1}^{t+1}(1-\iota_k)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)\iota_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(1-\iota_k)^{\varpi_k}}, \frac{\prod_{k=1}^{t+1}(1+(\xi-1)\eta_k)^{\varpi_k}-\prod_{k=1}^{t+1}(1-\eta_k)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)\eta_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(1-\eta_k)^{\varpi_k}}, \end{pmatrix}.$$

So, Eq. (16) holds for $r = t + 1$. Thus, the proof is complete. □

Definition 5.3. Let υ_k $(k = 1, 2, ..., r)$ be a collection of the CNEs. Then, neutrosophic cubic Hamacher weighted geometric (NCHWG) operator is defined as the mapping $NCHWG_{\varpi} : \mathcal{N}^r \to \mathcal{N}$ such that

$$NCHWG_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_r) = \bigotimes_{k=1}^{r}{}_{\hbar}\, \upsilon_k^{\varpi_k} \tag{17}$$

where $\mathcal{N}$ is the set of all NCEs and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_r)^T$ is weight vector of $(\upsilon_1, \upsilon_2, ..., \upsilon_r)$ such that $\varpi_k \in [0, 1]$ and $\sum_{k=1}^{r} \varpi_k = 1$.

Theorem 5.4. *The aggregation value of NCEs by using the NCHWG operator is still an NCE, and even*

$$NCHWG_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_r) = \bigotimes_{k=1}^{r}{}_{\hbar}\, \upsilon_k^{\varpi_k} =$$

$$\left(\begin{array}{l} \Big[\frac{\xi \prod_{k=1}^{r}(\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\mu}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\mu}_k^L)^{\varpi_k}}, \frac{\xi \prod_{k=1}^{r}(\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\mu}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\mu}_k^U)^{\varpi_k}}\Big], \\ \Big[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}_k^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\iota}_k^L)^{\varpi_k}}, \frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}_k^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\iota}_k^U)^{\varpi_k}}\Big], \\ \Big[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}_k^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\eta}_k^L)^{\varpi_k}}, \frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}_k^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\eta}_k^U)^{\varpi_k}}\Big], \\ \frac{\prod_{k=1}^{r}(1+(\xi-1)\mu_k)^{\varpi_k}-\prod_{k=1}^{r}(1-\mu_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\mu_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\mu_k)^{\varpi_k}}, \frac{\xi \prod_{k=1}^{r}(\iota_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\iota_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\iota_k)^{\varpi_k}}, \frac{\xi \prod_{k=1}^{r}(\eta_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\eta_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\eta_k)^{\varpi_k}} \end{array} \right) \quad (18)$$

Proof. It can be demonstrated similar to the proof of Theorem 5.2. □

Theorem 5.5. *(Idempotency)*
Let υ_k $(k = 1, 2, ..., r)$ be a collection of the NCEs. If $\upsilon_k = \upsilon$ for all $k = 1, 2, ..., r$ then

(1): $NCHWA_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_r) = \upsilon.$
(2): $NCHWG_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_r) = \upsilon.$

Proof. Let's prove (2), the other can be proved similar to this. Assume $\upsilon_k = \upsilon$ for all $k = 1, 2, ..., r$. By Theorem 5.4, we obtain that

$$NCHWG_{\Omega}(\upsilon_1, \upsilon_2, ..., \upsilon_r) = \bigotimes_{k=1}^{r}{}_{\hbar}\, \upsilon_k^{\varpi_j} = \bigotimes_{k=1}^{r}{}_{\hbar}\, \upsilon^{\varpi_j} =$$

$$\left(\begin{array}{l} \Big[\frac{\xi \prod_{k=1}^{r}(\widetilde{\mu}^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\mu}^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\mu}^L)^{\varpi_k}}, \frac{\xi \prod_{k=1}^{r}(\widetilde{\mu}^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\mu}^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\mu}^U)^{\varpi_k}}\Big], \\ \Big[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\iota}^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\iota}^L)^{\varpi_k}}, \frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\iota}^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\iota}^U)^{\varpi_k}}\Big], \\ \Big[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\eta}^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\eta}_k^L)^{\varpi_k}}, \frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}_k^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\eta}^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\eta}^U)^{\varpi_k}}\Big], \\ \frac{\prod_{k=1}^{r}(1+(\xi-1)\mu)^{\varpi_k}-\prod_{k=1}^{r}(1-\mu)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\mu)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\mu)^{\varpi_k}}, \frac{\xi \prod_{k=1}^{r}(\iota)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\iota))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\iota)^{\varpi_k}}, \frac{\xi \prod_{k=1}^{r}(\eta)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\eta))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\eta)^{\varpi_k}} \end{array} \right) = \upsilon.$$

□

Theorem 5.6. *(Monotonicity)*
Let υ_k and ν_k $(k = 1, 2, ..., r)$ be two collections of the NCEs. If $\upsilon_k \leq \upsilon'_k$ for all $k = 1, 2, ..., r$ then

(1): $NCHWA_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_r) \leq NCHWA_{\varpi}(\upsilon'_1, \upsilon'_2, ..., \upsilon'_r).$
(2): $NCHWG_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_r) \leq NCHWG_{\varpi}(\upsilon'_1, \upsilon'_2, ..., \upsilon'_r).$

Proof. (1) If $\upsilon_k \leq \upsilon'_k$ then we have

$$\left\{\begin{array}{l} \widetilde{\mu}_k^L \leq \widetilde{\mu'}_k^L,\ \widetilde{\mu}_k^U \leq \widetilde{\mu'}_k^U \\ \widetilde{\iota}_k^L \geq \widetilde{\iota'}_k^L,\ \widetilde{\iota}_k^U \geq \widetilde{\iota'}_k^U \\ \widetilde{\eta}_k^L \geq \widetilde{\eta'}_k^L,\ \widetilde{\eta}_k^L \geq \widetilde{\eta'}_k^L \\ \mu_k \geq \mu'_k,\ \iota_k \leq \iota'_k,\ \eta_k \leq \eta'_k \end{array}\right\}_{(k=1,2,...,r)}.$$

With these assumptions, we find that

$$\left(\begin{array}{l} \left[\frac{\prod_{k=1}^r(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}-\prod_{k=1}^r(1-\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^r(1-\widetilde{\mu}_k^L)^{\varpi_k}}, \frac{\prod_{k=1}^r(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}-\prod_{k=1}^r(1-\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^r(1-\widetilde{\mu}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^r(\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)(1-\widetilde{\iota}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^r(\widetilde{\iota}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^r(\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)(1-\widetilde{\iota}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^r(\widetilde{\iota}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^r(\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)(1-\widetilde{\eta}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^r(\widetilde{\eta}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^r(\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)(1-\widetilde{\eta}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^r(\widetilde{\eta}_k^U)^{\varpi_k}}\right], \\ \frac{\xi\prod_{k=1}^r(\mu_k)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)(1-\mu_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^r(\mu_k)^{\varpi_k}}, \frac{\prod_{k=1}^r(1+(\xi-1)\iota_k)^{\varpi_k}-\prod_{k=1}^r(1-\iota_k)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)\iota_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^r(1-\iota_k)^{\varpi_k}}, \frac{\prod_{k=1}^r(1+(\xi-1)\eta_k)^{\varpi_k}-\prod_{k=1}^r(1-\eta_k)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)\eta_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^r(1-\eta_k)^{\varpi_k}}, \end{array}\right)$$

$$\leq \left(\begin{array}{l} \left[\frac{\prod_{k=1}^r(1+(\xi-1)\widetilde{\mu'}_k^L)^{\varpi_k}-\prod_{k=1}^r(1-\widetilde{\mu'}_k^L)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)\widetilde{\mu'}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^r(1-\widetilde{\mu'}_k^L)^{\varpi_k}}, \frac{\prod_{k=1}^r(1+(\xi-1)\widetilde{\mu'}_k^U)^{\varpi_k}-\prod_{k=1}^r(1-\widetilde{\mu'}_k^U)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)\widetilde{\mu'}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^r(1-\widetilde{\mu'}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^r(\widetilde{\iota'}_k^L)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)(1-\widetilde{\iota'}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^r(\widetilde{\iota'}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^r(\widetilde{\iota'}_k^U)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)(1-\widetilde{\iota'}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^r(\widetilde{\iota'}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^r(\widetilde{\eta'}_k^L)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)(1-\widetilde{\eta'}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^r(\widetilde{\eta'}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^r(\widetilde{\eta'}_k^U)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)(1-\widetilde{\eta'}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^r(\widetilde{\eta'}_k^U)^{\varpi_k}}\right], \\ \frac{\xi\prod_{k=1}^r(\mu'_k)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)(1-\mu'_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^r(\mu'_k)^{\varpi_k}}, \frac{\prod_{k=1}^r(1+(\xi-1)\iota'_k)^{\varpi_k}-\prod_{k=1}^r(1-\iota'_k)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)\iota'_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^r(1-\iota'_k)^{\varpi_k}}, \frac{\prod_{k=1}^r(1+(\xi-1)\eta'_k)^{\varpi_k}-\prod_{k=1}^r(1-\eta'_k)^{\varpi_k}}{\prod_{k=1}^r(1+(\xi-1)\eta'_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^r(1-\eta'_k)^{\varpi_k}} \end{array}\right)$$

Then $\bigoplus_{\hbar\, k=1}^{r}(\varpi_k\upsilon_k) \leq \bigoplus_{\hbar\, k=1}^{r}(\varpi_k\upsilon'_k)$, so $NCHWA_\varpi(\upsilon_1,\upsilon_2,...,\upsilon_r) \leq NCHWA_\varpi(\upsilon'_1,\upsilon'_2,...,\upsilon'_r)$.
(2) It is shown similar to the proof of (1) by using Eq. (18). □

Theorem 5.7. *(Boundedness Property)*
Let υ_k $(k=1,2,...,r)$ be a collection of the NCEs. Then

(1): $\upsilon_{min} \leq NCHWA_\varpi(\upsilon_1,\upsilon_2,...,\upsilon_r) \leq \upsilon_{max}$

(2): $\upsilon_{min} \leq NCHWG_\varpi(\upsilon_1,\upsilon_2,...,\upsilon_r) \leq \upsilon_{max}$

where

$$\upsilon_{min} = \left\{\begin{array}{l} [min\{\widetilde{\mu}_k^L\}, min\{\widetilde{\mu}_k^U\}], \\ [max\{\widetilde{\iota}_k^L\}, max\{\widetilde{\iota}_k^U\}], \\ [max\{\widetilde{\eta}_k^L\}, max\{\widetilde{\eta}_k^U\}], \\ max\{\mu_k\}, min\{\iota_k\}, min\{\eta_k\} \end{array}\right\}_{(k=1,2,...,r)}$$

and

$$\upsilon_{max} = \left\{ \begin{array}{l} [max\{\widetilde{\mu}_k^L\}, max\{\widetilde{\mu}_k^U\}], \\ [min\{\widetilde{\iota}_k^L\}, min\{\widetilde{\iota}_k^U\}], \\ [min\{\widetilde{\eta}_k^L\}, min\{\widetilde{\eta}_k^U\}], \\ min\{\mu_k\}, max\{\iota_k\}, max\{\eta_k\} \end{array} \right\}_{(k=1,2,\ldots,r)}.$$

Proof. They can be proved using similar techniques, therefore omitted. □

6. The approaches to multiple-criteria decision making under neutrosophic cubic environment

Let o_i $(i = 1, 2, \ldots, p)$ be a fixed of alternatives, e_k $(k = 1, 2, \ldots, r)$ be a criterion and ϖ_k $(k = 1, 2, \ldots, r)$ be the weight of criterion e_k $(k = 1, 2, \ldots, r)$ respectively such that $\varpi_k \in [0, 1]$ and $\sum_{k=1}^{r} \varpi_k = 1$. Let υ_k^i denotes the neutrosophic cubic element (NCE) of the alternative o_i with respect to criterion e_k.

Algorithm 1.

Step 1. Obtain the aggregation value υ^i of neutrosophic cubic elements $\upsilon_1^i, \upsilon_2^i, \ldots, \upsilon_r^i$ by using of neutrosophic cubic Hamacher weighted averaging (NCHWA) operator or neutrosophic cubic Hamacher weighted geometric (NCHWG) operator, i.e., respectively

$$NCHWA_{\varpi}(\upsilon_1^i, \upsilon_2^i, \ldots, \upsilon_r^i) = \bigoplus_{k=1}^{r}{}_{\hbar} (\varpi_k \upsilon_k^i) \ \forall \ i = 1, 2, \ldots, p$$

or

$$NCHWG_{\varpi}(\upsilon_1^i, \upsilon_2^i, \ldots, \upsilon_r^i) = \bigotimes_{k=1}^{r}{}_{\hbar} (\upsilon_k^i)^{\varpi_k} \ \forall \ i = 1, 2, \ldots, p.$$

Step 2. Compute the value of score function $f_{scr}(\upsilon^i)$ $\forall$ $i = 1, 2, \ldots, p$. If $f_{scr}(\upsilon^{p_1}) = f_{scr}(\upsilon^{p_2})$ for any $p_1, p_2 \in \{1, 2, \ldots, p\}$, then compute the values of accuracy function $f_{acr}(\upsilon^{p_1})$ and $f_{scr}(\upsilon^{p_2})$ to compare these alternatives.

Step 3. Find the optimal alternative according to the values obtained in Step 2.

Example 6.1. In order to illustrate the proposed algorithm, the problem for logistic center location selection is described here. Assume that a new modern logistic center is required in a town. There are three locations o_1, o_2 and o_3. A committee of decision makers has been formed to choice the optimal location on the basis of three parameters (namely, cost (e_1), distance to customers (e_2), distance to suppliers (e_3), environmental impact (e_4), quality of service (e_5), transportation (e_6)) with respect to the evaluation of decision committee. As a result of the evaluation, the decision committee gives Table 1 with the neutrosophic cubic values.

Table 1. The collective evaluation values of location with respect to criteria.

$E/\mathcal{O}$	o_1	o_2	o_3
e_1	$\begin{pmatrix} [0.3, 0.7], [0.2, 1], \\ [0, 0.3], 0.4, 0.1, 1 \end{pmatrix}$	$\begin{pmatrix} [0.7, 0.8], [0.3, 0.3], \\ [0.4, 0.5], 0.5, 0.7, 0.4 \end{pmatrix}$	$\begin{pmatrix} [0.2, 0.3], [0.1, 0.5], \\ [0.3, 0.5], 0.2, 0, 0.6 \end{pmatrix}$
e_2	$\begin{pmatrix} [0.5, 0.5], [0, 0.5], \\ [0.2, 0.3], 0.4, 0.4, 0 \end{pmatrix}$	$\begin{pmatrix} [0.7, 0.9], [0.3, 0.3], \\ [0, 0.2], 0.4, 0.4, 0.5 \end{pmatrix}$	$\begin{pmatrix} [0, 0.4], [0.1, 0.4], \\ [0.5, 1], 0.5, 0, 0.7 \end{pmatrix}$
e_3	$\begin{pmatrix} [0.2, 0.6], [0.5, 0.7], \\ [0, 0.1], 0.6, 0.6, 0.6 \end{pmatrix}$	$\begin{pmatrix} [0.8, 1], [0.4, 0.5], \\ [0, 0.1], 0.5, 0.5, 0.4 \end{pmatrix}$	$\begin{pmatrix} [0.2, 0.6], [0.4, 0.4], \\ [0.2, 0.3], 0.4, 0.5, 0.4 \end{pmatrix}$
e_4	$\begin{pmatrix} [0.1, 0.2], [0.3, 0.6], \\ [0.1, 0.4], 0.6, 0.4, 0.1 \end{pmatrix}$	$\begin{pmatrix} [0.5, 1], [0.4, 0.6], \\ [0.5, 0.6], 0.5, 0.5, 0.4 \end{pmatrix}$	$\begin{pmatrix} [0.3, 0.3], [0.1, 0.6], \\ [0, 1], 0.4, 0.2, 0.4 \end{pmatrix}$
e_5	$\begin{pmatrix} [0.4, 0.7], [0.4, 0.5], \\ [0.1, 0.2], 0.5, 0, 0.5 \end{pmatrix}$	$\begin{pmatrix} [0.4, 1], [0.3, 0.8], \\ [0.2, 0.6], 0.6, 0.4, 0.3 \end{pmatrix}$	$\begin{pmatrix} [0.5, 0.5], [0.2, 0.7], \\ [0.7, 0.8], 0.6, 0.3, 0.4 \end{pmatrix}$
e_6	$\begin{pmatrix} [0.1, 0.3], [0.4, 0.7], \\ [0.4, 0.5], 0.3, 0.3, 0.3 \end{pmatrix}$	$\begin{pmatrix} [0.5, 0.7], [0.4, 0.5], \\ [0.4, 0.5], 0.4, 0.6, 0.9 \end{pmatrix}$	$\begin{pmatrix} [0.3, 0.6], [0.2, 0.8], \\ [0.5, 0.5], 0.1, 0.4, 0.1 \end{pmatrix}$

Also, the decision committee determines the weight of criteria as $\varpi = (0.1, 0.2, 0.1, 0.3, 0.2, 0.1)^T$. We are ready to apply the proposed approach to solve this problem based on the neutrosophic cubic information.

Step 1. By applying NCHWA operator with $q = 4$, we get the following aggregation values.

$\upsilon^1 = ([0.2666, 0.4733], [0, 0.6218], [0, 0.2913], 0.4846, 0.2941, 1)$,

$\upsilon^2 = ([0.5834, 1], [0.3476, 0.5184], [0, 0.4131], 0.4879, 0.4944, 0.4756)$,

$\upsilon^3 = ([0.2541, 0.4233], [0.1438, 0.5658], [0, 0.7997], 0.3837, 0.1721, 0.4583)$.

Step 2. Using Eq. (1) given in Definition 3.1, the value of score function are obtained as $f_{scr}(\upsilon^1) = 0.5623$, $f_{scr}(\upsilon^2) = 0.5928$ and $f_{scr}(\upsilon^3) = 0.6013$.

Step 3. Then, we obtain the ranking order of three locations as $o_3 \succ o_2 \succ o_1$. Therefore, we suggest o_3 as the optimal choice and so a new logistic center location.

Table 2 presents the ranking order of alternatives for some values of ξ.

Table 2. The ranking order according to NCHWA operator with some values of ξ.

ξ	$f_{scr}(\upsilon^1)$	$f_{scr}(\upsilon^2)$	$f_{scr}(\upsilon^3)$	ranking order
$\xi = 0.1$	0.5584	0.5898	0.5947	$o_3 \succ o_2 \succ o_1$
$\xi = 1$	0.5593	0.5927	0.5987	$o_3 \succ o_2 \succ o_1$
$\xi = 2$	0.5605	0.5929	0.5999	$o_3 \succ o_2 \succ o_1$
$\xi = 4$	0.5623	0.5928	0.6013	$o_3 \succ o_2 \succ o_1$
$\xi = 10$	0.5657	0.5925	0.6033	$o_3 \succ o_2 \succ o_1$
$\xi = 100$	0.5763	0.5922	0.6088	$o_3 \succ o_2 \succ o_1$

Figure 1 gives a graphical representation of score values for some values of ξ.

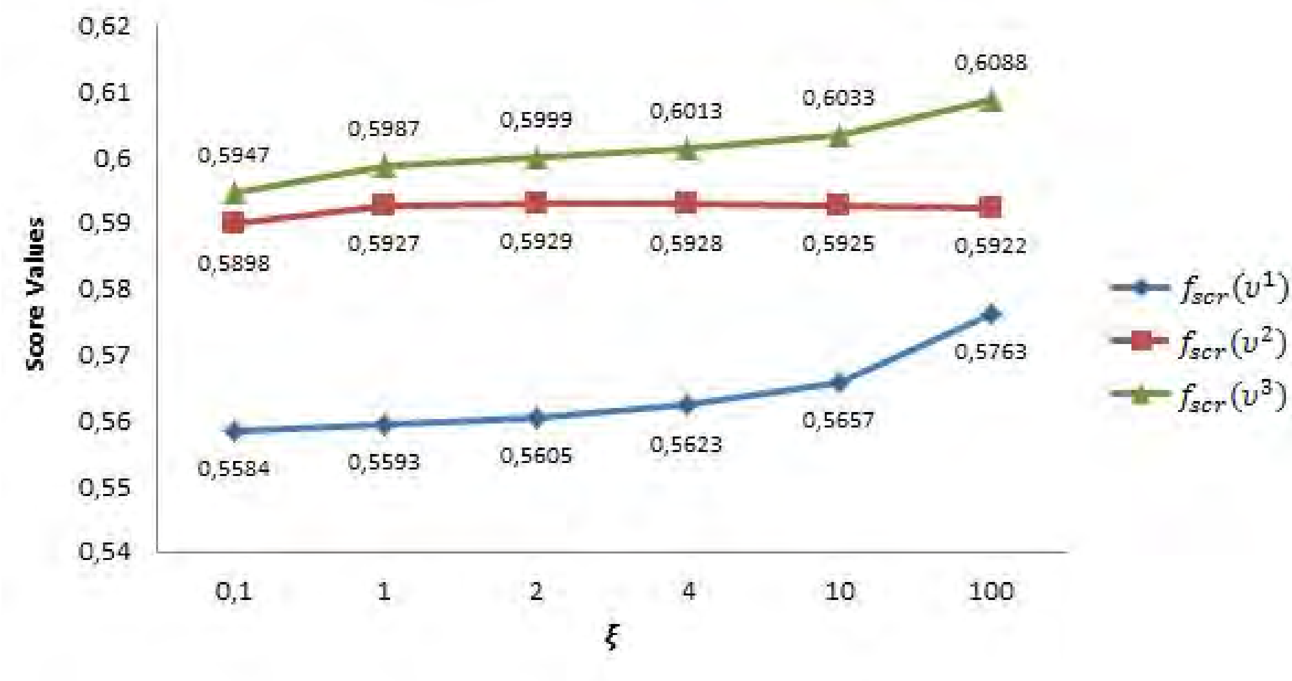

Figure 1. Graphical representation of score values for some values of ξ.

Algorithm 1 is efficient for decision making problems that include the evaluations of a single decision maker, but it cannot be used for decision systems with multiple experts. Now, we create a decision making model based on the neutrosophic cubic Hamacher weighted aggregation operators to deal with multi-criteria group decision making which includes the evaluations of two or more decision makers (experts).

Let o_i $(i = 1, 2, \ldots, p)$ be a fixed of alternatives, e_k $(k = 1, 2, \ldots, r)$ be a criterion and ϖ_k $(k = 1, 2, \ldots, r)$ be the weight of criterion e_k $(k = 1, 2, \ldots, r)$ respectively such that $\varpi_k \in [0, 1]$ and $\sum_{k=1}^{r} \varpi_k = 1$. Also, let D_j $(j = 1, 2, \ldots, v)$ be a fixed of decision makers and Ω_j $(j = 1, 2, \ldots, v)$ be the weight of decision maker D_j $(j = 1, 2, \ldots, v)$ respectively such that $\Omega_j \in [0, 1]$ and $\sum_{j=1}^{v} \Omega_j = 1$. Let $\upsilon_k^{(i,j)}$ denotes the neutrosophic cubic element (NCE) of the alternative o_i with respect to criterion e_k for the decision maker D_j.

Algorithm 2.

Step 1. Obtain the aggregation value $\upsilon^{(i,j)}$ of neutrosophic cubic elements $\upsilon_1^{(i,j)}, \upsilon_2^{(i,j)}, \ldots, \upsilon_r^{(i,j)}$ for each decision maker D_j by using the neutrosophic cubic Hamacher weighted averaging (NCHWA) operator or neutrosophic cubic Hamacher weighted geometric (NCHWG) operator.

For instance, for a decision making problem with two decision makers ($j = 1, 2$), obtain

$$NCHWA_{\varpi}(\upsilon_1^{(i,1)}, \upsilon_2^{(i,1)}, \ldots, \upsilon_r^{(i,1)}) = \bigoplus_{k=1}^{r}{}_{\hbar}(\varpi_k \upsilon_k^{(i,1)}) \ \forall \ i = 1, 2, \ldots, p,$$

$$NCHWA_{\varpi}(\upsilon_1^{(i,2)}, \upsilon_2^{(i,2)}, \ldots, \upsilon_r^{(i,2)}) = \bigoplus_{k=1}^{r}{}_{\hbar}(\varpi_k \upsilon_k^{(i,2)}) \ \forall \ i = 1, 2, \ldots, p$$

or

$$NCHWG_{\varpi}(\upsilon_1^{(i,1)}, \upsilon_2^{(i,1)}, \ldots, \upsilon_r^{(i,1)}) = \bigotimes_{k=1}^{r}{}_{\hbar}(\upsilon_k^{(i,1)})^{\varpi_k} \ \forall \ i = 1, 2, \ldots, p,$$

$$NCHWG_{\varpi}(\upsilon_1^{(i,2)}, \upsilon_2^{(i,2)}, \ldots, \upsilon_r^{(i,2)}) = \bigotimes_{k=1}^{r}{}_{\hbar}(\upsilon_k^{(i,2)})^{\varpi_k} \ \forall \ i = 1, 2, \ldots, p.$$

Step 2. Compute the value of score function $f_{scr}(\upsilon^{(i,j)}) \ \forall \ i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, v$ for each aggregation value $\upsilon^{(i,j)}$.

Step 3. Calculate the standardized score values for each decision maker by using the following formula:

$$\mathfrak{S}(i, j) = \Omega_j \frac{f_{scr}(\upsilon^{(i,j)})}{\sqrt{(f_{scr}(\upsilon^{(1,j)}))^2 + (f_{scr}(\upsilon^{(2,j)}))^2 + \ldots + (f_{scr}(\upsilon^{(p,j)}))^2}}.$$

Step 4. Calculate the decision value of each alternative by using the following formula:

$$\mathfrak{D}(i) = \frac{1}{v} \sum_{j=1}^{v} \mathfrak{S}(i, j).$$

If the decision values of any two alternatives are equal then in Step 3, the standardized accuracy values of these two alternatives are calculated (that is, f_{acr} substituted for f_{scr} in the formula $\mathfrak{S}(i, j)$).

Step 5. Find the optimal alternative according to the decision values obtained in Step 4.

Example 6.2. (adapted from [17]) Mobile companies play a major role in Pakistans stock market. The performance of these companies affects capital market resources and have become a common concern of creditors, shareholders, government authorities and other stakeholders. In this example, an investor company wants to invest the capital tax in listed companies. They acquire two types of decision makers (experts): Attorney and market maker.The attorney is acquired to look at the legal matters and the market maker is encouraged to provide his/her expertise in the capital market issues. The data are collected on the basis of stock market analysis and growth in different areas. Let the listed mobile companies be (o_1) Zong, (o_2) Jazz, (o_3) Telenor and (o_4) Ufone, which have higher ratios of earnings than the others available in the market, from the three alternatives of (e_1) stock market trends, (e_2) policy directions and (e_3) the annual performance. The two decision makers ($D_j \ j = 1, 2$) evaluated the mobile companies ($o_i, \ i = 1, 2, 3, 4$) with respect to the corresponding attributes ($e_k, \ k = 1, 2, 3$), and proposed their assessments consisting of neutrosophic cubic values in Table 3 and Table 4.

Assume that the weight of attributes is $\varpi = (0.35, 0.30, 0.35)^T$, and the weight of decision makers is $\Omega = (0.9, 0.1)^T$. Let's provide a solution for this decision making problem using the NCHWG operator on the attributes.

TABLE 3. The neutrosophic cubic values of attorney's assessment.

$E/\mathcal{O}$	o_1	o_2	o_3	o_4
e_1	$\begin{pmatrix}[0.2,0.6],[0.4,0.6],\\ [0.5,0.8],0.7,0.4,0.3\end{pmatrix}$	$\begin{pmatrix}[0.3,0.5],[0.6,0.9],\\ [0.3,0.6],0.3,0.6,0.7\end{pmatrix}$	$\begin{pmatrix}[0.6,0.9],[0.2,0.7],\\ [0.4,0.9],0.5,0.5,0.6\end{pmatrix}$	$\begin{pmatrix}[0.4,0.8],[0.5,0.9],\\ [0.3,0.8],0.5,0.8,0.5\end{pmatrix}$
e_2	$\begin{pmatrix}[0.1,0.4],[0.5,0.8],\\ [0.4,0.8],0.6,0.7,0.5\end{pmatrix}$	$\begin{pmatrix}[0.5,0.9],[0.1,0.3],\\ [0.4,0.8],0.8,0.3,0.6\end{pmatrix}$	$\begin{pmatrix}[0.2,0.6],[0.3,0.7],\\ [0.3,0.8],0.4,0.6,0.5\end{pmatrix}$	$\begin{pmatrix}[0.2,0.7],[0.4,0.9],\\ [0.5,0.7],0.6,0.4,0.5\end{pmatrix}$
e_3	$\begin{pmatrix}[0.4,0.6],[0.2,0.7],\\ [0.5,0.9],0.4,0.5,0.3\end{pmatrix}$	$\begin{pmatrix}[0.2,0.7],[0.1,0.6],\\ [0.4,0.7],0.5,0.4,0.7\end{pmatrix}$	$\begin{pmatrix}[0.5,0.9],[0.7,0.9],\\ [0.1,0.5],0.5,0.6,0.4\end{pmatrix}$	$\begin{pmatrix}[0.3,0.5],[0.5,0.9],\\ [0.3,0.7],0.3,0.3,0.8\end{pmatrix}$

TABLE 4. The neutrosophic cubic values of market maker's assessment.

$E/\mathcal{O}$	o_1	o_2	o_3	o_4
e_1	$\begin{pmatrix}[0.3,0.6],[0.2,0.6],\\ [0.2,0.6],0.8,0.7,0.2\end{pmatrix}$	$\begin{pmatrix}[0.2,0.5],[0.6,0.9],\\ [0.3,0.7],0.4,0.8,0.7\end{pmatrix}$	$\begin{pmatrix}[0.5,0.9],[0.2,0.6],\\ [0.3,0.8],0.7,0.7,0.8\end{pmatrix}$	$\begin{pmatrix}[0.3,0.5],[0.3,0.9],\\ [0.2,0.5],0.6,0.5,0.4\end{pmatrix}$
e_2	$\begin{pmatrix}[0.3,0.8],[0.4,0.8],\\ [0.3,0.8],0.6,0.7,0.4\end{pmatrix}$	$\begin{pmatrix}[0.4,0.9],[0.1,0.4],\\ [0.5,0.8],0.6,0.5,0.7\end{pmatrix}$	$\begin{pmatrix}[0.2,0.5],[0.2,0.7],\\ [0.5,0.8],0.6,0.7,0.2\end{pmatrix}$	$\begin{pmatrix}[0.4,0.7],[0.2,0.8],\\ [0.3,0.7],0.6,0.7,0.7\end{pmatrix}$
e_3	$\begin{pmatrix}[0.2,0.7],[0.2,0.6],\\ [0.3,0.8],0.5,0.3,0.5\end{pmatrix}$	$\begin{pmatrix}[0.4,0.9],[0.1,0.4],\\ [0.5,0.8],0.6,0.5,0.7\end{pmatrix}$	$\begin{pmatrix}[0.3,0.5],[0.3,0.9],\\ [0.2,0.5],0.6,0.5,0.4\end{pmatrix}$	$\begin{pmatrix}[0.2,0.6],[0.5,0.9],\\ [0.2,0.8],0.4,0.4,0.8\end{pmatrix}$

Steps 1-3. By using NCHWG operator with $q = 100$, the aggregation values, the score values and the standardized score values of alternatives are obtained as in Table 5.

TABLE 5. The aggregation values, score values and standardized score values.

j	$\upsilon^{(i,j)}$	$f_{scr}(\upsilon^{(i,j)})$	$\mathfrak{S}(i,j)$
$j=1$	$\upsilon^{(1,1)} = \begin{pmatrix}[0.2163,0.5402],\ [0.3849,0.7015],\\ [0.4696,0.8416],\ 0.5685,\ 0.5276,\ 0.3558\end{pmatrix}$	0.4787	0.4558
	$\upsilon^{(2,1)} = \begin{pmatrix}[0.3137,0.7198],\ [0.2205,0.6595],\\ [0.3636,0.7015],\ 0.5306,\ 0.4365,\ 0.6713\end{pmatrix}$	0.5113	0.4869
	$\upsilon^{(3,1)} = \begin{pmatrix}[0.4314,0.8382],\ [0.3926,0.7892],\\ [0.2397,0.7661],\ 0.4996,\ 0.5654,\ 0.4998\end{pmatrix}$	0.4736	0.4509
	$\upsilon^{(4,1)} = \begin{pmatrix}[0.2984,0.6761],\ [0.4696,0.9],\\ [0.3562,0.738],\ 0.4568,\ 0.5156,\ 0.6179\end{pmatrix}$	0.4223	0.4021
$j=2$	$\upsilon^{(1,1)} = \begin{pmatrix}[0.2671,0.5957],\ [0.3535,0.8279],\\ [0.3146,0.6665],\ 0.5778,\ 0.4689,\ 0.498\end{pmatrix}$	0.4877	0.0515
	$\upsilon^{(2,1)} = \begin{pmatrix}[0.3209,0.8035],\ [0.2205,0.6247],\\ [0.4266,0.7681],\ 0.5305,\ 0.6179,\ 0.7\end{pmatrix}$	0.4642	0.0491
	$\upsilon^{(3,1)} = \begin{pmatrix}[0.3288,0.6799],\ [0.232,0.7624],\\ [0.3146,0.7113],\ 0.6365,\ 0.634,\ 0.4799\end{pmatrix}$	0.4856	0.0513
	$\upsilon^{(4,1)} = \begin{pmatrix}[0.2882,0.5975],\ [0.3297,0.8758],\\ [0.2272,0.677],\ 0.5305,\ 0.5276,\ 0.6439\end{pmatrix}$	0.452	0.0478

Step 4-5. Consequently, we obtain the decision values of alternatives as $\mathfrak{D}(1) = 0.2536$, $\mathfrak{D}(2) = 0.268$, $\mathfrak{D}(3) = 0.2511$, $\mathfrak{D}(4) = 0.2249$. Then, the ranking order of alternatives is $o_2 \succ o_1 \succ o_3 \succ o_4$, and so the optimal choice is o_2.

In Table 6, we discuss the ranking order of alternatives for some values of ξ. Thus, we exhibit that the standardized score values and decision values show slight changes synchronous to the range of ξ.

TABLE 6. The ranking order according to NCHWG operator with some values of ξ.

ξ	i	$\mathfrak{S}(i,1)$	$\mathfrak{S}(i,2)$	$\mathfrak{D}(i)$	ranking order
$\xi = 0.1$	$i = 1$	0.4636	0.0541	0.2588	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i = 2$	0.4727	0.0456	0.2591	
	$i = 3$	0.4313	0.0514	0.2413	
	$i = 4$	0.4306	0.0484	0.2395	
$\xi = 1$	$i = 1$	0.4606	0.0532	0.2569	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i = 2$	0.4766	0.0463	0.2614	
	$i = 3$	0.4402	0.0516	0.2459	
	$i = 4$	0.4205	0.0484	0.2344	
$\xi = 2$	$i = 1$	0.4593	0.0525	0.2559	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i = 2$	0.4789	0.0477	0.2633	
	$i = 3$	0.4435	0.0514	0.2474	
	$i = 4$	0.4158	0.0481	0.2319	
$\xi = 4$	$i = 1$	0.4581	0.0522	0.2551	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i = 2$	0.4813	0.0482	0.2674	
	$i = 3$	0.4463	0.0514	0.2488	
	$i = 4$	0.4113	0.048	0.2296	
$\xi = 10$	$i = 1$	0.4569	0.0518	0.2543	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i = 2$	0.484	0.0486	0.2663	
	$i = 3$	0.4487	0.0513	0.25	
	$i = 4$	0.4067	0.0479	0.2273	
$\xi = 100$	$i = 1$	0.4558	0.0515	0.2536	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i = 2$	0.4869	0.0491	0.268	
	$i = 3$	0.4509	0.0513	0.2511	
	$i = 4$	0.4021	0.0478	0.2249	

In Figure 2, a figuration of the decision values of alternatives for some values of ξ is presented. Thus, the effect of the range of ξ on the selection priority is illustrated.

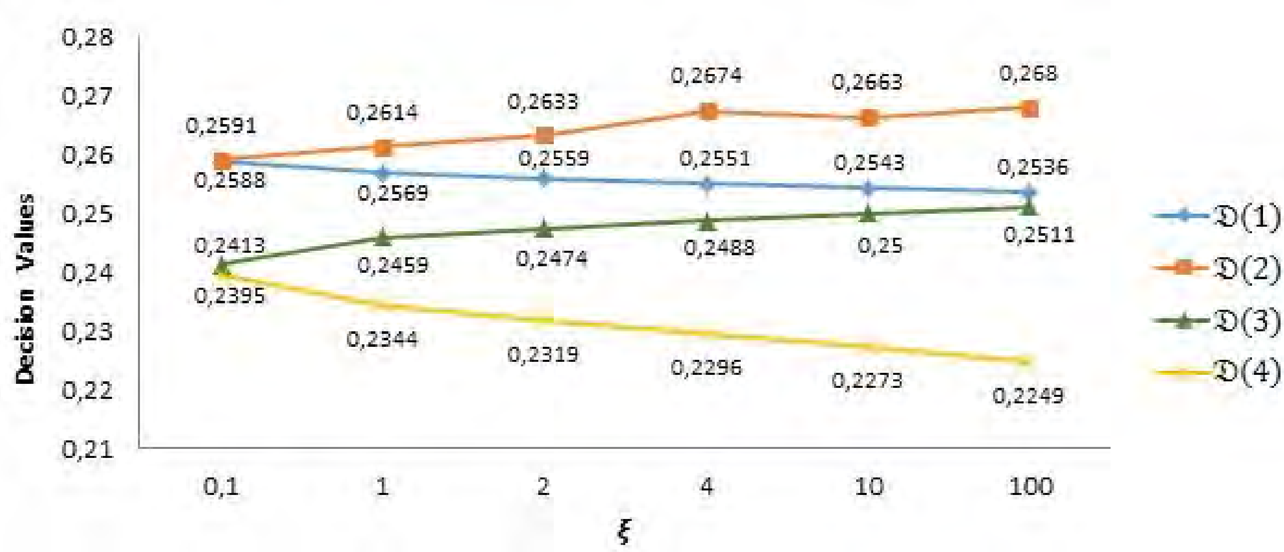

Figure 2. Graphical representation of decision values for some values of ξ.

Discussion and Comparison: If $\Omega = (0.4, 0.6)^T$ in Example 6.2, then this example is the same as the problem in "Application" (see: Section 6 on page 21) of [17]. For $\Omega = (0.4, 0.6)^T$, using NCHWG operator with $q = 100$, we rank the alternatives as $o_1 \succ o_2 \succ o_3 \succ o_4$. But it is proposed as a priority order of alternatives in [17] that $o_3 \succ o_2 \succ o_1 \succ o_4$. We think the reason for this order is the concepts of score function and accuracy function given in Definitions 18 and 19 of [17], and these functions should be improved. Let us demonstrate that the score and accuracy functions in Definitions 18 and 19 of [17] give erroneous outputs for some neutrosophic cubic elements. Let $\upsilon_1 = ([0.5, 0.7], [0.2, 0.5], [0.5, 0.6], 0.5, 0.8, 0.2)$ and $\upsilon_2 = ([0.4, 0.6], [0.3, 0.4], [0.4, 0.5], 0.8, 0.6, 0.5)$ be two NCEs. It is evident that υ_1 and υ_2 do not have identical values, i.e., $\upsilon_1 \neq \upsilon_2$. By Definition 18 in [17], the score values of υ_1 and υ_2 are $S(\upsilon_1) = 0.5 - 0.5 + 0.7 - 0.6 + 0.5 - 0.2 = 0.4$ and $S(\upsilon_2) = 0.4 - 0.4 + 0.6 - 0.5 + 0.8 - 0.5 = 0.4$, respectively. By Definition 19 in [17], the accuracy values of υ_1 and υ_2 are $H(\upsilon_1) = \frac{1}{9}(0.5 + 0.2 + 0.5 + 0.7 + 0.5 + 0.6 + 0.5 + 0.8 + 0.2) = 0.5$ and $H(\upsilon_2) = \frac{1}{9}(0.4 + 0.3 + 0.4 + 0.6 + 0.4 + 0.5 + 0.8 + 0.6 + 0.5) = 0.5$, respectively. By the comparison method in Definition 20 of [17], $\upsilon_1 = \upsilon_2$, which is against our intuition. By using the score function in Definition 3.1, we obtain $f_{scr}(\upsilon_1) = 0.5468$ and $f_{scr}(\upsilon_1) = 0.5968$, so $\upsilon_1 \prec \upsilon_2$. Also if the score function in Definition 3.1 is used for NCGW in Eq. (4) of [17]:

$$NCWG = \begin{pmatrix} \upsilon_1 \begin{pmatrix} [0.2375,0.6195], \\ [0.2885,0.7916], \\ [0.3567,0.8146], \\ 0.6315,\ 0.5757,\ 0.2851 \end{pmatrix} \\ \upsilon_2 \begin{pmatrix} [0.4426,0.7657], \\ [0.2165,0.5915], \\ [0.5382,0.7804], \\ 0.4827,\ 0.5729,\ 0.5282 \end{pmatrix} \\ \upsilon_3 \begin{pmatrix} [0.3500,0.6616], \\ [0.3335,0.8142], \\ [0.3131,0.7498], \\ 0.5791,\ 0.6133,\ 0.4439 \end{pmatrix} \\ \upsilon_4 \begin{pmatrix} [0.3327,0.6774], \\ [0.3630,0.7787], \\ [0.2888,0.7396], \\ 0.4906,\ 0.5359,\ 0.5692 \end{pmatrix} \end{pmatrix} \tag{15}$$

then it is calculated as $f_{scr}(\upsilon_1) = 0.4947$, $f_{scr}(\upsilon_2) = 0.4931$, $f_{scr}(\upsilon_3) = 0.4669$ and $f_{scr}(\upsilon_4) = 0.4589$. By these score values, we say that the ranking order of alternatives is $o_1 \succ o_2 \succ o_3 \succ o_4$. This result coincides with the output of Algorithm 2. Thereby, the efficiency of score function, accuracy function and decision making algorithms presented in this study are displayed.

7. Conclusions

In this study, we described a comparison strategy for two neutrosophic cubic elements. Some new aggregation operators for the neutrosophic cubic sets based on Hamacher t-norm and Hamacher t-conorm, which are a generalization of the operators based on algebraic t-norm and t-conorm or Einstein t-norm and Einstein t-conorm, were proposed and their basic properties were investigated. They were applied to solve the MCDM problems in which attribute values take the form of neutrosophic cubic elements. In addition, compared with the existing algorithm based on Einstein geometric aggregations under the neutrosophic cubic environment, the proposed algorithms can give the satisfactory sorting value of each alternative.

In further research, it is necessary and meaningful to give the applications of these aggregation operators to the other domains such as medical diagnosis, pattern recognition and selection of renewable energy.

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University of New Mexico

Plithogenic Soft Set

Shawkat Alkhazaleh

Department of Mathematics, Faculty of Science and Information Technology, Jadara University, Irbid, Jordan

shmk79@gmail.com

Abstract. In 1995, Smarandache initiated the theory of neutrosophic set as new mathematical tool for handling problems involving imprecise, indeterminacy, and inconsistent data. Molodtsov initiated the theory of soft set as a new mathematical tool for dealing with uncertainties, which traditional mathematical tools cannot handle. He has showed several applications of this theory for solving many practical problems in economics, engineering, social science, medical science, etc. In 2017 Smarandache initiated the theory of Plithogenic Set and their properties. He also generalized the soft set to the hypersoft set by transforming the function F into a multi-attribute function and introduced the hybrids of Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Hypersoft Set. In this research, for the first time we define the concept of Plithogenic soft set, and give it some generalizations and study some of its operations. Furthermore, We give examples for these concepts and operations. Finally, the similarities between two Plithogenic soft sets are also given.

Keywords: Soft set; Neutrosophic set; Neutrosophic soft set; Plithogenic set; Plithogenic soft set.

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1. Introduction

In our life not everything non vague or certain where many things around us are surrounded by uncertainty and vagueness , Zadeh [13] was successfully fulfilled the need to represent uncertain data by introducing the concept of fuzzy sets. Also Atanassov [4] extend Zadehs notion of fuzzy set to Intuitionistic fuzzy sets which proved to be a better model of uncertainty. The words neutrosophy and neutrosophic were introduced for the first time by Smarandache [9, 10] as a more general platform, which extends the concepts of the fuzzy set and intuitionistic fuzzy set. In 1999 Molodtsov [8] proposed a parameterised family of sets named "soft set", to deal with uncertainty in a parametric manner. Smarandache defined a concept of plithogenic set, where a plithogenic set P is a set whose elements are characterized by one or more attributes, and each attribute may have many values. Each attribute's value v has a corresponding degree of appurtenance $d(x, v)$ of the element x, to the set P, with respect to some given criteria. In order to obtain better accuracy for the plithogenic aggregation operators, a contradiction (dissimilarity) degree is defined between each attribute value and the dominant (most important) attribute value. However, there are cases when such dominant attribute value may not be taking into consideration or may not exist [therefore it is considered zero by default], or there may be many dominant attribute values. In such cases, either the contradiction degree function is suppressed, or another relationship function between attribute values should be established. The plithogenic aggregation operators (intersection, union, complement, inclusion and equality) are based on contradiction degrees between attributes values and the first two are linear combinations of the fuzzy operators t-norm and t-conorm. Plithogenic set is a generalization of the crisp set, fuzzy set, intuitionistic fuzzy set, and neutrosophic set, since these four types of sets are characterized by a single attribute value (appurtenance): which has one value (membership)for the crisp set and fuzzy set, two values (membership, and non-membership)for intuitionistic fuzzy set, or three values(membership, non-membership, and indeterminacy) for neutrosophic set. A plithogenic set, in general, may have elements characterized by attributes with four or more attributes. In 2019 Abdel-Basset et al. [1] suggested an approach constructed on the connotation of plithogenic theory technique to come up with a methodical procedure to assess the infirmary serving under a framework of plithogenic theory. In this research, they gave some definitions of the plithogenic environment, which is more general and comprehensive than fuzzy, intuitionistic fuzzy and neutrosophic ones. Abdel-Basset et al. [3] proposed method to increase the accuracy of the evaluation. This method is a combination of quality function deployment with plithogenic aggregation operations. They applied the aggregation operation to aggregate the decision makers opinions of requirements that are needed to evaluate the supply chain sustainability and the evaluation metrics based on the requirements. Also they applied the aggregation operation to aggregate the evaluation of information gathering difficulty. In 2020 Abdel-Basset and Rehab [2] proposed a methodology as a combination of plithogenic multi-criteria decision-making approach based on the Technique in Order of Preference by Similarity to Ideal Solution and Criteria Importance Through Inter-criteria Correlation methods to estimation of sustainable supply chain risk management.

2. **Preliminary**

In this section, we recall some definitions and properties required in this paper.

Definition 2.1. [9, 10] A neutrosophic set A on the universe of discourse X is defined as

$$A = \{< x; T_A(x); I_A(x); F_A(x) >; x \in X\}$$

where T; I; $F : X \to]^{-}0; 1^{+}[$ and $^{-}0 \leqslant T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$.

Remark 2.2. In a more general way, the summation can be less than 1 (for incomplete neutrosophic information), equal to 1 (for complete neutrosophic information), or greater than 1 (for paraconsistent/conflicting neutrosophic information).

Molodtsov defined soft set in the following way. Let U be a universe and E be a set of parameters. Let $P(U)$ denote the power set of U and $A \subseteq E$.

Definition 2.3. [8] A pair (F, A) is called a *soft set* over U, where F is a mapping

$$F : A \to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A, F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A).

Definition 2.4. [6] Let $P(U)$ denotes the set of all fuzzy sets of U. Let $A_i \subseteq E$ .A pair$(F_i, A_i)$is called a fuzzy soft set over U , where F_i is a mapping given by $F_i : A_i \longrightarrow P(U)$

Definition 2.5. [7] Let U be an initial universal set and let E be set of parameters. Let $P(U)$ denotes the set of all intuitionistic fuzzy sets of U. A pair (F, A) is called an intuitionistic fuzzy soft set over U if F is a mapping given by $F : A \longrightarrow P(U)$. We write an Intuitionistic fuzzy soft set shortly as IF soft set

Definition 2.6. [5] Let U be an initial universe set and E be a set of parameters. Consider $A \subset E$. Let $P(U)$ denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the neutrosophic soft set over U, where F is a mapping given by $F : A \to P(U)$.

3. Formal Definition of Single (Uni-Dimensional) Attribute Plithogenic Set

In this section we recall the definition of plithogenic set given by Smarandache [11, 12] and some definitions related to this concept as follows:
Let U be a universe of discourse, and P a non-empty set of elements, $P \subseteq U$.

3.1. *Attribute Value Spectrum*

Definition 3.1. Let $\mathfrak{A}$ be a non-empty set of uni-dimensional attributes $\mathfrak{A} = \{\alpha_1, \alpha_2, ..., \alpha_m\}$, $m \geq 1$; and $\alpha_i \in \mathfrak{A}$ be a given attribute whose spectrum of all possible values (or states) is the non-empty set S, where S can be a finite discrete set, $S = \{s_1, s_2, ..., s_l\}$,$1 \leq l \leq \infty$, or infinitely countable set $S = \{s_1, s_2, ..., s_\infty\}$, or infinitely uncountable (continuum) set $S =]a, b[, a < b$, where]...[is any open, semi-open, or closed interval from the set of real numbers or from other general set.

3.2. *Attribute Value Range*

Definition 3.2. Let V be a non-empty subset of S, where V is the range of all attribute's values needed by the experts for their application. Each element $x \in P$ is characterized by all attribute's values in $V = \{v_1, v_2, ..., v_n\}$, for $n \geq 1$.

3.3. *Dominant Attribute Value*

Definition 3.3. Into the attributes value set V, in general, there is a dominant attribute value, which is determined by the experts upon their application. Dominant attribute value is defined as the most important attribute value that the experts are interested in.

Remark 3.4. There are cases when such dominant attribute value may not be taking into consideration or not exist, or there may be many dominant (important) attribute values - when different approach should be employed.

3.4. *Attribute Value Truth-Value Degree Function*

Definition 3.5. The attribute value truth-value degree function is: $\forall P \in U, d : U \times V \longrightarrow \wp([0,1]^z)$, so $d(P, v)$ is a subset of $[0,1]^z$, where $\wp([0,1]^z)$ is the power set of the $[0,1]^z$ such that
$z = F$ (for fuzzy degree of truth-value),
$z = IF$ (for intuitionistic fuzzy degree of truth-value), or
$z = N$ (for neutrosophic degree de truth-value).

3.5. *Attribute Value Contradiction Degree Function*

Definition 3.6. The attribute value contradiction degree function between any two attribute values v_1 and v_2, denoted by $c(v_1, v_2)$ is a function define by $c : V \times V \longrightarrow [0,1]$ such that the cardinal $|V| \geq 1$ and satisfying the following axioms:

(1) $c(v_1, v_1) = 0$, the contradiction degree between the same attribute values is zero;
(2) $c(v_1, v_2) = c(v_2, v_1)$, commutativity.

3.6. *plithogenic set*

Definition 3.7. Let U be a universe of discourse, and P a non-empty set of elements, $P \subseteq U$. a is a (multi-dimensional in general) attribute, V is the range of the attributes values, d is the degree of appurtenance of each element xs attribute value to the set P with respect to some given criteria $(x \in P)$,and c stands for the degree of contradiction between attribute values. Then (P, a, V, d, c) is called a plithogenic set.

Remark 3.8. (1) d may stand for d_F , d_{IF} or d_N, when dealing with fuzzy degree of appurtenance, intuitionistic fuzzy degree of appurtenance, or neutrosophic degree of appurtenance respectively of an element x to the plithogenic set P;

(2) c may stand for c_F , c_{IF} or c_N, when dealing with fuzzy degree of contradiction, intuitionistic fuzzy degree of contradiction, or neutrosophic degree of contradiction between attribute values respectively;

(3) The functions $d(.,.)$ and $c(.,.)$ are defined in accordance with the applications the experts need to solve;

(4) One uses the notation: $x(d(x, V))$, where $d(x, v) = \{d(x, v)\forall v \in V\}$, $\forall x \in P$;

The plithogenic set is an extension of all: crisp set, fuzzy set, intuitionistic fuzzy set, and neutrosophic set. In this Paper we will study the plithogenic set as an extension of fuzzy set, intuitionistic fuzzy set, and neutrosophic set.

In **Single-Valued Fuzzy Set (SVFS)**, the attribute is $\alpha =$ "*appurtenance*"; the set of attribute values

$V = \{T\}$, whose cardinal $|V| = 1$; the dominant attribute value $= T$; the attribute value appurtenance degree function: $d : P \times V \longrightarrow [0, 1]$, $d(x, T) \in [0, 1]$

and the attribute value contradiction degree function:

$c : V \times V \longrightarrow [0, 1]$, $c(T, T) = 0$,

In **Single-Valued intuitionistic fuzzy set (SVIFS)**, the attribute is $\alpha =$ "*appurtenance*"; the set of attribute values $V = \{T, F\}$, whose cardinal $|V| = 2$;

the dominant attribute value $= T$; the attribute value appurtenance degree function:

$d : P \times V \longrightarrow [0, 1]$, $d(x, T) \in [0, 1]$, $d(x, F) \in [0, 1]$ with $0 \leq d(x, T) + d(x, F) \leq 1$; and the attribute value contradiction degree function: $c : V \times V \longrightarrow [0, 1]$, $c(T, T) = c(F, F) = 0$, $c(T, F) = 1$.

In **Single-Valued Neutrosophic Set (SVNS)**, the attribute is $\alpha =$ "*appurtenance*"; the set of attribute values

$V = \{T, I, F\}$, whose cardinal $|V| = 3$; the dominant attribute value $= T$; the attribute value appurtenance degree function:

$d : P \times V \longrightarrow [0, 1]$, $d(x, T) \in [0, 1]$, $d(x, I) \in [0, 1]$, $d(x, F) \in [0, 1]$,

with $0 \leq d(x, T) + d(x, I) + d(x, F) \leq 3$;

and the attribute value contradiction degree function:

$c : V \times V \longrightarrow [0, 1]$,

$c(T, T) = c(I, I) = c(F, F) = 0$,

$c(T, F) = 1$,

$c(T, I) = c(F, I) = 0.5$.

4. Plithogenic Intersection

In this section we recall the definition of plithogenic intersection over three cases: fuzzy, intuitionistic fuzzy and neutrosophic set which are defined for the first time by Smarandache in 2017 and 2018 [11, 12].

4.1. *Plithogenic Fuzzy Intersection*

Definition 4.1. Let $U = \{u_1, u_2, ..., u_n\}$. $A = \left\{\frac{u_1}{\langle\mu(u_1)}, \frac{u_2}{\mu(u_2)}, ..., \frac{u_n}{\mu(u_n)}\right\}$, and $B = \left\{\frac{u_1}{\nu(u_1)}, \frac{u_2}{\nu(u_2)}, ..., \frac{u_n}{\nu(u_n)}\right\}$ be any two plithogenic fuzzy sets over U. Then the plithogenic fuzzy intersection between A and B define as follows:

$$A \wedge_{FP} B = \left\{\frac{u_i}{(1-c_v)(\mu(u_i)\wedge_F \nu(u_i))+(c_v)(\mu(u_i)\vee_F \nu_F(u_i))}\right\}, \forall i = 1, 2, ..., n$$

Where $\wedge_F$ and $\vee_F$ represent fuzzy t-norm and fuzzy t-conorm (s-norm) respectively and c_v represent the degrees of contradictions.

4.2. *Plithogenic Intuitionistic Fuzzy Intersection*

Definition 4.2. Let $U = \{u_1, u_2, ..., u_n\}$. Suppose

$A = \left\{\frac{u_1}{(\mu_T(u_1),\mu_F(u_1))}, \frac{u_2}{(\mu_T(u_2),\mu_F(u_2))}, ..., \frac{u_n}{(\mu_T(u_n),\mu_F(u_n))}\right\}$, and

$B = \left\{\frac{u_1}{(\nu_T(u_1),\nu_F(u_1))}, \frac{u_2}{(\nu_T(u_2),\nu_F(u_2))}, ..., \frac{u_n}{(\nu_T(u_n),\nu_F(u_n))}\right\}$

be any two plithogenic intuitionistic fuzzy sets over U. Then the plithogenic intuitionistic fuzzy intersection between A and B is defined as follows:

$$A \wedge_{IP} B = \left\{\frac{u_i}{((1-c_v)(\mu(u_i)\wedge_F \nu(u_i))+(c_v)(\mu(u_i)\vee_F \nu_F(u_i)),(c_v)(\mu(u_i)\vee_F \nu(u_i))+(1-c_v)(\mu(u_i)\wedge_F \nu_F(u_i)))}\right\}, \forall i = 1, 2, ..., n$$

Where $\wedge_F$ and $\vee_F$ represent fuzzy t-norm and fuzzy t-conorm (s-norm) respectively and c_v represent the degrees of contradictions.

4.3. *Plithogenic Neutrosophic Intersection*

Definition 4.3. Let $U = \{u_1, u_2, ..., u_n\}$. Suppose

$A = \left\{\frac{u_1}{\langle\mu_T(u_1),\mu_I(u_1),\mu_F(u_1)\rangle}, \frac{u_2}{\langle\mu_T(u_2),\mu_I(u_2),\mu_F(u_2)\rangle}, ..., \frac{u_n}{\langle\mu_T(u_n),\mu_I(u_n),\mu_F(u_n)\rangle}\right\}$, and

$B = \left\{\frac{u_1}{\langle\nu_T(u_1),\nu_I(u_1),\nu_F(u_1)\rangle}, \frac{u_2}{\langle\nu_T(u_2),\nu_I(u_2),\nu_F(u_2)\rangle}, ..., \frac{u_n}{\langle\nu_T(u_n),\nu_I(u_n),\nu_F(u_n)\rangle}\right\}$

be any two plithogenic neutrosophic set over U. Then the plithogenic neutrosophic intersection between A and B is defined as follows:

$$A \wedge_{NP} B = \left\{\frac{u_i}{\langle(1-c_v)(\mu(u_i)\wedge_F \nu(u_i))+(c_v)(\mu(u_i)\vee_F \nu_F(u_i)),\frac{1}{2}[(\mu_I(u_1)\wedge_F \nu_T(u_1))+(\mu_I(u_1)\vee_F \nu_T(u_1))],(c_v)(\mu(u_i)\vee_F \nu(u_i))+(1-c_v)(\mu(u_i)\wedge_F \nu_F(u_i))\rangle}\right\},$$

$\forall i = 1, 2, ..., n$, where $\wedge_F$ and $\vee_F$ represent fuzzy t-norm and fuzzy t-conorm (s-norm) respectively and c_v represent the degrees of contradictions.

5. **Plithogenic Union**

In this section we recall the definition of plithogenic union over three cases: fuzzy, intuitionistic fuzzy and neutrosophic set which are defined for the first time by Smarandache in 2017 and 2018 [11, 12].

5.1. *Plithogenic Fuzzy Union*

Definition 5.1. Let U, A and B as defined in 4.1. Then the plithogenic fuzzy union between A and B is defined as follows:

$$A \vee_{FP} B = \left\{ \frac{u_i}{(1-c_v)(\mu(u_i) \vee_F \nu(u_i)) + (c_v)(\mu(u_i) \wedge_F \nu_F(u_i))} \right\}, \forall i = 1, 2, ..., n$$

Where $\wedge_F$ and $\vee_F$ represent fuzzy t-norm and fuzzy t-conorm (s-norm) respectively and c_v represent the degrees of contradictions.

5.2. *Plithogenic Intuitionistic Fuzzy Union*

Definition 5.2. Let U, A and B as defined in 4.2. Then the plithogenic intuitionistic fuzzy union between A and B is defined as follows:

$$A \vee_{IP} B = \left\{ \frac{u_i}{((1-c_v)(\mu(u_i) \vee_F \nu(u_i)) + (c_v)(\mu(u_i) \wedge_F \nu_F(u_i)), (c_v)(\mu(u_i) \wedge_F \nu(u_i)) + (1-c_v)(\mu(u_i) \vee_F \nu_F(u_i)))} \right\}, \forall i = 1, 2, ..., n$$

Where $\wedge_F$ and $\vee_F$ represent fuzzy t-norm and fuzzy t-conorm (s-norm) respectively and c_v represent the degrees of contradictions.

5.3. *Plithogenic Neutrosophic Union*

Definition 5.3. Let U, A and B as defined in 4.3. Then the plithogenic neutrosophic intersection between A and B is defined as follows:

$$A \vee_{NP} B = \left\{ \frac{u_i}{\langle (1-c_v)(\mu(u_i) \vee_F \nu(u_i)) + (c_v)(\mu(u_i) \wedge_F \nu_F(u_i)), \frac{1}{2}[(\mu_I(u_1) \wedge_F \nu_T(u_1)) + (\mu_I(u_1) \vee_F \nu_T(u_1))], (c_v)(\mu(u_i) \wedge_F \nu(u_i)) + (1-c_v)(\mu(u_i) \vee_F \nu_F(u_i)) \rangle} \right\},$$

$\forall i = 1, 2, ..., n$, where $\wedge_F$ and $\vee_F$ represent fuzzy t-norm and fuzzy t-conorm (s-norm) respectively and c_v represent the degrees of contradictions.

5.4. *Hypersoft set*

Definition 5.4. [11] Let U be a universe of discourse, $\wp(U)$ the power set of U. Let $a_1, a_2, \cdots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \cdots, A_n$, with $A_i \cap A_j = \varnothing$, for $j \neq i$, and $i, j \in \{1, 2, \cdots, n.\}$. Then the pair

$$(F, A_1 \times A_2 \times \cdots \times A_n)$$

where:

$$F : A_1 \times A_2 \times \cdots \times A_n \longrightarrow \wp(U)$$

is called a Hypersoft Set over U.

6. **Plithogenic Soft Set**

In this section we define the concept of plithogenic soft set(In General) as a generalization of soft set. We also, define its basic operations namely, union and intersection and study their properties.

Definition 6.1. Let U be a universe of discourse, $(U)^z$ the $z-$power set of U such that
$z = C$ (Power set of U),
$z = F$ (The set of all fuzzy set of U),
$z = IF$ (The set of all intuitionistic fuzzy set of U), or
$z = N$ (The set of all neutrosophic set of U). Let $a_1, a_2, \cdots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $V_1, V_2, \cdots, V_n$, with $V_i \cap V_j = \varnothing$, for $j \neq i$, and $i, j \in \{1, 2, \cdots, n.\}$. Suppose $V_i = \{v_{i1}, v_{i2}, \cdots, v_{im_i}\}$ and let $\Upsilon = V_1 \times V_2 \times \cdots \times V_n$. Let $D = (D_1, D_2, \cdots D_n)$ the dominant attribute element of A_i $\forall i$ and $c\left(D_i, v_{ij}\right), i \in \{1, 2, \cdots n\}, j \in \{1, 2, \cdots m_i\}$ the attribute value contradiction degree function such that $c_i : V_i \times V_i \longrightarrow [0, 1]$, Then the pair (F^z_P, Υ), where:

$$F^z_P : \Upsilon \longrightarrow [0, 1]_D \times (U)^z$$

is called a Plithogenic Soft Set (P-SS In short) over U.

To illustrate the above definition we give the following examples for all cases: crisp, fuzzy, intuitionistic and neutrosophic:

6.1. *Plithogenic Crisp Soft Set*

Example 6.2. Let $U = \{c_1, c_2, c_3, c_4\}$. Here U represents the set of cars. Let $a_1 = speed, a_2 = color, a_3 = model, a_4 = manufacturingyear$. Suppose their attributes values respectively:
Speed $\equiv A_1 = \{slow, fast, veryfast\}$,
Color $\equiv A_2 = \{white, yellow, red, black\}$,
Model $\equiv A_3 = \{model_1, model_2, model_3, model_4\}$,
manufacturing year $\equiv A_4 = \{2015\ and\ befor, 2016, 2017, 2018, 2019\}$.
Where the dominant attribute's value $D = (slow, red, model_3, 2019)$. Let $H \subseteq \Upsilon$ such that $H = \{\epsilon_1 = (slow, white, model_1, 2015\ and\ befor), \epsilon_2 = (slow, yellow, model_3, 2017),$
$\epsilon_3 = (fast, red, model_4, 2018)\}$. Define a function $F^z_P : H \longrightarrow [0, 1]_D \times (U)^C$ as follows:

$$F_P^C(\epsilon_1) = \left\{ \frac{(c_1,(0,0.4,1,1)_D)}{(1,1,1,1)}, \frac{(c_4,(0,0.4,1,1)_D)}{(1,1,1,1)} \right\},$$

$$F_P^C(\epsilon_2) = \left\{ \frac{(c_2,(0,0.5,0,1)_D)}{(1,1,1,1)} \right\},$$

$$F_P^C(\epsilon_3) = \left\{ \frac{(c_3,(0.5,0,1,1)_D)}{(1,1,1,1)} \right\},$$

Then we can find the plithogenic crisp soft set (P-CSS) $\left(F_P^C, H\right)$ as consisting of the following collection of approximations:

$$(F_P^C, H) = \left\{ \left(\epsilon_1, \left\{ \frac{(c_1,(0,0.4,1,1)_D)}{(1,1,1,1)}, \frac{(c_4,(0,0.4,1,1)_D)}{(1,1,1,1)} \right\}\right), \left(\epsilon_2, \left\{ \frac{(c_2,(0,0.5,0,1)_D)}{(1,1,1,1)} \right\}\right), \left(\epsilon_3, \left\{ \frac{(c_3,(0.5,0,1,1)_D)}{(1,1,1,1)} \right\}\right) \right\}.$$

6.2. *Plithogenic Fuzzy Soft Set*

Example 6.3. Consider Example 6.2 and suppose that the

$$F_P^F(\epsilon_1) = \left\{ \frac{(c_1,(0,0.4,1,1)_D)}{(0.6,0.7,1,1)}, \frac{(c_2,(0,0.4,1,1)_D)}{(0.7,0.3,0,0)}, \frac{(c_3,(0,0.4,1,1)_D)}{(0.1,0.3,0,0)}, \frac{(c_4,(0,0.4,1,1)_D)}{(0.7,0.6,1,1)} \right\},$$

$$F_P^F(\epsilon_2) = \left\{ \frac{(c_1,(0,0.5,0,1)_D)}{(0.6,0.2,0,0)}, \frac{(c_2,(0,0.5,0,1)_D)}{(0.5,0.7,1,1)}, \frac{(c_3,(0,0.5,0,1)_D)}{(0.1,0.1,0,0)}, \frac{(c_4,(0,0.5,0,1)_D)}{(0.7,0.4,0,0)} \right\},$$

$$F_P^F(\epsilon_3) = \left\{ \frac{(c_1,(0.5,0,1,1)_D)}{(0.1,0.2,0,0)}, \frac{(c_2,(0.5,0,1,1)_D)}{(0.1,0.3,0,0)}, \frac{(c_3,(0.5,0,1,1)_D)}{(0.8,0.7,1,1)}, \frac{(c_4,(0.5,0,1,1)_D)}{(0.1,0.1,0,0)} \right\},$$

Then we can find the plithogenic fuzzy soft set (P-FSS) $\left(F_P^F, H\right)$ as consisting of the following collection of approximations:

$$(F_P^F, H) = \left\{ \left(\epsilon_1, \left\{ \frac{(c_1,(0,0.4,1,1)_D)}{(0.6,0.7,1,1)}, \frac{(c_2,(0,0.4,1,1)_D)}{(0.7,0.3,0,0)}, \frac{(c_3,(0,0.4,1,1)_D)}{(0.1,0.3,0,0)}, \frac{(c_4,(0,0.4,1,1)_D)}{(0.7,0.6,1,1)} \right\}\right), \right.$$

$$\left(\epsilon_2, \left\{ \frac{(c_1,(0,0.5,0,1)_D)}{(0.6,0.2,0,0)}, \frac{(c_2,(0,0.5,0,1)_D)}{(0.5,0.7,1,1)}, \frac{(c_3,(0,0.5,0,1)_D)}{(0.1,0.1,0,0)}, \frac{(c_4,(0,0.5,0,1)_D)}{(0.7,0.4,0,0)} \right\}\right),$$

$$\left. \left(\epsilon_3, \left\{ \frac{(c_1,(0.5,0,1,1)_D)}{(0.1,0.2,0,0)}, \frac{(c_2,(0.5,0,1,1)_D)}{(0.1,0.3,0,0)}, \frac{(c_3,(0.5,0,1,1)_D)}{(0.8,0.7,1,1)}, \frac{(c_4,(0.5,0,1,1)_D)}{(0.1,0.1,0,0)} \right\}\right) \right\}.$$

6.3. *Plithogenic Intuitionistic Fuzzy Soft Set*

Example 6.4. Consider Example 6.2 and suppose that the

$$F_P^{IF}(\epsilon_1) = \left\{ \frac{(c_1,(0,0.4,1,1)_D)}{((0.6,0.2),(0.7,0.1),(1,0),(1,0))}, \frac{(c_2,(0,0.4,1,1)_D)}{((0.7,0.1),(0.3,0.5),(0,1),(0,1))}, \frac{(c_3,(0,0.4,1,1)_D)}{((0.1,0.8),(0.3,0.5),(0,1),(0,1))}, \frac{(c_4,(0,0.4,1,1)_D)}{((0.7,0.2),(0.6,0.3),(1,0),(1,0))} \right\},$$

$$F_P^{IF}(\epsilon_2) = \left\{ \frac{(c_1,(0,0.5,0,1)_D)}{((0.6,0.3),(0.2,0.6),(0,1),(0,1))}, \frac{(c_2,(0,0.5,0,1)_D)}{((0.5,0.4),(0.7,0.2),(1,0),(1,0))}, \frac{(c_3,(0,0.5,0,1)_D)}{((0.1,0.7),(0.1,0.8),(0,1),(0,1))}, \frac{(c_4,(0,0.5,0,1)_D)}{((0.7,0.2),(0.4,0.4),(0,1),(0,1))} \right\},$$

$$F_P^{IF}(\epsilon_3) = \left\{ \frac{(c_1,(0.5,0,1,1)_D)}{((0.1,0.7),(0.2,0.7),(0,1),(0,1))}, \frac{(c_2,(0.5,0,1,1)_D)}{((0.1,0.8),(0.3,0.5),(0,1),(0,1))}, \frac{(c_3,(0.5,0,1,1)_D)}{((0.8,0.1),(0.7,0.2),(1,0),(1,0))}, \frac{(c_4,(0.5,0,1,1)_D)}{((0.1,0.8),(0.1,0.7),(0,1),(0,1))} \right\},$$

Then we can find the plithogenic intuitionistic fuzzy soft set (P-IFSS) $\left(F_P^{IF},H\right)$ as consisting of the following collection of approximations:

$$\left(F_P^{IF},H\right)=\Big\{\Big(\epsilon_1,\Big\{\frac{(c_1,(0,0.4,1,1)_D)}{((0.6,0.2),(0.7,0.1),(1,0),(1,0))},\frac{(c_2,(0,0.4,1,1)_D)}{((0.7,0.1),(0.3,0.5),(0,1),(0,1))},\frac{(c_3,(0,0.4,1,1)_D)}{((0.1,0.8),(0.3,0.5),(0,1),(0,1))},\frac{(c_4,(0,0.4,1,1)_D)}{((0.7,0.2),(0.6,0.3),(1,0),(1,0))}\Big\}\Big),$$

$$\Big(\epsilon_2,\Big\{\frac{(c_1,(0,0.5,0,1)_D)}{((0.6,0.3),(0.2,0.6),(0,1),(0,1))},\frac{(c_2,(0,0.5,0,1)_D)}{((0.5,0.4),(0.7,0.2),(1,0),(1,0))},\frac{(c_3,(0,0.5,0,1)_D)}{((0.1,0.7),(0.1,0.8),(0,1),(0,1))},\frac{(c_4,(0,0.5,0,1)_D)}{((0.7,0.2),(0.4,0.4),(0,1),(0,1))}\Big\}\Big),$$

$$\Big(\epsilon_3,\Big\{\frac{(c_1,(0.5,0,1,1)_D)}{((0.1,0.7),(0.2,0.7),(0,1),(0,1))},\frac{(c_2,(0.5,0,1,1)_D)}{((0.1,0.8),(0.3,0.5),(0,1),(0,1))},\frac{(c_3,(0.5,0,1,1)_D)}{((0.8,0.1),(0.7,0.2),(1,0),(1,0))},\frac{(c_4,(0.5,0,1,1)_D)}{((0.1,0.8),(0.1,0.7),(0,1),(0,1))}\Big\}\Big)\Big\}.$$

6.4. *Plithogenic Neutrosophic Soft Set*

Example 6.5. Consider Example 6.2 and $U=\{c_1,c_2,c_3\}$, suppose that the

$$F_P^N(\epsilon_1)=\left\{\frac{(c_1,(0,0.4,1,1)_D)}{((0.6,0.3,0.1),(0.7,0.2,0.1),(1,0,0),(1,0,0))},\frac{(c_2,(0,0.4,1,1)_D)}{((0.7,0.1,0.2),(0.3,0.5,0.2),(0,0,1),(0,0,1))},\frac{(c_3,(0,0.4,1,1)_D)}{((0.1,0.1,0.8),(0.3,0.6,0.1),(0,0,1),(0,0,1))}\right\},$$

$$F_P^N(\epsilon_2)=\left\{\frac{(c_1,(0,0.5,0,1)_D)}{((0.6,0.3,0.1),(0.2,0.6,0.2),(0,0,1),(0,0,1))},\frac{(c_2,(0,0.5,0,1)_D)}{((0.5,0.2,0.3),(0.7,0.2,0.1),(1,0,0),(1,0,0))},\frac{(c_3,(0,0.5,0,1)_D)}{((0.1,0.2,0.7),(0.1,0.1,0.8),(0,0,1),(0,0,1))}\right\},$$

$$F_P^N(\epsilon_3)=\left\{\frac{(c_1,(0.5,0,1,1)_D)}{((0.1,0.2,0.7),(0.2,0.1,0.7),(0,0,1),(0,0,1))},\frac{(c_2,(0.5,0,1,1)_D)}{((0.1,0.1,0.8),(0.3,0.2,0.5),(0,0,1),(0,0,1))},\frac{(c_3,(0.5,0,1,1)_D)}{((0.8,0.1,0.1),(0.7,0.1,0.2),(1,0,0),(1,0,0))}\right\},$$

Then we can find the plithogenic neutrosophic soft set (P-NSS) $\left(F_P^N,H\right)$ as consisting of the following collection of approximations:

$$\left(F_P^N,H\right)=\Big\{\Big(\epsilon_1,\Big\{\frac{(c_1,(0,0.4,1,1)_D)}{((0.6,0.3,0.1),(0.7,0.2,0.1),(1,0,0),(1,0,0))},\frac{(c_2,(0,0.4,1,1)_D)}{((0.7,0.1,0.2),(0.3,0.5,0.2),(0,0,1),(0,0,1))},\frac{(c_3,(0,0.4,1,1)_D)}{((0.1,0.1,0.8),(0.3,0.6,0.1),(0,0,1),(0,0,1))}\Big\}\Big),$$

$$\Big(\epsilon_2,\Big\{\frac{(c_1,(0,0.5,0,1)_D)}{((0.6,0.3,0.1),(0.2,0.6,0.2),(0,0,1),(0,0,1))},\frac{(c_2,(0,0.5,0,1)_D)}{((0.5,0.2,0.3),(0.7,0.2,0.1),(1,0,0),(1,0,0))},\frac{(c_3,(0,0.5,0,1)_D)}{((0.1,0.2,0.7),(0.1,0.1,0.8),(0,0,1),(0,0,1))}\Big\}\Big),$$

$$\Big(\epsilon_3,\Big\{\frac{(c_1,(0.5,0,1,1)_D)}{((0.1,0.2,0.7),(0.2,0.1,0.7),(0,0,1),(0,0,1))},\frac{(c_2,(0.5,0,1,1)_D)}{((0.1,0.1,0.8),(0.3,0.2,0.5),(0,0,1),(0,0,1))},\frac{(c_3,(0.5,0,1,1)_D)}{((0.8,0.1,0.1),(0.7,0.1,0.2),(1,0,0),(1,0,0))}\Big\}\Big)\Big\}.$$

7. **Union and Intersection**

In this section, we introduce the definitions of union and intersection of plithogenic soft sets, derive their properties, and give some examples.

Definition 7.1. The *union* of two plithogenic soft sets (F_P^z,A) and (G_P^z,B) over U, denoted by
$(F_P^z,A)\bigvee_P^z(G_P^z,B)$, is the plithogenic soft set (H_P^z,Ω) where $\Omega=A\cup B$, and $\forall\,\varepsilon\in\Omega$,

$$H_P^z(\varepsilon)=\begin{cases}F_P^z(\varepsilon), & \text{if } \varepsilon\in A-B\\ G_P^z(\varepsilon), & \text{if } \varepsilon\in B-A\\ F_P^z(\varepsilon)\vee_P^z G_P^z(\varepsilon), & \text{if } \varepsilon\in A\cap B\end{cases}$$

where $\vee_P^z$ is a z-plithogenic union.

Definition 7.2. The *intersection* of two plithogenic soft sets (F_P^z,A) and (G_P^z,B) over U, denoted by

$(F_P^z, A) \bigwedge_P^z (G_P^z, B)$, is the plithogenic soft set (H_P^z, Ω) where $\Omega = A \cup B$, and $\forall \varepsilon \in \Omega$,

$$H_P^z(\varepsilon) = \begin{cases} F_P^z(\varepsilon), & \text{if } \varepsilon \in A - B \\ G_P^z(\varepsilon), & \text{if } \varepsilon \in B - A \\ F_P^z(\varepsilon) \wedge_P^z G_P^z(\varepsilon), & \text{if } \varepsilon \in A \cap B \end{cases}$$

where $\wedge_P^z$ is a z-plithogenic intersection.

7.1. *Plithogenic Fuzzy Soft Union*

Example 7.3. Consider Example 6.2 Let

$$A = \Big\{ a_1 = (slow, white, model_1, 2015\ and\ befor), a_2 = (slow, yellow, model_3, 2017), a_3 = (fast, red, model_4, 2018) \Big\} and$$

$$B = \Big\{ b_1 = (slow, white, model_1, 2015\ and\ befor), b_2 = (slow, yellow, model_3, 2017), b_3 = (fast, red, model_4, 2018), b_4 = (slow, red, model_1, 2019) \Big\}$$

Suppose (F_P^F, A) and (G_P^F, A) are two plithogenic fuzzy soft sets over U such that

$$(F_P^F, A) = \Bigg\{ \Bigg(a_1, \left\{ \frac{(c_1, (0, 0.4, 1, 1)_D)}{(0.6, 0.7, 1, 1)}, \frac{(c_2, (0, 0.4, 1, 1)_D)}{(0.7, 0.3, 0, 0)}, \frac{(c_3, (0, 0.4, 1, 1)_D)}{(0.1, 0.3, 0, 0)}, \frac{(c_4, (0, 0.4, 1, 1)_D)}{(0.7, 0.6, 1, 1)} \right\} \Bigg), \Bigg(a_2, \left\{ \frac{(c_1, (0, 0.5, 0, 1)_D)}{(0.6, 0.2, 0, 0)}, \frac{(c_2, (0, 0.5, 0, 1)_D)}{(0.5, 0.7, 1, 1)}, \frac{(c_3, (0, 0.5, 0, 1)_D)}{(0.1, 0.1, 0, 0)}, \frac{(c_4, (0, 0.5, 0, 1)_D)}{(0.7, 0.4, 0, 0)} \right\} \Bigg), \Bigg(a_3, \left\{ \frac{(c_1, (0.5, 0, 1, 1)_D)}{(0.1, 0.2, 0, 0)}, \frac{(c_2, (0.5, 0, 1, 1)_D)}{(0.1, 0.3, 0, 0)}, \frac{(c_3, (0.5, 0, 1, 1)_D)}{(0.8, 0.7, 1, 1)}, \frac{(c_4, (0.5, 0, 1, 1)_D)}{(0.1, 0.1, 0, 0)} \right\} \Bigg) \Bigg\}.$$

$$(G_P^F, B) = \Bigg\{ \Bigg(b_1, \left\{ \frac{(c_1, (0, 0.4, 1, 1)_D)}{(0.5, 0.6, 1, 1)}, \frac{(c_2, (0, 0.4, 1, 1)_D)}{(0.8, 0.4, 0, 0)}, \frac{(c_3, (0, 0.4, 1, 1)_D)}{(0.2, 0.2, 0, 0)}, \frac{(c_4, (0, 0.4, 1, 1)_D)}{(0.7, 0.5, 1, 1)} \right\} \Bigg), \Bigg(b_2, \left\{ \frac{(c_1, (0, 0.5, 0, 1)_D)}{(0.5, 0.3, 0, 0)}, \frac{(c_2, (0, 0.5, 0, 1)_D)}{(0.7, 0.7, 1, 1)}, \frac{(c_3, (0, 0.5, 0, 1)_D)}{(0.3, 0.2, 0, 0)}, \frac{(c_4, (0, 0.5, 0, 1)_D)}{(0.8, 0.5, 0, 0)} \right\} \Bigg), \Bigg(b_3, \left\{ \frac{(c_1, (0.5, 0, 1, 1)_D)}{(0.5, 0.6, 0, 0)}, \frac{(c_2, (0.5, 0, 1, 1)_D)}{(0.8, 0.3, 0, 0)}, \frac{(c_3, (0.5, 0, 1, 1)_D)}{(0.2, 0.7, 1, 1)}, \frac{(c_4, (0.5, 0, 1, 1)_D)}{(0.6, 0.3, 0, 0)} \right\} \Bigg), \Bigg(b_4, \left\{ \frac{(c_1, (0, 0, 1, 1)_D)}{(0.2, 0.3, 1, 0)}, \frac{(c_2, (0, 0, 1, 1)_D)}{(0.1, 0.3, 0, 0)}, \frac{(c_3, (0, 0, 1, 1)_D)}{(0.8, 0.7, 1, 1)}, \frac{(c_4, (0, 0, 1, 1)_D)}{(0.1, 0.1, 1, 0)} \right\} \Bigg) \Bigg\}.$$

By using basic fuzzy union (maximum) and basic fuzzy intersection (minimum) we have:

$\left(F_P^F, A\right) \bigvee_P^F \left(G_P^F, B\right) = \left(H_P^F, K\right)$ where

$$\left(H_P^F, K\right) = \left\{ \left(k_1, \left\{ \frac{(c_1,(0,0.4,1,1)_D)}{(0.6,0.66,1,1)}, \frac{(c_2,(0,0.4,1,1)_D)}{(0.8,0.36,0,0)}, \frac{(c_3,(0,0.4,1,1)_D)}{(0.2,0.26,0,0)}, \frac{(c_4,(0,0.4,1,1)_D)}{(0.7,0.56,1,1)} \right\}\right), \right.$$
$$\left(k_2, \left\{ \frac{(c_1,(0,0.5,0,1)_D)}{(0.6,0.25,0,0)}, \frac{(c_2,(0,0.5,0,1)_D)}{(0.7,0.7,1,1)}, \frac{(c_3,(0,0.5,0,1)_D)}{(0.3,0.15,0,0)}, \frac{(c_4,(0,0.5,0,1)_D)}{(0.8,0.45,0,0)} \right\}\right),$$
$$\left(k_3, \left\{ \frac{(c_1,(0.5,0,1,1)_D)}{(0.3,0.6,0,0)}, \frac{(c_2,(0.5,0,1,1)_D)}{(0.45,0.3,0,0)}, \frac{(c_3,(0.5,0,1,1)_D)}{(0.5,0.7,1,1)}, \frac{(c_4,(0.6,0,1,1)_D)}{(0.35,0.3,0,0)} \right\}\right),$$
$$\left.\left(k_4, \left\{ \frac{(c_1,(0,0,1,1)_D)}{(0.2,0.3,1,0)}, \frac{(c_2,(0,0,1,1)_D)}{(0.1,0.3,0,0)}, \frac{(c_3,(0,0,1,1)_D)}{(0.8,0.7,1,1)}, \frac{(c_4,(0,0,1,1)_D)}{(0.1,0.1,1,0)} \right\}\right) \right\}.$$

To describe the result in Example above let's compute $\frac{(c_1,(0,0.4,1,1)_D)}{(0.6,0.7,1,1)} \vee_P^F \frac{(c_1,(0,0.4,1,1)_D)}{(0.5,0.6,1,1)}$ as follows:

$(1-0)*max(0.6,0.5)+0*min(0.6,0.5) = 0.6$, $(1-0.4)*max(0.7,0.6)+0.4*min(0.7,0.6) = 0.66$, $(1-1)*max(1,1)+1*min(1,1) = 1$ and $(1-1)*max(1,1)+1*min(1,1) = 1$

7.2. *Plithogenic Fuzzy Soft Intersection*

Example 7.4. Consider Example 7.3 By using basic fuzzy union (maximum) and basic fuzzy intersection (minimum) we have: $\left(F_P^F, A\right) \bigwedge_P^F \left(G_P^F, B\right) = \left(H_P^F, K\right)$ where

$$\left(H_P^F, K\right) = \left\{ \left(k_1, \left\{ \frac{(c_1,(0,0.4,1,1)_D)}{(0.5,0.64,1,1)}, \frac{(c_2,(0,0.4,1,1)_D)}{(0.7,0.34,0,0)}, \frac{(c_3,(0,0.4,1,1)_D)}{(0.1,0.18,0,0)}, \frac{(c_4,(0,0.4,1,1)_D)}{(0.7,0.52,1,1)} \right\}\right), \right.$$
$$\left(k_2, \left\{ \frac{(c_1,(0,0.5,0,1)_D)}{(0.5,0.25,0,0)}, \frac{(c_2,(0,0.5,0,1)_D)}{(0.5,0.7,1,1)}, \frac{(c_3,(0,0.5,0,1)_D)}{(0.1,0.15,0,0)}, \frac{(c_4,(0,0.5,0,1)_D)}{(0.7,0.45,0,0)} \right\}\right),$$
$$\left(k_3, \left\{ \frac{(c_1,(0.5,0,1,1)_D)}{(0.3,0.2,0,0)}, \frac{(c_2,(0.5,0,1,1)_D)}{(0.45,0.3,0,0)}, \frac{(c_3,(0.5,0,1,1)_D)}{(0.5,0.7,1,1)}, \frac{(c_4,(0.5,0,1,1)_D)}{(0.35,0.1,0,0)} \right\}\right),$$
$$\left.\left(k_4, \left\{ \frac{(c_1,(0,0,1,1)_D)}{(0.2,0.3,1,0)}, \frac{(c_2,(0,0,1,1)_D)}{(0.1,0.3,0,0)}, \frac{(c_3,(0,0,1,1)_D)}{(0.8,0.7,1,1)}, \frac{(c_4,(0,0,1,1)_D)}{(0.1,0.1,1,0)} \right\}\right) \right\}.$$

To describe the result in Example above let's compute $\frac{(c_1,(0,0.4,1,1)_D)}{(0.6,0.7,1,1)} \wedge_P^F \frac{(c_1,(0,0.4,1,1)_D)}{(0.5,0.6,1,1)}$ as follows:

$(1-0)*min(0.6,0.5)+0*max(0.6,0.5) = 0.5$, $(1-0.4)*min(0.7,0.6)+0.4*max(0.7,0.6) = 0.64$, $(1-1)*min(1,1)+1*max(1,1) = 1$ and $(1-1)*min(1,1)+1*max(1,1) = 1$

7.3. *Plithogenic Intuitionistic Fuzzy Soft Union*

Example 7.5. Consider Example 7.3. Suppose $\left(F_P^{IF}, A\right)$ and $\left(G_P^{IF}, A\right)$ are two plithogenic intuitionistic fuzzy soft sets over U such that

$$(F_P^{IF}, A) = \left\{ \left(a_1, \left\{ \frac{(c_1,(0,0.4,1,1)_D)}{((0.6,0.3),(0.7,0.3),(1,0),(1,0))}, \frac{(c_2,(0,0.4,1,1)_D)}{((0.7,0.2),(0.3,0.5),(0,1),(0,1))}, \frac{(c_3,(0,0.4,1,1)_D)}{((0.1,0.8),(0.3,0.5),(0,1),(0,1))}, \frac{(c_4,(0,0.4,1,1)_D)}{((0.7,0.2),(0.6,0.2),(1,0),(1,0))} \right\}\right), \right.$$
$$\left(a_2, \left\{ \frac{(c_1,(0,0.5,0,1)_D)}{((0.6,0.3),(0.2,0.7),(0,1),(0,1))}, \frac{(c_2,(0,0.5,0,1)_D)}{((0.5,0.4),(0.7,0.3),(1,0),(1,0))}, \frac{(c_3,(0,0.5,0,1)_D)}{((0.1,0.8),(0.1,0.9),(0,1),(0,1))}, \frac{(c_4,(0,0.5,0,1)_D)}{((0.7,0.3),(0.4,0.5),(0,1),(0,1))} \right\}\right),$$
$$\left.\left(a_3, \left\{ \frac{(c_1,(0.5,0,1,1)_D)}{((0.1,0.8),(0.2,0.8),(0,1),(0,1))}, \frac{(c_2,(0.5,0,1,1)_D)}{((0.1,0.9),(0.3,0.7),(0,1),(0,1))}, \frac{(c_3,(0.5,0,1,1)_D)}{((0.8,0.1),(0.7,0.2),(1,0),(1,0))}, \frac{(c_4,(0.5,0,1,1)_D)}{((0.1,0.8),(0.1,0.9),(0,1),(0,1))} \right\}\right) \right\}.$$

$$\begin{aligned}(G_P^{IF},B)=\Bigg\{&\Bigg(b_1,\Bigg\{\frac{(c_1,(0,0.4,1,1)_D)}{((0.5,0.4),(0.6,0.3),(1,0),(1,0))},\frac{(c_2,(0,0.4,1,1)_D)}{((0.8,0.2),(0.4,0.6),(0,1),(0,1))},\frac{(c_3,(0,0.4,1,1)_D)}{((0.2,0.7),(0.2,0.8),(0,1),(0,1))},\frac{(c_4,(0,0.4,1,1)_D)}{((0.7,0.2),(0.5,0.4),(1,0),(1,0))}\Bigg\}\Bigg),\\ &\Bigg(b_2,\Bigg\{\frac{(c_1,(0,0.5,0,1)_D)}{((0.5,0.5),(0.3,0.7),(0,1),(0,1))},\frac{(c_2,(0,0.5,0,1)_D)}{((0.7,0.2),(0.7,0.2),(1,0),(1,0))},\frac{(c_3,(0,0.5,0,1)_D)}{((0.3,0.6),(0.2,0.8),(0,1),(0,1))},\frac{(c_4,(0,0.5,0,1)_D)}{((0.8,0.1),(0.5,0.5),(0,1),(0,1))}\Bigg\}\Bigg),\\ &\Bigg(b_3,\Bigg\{\frac{(c_1,(0.5,0,1,1)_D)}{((0.5,0.4),(0.6,0.4),(0,1),(0,1))},\frac{(c_2,(0.5,0,1,1)_D)}{((0.8,0.1),(0.3,0.7),(0,1),(0,1))},\frac{(c_3,(0.5,0,1,1)_D)}{((0.2,0.7),(0.7,0.2),(1,0),(1,0))},\frac{(c_4,(0.5,0,1,1)_D)}{((0.6,0.3),(0.3,0.7),(0,1),(0,1))}\Bigg\}\Bigg)\\ &\Bigg(b_4,\Bigg\{\frac{(c_1,(0,0,1,1)_D)}{((0.2,0.8),(0.3,0.6),(0,1),(0,1))},\frac{(c_2,(0,0,1,1)_D)}{((0.1,0.9),(0.3,0.7),(0,1),(0,1))},\frac{(c_3,(0,0,1,1)_D)}{((0.8,0.2),(0.7,0.3),(1,0),(1,0))},\frac{(c_4,(0,0,1,1)_D)}{((0.1,0.9),(0.1,0.8),(0,1),(0,1))}\Bigg\}\Bigg)\Bigg\}.\end{aligned}$$

By using basic fuzzy union (maximum) and basic fuzzy intersection (minimum) we have:
$\left(F_P^{IF},A\right)\bigvee_P^{IF}\left(G_P^{IF},B\right)=\left(H_P^{IF},K\right)$ where

$$\begin{aligned}(H_P^{IF},K)=\Bigg\{&\Bigg(k_1,\Bigg\{\frac{(c_1,(0,0.4,1,1)_D)}{((0.6,0.3),(0.66,0.3),(1,0),(1,0))},\frac{(c_2,(0,0.4,1,1)_D)}{((0.8,0.2),(0.4,0.54),(0,1),(0,1))},\frac{(c_3,(0,0.4,1,1)_D)}{((0.2,0.7),(0.26,0.63),(0,1),(0,1))},\frac{(c_4,(0,0.4,1,1)_D)}{((0.7,0.2),(0.56,0.26),(1,0),(1,0))}\Bigg\}\Bigg),\\ &\Bigg(k_2,\Bigg\{\frac{(c_1,(0,0.5,0,1)_D)}{((0.6,0.3),(0.25,0.7),(0,1),(0,1))},\frac{(c_2,(0,0.5,0,1)_D)}{((0.7,0.2),(0.7,0.25),(1,0),(1,0))},\frac{(c_3,(0,0.5,0,1)_D)}{((0.3,0.6),(0.15,0.85),(0,1),(0,1))},\frac{(c_4,(0,0.5,0,1)_D)}{((0.8,0.1),(0.45,0.5),(0,1),(0,1))}\Bigg\}\Bigg),\\ &\Bigg(k_3,\Bigg\{\frac{(c_1,(0.5,0,1,1)_D)}{((0.3,0.6),(0.6,0.4),(0,1),(0,1))},\frac{(c_2,(0.5,0,1,1)_D)}{((0.45,0.5),(0.3,0.7),(0,1),(0,1))},\frac{(c_3,(0.5,0,1,1)_D)}{((0.5,0.4),(0.7,0.2),(1,0),(1,0))},\frac{(c_4,(0.5,0,1,1)_D)}{((0.35,0.55),(0.3,0.7),(0,1),(0,1))}\Bigg\}\Bigg)\\ &\Bigg(k_4,\Bigg\{\frac{(c_1,(0,0,1,1)_D)}{((0.2,0.8),(0.3,0.6),(0,1),(0,1))},\frac{(c_2,(0,0,1,1)_D)}{((0.1,0.9),(0.3,0.7),(0,1),(0,1))},\frac{(c_3,(0,0,1,1)_D)}{((0.8,0.2),(0.7,0.3),(1,0),(1,0))},\frac{(c_4,(0,0,1,1)_D)}{((0.1,0.9),(0.1,0.8),(0,1),(0,1))}\Bigg\}\Bigg)\Bigg\}.\end{aligned}$$

7.4. *Plithogenic Intuitionistic Fuzzy Soft Intersection*

Example 7.6. Consider Example 7.5 By using basic fuzzy union (maximum) and basic fuzzy intersection (minimum) we have:
$\left(F_P^{IF},A\right)\bigwedge_P^{IF}\left(G_P^{IF},B\right)=\left(H_P^{IF},K\right)$ where

$$\begin{aligned}(H_P^{IF},K)=\Bigg\{&\Bigg(k_1,\Bigg\{\frac{(c_1,(0,0.4,1,1)_D)}{((0.6,0.4),(0.64,0.3),(1,0),(1,0))},\frac{(c_2,(0,0.4,1,1)_D)}{((0.7,0.2),(0.34,0.56),(0,1),(0,1))},\frac{(c_3,(0,0.4,1,1)_D)}{((0.1,0.8),(0.24,0.68),(0,1),(0,1))},\frac{(c_4,(0,0.4,1,1)_D)}{((0.7,0.2),(0.54,0.32),(1,0),(1,0))}\Bigg\}\Bigg),\\ &\Bigg(k_2,\Bigg\{\frac{(c_1,(0,0.5,0,1)_D)}{((0.5,0.4),(0.25,0.7),(0,1),(0,1))},\frac{(c_2,(0,0.5,0,1)_D)}{((0.5,0.4),(0.7,0.25),(1,0),(1,0))},\frac{(c_3,(0,0.5,0,1)_D)}{((0.1,0.8),(0.15,0.85),(0,1),(0,1))},\frac{(c_4,(0,0.5,0,1)_D)}{((0.7,0.3),(0.45,0.5),(0,1),(0,1))}\Bigg\}\Bigg),\\ &\Bigg(k_3,\Bigg\{\frac{(c_1,(0.5,0,1,1)_D)}{((0.3,0.6),(0.2,0.8),(0,1),(0,1))},\frac{(c_2,(0.5,0,1,1)_D)}{((0.45,0.5),(0.3,0.7),(0,1),(0,1))},\frac{(c_3,(0.5,0,1,1)_D)}{((0.5,0.4),(0.7,0.2),(1,0),(1,0))},\frac{(c_4,(0.5,0,1,1)_D)}{((0.35,0.55),(0.1,0.8),(0,1),(0,1))}\Bigg\}\Bigg)\\ &\Bigg(k_4,\Bigg\{\frac{(c_1,(0,0,1,1)_D)}{((0.2,0.8),(0.3,0.6),(0,1),(0,1))},\frac{(c_2,(0,0,1,1)_D)}{((0.1,0.9),(0.3,0.7),(0,1),(0,1))},\frac{(c_3,(0,0,1,1)_D)}{((0.8,0.2),(0.7,0.3),(1,0),(1,0))},\frac{(c_4,(0,0,1,1)_D)}{((0.1,0.9),(0.1,0.8),(0,1),(0,1))}\Bigg\}\Bigg)\Bigg\}.\end{aligned}$$

7.5. *Plithogenic Neutrosophic Soft Union*

Example 7.7. Consider Example 6.5. Let

$$\begin{aligned}A=\Bigg\{&a_1=(slow,white,model_1,2015\ and\ befor),a_2=(slow,yellow,model_3,2017),\\ &a_3=(fast,red,model_4,2018)\Bigg\}and\\ B=\Bigg\{&b_1=(slow,white,model_1,2015\ and\ befor),b_2=(slow,yellow,model_3,2017),\\ &b_3=(fast,red,model_4,2018)\Bigg\}\end{aligned}$$

Suppose $\left(F_P^N, A\right)$ and $\left(G_P^N, B\right)$ are two plithogenic neutrosophic soft sets over U such that

$$\begin{aligned}\left(F_P^N, A\right) = \Bigg\{ & \left(a_1, \left\{ \frac{\left(c_1,(0,0.4,1,1)_D\right)}{((0.6,0.3,0.1),(0.7,0.2,0.1),(1,0,0),(1,0,0))}, \frac{\left(c_2,(0,0.4,1,1)_D\right)}{((0.7,0.1,0.2),(0.3,0.5,0.2),(0,0,1),(0,0,1))}, \frac{\left(c_3,(0,0.4,1,1)_D\right)}{((0.1,0.1,0.8),(0.3,0.6,0.1),(0,0,1),(0,0,1))} \right\}\right), \\ & \left(a_2, \left\{ \frac{\left(c_1,(0,0.5,0,1)_D\right)}{((0.6,0.3,0.1),(0.2,0.6,0.2),(0,0,1),(0,0,1))}, \frac{\left(c_2,(0,0.5,0,1)_D\right)}{((0.5,0.2,0.3),(0.7,0.2,0.1),(1,0,0),(1,0,0))}, \frac{\left(c_3,(0,0.5,0,1)_D\right)}{((0.1,0.2,0.7),(0.1,0.1,0.8),(0,0,1),(0,0,1))} \right\}\right), \\ & \left(a_3, \left\{ \frac{\left(c_1,(0.5,0,1,1)_D\right)}{((0.1,0.2,0.7),(0.2,0.1,0.7),(0,0,1),(0,0,1))}, \frac{\left(c_2,(0.5,0,1,1)_D\right)}{((0.1,0.1,0.8),(0.3,0.2,0.5),(0,0,1),(0,0,1))}, \frac{\left(c_3,(0.5,0,1,1)_D\right)}{((0.8,0.1,0.1),(0.7,0.1,0.2),(1,0,0),(1,0,0))} \right\}\right)\Bigg\}.\end{aligned}$$

$$\begin{aligned}\left(G_P^N, B\right) = \Bigg\{ & \left(b_1, \left\{ \frac{\left(c_1,(0,0.4,1,1)_D\right)}{((0.5,0.2,0.3),(0.6,0.3,0.1),(1,0,0),(1,0,0))}, \frac{\left(c_2,(0,0.4,1,1)_D\right)}{((0.8,0.1,0.1),(0.2,0.3,0.5),(0,0,1),(0,0,1))}, \frac{\left(c_3,(0,0.4,1,1)_D\right)}{((0.3,0.1,0.6),(0.3,0.4,0.3),(0,0,1),(0,0,1))} \right\}\right), \\ & \left(b_2, \left\{ \frac{\left(c_1,(0,0.5,0,1)_D\right)}{((0.5,0.3,0.2),(0.4,0.4,0.2),(0,0,1),(0,0,1))}, \frac{\left(c_2,(0,0.5,0,1)_D\right)}{((0.4,0.3,0.3),(0.8,0.1,0.1),(1,0,0),(1,0,0))}, \frac{\left(c_3,(0,0.5,0,1)_D\right)}{((0.1,0.1,0.8),(0.2,0.2,0.6),(0,0,1),(0,0,1))} \right\}\right), \\ & \left(b_3, \left\{ \frac{\left(c_1,(0.5,0,1,1)_D\right)}{((0.3,0.2,0.5),(0.3,0.2,0.5),(0,0,1),(0,0,1))}, \frac{\left(c_2,(0.5,0,1,1)_D\right)}{((0.1,0.1,0.8),(0.5,0.1,0.4),(0,0,1),(0,0,1))}, \frac{\left(c_3,(0.5,0,1,1)_D\right)}{((0.6,0.2,0.2),(0.6,0.2,0.2),(1,0,0),(1,0,0))} \right\}\right)\Bigg\}.\end{aligned}$$

By using basic fuzzy union (maximum) and basic fuzzy intersection (minimum) we have:
$\left(F_P^N, A\right) \bigvee_P^N \left(G_P^N, B\right) = \left(H_P^N, K\right)$ where

$$\begin{aligned}\left(H_P^N, K\right) = \Bigg\{ & \left(k_1, \left\{ \frac{\left(c_1,(0,0.4,1,1)_D\right)}{((0.6,0.25,0.1),(0.66,0.25,0.1),(1,0,0),(1,0,0))}, \frac{\left(c_2,(0,0.4,1,1)_D\right)}{((0.8,0.1,0.1),(0.26,0.4,0.32),(0,0,1),(0,0,1))}, \frac{\left(c_3,(0,0.4,1,1)_D\right)}{((0.3,0.1,0.6),(0.3,0.5,0.18),(0,0,1),(0,0,1))} \right\}\right), \\ & \left(k_2, \left\{ \frac{\left(c_1,(0,0.5,0,1)_D\right)}{((0.6,0.3,0.1),(0.3,0.5,0.2),(0,0,1),(0,0,1))}, \frac{\left(c_2,(0,0.5,0,1)_D\right)}{((0.5,0.25,0.3),(0.75,0.15,0.1),(1,0,0),(1,0,0))}, \frac{\left(c_3,(0,0.5,0,1)_D\right)}{((0.1,0.15,0.7),(0.15,0.15,0.7),(0,0,1),(0,0,1))} \right\}\right), \\ & \left(k_3, \left\{ \frac{\left(c_1,(0.5,0,1,1)_D\right)}{((0.2,0.2,0.6),(0.3,0.15,0.5),(0,0,1),(0,0,1))}, \frac{\left(c_2,(0.5,0,1,1)_D\right)}{((0.1,0.1,0.8),(0.5,0.15,0.4),(0,0,1),(0,0,1))}, \frac{\left(c_3,(0.5,0,1,1)_D\right)}{((0.7,0.15,0.15),(0.7,0.15,0.2),(1,0,0),(1,0,0))} \right\}\right)\Bigg\}.\end{aligned}$$

7.6. *Plithogenic Neutrosophic Soft Intersection*

Example 7.8. Consider Example 7.7 By using basic fuzzy union (maximum) and basic fuzzy intersection (minimum) we have:
$\left(F_P^N, A\right) \bigwedge_P^N \left(G_P^N, B\right) = \left(H_P^N, K\right)$ where

$$\begin{aligned}\left(H_P^N, K\right) = \Bigg\{ & \left(k_1, \left\{ \frac{\left(c_1,(0,0.4,1,1)_D\right)}{((0.5,0.25,0.3),(0.64,0.25,0.1),(1,0,0),(1,0,0))}, \frac{\left(c_2,(0,0.4,1,1)_D\right)}{((0.7,0.1,0.2),(0.24,0.4,0.38),(0,0,1),(0,0,1))}, \frac{\left(c_3,(0,0.4,1,1)_D\right)}{((0.1,0.1,0.8),(0.3,0.5,0.22),(0,0,1),(0,0,1))} \right\}\right), \\ & \left(k_2, \left\{ \frac{\left(c_1,(0,0.5,0,1)_D\right)}{((0.5,0.3,0.2),(0.3,0.5,0.2),(0,0,1),(0,0,1))}, \frac{\left(c_2,(0,0.5,0,1)_D\right)}{((0.4,0.25,0.3),(0.75,0.15,0.1),(1,0,0),(1,0,0))}, \frac{\left(c_3,(0,0.5,0,1)_D\right)}{((0.1,0.15,0.8),(0.15,0.15,0.7),(0,0,1),(0,0,1))} \right\}\right), \\ & \left(k_3, \left\{ \frac{\left(c_1,(0.5,0,1,1)_D\right)}{((0.2,0.2,0.6),(0.2,0.15,0.7),(0,0,1),(0,0,1))}, \frac{\left(c_2,(0.5,0,1,1)_D\right)}{((0.1,0.1,0.8),(0.3,0.15,0.5),(0,0,1),(0,0,1))}, \frac{\left(c_3,(0.5,0,1,1)_D\right)}{((0.7,0.15,0.15),(0.6,0.15,0.2),(1,0,0),(1,0,0))} \right\}\right)\Bigg\}.\end{aligned}$$

Proposition 7.9. *Let F_P^z, G_P^z and H_P^z be any three PSSs over U. Then the following results hold:*

(1) $F_P^z \bigvee_P^z G_P^z = G_P^z \bigvee_P^z F_P^z$,
(2) $F_P^z \bigwedge_P^z G_P^z = G_P^z \bigwedge_P^z F_P^z$,
(3) $F_P^z \bigvee_P^z (G_P^z \bigvee_P^z H_P^z) = (F_P^z \bigvee_P^z G_P^z) \bigvee_P^z H_P^z$,
(4) $F_P^z \bigwedge_P^z (G_P^z \bigwedge_P^z H_P^z) = (F_P^z \bigwedge_P^z G_P^z) \bigwedge_P^z H_P^z$.

proof From union and intersection definitions and the fact that fuzzy set, intuitionistic fuzzy and neutrosophic set are commutative and associative, we can get the proof.

Proposition 7.10. *Let (F_P^z, E), (G_P^z, E) and (H_P^z, E) be any three PSSs over U. Then the following results hold:*

(1) $F_P^z \bigvee_P^z (G_P^z \bigwedge_P^z H_P^z) = (F_P^z \bigvee_P^z G_P^z) \bigwedge_P^z (F_P^z \bigvee_P^z H_P^z)$,

(2) $F_P^z \bigwedge_P^z (G_P^z \bigvee_P^z H_P^z) = (F_P^z \bigwedge_P^z G_P^z) \bigvee_P^z (F_P^z \bigwedge_P^z H_P^z)$.

proof. Let $z \equiv Fuzzy$, $\forall x \in E$, and without loss of generality suppose $c = 0$

(1)

$$\begin{aligned}
\lambda_{F_P^F(x) \bigvee_P^F (G_P^F(x) \bigwedge_P^F H_P^F(x))}(x) &= s\left\{\lambda_{F_P^F(x)}(x), \lambda_{(G_P^F(x) \bigwedge_P^F H_P^F(x))}(x)\right\} \\
&= s\left\{\lambda_{F_P^F(x)}(x), t\left(\lambda_{G_P^F(x)}(x), \lambda_{H_P^F(x)}(x)\right)\right\} \\
&= t\{s\left(\lambda_{F_P^F(x)}(x), \lambda_{G_P^F(x)}(x)\right), s\left(\lambda_{F_P^F(x)}(x), \lambda_{H_P^F(x)}(x)\right)\} \\
&= t\left\{\lambda_{(F_P^F(x) \bigvee_P^F G_P^F(x))}(x), \lambda_{(F_P^F(x) \bigvee_P^F H_P^F(x))}(x)\right\} \\
&= \lambda_{(F_P^F(x) \bigvee_P^F G_P^F(x)) \bigwedge_P^F (F_P^F(x) \bigvee_P^F H_P^F(x))}(x)
\end{aligned}$$

For $z \equiv IF$ and $z \equiv N$ use the same method.

(2) We can use the same method in (a).

8. Plithogenic Soft Similarity

In this section we introduce a measure of similarity between two P-FSSs ($z \equiv F$) and we leave the other similarity when ($z \equiv IF\, and\, N$) for future research. The set theoretic approach has been taken in this regard because it is popular and very easy for calculation.

Definition 8.1. Let (F_P^F, E) and (G_P^F, E) be two P-FSSs over (U, E) as in Definition 6.1. *Similarity* between (F_P^F, E) and (G_P^F, E), denoted by $S(F_P^F, G_P^F)$, is defined as follows:

$S(F_P^F, G_P^F) = \frac{1}{|E|}\sum_{k=1}^{|E|} M_k$ where

$$M_k = 1 - \frac{\sum_{j=1}^{|U|}\sum_{i=1}^{|e|} |F_j(e_{ik}) - G_j(e_{ik})|}{\sum_{j=1}^{|U|}\sum_{i=1}^{|e|} |F_j(e_{ik}) + G_j(e_{ik})|}.$$

Where $e \in E$.

Definition 8.2. Let (F_P^F, E) , (G_P^F, E) and (H_P^F, E) be two P-FSSs over (U, E). We say that (F_P^F, E) and (G_P^F, E) are *significantly similar* if $S(F_P^F, G_P^F) \geqslant \frac{1}{2}$.

Proposition 8.3. *Let (F_P^F, E) and (G_P^F, E) be any two P-FSSs over (U, E) such that F_P^F or G_P^F a non-zero P-FSS. Then the following holds:*

(1) $S(F_P^F, G_P^F) = S(G_P^F, F_P^F)$,

(2) $0 \leqslant S(F_P^F, G_P^F) \leqslant 1$,

(3) $F_P^F = G_P^F \Rightarrow S(F_P^F, G_P^F) = 1$,

(4) $F_P^F \subseteq G_P^F \subseteq H_P^F \Rightarrow S(F_P^F, H_P^F) \leqslant S(G_P^F, H_P^F)$,

(5) *If* $\left(F_P^F, A\right) \bigwedge_P^F \left(G_P^F, B\right) = \emptyset \Rightarrow S\left(F_P^F, G_P^F\right) = 0.$

proof. Proofs (a)-(d) follows from Definition 8.1, We will give the proof of (e).

For the left hand side we have $\left(F_P^F, A\right) \bigwedge_P^F \left(G_P^F, B\right) = \emptyset$, then

$t\left(F_P^F(e_{ik}), G_P^F(e_{ik})\right) = 0, \forall i, k.$ Now,

by using Definition 8.1 we have

$$M_k = 1 - \frac{\sum_{j=1}^{|U|}\sum_{i=1}^{|e|} |F_j(e_{ik}) - G_j(e_{ik})|}{\sum_{j=1}^{|U|}\sum_{i=1}^{|e|} |F_j(e_{ik}) + G_j(e_{ik})|}.$$

Since $t\left(F_P^F(e_{ik}), G_P^F(e_{ik})\right) = 0$, then $F_j(e_{ik}) - G_j(e_{ik}) = F_j(e_{ik}) + G_j(e_{ik})$ and this gives

$$\frac{\sum_{j=1}^{|U|}\sum_{i=1}^{|e|} |F_j(e_{ik}) - G_j(e_{ik})|}{\sum_{j=1}^{|U|}\sum_{i=1}^{|e|} |F_j(e_{ik}) - G_j(e_{ik})|} = 1. \text{ Then}$$

$M_k = 1 - 1 = 0.$

Example 8.4. Let $U = \{c_1, c_2, c_3, c_4\}$ and $a_1 =$ speed, $a_2 =$ color, $a_3 =$ model, $a_4 =$ manufacturing year. Suppose their attributes values respectively:

Speed $\equiv A_1 = \{slow, fast, veryfast\}$,
Color $\equiv A_2 = \{white, yellow, red, black\}$,
Model $\equiv A_3 = \{model_1, model_2, model_3, model_4\}$,
manufacturing year $\equiv A_4 = \{2015\ and\ befor, 2016, 2017, 2018, 2019\}$. Suppose

$$E = \Big\{ e_1 = (slow, white, model_1, 2015\ and\ befor), e_2 = (slow, yellow, model_3, 2017), \\ e_3 = (fast, red, model_4, 2018) \Big\}.$$

Suppose $\left(F_P^F, E\right)$ and $\left(G_P^F, E\right)$ are two plithogenic fuzzy soft sets over U such that

$$\begin{aligned}\left(F_P^F, E\right) = \Bigg\{ &\left(e_1, \left\{\frac{(c_1,(0,0.4,1,1)_D)}{(0.6,0.7,1,1)}, \frac{(c_2,(0,0.4,1,1)_D)}{(0.7,0.3,0,0)}, \frac{(c_3,(0,0.4,1,1)_D)}{(0.1,0.3,0,0)}, \frac{(c_4,(0,0.4,1,1)_D)}{(0.7,0.6,1,1)}\right\}\right), \\ &\left(e_2, \left\{\frac{(c_1,(0,0.5,0,1)_D)}{(0.6,0.2,0,0)}, \frac{(c_2,(0,0.5,0,1)_D)}{(0.5,0.7,1,1)}, \frac{(c_3,(0,0.5,0,1)_D)}{(0.1,0.1,0,0)}, \frac{(c_4,(0,0.5,0,1)_D)}{(0.7,0.4,0,0)}\right\}\right), \\ &\left(e_3, \left\{\frac{(c_1,(0.5,0,1,1)_D)}{(0.1,0.2,0,0)}, \frac{(c_2,(0.5,0,1,1)_D)}{(0.1,0.3,0,0)}, \frac{(c_3,(0.5,0,1,1)_D)}{(0.8,0.7,1,1)}, \frac{(c_4,(0.5,0,1,1)_D)}{(0.1,0.1,0,0)}\right\}\right) \Bigg\}.\end{aligned}$$

$$(G_P^F, E) = \left\{ \left(e_1, \left\{ \frac{(c_1, (0, 0.4, 1, 1)_D)}{(0.5, 0.6, 1, 1)}, \frac{(c_2, (0, 0.4, 1, 1)_D)}{(0.8, 0.4, 0, 0)}, \frac{(c_3, (0, 0.4, 1, 1)_D)}{(0.2, 0.2, 0, 0)}, \frac{(c_4, (0, 0.4, 1, 1)_D)}{(0.7, 0.5, 1, 1)} \right\} \right), \right.$$
$$\left(e_2, \left\{ \frac{(c_1, (0, 0.5, 0, 1)_D)}{(0.5, 0.3, 0, 0)}, \frac{(c_2, (0, 0.5, 0, 1)_D)}{(0.7, 0.7, 1, 1)}, \frac{(c_3, (0, 0.5, 0, 1)_D)}{(0.3, 0.2, 0, 0)}, \frac{(c_4, (0, 0.5, 0, 1)_D)}{(0.8, 0.5, 0, 0)} \right\} \right),$$
$$\left. \left(e_3, \left\{ \frac{(c_1, (0.5, 0, 1, 1)_D)}{(0.5, 0.6, 0, 0)}, \frac{(c_2, (0.5, 0, 1, 1)_D)}{(0.8, 0.3, 0, 0)}, \frac{(c_3, (0.5, 0, 1, 1)_D)}{(0.2, 0.7, 1, 1)}, \frac{(c_4, (0.5, 0, 1, 1)_D)}{(0.6, 0.3, 0, 0)} \right\} \right) \right\}.$$

Then we can find the similarity between F_P^F and G_P^F as follows:

$$M_1 = 1 - \frac{\sum_{j=1}^{4} \sum_{i=1}^{4} |F_j(e_{i1}) - G_j(e_{i1})|}{\sum_{j=1}^{4} \sum_{i=1}^{4} |F_j(e_{i1}) + G_j(e_{i1})|}$$

Now, for $j = 1$ we have

$$\sum_{i=1}^{4} |F_1(e_{i1}) - G_1(e_{i1})| = |(0.6 - 0.5)| + |(0.7 - 0.6)| + |(1 - 1)| + |(1 - 1)| = 0.2$$

and

$$\sum_{i=1}^{4} |F_1(e_{i1}) + G_1(e_{i1})| = |(0.6 + 0.5)| + |(0.7 + 0.6)| + |(1 + 1)| + |(1 + 1)| = 6.4.$$

For $j = 2$ we have

$$\sum_{i=1}^{4} |F_2(e_{i1}) - G_2(e_{i1})| = |(0.7 - 0.8)| + |(0.3 - 0.4)| + |(0 - 0)| + |(0 - 0)| = 0.2$$

and

$$\sum_{i=1}^{4} |F_2(e_{i1}) + G_2(e_{i1})| = |(0.7 + 0.8)| + |(0.3 + 0.4)| + |(0 + 0)| + |(0 + 0)| = 2.2.$$

For $j = 3$ we have

$$\sum_{i=1}^{4} |F_3(e_{i1}) - G_3(e_{i1})| = |(0.1 - 0.2)| + |(0.3 - 0.2)| + |(0 - 0)| + |(0 - 0)| = 0.2$$

and

$$\sum_{i=1}^{4} |F_3(e_{i1}) + G_3(e_{i1})| = |(0.1 + 0.2)| + |(0.3 + 0.2)| + |(0 + 0)| + |(0 + 0)| = 0.8.$$

For $j = 4$ we have

$$\sum_{i=1}^{4} |F_4(e_{i1}) - G_4(e_{i1})| = |(0.7 - 0.7)| + |(0.6 - 0.5)| + |(1 - 1)| + |(1 - 1)| = 0.1$$

and

$$\sum_{i=1}^{4} |F_4(e_{i1}) + G_4(e_{i1})| = |(0.7 + 0.7)| + |(0.6 + 0.5)| + |(1 + 1)| + |(1 + 1)| = 6.5.$$

Then $M_1 = 1 - \frac{\sum_{j=1}^{4}\sum_{i=1}^{4} |F_j(e_{i1}) - G_j(e_{i1})|}{\sum_{j=1}^{4}\sum_{i=1}^{4} |F_j(e_{i1}) + G_j(e_{i1})|} = 1 - \frac{[0.2 + 0.2 + 0.2 + 0.1]}{[6.4 + 2.2 + 0.8 + 6.5]} = 0.96$

Similarly we get $M_2 = 0.92$ and $M_3 = 0.78$. Then the similarity between the two P-FSSs F_P^F and G_P^F is given by

$$S\left(F_P^F, F_P^F\right) = \frac{1}{3}\sum_{k=1}^{3} M_k = \frac{0.96 + 0.92 + 0.78}{3} \cong 0.89.$$

9. Conclusions and future research

In this paper we have defined the concept of Plithogenic soft set, and gave some generalizations of this concept. We also gave examples for these concepts. We studied the basic properties of these operations. An important result of such a paper is that new questions can be used as an idea for further research, as such a research always unearths further questions. The work presented in this paper poses interesting new questions to researchers and provides the theoretical framework for further study on plithogenic soft sets. Based on the previous results we may suggest problems in relation to our research that we anticipate to venture elsewhere in the cases of operations in plithogenic soft sets, similarity in plithogenic soft sets and applications on plithogenic soft sets. We can investigate the following topics for further works:

(1) To define and study the "AND" and "OR" operations of plithogenic soft set.
(2) To define and study the application of similarity measure of plithogenic soft set on DM and MD.
(3) To define and study the application of plithogenic soft set operations on DM and MD.
(4) To generalized plithogenic soft sets to plithogenic soft multisets, expert set, possibility set and etc.

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Structures on Doubt Neutrosophic Ideals of BCK/BCI-Algebras under (S,T)-Norms

Anas Al-Masarwah[1,*] **and Abd Ghafur Ahmad**[2]

[1,2]School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia.

E-mail: [1] P90674@siswa.ukm.edu.my, almasarwah85@gmail.com, [2] ghafur@ukm.edu.my

*Correspondence: almasarwah85@gmail.com

Abstract. Smarandache implemented the idea of neutrosophic set theory as a method for dealing undetermined data. Neutrosophic set theory is commonly used in various algebric structures, such as groups, rings and BCK/BCI-algebras. At present, there exist no results on doubt neutrosophic ideals of BCK/BCI-algebras using t-conorm and t-norm. First, the notions of (S,T)- normed doubt neutrosophic subalgebras and ideals of BCK/BCI-algebras are introduced and the characteristic properties are described. Then, images and preimages of (S,T)- normed doubt neutrosophic ideals under homomorphism are considered. Moreover, the direct product and (S,T)- product of (S,T)- normed doubt neutrosophic ideals of BCK/BCI-algebras are also discussed.

Keywords: BCK/BCI-algebra; doubt neutrosophic subalgebra (ideal); (S,T)-normed doubt neutrosophic subalgebra (ideal).

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1. Introduction

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [1], and were applied to various mathematical fields, such as group theory, topology, functional analysis and probability theory, etc. In the same way, the concept of a BCI-algebra, which is a generalization of a BCK-algebra, was proposed by Iséki [2]. Zadeh [3] introduced the idea of fuzzy set theory in 1965, where the degree of membership is discussed, and Xi [4] introduced fuzzy subalgebras and ideals in BCK/BCI-algebras in 1991. Later on, fuzzy sets have been generalized to intuitionistic fuzzy sets [5] by adding a non-membership function by Atanassov in 1986 and this concept has been applied to BCK/BCI-algebras by Jun and Kim [6].

As anew idea and based on the concept defined by Xi [4], Jun [7] in 1994 introduced the notions of doubt fuzzy subalgebras and ideals in BCK/BCI-algebras. Bej and Pal [8] introduced the concepts of doubt intuitionistic fuzzy subalgebras and ideals in BCK/BCI-algebras.

Al-Masarwah and Ahmad [9] introduced the concepts of doubt bipolar fuzzy subalgebras and ideals in BCK/BCI-algebras. After that, many other researchers used these ideas and published numerous articles in different branches of algebraic structures [10–14].

Triangular norms were formulated by Schweizer and Sklar [15] to model the distances in probabilistic metric spaces. Triangular norms play an important role in many fields of mathematics, statistics, cooperative games, decision making and artificial intelligence [16]. In particular, in fuzzy set theory, t-conorm (S) and t-norm (T) have been widely used for fuzzy logic, fuzzy relation equations and fuzzy operations. In algebraic structures, Senapati [17] proposed the idea of (imaginable) T-fuzzy subalgebras and (imaginable) T-fuzzy closed ideals of BG-algebras. Kim [18] presented the intuitionistic (S, T)-normed fuzzy subalgebras in BCK/BCI-algebras using triangular norms. Also, Kutukcu and Tuna [19], presented a new classification of intuitionistic fuzzy subalgebras, ideals and implicative ideals in BCK/BCI-algebras.

Neutrosophy, [20, 21] a new branch of science that deals with indeterminacy, was launched by Smarandache in 1998. This concept is a generalization of the classical set, fuzzy set and intuitionistic fuzzy set. Neutrosophic set theory has been applied to several fields of mathematics including decision making [22–24], pattern recognition and medical diagnosis [25] and others [26–31]. In the aspect of algebraic structures, the papers [32–38] address neutrosophic algebraic structures in BCK/BCI-algebras.

As no studies have been reported so far to generalize the above mentioned concepts, so the aim of this present article is:

(1) To propose the concept of (S, T)-normed doubt neutrosophic subalgebras and (S, T)-normed doubt neutrosophic ideals of BCK/BCI-algebras as a generalization of (S, T)-normed intutionistic fuzzy subalgebras and ideals of BCK/BCI-algebras.

(2) To consider images and preimages of (S, T)- normed doubt neutrosophic ideals under homomorphism.

(3) To define and discuss the direct product and (S, T)- product of (S, T)- normed doubt neutrosophic ideals of BCK/BCI-algebras.

To do so, the rest of the article is structured as follows: In Section 2, we review some basic notions. In Section 3, we introduce the notions of (S, T)- normed doubt neutrosophic subalgebras and ideals of BCK/BCI-algebras and then describe some of the characteristic properties. Furthermore, we consider images and preimages of (S, T)- normed doubt neutrosophic ideals under homomorphism. In Section 4, we discuss the direct product and (S, T)- product of (S, T)- normed doubt neutrosophic ideals of BCK/BCI-algebras. Finally, in Section 5, we present the conclusion and future works of the study.

2. Preliminaries

In the current section, we remember some of the basic notions of BCK/BCI-algebras which will be very helpful in further study of the paper. Let X be a BCK/BCI-algebra in what follows, unless otherwise stated.

By a BCI-algebra, we mean a set X with a special element 0 and a binary operation $*$, for all $p, q, s \in X$, that satisfies the following axioms:

(I) $[(p * q) * (p * s)] * (s * q) = 0$,

(II) $[p * (p * q)] * q = 0$,

(III) $p * p = 0$,

(IV) $p * q = 0$ and $q * p = 0$ imply $p = q$.

If a BCI-algebra X satisfies $0 * p = 0$, then X is called a BCK-algebra. In a BCK/BCI-algebra, $p * 0 = p$ holds. A partial ordering $\leq$ on a BCK/BCI-algebra X can be defined by $p \leq q$ if and only if $p * q = 0$. A non-empty subset K of a BCK/BCI-algebra X is called a subalgebra of X if $p * q \in K, \forall p, q \in X$, and an ideal of X if $\forall p, q \in X$,

(1) $0 \in K$,

(2) $p * q \in K$ and $q \in K$ imply $p \in K$.

Definition 2.1. A neutrosophic set in a non-empty set X (see [14]) is a structure of the form:

$$B = \{\langle p; B_T(p), B_I(p), B_F(p)\rangle | p \in X\},$$

where $B_T, B_I, B_F : X \to [0,1]$. We shall use the symbol $B = (B_T, B_I, B_F)$, for the neutrosophic set $B = \{\langle p; B_T(p), B_I(p), B_F(p)\rangle | p \in X\}$.

If $B = (B_T, B_I, B_F)$ is a neutrosophic set in X, then $\Box B = (B_T, B_I, B_T^c)$ and $\Diamond B = (B_F^c, B_I, B_F)$ are also neutrosophic sets in X.

Definition 2.2 ([15]). A function $T : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm, if it satisfies the following conditions: $\forall p, q.s \in [0,1]$,

(1) $T(0,0) = 0, T(1,1) = 1$,

(2) $T(p, T(q, s)) = T(T(p, q), s)$,

(3) $T(p, q) = T(q, p)$,

(4) $T(p, q) \leq T(p, s)$ if $q \leq s$.

If $T(p, 0) = p$ and $T(p, 1) = p$ for all $p \in [0,1]$, then T is called a t-conorm and a t-norm, respectively. Throughout this paper, denote S and T as a t-conorm and a t-norm, respectively.

Some examples of t-conorms and t-norms are:

(1) $S_M(p, q) = \max\{p, q\}$ and $T_M(p, q) = \min\{p, q\}$.

(2) $S_L(p, q) = \min\{p + q, 1\}$ and $T_L(p, q) = \max\{p + q - 1, 0\}$.

(3) $S_P(p,q) = p + q - pq$ and $T_P(p,q) = pq$.

A t-conorm S and a t-norm T are associated [39], i.e., $S(p,q) = 1 - T(1-p, 1-q), \forall p,q \in [0,1]$.

Lemma 2.3 ([40]). *For any* $p,q \in [0,1]$, *we have* $0 \leq \max\{p,q\} \leq S(p,q) \leq 1$ *and* $0 \leq T(p,q) \leq \min\{p,q\} \leq 1$.

Definition 2.4 ([7]). A fuzzy set λ of X is called a doubt fuzzy subalgebra of X if $\lambda(p*q) \leq \max\{\lambda(p), \lambda(q)\}$ $\forall p,q \in X$, and a doubt fuzzy ideal of X if $\lambda(0) \leq \lambda(p) \leq \max\{\lambda(p*q), \lambda(q)\}$ $\forall p,q \in X$.

Definition 2.5 ([41]). A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a neutrosophic subalgebra of X if for all $p,q \in X$,

(1) $B_T(p*q) \geq \min\{B_T(p), B_T(q)\}$,
(2) $B_I(p*q) \geq \min\{B_I(p), B_I(q)\}$,
(3) $B_F(p*q) \leq \max\{B_F(p), B_F(q)\}$.

Definition 2.6 ([41]). A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a neutrosophic ideal of X if for all $p,q \in X$,

(1) $B_T(0) \geq B_T(p) \geq \min\{B_T(p*q), B_T(q)\}$,
(2) $B_I(0) \geq B_I(p) \geq \min\{B_I(p*q), B_I(q)\}$,
(3) $B_F(0) \leq B_F(p) \leq \max\{B_F(p*q), B_F(q)\}$.

3. (S,T)-Normed doubt neutrosophic ideals

Definition 3.1. A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a doubt neutrosophic subalgebra of X if for all $p,q \in X$,

(1) $B_T(p*q) \leq \max\{B_T(p), B_T(q)\}$,
(2) $B_I(p*q) \leq \max\{B_I(p), B_I(q)\}$,
(3) $B_F(p*q) \geq \min\{B_F(p), B_F(q)\}$.

Definition 3.2. A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a doubt neutrosophic subalgebra of X with respect to a t-conorm S and a t-norm T (or simply, an (S,T)-normed doubt neutrosophic subalgebra of X) if for all $p,q \in X$,

(1) $B_T(p*q) \leq S(B_T(p), B_T(q))$,
(2) $B_I(p*q) \leq S(B_I(p), B_I(q))$,
(3) $B_F(p*q) \geq T(B_F(p), B_F(q))$.

Definition 3.3. A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a doubt neutrosophic ideal of X if for all $p,q \in X$,

(1) $B_T(0) \leq B_T(p) \leq \max\{B_T(p*q), B_T(q)\}$,

(2) $B_I(0) \leq B_I(p) \leq \max\{B_I(p * q), B_I(q)\}$,

(3) $B_F(0) \geq B_F(p) \geq \min\{B_F(p * q), B_F(q)\}$.

Definition 3.4. A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a doubt neutrosophic ideal of X with respect to a t-conorm S and a t-norm T (or simply, an (S,T)-normed doubt neutrosophic ideal of X) if for all $p, q \in X$,

(1) $B_T(0) \leq B_T(p) \leq S(B_T(p * q), B_T(q))$,

(2) $B_I(0) \leq B_I(p) \leq S(B_I(p * q), B_I(q))$,

(3) $B_F(0) \geq B_F(p) \geq T(B_F(p * q), B_F(q))$.

Example 3.5. Consider a given BCK-algebra $X = \{0, k, l, m\}$ in Table 1:

Table 1. Tabular representation of a BCK-algebra $X = \{0, k, l, m\}$.

$*$	0	k	l	m
0	0	0	0	0
k	k	0	0	k
l	l	k	0	l
m	m	m	m	0

Define a neutrosophic set $B = (B_T, B_I, B_F)$ of X by Table 2:

Table 2. Neutrosophic set $B = (B_T, B_I, B_F)$.

X	$B_T(p)$	$B_I(p)$	$B_F(p)$
0	0	0	1
k	0.50	0.40	0.33
l	0.50	0.40	0.33
m	1	0.90	0

Clearly, $B_T(0) \leq B_T(p) \leq S_M(B_T(p * q), B_T(q))$, $B_I(0) \leq B_I(p) \leq S_M(B_I(p * q), B_I(q))$ and $B_F(0) \geq B_F(p) \geq T_L(B_F(p * q), B_F(q))$ for all $p, q \in X$. Hence, $B = (B_T, B_I, B_F)$ is an (S_M, T_L)-normed doubt neutrosophic ideal of X. Also, note that a t-conorm S_M and a t-norm T_L are not associated.

Remark 3.6. Example 3.5 holds even with the t-conorm S_M and t-norm T_M. Hence, $B = (B_T, B_I, B_F)$ is an (S_M, T_M)-normed doubt neutrosophic ideal of X.

Remark 3.7. Every doubt neutrosophic ideal of X is an (S,T)-normed doubt neutrosophic ideal of X, but the converse is not true.

Example 3.8. Consider a given BCK-algebra $X = \{0, 1, 2, 3, 4\}$ in Table 3:

TABLE 3. Tabular representation of a BCK-algebra $X = \{0, 1, 2, 3, 4\}$.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	4	4	4	0

Define a neutrosophic set $B = (B_T, B_I, B_F)$ of X by Table 4:

TABLE 4. Neutrosophic set $B = (B_T, B_I, B_F)$.

X	$B_T(p)$	$B_I(p)$	$B_F(p)$
0	0.50	0.50	0.33
1	0.50	0.50	0.33
2	0.50	0.50	0.33
3	0.75	0.75	0.25
4	0.75	0.75	0.25

Clearly, $B_T(0) \leq B_T(p) \leq S_L(B_T(p*q), B_T(q))$, $B_I(0) \leq B_I(p) \leq S_L(B_I(p*q), B_I(q))$ and $B_F(0) \geq B_F(p) \geq T_P(B_F(p*q), B_F(q))$ for all $p, q \in X$. Hence, $B = (B_T, B_I, B_F)$ is an (S_L, T_P)-normed doubt neutrosophic ideal of X, but it is not a doubt neutrosophic ideal of X.

Lemma 3.9. *If $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of X, then so is $\Box B = (B_T, B_I, B_T^c)$, where a t-conorm S and a t-norm T are associated.*

Proof. Let $B = (B_T, B_I, B_F)$ be an (S, T)-normed doubt neutrosophic ideal of X. Then, $B_T(0) \leq B_T(p) \forall p \in X$ and so $1 - B_T^c(0) \leq 1 - B_T^c(p)$. Hence, $B_T^c(0) \geq B_T^c(p)$. Also, for all $p, q \in X$, we have $B_T(p) \leq S(B_T(p*q), B_T(q))$ and so $1 - B_T^c(p) \leq S(1 - B_T^c(p*q), 1 - B_T^c(q))$ which implies $B_T^c(p) \geq 1 - S(1 - B_T^c(p*q), 1 - B_T^c(q))$. Since S and T are associated, we have $B_T^c(p) \geq T(B_T^c(p*q), B_T^c(q))$. Thus, $\Box B = (B_T, B_I, B_T^c)$ is an (S, T)-normed doubt neutrosophic ideal of X. □

Lemma 3.10. *If $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of X, then so is $\Diamond B = (B_F^c, B_I, B_F)$, where a t-conorm S and a t-norm T are associated.*

Proof. The proof is similar to the proof of Lemma 3.9. □

Combining Lemmas 3.9 and 3.10, we deduce that:

Theorem 3.11. *A neutrosophic set $B = (B_T, B_I, B_F)$ is an (S,T)-normed doubt neutrosophic ideal of X if and only if $\Box B$ and $\Diamond B$ are (S,T)-normed doubt neutrosophic ideals of X, where a t-conorm S and a t-norm T are associated.*

Lemma 3.12. *Every (S,T)-normed doubt neutrosophic ideal $B = (B_T, B_I, B_F)$ of X satisfies: for all $p, q \in X$,*

$$p \leq q \Rightarrow B_T(p) \leq B_T(q), B_I(p) \leq B_I(q) \text{ and } B_F(q) \geq B_F(p).$$

Proof. Let $p, q \in X$ be such that $p \leq q$. Then, $p * q = 0$ and so

$$B_T(p) \leq S(B_T(p * q), B_T(q)) = S(B_T(0), B_T(q)) \leq B_T(q),$$
$$B_I(p) \leq S(B_I(p * q), B_I(q)) = S(B_I(0), B_I(q)) \leq B_I(q),$$

and

$$B_F(p) \geq T(B_F(p * q), B_F(q)) = T(B_F(0), B_F(q)) \geq B_F(q).$$

This completes the proof. □

Theorem 3.13. *Every (S,T)-normed doubt neutrosophic ideal of X is an (S,T)-normed doubt neutrosophic subalgebra of X.*

Proof. Let $B = (B_T, B_I, B_F)$ be an (S,T)-normed doubt neutrosophic ideal of X. Since $p*q \leq p$ $\forall p, q \in X$, it follows from Lemma 3.12 that $B_T(p * q)) \leq B_T(p), B_I(p * q)) \leq B_I(p)$ and $B_F(p * q)) \geq B_F(p)$. Then,

$$B_T(p * q)) \leq B_T(p) \leq S(B_T(p * q), B_T(q)) \leq S(B_T(p), B_T(q)),$$
$$B_I(p * q)) \leq B_I(p) \leq S(B_I(p * q), B_I(q)) \leq S(B_I(p), B_I(q)),$$

and

$$B_F(p * q)) \geq B_F(p) \geq T(B_F(p * q), B_F(q)) \geq T(B_F(p), B_F(q)).$$

Hence, $B = (B_T, B_I, B_F)$ is an (S,T)-normed doubt neutrosophic subalgebra of X. □

Remark 3.14. The converse of Theorem 3.13 is not hold in general.

Example 3.15. Reconsider the BCK-algebra X given in Example 3.5. Define a neutrosophic set $B = (B_T, B_I, B_F)$ of X by Table 5:

TABLE 5. Neutrosophic set $B = (B_T, B_I, B_F)$.

X	$B_T(p)$	$B_I(p)$	$B_F(p)$
0	0	0	1
k	0.50	0.50	0.33
l	1	1	0
m	1	1	0

Clearly, $B = (B_T, B_I, B_F)$ is an (S_M, T_M)-normed doubt neutrosophic subalgebra of X, but it is not (S_M, T_M)-normed doubt neutrosophic ideal of X, since

$$B_T(l) = 1 > \max\{B_T(l * k), B_T(k)\},$$
$$B_I(l) = 1 > \max\{B_I(l * k), B_I(k)\},$$

and

$$B_F(l) = 0 < \min\{B_F(l * k), B_F(k)\}.$$

Definition 3.16. A mapping $\theta : X \to Y$ of BCK/BCI-algebras is said to be a homomorphism if $\theta(p * q) = \theta(p) * \theta(q) \forall p, q \in X$. If $\theta : X \to Y$ is a homomorphism, then $\theta(0) = 0$.

Let $\theta : X \to Y$ be a homomorphism of BCK/BCI-algebras. For any neutrosophic set $B = (B_T, B_I, B_F)$ in Y, we define a new neutrosophic set $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ such that for all $p \in X$,

$$B_T[\theta] : X \to [0,1], B_T[\theta](p) = B_T(\theta(p)),$$
$$B_I[\theta] : X \to [0,1], B_I[\theta](p) = B_I(\theta(p)),$$
$$B_F[\theta] : X \to [0,1], B_F[\theta](p) = B_F(\theta(p)).$$

Theorem 3.17. *Let $\theta : X \to Y$ be a homomorphism of BCK/BCI-algebras. If $B = (B_T, B_I, B_F)$ is an (S,T)-normed doubt neutrosophic ideal of Y, then $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ is an (S,T)-normed doubt neutrosophic ideal of X.*

Proof. We first have

$$B_T[\theta](0) = B_T(\theta(0)) = B_T(0) \le B_T(\theta(p)) = B_T[\theta](p),$$
$$B_I[\theta](0) = B_I(\theta(0)) = B_I(0) \le B_I(\theta(p)) = B_I[\theta](p),$$
$$B_F[\theta](0) = B_F(\theta(0)) = B_F(0) \ge B_F(\theta(p)) = B_F[\theta](p)$$

for all $p, q \in X$. Let $p, q \in X$. Then,

$$\begin{aligned} B_T[\theta](p) = B_T(\theta(p)) &\leq S(B_T(\theta(p) * \theta(q)), B_T(\theta(q))) \\ &= S(B_T(\theta(p * q)), B_T(\theta(q))) \\ &= S(B_T[\theta](p * q), B_T[\theta](q)), \end{aligned}$$

$$\begin{aligned} B_I[\theta](p) = B_I(\theta(p)) &\leq S(B_I(\theta(p) * \theta(q)), B_I(\theta(q))) \\ &= S(B_I(\theta(p * q)), B_I(\theta(q))) \\ &= S(B_I[\theta](p * q), B_I[\theta](q)) \end{aligned}$$

and

$$\begin{aligned} B_F[\theta](p) = B_F(\theta(p)) &\geq T(B_F(\theta(p) * \theta(q)), B_F(\theta(q))) \\ &= T(B_F(\theta(p * q)), B_F(\theta(q))) \\ &= T(B_F[\theta](p * q), B_F[\theta](q)). \end{aligned}$$

Therefore, $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ is an (S, T)-normed doubt neutrosophic ideal of X. □

Theorem 3.18. *Let $\theta : X \to Y$ be an onto homomorphism of BCK/BCI-algebras and let $B = (B_T, B_I, B_F)$ be a neutrosophic set of Y. If $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ is an (S, T)-normed doubt neutrosophic ideal of X, then $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of Y.*

Proof. For any $b \in Y$, there exists $a \in X$ such that $\theta(a) = b$. Then,

$$\begin{aligned} B_T(0) = B_T(\theta(0)) = B_T[\theta](0) \leq B_T[\theta](a) = B_T(\theta(a)) = B_T(b), \\ B_I(0) = B_I(\theta(0)) = B_I[\theta](0) \leq B_I[\theta](a) = B_I(\theta(a)) = B_I(b), \\ B_F(0) = B_F(\theta(0)) = B_F[\theta](0) \geq B_F[\theta](a) = B_F(\theta(a)) = B_F(b). \end{aligned}$$

Let $p, q \in Y$. Then, $\theta(a) = p$ and $\theta(b) = q$ for some $a, b \in X$. It follows that

$$\begin{aligned} B_T(p) = B_T(\theta(a)) &= B_T[\theta](a) \\ &\leq S(B_T[\theta](a * b), B_T[\theta](b)) \\ &= S(B_T(\theta(a * b)), B_T(\theta(b))) \\ &= S(B_T(\theta(a) * \theta(b)), B_T(\theta(b))) \\ &= S(B_T(p * q), B_T(q)), \end{aligned}$$

$$\begin{aligned}
B_I(p) = B_I(\theta(a)) &= B_I[\theta](a) \\
&\leq S(B_I[\theta](a * b), B_I[\theta](b)) \\
&= S(B_I(\theta(a * b)), B_I(\theta(b))) \\
&= S(B_I(\theta(a) * \theta(b)), B_I(\theta(b))) \\
&= S(B_I(p * q), B_I(q))
\end{aligned}$$

and

$$\begin{aligned}
B_F(p) = B_F(\theta(a)) &= B_F[\theta](a) \\
&\geq T(B_F[\theta](a * b), B_F[\theta](b)) \\
&= T(B_F(\theta(a * b)), B_F(\theta(b))) \\
&= T(B_F(\theta(a) * \theta(b)), B_F(\theta(b))) \\
&= T(B_F(p * q), B_F(q)),
\end{aligned}$$

Therefore, $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of Y. □

4. Product of (S, T)-normed doubt neutrosophic ideals

In this section, we discuss the direct product and (S, T)- product of (S, T)- normed doubt neutrosophic ideals of BCK/BCI-algebras.

Lemma 4.1 ([16]). *Let S and T be a t-conorm and t-norm, respectively. Then,*

$$S(S(p, q), S(a, b)) = S(S(p, a), S(q, b)),$$
$$T(T(p, q), T(a, b)) = T(T(p, a), T(q, b))$$

for all $p, q, a, b \in [0, 1]$.

Theorem 4.2. *Let $X = P_1 \times P_2$ be the direct product BCK/BCI-algebra of two BCK/BCI-algebras P_1 and P_2. If $B = (B_T, B_I, B_F)$ (resp., $C = (C_T, C_I, C_F)$) is an (S, T)-normed doubt neutrosophic ideal of P_1 (resp., P_2), then $D = (D_T, D_I, D_F)$ is an (S, T)-normed doubt neutrosophic ideal of X defined by $D_T = B_T \times C_T, D_I = B_I \times C_I$, and $D_F = B_F \times C_F$ such that*

$$D_T(p_1, p_2) = (B_T \times C_T)(p_1, p_2) = S(B_T(p_1), C_T(p_2)),$$
$$D_I(p_1, p_2) = (B_I \times C_I)(p_1, p_2) = S(B_I(p_1), C_I(p_2)),$$
$$D_F(p_1, p_2) = (B_F \times C_F)(p_1, p_2) = T(B_F(p_1), C_F(p_2))$$

for all $(p_1, p_2) \in X$.

Proof. Let $(p_1, p_2), (q_1, q_2) \in P_1 \times P_2$. Since $X = P_1 \times P_2$ is a BCK/BCI-algebra, we have

$$\begin{aligned} D_T(0,0) = (B_T \times C_T)(0,0) &= S(B_T(0), C_T(0)) \\ &\leq S(B_T(p_1), C_T(p_2)) \\ &= S((B_T \times C_T)(p_1, p_2)) \\ &= D_T(p_1, p_2), \end{aligned}$$

$$\begin{aligned} D_I(0,0) = (B_I \times C_I)(0,0) &= S(B_I(0), C_I(0)) \\ &\leq S(B_I(p_1), C_I(p_2)) \\ &= S((B_I \times C_I)(p_1, p_2)) \\ &= D_I(p_1, p_2), \end{aligned}$$

and

$$\begin{aligned} D_F(0,0) = (B_F \times C_F)(0,0) &= T(B_F(0), C_F(0)) \\ &\geq T(B_F(p_1), C_F(p_2)) \\ &= T((B_F \times C_F)(p_1, p_2)) \\ &= D_F(p_1, p_2). \end{aligned}$$

Also,

$$\begin{aligned} D_T(p_1, p_2) = (B_T \times C_T)(p_1, p_2) &= S(B_T(p_1), C_T(p_2)) \\ &\leq S\Big(S(B_T(p_1 * q_1), B_T(q_1)), S(C_T(p_2 * q_2), C_T(q_2))\Big) \\ &= S\Big(S(B_T(p_1 * q_1), C_T(p_2 * q_2)), S(B_T(q_1), C_T(q_2))\Big) \\ &= S\Big((B_T \times C_T)(p_1 * q_1, p_2 * q_2), (B_T \times C_T)(q_1, q_2)\Big) \\ &= S\Big((B_T \times C_T)((p_1, p_2) * (q_1, q_2)), (B_T \times C_T)(q_1, q_2)\Big) \\ &= S\Big(D_T((p_1, p_2) * (q_1, q_2)), D_T(q_1, q_2)\Big), \end{aligned}$$

$$\begin{aligned} D_I(p_1, p_2) = (B_I \times C_I)(p_1, p_2) &= S(B_I(p_1), C_I(p_2)) \\ &\leq S\Big(S(B_I(p_1 * q_1), B_I(q_1)), S(C_I(p_2 * q_2), C_I(q_2))\Big) \\ &= S\Big(S(B_I(p_1 * q_1), C_I(p_2 * q_2)), S(B_I(q_1), C_I(q_2))\Big) \\ &= S\Big((B_I \times C_I)(p_1 * q_1, p_2 * q_2), (B_I \times C_I)(q_1, q_2)\Big) \\ &= S\Big((B_I \times C_I)((p_1, p_2) * (q_1, q_2)), (B_I \times C_I)(q_1, q_2)\Big) \\ &= S\Big(D_I((p_1, p_2) * (q_1, q_2)), D_I(q_1, q_2)\Big), \end{aligned}$$

and

$$\begin{aligned} D_F(p_1, p_2) = (B_F \times C_F)(p_1, p_2) &= T(B_F(p_1), C_F(p_2)) \\ &\geq T\Big(T(B_F(p_1 * q_1), B_F(q_1)), T(C_F(p_2 * q_2), C_F(q_2))\Big) \\ &= T\Big(T(B_F(p_1 * q_1), C_F(p_2 * q_2)), T(B_F(q_1), C_F(q_2))\Big) \\ &= T\Big((B_F \times C_T)(p_1 * q_1, p_2 * q_2), (B_F \times C_F)(q_1, q_2)\Big) \\ &= T\Big((B_F \times C_F)((p_1, p_2) * (q_1, q_2)), (B_F \times C_F)(q_1, q_2)\Big) \\ &= T\Big(D_F((p_1, p_2) * (q_1, q_2)), D_F(q_1, q_2)\Big). \end{aligned}$$

This completes the proof. □

Definition 4.3. Let $B = (B_T, B_I, B_F)$ and $C = (C_T, C_I, C_F))$ be two neutrosophic sets of a BCK/BCI-algebra X. Then, (S, T)- product of B and C, written as $[B.C]_{(S,T)}$, are defined by

$$[B.C]_{(S,T)} = ([B_T.C_T]_S, [B_I.C_I]_S, [B_F.C_F]_T),$$

where

$$[B_T.C_T]_S(p) = S(B_T(p), C_T(p)),$$
$$[B_I.C_I]_S(p) = S(B_I(p), C_I(p))$$

and

$$[B_F.C_F]_T(p) = T(B_F(p), C_F(p))$$

for all $p \in X$.

Theorem 4.4. *Let S and T be a t-conorm and s-norm, respectively. Let $B = (B_T, B_I, B_F)$ and $C = (C_T, C_I, C_F))$ be two (S, T)- normed doubt neutrosophic ideals of X. If S_1 is a t-conorm which dominates S, i.e.,*

$$S_1(S(p, q), S(a, b)) \leq S(S_1(p, a), S_1(q, b))$$

and T_1 is a t-norm which dominates T, i.e.,

$$T_1(T(p, q), T(a, b)) \geq T(T_1(p, a), T_1(q, b))$$

for all $p, q, a, b \in [0, 1]$, then $([B_T.C_T]_{S_1}, [B_I.C_I]_{S_1}, [B_F.C_F]_{T_1})$ is an (S, T)-normed doubt neutrosophic ideal of X.

Proof. For any $p \in X$, we have

$$[B_T.C_T]_{S_1}(0) = S_1(B_T(0), C_T(0)) \leq S_1(B_T(p), C_T(p)) = [B_T.C_T]_{S_1}(p),$$
$$[B_I.C_I]_{S_1}(0) = S_1(B_I(0), C_I(0)) \leq S_1(B_I(p), C_I(p)) = [B_I.C_I]_{S_1}(p),$$
$$[B_F.C_F]_{T_1}(0) = T_1(B_F(0), C_F(0)) \geq T_1(B_F(p), C_F(p)) = [B_F.C_F]_{T_1}(p).$$

Also, for all $p, q \in X$, we have

$$\begin{aligned}[B_T.C_T]_{S_1}(p) &= S_1(B_T(p), C_T(p)) \\ &\leq S_1\Big(S(B_T(p*q), B_T(q)), S(B_T(p*q), B_T(q))\Big) \\ &\leq S\Big(S_1(B_T(p*q), B_T(p*q)), S_1(B_T(q), B_T(q))\Big) \\ &= S\Big([B_T.C_T]_{S_1}(p*q), [B_T.C_T]_{S_1}(q)\Big),\end{aligned}$$

$$\begin{aligned}[B_I.C_I]_{S_1}(p) &= S_1(B_I(p), C_I(p)) \\ &\leq S_1\Big(S(B_I(p*q), B_I(q)), S(B_I(p*q), B_I(q))\Big) \\ &\leq S\Big(S_1(B_I(p*q), B_I(p*q)), S_1(B_I(q), B_I(q))\Big) \\ &= S\Big([B_I.C_I]_{S_1}(p*q), [B_I.C_I]_{S_1}(q)\Big),\end{aligned}$$

and

$$\begin{aligned}[B_F.C_F]_{T_1}(p) &= T_1(B_F(p), C_F(p)) \\ &\geq T_1\Big(T(B_F(p*q), B_F(q)), T(B_T(p*q), B_F(q))\Big) \\ &\geq T\Big(T_1(B_F(p*q), B_F(p*q)), T_1(B_F(q), B_F(q))\Big) \\ &= T\Big([B_F.C_F]_{T_1}(p*q), [B_F.C_F]_{T_1}(q)\Big).\end{aligned}$$

This completes the proof. □

5. **Conclusions**

In this paper, we have introduced the notions of (S,T)- normed doubt neutrosophic subalgebras and ideals of BCK/BCI-algebras and described the characteristic properties. Then, we have considered images and preimages of (S,T)- normed doubt neutrosophic ideals under homomorphism. Moreover, we have discussed the direct product and (S,T)- product of (S,T)- normed doubt neutrosophic ideals of BCK/BCI-algebras. We aim to extend our notions to

(1) (S,T)- normed doubt generalized neutrosophic positive implicative ideals of BCK-algebras.

(2) (S,T)- normed doubt generalized neutrosophic ideals of BCK/BCI-algebras.

(3) (S,T)- normed doubt cubic neutrosophic ideals of BCK/BCI-algebras.

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Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-)HyperAlgebra

Florentin Smarandache [1,*]

[1] University of New Mexico, 705 Gurley Ave., Gallup Campus, New Mexico 87301, United States

* Correspondence: smarand@unm.edu

Abstract: We recall and improve our 2019 concepts of *n-Power Set of a Set, n-SuperHyperGraph, Plithogenic n-SuperHyperGraph,* and *n-ary HyperAlgebra, n-ary NeutroHyperAlgebra, n-ary AntiHyperAlgebra* respectively, and we present several properties and examples connected with the real world.

Keywords: n-Power Set of a Set, n-SuperHyperGraph (n-SHG), n-SHG-vertex, n-SHG-edge, Plithogenic n-SuperHyperGraph, n-ary HyperOperation, n-ary HyperAxiom, n-ary HyperAlgebra, n-ary NeutroHyperOperation, n-ary NeutroHyperAxiom, n-ary NeutroHyperAlgebra, n-ary AntiHyperOperation, n-ary AntiHyperAxiom, n-ary AntiHyperAlgebra

1. Introduction

In this paper, with respect to the classical HyperGraph (that contains HyperEdges), we add the SuperVertices (a group of vertices put all together form a SuperVertex), in order to form a SuperHyperGraph (SHG). Therefore, each SHG-vertex and each SHG-edge belong to $P(V)$, where V is the set of vertices, and $P(V)$ means the power set of V.

Further on, since in our world we encounter complex and sophisticated groups of individuals and complex and sophisticated connections between them, we extend the SuperHyperGraph to n-SuperHyperGraph, by extending $P(V)$ to $P^n(V)$ that is the n-power set of the set V (see below).
Therefore, the n-SuperHyperGraph, through its n-SHG-vertices and n-SHG-edges that belong to $P^n(V)$, can the best (so far) to model our complex and sophisticated reality.
In the second part of the paper, we extend the classical HyperAlgebra to n-ary HyperAlgebra and its alternatives n-ary NeutroHyperAlgebra and n-ary AntiHyperAlgebra.

2. n-Power Set of a Set

Let U be a universe of discourse, and a subset $V \subseteq U$. Let $n \geq 1$ be an integer.
Let $P(V)$ be the *Power Set of the Set V* (i.e. all subsets of V, including the empty set ϕ and the whole set V). This is the classical definition of power set.
For example, if $V = \{a, b\}$, then $P(V) = \{\phi, a, b, \{a, b\}\}$.
But we have extended the power set to *n-Power Set of a Set* [1].

For $n = 1$, one has the notation (identity): $P^1(V) \equiv P(V)$.

For $n = 2$, the 2-Power Set of the Set V is defined as follows:

$P^2(V) = P(P(V)$.

In our previous example, we get:

$P^2(V) = P(P(V) = P(\{\phi, a, b, \{a, b\}\}) = \{\phi, a, b, \{a, b\};\ \{\phi, a\}, \{\phi, b\}, \{\phi, \{a, b\}\}, \{a, \{a, b\}\}, \{b, \{a, b\}\};$
$\{\phi, a, b\}, \{\phi, a, \{a, b\}\}, \{\phi, b, \{a, b\}\}, \{a, b, \{a, b\}\};\ \{\phi, a, b, \{a, b\}\}\}$.

Definition of n-Power Set of a Set

In general, the **n-Power Set of a Set V** is defined as follows:

$$P^{n+1}(V) = P(P^n(V)), \text{ for integer } n \geq 1.$$

3. Definition of SuperHyperGraph (*SHG)*

A **SuperHyperGraph** (*SHG)* [1] is an ordered pair $SHG = (G \subseteq P(V), E \subseteq P(V))$, where

(i) $V = \{V_1, V_2, \ldots, V_m\}$ is a finite set of $m \geq 0$ vertices, or an infinite set.

(ii) P(V) is the power set of V (all subset of V). Therefore, an ***SHG-vertex*** may be a *single* (classical) *vertex*, or a *super-vertex* (a subset of many vertices) that represents a group (organization), or even an *indeterminate-vertex* (unclear, unknown vertex); ϕ represents the *null-vertex* (vertex that has no element).

(iii) $E = \{E_1, E_2, \ldots, E_m\}$, for $m \geq 1$, is a family of subsets of V, and each E_j is an SHG-edge, $E_i \in P(V)$. An ***SHG-edge*** may be a (classical) *edge*, or a *super-edge* (edge between super-vertices) that represents connections between two groups (organizations), or *hyper-super-edge*) that represents connections between three or more groups (organizations), *multi-edge*, or even *indeterminate-edge* (unclear, unknown edge); ϕ represents the *null-edge* (edge that means there is no connection between the given vertices).

4. **Characterization of the SuperHyperGraph**

Therefore, a **SuperHyperGraph** (*SHG)* may have any of the below:

- *SingleVertices* (V_i), as in classical graphs, such as: V_1, V_2, etc.;
- *SuperVertices (or SubsetVertices)* (SV_i), belonging to $P(V)$, for example: $SV_{1,3} = V_1V_3$, $SV_{2,57} = V_2V_{57}$, etc. that we introduce now for the first time. A super-vertex may represent a group (organization, team, club, city, country, etc.) of many individuals;

  The comma between indexes distinguishes the single vertexes assembled together into a single SuperVertex. For example $SV_{12,3}$ means the single vertex S_{12} and single vertex S_3 are put together to form a super-vertex. But $SV_{1,23}$ means the single vertices S_1 and S_{23} are put together; while $SV_{1,2,3}$ means S_1, S_2, S_3 as single vertices are put together as a super-vertex.

  In no comma in between indexes, i.e. SV_{123} means just a single vertex V_{123}, whose index is 123, or $SV_{123} \equiv V_{123}$.
- *IndeterminateVertices* (i.e. unclear, unknown vertices*);* we denote them as: IV_1, IV_2, etc. that we introduce now for the first time;
- *NullVertex* (i.e. vertex that has no elements, let's for example assume an abandoned house, whose all occupants left), denoted by ϕV .

- *SingleEdges*, as in classical graphs, i.e. edges connecting only two single-vertices, for example: $E_{1,5} = \{V_1, V_5\}$, $E_{2,3} = \{V_2, V_3\}$, etc.;
- *HyperEdges*, i.e. edges connecting three or more single-vertices, for example $HE_{1,4,6} = \{V_1, V_4, V_6\}$, $HE_{2,4,5,7,8,9} = \{V_2, V_4, V_5, V_7, V_8, V_9\}$, etc. as in hypergraphs;
- *SuperEdges* (or *SubsetEdges*), i.e. edges connecting only two SHG-vertices (and at least one vertex is SuperVertex), for example $SE_{(13,6),(45,79)} = \{SV_{13,6}, SV_{45,79}\}$ connecting two SuperVertices, $SE_{9,(2,345)} = \{V_9, SV_{2,345}\}$ connecting one SingleVertex V_9 with one SuperVertex, $SV_{2,345}$, etc. that we introduce now for the first time;
- *HyperSuperEdges* (or *HyperSubsetEdges*), i.e. edges connecting three or more vertices (and at least one vertex is SuperVertex, for example $HSE_{3,45,236} = \{V_3, V_{45}, V_{236}\}$, $HSE_{1234,456789,567,5679} = \{SV_{1234}, SV_{456789}, SV_{567}, SV_{5679}\}$, etc. that we introduce now for the first time;
- *MultiEdges*, i.e. two or more edges connecting the same (single-/super-/indeterminate-) vertices; each vertex is characterized by many attribute values, thus with respect to each attribute value there is an edge, the more attribute values the more edges (= multiedge) between the same vertices;
- *IndeterminateEdges* (i.e. unclear, unknown edges; either we do not know their value, or we do not know what vertices they might connect): IE_1, IE_2, etc. that we introduce now for the first time;
- *NullEdge* (i.e. edge that represents no connection between some given vertices; for example two people that have no connections between them whatsoever): denoted by ϕE .

5. Definition of the n-SuperHyperGraph (*n-SHG*)

A **n-SuperHyperGraph** (*n-SHG*) [1] is an ordered pair $n\text{-}SHG = (G_n \subseteq P^n(V), E_n \subseteq P^n(V))$, where $P^n(V)$ is the *n*-power set of the set *V*, for integer $n \geq 1$.

6. Examples of 2-SuperHyperGraph, SuperVertex, IndeterminateVertex, SingleEdge, Indeterminate Edge, HyperEdge, SuperEdge, MultiEdge, 2-SuperHyperEdge

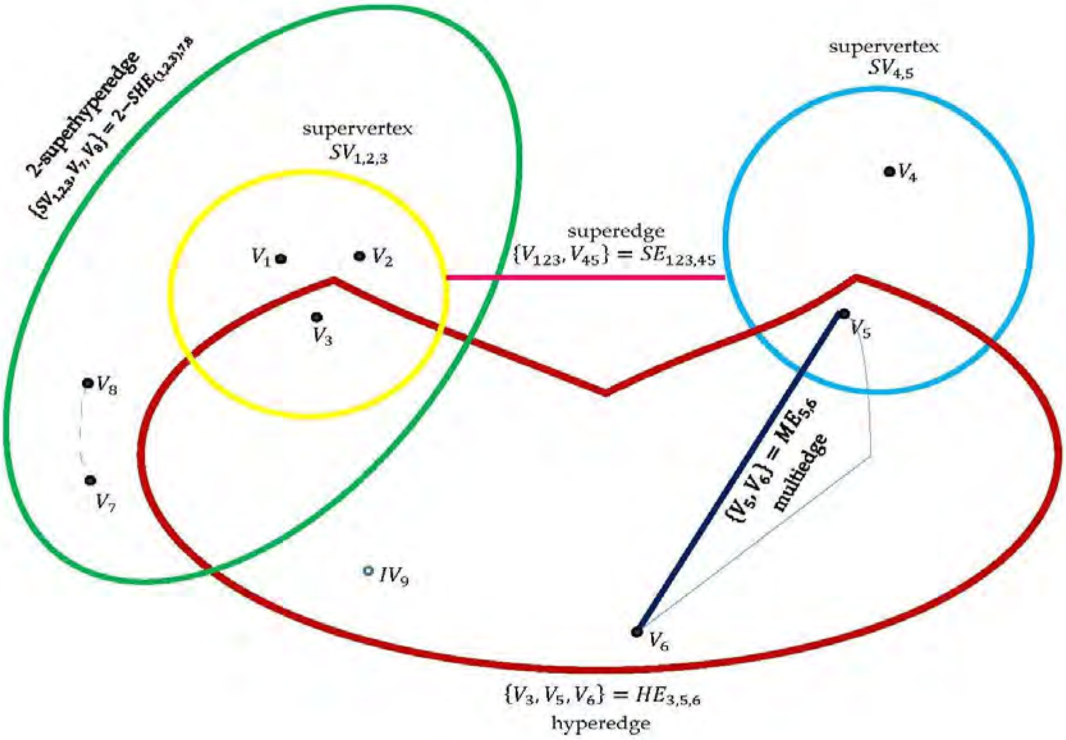

Figure 1. 2-SuperHyperGraph,
($IE_{7,8}$ = Indeterminate Edge between single vertices V_7 and V_8, since the connecting curve is dotted,
IV_9 is an Indeterminate Vertex (since the dot is not filled in),
while $ME_{5,6}$ is a MultiEdge (double edge in this case) between single vertices V_5 and V_6.

Let V_1 and V_2 be two single-vertices, characterized by the attributes a_1 = *size,* whose attribute values are *{short, medium, long}*, and a_2 = *color*, whose attribute values are *{red, yellow}.*

Thus we have the attributes values *(Size{short, medium, long}, Color{red, yellow}),* whence: $V_1(a_1\{s_1, m_1, l_1\}, a_2\{r_1, y_1\})$, where s_1 is the degree of short, m_1 degree of medium, l_1 degree of long, while r_1 is the degree of red and y_1 is the degree of yellow of the vertex V_1.

And similarly $V_2(a_1\{s_2, m_2, l_2\}, a_2\{r_2, y_2\})$.

The degrees may be fuzzy, neutrosophic etc.

Example of fuzzy degree:

$V_1(a_1\{0.8, 0.2, 0.1\}, a_2\{0.3, 0.5\})$.

Example of neutrosophic degree:

$V_1(a_1\{ (0.7,0.3,0.0), (0.4,0.2,0.1),(0.3,0.1,0.1) \}, a_2\{ (0.5,0.1,0.3), (0.0,0.2,0.7) \})$.

Examples of the SVG-edges connecting single vertices V_1 and V_2 are below:

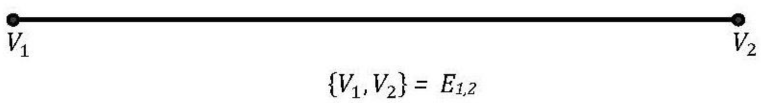

Figure 2. SingleEdge with respect to attributes a_1 and a_2 all together

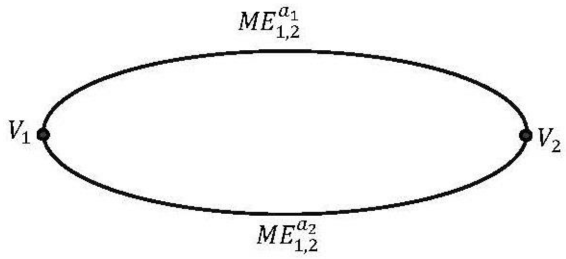

Figure 3. MultiEdge: top edge with respect to attribute a_1, and bottom edge with respect to attribute a_2

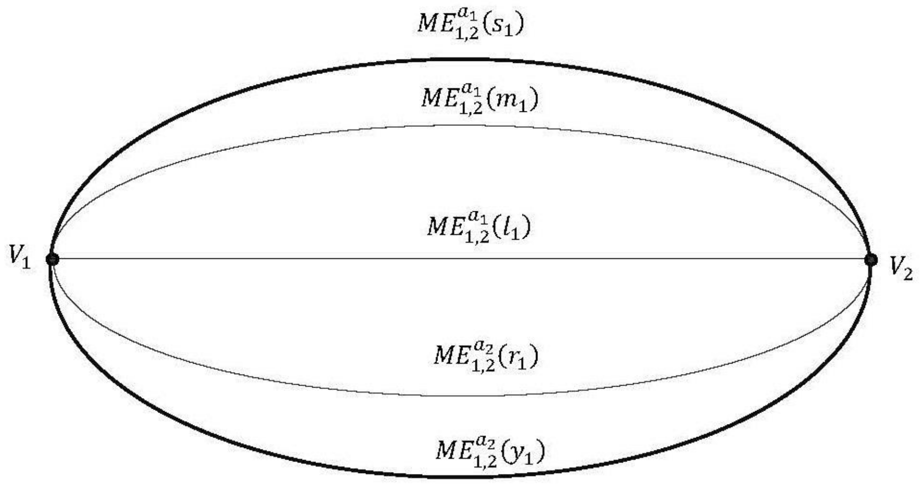

Figure 4. MultiEdge (= Refined MultiEdge from Figure 3):
the top edge from Figure 3, corresponding to the attribute a_1, is split into three sub-edges with respect to the attribute a_1 values s_1, m_1, and l_1;
while the bottom edge from Figure 3, corresponding to the attribute a_2, is split into two sub-edges with respect to the attribute a_2 values r_1, and y_1.

Depending on the application and on experts, one chooses amongst SingleEdge, MultiEdge, Refined-MultiEdge, Refined RefinedMultiEdge, etc.

7. Plithogenic n-SuperHyperGraph

As a consequence, we introduce for the first time the Plithogenic n-SuperHyperGraph.

A **Plithogenic n-SuperHyperGraph (n-PSHG)** is a n-SuperHyperGraph whose each *n-SHG-vertex* and each *n-SHG-edge* are characterized by many distinct attributes values $(a_1, a_2, \ldots, a_p, p \geq 1)$.

Therefore one gets *n-SHG-vertex*$(a_1, a_2, \ldots, a_p)$ and *n-SHG-edge*$(a_1, a_2, \ldots, a_p)$.

The attributes values degrees of appurtenance to the graph may be crisp / fuzzy / intuitionistic fuzzy / picture fuzzy / spherical fuzzy / etc. / neutrosophic / refined neutrosophic / degrees with respect to each *n-SHG-vertex* and each *n-SHG-edge* respectively.

For example, one has:

Fuzzy-*n-SHG-vertex*$(a_1(t_1), a_2(t_2), \ldots, a_p(t_p))$ and Fuzzy-*n-SHG-edge*$(a_1(t_1), a_2(t_2), \ldots, a_p(t_p))$;

Intuitionistic Fuzzy-*n-SHG-vertex*$(a_1(t_1, f_1), a_2(t_2, f_2), \ldots, a_p(t_p, f_p))$

and Intuitionistic Fuzzy-*n-SHG-edge*($a_1(t_1, f_1), a_2(t_2, f_2), \ldots, a_p(t_p, f_p)$);

Neutrosophic-*n-SHG-vertex*($a_1(t_1, i_1, f_1), a_2(t_2, i_2, f_2), \ldots, a_p(t_p, i_p, f_p)$)

and Neutrosophic-*n-SHG-edge*($a_1(t_1, i_1, f_1), a_2(t_2, i_2, f_2), \ldots, a_p(t_p, i_p, f_p)$);

etc.

Whence we get:

8. The Plithogenic (Crisp / Fuzzy / Intuitionistic Fuzzy / Picture Fuzzy / Spherical Fuzzy / etc. / Neutrosophic / Refined Neutrosophic) **n-SuperHyperGraph.**

9. Introduction to n-ary HyperAlgebra

Let U be a universe of discourse, a nonempty set $S \subset U$. Let $P(S)$ be the power set of S (i.e. all subsets of S, including the empty set ϕ and the whole set S), and an integer $n \geq 1$.

We formed [2] the following neutrosophic triplets, which are defined in below sections:

(n-ary HyperOperation, n-ary NeutroHyperOperation, n-ary AntiHyperOperation),

(n-ary HyperAxiom, n-ary NeutroHyperAxiom, n-ary AntiHyperAxiom), and

(n-ary HyperAlgebra, n-ary NeutroHyperAlgebra, n-ary AntiHyperAlgebra).

10. n-ary HyperOperation (n-ary HyperLaw)

A *n-ary HyperOperation (n-ary HyperLaw)* $*_n$ is defined as:

$*_n : S^n \rightarrow P(S)$, and

$\forall a_1, a_2, \ldots, a_n \in S$ one has $*_n(a_1, a_2, \ldots, a_n) \in P(S)$.

The n-ary HyperOperation (n-ary HyperLaw) is well-defined.

11. n-ary HyperAxiom

A *n-ary HyperAxiom* is an axiom defined of S, with respect the above *n-ary* operation $*_n$, that is true for all *n-plets* of S^n.

12. n-ary HyperAlgebra

A *n-ary HyperAlgebra* $(S, *_n)$, is the S endowed with the above n-ary well-defined HyperOperation $*_n$.

13. Types of n-ary HyperAlgebras

Adding one or more n-ary HyperAxioms to S we get different types of *n-ary* HyperAlgebras.

14. n-ary NeutroHyperOperation (n-ary NeutroHyperLaw)

A *n-ary NeutroHyperOperation* is a *n-ary* HyperOperation $*_n$ that is well-defined for some *n-plets* of S^n

[i.e. $\exists (a_1, a_2, \ldots, a_n) \in S^n, *_n(a_1, a_2, \ldots, a_n) \in P(S)$],

and indeterminate [i.e. $\exists (b_1, b_2, \ldots, b_n) \in S^n, *_n(b_1, b_2, \ldots, b_n) = indeterminate$]

or outer-defined [i.e. $\exists(c_1, c_2, ..., c_n) \in S^n, *_n(c_1, c_2, ..., c_n) \notin P(S)$] (or both), on other *n-plets* of S^n.

15. n-ary NeutroHyperAxiom

A *n-ary NeutroHyperAxiom* is an n-ary HyperAxiom defined of *S*, with respect the above *n-ary* operation $*_n$, that is true for some *n-plets* of S^n, and indeterminate or false (or both) for other *n-plets* of S^n.

16. n-ary NeutroHyperAlgebra is an n-ary HyperAlgebra that has some n-ary NeutroHyper-Operations or some n-ary NeutroHyperAxioms

17. n-ary AntiHyperOperation (n-ary AntiHyperLaw)

A *n-ary AntiHyperOperation* is a *n-ary* HyperOperation $*_n$ that is outer-defined for all *n-plets* of S^n [i.e. $\forall(s_1, s_2, ..., s_n) \in S^n, *_n(s_1, s_2, ..., s_n) \notin P(S)$].

18. n-ary AntiHyperAxiom

A *n-ary AntiHyperAxiom* is an n-ary HyperAxiom defined of *S*, with respect the above *n-ary* operation $*_n$ that is false for all *n-plets* of S^n.

19. n-ary AntiHyperAlgebra is an n-ary HyperAlgebra that has some n-ary AntiHyperOperations or some n-ary AntiHyperAxioms.

20. Conclusion

We have recalled our 2019 concepts of n-Power Set of a Set, n-SuperHyperGraph and Plithogenic n-SuperHyperGraph [1], afterwards the n-ary HyperAlgebra together with its alternatives n-ary NeutroHyperAlgebra and n-ary AntiHyperAlgebra [2], and we presented several properties, explanations, and examples inspired from the real world.

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Neutrosophic Triplet Partial Bipolar Metric Spaces

Memet Şahin[1], Abdullah Kargın[2,*] and Merve Sena Uz[3]

[1]Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey. mesahin@gantep.edu.tr
[2,*] Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey. abdullahkargin27@gmail.com
[3]Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey. evrem_anes@yahoo.com
*Correspondence: abdullahkargin27@gmail.com; Tel.:+9005542706621

Abstract: In this article, neutrosophic triplet partial bipolar metric spaces are obtained. Then some definitions and examples are given for neutrosophic triplet partial bipolar metric space. Based on these definitions, new theorems are given and proved. In addition, neutrosophic triplet partial bipolar metric spaces have been shown to be different from classical partial metric space, neutrosophic triplet partial metric space and neutrosophic triplet metric space. Thus, we add a new structure in neutrosophic triplet theory.

Keywords: triplet set, neutrosophic triplet metric space, bipolar metric space, neutrosophic triplet bipolar metric space, neutrosophic triplet partial metric space, neutrosophic triplet partial bipolar metric

1 Introduction

Smarandache obtained neutrosophic logic and set [1]. In neutrosophic theory, there is a degree of membership (t), there is a degree of indeterminacy (ı) and there is a degree of non-membership (f). These degrees are defined independently of each other. Therefore, neutrosophic logic and neutrosophic set help us to explain many uncertainties in our lives. In addition, many researchers have made studies on this theory [2-27]. Recently, some researchers studied neutrosophic theory [50 - 53]. Also, Tey et al. studied novel neutrosophic data analytic hierarchy process for multi-criteria decision making method [54], Son et al. obtained on the stabilizability for a class of linear time-invariant systems under uncertainty [55], Tanuwijaya et al. introduced novel single valued neutrosophic hesitant fuzzy time series model [56].

In fact, neutrosophic set is a generalized state of fuzzy [28] and intuitionistic fuzzy set [29].

Also, Smarandache and Ali obtained neutrosophic triplet set (NTS) and neutrosophic triplet groups (NTG) [30]. For every element "x" in NTS A, there exist a neutral of "x" and an opposite of "x". Also, neutral of "x" must be different from the classical neutral element. Therefore, the NTS is different from the classical set. Furthermore, a neutrosophic triplet (NT) "x" is showed by <x, neut(x), anti(x)>. Also, many researchers have obtained NT structures [31-44]. Recently, Şahin, Kargın Uz and Kılıç have discussed neutrosophic triplet bipolar metric space [45].

Mutlu and Gürdal introduced bipolar metric space [46] in 2016. Bipolar metric space is a generalization of metric space. Also, bipolar metric spaces have an important role in fixed point theory. Recently, Mutlu, Özkan and Gürdal studied fixed point theorems on bipolar metric spaces [47]; Kishore, Agarwal, Rao, and Rao introduced contraction and fixed point theorems in bipolar metric spaces with applications [48]; Rao, Kishore and Kumar obtained Geraghty type contraction and common coupled fixed point theorems in bipolar metric spaces with applications to homotopy [49].

In this section, neutrosophic triplet partial bipolar metric space is introduced. Chapter 2 provides definitions and properties for bipolar metric space [46], neutrophic triplet sets [30], neutrophic triplet metric spaces [32], neutrosophic triplet partial metric space [36] and neutrosophic triplet b - metric space [45]. In chapter 3, neutrosophic triplet partial bipolar metric space is described and some properties are given for neutrosophic triplet partial bipolar metric space. In addition, neutrosophic triplet partial bipolar metric spaces are shown to be different from classical partial metric space, neutrosophic triplet partial metric space and neutrosophic triplet metric space. We give conclusions in Chapter 4.

2 Preliminaries

Definition 2.1: [30] Let # be a binary operation. An NTS (X, #) is a set such that for x ∈ X,

i) There exists neutral of "x" such that x#neut(x) = neut(x)#x = x,

ii) There exists anti of "x" such that x#anti(x) = anti(x)#x = neut(x).

Also, a neutrosophic triplet "x" is denoted by (x, neut(x), anti(x)).

Definition 2.2: [32] Let (N,*) be an NTS and d_N:NxN→ $\mathbb{R}^+$∪{0} be a function. If d_N:NxN→ $\mathbb{R}^+$∪{0} and (N, *) satisfies the following conditions, then d_N is called NTM.

a) x*y ∈ N ,

b) d_N(x, y) ≥ 0 ,

c) If x = y, then d_N(x, y) = 0,

d) d_N(x, y) = d_N(y, x) ,

e) If there exits at least a y ∈ N for each x, z ∈N such that d_N(x, z) ≤ d_N(x, z*neut(y)), then d_N(x, z*neut(y)) ≤ d_N(x, y) + d_N(y, z).

In this case, ((N,*), d_N) is called an NTMS.

Definition 2.3: [36] Let (N,*) be a NTS. If d_p:NxN→ $\mathbb{R}^+$∪{0} satisfies the following conditions, then d_p is a NTpM. For all x, y, z ∈ N,

a) x*y ∈ N,

b) d_p (x, y) ≥ d_p (x, x) ≥ 0,

c) If d_p (x, y) = d_p (x, x) = d_p (y, y) = 0, then there exists at least one pair of elements x, y ∈N such that x = y.

d) d_p (x, y) = d_p (y, x),

e) If for each pair of x, z ∈N, there exists at least one y ∈N such that d_p (x, z) ≤ d_p (x, z*neut(y)), then d_p (x, z*neut(y)) ≤ d_p (x, y) + d_p (y, z) - d_p (y, y).

In this case, ((N,*), d_p) is called a NTpMS.

Definition 2.4: [46] Let *X* and *Y* be nonempty sets and *d*:NxN→ $\mathbb{R}^+$∪{0} be a function. If d satisfies the following conditions, then d is called a bipolar metric (bM).

i) For $\forall(x, y) \in X \times Y$, if *d(x, y)* = 0, then *x* = *y*,

ii) For $\forall u \in X \cap Y$, *d(u, u)* = 0,

iii) For $\forall u \in X \cap Y$, $d(u, v) = d(v, u)$,

iv) For $(x, y), (x', y') \in X \times Y$, $d(x, y) \leq d(x, y') + d(x', y') + d(x', y)$.

In this case, (*X, Y, d*) is called a bipolar metric space (bMS).

Definition 2.5: [45] Let $(X,*)$ and $(Y,*)$ be two NTSs and let $d: X \times Y \to \mathbb{R}^+ \cup \{0\}$ be a function. If d, $(X, *)$ and $(Y,*)$ satisfy the following conditions, then d is called a neutrosophic triplet bipolar metric (NTbM).

i) For $\forall a, b \in X$, $a * b \in X$,

for $\forall c, d \in Y$, $c * d \in Y$,

ii) For $\forall a \in X$ and $\forall b \in Y$, if $d(a,b) = 0$, then $a = b$,

iii) For $\forall u \in X \cap Y$, d(u,u)=0,

iv) For $\forall u, v \in X \cap Y$, $d(u,v) = d(v,u)$.

v) Let $(x,y), (x',y') \in X \times Y$. For each (x,y), if there exists at least one (x',y') such that

$d(x,y) \le d(x, y * neut(y')) \le d(x * neut(x'), y * neut(y'))$ and

$d(x,y) \le d(x * neut(x'), y) \le d(x * neut(x'), y * neut(y'))$,

then

$d(x * neut(x'), y * neut(y')) \le d(x,y') + d(x',y') + d(x',y)$.

In this case, $(((X,Y),*), d)$ is called a neutrosophic triplet bipolar metric space (NTbMS).

Definition 2.6: [45] Let $(((X, Y), *), d)$ be a NTbMS. A left sequence (x_n) converges to a right point y (symbolically $(x_n) \rightharpoondown y$ or $\lim_{n\to\infty}(x_n) = y$) if and only if for every ε > 0 there exists an $n_0 \in \mathbb{N}$, such that $d(x_n, y) < \varepsilon$ for all $n \ge n_0$. Similarly, a right sequence (y_n) converges to a left point x (denoted as $y_n \rightharpoonup x$ or $\lim_{n\to\infty}(y_n) = x$) if and only if, for every ε > 0 there exists an $n_0 \in \mathbb{N}$ such that, whenever $n \ge n_0$, $d(x, y_n) < \varepsilon$. Also, if $(u_n) \rightharpoondown u$ and $(u_n) \rightharpoonup u$, then (u_n) converges to point u ((u_n) is a central sequence).

Definition 2.7: [45] Let $(((X, Y), *), d)$ be an NTbMS, (x_n) be a left sequence and (y_n) be a right sequence in this space. (x_n, y_n) is called an NT bisequence. Furthermore, if (x_n) and (y_n) are convergent, then (x_n, y_n) is called an NT convergent bisequence. Also, if (x_n) and (y_n) converge to the same point, then (x_n, y_n) is called an NT biconvergent bisequence.

Definition 2.8: [45] Let $(((X, Y), *), d)$ be an NTbMS and (x_n, y_n) be an NT bisequence. (x_n, y_n) is called an NT Cauchy bisequence if and only if for every ε > 0 there exists an $n_0 \in \mathbb{N}$, such that $d(x_n, y_n) < \varepsilon$ for all $n \ge n_0$.

3 Neutrosophic Triplet Partial Bipolar Metric Space

Definition 3.1: Let $(X, *)$ and $(Y,*)$ be two NTSs and let $d_{pb}: X \times Y \to \mathbb{R}^+ \cup \{0\}$ be a function. If d_{pb}, (X, ∗) and (Y, ∗) satisfy the following conditions, then d_{pb} is called a NT partial bipolar metric (NTpbM).

i-) For all $a, b \in X$, $a * b \in X$,

for all $c, d \in Y$, $c * d \in Y$,

ii-) For all $x \in X$ and $y \in Y$,

$d_{pb}(x,y) \ge d_{pb}(x,x) \ge 0$ and $d_{pb}(x,y) \ge d_{pb}(y,y) \ge 0$,

iii-) If $d_{pb}(x,y) = d_{pb}(x,x) = d_{pb}(y,y) = 0$, there exists at least one pair of elements $x, y \in X \cap Y$ such that $d_{pb}(x,y) = 0$, iv-) For all $x, y \in X \cap Y$, $d_{pb}(x, y) = d_{pb}(y, x)$,

v-) Let $(x, y), (x', y') \in X \times Y$. For each (x, y), if there exists at least one (x', y') such that

$d_{pb}(x, y) \leq d_{pb}(x, y * neut(y')) \leq d_{pb}(x * neut(x'), y * neut(y'))$ and

$d_{pb}(x, y) \leq d_{pb}(x * neut(x'), y) \leq d_{pb}(x * neut(x'), y * neut(y'))$,

then

$d_{pb}(x * neut(x'), y * neut(y')) \leq d_{pb}(x, y') + d_{pb}(x', y') + d_{pb}(x', y) - \min\{d_{pb}(x', x'), d_{pb}(y', y')\}$.

In this case, $\left(((X, Y), *), d_{pb}\right)$ is called a NTpbM space (NTpbMS).

Example 3.2: Let $X = \{0, 3, 6, 9, 10, 12\}$ and $Y = \{0, 5, 6, 10\}$. We show that $(X, .)$ and $(Y, .)$ are NTSs in $(\mathbb{Z}_{15}, .)$.

For $(X, .)$, NTs are (0, 0, 0), (3, 6, 12), (6, 6, 6), (9, 6, 9), (10,10,10), (12, 6, 3).

Also, for $(Y, .)$, NTs are (0, 0, 0), (5, 10, 5), (6, 6, 6), (10, 10, 10).

Thus, $(X, .)$ and $(Y, .)$ are NTSs.

Furthermore, we define the $d_{pb}: X \times Y \to \mathbb{R}^+ \cup \{0\}$ function such that $d_{pb}(s, r) = max\{|3^s - 1|, |3^r - 1|\}$. We show that d is a NTpbM.

i-) $0.0 = 0 \in$, $0.3 = 0 \in X$, $0.6 = 0 \in X$, $0.9 = 0 \in X$, $0.10 = 0 \in X$, $0.12 = 0 \in X$, $3.3 = 9 \in X$, $3.6 = 3 \in X$, $3.9 = 12 \in X$, $3.10 = 0 \in X$, $3.12 = 6 \in X$, $6.6 = 6 \in X$, $6.9 = 9 \in X$, $6.10 = 0 \in X$, $6.12 = 12 \in X$, $9.9 = 6 \in X$, $9.10 = 0 \in X$, $9.12 = 3 \in X$, $10.10 = 10 \in X$, $12.10 = 0 \in X$, $12.12 = 9 \in X$.

Thus, for all $a, b \in X$, $a.b \in X$.

Also, $0.0 = 0 \in Y$, $0.5 = 0 \in Y$, $0.6 = 0 \in Y$, $0.10 = 0 \in Y$, $5.5 = 10 \in Y$, $5.10 = 5 \in Y$, $5.6 = 0 \in Y$, $10.10 = 10 \in Y$, $10.6 = 0 \in Y$, $6.6 = 6 \in Y$

Thus, for all $c, d \in Y$, $c.d \in Y$.

ii-)For all $x \in X$, $y \in Y$, if

$d_{pb}(x, y) = \max\{|3^x - 1|, |3^y - 1|\}$,

$d_{pb}(x, x) = \max\{|3^x - 1|, |3^x - 1|\}$,

$d_{pb}(y, y) = \max\{|3^y - 1|, |3^y - 1|\}$,

then it is clear that

$d_{pb}(x, y) \geq d_{pb}(x, x) \geq 0$ and

$d_{pb}(x, y) \geq d_{pb}(y, y) \geq 0$.

iii-)For $d_{pb}(x, y) = d_{pb}(x, x) = d_{pb}(y, y) = 0$, if $d_{pb}(x, y) = 0$, then there exists at least one pair $x, y \in X \cap Y$.

If $d_{pb}(x, y) = \max\{|3^x - 1|, |3^y - 1|\} = 0$, then $3^x - 1 = 0$ and $3^y - 1 = 0$.

If $3^x = 1$ and $3^y = 1$, $x, y \in X \cap Y$ are pairs of elements, since $x = 0 \in X$ and $y = 0 \in Y$.

iv) For all $, y \in X \cap Y$, $d_{pb}(x, y) = d_{pb}(y, x)$.

$d_{pb}(x, y) = \max\{|3^x - 1|, |3^y - 1|\} = \max\{|3^y - 1|, |3^x - 1|\} = d_{pb}(y, x)$.

v) It is clear that

$$d_{pb}(0,0) = 0 \le d_{pb}\big(0\,,0.neut(6)\big) = 0 \le d_{pb}\big(0.neut(3), 0.neut(6)\big) = d_{pb}(0,0) = 0,$$

$$d_{pb}(0,0) = 0 \le d_{pb}(0.neut(3), 0) = 0 \le d_{pb}\big(0.neut(3), 0.neut(6)\big) = d_{pb}(0,0) = 0.$$

Also,

$d_{pb}\big(0.neut(3), 0.neut(6)\big) = d_{pb}(0,0) = 0 \le \; d_{pb}(0,6) + d_{pb}(3,6) + d_{pb}(3,0) -$

$\min\{d_{pb}(3,3), d_{pb}(6,6)\}$.

It is clear that

$$d_{pb}(0,5) = 0 \le d_{pb}\big(0,5.neut(10)\big) = d_{pb}(0,5) = max\{|3^0 - 1|, |3^5 - 1|\}$$
$$\le d_{pb}\big(0.neut(6), 5.neut(10)\big) = d_{pb}(0,5),$$
$$d_{pb}(0,5) = 0 \le d_{pb}(0.neut(6), 5) = d_{pb}(0,5) = max\{|3^0 - 1|, |3^5 - 1|\}$$
$$\le d_{pb}\big(0.neut(6), 5.neut(10)\big) = d_{pb}(0,5).$$

Also,

$d_{pb}\big(0.neut(6), 5.neut(10)\big) = d_{pb}(0,5) \le$

$d_{pb}\,(0,10) + d_{pb}(6,10) + d_{pb}(6,5) - \; min\{d_{pb}(6,6), d_{pb}(10,10)\}$.

It is clear that

$$d_{pb}(0,10) \le d_{pb}\big(0,10.neut(5)\big) = d_{pb}(0,10) = max\{|3^0 - 1|, |3^{10} - 1|\}$$
$$\le d_{pb}\big(0.neut(3), 10.neut(5)\big) = d_p(0,10),$$
$$d_{pb}(0,10) = |3^{10} - 1| \le d_{pb}(0.neut(3), 10) = d_{pb}(0,10) = max\{|3^0 - 1|, |3^{10} - 1|\}$$
$$\le d_{pb}\big(0.neut(3), 10.neut(5)\big) = d_{pb}(0,10)$$

Also,

$d_{pb}\big(0.neut(3), 10.neut(5)\big) = d_{pb}(0,10) \le \; d_{pb}(0,5) + d_{pb}(3,5) + d_p(3,10) - \; min\{d_{pb}(3,3), d_{pb}(5,5)\}$

It is clear that

$$d_{pb}(0,6) = |3^6 - 1| \le d_{pb}\big(0,6.neut(6)\big) = d_{pb}(0,6) = max\{|3^0 - 1|, |3^6 - 1|\}$$
$$\le d_{pb}\big(0.neut(3), 6.neut(6)\big) = d_{pb}(0,6),$$
$$d_{pb}(0,6) = |3^6 - 1| \le d_{pb}(0.neut(3), 6) = d_{pb}(0,6) = max\{|3^0 - 1|, |3^6 - 1|\}$$
$$\le d_{pb}\big(0.neut(3), 6.neut(6)\big) = d_{pb}(0,6).$$

Also,

$$d_{pb}\big(0.neut(3), 6.neut(6)\big) = d_{pb}(0,6) = 728$$
$$\le \; d_{pb}(0,6) + d_{pb}(3,6) + d_{pb}(3,6) - \; min\{d_{pb}(3,3), \; d_{pb}(6,6)\}$$

It is clear that

$$d_{pb}(3,0) = |3^3 - 1| \le d_{pb}\big(3, 0.neut(5)\big) = d_{pb}(3,0) = max\{|3^3 - 1|, |3^0 - 1|\}$$
$$\le d_{pb}\big(3.neut(6), 0.neut(5)\big) = d_{pb}(3,0),$$
$$d_{pb}(3,0) = |3^3 - 1| = 26 \le d_{pb}(3.neut(6), 0) = d_{pb}(3,0) = max\{|3^3 - 1|, |3^0 - 1|\} = 26$$
$$\le d_{pb}\big(3.neut(6), 0.neut(5)\big) = d_{pb}(3,0).$$

Also,

$$d_{pb}\big(3.neut(6), 0.neut(5)\big) = d_{pb}(3,0) = 26$$
$$\leq\ d_{pb}(3,5) + d_{pb}(6,5) + d_{pb}(6,0) -\ min\big\{d_{pb}(6,6), d_{pb}(5,5)\big\}.$$

It is clear that

$$d_{pb}(3,5) = |3^5 - 1| \leq d_{pb}\big(3, 5.neut(10)\big) = d_{pb}(3,5) = max\{|3^3 - 1|, |3^5 - 1|\}$$
$$\leq d_{pb}\big(3.neut(6), 5.neut(10)\big) = d_{pb}(3,5),$$
$$d_{pb}(3,5) = |3^5 - 1| \leq d_{pb}(3.neut(6), 5) = d_{pb}(3,5) = max\{|3^3 - 1|, |3^5 - 1|\}$$
$$\leq d_{pb}\big(3.neut(6), 5.neut(10)\big) = d_{pb}(3,5).$$

Also,

$$d_{pb}\big(3.neut(6), 5.neut(10)\big) \leq\ d_{pb}(3,10) + d_{pb}(6,10) + d_{pb}(6,5) -\ min\big\{d_{pb}(6,6), d_{pb}(10,10)\big\}.$$

It is clear that

$$d_{pb}(3,10) = |3^{10} - 1| \leq d_{pb}\big(3, 10.neut(5)\big) = d_{pb}(3,10) = max\{|3^3 - 1|, |3^{10} - 1|\}$$
$$\leq d_{pb}\big(3.neut(9), 10.neut(5)\big) = d_{pb}(3,10),$$
$$d_{pb}(3,10) = |3^{10} - 1| \leq d_{pb}(3.neut(9), 10) = d_{pb}(3,10) = max\{|3^3 - 1|, |3^{10} - 1|\}$$
$$\leq d_{pb}\big(3.neut(9), 10.neut(5)\big) = d_{pb}(3,10).$$

Also, $d_{pb}\big(3.neut(9), 10.neut(5)\big) \leq\ d_{pb}(3,5) + d_{pb}(9,5) + d_{pb}(9,10) -\ min\big\{d_{pb}(9,9),\ d_{pb}(5,5)\big\}$

It is clear that

$$d_{pb}(3,6) = |3^6 - 1| \leq d_{pb}\big(3, 6.neut(6)\big) = d_{pb}(3,6) = max\{|3^3 - 1|, |3^6 - 1|\}$$
$$\leq d_{pb}\big(3.neut(9), 6.neut(6)\big) = d_{pb}(3,6),$$
$$d_{pb}(3,6) = |3^6 - 1| \leq d_{pb}(3.neut(9), 6) = d_{pb}(3,6) = max\{|3^3 - 1|, |3^6 - 1|\}$$
$$\leq d_{pb}\big(3.neut(9), 6.neut(6)\big) = d_{pb}(3,6).$$

Also,

$$d_{pb}\big(3.neut(9), 6.neut(6)\big) = d_{pb}(3,6) \leq\ d_{pb}(3,6) + d_{pb}(9,6) + d_{pb}(9,6) -\ min\big\{d_{pb}(9,9), d_{pb}(6,6)\big\}$$

It is clear that

$$d_{pb}(6,0) = |3^6 - 1| \leq d_{pb}\big(6, 0.neut(5)\big) = d_{pb}(6,0) = max\{|3^6 - 1|, |3^0 - 1|\}$$
$$\leq d_{pb}\big(6.neut(9), 0.neut(5)\big) = d_{pb}(6,0),$$
$$d_{pb}(6,0) = |3^6 - 1| \leq d_{pb}(6.neut(9), 0) = d_{pb}(6,0) = max\{|3^6 - 1|, |3^0 - 1|\}$$
$$\leq d_{pb}\big(6.neut(9), 0.neut(5)\big) = d_{pb}(6,0).$$

Also,

$$d_{pb}\big(6.neut(9), 0.neut(5)\big) = d_{pb}(6,0) \leq\ d_{pb}(6,5) + d_{pb}(9,5) + d_{pb}(9,0) -\ min\big\{d_{pb}(9,9), d_{pb}(5,5)\big\}.$$

It is clear that

$$d_{pb}(6,5) = |3^6 - 1| \leq d_{pb}\big(6, 5.neut(0)\big) = d_{pb}(6,0) = max\{|3^6 - 1|, |3^0 - 1|\}$$
$$\leq d_{pb}\big(6.neut(12), 5.neut(0)\big) = d_{pb}(6,0),$$

$$d_{pb}(6,5) = |3^6 - 1| \le d_{pb}(6.neut(12), 5) = d_{pb}(6,5) = max\{|3^6 - 1|, |3^5 - 1|\}$$
$$\le d_{pb}\big(6.neut(12), 5.neut(0)\big) = d_{pb}(6,0).$$

Also,

$$d_{pb}\big(6.neut(12), 5.neut(0)\big)$$
$$\le d_{pb}(6,0)\ d_{pb}(6,0) + d_{pb}(12,0) + d_{pb}(12,5) -\ min\{d_{pb}(12,12), d_{pb}(0,0)\}.$$

It is clear that

$$d_{pb}(6,10) = |3^{10} - 1| \le d_{pb}\big(6, 10.neut(5)\big) = d_{pb}(6,10) = max\{|3^6 - 1|, |3^{10} - 1|\}$$
$$\le d_{pb}\big(6.neut(0), 10.neut(5)\big) = d_{pb}(0,10),$$
$$d_{pb}(6,10) = |3^{10} - 1| \le d_{pb}(6.neut(0), 10) = d_{pb}(0,10) = max\{|3^0 - 1|, |3^{10} - 1|\}$$
$$\le d_{pb}\big(6.neut(0), 10.neut(5)\big) = d_{pb}(0,10).$$

Also,

$$d_{pb}\big(6.neut(0), 10.neut(5)\big) = d_{pb}(0,10) \le\ d_{pb}(6,5) + d_p(0,5) + d_{pb}(0,10) -\ min\{d_{pb}(0,0), d_{pb}(5,5)\}$$

It is clear that

$$d_{pb}(6,6) = |3^6 - 1| \le d_{pb}\big(6, 6.neut(0)\big) = d_{pb}(6,0) = max\{|3^6 - 1|, |3^0 - 1|\}$$
$$\le d_{pb}\big(6.neut(3), 6.neut(0)\big) = d_{pb}(6,0),$$
$$d_{pb}(6,6) = |3^6 - 1| \le d_{pb}(6.neut(3), 6) = d_{pb}(6,6) = max\{|3^6 - 1|, |3^6 - 1|\}$$
$$\le d_{pb}\big(6.neut(3), 6.neut(0)\big) = d_{pb}(6,0).$$

Also,

$$d_{pb}\big(6.neut(3), 6.neut(0)\big) = d_{pb}(6,0) \le\ d_{pb}(6,0) + d_{pb}(3,0) + d_{pb}(3,6) -\ min\{d_{pb}(3,3), d_{pb}(0,0)\}$$

It is clear that

$$d_{pb}(9,0) = |3^9 - 1| \le d_{pb}\big(9, 0.neut(5)\big) = d_{pb}(9,0) = max\{|3^9 - 1|, |3^0 - 1|\}$$
$$\le d_{pb}\big(9.neut(12), 0.neut(5)\big) = d_p(9,0),$$
$$d_{pb}(9,0) = |3^9 - 1| \le d_{pb}(9.neut(12), 0) = d_{pb}(9,0) = max\{|3^9 - 1|, |3^0 - 1|\}$$
$$\le d_{pb}\big(9.neut(12), 0.neut(5)\big) = d_{pb}(9,0).$$

Also,

$$d_{pb}\big(9.neut(12), 0.neut(5)\big) = d_{pb}(9,0) \le$$
$$d_{pb}(9,5) + d_{pb}(12,5) + d_{pb}(12,0) -\ min\{d_{pb}(12,12), d_{pb}(5,5)\}.$$

It is clear that

$$d_{pb}(9,5) = |3^9 - 1| \le d_{pb}\big(9, 5.neut(0)\big) = d_{pb}(9,0) = max\{|3^9 - 1|, |3^0 - 1|\}$$
$$\le d_{pb}\big(9.neut(12), 5.neut(0)\big) = d_{pb}(9,0),$$
$$d_{pb}(9,5) = |3^9 - 1| \le d_{pb}(9.neut(12), 5) = d_{pb}(9,5) = max\{|3^9 - 1|, |3^5 - 1|\}$$
$$\le d_{pb}\big(9.neut(12), 5.neut(0)\big) = d_{pb}(9,0).$$

Also,

$$d_{pb}\big(9.neut(12), 5.neut(0)\big) = d_{pb}(9,0)$$
$$\leq\ d_{pb}(9,0) + d_{pb}(12,0) + d_{pb}(12,5) -\ min\{d_{pb}(12,12), d_{pb}(0,0)\}.$$

It is clear that

$$d_{pb}(9,10) = |3^{10} - 1| \leq d_{pb}\big(9, 10.neut(5)\big) = d_{pb}(9,10) = max\{|3^{9} - 1|, |3^{10} - 1|\}$$
$$\leq d_{pb}\big(9.neut(0), 10.neut(5)\big) = d_{pb}(0,10),$$
$$d_{pb}(9,10) = |3^{10} - 1| \leq d_{pb}(9.neut(0), 10) = d_{pb}(0,10) = max\{|3^{0} - 1|, |3^{10} - 1|\}$$
$$\leq d_{pb}\big(9.neut(0), 10.neut(5)\big) = d_{pb}(0,10).$$

Also,

$$d_{pb}\big(9.neut(0), 10.neut(5)\big) = d_{pb}(0,10) \leq$$
$$d_{pb}(9,5) + d_{pb}(0,5) + d_{pb}(0,10) -\ min\{d_{pb}(0,0), d_{pb}(5,5)\}.$$

It is clear that

$$d_{pb}(9,6) = |3^{9} - 1| \leq d_{pb}\big(9, 6.neut(5)\big) = d_{pb}(9,0) = max\{|3^{9} - 1|, |3^{0} - 1|\}$$
$$\leq d_{pb}\big(9.neut(3), 6.neut(5)\big) = d_{pb}(9,0),$$
$$d_{pb}(9,6) = |3^{9} - 1| \leq d_{pb}(9.neut(3), 6) = d_{pb}(9,6) = max\{|3^{9} - 1|, |3^{6} - 1|\}$$
$$\leq d_{pb}\big(9.neut(3), 6.neut(5)\big) = d_{pb}(9,0).$$

Also,

$$d_{pb}\big(9.neut(3), 6.neut(5)\big) = d_{pb}(9,0) \leq\ d_{pb}(9,5) + d_{pb}(3,5) + d_{pb}(3,6) -\ min\{d_{pb}(3,3), d_{pb}(5,5)\}$$

It is clear that

$$d_{pb}(10,0) = |3^{10} - 1| \leq d_{pb}\big(10, 0.neut(6)\big) = d_{pb}(10,0) = max\{|3^{10} - 1|, |3^{0} - 1|\}$$
$$\leq d_{pb}\big(10.neut(10), 0.neut(6)\big) = d_{pb}(10,0),$$
$$d_{pb}(10,0) = |3^{10} - 1| \leq d_{pb}(10.neut(10), 0) = d_{pb}(10,0) = max\{|3^{10} - 1|, |3^{0} - 1|\}$$
$$\leq d_{pb}\big(10.neut(10), 0.neut(6)\big) = d_{pb}(10,0).$$

Also,

$$d_{pb}\big(10.neut(10), 0.neut(6)\big) = d_{pb}(10,0) \leq$$
$$d_{pb}(10,6) + d_{pb}(10,6) + d_{pb}(10,0) -\ min\{d_{pb}(10,10), d_{pb}(6,6)\}.$$

It is clear that

$$d_{pb}(10,5) = |3^{10} - 1| \leq d_{pb}\big(10, 5.neut(6)\big) = d_{pb}(10,0) = max\{|3^{10} - 1|, |3^{0} - 1|\}$$
$$\leq d_{pb}\big(10.neut(10), 5.neut(6)\big) = d_{pb}(10,0),$$
$$d_{pb}(10,5) = |3^{10} - 1| \leq d_{pb}(10.neut(10), 5) = d_{pb}(10,5) = max\{|3^{10} - 1|, |3^{5} - 1|\}$$
$$\leq d_{pb}\big(10.neut(10), 5.neut(6)\big) = d_{pb}(10,0).$$

Also,

$$d_{pb}\big(10.neut(10), 5.neut(6)\big) = d_{pb}(10,0) \leq$$
$$d_{pb}(10,6) + d_{pb}(10,6) + d_{pb}(10,5) -\ min\{d_{pb}(10,10), d_{pb}(6,6)\}.$$

It is clear that

$$d_{pb}(10,10) = |3^{10} - 1| \leq d_{pb}\big(10, 10.neut(5)\big) = d_{pb}(10,10) = max\{|3^{10} - 1|, |3^{10} - 1|\}$$
$$\leq d_{pb}\big(10.neut(3), 10.neut(5)\big) = d_{pb}(0,10),$$
$$d_{pb}(10,10) = |3^{10} - 1| \leq d_{pb}(10.neut(3), 10) = d_{pb}(0,10) = max\{|3^{0} - 1|, |3^{10} - 1|\}$$
$$\leq d_{pb}\big(10.neut(3), 10.neut(5)\big) = d_{pb}(0,10).$$

Also,

$d_{pb}\big(10.neut(3), 10.neut(5)\big) = d_{pb}(0,10) \leq$

$d_{pb}(10,5) + d_{pb}(3,5) + d_{pb}(3,10) - \ min\{d_{pb}(3,3), d_{pb}(5,5)\}$.

It is clear that

$$d_{pb}(10,6) = |3^{10} - 1| \leq d_{pb}\big(10, 6.neut(0)\big) = d_{pb}(10,0) = max\{|3^{10} - 1|, |3^{0} - 1|\}$$
$$\leq d_{pb}\big(10.neut(5), 6.neut(0)\big) = d_{pb}(10,0),$$
$$d_{pb}(10,6) = |3^{10} - 1| \leq d_{pb}(10.neut(5), 6) = d_{pb}(10,6) = max\{|3^{10} - 1|, |3^{6} - 1|\}$$
$$\leq d_{pb}\big(10.neut(5), 6.neut(0)\big) = d_{pb}(10,0).$$

Also,

$d_{pb}\big(10.neut(5), 6.neut(0)\big) = d_{pb}(10,0) \leq$

$d_{pb}(10,0) + d_{pb}(5,0) + d_{pb}(5,6) - \ min\{d_{pb}(5,5), d_{pb}(0,0)\}$.

It is clear that

$$d_{pb}(12,0) = |3^{12} - 1| \leq d_{pb}\big(12, 0.neut(5)\big) = d_{pb}(12,0) = max\{|3^{12} - 1|, |3^{0} - 1|\}$$
$$\leq d_{pb}\big(12.neut(6), 0.neut(5)\big) = d_{pb}(12,0),$$
$$d_{pb}(12,0) = |3^{12} - 1| \leq d_{pb}(12.neut(6), 0) = d_{pb}(12,0) = max\{|3^{12} - 1|, |3^{0} - 1|\}$$
$$\leq d_{pb}\big(12.neut(6), 0.neut(5)\big) = d_{pb}(12,0).$$

Also,

$d_{pb}\big(12.neut(6), 0.neut(5)\big) = d_{pb}(12,0) \leq$

$d_{pb}(12,5) + d_{pb}(6,5) + d_{pb}(6,0) - \ min\{d_{pb}(6,6), d_{pb}(5,5)\}$.

It is clear that

$$d_{pb}(12,5) = |3^{12} - 1| \leq d_{pb}\big(12, 5.neut(10)\big) = d_{pb}(12,5) = max\{|3^{12} - 1|, |3^{5} - 1|\}$$
$$\leq d_{pb}\big(12.neut(3), 5.neut(10)\big) = d_{pb}(12,5),$$
$$d_{pb}(12,5) = |3^{12} - 1| \leq d_{pb}(12.neut(3), 5) = d_{pb}(12,5) = max\{|3^{12} - 1|, |3^{5} - 1|\}$$
$$\leq d_{pb}\big(12.neut(3), 5.neut(10)\big) = d_{pb}(12,5).$$

Also,

$d_{pb}\big(12.neut(3), 5.neut(10)\big) = d_{pb}(12{,}5) \leq$

$d_{d_{pb}}(12{,}10) + d_{pb}(3{,}10) + d_{pb}(3{,}5) - \ min\{d_{pb}(3{,}3), d_{pb}(10{,}10)\}$.

It is clear that

$$d_{pb}(12,10) = |3^{12}-1| \le d_{pb}\big(12, 10.neut(0)\big) = d_{pb}(12,0) = max\{|3^{12}-1|, |3^{0}-1|\}$$
$$\le d_{pb}\big(12.neut(6), 10.neut(0)\big) = d_{pb}(12,0),$$
$$d_{pb}(12,10) = |3^{12}-1| \le d_{pb}(12.neut(6), 10) = d_{pb}(12,10) = max\{|3^{12}-1|, |3^{10}-1|\}$$
$$\le d_{pb}\big(12.neut(6), 10.neut(0)\big) = d_{pb}(12,0).$$

Also,

$d_{pb}\big(12.neut(6), 10.neut(0)\big) = d_{pb}(12,0) \le$

$d_{pb}(12,0) + d_{pb}(6,0) + d_{pb}(6,10) - \ min\{d_{pb}(6,6), d_{pb}(10,10)\}$.

It is clear that

$$d_{pb}(12,6) = |3^{12}-1| \le d_{pb}\big(12, 6.neut(10)\big) = d_{pb}(12,0) = max\{|3^{12}-1|, |3^{0}-1|\}$$
$$\le d_{pb}\big(12.neut(9), 6.neut(10)\big) = d_{pb}(12,0),$$
$$d_{pb}(12,6) = |3^{12}-1| \le d_{pb}(12.neut(9), 6) = d_{pb}(12,6) = max\{|3^{12}-1|, |3^{6}-1|\}$$
$$\le d_{pb}\big(12.neut(9), 6.neut(10)\big) = d_{pb}(12,0).$$

Also,

$d_{pb}\big(12.neut(9), 6.neut(10)\big) = d_{pb}(12,0) \le$

$d_{pb}(12,10) + d_{pb}(9,10) + d_{pb}(9,6) - \ min\{d_{pb}(9,9), d_{pb}(10,10)\}$.

Thus, for each (x, y), if there exists at least a (x', y') such that

$d_{pb}(x,y) \le d_{pb}\big(x, y * neut(y')\big) \ \le d_{pb}(x * neut(x'), y * neut(y'))$

$d_{pb}(x,y) \le d_{pb}(x * neut(x'), y) \le d_{pb}(x * neut(x'), y * neut(y'))$,

then

$d_{pb}(x * neut(x'), y * neut(y')) \le \ d_{pb}(x, y') + d_{pb}(x', y') + d_{pb}(x', y) - min\{d_{pb}(x', x'), d_{pb}(y', y')\}$.

Therefore, d_{pb} is an NTpbM and $\big(((X,Y),*), d_{pb}\big)$ is an NTpbMS.

Corollary 3.3:

1) The NTpbMS differs from the NTPMS due to the i -), ii-) and v-) conditions in the NTpbMS.

2) The NTpbMS differs from the NTMS. Because the triangle inequality in the NTMS differs from the triangle inequality in the NTpbMS.

3) The NTpbMS differs from the NTbMS. Because the triangle inequality in the NTbMS differs from the triangle inequality in the NTpbMS. Also, in a NTpbMS, it can be that d_{pb}(x, x) ≠ 0.

Theorem 3.4: Let $(\big((X,Y),*\big),\ d_{pb})$ be a NTpbMS. If the following conditions are satisfied, then $((X,*),\ d_{pb})$ is a NTpMS.

a) $Y = X$.

b) $y' = x'$, by the triangle equality in Definition 3.1.

Proof:

i) $(\big((X,Y),*\big), d_{pb})$ is a NTpbMS implies that for all $a, b \in X$, $a * b \in X$ and for all $c, d \in Y$, $c * d \in Y$. Also, from condition a) it is clear that for all $a, c \in X$ = Y, $a * c \in X$ = Y.

ii) Since $(((X,Y),\ *),d_{pb})$ is a NTpbMS, for all $a \in X$ and for all $b \in Y$,

$d_{pb}(a,b) \geq d_{pb}(a,a) \geq 0$ and

$d_{pb}(a,b) \geq d_{pb}(b,b) \geq 0$ and is obvious by condition (a)

For all $,b \in X$, $d_{pb}(a,b) \geq d_{pb}(a,a) \geq 0.$

iii) Since $(((X,Y),*),d_{pb})$ is a NTpbMS, for $d_{pb}(a,b) = d_{pb}(a,a) = d_{pb}(b,b) = 0$, if $d_{pb}(a,b) = 0$; then there exists at least one pair of $a,b \in X \cap Y$, and also from condition a), if $y = x$, then $d_{pb}(a,b) = d_{pb}(a,a) = d_{pb}(b,b) = 0$ there exists at least one pair of $a,b \in X \cap X = X$ such that $d_{pb}(a,b) = 0$.

iv) Since $(((X,Y),*),d_{pb})$ is a NTpbMS, we have for all $a,b \in X \cap Y$, $d_{pb}(a,b) = d_{pb}(b,a)$. Also, from condition a), we can write $X \cap Y$ = X. Thus, for all $x,y \in X$, $d_{pb}(a,b) = d_{pb}(b,a)$.

v) Since $(((X,Y),*),d_{pb})$ is a NTpbMS, for each (a,b), if there exists at least a (a',b') such that

$d_{pb}(a,b) \leq d_{pb}(a, b * neut(b')) \leq d_{pb}(a * neut(a'), b * neut(b'))$ and

$d_{pb}(a,b) \leq d_{pb}(a * neut(a'), b) \leq d_{pb}(a * neut(a'), b * neut(b'))$,

then

$d_{pb}(a * neut(a'), b * neut(b')) \leq d_{pb}(a,b') + d_{pb}(a',b') + d_{pb}(a',b) - \min\{d_{pb}(a',a'),\ d_{pb}(b',b')\}$.

From condition b), we can write that

$$d_{pb}(a,b) \leq d_{pb}(a, b * neut(b')) \leq$$

$d_{pb}(a * neut(a'), b * neut(b')) \leq d_{pb}(a,b') + d_{p}(a',a') + d_{pb}(a',b) - \min\ \{d_{pb}(a',a'), d_{pb}(b',b')\} = d_{pb}(a,b') + d_{pb}(a',b) - \min\{d_{pb}(a',a'), d_{pb}(b',b')\}$.

Also, from condition a), if there exists at least a $b' \in$ Y = X for each a, b ∈ Y = X such that

$d_{pb}(a$, b) ≤ $d_{pb}(a$, b*neut(b')), then

$d_{pb}(a$, b*neut(b')) ≤ $d_{pb}(a, a')$ + $d_{pb}(a'$, b)- min $\{d_{pb}(a',a'),\ d_{pb}(a',a')\}$= $d_{pb}(a, a')$ + $d_{pb}(a'$, b) .

Thus, $((X,*),d_{pb})$ is a NTpMS.

Theorem 3.5: Let $(((X,\ Y),\ *),\ d_{pb})$ be a NTpbMS. If (X∩Y, ∗) is a NTS, then $(((X \cap Y,\ X \cap Y),\ *),d_{pb})$ is a NTpbMS**.**

Proof: We suppose that (X∩Y, ∗) is a NTS.

i) Since $(((X,Y),*),d_{pb})$ is a NTpbMS, for all $a,b \in X$, $a * b \in X$ and for all $c,d \in Y$, $c * d \in Y$. Thus, it is clear that $\forall a,c \in X \cap Y,\ a * c \in X \cap Y$.

ii) Since $(((X,Y),*),d_{pb})$ is a NTpbMS, for all $a \in X$ and for all $c \in Y$, if $d_{pb}(a,c) \geq d_{pb}(a,a) \geq 0$, then $d_{pb}(a,c) \geq d_{pb}(c,c) \geq 0$.

Thus, it is clear that for all $a \in X \cap Y,\ c \in X \cap Y$;

$d_{pb}(a,c) \geq d_{pb}(a,a) \geq 0$ and $d_{pb}(a,c) \geq d_{pb}(c,c) \geq 0$.

iii) Since $(((X,Y),*),d_{pb})$ is a NTpbMS, if $d_{pb}\ (a,b)$ = $d_{pb}(a,a)$ = $d_{pb}(b,b) = 0$, then there exists at least one pair of elements $a,b \in X \cap Y$ such that $d_{pb}\ (\ a,b\ )\ =\ 0$. Thus, it is clear that for $d_{pb}\ (a,b)$ = $d_{pb}(a,a)$ = $d_{pb}(b,b) = 0$, there exists at least one pair of $a,b \in (X \cap Y) \times (X \cap Y)$.

iv) Since $(((X,Y),*),d_{pb})$ is a NTpbMS, we have for all $a,b \in X \cap Y$, $d_p b(a,b) = d_{pb}(b,a)$. Thus, it is clear that for all $a,b \in (X \cap Y) \cap (X \cap Y) = X \cap Y$, $d_{pb}(a,b) = d_{pb}(b,a)$.

v) Now, let $(a,b),(a',b') \in X \times Y$. For each (a,b), if there exists at least one (a',b') such that

$d_{pb}(a,b) \le d_{pb}(a, b * neut(b')) \le d_{pb}(a * neut(a'), b * neut(b'))$ and

$d_{pb}(a,b) \le d_{pb}(a * neut(a'), b) \le d_{pb}(a * neut(a'), b * neut(b'))$,

then

$d_{pb}(a * neut(a'), b * neut(b')) \le d_{pb}(a,b') + d_{pb}(a',b') + d_{pb}(a',b) - \min\{d_{pb}(a',a'), d_{pb}(b',b')\}$.

Thus, it is clear that for each $(a,b) \in (X \cap Y) \times (X \cap Y)$, if there exists at least one $(a',b') \in (X \cap Y) \times (X \cap Y)$ such that

$d_{pb}(a,b) \le d_{pb}(a, b * neut(b'))$,

$d_{pb}(a,b) \le d_{pb}(a * neut(a'), b)$ and

$d_{pb}(a,b) \le d_{pb}(a * neut(a'), b * neut(b'))$, then

$d_{pb}(a * neut(a'), b * neut(b')) \le d_{pb}(a,b') + d_{pb}(a',b') + d_{pb}(a',b) - \min\{d_{pb}(a',a'), d_{pb}(b',b')\}$.

Thus, $(((X \cap Y, X \cap Y), *), d_{pb})$ is an NTpbMS**.**

Theorem 3.6: Let $(((X,Y),*),d)$ a NTbMS. Then, for $k \in \mathbb{R}^+$, $d_{kp}(x,y) = d(x,y) + k$ is a NTpbM.

Proof:

i) For all $, b \in X$, $a * b \in X$ and for all $c,d \in Y$,$c * d \in Y$ since d is a neutrosophic triplet bipolar metric.

ii) For all $x \in X$ and for all $y \in Y$,

$d_{kp}(x,y) \ge d(x,x) \ge 0$ and

$d_{kp}(x,y) \ge d(y,y) \ge 0$.

Because;

$d_{kp}(x,y) = d(x,y) + k$,

$d_{kp}(x,x) = d(x,x) + k$,

$d_{kp}(y,y) \ge d(y,y) + k$,

$d(x,x) = 0$ and $d(y,y) = 0$ for all $x,y \in X \cap Y$.

iii) If $d_{kp}(x,y) = d_{kp}(x,x) = d_{kp}(y,y) \ne 0$ the proof is straightforward, since

$d_{kp}(x,x) = d(x,x) + k > 0$ $(d(x,x) = 0)$

$d_{kp}(y,y) = d(y,y) + k > 0$ $(d(y,y) = 0)$.

iv) For all $x,y \in X \cap Y$, $d_{kp}(x,y) = d_{kp}(y,x)$.

This is because of the fact that $d_{kp}(x,y) = d(x,y) + k$ and $d(x,y) = d(y,x)$ for $\forall x,y \in X \cap Y$.

v) Let $\forall (x,y),(x',y') \in X \times Y$. For each $(x,y) \in X \times Y$, If there exists at least one $(x',y') \in X \times Y$ such that

$d_{kp}(x,y) \le d_{kp}(x, y * neut(y'))$,

$d_{kp}(x,y) \le d_{kp}(x * neut(x'), y)$,

$d_{kp}(x,y) \le d_{kp}(x * neut(x'), y * neut(y'))$,

then

$d_{kp}(x,y) \le d_{kp}\big(x * neut(x'), y * neut(y')\big) \le$
$d_{kp}(x,y') + d_{kp}(x',y') + d_{kp}(x',y) - \min\{d_{kp}(x',x'), d_{kp}(y',y')\}$.

As $d_{kp}(x,y) = d(x,y) + k$ and $d(x * neut(x'), y * neut(y')) \le d(x,y') + d(x',y') + d(x',y)$, we obtain that

$d_{kp}\big(x * neut(x'), y * neut(y')\big) \le$
$d(x,y') + k + d(x',y') + k + d(x',y) + k - \min\{d_{kp}(x',x'), d_{kp}(y',y')\} =$
$d(x,y') + d(x',y') + d(x',y) + 3k - \min\{d_{kp}(x',x'), d_{kp}(y',y')\} =$
$d(x,y') + d(x',y') + d(x',y) + 3k - \min\{d(x',x') + k, d(y',y') + k\} =$
$d(x,y') + d(x',y') + d(x',y) + 3k - \min\{k,k\} =$
$d(x,y') + d(x',y') + d(x',y) + 3k - k =$
$d(x,y') + d(x',y') + d(x',y) + 2k$.

In this case,

$d_{kp}\big(x * neut(x'), y * neut(y')\big) \le d_{kp}(x,y') + d_{kp}(x',y') + d_{kp}(x',y) - \min\{d_{kp}(x',x'), d_{kp}(y',y')\}$.

Corollary 3.7: A NTpbMS can be obtained from a NTbMS.

Definition 3.8: Let $\big(((X,Y),*), d_{pb}\big)$ be a NTpbMS. A left sequence (x_n) converges to a right point y (symbolically $(x_n) \rightharpoonup y$ or $\lim_{n\to\infty}(x_n) = y$) if and only if for every ε > 0 there exists an $n_0 \in \mathbb{N}$, such that $d_{pb}(x_n, y) < \varepsilon$ -min $\{d_{pb}(x,x), d_{pb}(y,y)\}$for all $n \ge n_0$. Similarly, a right sequence (y_n) converges to a left point x (denoted as $y_n \rightharpoonup x$ or $\lim_{n\to\infty}(y_n) = x$) if and only if, for every ε > 0 there exists an $n_0 \in \mathbb{N}$ such that, whenever $n \ge n_0$, $d_{pb}(x, y_n) < \varepsilon$ - min $\{d_{pb}(x,x), d_{pb}(y,y)\}$. Also, if $(u_n) \rightharpoonup u$ and $(u_n) \rightharpoonup u$, then (u_n) converges to point u ((u_n) is a central sequence).

Definition 3.9: Let $(((X,\ Y),\ *), d_{pb})$ be a NTpbMS, (x_n) be a left sequence and (y_n) be a right sequence in this space. (x_n, y_n) is called a NT partial bisequence. Furthermore, if (x_n) and (y_n) are convergent, then (x_n, y_n) is called a NT partial convergent bisequence. Also, if (x_n) and (y_n) converge to same point, then (x_n, y_n) is called a NT partial biconvergent bisequence.

Definition 3.10: Let $(((X,\ Y),\ *), d_{pb})$ be a NTpbMS and (x_n, y_n) be a NT partial bisequence. (x_n, y_n) is called an NT partial Cauchy bisequence if and only if for every ε > 0, there exists an $n_0 \in \mathbb{N}$, such that $d_{pb}(x_n, y_m) < \varepsilon$ -min $\{d_{pb}(x,x), d_{pb}(y,y)\}$ for all $n, m \ge n_0$.

Definition 3.11: Let $\big(((X,Y),*), d_{pb}\big)$ be a NTpbMS. In this space, if each (x_n, y_n) NT partial Cauchy bisequence is a NT partial convergent Cauchy bisequence, then $\big(((X,Y),*), d_{pb}\big)$ is called complete NT partial bipolar metric space.

Conclusion

In this study we first obtained NTpbMS. We show that NTpbMS is different from NTpMS and NTMS. Also, we show that a NTpbMS will provide the properties of a NTbMS under which conditions are met. Thus, we added a new structure to neutrosophic triple structures. Also, thanks to this study, researchers can obtain new fixed point theories, neutrosophic triplet partial bipolar normed space, neutrosophic triplet partial bipolar inner product space.

Abbreviations

bM: bipolar metric

bMS: bipolar metric space

pMS: partial metric space

NT: Neutrosophic triplet

NTS: Neutrosophic triplet set

NTM: Neutrosophic triplet metric

NTMS: Neutrosophic triplet metric space

NTpM: Neutrosophic triplet partial metric

NTpMS: Neutrosophic triplet partial metric space

NTbM: Neutrosophic triplet bipolar metric

NTbMS: Neutrosophic triplet bipolar metric space

NTpbMS: Neutrosophic triplet partial bipolar metric space

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The Neutrosophic Triplet of BI-algebras

Akbar Rezaei [1]* and Florentin Smarandache [2]

1 Akbar Rezaei, Department of Mathematics, Payame Noor University, P.O..Box. 19395-3697, Tehran, Iran. E-mail: rezaei@pnu.ac.ir
2 Florentin Smarandache, Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA. E-mail: smarand@unm.edu

* Correspondence: rezaei@pnu.ac.ir; Tel:+989131405381, Fax: +983433353624

Abstract: In this paper, the concepts of a Neutro-BI-algebra and Anti-BI-algebra are introduced, and some related properties are investigated. We show that the class of Neutro-BI-algebra is an alternative of the class of BI-algebras.

Keywords: BI-algebra; Neutro-BI-algebra; sub-Neutro-BI-algebra; Anti-BI-algebra; sub-Anti-BI-algebra; Neutrosophic Triplet of BI-algebra.

1. Introduction

1.1. BI-algebras

In 2017, A. Borumand Saeid et al. introduced BI-algebras as an extension of both a (dual) implication algebras and an implicative BCK-algebra, and they investigated some ideals and congruence relations [1]. They showed that every implicative BCK-algebra is a BI-algebra, but the converse is not valid in general. Recently, A. Rezaei et al. introduced the concept of a (branchwise) commutative BI-algebra and showed that commutative BI-algebras form a class of lower semilattices and showed that every commutative BI-algebra is a commutative BH-algebra [2].

1.2 Neutrosophy

Neutrosophy is a new branch of philosophy that generalized the dialectics and took into consideration not only the dynamics of opposites, but the dynamics of opposites and their neutrals introduced by Smarandache in 1998 [5]. Neutrosophic Logic / Set / Probability / Statistics etc. are all based on it.

One of the most striking trends in the neutrosophic theory is the hybridization of neutrosophic set with other potential sets such as rough set, bipolar set, soft set, vague set, etc. The different hybrid structures such as rough neutrosophic set, single valued neutrosophic rough set, bipolar neutrosophic set, single valued neutrosophic vague set, etc. are proposed in the literature in a short period of time. Neutrosophic set has been a very important tool in all various areas of data mining, decision making, e-learning, engineering, computer science, graph theory, medical diagnosis, probability theory, topology, social science, etc.

1.3 NeutroLaw, NeutroOperation, NeutroAxiom, and NeutroAlgebra

In this section, we review the basic definitions and some elementary aspects that are necessary for this paper.

The Neutrosophy's Triplet is (<A>, <neutroA>, <antiA>), where <A> may be an item (concept, idea, proposition, theory, structure, algebra, etc.), <antiA> the opposite of <A>, while <neutroA> {also the notation <neutA> was employed before} the neutral between these opposites.

Based on the above triplet the following Neutrosophic Principle one has: a law of composition defined on a given set may be true (T) for some set's elements, indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroLaw.

A law of composition defined on a given sets, such that the law is false (F) for set's elements is called AntiLaw.

Similarly, an operation defined on a given set may be well-defined for some set's elements, indeterminate for other set's elements, and outer-defined for the remainder of the set's elements; we call it NeutroOperation.

While, an operation defined on a given set that is outer-defined for all set's elements is called AntiOperation.

In classical algebraic structures, the laws of compositions or operations defined on a given set are automatically well-defined [i.e. true (T) for all set's elements], but this is idealistic.

Consequently, an axiom (let's say Commutativity, or Associativity, etc.) defined on a given set, may be true (T) for some set's elements, indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroAxiom.

In classical algebraic structures, similarly an axiom defined on a given set is automatically true (T) for all set's elements, but this is idealistic too.

A NeutroAlgebra is a set endowed with some NeutroLaw (NeutroOperation) or some NeutroAxiom.

The NeutroLaw, NeutroOperation, NeutroAxiom, NeutroAlgebra and respectively AntiLaw, AntiOperation, AntiAxiom and AntiAlgebra were introduced by Smarandache in 2019 [4] and afterwards he recalled, improved and extended them in 2020 [5].

2. **Neutro-BI-algebras, Anti-BI-Algebras**

In this section, we apply Neutrosophic theory to generalize the concept of a BI-algebra. Some new concepts as, Neutro-sub- BI -algebra, Anti-sub- BI -algebra, Neutro- BI -algebra, sub-Neutro- BI -algebra, NutroLow-sub-Neutro- BI -algebra, AntiLow-sub-Neutro- BI -algebra, Anti- BI -algebra, sub-Anti- BI -algebra, NeutroLow-sub-Anti- BI -algebra and AntiLow-sub-Anti-BI-algebra are proposed.

Definition 2.1. (Definition of classical BI-algebras [1])

An algebra $(X,*,0)$ of type $(2,0)$ (i.e. X is a nonempty set, $*$ is a binary operation and 0 is a constant element of X) is said to be a *BI-algebra* if it satisfies the following axioms:

(B) $(\forall x \in X)(x * x = 0)$,

(BI) $(\forall x, y \in X)(x * (y * x) = x)$.

Example 2.2. ([1])

(i) Let X be a set with $0 \in X$. Define a binary operation $*$ on X By

$$x * y = \begin{cases} 0 & if\ x = y; \\ x & if\ x \neq y. \end{cases}$$

Then $(X,*,0)$ is a BI-algebra.

(ii) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Then $(\mathcal{P}(S), -, \emptyset)$ is a BI-algebra. Since $A - A = \emptyset$ and for every $A \in \mathcal{P}(S)$. Also, $A - (B - A) = A \cap (B \cap A^c)^c = A \cap (B^c \cup A^{cc}) = A$, for every $A, B \in \mathcal{P}(S)$. Thus, (B) and (BI) hold.

Definition 2.3. (Definition of classical sub-BI-algebras)

Let $(X,*,0)$ be a BI-algebra. A nonempty set S of X is said to be a *sub-BI-algebra* of X if $(\forall x,y \in S)(x * y \in S)$.

We note that X and $\{0\}$ are sub-BI-algebra.

Example 2.4. Let $X := \{0,a,b,c\}$ be a set with the following table.

Table 1

*	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then $(X,*,0)$ is a BI-algebra. We can see that $S = \{0,a,b\}$ is a sub-algebra of X, $T = \{0,a,c\}$ is not a sub-algebra, since, $a,c \in T$, but $c * a = b \notin T$.

Definition 2.5. (Definition of Neutro-sub-BI-algebras)

Let $(X,*,0)$ be a BI-Algebra. A nonempty set NS of X is said to be a *Neutro-sub-BI-algebra* of X if $(\exists x,y \in NS)(x * y \in NS)$and $(\exists x,y \in NS)$ such that $x * y \notin NS$ or $x * y$ = indeterminate.

We note that X and $\{0\}$ are not Neutro-sub-BI-algebras. Since $*$ is a binary operation, and so $x * y \in X$, for all $x,y \in X$. Also, there are no $x,y \in \{0\}$ such that $x * y \notin \{0\}$.

Example 2.6. Consider the BI-algebra $(X,*,0)$ given in Example 2.4. $S = \{0,a,c\}$ is a Neutro-sub-BI-algebra, since $0 * a = 0 \in S$, $a * 0 = a \in S$ and $c * 0 = c \in S$, but $c * a = b \notin S$.

Definition 2.7. (Definition of Anti-sub-BI-algebras)

Let $(X,*,0)$ be a BI-algebra. A nonempty set AS of X is said to be an *Anti-sub-BI-algebra of X* if $(\forall x,y \in AS)(x * y \notin AS)$.

We note that X and $\{0\}$ are not Anti-sub-BI-algebra. Since $*$ is a binary operation, and so $x * y \in X$, for all $x,y \in X$. Also, $(\forall x,y \in \{0\})(x * y \in \{0\})$.

Example 2.8. Consider the BI-algebra $(X,*,0)$ given in Example 2.4. $S = \{c\}$ is an Anti-sub-BI-algebra, since $c * c = 0 \notin S$.

In classical algebraic structures, a Law (Operation) defined on a given set is automatically well-defined (i.e. true for all set's elements), but this is idealistic; in reality we have many more cases where the law (or operation) are not true for all set's elements. In NeutroAlgebra, a law (operation) may be well-defined (T) for some set's elements, indeterminate (I) for other set's elements, and outer-defined (F) for the other set's elements. We call it NeutroLaw (NeutroOperation).

In classical algebraic structures, an Axiom defined on a given set is automatically true for all set's elements, but this is idealistic too. In NeutroAlgebra, an axiom may be true for some of the set's elements, indeterminate (I) for other set's elements, and false (F) for other set's elements.

We call it NeutroAxiom.

A NeutroAlgebra is a set endowed with some NeutroLaw (NeutroOperation) or NeutroAxiom. NeutroAlgebra better reflects our imperfect, partial, indeterminate reality.

There are several NeutroAxioms that can be defined on a BI-algebra. We neutrosophically convert its first two classical axioms: (B) into (NB), and (BI) into (NBI). Afterwards, the classical axiom (BI) is completed negated in two different ways $(ABI1)$ and $(ABI2)$ respectively.

- (NB) $(\exists x \in NX)(x *_N x = 0)$ and $(\exists x \in NX)(x *_N x \neq 0)$,
- (NBI) $(\exists x, y \in NX)(x *_N (y *_N x) = x)$ and $(\exists x, y \in NX)(x *_N (y *_N x) \neq x)$,
- $(ABI1)$ $(\forall x \in NX, \exists y \in NX)(x *_N (y *_N x) \neq x)$,
- $(ABI2)$ $(\exists x \in NX, \forall y \in NX)(x *_N (y *_N x) \neq x)$.

In this paper we consider the following:

Definition 2.9. (Definition of Neutro-BI-algebras)

An algebra $(NX, *_N, 0_N)$ of type $(2, 0)$ (i.e. NX is a nonempty set, $*_N$ is a binary operation and 0_N is a constant element of X) is said to be a *Neutro-BI-algebra* if it satisfies the following NeutroAxioms:

(NB) $(\exists x \in NX)(x *_N x = 0_N)$ and $(\exists x \in NX)(x *_N x \neq 0_N$ or indeterminate$)$,

(NBI) $(\exists x, y \in NX)(x *_N (y *_N x) = x)$ and $(\exists x, y \in NX)(x *_N (y *_N x) \neq x$ or indeterminate$)$.

Example 2.10.

(i) Let $NX := \{0_N, a, b, c\}$ be a set with the following table.

Table 2

$*_N$	0_N	a	b	c
0_N	0_N	0_N	0_N	0_N
a	a	0_N	a	b
b	b	b	a	b
c	c	b	b	0_N

Then $(NX, *_N, 0_N)$ is a Neutro-BI-algebra. Since $a *_N a = 0_N$ and $b *_N b = a \neq 0_N$. Also, $a *_N (b *_N a) = a *_N b = a$ and $c *_N (b *_N c) = c *_N b = b \neq c$.

(ii) Let $\mathbb{R}$ be the set of real numbers. Define a binary operation $*_N$ on $\mathbb{R}$ by $x *_N y = x + y + 1$. Then $(\mathbb{R}, *_N, 0)$ is a Neutro-BI-algebra. Since if $x = 0$, then $0 *_N 0 = 0 + 0 + 1 = 1 \neq 0$, and if $x = -0.5$, then $x *_N x = x + x + 1 = 2x + 1 = -1 + 1 = 0$, so (NB) holds. For (NBI), let $x \in \mathbb{R}$. If $y = -x - 2$, then $x *_N (y *_N x) = x$, and if $y \neq -x - 2$, then $x *_N (y *_N x) \neq x$.

(iii) Consider the BI-algebra given in Example 2.2 (ii), it is not a Neutro-BI-algebra. Since (NB) and (NBI) are not valid.

(iv) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Then $(\mathcal{P}(S), \cap, \emptyset)$ is a Neutro-BI-algebra. Since $\emptyset \cap \emptyset = \emptyset$, and for every $A \neq \emptyset$, $A \cap A = A \neq \emptyset$. Further, if $A \subseteq B$, then $A \cap (B \cap A) = A \cap A = A$. Also, since $A, A^c \in \mathcal{P}(S)$, we get $A \cap (A^c \cap A) = A \cap \emptyset = \emptyset \neq A$. Thus, (NB) and (NBI) hold. Moreover, by a similar argument $(\mathcal{P}(S), \cup, \emptyset)$, is not a BI-algebra, but is a Neutro-BI-algebra.

(v) Similarly, $(\mathcal{P}(S), \cap, S)$ and $(\mathcal{P}(S), \cup, S)$ are Neutro-BI-algebras.

(vi) Let $\mathbb{R}$ be the set of real numbers. Define a binary operation $*_N$ on $\mathbb{R}$ by $x *_N y = x^2 - y$. Then $(\mathbb{R}, *_N, 0)$ is not a BI-algebra. Since $3 *_N 3 = 3^2 - 3 = 6 \neq 0$, so (B) is not valid. If $x \in \{0,1\}$, then $x *_N x = 0$. If $x \notin \{0,1\}$, $x *_N x \neq 0$. Hence (NB) holds. If $x \in \{-y, y\}$, then $x *_N (y *_N x) = x$. If $x \notin \{-y, y\}$, then $x *_N (y *_N x) \neq x$. Thus, (NBI) is valid. Therefore, $(\mathbb{R}, *_N, 0)$ is a Neutro-BI-algebra.

(vii) Let $\mathbb{R}$ be the set of real numbers. Define a binary operation $*_N$ on $\mathbb{R}$ by $x *_N y = x^3 - y$. Then $(\mathbb{R}, *_N, 0)$ is not a BI-algebra. Since $3 *_N 3 = 3^3 - 3 = 24 \neq 0$, so (B) is not valid. If $x \in \{-1,0,1\}$, then $x *_N x = 0$. If $x \notin \{-1,0,1\}$, $x *_N x \neq 0$. Hence (NB) holds. If $x = y$, then $x *_N (y *_N x) = x$. If $x \neq y$, then $x *_N (y *_N x) \neq x$. Thus, (NBI) is valid. Therefore, $(\mathbb{R}, *_N, 0)$ is a Neutro-BI-algebra.

Definition 2.11. (Definition of sub-Neutro-BI-algebras)

Let $(NX, *_N, 0)$ be a Neutro-BI-algebra. A nonempty set NS of NX is said to be a *sub-Neutro-BI-algebra* of NX if $(\forall x, y \in NS)(x *_N y \in NS)$ and NS is itself a Neutro-BI-algebras.

Note that NX is a sub-Neutro-BI-algebra, because $*_N$ is a binary operation, and so it is close. $\{0_N\}$ is not a sub-Neutro-BI-algebra, since it is not a Neutro-BI-algebra because $0_N = 0_N *_N 0_N \in \{0_N\}$.

Example 2.12. Consider the Neutro-BI-algebra $(NX, *_N, 0_N)$ given in Example 2.10 (i). $NS = \{0_N, a, b\}$ is a sub-Neutro-BI-algebra of NX, but $NT = \{0_N, b, c\}$ is not a sub-Neutro-BI-algebra, since $b \in NT$, $b *_N b = a \notin NT$.

Definition 2.13. (Definition of NeutroLaw-sub-Neutro-BI-algebras)

Let $(NX, *_N, 0_N)$ be a Neutro-BI-algebra. A nonempty set NS of NX is said to be a *NeutroLaw-sub-Neutro-BI-algebra* of NX if $(\exists x, y \in NS)(x *_N y \in NS)$and $(\exists x, y \in NS)(x *_N y \notin NS)$.

{As a parenthesis, we recall that NS had to be itself a Neutro-BI-algebra, and this could occur by NS satisfying one or more of the following: the (NB) NeutroAxiom, the (NBI) NeutroAxiom, or the NeutroLaw. We chose, as a particular definition, the NeutroLaw.}

We note that neither NX nor $\{0\}$ are NeutroLaw-sub-Neutro-algebra.

Example 2.14. From Example 2.12, $NT = \{0_N, b, c\}$ is a NeutroLaw-sub-Neutro-BI-algebra. Since $b *_N c = b \in NT$ and $b *_N b = a \notin NT$.

Definition 2.15. (Definition of AntiLaw-sub-Neutro-BI-algebras)

Let $(NX, *_N, 0_N)$ be a Neutro-BI-algebra. A nonempty set AS of NX is said to be an *AntiLaw-sub-Neutro-BI-algebra* of X if $(\forall x, y \in AS)(x *_N y \notin AS)$.

{Similarly, as a parenthesis, we recall that AS had to be itself an Anti-BI-algebra, and this could occur by AS satisfying one or more of the following: the (AB) AntiAxiom, the (NBI) AntiAxiom, or the AntiLaw. We chose, as a particular definition, the AntiLaw.}

In this case NX is not an AntiLaw-sub-Neutro-BI-algebra, but $\{0_N\}$ may or may not be an AntiLaw-sub-Neutro-algebra. If $0_N *_N 0_N \in \{0_N\}$, then it is not an AntiLaw-sub-Neutro-algebra. If $0_N *_N 0_N \notin \{0_N\}$, then it is.

Example 2.16. Let $NX:=\{0_N, a, b, c\}$ be a set with the following table.

Table 3

$*_N$	0_N	a	b	c
0_N	0_N	0_N	0_N	0_N
a	a	0_N	a	b
b	b	b	a	a
c	c	b	a	a

Then $(NX,*_N, 0_N)$ is a Neutro-BI-algebra. $AS=\{b,c\}$ is an AntiLaw-sub-Neutro-BI-algebra, because $b *_N b = b *_N c = c *_N b = c *_N c = a \notin AS$.

Definition 2.17. (Definition of Anti-BI-algebras)

An algebra $(AX,*_A,0_A)$ of type $(2,0)$ (i.e. AX is a nonempty set, $*_A$ is a binary operation and 0_A is a constant element of AX) is said to be an *Anti-BI-algebra* if it satisfies the following AntiAxioms,

(AB) $(\forall x \in AX)(x *_A x \neq 0_A)$,

(ABI) $(\forall x,y \in AX)(x *_A (y *_A x) \neq x)$.

Example 2.18.

(i) Let $\mathbb{N}$ be the natural number and $AX:=\mathbb{N} \cup \{0\}$. Define a binary operation $*$ on AX by $x *_A y = x+y+1$. Then $(AX,*_A,0)$ is an Anti-BI-algebra. Since $x *_A x = x+y+1 \neq 0$, for all $x \in AX$, and $x *_A (y *_A x) = x *_A (y+x+1) = x+(x+y+1)+1 = 2x+y+2 \neq 0$, for all $x,y \in AX$.

(ii) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Define the binary operation Δ (i.e. symmetric difference) by $A\Delta B = (A\cup B)-(A\cap B)$ for every $A,B \in \mathcal{P}(S)$. Then $(\mathcal{P}(S),\Delta,S)$ is not a BI-algebra neither Neutro-BI-algebra nor Anti-BI-algebra. Since $A\Delta A = \emptyset \neq S$ for every $A \in \mathcal{P}(S)$ we get (AB) hold, and so (B) and (NB) are not valid. Also, for every $A,B \in \mathcal{P}(S)-\{\emptyset\}$, we have $A\Delta(B\Delta A) = B \neq A$, and since $\emptyset \in \mathcal{P}(S)$, we get $\emptyset\Delta(\emptyset\Delta\emptyset)=\emptyset$. Thus, (ABI) is not valid.

(iii) Similarly, $(\mathcal{P}(S),\Delta,\emptyset)$ is not a BI-algebra neither Neutro-BI-algebra nor Anti-BI-algebra.

(iv) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S. Define the binary operation ∇ as $A\nabla B = (A\cup B)\cup C$, for every $A,B \in \mathcal{P}(S)$, where C is a given set of $P(S)$ and $C \notin \{\emptyset, A, B\}$. Then $(\mathcal{P}(S)-\{S\},\nabla,\emptyset)$ is an Anti-BI-algebra. Since $A\nabla A = (A\cup A)\cup C = A\cup C$, which can never be equal to $\emptyset$ since $C \neq \emptyset$. Hence (AB) holds. Also, $A\nabla(B\nabla A) \neq A$ and so (ABI) holds.

(v) Let $\mathbb{R}$ be the set of real numbers. Define a binary operation $*_A$ on $\mathbb{R}$ by $x *_A y = x^2+1$. Then $(\mathbb{R},*_A,0)$ is not a BI-algebra. Since $3 *_A 3 = 3^2+1 = 10 \neq 0$, so (B) is not valid. Let $x,y \in \mathbb{R}$, then $x *_A x = x^2+1 \neq 0$ and $x *_A (y *_A x) = x *_A (y^2+1) = x^2+1 \neq 0$. Thus, $(\mathbb{R},*_A,0)$ is an Anti-BI-algebra.

(vi) Let $\mathbb{R}$ be the set of real numbers. Define a binary operation $*_A$ on $\mathbb{R}$ by $x *_A y = x^2+1$. Then $(\mathbb{R},*_A,0)$ is not a BI-algebra. Since $3 *_A 3 = 3^2+1 = 10 \neq 0$, so (B) is not valid. Let $x,y \in \mathbb{R}$, then $x *_A x = x^2+1 \neq 0$, thus one has (AB), and $x *_A (y *_A x) = x *_A (y^2+1) = x^2+1 \neq 0$, or one has (ABI). Therefore, $(\mathbb{R},*_A,0)$ is an Anti-BI-algebra.

Definition 2.19. (Definition of sub-Anti-BI-algebras)

Let $(AX,*_A,0_A)$ be an Anti-BI-algebra. A nonempty set AS of AX is said to be a *sub-Anti-BI-algebra* of X if $(\forall x,y \in AS)(x *_A y \in AS)$.

We note that AX is a sub-Anti-BI-algebra, but $\{0_A\}$ is not a sub-Anti-BI-algebra, since

$0_A *_A 0_A \notin \{0_A\}$.

Example 2.20. Consider the Anti-BI-algebra $(AX,*_A,0)$ given in Example 2.18 (i). $\mathbb{N}$ is a sub-Anti-BI-algebra of AX. Since $x *_A y = x + y + 1 \in \mathbb{N}$, for all $x, y \in \mathbb{N}$.

Definition 2.21. (Definition of NeutroLaw-sub-Anti-BI-algebras)

Let $(AX,*_A,0_A)$ be an Anti-BI-algebra. A nonempty set AS of AX is said to be a *NeutroLaw-sub-Anti-BI-algebra* of X if $(\exists x, y \in AS)(x *_A y \in AS)$ and $(\exists x, y \in AS)(x *_A y \notin AS)$.

In this case AX and $\{0_A\}$ are not NeutroLaw-sub-Anti-BI-algebras. Since $\nexists x, y \in AX$ such that $x *_A y \notin AX$, and similarly for $\{0_A\}$.

Example 2.22. Let $AX := \{0_A, a, b, c\}$ be a set with the following table.

Table 4

$*_A$	0_A	a	b	c
0_A	b	a	c	a
a	a	c	b	b
b	b	c	a	a
c	c	b	a	a

Then $(AX,*_A,0_A)$ is an Anti-BI-algebra. $NS = \{a, b\}$ is a NeutroLaw-sub-Anti-BI-algebra, since $a *_A b = b \in NS$ and $b *_A a = c \notin NS$.

Definition 2.23. (Definition of AntiLaw-sub-Anti-BI-algebras)

Let $(AX,*_A,0)$ be an Anti-BI-algebra. A nonempty set AS of AX is said to be an *AntiLaw-sub-Anti-BI-algebra* of X if $(\forall x, y \in AS)(x *_A y \notin AS)$.

In this case AX is not an AntiLaw-sub-Anti-BI-algebra, but $\{0_A\}$ may or may not be an AntiLaw-sub-Anti-BI-algebra. If $0_A *_A 0_A \in \{0_A\}$, then it is not an AntiLaw-sub-Anti-algebra. If $0_A *_A 0_A \notin \{0_A\}$, then it is.

Example 2.24. Consider the Anti-BI-algebra $(AX,*_A *,0_A)$ given in Example 2.22. $AS = \{0_A\}$ is an AntiLaw-sub-Anti-BI-algebra of AX, since $0_A *_A 0_A = b \notin AS$.

Note. It is obvious that the concepts of BI-algebra and Anti-BI-algebra are different. In the following example we show that the concept of Neutro-BI-algebra is different from the concepts of BI-algebra and Anti-BI-algebra.

Example 2.25. Let $X = \mathbb{R} - \{0\}$, endowed with the real division $\div$ of numbers. $(X,\div)$ is well defined, since there is no division by zero. Put $x := 3$ and $y := 2$, we obtain $2 \div (3 \div 2) = \frac{4}{3} \neq 2$, and so (BI) is not valid. Then $(X,\div,-1)$ is not a BI-algebra, but it is a Neutro-BI-algebra, since if $x = y := \pm 1$, then $x \div y = (\pm 1) \div (\pm 1) = 1 \neq -1$. If $x := 3$ and $y := -3$, then $x \div y = 3 \div (-3) = -1$, and so (NB) holds. For (NBI), again $x = y := -1$, we get $(-1) \div \big((-1) \div (-1)\big) = -1$, and if $x := 4$ and $y := 7$, we have $4 \div (7 \div 4) = \frac{16}{7} \neq 4$, so (NBI) holds. Also, we can see that $(X,\div,-1)$ is not an Anti-BI-algebra, since (AB) and (ABI) are not valid.

3. The Neutrosophic Triplet of BI-algebra

In 2020, F. Smarandache defined a novel definition of Neutrosophic Triplet of (*Algebra*, *NeutroAlgebra*, *AntiAlgebra*) [4]. In this section we give a particular example, when the Algebra is replaced by a BI-algebra, and we get (BI-*algebra*, *Neutro*-BI-*algebra*, *Anti*-BI-*algebra*) as below.

Definition 3.1. Let $\mathcal{U}$ be a nonempty universe of discourse, and X, NX and AX be nonempty sets of $\mathcal{U}$, and an operation $*$ defined on the set X, and the same operation restrained to the set NX (denoted as $*_N$) and to the set AX (denoted as $*_A$) respectively. A triplet (X, NX, AX) endowed with a triplet of binary operations $(*, *_N, *_A)$ and a triplet of constants $(0, 0_N, 0_A)$ is said to be The ***Neutrosophic Triplet of BI-algebra*** for briefly ***NT-BI-algebra*** if it satisfies the following **Axioms** $\{(B), (BI)\}$, **NeutroAxioms** $\{(NB), (NBI)\}$, or **AntiAxioms** $\{(AB), (ABI)\}$ respectively:

(B) $(\forall x \in X)(x * x = 0)$,

(BI) $(\forall x, y \in X)(x * (y * x) = x)$,

(NB) $(\exists x \in NX)(x *_N x = 0_N)$ and $(\exists x \in NX)(x *_N x \neq 0_N$ or is $indeterminate)$,

(NBI) $(\exists x, y \in NX)(x *_N (y *_N x) = x)$ and

$(\exists x, y \in NX)(x *_N (y *_N x) \neq x$ or is $indeterminate)$,

(AB) $(\forall x \in AX)(x *_A x \neq 0_A)$,

(ABI) $(\forall x, y \in AX)(x *_A (y *_A x) \neq x)$.

Definition 3.2. A triplet $((S, *, 0), (NS, *_N, 0_N), (AS, *_A, *_A))$, where $S \subseteq X$, $NS \subseteq NX$ and $AS \subseteq AX$ is said to be a *sub-NT-BI-algebra* of NT-BI-algebra $((X, *, 0), (NX, *_N, 0_N), (AX, *_A, *_A))$ if:

(i) $(S, *, 0)$ is a sub-BI-algebra of $(X, *, 0)$,

(ii) $(NS, *_N, 0_N)$ is a sub-Neutro-BI-algebra of $(NX, *_N, 0_N)$,

(iii) $(NS, *_A, 0_A)$ is an sub-Anti-BI-algebra of $(AX, *_A, 0_A)$.

4. Conclusions

In this paper, we introduced the notions of new types of sub-BI-algebras. Also, Neutro-BI-algebras, sub-Neutro-BI-algebras, NeutroLow-sub-Neutro-BI-algebras, AntiLow-sub-Neutro-BI-algebras, Anti-BI-algebras, sub-Anti-BI-algebras, NeutroLow-sub-Anti-BI-algebras, AntiLow-sub-Anti-BI-algebras are studied and by several examples showed that the notions are different. Finally, the concept of a Neutrosophic Triplet of BI-algebra is defined. For future work we would define some types of NeutroFilters, NeutroIdeals, AntiFilters, AntiIdeals in the Neutrosophic Triplet of BI-algebras.

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A Novel Plithogenic MCDM Framework for Evaluating the Performance of IoT Based Supply Chain

Mohamed Grida*[1], Rehab Mohamed[2] and Abdel Nasser H. Zaied[3]

[1]Faculty of Engineering, Zagazig University; Zagazig, Sharkia, Egypt, Email: mogrida@zu.edu.eg
[2]Faculty of Computers and Informatics, Zagazig University; Sharkia, Egypt, Email: rehab_argawy@zu.edu.eg
[3]Faculty of Computers and Informatics, Zagazig University; Sharkia, Egypt, Email: nasserhr@zu.edu.eg
*Correspondence: mogrida@zu.edu.eg; Tel.:+201009996160

Abstract: The Internet of Things (IoT) is used in the Supply chain management (SCM) systems to respond to the globalization of complex and dynamic markets and competitiveness in various supply chain scopes. Despite the current buzz about IoT and its role in the supply chain, there is not enough empirical data or extensive expertise to guide its implementation. Therefore, this paper addresses the ambiguity of assessing the performance of the IoT based supply chain by integrating plithogenic set with both Best-Worst (BWM) and Vlse Kriterijumska Optimizacija Kompromisno Resenje (VIKOR) methods in a decision-making framework tailored for this field. The framework is based on 23 criteria that measure different aspects of the performance. The performance of the framework is assessed according to the plithogenic set theory and to the neutrosophic set theory using a case study of comparing the performance of IoT implantation with the SC of five e-commerce companies using three experts. The case study shows that the proposed framework has more consideration of the contradiction degree of each criteria to improve the accuracy of the evaluation results.

Keywords: supply chain management (SCM), Internet of Things (IoT), Multi-criteria decision-making (MCDM), VIKOR method, BWM, Plithogenic set

1. Introduction

Intensive competition is generated as a result of the globalization of international trade market, which increases challenging marketplace requirements. In order to obtain the competition requirements, it is necessary to develop an efficient and coordinated supply chain.

The supply chain (SC) is an integration of business processes (i.e., supplying, producing, distributing, and storing), that is converting raw material to final product or service that is utilized by customers which satisfy their needs. The SC is usually depicted by the flow of information, finance, and material through its stages, while the SCM is the organizing, implementing and monitoring of the networks [1]. Many supply chains are suffering from supply-demand incompatibility, overstocking, delivery delays, and many other issues. That is why traditional supply chains seem to be more complex, uncertain, and susceptible [2]. Thus, it becomes significant to develop a smarter and coordinated supply chain that integrates data, information, physical entities, and business processes altogether.

The variety of organization's standards, purposes, interests, and market strategies leads to ambiguous definitions of the IoT. Kevin Ashton (1999) imagined an interconnected physical world through the internet that enables sensors and platforms which allow a real-time feedback, in order to consolidate the monitoring and to secure communication [3]. The IoT may be defined as: "an intelligent infrastructure linking objects, information, and people through the computer networks, and where the RFID technology found the basis for its realization [4]". The main steps toward IoT are data collection, the transmission of data across the network, and data processing [5]. Data collection is the first step that is responsible for gathering the data about the network objects through main technologies such as sensors, RFID technology, or Near Field Communication (NFC) technology [3]. The moment that sensing technologies collected the data, it must be transmitted across the network through wired (e.g., coaxial cables, and optical fibers) or wireless (e.g., Wi-Fi) technologies [6]. In the last phase, transmitted data must be processed and then forwarded to the application.

The SCM based on IoT refers to the connection of physical objects in order to monitor the interaction of a firm with its supply chain, by focusing on information sharing toward facilitating the control and the coordination of supply chain processes [2]. The IoT based supply chain would possess the ability to have great connection across all supply chain phases, and provide intelligent decision-making in order to meet the customer expectations. The IoT is applied in a variety of fields such as transportation, energy, healthcare, retail, manufacturing, agriculture, and others.

Understanding the performance of IoT based SC requires effectively measuring the performance of all alternatives according to several sets of criteria. The evaluation of IoT based supply chain requires considering several aspects such as security, technological infrastructure, the functionality of the supply chain, and others that distinguish it from the traditional supply chain. As in many evaluations and decision-making problems, there is a defect of uncertain, vague, and incomplete information that may lead to a non-optimal decision. Thus, integrating the plithogenic set with neutrosophic set's triple components (truth-membership, falsity-membership, and indeterminacy-membership) should provide more accurate assessment results. Plithogenic set increases the accuracy and efficiency of decision-making. Plithogeny, introduced by Florentin Smarandache in 2017, is a generalization of neutrosophy. The plithogenic set is a set of elements, such that each element x is characterized by attribute values v that have a corresponding contradiction degree c(v, D) between them and a dominant attribute value D, and by an appurtenance degree d(x, v) of element x to the plithogenic set [7, 22].

For measuring the performance of IoT based SC, this research proposed a framework that integrates the best-worst method (BWM) and Vlse Kriterijumska Optimizacija Kompromisno Resenje (VIKOR) method under plithogenic environment. We present the BWM and VIKOR in the plithogenic environment because most of the evaluations face a problem of uncertainty of expert's judgment, where contradiction degree provides more accurate aggregation results of their evaluations. Thus, the features of the plithogenic set should efficiently lessen the problem of ambiguity and take into consideration the different judgments of decision-makers, helping to choose the optimal decision and obtain the best assessment of the IoT based supply chains.

The present research is organized as follows: Section 2 reviews some literature regarding the internet of things and its effect on the supply chain. Section 3 presents the background of the methods. Section 4 presents the steps of the proposed integrated framework for measuring the IoT based supply chain. Section 5 presents a case study of the proposed framework to evaluate the Ecommerce supply chain based on the IoT. The conclusions and the future directions of the research are presented in Section 6.

2. Literature review

There are many studies that focus on IoT and supply chain. For instance, Musa et al. (2016) reviewed the importance of RFID technology in SCM [8]. Zhou et al. (2015) introduce a framework of traceability of the supply chain based on IoT [9]. Zhang et al. (2017) studied the importance of real-time data acquiring based on IoT in the field of perishable foods [10]. Papert et al. (2017) developed a hypothetical IoT ecosystem model in order to assess the firms to establish their own ecosystem [11]. Li et al. (2017) proposed an efficient management platform to track and trace the pre-packaged food SC based on IoT [12]. Chen (2019) evaluated the performance of IoT based supply chain finance risk management performance using the fuzzy QFD method [13].

Different MCDM techniques have been applied in IoT context. Mashal et al. (2019) applied the AHP method to evaluate smart objects, applications, and providers of IoT [14]. Uslu et al. (2019) applied AHP and ANP methods in order to evaluate the difficulties faced by enterprises when adopting the IoT [15]. In order to evaluate the internet of cloud sensors search and selection, Nunes et al. (2017) used SAW, TOPSIS, and VIKOR methods [16]. Mohammadzadeh et al. (2018) applied the ANP method under fuzzy environment to recognise the most significant IoT technology challenges in Iran [17]. Nabeeh et al. (2019) applied neutrosophic AHP in order to evaluate the influential factors of IoT in enterprises as shown in Figure 1 [18]. On the other side, Ly et al. (2018) evaluate the success factors of IoT systems using fuzzy AHP method [19].

One of the major issues is uncertainty in the evaluation problems that may confuse decision-makers. As a generalization of the fuzzy set and intuitionistic fuzzy set, Florentin Smarandache introduced the neutrosophic set (1998) [20]. Van et al. (2018) proposed the application of neutrosophic QFD in order to solve the problem of green supplier selection [21]. They also studied the influence of IoT on the SC using neutrosophic AHP and neutrosophic DEMATEL [2]. The characteristics of the neutrosophic set are clearly detailed as follows.

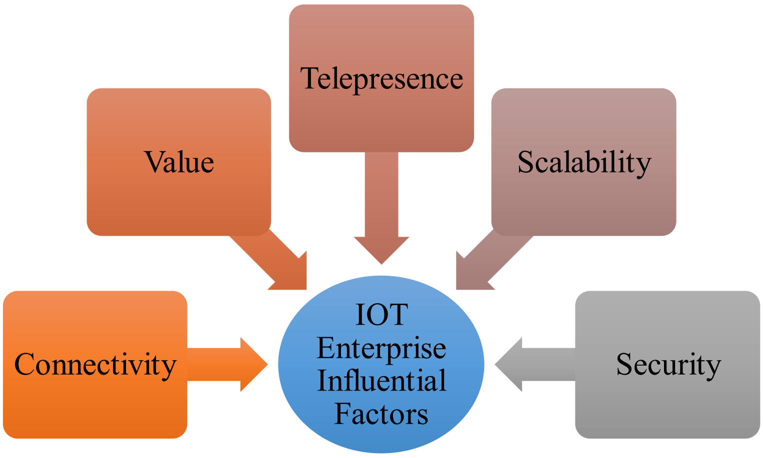

Figure 1: Effective Factors for IoT enterprise adoption [18]

3. Methods

In this study, two MCDM methods (BWM and VIKOR) are employed in order to measure IoT based supply chain performance. These methods are based on the plithogenic set in order to increase the precision of the evaluation procedure and solve the uncertainty problem in the assessment.

3.1 Basic concepts of the neutrosophic set

Definition 1. Let X be a universe of discourse. A single valued neutrosophic set (SVNS) N over X is an object with the form $N = \{\langle x, T_N(x),\ I_N(x),\ F_N(x)\rangle : x \in X\}$, where $T_N(x): X \to [0,1]$, $I_N(x): X \to [0,1]\ and\ F_N(x): X \to [0,1]$ with $0 \leq\ T_N(x) +\ I_N(x) +\ F_N(x) \leq 3$ for all $x \in X$, where TN(x), IN(x) and FN(x) represent the truth-membership function, indeterminacy-membership function, and falsity-membership function, respectively. A Single Valued Neutrosophic (SVN) number is represented as $A = (a, b, c)\ where\ a, b, c\ \in [0,1]\ and\ a + b + c \leq 3$.

Definition 2. Let $\tilde{a} = \langle (a1, a2, a3); \alpha, \theta, \beta \rangle$ be a SVNS, with truth membership Ta(x), indeterminate membership Ia(x), and falsity membership function Fa(x) as follows:

$$T_a(\mathrm{x}) = \begin{cases} \alpha_a\left(\frac{x-a_1}{a_2-a_1}\right) & if\ a_1 \le x \le a_2 \\ \alpha_a & if\ x = a_2 \\ \alpha_a\left(\frac{a_3-x}{a_3-a_2}\right) & if\ a_2 \le x \le a_3 \\ o & otherwise \end{cases} \quad (1)$$

$$I_a(\mathrm{x}) = \begin{cases} \frac{(a_2-x)}{(a_2-a_1)}\theta_a & if\ a_1 \le x \le a_2 \\ \theta_a & if\ x = a_2 \\ \frac{(x-a_3)}{(a_3-a_2)}\theta_a & if\ a_2 \le x \le a_3 \\ 1 & otherwise \end{cases} \quad (2)$$

$$F_a(\mathrm{x}) = \begin{cases} \frac{(a_2-x)}{(a_2-a_1)}\beta_a & if\ a_1 \le x \le a_2 \\ \beta_a & if\ x = a_2 \\ \frac{(x-a_3)}{(a_3-a_2)}\beta_a & if\ a_2 < x \le a_3 \\ 1 & otherwise \end{cases} \quad (3)$$

Definition 3. Let $\tilde{a} = \langle(a1, a2, a3); \alpha_a, \theta_a, \beta_a\rangle$ and $\tilde{b} = \langle(b1, b2, b3); \alpha_b, \theta_b, \beta_b\rangle$ be two triangular neutrosophic numbers (TNN). Then, we have:

- Addition of two TNN :

$$\tilde{a} + \tilde{b} = \langle(a1 + b1, a2 + b2, a3 + b3); \alpha_a \cap \alpha_b, , \theta_a \cup \theta_b, \beta_a \cup \beta_b\rangle \quad (4)$$

Subtraction of two TNN :

$$\tilde{a} - \tilde{b} = \langle(a1 - b3, a2 - b2, a3 - b1); \alpha_a \cap \alpha_b, , \theta_a \cup \theta_b, \beta_a \cup \beta_b\rangle \quad (5)$$

- Inverse of two TNN :

$$\tilde{a}^{-1} = \langle\left(\frac{1}{a3}, \frac{1}{a2}, \frac{1}{a1}\right); \alpha_a, \theta_a, \beta_a\rangle, Where\ (\tilde{a} \neq 0) \quad (6)$$

- Multiplication of two TNN:

$$\tilde{a}\tilde{b} = \begin{cases} \langle(a1b1, a2b2, a3b3); \alpha_a \cap \alpha_b, , \theta_a \cup \theta_b, \beta_a \cup \beta_b\rangle\ if\ (a3 > 0, b3 > 0) \\ \langle(a1b3, a2b2, a3b1); \alpha_a \cap \alpha_b, , \theta_a \cup \theta_b, \beta_a \cup \beta_b\rangle\ if\ (a3 < 0, b3 > 0) \\ \langle(a3b3, a2b2, a1b1); \alpha_a \cap \alpha_b, , \theta_a \cup \theta_b, \beta_a \cup \beta_b\rangle\ if\ (a3 < 0, b3 < 0) \end{cases} \quad (7)$$

- Division of two TNN:

$$\frac{\tilde{a}}{\tilde{b}} = \begin{cases} \langle\left(\frac{a1}{b3}, \frac{a2}{b2}, \frac{a3}{b1}\right); \alpha_a \cap \alpha_b, , \theta_a \cup \theta_b, \beta_a \cup \beta_b\rangle\ if\ (a3 > 0, b3 > 0) \\ \langle\left(\frac{a3}{b3}, \frac{a2}{b2}, \frac{a1}{b1}\right); \alpha_a \cap \alpha_b, , \theta_a \cup \theta_b, \beta_a \cup \beta_b\rangle\ if\ (a3 < 0, b3 > 0) \\ \langle\left(\frac{a3}{b1}, \frac{a2}{b2}, \frac{a1}{b3}\right); \alpha_a \cap \alpha_b, , \theta_a \cup \theta_b, \beta_a \cup \beta_b\rangle\ if\ (a3 < 0, b3 < 0) \end{cases} \quad (8)$$

3.2 Basic concepts of the plithogenic set

Smarandache (2017) introduced a generalization of neutrosophy that denotes to genesis, construction, improvement and advances of new objects from syntheses of conflicting or non-conflicting multiple old objects [22] which is known as plithogeny. The plithogenic set operations are plithogenic intersection $\wedge_P$, plithogenic union $\vee_P$, plithogenic complement $\neg_P$, plithogenic inclusion →, and plithogenic equality ↔.

In order to obtain more accurate results, the plithogenic set provides high consideration of uncertainty of information due to its two main features, the contradiction degree and the appurtenance degree. Contradiction (dissimilarity) degree function c(v,D) distinguishes between each attribute value and the dominant (greatest preferred) attribute value. The attribute value contradiction degree function c(v1, v2) is c: V×V → [0, 1], sustaining the next axioms:

- c(v1, v1) = 0, contradiction degree between the same the attribute values is zero;
- c(v1, v2) = c(v2, v1), symbolizing the distinction between two attribute values v1 and v2.

Abdel-Basset et al. (2019) proposed a model to be applied to measure the performance of hospitals in Zagazig city in Egypt using the VIKOR method according to 11 evaluation standards based on plithogenic set [23]. Another application of the plithogenic set was applied in SC sustainability evaluation based on QFD [24].

Definition 4. [25] Let $\tilde{a} = (a1, a2, a3)$ and $\tilde{b} = (b1, b2, b3)$ be two plithogenic sets; operations are:

- Plithogenic intersection:

$$\left((a_{i1}, a_{i2}, a_{i3}), 1 \le i \le n\right) \wedge_p \left((b_{i1}, b_{i2}, b_{i3}), 1 \le i \le n\right)$$
$$= \left(\left(a_{i1} \wedge_F b_{i1}, \frac{1}{2}(a_{i2} \wedge_F b_{i2}) + \frac{1}{2}(a_{i2} \vee_F b_{i2}), a_{i2} \vee_F b_{i3}\right)\right), 1 \le i \le n. \quad (9)$$

- Plithogenic union:

$$\left((a_{i1}, a_{i2}, a_{i3}), 1 \le i \le n\right) \vee_p \left((b_{i1}, b_{i2}, b_{i3}), 1 \le i \le n\right)$$
$$= \left(\left(a_{i1} \vee_F b_{i1}, \frac{1}{2}(a_{i2} \wedge_F b_{i2}) + \frac{1}{2}(a_{i2} \vee_F b_{i2}), a_{i2} \wedge_F b_{i3}\right)\right), 1 \le i \le n. \quad (10)$$

where

$$a_{i1} \wedge_p b_{i1} = [1 - c(v_D, v_1)].t_{norm}(v_D, v_1) + c(v_D, v_1).t_{conorm}(v_D, v_1) \quad (11)$$
$$a_{i1} \vee_p b_{i1} = [1 - c(v_D, v_1)].t_{conorm}(v_D, v_1) + c(v_D, v_1).t_{norm}(v_D, v_1) \quad (12)$$

where, tnorm= $a \wedge_F b = ab$, tconorm $a \vee_F b = a + b - ab$

- Plithogenic complement (negation):

$$\neg\left((a_{i1}, a_{i2}, a_{i3}), 1 \le i \le n\right) = \left((a_{i3}, a_{i2}, a_{i1}), 1 \le i \le n\right) \quad (13)$$

The appurtenance degree d(x,v) of attribute value v is: $\forall x \in$ P, d: P×V→ P ([0, 1]z), so d(x, v) is a subset of [0, 1]z, and P([0, 1] z) is the power set of [0, 1] z, where z = 1, 2, 3, for fuzzy, intuitionistic fuzzy, and neutrosophic degrees of appurtenance respectively.

3.3 The Best-Worst Method (BWM)

BWM is one of the most efficient and useful methods in multi-criteria decision-making. The model of this method is used to find the weight of each selection criteria. The BWM was applied in several fields of research such as engineering sustainability [26], financial performance evaluation [27], sustainable supplier selection and order allocation [28], evaluating the community sustainability of supply chains [29], and Location Selection for Wind Farms [30].

In addition, there are several researches that used the BWM under the neutrosophic environment and applied it in different topics. For instance, Yucesan et al. (2019) applied neutrosophic BWM in the evaluation of the implant manufacturing according to five groups of criteria [31], while Lou et al. (2019) proposed an integrated MCDM framework based on the BWM in order to solve the personnel selection problem [32].

The best-worst method is based on pairwise comparisons of the selection standards on the basis of the decision-maker's preference. Thus, the BWM value is requiring fewer comparisons than the analytic hierarchy process (AHP). In order to handle the drawback of discrepancy in AHP comparison, decision-makers should identify the most preferred criterion (best) and the least preferred criterion (worst) and then stratify the pairwise comparison between these two criteria and the other criteria [33]. Moreover, BWM consists of less complexity of comparisons as it exploits only whole numbers. Finally, BWM is notable because the redundant comparisons are eradicated. [34]. The phases of the BWM are as follows:

- Step 1. The first step decision-maker identifies the set of selection criteria based on the problem nature N = {c_1, c_2, …, c_n}
- Step 2. Determine the best and the worst criteria.
- Step 3. Obtain the best-to-other vector AB = (aB1, aB2, … , aBn), which is decision-maker's judgment of the best criterion in comparison with other criteria - using a (1-9) scale, where aBn designates the judgment of the best criterion over criterion n . It is obvious that aBB = 1.
- Step 4. Establish the others-to-worst vector Aw = (aw1, aw2, … , awn), which is decision-maker's judgment of all criteria in comparison with the worst one - using a (1-9) scale, where awn designates the preference of criterion n over the worst criterion. It is also obvious that aww = 1.
- Step 5. Use the nonlinear programming model to find the optimal criteria weights (W1* , W2*, … , Wn*).

  Min ε

s.t.

$\left|\frac{w_B}{w_j} - a_{Bj}\right| \leq \varepsilon$, for all j

$\left|\frac{w_j}{w_w} - a_{jW}\right| \leq \varepsilon$, for all j

$\sum_j w_j = 1$ (14)

$w_j \geq 0$, for all j

3.4 The Vlse Kriterijumska Optimizacija Kompromisno Resenje (VIKOR) method

VIKOR, proposed by Opricovic (1998), is a useful method to solve complex MCDM problems with inconsistent criteria that can assist decision maker to find the optimal alternative. There are several studies that applied the VIKOR method in different topics under uncertain environment. Hussain et al. (2019) integrated the VIKOR method with interval neutrosophic numbers in situations that need consideration of indeterminacy along with the certainty and uncertainty [35]. Wang et al. (2019) proposed a framework according to the VIKOR method based on the linguistic neutrosophic set, and it was applied in selecting problems of fault handling point [36].

The ranking of the alternatives is based on their distance to the ideal alternative. The main steps of VIKOR method are described as follows and illustrated in Figure 2:

- Step 1: Decision-maker evaluates the alternatives based on the selection standards. Build the decision matrix based on the decision-maker's assessment according to the weight of each criterion in contrast to the alternatives to be assessed.
- Step 2: Normalize the decision matrix using Equation 15.

  $$(f_{ij})_{m\times n} = x_{ij} \Big/ \left(\sqrt{\sum_{i=1}^{m} x_{ij}^2}\right) \quad (15)$$

  where m is the number of alternatives and n is the number of criteria.
- Step 3: Distinguish the beneficial and non-beneficial criteria based on the problem nature and the decision-maker's preference. Determine the best values $f_j^*$and worst values f_j^- of criteria. If f_j is beneficial criteria, then $f_j^* = \max(f_{ij})$ and $f_j^- = \min(f_{ij})$. On the other hand, if f_j is non-beneficial criteria, then $f_j^* = \min(f_{ij})$ and $f_j^- = \max(f_{ij})$.
- Step 4: Calculate the values of S_i (maximum group utility) and R_i (minimum individual regret of the opponent) by Equation 16 and 17:

  $$S_i = \sum_{j=1}^{n} w_j * \frac{f_j^* - f_{ij}}{f_j^* - f_j^-} \quad (16)$$

  $$R_i = \max\left[w_j * \frac{f_j^* - f_{ij}}{f_j^* - f_j^-}\right] \quad (17)$$

  where w_j id the weight of criteria expressing their importance.
- Step 5: Calculate the value of concordance index Q_i by Equation 18.

  $$Q_i = v\left[\frac{S_i - S^*}{S^- - S^*}\right] + (1 - v)\left[\frac{R_i - R^*}{R^- - R^*}\right] \quad (18)$$

  where $S^- = max_i S_i$, $S^* = min_i S_i$, $R^- = max_i R_i$, $R^* = min_i R_i$, and v is the weight of strategy of maximum group utility, usually equal to 0.5.
- Step 6: The alternatives are ranked according to Q_i descending order, where the optimal alternative has the minimum Q value.
- Step 7: There are two conditions that should be satisfied in regard to this rank:

  Condition 1 (acceptable advantage):

  $$Q(A^2) - Q(A^1) \geq \frac{1}{m-1} \quad (19)$$

  where A^1 is the first alternative in Q ranking and A^2 is the second, and m is the number of alternatives.

  Condition 2 (acceptable stability): as the ranking of Q, A^1 must be the superior in the ranking of S and R. In case that one condition is not satisfied, a set of alternatives is proposed:

  If condition 2 is not satisfied, then A^1 and A^2 are compromise solutions;

If condition 1 is not satisfied, then $A^1, A^2, \dots, A^m$ are compromise solutions, where A^m is determined by Equation 20.

$$Q(A^m) - Q(A^1) < \frac{1}{m-1} \tag{20}$$

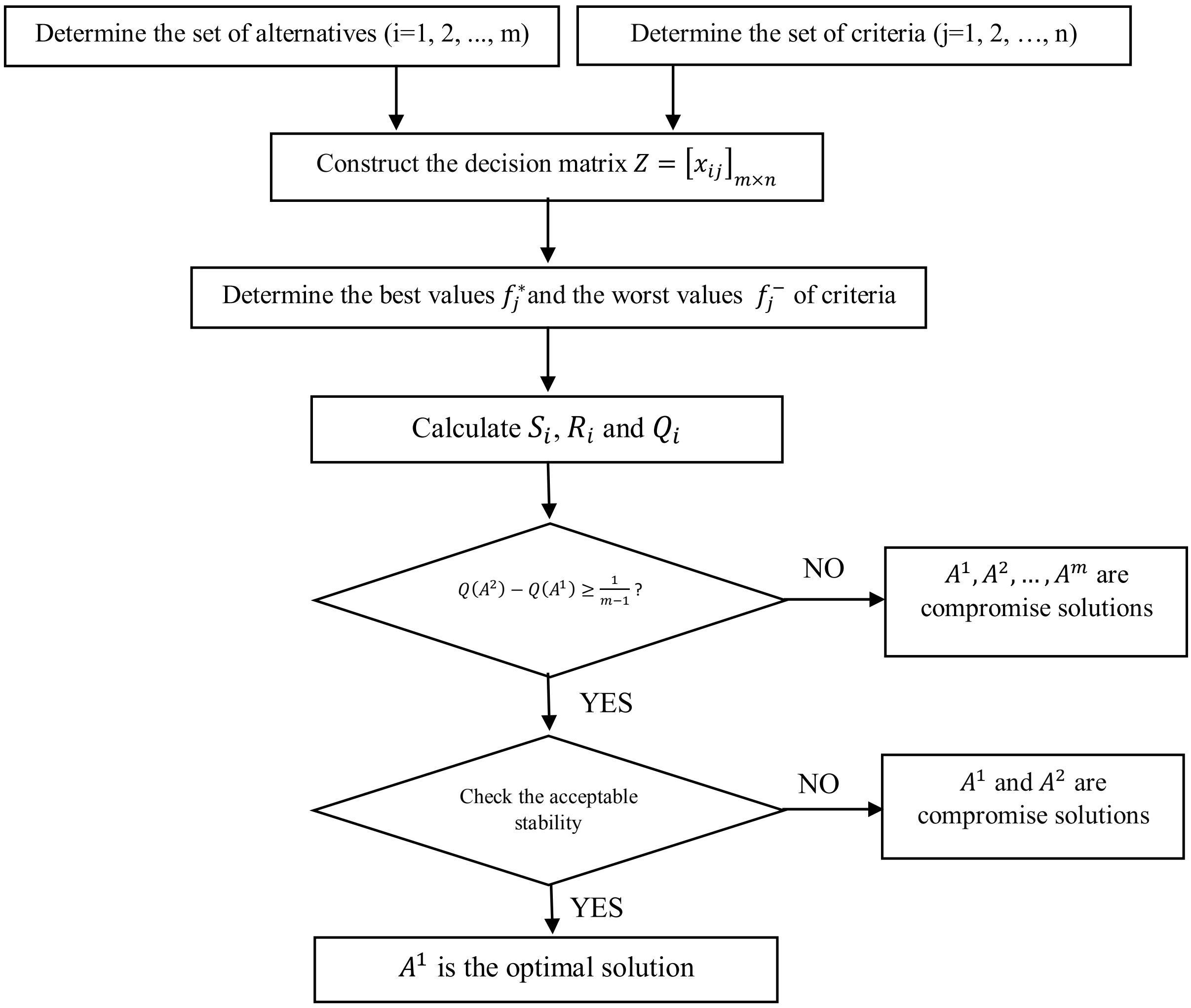

Figure 2: Flow Chart of VIKOR Method

4. Proposed framework

This research proposes an integrated framework to assess the performance of the supply chain based on the IoT under uncertainty environment. The BWM method identifies the weights of the performance criteria based on the pairwise comparison of the best and worst criteria among the rest of the criteria, while VIKOR method evaluates the performance of the IoT based supply chains according to the selection criteria. The importance of this approach lies in handling the high level of uncertainty resulted from the scarce of expertise in the field. Plithogenic set it powerful in handling uncertain judgments by considering the truth-membership function, falsity-membership function, and indeterminacy-membership function. In addition, the features of the plithogenic set operations provide more accurate results. This framework utilizes the advantages of plithogenic set operation, BWM, and VIKOR method to provide a more accurate evaluation. The steps of the proposed framework are as described below and illustrated in Figure 3:

- Phase 1: As in all evaluation problems, acquire the evaluation information by integrating a committee of decision-makers. $DM = \{ d_1, d_2, \dots, d_k\}$. Define a set of criteria that measures the performance of the IoT based SC. $C = \{c_1, c_2, \dots, c_n\}$, and the alternatives (IoT based supply chains) that need to be evaluated $A = \{a_1, a_2, \dots, a_m\}$.

In this research, the proposed framework examines the performance of the IoT based supply chain according to 23 criteria (Table 2) that measure the financial cost, service quality, resource consumption, degree of customer satisfaction, functionality, technological infrastructure, and security. For validating the proposed framework we rank five Ecommerce companies that are managed according to IoT based supply chain rendering to their performance.

Table 1: Evaluation Criteria of IoT based supply chain [37, 38]

Main aspects	Criteria
Financial cost A	Hardware costs A1
	Software costs A2
	Implementation costs A3
	Maintenance cost A4
Service quality B	Service level B1
	Service flexibility B2
	System reliability B3
	Distribution network quality B4
Resource consumption C	Total number of services C1
	Rate of actual work C2
	Request frequency /min C3
Degree of customer satisfaction D	Time delivery rate D1
	Order accuracy D2
	Complaint response time D3
	After-sales support D4
Functionality E	Technical compliance of the devices E1
	Operational feasibility of devices E2
Technological infrastructure F	Competence of the system F1
	the association abilities with other systems F2
	the transferability of the system F3
Security G	Level of access control G1
	the level of device verification G2
	the level of encryption G3

- Phase 2: Apply the BWM (as discussed in section 3.3) to compute the weights of the criteria that measure the performance of the IoT based supply chain.
  - Step 1: Regulate the most preferred and the least preferred criteria according to the decision-maker's preference.
  - Step 2: Construct the Best-to-Other vector and Others-to-Worst vector.
  - Step 3: Use the BWM model (14) to find the weight vector.
- Phase 3: Construct the evaluation matrix based on plithogenic aggregation operation.

Table 2: Triangular neutrosophic scale for decision matrix

Importance Linguistic variable	Triangular neutrosophic scale
Very Weakly important (VWI)	((0.10, 0.2,0.3), 0.1,0.2,0.15)
Weakly important (WI)	((0.15,0.3,0.50), 0.6,0.2,0.3)
Partially important (PI)	((0.40,0.35,0.50), 0.6,0.1,0.2)
Equal important (EI)	(0.5,0.6,0.70),0.8,0.1,0.1)
Strong important (SI)	((0.65,0.7,0.80),0.9,0.2,0.1)
Very strongly important (VSI)	((0.8,0.75,0.95),0.7,0.2,0.2)
Absolutely important (AI)	((0.95,0.90,0.95),0.9,0.10,0.10)

- Step 1: Construct the evaluation matrices in order to evaluate alternatives according to the corresponding criteria by decision-makers based on triangular neutrosophic scale as shown in Table 1.
- Step 2: In this step, aggregate the evaluation matrices using plithogenic operator as shown in Equations 10, 11, and 12. In this step, the contradiction degree of each criterion should be considered in order to provide more accurate aggregation results.
- Step 3: To make the computations easier, apply the de-neutrosophication of the aggregated evaluation matrix using Equation 21

$$S(a) = \frac{1}{8}(a_1 + b_1 + c_1) \times (2 + \alpha - \theta - \beta) \quad (21)$$

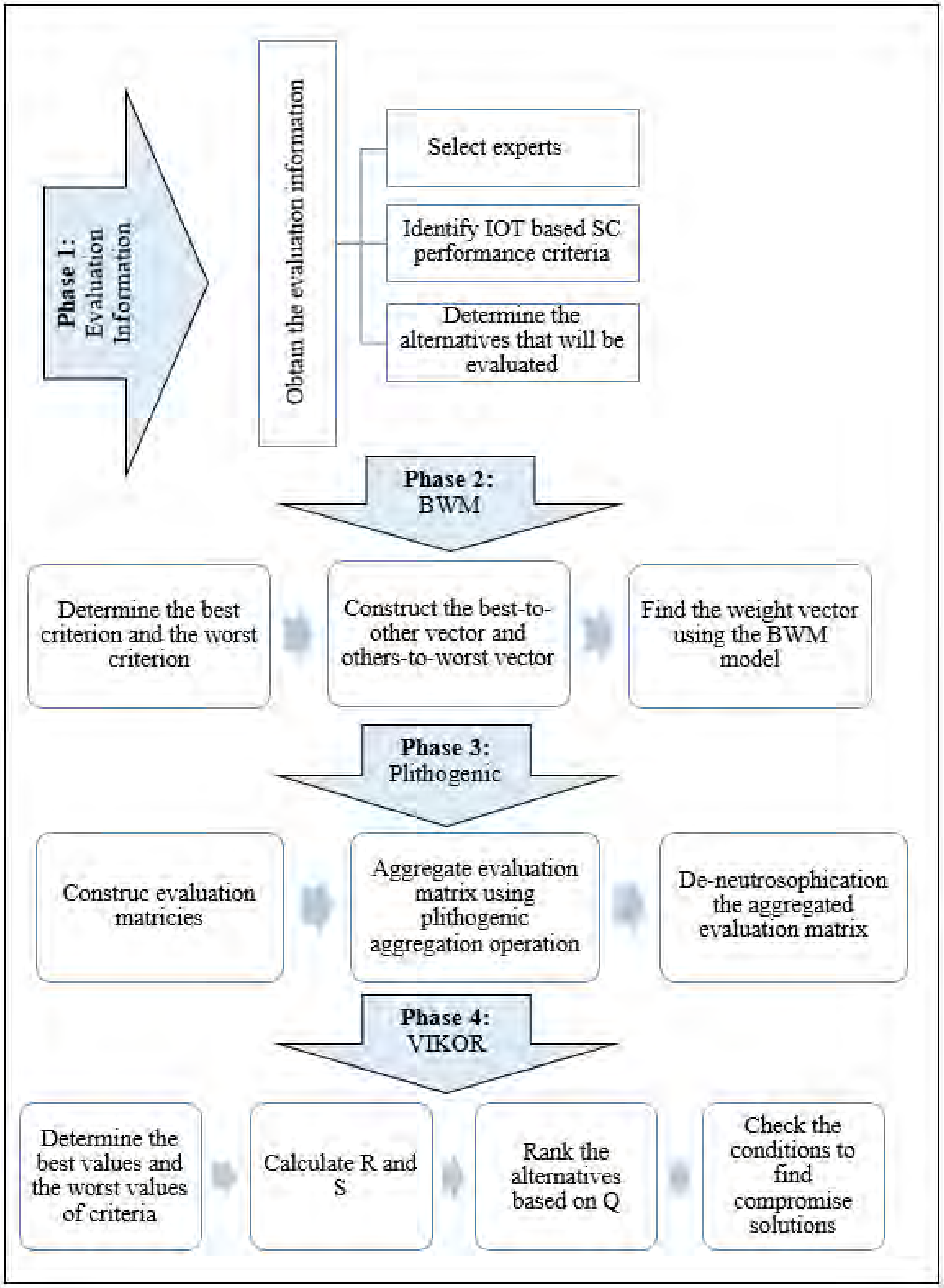

Figure 3: Phases of the Proposed Framework

- Phase 4: Using VIKOR method, rank the IoT based SCs based on their performance evaluation.
  - Step 1: Define the best values and the worst values of criteria.
  - Step 2: Calculate S_i and R_i.
  - Step 3: Rank the alternatives based on the concordance index Q_i
  - Step 4: Check the conditions to find compromise solutions.

5. Case study: Evaluation of IoT based Ecommerce supply chains

The proposed framework based on the plithogenic set is used to measure the IoT based Ecommerce supply chains performance. The evaluation is obtained by a group of three experts (e): Ecommerce management expert (e_1), IT expert (e_2), and supply chain expert (e_3). After the response to questions are collected, the assessment of Ecommerce companies managed according to IoT based supply chain is conducted as follows:

- Phase 1: The performance evaluation of the IoT based Ecommerce supply chain is based on 23 criteria. Evaluate the five companies according to their performance. The judgment of the performance is based on three experts.
- Phase 2: In order to evaluate the weights of the 23 criteria, BWM is applied. Experts define the competence of the system as the most sufficient criterion, and the level of device verification as the least important criterion. According to the importance rating scale, best-to-other and others-to-worst vectors where determined as in Table 3 and 4. After applying BWM model, the weight vector resulted is presented in Table 5 and Figure 4. As results show, the competence of the system (F1) has the highest weight (0.10961), while the level of device verification (G1) has the lowest weight (0.00645).

Table 3: Best-to-Others Vector

Best-to-Others	A1	A2	A3	A4	B1	B2	B3	B4	C1	C2	C3	
F1	0.1	0.5	0.2	0.5	0.6	0.2	0.3	0.6	0.4	0.6	0.8	
	D1	D2	D3	D4	E1	E2	F1	F2	F3	G1	G2	G3
	0.8	0.7	0.4	0.4	0.5	0.4	0.1	0.5	0.3	0.3	0.9	0.9

Table 4: Others-to-Worst Vector

Others-to-Worst		G2	
A1	0.9	D1	0.2
A2	0.4	D2	0.2
A3	0.9	D3	0.6
A4	0.4	D4	0.5
B1	0.3	E1	0.5
B2	0.8	E2	0.6
B3	0.8	F1	0.9
B4	0.3	F2	0.4
C1	0.6	F3	0.7
C2	0.3	G1	0.7
C3	0.2	G2	0.1
		G3	0.2

Table 5: Evaluation Criteria Weights

Criteria	weight	Criteria	Weight
A1	0.10861	D1	0.02015
A2	0.03224	D2	0.02303
A3	0.08059	D3	0.04030

A4	0.03224	D4	0.04030
B1	0.02686	E1	0.03224
B2	0.08059	E2	0.04030
B3	0.05373	F1	0.10961
B4	0.02686	F2	0.03224
C1	0.04030	F3	0.05373
C2	0.02686	G1	0.05373
C3	0.02015	G2	0.00645
		G3	0.01791

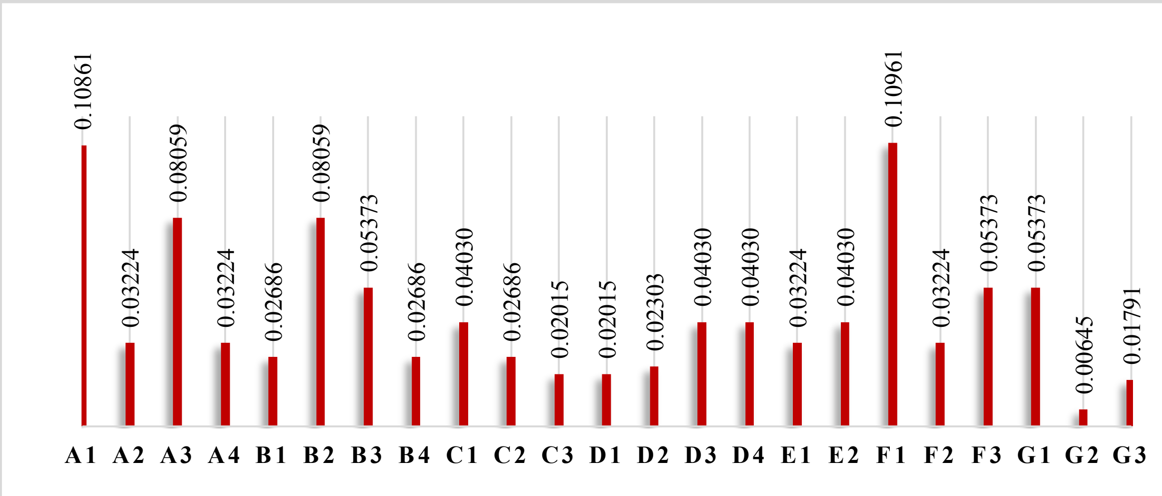

Figure 4: Weights of the Criteria

- Phase 3: Construct the evaluation matrix according to the three expert's judgments based on the triangular neutrosophic linguistic scale in Table 1, as shown in Table 6. Then, the plithogenic aggregation operator is used in combining the evaluations of the three experts. The equidistant contradiction degree (the dominant attribute value is 0) of the criteria was defined to ensure more accurate aggregation, as shown in Table 7. Using Equation 21, calculate the crisp evaluation matrix as shown in Table 8.
- Phase 4: In this phase, the target is to rank the five Ecommerce companies by VIKOR. Table 9 shows the normalized evaluation matrix. The values of S_i, R_i and Q_i were calculated as shown in Table 10 using Equations 16, 17, and 18 respectively. The w_j values, found by BWM in phase 2, was determined from Table 5. As the results show, company 5 in the top of the ranking, while company 3 at the end. According to VIKOR conditions, company 5 has the best rank, and it satisfies condition 1 $(0.27649 - 0 > 1/4)$, and also satisfies condition 2 (company 1 is superior in ranking of S and R as well as Q), so company 1 is the optimal solution.

5.1 Results Discussion and Sensitivity Analysis

- As the results of BWM show, the competence of the system and the hardware costs are the most important metric considered to evaluate the IoT based Ecommerce supply chains. The second level of IoT based Ecommerce supply chains performance measure is implementation costs and service flexibility. The last level of criteria consist of the level of device verification and level of encryption, with weights 0.00645 and 0.01791, respectively.
- According to VIKOR method results, the ranking of companies varies based on parameter v. The ranking of Ecommerce companies based on their performance as follows (v=0.5): company 5> company 1 > company 4 > company 2 > company 3, as Figure 5 shows.

- It is important to mention the impact of chaining the weight of the strategy v within interval [0, 1]. The sensitivity analysis on v is shown in Table 11. As Figure 6 shows, usually company 5 is in the top of ranking while company 2 is at the end.
- One of the main contributions of this framework is using the plithogenic aggregation operation based on the contradiction degree between the criteria. In order to show the importance of the proposed framework, a comparison with neutrosophic set is constructed on the same steps of the framework (Figure 7). The results of the proposed framework under neutrosophic set show that the ranking of Ecommerce companies based on their performance are as follows (v=0.5) company 2> company 3 > company 1 > company 4 > company 5 (Table 12).

Table 6: Three Expert's Evaluation Matrix

		A1	A2	A3	A4	B1	B2	B3	B4	C1	C2	C3	D1	D2	D3	D4	E1	E2	F1	F2	F3	G1	G2	G3
Company 1	DM1	AI	EI	VSI	AI	EI	PI	SI	VWI	EI	SI	SI	VSI	VSI	SI	AI	VSI	VSI	SI	PI	WI	AI	VS	SI
	DM2	PI	PI	WI	EI	SI	VSI	VSI	AI	SI	SI	SI	VS	AI	AI	VSI	VSI	VSI	AI	EI	VSI	AI	AI	AI
	DM3	VSI	EI	WI	EI	VWI	VWI	EI	VSI	VSI	SI	SI	SI	AI	AI	AI	VWI	VWI	EI	VWI	VWI	EI	EI	VWI
Company 2	DM1	SI	PI	EI	AI	PI	EI	EI	WI	PI	PI	EI	SI	AI	EI	VSI	SI	EI	VSI	WI	WI	VSI	EI	PI
	DM2	WI	WI	WI	EI	SI	VSI	VSI	AI	VSI	VSI	VSI	VSI	AI	AI	VSI	VSI	VSI	AI	EI	VSI	AI	AI	AI
	DM3	VSI	EI	AI	AI	VWI	VSI	EI	VSI	VSI	SI	EI	SI	VSI	AI	AI	VWI	VWI	EI	VWI	VWI	EI	EI	VWI
Company 3	DM1	VSI	AI	SI	VSI	SI	SI	SI	AI	EI	SI	VSI	VSI	AI	PI	AI	EI	EI	SI	EI	EI	PI	WI	VWI
	DM2	SI	SI	SI	SI	EI	SI	SI	VSI	EI	SI	VSI	VSI	VSI	AI	AI	VSI	VSI	AI	EI	VSI	AI	VSI	AI
	DM3	VSI	EI	VSI	VSI	VWI	VWI	EI	VSI	VSI	SI	AI	SI	EI	WI	WI	VWI	VWI	EI	VWI	VWI	EI	EI	VWI
Company 4	DM1	AI	VSI	VSI	AI	EI	EI	VSI	VSI	PI	EI	EI	AI	AI	VSI	AI	SI	VSI	EI	PI	PI	WI	WI	VWI
	DM2	SI	VSI	VSI	SI	EI	SI	SI	VSI	EI	SI	VSI	VSI	VSI	AI	AI	VSI	VSI	AI	EI	VSI	AI	VSI	AI
	DM3	VSI	VWI	VSI	VI	VWI	VWI	EI	VSI	VSI	WI	VSI	AI	AI	SI	AI	VWI	VWI	EI	EI	AI	EI	EI	EI
Company 5	DM1	VSI	WI	SI	VSI	WI	PI	PI	VWI	WI	WI	WI	EI	SI	SI	EI	EI	EI	EI	PI	PI	VSI	EI	EI
	DM2	WI	WI	WI	PI	SI	VSI	VSI	AI	SI	SI	SI	VSI	AI	AI	VSI	VSI	VSI	AI	EI	VSI	AI	AI	AI
	DM3	VSI	EI	AI	AI	VWI	VWI	EI	VSI	VSI	SI	SI	SI	AI	WI	WI	AI	VSI	EI	VWI	VWI	EI	EI	AI

Table 7: Aggregated Evaluation Matrix

Contradiction degree	**0**	**0.043**		**0.957**
	A1	A2	…	G3
Company 1	⟨(0.304,0.688,0.99); 0.75,0.15, 0.175⟩	⟨(0.132,0.54,0.93); 0.75,0.1, 0.125⟩	…	⟨(0.944,0.513,0.3); 0.45,0.175, 0.15⟩
Company 2	⟨(0.078,0.625,995); 0.73,0.2, 0.200⟩	⟨(0.06,0.463,0.9); 0.7,0.125, 0.175⟩	…	⟨(0.913,0.413,0.17); 0.425,0.15, 0.15⟩
Company 3	⟨(0.416,0.74,1); 0.75,0.2, 0.175⟩	⟨(0.338,0.7,0.981); 0.9,0.125, 0.1⟩	…	⟨(0.89,0.375,0.113); 0.3,0.175, 0.138⟩
Company 4	⟨(0.494,0.775,1); 0.8,0.175, 0.15⟩	⟨(0.092,0.475,0.965); 0.4,0.2, 0.175⟩	…	⟨(0.938,0.575,0.245); 0.65,0.125, 0.113⟩
Company 5	⟨(0.096,0.638,0.99); 0.675,0.2, 0.23⟩	⟨(0.038,0.45,0.901); 0.7,0.15, 0.2⟩	…	⟨(0.994,0.825,0.659); 0.875,0.1, 0.1⟩

Table 8: Crisp Evaluation Matrix

	A1	A2	A3	A4	B1	B2	…	F1	F2	F3	G1	G2	G3
Company 1	0.59708	0.50577	0.37252	0.64020	0.36431	0.32495	…	0.71670	0.28558	0.33604	0.78219	0.73973	0.46672
Company 2	0.49348	0.42727	0.57815	0.79660	0.32154	0.65168	…	0.72738	0.25231	0.33604	0.73672	0.69531	0.39744
Company 3	0.63917	0.67507	0.69131	0.66567	0.43550	0.45347	…	0.71670	0.40111	0.52102	0.58052	0.48615	0.34233
Company 4	0.70182	0.38793	0.66507	0.72577	0.34632	0.36052	…	0.68751	0.52324	0.71330	0.59427	0.54268	0.52990
Company 5	0.48720	0.40808	0.59677	0.67318	0.30154	0.32495	…	0.68751	0.28558	0.40886	0.73672	0.69531	0.82862

Table 9: Normalized Evaluation Matrix

	A1	A2	A3	…	G1	G2	G3
Company 1	0.0405045	0.029064	0.015767	…	0.069514	0.062172	0.024749
Company 2	0.0283441	0.021248	0.038905	…	0.063173	0.056269	0.018385
Company 3	0.0487870	0.05442	0.05707	…	0.040245	0.028223	0.013994
Company 4	0.0533448	0.016299	0.047904	…	0.038248	0.031895	0.030411
Company 5	0.0288259	0.020224	0.043251	…	0.065915	0.058712	0.083386
Best (f^+)	0.0283441	0.016299	0.015767	…	0.038248	0.028223	0.013994
Worst (f^-)	0.0533448	0.05442	0.05707	…	0.069514	0.062172	0.083386

Table 10: VIKOR Method Results

Alternatives	S_i	Rank (S)	R_i	Rank (R)	Q_i (v=0.5)	Rank (Q)
Company 1	0.428163	2	0.0756318	2	0.27649	2
Company 2	0.482287	4	0.10960727	5	0.76355	4
Company 3	0.608172	5	0.10706501	3	0.97116	5
Company 4	0.438928	3	0.10860727	4	0.67077	3
Company 5	0.341968	1	0.06552981	1	0.00000	1

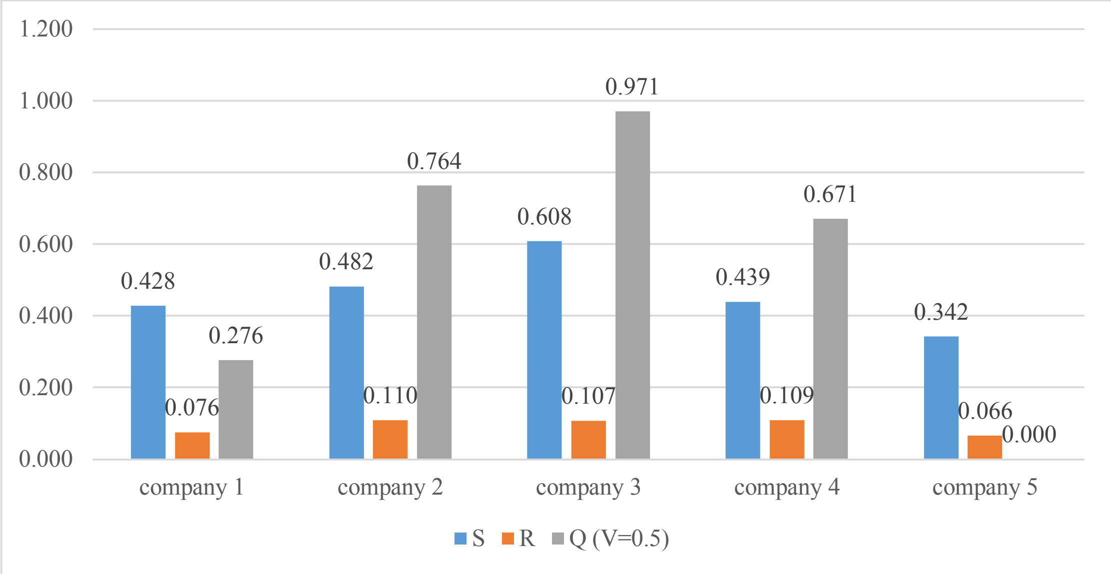

Figure 5: Ranking of 5 Companies

Table 11: The Ranking of Companies using different v values

	v=0		**v=0.25**		**v=0.5**		**v=0.75**		**v=1**	
Alternatives	Q_i	Rank	Q_i	Rank	Q_i	Rank	Q_i	Rank	Q_i	Rank
Company 1	0.22919	2	0.25284	2	0.27649	2	0.30014	2	0.32379	2
Company 2	1.00000	5	0.88178	4	0.76355	4	0.64533	4	0.52711	4
Company 3	0.94232	3	0.95674	5	0.97116	5	0.98558	5	1.00000	5
Company 4	0.97731	4	0.82404	3	0.67077	3	0.51750	3	0.36423	3
Company 5	0.00000	1	0.00000	1	0.00000	1	0.00000	1	0.00000	1

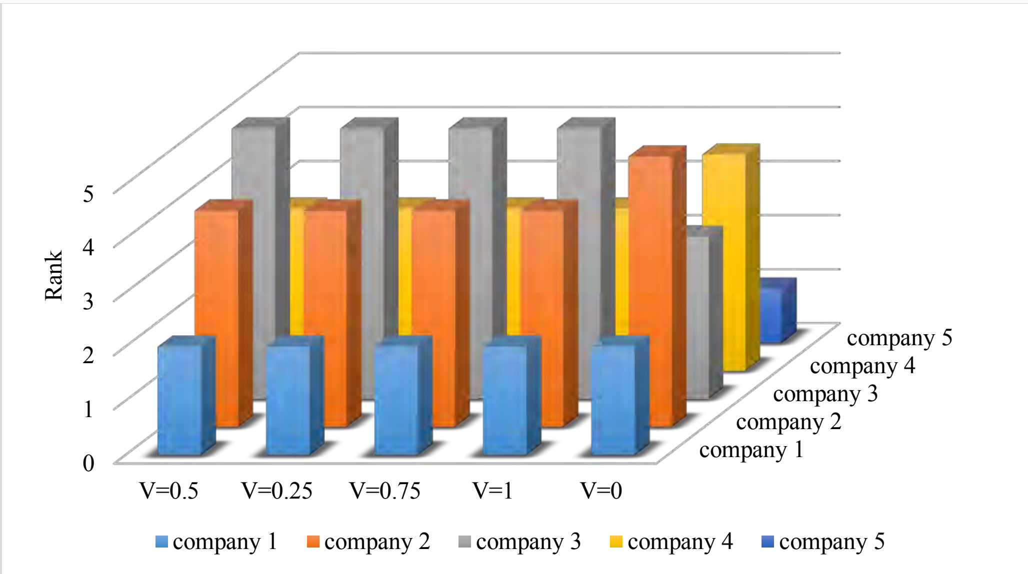

Figure 6: The Ranking of Companies using different v values

Table 12: The Ranking of Companies using different v values according to neutrosophic set theory

	v=0		v=0.25		v=0.5		v=0.75		v=1	
Alternatives	Q_i	Rank	Q_i	Rank	Q_i	Rank	Q_i	Rank	Q_i	Rank
Company 1	0.61344	3	0.58151	3	0.54959	3	0.51766	4	0.48574	4
Company 2	0.00000	1	0.00000	1	0.00000	1	0.00000	1	0.00000	1
Company 3	0.32677	2	0.28984	2	0.25291	2	0.21597	2	0.17904	2
Company 4	0.98666	4	0.79200	4	0.59735	4	0.40269	3	0.20803	3
Company 5	1.00000	5	1.00000	5	1.00000	5	1.00000	5	1.00000	5

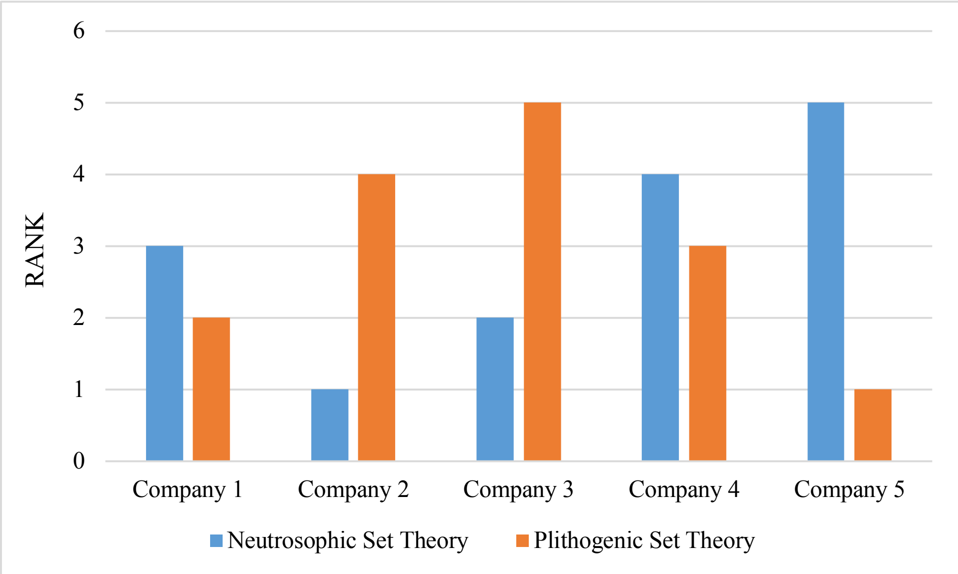

Figure 7: Comparison of Companies ranking using plithogenic set theory vs. neutrosophic set theory

6. Conclusion

This study formulates the problem of performance evaluation of IoT based supply chains as an MCDM by using a hybrid BWM and VIKOR methods. Most evaluation problems present insufficiencies from the existence of different decision-makers, alternatives, and criteria. That is why the proposed framework is based on the plithogenic set. The neutrosophic theory provides highly accurate results in vague, uncertain, inconsistent and incomplete information which exists in real life judgments. Meanwhile, it takes into account the truth, indeterminacy and falsity degrees for each evaluation.

VIKOR method helps evaluating the alternatives weights compared to the evaluation criteria. The weights of the criteria were calculated using the BWM. The proposed framework presents an accurate result which is useful in large scale problems with large criteria and alternatives. The first phase of this framework defines the evaluation information, such as a group of experts, criteria, and alternatives. The second phase comprises the calculation of weights by using BWM method based on the plithogenic set. The final phase, based also on the plithogenic set, ranks the alternatives according to their performance.

A case study of IoT based Ecommerce supply chain assessment validates the accuracy and reliability of the suggested framework. Based on the literature, there are 23 criteria that measure the performance of the five Ecommerce companies. According to three experts' judgments and using a proposed framework based on the BWM, the results show that the top three evaluation criteria are: competence of the system F1, hardware costs A1, implementation costs A3. These criteria have a higher priority to be considered in the evaluation of IoT based supply chains.

In this proposed framework, the weight of the decision-makers is not considered. So, decision-makers' weights should be considered to have a more accurate judgment in such evaluation processes. In addition, to prove the validity and to improve the accuracy of the proposed framework, it can be applied to other fields.

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Prof. Dr. Florentin Smarandache
Department of Mathematics and Science
University of New Mexico
705 Gurley Avenue
Gallup, NM 87301, USA
E-mail: smarand@unm.edu

Dr. Mohamed Abdel-Basset
Department of Computer Science
Faculty of Computers and Informatics
Zagazig University
Zagazig, Ash Sharqia 44519, Egypt
E-mail:mohamed.abdelbasset@fci.zu.edu

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